US
UNIVERSITY OF SUSSEX

Sussex Research

Full characterization of the fractional Poisson process

Mauro Politi, Taisei Kaizoji, Enrico Scalas

Publication date
29-09-2011

Licence
This work is made available under the Copyright not evaluated licence and should only be used in accordance with that licence. For more information on the specific terms, consult the repository record for this item.

Citation for this work (American Psychological Association 7th edition)
Politi, M., Kaizoji, T., & Scalas, E. (2011). *Full characterization of the fractional Poisson process* (Version 1). University of Sussex. https://hdl.handle.net/10779/uos.23398856.v1

Published in
Europhysics Letters

Link to external publisher version
https://doi.org/10.1209/0295-5075/96/20004

Copyright and reuse:
This work was downloaded from Sussex Research Open (SRO). This document is made available in line with publisher policy and may differ from the published version. Please cite the published version where possible. Copyright and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners unless otherwise stated. For more information on this work, SRO or to report an issue, you can contact the repository administrators at sro@sussex.ac.uk. Discover more of the University's research at https://sussex.figshare.com/

Full characterization of the fractional Poisson process

Mauro Politi and Taisei Kaizoji
SSRI & Department of Economics and Business,
International Christian University, 3-10-2 Osawa, Mitaka, Tokyo, 181-8585 Japan.

Enrico Scalas
Dipartimento di Scienze e Tecnologie Avanzate,
Università del Piemonte Orientale "Amedeo Avogadro",
Viale T. Michel 11, 15121 Alessandria, Italy.
(Dated: April 22, 2011)

The fractional Poisson process (FPP) is a counting process with independent and identically distributed inter-event times following the Mittag-Leffler distribution. This process is very useful in several fields of applied and theoretical physics including models for anomalous diffusion. Contrary to the well-known Poisson process, the fractional Poisson process does not have stationary and independent increments. It is not a Lévy process and it is not a Markov process. In this letter, we present formulae for its finite-dimensional distribution functions, fully characterizing the process. These exact analytical results are compared to Monte Carlo simulations.

arXiv:1104.4234v1 [math.PR] 21 Apr 2011

From a loose mathematical point of view, counting processes $N(A)$ are stochastic processes that count the random number of points in a set A. They are used in many fields of physics and other applied sciences. In this letter, we will consider one-dimensional real sets with the physical meaning of time intervals. The points will be incoming events whose duration is much smaller than the inter-event or inter-arrival waiting time. For instance, counts from a Geiger-Müller counter can be described in this way. The number of counts, $N(\Delta t)$, in a given time interval Δt is known to follow the Poisson distribution

$$\mathbb{P}(N(\Delta t) = n) = \exp(-\lambda \Delta t)\frac{(\lambda \Delta t)^n}{n!}, \qquad (1)$$

where λ is the constant rate of arrival of ionizing particles. Together with the assumption of independent and stationary increments, Eq. (1) is sufficient to define the *homogeneous* Poisson process. Curiously, one of the first occurrences of this process in the scientific literature was connected to the number of casualties by horse kicks in the Prussian army cavalry [1]. The Poisson process is strictly related to the exponential distribution. The inter-arrival times τ_i identically follow the exponential distribution and are independent random variables. This means that the Poisson process is a prototypical *renewal* process. A justification for the ubiquity of the Poisson process has to do with its relationship with the binomial distribution. Suppose that the time interval of interest $(t, t + \Delta t)$ is divided into n equally spaced sub-intervals. Further assume that a counting event appears in such a sub-interval with probability p and does not appear with probability $1 - p$. Then, $\mathbb{P}(N(\Delta t) = k) = \mathrm{Bin}(k; p, n)$ is a binomial distribution of parameters p and n and the expected number of events in the time interval is given by $\mathbb{E}[N(\Delta t)] = np$. If this expected number is kept constant for $n \to \infty$, the binomial distribution converges to the Poisson distribution of parameter $\lambda = \mathbb{E}[N(\Delta t)]/\Delta t$, while, in the meantime, $p \to 0$. However, it can be shown that many counting processes with non-stationary increments converge to the Poisson process after a transient period. It is sufficient to require that they are renewal process (i.e. they have independent and identically distributed (iid) inter-arrival times) and that $\mathbb{E}(\tau_i) < \infty$. In other words, many counting processes with non-independent and non-stationary increments behave as the Poisson process if observed long after the transient period.

In recent times, it has been shown that heavy-tailed distributed inter-arrival times (for which $\mathbb{E}(\tau_i) = \infty$) do play a role in many phenomena such as blinking nanodots [2, 3], human dynamics [4, 5] and the related inter-trade times in financial markets [6, 7].

Among the counting processes with non-stationary increments, the so-called *fractional Poisson process* [8], $N_\beta(t)$, is particularly important because it is the thinning limit of counting processes related to renewal processes with power-law distributed inter-arrival times [9, 10]. Moreover, it can be used to approximate anomalous diffusion ruled by space-time fractional diffusion equations [9, 11–16]. It is a straightforward generalization of the Poisson process defined as follows. Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed positive random variables with the meaning of inter-arrival times and let their common cumulative distribution function (cdf) be

$$F_\tau(t) = \mathbb{P}(\tau \leq t) = 1 - E_\beta(-t^\beta), \qquad (2)$$

where $E_\beta(-t^\beta)$ is the one-parameter Mittag-Leffler function, $E_\beta(z)$, defined in the complex plane as

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta + 1)} \qquad (3)$$

evaluated in the point $z = -t^\beta$ and with the prescription $0 < \beta \leq 1$. In equation (3), $\Gamma(\cdot)$ is Euler's Gamma

function. The sequence of the *epochs*, $\{T_n\}_{n=1}^{\infty}$, is given by the sums of the inter-arrival times

$$T_n = \sum_{i=1}^{n} \tau_i. \tag{4}$$

The epochs represent the times in which events arrive or occur. Let $f_\tau(t) = dF_\tau(t)/dt$ denote the probability density function (pdf) of the inter-arrival times, then the probability density function of the n-th epoch is simply given by the n-fold convolution of $f_\tau(t)$, written as $f_\tau^{*n}(t)$. In Ref. [10], it is shown that

$$f_{T_n}(t) = f_\tau^{*n}(t) = \beta \frac{t^{n\beta-1}}{(n-1)!} E_\beta^{(n)}(-t^\beta), \tag{5}$$

where $E_\beta^{(n)}(-t^\beta)$ is the n-th derivative of $E_\beta(z)$ evaluated in $z = -t^\beta$. The counting process $N_\beta(t)$ counts the number of epochs (events) up to time t, assuming that $T_0 = 0$ is an epoch as well, or, in other words, that the process begins from a renewal point. This assumption will be used all over this paper. $N_\beta(t)$ is given by

$$N_\beta(t) = \max\{n : \ T_n \leq t\}. \tag{6}$$

In Ref. [9], the fractional Poisson distribution is derived and it is given by

$$\mathbb{P}(N_\beta(t) = n) = \frac{t^{\beta n}}{n!} E_\beta^{(n)}(-t^\beta). \tag{7}$$

Eq. (7) coincides with the Poisson distribution of parameter $\lambda = 1$ for $\beta = 1$. In principle, equations (3) and (7) can be directly used to derive the fractional Poisson distribution, but convergence of the series is slow. Fortunately, in a recent paper, Beghin and Orsingher proved that

$$E_\beta^{(n)}(-t^\beta) = \frac{n!}{t^{\beta n}} \int_0^\infty F_{S_\beta}(t;u) \left[\frac{\exp(-u)u^{n-1}}{(n-1)!} - \frac{\exp(-u)u^n}{n!} \right] du, \tag{8}$$

where $F_{S_\beta}(t;u)$ is the cdf of a stable random variable $S_\beta(\nu, \gamma, \delta)$ with index β, skewness parameter $\nu = 1$, scale parameter $\gamma = (u \cos \pi\beta/2)^{1/\beta}$ and location $\delta = 0$ [17]. The integral in equation (8) can be evaluated numerically and Fig. 11 shows $\mathbb{P}(N_\beta(t) = n)$ for three different values of β. The Monte Carlo simulation of the fractional Poisson process is based on the algorithm presented in equation (20) of Ref. [14].

As a consequence of Kolmogorov's extension theorem, in order to fully characterize the stochastic process $N_\beta(t)$, one has to derive its finite dimensional distributions. A further requirement on the process' paths uniquely determines the process, namely that they are right-continuous step functions with left limits [18]. The finite-dimensional

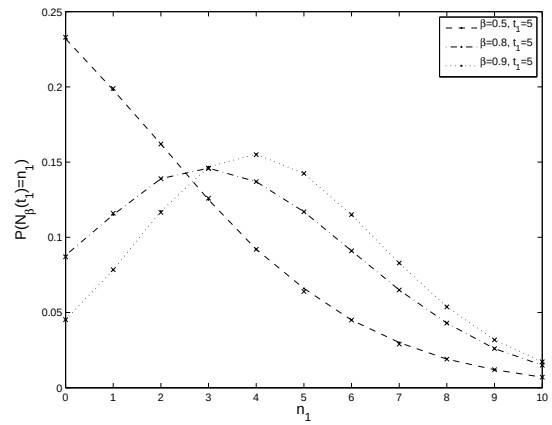

Figure 1: $P(N_\beta(T_1) = n_1)$ as function of n_1 for three different values of β. The crosses are estimations obtained from 10^5 Monte Carlo samples and the lines are given to guide the eye.

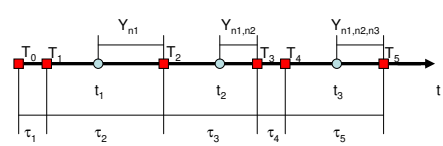

Figure 2: (Color online) Pictorial illustration of the random variables used in the text. The light blue dots represent the observation points t_1, t_2 and t_3. The red squares are the epochs $T_0 = 0, T_1, \ldots, T_5$. The conditional residual life-time is the time elapsed between t_i and the next epoch T_{n_i+1}. It depends on previous values of n_i, this is the number of events between 0 and t_i, with the event at $t = T_0 = 0$ not considered. Here, we have $n_1 = 1$, $n_2 = 2$ and $n_3 = 4$. All the equations in this paper can be derived by analyzing this figure.

distributions are the multivariate probability distribution functions $\mathbb{P}(N_\beta(t_1)=n_1, N_\beta(t_2)=n_2,\ldots,N_\beta(t_k)=n_k)$ with $t_1<t_2<\ldots<t_k$ and $n_1\le n_2\le\ldots\le n_k$. We have already given the formula for the one-point functions in Eq. (7). The general finite dimensional distribution can be computed observing that the event $\{N_\beta(t_1)=n_1, N_\beta(t_2)=n_2,\ldots,N_\beta(t_k)=n_k\}$ is equivalent to $\{0<T_{n_1}<t_1, T_{n_1+1}>t_1, t_1<T_{n_2}<t_2, T_{n_2+1}>t_2,\ldots,t_{k-1}<T_{n_k}<t_k, T_{n_k+1}>t_k\}$. Therefore, we find

$$\begin{aligned}&\mathbb{P}(N_\beta(t_1)=n_1, N_\beta(t_2)=n_2,\ldots,N_\beta(t_k)=n_k)=\\&\mathbb{P}(0<T_{n_1}<t_1, T_{n_1+1}>t_1, t_1<T_{n_2}<t_2, T_{n_2+1}>t_2,\\&\ldots,t_{k-1}<T_{n_k}<t_k, T_{n_k+1}>t_k)=\\&\int_0^{t_1}du_1 f_\tau^{*n_1}(u_1)\int_{t_1-u_1}^{\infty}du_2 f_\tau(u_2)\\&\int_{t_1-u_1-u_2}^{t_2-u_1-u_2}du_3 f_\tau^{*(n_2-n_1-1)}(u_3)\int_{t_2-u_1-u_2-u_3}^{\infty}du_4 f_\tau(u_4)\\&\cdots\int_{t_{k-1}-\sum_{i=1}^{2k-2}u_i}^{t_k-\sum_{i=1}^{2k-2}u_i}du_{2k-1} f_\tau^{*(n_k-n_{k-1}-1)}(u_{2k-1})\\&\left[1-F_\tau\left(t_k-\sum_{i=1}^{2k-1}u_i\right)\right].\end{aligned}\tag{9}$$

For instance, the two point function is given by

$$\begin{aligned}&\mathbb{P}(N_\beta(t_1)=n_1, N_\beta(t_2)=n_2)=\\&\mathbb{P}(0<T_{n_1}<t_1, T_{n_1+1}>t_1, t_1<T_{n_2}<t_2, T_{n_2+1}>t_2)=\\&\int_0^{t_1}du_1 f_\tau^{*n_1}(u_1)\int_{t_1-u_1}^{\infty}du_2 f_\tau(u_2)\\&\int_{t_1-u_1-u_2}^{t_2-u_1-u_2}du_3 f_\tau^{*(n_2-n_1-1)}(u_3)\\&[1-F_\tau(t_2-u_1-u_2-u_3)].\end{aligned}\tag{10}$$

Let us focus on the two-point case for the sake of illustration. As $N_\beta(t)$ is a counting process, one has $\mathbb{P}(N_\beta(t_1)=n_1, N_\beta(t_2)=n_2)=\mathbb{P}(N_\beta(t_1)=n_1, N_\beta(t_2)-N_\beta(t_1)=n_2-n_1)$ and, as a consequence of the definition of conditional probability

$$\begin{aligned}&\mathbb{P}(N_\beta(t_1)=n_1, N_\beta(t_2)-N_\beta(t_1)=n_2-n_1)=\\&\mathbb{P}(N_\beta(t_2)-N_\beta(t_1)=n_2-n_1|N_\beta(t_1)=n_1)\times\\&\times\mathbb{P}(N_\beta(t_1)=n_1).\end{aligned}\tag{11}$$

For $\beta=1$, when the fractional Poisson process coincides with the standard Poisson process, the increments are iid random variables and one has

$$\begin{aligned}&\mathbb{P}(N_1(t_2)-N_1(t_1)=n_2-n_1|N_1(t_1)=n_1)=\\&\mathbb{P}(N_1(t_2)-N_1(t_1)=n_2-n_1)=\\&\exp(-(t_2-t_1))\frac{(t_2-t_1)^{(n_2-n_1)}}{(n_2-n_1)!}.\end{aligned}\tag{12}$$

On the contrary, for $0<\beta<1$, the increment $N_\beta(t_2)-N_\beta(t_1)$ and $N_\beta(t_1)$ are not independent. Note that $N_\beta(t_1)$ can be seen as an increment as $N_\beta(0)=0$ by definition. However from Eq. (11), the conditional probability of having n_2-n_1 epochs in the interval (t_1,t_2) conditional on the observation of n_1 epochs in the interval $(0,t_1)$ can be written as a ratio of two finite dimensional distribution:

$$\begin{aligned}&\mathbb{P}(N_\beta(t_2)-N_\beta(t_1)=n_2-n_1|N_\beta(t_1)=n_1)=\\&\frac{\mathbb{P}(N_\beta(t_1)=n_1, N_\beta(t_2)=n_2)}{\mathbb{P}(N_\beta(t_1)=n_1)}.\end{aligned}\tag{13}$$

This probability can be evaluated by means of an alternative method, more appealing for a direct and practical understanding of the dependence structure. Let

$$Y_{n_1}\stackrel{\text{def}}{=}[T_{n_1+1}-t_1|N_\beta(t_1)=n_1]\tag{14}$$

denote the residual lifetime at time t_1 (that is the time to the next epoch or renewal) conditional on $N_\beta(t_1)=n_1$. With reference to Fig. 2, one can see that the conditional probability $\mathbb{P}(N_\beta(t_2)-N_\beta(t_1)=n_2-n_1|N_\beta(t_1)=n_1)$ is given by the following convolution integral for $n_2-n_1\ge 1$

$$\begin{aligned}&\mathbb{P}(N_\beta(t_2)-N_\beta(t_1)=n_2-n_1|N_\beta(t_1)=n_1)=\\&\int_0^{t_2-t_1}\mathbb{P}(N_\beta(t_2-t_1-y)=n_2-n_1-1)f_{Y_{n_1}}(y)\,dy,\end{aligned}\tag{15}$$

where $f_{Y_{n_1}}(t)$ is the pdf of Y_{n_1}. In the case $n_2-n_1=0$, one has

$$\mathbb{P}(N_\beta(t_2)-N_\beta(t_1)=0|N_\beta(t_1)=n_1)=1-F_{Y_{n_1}}(t_2-t_1)\tag{16}$$

where $F_{Y_{n_1}}(y)$ is the cdf of Y_{n_1}. The distribution of the conditional residual lifetime Y_{n_1} can be evaluated in several ways. For instance, one can notice that it can be decomposed as follows

$$Y_{n_1}=\tilde{\tau}_{n_1+1}+U_{n_1}\tag{17}$$

where U_{n_1} is defined as

$$U_{n_1}\stackrel{\text{def}}{=}[T_{n_1}|N_\beta(t_1)=n_1],\tag{18}$$

and is the position of the last epoch before t_1 conditional on $N_\beta(t_1)=n_1$, and

$$\tilde{\tau}_{n_1+1}\stackrel{\text{def}}{=}[\tau_{n_1+1}-t_1|T_{n_1+1}>t_1]\tag{19}$$

is the difference between τ_{n_1+1} and t_1 conditional on $T_{n_1+1}>t_1$. The pdf of U_{n_1} is given by the following

chain of equalities

$$\begin{aligned} f_{U_{n_1}}(t)dt =& \mathbb{P}(T_{n_1} \in dt | N_\beta(t_1) = n_1) \\ =& \mathbb{P}(T_{n_1} \in dt | T_{n_1} < t_1, T_{n_1} + \tau_{n_1+1} > t_1) \\ =& \mathbb{P}(T_{n_1} \in dt | T_{n_1} < t_1, \tau_{n_1+1} > t_1 - T_{n_1}) \\ \stackrel{\star}{=}& \frac{\mathbb{P}(T_{n_1} \in dt) \int_{t_1-t}^{\infty} \mathbb{P}(\tau_{n_1+1} \in dw)}{\mathbb{P}(T_{n_1} < t_1, \tau_{n_1+1} > t_1 - T_{n_1})} \\ \stackrel{*}{=}& \frac{f_\tau^{*n_1}(t)[1 - F_\tau(t_1 - t)]dt}{\int_0^{t_1} du f_\tau^{*n_1}(u)[1 - F_\tau(t_1 - u)]}, \end{aligned} \tag{20}$$

where we used the independence between T_{n_1} and τ_{n_1+1} ($\star$) and $f_{T_{n_1}}(x) = f_\tau^{*n_1}(x)$ ($*$). The pdf of $\tilde{\tau}_{n_1+1}$ is

$$\begin{aligned} f_{\tilde{\tau}_{n_1+1}}(t|U_{n_1})dt =& \mathbb{P}(\tau_{n_1+1} - t_1 \in dt | T_{n_1+1} > t_1) \\ =& \frac{\mathbb{P}(\tau_{n_1+1} \in dt + t_1)}{\mathbb{P}(\tau_{n_1+1} > t_1 - U_{n_1})} \\ =& \frac{f_\tau(t + t_1)dt}{1 - F_\tau(t_1 - U_{n_1})}. \end{aligned} \tag{21}$$

From Eq. (17), one can write that

$$f_{Y_{n_1}}(t) = \int_0^{t_1} f_{\tilde{\tau}_{n_1+1}}(t - u|u) f_{U_{n_1}}(u)du \tag{22}$$

and this equation leads to

$$f_{Y_{n_1}}(t) = \frac{\int_0^{t_1} du f_\tau^{*n_1}(u) f_\tau(t + t_1 - u)}{\int_0^{t_1} du f_\tau^{*n_1}(u)[1 - F_\tau(t_1 - u)]} \tag{23}$$

that, together with Eq. (7), gives us the probability of the conditional increments in Eq. (15). Notice that, for $n_1 = 0$, one has $f_\tau^{*0}(u) = \delta(u)$ and Eq. (23) reduces to the familiar equation for the residual life-time pdf in the absence of previous renewals

$$f_{Y_0}(t) = \frac{f_\tau(t + t_1)}{1 - F_\tau(t_1)}. \tag{24}$$

This method can be applied to the general multidimensional case. As in Eq. (11) we can write

$$\begin{aligned} &\mathbb{P}(N_\beta(t_1) = n_1, \ldots, N_\beta(t_k) = n_k, N_\beta(t_{k+1}) = n_{k+1}) = \\ &\quad \mathbb{P}(N_\beta(t_{k+1}) - N_\beta(t_k) = n_{k+1} - n_k | \\ &\quad N_\beta(t_1) = n_1, \ldots, N_\beta(t_k) = n_k) \times \\ &\quad \times \mathbb{P}(N_\beta(t_1) = n_1, \ldots, N_\beta(t_k) = n_k) \end{aligned} \tag{25}$$

and the predictive probabilities can be evaluated as

$$\begin{aligned} &\mathbb{P}(N_\beta(t_{k+1}) - N_\beta(t_k) = n_{k+1} - n_k | \ldots \\ &\quad | N_\beta(t_1) = n_1, \ldots, N_\beta(t_k) = n_k) = \\ &\int_0^{t_{k+1}-t_k} \mathbb{P}(N_\beta(t_{k+1} - t_k - y) = n_{k+1} - n_k - 1) \times \\ &\quad \times f_{Y_{n_1,\ldots,n_k}}(y)dy, \end{aligned} \tag{26}$$

where we defined

$$Y_{n_1,\ldots,n_k} \stackrel{\text{def}}{=} [T_{n_k+1} - t_k | N_\beta(t_1) = n_1, \ldots, N_\beta(t_k) = n_k]. \tag{27}$$

Again, we can use a decomposition of $Y_{n_1,\ldots,n_k}$

$$Y_{n_1,\ldots,n_k} = \tilde{\tau}_{n_k+1} + U_{n_k}, \tag{28}$$

where

$$U_{n_k} \stackrel{\text{def}}{=} [T_{n_k} | N_\beta(t_1) = n_1, \ldots, N_\beta(t_k) = n_k], \tag{29}$$

and

$$\tilde{\tau}_{n_k+1} \stackrel{\text{def}}{=} [\tau_{n_k+1} - t_k | T_{n_k+1} > t_k]. \tag{30}$$

The difference with the two-point case is that $U_{n_1} = [T_{n_1} | N_\beta(t_1) = n_1] = [\sum_{i=1}^{n_1} \tau_i | N_\beta(t_1) = n_1]$ must be replaced by

$$U_{n_k} = t_{k-1} + Y_{n_1,\ldots,n_{k-1}} + \left[\sum_{i=n_{k-1}+1}^{n_k} \tau_i | N_\beta(t_k) = n_k \right]. \tag{31}$$

The time between t_{k-1} and the next renewal epoch is $Y_{n_1,\ldots,n_{k-1}}$ and it is independent from $\sum_{i=n_{k-1}+1}^{n_k} \tau_i$. Therefore, the convolution

$$q(n_1, \ldots, n_k; t) = f_{Y_{n_1,\ldots,n_{k-1}}} * f_\tau^{*(n_k - n_{k-1} - 1)}(t) \tag{32}$$

replaces $f_\tau^{*n_1}(t)$ in Eq. (20). This leads to

$$f_{U_{n_k}}(z) = \frac{q(n_1, \ldots, n_k; t + t_{k-1})[1 - F_\tau(t_k - t)]}{\int_{t_{k-1}}^{t_k} q(n_1, \ldots, n_k; u + t_{k-1})[1 - F_\tau(t_k - u)]du}. \tag{33}$$

On the other hand, $f_{\tilde{\tau}_{n_k+1}}(t)$ has the same functional form as $f_{\tilde{\tau}_{n_1+1}}(t)$ given in Eq. (21) with U_{n_k} replacing U_{n_1}. Therefore, $Y_{n_1,\ldots,n_k}$ has the following pdf

$$f_{Y_{n_1,\ldots,n_k}}(t) = \frac{\int_{t_{k-1}}^{t_k} du\, q(n_1, \ldots, n_k; u + t_{k-1}) f_\tau(t + t_k - u)}{\int_{t_{k-1}}^{t_k} du\, q(n_1, \ldots, n_k; u + t_{k-1})[1 - F_\tau(t + t_k - u)]}. \tag{34}$$

In practice, the random variable $Y_{n_1,\ldots,n_{k-1}}$ carries the memory of the observations made at times $t_1, \ldots, t_{k-1}$; the knowledge of $f_{Y_{n_1,\ldots,n_{k-1}}}$ allows the computation of $f_{Y_{n_1,\ldots,n_k}}$, and, via Eqs. (25) and (26), the $k+1$-dimensional distribution can be derived as well.

Figs. 3 and 4 compare the theoretical results of Eqs. (20), (23) and (24) with those of a Monte Carlo simulation based on the algorithm presented in equation (20) of Ref. [14].

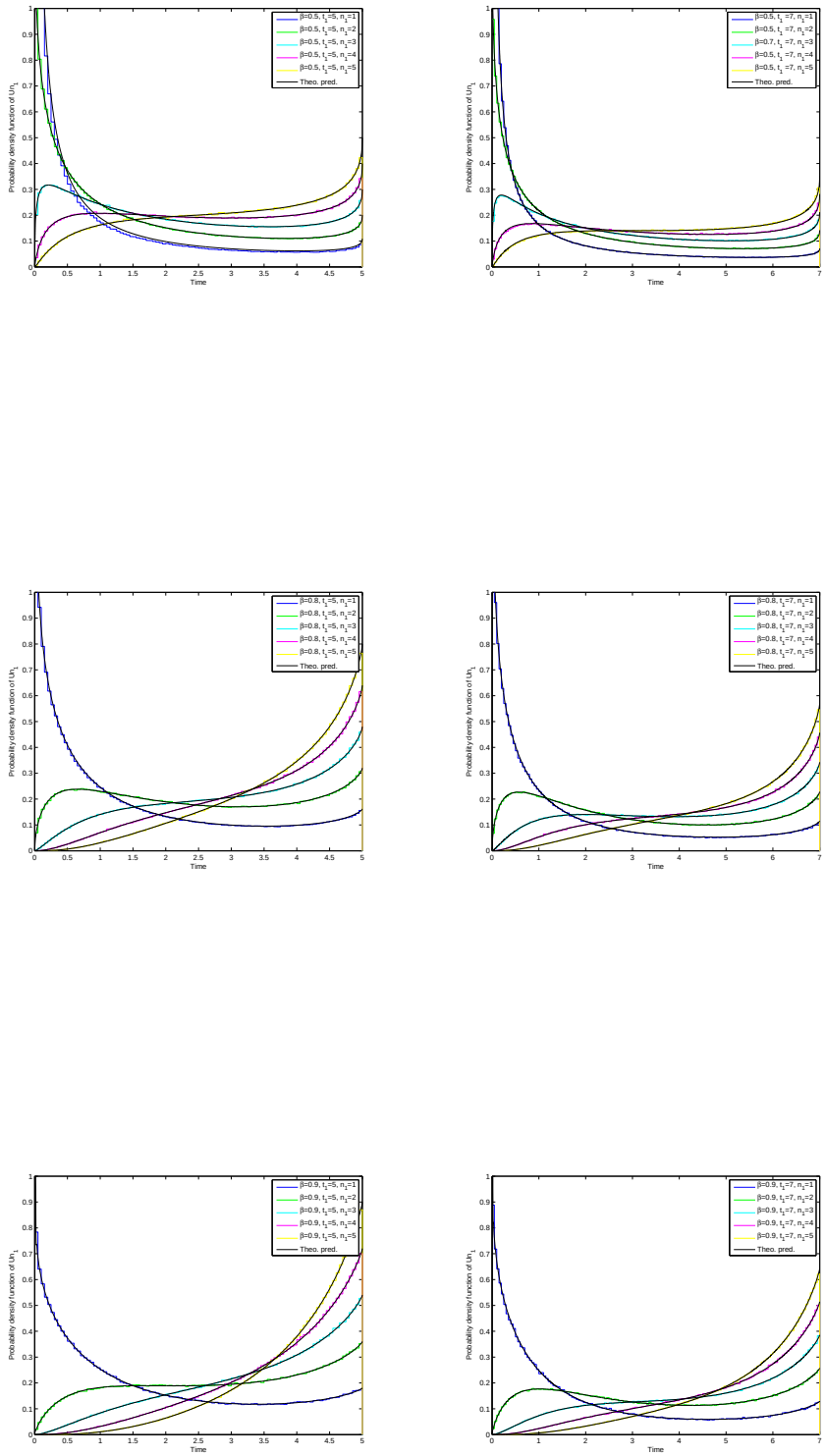

Figure 3: (Color online) Pdf of the random variable U_{n_1} as given in Eq. (20) (solid black lines) compared to Monte Carlo simulations (colored step lines) for three values of β and two different values of t_1. 10^7 different paths were simulated for each value of β and the bin width is 0.05. Time is in arbitrary units.

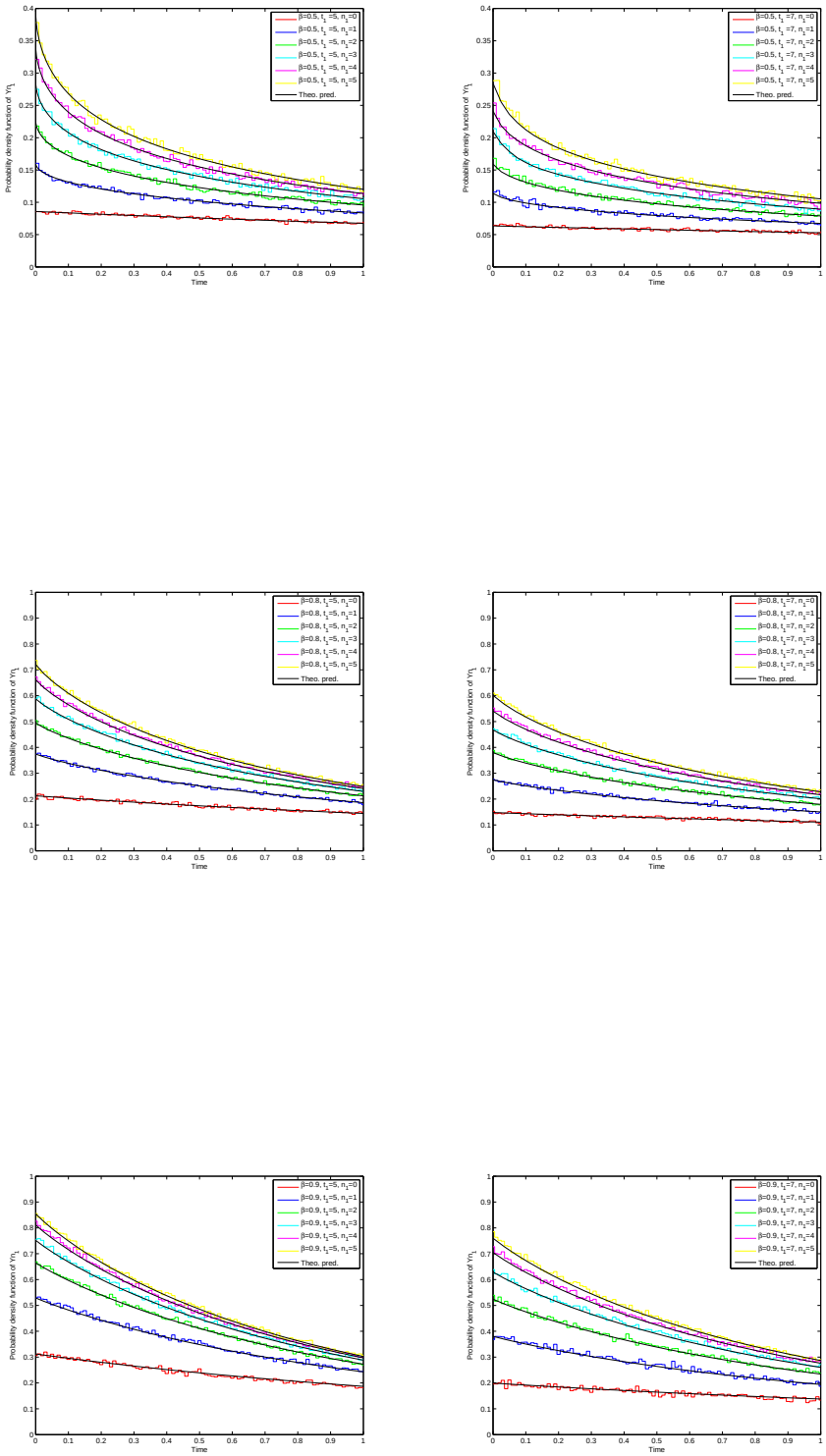

Figure 4: (Color online) Pdf of the random variable Y_{n_1} as given in Eqs. (23) and (24) (solid black lines) compared to Monte Carlo simulations (colored step lines) for three values of β and two different values of t_1. 10^7 different paths were simulated for each value of β and the bin width is 0.01. Time is in arbitrary units.

The Japanese Society for the Promotion of Science (grant N. PE09043) supported MP during his stay at the International Christian University in Tokyo, Japan.

[1] L. J. Bortkiewicz, *Das Gesetz der kleinen Zahlen* (B. G. Teubner, 1898).
[2] G. Margolin and E. Barkai, Physical Review Letters **94**, 080601 (2005).
[3] G. Margolin, V. Protasenko, M. Kuno, and E. Barkai, Journal of Physical Chemistry B **110**, 19053 (2006).
[4] A.-L. Barabási, Nature **435**, 207 (2005).
[5] A.-L. Barabási, *Bursts: The Hidden Pattern Behind Everything We Do* (Dutton, 2010).
[6] E. Scalas, R. Gorenflo, H. Luckock, F. Mainardi, M. Mantelli, and M. Raberto, Quantitative Finance **4**, 695 (2004).
[7] E. Scalas, T. Kaizoji, M. Kirchler, J. Huber, and A. Tedeschi, Physica A **366**, 463 (2006).
[8] N. Laskin, Communications in Nonlinear Science and Numerical Simulation **8**, 201 (2003).
[9] E. Scalas, R. Gorenflo, and F. Mainardi, Physical Review E **69**, 011107 (2004).
[10] F. Mainardi, R. Gorenflo, and E. Scalas, Vietnam Journal of Mathematics **32**, 53 (2004).
[11] R. Metzler and J. Klafter, Journal of Physics A: Mathematical and General **37**, R161 (2004).
[12] E. Heinsalu, M. Patriarca, I. Goychuk, G. Schmid, and P. Hänggi, Physical Review E **73**, 046133 (2006).
[13] M. Magdziarz, A. Weron, and K. Weron, Physical Review E **75**, 016708 (2007).
[14] D. Fulger, E. Scalas, and G. Germano, Physical Review E **77**, 021122 (2008).
[15] G. Germano, M. Politi, E. Scalas, and R. L. Schilling, Physical Review E **79**, 066102 (2009).
[16] M. M. Meerschaert, E. Nane, and P. Vellaisamy, `http://arxiv.org/abs/1007.5051` (2010).
[17] L. Beghin and E. Orsingher, Electronic Journal of Probability **14**, 1790 (2009).
[18] P. Billingsley, *Probability and Measure*, 2nd ed. (John Wiley & Sons, 1986).