University of Sussex

A University of Sussex PhD thesis

Available online via Sussex Research Online:

http://sro.sussex.ac.uk/

This thesis is protected by copyright which belongs to the author.

This thesis cannot be reproduced or quoted extensively from without first obtaining permission in writing from the Author

The content must not be changed in any way or sold commercially in any format or medium without the formal permission of the Author

When referring to this work, full bibliographic details including the author, title, awarding institution and date of the thesis must be given

Please visit Sussex Research Online for more information and further details



Observational prospects for gravitational waves from cosmological first order phase transitions at LISA

Chloe Gowling

Submitted for the degree of Doctor of Philosophy University of Sussex 30th September 2022

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

The work presented in this thesis has been completed in collaboration with Guillaume Boileau, Nelson Christensen, Mark Hindmarsh, Deanna C. Hooper, Renate Meyer and Jesús Torrado Cacho, and is comprised of the following papers.

• C. Gowling and M. Hindmarsh, Observational prospects for phase transitions at LISA: Fisher matrix analysis, 2106.05984

For this paper I performed the analytic and numerical calculations for the Fisher matrix, conducted the exploration of the spectral parameter space that corresponds to the region of thermodynamic parameter space we considered. I also created a function that interpolates and extrapolates the GW suppression factor in the sound shell model. I wrote the first draft of the paper and generated all the figures. Edits and revisions were made collaboratively with Mark Hindmarsh who supervised throughout this project.

 G. Boileau, N. Christensen, C. Gowling, M. Hindmarsh and R. Meyer, Prospects for LISA to detect a gravitational-wave background from first order phase transitions, 2209.13277

In this work I corroborated the presented results for the DIC and the Fisher matrix results exploring the impact of different galactic foregrounds. I wrote a first draft for the introduction, the description of the SGWB from a first order phase transition, and for the DIC results section. Edits were made collaboratively with Guillaume Boileau, Nelson Christensen, Mark Hindmarsh, and Renate Meyer. • C. Gowling, M. Hindmarsh, D.C. Hooper and J. Torrado, *Reconstructing physical parameters from template gravitational wave spectra at LISA: first order phase transitions*, 2209.13551

For this project I constructed the thermodynamic parameter reconstruction method, generated the minimizer fit array, and performed all the MCMC inferences and data analysis of the results. I wrote the first draft and created all the figures for the paper. Edits were then made with Mark Hindmarsh, Deanna C. Hooper, and Jesús Torrado all of whom supervised throughout the project.

Signature:

Chloe Gowling

UNIVERSITY OF SUSSEX

CHLOE GOWLING, DOCTOR OF PHILOSOPHY

OBSERVATIONAL PROSPECTS FOR FIRST ORDER PHASE TRANSITIONS AT LISA

SUMMARY

The era of gravitational wave (GW) observations began with the ground-breaking detection at the Laser interferometer Gravitational Wave detector (LIGO), we are now exploring more of the GW power spectrum. Upcoming space-based detectors such as the Laser Interferometer Space Antenna (LISA) will probe, for the first time, the millihertz window of the GW spectrum with the hope of detecting astrophysical and cosmological sources. The source of interest in this thesis is a cosmological first order phase transition at the electroweak scale. Detecting these GWs from the early universe would provide the opportunity to delve further back in the history of the universe than ever before.

In theories beyond the Standard Model a cosmological first order phase transition occurs when, below a critical temperature, bubbles of stable phase spontaneously nucleate in the surrounding metastable phase. These bubbles expand, collide, and merge until only the stable phase remains. This phenomenon produces a stochastic gravitational wave background (SGWB) that, once scaled due to the expansion of the universe, peaks in the frequency band corresponding to LISA. In the Standard Model this transition is a crossover and no GWs are produced. Thus, a detection of such a SGWB would be a discovery of new physics.

In the following thesis, using the latest analytical model of a SGWB from a first order phase transition (the Sound Shell Model) and information from cutting edge simulations I explore LISA's ability to perform parameter estimation on key phase transition parameters. I focus on two different parameterisations of the phase transition SGWB, the thermodynamic and spectral. The thermodynamic parameters are derived from the physics of the phase transition and consequently are related to the beyond standard model theory they inhabit. The spectral parameterisation is computationally cheaper than the thermodynamic parameterisation, which is advantageous when performing MCMC simulations. However, the connection from the spectral to thermodynamic parameters is an unanswered question. Here, we present a method for reconstructing the thermodynamic parameters from the spectral parameters. I use statistical methods including Fisher analysis and Markov chain Monte Carlo (MCMC) simulations to estimate parameter uncertainties on the two parameterisations. The impact of astrophysical foregrounds on resolving a SGWB for a first order phase transition are also investigated.

Acknowledgements

First and foremost a tremendous thank you to my supervisor Mark Hindmarsh I am forever grateful for your careful explanations, seemingly never ending patience, and your generosity with your time and wisdom. It has been a privilege to have you as a supervisor.

I would also like to thank my collaborators Guillaume Boileau, Nelson Christensen, Deanna Hooper, Renate Meyer, and Jesús Torrado it was a pleasure to work with you.

To the 5A3 crew, thanks for all the laughs and a space where there are no stupid questions: Fëanor Reuben Ares, Charlie Cresswell-Hogg, Nahzaan Riyaz, Mike Soughton and Ryan Wood. Thanks to the wider TPP community and to Jeff Kost, Folkert Kuipers, Yannick Kluth, Bradley Garland, Manuel Reichert and Nathan Sherrill.

Charlie a special thank you to you, a constant in the pre and post pandemic times. It has meant so much to have a friend like you to walk the PhD path with me and all the highs and lows that it entails. Fëanor Reuben Ares, a fellow phase transitioner it been a pleasure to have you as a PhD brother, your encouragement and friendship kept me going.

I was fortunate enough to spend an extended amount of time at the Helsinki during my PhD. I am grateful for all the lovely people at Helsinki who made me feel welcome: Daniel Cutting, Oliver Gould, Joonas Hirvonen, Deanna Hooper, Anna Kormu, Tiina Minkkinen, Saga Säppi, Satumaaria Sukuvaara, Essi Vilhonen - thanks for the Hyvia Entetia. A huge thank you to David Weir for your incredible generosity with your time, sharing your knowledge of cluster computing, and for saving my belongings from storage when the pandemic hit.

Beth Wells - the best cheerleader anyone could every hope for. I am grateful for the friendship of Jade Booth, Amy-Rose Collins, Hannah Eves and Laura Macdonald.

To my sister Hannah thank you for your unwavering belief in me. To Hannah and

Jay - thank you for bringing Archie into the world, he has brought me so much joy as I have written this thesis. Mum and Dad, I can't thank you enough for everything you have done and continue to do for me.

Michael, you brighten all of my days, thank you for everything.

Contents

Li	st of	Tables	xii									
Li	st of	Figures x	ix									
1	Intr	Introduction										
	1.1	Thesis outline	2									
2	Ger	eral Relativity, Cosmology and Gravitational waves	5									
	2.1	General Relativity	5									
	2.2	Modern Cosmology	7									
	2.3	Gravitational waves	10									
		2.3.1 Plane wave solutions in linearized GR	11									
		2.3.2 Gravitational waves in curved spacetime	14									
3	Gra	vitational waves from a cosmological first order phase transition	18									
	3.1	Bubble nucleation	20									
	3.2	Bubble wall speed	21									
	3.3	Fluid shell profiles	22									
	3.4	Energy	26									
	3.5	Gravitational waves from phase transitions	27									
		3.5.1 Scalar field contribution	27									
		3.5.2 Sound waves	28									
		3.5.3 Shocks and turbulence	28									
		3.5.4 Sound shell model	30									
		3.5.5 Double broken power law	32									
4	Gra	vitational waves: Experiments	34									
	4.1	LISA	35									

	4.2	Time	delay interferometry	36										
	4.3	Detec	tor response function	42										
	4.4	Signal	to noise ratio	43										
5	Dat	a anal	ysis methods	44										
	5.1	Fisher	$^{\circ}$ Matrix	45										
	5.2	Marko	ov chain Monte Carlo inference	46										
6	Observational prospects for phase transitions at LISA: Fisher matrix													
	ana	lysis		49										
	6.1	Introd	luction	50										
	6.2	Gravit	tational waves from a first order cosmological phase transition	54										
		6.2.1	Cosmological phase transitions	54										
		6.2.2	Gravitational waves from a first order phase transition	56										
		6.2.3	Double broken power law	60										
	6.3	Noise	model	63										
		6.3.1	LISA sensitivity curve	63										
		6.3.2	Extragalactic compact binaries	64										
		6.3.3	Unresolved galactic compact binaries	65										
		6.3.4	Signal-to-noise ratio	66										
	6.4	Fisher	matrix analysis	66										
		6.4.1	LISA likelihood model	67										
		6.4.2	Principal components	71										
	6.5	Fisher	matrix calculation and relative uncertainties	71										
		6.5.1	Double broken power law model	71										
		6.5.2	Sound shell model	73										
		6.5.3	Principal component analysis	76										
		6.5.4	Sound shell model with fixed nucleation temperature	78										
	6.6	Discus	ssion	80										
	6.7	Apper	ndix	85										
		6.7.1	Kinetic energy suppression in the sound shell model	85										
		6.7.2	Broken power law approximations to the GW spectrum	86										
7	Pro	\mathbf{spects}	for LISA to detect a gravitational-wave background from first	5										
	ord	er pha	se transitions	88										
	7.1	Introd	uction	89										

	7.2	Model Components
		7.2.1 Cosmological background from a first order phase transition 93
		7.2.2 LISA noise model
		7.2.3 Double white dwarf foreground
		7.2.4 Extragalactic compact binary foreground
		7.2.5 Model illustration
		7.2.6 Simulation
		7.2.7 Fisher information and Deviance Information Criterion 100
	7.3	Results
		7.3.1 DIC results
		7.3.2 Fisher matrix and MCMC comparison
	7.4	Conclusions
	7.5	Appendix
		7.5.1 Parameter correlations
•	Doc	constructing physical parameters from templete gravitational wave
•	net	at reacting physical parameters from template gravitational wave
	8 1	Introduction 112
	8.2	SCWB from cosmological first order phase transition
	0.2	8.2.1 Gravitational wave power spectrum in the SSM
		8.2.2 Double broken power law 11
	8.3	LISA instrument noise model
	8.4	Parameter inference from mock LISA data 110
	0.1	8 4 1 Data model and likelihood 120
		84.2 Priors on thermodynamic parameters
		84.3 Initial priors on spectral parameters
		8.4.4 Markov chain Monte Carlo inference
	8.5	Reconstructing thermodynamic parameter posteriors
		8.5.1 Constructing the map between spectral and thermodynamic para-
		meters
		8.5.2 Induced prior on the spectral parameters
		8.5.3 Reconstruction of the thermodynamic parameters
		8.5.4 Properties of the reconstructed thermodynamic posterior
	8.6	Results
		8.6.1 Deflagration fiducial model

9	Conclusions															-	139			
	8.7	Conclu	isions									•			•	 •	•	 •	•	136
		8.6.2	Detor	ation	fidu	cial	mo	del			•				•	 •				132

List of Tables

7.1	Parameter values used in the data simulation described by Eq. 7.19, exclud-
	ing the four phase transition parameters
8.1	Ranges for the uniform priors on the thermodynamic parameters
8.2	Ranges for the uniform priors on the spectral parameters. $\ldots \ldots \ldots$
8.3	Regular grid of thermodynamic parameters used to construct the fit array.
	Notice that the scaling corresponds to the prior density in Table 8.1 124
8.4	Means and 68% credible intervals for the spectral parameters, deflagration
	fiducial model
8.5	Thermodynamic parameters for the fiducial deflagration model, and the
	thermodynamic parameters inferred from the MCMC samples. "Direct" uses
	chains sampled directly on the thermodynamic parameters, "reconstructed"
	uses chains sampled on the spectral parameters, and reconstructs the corres-
	ponding thermodynamic parameters using the method described in Section
	8.5. Values given are means and 68% confidence intervals
8.6	Means and 68% credible intervals for the spectral parameters, detonation
	fiducial model
8.7	Thermodynamic parameters for the fiducial detonation model, and the ther-
	modynamic parameters inferred from the MCMC samples. "Direct" uses
	chains sampled directly on the thermodynamic parameters, "reconstructed"
	uses chains sampled on the spectral parameters, and reconstructs the corres-
	ponding thermodynamic parameters using the method described in Section
	8.5. Values given are means and 68% credible intervals

List of Figures

The evolution of the effective potential $V(\phi)$ for a first order phase trans-	
ition. $T_{\rm c}$ is the critical temperature where two degenerate minima are	
present	20
Schematic diagram of the possible fluid profiles: deflagration (left), hybrid	
(middle) and detonation (right). The black circle marks the bubble wall and	
the black arrows show the radial direction of the wall speed $v_{\rm w}$. The speed	
of sound is $c_{\rm s}$, and the Chapman-Jouguet speed $c_{\rm J}$ is defined in Eq. 3.18.	
The regions of fluid with non-zero velocities in the universe frame are shown	
in the coloured shells, where yellow denotes small velocities and orange large	
fluid velocities. Credit: D. Cutting.	25
The fluid velocity v and the enthalpy w given as functions of the dimen-	
sionless coordinate $\xi = r/t$. From left to right the panels show: subsonic	
deflagration $v_{\rm w} = 0.55$, hybrid (supersonic deflagration) $v_{\rm w} = 0.7$ and det-	
onation $v_{\rm w} = 0.88$. The dashed line represent curves which must intersect	
the shock front. The dashed dotted lines denote the maximum fluid velocity	
behind the phase boundary. Here the parameters chosen represent fidu-	
cial models we go on to use in our data analysis. Figures generated using	
PTtools [4]	26
LISA instrument design schematics.	37
The paths taken by the laser used to construct the Michelson signal (left)	
and the time delay interferometry channels (right). Figure taken from $\left[5\right]$	37
The response function for the A, E, T TDI channels. The solid lines show	
the numerical approximation and the dashed lines show the analytic fits.	
Figure taken from [5].	42
	The evolution of the effective potential $V(\phi)$ for a first order phase transition. T_c is the critical temperature where two degenerate minima are present

- 6.1 Gravitational wave power spectra for a first order phase transition calculated using the sound shell model, Eq.(6.10). In each panel we vary one of the thermodynamic parameters $v_{\rm w}$ (wall speed), α (phase transition strength), r_* (Hubble-scaled bubble spacing) and $T_{\rm n}$ (nucleation temperature). Shown also in solid black is the LISA instrument noise given by the science requirements (SR) document sensitivity curve (Eq. (6.26), [6]). The dashed line shows the predicted foreground from extragalactic binaries, Eq. (6.27), along with a grey uncertainty band. The dash-dotted line shows the estimated foreground from unresolved galactic binaries, Eq. (6.29). Signalto-noise ratios for $T_{\rm n} = 100$ GeV and $r_* = 0.1, 0.01$ are given in Fig. 6.4.
- 6.2 The peak power today $\Omega_{p,0}^{ssm}$ and the peak frequency today $f_{p,0}^{ssm}$ calculated with the sound shell model, for a range of wall speeds, $0.4 < v_w < 0.9$, and phase transition strengths, $0.01 < \alpha < 0.5$, The Hubble-scaled mean bubble spacing $r_* = 0.1$ and nucleation temperature T_n . The turquoise dashed line shows the Jouguet speed Eq. (6.4).

59

62

67

- 6.3 The best fit spectral parameters from fitting the double broken power law model to the power spectra from the sound shell model Eq. (6.10). $\Omega_{p,0}$ is the peak power today with the Hubble-scaled mean bubble spacing $r_* = 0.1$ and $T_n = 100$ GeV, $f_{p,0}$ is the corresponding position of the peak (scaled using Eq.(6.18)), r_b is the ratio of the frequency positions of the two breaks in the spectrum and b the spectral slope of the power law between the two breaks. The turquoise dashed line is the Jouguet speed, Eq. (6.4). 62
- 6.4 The signal-to-noise ρ for different combinations of the wall speed v_w , phase transition strength α , Hubble-scaled mean bubble spacing r_* , with the nucleation temperature $T_n = 100$ GeV. In the left column the noise model includes the LISA instrument noise - Eq. (6.26), the foreground from unresolved stellar origin black hole binaries - Eq. (6.27). In the right hand column we also include the unresolved galactic binary foreground- Eq. (6.29). The turquoise dashed line shows the Jouguet detonation speed, the minimum speed of a detonation for each α , given in Eq. (6.4).

- 6.5 Coloured contours show relative uncertainties calculated from the Fisher matrix for the parameters of the double broken power law model (Eq. 7.4): peak power $\Omega_{p,0}$, peak frequency $f_{p,0}$, break ratio r_b and intermediate power law b. The line styles indicate the break ratio values r_b . The black lines show contours of signal-to-noise ratio $\rho = 20$ for different r_b , with the same line styles. The grey shaded area indicates the region where the peak signal power is above the combined instrumental noise and foregrounds. In the upper panel the noise model consists of the LISA instrument noise, Eq. (6.26), foreground from compact binaries, Eq. (6.27) and the galactic binary foreground is removed.
- 6.6 Contours of relative uncertainty in the thermodynamic parameters wall speed $v_{\rm w}$, transition strength α , scaled mean bubble spacing r_* and nucleation temperature $T_{\rm n}$. In each sub-figure, the upper and lower panels have Hubble-scaled bubble spacing r_* as annotated. In both panels $T_{\rm n} = 100$ GeV. The black solid line shows contours of signal-to-noise ratio ρ . The turquoise dashed line is the Jouguet speed, the minimum for a detonation.

72

75

77

78

- 6.7 Contours of standard deviation $(1/\sqrt{\lambda_n})$ for the principal components constructed from the eigenvectors of the Fisher matrix evaluated across the wall speed v_w and phase transition strength α parameter space. In each sub-figure, the upper and lower panels have Hubble-scaled bubble spacing r_* as annotated. In both panels $T_n = 100$ GeV. The black solid line shows contours of signal-to-noise ratio ρ . The turquoise dashed line is the Jouguet speed, the minimum for a detonation.

- 6.11 The mean squared relative deviation, $\delta_{\mathcal{P}}^2$ defined in Eq. (6.59), for two analytic fits to the sound shell model gravitational wave power spectrum. On the left is the single broken power law used by the LISA cosmological working group [8], given in Eq. (6.60). On the right is the general double broken power law given in Eq. (7.5). The turquoise dashed line is the Jouguet speed, Eq. (6.4), the minimum speed of the phase boundary in a detonation. Note the difference in the colour scales.
- 7.1 The LISA sensitivity curve [6, 9], black line, in terms of the dimensionless energy spectral density $\Omega_{\rm gw}(f)$. The DWD foreground models are also presented. The light and dark blue lines are respectively the Galactic foreground from the Lamberts *et al.* catalogue [10] and the analytic galactic foreground fit of Boileau *et al.* [11]. The red line is the Galactic confusion noise from Robson *et al.* [12]. The green line is the estimated extragalactic compact binary foreground from the LIGO-Virgo 02 data [13], while the yellow curve is estimation from the LIGO-Virgo 03 data [14]. The pink and orange lines are PT broken power law models with $\Omega_{\rm p} = 3 \times 10^{-11}$ and $\Omega_{\rm p} = 1 \times 10^{-10}$.

87

98

- xvii

- 7.4 Uncertainty estimates for the peak amplitude $\Omega_{\rm p}$ and the peak frequency $f_{\rm p}$, calculated with the Fisher information (continuous lines) and MCMC simulations (points). The solid line corresponds to a model of LISA instrument noise, phase transition signal, astrophysical background and a Robson *et al.* [12] DWD foreground model. The dashed lines are identical but instead consider the Lamberts *et al.* model for the DWD foreground [10]. The relative uncertainties as calculated from the Fisher matrix for $\Omega_{\rm p}$ when only the LISA noise and phase transition signal are considered are show in blue in Fig. 7.4a. In both cases $r_b = 0.4$, b = 1 and in Fig. 7.4a $f_{\rm p} = 1$ mHz. . . . 107
- 7.5 Uncertainty estimates in the break ratio r_b and the intermediate slope b, calculated with the Fisher information (continuous lines) and MCMC simulations (points). The model: LISA instrument noise, first order phase transition signal modelled as a double broken power law, astrophysical background and a Robson *et al.* [12] DWD foreground model. In both figures $f_p = 1 \text{ mHz}$, in Fig. 7.5a b = 1 and in Fig. 7.5b $r_b = 0.4. \dots \dots \dots 108$
- 7.6 Corner plot for an example adaptive MCMC with an injected phase transition signal characterised by $(\log_{10}(\Omega_p), \log_{10}(f_p), rb, b) = (-9, -2, 7, 0.4, 1)$ and a data model described by Eq. 7.19. The vertical dashed lines on the posterior distribution represent from left to right the quantiles [16%, 50%, 84%]. The red, green and blues lines are respectively the mean, the median of the posterior distribution and the input parameter values on the simulation. . . 111

- 8.1 2D projections of the induced priors on the spectral parameters, where red and blue regions correspond to high and low probability respectively. Notice the difference between this prior density and the one described in Sec. 8.4.3, which remarks the need for the use of the induced prior in order for the recovered thermodynamic parameter constraints to be physically meaningful. 126
- 8.3 Triangle plots for the deflagration fiducial model $\alpha = 0.4$, $v_{\rm w} = 0.55$, $r_* = 0.1$, $T_{\rm n} = 120$ GeV, for MCMCs sampling on spectral parameters 8.3a and thermodynamic parameters 8.3b. On the left, the spectral MCMC samples with flat priors (blue) and with induced priors (purple). The cross hairs in the spectral triangle plots mark the best fit to the injected spectrum calculated using the optimisation procedure described in Section 8.5. On the right are the corresponding samples on the thermodynamic parameters (green) and thermodynamic parameters reconstructed from the spectral sample (purple). The cross hairs in the thermodynamic triangle plot show the injected thermodynamic parameters. The grey shading in the $v_{\rm w}$ - α plot shows the region excluded by the physical prior, described in Eq. (8.26). . . 130

8.6 Injected and best fit spectra for the detonation fiducial model with $v_{\rm w} = 0.88$, $\alpha = 0.2$, $r_* = 0.1$, $T_{\rm n} = 200$ GeV. The light and dark grey bands show the 1 and 2 sigma spread on the power spectra for the MCMC sample with the induced prior. In the spectral parametrisation (a) the best fit spectrum with the uniform prior is shown in blue, and the induced prior is shown in purple. In the thermodynamic parametrisation (b): the best fit spectrum for the direct sampling is shown in green, and the reconstructed sampling in purple. In both cases the injected spectrum is shown in yellow. 136

Chapter 1

Introduction

The ground breaking first direct detection of gravitational waves (GWs) at the Laser Interferometer Gravitational wave Observatory (LIGO) in 2015 provided us with a new window to the cosmos [15]. The GW spectrum is populated with a diverse range astrophysical and cosmological signatures across the frequency decades. GWs provide a particularly exciting opportunity as they propagate from times when the university was still opaque to electromagnetic radiation. The GW source that takes centre stage in this thesis is a stochastic gravitational wave background (SGWB) from a first order phase transition at the electroweak scale, which corresponds to $\sim 10^{-11}$ s after the Big Bang.

Phase transitions describe the change of state of a system and are a common feature in some, but not all, gauge field theories. In general a phase transition in a gauge theory is a first order phase transition and proceeds via the nucleation of bubbles of new phase within the surrounding old phase which expand, collide, and merge until the phase transition completes [16–19]. In the Standard Model (SM) the transition between phases at the electroweak scale occurs smoothly and there is no departure from equilibrium. This is known as a crossover [20].

Although there is no first order phase transition in the SM in many extensions of the SM, such as those with additional scalar fields, there are first order phase transitions (for a review see [8]). One motivation to study the electroweak phase transition is the departure from equilibrium that occurs during a first order phase transition is one of the requirements needed to explain the matter-antimatter asymmetry in the universe [21–23]. A first order phase transition in the early universe in the presence of a cosmic plasma leads to the production of GWs in a number of ways: the collision of bubble walls, the overlap of sound waves established in the plasma, and acoustic and vortical turbulence. Of these GW contributions numerical simulations have shown that the acoustic GWs dominate the total SGWB from a first order phase transition [24–26]. The most sophisticated model for the acoustic contribution to the SGWB is the sound shell model (SSM) [4, 27]. The SSM calculates the GW power spectrum from a few key thermodynamic parameters of the phase transition, in turn these parameters can be directly calculated from the underlying particle physics model which describes the transition. A GW signature from a first order phase transition at the electroweak scale (100-1000 GeV) would peak in millihertz region today. In the 2030s space-based millihertz GW observatories like the Laser Interferometer Space Antenna (LISA) are planned to start operation [28]. The possibility of detecting a SGWB from a first order phase transition and constraining the thermodynamic parameters provides an exciting opportunity to search for new physics (that is complimentary to collider experiments).

Before millihertz GW detectors come online there is a bounty of interesting research to conduct including (but not exclusively): exploring the connection of extensions of the SM to the thermodynamic parameters of a first order phase transition, extending numerical simulations to cover a wider range of thermodynamic parameter space, and developing a thorough understanding of possible data analysis methods and the how the parameterisation of models impacts this. In thesis I hope to contribute to this endeavour by focusing on the last segment and investigating LISA's ability to estimate the parameters associated with a SGWB from a first order phase transition at the electroweak scale.

1.1 Thesis outline

This thesis focuses on gravitational waves from first order phase transitions and the possibility of detecting them at upcoming space-based GW detectors. The preliminary material contains a brief overview of the production of GWs from first order phase transitions and the experiments that aim to detect them. I then present the articles written during my time as a doctoral candidate. The main focus of the articles is LISA's ability to constrain the parameters associated with a GW power spectrum of a first order phase transition and subsequently the underlying parameters associated with new physics. The outline of this thesis as follows. In Chapter 2, I cover some key concepts of general relativity, the production of GWs and cosmology. Chapter 3 contains a summary of the key thermodynamic parameters and the GW power spectrum produced by a first order phase transition. In Chapter 4 I review GW observation methods and introduce LISA. The final chapter of the background, Chapter 5, describes the data analysis methods we will go on to employ in the subsequent articles.

In the first paper, presented in Chapter 6, we use the sound shell model to characterise the gravitational wave power spectrum from a first order phase transition, and the Fisher matrix to estimate uncertainties in the parameters associated with such a transition [1]. We explore a parameter space with transition strengths $0.01 < \alpha < 0.5$ and phase boundary speeds $0.4 < v_w < 0.9$, with mean bubble spacings 0.1 and 0.01 of the Hubble length, and sound speed $c/\sqrt{3}$. In this work, for simplicity we restrict ourselves to a fixed $T_n = 100$ GeV. Here, it is shown that the power spectrum in the sound shell model can be well approximated by a four-parameter double broken power law. We calculate the relative uncertainties for the thermodynamic parameters and the spectral parameters of the double broken power law fit to the sound shell model. We also consider the corresponding uncertainties to the principal components of the Fisher matrix.

Chapter 7 contains the second paper [2] which builds upon the previous work by studying LISA's ability to observe a GW background from phase transitions in the presence of an extragalactic foreground from binary black hole mergers throughout the universe, a galactic foreground from white dwarf binaries, and LISA noise. Here we model the phase transition gravitational wave background as a double broken power law, we use the deviance information criterion as a detection statistic, and Fisher matrix and Markov Chain Monte Carlo (MCMC) methods to assess the measurement accuracy of the parameters of the power spectrum.

In Chapter 8, which contains the third paper [3], we introduce and test a method for investigating LISA's sensitivity to gravitational waves from a first order phase transition using parametrised templates as an approximation to a more complete physical model. Starting from a map between the physical parameters and the parameters of an empirical template, we first construct a prior on the empirical parameters that contains the necessary information about the physical parameters; we then use the inverse mapping to reconstruct approximate posteriors on the physical parameters from a fast MCMC on the empirical template. We test the method on a double broken power law approximation to spectra in the sound shell model.

Finally, I present the conclusions from this body of work in Chapter 9.

Chapter 2

General Relativity, Cosmology and Gravitational waves

2.1 General Relativity

Einstein's theory of general relativity (GR) revolutionised the way we think about the universe, Newton's theory of gravity which models gravity as an attractive force between masses served us well for centuries and can functionally describe the motion of objects, from an apple falling to the ground to the orbits of the planets. The problems arise when we look at the causal nature of gravity, how does one massive object know the force to feel from another distant object? To develop a causal gravitational theory that explains how distant objects influence one another Einstein reimagined gravity as geometry. Einstein introduced a background field, spacetime, that places time and space on equal footing. In this framework a gravitational field that is present at all points in space and time that interacts with mass. An intuition of what gravity as geometry means can be gleaned from John Wheeler's quote "Space-time tells matter how to move; matter tells space-time how to curve".

The first observational divergence from Newtonian gravity, which was known of before Einstein introduce GR, was the perihelion precession of Mercury. GR is able to describe this phenomenon. Further predictions of general relativity include: gravitational time dilation, gravitational lensing, gravitational redshifting of light and black holes all of which have been confirmed. Another prediction of general relativity, and a central pillar of this thesis, is gravitational waves which we will discuss in detail in Sec. 2.3. In GR spacetime curvature is encoded in the metric $g_{\mu\nu}$ which we define by considering the infinitesimal spacetime separation between two points, this is known as the line element

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{2.1}$$

where $g_{\mu\nu}$ is the metric, a symmetric two-index tensor which tells you how the distance between two points varies with your spacetime coordinates. For a flat spacetime in the absence energy-momentum the line element takes the simple form

$$ds^{2} = -dt^{2} + dx^{2} + dy^{2} + dz^{2}.$$
(2.2)

In this special case $g_{\mu\nu}$ is known as the Minkowski metric

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1).$$
(2.3)

Here (and throughout this thesis) we use the (-, +, +, +) convention.

There are three distinct cases for ds^2 depending on how the two points are separated. Firstly for $ds^2 < 0$ the points are said to be timelike separated, these points are causally connected. Secondly, lightlike separated points $ds^2 = 0$. Finally, $ds^2 > 0$ corresponds to spacelike separated points, which are causally disconnected.

The path a particle takes through 4-D spacetime is known as its worldline, where the distinction from a trajectory is made as this is also a path through time. A worldline is specified by $x^{\mu}(\lambda)$ and is designated as timelike, lightlike or spackelike by its tangent vector $dx^{\mu}/d\lambda$. For an observer journeying along a timelike path it is useful to introduce the concept of proper time τ

$$d\tau^2 \equiv -ds^2. \tag{2.4}$$

The proper time τ along a timelike path is computed as follows

$$\tau = \int \sqrt{-ds^2} = \int \sqrt{-g_{\mu\nu}} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} d\lambda.$$
(2.5)

A requirement when building GR was that the form of equations are coordinate invariant (i.e that they are covariant and transform linearly, as tensors do). It can be shown that the partial derivative ∂_{μ} acting on a vector V^{μ} is not invariant under coordinate transformation, instead the covariant derivative is defined.

$$\nabla_{\mu}V^{\mu} = \partial_{\mu}V^{\mu} + \Gamma^{\nu}_{\mu\lambda}v^{\lambda}, \qquad (2.6)$$

where the $\Gamma^{\nu}_{\mu\lambda}$ are called the connection coefficients, also known as the Christoffel symbols

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu} \right).$$
(2.7)

As gravity manifests as curvature within GR we need to be able to describe said curvature. Information about the curvature is contained within the metric but more work is required to extract it into a useful form. For this we introduce the Riemann curvature tensor, a four index tensor, that is constructed from the Christoffel symbols (which as can be seen in Eq. 2.7 is built from the metric)

$$R^{\sigma}_{\mu\alpha\beta} \equiv \partial_{\alpha}\Gamma^{\sigma}_{\mu\beta} - \partial_{\beta}\Gamma^{\sigma}_{\mu\alpha} + \Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\beta} - \Gamma^{\sigma}_{\beta\lambda}\Gamma^{\lambda}_{\mu\alpha}.$$
 (2.8)

A unique quality of flat space is that all the components of the Riemann tensor vanish.

There are two useful contractions of Riemann tensor, the Ricci tensor

$$R_{\alpha\beta} = R^{\lambda}{}_{\alpha\lambda\beta},\tag{2.9}$$

and the Ricci scalar

$$R = R^{\lambda}{}_{\lambda} = g^{\mu\nu}R_{\mu\nu}.$$
(2.10)

We can then introduce the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}.$$
 (2.11)

So far we have only looked at the curvature side of the GR coin, the connection to energy and momentum is made through the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \qquad (2.12)$$

where G is the gravitational constant and $T_{\mu\nu}$ is a symmetric two-index tensor know as the energy-momentum tensor. $T_{\mu\nu}$ contains information about the energy and momentum of matter that sources gravity.

The Einstein field equations can be thought of as the equations of motion of the metric. We have reached the point where we have the mathematical formulation of the underlying concept of GR: curvature of spacetime is governed by the left hand side of Eq. 2.12 and the right hand side describes the energy and momentum in that spacetime.

2.2 Modern Cosmology

For the majority of human history we have thought of ourselves, on planet Earth, at the centre of the universe, Copernicus' paradigm shattering postulation in the 15th century that the opposite was true, has since been confirmed with observational evidence. The key foundation of modern cosmology, which describes the universe on the largest scales conceivable, is often called the cosmological principle and is built on the Copernican principle: on the largest scales the universe is statistically homogenous and isotropic. Statistical homogeneity requires that the universe would appear the same at every point in space (at a set time) and isotropy means the universe looks the same in every direction. Working on these scales and assumptions will simplify the problem of trying to describe how the universe evolves.

When we look on local or even galactic scales it is hard to see how the great diversity can allow for the cosmological principle to hold, it is only on cosmological scales we can see the supporting evidence. Observational support for the cosmological principle has been found: the cosmic microwave background (CMB) observations show that they early universe was largely statistically isotropic [29, 30], and studies of galaxy distributions show homogeneity on scales $\gtrsim 100$ Mpc.

The dominant force on cosmological scales is gravity, to describe the evolution of the universe we must employ GR. As introduced in Sec. 2.1 we can use geometry, in the form of a metric to describe the universe on large scales. The cosmological principle is encapsulated by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric

$$ds^{2} = -dt^{2} + a(t)^{2} \left[\frac{dr^{2}}{1 - kr^{2}} + r^{2} d\Omega \right], \qquad (2.13)$$

where $d\Omega = d\theta^2 + \sin^2 \theta d\phi^2$. k is defined as the curvature and the cosmological principle is fulfilled for the following three cases k < 0, k = 0, k > 0 which correspond to open, flat, and closed universes respectively. Here, a(t) is the scale factor which will describe how distances evolve as the universe evolves, the scale factor is commonly normalised such that today $t = t_0$ the scale factor is $a(t_0) = a_0 = 1$. Our aim is to be able to describe how the universe evolves depending on what it is made of, for this we need to solve the Einstein equations.

Applying the cosmological principle we require that the energy momentum tensor must meet the following conditions

$$T_0^{\mu} = 0, \quad T_1^1 = T_2^2 = T_3^3,$$
 (2.14)

furthermore, none of the elements can depend on space.

On cosmological scales we model the distribution of matter as as a perfect fluid, that does not conduct heat and has no viscosity. For the general case, where the observer is moving with respect to the rest frame of the fluid, the covariant form of the energymomentum tensor is

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)U_{\mu}U_{\nu} \tag{2.15}$$

where $\rho(t)$ is the energy density, p(t) is the pressure of the perfect fluid and U_{μ} is the fluid velocity four-vector. The pressure and the energy density are related as follows through the equation of state

$$p = w\rho, \tag{2.16}$$

the constant w takes the values 0, 1/3, -1 for pressureless matter, radiation and vacuum energy respectively. Now we have the energy momentum tensor we can begin to solve the Einstein equations. Remembering that the energy momentum tensor is covariantly conserved we arrive at

$$T^{\mu\nu}{}_{;\mu} = 0 = G^{\mu\nu}{}_{;\mu}.$$
 (2.17)

Considering the time component of the conservation of energy momentum tensor and the equation of state Eq. 2.16

$$T^{\mu\nu}{}_{0;\mu} = \dot{\rho} + \Gamma^{i}_{i0} \left(\rho + p\right) = \dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)\left(\rho + p\right) = 0, \qquad (2.18)$$

which can be written in the more familiar form as the first law of thermodynamics

$$d(\rho a^3) = -pda^3. (2.19)$$

For the equation of state given in Eq. 2.16 this can be written as

$$\frac{\dot{\rho}}{\rho} = -3(1+w)H,$$
 (2.20)

where H is the Hubble parameter

$$H(t) = \frac{\dot{a}}{a} \tag{2.21}$$

and $\dot{a} = da/dt$. Eq. 2.20 can be rearranged to form the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.22}$$

Writing H in terms of the scale factor and integrating Eq. 2.20 we can solve for a(t)

$$\rho \propto a(t)^{-3(1+w)} \begin{cases} w = 0 & \text{pressureless matter} \\ w = 1/3 & \text{radiation} \\ w = -1 & \text{vacuum energy} . \end{cases}$$
(2.23)

The difference between the rate at which radiation and matter dilutes is because the energy density in radiation decreases more rapidly than matter. This is due to the fact as the universe expands the wavelength of light (and thus the energy) is redshifted. In an expanding universe, such as our own, first there is a period of radiation domination, followed by matter domination, if there is a cosmological constant (vacuum energy) eventually this will come to dominate. Currently the universe is undergoing an era of accelerated expansion that is thought to be driven by a cosmological constant.

The non-zero components of the Ricci tensor and scalar are R_{00} , R_{ij} and R. It follows that the 00 component of the Einstein equation gives the well known first Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho_{\rm tot} - \frac{k}{a^3},$$
(2.24)

where ρ_{tot} is the sum of all contributions to the energy density. Considering the *ii* components of the Einstein equations and substituting in the first Friedmann equation we reach the second Friedmann equation

$$\dot{H} + H^2 = -\frac{4\pi G}{3} \left(\rho + 3p\right). \tag{2.25}$$

To determine the critical density ρ_c required for the universe to be flat insert k = 0 into the first Friedmann equation, we obtain

$$\rho_c = \frac{3H^2}{8\pi G}.\tag{2.26}$$

Current observations of ρ_{tot}/ρ_c are extremely close to 1, in this thesis we assume that the background curvature of the Universe is flat.

2.3 Gravitational waves

Within relativistic theories, such as Einstein's theory of GR, the transmission of information is subject to the speed limit of the speed of light. In the context of a field theory a consequence of a moving mass or charge is the surrounding fields must reconfigure to account for this change, with this reconfiguration we can expect some energy and momentum is radiated away (what we go on to introduce as gravitational waves). Analogously to electromagnetism when we think of gravity in terms of GR we can expect radiative solutions, that travel at a finite speed (the speed of light). This phenomena is unique to GR as in Newtonian gravity there are no constraints on how fast information can travel and therefore is modelled as happening instantaneously. A key difference between GWs and electromagnetic radiation is that the leading contribution to GWs comes from an accelerating quadrupole moment. The leading order contribution to electromagnetic radiation comes from an accelerating dipole moment, for example as is created by the oscillation of charges in a metal rod. In the context of gravity instead of a charge distribution we consider a mass distribution and first investigate the mass dipole moment. Any GW emission from a varying dipole moment would depend on the second order time derivative of the mass dipole moment. However, the first order time derivative of the mass dipole is equal to the total linear momentum of the system; as this quantity must be conserved it follows that there is no gravitational dipole radiation. Binary systems orbiting their common centre of mass are an example of a source that has a non-zero accelerating quadrupole moment and produce GWs. On the other hand, spherically symmetric systems have zero quadrupole moment and therefore do not produce GWs.

GWs were first proposed by Henri Poincaré in 1905 and later formalised as a prediction of Einstein's GR in 1915. Although, their existence was a subject of debate for years until they were accepted as physical solutions and not just coordinate artefacts [31]. GWs have proved elusive and it wasn't until nearly 100 years later that they were first detected at the Laser Interferometer Gravitational wave Observatory (LIGO) in 2015 from the inspiral and merger of a pair of black holes [15].

Following the steps outlined in [32], in this section we will look how to obtain the wave equation from Einstien's field equations by considering the simple case of linearized gravity, in which we expand Einstein's equations around a flat Minkowski metric. This solution can be further simplified with a careful gauge choice. We then describe the nature of the energy and momentum carried away by these GWs and how to convince oneself that they are in fact a physical phenomena that cannot simply be eradicated with a gauge choice.

2.3.1 Plane wave solutions in linearized GR

To begin we consider a small gravitational perturbation in a region of spacetime that is far away from any other sources of mass. In this case we can model the spacetime as flat Minkowski $\eta_{\mu\nu}$ with an additional small fluctuation $h_{\mu\nu}$ We assume that $|h_{\mu\nu}| \ll 1$ and we only need to consider linear contributions in $h_{\mu\nu}$. We can then write the Riemann tensor up to linear orders in $h_{\mu\nu}$ as

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left(\partial_{\nu}\partial_{\rho}h_{\mu\sigma} + \partial_{\mu}\partial_{\sigma}h_{\nu\rho} - \partial_{\mu}\partial_{\rho}h_{\nu\sigma} - \partial_{\nu}\partial_{\sigma}h_{\mu\rho} \right)$$
(2.28)

and the Ricci tensor is

$$R_{\mu\nu} = \frac{1}{2} \left(\partial^{\alpha} \partial_{\mu} h_{\alpha\nu} + \partial^{\alpha} \partial_{\nu} h_{\alpha\mu} - \partial^{2} h_{\mu\nu} - \partial_{\mu} \partial_{\nu} h \right)$$
(2.29)

noting that the trace of $h_{\mu\nu}$ is given by $h = h^{\mu}{}_{\mu} = \eta^{\mu\nu} h_{\mu\nu}$.

The physical situation we are interested in is a reference frame in which Eq. 2.27 holds for a large enough region of space. Choosing a reference frame breaks the invariance of GR under coordinate transformations, which is a handy way of reducing degrees of freedom in order to uncover the underlying physics of the theory. Even with this choice of reference frame there still remains a residual gauge symmetry. If we consider a coordinate transformation in the form of a slight distortion from cartesian, such as

$$x^{\mu} \to x^{\mu} + \xi^{\mu} \tag{2.30}$$

where $|\xi^{\mu}| \ll |x^{\mu}|$. The small fluctuations on the flat metric in the new coordinate system are given by

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \eta_{cb}\partial_a\xi^c - \eta_{ad}\partial_b\xi^d.$$
(2.31)

If at most $|\partial_a \xi^c| \sim \mathcal{O}(|h_{ab}|)$ then the condition $|h_{ab}| \ll 1$ is conserved, this means that these slowly varying coordinate transformations are a symmetry of the linearized theory.

The Ricci tensor can be simplified by instead of using h_{ab} it is replaced with its trace reverse \bar{h}_{ab}

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2} \eta_{\mu\nu}.$$
(2.32)

The Ricci tensor can now be written as

$$R_{\mu\nu} = \frac{1}{2} \left(\partial^{\alpha} \partial_{\mu} \bar{h}_{\alpha\nu} + \partial^{\alpha} \partial_{\nu} \bar{h}_{\alpha\mu} - \partial^{2} \bar{h}_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \partial^{2} \bar{h} \right).$$
(2.33)

We can now make use of the gauge freedom to choose the Lorenz gauge 1

$$\partial^{\nu}\bar{h}_{\mu\nu} = 0, \qquad (2.34)$$

which further simplifies the Ricci tensor

$$R_{\mu\nu} = \frac{1}{2} \left(\frac{1}{2} \eta_{\mu\nu} \partial^2 \bar{h} - \partial^2 \bar{h}_{\mu\nu} \right).$$
(2.35)

¹Due to historical inaccuracies often called the Lorentz gauge.

Th Ricci scalar can then be calculated

$$R = \frac{1}{2} \left(2\partial^2 \bar{h} - \partial^2 \bar{h} \right) = \partial^2 \bar{h}.$$
 (2.36)

Using the Lorenz gauge Eq. 2.34 reduces the 10 independent degrees of freedom in the symmetric 4x4 matrix $h_{\mu\nu}$ to 6 degrees of freedom. The Einstein tensor $G_{\mu\nu}$ is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = \frac{1}{2}\left(\frac{1}{2}\eta_{\mu\nu}\partial^{2}\bar{h} - \partial^{2}\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial^{2}\bar{h}\right) = -\frac{1}{2}\partial^{2}\bar{h}_{\mu\nu}$$
(2.37)

We now have all the elements required to construct the linearized Einstein equations

$$-\frac{1}{2}\partial^2 \bar{h}_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$
 (2.38)

If we consider a point in a vacuum outside the source where $T_{bd} = 0$ we are left with

$$\partial^2 \bar{h}_{\mu\nu} = \Box \bar{h}_{\mu\nu} = \left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \bar{h}_{\mu\nu} = 0, \qquad (2.39)$$

which is the wave equation with plane wave solutions analogous to electromagnetism, $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$ this implies that GWs propagate at the speed of light.

Before discussing the plane wave solutions further we again need to discuss gauge choices, so far we have not completely fixed our gauge we are free to make a further coordinate transformation as in Eq. 2.30. Under such a coordinate transformation the Lorenz gauge will remain untouched as long as

$$\Box \xi_{\mu} = 0. \tag{2.40}$$

From this condition it follows that $\Box \xi_{\mu\nu} = 0$, where

$$\xi_{\mu\nu} \equiv \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \eta_{\nu\mu}\partial_{\rho}\xi^{\rho}.$$
 (2.41)

We are then free to choose functions of ξ_{μ} that impose four conditions on $h_{\mu\nu}$, this fixes a further 4 degrees of freedom. A common choice is to adopt the transverse traceless (TT) gauge which corresponds to the following conditions

$$h^{0\mu} = 0, \quad h^i{}_i = 0, \quad \partial^j h_{ij} = 0$$
 (2.42)

Eq. 2.39 has plane wave solutions which can be written as

$$h_{ij}^{TT}(x) = e_{ij}(\mathbf{k})e^{ikx}, \qquad (2.43)$$

where $e_{ij}(\mathbf{k})$ is the polarisation tensor and wavevector \mathbf{k} . From Eq. 2.42 we see that the only non-zero component of h_{ij}^{TT} are those in the plane transverse to the direction of travel \hat{n} . Setting \hat{n} along the z axis a simplified version of h_{ij} can be written

$$h_{ab}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}_{ab} \cos[\omega(t-z/c)]$$
(2.44)

where a, b = 1, 2 are indices in the x, y plane, here we introduce the "plus" and "cross" polarisation h_+ and h_{\times} respectively.

2.3.2 Gravitational waves in curved spacetime

So far we have only considered GWs in a flat background, this does not account for the possibility that the GWs themselves may curve spacetime as they propagate. To allow for this scenario we must instead allow the background spacetime to be dynamical. In terms of our metric this is written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \qquad (2.45)$$

where $\bar{g}_{\mu\nu}(x)$ is our curved, dynamical background metric. Again we require the $h_{\mu\nu} \ll 1$. We are now faced with the challenge of separating the background from the fluctuations, in the linearized theory this was easy as our background was chosen to be the Minkowski metric for all space and time. In the current scenario there is no unambiguous way to distinguish the source of changes to $g_{\mu\nu}$ due to the background metric $\bar{g}_{\mu\nu}(x)$ from the fluctuations $h_{\mu\nu}$ we are interested in.

One approach for separating GWs from the background is to consider the case where there is a distinct separation of scales. If we can define a typical length scale L_B for the variations in the background $\bar{g}_{\mu\nu}$, on top of which the small amplitude fluctuations have a relatively small wavelength λ such that

$$\lambda = \frac{\lambda}{2\pi} \ll L_B,\tag{2.46}$$

where λ is the reduced wavelength. This scenario can physically be interpreted as small ripples $h_{\mu\nu}$ on a smooth background.

We can approach this from a different (and independent) perspective by considering the frequency domain. Consider a background metric $\bar{g}_{\mu\nu}$ where the largest frequency is f_B whereas $h_{\mu\nu}$ is peaked around f where Here $h_{\mu\nu}$ can be visualised as a small amplitude and high frequency perturbation around a slowly varying background. Both of these approaches are equally valid and the choice of wavelength or frequency approach is made dependent on which condition is satisfied.

Our aim is to understand the energy momentum carried by GWs at large distances from their source, for example at a detector. To investigate how the GWs propagate on a curved background we expand the background metric $\bar{g}_{\mu\nu}$. In this case we have two small expansion parameters $h \equiv \mathcal{O}(h_{\mu\nu})$ and either λ/L_B or f_B/f depending on whether Eq. 2.46 or Eq. 2.47 is satisfied. These two scenarios can be treated in parallel with the appopriate changes in notation, here we will refer to both as the small wave expansion. We now expand the Einstein equations to quadratic order in $h_{\mu\nu}$.

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \qquad (2.48)$$

where $T_{\mu\nu}$ is the energy-momentum tensor for matter and T its trace. We then expand the Ricci tensor to quadratic order in h

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \dots, \qquad (2.49)$$

here we have grouped together $\bar{g}_{\mu\nu}$ terms in $\bar{R}_{\mu\nu}$, linear and quadratic order terms of $h_{\mu\nu}$ are collected in $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ respectively, where

$$\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{low}} + \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{low}},\qquad(2.50)$$

$$R_{\mu\nu}^{(1)} = -[R_{\mu\nu}^{(2)}]^{\text{high}} + \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{high}}.$$
 (2.51)

Here low relates to the projection on to low momenta or low frequencies depending on whether Eq. 2.46 or Eq. 2.47 applies. From Eq. 2.50 we can describe the energy-momentum tensor of GWs and from Eq. 2.51 how GWs propagate on the background spacetime.

Consider Eq. 2.50, when we have a clear separation between the two length scales, by introducing an intermediate length scale \bar{l} where $\lambda \ll \bar{l} \ll L_B$ we can perform a projection over the long wavelength modes. By averaging over \bar{l} the L_B modes are left unaffected and λ averages to zero. We can then write Eq. 2.50 as

$$\bar{R}_{\mu\nu} = \left\langle R^{(2)}_{\mu\nu} \right\rangle + \frac{8\pi G}{c^4} \left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right\rangle, \qquad (2.52)$$

where $\langle ... \rangle$ describes averaging over the reduced wavelength λ , if λ is much smaller than the background wavelength scale, or a temporal average if $f \gg f_B$. We now define an effective energy-momentum tensor of matter $\bar{T}_{\mu\nu}$

$$\left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right\rangle = \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T},\qquad(2.53)$$

where $\bar{T} = \bar{g}_{\mu\nu}T^{\mu\nu}$ is the trace. The construction of $\bar{T}_{\mu\nu}$ means it only contains low frequency (or low momentum/ long wavelength) modes and can be thought of as a smoothed form of the energy momentum tensor $T_{\mu\nu}$. We can also write down the energy momentum tensor for GWs

$$t_{\mu\nu} = -\frac{c^4}{8\pi G} \left\langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \right\rangle, \qquad (2.54)$$

where $R^{(2)} = \bar{g}^{\mu\nu} R^{(2)}_{\mu\nu}$. The "coarse-grained" Einstein equations which describe how the background metric evolves can be introduced as

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4} \left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right).$$
(2.55)

As discussed when considering the linearized theory the matrix $h_{\mu\nu}$ has both physical and gauge modes, we can follow the same approach here to rid ourselves of the non-physical gauge modes. Remembering that outside the source $\Box h_{\alpha\beta} = 0$

$$t_{\mu\nu} = \frac{c^4}{32\pi G} \left\langle \partial_{\mu} h_{\alpha\beta} \partial_{\nu} h^{\alpha\beta} \right\rangle.$$
 (2.56)

Finally, by choosing the Lorenz gauge Eq. 2.34 and the transverse traceless gauge as outlined in Eq. 2.42 we are left with the gauge invariant energy density for GWs travelling in a vacuum

$$t^{00} = \frac{c^4}{32\pi G} \left\langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \right\rangle \tag{2.57}$$

where $\dot{h}_{ij}^{TT} = \partial_t h_{ij}^{TT}$. This can be written in terms of the two polarisations of the GWs

$$t^{00} = \frac{c^4}{16\pi G} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle.$$
 (2.58)

 t^{00} is often written as $t^{00} = \rho_{gw}$ as we will go in to discuss GWs in a cosmological context it will be useful to define the fractional energy density in GWs

$$\Omega_{\rm gw} = \frac{\rho_{gw}}{\rho_c},\tag{2.59}$$

where ρ_c is the critical energy density given in Eq. 2.26.

The propagation of GWs in an expanding universe can be described by the Einstein equation linearized to first order over a spatially flat FLRW background (see Eq. 2.13). In this case the metric perturbation is written $a^2h_{\mu\nu}$ which gives the GW equations of motion

$$\ddot{h}_{ij}(\mathbf{x},t) + 3H\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}h_{ij}(\mathbf{x},t) = 16\pi G\Pi_{ij}^{TT}(\mathbf{x},t)$$
(2.60)

where $\nabla^2 = \partial_i \partial_i$, and Π_{ij}^{TT} is the transverse traceless part of the anisotropic stress. The anisotropic stress is given by

$$a^{2}\Pi_{ij} = T_{ij} - pa^{2}(d_{ij} + h_{ij}), \qquad (2.61)$$
where T_{ij} denotes the spatial part of the energy-momentum tensor of the source, and p is the background pressure. Working with conformal time $d\eta = dt/a(t)$ and defining the

$$H_{ij}(k,\eta) = ah_{ij}(k,\eta) \tag{2.62}$$

where k is the comoving wavenumber. The wave equation becomes

$$H_{ij}''(k,\eta) + \left(k^2 - \frac{a''}{a}\right) H_{ij}(k,\eta) = 16\pi G a^3 \Pi_{ij}^{TT}(k,\eta)$$
(2.63)

where primes denotes derivatives with respect to η . The majority of cosmological sources only generate GWs for a finite amount of time, after the source has ceased the GWs are free to propagate through the FLRW spacetime. This corresponds to setting the source term $\Pi_{ij}^{TT}(k,\eta) = 0$ in Eq. 2.63. The scale factor can be described as $a(\eta) = a_n \eta^n$ where for radiation n = 1 which means the a''/a term vanishes and H_{ij} obeys the same equation as a perturbation around Minkowski space (see Eq. 2.39).

Chapter 3

Gravitational waves from a cosmological first order phase transition

Cosmological first order phase transitions are dramatic events that can lead to exciting observational signatures such as GWs. We can gain intuition about a cosmological first order phase transition by considering a more down to earth phenomena such as the boiling of water. A first order phase transition is characterised by a critical temperature, latent heat, and mixed phases separated by phase boundaries. The two phases are distinguished by an order parameter. In the case of boiling water the order parameter is the density which under lab conditions (temperature 373K and 1 bar pressure) changes by a factor of about 1000 at the phase transition.

Here we focus on the phase transition associated with the symmetry breaking at the electroweak scale. The order parameter for electroweak symmetry breaking is the square root of vacuum expectation value of the Higgs field $\phi = \sqrt{\langle HH^{\dagger} \rangle}$. At temperatures well above a critical temperature $T \gg T_c$ the thermal Higgs potential has only one minimum at $\phi \sim 0$ [33] and the Higgs is in the symmetric phase everywhere. As the universe expands and cools the Higgs potential evolves. In the Standard Model the evolution of the Higgs potential is smooth and ϕ gradually rolls down the potential to the new minimum in a crossover, there is no departure from equilibrium and no GWs are generated [20, 34]. In theories with extensions to the SM Higgs sector it is possible for a first order phase

transition to take place, an illustration of how the potential evolves in this case is given in Fig. 3.1. In this scenario a second minimum forms in the Higgs potential, and the two minima are separated by a potential barrier. As temperature continues to drop at some critical temperature T_c the minima are degenerate. When $T < T_c$ it becomes energetically favourable for the scalar field to reside at the new minimum where $\phi = \phi_b$ (where *b* denotes the broken phase). The challenge for the scalar field to overcome the potential barrier still remains, this can be surpassed by either thermal fluctuations [35] or quantum tunnelling [36, 37]. We consider the thermal case.

The extensions to the SM that exhibit these first order phase transitions can, in a number of cases, resolve outstanding questions in physics such as the baryon number asymmetry (baryogenesis) and the nature of dark matter [22, 23, 38]. For a summary of such models see [8]. Collider experiments that probe high energy physics such as the large hadron collider (LHC) are exploring the electroweak scale but have not yet constrained the order parameter ϕ of the electroweak phase transition as we still know very little about the Higgs sector. A detection of the GW signature for a first order phase transition at the electroweak scale would compliment collide experiments.

In regions of space where the scalar field successfully overcomes the potential barrier, islands of broken phase form in a sea of metastable phase. Spherical bubbles are the most probable shape of regions of broken phase, which if larger than a critical size R_c will expand into the metastable region. The bubble will expand if the interior pressure is greater enough to overcome the force due to the surface tension of the bubble. The boundary between the two phases is known as the bubble wall.

The GW signal from a first order phase transition depends on a few key parameters. These key parameters are determined by the underlying particle physics of the phase transition and we will refer to them as the thermodynamic parameters. These parameters dictate the evolution of the bubbles, how the bubble wall interacts with the surrounding plasma and the amount of energy available to go into GW production. As there are a wide range of models that could lead to such a first order phase transition it is useful to work in a model-independent framework, where we take the thermodynamic parameters as the starting point for calculating the GW power spectra.

In this chapter we cover the definitions of the key thermodynamic parameters of a first order phase transition, the nature of the fluid shells that are established around the expanding bubbles and the GW sources that are possible from such a transition.

3.1 Bubble nucleation

The temperature at which the probability of one bubble nucleating in one Hubble volume is of order one is known as the nucleation temperature T_n which is one of the key thermodynamic parameters that determine the GW signature. The nucleation rate

$$\Gamma(t) = A(t)e^{-S(t)},\tag{3.1}$$

where A is a prefactor with dimension of energy to the fourth power and S is the Euclidean action of a critical bubble. An approximate inverse time duration of the phase transition can then be defined as the rate of variation of the bubble nucleation rate

$$\beta \equiv \frac{dS}{dt} \Big|_{t_*} \simeq \frac{\dot{\Gamma}}{\Gamma} \Big|_{t_*}, \tag{3.2}$$

where t_* denotes the time when the GWs are produced, this approximately coincides with $T_* \sim T_n$, in phase transitions without significant supercooling. This quantity β is often parameterised as a ratio to the Hubble rate at the time of GW production H_n

$$\tilde{\beta} = \frac{\beta}{H_n}.$$
(3.3)



Figure 3.1: The evolution of the effective potential $V(\phi)$ for a first order phase transition. $T_{\rm c}$ is the critical temperature where two degenerate minima are present.

3.2 Bubble wall speed

Once the bubble has surpassed its critical size the system is best described by hydrodynamics. The speed of the bubble wall v_w is an important quantity as the nature of the fluid flow depends on it. As we will go on to see, the fluid profiles are vital for estimating GW power spectrum for a first order phase transition. The flows are also important for baryogenesis [39, 40]. We note the phase transition strength, which we go on to define, will also be vital in characterising the flow around the bubble. As done in [41], in the following discussion we assume that the transition completes within one Hubble time, so we can neglect the expansion of the universe.

In order to describe the bubble wall speed the equations of motion for the fluid-field system must be derived. This is done by considering the energy-momentum of the system

$$T_{\mu\nu} = T_{\mu\nu}^{\text{field}} + \sum_{i} T_{i,\mu\nu}^{\text{fluid}}, \qquad (3.4)$$

where the cosmic fluid is modelled as a perfect fluid with

$$T_{\mu\nu}^{\text{fluid}} = (e+p)U_{\mu}U_{\nu} + g_{\mu\nu}p.$$
(3.5)

The total energy-momentum of the fluid-field system is conserved $(\partial^{\nu}T_{\mu\nu} = 0)$ but those of the two subsystems are not. The relationship between the scalar field and the plasma is controlled via the dissipative phenomenological friction term $\eta(\phi)$, the η function requires an out-of-equilibrium calculation. Once the equations of motion have been derived one can solve for how the bubbles evolve, given the parameters of the phase transition and the friction parameter. As the bubble expands it comes into contact with the fluid which exerts a frictional force slowing the bubble wall until it reaches a terminal velocity v_w . Full analysis of the microscopic interactions between the bubble wall and the fluid need to be carried out to determine v_w [42, 43].

The wall speed, the inverse duration of the phase transition and the mean bubble spacing R_* are related as follows

$$\tilde{\beta} = \frac{\beta}{H_n} \sim \frac{v_{\rm w}}{H_n R_*}.\tag{3.6}$$

The constant of proportionality $(8\pi)^{1/3}$ given for this equation in [44] is reasonable for wall speeds much greater than the speed of sound, but for wall speeds below the speed of sound it is also dependent on the phase transition strength and the bubble wall speed (as the nucleation rate is reduced by the reheating of the fluid in front of the bubble wall [45]). In view of this uncertainty, it is more convenient to work in terms of R_* , and more precisely the Hubble-scaled mean bubble spacing

$$r_* = H_n R_*. \tag{3.7}$$

 r_* will be one of our key thermodynamic parameters used to describe the GW power spectrum from a first order phase transition, as we will see it controls the amplitude and overall frequency scale of the GW power spectrum.

3.3 Fluid shell profiles

Assuming that the wall speed is known we now turn to discussing the hydrodynamic solutions for the fluid flows surrounding the expanding bubbles, we follow the approaches covered in [4, 40, 41]. Expanding bubbles deposit kinetic and thermal energy into the fluid they encounter, leading to bulk motion of the fluid and fluid shells being established around the bubbles. The nature of these fluid shells is vital in the prediction of the GWs from a first order phase transition.

To start we will describe the fluid velocities in front and behind the bubble wall, these two regions will be distinguished using \pm notation, plus for just in front of wall and minus for just behind. In the following discussion it will be useful to switch between two frames, the universe frame where the bubble wall moves at v_w and the wall rest frame which corresponds to $\tilde{v}_w = 0$. Quantities in the wall rest frame will be decorated as follows \tilde{v}_+ where this is the fluid velocity in front of the bubble wall in the wall rest frame.

We begin by considering the conservation of energy-momentum across the phase boundary in the wall rest frame

$$w_{-}\tilde{\gamma}_{-}^{2}\tilde{v}_{-} = w_{+}\tilde{\gamma}_{+}^{2}\tilde{v}_{+}, \quad w_{-}\tilde{\gamma}_{-}^{2}\tilde{v}_{-}^{2} + p_{-} = w_{+}\tilde{\gamma}_{+}^{2}\tilde{v}_{+}^{2} + p_{+}, \tag{3.8}$$

where w is the enthalpy of the fluid w = e + p, e, p are energy density and pressure respectively. $\tilde{\gamma}_{\pm} = (1 - \tilde{v}_{\pm}^2)^{-1/2}$. Note the enthalpy density w and the pressure p are both scalar quantities so are the same in this frame and in the universe rest frame. This set of equations are known as the bubble wall junction conditions and they can be rearranged as follows

$$\tilde{v}_{+}\tilde{v}_{-} = \frac{p_{+} - p_{-}}{e_{+} - e_{-}}, \quad \frac{\tilde{v}_{+}}{\tilde{v}_{-}} = \frac{e_{-} + p_{+}}{e_{+} + p_{-}}.$$
(3.9)

We now introduce the trace anomaly

$$\theta = \frac{1}{4}(e - 3p), \tag{3.10}$$

which is proportional to the trace of the energy-momentum tensor. The trace anomaly allows us to define the first form of another of our key parameters, the transition strength α_+

$$\alpha_{+} = \frac{4\Delta\theta}{3w_{+}},\tag{3.11}$$

where $\Delta \theta = \theta_+ - \theta_-$.

The junction conditions given in Eq. 3.9 can be recast in terms α_+ , and then solving for either \tilde{v}_+ as a function of \tilde{v}_- or vice versa we find

$$\tilde{v}_{+} = \frac{1}{1+\alpha_{+}} \left(\frac{\tilde{v}_{-}}{2} + \frac{1}{6\tilde{v}_{-}} \pm \sqrt{\left(\frac{\tilde{v}_{-}}{2} - \frac{1}{6\tilde{v}_{-}}\right)^{2} + \frac{2}{3}\alpha_{+}} + \alpha_{+}^{2} \right).$$
(3.12)

Different sets of physical solutions are found depending on the fluid velocity relative to the speed of sound $c_s = 1/\sqrt{3}$. If $\tilde{v}_- < 1/\sqrt{3}$ only the upper sign returns physical solutions and conversely for $\tilde{v}_- > 1/\sqrt{3}$. These two regimes correspond to subsonic deflagrations and supersonic deflagrations. A consequence of this is the fluid must be in the same regime in front and behind the wall.

The continuity equations are obtained by projecting the conservation equation

$$0 = u_{\mu}\partial_{\nu}T^{\mu\nu} = -\partial_{\mu}(wu^{\mu}) + u^{\mu}\partial_{\mu}p, \qquad (3.13)$$

$$0 = \bar{u}_{\mu}\partial_{\nu}T^{\mu\nu} = w\bar{u}^{\nu}\partial_{\mu}u_{\nu} + \bar{u}^{\mu}\partial_{\mu}p. \qquad (3.14)$$

To simplify the continuity equations we assume the bubbles have spherical symmetry prior to collision and the bubble radius is $R = v_w t$, where nucleation time is set to t' = 0. Due to the spherical symmetry there is no length scale involved in the problem and there will be similarity solutions that depend on the dimensionless quantity $\xi = r/t$. The fluid velocity can now be written in terms of the new coordinate ξ as $\vec{v} = v(r, t)\vec{r}$, where \vec{r} is a unit radial vector. The continuity equations can now be rearranged

$$\frac{dv}{d\xi} = \frac{2v(1-v^2)}{\xi(1-\xi v)} \left(\frac{\mu^2}{c_{\rm s}^2} - 1\right)^{-1},\tag{3.15}$$

$$\frac{dw}{d\xi} = w\left(1 + \frac{1}{c_{\rm s}^2}\right)\gamma^2 \mu \frac{dv}{d\xi}.$$
(3.16)

Here, the speed of sound squared $c_s^2 = dp/de$ and the fluid velocity μ at ξ in a frame that moves outward with speed ξ is

$$\mu = \frac{\xi - v}{1 - \xi v}.$$
(3.17)

In general these equations are solved using numerical techniques taking into account a series of boundary conditions. The fluid velocity must vanish v = 0 at the centre of the bubble $\xi = 0$ due to spherical symmetry and at $\xi = 1$ due to causality (because we assume the fluid is at rest until it is disturbed by the bubble wall). At the bubble wall $v \to v_{\pm} = \mu(\xi_w, \tilde{v}_{\pm})$ as $\xi \to \xi_w^{\pm}$ where $\xi_w^{\pm} = v_w \pm \delta$ for infinitesimally small δ .

Depending on how the boundary conditions are fufilled three distinct regimes for the asymptotic fluid profiles are defined: deflagration, detonations and hybrids which are discussed qualitatively below and schematic illustrations are shown in Fig. 3.2.

Deflagrations: Solution with wall speeds less than the speed of sound $v_{\rm w} < c_{\rm s}$ are known as (subsonic) deflagrations. For this case the fluid in front of the bubble wall has time to react to the approaching phase boundary. The fluid in front of the bubble wall gets compressed $p_+ > p_-$ and set in motion, the fluid velocity increases as the bubble wall gets closer, reaching its maximum value as it crosses the phase boundary. The fluid then cools and decelerates until at rest inside the bubble, this corresponds to $\tilde{v}_- = v_{\rm w}$ in the wall rest frame. At a distance from the phase boundary there is a discontinuity in the fluid velocity in the form of a shock front, after which the fluid is at rest (in the universe frame). See Fig. 3.3a for an example fluid velocity profile for a deflagration. **Detonations:** There is a minimum wall speed for a detonation, which in the bag model (used by the sound shell model) is equal to the Chapman-Jouguet speed $c_{\rm I}$

$$c_{\rm J} = \frac{1}{\sqrt{3}} \left(\frac{1 + \sqrt{\alpha_+ + 3\alpha_+^2}}{1 + \alpha_+} \right), \qquad (3.18)$$

In the universe frame $v_{+} = 0$, which in the wall rest frame his corresponds to $\tilde{v}_{+} = v_{\rm w}$. This relationship and the condition on \tilde{v}_{+} translates into a minimum wall speed $v_{\rm w} > c_{\rm J}$. When the bubble wall travels supersonically, the fluid has no time to react to the approaching phase boundary and the fluid ahead of the bubble wall is at rest $v_{+} = 0$, which in the wall rest frame this corresponds to $\tilde{v}_{+} = v_{\rm w}$. After the fluid crosses the bubble wall the fluid is compressed and accelerated $p_{+} < p_{-}$, then the fluid velocity smoothly decrease until it is at rest at $\xi = c_{\rm s}$ forming a rarefaction wave behind the bubble wall. See Fig. 3.3c for an example fluid velocity profile for a detonation.

Supersonic deflagrations (hybrids): A hybrid between the two previous cases is a possible solution, if $c_s > v_w \le c_J$ the fluid profile in front of the bubble wall is the same as a deflagration and the fluid decelerates in a rarefaction wave after the phase boundary as in a detonation. The existence of hybrid solutions is debated [40][46]. This can only occur if for a wall speed $v_{\rm w} > 1/\sqrt{3}$, $\tilde{v}_{-} = 1/\sqrt{3}$. physical solution if $\tilde{v}_{+} > v_{\rm w}$ to ensure v_{+} is positive. See Fig. 3.3b for an example fluid velocity profile for a hybrid.

The asymptotic fluid shell profiles can be calculated for a given wall speed and phase transition strength using PTtools (see Sec. 3.5.4 for more details), examples of fluid velocity and enthalpy profiles are shown in Fig. 3.3.

Previously we introduced α_+ as the phase transition strength, which depends on the quantities in front of and behind the bubble wall (which are perturbative quantities). We now define the phase transition strength as α_n , which is described in terms of background quantities

$$\alpha_n = \frac{4}{3} \frac{\Delta \theta}{w_{\rm s}} \Big|_{T=T_{\rm n}}.$$
(3.19)

Here, $\Delta \theta = \theta_s - \theta_b$, note in the following papers when we refer to α we are referring to α_n . Again, θ is the trace anomaly

$$\theta = V_T(\phi) - \frac{1}{4}T\frac{\partial V_T}{\partial T}, \qquad (3.20)$$

where $V_T(\phi)$ is the scalar field potential energy. The phase transition strength α_n is important in GW production as it depends on the trace anomaly (see Eq. 3.19) which determines how much energy is available to be converted to shear stress energy.



Figure 3.2: Schematic diagram of the possible fluid profiles: deflagration (left), hybrid (middle) and detonation (right). The black circle marks the bubble wall and the black arrows show the radial direction of the wall speed $v_{\rm w}$. The speed of sound is $c_{\rm s}$, and the Chapman-Jouguet speed $c_{\rm J}$ is defined in Eq. 3.18. The regions of fluid with non-zero velocities in the universe frame are shown in the coloured shells, where yellow denotes small velocities and orange large fluid velocities. Credit: D. Cutting.

To summarise the key parameters of the first order phase transition are the nucleation temperature T_n , the phase transition strength α , the wall speed v_w and the Hubble-scaled mean bubble spacing r_* . We will refer to these as the thermodynamic parameters.

3.4 Energy

As the bubble of broken phase expands into the fluid the potential energy of the scalar is field converted to kinetic energy and heat. The kinetic energy fraction (the amount of initial energy contained in the bubble that is converted into kinetic energy) is an important quantity as it is good estimate for the power in GWs. Assuming that the kinetic energy fraction estimate for one bubble has been found to be a reasonable estimate the kinetic energy fraction for the entire fluid [4, 26]. This means the GW power can be calculated by studying a single bubble. However, for strong transitions it has recently shown that the efficiency of transfer of kinetic energy to the fluid is less than theoretically predicted [7].

The kinetic energy fraction of the fluid is given by the trace of the energy momentum tensor minus the trace in the rest frame of the fluid

$$K = \frac{1}{\mathcal{V}\bar{e}} \int d^2 x w \gamma^2 v^2 \tag{3.21}$$



Figure 3.3: The fluid velocity v and the enthalpy w given as functions of the dimensionless coordinate $\xi = r/t$. From left to right the panels show: subsonic deflagration $v_{\rm w} = 0.55$, hybrid (supersonic deflagration) $v_{\rm w} = 0.7$ and detonation $v_{\rm w} = 0.88$. The dashed line represent curves which must intersect the shock front. The dashed dotted lines denote the maximum fluid velocity behind the phase boundary. Here the parameters chosen represent fiducial models we go on to use in our data analysis. Figures generated using PTtools [4].

where \mathcal{V} is the averaging volume and \bar{e} is the mean energy density. For a single bubble this is given by

$$K_1 = \frac{3}{\xi_{\rm w} e_{\rm s}} \int d\xi \xi^2 w \gamma^2 v^2 \tag{3.22}$$

where e_s is the energy density in the symmetric phase. Due to energy conservation and the speed of the transition (relative to the Hubble rate) $e_s = e_b$. Our assumption that kinetic energy fraction for a single bubble can describe the kinetic energy fraction for the entire fluid means $K = K_1$.

3.5 Gravitational waves from phase transitions

During and in the aftermath of a first order phase transition GWs can be sourced via three mechanisms: the collisions of bubble walls, the overlap of sound waves established in the plasma after bubbles have collided and turbulence in the form of vorticity and shocks that develop after the bubbles have collided. In what follows we present a brief overview of each component, summaries of the GW sources associated with a first order phase transition can be found in [8, 47].

3.5.1 Scalar field contribution

As bubbles of stable phase collide and merge, the collision of bubbles leads to the production of GWs [48]. This contribution to the GW spectrum has been understood using the "envelope approximation", which models the expanding bubbles walls as the propagation of spherical, infinitesimally thin shapes using numerical simulations (instead of using the Klein Gordon equation to describe the evolution of the scalar field) [49–51]. In this approximation, a fraction κ_{ϕ} of the latent heat of the phase transition is deposited in a thin shell near the bubble wall. The energy in each shell is expected to disperse quickly after the bubbles walls collide, therefore most of the energy is stored in uncollided bubble walls. The most recent numerical simulations have been able to improve numerical accuracy for high frequency behaviour of the GW power spectrum [50]. The contribution from bubble collisions in terms of energy density is given by

$$h^{2}\Omega_{\rm bc} = 1.67 \times 10^{-5} \left(\frac{H_{*}}{\beta}\right)^{2} \left(\frac{\kappa_{\phi}\alpha}{1+\alpha}\right)^{2} \left(\frac{100}{g_{*}(T_{*})}\right)^{1/3} \left(\frac{0.11v_{\rm w}^{3}}{0.42+v_{\rm w}^{2}}\right) \frac{3.8(f/f_{\phi})^{2.8}}{1+2.8(f/f_{\phi})^{3.8}},$$
(3.23)

where g_* is the effective number of relativistic degrees of freedom at GW production, and f_{ϕ} is the peak frequency of the contribution from the scalar field contribution [47]. As this portion of the phase transition GW production has a relatively short duration it will

not contribute significantly to the overall GW signature, except if there is very strong supercooling [52, 53].

3.5.2 Sound waves

In thermal phase transitions fluid shells form around the expanding bubble wall due to the interplay between the scalar field and the fluid. These fluid shells induce sound waves which once the bubbles have collided these sound waves overlap and produce a long lasting source GWs which we label Ω_{sw} . The sound waves remain (and are a persistent source of GWs) until they are damped by viscosity or they generate shocks [54]. Simulations that model the interaction between the fluid and the scalar field via a phenomenological friction parameter have shown that there are sound waves established in the fluid surrounding the bubble wall [24–26]. These studies have also shown the sounds waves continue to source GWs long after the phase transition completes. Their long-lasting nature boosts the associated GW signature, making them a significant contribution to the overall GW spectrum. A general form of the contribution to the GW spectrum from the sound waves is given by

$$\Omega_{\rm sw} = 2.65 \times 10^{-6} \left(\frac{H_*}{\beta}\right) \left(\frac{\kappa_v \alpha}{1+\alpha}\right)^2 \left(\frac{100}{g_*(T_*)}\right)^{1/3} v_{\rm w} C(f, f_{\rm sw}), \qquad (3.24)$$

where the efficiency κ_v denotes the fraction of trace anomaly that is transformed into bulk motion of the fluid, C is a generic spectral fit around the peak of the power spectra from numerical simulations, with the form

$$C(f, f_{\rm sw}) = \left(\frac{f}{f_{\rm sw}}\right)^3 \left(\frac{7}{4+3(f/f_{\rm sw})}\right)^{7/2},\tag{3.25}$$

 $f_{\rm sw}$ is the frequency that corresponds to the peak of the acoustic GW power spectrum [47]. We will refer to C as the LISA cosmology working group spectral fit, which takes the form of a single broken power law. An improvement on Eq. 3.24 that agrees with a wider range of numerical simulations is the sound shell model which we introduce in sec. 3.5.4 and is a physical model instead of a spectral fit.

3.5.3 Shocks and turbulence

In the aftermath of a first order phase transition shocks and vortical modes are established in the bulk motion of the plasma, both of which can lead to the production of GWs.

The evolution of shocks has been studied, and for strong transitions the power spectrum around the peak will not be completely described by the sound shell model [55]. The timescale for the generation of shocks is $\tau_{\rm sh} \sim \xi_{\parallel *}/\bar{v}_{\parallel *}$, where $\xi_{\parallel *}$ is the initial value of the integral length scale of the longitudinal component, and $\bar{v}_{\parallel *}$ is the longitudinal rms velocity at GW generation. If shocks appear before one Hubble time they may have significance for the GW spectrum. In particular, they would modify the low wavenumber region of the GW power with a spectral form as given below

$$\frac{1}{\left(\mathcal{H}_{*}\xi_{\parallel*}\right)^{2}}\mathcal{P}_{\mathrm{gw}} \propto k^{\frac{3\beta-1}{2}},\tag{3.26}$$

where \mathcal{H}_* is the comoving Hubble parameter evaluated at the time of the phase transition, and k is the wavenumber. Here $\beta \sim 4$ is the low-k power law index [55] (not the inverse duration of the phase transition).

The vortical modes can become turbulent which would leave an imprint on the GW spectrum from a first order phase transition. Turbulence is expected to set in after of the order of a few eddy turn over times, $\tau_{tu} \sim \xi_{\perp *}/\bar{v}_{\perp *}$, where $\xi_{\perp *}$ is the integral scale of the vortical component of the flow and $\bar{v}_{\perp *}$ is the transverse rms velocity at GW generation. The vorticity is small for all but very strong transitions which implies the turbulent contribution the GW power spectrum is also small [7]. For cases where there is significant vorticity the GW spectrum from fully developed turbulence can be approximated as follows [56]

$$\Omega_{\rm tu}(K \ll 1) = c_1 \frac{\mathcal{N}_{\rm cut}(\mathcal{H}_* \xi_{\perp *})^2}{\bar{v}_{\perp *} + \mathcal{N}_{\rm cut} \mathcal{H}_* \xi_{\perp *}} \bar{v}_{\perp *}^6 \left(\frac{K}{\bar{v}_{\perp *}}\right)^3 \tag{3.27}$$

$$\Omega_{\rm tu}(K \gg 1) = \frac{\mathcal{N}_{\rm cut}(\mathcal{H}_* \xi_{\perp *})^2}{\bar{v}_{\perp *} + \mathcal{N}_{\rm cut} \mathcal{H}_* \xi_{\perp *}} \left[c_2 \bar{v}_{\perp *}^6 \left(\frac{K}{\bar{v}_{\perp *}}\right)^{-\frac{7}{3}} + c_3 \bar{v}_{\perp *}^{-4/3} \exp\left(-\frac{3}{2\bar{v}_{\perp *}^2}\right) \left(\frac{K}{\bar{v}_{\perp *}}\right)^{-\frac{5}{3}} \right] \tag{3.28}$$

where \mathcal{N}_{cut} describes the duration of the turbulence in terms of the number of eddy turnover times the turbulence lasts for. The numerical coefficients c_i are given in Appendix B of [56] and are $c_1 \simeq 0.3$, $c_2 \simeq 23$ and $c_3 \simeq 57$. The term $K = \mathcal{A}\xi k$, where ξ is the integral scale of the flow, $\mathcal{A} \simeq 4.02$ is a numerical coefficient described in Eq.(3.14) of [56].

To summarise the total GW signature is the sum of each contribution

$$\Omega_{\rm gw} = \Omega_{\rm bc} + \Omega_{\rm sw} + \Omega_{\rm tu}. \tag{3.29}$$

In this work we study phase transitions for which the impact of shocks and turbulence is not expected to be important and will neglect their contribution to the GW power spectrum. Here we will focus on GWs from sound waves as numerical simulations have shown them to be the dominant source of GWs from a first order phase transition [24–26].

So far we have considered the GW power spectrum when it is generated, in order to translate these quantities into present day observables we need to understand how the GW frequency and intensity evolve. The characteristic wavenumber k_* of the GW power spectrum is defined by the inverse length scale of the problem which here is set by the bubble size $k_*/a_* \simeq 2\pi/R_*$ (we have included the scale factor a_* as we are comparing a comoving wavenumber to physical quantities). Using the Friedmann equation, the conservation of entropy, and a photon temperature today of $T_{\gamma,0} = 2.725$ K the frequency today due to redshifting is

$$f_{*,0} \simeq 2.62 \left(\frac{1}{H_n R_*}\right) \left(\frac{T_{\rm n}}{10^2 {\rm GeV}}\right) \left(\frac{g_*}{100}\right)^{\frac{1}{6}} \mu {\rm Hz}.$$
 (3.30)

From the definition of $f_{*,0}$ we can see a millihertz GW detector such as LISA will be most sensitive to GWs from a first order phase transition at temperatures between $T_{\rm n} \simeq 100 -$ 1000GeV (which corresponds to the electroweak scale) and bubble separations between 10^{-2} and 10^{-3} of the Hubble length [41]. The possible values for the bubble spacings are model dependent. Here we do not consider a specific model, instead we select a range of bubble spacings $(10^{-2} - 10^{-3})$ that lead to the GW power spectrum peaking in LISA's sensitivity window today. In ongoing work, not covered here, we investigate LISA's ability to constrain parameters associated with an additional triplet in the Standard Model, in that case the Hubble-scaled mean bubble spacing ranges from $\sim 10^{-5} - 10$.

To take into account the dilution of the energy density we must consider the history of the Universe. When the Universe was dominated by radiation, as the energy densities of both radiation and GWs redshift in the same way, the energy density of GWs remains constant. When the Universe entered the matter domination epoch the energy density of GWs decreased. Today we can account for this with a dilution factor.

$$F_{\rm gw,0} = \Omega_{\gamma,0} \left(\frac{g_{s0}}{g_{s*}}\right)^{\frac{4}{3}} \frac{g_*}{g_0} = (3.57 \pm 0.05) \times 10^{-5} \left(\frac{100}{g_*(T_{\rm n})}\right)^{\frac{1}{3}}.$$
 (3.31)

In practice, g_* will vary slightly during the phase transition and depending on T_n here we assume it to be constant as its contribution to the overall dilution is small relative to subsequent dilution during matter domination.

3.5.4 Sound shell model

The sound shell model (SSM) has been proposed as tool for predicting the shape and power of the GW power spectrum for acoustically produced GWs when given the thermodynamic parameters of the phase transition [4, 27]. There are two characteristic length scales in the acoustic GW power spectrum from a first order phase transition: the mean bubble separation and the sound shell thickness. In particular, the shell thickness depends strongly on the wall speed, as can be seen in Fig. 3.3. As the wall speed approaches the speed of sound the shell becomes thin which leads to a broad GW power spectrum. Conversely, away from the speed of sound the fluid profiles are relatively broad and the GW power spectrum have narrower peaks.

The power spectrum from colliding sound waves can be calculated if the velocity power spectrum is known and is gaussian. The SSM calculates the velocity power spectrum from the sound shells that surround the expanding bubbles of broken phase, these are the fluid shells calculated from relativistic hydrodynamics in Sec. 3.3.

The GW power spectrum predicted by the SSM depends on the mean bubble spacing R_* , phase transition strength α , wall speed v_w and the frequency scale is set by the nucleation temperature T_n . The gravitational wave power spectrum predicted by the SSM is as follows

$$\Omega_{\rm gw}(z) = 3K^2(v_{\rm w}, \alpha)(H_n \tau_v)(H_n R_*) \frac{z^3}{2\pi^2} \tilde{P}_{\rm gw}(z).$$
(3.32)

The quantity specifically calculated sing the SSM and is one of the outputs of the PTtools Python module is

$$\hat{\mathcal{P}}_{\rm gw}(z, v_{\rm w}, \alpha) = 3K^2 \frac{z^3}{2\pi^2} \tilde{P}_{\rm gw}(z).$$
(3.33)

When comparing the SSM with numerical simulations the gravitational wave peak power and frequency are in good agreement with the numerical simulations [4]. For a comparison between sound shell model predictions and parameters derived from 3D hydrodynamic simulation data, presented in [26], see Table 2 in [4]. The SSM over predicts the gravitational wave peak power for deflagrations and over estimates the kinetic energy when compared with recent simulations [7], we account for this by introducing a numerical kinetic energy suppression factor in Chapter 6.

We introduce an overall efficiency factor $\tilde{\Omega}_{gw}$ quantifying the efficiency with which shear stress is converted into GWs during a first order phase transition. This places an estimate of the maximum energy density in GWs at the time of emission and has been found to be $\tilde{\Omega}_{gw} \sim 10^{-2}$ [4, 26]. See Table 2 of [4] for $\tilde{\Omega}_{gw}$ calculated across a range of thermodynamic parameters.

3.5.5 Double broken power law

In the SSM there are two characteristic length scales, the mean bubble separation and the sound shell thickness, which motivate a simplified description in terms of a function with two frequency scales and three power law indices - a double broken power law [4]. The power spectrum today for the double broken power law fit can be described as

$$\Omega_{\rm gw,0}^{\rm dbp}(f,\Omega_{\rm p},f_{\rm p},r_{\rm b},b) = \Omega_{\rm p}M(s,r_{\rm b},b)$$
(3.34)

where $\Omega_{\rm p}$ is the peak of the power spectrum, $s = f/f_{\rm p}$, $f_{\rm p}$ is the frequency corresponding to $\Omega_{\rm p}$ and $r_{\rm b} = f_{\rm b}/f_{\rm p}$ is the ratio between the two breaks in the spectrum. The parameter b defines the spectral slope between the two breaks. The spectral shape $M(s, r_{\rm b}, b)$ is a double broken power law with a spectral slope 9 at low frequencies and -4 at high frequencies ¹, a form that was chosen to best describe the SSM [4].

$$M(s, r_{\rm b}, b) = s^9 \left(\frac{1 + r_{\rm b}^4}{r_{\rm b}^4 + s^4}\right)^{(9-b)/4} \left(\frac{b+4}{b+4 - m + ms^2}\right)^{(b+4)/2}.$$
 (3.35)

Within $M(s, r_{\rm b}, b)$, m has been chosen to ensure that for $r_{\rm b} < 1$ the peak occurs at s = 1and $M(1, r_{\rm b}, b) = 1$, giving

$$m = (9r_{\rm b}{}^4 + b) / (r_{\rm b}{}^4 + 1).$$
(3.36)

In the sound shell model the low wave number k power law index is k^9 and for high k there is a k^{-3} spectral slope [27]. In this thesis we use a double broken power law fit to the SSM with a high k power law index of k^{-4} , this choice of high k^{-4} provides better agreement with the "domed" nature around the peak of the SSM GW power spectra.

In the SSM the GW power spectra are calculated from the velocity power spectra [27]. In general, the GW power law index n_{gw} is determined by the velocity power law index n_v via the following relationship $n_{gw} = 2n_v - 1$.

For low k the velocity power law index is $n_v = 2 + 3$, this leads to $n_{gw} = 9$. This can be partly explained by the fact that the velocity is the gradient of the pressure which gives a velocity power law index contribution of 2 [57]. In the low k region of the power spectrum $(kR_* < 1)$ simulations of the GW power spectrum are limited by the computational volume,

¹In practice, the SSM's predicted high-frequency power law of -3 emerges only slowly, and -4 provides a better fit around the peak [4, 26].

they are not currently large enough to resolve the low wave number regime, so for now we use the SSM estimate for the low k region, with k^9 .

For the high k velocity power law index there are shocks, which leads to sharp edges and automatically the velocity power law index $n_v = -1$ [55] which subsequently leads to a GW high-k power law index of $n_{gw} = -3$. Hydrodynamic simulations have shown a GW high-k power law index of $n_{gw} = -3$ for detonations (in agreement with SSM) and slightly steeper for deflagrations (not in agreement with SSM) [26]. Comparisons between simulations and the SSM over a wide range of thermodynamic parameters have been performed but have not yet been published. In general, the universal use of 9 and -4 for the double broken power law provide reasonable fits to the SSM (and current simulations), as understanding improves these approximations are expected to evolve.

Chapter 4

Gravitational waves: Experiments

The direct detection of gravitational waves, predicted by Einstein's theory of general relativity in 1916 proved elusive until almost 100 years later. In the 1970's and 80's the observation of a binary pulsar by Hulse and Taylor [58] and the following work showing its energy loss in the form of gravitational radiation by Taylor and Weisberg [59] confirmed the existence of GWs. On September 14th 2015 the Laser Interferometer Gravitationalwave Observatory (LIGO) recorded the first ever direct detection of GWs. LIGO detected the inspiral and merger of a pair of black holes, this event is labelled GW150914 [15]. The detection of GWs is challenging as their amplitudes are extremely small and even a dramatic event such as GW150914 required LIGO to have sensitivity to changes in length of 1/10000 fraction of the width of a proton.

The majority of current and future GW experiments use interferometers to measure GWs. Interference patterns generated by the superposition of laser beams that have travelled along different optical paths are used to measure differences in arm lengths induced by passing GWs. Interferometers are favourable for GW experiments as they can measure microscopic changes in arm length. The interferometer arm length determines the GW wavelength sensitivity of the experiment.

LIGO, now with the addition of Virgo and Kagra, probes the 10 Hz - 10 kHz window of the GW spectrum, so far they have detected 90 compact binary coalescences [60]. The next generation of ground based detectors will achieve greater sensitivity to GWs by using a longer arm length. In particular, the Einstein telescope has been proposed to increase sensitivity by a factor of the order of 10, with a triangular configuration and arm length of 10 km [61–63].

For the millihertz frequency range in the GW spectrum, as studied in this thesis, interferometers with even longer arms are required. These longer arm lengths are not possible at ground based detectors due to the curvature and size of the Earth. Upcoming millihertz GW observatories will take to the skies (space) where the vast distances allow for longer arm lengths that provide access to the 10^{-4} Hz to 10^{-1} Hz range of the GW spectrum. The next few decades promise to be an exciting time for this window of the GW spectrum with the ESA-NASA mission Laser Interferometer Space Antenna (LISA) [28], Taiji [64] and TianQin [65], all aiming for launch in the mid-2030s. LISA and TianQin have both launched technology test satellites: LISA pathfinder [66] and TianQin-1 [67], both of which were successful proof of concepts.

Another area of the GW spectrum currently being explored by pulsar timing arrays (PTAs) is the 10^{-9} Hz to 10^{-7} Hz frequency band. By monitoring the arrival times of radio pulses from an array of galactic millisecond pulsars over many years one can observe a passing gravitational wave which induces correlated modulations in the arrival times of the radio pulses [68, 69]. PTAs are most sensitive to the period corresponding to the observation time. The nanohertz frequency band is populated with supermassive black hole binaries and potentially a stochastic background from cosmic strings and other exotic signatures. Current experiments such as Nanograv [70] are searching for (and are potentially on the cusp detecting) a SGWB [71].

In this chapter the we outline the LISA mission profile and instrument specification, how a SGWB and the instrument noise will appear in the LISA data channels is also discussed. We then introduce the time delay interferometry (TDI) variables that reduce the contribution from the LISA instrument noise. The instrument response functions and the formulation of the LISA signal in terms of energy density (in order to compare with the SGWB introduced in the previous chapter) are presented.

4.1 LISA

LISA will be made up of three satellites that form an equilateral triangle constellation following an earth-trailing orbit around the Sun. The distance between LISA and Earth will be 50-60 million km, see Fig. 4.1a. LISA's arms will be 2.5 million km long, meaning LISA's sensitivity window is between 10^{-4} Hz to 10^{-1} Hz [28]. To give a sense of perspective the Sun would fit snugly inside the LISA constellation. The LISA mission has a planned mission lifetime of 4 years with the possibility to be extended to 10 years [28]. Due to repointing of the antennas and re-configuration of the laser locking the conservative estimate for science operation mode will be 75% of the mission duration.

Within each spacecraft there will be: two lasers one for each of the optical benches, two test masses each one for a specific interferometer arm acting as a geodesic reference frame end mirror, and two telescopes with an aperture of 30cm to receive light from the neighbouring satellites. See Fig. 4.1a for a simplified illustration of the setup inside a LISA spacecraft.

The test masses will be in free fall inside the spacecraft and micro-newton thrusters will be used to ensure the spacecraft follows along with the test masses. LISA will use interferometry to measure differential optical pathlength modulations induced by passing GWs. The expected variations in pathlength due to GWs is of the order of pm to nm. Due to beam divergence over the long arm length instead of reflecting the laser beam at a neighbouring spacecraft a fresh phase-locked laser beam will be transmitted for the journey back to the original spacecraft. 6 interferometric readings (2 from each spacecraft) containing the information in change in distance (light travel times between test masses) produced from the 6 optical benches will be post-processed upon their return to earth to build the TDI channels that will be used in the data analysis. The data will transmitted as a time series with a sampling rate of $f_s = 4$ Hz.

LISA will detect GW signatures from a medley of astrophysical and cosmological sources including: massive black hole coalescences[72], compact galactic binaries [73], extreme mass ratio inspirals [74], and precursors to binary black hole and binary neutron star mergers (LIGO-type signatures) [75]. SGWBs from cosmological sources may include cosmic strings, inflation and first order phase transitions [76].

4.2 Time delay interferometry

LISA's 3 spacecrafts will form a single GW detector, meaning we cannot use multiple detectors to reduce the instrument noise as is done at ground based GW detectors [77]. Therefore, a careful understanding of how the GW signatures and instrument noise appear in the interferometer data is required. The laser phase noise, which if left untreated would be the main source of noise, is expected to be eliminated by constructing Time Delay Interferometry (TDI) channels from the interferometer data. In practice it remains to be seen whether the laser phase noise will be completely eliminated. Assuming the laser phase noise has been removed there are currently two further main expected sources of instrument noise: the position noise, due to uncertainties in measurements of the phase of the laser light, and the noise due to residual jitter of the test masses known, as the acceleration noise. The two sources of noise introduced here are umbrella terms, each with multiple constituent parts. As we approach LISA's launch we will have to consider more detailed breakdown of the instrument noise. In the following we introduce the concept of TDI variables first presented by Armstong, Estabrook and Tinto in [78, 79]. To illustrate how they allow the cancellation of the laser noise we first follow the approach outlined in



(a) LISA spacecraft optical bench, figure taken(b) The LISA planned orbit and design specificfrom [28].ations, figure taken from [28].





Figure 4.2: The paths taken by the laser used to construct the Michelson signal (left) and the time delay interferometry channels (right). Figure taken from [5]

[80]. Starting with the phase output for a single link between spacecraft i and j,

$$\Phi_{ij}(t) = C_i(t - L_{ij}) - C_j(t) + \varphi_{ij}(t) + n_{ij}(t)$$
(4.1)

where C_i are the laser phase noises, φ_{ij} is the gravitational strain, n_{ij} contains the noise contributions from the position and acceleration noises.

We can then construct a Michelson signal M_{ijk} at any of the vertices by combining the phase at that detector with the time-delayed signal from the spacecrafts at the end of the two adjoining arms (left hand figure in Fig. 4.2)

$$M_{A_{BC}}(t) = \Phi_{AB}(t)(t - L_{AB}) + \Phi_{BA}(t) - \Phi_{AC}(t - L_{AC}) - \Phi_{CA}(t), \qquad (4.2)$$

where the label A_{BC} denotes the signal at the vertex between the arms AB and AC. For an equal arm length LISA TDI channels that cancel the laser phase noise C_i can be constructed by subtracting a time-delayed Michelson signal from the Michelson signal at a vertex (right hand figure in Fig. 4.2). Here we introduce the first TDI variable, know as the X channel

$$X(t) = \Phi_{A_{BC}} = M_{A_{BC}}(t) - M_{A_{BC}}(t - 2L).$$
(4.3)

Using Eq. 4.1 and assuming equal arm lengths,

$$X(t) = \Phi_{AB}(t - L) + \Phi_{BC}(t) - \Phi_{AC}(t - L) - \Phi_{CA}(t)$$

$$- (\Phi_{AB}(t - 3L) + \Phi_{BC}(t - 2L) - \Phi_{AC}(t - 3L) - \Phi_{CA}(t - 2L)).$$
(4.4)

The cancellation of the laser phase noises can be seen in the following, where for simplicity the gravitational wave φ_{ij} and other noise components n_{ij}^p and \vec{n}_{ij}^a are ignored

$$X(t) = C_A(t - 2L) - C_B(t - L) + C_B(t - L) - C_A(t)$$
(4.5)

$$-C_A(t-2L) + C_B(t-L) - C_C(t-L) + C_A(t)$$
(4.6)

$$-C_A(t-4L) + C_B(t-3L) - C_B(t-3L) + C_A(t-2L)$$
(4.7)

$$+C_A(t-4L) - C_C(t-3L) + C_C(t-3L) - C_A(t-2L) = 0.$$
(4.8)

Two further TDI channels, Y and Z can be generated by permuting the indices in Eq. 4.4. For an unequal arm interferometer, as LISA will be in practice, see [81].

Now we have the TDI variables we turn our attention to how the GW signature appears in these TDI data channels. For a general GW incident on the detector

$$h_{ab}(\vec{x},t) = \int_{-\infty}^{\infty} df \int d^2 \hat{n} \sum_{P} h_P(f,\hat{n}) e^P_{ab}(\hat{n}) e^{i2\pi f \left(t - \hat{n} \cdot \vec{x}/c\right)}, \tag{4.9}$$

where e_{ab}^{P} is the polarisation tensor and we sum over the two polarisation modes $p = +, \times$ of a gravitational wave, the indices *ab* correspond to the space indices.

$$\hat{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \tag{4.10}$$

$$e_{ab}^{+}(\hat{n}) = \hat{m}_{a}\hat{m}_{b} - \hat{n}_{a}\hat{m}_{b} \tag{4.11}$$

$$e_{ab}^{\times}(\hat{n}) = \hat{\mathfrak{m}}_a \hat{\mathfrak{m}}_b - \hat{\mathfrak{m}}_a \hat{\mathfrak{m}}_b \tag{4.12}$$

$$\hat{\mathbf{m}} = (\sin\phi, -\cos\phi, 0) \tag{4.13}$$

$$\hat{\mathbf{n}} = (\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta) \tag{4.14}$$

where $e_{ab}^{P}(\hat{n})e^{P'ab}(\hat{n}) = 2\delta_{PP'}$ and \hat{m} , \hat{m} are basis vectors that define the coordinate system in the plane transverse to the direction of propagation.

Assuming the laser phase noise has been eliminated we can reintroduce the measured phase difference at an interferometer vertex Φ (Eq. 4.1), which is now described completely in terms of the phase shifts due to the incident GW φ and the noise *n*, following the notation in [5] we can write

$$\Phi_{A_{BC}}(t) = \varphi_{A_{BC}}(t) + n_{A_{BC}}(t), \qquad (4.15)$$

where the label A_{BC} denotes the signal at the vertex between the arms AB and AC. We leave the labelling in this general form instead of specifying to the TDI signal so later it becomes clear how to switch between the Michelson and TDI framework.

The connection between the incident GW $h_{ab}(t)$ and detector output is made with the detector tensor D^{ab}

$$\varphi_{A_{BC}}(t) = D^{ab} h_{ab}(t). \tag{4.16}$$

Using Eq. 4.9 this can be written as (see [82] for the details)

$$\varphi_{A_{BC}}(t) = \int_{-\infty}^{\infty} df \int d^2 \hat{n} \sum_{P} h_P(f, \hat{n}) e^{i2\pi f t} F^P_{A_{BC}}(\hat{n}, f; t)$$
(4.17)

where

$$F_{A_{BC}}^{P}(\hat{n}, f; t) = \frac{1}{2} e^{-i2\pi f \hat{n} \cdot \vec{x}_{A}(t)/c} e_{ab}^{P}(\hat{n}) \left[\mathcal{F}^{ab}\left(\hat{\ell}_{AB}(t) \cdot \hat{n}, f\right) - \mathcal{F}^{ab}\left(\hat{\ell}_{AC}(t) \cdot \hat{n}, f\right) \right]$$
(4.18)

$$\begin{aligned} \mathcal{F}^{ab}(\hat{\ell} \cdot \hat{n}, f) &= \left(\operatorname{sinc} \left[\frac{f}{2f_*} (1 - \hat{\ell} \cdot \hat{n}) \right] e^{-i \frac{f}{2f_*} (3 + \hat{\ell} \cdot \hat{n})} + \operatorname{sinc} \left[\frac{f}{2f_*} (1 + \hat{\ell} \cdot \hat{n}) \right] e^{-i \frac{f}{2f_*} (1 + \hat{\ell} \cdot \hat{n})} \right) \\ &\times \frac{1}{2} W \left(f, f_* \right) \hat{\ell}^a \hat{\ell}^b \end{aligned}$$

describes the gain of the detector vertex. The unit vector $\hat{\ell}_{AB}$ points from vertex A to B, $f_* = c/(2\pi L)$ and \hat{n} points in the direction of GW propagation. In Eq. 4.18 W represents the interference induced by a return journey along one arm, which allows one to change between the Michelson signal and TDI variables. When W = 1 the signal is described as in Eq. 4.2 and the path is shown in the left hand side of Fig. 4.2. As we saw in Eq. 4.4 it is possible to construct TDI variables that (in the equal arm length approximation) remove the laser noise, this corresponds to $W(f, f_*) = 1 - e^{(-2if/f_*)}$ as described by Eq. 4.3 and the right hand side of Fig 4.2.

For a Gaussian distributed stochastic GW source the mean vanishes, we consider instead the variance which we refer to as the power spectrum and is given by

$$\langle J(t)J'(t')\rangle = \frac{1}{2} \int_{-\infty}^{\infty} df e^{i2\pi f(t-t')} \left[\mathcal{R}_{JJ'}(f;t,t')I(f) + N_{JJ'}(f) \right],$$
 (4.19)

where J, J' are the TDI channels $J, J' = \{X, Y, Z\}$ that include both the stochastic GW source and the instrument noise (assuming noise and signal are uncorrelated). Here, $N_{JJ'}$ is the correlated noise power spectrum between the two vertices, and I is the intensity of the SGWB. The instrument response to gravitational waves is accounted for with $\mathcal{R}_{JJ'}$, the response function,

$$\mathcal{R}_{JJ'}(f;t,t') = \int \frac{d^2\hat{n}}{4\pi} \left[F_J^+(\hat{n},f;t) F_{J'}^{*+}(\hat{n},f;t') + F_J^{\times}(\hat{n},f;t) F_{J'}^{*\times}(\hat{n},f;t') \right].$$
(4.20)

Assuming the noise spectra in all the links are uncorrelated, stationary and identical

$$N_{XX} = N_{YY} = N_{ZZ}, \quad N_{XY} = N_{YZ} = N_{XZ},$$
 (4.21)

and

$$\mathcal{R}_{XX} = \mathcal{R}_{YY} = \mathcal{R}_{ZZ}, \quad \mathcal{R}_{XY} = \mathcal{R}_{YZ} = \mathcal{R}_{XZ}.$$
 (4.22)

For completeness the corresponding terms for the GW signal \mathcal{S}

$$S_{XX} = S_{YY} = S_{ZZ}, \quad S_{XY} = S_{YZ} = S_{XZ},$$

$$(4.23)$$

where $S_{XX} = \mathcal{R}_{XX}(f)I(f)$. Under the further assumption that the LISA constellation will have equal arm length L the noise auto correlation is given by [5, 83]

$$N_{XX}(f, N_{\rm acc}, N_{\rm oms}) = \left[8\left(1 + \cos^2\left(\frac{\pi fL}{c}\right)\right) P_{\rm acc}(f) + 4P_{\rm oms}(f)\right] |W|^2, \qquad (4.24)$$

and the cross-spectra

$$N_{XY}(f, N_{\rm acc}, N_{\rm oms}) = -\left[8P_{\rm acc}(f) + 2P_{\rm oms}(f)\right]\cos\left(f/f_*\right)|W|^2 \tag{4.25}$$

where

$$P_{\rm oms} = N_{\rm oms},\tag{4.26}$$

here $N_{\rm oms} = 3.6 \times 10^{-41} {\rm Hz}^{-1}$ is the optical metrology system noise,

$$P_{\rm acc} = \frac{N_{\rm acc}}{(2\pi f)^4} \left(1 + \left(\frac{f_1}{f}\right)^2 \right), \qquad (4.27)$$

where $f_1 = 0.4$ mHz, the acceleration noise $N_{\rm acc} = 1.44 \times 10^{-48} s^{-4} \text{Hz}^{-1}$. The instrument noise requirements $N_{\rm acc}$ and $N_{\rm oms}$ are given in the LISA science requirements document[6].

Exploiting the symmetries of the $\langle J(t)J'(t')\rangle$ matrix it can be diagonalised and a further three orthogonal TDI channels can be constructed from the resulting eigenvectors. They are the A, E, T TDI channels [84]

$$A = \frac{1}{\sqrt{6}} \left(X - 2Y + Z \right) \tag{4.28}$$

$$E = \frac{1}{\sqrt{2}} \left(X - Z \right)$$
 (4.29)

$$T = \frac{1}{\sqrt{3}} \left(X + Y + Z \right). \tag{4.30}$$

The TDI signal T is insensitive to the GW signal at low frequencies [85], meaning T can be used to characterise the LISA instrument noise and subsequently reduce the impact of the noise in the A and E channels.

The resulting noise spectral densities for the A, E, T channels are described completely by X, Y, Z auto-correlation and the cross spectra of the X, Y, Z TDI channels

$$N_A = N_E = \langle AA^* \rangle_{\text{noise}} = \langle EE^* \rangle_{\text{noise}} = N_{XX} - N_{XY}, \qquad (4.31)$$

$$N_T = \langle TT^* \rangle_{\text{noise}} = N_{XX} + 2N_{XY}. \tag{4.32}$$

Similarly for the GW signal

$$\mathcal{S}_A = \mathcal{S}_E = \langle AA^* \rangle_{\text{sig}} = \langle EE^* \rangle_{\text{sig}} = \mathcal{S}_{XX} - \mathcal{S}_{XY}, \qquad (4.33)$$

$$\mathcal{S}_T = \langle TT^* \rangle_{\text{sig}} = \mathcal{S}_{XX} + 2\mathcal{S}_{XY}. \tag{4.34}$$

Noting that assuming the X, Y, Z noises uncorrelated leads to $\langle AE^* \rangle = \langle AT^* \rangle = \langle ET^* \rangle = 0.$

4.3 Detector response function

To relate the power spectral density of the incident GWs to the power spectral density recorded in the detector we use the response function $\mathcal{R}(f)$, which was introduced generally in Eq. 4.20. Upon substituting all the components into Eq. 4.20 it is possible to achieve an integral form of the solution for the response function¹ which currently is solved numerically². We employ the analytic fit to the numerical solutions presented in [5]. For the A and E channels

$$\mathcal{R}_{A,E}^{\text{fit}} \simeq \frac{9}{20} |W|^2 \left[1 + \left(\frac{3f}{4f_*}\right)^2 \right]^{-1},$$
(4.35)

and for the T channel

$$\mathcal{R}_T^{\text{fit}} \simeq \frac{1}{4032} \left(\frac{f}{f_{\text{t}}}\right)^6 |W|^2 \left[1 + \frac{5}{16128} \left(\frac{f}{f_{\ast}}\right)^8\right]^{-1}.$$
(4.36)

A comparison between the numerical functions and the analytic fits is presented in Fig. 4.3.



Figure 4.3: The response function for the A, E, T TDI channels. The solid lines show the numerical approximation and the dashed lines show the analytic fits. Figure taken from [5].

The spectral densities introduced above are related to the energy density as follows

$$\Omega_{\rm gw,i}(f) = \frac{4\pi^2}{3H_0^2} f^3 \left(\frac{N_i(f)}{\mathcal{R}_i(f)} + \frac{\mathcal{S}_i(f)}{\mathcal{R}_i(f)} \right), \tag{4.37}$$

where i = A, E, T.

¹See [86] appendix A.3 for the complete expanded form of the response function.

²Although, work is being undertaken to find analytic forms [87].

4.4 Signal to noise ratio

The signal to noise ratio ρ is given by

$$\rho = \frac{\mu}{\sigma} \simeq \sqrt{\frac{T_{\text{obs}}}{2}} \frac{\sum_{i=A,E} \int_{-\infty}^{\infty} df \mathcal{S}_i(f) Q_i(f)}{\sqrt{\sum_{i=A,E} \int_{-\infty}^{\infty} df N_i^2(f) Q_i^2(f)}},$$
(4.38)

where μ is the expectation value of the optimal statistic and σ is the expectation value of the square of the optimal statistic. Q is a weight function that is chosen to maximise the signal to noise ratio. As the noise is the same in both the A and E channels this can be written as

$$\rho = \sqrt{T_{\text{obs}}} \frac{\int_{-\infty}^{\infty} df \mathcal{S}_A(f) Q_A(f)}{\sqrt{\int_{-\infty}^{\infty} df N_A^2(f) Q_A^2(f)}}.$$
(4.39)

A function Q_A can be found to maximise the signal to noise ratio. In order to only consider positive frequencies in the final form of the signal to noise ratio we move to single-sided power spectra. This leads to a change of the integral limits and multiplying the integrand by a factor of two, this leaves us with

$$\rho = \left[2T_{\rm obs} \int_0^\infty df \frac{S_i^2(f)}{N_i^2(f)}\right]^{1/2}.$$
(4.40)

Chapter 5

Data analysis methods

In the previous chapters we have worked towards describing a model for the SGWB from a first order phase transition that depends on the thermodynamic parameters, we now discuss how we can use data analysis to test such models. If we have a model $\mathcal{M}(\theta)$ that depends on parameters θ we can take a Bayesian approach to determine the posterior distributions of model parameters given some observational data. For a background on Bayesian analysis see [88]. The Bayes theorem summarises this methodology by introducing the posterior distribution $\mathcal{P}(\theta|\mathcal{D}, \mathcal{M})$ as follows

$$\mathcal{P}(\theta|\mathcal{D},\mathcal{M}) = \frac{\mathcal{L}(\mathcal{D}|\mathcal{M}(\theta))\pi(\theta|\mathcal{M})}{\mathcal{P}(\mathcal{D}|\mathcal{M})}$$
(5.1)

where \mathcal{D} is the data, \mathcal{M} is the model, and the likelihood $\mathcal{L}(\mathcal{D}|\mathcal{M}(\theta))$ gives the probability of the data given the model \mathcal{M} with parameter values θ . Here, $\pi(\theta|\mathcal{M})$ is the prior distribution, which reads as the probability of parameters θ given the model \mathcal{M} . The prior contains the prior knowledge about the parameters, this can be in the form of theoretical or observational bounds. The normalisation $\mathcal{P}(\mathcal{D}|\mathcal{M})$ is the marginal likelihood which is given by

$$\mathcal{P}(\mathcal{D}|\mathcal{M}) = \int \mathcal{L}(\mathcal{D}|\mathcal{M}(\theta))\pi(\theta|\mathcal{M})d\theta.$$
(5.2)

In this chapter we will discuss the two key methods we use when exploring LISA's sensitivity to a SGWB from a first order phase transition: Fisher analysis and Markov chain Monte Carlo (MCMC) methods.

5.1 Fisher Matrix

Before experiments come online it is useful to understand how well the experiment will be able to constrain model parameters. This can be done with the Fisher information which is related to the asymptotic distribution of the posterior [89]. The Fisher information can be used to calculate the covariance matrices and thus the expected uncertainties on model parameters when one does go on to perform a maximum likelihood estimation. One benefit of the Fisher matrix is that it is computationally cheap to evaluate relative to simulating data and performing MCMC runs to obtain predicted uncertainties. The Fisher matrix relies on a Gaussian approximation of the likelihood, meaning if there are non-Gaussian degeneracies between parameters the Fisher matrix will only provide a rough approximation to the likelihood (or posterior) contours.

Formally, the Fisher information is defined as the variance of the score, where the score is

$$s(\theta) \equiv \frac{\partial l(\theta)}{\partial \theta_i},\tag{5.3}$$

 $l = \ln(\mathcal{P})$ is the log-likelihood and θ is the parameter vector with *i* model parameters θ_i .

As we would like to estimate the uncertainties of a maximum likelihood estimate we will consider the Fisher matrix evaluated at the maximum of the likelihood where $\partial l/\partial \theta_i|_{\theta=\theta_0} = 0$. We now introduce the general form of the Fisher matrix

$$F_{ij} = \frac{\partial^2 l}{\partial \theta_i \partial \theta_j} \bigg|_{\theta = \theta_0}.$$
(5.4)

When we refer to the Fisher matrix it is Eq. 5.4 we are referring to.

In our analysis in Chapter 6 we will perform the Fisher analysis for a GW spectrum from a first order phase transition $\Omega_{gw}(f)$. We reduce the evaluation time by binning our raw frequencies f into $N_{\rm b}$ bins each with frequency $f_{\rm b}$. There are $n_{\rm b}$ raw frequencies in each bin

$$n_{\rm b} = (f_{\rm b} - f_{\rm b-1})T_{\rm obs},$$
 (5.5)

as $n_b \gg 1$ we apply the central limit theorem and can use a Gaussian likelihood

$$\mathcal{L} = \exp\left(-\frac{1}{2}\chi^2\right). \tag{5.6}$$

Here, χ^2 is the chi-squared which in terms of the observables at LISA is given by

$$\chi^2 = T_{\rm obs} \int df_{\rm b} \frac{\left(\Omega_{\rm t}\left(\theta, f_{\rm b}\right) - \Omega_{\rm d}(f_{\rm b})\right)^2}{\Omega_{\rm n}^2(f_{\rm b})},\tag{5.7}$$

where $T_{\rm obs}$ is the observation time, $\Omega_{\rm t}$ is the theoretical model for the gravitational energy density which includes a phase transition GW signature and LISA instrument noise. Here, $\Omega_{\rm d}$ is the data and the variance is given by $\Omega_{\rm n}^2(f)$. The appearance of $T_{\rm obs}$ arises due to the use of Eq. 5.5 to take into account of the $n_{\rm b}$ realisations in each frequency bin $f_{\rm b}$.

Evaluating the Fisher matrix at the maximum of the likelihood $\theta = \theta_0$

$$F_{ij} = \frac{\partial^2 l}{\partial \theta_i \partial \theta_j} \bigg|_{\theta = \theta_0} = T_{\text{obs}} \int df \frac{1}{\Omega_n^2} \frac{\partial^2 \Omega_{\text{t}}(\theta, f)}{\partial \theta_i \partial \theta_j}.$$
(5.8)

The inverse of the FM is the covariance matrix, the diagonal elements of which give an approximation of the uncertainty in any given parameter. We use Fisher matrix analysis in preliminary exploration of LISA's sensitivity to phase transition parameters presented in Chapter 6.

5.2 Markov chain Monte Carlo inference

To go beyond the Fisher matrix analysis we can use Markov chain Monte-Carlo (MCMC) methods to evaluate the posterior distribution. The posterior defined via Bayes theorem Eq. 5.1 is often difficult to evaluate. The likelihood \mathcal{L} may not be analytically defined or be a complicated function that may be difficult (or impossible) to integrate and is often a high dimensional object. The normalisation (marginal likelihood) defined in Eq. 5.2 is also difficult to evaluate; this means one often knows only the posterior up to a constant.

Quantities derived from the posterior, such as the variance, are difficult to calculate directly. Instead one can sample the posterior using Monte Carlo (MC) methods [90–92]. In general the MC approach works by randomly sampling many points in the posterior to build up a picture of the posterior distribution around its local maxima which allows the estimation of the mean, variance and covariances. We will use a specific MC approach Markov chain Monte Carlo methods, which construct a chain of points that meet a characteristic criteria (described below) and sample for a long time. The age of computers has made (MC)MC methods a tractable way of solving statistical problems.

A major benefit of MCMC methods is they don't require a fully analytic function of the properly normalised pdf $\mathcal{P}(\mathcal{D}|\mathcal{M})$ Eq. 5.1. This is avoided by only calculating ratios of pdfs at pairs of locations in the chain. Other benefits of MCMC inferences are they are easy to implement, efficient, scalable and parallelisable. Different MCMC sampler codes have variations in specifications but in general they are based on Metropolis-Hastings samplers. In what follows we describe the Metropolis-Hastings (MH) algorithm as done in [92]. An MCMC chain is a list of sampled parameter values starting from a randomly selected point in the parameter space θ_0 . At the n^{th} step in the chain we have the parameter θ_n . In the MH algorithm a candidate θ' for the subsequent point is generated from an an arbitrary probability density known as the proposal density $q(\theta'|\theta_n)$. The proposal density is often a Gaussian centred on θ_n . An overview of how the next step in the chain θ_{n+1} is accepted is as follows:

- Draw a candidate θ' from the proposal distribution $q(\theta'|\theta_n)$.
- Draw a random number 0 < r < 1 from the uniform distribution.
- Evaluate the posterior at the candidate θ' and at the previous sample θ_n , $\mathcal{P}(\theta'|\mathcal{D}, \mathcal{M})$ and $\mathcal{P}(\theta_n|\mathcal{D}, \mathcal{M})$ (see Eq. 5.1).
- The acceptance rejection criteria for θ' is governed by the quantity:

$$a = \frac{\mathcal{P}(\theta'|\mathcal{D}, \mathcal{M})}{\mathcal{P}(\theta_n | \mathcal{D}, \mathcal{M})}.$$
(5.9)

• If a > r the new candidate is accepted and $\theta_{n+1} = \theta'$, otherwise the previous sample is repeated and $\theta_{n+1} = \theta_n$.

From the proposal distribution $q(\theta'|\theta_n)$ we can see each subsequent point θ_{n+1} is chosen from a distribution about the current point θ_n , meaning neighbouring points in a chain are correlated and distant ones are not. An implication of this is the chain will eventually forget its starting point. In practice the beginning of chains are often referred to as the burn-in and are discarded in post-processing (to remove impact of start point). As the quantity *a* uses a ratio of values of the posterior distribution we see how the MCMC avoids the necessity to have a normalised marginal posterior, avoiding having to perform the integral of the marginal likelihood.

For the above method to work it must be equally probable for the chain to take steps in one direction of parameter space as any other. This is fulfilled with the detailed balance-condition on the proposal pdf

$$q(\theta'|\theta) = q(\theta|\theta'). \tag{5.10}$$

A key question when performing a MCMC simulation is how long should it run for in order for it to have sufficiently sampled the posterior distribution? In practice this is challenging, for an example the "true" posterior may have multiple modes of high probability but it is possible that the MCMC has only sampled a subset of these high probability modes. In general qualitative terms there are a few ways one can assume the chain is well converged: if the chain has traversed the high probability region of the posterior many times in the length of the chain, if the shape of the posterior from the first half of the chain is comparable to that of the second half of the chain, or if the posteriors from multiple independent starting points are similar.

In this thesis we will use the a code for Bayesian analysis in Cosmology (Cobaya) as our MCMC sampler [93]. For the convergence checks of MCMC runs Cobaya uses a generalised version of the Gelman-Rubin statistic R-1, implemented as described in [94]. The R-1 statistic measures the variance between the means of different chains from the same run (or different segments of the same chain if there is only one). When R-1 is smaller than the stopping criteria twice in a row a further check on the bounds of the percentage confidence interval is performed, if it passes this second check the run will end.

Prior distributions $\pi(\theta|\mathcal{M})$ are used to refine the MCMC search and allow any prior knowledge about the parameter space to be passed to the sampler. This can prevent the chain exploring any theoretically forbidden regions of parameter space, or restrict the chain to explore the parameter space that is relevant for the given experiment. The implementation of sensible priors can speed up the evaluation time of an MCMC chain.

Chapter 6

Observational prospects for phase transitions at LISA: Fisher matrix analysis

Abstract

A first order phase transition at the electroweak scale would lead to the production of gravitational waves that may be observable at upcoming space-based gravitational wave (GW) detectors such as LISA (Laser Interferometer Space Antenna). As the Standard Model has no phase transition, LISA can be used to search for new physics by searching for a stochastic gravitational wave background. In this work we investigate LISA's sensitivity to the thermodynamic parameters encoded in the stochastic background produced by a phase transition, using the sound shell model to characterise the gravitational wave power spectrum, and the Fisher matrix to estimate uncertainties. We explore a parameter space with transition strengths $0.01 < \alpha < 0.5$ and phase boundary speeds 0.4 < $v_{\rm w}$ < 0.9, for transitions nucleating at $T_{\rm n}$ = 100 GeV, with mean bubble spacings 0.1 and 0.01 of the Hubble length, and sound speed $c/\sqrt{3}$. We show that the power spectrum in the sound shell model can be well approximated by a four-parameter double broken power law, and find that the peak power and frequency can be measured to approximately 10% accuracy for signal-to-noise ratios (SNRs) above 20. Determinations of the underlying thermodynamic parameters are complicated by degeneracies, but in all cases the phase boundary speed will be the best constrained parameter. Turning to the principal components of the Fisher matrix, a signal-to-noise ratio above 20 produces a relative uncertainty less than 3%in the two highest-order principal components, indicating good prospects for combinations of parameters. The highest-order principal component is dominated by the wall speed. These estimates of parameter sensitivity provide a preliminary accuracy target for theoretical calculations of thermodynamic parameters.

6.1 Introduction

With the initial LIGO detection of a black hole merger [15], the multi-messenger observation of a merging neutron star binary [95] and the recent observation of an intermediate mass black hole merger [96] we are beginning to realise the discovery potential of gravitational waves (GWs). LIGO and other ground-based detectors are optimised for stellarorigin black holes and sensitive to the 10 Hz -10 kHz frequency band. The exploration of signals from the merger of the much larger black holes at the centre of galaxies will take place at future space-based GW observatories, where longer arm lengths open up the 10^{-4} Hz to 10^{-1} Hz range of the GW spectrum. Such experiments include the ESA-NASA mission LISA [28], Taiji [64] and TianQin [65], all aiming for launch in the mid-2030s. LISA and TianQin have both launched test satellites, the final LISA pathfinder mission results [66] and the initial results from TianQin-1 [67] are promising.

As well as massive black hole mergers, expected astrophysical sources in the millihertz band include galactic binaries [73], extreme mass ratio binaries [74] and precursors for stellar origin black hole mergers [75]. Cosmological sources could include stochastic gravitational wave backgrounds (SGWBs) from inflation, cosmic strings and cosmological phase transitions [76].

In this paper we focus on a SGWB from cosmological phase transitions, specifically those from around 10 picoseconds after the Big Bang, when it is expected the electroweak symmetry broke. In the Standard Model this process occurs via crossover [20, 34], and the GW signal is expected to be negligible at observable frequencies [97]. However, there are numerous extensions to the Standard Model in which a first order phase transition is possible. A review of possible extensions can be found in Ref. [8].

In such theories, below the critical temperature, bubbles of the stable phase spontaneously nucleate in the surrounding metastable phase. These bubbles expand, collide and merge until only the stable phase remains, leaving behind a characteristic spectrum of sound waves, which are a persistent source of GWs [24–26]. The collision of bubble walls [48, 98–100] and turbulent flows [101–104] also generate GWs. Here, we consider only the contribution from sound waves as they are currently expected to be the dominant source over a wide range of parameters [8].

If the critical temperature is in the range 100 – 1000 GeV, the peak frequency of the GW power spectrum can be in the millihertz range, and potentially detectable at a space-based observatory. This means that the discovery potential of GW observations includes fundamental physics beyond the Standard Model. The new physics may include a mechanism for baryogenesis [38], a strong motivation for considering Standard Model extensions with a first order phase transition. For a recent introduction to baryogenesis see Ref. [23], and to phase transitions in the early Universe see Refs. [41, 105]. For a review of the prospects for probing physics beyond the Standard Model, see Ref. [8].

Numerical simulations for the acoustic contribution to the GW signature from a first order phase transition [24–26] have shown that the sound waves generated by the expanding bubble determine the GW power spectrum. The simulations motivate a simple broken power-law model for the sound wave contribution used by the LISA cosmology working group [8]. The uncertainties that arise from various levels of approximations have been explored in [106].

Currently the most sophisticated model for computing the GW power spectrum from sound waves is the sound shell model (SSM) [4, 27]. The SSM shows how the GW power spectrum can be computed from the velocity power spectrum of the fluid, which in turn is dependent on a few key thermodynamic parameters that can be calculated from the underlying theory. These key parameters effect the overall amplitude, frequency scale and the detailed shape of the power spectra. In its simplest form, there are four thermodynamic parameters: the bubble nucleation temperature, the transition rate, the transition strength, and the bubble wall speed. All are in principle computable from an underlying theory, making them the interface between observation and theory. At the moment there are significant uncertainties in these calculations [107]. Our work can be used to set targets for future developments of theoretical methods.

The SSM predicts two important frequency scales in the power spectrum, and a double broken power law has been proposed as an analytic fit [4]. The functional form depends on the peak power, peak frequency, the ratio of the frequencies of the two breaks and the slope between the two breaks. We call these the spectral parameters, and distinguish them from the thermodynamic parameters discussed above. We show that the double broken power law form is much closer to the SSM prediction than the single broken power law fit given in [8].

In this work we use the Fisher matrix [89] to explore LISA's ability to extract parameters that describe a SGWB from a first order phase transition, also examining the effect of the expected foregrounds from galactic and extragalactic compact binaries. The Fisher matrix is known to overestimate uncertainties, especially when there are degeneracies amongst parameters, as is thought to be the case with the thermodynamic parameters. Despite this, we can expect the Fisher matrix will give an insight into parameter sensitivity and provide a better understanding of the degeneracies themselves.

We calculate the relative uncertainty both of the spectral parameters and the thermodynamic parameters as described above, with and without foregrounds, over a range of fiducial models. We focus on LISA but the methods could be easily adapted to other missions by altering the noise model. This complements general power law searches in mock LISA data [108–110] Fisher matrix analysis for a single broken power law with LISA, DECIGO and BBO mock data [111], searches for cosmological phase transition SGWB in LIGO and NANOGrav data [112, 113], and methods for general SGWB searches where the search is agnostic about the spectral shape of the GW background [83, 114].

For our fiducial models we focus on a thermodynamic parameter space motivated by an electroweak-scale transition, by relevance for observation, and also by the reliability of predictions. The electroweak scale motivates the choice of nucleation temperature $T_{\rm n} = 100$ GeV. Relevance for observation motivates examining supercooled transitions with mean bubble spacing to Hubble length ratio $r_* = 10^{-1}$ and 10^{-2} , as much smaller values would render the signal too weak. The reliability of the sound shell model predictions can be tested against numerical simulations [7] in the range of wall speeds $0.24 < v_{\rm w} < 0.92$ and with transition strength parameter $0.01 < \alpha < 0.5$. We study the range $0.4 < v_{\rm w} < 0.9$, as lower wall speeds will also probably not be observable at LISA.

This parameter space produces signals with gravitational wave density fraction today up to $\Omega_{\rm p,0} \sim 10^{-10}$ and peak frequencies $f_{\rm p,0}$ in the range 10^{-2} mHz to 5 mHz, which can produce SNRs well over 100. A transition with $r_* = 0.1$ should produce an observable signal over most of the parameter space.

Of the spectral parameters, LISA will be most sensitive to the peak power and peak
frequency, reaching approximately 10% uncertainty in the peak power and frequency for signal-to-noise ratio (SNR) above 20. For $r_* = 0.1$, SNR 20 can be reached over most of the range $0.5 < v_w < 0.8$ and $\alpha > 0.2$. For $r_* = 0.01$, stronger transitions are required to reach the same SNR.

Of the thermodynamic parameters, there is greatest sensitivity to the wall speed. At SNR = 20 the relative uncertainty in the wall speed is 10% in some regions of the thermodynamic parameter space, but the sensitivity to the other parameters is reduced by degeneracies. Examining the principal components, one finds an uncertainty of 3% or better for the two highest-order components, at SNR = 20. Hence there are good prospects for combinations of parameters. The best-determined principal component is dominated by the wall speed. The second-best has α as the most important contribution, but other parameters also contribute.

If a parameter combination could be predicted in the light of other data, the prospects for estimating the other parameters would be much better. As a simple example, we consider a case where the nucleation temperature is known, for a transition in which the mean bubble spacing parameter is $r_* = 0.1$. Here, the phase transition strength and the mean bubble spacing can be constrained to 10% and 30% respectively. If the galactic binary foreground can be removed, the uncertainty in the phase transition strength can be as low as 10% for transitions with $\alpha \simeq 0.1$.

The paper is organised as follows. In Sec. 6.2 we review the production of GWs from a first order phase transition in the early universe, how they relate to the underlying thermodynamic parameters, and introduce the SSM [27]. The setup we consider for LISA and the noise model is outlined in Sec. 8.3. In Sec. 6.4 we describe our method for calculating the Fisher matrix, relative uncertainties and principal components. The relative uncertainties in the spectral and thermodynamic parameters are presented in Sec. 8.6. The discussion of the results are given in Sec. 6.6.

In this work we set c = 1 and $k_{\rm B} = 1$, unless otherwise specified.

6.2 Gravitational waves from a first order cosmological phase transition

6.2.1 Cosmological phase transitions

As the universe expanded and cooled, significant changes of in the equation of state must have occurred at temperatures of around 100 GeV, when elementary particle rest masses were generated, and at 100 MeV, when quarks and gluons became confined into hadrons.

It is known that both of these changes happened via a smooth cross-over in the Standard Model [115], [20, 34], but in extensions of the Standard Model first order electroweakscale transitions are common (see Ref. [8] for a survey). Such phase transitions are often associated with a change in the symmetry of the plasma, accompanied by a change in the value of an order parameter, which in the case of the electroweak transition is the magnitude of the Higgs field.

In a symmetry-breaking phase transition such as the electroweak transition, one often refers to the high-temperature phase as the "symmetric" phase and the low-temperature phase as the "broken" phase.

In a first order phase transition, at a critical temperature T_c there are two degenerate minima of the free energy separated by a barrier. As the temperature cools below T_c , the broken phase becomes lower in free energy, and the system can move to it via localised thermal or quantum fluctuations. This leads to bubbles of broken phase nucleating within the symmetric region. These bubbles expand due to the pressure difference between the interior and exterior, inevitably present as the pressure is minus the free energy density. The bubbles collide and merge until only the broken phase remains. Some of the latent heat of the transition is converted into kinetic energy of the cosmological fluid surrounding the bubbles, which is a source of shear stress, leading to the production of gravitational waves.

Now we introduce the key thermodynamic parameters that determine the gravitational wave signature from a first order phase transition (see e.g. [41]). The first of these is the nucleation temperature T_n , which we define as the peak of the globally-averaged bubble nucleation rate. The Hubble rate at the nucleation temperature sets the frequency scale of the GW power spectrum. The second one is the nucleation rate parameter β , which is often given as a ratio with H_n , the Hubble parameter at T_n

$$\tilde{\beta} = \frac{\beta}{H_{\rm n}} \sim \frac{v_{\rm w}}{H_{\rm n}R_*},\tag{6.1}$$

where $v_{\rm w}$ is the speed of the expanding bubble wall and R_* is the mean bubble spacing. From this we see that $\tilde{\beta}$ controls R_* , which in turn sets the characteristic wavelength of the gravitational radiation. The constant of proportionality is $(8\pi)^{1/3}$ [44] for detonations, but for deflagrations it is also dependent on α and $v_{\rm w}$, as the nucleation rate is reduced by the reheating of the fluid in front of the bubble wall. In view of this uncertainty, it is more convenient to work in terms of R_* , and more precisely the Hubble-scaled mean bubble spacing

$$r_* = H_{\rm n} R_*. \tag{6.2}$$

Note that β^{-1} is the time taken for a bubble wall to move a distance R_* , and therefore has an interpretation as the duration of the phase transition.

Another key parameter in the generation of GWs is the phase transition strength parameter α

$$\alpha = \frac{4}{3} \frac{\Delta \theta}{w_{\rm s}} \Big|_{T=T_{\rm n}} \tag{6.3}$$

where w_s is the enthalpy of the fluid in the symmetric phase, and $\Delta \theta = \theta_s - \theta_b$, where θ is a quarter of the trace of the energy-momentum tensor, and subscripts s and b denote symmetric and broken phases. The trace difference is the energy available to be converted to shear stress energy and thus GW power. A stronger transition means more energy is converted to shear stress energy and a larger overall amplitude for the GW signal.

The fourth parameter is the wall speed, v_w , which (with α) determines the motion of the surrounding plasma induced by the passing bubble wall. Wall speeds are split into three categories relative to the speed of sound c_s . Deflagrations occur when $v_w < c_s$, where the surrounding fluid is pushed in front of the expanding phase transition wall. When v_w is greater than a certain critical speed c_J , the Jouguet speed, the motion in the plasma is entirely behind the bubble wall, and the fluid configuration is called a detonation. The Jouguet speed is given by

$$c_{\rm J} = c_{\rm s} \frac{\left(1 + \sqrt{\alpha(2+3\alpha)}\right)}{(1+\alpha)} \tag{6.4}$$

If the wall speed is between the sound speed and the Jouguet speed, the velocity profile is a mix between deflagrations and detonations, non-zero both in front and behind the bubble wall. These supersonic deflagrations [46], sometimes called hybrids [40], are very finely tuned, and it is not clear that they exist in a real fluid.

The sound speeds in the two phases are also potentially important parameters [116, 117]. To simplify this first analysis, we will take them both to be the ultrarelativistic value $c_{\rm s} = 1/\sqrt{3}$, to focus on LISA's sensitivity to the four parameters $(T_{\rm n}, \alpha, r_*, v_{\rm w})$.

6.2.2 Gravitational waves from a first order phase transition

In a first order transition driven by thermal fluctuations, sound waves created by the expanding bubbles are the dominant source of gravitational waves [24-26].

Approximate fits to the numerical power spectra are available in [26]. They have a fixed broken power law shape, with peak intensity and frequency depending on the four thermodynamic parameters in an easily computable way [8]. The peak intensity depends on α , r_* and v_w , while the peak frequency depends on T_n and r_* . It is clear that there are likely to be degeneracies in the power spectrum with respect to the thermodynamic parameters, which would intrinsically limit LISA's ability to measure them individually.

However, the simulations make it clear that the shape of the GW power spectrum also depends on wall speed and transition strength, and such dependence is found in a more sophisticated theoretical framework, the sound shell model (SSM) [4, 27]. We therefore use the sound shell model to model the GW power spectrum from phase transitions, and investigate LISA's constraining power on its parameters. While the sound shell model has not been tested in detail against a wide range of numerical simulations, it can act as guidance for data analysis techniques aimed at extracting phase transition parameters from phase transitions.

To characterise how the energy density in GWs is distributed over frequencies today we introduce the gravitational wave power spectrum [118]

$$\Omega_{\rm gw,0}(f) \equiv \frac{1}{\rho_{\rm c,0}} \frac{d\rho_{\rm gw,0}}{d\ln f},\tag{6.5}$$

where f is frequency and $d\rho_{gw}$ is the gravitational wave energy density within a frequency interval df. The critical density is $\rho_c = 3H^2/8\pi G$, where H is the Hubble rate, G is the gravitational constant and c is the speed of light. Quantities evaluated at the present day are given the subscript 0. For the Hubble constant H_0 we take the central value measured by the Planck satellite $H_0 = 67.4 \,\mathrm{km \ s^{-1} Mpc^{-1}}$ as given in [29]. The general form of the gravitational wave power spectrum from a first order phase transition is

$$\Omega_{\rm gw}(z) = 3K^2(v_{\rm w}, \alpha) \left(H_{\rm n}\tau_{\rm v}\right) \left(H_{\rm n}R_*\right) \frac{z^3}{2\pi^2} \tilde{P}_{\rm gw}\left(z\right), \tag{6.6}$$

where R_* is the mean bubble spacing, $z = kR_*$, k is comoving wavenumber and $K(v_w, \alpha)$ is the fraction of the total energy converted into kinetic energy of the fluid. The Hubble rate at nucleation is H_n , τ_v is the lifetime of the shear stress source, the factor R_* appears as an estimate of the source coherence time and $\tilde{P}_{gw}(z)$ is the dimensionless spectral density. Eq. (6.6) can be regarded as the definition of \tilde{P}_{gw} . Its integral (denoted $\tilde{\Omega}_{gw}$ in Refs. [4, 25, 26]) depends only weakly on the thermodynamic parameters, taking values of order 10^{-2} .

As the notation of Eq. (6.6) suggests, the important parametric dependences of the total power are through the kinetic energy fraction, the source lifetime, and the source coherence time. The kinetic energy fraction depends only on the transition strength α and the wall speed v_w . The lifetime of the GW source τ_v is the shorter of the two timescales, the Hubble time H_n^{-1} and the fluid flow lifetime τ_v , which is estimated as R_*/\sqrt{K} , the timescale for non-linearities to become important. Denoting the ratio of the two timescales by $x = H_n R_*/\sqrt{K}$, we approximate the Hubble-scaled source lifetime as [119]

$$H_{\rm n}\tau_v \simeq \left(1 - \frac{1}{\sqrt{1+2x}}\right). \tag{6.7}$$

From this we see that even if the flow persists over many Hubble times it does not continue to contribute to the GW power spectrum. For future convenience we will combine the factors of the source lifetime and source coherence time into one,

$$J = H_n R_* H_n \tau_v = r_* \left(1 - \frac{1}{\sqrt{1+2x}} \right).$$
 (6.8)

The sound shell model [4, 27] predicts the gravitational wave power spectrum as a numerical function of a given set of thermodynamic parameters (T_n, α, r_*, v_w) and scaled wavenumbers z. We denote this prediction $\Omega_{gw}^{ssm}(z)$. The shape of the power spectrum has significant dependencies on v_w and α .

Recent 3d-hydro simulations for α up to $\mathcal{O}(1)$ (strong transitions) found that as transition strength increases the efficiency of fluid kinetic energy production becomes less than previously expected [7]. For deflagrations this is thought to be due to reheating which occurs in front of the expanding bubbles, which leads to a reduction in pressure difference, and a slowing of the bubble wall. The reduction in kinetic energy production leads to a suppression in gravitational waver power, which we approximate by a factor $\Sigma(v_{\rm w}, \alpha)$. The estimation of this suppression factor from the numerical simulations is described in Appendix 6.7.1.

The gravitational wave power spectrum at dimensionless comoving wavenumber z just after the transition, and before any further entropy production, is then

$$\Omega_{\rm gw}(z) = \Omega_{\rm gw}^{\rm ssm}(z) \Sigma(v_{\rm w}, \alpha), \tag{6.9}$$

where $\Omega_{\rm gw}^{\rm ssm}(z)$ is the sound shell model prediction.

Today the power spectrum at physical frequency f is

$$\Omega_{\mathrm{gw},0}(f) = F_{\mathrm{gw},0}\Omega_{\mathrm{gw}}(z(f)), \qquad (6.10)$$

where

$$F_{\rm gw,0} = \Omega_{\gamma,0} \left(\frac{g_{s0}}{g_{s*}}\right)^{\frac{4}{9}} \frac{g_*}{g_0} = (3.57 \pm 0.05) \times 10^{-5} \left(\frac{100}{g_*}\right)^{\frac{1}{3}},\tag{6.11}$$

is the power attenuation following the end of the radiation era. Here, $\Omega_{\gamma,0}$ is the photon energy density parameter today, g_s denotes entropic degrees of freedom and g describes the pressure degrees of freedom. In both cases the subscripts 0 and * refer to their value today and the value at the time the GWs were produced respectively. We evaluate $F_{\text{gw},0}$ with the values given in [8], and use a reference value $g_* = 100$.

We convert from dimensionless wavenumber z to frequency today by taking into account redshift

$$f = \frac{z}{r_*} f_{*,0} \tag{6.12}$$

where [8]

$$f_{*,0} = 2.6 \times 10^{-6} \,\mathrm{Hz} \left(\frac{T_{\mathrm{n}}}{100 \,\mathrm{GeV}}\right) \left(\frac{g_{*}}{100}\right)^{\frac{1}{6}}$$
 (6.13)

is the Hubble rate at the phase transition redshifted to today. We assume the phase transition takes place well within one Hubble time so all frequencies throughout the transition have the same redshift.

We compute the scale-free gravitational wave spectral density

$$\hat{\mathcal{P}}_{\rm gw}(z) = 3K^2 \frac{z^3}{2\pi^2} \tilde{P}_{\rm gw}(z) \,, \tag{6.14}$$

using the PTtools python module, which uses the SSM to directly compute $\hat{\mathcal{P}}_{gw}$ for a given v_w and α [4]. In this work PTtools was used to evaluate the power spectra at 100

logarithmic spaced z values between 1 and 1000. Within PTtools the number of points used in the fluid shell profiles was set to be 70000, with 7000 wavevectors used in in the velocity convolution integrations. The bubble lifetime distribution, taken to be exponential, was integrated with 200 linearly spaced values between 0 and $20\beta^{-1}$. The high wavenumber resolution was used to ensure the integration over the velocity power spectrum converges. The above is encoded in PTtools and returns Eq.(6.14). We explore the prospects for



(a) Fixed: $\alpha = 0.2$, $r_* = 0.1$, $T_n = 100$ GeV. (b) Fixed: $v_w = 0.6$, $r_* = 0.1$, $T_n = 100$ GeV.



(c) Fixed: $v_{\rm w} = 0.6$, $\alpha = 0.2$, $T_{\rm n} = 100$ GeV. (d) Fixed: $v_{\rm w} = 0.6$, $\alpha = 0.2$, $r_* = 0.1$.

Figure 6.1: Gravitational wave power spectra for a first order phase transition calculated using the sound shell model, Eq.(6.10). In each panel we vary one of the thermodynamic parameters $v_{\rm w}$ (wall speed), α (phase transition strength), r_* (Hubble-scaled bubble spacing) and $T_{\rm n}$ (nucleation temperature). Shown also in solid black is the LISA instrument noise given by the science requirements (SR) document sensitivity curve (Eq. (6.26), [6]). The dashed line shows the predicted foreground from extragalactic binaries, Eq. (6.27), along with a grey uncertainty band. The dash-dotted line shows the estimated foreground from unresolved galactic binaries, Eq. (6.29). Signal-to-noise ratios for $T_{\rm n} = 100$ GeV and $r_* = 0.1, 0.01$ are given in Fig. 6.4.

estimation in the parameter space $0.4 < v_w < 0.9$, $\alpha < 0.5$, $r_* = 0.01, 0.1$ and $T_n = 100$ GeV.

We show, using this framework to calculate the GW power spectra, how varying the thermodynamic parameters effects the shape, frequency scales and amplitudes of the power spectrum in Fig. 6.1.

From Fig. 6.1a we see the wall speed $v_{\rm w}$ has a strong effect on the shape of the power spectrum, especially between the sound speed and the Jouguet speed. At low $v_{\rm w}$ the power spectrum is narrow and as $v_{\rm w}$ approaches the speed of sound the peak broadens, due to the narrowing of the sound shell around expanding bubbles. Once $v_{\rm w} > c_{\rm s}$ the peak begins to narrow again. As $v_{\rm w}$ increases we also see a decrease in overall amplitude, because the efficiency of converting latent heat into fluid motion depends on $v_{\rm w}$.

As the strength of the phase transition, α , increases so does the overall amplitude of the GW power spectrum, as more kinetic energy is deposited into the plasma (see Fig. 6.1b). In Fig. 6.1c we note that r_* contributes both to the frequency scale and overall amplitude of the power spectrum. In Fig. 6.1d we see that the nucleation temperature T_n affects only the frequency scale see Eq.(6.13).

We note that there is more structure in these power spectra than can be captured by a broken power law, motivating an improved approximation in the next section. The precise functional dependence on the thermodynamic parameters is likely to change as our understanding improves, but our analysis can be easily adapted to include future developments. We believe that the double broken power law form is likely to remain adequate.

6.2.3 Double broken power law

The full calculation in the SSM can be computationally intensive when one is calculating many power spectra over a large parameter space. This motivates the use of an analytic fit that that can be used for rapid evaluation. The LISA Cosmology Working Group put forward a single broken power law to describe the acoustic contribution to the GW power spectrum, with two parameters, the peak amplitude Ω_p and the peak frequency f_p , whose scale is set by the bubble spacing R_* [8].

In the SSM there are in fact two characteristic length scales, R_* and the width of the

sound shell $\Delta R_* \sim |v_w - c_s|/\beta$, which indicate a double broken power law may be a good fit for the power spectrum [4]. A general form for such a double broken power law can be defined by four spectral parameters (Ω_p, f_p, r_b, b), with the power spectrum taking the form

$$\Omega_{\rm gw}^{\rm fit} = F_{\rm gw,0} \Omega_{\rm p} M(s, r_{\rm b}, b) \tag{6.15}$$

where $\Omega_{\rm p}$ is the peak power of the power spectrum, $s = f/f_{\rm p}$, $f_{\rm p}$ is the frequency corresponding to $\Omega_{\rm p}$ and $r_{\rm b} = f_{\rm b}/f_{\rm p}$ describes the ratio between the two breaks in the spectrum. The parameter *b* defines the spectral slope between the two breaks. The spectral shape $M(s, r_{\rm b}, b)$ is a double broken power law with a spectral slope 9 at low frequencies and -4 at high frequencies.

$$M(s, r_{\rm b}, b) = s^9 \left(\frac{1 + r_{\rm b}^4}{r_{\rm b}^4 + s^4}\right)^{(9-b)/4} \left(\frac{b+4}{b+4 - m + ms^2}\right)^{(b+4)/2}.$$
 (6.16)

In this function, m has been chosen to ensure that for $r_{\rm b} < 1$ the peak occurs at s = 1 and $M(1, r_{\rm b}, b) = 1$, giving

$$m = (9r_{\rm b}{}^4 + b) / (r_{\rm b}{}^4 + 1).$$
(6.17)

Ultimately, we want to connect these spectral parameters quantitatively with the thermodynamic parameters in order to understand the underlying theory, however these relationships are not straightforward. An outline of how the spectral parameters depend on the thermodynamic parameters is as follows

$$\Omega_{p,0} = F_{gw,0}J(r_*, K(\alpha, v_w)) \hat{\Omega}_p(\alpha, v_w) \Sigma_{ssm}(v_w, \alpha)$$

$$f_{p,0} = f_{*,0}(T_n) z_p(\alpha, v_w) / r_*$$

$$r_b = r_b(\alpha, v_w)$$

$$b = b(\alpha, v_w),$$
(6.18)

where $\hat{\Omega}_{p}$ is the maximum of $\Omega_{gw}^{ssm}(z)$, z_{p} is z at the peak power and z_{b} is the scale-free wavenumber at the second break. J is the timescale pre-factor defined in Eq. (6.8).

In Fig. 6.2 we show the peak power today $\Omega_{p,0}^{\text{ssm}}$ and the corresponding peak frequency $f_{p,0}^{\text{ssm}}$ calculated in the SSM, using Eq.(6.10), for the v_{w} and α parameter space of interest, with $r_* = 0.1$ and $T_n = 100$ GeV. Fig. 6.3 shows the corresponding best fit spectral parameters for the double broken power law model.

A comparison of the quality of the fit of the single broken power law and double broken power law models to the GW power spectrum from a first order phase transition, as described by the sound shell model, can be found in Appendix 6.7.2.



62

Figure 6.2: The peak power today $\Omega_{p,0}^{ssm}$ and the peak frequency today $f_{p,0}^{ssm}$ calculated with the sound shell model, for a range of wall speeds, $0.4 < v_w < 0.9$, and phase transition strengths, $0.01 < \alpha < 0.5$, The Hubble-scaled mean bubble spacing $r_* = 0.1$ and nucleation temperature T_n . The turquoise dashed line shows the Jouguet speed Eq. (6.4).



Figure 6.3: The best fit spectral parameters from fitting the double broken power law model to the power spectra from the sound shell model Eq. (6.10). $\Omega_{p,0}$ is the peak power today with the Hubble-scaled mean bubble spacing $r_* = 0.1$ and $T_n = 100$ GeV, $f_{p,0}$ is the corresponding position of the peak (scaled using Eq.(6.18)), r_b is the ratio of the frequency positions of the two breaks in the spectrum and b the spectral slope of the power law between the two breaks. The turquoise dashed line is the Jouguet speed, Eq. (6.4).

6.3 Noise model

6.3.1 LISA sensitivity curve

The sensitivity of a gravitational wave detector can be characterised by the effective noise power spectral density¹ (PSD) S(f), which is the gravitational strain spectral density required to produce a signal equal to the instrument noise N(f). If $\mathcal{R}(f)$ is the detector response function for gravitational waves,

$$S(f) \equiv \frac{N(f)}{\mathcal{R}(f)}.$$
(6.19)

LISA [28] is designed to be a triangular constellation of spacecraft connected by three pairs of laser links, through which changes in the distance between three pairs of freefalling test masses can be measured. The changes in distance are monitored through the differences in phase between the local oscillators and the remote spacecraft oscillators, communicated by the laser. The phase differences can be combined in different ways with different time delays to eliminate the laser noise [79, 120], by using the technique of time delay interferometry (TDI).

We work with the three noise-orthogonal TDI variables A, E and T as described in [84, 121]. The T channel is insensitive to GWs at low frequencies. We will make the simplifying assumption that the T channel allows us to completely characterise the instrument noise.

The instrument noise in LISA is expected to be dominated by two main sources: the test mass acceleration noise (acc), due to local disturbances of the test mass, and the optical metrology noise (oms) which includes shot noise. As outlined in the LISA Science Requirements Document [6] the target for the single link optical path-length fluctuations is

$$P_{\rm oms}(f) = \left(\frac{1.5 \times 10^{-11} {\rm m}}{L}\right)^2 {\rm Hz}^{-1}, \tag{6.20}$$

where $L = 2.5 \times 10^9$ m is the constellation arm length. The single test mass acceleration noise target is

$$P_{\rm acc}(f) = \left(\frac{3 \times 10^{-15} \,\mathrm{m\,s}^{-2}}{(2\pi f)^2 L}\right)^2 \left(1 + \left(\frac{0.4 \,\mathrm{mHz}}{f}\right)^2\right) \,\mathrm{Hz}^{-1}.$$
 (6.21)

¹In this paper we consider two-sided power spectral densities, meaning the frequencies range from $-f_{\text{max}}$ to $+f_{\text{max}}$

In the A and E channels the instrument noise is then (see e.g. [5])

$$N_A = N_E = \left[\left(4 + 2\cos\left(f/f_{\rm t}\right) \right) P_{\rm oms} + 8 \left(1 + \cos\left(f/f_{\rm t}\right) + \cos^2\left(f/f_{\rm t}\right) \right) P_{\rm acc} \right] |W|^2, \quad (6.22)$$

where $f_t = c/(2\pi L)$ is the transfer frequency and c is the speed of light. The function $W(f, f_t) = 1 - \exp(-2if/f_t)$ is the modulation caused by one round trip of a signal along a link. We use a simplified version of Eq. (6.3.1) with $\cos(f/f_t) = 1$,

$$N_A(f) \simeq (6P_{\rm oms}(f) + 24P_{\rm acc}(f)) |W(f)|^2, \tag{6.23}$$

which gives a reasonable fit to the true sensitivity curve.

The gravitational wave response function for the A and E channels is known only numerically,² but an approximate fit is

$$\mathcal{R}_A^{\text{Fit}} = \mathcal{R}_E^{\text{Fit}} \simeq \frac{9}{20} |W|^2 \left[1 + \left(\frac{f}{4f_{\text{t}}/3}\right)^2 \right]^{-1}.$$
 (6.24)

We can now construct the approximate noise power spectral density for the A and E channels using Eqs. (6.3.1), (6.23) and (6.24):

$$S_{\rm A} = S_{\rm E} = \frac{N_A}{\mathcal{R}_A} \simeq \frac{40}{3} \left(P_{\rm oms} + 4P_{\rm acc} \right) \left[1 + \left(\frac{f}{4f_{\rm t}/3} \right)^2 \right],$$
 (6.25)

in this work we will be interested in the sensitivity to the GW fractional energy density power spectrum, which is related to the PSD by

$$\Omega_{\rm ins} = \left(\frac{4\pi^2}{3H_0^2}\right) f^3 S_A(f) \tag{6.26}$$

we will refer to this as the LISA instrument noise

6.3.2 Extragalactic compact binaries

A stochastic gravitational wave background (SGWB) from a superposition of unresolved extragalactic compact binaries is expected in the millihertz GW frequency band [122]. This signature is expected to be stationary, Gaussian and isotropic, distinguishable only by its frequency spectrum from cosmological signatures, such as a SGWB from a first order phase transition. It is composed of signals from stellar origin black hole binaries, neutron star binaries, and white dwarf binaries. These objects include precursors to compact

²Recently, an analytic expression for the response function in the TDI X channel has been derived [87], but the A and E channels also require the response function of the XY cross-correlation.

binary mergers seen by the LIGO-Virgo collaboration [123]. We will refer to this as the extragalactic binary foreground (eb), which has the GW power spectrum

$$\Omega_{\rm eb}(f) = \Omega_{\rm ref,eb} \left(\frac{f}{f_{\rm ref,eb}}\right)^{\frac{2}{3}}.$$
(6.27)

We will assume that it is dominated by stellar origin black hole binaries, and take $\Omega_{\text{ref,eb}}$ to be the energy density of the LIGO-Virgo compact binaries at the reference frequency $f_{\text{ref,eb}} = 25$ Hz. The current estimate is $\Omega_{\text{ref,eb}} = 8.9^{+12.6}_{-5.6} \times 10^{-10}$ [123]. It is well below the instrument noise, and therefore not a significant contributor to the overall noise relevant for stochastic backgrounds. This foreground is shown in Fig.(6.1). The contribution to the amplitude $\Omega_{\text{ref,eb}}$ from black hole and neutron star binaries will be more accurately measured by LIGO/Virgo once it reaches design sensitivity [124, 125], and by future ground-based detectors that may be online at a similar time to LISA [61, 62].

6.3.3 Unresolved galactic compact binaries

A significant noise source for LISA is due to the large number of white dwarf binaries located within our galaxy [126, 127]. Some loud binaries will be individually resolvable, and as the mission progresses more will be identified. At any mission stage, unresolved binaries will produce a confusion noise, which can be estimated using an iterative subtraction procedure outlined in [128]. After a 4-year mission, estimates suggest around 20,000 of the estimated 20 million galactic binaries (gb) will be resolved, leaving a foreground³

$$S_{\rm gb}(f) = A\left(\frac{1\,{\rm mHz}}{f}\right)^{-7/3} \exp\left(-\left(\frac{f}{f_{\rm ref,gb}}\right)^a - bf\sin(cf)\right) \left[1 + \tanh\left(d\left(f_k - f\right)\right)\right],\tag{6.28}$$

where $A = 9 \times 10^{-38} \text{ mHz}^{-1}$ and $f_{\text{ref,gb}} = 1000 \text{ mHz}$. The parameters a, b, c and f_k depend on the observation time: for a 4-year observation period, a = 0.138, $b = -0.221 \text{ mHz}^{-1}$, $c = 0.521 \text{ mHz}^{-1}$, $d = 1.680 \text{ mHz}^{-1}$ and the frequency of the knee of the power spectrum is $f_k = 1.13 \text{ mHz}$. S_{gb} can be expressed in terms of energy density,

$$\Omega_{\rm gb} = \left(\frac{4\pi^2}{3H_0^2}\right) f^3 S_{\rm gb}(f), \tag{6.29}$$

which we will refer to as the galactic binary foreground, and show in Fig. 6.1. There is potential for this foreground to be extracted separately, due to the annual modulation in the signal as LISA's direction of maximum sensitivity sweeps past the galactic plane [108]. If no attempt to remove the annually modulated stochastic signals is made, galactic binaries will be a significant source of noise around 1 mHz. We will consider parameter

³In this work we use the correction to the sign of the coefficient b given in [129].

sensitivity both with and without the galactic binary foreground, to estimate upper and lower bounds.

In addition to the foregrounds considered here there are a number of other sources that may need to be considered when trying to extract a stochastic GW background from a first order phase transition. These include confusion noise from unresolved extreme mass ratio inspirals [130] and a foreground from unresolved massive black hole binaries [131]. In addition, extragalactic white dwarf binaries could contribute significantly to the compact binary foreground [132]. We choose to leave the inclusion of these foregrounds for future work: current models are not as well characterised, and, at least in the case of massive black hole binaries, are expected to be less significant.

6.3.4 Signal-to-noise ratio

As a first assessment of whether a signal is observable or not, one can calculate the signalto-noise ratio ρ by comparing the signal Ω_{gw} with the noise model Ω_n [82, 118]:

$$\rho = \sqrt{T_{\rm obs} \int_{f_{\rm min}}^{f_{\rm max}} df \frac{h^2 \Omega_{\rm gw,0}^2}{h^2 \Omega_{\rm n}^2}},\tag{6.30}$$

where Ω_n is the sum of all sources of noise. Our base noise model consists of the LISA instrument noise as given in Eq. (6.26), the extragalactic background Eq. (6.27) and the galactic binaries Eq. (6.29), so that

$$\Omega_{\rm n} = \Omega_{\rm ins} + \Omega_{\rm eb} + \Omega_{\rm gb}. \tag{6.31}$$

In this work we take the observation time $T_{\rm obs} = 4$ years, which is LISA's designed mission lifetime [28], so that we can use our noise model in combination with the prediction of the galactic binary foreground given in [128]. LISA's science operational time is expected to be $\approx 75\%$ to the total mission lifetime, but the mission may last up to 10 years.

In Fig. 6.4, we calculate ρ for the thermodynamic parameter space explored in this paper. The power spectra are described by Eq. (6.10). Generally, ρ is larger for stronger phase transitions, corresponding to larger α , and sensitivity to the wall speeds $v_{\rm w}$ peaks in the region of the speed of sound $c_{\rm s}$.

6.4 Fisher matrix analysis

An estimation of LISA's sensitivity to parameters that describe a first order phase transition can be obtained by Fisher matrix (FM) analysis. The FM is the curvature of a



Figure 6.4: The signal-to-noise ρ for different combinations of the wall speed $v_{\rm w}$, phase transition strength α , Hubble-scaled mean bubble spacing r_* , with the nucleation temperature $T_{\rm n} = 100$ GeV. In the left column the noise model includes the LISA instrument noise - Eq. (6.26), the foreground from unresolved stellar origin black hole binaries - Eq. (6.27). In the right hand column we also include the unresolved galactic binary foreground- Eq. (6.29). The turquoise dashed line shows the Jouguet detonation speed, the minimum speed of a detonation for each α , given in Eq. (6.4).

Gaussian approximation to the posterior likelihood around the maximum. The inverse of the FM is the covariance matrix, the diagonal elements of which give an approximation of the uncertainty in any given parameter.

6.4.1 LISA likelihood model

Here we outline how we model the LISA data, explain the assumptions made, and define the likelihood used. The LISA data is expected to be a $T_{obs} = 4$ yr stream with a regular data sampling interval $T_{samp} = 5$ s, not taking into account scheduled maintenance breaks. The frequency domain gravitational wave strain amplitude h(f) is the Fourier transform

67

of the strain time series h(t):

$$h(f_n) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} h(t) \exp\left(-2\pi i f_n t_m\right), \tag{6.32}$$

where $t_m = mT_{\text{samp}}$ and $f_n = n/T_{\text{obs}}$, with $-N/2 < n \le N/2$, and $N = T_{\text{obs}}/T_{\text{samp}}$. The strain amplitude is related to the gravitational wave power spectrum by

$$\Omega_{\text{gw},0}(f_n) = \left(\frac{4\pi^2}{3H_0^2}\right) f_n^3 |h(f_n)|^2.$$
(6.33)

In this analysis we consider the A and E TDI channels in the frequency domain with power spectral densities $D_n^A = |A(f_n)|^2$ and $D_n^E = |E(f_n)|^2$. The variances of the Fourier amplitudes in the A and E channels are taken to be identical and independent, and written S_n , and to depend on a vector of model parameters $\vec{\theta}$.

As $N \simeq 2 \times 10^7$ it saves computation time to group the data by frequency binning, which is a crude way of grouping the data, but sufficient for the level of analysis we carry out. We split the frequency range into a set of N_b logarithmically spaced positive frequency bins, with bin boundaries f_b , where b ranges from 0 to N_b . In each bin there are

$$n_b = [(f_b - f_{b-1})T_{\rm obs}] \tag{6.34}$$

different frequencies, where the square brackets denote the integer part.

First considering D_n^A we define \bar{D}_b^A to be the weighted mean value for D_n^A in bin b, so that

$$\bar{D}_b^A = \frac{S_b}{n_b} \sum_{n \in I_b} \frac{D_n^A}{S_n},\tag{6.35}$$

with

$$\frac{1}{S_b} = \frac{1}{n_b} \sum_{n \in I_b} \frac{1}{S_n},\tag{6.36}$$

where I_b the set of integers n such that $|f_n|$ is in frequency bin b and S_n is the mean value of D_n^A .

As \bar{D}_b^A is the average of squares of $2n_b$ normally distributed real random variables, the likelihood is a chi-squared distribution ⁴

$$p\left(\bar{D}_{b}^{A} \mid S_{b}\right) = \prod_{b=1}^{N_{b}} \frac{1}{(n_{b}-1)!} \frac{n_{b}}{S_{b}} \left(n_{b} \frac{\bar{D}_{b}^{A}}{S_{b}}\right)^{n_{b}-1} \exp\left(-n_{b} \frac{\bar{D}_{b}^{A}}{S_{b}}\right).$$
(6.37)

⁴In this paper we echo the notation used in [114].

It will be convenient to approximate the likelihood with a Gaussian distribution, using the central limit theorem with the assumption $n_b \gg 1$ for all bins. The distribution for \bar{D}_b^A has mean S_b and variance S_b^2/n_b , giving the Gaussian approximation

$$p(\bar{D}_b^A|S_b) = \prod_{b=1}^{N_b} \left(\frac{n_b}{2\pi S_b^2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\frac{n_b\left(\bar{D}_b^A - S_b\right)^2}{S_b^2}\right).$$
(6.38)

As the A and E channels are uncorrelated and are assumed to a first approximation have identical noise, we can combine them into an average variable $\bar{D}_b = (\bar{D}_b^A + \bar{D}_b^E)/2$ with variance $S_b^2/2n_b$. The likelihood for the binned average spectral density \bar{D}_b is then

$$p(\bar{D}_b|S_b) = \prod_{b=1}^{N_b} \left(\frac{2n_b}{2\pi S_b^2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\frac{2n_b\left(\bar{D}_b - S_b\right)^2}{S_b^2}\right).$$
(6.39)

The Gaussian approximation is known to be biased [133–135], and one could improve the accuracy of the likelihood with a log normal correction [83, 134, 135]. One can also evaluate the Fisher matrix directly with the chi-squared distribution. On the other hand, using the Gaussian approximation simplifies the calculations. As we now show, we always have $2n_b \gtrsim O(10^2)$, for which the Gaussian approximation is sufficiently accurate.

Firstly, when working with the double broken power law model, we take $N_{\rm b} = 100$ logarithmically spaced frequency bins, for which $n_b \gtrsim 155$. The frequency binning for calculations in the sound shell model is a little more complicated, as we calculate the theoretical model in terms of the angular frequency scaled by the mean bubble separation $z = kR_*$, rather than an absolute frequency. This avoids having to recompute power spectra for different r_* and $T_{\rm n}$, as the shape of the GW power spectrum depends only on the thermodynamic parameters $v_{\rm w}$ and α .

The transformation from z to a frequency today is

$$f = f_{*,0} z / r_*, \tag{6.40}$$

where $f_{*,0}$ is given in Eq. (6.13), and we recall that $r_* = R_*H_n$. In this case, the number of data frequencies in bin b is

$$n_b = f_{*,0} T_{\rm obs} \Delta z_b / r_* \tag{6.41}$$

where $\Delta z_b = z_b - z_{b-1}$. In this work we compute 100 z values with logarithmic spacing between 1 and 1000. For the LISA data described here, the minimum $n_b \simeq 42$.

The Fisher matrix is

$$F_{ij} = \left\langle \frac{\partial l_G}{\partial \theta_i} \frac{\partial l_G}{\partial \theta_j} \right\rangle, \tag{6.42}$$

where θ_i denotes the *i*th component of the vector of model parameters $\vec{\theta}$, and the Gaussian approximation to the log-likelihood is

$$l_G = \ln(p) = -\frac{1}{2} \sum_{b=1}^{N_{\rm b}} \frac{2n_b \left(\bar{D}_b - S_b\right)^2}{S_b^2} - \sum_{b=1}^{N_{\rm b}} \ln S_b + \text{const.}$$
(6.43)

Hence the Gaussian approximation to the Fisher matrix is

$$F_{ij}^G = \sum_{b=1}^{N_{\rm b}} \frac{2n_b}{S_b^2} \frac{\partial S_b}{\partial \theta_i} \frac{\partial S_b}{\partial \theta_j}.$$
(6.44)

One can show that the Fisher matrix calculated using the full chi-squared distribution is

$$F_{ij} = \left(1 + \frac{1}{2n_b}\right) F_{ij}^G,\tag{6.45}$$

Hence with smallest value of $2n_b = 84$, the difference is minimal, and we take F_{ij} to be its Gaussian approximation from now on.

We will also use the power spectra, $\Omega_t(f_b, \vec{\theta})$, rather than the spectral densities S_b to formulate the theoretical model of the data:

$$\Omega_{\rm t}(f_b, \vec{\theta}) = \Omega_{\rm ins}(f_b) + \Omega_{\rm fg}(f_b) + \Omega_{\rm pt}(f_b, \vec{\theta}).$$
(6.46)

In this analysis we assume that the instrument noise Ω_{ins} and the foregrounds Ω_{fg} are much better known than the parameters of the phase transition, meaning we can use the analytic functions given in the previous section. Therefore, the parameters in the Fisher matrix are just those describing the phase transition. The instrument noise is described by Eq.(6.26) and we consider two kinds of foregrounds: one from extragalactic binaries Eq. (6.27), and one with both extragalactic and galactic binaries Eq. (6.29). As only the ratio of spectra appears, the Fisher matrix in terms of the power spectrum is simply

$$F_{ij} = T_{\rm obs} \sum_{b=1}^{N_{\rm b}} \frac{2\Delta f_b}{\Omega_{\rm t}^2} \frac{\partial \Omega_{\rm t}}{\partial \theta_i} \frac{\partial \Omega_{\rm t}}{\partial \theta_j}.$$
(6.47)

The sum on the right-hand side can be viewed as a numerical approximation to an integral over frequencies. The covariance matrix is the inverse of the Fisher matrix,

$$C_{ij} = F_{ij}^{-1}, (6.48)$$

where the square roots of diagonal entries give the uncertainty in the *i*th parameter $\Delta \theta_i$. These uncertainties include correlations between parameters. We define θ_i to be the logarithmic model parameter, so the square roots of the diagonal entries in the covariance matrix are the relative uncertainty in the parameter.

6.4.2 Principal components

The Fisher matrix can be used to construct a set of uncorrelated and orthonormal observables, the principal components. As the Fisher matrix is a symmetric $n \times n$ matrix we can find its eigenvectors and eigenvalues

$$F = U\Lambda U^{\dagger}, \quad \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

$$(6.49)$$

where U is a matrix of the orthonormal eigenvectors of the Fisher matrix, u_n is the n^{th} eigenvector and λ_n is the n^{th} eigenvalue. We can then construct a new set of variables $\vec{X} = (X_1, X_2, X_3, X_4)$, each X_n being a linear combination of our original parameters calculated by using U^{\dagger} as a projection vector,

$$\vec{X} = U^{\dagger} \bar{\theta}. \tag{6.50}$$

 X_n is the n^{th} principal component and the standard deviation in X_n is $\lambda_n^{-1/2}$. The principal components are ordered according to the size of their corresponding eigenvalues, meaning X_1 is the highest-order and best constrained parameter, and X_4 is the lowest order parameter and worst constrained [136].

6.5 Fisher matrix calculation and relative uncertainties

In this section we calculate the Fisher matrix and the relative uncertainties as described in the previous section, for several scenarios. Firstly, the FM for the spectral parameters from the double broken power law fit to the SSM. Then we evaluate the FM for the four key thermodynamic parameters in the SSM $(v_{\rm w}, \alpha, r_*, T_{\rm n})$ for two cases: free and fixed nucleation temperature $T_{\rm n}$. We also calculate the expected sensitivity to the principal components of the GW power spectrum calculated with the SSM. In each case we use the Fisher matrix Eq. (6.47) with a data model $\Omega_t(f_b, \vec{\theta})$ as given in Eq.(6.46) that assumes the LISA instrument noise and foregrounds are known perfectly, we also investigate the impact of including the foreground from galactic binaries.

6.5.1 Double broken power law model

First we look at the relative uncertainty for the spectral parameters as described in the proposed double broken power law model given in Eq. (7.4). In this case the parameters are $\vec{\theta} = (\ln(\Omega_{p,0}), \ln(f_{p,0}), \ln(r_b), \ln(b))$. The Fisher matrix entries can be calculated analytically. The gravitational wave power spectrum is evaluated at 100 frequencies with



Figure 6.5: Coloured contours show relative uncertainties calculated from the Fisher matrix for the parameters of the double broken power law model (Eq. 7.4): peak power $\Omega_{p,0}$, peak frequency $f_{p,0}$, break ratio r_b and intermediate power law b. The line styles indicate the break ratio values r_b . The black lines show contours of signal-to-noise ratio $\rho = 20$ for different r_b , with the same line styles. The grey shaded area indicates the region where the peak signal power is above the combined instrumental noise and foregrounds. In the upper panel the noise model consists of the LISA instrument noise, Eq. (6.26), foreground from compact binaries, Eq. (6.27) and the galactic binary foreground, Eq. (6.29). In the lower panel the galactic binary foreground is removed.

logarithmic spacing between 10^{-5} Hz and 1 Hz. We sample the parameter space as follows: 200 peak powers $\Omega_{\rm p,0}$, with logarithmic spacing between 10^{-13} and 10^{-8} ; 200 peak frequencies $f_{\rm p,0}$, with logarithmic spacing between 10^{-5} and 1; 4 frequency break ratios $r_{\rm b} = [0.1, 0.2, 0.3, 0.4]$; and intermediate power law with spectral slope b = 1, the generic value, as explained in [4]. The range of spectral parameters at which we evaluate the relative uncertainties was chosen such that they could be produced by thermodynamic parameters currently explored in models and simulations (as displayed in Fig. 6.3). The resulting relative uncertainties, with and without the foreground from unresolved galactic binaries, are shown in Fig. 6.5 as contours in the ($\Omega_{p,0}, f_{p,0}$) plane. The line style shows the frequency break ratio r_b , and the colour the relative uncertainty. Also plotted is the curve at which the signal-to-noise ratio ρ is 20, and for comparison, the noise model (which includes foregrounds) $\Omega_n(f)$ as a black line.

It can be seen that $\rho = 20$ is reached for peak powers well below the noise level, which is an effect of the integration over frequencies. One can regard the $\rho = 20$ line as a peak-integrated sensitivity [129], which generalises the idea of power law sensitivity [137] to peaked power spectra.

The results for the relative uncertainty in $\Omega_{p,0}$ and $f_{p,0}$ are consistent with those in Ref. [111], which studied the two-parameter single broken power law model advocated by the LISA Cosmology Working group [8]. One can summarise the conclusion in a parameterindependent way by the statement that a SNR of about 20 allows a measurement of the peak power and peak frequency at around a 10% level of uncertainty. If the unresolved galactic binaries are not removed, the parameter space required to achieve $\rho = 20$ is reduced.

A 10% measurement of $r_{\rm b}$, which encodes information about the wall speed, requires higher signal-to-noise ratios, with the best resolved break ratio being $r_{\rm b} = 0.4$. This is the value of $r_{\rm b}$ giving a power spectrum with the narrowest peak, and so the whole peak is likely to be in the sensitivity window of the detector.

6.5.2 Sound shell model

In the simplest version of the sound shell model we study, the parameters are the logarithms of the wall speed, the phase transition strength, the Hubble-scaled mean bubble spacing and the nucleation temperature, giving a parameter vector $\vec{\theta} = (\ln v_{\rm w}, \ln \alpha, \ln r_*, \ln(T_{\rm n}/{\rm GeV})).$

We evaluate the Fisher matrix at all combinations of our parameter space using Eq. (6.47). The parameter space was sampled with 50 wall speeds $v_{\rm w}$ in the range $0.4 \leq v_{\rm w} \leq 0.9$, 51 phase transition strengths α logarithmically spaced between 0.01 and 0.5, 2 Hubble-scaled mean bubble spacings $r_* = 0.01, 0.1$, and nucleation temperature $T_{\rm n} =$ 100 GeV.

To construct the Fisher matrix we need to calculate the partial differentials of the GW power spectrum with respect to each of our thermodynamic parameters. The gradients with respect to $v_{\rm w}$ and α were computed numerically. The derivatives with respect to the Hubble-scaled mean bubble spacing r_* and the nucleation temperature $T_{\rm n}$ are calculated as follows.

With the phase transition model spectrum Ω_{pt} given by $\Omega_{gw,0}$ in Eq. (6.10), we recall that

$$J = H_n R_* H_n \tau_v = r_* \left(1 - \frac{1}{\sqrt{1 + 2x}} \right), \tag{6.51}$$

where $x = r_*/\sqrt{K(\alpha, v_w)}$. The gravitational wave frequency today f is related to the dimensionless wavenumber z through $z = r_*(f/f_{*,0})$, with the reference frequency depending on T_n through Eq. (6.13). Hence we find

$$\frac{\partial \Omega_{\rm t}(f)}{\partial \ln r_*} = \Omega_{\rm gw,0} \left(\frac{\partial \ln J}{\partial \ln r_*} + \gamma_{\rm gw}(z) \right), \tag{6.52}$$

where

$$\frac{\partial \ln J}{\partial \ln r_*} = 1 + \frac{r_*}{J} \frac{x}{\left(1 + 2x\right)^{3/2}},\tag{6.53}$$

and $\gamma_{\rm gw} = d \ln \Omega_{\rm gw}/d \ln z$ is the local power law index of the gravitational wave power spectrum, which we compute numerically.

The partial differential with respect to T_n is then

$$\frac{\partial \Omega_{\rm t}(f)}{\partial \ln(T_{\rm n}/{\rm GeV})} = -\Omega_{\rm gw,0} \gamma_{\rm gw}(z).$$
(6.54)

The resulting relative uncertainties are shown in Figs. 6.6a with the galactic binary foreground and 6.6b without the galactic binary foreground. Below the Jouguet speed, indicated by a dashed line, the fluid shell becomes a supersonic deflagration, with a significant change in the sound wave power spectrum, and hence the gravitational wave power spectrum [4]. Thus one expects to see features in the signal-to-noise ratio and the relative uncertainties to the left of this line. The intricate shape of the contours is also partly due to the complex degeneracies, discussed below, and inaccuracies in the interpolation of the numerically-determined GW suppression factor.

A general conclusion is that, even when $\rho = 20$, the only parameter which has relative uncertainty less than 1 is the wall speed. That the wall speed $v_{\rm w}$ is the best determined parameter is perhaps surprising, but it can be understood as follows.



(a) Noise model: LISA instrument noise, foregrounds from extragalactic compact binaries Eq. (6.27) and unresolved galactic compact binaries (6.29).



(b) Noise model: same as above, with the foreground from unresolved galactic binaries removed.

Figure 6.6: Contours of relative uncertainty in the thermodynamic parameters wall speed $v_{\rm w}$, transition strength α , scaled mean bubble spacing r_* and nucleation temperature $T_{\rm n}$. In each sub-figure, the upper and lower panels have Hubble-scaled bubble spacing r_* as annotated. In both panels $T_{\rm n} = 100$ GeV. The black solid line shows contours of signal-to-noise ratio ρ . The turquoise dashed line is the Jouguet speed, the minimum for a detonation.

75

Looking at the upper left panel of Fig. 6.1, one can see that varying the wall speed significantly changes the shape of the power spectrum, which none of the other parameters do. On the other hand, the other parameters have complex degeneracies. For example, r_* and T_n both affect the overall frequency scale, and α and r_* both affect the overall amplitude of the power spectrum. Increasing T_n (see Fig. 6.1, bottom right panel) shifts the peak frequency, which can be compensated by a combination of increasing r_* (Fig. 6.1, bottom left panel) and reducing α .

Another general conclusion, clear from the comparison between Figs. 6.6a and 6.6b, is the importance of removing the galactic binary foreground for parameter estimation. The two figures represent the extremes of what can be achieved in practice. The study of Ref. [108] indicates that the annual variation of the galactic binary foreground will enable its near-complete removal, and so Fig. 6.6b is likely to be a better approximation.

6.5.3 Principal component analysis

The degeneracy between α , r_* and T_n gives the impression that they will be virtually no sensitivity to these parameters, even at high signal-to-noise ratio. The Fisher matrix may be overestimating the uncertainties, so we look to the principal components to see if there is greater sensitivity to linear combinations of the thermodynamic parameters. The contours of the standard deviation of our principal components $\lambda_n^{-1/2}$ can be seen in Fig. 6.7

Comparing Figs. 6.6a and 6.7 it is immediately obvious that there is greater sensitivity to the principal components over a broader region of parameter space, even when the foreground from galactic binaries is present. In general, for GW power spectra with $\rho > 20$ the two highest-order principal components reach $1/\sqrt{\lambda_n} < 3\%$ for both values of the Hubble-scaled mean bubble spacing. For GW power spectra with $r_* = 0.01$, whilst there is only a small region of sensitivity to v_w , there is broad sensitivity to the two highest-order principal components. In the $r_* = 0.1$ case $1/\sqrt{\lambda_n} < 30\%$ for the majority of the parameter space for the two highest-order principal components (see Fig. 6.7).



(a) Noise model: LISA instrument noise, foregrounds from extragalactic compact binaries Eq. (6.27) and unresolved galactic compact binaries (6.29).

Figure 6.7: Contours of standard deviation $(1/\sqrt{\lambda_n})$ for the principal components constructed from the eigenvectors of the Fisher matrix evaluated across the wall speed v_w and phase transition strength α parameter space. In each sub-figure, the upper and lower panels have Hubble-scaled bubble spacing r_* as annotated. In both panels $T_n = 100$ GeV. The black solid line shows contours of signal-to-noise ratio ρ . The turquoise dashed line is the Jouguet speed, the minimum for a detonation.

To investigate the contribution of the principal components to the thermodynamic parameters, we assigned to each of the first three principal components the colours red, green and blue respectively. We took the four thermodynamic parameter eigenvectors in the principal component basis, and constructed an RGB colour from the square of the corresponding entry in the eigenvector. A significant mixture of the fourth principal component would then appear as a dark colour.

We show the result in Fig. 6.8. We see the wall speed v_w is predominantly red, meaning the first principal component provides the largest contribution, which confirms that we would expect greatest sensitivity to v_w . The other parameters show an interesting mix of colours, which is partly noise introduced when we interpolate the kinetic energy suppression data (see Appendix 6.7.1). We believe the remaining sudden changes of colour comes from the degeneracy between parameters, in particular the streak originating around the speed of sound on the wall speed axis.



Figure 6.8: The contributions of the first three principal components to the thermodynamic parameters wall speed $v_{\rm w}$, transition strength α , scaled mean bubble spacing r_* and nucleation temperature $T_{\rm n}$. Red, green and blue correspond to the first, second and third principal components respectively. The upper and lower panels have Hubble-scaled bubble spacing r_* as annotated. In both panels $T_{\rm n} = 100$ GeV. Noise model: LISA instrument noise, foregrounds from extragalactic compact binaries Eq. (6.27) and unresolved galactic compact binaries (6.29).

6.5.4 Sound shell model with fixed nucleation temperature

For the final analysis we explore the impact of information from particle physics data. While the information is likely to constrain a combination of parameters, we take as a limiting example a known nucleation temperature $T_n = 100$ GeV. The nucleation temperature is likely to be close to the critical temperature of the phase transition, which is the most straightforward thermodynamic parameter to calculate from an underlying theory,

The Fisher matrix, covariance matrix and relative uncertainties are calculated following the same procedure as above for two scenarios: first with our base noise model Fig. 6.9a and then with the unresolved galactic binaries foreground removed, Fig. 6.9b.



(a) Noise model: LISA instrument noise, foregrounds from extragalactic compact binaries Eq. (6.27) and unresolved galactic compact binaries (6.29).



(b) Noise model same as above, with the foreground from unresolved galactic binaries removed.

Figure 6.9: Contours of relative uncertainty in the thermodynamic parameters wall speed $v_{\rm w}$, transition strength α and scaled mean bubble spacing r_* with nucleation temperature $T_{\rm n} = 100$ GeV, for gravitational wave power spectra calculated using the sound shell model, Eq. (6.10). In each sub-figure, the upper and lower panels have Hubble-scaled bubble spacing r_* as annotated. The turquoise dashed line is the Jouguet speed, the minimum for a detonation.

Prior knowledge of the nucleation temperature T_n greatly improves the power of LISA to estimate other parameters, as the degeneracies become partly broken. With fixed T_n for GW power spectra with $\rho > 20$ the wall speed has relative uncertainty of less than 10%. Fixed T_n also improves sensitivity to the phase transition strength, α , and the Hubblescaled mean bubble spacing, r_* . For example, if the phase transition has $r_* = 0.1$, one can achieve relative uncertainties $\Delta \alpha / \alpha < 10\%$ and $\Delta r_* / r_* < 30\%$ with a signal-to-noise ration greater than 50.

There is an interesting feature in the relative uncertainty contours at $r_* = 0.1$, where the SNR is higher: a small ridge of lower uncertainty in α , for wall speeds just over 0.6. This is accompanied by reduction in the sensitivity to v_w .

The origin of this ridge is perhaps as follows. Referring to Fig. 6.1, one can see that at around $v_{\rm w} = 0.6$ at $r_* = 0.1$, changes in the wall speed and r_* have the effect of moving the closest part of the signal to the sensitivity curve in a direction tangent to the sensitivity curve, without changing the shape. This would mean that the likelihood changes little in these directions, and it would be difficult to distinguish between possible parameter values, this would lead to a reduction in sensitivity. Changes in α , on the other hand, change the signal power, and will change the likelihood. Thus the likelihood is most sensitive to changes in α in this region.

6.6 Discussion

In this paper we have explored the prospect of extracting the model parameters of a stochastic gravitational wave background from a first order phase transition at future space-based gravitational wave observatories. We focused on LISA, and the impact of including expected foregrounds from compact binaries. Here we studied the gravitational wave power spectra predicted by the sound shell model (SSM), and used Fisher matrix analysis to investigate the sensitivity both to the four parameters of a double broken power law approximation, and to the underlying thermodynamic parameters in the SSM. The key thermodynamic parameters are the nucleation temperature T_n , the transition strength α , the mean bubble spacing in Hubble units r_* and the wall (phase boundary) speed v_w . We assumed a sound speed $c_s = 1/\sqrt{3}$ in both phases, and leave an investigation of sensitivity to this parameter to future work. The fact that different sound speeds significantly change the kinetic energy fraction of the fluid [116, 117] suggests that there will be sensitivity to this parameter as well.

In Sec. 6.2.3 we studied the double broken power law approximation to the sound shell model gravitational wave power spectrum, which was advocated in Ref. [4]. It has parameters characterising the peak $\Omega_{\rm p}$ and two frequency scales, the peak frequency $f_{\rm p}$, and a lower "break" frequency $f_{\rm b} = r_{\rm b}f_{\rm p}$. In its original form, the indices of the three power laws were fixed by arguments based on the limits of certain integrals. We introduced a fourth parameter, the spectral slope of the intermediate power law b, to improve the fit between $f_{\rm b}$ and $f_{\rm p}$ for phase transitions proceeding by supersonic deflagrations. This form, given in Eq. (7.4), is a significant improvement on the single broken power law in fitting the predictions of the sound shell model (see Fig. 6.11).

We performed a Fisher matrix analysis to calculate the relative uncertainty for the four parameters of the double broken power law spectrum. In Fig. 6.5 we see that the $\Omega_{\rm p}$ and the peak frequency $f_{\rm p}$ are expected to be best constrained, with a signal-to-noise ratio of 20 delivering determinations to around 10% for peak frequencies between 10^{-4} and 10^{-2} Hz. The other parameters, the break frequency ratio $r_{\rm b}$ and the intermediate power law b, are less well determined, but can be determined with less than 1% relative uncertainty for signals with peak power and peak frequencies that lie on LISA's sensitivity curve, that is, signals at the same level as or above the instrument noise.

The extragalactic compact binary foreground expected from LIGO/Virgo data [138] is not an important contributor to the total noise, but the galactic binary foreground would be significant if it could not be removed. The main effect is to somewhat reduce the range over which parameters can be determined within a given uncertainty; the magnitude of the effect can be judged in the difference between Figs. 6.6a and 6.6b. However, it is expected that the galactic binary foreground will be at least partially removable through its annual modulation [108].

We also studied LISA's sensitivity to the four principal thermodynamic parameters of a first order phase transition, as described above. The GW power spectrum model used was the sound shell model, Eq. (6.10), incorporating kinetic energy suppression in slow deflagrations [7].

We investigated scenarios with a nucleation temperature $T_{\rm n} = 100$ GeV and Hubblescaled mean bubble spacing $r_* = 0.1, 0.01$, scanning over a range in $(v_{\rm w}, \alpha)$ space with $0.4 \leq v_{\rm w} \leq 0.9, 0.005 \leq \alpha \leq 0.5$, where numerical simulations have been performed [7] and the model can be calibrated. We observed that in order to match the total power in the simulations a suppression factor had to be applied to the sound shell model (see Appendix 6.7.1).

We found that the wall speed $v_{\rm w}$ would be the best constrained parameter, with a relative uncertainty of better than 30% provided the signal-to-noise ratio is above 20, and the wall is supersonic, even in the worst-case scenario where the foreground from unresolved galactic binaries cannot be removed. As the Hubble-scaled mean bubble spacing r_* gets smaller the signal power decreases, this leads to a reduction in the region of parameter space over which a signal-to-noise ratio of 20 can be achieved. For example, with $r_* = 0.1$, SNR 20 can be produced by phase transition strengths down to about $\alpha \simeq 0.13$, while at $r_* = 0.01$ the corresponding figure is $\alpha \simeq 0.35$.

There is limited sensitivity to the parameters α , r_* and T_n due to degeneracies. For example, the peak frequency is left unchanged by simultaneous changes in the nucleation temperature T_n and the mean bubble spacing r_* . Changing r_* changes the peak power, which can be brought back to its original value, without changing the peak frequency much, by a change in the transition strength α . This will mean that the parameters most easily computable from underlying models, T_n , $\tilde{\beta}$, and α , will not be individually well determined.

However, there is much better sensitivity to the principal components across the explored parameter space. The two highest-order principal components have relative uncertainty less than 3% for GW power spectra with $\rho > 20$, for both values of the Hubble-scaled mean bubble spacing. The highest-order component is found to be dominated by the wall speed, as is consistent with the wall speed being the best determined parameter.

If one of the parameters is known, the other thermodynamic parameters are much better constrained. For example, with a known nucleation temperature $T_n = 100$ GeV, the wall speed would have an estimated uncertainty of less than 10% for the majority of the parameter space, and the phase transition strength would be almost as accurately measured as the wall speed. The mean bubble spacing would be less accurately measured. A more realistic situation would involve constraints on masses and coupling constants in an underlying particle physics model, which we will explore elsewhere.

The parameter degeneracies we have found mean that one must consider the reliability of the Fisher matrix as an indicator of parameter uncertainties. For the spectral parameters of the double broken power law fit the Fisher matrix can be trusted, as there is little or no degeneracy between parameters and the Gaussian approximation made with the Fisher matrix is reasonable. To check, we carried out preliminary Markov Chain Monte Carlo (MCMC) estimation using the Gaussian approximation to the likelihood, which returned ellipsoidal posteriors for the spectral parameters. We also found approximate agreement between the marginalised posteriors and the uncertainties predicted by Fisher analysis. On the other hand the thermodynamic parameters do have significant degeneracies. In some regions of parameter space where the signal to noise ratio is high, preliminary MCMC has shown ellipsoidal posteriors, supporting the case that the principal component analysis of the Fisher matrix should be a reasonably good indicator. The general tendency of the Fisher matrix is to over-estimate uncertainties [136], and so we expect that the uncertainty estimates presented here are conservative.

Another source of uncertainty in the results presented here are future developments in the understanding of the power spectrum for strong phase transitions. We argue that the double broken power law fit is generic as we expect to the presence of the two length scales to persist. There may be new spectral shape parameters for the infrared and ultraviolet power laws but they too will be functions of $v_{\rm w}$ and α . Therefore, we believe such refinements in the shape of the power spectrum at high α are unlikely to lead to a significant loss of sensitivity to the thermodynamic parameters.

In summary, we have presented the relative uncertainties calculated with the Fisher matrix as a preliminary guide to the expected power of LISA to resolve the spectral and thermodynamic parameters of a first order phase transition in the early Universe. The analysis makes it clear there are significant degeneracies that limit the accuracy of the direct determination of the thermodynamic parameters. However, the principal components show that at least two combinations of parameters can be well-determined, and the wall speed is will be the best measured phase transition parameter. For GW signals with signal-to-noise ratio greater than 20 we found the relative uncertainty for the two highest-order principal components to be less than 3%. This provides a target for the accuracy required from theoretical models. We plan to carry out a more detailed MCMC analysis, exploring more realistic noise models, and with the sound speeds as parameters to an extended sound shell model.

Acknowledgements

We thank Oliver Gould, Elina Keihänen, Antony Lewis, Jesús Torrado, Ville Vaskonen, Essi Vilhonen and Graham White for useful comments and feedback. We are grateful to Daniel Cutting for supplying data from Ref. [7] and Fëanor Reuben Ares for helpful discussions. CG (ORCID ID 0000-0002-7955-4465) is supported by a STFC Studentship. MH (ORCID ID 0000-0002-9307-437X) acknowledges support from the Academy of Finland (grant number 333609).

6.7 Appendix

6.7.1 Kinetic energy suppression in the sound shell model

Here, we outline the implementation of kinetic energy suppression in our model of the GW spectrum, which brings the sound shell model [4] into better agreement with numerical simulations [7] at low wall speeds and high transition strengths.

The gravitational wave power spectrum takes the form (6.6). We can remove all dependence on the mean bubble separation and the nucleation temperature by considering the scale-free power spectrum

$$\hat{\mathcal{P}}_{\rm gw}(z) \equiv (H_{\rm n}\tau_{\rm v})^{-1} (H_{\rm n}R_{*})^{-1} \Omega_{\rm gw}(z) = 3K^{2}(v_{\rm w},\alpha) \frac{z^{3}}{2\pi^{2}} \tilde{P}_{\rm gw}(z) \,.$$
(6.55)

Integrating this quantity over $\ln(z)$, we define a dimensionless gravitational wave production efficiency parameter $\tilde{\Omega}_{gw}$ by dividing by square of the kinetic energy fraction around a single self-similar bubble, $K_1(v_w, \alpha)$, giving

$$\int \frac{dz}{z} \hat{\mathcal{P}}_{gw}(z) = K_1^2(v_w, \alpha) \tilde{\Omega}_{gw}.$$
(6.56)

This integral can be compared between the sound shell model and the numerical simulations. We will assume that the numerical simulations give a better estimate of $\tilde{\Omega}_{gw}$ than the sound shell model, and so we scale the sound shell model power spectrum by the ratio of the production efficiency parameters.

A complication is that in the simulations, the bubbles have not yet reached selfsimilar profiles and their full kinetic energy fraction when they collide, and so the total gravitational wave power is underestimated. To compensate for this effect, the estimate of $\tilde{\Omega}_{gw}^{sim}$ is made by dividing by the kinetic energy fraction around a numerical solution of the 1d hydro equations at a bubble size of R_*^{sim} , the mean bubble size found in the simulation. We denote this kinetic energy fraction by $K_1(v_w, \alpha; R_*^{sim})$. It is related to the RMS fluid velocity $\bar{U}_{f,exp}$ given in the 7th column of Table 1 of Ref. [7] by

$$K_1(v_{\rm w},\alpha;R_*^{\rm sim}) = \frac{4}{3}\bar{U}_{\rm f,exp}^2.$$
 (6.57)

The suppression factor is then

$$\Sigma(v_{\rm w},\alpha) = \frac{\int d\ln(z)\,\hat{\mathcal{P}}_{\rm gw}^{\rm sim}(z)}{\int d\ln(z)\,\hat{\mathcal{P}}_{\rm gw}^{\rm ssm}(z)} \left(\frac{K_1(v_{\rm w},\alpha)}{K_1(v_{\rm w},\alpha;R_*^{\rm sim})}\right)^2. \tag{6.58}$$

The simulations of Ref. [7] covered a region $0.24 \lesssim v_{\rm w} \lesssim 0.92$ and $0.005 \lesssim \alpha \lesssim 0.5$ with around 60 samples, and gave $\int d \ln(z) \hat{\mathcal{P}}_{\rm gw}^{\rm sim}(z)$ in the second last column of Table 1. The resulting Σ is plotted as coloured contours in the (v_w, α) plane in Fig. 6.10. Intermediate values are obtained by linear interpolation.



Figure 6.10: Contours showing the gravitational wave power suppression factor Σ (6.58) in the $(v_{\rm w}, \alpha)$ plane. The suppression factor is constructed to make the total gravitational wave power in the sound shell model [4] agree with the simulations of Ref. [7]. The turquoise dashed line shows the Jouguet speed, the minimum speed for a detonation.

6.7.2 Broken power law approximations to the GW spectrum

Here we compare the single broken power law and the double broken power law approximations to the GW power spectrum from a first order phase transition as described by the sound shell model. For each pair of thermodynamic parameters (v_w, α) we find the best fit spectral parameters $\vec{\theta}$ for each analytic form, obtained by least squares on the part of the spectrum within a factor 10 of the peak.

We evaluate the minimum of the mean-squared relative deviation of the scale-free power spectra (see Eq. 6.55),

$$\delta_{\mathcal{P}}^{2} = \min_{\vec{\theta}} \int \frac{dz}{z} \left(\frac{\hat{\mathcal{P}}_{gw}^{\text{fit}}(z,\vec{\theta}) - \hat{\mathcal{P}}_{gw}^{\text{ssm}}(z;v_{w},\alpha)}{\hat{\mathcal{P}}_{gw}^{\text{ssm}}(z;v_{w},\alpha)} \right)^{2}, \tag{6.59}$$

where $\hat{\mathcal{P}}_{gw}^{\text{fit}}$ is the fit and $\hat{\mathcal{P}}_{gw}^{\text{ssm}}$ the scaled power spectrum calculated in the SSM. We evaluate this quantity for the two different fit functions: the four-parameter double broken power law Eq. (7.5) and the two-parameter broken power law given by the LISA Cosmology

working group [8], where $M(s, r_{\rm b}, b)$ is replaced by

$$C(s) = \frac{7s^3}{(4+3s^2)^{7/2}}.$$
(6.60)

The result is plotted in Fig. 6.11.



Figure 6.11: The mean squared relative deviation, $\delta_{\mathcal{P}}^2$ defined in Eq. (6.59), for two analytic fits to the sound shell model gravitational wave power spectrum. On the left is the single broken power law used by the LISA cosmological working group [8], given in Eq. (6.60). On the right is the general double broken power law given in Eq. (7.5). The turquoise dashed line is the Jouguet speed, Eq. (6.4), the minimum speed of the phase boundary in a detonation. Note the difference in the colour scales.

One sees that the two-parameter fit is poor for supersonic deflagrations, where the peak in the power spectrum is broad, while the four-parameter double broken power law has a mean-squared deviation less than 0.05 almost everywhere, note the different colour scales between the left and right plots of Fig. 6.11 . The region where $\delta_{\mathcal{P}}^2 > 0.05$ for the double broken power law fit corresponds to supersonic deflagrations at weak transition strengths. Here, differences between the leading and trailing portions of the velocity profiles introduce two length scales, and thus deviations from the double broken power law, this effect is greatest for low α .

Chapter 7

Prospects for LISA to detect a gravitational-wave background from first order phase transitions

Abstract

First order phase transitions in the early universe could produce a gravitationalwave background that might be detectable by the Laser Interferometer Space Antenna (LISA). Such an observation would provide evidence for physics beyond the Standard Model. We study the ability of LISA to observe a gravitational-wave background from phase transitions in the presence of an extragalactic foreground from binary black hole mergers throughout the universe, a galactic foreground from white dwarf binaries, and LISA noise. Modelling the phase transition gravitational wave background as a double broken power law, we use the deviance information criterion as a detection statistic, and Fisher matrix and Markov Chain Monte Carlo methods to assess the measurement accuracy of the parameters of the power spectrum. While estimating all the parameters associated with the gravitational-wave backgrounds, foregrounds, and LISA noise, we find that LISA could detect a a gravitational-wave background from phase transitions with a peak frequency of 1 mHz and normalized energy density amplitude of $\Omega_{\rm p} \simeq 3 \times 10^{-11}$. With $\Omega_{\rm p} \simeq 10^{-10}$, the signal is detectable if the peak frequency is in the range 4×10^{-4} to 9×10^{-3} Hz, and the peak amplitude and frequency can be estimated to an accuracy of 10% to 1%.
7.1 Introduction

The Laser Interferometer Space Antenna (LISA) [28] will be sensitive to the millihertz gravitational wave (GW) frequency range, and will simultaneously observe signals from numerous independent sources, both astrophysical [139] and cosmological [140]. Of particular interest is a search for a cosmological stochastic GW background, which could come from many different processes in the early universe, such as cosmic strings, inflation, or phase transitions [141]. Here we focus on the cosmological GW background from a first order phase transition at the electroweak scale (see e.g. [41, 105] for reviews).

Any cosmological GW background will compete with numerous foregrounds. Foregrounds from large numbers of astrophysical objects with low signal-to-noise ratio will also produce stochastic signals, which need to be separated from the cosmological signal of interest. A first component to consider is the foreground from double white dwarf (DWD) binaries in our galaxy [10], whose amplitude will be annually modulated by the orbit of the LISA constellation around the Sun [108]. This orbital modulation aids in the separation of the galactic foreground using the LISA data [109].

From the LIGO-Virgo observations of binary black hole (BBH) and binary neutron star (BNS) mergers one knows that there will be a foreground created from mergers of extragalactic compact binaries (ECB) over the history of the universe; LIGO and Virgo predict a background at the level (normalized energy density) of $\Omega_{\rm ECB}(f) = A_{\rm ECB}(f/25 {\rm Hz})^{2/3}$, where $A_{\rm ECB} = 6.8^{+3.6}_{-2.2} \times 10^{-10}$ [14]. Other studies based on the LIGO-Virgo observations predict $A_{\rm ECB} \simeq 1.8 \times 10^{-9} - 2.5 \times 10^{-9}$ [142], and population simulations populations predict $A_{\rm ECB} \simeq 5.0 \times 10^{-9} - 2.6 \times 10^{-8}$ [143].

An important line of study is to investigate LISA's ability to separate a GW background of cosmological origin from the numerous astrophysical sources and LISA noise. In this paper we consider GWs emitted from a first order phase transition at the electroweak energy scale. Such a transition would have happened at around 10 picoseconds after the Big Bang, and generated a signal which could fall within LISA's peak sensitivity window, in the range 1 - 10 mHz.

In the Standard Model electroweak symmetry-breaking is not associated with a first order phase transition: there is instead a smooth crossover [20, 34]. However, in numerous extensions to the Standard model a first order phase transition is possible (for a summary see [8]) turning a search for GW background into a search for physics beyond the Standard Model, which is needed for explanations for the dark matter and baryon asymmetry of the Universe.

The production of GWs during a first order phase transition occurs via the collision of bubbles of the stable phase, and the subsequently produced sound waves and turbulent flows. In a first order transition driven by thermal fluctuations, sound waves created by the expanding bubbles are the dominant source of GWs [24–26]; production by bubble collisions [98–100, 144–146] can become relevant if there is very strong supercooling [52, 53].

Here we assume that the sound wave component is dominant, and model the GW background component with the Sound Shell Model (SSM) [4, 27]. We include an empirical factor accounting for the kinetic energy suppression in strong transitions [7], and assume that the transition is not so strong that the modifications to the spectrum from shocks [55] and vortical turbulence [56] become important.

The GW power spectrum in the SSM is determined by a few key thermodynamic parameters: the bubble nucleation temperature, $T_{\rm n}$, the phase transition strength, α , the bubble wall speed, $v_{\rm w}$, and the mean bubble spacing in units of the Hubble length, r_* . The sound speeds in the two phases are also important [116, 117]. These thermodynamic parameters are directly related to the underlying physics model.

As it is computationally intensive to calculate a power spectrum with the full SSM it is useful to investigate LISA's sensitivity to a parametrised spectral shape that approximates the SSM and can be used for the rapid evaluations needed in Markov Chain Monte Carlo (MCMC) searches. Here we use a double broken power law with four spectral parameters: the peak amplitude, Ω_p , the peak frequency, f_p , the ratio r_b between the peak frequency and the break frequency and the slope between the two characteristic frequency scales b. The SSM predicts that, where long-lived sound waves are the dominant source of GWs, the low frequency and high frequency spectral slopes are fixed at 9 and -3. As discussed in [1] the relationship between the spectral and thermodynamic parameters is complicated and there are numerous degeneracies. In this work we focus on LISA's ability to constrain the spectral parameters and leave the connection between thermodynamic and spectral parameters for future work.

The Fisher matrix study performed in [1] estimates LISA's sensitivity to a first order phase transition signature described by the SSM. In that work two GW power spectra models were considered: the SSM itself, and the double broken power law fit to the SSM. Relative uncertainties for the thermodynamic and spectral parameters were calculated using Fisher analysis and a data model that takes into account LISA noise, a stationary DWD foreground and an extragalactic astrophysical background.

Another study looked at LISA's ability to detect to a general double broken power law that has the low and high frequency spectral slopes unspecified [147], instead of fixed at the SSM values. The Akaike Information Criterion was used to determine whether, in the presence of LISA instrument noise, one is able to identify the break frequencies. This work also calculated the uncertainties in spectral parameters using MCMC simulations for several example cases. The noise model included LISA instrument noise built out of one TDI channel in mock data generation, but did not include any astrophysical foregrounds.

In this present paper we investigate LISA's ability to detect a GW background from a first order phase transition, in the presence of LISA noise, the galactic foreground, and the foreground from extragalactic compact binaries. To do this we use the difference in the deviance information criterion (DIC) between models with and without phase transition, calculated using MCMC methods, as a detection statistic [148–150]. We consider a value of Δ DIC > 5 to be a detection as discussed in more detail in Section 7.2.7. In comparison to [1, 147] we use the *AET* time delay interferometery (TDI) channels to build our data model and use the *T* channel to constrain the LISA instrument noise in our MCMC simulations. We then perform a systematic scan over the spectral parameter space using Fisher matrix and MCMC methods to determine how well the four spectral parameters of the first order phase transition can be estimated with LISA.

The rest of this work is organised as follows. In Section 7.2 we describe the GW background from a phase transition, the LISA noise model, the DWD foreground and the extragalactic compact binary foreground used in this analysis. In Section 7.2.6 we discuss how we simulate the data. In Section 7.2.7 we outline how the estimates based on the Fisher matrix and DIC are evaluated. The results and conclusions from this study are presented in Section 7.3 and Section 8.7.

7.2 Model Components

The parameter estimation methods used here follow those outlined in [109], which explored parameter estimation with GW backgrounds with spectra in the form of a simple power law. We consider a cosmological GW background described by the double broken power law discussed in [1] which models the signal expected from a first order phase transition. In this Section we outline our model for the combined power spectrum from a first order phase transition, LISA noise, the DWD foreground and the extragalactic compact binary foreground.

With LISA's triangular geometry, the interferometric phase differences can be combined in different ways with different time delays to eliminate the laser frequency noise [120, 121], using the technique of time delay interferometry (TDI). This leads to the construction of three GW measurement channels known as the X, Y, Z TDI channels [151]. Here we assume that the GW background signal, observed in the X, Y, Z channels, is stationary and uncorrelated with the stationary LISA instrument noise. Furthermore, we make the simplifying assumption that the instrument noise consists of two components: test mass acceleration noise and optical path length fluctuation noise; that these instrumental noises are identical in each spacecraft, and that arm lengths are the same so that the LISA instruments form an equilateral triangle. Under these assumptions, the crossspectra and response functions of the X, Y, Z channel combinations are identical [86]. To be conservative, we ignore the annual modulation of the galactic binary foreground.

It is possible to work with linear combinations of these channels for convenience: we choose the two "noise orthogonal" channels A and E, and the "null" channel T which has a reduced sensitivity to GWs, which are defined as

$$\begin{cases}
A = \frac{1}{\sqrt{2}}(Z - X), \\
E = \frac{1}{\sqrt{6}}(X - 2Y + Z), \\
T = \frac{1}{\sqrt{3}}(X + Y + Z).
\end{cases}$$
(7.1)

For ease of calculation, we use the approximation for the GW response of the A, E and T channels given in Ref. [5]:

$$\mathcal{R}_{A}^{\text{Fit}}(f) = \mathcal{R}_{E}^{\text{Fit}}(f) = \frac{9}{20} |W(f)|^{2} \left[1 + \left(\frac{f}{4f_{*}/3}\right)^{2} \right]^{-1},$$
(7.2)

$$\mathcal{R}_T^{\text{Fit}} \simeq \frac{1}{4032} \left(\frac{f}{f_*}\right)^6 |W(f)|^2 \left[1 + \frac{5}{16128} \left(\frac{f}{f_*}\right)^8\right]^{-1}$$
(7.3)

where $W(f) = 1 - e^{-2if/f_*}$ with $f_* = c/2\pi L$, and we take $L = 2.5 \times 10^9$ m as appropriate for the LISA arms.

7.2.1 Cosmological background from a first order phase transition

The GW power spectrum from a first order cosmological phase transition is thought to be well approximated by the Sound Shell Model [4], at least for transitions which are not too strong and have high enough wall speeds [7]. In the model there are two characteristic length scales, the mean bubble separation and the sound shell thickness, which motivate a simplified description in terms of a function with two frequency scales and three power law indices - a double broken power law [4]. In this work where we address a GW background from a first order phase transition, we use the double broken power law fit to the SSM put forward in [1] and shown to be a good fit over a wide range of wall speeds and transition strengths. In this fit, the power spectrum takes the form

$$\Omega_{\rm PT}(f;\Omega_{\rm p}, f_{\rm p}, r_b, b) = \Omega_{\rm p} M(f; f_{\rm p}, r_{\rm b}, b)$$
(7.4)

where $\Omega_{\rm p}$ is the peak of the power spectrum, $f_{\rm p}$ is the frequency corresponding to $\Omega_{\rm p}$ and $r_{\rm b} = f_{\rm b}/f_{\rm p}$ describes the ratio between the two breaks in the spectrum. The parameter b defines the spectral slope between the two breaks. The spectral shape $M(f, f_{\rm p}, r_{\rm b}, b)$ is a double broken power law with a spectral slope 9 at low frequencies and -4 at high frequencies¹

$$M(f; f_{\rm p}, r_{\rm b}, b) = \left(\frac{f}{f_{\rm p}}\right)^9 \left(\frac{1+r_{\rm b}^4}{r_{\rm b}^4 + \left(\frac{f}{f_{\rm p}}\right)^4}\right)^{(9-b)/4} \left(\frac{b+4}{b+4-m+m\left(\frac{f}{f_{\rm p}}\right)^2}\right)^{(b+4)/2}.$$
 (7.5)

Within $M(f; f_p, r_b, b)$, m has been chosen to ensure that for $r_b < 1$ the peak occurs at $f = f_p$ and $M(f_p; f_p, r_b, b) = 1$, giving

$$m = (9r_{\rm b}{}^4 + b) / (r_{\rm b}{}^4 + 1).$$
(7.6)

There are regions of the spectral parameter space that lead to Eq. 7.5 diverging. In particular, when the denominator of the final factor becomes negative, i.e.

$$b+4-m+m\left(\frac{f}{f_{\rm p}}\right)^2 \le 0. \tag{7.7}$$

We restrict ourselves to working within the region of parameter space that is well-defined.

Here, we outline the key thermodynamic parameters and their connection to the spectral ones. The first of the thermodynamic parameters is the nucleation temperature

¹In practice, the SSM's predicted high-frequency power law of -3 emerges only slowly, and -4 provides a better fit around the peak [4, 26].

 $T_{\rm n}$, which we define as the temperature corresponding to the peak of the globally-averaged bubble nucleation rate. The Hubble rate at $T_{\rm n}$ sets the frequency scale of the GW spectrum.

The second thermodynamic parameter is the nucleation rate parameter β . As discussed in [1] due to uncertainties in the calculation of β , we instead consider the related quantity, the mean bubble spacing R_* . We note that β^{-1} is the time for the bubble wall to move a distance R_* and therefore has the interpretation of the duration of the phase transition. In this work we refer to the Hubble-scaled mean bubble spacing $r_* = H_n R_*$ which contributes to the frequency scale and amplitude of the GW power spectrum.

Our third key thermodynamic parameter is the phase transition strength α , which we define as the ratio between the trace anomaly and the thermal energy, where the trace anomaly describes the amount of energy available to convert to shear stress energy. A stronger transition means more energy is converted to shear stress energy and a larger overall amplitude for the GW signal.

The final parameter to introduce is the wall speed $v_{\rm w}$ which, along with α , determines the motion of the plasma surrounding the bubble wall. The value of the wall speed relative to speed of sound $c_{\rm s}$ determines the width of the GW power spectrum, here we assume is the ultrarelativistic value $c_{\rm s} = 1/\sqrt{3}$ (see [116, 117] for other scenarios). For wall speeds close to $c_{\rm s}$ the power spectra are broad and r_b is small, in the alternate case the power spectra are narrow.

To summarise, the peak amplitude is controlled by the phase transition strength, the Hubble-scaled mean bubble spacing and the bubble wall speed in rough order of efficacy from high to low. For the peak frequency all thermodynamic parameters contribute to varying degrees. It is worth noting the nucleation temperature only impacts the overall frequency scale whereas all the other thermodynamic parameters play a role in numerous spectral parameters. The break ratio and the intermediate spectral slope are related to the phase transition strength and the wall speed parameters. Fig. 3 of [1] demonstrates how changing the thermodynamic parameters affects the shape of the GW power spectrum.

7.2.2 LISA noise model

We take the LISA noise model to be that given in the LISA Science Requirement Document [6] and [152]. The model assumes constant equal noise in all channels, and has only two parameters: the acceleration noise level $N_{\rm acc} = 1.44 \times 10^{-48} \text{ s}^{-4} \text{Hz}^{-1}$ and the optical path

length fluctuation noise level $N_{\text{pos}} = 3.6 \times 10^{-41} \text{ Hz}^{-1}$. The noise model is then specified by the spectral density of the X channel and the cross spectral density of the channel X and Y, which are

$$\begin{cases} N_X(f) = \left(4P_s(f) + 8\left[1 + \cos^2\left(\frac{f}{f_*}\right)\right] P_a(f)\right) |W(f)|^2\\ N_{XY}(f) = -\left[2P_s(f) + 8P_a(f)\right] \cos\left(\frac{f}{f_*}\right) |W(f)|^2. \end{cases}$$
(7.8)

We also define the functions

$$\begin{cases} P_s(f) = N_{\text{pos}} \\ P_a(f) = \frac{N_{\text{acc}}}{(2\pi f)^4} \left(1 + \left(\frac{0.4 \text{ mHz}}{f}\right)^2\right), \end{cases}$$
(7.9)

with $P_s(f)$ the single optical path-length fluctuation noise (which is frequency-independent) and $P_a(f)$ the single test mass acceleration noise. The noise models for the *AET* channel power spectral densities are given by the diagonalization of the covariance matrix of the *XYZ* channels (see e.g. [5]). The diagonal entries are then

$$N_A(f) = N_E(f) = N_X(f) - N_{XY}(f),$$

$$N_T(f) = N_X(f) + 2N_{XY}(f),$$
(7.10)

using the assumption that the correlation noise is the same for all interferometers.

Rather than comparing the detector response to a stochastic GW signal to the noise, it is more convenient to introduce noise spectral densities $S_A(f)$ and $S_E(f)$ by dividing by the GW response function,

$$S_A(f) = S_E(f) = \frac{N_A(f)}{\mathcal{R}_{A,E}^{\text{Fit}}(f)},\tag{7.11}$$

where $\mathcal{R}_{A,E}^{\text{Fit}}(f)$ is given by Eq. 7.2. For completeness the noise spectral density in the T channel is

$$S_T(f) = \frac{N_T(f)}{\mathcal{R}_T^{\text{Fit}}(f)},\tag{7.12}$$

where $\mathcal{R}_T^{\text{Fit}}(f)$ is given in Eq. 7.3. From the noise spectral densities, the equivalent energy spectral density is given by

$$\Omega_A(f) = \Omega_E(f) = S_A(f) \frac{4\pi^2 f^3}{3H_0^2}.$$
(7.13)

These power spectra have the interpretation as an isotropic GW signal which would have unit signal-to-noise ratio at every frequency.

7.2.3 Double white dwarf foreground

A foreground from DWD binaries in our galaxy [10, 153–157], will be observed as a modulated waveform due to LISA's orbit around the Sun. The large majority of the DWD will not be resolved, and the superposition of all GWs received by LISA constitutes the galactic foreground (see Eq. 7.14). We assume that the waveform of each binary can be modelled as a pseudo-monochromatic signal. Thus, we can build the superposition of their GW signals s(t):

$$s(t) = \sum_{i=1}^{N} \sum_{P=+,\times} h_{A,i}(f_{orb,i}, M_{1i}, M_{2i}, \mathbf{r}_i, t) \times F_P(\theta, \phi, t) \mathbf{D}(\theta, \phi, f)_P : \mathbf{e}_P$$
(7.14)

with *i* labelling the binaries. The masses of the two stars are M_{1i} for the larger mass and M_{2i} for the smaller; the orbital frequency of the binary is f_{orb_i} ; the Cartesian position in the Galaxy \mathbf{r}_i and the position in the sky θ, ϕ ; F_A is the beam pattern function for the polarization $A = +, \times, \mathbf{h}_{A,i} = h_{A,i}\mathbf{e}_A$ the tensor of the amplitude of the GW; **D** the one-arm detector tensor; and $\mathbf{h}_{A,i}$ the dimensionless GW amplitude. An initial description has been done with resolved sources to provide the modulation of the foreground from the LISA orbit [108].

The dimensionless energy spectral density of the DWD foreground can be approximated by a broken power law. The broken power law model for the galactic foreground from Lambert *et al.* used in Boileau *et al.* [11] is given by

$$\Omega_{\rm DWD}(f) = \frac{A_1 \left(f/f_* \right)^{\alpha_1}}{1 + A_2 \left(f/f_* \right)^{\alpha_2}} \tag{7.15}$$

with $\alpha_1 - \alpha_2 = \alpha \simeq 2/3$ at low-frequencies and the frequency reference $f_* = c/2\pi L$. The spectral shape of the DWD foreground is a broken power law because at high-frequencies ($\simeq 0.1 \text{ Hz}$) the number of DWDs decreases due to the physical limitation from the respective radii of the two white dwarfs in each binary.

A different model of the DWD foreground is used in Ref. [12]. Here, the frequency break is much higher, and the signal is approximated as a simple power law over the LISA frequency range. With this model, it is important to account for the resolved binaries, which are then removed, leaving behind a confusion noise $S_c(f)$ from the unresolved binaries. For a LISA mission duration of 4 years, the confusion noise from unresolved DWDs is approximated by:

$$S_c(f) = A f_{\rm Hz}^{7/3} e^{-f_{\rm Hz}^{\alpha} + \beta f_{\rm Hz} \sin(\kappa f_{\rm Hz})} \left[1 + \tanh(\gamma (f_k - f_{\rm Hz}))\right]$$
(7.16)

with $f_{\text{Hz}} = f/(1 \text{ Hz})$, $\alpha = 0.138$, $\beta = -221$, $\gamma = 1680$ and $f_k = 0.00113^{-2}$. The corresponding dimensionless energy spectral density is then

$$\Omega_c(f) = S_c(f) \frac{4\pi^2 f^3}{3H_0^2} .$$
(7.17)

7.2.4 Extragalactic compact binary foreground

A background from compact binaries consisting of black holes and neutron stars in other galaxies is expected. As this background has not yet been detected at ground-based GW observatories for this work we estimate its amplitude from the LIGO-Virgo observations, as outlined in [13]. Our model for this energy spectral density is a power law

$$\Omega_{\rm ECB}(f) = A_{\rm ECB} \left(\frac{f}{f_{\rm ref}}\right)^{\alpha_{\rm ECB}}.$$
(7.18)

In our simulations we inject an extragalactic compact binary foreground with a spectral slope $\alpha_{\rm ECB} = \frac{2}{3}$ and $f_{\rm ref} = 25$ Hz as the reference frequency [132]. For the amplitude $A_{\rm ECB}$ we use the upper value from the LIGO-Virgo O2 limit distribution, $A_{\rm ECB} = \Omega_{\rm ECB}(25 \text{ Hz}) = 2.15 \times 10^{-9}$ [123].

7.2.5 Model illustration

The various contributions to the dimensionless energy density power spectrum $\Omega_{\rm gw}(f)$ in the LISA observational band are displayed in Fig. 7.1. The black line is the LISA design sensitivity [9]. We display three models for the galactic foreground: the Lamberts et al. catalogue DWD (light blue line) [10], the Boileau et al. broken power law (dark blue) [11], and the galactic confusion noise of Robson et al. (red line) [159]. The green line is the LIGO-Virgo O2 observations $\Omega_{\rm ECB}(25 \text{ Hz}) = 8.9^{+12.6}_{-5.6} \times 10^{-10}$ [123], and the yellow is the LIGO-Virgo O3 observation measurement $\Omega_{\rm ECB}(25 \text{ Hz}) = 7.2^{+3.3}_{-2.3} \times 10^{-10}$ [14]. The pink and orange lines are respectively the dimensionless energy power spectrum of the PT for $\Omega_{\rm p} = 3 \times 10^{-11}$ and $\Omega_{\rm p} = 1 \times 10^{-10}$. The two curves are given by Eq. 7.4 with $f_{\rm p} = 1 \text{mHz}$, $r_b = 0.4$ and b = 1.

7.2.6 Simulation

To simulate the data in the frequency domain, we use the fractional energy density power spectrum of a first order phase transition $\Omega_{\text{PT}}(f)$ (see Section 7.2.1). The phase transition model is parametrised by four parameters; the peak power Ω_{p} , the peak frequency f_{p} , the

 $^{^{2}}$ We note that the models for galactic confusion noise continue to be improved [158], and we will incorporate these advances in future work.



Figure 7.1: The LISA sensitivity curve [6, 9], black line, in terms of the dimensionless energy spectral density $\Omega_{\rm gw}(f)$. The DWD foreground models are also presented. The light and dark blue lines are respectively the Galactic foreground from the Lamberts *et al.* catalogue [10] and the analytic galactic foreground fit of Boileau *et al.* [11]. The red line is the Galactic confusion noise from Robson *et al.* [12]. The green line is the estimated extragalactic compact binary foreground from the LIGO-Virgo 02 data [13], while the yellow curve is estimation from the LIGO-Virgo 03 data [14]. The pink and orange lines are PT broken power law models with $\Omega_{\rm p} = 3 \times 10^{-11}$ and $\Omega_{\rm p} = 1 \times 10^{-10}$.

break ratio r_b and the intermediate power law b as described in Eq. 7.4. We also include the DWD foreground $\Omega_{\text{DWD}}(f)$ (see Eq. 7.15), and a power law for the extragalactic compact binary foreground, $\Omega_{\text{ECB}}(f)$ (see Eq. 7.18). The total GW background is the sum

$$\Omega_{\rm gw}(f) = \Omega_{\rm DWD}(f; A_1, A_2, \alpha_1, \alpha_2) + \Omega_{\rm ECB}(f; A_{\rm ECB}, \alpha_{\rm ECB}) + \Omega_{\rm PT}(f; \Omega_{\rm p}, f_{\rm p}, r_b, b).$$
(7.19)

The total GW power spectrum model then has ten independent parameters given by $\theta_{gw} = (\Omega_p, f_p, r_b, b, A_1, \alpha_1, A_2, \alpha_2, A_{ECB}, \alpha_{ECB})$. We also model and simulate the LISA noise with the two magnitude parameters $\theta_{LISA} = (N_{acc}, N_{pos})$. The parameter vector $\theta = \theta_{gw} \cup \theta_{LISA}$ used in this study has 12 components from the GW background model and the LISA noise model. For the MCMC runs, data are simulated in the frequency domain, with a linear frequency vector of 100,000 points in the frequency band $[1 \times 10^{-5}, 1]$ Hz.

$N_{ m acc}$	$1.44 \times$		
	$10^{-48} \mathrm{s}^{-4} \mathrm{Hz}^{-1}$		
$N_{ m pos}$	$3.6 \times 10^{-41} \text{ Hz}^{-1}$		
A_1	7.44×10^{-14}		
A_2	2.96×10^{-7}		
α_1	-1.98		
α_2	-2.6		
$lpha_{ m ECB}$	2/3		
$A_{ m ECB}$	2.15×10^{-9}		

Table 7.1: Parameter values used in the data simulation described by Eq. 7.19, excluding the four phase transition parameters.

The data are produced in the frequency domain by generating $N = 10^5$ independent 3-component Gaussian random vectors with mean zero and covariance matrix given by the spectral density matrix $C_{IJ}(\theta, f_k)$ defined in Eq. 7.21 for equally spaced f_k between 5×10^{-6} Hz and Nyquist frequency $1/2\Delta t = 0.5$ Hz, with a frequency resolution of 5×10^{-6} Hz and a time resolution of $\Delta t = 1$ s. $C_{IJ}(\theta, f_k)$ corresponds to the noise energy spectral density matrix, rescaled by the factor of $N/N_{T_{obs}}$ where $N_{T_{obs}}$ denotes the total number of Fourier frequencies for a time series of 4 years sampled at 1 Hz. This corresponds to segmenting a 4 year long data set sampled at 1 Hz into segments of 1.16 days and averaging over the spectra of the individual segments.

The astrophysical background is derived from the non-continuous compact binary

merger signals, known as a "popcorn" background [160]. This background is non Gaussian. If the merger signals are long in duration and large in their rate, the signals overlap and produce a more continuous signal. This continuous signal approaches Gaussianity via the central limit theorem. It has previously been demonstrated that even with such a non Gaussian background the standard GW background searches still can detect the signal [160]. For our data in this present study we have assumed ideal stationary noise, no glitches, the absence of instrumental lines and gaps. The galactic foreground is modulated in amplitude [109] because of the LISA constellation orbit around the Sun. We have averaged over small segments of time (1.16 days); it has previously demonstrated that the modulation can be assumed as constant within small segments [109]. The phase transition GW background we considered is generated from the overlap of many sound waves, and can be considered as Gaussian.

For the different MCMC analyses we simulate numerous different sets of data with an independent variation on the phase transition parameters ($\Omega_{\rm p}, f_{\rm p}, r_b, b$). The goal is to estimate the impact each phase transition parameter has on the overall observability and parameter estimation. We use the same parameter estimation methods as in Boileau *et al.* [11] treating all parameters of the GW background $\theta_{\rm gw}$ and LISA noise θ_{LISA} as unknown and estimating these simultaneously.

7.2.7 Fisher information and Deviance Information Criterion

The likelihood function with the data $\mathcal{D} = (d_A, d_E, d_T)$ uses the Fourier transform vectors for the channels *AET*. The data is in the frequency domain, given the model parameters θ , and gives the likelihood

$$\mathcal{L}(\mathcal{D}|\theta) = \prod_{k=0}^{N} \frac{1}{\sqrt{\det\left(2\pi \mathcal{C}(\theta, f_k)\right)}} e^{-\frac{1}{2}\mathcal{D}_k^{*T}\mathcal{C}^{-1}(\theta, f_k)\mathcal{D}_k} , \qquad (7.20)$$

where the product is over the of Fourier frequencies f_k , and C denotes the cross powers spectral covariance matrix with components

$$\mathcal{C}(\theta, f) = \frac{3H_0^2}{4\pi^2 f^3} \begin{pmatrix} (\Omega_A(f) + \Omega_{\rm gw}(f)) \mathcal{R}_{A,E}^{\rm Fit} & 0 & 0 \\ 0 & (\Omega_E(f) + \Omega_{\rm gw}(f)) \mathcal{R}_{A,E}^{\rm Fit} & 0 \\ 0 & 0 & \Omega_T(f) \mathcal{R}_T^{\rm Fit} \end{pmatrix}.$$
(7.21)

The dimensionless energy spectral density of the GW signal contributes equally to channels I = [A, E], and we neglect the response of the T channel to GWs. In this work we use a

simple model for the T channel that contains, by definition, no GW signal. We understand that the real data channels will not be as simple and leave the inclusion of second generation TDI channels for future work. Here, the focus is on whether we can separate the PT GW background from the instrument noise and astrophysical foregrounds. We note too that a recent study of the triangular configuration for non-equal detector noise and non-equal noise correlations changes the properties of the A and E channels [161]. To keep the notation compact we have omitted explicit notation for the sum over frequency bins k.

The Fisher information matrix F_{ab} is used to estimate the parameters with the uncertainty $\sqrt{F_{aa}^{-1}}$ of the Fisher information (see Eq. 7.22) from the likelihood (see Eq. 7.20):

$$F_{ab} = \frac{1}{2} \sum_{I=A,E,T} \sum_{k=0}^{N} T_{\text{obs}} \Delta f_k \frac{\partial \ln \mathcal{C}_{II}(f_k)}{\partial \theta_a} \frac{\partial \ln \mathcal{C}_{II}(f_k)}{\partial \theta_b}.$$
 (7.22)

Here, T_{obs} is the time duration of observations for the LISA mission, assumed to be 4 years, and $\Delta f_k = f_k - f_{k-1}$. To reduce the number of calculations, we can assume that parameters from different sources are independent and that θ can be grouped into LISA noise, extra galactic compact binary, DWD and phase transition parameters as

$$\theta = (N_{\text{acc}}, N_{\text{pos}}, A_{\text{ECB}}, \alpha_{\text{ECB}}, A_1, \alpha_1, A_2, \alpha_2, \Omega_{\text{p}}, f_{\text{p}}, r_b, b).$$
(7.23)

The Fisher information matrix is then a block diagonal matrix

$$F_{ab}(\theta) = \begin{pmatrix} \Gamma_{LISA} & 0 & 0 & 0 \\ 0 & \Gamma_{\rm ECB} & 0 & 0 \\ \hline 0 & 0 & \Gamma_{\rm DWD} & 0 \\ \hline 0 & 0 & 0 & \Gamma_{\rm PT} \end{pmatrix},$$
(7.24)

with respectively the Fisher information matrix of the LISA noise Γ_{LISA} , the extragalactic compact binary background Γ_{ECB} , the DWD foreground Γ_{DWD} and the phase transition background Γ_{PT} . Thus, the inverse of the Fisher matrix is

$$F_{ab}^{-1}(\theta) = \begin{pmatrix} \Gamma_{LISA}^{-1} & 0 & 0 & 0 \\ 0 & \Gamma_{ECB}^{-1} & 0 & 0 \\ \hline 0 & 0 & \Gamma_{DWD}^{-1} & 0 \\ \hline 0 & 0 & 0 & \Gamma_{PT}^{-1} \end{pmatrix}$$
(7.25)

The uncertainty in parameter θ_a is estimated as $\sigma_a = \sqrt{F_{aa}^{-1}}$. In the following, we will study only the sub-matrix of the phase transition parameters $\Gamma_{\rm PT}^{-1}$. As a cross check, we

also use MCMC methods to estimate the posterior distribution of the signal parameters, $p(\theta|\mathcal{D}) \propto p(\theta)\mathcal{L}(\mathcal{D}|\theta).$

The Fisher information matrix is obtained by calculating the second order partial derivatives of the log-likelihood function with respect to unknown parameters. There will be non-zero off-diagonal entries of the Fisher information matrix. However, for the sake of faster computation, we assume a block diagonal matrix where parameters within each block Γ_{LISA} , Γ_{ECB} , Γ_{DWD} and Γ_{PT} , are potentially dependent but parameters from different blocks are independent. We refer to our MCMC results in Section 7.3, where indeed the parameters from different blocks have negligible posterior correlation; see Figure 7.6 in Appendix 7.5.1.

The posterior distribution of the parameter vector θ is obtained by combining the likelihood in Eq. 7.20 with independent general priors for each of the parameters. We specify independent Gaussian priors

$$p(\theta) = \prod_{i} \exp\left(-\frac{(\theta_i - \mu_i)^2}{2\sigma_i^2}\right)$$
(7.26)

for the GW background, DWD and LISA noise parameters where μ_i is the true value and $\sigma_i = 1$. For example, for the parameter Ω_p , we sample on $\log(\Omega_p)$ with a Gaussian prior centred on the "true" value $\log(\Omega_p)$, with $\log(\sigma_{\Omega_p}) = 1$. We use log-parameters for N_{acc} , N_{pos} , A_{ECB} , A_1 , A_2 , Ω_p , and f_p and sample directly with $\alpha_1, \alpha_2, \alpha_{\text{ECB}}, r_{\text{b}}$ and b. We use the MCMC algorithm of [162]. This is an adaptive Metropolis-Hastings algorithm with a proposal distribution:

$$Q_n(\theta) = (1 - \beta)\mathcal{N}(\theta, (2.28)^2 \Sigma_n/d) + \beta \mathcal{N}(\theta, (0.1)^2 I_d/d) , \qquad (7.27)$$

where Σ_n is the current empirical estimate of the covariance matrix of the parameter vector θ (based on the previous MCMC samples), $\beta = 0.01$, d the number of parameters, I_d the identity matrix and \mathcal{N} the multi-normal distribution.

The ultimate aim is to study whether the model that includes phase transitions provides a substantially better fit than a model without phase transitions where both models include the LISA noise, DWD foreground and the compact binary produced GW background. Within the Bayesian framework, this model comparison could be performed by computing Bayes factors. However, when using improper priors or even very vague priors, these are not well defined. The sensitivity of the Bayes factor to the choice of increasingly diffuse priors is well known and often referred to as *Lindley's paradox* [163, 164]. It is illustrated for instance in [165] for an example of a Gaussian likelihood with unknown mean θ and unknown variance σ^2 where a Normal $(0,\tau^2)$ prior is put on the variance parameter σ^2 . With increasing τ^2 , the marginal likelihood of the null model $\theta = \theta_0$ and that of the alternative will converge to 1 and zero, respectively, no matter the value of the data. Thus the Bayes factor for comparing the null to the alternative model will go to infinity even if the observed data value is far away from θ_0 . Therefore, we use the DIC [148, 149] which can be regarded as the Bayesian analogue of the AIC/BIC and a Bayes factor approximation, and can be used even if improper priors have been specified. The DIC is a very popular choice for practical model comparison as it is easy to compute when a MCMC sample of the posterior distribution is available [150].

The DIC combines a model fit statistic with a term that penalizes the model complexity. It is based on the deviance $D(\theta)$ defined as $D(\theta) = -2 \log \mathcal{L}(\mathcal{D}|\theta)$, and evaluated at the posterior mean $\bar{\theta}$ of θ (the average of the posterior samples from the MCMC). The penalty term is given by $p_D = \bar{D} - D(\bar{\theta})$ where \bar{D} denotes the posterior mean of the deviance. The DIC is given by

$$DIC = D(\overline{\theta}) + 2p_D. \tag{7.28}$$

We calculate the difference in DIC for the models with a phase transition and without. We follow the general rule of thumb that a difference in the DIC of Δ DIC > 5 there is substantial evidence for the model with a phase transition, and we have strong and decisive evidence for Δ DIC > 10 [166]. In this study we use the level of Δ DIC > 5 as the threshold for detectability.

7.3 Results

7.3.1 DIC results

We use the DIC to investigate LISA's sensitivity to a GW background from a first order phase transition in the presence of foregrounds from DWDs in the galaxy and extragalactic compact binaries. We explore how this sensitivity varies as a function of the parameters of the fit to the phase transition signal. The peak amplitude Ω_p and the peak frequency f_p are the parameters that play the greatest role in determining whether one can distinguish between models with or without a phase transition signature. In Fig. 7.2a and Fig. 7.2b we show Δ DIC as a function of these parameters. For a signal peaking at 1 mHz, which is the most favourable frequency for detection by LISA, Δ DIC is above 5 for peak amplitudes $\Omega_p \gtrsim 3 \times 10^{-11}$, and above 10 for peak amplitudes $\Omega_p \gtrsim 1 \times 10^{-10}$. A signal of with



Figure 7.2: Fig. 7.2a shows the changes in the deviance information criterion (ΔDIC) as the peak amplitude $\Omega_{\rm p}$ is varied, when the peak frequency $f_{\rm p} = 1 \times 10^{-3}$ Hz. Fig. 7.2b shows the changes in the ΔDIC as the peak frequency $f_{\rm p}$ is varied, for three values of peak amplitude $\Omega_{\rm p} = 1 \times 10^{-9}$ (red), 1×10^{-10} (blue) and 1×10^{-11} (green). In both cases the break ratio and the intermediate slope are fixed to $r_b = 0.4$, and b = 1.

magnitude $\Omega_{\rm p} = 1 \times 10^{-10}$ has $\Delta \text{DIC} > 5$ over a band from 3×10^{-4} to 10^{-2} Hz, where we use the level of $\Delta \text{DIC} > 5$ as the threshold for detectability for the model with a phase transition.

In Fig. 7.3a and Fig. 7.3b we see that varying the break ratio r_b and the intermediate slope b have little impact on the overall observational prospects of the phase transition signal. Instead we again see the importance of the peak amplitude parameter.

As discussed in Sec. 7.2.1 the relationship between the spectral parameters and the thermodynamic parameters of a first order phase transition is complicated, which makes it challenging to say anything concrete about LISA's sensitivity to the thermodynamic parameters from these results alone. The DIC analysis has shown the spectral parameters with the biggest impact on resolving a PT signature are the peak amplitude which relates to (α, v_w, r_*) and the peak frequency which all thermodynamic parameters contribute to. For a more quantitative description of how the uncertainties in the spectral parameters translate into uncertainties in thermodynamic parameters see Gowling et al. [3].

7.3.2 Fisher matrix and MCMC comparison

Here, we compare the uncertainties in the measurements of spectral parameters when calculated with the Fisher matrix [1] to those computed with MCMC simulations. In the Fisher method, the relative uncertainties are calculated using the Fisher matrix F_{ab} , as outlined in Eq. 7.22, and are given by $\sqrt{F_{aa}^{-1}}$. The Fisher matrix is evaluated with 200



Figure 7.3: Fig. 7.3a displays the changes in the deviance information criterion (Δ DIC) as the break ratio $r_{\rm b}$ is varied and the intermediate slope b = 1. Fig. 7.3b shows the changes in Δ DIC as the intermediate slope b is varied; here $r_{\rm b} = 0.4$. In both cases we considered three values of peak amplitude $\Omega_{\rm p} \ 1 \times 10^{-9}$ (red), 1×10^{-10} (blue) and 1×10^{-11} (green) and the peak frequency $f_{\rm p} = 1$ mHz.

points in a log-frequency band between $[1 \times 10^{-5}, 1]$ Hz according to Eq. 4.3 in [1].

For the MCMC method, we use the same Adaptive-MCMC algorithm as previously presented in [11, 110]. We define the uncertainty on a parameter to be the standard deviation of the marginalised posterior distribution. In the following, unless stated otherwise, the total GW model used is described by Eq. 7.19. We look at each of the spectral parameters in turn, showing the results for each of the phase transition parameters in Figs. 7.4a, 7.4b, 7.5a and 7.5b. The relative uncertainties calculated from the MCMC results are shown as dots with a 1- σ error bar, and those from the Fisher information are denoted by continuous lines.

The Fisher information matrix is much faster to evaluate than an MCMC, and allows us to explore LISA's sensitivity to a wide range of parameter space associated with a first order phase transition; to explore the same parameter space with MCMC methods would take significantly longer. For this work we aimed to investigate the similarities (and differences) between the Fisher information matrix and MCMC results, which gives insight into how to interpret the Fisher information matrix results for the broader parameter space explored in [1].

In Fig. 7.4a, the relative uncertainties in the peak amplitude $\Omega_{\rm p}$ are shown, with the other spectral parameters being $f_{\rm p} = 1 \times 10^{-3}$ Hz, $r_b = 0.4$ and b = 1. The agreement between the two ways of estimating the relative uncertainties is very good. We also see that for $\Omega_{\rm p} = 3 \times 10^{-11}$, which for this combination of parameters we found to be the

threshold for detectability in our DIC analysis (see Fig. 7.2a), the relative uncertainty of $\Delta\Omega_p/\Omega_p < 0.1$ is reached, consistent with interpreting Δ DIC > 5 as a threshold for distinguishing the models. We see in Fig. 7.4a that the relative uncertainty decreases as $1/\Omega_p$, before saturating. As the signal-to-noise ratio (SNR) should be proportional to the peak amplitude Ω_p , this is consistent with the expectation that the relative uncertainty is inversely proportional to the SNR. The saturation occurs when the signal dominates the noise, and there is little further change in the Fisher matrix.

Fig. 7.4a also shows the impact of the DWD foreground model on the phase transition measurement. The nature of the DWD foreground spectral density is an open question and as shown in Fig. 22 of [11] the position of the frequency break of the galactic foreground has a large impact on the constraints one is able to place on a flat GW background. Due to the computationally intensive nature of MCMC computations we only evaluate the relative uncertainties with the different foreground model using the Fisher matrix.

The two red lines in Fig. 7.4a are the two DWD models considered: the dashed line is the analytic fit [11] to the galactic foreground from the Lamberts *et al.* catalogue [10] (Eq. 7.15) and the solid line is the galactic confusion noise model from Robson *et al.* [12] (Eq. 7.16). We also show the case where all foregrounds are removed with the blue dashed line; here the signal fluctuations are the sum of the LISA instrument noise and the fluctuations in the phase transition GW background. As in [11] we see a drop in the limiting performance for low values of Ω_p . In this case, the drop is larger for the Robson *et al.* model, which can be traced to the model having a higher amplitude at 1 mHz, where the chosen phase transition model power peaks. The difference in impact of the models decreases with amplitude, as the phase transition signal starts to exceed the power in both foreground models. In all cases the difference is within the 68% error posterior credible interval, which suggests that the modelling of the DWD foreground is not quite as critical as might be expected.

For the remaining three spectral parameters $f_{\rm p}$, r_b and b, as well as varying these parameters we have studied the measurement uncertainty coming from the MCMC and Fisher analysis for different values of $\Omega_{\rm p}$, $(1 \times 10^{-9}, 1 \times 10^{-10} \text{ and } 1 \times 10^{-11})$ they are shown in green, blue and red respectively in Fig. 7.4b, Fig. 7.5a and Fig. 7.5b. It is evident that the uncertainty in all parameters increases when the amplitude is lower.

In Fig. 7.4a the solid line displays a total GW model that uses the Robson *et al.*



Figure 7.4: Uncertainty estimates for the peak amplitude $\Omega_{\rm p}$ and the peak frequency $f_{\rm p}$, calculated with the Fisher information (continuous lines) and MCMC simulations (points). The solid line corresponds to a model of LISA instrument noise, phase transition signal, astrophysical background and a Robson *et al.* [12] DWD foreground model. The dashed lines are identical but instead consider the Lamberts *et al.* model for the DWD foreground [10]. The relative uncertainties as calculated from the Fisher matrix for $\Omega_{\rm p}$ when only the LISA noise and phase transition signal are considered are show in blue in Fig. 7.4a. In both cases $r_b = 0.4$, b = 1 and in Fig. 7.4a $f_{\rm p} = 1$ mHz.

galactic foreground model and the dashed line is the analytic model [11] fit on the Galactic foreground from the Lamberts *et al.* catalogue [10]. As one might expect, the peak frequency is less well determined in the louder foreground model; however, the effect is not large. It appears that the good overlap between the Fisher study and the Bayesian MCMC analysis disappears when the amplitude Ω_p decreases. Indeed, in view of the DIC study, when the peak amplitude is below $\Omega_p = 3 \times 10^{-11}$, we are unable to state with certainty that the model including the phase transition signal is a better fit. It is therefore not surprising that the different methods for estimating the parameter uncertainty give different results below this value.

In Fig. 7.4b we show the effect of varying the peak frequency, for break ratio $r_b = 0.4$ and intermediate slope b = 1. We see that with a peak amplitude $\Omega_{\rm p} > 1 \times 10^{-10}$, we achieve a relative uncertainty $\Delta f_{\rm p}/f_{\rm p} < 0.2$ for peak frequency between $f_{\rm p} = 2 \times 10^{-4}$ Hz and 2×10^{-2} Hz. We also note an effect of the different galactic models on the measurement of the peak frequency of the phase transition signal.

Finally, in Fig. 7.5, we display the Fisher and MCMC results for the two remaining spectral parameters, r_b (the ratio between the breaks in the power laws) and b (the intermediate power law), for spectra with peak frequency $f_p = 1$ mHz. There is no systematic



Figure 7.5: Uncertainty estimates in the break ratio r_b and the intermediate slope b, calculated with the Fisher information (continuous lines) and MCMC simulations (points). The model: LISA instrument noise, first order phase transition signal modelled as a double broken power law, astrophysical background and a Robson *et al.* [12] DWD foreground model. In both figures $f_p = 1$ mHz, in Fig. 7.5a b = 1 and in Fig. 7.5b $r_b = 0.4$.

trend in measurement performance as the parameters are varied, except at low r_b , where, for this peak frequency choice, the lower break frequency moves out of the LISA sensitivity window, and the uncertainty quickly increases. In Fig. 7.5a we see a dip in sensitivity at $r_b = 0.7$, this feature is due to the complicated nature of the differential that goes into the Fisher matrix, as opposed to anything special about the spectrum at this combination of spectral parameters. When other parameter combinations are considered and r_b is varied this dip appears at different r_b values.

For the parameters r_b and b, when we compare the parameter estimation results with different galactic foreground models we again see the sensitivity to r_b and b is reduced for the louder galactic foreground model. In Fig. 7.5b the vertical brown line shows the point where the double broken power law, Eq. 7.4, becomes ill-defined due to the limitations of the double broken power law discussed in Sec. 7.2.1. The relative uncertainty is ill-defined at b = 0 so here we instead consider Δb .

7.4 Conclusions

In this paper we have investigated the ability of LISA to observe a GW background produced by a first order phase transition in the early universe. We have considered the presence of GW foregrounds from DWD binaries in our galaxy, from compact binary mergers throughout the universe, and LISA noise. For a phase transition GW spectrum with break ratio $r_b = 0.4$ and intermediate spectral slope b = 1, we show that signals with peak frequency 1 mHz can be detected for $\Omega_{\rm p} \geq 3 \times 10^{-11}$. Signals with peak amplitude $\Omega_{\rm p} = 10^{-10}$ achieve the detection threshold $\Delta \text{DIC} > 5$ with peak frequency between $f_{\rm p} = 4 \times 10^{-4}$ to 9×10^{-3} Hz. For phase transition signals with a larger peak amplitude than $\Omega_{\rm p} = 10^{-10}$ there would be a broader frequency window of detectability, including phase transition signals with peak frequencies between $f_{\rm p} \approx 1 \times 10^{-4}$ to $\approx 2 \times 10^{-2}$ Hz.

We then used Fisher Matrix and MCMC methods to show how well the four parameters associated with the first order phase transition could be estimated, simultaneously with noise and GW foregrounds. For example, with a GW background of peak amplitude $\Omega_{\rm p} = 10^{-10}$ the parameter estimation accuracies are $\Delta\Omega_{\rm p}/\Omega_{\rm p} \approx 10^{-2}$, $\Delta f_{\rm p}/f_{\rm p} \approx 10^{-2}$ at $f_{\rm p} = 3 \times 10^{-3}$, $\Delta r_{\rm b}/r_{\rm b} \approx 0.1$ at $r_{\rm b} = 0.2$, and $\Delta b/b \approx 0.1$ at b = 1. The Fisher Matrix and MCMC methods give similar results for $\Omega_{\rm p} > 3 \times 10^{-11}$, where the signal becomes detectable.

We have modelled the GW background from a first order phase transition as a double broken power law, which is a good fit to the GW power spectrum calculated from the thermodynamic parameters for the majority of the thermodynamic parameter space. However, subtleties in the characteristics of the GW power spectra from thermodynamic parameters are not encapsulated in the double broken power law, for example the double broken power law struggles to describe the spectra for wall speeds around the speed of sound, see Fig. 11 in [1]. Another challenge for the parameter estimation of first order phase transitions at LISA is, as discussed in Sec. 7.2.1, the relationship between the spectral and thermodynamic parameters is complicated. See [3] for a discussion on the connection between the spectral and thermodynamic parameters and how to reconstruct thermodynamic parameters from MCMC samples on the spectral parameters (like those performed here). These differences and challenges mean that as we improve our understanding of phase transition physics and develop better spectral fits, the findings presented here will evolve.

We have used a basic model for the LISA noise based on only two parameters [5, 6], and it will be important to incorporate more sophisticated noise models in order to better understand the prospects for cosmological GW background detection and parameter estimation.

We have been conservative in not using annual modulation to improve the estimation of the DWD foreground parameters. In addition, other GW wave signals will be present in the data, such as identifiable galactic binaries, massive black hole binaries, and extreme mass ratio inspirals [139]. Searches may also need to allow for the presence of other cosmological GW backgrounds in the LISA data [141], for example from inflation [76] or cosmic strings [11, 167]. More advanced parameter estimation methods will need to be developed, and realistic early-universe signal models will need to be included in global fits for the LISA GW signals [168].

Acknowledgements

G. B. and N. C. thank the Centre national d'études spatiales for support for this research. R. M. acknowledges support by the Marsden Fund Council from Government funding, administered by the Royal Society of New Zealand and DFG Grant No. KI 1443/3-. C.G. is supported by a STFC Studentship. MH (ORCID ID 0000-0002-9307-437X) acknowledges support from the Academy of Finland (grant number 333609).

7.5 Appendix

7.5.1 Parameter correlations

When evaluating the Fisher matrix we assumed that parameters from different sources are independent. In Figure 7.6 we present a corner plot for the MCMC results presented in Section 7.3. Parameters from different sources can be grouped into different parameter blocks: Γ_{LISA} , Γ_{ECB} , Γ_{DWD} and Γ_{PT} . The parameters within a particular block can exhibit small correlations, but parameters from different blocks are independent.



Figure 7.6: Corner plot for an example adaptive MCMC with an injected phase transition signal characterised by $(\log_{10}(\Omega_p), \log_{10}(f_p), rb, b) = (-9, -2, 7, 0.4, 1)$ and a data model described by Eq. 7.19. The vertical dashed lines on the posterior distribution represent from left to right the quantiles [16%, 50%, 84%]. The red, green and blues lines are respectively the mean, the median of the posterior distribution and the input parameter values on the simulation.

Chapter 8

Reconstructing physical parameters from template gravitational wave spectra at LISA: first order phase transitions

Abstract

A gravitational wave background from a first order phase transition in the early universe may be observable at millihertz gravitational wave (GW) detectors such as the Laser Interferometer Space Antenna (LISA). In this paper we introduce and test a method for investigating LISA's sensitivity to gravitational waves from a first order phase transition using parametrised templates as an approximation to a more complete physical model. The motivation for developing the method is to provide a less computationally intensive way to perform Markov Chain Monte Carlo (MCMC) inference on the thermodynamic parameters of a first order phase transition, or on generally computationally intensive models. Starting from a map between the physical parameters and the parameters of an empirical template, we first construct a prior on the empirical parameters that contains the necessary information about the physical parameters; we then use the inverse mapping to reconstruct approximate posteriors on the physical parameters from a fast MCMC on the empirical template. We test the method on a double broken power law approximation to spectra in the sound shell model. The reconstruction method substantially reduces the proposal evaluation time, and despite requiring some precomputing of the mapping, this method is still

cost-effective overall. In two test cases, with signal-to-noise ~ 40, the method recovers the physical parameters and the spectrum of the injected gravitational wave power spectrum to 95% confidence. In previous Fisher matrix analysis we found the phase boundary speed $v_{\rm w}$ was expected to be the best constrained of the thermodynamic parameters. In this work, for an injected phase transition GW power spectrum with $v_{\rm w} = 0.55$, with a direct sample on the thermodynamic parameters we recover $0.630^{+0.17}_{-0.059}$ and for our reconstructed sample $0.646^{+0.098}_{-0.075}$.

8.1 Introduction

The Laser Interferometer Space Antenna (LISA), due to launch in the 2030s [28] will probe the previously unexplored millihertz region of the gravitational wave (GW) spectrum. The LISA sensitivity window, 10^{-4} Hz to 10^{-1} Hz, has an abundance of GW sources ranging from astrophysical: black hole mergers, galactic binaries [73], extreme mass ratio binaries [74] and precursors for stellar origin black hole mergers [75]; to the cosmological: cosmic strings, inflation and phase transitions [76, 169]. Here we focus on LISA's sensitivity to the cosmological stochastic GW background (SGWB) from a first order phase transition.

The early universe was hot and dense; as it expanded and cooled the universe may have undergone several phase transitions. In particular, we are interested in a possible first order phase transition associated with electroweak symmetry breaking. In the Standard Model this process occurs via a crossover and no GWs are produced [20, 34]. Alternatively, in numerous extensions to the Standard Model, in some cases motivated as explanations for dark matter or the baryon asymmetry of the universe, a first order phase transition and the production of GWs is possible. See [8] for a review of models.

In a first order phase transition, once a critical temperature is reached, bubbles of the broken phase nucleate in the symmetric phase; these bubbles expand, collide, and percolate until the phase transition is complete. During this process GWs are produced via the collisions of bubble wall, the subsequently produced sound waves and turbulent flows. For a review of first order phase transitions see [41, 105, 170].

If the first order phase transition is driven by thermal fluctuations, the acoustic source of GWs dominates [24–26]; production by bubble collisions [98–100, 144–146] can become relevant if there is very strong supercooling [52, 53]. Here we assume that the sound wave component is dominant, and model the GW background component with the Sound Shell Model (SSM) [4, 27]. The SGWB from a first order phase transition is determined by key thermodynamic parameters: the nucleation temperature $T_{\rm n}$, the phase transition strength α , the wall speed $v_{\rm w}$ and the Hubble-scaled mean bubble spacing r_* . The speed of sound, which can take different values in the two phases, also impacts the SGWB produced [116, 117]; however for this first analysis we take it to be the ultra-relativistic value of $1/\sqrt{3}$.

Calculating numerous GW power spectra for a first order phase transition using the SSM, as when one conducts Markov Chain Monte Carlo (MCMC) analyses, is computationally expensive. This motivates the use of a fit to the phase transition SGWB that is quick to evaluate: here we use a double broken power law, which provides a good fit to the SSM over most of the thermodynamic parameter space [1, 27]. The double broken power law is characterised by four "spectral" parameters: the peak amplitude Ω_p , the peak frequency f_p , the ratio r_b between the peak frequency and the break frequency, and the slope between the two characteristic frequency scales b. The ultimate goal is to infer the thermodynamic parameters of a supposed SGWB signal by fitting to it a computationally cheap double broken power law.

To achieve this, we require a robust method for transforming information about the spectral parameters into constraints on the thermodynamic ones. The first step is to transform the physically-motivated prior density on the thermodynamic parameters into an induced prior on the spectral ones, which is achieved by weighting an initial spectral prior with the density of the image of a prior-consistent grid of thermodynamic parameters in the spectral parameter space. Constraints on the spectral parameters obtained with such a prior can then be translated back to the thermodynamic parameter space by using the inverse of the projection that we just described. This reconstruction method is a general cost-effective preliminary parameter estimation framework that can be applied to any model for which computing the SGWB is expensive, but for which there exists a reasonably good empirical approximation.

As a demonstration, in this study we consider two fiducial models with different thermodynamic parameters, and use MCMC methods to estimate LISA's ability to perform parameter estimation for both the spectral and the (much slower) thermodynamic parameterisations. We then compare the latter result with the constraints on the thermodynamic parameters derived from the spectral parameter sample using our reconstruction methodology. We consider a data model made of the phase transition SGWB and LISA noise. In a global fit the impact of astrophysical foregrounds from the extragalactic black holes, binary neutron stars and double white dwarf populations should also be considered; in this first investigation of the method we ignore these foregrounds. However, their impact on the MCMC estimation of spectral parameters has recently been considered in [171]. Parameterised templates with more general spectral forms have been explored in [147]; although no reconstruction of the underlying parameters was attempted.

This paper is structured as follows: in Section 8.2 we describe the expected SGWB spectrum from cosmological first order phase transitions in the SSM; in Section 8.3 we describe the LISA noise model, and in Section 8.4 we go on presenting our data model, and the likelihood and base priors that will be used; in Section 8.5 we present the reconstruction algorithm, and finally in Section 8.6 we apply it to the aforementioned fiducial models. We lay out our conclusions and discuss some future prospects in Section 8.7.

8.2 SGWB from cosmological first order phase transition

The GW power spectrum from a first order phase transition can be characterised by the thermodynamic parameters (T_n, r_*, α, v_w) . Firstly, the nucleation temperature T_n , is the temperature corresponding to the peak of the globally-averaged bubble nucleation rate. The Hubble rate at the nucleation temperature H_n sets the frequency scale of the GW spectrum.

The second thermodynamic parameter is the nucleation rate parameter β . As discussed in [1], due to uncertainties in the calculation of β , we instead consider the related quantity, the mean bubble spacing R_* . We note that β^{-1} is the time for the bubble wall to move a distance R_* and therefore has the interpretation of the duration of the phase transition. In this work we refer to the Hubble-scaled mean bubble spacing $r_* = H_n R_*$ which contributes to the frequency scale and amplitude of the GW power spectrum.

Our third key thermodynamic parameter is the phase transition strength α , which we define as the ratio between the trace anomaly and the thermal energy, where the trace anomaly describes the amount of energy available for conversion to shear stress energy. A stronger transition means more energy is converted to shear stress energy and a larger overall amplitude for the GW signal.

The final parameter to introduce is the wall speed v_w which, along with α , determines the motion of the plasma surrounding the bubble wall. The value of the wall speed relative to the speed of sound c_s determines the width of the GW power spectrum, here we assume the ultrarelativistic value $c_s = 1/\sqrt{3}$ (see [116, 117] for other scenarios). For wall speeds close to c_s the power spectra are broad and r_b (the ratio between the peak frequency and the break frequency) is small, in the opposite case the power spectra are narrow.

The general form of the gravitational wave power spectrum from a thermal first order phase transition is

$$\Omega_{\rm gw}(z) = 3K^2(v_{\rm w},\alpha) \left(H_{\rm n}\tau_{\rm v}\right) \left(H_{\rm n}R_*\right) \frac{z^3}{2\pi^2} \tilde{P}_{\rm gw}\left(z\right),\tag{8.1}$$

where R_* is the mean bubble spacing, $z = kR_*$, k is the comoving wavenumber and $K(v_w, \alpha)$ is the fraction of the total energy converted into kinetic energy of the fluid. The Hubble rate at nucleation is H_n , τ_v is the lifetime of the shear stress source, the factor R_* appears as an estimate of the source coherence time and $\tilde{P}_{gw}(z)$ is the dimensionless shape spectral density. Eq. (8.1) can be regarded as the definition of \tilde{P}_{gw} . As introduced and discussed in [1], for simplicity we define

$$J = H_{\rm n} R_* H_{\rm n} \tau_v = r_* \left(1 - \frac{1}{\sqrt{1+2x}} \right).$$
(8.2)

where $x = H_{\rm n}R_*/\sqrt{K}$ is the ratio of the Hubble time $H_{\rm n}^{-1}$ and the fluid shock appearance time $\tau_{\rm sh} = R_*/\sqrt{K}$ [55]. The second equality is a model for the lifetime of the shear stress source in an expanding universe [119].

8.2.1 Gravitational wave power spectrum in the SSM

Here we focus on the contribution from the sound waves and use the Sound Shell Model [4, 27], which limits us to transitions which are not so strong that the modifications to the spectrum from shocks [55] and vortical turbulence [56] become important. We use the PTtools¹ module which uses the SSM to directly compute the scale-free gravitational wave power spectrum $\hat{\mathcal{P}}_{gw}$ for a given v_w and α [4], defined as

$$\hat{\mathcal{P}}_{\rm gw}(z) = 3K^2 \frac{z^3}{2\pi^2} \tilde{P}_{\rm gw}(z) \,. \tag{8.3}$$

The specifications of the calculations done with PTtools are the same as used in our previous work [1]. We now introduce

$$\Omega_{\rm gw}^{\rm ssm}(z) = J\hat{\mathcal{P}}_{\rm gw}(z). \tag{8.4}$$

As discussed in [1], recent 3d-hydro simulations for α up to $\mathcal{O}(1)$ (strong transitions) found that as transition strength increases, the efficiency of fluid kinetic energy production

¹Code available on request to MH.

is less than previously expected [7]. We estimate suppression in gravitational waver power observed in the numerical simulations, as a factor $\Sigma(v_{\rm w}, \alpha)$. For a complete outline of how we calculate Σ see Appendix A in [1]. The gravitational wave power spectrum at dimensionless comoving wavenumber z just after the transition, and before any further entropy production, is then

$$\Omega_{\rm gw}(z) = \Omega_{\rm gw}^{\rm ssm}(z) \Sigma(v_{\rm w}, \alpha). \tag{8.5}$$

Today the power spectrum at physical frequency f is

$$\Omega^{\rm ssm}_{\rm gw,0}(f) = F_{\rm gw,0} \Omega_{\rm gw}(z(f)), \tag{8.6}$$

where

$$F_{\rm gw,0} = \Omega_{\gamma,0} \left(\frac{g_{s0}}{g_{s*}}\right)^{\frac{4}{9}} \frac{g_*}{g_0} = (3.57 \pm 0.05) \times 10^{-5} \left(\frac{100}{g_*}\right)^{\frac{1}{3}}$$
(8.7)

is the power attenuation following the end of the radiation era. Here $\Omega_{\gamma,0}$ is the photon energy density parameter today, g_s denotes entropic degrees of freedom and g describes the pressure degrees of freedom. In both cases the subscripts 0 and * refer to their value today and the value at the time the GWs were produced respectively. We evaluate $F_{\text{gw},0}$ with the values given in [8], and use a reference value $g_* = 100$.

We convert from dimensionless wavenumber z to frequency today by taking into account redshift

$$f = \frac{z}{r_*} f_{*,0},$$
(8.8)

where

$$f_{*,0} = 2.6 \times 10^{-6} \,\mathrm{Hz}\left(\frac{T_{\mathrm{n}}}{100 \,\mathrm{GeV}}\right) \left(\frac{g_{*}}{100}\right)^{\frac{1}{6}},$$
(8.9)

is the Hubble rate at the phase transition redshifted to today [8]. We assume the phase transition takes place well within one Hubble time so all frequencies throughout the transition have the same redshift.

8.2.2 Double broken power law

In the SSM there are two characteristic length scales, the mean bubble separation and the sound shell thickness, which motivate a simplified description in terms of a function with two frequency scales and three power law indices - a double broken power law [4]. The power spectrum today for the double broken power law fit can be described as

$$\Omega_{\mathrm{gw},0}^{\mathrm{dbp}}(f,\Omega_{\mathrm{p}},f_{\mathrm{p}},r_{\mathrm{b}},b) = \Omega_{\mathrm{p}}M(s,r_{\mathrm{b}},b)$$
(8.10)

where $\Omega_{\rm p}$ is the peak of the power spectrum, $s = f/f_{\rm p}$, $f_{\rm p}$ is the frequency corresponding to $\Omega_{\rm p}$ and $r_{\rm b} = f_{\rm b}/f_{\rm p}$ is the ratio between the two breaks in the spectrum. The parameter b defines the spectral slope between the two breaks. The spectral shape $M(s, r_{\rm b}, b)$ is a double broken power law with a spectral slope 9 at low frequencies and -4 at high frequencies, a form that was chosen to best describe the SSM [4].

$$M(s, r_{\rm b}, b) = s^9 \left(\frac{1 + r_{\rm b}^4}{r_{\rm b}^4 + s^4}\right)^{(9-b)/4} \left(\frac{b+4}{b+4 - m + ms^2}\right)^{(b+4)/2}.$$
(8.11)

Within $M(s, r_{\rm b}, b)$, m has been chosen to ensure that for $r_{\rm b} < 1$ the peak occurs at s = 1and $M(1, r_{\rm b}, b) = 1$, giving

$$m = (9r_{\rm b}{}^4 + b) / (r_{\rm b}{}^4 + 1).$$
(8.12)

8.3 LISA instrument noise model

LISA will be a triangular constellation of three spacecraft connected via lasers with arm length of 2.5 million km. Passing GWs will induce a distance modulation in the instrument arm length that is measured via the phase differences between lasers on the local and remote spacecraft. The phase differences (interferometer signals) can be combined in different ways with different time delays to eliminate the laser noise [120, 121]. We follow the convention for the three noise-orthogonal time delay interferometry (TDI) variables A, E and T, as described in [5]. The T variable can be approximated as being insensitive to GWs. Here we assume the instrument noise is completely known and build our data model combining the A and E channels.

We construct the instrument power spectral density following the conventions given in [5] and used in [1]. For the LISA instrument noise model we use the functions and parameter values given in the LISA Science Requirements Document [6]. In the A and E TDI channels the instrument noise spectral density arising from the optical metrology system noise (oms) and the test mass acceleration noise (acc) is given by

$$N_A = N_E = N_1 - N_2 \simeq (6P_{\rm oms} + 24P_{\rm acc})|W(f)|^2, \tag{8.13}$$

where

$$N_1 = [4P_{\rm oms}(f) + 8 [1 + \cos^2(f/f_*)] P_{\rm acc}(f)] |W(f)|^2, \qquad (8.14)$$

$$N_2 = -[P_{\rm oms}(f) + 8P_{\rm acc}]\cos(f/f_*)|W(f)|^2, \qquad (8.15)$$

and $W(f) = 1 - \exp(2if/f_*)$, representing the interference induced by a return journey along one arm. In the above $f_* = c/(2\pi L)$ is the transfer frequency, $L = 2.5 \times 10^9$ m is the constellation arm length, c is the speed of light, and the model for the noise is

$$P_{\rm oms} = N_{\rm pos}, \tag{8.16}$$

$$P_{\rm acc} = \frac{N_{\rm acc}}{(2\pi f)^4} \left(1 + \left(\frac{f_1}{f}\right)^2 \right), \qquad (8.17)$$

with $N_{\rm acc} = 1.44 \times 10^{-48} \text{ s}^{-4} \text{Hz}^{-1}$, $N_{\rm pos} = 3.6 \times 10^{-41} \text{ Hz}^{-1}$ and $f_1 = 0.4 \text{ mHz}$ [6].

To take into account the detector response to incident GWs, we consider the sensitivity S for the A and E channels,

$$S_A = S_E = \frac{N_A}{\mathcal{R}_A} \simeq \frac{40}{3} \left(P_{\text{oms}} + 4P_{\text{acc}} \right) \left[1 + \left(\frac{f}{4f_*/3} \right)^2 \right],$$
 (8.18)

where \mathcal{R} is the detector response to isotropic stochastic GWs. In general, \mathcal{R} must be evaluated numerically; here we use the simpler analytic fits presented in [5]

$$\mathcal{R}_{A}^{\text{Fit}}(f) = \mathcal{R}_{E}^{\text{Fit}}(f) = \frac{9}{20} |W(f)|^{2} \left[1 + \left(\frac{f}{4f_{*}/3}\right)^{2} \right]^{-1}.$$
(8.19)

The sensitivities can be thought of as GW signals with unit signal-to-noise ratio at all frequencies.

In this work we will be interested in the sensitivity expressed as a GW fractional energy density power spectrum, related to the sensitivity by

$$\Omega_{\rm ins} = \left(\frac{4\pi^2}{3H_0^2}\right) f^3 S_A(f),\tag{8.20}$$

which we will refer to as the LISA instrument noise. The fiducial models have a signalto-noise ratio ρ of approximately $\rho \approx 40$. As we will show in the next section, our data model will combine the A and E channels, and the corresponding signal-to-noise ratio is given by [5]

$$\rho = \sqrt{2T_{\rm obs} \int_0^\infty df \frac{\Omega_{\rm gw}^2}{\Omega_{\rm ins}^2}} \ . \tag{8.21}$$

As the T channel is insensitive to GW signatures at low frequencies, it allows the instrument noise at low frequencies to be better characterised.

8.4 Parameter inference from mock LISA data

In this section we describe the data model used for LISA, the likelihood used for parameter inference from an injected SGWB, and priors for both thermodynamic and spectral parameters.

8.4.1 Data model and likelihood

Here we outline how we model the LISA data, explain the assumptions made, and define the likelihood used. The LISA data is expected to be a $T_{\rm obs} = 4$ yr stream with a regular data sampling interval $T_{\rm samp} = 5$ s, not taking into account scheduled maintenance breaks. We use the data model as described in our previous work [1].

In this analysis we consider the A and E TDI channels in the frequency domain, binned into $N_{\rm b} = 1000$ logarithmically spaced positive frequency bins, with power spectral densities \bar{D}_b^A , \bar{D}_b^E . The variance of the A and E channels are taken to be independent and identical. Within each bin there are n_b frequencies

$$n_b = [(f_b - f_{b-1})T_{\text{obs}}] \tag{8.22}$$

where the square brackets denote the integer part, and here $n_b \gg 1$, which justifies the use of a Gaussian likelihood. We combine the A and E data channels $\bar{D}_b = (\bar{D}_b^A + \bar{D}_b^E)/2$, so that the log-likelihood for the spectral parameter case is then given by

$$l = -\frac{1}{2} \sum_{b=1}^{N_{\rm b}} \frac{2n_b \left(\Omega_{\rm t}(f_b, \theta) - \Omega_{\rm fid}(f_b, \tilde{\theta}_{\rm fid})\right)^2}{\Omega_{\rm t}(f_b, \theta)^2},\tag{8.23}$$

where $\Omega_{\rm fid}$, $\Omega_{\rm t}$ are related to the power spectral densities as described in Eq. 8.20 and $\theta_{\rm fid}$ describes the fiducial model. The theoretical model of the data is given by

$$\Omega_{\rm t}(f_b,\theta) = \Omega_{\rm ins}(f_b) + \Omega_{\rm pt}(f_b,\theta), \qquad (8.24)$$

where $\Omega_{\rm pt}(f_b, \theta)$ is described by Eq. (8.10). The thermodynamic case is obtained by replacing $\Omega_{\rm pt}(f_b, \theta)$ with $\Omega_{\rm pt}(f_b, \tilde{\theta})$ which is described by Eq. (8.6). The instrument noise $\Omega_{\rm ins}(f_b)$ is described by Eq. (8.20).

Irrespective of the parameters on which the MCMC samples, the injected fiducial is calculated using the thermodynamic parameters as follows:

$$\Omega_{\rm fid} = \Omega_{\rm ins}(f_b) + \Omega_{\rm pt}(f_b, \hat{\theta}_{\rm fid}) \tag{8.25}$$

are generated in the frequency domain using 1000 frequency logarithmic spaced points, $\Omega_{\rm pt}(f_b, \tilde{\theta}_{\rm fid})$ is described by Eq. (8.6) and $\Omega_{\rm ins}$ by Eq. (8.20). The injected power spectrum is a Gaussian draw around the theoretical fiducial model. In this work we do not consider any astrophysical foregrounds, as our focus is on the reconstruction of parameters. Furthermore, the fiducial models we go on to consider are strong enough that we expect the foregrounds to have little impact. For an exploration of the impact of foregrounds on LISA's ability to detect a SGWB from a first order phase transition see [171].

8.4.2 Priors on thermodynamic parameters

The priors on the four thermodynamic parameters are chosen based on constraints from theory, simulations, the corresponding signal-to-noise ratio ρ of the GW signals they produce, and trustworthiness of the SSM.

The prior on the nucleation temperature T_n was chosen so the temperature scale is relevant to the electroweak scale. Due to the large range of scales involved, we impose a log-uniform prior between $T_n = 10$ GeV – 50 TeV.

For the phase transition strength α , which we remind the reader is the ratio of potential energy to thermal energy, we place a lower bound of $\alpha = 0.01$, which corresponds roughly to the lowest phase transition strength with signal-to-noise ratio $\rho > 1$ for the (r_*, T_n) cases we consider. For the upper bound we use $\alpha = 0.67$, which is the highest phase transition strength used in current simulations [7]. We impose a log-uniform prior for α .

We place a log-uniform prior on the Hubble-scaled mean bubble spacing r_* with a lower bound $r_* = 0.0005$, as lower signals are not observable i.e. $\rho < 1$ even for largest phase transition strength. The upper bound in general could be up to $r_* \simeq 1$, otherwise the bubbles would be bigger than the observable universe. The SSM assumes the phase transition completes much faster than one Hubble time, which corresponds to $r_* \ll 1$. In practice we use an upper bound of $r_* = 0.5$.

Theoretically, the wall speed $v_{\rm w}$ could take any value between 0 and 1 (where 1 indicates the speed of light in natural units). Here we choose to use the current region explored by simulations and apply a flat uniform prior between $v_{\rm w} = 0.24$ and 0.92.

We also include a joint prior on α and $v_{\rm w}$ that arises from the maximum phase transition strength $\alpha_{\rm max}$ for a given wall speed [40]. We use an approximate form of this relationship

$$\alpha_{\max} = \frac{1}{3} \frac{\left(1 + 3v_{\rm w}^2\right)}{\left(1 - v_{\rm w}^2\right)}.$$
(8.26)

We summarise the priors on the thermodynamic parameters in Table 8.1.

8.4.3 Initial priors on spectral parameters

The naive priors on the spectral parameters are chosen to allow for a generous spread around what we take to be observable, spectra with $\rho > 1$. We do this to give the optimiser a wide range of spectral parameters when fitting to the thermodynamic parameters. The

Parameter	Min	Max	
$\log_{10}(T_{\rm n}/{\rm GeV})$	$\log_{10}(10)$	$\log_{10}(50 \times 10^3)$	
$\log_{10} \alpha$	$\log_{10}(0.01)$	$\log_{10}(0.67)$	
$\log_{10} r_*$	$\log_{10}(0.0005)$	$\log_{10}(0.5)$	
$v_{\rm w}$	0.24	0.92	

Table 8.1: Ranges for the uniform priors on the thermodynamic parameters.

spectral priors are summarised in Table 8.2. The prior on the break ratio $r_{\rm b}$ is chosen to be linear as $r_{\rm b}$ is closely related to the wall speed $v_{\rm w}$, which has a linear prior. For the intermediate slope b we use a prior range that encompasses the range we found when fitting the double broken power law to a range of SSM spectra in [1].

Parameter	Min	Max	
$\log_{10}\Omega_{\rm p}$	$\log_{10}(1 \times 10^{-20})$	$\log_{10}(1\times10^{-7})$	
$\log_{10}(f_{\rm p}/{\rm Hz})$	$\log_{10}(1 \times 10^{-7})$	$\log_{10}(1)$	
$r_{\rm b}$	1×10^{-7}	1	
b	-2	2	

Table 8.2: Ranges for the uniform priors on the spectral parameters.

8.4.4 Markov chain Monte Carlo inference

We sample from the likelihood described above, combined with different priors, using the adaptive Markov chain Monte-Carlo (MCMC) algorithm [172] included in Cobaya [93]. The resulting chains are analysed using GetDist [173] in order to produce posterior density plots and credible intervals.

For each of our fiducial models we consider three set-ups: sampling on the spectral parameters $\theta = (\log_{10} \Omega_{\rm p}, \log_{10}(f_{\rm p}/{\rm Hz}), r_{\rm b}, b)$ with flat priors, sampling on the spectral parameters with the induced priors described in 8.5.2 (in order to reconstruct the thermodynamic parameters), and finally, as a benchmark, sampling directly on the thermodynamic parameters $\tilde{\theta} = (\log_{10}(T_{\rm n}/{\rm Gev}), \log_{10} \alpha, \log_{10} r_*, v_{\rm w}).$

122

8.5 Reconstructing thermodynamic parameter posteriors

To take advantage of the computationally cheaper double broken power law, we introduce a method for transforming spectral parameters into the corresponding thermodynamic parameters. We generate a map Θ between the two parameter spaces by fitting the spectral parameters over a regular grid of thermodynamic parameters. This map is then used to generate an induced prior on the spectral parameters that is informed by our chosen thermodynamic parameter space. Finally, we introduce our reconstruction method using Θ and comment on the utility and interpretation of the reconstructed posterior.

8.5.1 Constructing the map between spectral and thermodynamic parameters

Here we aim to make a map between spectral and thermodynamic parameters, as an analytic expression connecting the two sets of parameters does not exist. We wish to find the spectral parameters giving the best fit for a GW power spectrum defined by a given set of thermodynamic parameters. To do that, we could use least-squares curve fitting between the two spectra. Assuming that there will be imperfections in the mapping (e.g. regions in the thermodynamic parameter producing features that cannot be represented by the simpler spectroscopic template) there is a decision to be made about which parts of the power spectra should be allowed to fit best. A natural prescription would be favouring the frequencies to which LISA is most sensitive, which could be implemented by weighting frequency bins during the fitting with the respective sensitivities. We accomplish this with a maximisation of the log-likelihood of Eq. (8.23), where a thermodynamic template is injected as the fiducial model and a spectroscopic one is fitted to it. We use the optimiser code in Cobaya which uses Py-BOBYQA [174, 175]. This defines the map and its numerical approximation.

We evaluate the map by using the above procedure to fit the gravitational wave power spectra for a regular 4D grid of thermodynamic parameters; each evaluation returns a vector of spectroscopic parameters. These vectors are assembled into a 4D array of 4-component vectors, Θ , which we refer to as the fit array.

The underlying regular grid of thermodynamic parameters is summarised in Table 8.3. The fact that it is regularly-spaced according to the uniform density of the set of thermodynamic parameters $(\log_{10}(T_n/\text{Gev}), \log_{10} \alpha, \log_{10} r_*, v_w)$ will make the computation of the induced prior simpler, as we will see below. The reader will note that the lower bounds for $v_{\rm w}$, r_* , and $T_{\rm n}$ in the regular grid of thermodynamic parameters (see Table 8.3) do not directly correspond to the ranges used for the priors in Table 8.1. As we will go on to consider high signal-to-noise ratio fiducial models we do not expect the MCMC chain to explore these relatively low signal-to-noise ratio regions. In order to reduce the computation time of the fit array and focus on a denser population of points in the regions of parameter space we expect the chains to explore, we trim the lower bounds on $v_{\rm w}$, r_* , and $T_{\rm n}$.

Parameter	Min.	Max.	No. of points	Scale
$T_{\rm n}$	$50 { m GeV}$	$5000 {\rm GeV}$	20	logarithmic
α	0.01	0.67	44	logarithmic
r_*	0.05	0.5	19	logarithmic
$v_{\rm w}$	0.4	0.9	43	linear

Table 8.3: Regular grid of thermodynamic parameters used to construct the fit array. Notice that the scaling corresponds to the prior density in Table 8.1.

The starting point for each fit was the following: $\Omega_{\rm p}$: peak value for injected GW power spectra, $f_{\rm p}$: frequency corresponding to the peak amplitude, $r_{\rm b}$: 0.5, and b: 0.4. The values for $r_{\rm b}$ and b were chosen to be generic starting points. In order to improve efficiency of the fit array generation, we chose convergence criteria $\rho_{\rm end}$, which corresponds to the minimum allowed value of the trust region radius, to depend on signal-to-noise ratio ρ :

$$\rho \le 0.001 \qquad \rho_{\text{end}} = 0.01,
0.001 < \rho \le 1 \qquad \rho_{\text{end}} = 0.001,
\rho > 1 \qquad \rho_{\text{end}} = 0.00001.$$
(8.27)

The fit array described here, which is a catalogue of thermodynamic parameters and their corresponding spectral parameters, forms the basis of both the theory-informed induced prior on the spectral parameters presented in Sec. 8.5.2, and our reconstruction algorithm presented in Sec. 8.5.3.

The computational cost to generate the fit array can be split into two parts. Firstly, we have to evaluate the theoretical GW power spectra for all parameter combinations, and then we have to perform the optimiser fits. For the SSM the first part is relatively quick because the GW power spectrum for different r_* and T_n combinations can be rapidly evaluated by rescaling according to Eq. 8.4. The 718960 optimiser fits to these spectra
took 3000 core hours and form the main upfront cost of the method.

8.5.2 Induced prior on the spectral parameters

In this section we suppose that there is some physically motivated prior imposed on the thermodynamic parameters, $\pi(\tilde{\theta})$, and address the problem of finding the prior induced by the map on the spectral parameters, $\pi(\theta)$. In the case that there are the same number of parameters m in each space, and that the map is differentiable, the induced prior is the imposed prior multiplied by the Jacobian determinant of the map,

$$\pi(\theta) = \pi(\tilde{\theta}) \left| \frac{\partial(\tilde{\theta}_1, \dots, \tilde{\theta}_m)}{\partial(\theta_1, \dots, \theta_m)} \right|.$$
(8.28)

The Jacobian determinant gives the ratio between a volume element in the original θ space, and its image in the θ space. Unfortunately, the mapping between spectroscopic
and thermodynamic parameters is not analytic in most of the cases that we would consider
in this context, so we must resort to the alternative approach. We have already obtained a
sample of the thermodynamic parameters in the last section: the fit array Θ . The density
of the sample is proportional to the prior density, that is, both the grid and the prior are
uniform in either the parameter or its logarithm. Hence we can directly compute the prior
for the spectral parameters as

$$\pi(\theta) = \Delta(\theta) , \qquad (8.29)$$

where $\Delta(\theta)$ is the density in the spectral parameter space induced by the mapping of the regular grid. As the fit array Θ is discrete, we use it to generate a frequency histogram on the spectral parameter space, and smooth the histogram value using a kernel density estimator (from scipy.stats [176]) to approximate the density $\Delta(\theta)$. This is the prior that we will use in the MCMC runs which are aimed at recovering the thermodynamic from the spectral parameters. Notice that possible exclusion regions in the thermodynamic parameter space (such as that on the (α, v_w) described in Section 8.4.2) are automatically accounted for in the mapping fit array Θ , simply by the corresponding region having been excluded from the original grid.

The 2D projections of the induced prior probability density functions are shown in Fig. 8.1, where the red and blue regions correspond to high and low prior probability respectively. The prior bounds we implement for the thermodynamic parameter space approximately correspond to the region of thermodynamic parameter space where the SGWB has signal-to-noise ratio $\rho > 1$. This means the induced priors shown in Fig. 8.1

contain the spectral parameter space for a first order phase transition observable at LISA. These priors are clearly different from the naive uniform priors that we started from for the spectroscopic parameters, i.e. uniform on $(\log_{10} \Omega_{\rm p}, \log_{10}(f_{\rm p}/{\rm Hz}), r_{\rm b}, b)$. This difference remarks the need to account for the mapping by using the induced prior of Eq. (8.29), or we would be inadvertently imposing a very non-physical prior on the thermodynamic parameters when recovered as explained in the next section.



Figure 8.1: 2D projections of the induced priors on the spectral parameters, where red and blue regions correspond to high and low probability respectively. Notice the difference between this prior density and the one described in Sec. 8.4.3, which remarks the need for the use of the induced prior in order for the recovered thermodynamic parameter constraints to be physically meaningful.

8.5.3 Reconstruction of the thermodynamic parameters

As a last step to produce constraints on the thermodynamic parameters from a sample of the spectral ones, we need to map the spectral parameters in the sample back to their corresponding thermodynamic ones. The fit array Θ cannot simply be inverted, since it is not regularly spaced in the spectral parameter space, and in any case we would need to interpolate to obtain mappings of arbitrary points that are not in the grid. Here we describe a procedure to do both the inversion and interpolation at once.

For a set of spectral parameters $\theta = (\log_{10} \Omega_{\rm p}, \log_{10}(f_{\rm p}/{\rm Hz}), r_{\rm b}, b)$, the aim is to find a

unique set of thermodynamic parameters $\tilde{\theta} = (\log_{10}(T_n/\text{Gev}), \log_{10}\alpha, \log_{10}r_*, v_w)$. We do this by finding a weighted nearest neighbour. The displacement of θ from a given element of Θ is given by

$$\Delta \theta = \theta - \Theta = (\Delta \log_{10} \Omega_{\rm p}, \Delta \log_{10} f_{\rm p}, \Delta r_{\rm b}, \Delta b).$$
(8.30)

The distance d, in spectral parameter space, between the input point θ and a point in the fit array Θ is given by

$$d = \sqrt{\Delta \log_{10} \Omega_{\rm p}^2 + \Delta \log_{10} f_{\rm p}^2 + \Delta r_{\rm b}^2 + \Delta b^2} + \epsilon, \qquad (8.31)$$

where ϵ is a small value used as a regulator preventing divide by zero errors. We take the 5 smallest values of distances to build the array d_a of the 5 nearest neighbours. The 5 corresponding sets of thermodynamics parameters $\tilde{\theta}_a$ are then averaged with the inverse square of the distance (Eq. (8.31))

$$\tilde{\theta} = \frac{\sum_{a=1}^{N} \tilde{\theta}_a / d_a^2}{\sum_{a=1}^{N} 1 / d_a^2}.$$
(8.32)

This is the reconstructed thermodynamic parameter, illustrated by the filled triangle in Fig. 8.2. The evaluation time of the reconstructed parameters as described in Eq. 8.32 method is minimal so we can calculate them as we sample on the spectral parameters.



Figure 8.2: A regular grid of thermodynamic parameters $\tilde{\theta}_n$ shown with filled points, Θ is the fit array that connects the spectral parameters θ_n to the corresponding to $\tilde{\theta}_n$. The irregular grid of spectral parameters θ_n found using the optimiser fit are shown here as unfilled points. d_a is the distance between set of spectral parameters θ to reconstruct, shown here as a triangle, and one of the five nearest neighbours in the θ_n grid. The filled triangle in the thermodynamic parameter space on the left represents the reconstructed thermodynamic parameters.

An important feature of this reconstruction technique is that excluded regions of the parameter space can never be accessed by the reconstructed parameters (as long as the mapping is well-behaved, which it will be if the spectroscopic template is a good enough approximation to the physical one). As the grid will not contain points in the excluded regions, all points $\tilde{\theta}_a$ corresponding to the nearest neighbours will necessarily be allowed values, and (provided the allowed region is convex) their weighted sum will too.

8.5.4 Properties of the reconstructed thermodynamic posterior

It would be desirable if the induced prior on the spectral parameters would approach the original thermodynamic prior when reconstructing the thermodynamic parameters on finer and finer grids. However, in order for this to be achievable, every possible physical template must be reproduced exactly by the spectroscopic template with some unique combination of the spectral parameters, the mapping between the two sets of parameters must be one-to-one, and the optimiser must find the precise correspondence every time.

These conditions are not generally satisfied, and so it is to be expected that the recovered thermodynamic parameters will not be distributed according to the exact physical prior, and thus the reconstructed posteriors will not be equivalent to the ones we would obtain by sampling directly on the thermodynamic parameters. Nevertheless, reasonably small deviations from these conditions (e.g. the spectroscopic template may miss some corner-case physical features, the fit array grid is fine but finite, or the optimiser fails to find the best fitting function) will still produce priors with useful properties: parameter values for physically excluded regions can never be recovered (as explained in the last section), the base density for the thermodynamic parameters (e.g. uniform, log-uniform...) is preserved; and on data containing an actual signal, the best-fit model of a hypothetical thermodynamic sample has high likelihood of being contained within the reconstructed contours.

The inevitable differences in the prior indicates that the reconstructed posteriors should not be interpreted as a direct reconstruction of the actual ones, but these nice properties guarantee that they provide a sound but much cheaper first order approximation to parameter constraints in the physical parameters, which is physically reasonable (reproduces exclusions and densities) that can be used e.g. to refine the spectroscopic formula or the fit array in the region of interest to get an even better approximation.

In the next section we will find some of these differences and test the soundness of

the reconstructed posteriors using to benchmark cases.

8.6 Results

We perform MCMC inference for two fiducial models: a deflagration and a detonation, each with signal-to-noise $\rho \sim 40$. In each case the injected signal contains the SGWB from a first order phase transition, as described by the SSM Eq. (8.6), and the LISA instrument noise, as described by Eq. (8.20).

For each fiducial model we perform three MCMC runs: sampling on the spectral parameters with the flat priors given in Table 8.2, sampling on the spectral parameters with the induced priors as described in Sec. 8.5.2 and Fig. 8.1, and sampling on the thermodynamic parameters with the priors given in Table 8.1.

The MCMC runs are implemented using Cobaya [93] with a log-likelihood described by Eq. (8.23). For the MCMCs that sample on the spectral parameters the SGWB from a phase transition is described by Eq. (8.10) and $\theta = (\log_{10} \Omega_{\rm p}, \log_{10}(f_{\rm p}/{\rm Hz}), r_{\rm b}, b)$. When sampling directly on the thermodynamic parameters the phase transition signature is described by Eq. (8.6) and $\tilde{\theta} = (\log_{10}(T_{\rm n}/{\rm Gev}), \log_{10}\alpha, \log_{10}r_*, v_{\rm w})$.

The set-up for the MCMC runs is as follows. Cobaya uses the Gelman-Rubin statistic R-1 as the convergence criteria, specifically we use $R-1 \leq 0.001$ for the spectral samples and $R-1 \leq 0.01$ for the thermodynamic samples (as they take longer to evaluate). The maximum number of tries at each point in the chain is 100000. For the runs on spectral parameters we use the optimiser fit (as described in Sec. 8.5.1) to the injected phase transition signal as the starting point of the chain. For the thermodynamic samples the starting point of the chain is taken from a Gaussian draw centred around the fiducial model values.

For the MCMC samples performed here with a single chain and four threads, the spectral sample with the induced prior took ~ 5 days to converge with $\sim 200,000$ points in the chain. The corresponding direct sample on the thermodynamic parameters took ~ 16 days to reach $R - 1 \leq 0.01$ with $\sim 70,000$ points in the chain.

8.6.1 Deflagration fiducial model

For the deflagration fiducial model we use $v_{\rm w} = 0.55$, $\alpha = 0.4$, $r_* = 0.1$ and $T_{\rm n} = 120$ GeV, which has a signal-to-noise ratio $\rho = 40.1$.

In Fig. 8.3a we present the results for the spectral samples in the deflagration case. The blue regions show the posterior with the uniform spectral priors given in Table 8.2. The purple regions show the posteriors when the induced prior informed by the thermodynamic parameter space is included. The cross-hairs show the start point of the chain, which corresponds to the optimiser fit to the spectrum generated from the thermodynamic fiducial model.



Figure 8.3: Triangle plots for the deflagration fiducial model $\alpha = 0.4$, $v_w = 0.55$, $r_* = 0.1$, $T_n = 120$ GeV, for MCMCs sampling on spectral parameters 8.3a and thermodynamic parameters 8.3b. On the left, the spectral MCMC samples with flat priors (blue) and with induced priors (purple). The cross hairs in the spectral triangle plots mark the best fit to the injected spectrum calculated using the optimisation procedure described in Section 8.5. On the right are the corresponding samples on the thermodynamic parameters (green) and thermodynamic parameters reconstructed from the spectral sample (purple). The cross hairs in the thermodynamic triangle plot show the injected thermodynamic parameters. The grey shading in the v_w - α plot shows the region excluded by the physical prior, described in Eq. (8.26).

For each point in the spectral parameter chain with the induced priors we perform

the reconstruction algorithm Eq. (8.32) to build a corresponding chain of reconstructed thermodynamic parameters. The distributions corresponding to reconstructed thermodynamic parameters are shown in purple in Fig. 8.3b. The posterior from the MCMC sampling directly on the thermodynamic parameters is shown in green in Fig. 8.3b.

The marginalised 1D and 2D posteriors in Fig. 8.3a from the flat and induced priors are in good agreement. The 2D posteriors for the MCMC sample with the induced priors (purple) cover a smaller area. The difference is mostly due to a prior cut for large r_b , which is not favoured by the SSM, despite being allowed by the data. The disfavouring of large r_b values can be seen as a sharp fall for $r_b \simeq 0.5$ in the induced prior of Fig. 8.1. The break ratio r_b is the hardest for the MCMC to estimate as it requires knowledge of both breaks in the GW power spectrum. In this case (and in general) one of the breaks is at low or high frequencies and out of LISA's peak sensitivity region. The means and 68% credible intervals for the spectral parameters for the flat and induced priors are summarised in Table 8.4.

	$\log_{10}\Omega_{\rm p}$	$\log_{10}(f_{\rm p}/{\rm Hz})$	$r_{ m b}$	b
Flat priors	$-9.791\substack{+0.044\\-0.075}$	$-3.78^{+0.12}_{-0.036}$	< 0.368	$0.78\substack{+0.58 \\ -0.67}$
Induced priors	$-9.779_{-0.063}^{+0.046}$	$-3.81^{+0.11}_{-0.048}$	< 0.267	$0.70_{-0.47}^{+0.30}$

Table 8.4: Means and 68% credible intervals for the spectral parameters, deflagration fiducial model.

We now consider the results for the posteriors on the thermodynamic parameters and compare the results from the direct sample and the reconstructed sample. In Fig. 8.3b there is general agreement between the two sets of 2D posteriors. In particular, we note the directions of the correlations in the 2D posteriors are recovered well in the reconstructed sample. The largest difference appears for the Hubble-scaled mean bubble spacing r_* , which has a tighter lower bound and more defined peak than the posterior from the directly sampled thermodynamic parameters. This difference is not surprising, since the direct thermodynamic sample also fails to recover r_* . This is because the injected r_* value is hard to distinguish from higher ones: the SGWB for this deflagration has a plateau peaking at a frequency lower than LISA's peak sensitivity, and increasing r_* displaces the signal peak towards lower frequencies at the same time as increasing the amplitude, keeping the signal-to-noise approximately constant (see Fig. 1c of [1]). This effect can also be seen as a degeneracy between Ω_p and f_p ; the reconstruction simply selects from the long tails the values that are more likely to be reproduced by a spectroscopic template. The wall speed posterior is bi-modal because away from the speed of sound detonations and deflagrations have similar spectral shape (this can be seen in Fig. 1 in [1]).

	$v_{ m w}$	$\log_{10} \alpha$	$\log_{10} r_*$	$\log_{10}(T_{\rm n}/{\rm GeV})$
Fiducial model	0.55	-0.398	-1	2.08
Direct	$0.630\substack{+0.17 \\ -0.059}$	> -0.595	> -0.890	$2.03_{-0.54}^{+0.27}$
Reconstructed	$0.646^{+0.098}_{-0.075}$	$-0.52^{+0.12}_{-0.15}$	$-0.59^{+0.22}_{-0.13}$	$2.15_{-0.36}^{+0.14}$

The means and 68% credible intervals for the thermodynamic parameters for the direct and reconstructed samples for the deflagration case are summarised in Table 8.5.

Table 8.5: Thermodynamic parameters for the fiducial deflagration model, and the thermodynamic parameters inferred from the MCMC samples. "Direct" uses chains sampled directly on the thermodynamic parameters, "reconstructed" uses chains sampled on the spectral parameters, and reconstructs the corresponding thermodynamic parameters using the method described in Section 8.5. Values given are means and 68% confidence intervals.

In Fig. 8.4a we compare GW power spectra for the injected deflagration fiducial model (orange line) with the best fit spectra for the MCMC inferences, with flat and induced priors on the spectral parameters, shown in blue and purple respectively. The light grey and dark grey bands highlight the 68% and 95% confidence intervals on the GW spectra from the MCMC simulation which samples on the spectral parameters with the induced prior. In the frequency window that corresponds to LISA's peak sensitivity the spectra agree well. In the low frequency region the best fit for the induced prior run does not match with the injected phase transition signal so well; here LISA has little constraining power because of the low sensitivity, and the induced prior does not prevent sampling on very low values of $r_{\rm b}$. For the MCMC on the thermodynamic parameters the best fit spectra are shown in purple and green for the reconstructed and direct samples respectively in Fig. 8.4b. Here we see the spectrum from the best fit of the reconstructed thermodynamic parameters sample falls within the 95% confidence band over the majority of the frequency band.

8.6.2 Detonation fiducial model

For the detonation fiducial model we use $v_{\rm w} = 0.88$, $\alpha = 0.3$, $r_* = 0.1$ and $T_{\rm n} = 200$ GeV, which has a signal-to-noise ratio $\rho = 38.6$. In this case, the chosen wall speed is close to



Figure 8.4: Injected and best fit spectra for the detonation fiducial model with $v_{\rm w} = 0.88$, $\alpha = 0.2$, $r_* = 0.1$, $T_{\rm n} = 200$ GeV. The light and dark grey bands show the 1 and 2 sigma spread on the power spectra for the MCMC sample with the induced prior. In the spectral parametrisation (a) the best fit spectrum with the uniform prior is shown in blue, and the induced prior is shown in purple. In the thermodynamic parametrisation (b): the best fit spectrum for the direct sampling is shown in green, and the reconstructed sampling in purple. In both cases the injected spectrum is shown in yellow.

the upper bound on the prior, and so we expect this choice to test the edge effects in the reconstruction method.

We follow the same approach for the detonation as for the deflagration fiducial model. In Fig. 8.5a we present the triangle plots for the spectral parameters with flat priors (blue) and induced priors (purple). Again, there is good agreement between the 1D and 2D posteriors from the flat and induced priors. Here, unlike the deflagration case, the 2D posteriors for MCMC runs with the induced priors cover a larger area than those for the flat priors. In this case the spectral best fit has a large negative b, which is disfavoured by the induced prior, so the sampling is predominantly on less negative values of b. The strong correlation between b and $r_{\rm b}$ increases the apparent area wherever one of these parameters appears. The means and 68% credible intervals of the chains are presented in Table 8.6.

	$\log_{10}\Omega_{\rm p}$	$\log_{10}(f_{\rm p}/{\rm Hz})$	$r_{ m b}$	b
Flat priors	$-10.332_{-0.11}^{+0.050}$	$-3.58^{+0.12}_{-0.042}$	$0.617\substack{+0.031\\-0.019}$	$-1.29^{+0.20}_{-0.25}$
Induced priors	$-10.326^{+0.057}_{-0.12}$	$-3.64^{+0.16}_{-0.059}$	$0.585^{+0.047}_{-0.033}$	$-1.04^{+0.33}_{-0.28}$

Table 8.6: Means and 68% credible intervals for the spectral parameters, detonation fiducialmodel.

133



Figure 8.5: Same as Fig. 8.3 but for the detonation fiducial model with $v_{\rm w} = 0.88$, $\alpha = 0.2$, $r_* = 0.1$, $T_{\rm n} = 200$ GeV.

In Fig. 8.5b we present the triangle plot for the reconstructed (purple) and direct (green) samples on the thermodynamic parameters. Again there is general agreement in the 1D and 2D posteriors. For the 2D posteriors that include the wall speed there is a greater spread in the reconstructed samples, as the correlations with the other three parameters are not reproduced. This is a rather special region of thermodynamic parameter space: small changes in the wall speed make large changes in the power spectrum at higher frequencies, as can be seen in Fig. 1a of Ref. [1]. This accounts for the broadness of the 68% and 95% bands of the gravitational wave power spectrum for the reconstructed sample in Fig. 8.6b. The GW power spectra possible in this region of parameter space are also not well described by the double broken power law. This results in a wide range of wall speeds being mapped onto the small range of spectral parameters in the fit array, which subsequently causes the broad spread in the reconstructed posteriors.

An edge effect is also on display in the 1D and 2D posteriors for v_w : the sampling on the thermodynamic parameters explores the region all the way up to the upper bound, while there is a cut-off in the posterior reconstructed from the sampling with spectral parameters. This can be ascribed to the kernel density estimate smoothing the prior at the boundaries. We would expect to reduce the edge effect by refining the grid near the boundary.

134

	$v_{ m w}$	$\log_{10} \alpha$	$\log_{10} r_*$	$\log_{10}(T_{\rm n}/{\rm GeV})$
Fiducial model	0.88	-0.52	-1	2.30
Direct	> 0.843	$-0.60^{+0.21}_{-0.17}$	$-0.85^{+0.18}_{-0.37}$	$2.41_{-0.31}^{+0.25}$
Reconstructed	$0.840^{+0.041}_{-0.025}$	$-0.59^{+0.15}_{-0.18}$	$-0.68^{+0.20}_{-0.13}$	$2.54_{-0.20}^{+0.23}$

The means and 68% credible intervals for the thermodynamic parameters in the case of the detonation fiducial model are displayed in Table 8.7.

Table 8.7: Thermodynamic parameters for the fiducial detonation model, and the thermodynamic parameters inferred from the MCMC samples. "Direct" uses chains sampled directly on the thermodynamic parameters, "reconstructed" uses chains sampled on the spectral parameters, and reconstructs the corresponding thermodynamic parameters using the method described in Section 8.5. Values given are means and 68% credible intervals.

In Fig. 8.6a we show the 68% and 95% confidence band on the GW spectra from the MCMC simulation which samples on the spectral parameters with the induced prior. The injected signal falls within the 95% confidence band in both the spectral parametrisation and in the thermodynamic parametrisation. In the thermodynamic parametrisation, shown in Fig. 8.6b, the best fit spectra coincide very well with the injected spectrum, for both direct (green) and reconstructed sampling (purple).



Figure 8.6: Injected and best fit spectra for the detonation fiducial model with $v_{\rm w} = 0.88$, $\alpha = 0.2$, $r_* = 0.1$, $T_{\rm n} = 200$ GeV. The light and dark grey bands show the 1 and 2 sigma spread on the power spectra for the MCMC sample with the induced prior. In the spectral parametrisation (a) the best fit spectrum with the uniform prior is shown in blue, and the induced prior is shown in purple. In the thermodynamic parametrisation (b): the best fit spectrum for the direct sampling is shown in green, and the reconstructed sampling in purple. In both cases the injected spectrum is shown in yellow.

8.7 Conclusions

In this paper we introduced and tested a method for investigating LISA's sensitivity to a SGWB from a first order phase transition using parametrised templates as an approximation to the more complete sound shell model (SSM) of gravitational wave production. The parametrised template took the form of a double broken power law, a function of four "spectral" parameters. We investigated what information about the thermodynamic parameters of the sound shell model (wall speed v_w , phase transition strength α , Hubble-scaled mean bubble spacing r_* , and nucleation temperature T_n) can be obtained from sampling on the spectral parameters. The double broken power law is advantageous as it is a simple function, and therefore much faster to evaluate than the SSM, which involves a rather complex sequence of operations [4]. However, the mapping from the spectral to the thermodynamic parameters is not straightforward, as discussed in [1]. Here we have proposed a reconstruction method as a solution to this problem.

The motivation for developing the reconstruction algorithm was to provide a less computationally intense way to perform MCMC runs that constrain the thermodynamic parameters of a first order phase transition. The evaluation time of a proposal in the reconstructed chain is $\mathcal{O}(1000)$ times quicker than in the direct chain. This reconstruction method could be applied to other data analysis problems where the connection between a computationally intensive theoretical model and an analytic fit is required.

A key component of the reconstruction method is the construction of the prior induced on the spectral parameter space by the mapping from the "physical" prior on the thermodynamic parameter space. The other is the construction of the inverse mapping.

To illustrate and test the method, we consider two thermodynamic fiducial models: a deflagration and a detonation, each with signal-to-noise ratio around 40. For each fiducial model we perform 3 MCMC runs: the first samples on the spectral parameters with uniform priors, the second samples on the spectral parameters with an induced prior that is informed by the thermodynamic parameter space, and the last one direct samples on the thermodynamic parameters. For the MCMC runs sampling on the spectral parameters with the induced priors, using our reconstruction method, we also constructed a derived chain of reconstructed thermodynamic parameters.

The success of the method can be judged by its ability to recover the physical parameters and the spectrum of the injected SGWB to 95% confidence. For example, for the deflagration model with $v_{\rm w} = 0.55$, $\alpha = 0.4$, $r_* = 0.1$, $\log_{10}(T_{\rm n}/{\rm GeV}) = 2.079$ the best constrained thermodynamic parameters are, as can be seen in Table 8.5, the wall speed $v_{\rm w} = 0.630^{+0.17}_{-0.059}$ and the nucleation temperature $\log_{10}(T_{\rm n}/{\rm GeV}) = 2.03^{+0.27}_{-0.54}$. The corresponding reconstructed thermodynamic parameters are $v_{\rm w} = 0.646^{+0.098}_{-0.075}$ and $\log_{10}(T_{\rm n}/{\rm GeV}) = 2.15^{+0.14}_{-0.36}$. In general the reconstruction method successfully reconstructed the shape of the 1D posterior distributions. The reconstruction could be further improved with a finer grid in the space of thermodynamic parameters used to generate the fit array.

Finally, we highlight the reconstruction method presented here is easily adaptable to different likelihood models (e.g. one with the astrophysical foregrounds included) without the need to recalculate the physical set of SGWBs. More importantly, in the likely scenario that LISA will release a set of posteriors on generic spectroscopic templates, this method would allow us to extract sound constraints on physical parameters.

Acknowledgements

C.G. is supported by a STFC Studentship. MH (ORCID ID 0000-0002-9307-437X) acknowledges support from the Academy of Finland (grant number 333609). DCH (ORCID ID 0000-0003-2811-0917) acknowledges support from the Academy of Finland (grant numbers 328958 and 353131).

Chapter 9

Conclusions

In this thesis we have investigated LISA's ability to constrain parameters associated with a stochastic gravitational wave background from a first order phase transition at the electroweak scale.

We began in Chapter 2 with a review of general relativity, cosmology and the theory of gravitational waves. We outlined the mechanisms for producing GWs during (and shortly after) a first order phase transition in Chapter 3. The key takeaways from this chapter were: the SGWB for a first order phase transition can be described by four key thermodynamic parameters (the nucleation temperature T_n , the phase transition strength α , the wall speed v_w and the Hubble-scaled mean bubble spacing r_*); the sound wave contribution is expected to dominate the total GW signature from such a transition; and the most sophisticated model for calculating this acoustic contribution is the sound shell model. We then described the GW experiments sensitive to a SGWB from a first order phase transition in Chapter 4, focusing on the ESA-NASA mission LISA which is due to launch in 2035. The final chapter of the preliminary material, Chapter 5, focuses on the Fisher matrix analysis and Markov chain Monte Carlo methods which we utilised in Chapters 6-8.

The first article of this thesis was presented in Chapter 6, where we used the sound shell model to characterise the gravitational wave power spectrum, and the Fisher matrix to estimate uncertainties in the parameters of the GW power spectrum. We explored a thermodynamic parameter space that corresponds to the region explored so far by numerical simulations. We showed that the power spectrum in the sound shell model can be well approximated by a four-parameter double broken power law, and find that the peak power and frequency can be measured to approximately 10% accuracy for signal-to-noise ratios above 20. Determinations of the underlying thermodynamic parameters are complicated by degeneracies, but in all cases the phase boundary speed will be the best constrained parameter. For the principal components of the Fisher matrix, a signal-to-noise ratio above 20 produces a relative uncertainty less than 3% in the two highest-order principal components, indicating good prospects for combinations of parameters. The highest-order principal component is dominated by the wall speed. These estimates of parameter sensitivity provide a preliminary accuracy target for theoretical calculations of thermodynamic parameters.

In Chapter 7 we studied the ability of LISA to observe a gravitational-wave background from phase transitions in the presence of an extragalactic foreground from binary black hole mergers throughout the universe, a galactic foreground from white dwarf binaries, and LISA noise. We modelled the phase transition gravitational wave background as a double broken power law and used the deviance information criterion as a detection statistic. We used Fisher matrix and Markov Chain Monte Carlo methods to assess the measurement accuracy of the parameters of the power spectrum. The general agreement found between the FM and MCMC results in this work allowed us to confirm the results for the spectral parameters from paper 1. While estimating all the parameters associated with the gravitational-wave backgrounds, foregrounds, and LISA noise, we found that LISA could detect a gravitational-wave background from phase transitions with a peak frequency of 1 mHz and normalized energy density amplitude of $\Omega_{\rm p} \simeq 3 \times 10^{-11}$. With $\Omega_{\rm p} \simeq 10^{-10}$, the signal is detectable if the peak frequency is in the range 4×10^{-4} to 9×10^{-3} Hz, and the peak amplitude and frequency can be estimated to an accuracy of 10% to 1%.

In Chapter 8 we introduced and tested a method for investigating LISA's sensitivity to gravitational waves from a first order phase transition using parametrised templates as an approximation to a more complete physical model. The motivation for developing the method was to provide a less computationally intensive way to perform Markov Chain Monte Carlo (MCMC) inference on the thermodynamic parameters of a first order phase transition, or on generally computationally intensive models. We constructed a prior on the empirical parameters that contains the necessary information about the physical parameters; we then used the inverse mapping to reconstruct approximate posteriors on the physical parameters from a fast MCMC on the empirical template. We tested the method on a double broken power law approximation to spectra in the sound shell model. The reconstruction method substantially reduces the proposal evaluation time, and despite requiring some precomputing of the mapping, this method is still cost-effective overall. In two test cases, with signal-to-noise ~ 40, the method recovered the physical parameters and the spectrum of the injected gravitational wave power spectrum to 95% confidence. In previous Fisher matrix analysis we found the phase boundary speed v_w was expected to be the best constrained of the thermodynamic parameters. In this work, for an injected phase transition GW power spectrum with $v_w = 0.55$, with a direct sample on the thermodynamic parameters we recovered $0.630^{+0.17}_{-0.059}$ and for our reconstructed sample $0.646^{+0.098}_{-0.075}$.

This thesis has focused on the observational prospects for a SGWB from a first order phase transition at the upcoming GW observatory LISA. In the papers presented here we have shown that it is possible that the spectral shape and the parameters associated with a phase transition could be recovered at LISA, for a sufficiently loud SGWB, even in the presence of astrophysical foregrounds. It is not straight forward to give a single signal-tonoise ratio that would allow for the detection and reconstruction of the SGWB from a first order phase transition as this depends on the location of the SGWB relative to the LISA sensitivity curve and astrophysical foregrounds, and the nature of the spectral shape of the SGWB itself. In general one could expect to detect a SGWB with a signal-to-noise ratio between 20 - 40 if the peak frequency falls within LISA's peak sensitivity region.

The method for reconstructing the physical parameters from the empirical parameters of a spectral template presented here provides a promising method for performing quick preliminary scans in LISA data that would generate initial constraints on the thermodynamic parameters. This could then be followed up with a focused direct exploration of the thermodynamic parameters.

In this thesis we have used a double broken power law as a spectral template for the sound shell model, whilst this fit works well for a broad range of the thermodynamic parameter space there is room for improvement. As the understanding of shocks and turbulence improves it will become feasible to include these contributions when modelling the GW power spectrum from a first order phase transition. The double broken power law could be adapted to allow the inclusion of such GW sources by relaxing the constraints on the high and low frequency spectral slopes.

To further understand the robustness of the reconstruction method one could perform the analysis for a wider set of thermodynamic parameters, in particular for SGWB that peak at higher frequencies. Whilst we have been able to compare the Fisher matrix and MCMC results for the spectral parameters a thorough comparison has not been completed for the thermodynamic results. These results will be especially informative as the MCMC samples we have obtained seem to have performed better than expected for the nucleation temperature. For the deflagration fiducial model with signal-to-noise ratio ~ 40 the relative uncertainty in the nucleation temperature, as found by the MCMC inference on the thermodynamic parameters, is $\Delta T_n/T_n \sim 0.4$; in the Fisher matrix analysis a comparable set of thermodynamic parameters resulted in $\Delta T_n/T_n > 1$.

The reconstruction method developed here could be applied to other problems; one possible example is investigating LISA's ability to constrain the underlying physics parameters. There is on going work, which considers a Standard Model extension with a Higgs triplet, investigating how the bounds on the thermodynamic parameters translate into bounds on model parameters (masses and couplings). The reconstruction method would lend itself well to this work as calculating the thermodynamic parameters from the underlying parameters is computationally expensive.

With the launch of LISA planned for 2035 it is an exciting time for GW science and I hope that the work presented in this thesis will contribute to the effort to use LISA as a probe of the very early universe.

Bibliography

- C. Gowling and M. Hindmarsh, Observational prospects for phase transitions at LISA: Fisher matrix analysis, 2106.05984.
- [2] G. Boileau, N. Christensen, C. Gowling, M. Hindmarsh and R. Meyer, Prospects for LISA to detect a gravitational-wave background from first order phase transitions, 2209.13277.
- [3] C. Gowling, M. Hindmarsh, D.C. Hooper and J. Torrado, Reconstructing physical parameters from template gravitational wave spectra at LISA: first order phase transitions, 2209.13551.
- [4] M. Hindmarsh and M. Hijazi, Gravitational waves from first order cosmological phase transitions in the Sound Shell Model, JCAP 12 (2019) 062 [1909.10040].
- [5] T.L. Smith and R. Caldwell, LISA for Cosmologists: Calculating the Signal-to-Noise Ratio for Stochastic and Deterministic Sources, Phys. Rev. D 100 (2019) 104055 [1908.00546].
- [6] LISA Science Requirements Document, Tech. Rep. ESA-L3-EST-SCI-RS-001 (2019).
- [7] D. Cutting, M. Hindmarsh and D.J. Weir, Vorticity, kinetic energy, and suppressed gravitational wave production in strong first order phase transitions, Phys. Rev. Lett. 125 (2020) 021302 [1906.00480].
- [8] C. Caprini et al., Detecting gravitational waves from cosmological phase transitions with LISA: an update, JCAP **03** (2020) 024 [1910.13125].
- [9] S. Babak, A. Petiteau and M. Hewitson, *LISA Sensitivity and SNR Calculations*, 2108.01167.

- [10] A. Lamberts, S. Blunt, T.B. Littenberg, S. Garrison-Kimmel, T. Kupfer and R.E. Sanderson, Predicting the LISA white dwarf binary population in the Milky Way with cosmological simulations, Mon. Not. Roy. Astron. Soc. 490 (2019) 5888
 [1907.00014].
- [11] G. Boileau, A.C. Jenkins, M. Sakellariadou, R. Meyer and N. Christensen, Ability of LISA to detect a gravitational-wave background of cosmological origin: The cosmic string case, Phys. Rev. D 105 (2022) 023510 [2109.06552].
- [12] T. Robson, N.J. Cornish and C. Liu, The construction and use of LISA sensitivity curves, Classical and Quantum Gravity 36 (2019) 105011.
- [13] Z.-C. Chen, F. Huang and Q.-G. Huang, Stochastic Gravitational-wave Background from Binary Black Holes and Binary Neutron Stars and Implications for LISA, Astrophys. J. 871 (2019) 97 [1809.10360].
- [14] R. Abbott et al., Upper limits on the isotropic gravitational-wave background from Advanced LIGO and Advanced Virgo's third observing run, Phys. Rev. D 104 (2021) 022004 [2101.12130].
- [15] LIGO SCIENTIFIC, VIRGO collaboration, Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. 116 (2016) 061102 [1602.03837].
- [16] A.H. Guth and E.J. Weinberg, Cosmological Consequences of a First Order Phase Transition in the SU(5) Grand Unified Model, Phys. Rev. D 23 (1981) 876.
- [17] E. Witten, Cosmological Consequences of a Light Higgs Boson, Nucl. Phys. B 177 (1981) 477.
- [18] P.J. Steinhardt, Relativistic Detonation Waves and Bubble Growth in False Vacuum Decay, Phys. Rev. D 25 (1982) 2074.
- [19] P.J. Steinhardt, The Weinberg-Salam Model and Early Cosmology, Nucl. Phys. B 179 (1981) 492.
- [20] K. Kajantie, M. Laine, K. Rummukainen and M.E. Shaposhnikov, Is there a hot electroweak phase transition at m(H) larger or equal to m(W)?, Phys. Rev. Lett. 77 (1996) 2887 [hep-ph/9605288].

- [21] M.E. Shaposhnikov, Baryon Asymmetry of the Universe in Standard Electroweak Theory, Nucl. Phys. B 287 (1987) 757.
- [22] J.M. Cline and K. Kainulainen, Electroweak baryogenesis and dark matter from a singlet Higgs, JCAP 01 (2013) 012 [1210.4196].
- [23] J.M. Cline, TASI Lectures on Early Universe Cosmology: Inflation, Baryogenesis and Dark Matter, PoS TASI2018 (2019) 001 [1807.08749].
- [24] M. Hindmarsh, S.J. Huber, K. Rummukainen and D.J. Weir, Gravitational waves from the sound of a first order phase transition, Phys. Rev. Lett. 112 (2014) 041301
 [1304.2433].
- [25] M. Hindmarsh, S.J. Huber, K. Rummukainen and D.J. Weir, Numerical simulations of acoustically generated gravitational waves at a first order phase transition, Phys. Rev. D 92 (2015) 123009 [1504.03291].
- [26] M. Hindmarsh, S.J. Huber, K. Rummukainen and D.J. Weir, Shape of the acoustic gravitational wave power spectrum from a first order phase transition, Phys. Rev. D 96 (2017) 103520 [1704.05871].
- [27] M. Hindmarsh, Sound shell model for acoustic gravitational wave production at a first-order phase transition in the early Universe, Phys. Rev. Lett. 120 (2018) 071301 [1608.04735].
- [28] LISA collaboration, Laser Interferometer Space Antenna, 1702.00786.
- [29] PLANCK collaboration, Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. 641 (2020) A6 [1807.06209].
- [30] PLANCK collaboration, Planck 2015 results. XVI. Isotropy and statistics of the CMB, Astron. Astrophys. 594 (2016) A16 [1506.07135].
- [31] J.L. Cervantes-Cota, S. Galindo-Uribarri and G.-F. Smoot, A Brief History of Gravitational Waves, Universe 2 (2016) 22 [1609.09400].
- M. Maggiore, Gravitational Waves. Vol. 1: Theory and Experiments, no. 978-0-19-857074-5, 978-0-19-852074-0 in Oxford Master Series in Physics, Oxford University Press (2007).

- [33] M. D'Onofrio and K. Rummukainen, Standard model cross-over on the lattice, *Phys. Rev. D* 93 (2016) 025003 [1508.07161].
- [34] K. Kajantie, M. Laine, K. Rummukainen and M.E. Shaposhnikov, A Nonperturbative analysis of the finite T phase transition in SU(2) x U(1) electroweak theory, Nucl. Phys. B 493 (1997) 413 [hep-lat/9612006].
- [35] A.D. Linde, Fate of the False Vacuum at Finite Temperature: Theory and Applications, Phys. Lett. B 100 (1981) 37.
- [36] S.R. Coleman, The Fate of the False Vacuum. 1. Semiclassical Theory, Phys. Rev. D 15 (1977) 2929.
- [37] C.G. Callan, Jr. and S.R. Coleman, The Fate of the False Vacuum. 2. First Quantum Corrections, Phys. Rev. D 16 (1977) 1762.
- [38] V. Kuzmin, V. Rubakov and M. Shaposhnikov, On the Anomalous Electroweak Baryon Number Nonconservation in the Early Universe, Phys. Lett. B 155 (1985) 36.
- [39] D.E. Morrissey and M.J. Ramsey-Musolf, *Electroweak baryogenesis*, *New J. Phys.* 14 (2012) 125003 [1206.2942].
- [40] J.R. Espinosa, T. Konstandin, J.M. No and G. Servant, Energy Budget of Cosmological First-order Phase Transitions, JCAP 06 (2010) 028 [1004.4187].
- [41] M.B. Hindmarsh, M. Lüben, J. Lumma and M. Pauly, Phase transitions in the early universe, 2008.09136.
- [42] G.D. Moore and T. Prokopec, How fast can the wall move? A Study of the electroweak phase transition dynamics, Phys. Rev. D 52 (1995) 7182
 [hep-ph/9506475].
- [43] J. Kozaczuk, Bubble Expansion and the Viability of Singlet-Driven Electroweak Baryogenesis, JHEP 10 (2015) 135 [1506.04741].
- [44] K. Enqvist, J. Ignatius, K. Kajantie and K. Rummukainen, Nucleation and bubble growth in a first order cosmological electroweak phase transition, Phys. Rev. D 45 (1992) 3415.

- [45] M.A. Ajmi and M. Hindmarsh, Thermal suppression of bubble nucleation at first-order phase transitions in the early Universe, Phys. Rev. D 106 (2022) 023505
 [2205.04097].
- [46] H. Kurki-Suonio and M. Laine, Supersonic deflagrations in cosmological phase transitions, Phys. Rev. D 51 (1995) 5431 [hep-ph/9501216].
- [47] C. Caprini et al., Science with the space-based interferometer eLISA. II: Gravitational waves from cosmological phase transitions, JCAP 04 (2016) 001
 [1512.06239].
- [48] A. Kosowsky, M.S. Turner and R. Watkins, Gravitational radiation from colliding vacuum bubbles, Phys. Rev. D 45 (1992) 4514.
- [49] A. Kosowsky and M.S. Turner, Gravitational radiation from colliding vacuum bubbles: envelope approximation to many bubble collisions, Phys. Rev. D 47 (1993) 4372 [astro-ph/9211004].
- [50] S.J. Huber and T. Konstandin, Gravitational Wave Production by Collisions: More Bubbles, JCAP 09 (2008) 022 [0806.1828].
- [51] M. Kamionkowski, A. Kosowsky and M.S. Turner, Gravitational radiation from first order phase transitions, Phys. Rev. D 49 (1994) 2837 [astro-ph/9310044].
- [52] J. Ellis, M. Lewicki, J.M. No and V. Vaskonen, Gravitational wave energy budget in strongly supercooled phase transitions, JCAP 06 (2019) 024 [1903.09642].
- [53] J. Ellis, M. Lewicki and V. Vaskonen, Updated predictions for gravitational waves produced in a strongly supercooled phase transition, JCAP 11 (2020) 020
 [2007.15586].
- [54] U.-L. Pen and N. Turok, Shocks in the Early Universe, Phys. Rev. Lett. 117 (2016)
 131301 [1510.02985].
- [55] J. Dahl, M. Hindmarsh, K. Rummukainen and D.J. Weir, Decay of acoustic turbulence in two dimensions and implications for cosmological gravitational waves, Phys. Rev. D 106 (2022) 063511 [2112.12013].
- [56] P. Auclair, C. Caprini, D. Cutting, M. Hindmarsh, K. Rummukainen, D.A. Steer

et al., Generation of gravitational waves from freely decaying turbulence, JCAP **09** (2022) 029 [2205.02588].

- [57] R. Durrer and C. Caprini, Primordial magnetic fields and causality, JCAP 11 (2003) 010 [astro-ph/0305059].
- [58] R.A. Hulse and J.H. Taylor, Discovery of a pulsar in a binary system., .
- [59] J.H. Taylor and J.M. Weisberg, A new test of general relativity Gravitational radiation and the binary pulsar PSR 1913+16, .
- [60] LIGO SCIENTIFIC, VIRGO, KAGRA collaboration, GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo During the Second Part of the Third Observing Run, 2111.03606.
- [61] M. Maggiore et al., Science Case for the Einstein Telescope, JCAP 03 (2020) 050[1912.02622].
- [62] B. Sathyaprakash et al., Scientific Objectives of Einstein Telescope, Class. Quant. Grav. 29 (2012) 124013 [1206.0331].
- [63] M. Punturo et al., The Einstein Telescope: A third-generation gravitational wave observatory, Class. Quant. Grav. 27 (2010) 194002.
- [64] W.-H. Ruan, Z.-K. Guo, R.-G. Cai and Y.-Z. Zhang, *Taiji program:* Gravitational-wave sources, Int. J. Mod. Phys. A 35 (2020) 2050075 [1807.09495].
- [65] TIANQIN collaboration, The TianQin project: current progress on science and technology, 2008.10332.
- [66] ST7, LISA PATHFINDER collaboration, Experimental results from the ST7 mission on LISA Pathfinder, Phys. Rev. D 98 (2018) 102005 [1809.08969].
- [67] J. Luo et al., The first round result from the TianQin-1 satellite, Class. Quant. Grav. 37 (2020) 185013 [2008.09534].
- [68] S.L. Detweiler, Pulsar timing measurements and the search for gravitational waves, Astrophys. J. 234 (1979) 1100.
- [69] R.w. Hellings and G.s. Downs, UPPER LIMITS ON THE ISOTROPIC

GRAVITATIONAL RADIATION BACKGROUND FROM PULSAR TIMING ANALYSIS, Astrophys. J. Lett. **265** (1983) L39.

- [70] NANOGRAV collaboration, Science with the Next-Generation VLA and Pulsar Timing Arrays, 1810.06594.
- [71] NANOGRAV collaboration, The NANOGrav 12.5 yr Data Set: Search for an Isotropic Stochastic Gravitational-wave Background, Astrophys. J. Lett. 905 (2020)
 L34 [2009.04496].
- [72] A. Sesana, F. Haardt, P. Madau and M. Volonteri, Low frequency gravitational radiation from coalescing massive black hole binaries in hierarchical cosmologies, Astrophys. J. 611 (2004) 623 [astro-ph/0401543].
- [73] K.A. Postnov and L.R. Yungelson, The Evolution of Compact Binary Star Systems, Living Rev. Rel. 17 (2014) 3 [1403.4754].
- [74] P. Amaro-Seoane, J.R. Gair, M. Freitag, M. Coleman Miller, I. Mandel, C.J. Cutler et al., Astrophysics, detection and science applications of intermediate- and extreme mass-ratio inspirals, Class. Quant. Grav. 24 (2007) R113 [astro-ph/0703495].
- [75] A. Sesana, Prospects for Multiband Gravitational-Wave Astronomy after GW150914, Phys. Rev. Lett. 116 (2016) 231102 [1602.06951].
- [76] C. Caprini and D.G. Figueroa, Cosmological Backgrounds of Gravitational Waves, Class. Quant. Grav. 35 (2018) 163001 [1801.04268].
- [77] LIGO SCIENTIFIC, VIRGO collaboration, A guide to LIGO-Virgo detector noise and extraction of transient gravitational-wave signals, Class. Quant. Grav. 37 (2020) 055002 [1908.11170].
- [78] M. Tinto and J.W. Armstrong, Cancellation of laser noise in an unequal-arm interferometer detector of gravitational radiation, Phys. Rev. D 59 (1999) 102003.
- [79] F.B. Estabrook, M. Tinto and J.W. Armstrong, Time delay analysis of LISA gravitational wave data: Elimination of spacecraft motion effects, Phys. Rev. D 62 (2000) 042002.
- [80] M.R. Adams and N.J. Cornish, Discriminating between a Stochastic Gravitational

Wave Background and Instrument Noise, Phys. Rev. D 82 (2010) 022002 [1002.1291].

- [81] M. Tinto and S.V. Dhurandhar, Time-delay interferometry, Living Rev. Rel. 24 (2021) 1.
- [82] M. Maggiore, Gravitational wave experiments and early universe cosmology, Phys. Rept. 331 (2000) 283 [gr-qc/9909001].
- [83] R. Flauger, N. Karnesis, G. Nardini, M. Pieroni, A. Ricciardone and J. Torrado, Improved reconstruction of a stochastic gravitational wave background with LISA, JCAP 01 (2021) 059 [2009.11845].
- [84] T.A. Prince, M. Tinto, S.L. Larson and J.W. Armstrong, The LISA optimal sensitivity, Phys. Rev. D 66 (2002) 122002 [gr-qc/0209039].
- [85] C.J. Hogan and P.L. Bender, Estimating stochastic gravitational wave backgrounds with Sagnac calibration, Phys. Rev. D 64 (2001) 062002 [astro-ph/0104266].
- [86] R. Flauger, N. Karnesis, G. Nardini, M. Pieroni, A. Ricciardone and J. Torrado, Improved reconstruction of a stochastic gravitational wave background with LISA, Journal of Cosmology and Astroparticle Physics 2021 (2021) 059.
- [87] C. Zhang, Q. Gao, Y. Gong, B. Wang, A.J. Weinstein and C. Zhang, Full analytical formulas for frequency response of space-based gravitational wave detectors, Phys. Rev. D 101 (2020) 124027 [2003.01441].
- [88] J.D. Romano and N.J. Cornish, Detection methods for stochastic gravitational-wave backgrounds: a unified treatment, Living Rev. Rel. 20 (2017) 2 [1608.06889].
- [89] R.A. Fisher, On the Mathematical Foundations of Theoretical Statistics, Phil. Trans. Roy. Soc. Lond. A 222 (1922) 309.
- [90] N. Metropolis and S. Ulam, The monte carlo method, Journal of the American Statistical Association 44 (1949).
- [91] D.W. Hogg and D. Foreman-Mackey, Data analysis recipes: Using Markov Chain Monte Carlo, Astrophys. J. Suppl. 236 (2018) 11 [1710.06068].

- [92] W. Gilks, S. Richardson and D. Spiegelhalter, Markov Chain Monte Carlo in Practice, Taylor and Francis (1995).
- [93] J. Torrado and A. Lewis, Cobaya: Code for Bayesian Analysis of hierarchical physical models, JCAP 05 (2021) 057 [2005.05290].
- [94] A. Lewis, Efficient sampling of fast and slow cosmological parameters, Phys. Rev. D 87 (2013) 103529 [1304.4473].
- [95] LIGO SCIENTIFIC, VIRGO, FERMI GBM, INTEGRAL, ICECUBE, ASTROSAT CADMIUM ZINC TELLURIDE IMAGER TEAM, IPN, INSIGHT-HXMT, ANTARES, SWIFT, AGILE TEAM, 1M2H TEAM, DARK ENERGY CAMERA GW-EM, DES, DLT40, GRAWITA, FERMI-LAT, ATCA, ASKAP, LAS CUMBRES
 OBSERVATORY GROUP, OZGRAV, DWF (DEEPER WIDER FASTER PROGRAM), AST3, CAASTRO, VINROUGE, MASTER, J-GEM, GROWTH, JAGWAR, CALTECHNRAO, TTU-NRAO, NUSTAR, PAN-STARRS, MAXI TEAM, TZAC CONSORTIUM, KU, NORDIC OPTICAL TELESCOPE, EPESSTO, GROND, TEXAS TECH UNIVERSITY, SALT GROUP, TOROS, BOOTES, MWA, CALET, IKI-GW FOLLOW-UP, H.E.S.S., LOFAR, LWA, HAWC, PIERRE AUGER, ALMA, EURO VLBI TEAM, PI OF SKY, CHANDRA TEAM AT MCGILL UNIVERSITY, DFN, ATLAS TELESCOPES, HIGH TIME RESOLUTION UNIVERSE SURVEY, RIMAS, RATIR, SKA SOUTH AFRICA/MEERKAT collaboration, *Multi-messenger Observations of a Binary Neutron Star Merger, Astrophys. J. Lett.* 848 (2017) L12 [1710.05833].
- [96] LIGO SCIENTIFIC, VIRGO collaboration, GW190814: Gravitational Waves from the Coalescence of a 23 Solar Mass Black Hole with a 2.6 Solar Mass Compact Object, Astrophys. J. Lett. 896 (2020) L44 [2006.12611].
- [97] J. Ghiglieri, G. Jackson, M. Laine and Y. Zhu, Gravitational wave background from Standard Model physics: Complete leading order, JHEP 07 (2020) 092
 [2004.11392].
- [98] D. Cutting, E.G. Escartin, M. Hindmarsh and D.J. Weir, Gravitational waves from vacuum first order phase transitions II: from thin to thick walls, 2005.13537.
- [99] M. Lewicki and V. Vaskonen, Gravitational wave spectra from strongly supercooled phase transitions, Eur. Phys. J. C 80 (2020) 1003 [2007.04967].

- [100] M. Lewicki and V. Vaskonen, Gravitational waves from colliding vacuum bubbles in gauge theories, Eur. Phys. J. C 81 (2021) 437 [2012.07826].
- [101] A. Kosowsky, A. Mack and T. Kahniashvili, Gravitational radiation from cosmological turbulence, Phys. Rev. D 66 (2002) 024030 [astro-ph/0111483].
- [102] G. Gogoberidze, T. Kahniashvili and A. Kosowsky, The Spectrum of Gravitational Radiation from Primordial Turbulence, Phys. Rev. D 76 (2007) 083002 [0705.1733].
- [103] C. Caprini, R. Durrer and G. Servant, The stochastic gravitational wave background from turbulence and magnetic fields generated by a first-order phase transition, JCAP 12 (2009) 024 [0909.0622].
- [104] A. Roper Pol, S. Mandal, A. Brandenburg, T. Kahniashvili and A. Kosowsky, Numerical simulations of gravitational waves from early-universe turbulence, Phys. Rev. D 102 (2020) 083512 [1903.08585].
- [105] A. Mazumdar and G. White, Review of cosmic phase transitions: their significance and experimental signatures, Rept. Prog. Phys. 82 (2019) 076901 [1811.01948].
- [106] H.-K. Guo, K. Sinha, D. Vagie and G. White, The Benefits of Diligence: How Precise are Predicted Gravitational Wave Spectra in Models with Phase Transitions?, 2103.06933.
- [107] D. Croon, O. Gould, P. Schicho, T.V.I. Tenkanen and G. White, Theoretical uncertainties for cosmological first-order phase transitions, JHEP 04 (2021) 055 [2009.10080].
- [108] M.R. Adams and N.J. Cornish, Detecting a Stochastic Gravitational Wave Background in the presence of a Galactic Foreground and Instrument Noise, Phys. Rev. D 89 (2014) 022001 [1307.4116].
- [109] G. Boileau, A. Lamberts, N. Christensen, N.J. Cornish and R. Meyer, Spectral separation of the stochastic gravitational-wave background for LISA in the context of a modulated Galactic foreground, 2105.04283.
- [110] G. Boileau, N. Christensen, R. Meyer and N.J. Cornish, Spectral separation of the stochastic gravitational-wave background for LISA: Observing both cosmological and astrophysical backgrounds, Phys. Rev. D 103 (2021) 103529 [2011.05055].

- [111] K. Hashino, R. Jinno, M. Kakizaki, S. Kanemura, T. Takahashi and M. Takimoto, Selecting models of first-order phase transitions using the synergy between collider and gravitational-wave experiments, Phys. Rev. D 99 (2019) 075011 [1809.04994].
- [112] A. Romero, K. Martinovic, T.A. Callister, H.-K. Guo, M. Martínez,
 M. Sakellariadou et al., Implications for first-order cosmological phase transitions from the third LIGO-Virgo observing run, Phys. Rev. Lett. 126 (2021) 151301
 [2102.01714].
- [113] NANOGRAV collaboration, Searching For Gravitational Waves From Cosmological Phase Transitions With The NANOGrav 12.5-year dataset, 2104.13930.
- [114] M. Pieroni and E. Barausse, Foreground cleaning and template-free stochastic background extraction for LISA, JCAP 07 (2020) 021 [2004.01135].
- [115] S. Borsanyi et al., Calculation of the axion mass based on high-temperature lattice quantum chromodynamics, Nature 539 (2016) 69 [1606.07494].
- [116] F. Giese, T. Konstandin and J. van de Vis, Model-independent energy budget of cosmological first-order phase transitions—A sound argument to go beyond the bag model, JCAP 07 (2020) 057 [2004.06995].
- [117] F. Giese, T. Konstandin, K. Schmitz and J. Van De Vis, Model-independent energy budget for LISA, JCAP 01 (2021) 072 [2010.09744].
- [118] B. Allen and J.D. Romano, Detecting a stochastic background of gravitational radiation: Signal processing strategies and sensitivities, Phys. Rev. D 59 (1999) 102001 [gr-qc/9710117].
- [119] H.-K. Guo, K. Sinha, D. Vagie and G. White, Phase Transitions in an Expanding Universe: Stochastic Gravitational Waves in Standard and Non-Standard Histories, 2007.08537.
- [120] M. Tinto, J.W. Armstrong and F.B. Estabrook, Discriminating a gravitational wave background from instrumental noise in the LISA detector, Phys. Rev. D 63 (2001) 021101.
- [121] M. Tinto, F.B. Estabrook and J.W. Armstrong, Time delay interferometry for LISA, Phys. Rev. D 65 (2002) 082003.

- [122] T. Regimbau, The astrophysical gravitational wave stochastic background, Res. Astron. Astrophys. 11 (2011) 369 [1101.2762].
- [123] B.P. Abbott et al., Search for the isotropic stochastic background using data from Advanced LIGO's second observing run, Phys. Rev. D 100 (2019) 061101
 [1903.02886].
- [124] B.P. Abbott et al., Sensitivity of the Advanced LIGO detectors at the beginning of gravitational wave astronomy, Phys. Rev. D 93 (2016) 112004 [1604.00439].
- [125] LIGO SCIENTIFIC, VIRGO collaboration, GW150914: Implications for the stochastic gravitational wave background from binary black holes, Phys. Rev. Lett. 116 (2016) 131102 [1602.03847].
- [126] P.L. Bender and D. Hils, Confusion noise level due to galactic and extragalactic binaries, Class. Quant. Grav. 14 (1997) 1439.
- [127] C.R. Evans, I. Iben and L. Smarr, Degenerate dwarf binaries as promising, detectable sources of gravitational radiation, Astrophys. J. 323 (1987) 129.
- [128] N. Cornish and T. Robson, Galactic binary science with the new LISA design, J. Phys. Conf. Ser. 840 (2017) 012024 [1703.09858].
- [129] K. Schmitz, LISA Sensitivity to Gravitational Waves from Sound Waves, Symmetry
 12 (2020) 1477 [2005.10789].
- [130] M. Bonetti and A. Sesana, Gravitational wave background from extreme mass ratio inspirals, Phys. Rev. D 102 (2020) 103023 [2007.14403].
- [131] A. Sesana, F. Haardt, P. Madau and M. Volonteri, The gravitational wave signal from massive black hole binaries and its contribution to the LISA data stream, Astrophys. J. 623 (2005) 23 [astro-ph/0409255].
- [132] A.J. Farmer and E.S. Phinney, The gravitational wave background from cosmological compact binaries, Mon. Not. Roy. Astron. Soc. 346 (2003) 1197
 [astro-ph/0304393].
- [133] J.R. Bond, A.H. Jaffe and L.E. Knox, Radical compression of cosmic microwave background data, Astrophys. J. 533 (2000) 19 [astro-ph/9808264].

- [134] WMAP collaboration, First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Parameter estimation methodology, Astrophys. J. Suppl. 148 (2003) 195 [astro-ph/0302218].
- [135] S. Hamimeche and A. Lewis, Likelihood Analysis of CMB Temperature and Polarization Power Spectra, Phys. Rev. D 77 (2008) 103013 [0801.0554].
- [136] G. Efstathiou and J.R. Bond, Cosmic confusion: Degeneracies among cosmological parameters derived from measurements of microwave background anisotropies, Mon. Not. Roy. Astron. Soc. 304 (1999) 75 [astro-ph/9807103].
- [137] E. Thrane and J.D. Romano, Sensitivity curves for searches for gravitational-wave backgrounds, Phys. Rev. D 88 (2013) 124032 [1310.5300].
- [138] LIGO SCIENTIFIC, VIRGO collaboration, GW170817: Implications for the Stochastic Gravitational-Wave Background from Compact Binary Coalescences, Phys. Rev. Lett. 120 (2018) 091101 [1710.05837].
- [139] P. Amaro-Seoane et al., Astrophysics with the Laser Interferometer Space Antenna, 2203.06016.
- [140] N. Christensen, Stochastic Gravitational Wave Backgrounds, Rept. Prog. Phys. 82 (2019) 016903 [1811.08797].
- [141] P. Auclair et al., Cosmology with the Laser Interferometer Space Antenna, 2204.05434.
- [142] Z.-C. Chen, F. Huang and Q.-G. Huang, Stochastic Gravitational-wave Background from Binary Black Holes and Binary Neutron Stars and Implications for LISA, Astrophys. J. 871 (2019) 97 [1809.10360].
- [143] C. Périgois, C. Belczynski, T. Bulik and T. Regimbau, Startrack predictions of the stochastic gravitational-wave background from compact binary mergers, Phys. Rev. D 103 (2021) 043002.
- [144] D. Cutting, M. Hindmarsh and D.J. Weir, Gravitational waves from vacuum first-order phase transitions: from the envelope to the lattice, Phys. Rev. D 97 (2018) 123513 [1802.05712].

- [145] B.-H. Lee, W. Lee, D.-h. Yeom and L. Yin, Gravitational waves from the vacuum decay with LISA *, Chin. Phys. C 46 (2022) 075101 [2106.07430].
- [146] M. Lewicki and V. Vaskonen, Gravitational waves from bubble collisions and fluid motion in strongly supercooled phase transitions, 2208.11697.
- [147] F. Giese, T. Konstandin and J. van de Vis, Finding sound shells in LISA mock data using likelihood sampling, JCAP 11 (2021) 002 [2107.06275].
- [148] D.J. Spiegelhalter, N.G. Best, B.P. Carlin and A. van der Linde, Bayesian measures of model complexity and fit, J. Roy. Statist. Soc. B 64 (2002) 583.
- [149] D.J. Spiegelhalter, N.G. Best, B.P. Carlin and A. van der Linde, The deviance information criterion: 12 years on, Journal of the Royal Statistical Society. Series B (Statistical Methodology) 76 (2014) 485.
- [150] R. Meyer, Deviance information criterion (dic), in Wiley StatsRef: Statistics Reference Online, pp. 1–6, Wiley & Sons (2016), DOI.
- [151] M. Vallisneri and C.R. Galley, Non-sky-averaged sensitivity curves for space-based gravitational-wave observatories, Classical and Quantum Gravity 29 (2012) 124015.
- [152] J. Baker et al., The Laser Interferometer Space Antenna: Unveiling the Millihertz Gravitational Wave Sky, 1907.06482.
- [153] G. Nelemans, L.R. Yungelson, S.F. Portegies Zwart and F. Verbunt, Population synthesis for double white dwarfs, Astronomy & Astrophysics 365 (2001) 491.
- [154] N.J. Cornish and T.B. Littenberg, Tests of bayesian model selection techniques for gravitational wave astronomy, Phys. Rev. D 76 (2007) 083006.
- [155] A.J. Ruiter, K. Belczynski, M. Benacquista, S.L. Larson and G. Williams, The LISA Gravitational Wave Foreground: A Study of Double White Dwarfs, Astrophys. J. 717 (2010) 1006 [0705.3272].
- [156] M.S. Hernandez, M.R. Schreiber, S.G. Parsons, B.T. Gänsicke, F. Lagos, R. Raddi et al., The white dwarf binary pathways survey – iv. three close white dwarf binaries with g-type secondary stars, Monthly Notices of the Royal Astronomical Society 501 (2020) 1677.

- [157] C. Ungarelli and A. Vecchio, Studying the anisotropy of the gravitational wave stochastic background with lisa, Phys. Rev. D 64 (2001) 121501.
- [158] N. Karnesis, S. Babak, M. Pieroni, N. Cornish and T. Littenberg, Characterization of the stochastic signal originating from compact binary populations as measured by LISA, Phys. Rev. D 104 (2021) 043019 [2103.14598].
- [159] T. Robson, N.J. Cornish and C. Liu, The construction and use of LISA sensitivity curves, Class. Quant. Grav. 36 (2019) 105011 [1803.01944].
- [160] D. Meacher, M. Coughlin, S. Morris, T. Regimbau, N. Christensen, S. Kandhasamy et al., Mock data and science challenge for detecting an astrophysical stochastic gravitational-wave background with Advanced LIGO and Advanced Virgo, Phys. Rev. D 92 (2015) 063002 [1506.06744].
- [161] K. Janssens, G. Boileau, M.-A. Bizouard, N. Christensen, T. Regimbau and N. van Remortel, Formalism for power spectral density estimation for non-identical and correlated noise using the null channel in Einstein Telescope, 2205.00416.
- [162] G.O. Roberts and J.S. Rosenthal, Examples of adaptive mcmc, Journal of Computational and Graphical Statistics 18 (2009) 349
 [https://doi.org/10.1198/jcgs.2009.06134].
- [163] D. Lindley, A problem in forensic science, Biometrika 64 (1977) 207.
- [164] G. Shafer, Lindley's paradox, Journal of the American Statistical Association 77 (1982) 325.
- [165] C. Mauri and M.G. Paris, The lindley paradox in optical interferometry, Physics Letters A 380 (2016) 570.
- [166] D. Lunn, The BUGS Book: A Practical Introduction to Bayesian Analysis, Texts in statistical science (2013).
- [167] P. Auclair et al., Probing the gravitational wave background from cosmic strings with LISA, JCAP 04 (2020) 034 [1909.00819].
- [168] S. Babak, "enchilada" is back on the menu, Journal of Physics: Conference Series 012026.

- [169] LISA COSMOLOGY WORKING GROUP collaboration, Cosmology with the Laser Interferometer Space Antenna, 2204.05434.
- [170] D.J. Weir, Gravitational waves from a first order electroweak phase transition: a brief review, Phil. Trans. Roy. Soc. Lond. A 376 (2018) 20170126 [1705.01783].
- [171] G. Boileau, N. Christensen, C. Gowling, M. Hindmarsh, R. Meyer and N.J. Cornish, Prospects for LISA to detect a gravitational-wave background from first order phase transitions, In preparation (2022).
- [172] A. Lewis and S. Bridle, Cosmological parameters from CMB and other data: A Monte Carlo approach, Phys. Rev. D66 (2002) 103511 [astro-ph/0205436].
- [173] A. Lewis, GetDist: a Python package for analysing Monte Carlo samples, 1910.13970.
- [174] C. Cartis, L. Roberts and O. Sheridan-Methven, Escaping local minima with derivative-free methods: a numerical investigation, 2018. 10.48550/ARXIV.1812.11343.
- [175] C. Cartis, J. Fiala, B. Marteau and L. Roberts, Improving the flexibility and robustness of model-based derivative-free optimization solvers, 2018. 10.48550/ARXIV.1804.00154.
- [176] P. Virtanen et al., SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python, Nature Methods 17 (2020) 261.