Can you beat Black-Scholes at delta hedging?

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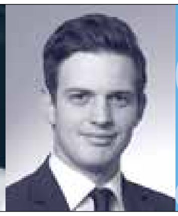

Carol Alexander and Arben Imeraj

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Can You Beat Black-Scholes at Delta Hedging?

Not a bad question given Black-Scholes assumes zero correlation between underlying price and volatility, the consequence of which is an entirely flat volatility surface.

The Black and Scholes (1973) (BS) model is pretty robust for delta-hedging European call and put options, which is surprising because it assumes a zero correlation between the underlying price and its volatility, and consequently an entirely flat volatility surface. Yet it is well known that there is a pronounced smile or skew in implied volatilities for most types of underlyings, equities in particular, but also commodities, currencies, and other tradable assets. So, accounting for a non-zero price-volatility correlation should be important for optimal delta hedging. But the BS delta is just the derivative of the option price with respect to the underlying price — because of the zero price-volatility correlation, there is no additional vega term in the total derivative.

Non-zero price-volatility correlation is a common feature of almost all stochastic volatility price processes. However, for modeling options on tradable assets, there is no theoretical difference between their price hedge ratios. That is, the first partial derivative (delta) of the option price with respect to the underlying price is model-free — the deltas are identical for all stochastic volatility models. Likewise, every model has exactly the same second partial derivative (gamma). The reason is that every tradable asset must be modeled using a scale-invariant (SI) price process (Alexander and Nogueira, 2007b), so all stochastic — and indeed all local volatility pricing models — for options on a tradable asset should fall into the SI class. It doesn't matter how complex the additional features such as jumps or Lévy processes are, if the delta or gamma hedge ratios differ between two different SI models, this is purely due to calibration error. Any two models for price processes of a tradable asset have theoretically identical partial price hedge ratios.

Alexander and Nogueira (2007a) use the slope of the volatility smile to imply an adjustment to the BS delta which is model-free, in the sense that it is the same for any SI model. This and other simple model-free, smile-implied adjustments to the BS delta are very popular with practitioners, as evidenced by numerous articles and forums.[1] But results from previous empirical studies of these BS-adjusted deltas are focused on equity options, and the results are mixed. Among others, Vähäamaa (2004) and Crépey (2004) both find that the BS model can only be outperformed during excessively volatile periods. Confirming this, Alexander *et al.* (2012) show that it is only possible to improve on the BS delta consistently using a regime-dependent framework where the size and sign of the delta adjustment depends on a Markov-switching model.

Almost all previous research on smile-adjusted delta hedges focuses on equity index options. But a recent paper by Alexander and Imeraj (2023) examines how bitcoin options fare with smile-adjusted dynamic delta hedging. The vast majority of bitcoin option trades are on the Deribit exchange. According to The Block, over 95% of bitcoin options trading is on this exchange, with about US$6 billion in open interest at the time of writing and a trading volume of well over US$16 billion in February 2023. Unlike traditional options, most of the trading in bitcoin options is on very short-term options with anything from a day to a few weeks to expiry. Indeed, Alexander and Imeraj (2023) find that Deribit bitcoin options with maturity between one and three months represent only 15% of total trading volume and roughly 85% of all trading volume on bitcoin options is on options that expire in 30 days or less.

For this reason, the paper only studies the delta hedging performance of synthetic options with 10, 20 and 30 days to expiry but moneyness has a fairly wide range of between 0.7 and 1.3. The hedging instrument also differs from the typical calendar futures contract used in traditional markets. Over 95% of trading in bitcoin futures is on perpetual contracts, and the Deribit bitcoin-US dollar perpetual is the ideal choice for delta hedging because the basis risk between this and the settlement index spot price is tiny. Another important difference between bitcoin options and those traded in traditional financial markets is the behavior of bitcoin implied volatility. Alexander (2022) shows that bitcoin options have very much higher implied volatilities than equity index options, in general. The shape of the curve also varies considerably over time. It can take a left or right hockey stick shape, a flat symmetric smile shape when bitcoin prices are range-bounded, a positive skew associated with large upward price jumps or a negative skew during downward trending markets. So, it is not obvious that the BS hedge ratio should be as difficult to beat when hedging bitcoin options, as it is

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for delta hedging options in traditional markets.

The study by Alexander and Imeraj (2023) examines a variety of smile-adjusted hedge ratios, which are summarized in the following:

$$\delta^{adj} = \begin{cases} \delta^{BS} & \text{(SS/BS)} \\ \delta^{BS} + \nu^{BS}\theta_m \dfrac{1}{F} & \text{(ST)} \\ \delta^{BS} - \nu^{BS}\theta_m \dfrac{m}{F} & \text{(SM/SI)} \\ \delta^{BS} + \nu^{BS}\theta_m \dfrac{m}{F} & \text{(MV)} \\ \delta^{BS} + \nu^{BS}\dfrac{1}{F\sqrt{\tau}}(a+b)\delta^{BS} + c(\delta^{BS})^2) & \text{(HW)} \end{cases}$$

Here δ^{BS} is the BS delta, ν^{BS} is the BS vega, $m = K/F$ is the moneyness of the option, K its strike, F the perpetual price, and θ_m is the slope of the smile in the moneyness metric. The different adjustments to the BS delta were originally motivated by Derman (1999), as regime-dependent deltas for hedging equity index options. He introduced three different "sticky models" to approximate the behavior of local volatility in different market regimes. The sticky strike model (SS) describes a trending market situation with zero price-volatility correlation. Each option has its own implied tree in which volatility depends on the strike. In a range-bound market Derman proposed the sticky moneyness (SM) model where an option's volatility depends on its moneyness — so one has to float between different trees to price an option accordingly as its moneyness changes. The only model that has a single tree for pricing all options is the sticky tree (ST) model which captures local volatility behavior when there is a strong negative correlation between volatility and the underlying price, as in a market crash. Again, the local volatility is a deterministic function, but it can be different at each node in the tree, and the same tree is used to price all options.

The smile-implied, scale-invariant delta of Alexander and Nogueira (2007a) is identical to the sticky moneyness (SM) approximation. In addition, there is the minimum variance (MV) delta δ^{mv}, i.e., the delta that minimizes the instantaneous variance of a delta-hedged portfolio. Lee (2001) shows that this MV hedge ratio has an adjustment of the same size as the (SM) smile-implied delta but with the opposite sign. Finally, proposed by Hull and White (2017) (HW) for currency options, we consider a smile adjustment derived from a regression of the absolute value of the daily profit and loss of a BS delta-hedged portfolio on a quadratic function of the BS delta.

The results examine dynamic delta hedging of 10-, 20- and 30-day options with a broad range of moneyness, rebalancing the hedge either daily or every eight hours. Because bitcoin prices move very much faster than those of traditional financial assets, the eight-hour frequency is often the base frequency of choice, rather than one day. We select the rebalancing times to coincide with the time of the perpetual funding payments, because volatility spikes are evident at these times as shown by Alexander *et al.* (2022). A comparison of hedging instruments shows that perpetuals offer significant improvements over calendar futures, especially for 20-day and 30-day options, and at both eight-hour and daily rebalancing frequencies. But little, if any, improvement on the BS hedge ratios is offered by using smile-adjusted deltas. For some out-of-the-money options the SI/SM delta can provide a significantly better hedge than a standard BS delta. For example, relative to the BS delta, efficiency gains of over 40% are possible for short-term out-of-the-money calls at times when the slope of the implied volatility curve is positive. The MV delta is sometimes also better than the BS delta, but only for at-the-money options, where it coincides with the ST delta. No other smile-adjusted delta can consistently improve on the BS delta.

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ENDNOTE

1. See for instance, this recent CAIA article, another one on medium, and several quantitative finance forums such as risklatte and stackexchange.

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