Caps on Hermitian varieties and maximal curves

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Dedicated to Adriano Barlotti on the occasion of his 80-th birthday

Abstract

A lower bound for the size of a complete cap of the polar space $H(n, q^2)$ associated to the non-degenerate Hermitian variety $\mathcal{U}_n$ is given; this turns out to be sharp for even q when $n = 3$. Also, a family of caps of $H(n, q^2)$ is constructed from $\mathbf{F}_{q^2}$-maximal curves. Such caps are complete for q even, but not necessarily for q odd.

1 Introduction

Let $\mathcal{U}_n$ be the non-degenerate Hermitian variety of the n-dimensional projective space $\mathrm{PG}(n, q^2)$ coordinatised by the finite field $\mathbf{F}_{q^2}$ of square order q^2. An *ovoid* of the polar space $H(n, q^2)$ arising from the non-degenerate Hermitian variety $\mathcal{U}_n$ with $n \geq 3$ is defined to be a point set in $\mathcal{U}_n$ having exactly one common point with every generator of $\mathcal{U}_n$. For n even, $\mathcal{U}_n$ has no ovoid; see [24]. For n odd, the existence problem for ovoids of $\mathcal{U}_n$ has been solved so far only in the smallest case $n = 3$; see [20].

A natural generalization of an ovoid is a *cap* (also called a *partial ovoid*). A cap of $\mathcal{U}_n$ is a point set in $\mathcal{U}_n$ which has at most one common point with every generator of $\mathcal{U}_n$. Equivalently, a cap is a point set consisting of pairwise non–conjugate points of $\mathcal{U}_n$. A cap is called *complete* if it is not contained in a larger cap of $\mathcal{U}_n$.

The size of a cap is at most $q^n + 1$ for odd n and q^n for even n; equality holds if and only if the cap is an ovoid. The following upper bound for the size k of a cap different from an ovoid is due to Moorhouse [19]:

$$k \leq \left[\binom{p+n-1}{n}^2 - \binom{p+n-2}{n}^2 \right]^h + 1, \qquad q = p^h. \tag{1.1}$$

A lower bound for k is given in Section 2 by proving that $k \geq q^2 + 1$.

In this paper a family of caps of $\mathcal{U}_n$ that are not ovoids is constructed, and it is shown that they are complete provided that $n = 3$ and q is even. The construction relies on an interesting property of $\mathbf{F}_{q^2}$–maximal curves of $\mathrm{PG}(n, q^2)$ that is stated in §3: the $\mathbf{F}_{q^2}$–rational points of an $\mathbf{F}_{q^2}$-maximal curve naturally embedded in a Hermitian variety $\mathcal{U}_n$ are pairwise non-conjugate under the associated unitary polarity. Hence the set $\mathcal{X}(\mathbf{F}_{q^2})$ of all $\mathbf{F}_{q^2}$–rational points of an $\mathbf{F}_{q^2}$–maximal curve is a cap of $\mathcal{U}_n$. The main result is that $\mathcal{X}(\mathbf{F}_{q^2})$ is a complete cap for $n = 3$ and q even.

For $n = 3$ and q odd, there exist $\mathbf{F}_{q^2}$-maximal curves such that $\mathcal{X}(\mathbf{F}_{q^2})$ is a cap of size $\frac{1}{2}(q^3 - q)$ contained in an ovoid of $\mathcal{U}_3$; see Example 4.8.

2 A lower bound for the size of a complete cap of $\mathcal{U}_n$

A (non–degenerate) Hermitian variety $\mathcal{U}_n$ is defined as the set of all self–conjugate points of a non–degenerate unitary polarity of a projective space $\mathrm{PG}(n, q^2)$. Hermitian varieties of $\mathrm{PG}(n, q^2)$ are projectively equivalent, as they can be reduced to the canonical form

$$X_0^{q+1} + \ldots + X_n^{q+1} = 0$$

by a non-singular linear transformation of $\mathrm{PG}(n, q^2)$. A *generator* of $\mathcal{U}_n$ is defined to be a projective subspace of maximum dimension lying on $\mathcal{U}_n$, namely of dimension $\lfloor \frac{1}{2}(n-1) \rfloor$. General results on Hermitian varieties are due to Segre [22]; see also [15], [14], [16]. Here, some basic facts from [16, Section 23.2] are recalled. Let μ_n denote the number of points on $\mathcal{U}_n$.

Result 2.1 (1) $\mu_n = (q^{n+1} + (-1)^n)(q^n - (-1)^n)/(q^2 - 1)$.

(2) *For any point $P \in \mathcal{U}_n$, the number of lines through P and contained in $\mathcal{U}_n$ is equal to μ_{n-2}.*

(3) *The tangent hyperplane at $P \in \mathcal{U}_n$ meets $\mathcal{U}_n$ in $q^2\mu_{n-2} + 1$ points.*

Now we give a lower bound for the size of complete caps which does not depend on n.

Theorem 2.2 *The size k of a complete cap of $\mathcal{U}_n$ satisfies $k \geq q^2 + 1$.*

Proof The assertion is true for ovoids. Let $\mathcal{K}$ be a complete cap of $\mathcal{U}_n$ that is not an ovoid. Take a generator H of $\mathcal{U}_n$ disjoint from $\mathcal{K}$. For any point $P \in \mathcal{K}$, the tangent hyperplane Π_P to $\mathcal{U}_n$ at P does not contain H. In fact, some point of H is not conjugate to P, as H is a projective subspace of maximum dimension contained in $\mathcal{U}_n$. This implies that $\Pi_P \cap H$ is a hyperplane $H(P)$ of H. As $\mathcal{K}$ is a complete cap of $\mathcal{U}_n$, the projective subspaces $H(P)$ cover H as P ranges over $\mathcal{K}$. Since H is a projective space of dimension $r = \lfloor \frac{1}{2}(n-1) \rfloor$, this yields

$$1 + q^2 + \ldots + q^{2r} \leq k(1 + q^2 + \ldots + q^{2(r-1)}).$$

Hence

$$k \geq q^2 + 1/(1 + q^2 + \ldots + q^{2(r-1)}).$$

Since k is an integer, this is only possible for $k \geq q^2 + 1$. □

The above lower bound is sharp for $n = 3$ and even q; see Example 3.6 and Theorem 4.1 with $g = 0$. For the classification of transitive ovoids when $n = 3$ and q is even, see [5]. It is not known whether the lower bound remains true for $n > 3$ or for $n = 3$ and arbitrary odd q. To the best of our knowledge, the smallest complete cap of $\mathcal{U}_n$ for q is that described in the following theorem.

Theorem 2.3 *Let α be a plane of* $\mathrm{PG}(n, q^2)$ *which meets $\mathcal{U}_n$ in a non-degenerate Hermitian curve $\mathcal{U}_2$. Then $\mathcal{U}_2$ is a complete cap of $\mathcal{U}_n$ of size $q^3 + 1$.*

Proof First, $\mathcal{U}_2$ is a cap of $\mathcal{U}_n$. Let $A \in \mathcal{U}_n$ be any point. The tangent hyperplane Π_A to $\mathcal{U}$ at A either contains α or meets it in a line ℓ. It turns out in both cases that Π_A has a common point with $\mathcal{U}_2$, whence the assertion follows. □

3 Hermitian varieties and maximal curves

In algebraic geometry in positive characteristic the Hermitian variety is defined to be the hypersurface $\overline{\mathcal{U}}_n$ of homogeneous equation

$$X_0^{q+1} + \ldots + X_n^{q+1} = 0,$$

viewed as an algebraic variety in $\mathrm{PG}(n, \overline{\mathbf{F}})$ where $\overline{\mathbf{F}}$ is the algebraic closure of $\mathbf{F}_{q^2}$. Points of $\mathcal{U}_n$ are the points of $\overline{\mathcal{U}}_n$ with coordinates in $\mathbf{F}_{q^2}$, usually called $\mathbf{F}_{q^2}$-rational points of $\overline{\mathcal{U}}_n$. For a point $A = (a_0, a_1, \ldots, a_n)$ of $\overline{\mathcal{U}}_n$, the tangent hyperplane to $\overline{\mathcal{U}}_n$ at A has equation

$$a_0^q X_0 + a_1^q X_1 + \ldots + a_n^q X_n = 0.$$

In this paper, the term *algebraic curve defined over* $\mathbf{F}_{q^2}$ stands for a projective, geometrically irreducible, non-singular algebraic curve $\mathcal{X}$ of $\mathrm{PG}(n, q^2)$ viewed as a curve of $\mathrm{PG}(n, \overline{\mathbf{F}})$. Further, $\mathcal{X}(\mathbf{F}_{q^{2i}})$ denotes the set of points of $\mathcal{X}$ with all coordinates in $\mathbf{F}_{q^{2i}}$, called $\mathbf{F}_{q^{2i}}$-rational points of $\mathcal{X}$. For a point $P = (x_0, \ldots, x_n)$ of $\mathcal{X}$, the Frobenius image of P is defined to be the point $\Phi(P) = (x_0^{q^2}, \ldots, x_n^{q^2})$. Then $P = \Phi(P)$ if and only if $P \in \mathcal{X}(\mathbf{F}_{q^2})$.

An algebraic curve $\mathcal{X}$ defined over $\mathbf{F}_{q^2}$ is called $\mathbf{F}_{q^2}$*-maximal* if the number N_{q^2} of its $\mathbf{F}_{q^2}$-rational points attains the Hasse–Weil upper bound, namely $N_{q^2} = q^2 + 1 + 2gq$, where g denotes the genus of $\mathcal{X}$. In recent years, $\mathbf{F}_{q^2}$-maximal curves have been the subject of numerous papers; a motivation for their study comes from coding theory based on algebraic curves having many points over a finite field. Here, only results on maximal curves which play a role in the present investigation are gathered.

Result 3.1 (Natural embedding theorem [17]) *Up to* $\mathbf{F}_{q^2}$*-isomorphism, the* $\mathbf{F}_{q^2}$*-maximal curves of* $\mathrm{PG}(n, q^2)$ *are the algebraic curves defined over* $\mathbf{F}_{q^2}$ *of degree* $q + 1$ *and contained in the non-degenerate Hermitian variety* $\overline{\mathcal{U}}_n$.

Remark 3.2 The $\mathbf{F}_{q^2}$-maximality of $\mathcal{X}$ implies that $(q+1)P \equiv qQ + \Phi(Q)$ for every $Q \in \mathcal{X}$, and the natural embedding arises from the smallest linear series Σ containing all such divisors. Apart from some exceptions, Σ is complete and hence $\Sigma = |(q+1)P_0|$ for any $P_0 \in \mathbf{F}_{q^2}$. By the Riemann–Roch theorem, $\dim \Sigma = q + 1 - g + i$ where i is the index of speciality. In many situations, for instance when $q+1 > 2g-2$, we have $i = 0$, and hence $\dim \Sigma = q + 1 - g$. With our notation, $n = \dim \Sigma$.

This, together with some more results from [17], gives the following.

Result 3.3 *Let* $\mathcal{X}$ *be an* $\mathbf{F}_{q^2}$*-maximal curve naturally embedded in* $\overline{\mathcal{U}}_n$*. For a point* $P \in \mathcal{X}$*, let* Π_P *be the tangent hyperplane to* $\overline{\mathcal{U}}_n$ *at* P*. Then* Π_P *coincides with the hyper-osculating hyperplane to* $\mathcal{X}$ *at* P*, and*

$$\Pi_P \cap \mathcal{X} = \begin{cases} \{P\} & \text{for } P \in \mathcal{X}(\mathbf{F}_{q^2}), \\ \{P, \Phi(P)\} & \text{for } P \in \mathcal{X} \setminus \mathcal{X}(\mathbf{F}_{q^2}). \end{cases} \tag{3.1}$$

More precisely, for the intersection divisor D cut out on $\mathcal{X}$ by Π_P,

$$D = \begin{cases} (q+1)P & \text{for } P \in \mathcal{X}(\mathbf{F}_{q^2}), \\ qP + \Phi(P) & \text{for } P \in \mathcal{X}\backslash\mathcal{X}(\mathbf{F}_{q^2}). \end{cases} \tag{3.2}$$

Theorem 3.4 *Let $\mathcal{X}$ be an $\mathbf{F}_{q^2}$-maximal curve naturally embedded in $\overline{\mathcal{U}}_n$. For a point $A \in \mathcal{U}_n\backslash\mathcal{X}$, let Π_A be the tangent hyperplane to $\mathcal{U}_n$ at A. If $n = 3$ and q is even, then Π_A has a common point with $\mathcal{X}(\mathbf{F}_{q^2})$.*

Proof Let ℓ be a line of $\mathcal{U}_n$. Then ℓ, viewed as a line of $\mathrm{PG}(n, \overline{\mathbf{F}})$, is contained in $\overline{\mathcal{U}}_n$. Let $Q \in \ell \cap \mathcal{X}$; then it must be shown that $Q \in \mathcal{X}(\mathbf{F}_{q^4})$.

Assume, on the contrary, that $Q \in \mathcal{X}(\mathbf{F}_{q^{2i}})$ with $i \geq 3$. Then the three points $Q, \Phi(Q), \Phi(\Phi(Q))$ are distinct points of $\mathcal{X}$. Since ℓ is defined over $\mathbf{F}_{q^2}$, so ℓ contains not only Q but also $\Phi(Q)$ and $\Phi(\Phi(Q))$. By (3.1), the hyperosculating hyperplane Π_Q to $\mathcal{X}$ at Q contains $\Phi(Q)$, and hence Π_Q contains the line ℓ. But then Π_Q must contain $\Phi(\Phi(Q))$, contradicting (3.1).

Assume now that $Q \in \mathcal{X}(\mathbf{F}_{q^4})$. The previous argument also shows that ℓ contains both Q and $\Phi(Q)$ but no more points from $\mathcal{X}$. Also, ℓ cannot contain more than one point from $\mathcal{X}(\mathbf{F}_{q^2})$, again by (3.1). Hence, if $\ell \cap \mathcal{X}$ is non–trivial, then either $\ell \cap \mathcal{X}$ is a single $\mathbf{F}_{q^2}$-rational point or $\ell \cap \mathcal{X}$ consists of two distinct points, Frobenius images of each other, both in $\mathcal{X}(\mathbf{F}_{q^4})\backslash\mathcal{X}(\mathbf{F}_{q^2})$.

Let $Q \in \overline{\mathcal{U}}_n$ be any point in $\Pi_A \cap \mathcal{X}$. Then the line ℓ through A and Q is contained in $\overline{\mathcal{U}}_n$. Now, assume that $n = 3$; then such a line is contained in $\mathcal{U}_n$. By the above assertions, the points in $\Pi_A \cap \mathcal{X}$ are $\mathbf{F}_{q^4}$-rational points of $\mathcal{X}$. For a point $Q \in \mathcal{X}$, let $I(\mathcal{X}, \Pi_A; Q)$ denote the intersection multiplicity of $\mathcal{X}$ and Π_A at Q. By Bézout's theorem, $\sum_Q I(\mathcal{X}, \Pi_A; Q) = q + 1$ where Q ranges over all points of $\mathcal{X}$. Write

$$\sum_Q I(\mathcal{X}, \Pi_A; Q) = {\sum_Q}' I(\mathcal{X}, \Pi_A; Q) + {\sum_Q}'' I(\mathcal{X}, \Pi_A; Q),$$

where the summation $\sum'$ is over $\mathcal{X}(\mathbf{F}_{q^2})$ while $\sum''$ is over $\mathcal{X}(\mathbf{F}_{q^4})\backslash\mathcal{X}(\mathbf{F}_{q^2})$. Since both Π_A and $\mathcal{X}$ are defined over $\mathbf{F}_{q^2}$,

$$I(\mathcal{X}, \Pi_A; Q) = I(\mathcal{X}, \Pi_A; \Phi(Q)).$$

Hence $\sum''_Q I(\mathcal{X}, \Pi_A; Q) \equiv q + 1 \pmod 2$. For q even, this implies that $I(\mathcal{X}, \Pi_A; Q) > 0$ for at least one point $Q \in \mathcal{X}(\mathbf{F}_{q^2})$, whence the assertion follows. □

Remark 3.5 Theorem 3.4 might not extend to $n > 3$. For a point $A \in \mathcal{U}_n$, let $Q \in \overline{\mathcal{U}}_n$ be a point other than A in the tangent hyperplane Π_A of $\mathcal{U}_n$ at A. If $n = 3$, then the line ℓ through A and Q is $\mathbf{F}_{q^2}$-rational. But this assertion does not hold true for $n > 3$.

In fact, let $\mathcal{U}_n$ be given in its canonical form

$$X_0^q X_n + X_0 X_n^q + X_1^{q+1} + \ldots + X_{n-1}^{q+1} = 0.$$

It may be assumed that $A = (0, \ldots, 0, 1)$. Then Π_A has equation $X_0 = 0$ and $Q = (0, a_1, \ldots, a_{n-1}, 1)$ with $a_1^{q+1} + \ldots + a_{n-1}^{q+1} = 0$. The line ℓ is $\mathbf{F}_{q^2}$-rational if and only if $\Phi(Q)$ also lies on ℓ. This happens when $a_i^{q^2} = \lambda a_i$, $i = 1, \ldots, n-1$, for a suitable element $\lambda \in \overline{\mathbf{F}}$, or, equivalently, when $a_i^{q^2-1} = a_j^{q^2-1}$ for all i, j with $1 \leq i, j \leq n-1$ and $a_i, a_j \neq 0$. Now, $a_1^{q+1} = -a_2^{q+1}$ implies $(a_1^{q+1})^{q-1} = (a_2^{q+1})^{q-1}$, whence the assertion follows for $n = 3$. Unfortunately, as soon as $n > 3$, $a_1^{q+1} + \ldots + a_{n-1}^{q+1} = 0$ does not imply $a_i^{q^2-1} = a_j^{q^2-1}$ for any i, j with $1 \leq i, j \leq n-1$ and $a_i, a_j \neq 0$. Thus the assertion is not valid for $n > 3$.

The following example illustrates property (3.1).

Example 3.6 Still with q even, write the equation of $\mathcal{U}_3$ in the form

$$X_0^q X_3 + X_0 X_3^q = X_1^{q+1} + X_2^{q+1}.$$

The rational algebraic curve $\mathcal{X}$ of degree $q+1$, consisting of all points

$$A(t) = \{(1, t, t^q, t^{q+1}) \mid t \in \overline{\mathbf{F}}\}$$

together with the point $A(\infty) = (0, 0, 0, 1)$, lies on $\mathcal{U}_3$. The morphism

$$(1, t) \to (1, t, t^q, t^{q+1})$$

is a natural embedding. We note that the tangent hyperplane $\Pi_{A(t)}$ to $\overline{\mathcal{U}}_3$ at $A(t)$ has equation

$$t^{q(q+1)} X_0 + X_3 + t^q X_1 + t^{q^2} X_2 = 0.$$

To show that (3.1) holds for $A(t)$, it is necessary to check that the equation

$$t^{q(q+1)} + u^{q+1} + t^q u + t^{q^2} u^q = 0$$

has only two solutions in u, namely $u = t$ and $u = t^{q^2}$. Replacing u by $v + t$, the equation becomes $v^{q+1} + v^q t + t^{q^2} v^q = 0$. For $v \neq 0$, that is, for $u \neq t$, this implies $v = t^{q^2} + t$, proving the assertion. For $A(\infty)$, the tangent hyperplane $\Pi_{A(\infty)}$ has equation $X_0 = 0$. Hence it does not meet $\mathcal{X}$ outside $A(\infty)$, showing that (3.1) also holds for $A(\infty)$.

4 Caps of the Hermitian variety arising from maximal curves

From the results stated in Section 3 we deduce the following theorem.

Theorem 4.1 *Let $\mathcal{X}$ be an $\mathbf{F}_{q^2}$-maximal curve naturally embedded in $\overline{\mathcal{U}}_n$. Then*

(i) *$\mathcal{X}(\mathbf{F}_{q^2})$ is a cap of $\mathcal{U}_n$ of size $q^2+1+2gq$;*

(ii) *when q is even and $n=3$, such a cap is complete.*

Proof Let $P \in \mathcal{X}(\mathbf{F}_{q^2})$. By (3.1), no further point from $\mathcal{X}$ is in Π_P. Hence no point in $\mathcal{X}(\mathbf{F}_{q^2})$ is conjugate to P. This shows that $\mathcal{X}(\mathbf{F}_{q^2})$ is a cap of $\mathcal{U}_n$ whose size is equal to $q^2+1+2gq$ by the $\mathbf{F}_{q^2}$-maximality of $\mathcal{X}$. Completeness for even q and $n=3$ follows from Theorem 3.4. □

In applying Theorem 4.1 it is essential to have information on the spectrum of the genera g of $\mathbf{F}_{q^2}$-maximal curves. However, it would be inappropriate in the present paper to discuss the spectrum in all details; so we shall content ourselves with a summary of the relevant results in characteristic 2. For this reason, q will denote a power of 2 in the rest of the paper, apart from Example 4.8.

Result 4.2 (1) *The lower limit of the spectrum of genera is* 0, *which is only attained by rational algebraic curves.*

(2) *The upper limit of the spectrum is $\frac{1}{2}(q^2-q)$, which is only attained by the Hermitian curve over $\mathbf{F}_{q^2}$; see* [23, Proposition V.3.3].

Result 4.3 [1, 10, 18]

(1) *The second largest value in the spectrum of genera is $\frac{1}{4}(q^2-2q)$, which is only attained by Example* 4.5.

(2) *In the interval $[\frac{1}{8}(q^2-4q+3), \frac{1}{4}(q^2-q)]$, there are* 12 *known examples.*

Result 4.4 [18] *The third largest value in the spectrum is $\lfloor\frac{1}{6}(q^2-q+4)\rfloor$. Examples* 4.6 *and* 4.7 *are the only known examples with this genus.*

Example 4.5 ([9]). The absolutely irreducible plane curve $\mathcal{C}$ with equation

$$y + y^2 + \ldots + y^{q/2} + x^{q+1} = 0$$

has genus $\frac{1}{4}q(q-2)$. A non-singular model $\mathcal{X}$ of $\mathcal{C}$ is the $\mathbf{F}_{q^2}$-maximal curve defined by the morphism $\pi : \mathcal{C} \to \mathrm{PG}(3, q^2)$ with coordinate functions

$$f_0 = 1, \; f_1 = x, \; f_2 = y, \; f_3 = x^2.$$

The curve $\mathcal{X}$ lies on the Hermitian variety $\overline{\mathcal{U}}_3$ with equation

$$X_2^q X_0 + X_2 X_0^q + X_1^{q+1} + X_3^{q+1} = 0.$$

Also, $\mathcal{X}$ lies on the quadric cone with equation $X_3 X_0 = X_1^2$. The size of the corresponding complete cap $\mathcal{X}(\mathbf{F}_{q^2})$ of $\mathcal{U}_3$ is $\frac{1}{2}(q^3+2)$.

Example 4.6 ([7, Theorem 2.1.(IV)(2)]) Let $q \equiv 2 \pmod 3$. The absolutely irreducible plane curve $\mathcal{C}$ with equation $x^{(q+1)/3} + x^{2(q+1)/3} + y^{q+1} = 0$ has genus $g = \frac{1}{6}(q^2 - q + 4)$. A non-singular model $\mathcal{X}$ of $\mathcal{C}$ is the $\mathbf{F}_{q^2}$-maximal curve defined by the morphism $\pi : \mathcal{C} \to \mathrm{PG}(3, q^2)$ with coordinate functions

$$f_0 = x, \; f_1 = x^2, \; f_2 = y^3, \; f_3 = xy.$$

The curve $\mathcal{X}$ lies on the Hermitian variety $\overline{\mathcal{U}}_3$ given by the usual canonical equation

$$X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0.$$

Also, $\mathcal{X}$ lies on the cubic surface with equation

$$X_3^3 + w^3 X_0 X_1 X_2 = 0$$

with $w^{q+1} = -3$. The size of the corresponding complete cap $\mathcal{X}(\mathbf{F}_{q^2})$ of $\mathcal{U}_3$ is $\frac{1}{3}(q^3 + 2q^2 + 4q + 3)$.

Example 4.7 ([6, §6]) A similar but non-isomorphic example is given in [6]. Again, assume that $q \equiv 2 \pmod 3$. The absolutely irreducible plane curve $\mathcal{C}$ with equation

$$yx^{(q-2)/3} + y^q + x^{(2q-1)/3} = 0$$

has genus $\frac{1}{6}(q^2 - q - 2)$. A non-singular model $\mathcal{X}$ of $\mathcal{C}$ is the $\mathbf{F}_{q^2}$-maximal curve defined by the morphism $\pi : \mathcal{C} \to \mathrm{PG}(3, q^2)$ with coordinate functions

$$f_0 = x, \; f_1 = x^2, \; f_2 = y^3, \; f_3 = -3xy.$$

The curve $\mathcal{X}$ lies on the Hermitian variety Σ_{q+1} with equation

$$X_0^q X_1 + X_1^q X_2 + X_2^q X_0 - 3X_3^{q+1} = 0.$$

Also, $\mathcal{X}$ is contained in the cubic surface with equation

$$X_3^3 + 27X_0X_1X_2 = 0.$$

It is worth noting that Σ_{q+1} is projectively equivalent to $\mathcal{U}_3$ in $\mathrm{PG}(3, q^6)$ but not in $\mathrm{PG}(3, q^2)$. Nevertheless, the projective transformation taking Σ_{q+1} to $\mathcal{U}_3$ maps $\mathcal{X}$ to an $\mathbf{F}_{q^2}$-maximal curve lying on $\mathcal{U}_3$. The size of the corresponding complete cap $\mathcal{X}(\mathbf{F}_{q^2})$ of $\mathcal{U}_3$ is $\frac{1}{3}(q^3 + 2q^2 - 2q + 3)$.

We end the paper with an example for q odd which shows that assertion (ii) in Theorem 4.1 does not hold for q odd.

Example 4.8 Let q be odd and let $\mathcal{C}(\mathbf{F}_{q^2})$ be the absolutely irreducible plane curve with equation

$$y^q + y + x^{(q+1)/2} = 0;$$

it has genus $\frac{1}{4}(q-1)^2$. A non-singular model $\mathcal{X}$ of $\mathcal{C}$ is the $\mathbf{F}_{q^2}$-maximal curve defined by the morphism $\pi : \mathcal{C} \to \mathrm{PG}(3, q^2)$ with coordinate functions

$$f_0 = 1, \;\; f_1 = x, \;\; f_2 = y, \;\; f_3 = y^2.$$

The curve $\mathcal{X}$ lies on the Hermitian surface $\mathcal{U}_3$ with equation

$$X_3^q X_0 + X_3 X_0^q + 2X_2^{q+1} - X_1^{q+1} = 0.$$

Also, $\mathcal{C}$ lies on the quadric cone $\mathcal{Q}$ with equation $X_2^2 - X_0X_3 = 0$. The size of the corresponding cap $\mathcal{K}$ of $\mathcal{U}_3$ is $q^2 + 1 + \frac{1}{2}q(q-1)^2 = \frac{1}{2}(q^3 + q + 2)$. The cap $\mathcal{K}$ is incomplete, since it is contained in an ovoid of $\mathcal{U}_3$; see [13].

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