Topological Methods for Strong Local Minimizers and Extremals of Multiple Integrals in the Calculus of Variations

by

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A thesis submitted for the degree of

Doctor of Philosophy

in the

University of Sussex
Department of Mathematics

January 2011

UNIVERSITY OF SUSSEX

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider the energy functional

$$\mathbb{F}[u, \Omega] := \int_\Omega \mathbf{F}(\nabla u(x))\, dx,$$

over the space $\mathcal{A}_p(\Omega) := \{u \in W^{1,p}(\Omega, \mathbb{R}^n) : u|_{\partial\Omega} = x, \det \nabla u > 0 \text{ } a.e. \text{ in } \Omega\}$, where the integrand $\mathbf{F} : \mathbb{M}_{n\times n} \to \mathbb{R}$ is quasiconvex, sufficiently regular and satisfies a p-coercivity and p-growth for some exponent $p \in [1, \infty[$. A motivation for the study of above energy functional comes from nonlinear elasticity where $\mathbb{F}$ represents the elastic energy of a homogeneous hyperelastic material and $\mathcal{A}_p(\Omega)$ represents the space of orientation preserving deformations of Ω fixing the boundary pointwise. The aim of this thesis is to discuss the question of multiplicity *versus* uniqueness for extremals and strong local minimizers of $\mathbb{F}$ and the relation it bares to the domain topology. Our work, building upon previous works of others, explicitly and quantitatively confirms the significant role of domain topology, and provides explicit and new examples as well as methods for constructing such maps.

Our approach for constructing strong local minimizers is topological in nature and is based on defining suitable homotopy classes in $\mathcal{A}_p(\Omega)$ (for $p \geq n$), whereby minimizing $\mathbb{F}$ on each class results in, modulo technicalities, a strong local minimizer. Here we work on a prototypical example of a topologically non-trivial domain, namely, a generalised *annulus*, $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$, with $0 < a < b < \infty$. Then the associated homotopy classes of $\mathcal{A}_p(\Omega)$ are infinitely many when $n = 2$ and two when $n \geq 3$. In contrast, for constructing explicitly and directly solutions to the system of Euler-Lagrange equations associated to $\mathbb{F}$ we introduce a topological class of maps referred to as *generalised* twists and relate the problem to extremising an associated energy on the compact Lie group $\mathbf{SO}(n)$. The main result is a surprising discrepancy between *even* and *odd* dimensions. In even dimensions the latter system of equations admits *infinitely* many smooth solutions, modulo isometries, amongst such maps whereas in odd dimensions this number reduces to *one*. Even more surprising is the fact that in odd dimensions the functional $\mathbb{F}$ admits strong local minimizers yet no solution of the Euler-Lagrange equations can be in the form of a generalised twist. Thus the strong local minimizers here do not have the symmetry one intuitively expects!

Acknowledgements

It is a pleasure to begin by thanking my supervisor Dr. Ali Taheri, whose guidance, humour and frequent prodding kept me going in the right direction. I am also grateful to Dr. Miroslav Chlebik for his support. My next thanks go to the University of Sussex and to ORSAS for financially supporting my research during this course.

Finally and foremost I am greatly indebted my closest mathematical friends, who supported me through many of the more times. Special mention in this regard must go to Adrian Martin and Raouf Mobasheri at the University of Sussex for all the fun we have had.

To my family & friends

Contents

Notation

$\mathbf{I}_n$	Identity matrix of size n
$\mathbf{O}(n)$	Orthogonal group
$\mathbf{SL}(n)$	Special linear group
$\mathbf{SO}(n)$	Special orthogonal group
$\mathbb{Z}_2$	Cyclic group of order 2
$\mathbb{Z}$	Group of integers
$\mathbb{C}$	Field of complex numbers
$\mathbb{R}^n$	Euclidean real n-space
$\mathbb{C}^n$	Euclidean complex n-space
$\mathbb{M}_{n\times n}$	Algebra of n-by-n matrices
θ	Unit normal vector
$\mathbb{B}$	Unit open ball in $\mathbb{R}^n$
$\mathbb{S}^{n-1}$	Unit sphere in $\mathbb{R}^n$
$\mathcal{L}^n$	Lebesgue measure
ω_n	$\mathcal{L}^n(\mathbb{B})$:= The Lebesgue measure of $\mathbb{B}$
$\mathcal{H}^{n-1}$	$(n-1)$-dimensional Hausdorff measure
$L^p(X,Y)$	L^p space
$W^{1,p}(X,Y)$	Sobolev space
$\|\cdot\|_{L^p}$	L_p norm
$\|\cdot\|_{W^{1,p}}$	$W^{1,p}$ norm
$\mathbf{C}(X,Y)$	Space of continuous functions from X to Y
$\mathbf{C}_c(X,Y)$	Space of functions in $\mathbf{C}(X,Y)$ with compact support
$\mathbf{1}_E$	Characteristic function of E
∇u	Gradient of u
$\mathrm{div}\,\mathrm{u}$	Divergence of u
Δu	Laplacian of u

$\Delta_p u$	p-Laplacian of u
$\mathbf{A}^t$	Transpose of $\mathbf{A}$
$tr\,\mathbf{A}$	Trace of $\mathbf{A}$
$\mathbf{A} : \mathbf{B}$	Inner product of matrices $\mathbf{A}$ and $\mathbf{B}$
$\det \mathbf{A}$	Determinant of $\mathbf{A}$
$\operatorname{cof} \mathbf{A}$	Cofactor of $\mathbf{A}$
$e^{\mathbf{A}}$	Exponential of $\mathbf{A}$
$\langle x, y \rangle$	Inner product of vectors x and y
$x \otimes y$	Tensor product of vectors x and y
$\{e_1, \ldots, e_n\}$	Standard basis of $\mathbb{R}^n$
$\partial\Omega$	Frontier or boundary of Ω
$\bar{\Omega}$	Closure of Ω
$\pi_l[X]$	l-th homotopy group of X
$\lfloor q \rfloor$	Floor of q

Chapter 1

Introduction

A large number of problems in mathematics, physics and engineering sciences naturally lead to minimizing an energy functional $\mathbb{F} : \mathcal{A} \to \mathbb{R}$ over a set $\mathcal{A}$. Problems of this type appear in a variety of areas ranging from analysis and geometry, e.g., harmonic maps, minimal surfaces and their higher dimensional counterparts to more applied branches in economy, optimization, materials science, e.g., nonlinear elasticity, optimal-shape design, modeling of solid-solid phase transitions and liquid crystals.

A general strategy for proving existence of a minimizer is the direct methods of the calculus of variations. It is based on the observation that if the set $\mathcal{A}$ admits a topology τ with respect to which the following two properties hold:

[i] $\mathbb{F}$ is τ-coercive, [1]

[ii] $\mathbb{F}$ is τ-lower semicontinuous, [2]

then there exists an $a \in \mathcal{A}$ such that $\mathbb{F}(a) = \inf_{\mathcal{A}} \mathbb{F}[\cdot]$.

1.1 Background

In continuum theories of solid mechanics, specifically elasticity theory, the response of a hyperelastic material subject to external excitations, in the form of applied forces: body and surface forces, as well as boundary displacement, is described by minimization of the total elastic energy

$$\mathbb{F}[u, \Omega] := \int_{\Omega} \mathbf{F}(\nabla u(x))\, dx. \tag{1.1}$$

[1] For every $t \in \mathbb{R}$ there exists a τ-compact set $K_t \subset \mathcal{A}$ such that $\{a \in \mathcal{A} : \mathbb{F}(a) \leq t\} \subset K_t$.

[2] For every $t \in \mathbb{R}$ the set $\{a \in \mathcal{A} : \mathbb{F}(a) \leq t\}$ is τ-closed.

Here $\Omega \subset \mathbb{R}^n$ is the region occupied by the body [3], $u : \Omega \to \mathbb{R}^n$ represents the deformation which is described on parts or whole of the boundary $\partial\Omega$, $\nabla u : \Omega \to \mathbb{M}_{n\times n}$ is the deformation gradient and $\mathbf{F} : \mathbb{M}_{n\times n} \to \mathbb{R}$ is the stored energy density. As matter can not interpenetrate itself the deformation is taken orientation preserving and thus locally invertible, that is, $\det \nabla u > 0$ (almost) everywhere in the domain. Moreover to comply with physics and to avoid unrealistic hypotheses the stored energy density $\mathbf{F}$ is taken *quasiconvex* or often *polyconvex* but strictly non-convex. (See Ball [6], [7] or Dacarogna [22].)

The aim of this thesis is to investigate the effect of domain topology and geometry on multiplicity *versus* uniqueness of minimizers (local or global) as well as extremals of $\mathbb{F}$. The earliest example of non-uniqueness of extremals for energies of the type described and over spaces of deformations keeping the boundary pointwise fixed, i.e., agreeing with the linear map identity, is a heuristic example of John [41] and [42]. Indeed John considers a two dimensional annulus as the underlying domain and argues that by considering deformations, where a typical representative of each is one keeping the inner boundary fixed while rotating the outer boundary by an integer multiple of 2π, one can define infinitely many distinct classes of non-homotopic self-maps of the annulus and thus arrive at multiple equilibria, see the work by John [41] and [42]. [4]

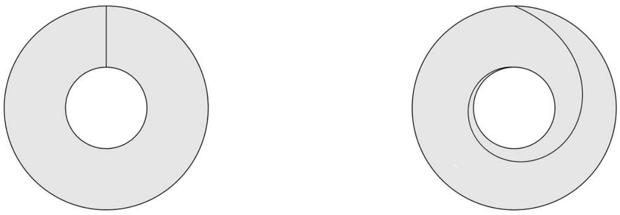

FIGURE 1.1: The image of a line segment under a self map of the annulus that keeps the inner boundary fixed and rotates the outer boundary anti-clockwise by 2π.

A more rigorous treatment of this example is due to Post & Sivalogonathan [52] where the authors use the notion of winding number of planar curves to define suitable homotopy classes in the corresponding Sobolev spaces and then proceed by minimizing the energy on each such homotopy class. It is important to note that the use of winding numbers although works well in this example can not be immediately extended to more complicated plane geometries as well as to higher dimensions as faced and expressed by the authors. The difficulty stems from the fact that in higher dimensions a simply connected domain (i.e., one in which every closed curve is homotopic

[3]Realistically $n = 2$ or 3, however, in this exposition and throughout this thesis, unless otherwise specified, we do not restrict the dimension to these two integers.

[4]The example of the two dimensional annulus and its infinite homotopy classes of self-maps was known to topologists much earlier in the century (cf. Dehn [23]). These are nowadays known as Dehn-twists and are instrumental in the study of mapping class groups of surfaces.

to a point) can still have a non-trivial topology as far as the space of self-maps of the domain is concerned and so the device of winding number of curves is not capable of confronting the task. (See the work by Taheri [70] and [72] where these problems are discussed and resolved.)

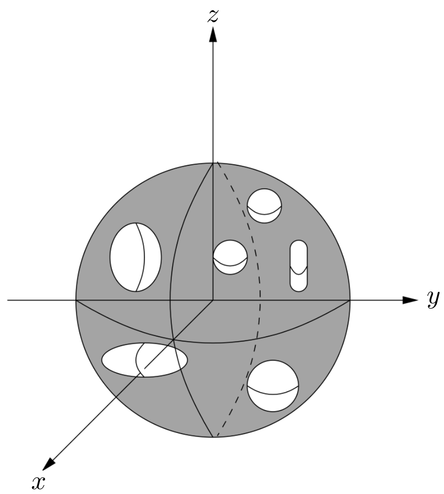

FIGURE 1.2: An example of a smooth simply connected domain in $\mathbb{R}^3$ whose spaces of self-maps admits multiple homotopy classes.

In contrast to the non-uniqueness results expressed above there are also examples where by imposing stringent conditions on the domain one can arrive at uniqueness of minimizers and extremals. The first result in this direction is the work of Knops & Stuart [44] where for similar type of energies subject to the domain being *starshaped* any linear map is the unique minimizer as well as the unique extremal of the energy subject to its own boundary condition. That is any other sufficiently regular extremal must coincide with the latter map. (See Taheri [71] for a different proof and for an analogous result for strong local minimizers. Also Ball [2] and Spadaro [61].)

In this thesis we aim to analyze this distinction more closely and examine a particular geometry, with no restriction on the dimension, where the energy functional $\mathbb{F}$ admits infinitely many smooth extremals as well as multiple local minimizers. Indeed, the thesis can be divided into roughly two parts: the first half focuses on domains Ω with a non-trivial topology, and as a prototype example of such domains, we restrict to generalised *annuli*, that is, domains in the form $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ with $0 < a < b < \infty$. We proceed by introducing a class of maps, referred to as *generalised* twists (*see* Definition 3.1.1) and examine them as possible solutions to the system of Euler-Lagrange equations associated with $\mathbb{F}$ (both in the so-called compressible and incompressible cases); the second half focuses on the other extreme, that is, domains Ω with a trivial topology, where, here, the prototype example are *starshaped* domains. We address the question of uniqueness of extremals and strong local minimizers using a method reminiscent of that in [71] by Taheri.

1.2 Outline of the Thesis

To outline in more detail the plan of the thesis and a discussion of the results. In the *second* chapter bring together some basic properties of the space of *self-maps* of generalised annuli that are required for the development of the thesis. In the *third* chapter, we take $\Omega \subset \mathbb{R}^n$ such *annulus* and consider the energy functional (1.1) over the space of *admissible* maps

$$\mathcal{A}_p(\Omega) := \Big\{ u \in W^{1,p}_\varphi(\Omega, \mathbb{R}^n) : \det \nabla u = 1 \ a.e. \text{ in } \Omega \Big\}, \tag{1.2}$$

where

$$W^{1,p}_\varphi(\Omega, \mathbb{R}^n) = \Big\{ u \in W^{1,p}(\Omega, \mathbb{R}^n) : u|_{\partial\Omega} = \varphi \Big\}.$$

Here $W^{1,p}(\Omega, \mathbb{R}^n)$ is the standard Sobolev space of vector valued L^p integrable functions defined on Ω, having L^p integrable distributional derivatives, and equipped with the norm

$$\|u\|^p_{W^{1,p}} := \|u\|^p_{L_p} + \|\nabla u\|^p_{L_p}.$$

Our terminology throughout is in agreement with that used by Adams [1] and Ziemer [79]. The boundary map φ is taken *linear*; indeed the case $\varphi = x$ (identity) is of particular interest to us. With regards to the integrand $\mathbf{F}$ we assume $\mathbf{F} : \mathbb{M}_{n\times n} \to \mathbb{R}$ to be *continuous* and to satisfy the following set of hypotheses:

[**H1**] (*Growth* condition) There exists $c_1 > 0$ such that for all $\xi \in \mathbb{M}_{n\times n}$ we have that

$$|\mathbf{F}(\xi)| \leq c_1(1 + |\xi|^p).$$

[**H2**] (*Coercivity* condition) There exists $c_2 > 0$ such that for all $\xi \in \mathbb{M}_{n\times n}$ we have that

$$c_2|\xi|^p - c_1 \leq \mathbf{F}(\xi).$$

[**H3**]$_\xi$ (*Quasiconvexity* at ξ) For *fixed* $\xi \in \mathbb{M}_{n\times n}$ and all $\phi \in \mathbf{C}^\infty_c(\Omega, \mathbb{R}^n)$ we have that

$$\int_\Omega \Big(\mathbf{F}(\xi + \nabla\phi(x)) - \mathbf{F}(\xi) \Big)\, dx \geq 0.$$

If, additionally, the inequality is *strict* for $\phi \neq 0$ then $\mathbf{F}$ is referred to as being *strictly* quasiconvex at ξ. (If the subscript ξ is *omitted* $\mathbf{F}$ is taken quasiconvex everywhere.)

In Chapter 3 we are primarily concerned with the problem of *extremising* the energy functional

(1.1) over the space (1.2) and *examining* a class of maps of *topological* significance as *solutions* to the associated system of Euler-Lagrange equations

$$\begin{cases} \operatorname{div} \mathfrak{S}[x, \nabla u(x)] = 0 & x \in \Omega, \\ \det \nabla u(x) = 1 & x \in \Omega, \\ u(x) = \varphi(x) & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where, we have set

$$\begin{aligned} \mathfrak{S}[x, \xi] &= \mathbf{F}_\xi(\xi) - \mathfrak{p}(x)\xi^{-t} \\ &=: \mathfrak{T}[x, \xi]\xi^{-t}, \end{aligned} \tag{1.4}$$

for $x \in \Omega$, $\xi \in \mathbb{M}_{n\times n}$ satisfying $\det \xi = 1$ and $\mathfrak{p}$ a suitable Lagrange multiplier while

$$\mathfrak{T}[x, \xi] = \mathbf{F}_\xi(\xi)\xi^t - \mathfrak{p}(x)\mathbf{I}_n. \tag{1.5}$$

In the language of elasticity, the *tensor* fields (1.4) and (1.5) are referred to as the *Piola-Kirchhoff* and the *Cauchy* stress tensors respectively and the *Lagrange* multiplier $\mathfrak{p}$ is better known as the *hydrostatic* pressure i.e., *see* Ciarlet [20].

While the *linear* map $u = \varphi$ serves as the *unique* minimizer of $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$ little is known about the *structure* and *features* of the solution *set* to this system of Euler-Lagrange equations [e.g., multiplicity *versus* uniqueness, existence of *strong* local minimizers, *partial* regularity, the nature and form of *singularities*, symmetries, *etc.* (*see*, e.g., [4], [8], [9], [24], [44], [52], [55], [68]).

We contribute towards understanding aspects of these questions by way of presenting *multiple* solutions to the above system of equations. For most of Chapter three we specialise to $\mathbf{F}(\xi) = p^{-1}|\xi|^p$ ($p > 1$), that is, the so-called p-Dirichlet energy and proceed by introducing a class of maps, referred to as *generalised* twists, characterised and defined by

$$u(x) = \mathbf{Q}(r)x,$$

where $\mathbf{Q} \in \mathbf{C}([a, b], \mathbf{SO}(n))$ and $r = |x|$. To ensure *admissibility*, i.e., $u \in \mathcal{A}_p(\Omega)$ it suffices to impose a *further* p-summability on $\dot{\mathbf{Q}} := d\mathbf{Q}/dr$ along with $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$. Restricting the p-energy to the space of such *twists* we can write

$$\mathbb{E}_p[\mathbf{Q}] := p\mathbb{F}_p[\mathbf{Q}(r)x, \Omega]$$

$$= \int_a^b \mathbf{E}(r, \dot{\mathbf{Q}}) r^{n-1}\, dr,$$

where the *integrand* itself is given through an integral over the *unit* sphere, i.e.,

$$\mathbf{E}(r, \xi) := \int_{\mathbb{S}^{n-1}} (n + r^2|\xi\theta|^2)^{\frac{p}{2}}\, d\mathcal{H}^{n-1}(\theta).$$

Here, the Euler-Lagrange equation can be shown to be the *second* order ordinary *differential* equation

$$\frac{d}{dr}\left\{ r^{n-1}\left[\mathbf{E}_\xi(r, \dot{\mathbf{Q}})\mathbf{Q}^t - \mathbf{Q}\mathbf{E}_\xi^t(r, \dot{\mathbf{Q}})\right]\right\} = 0.$$

Now in order to characterise among solutions to the above equation, *all* those which grant a solution to the Euler-Lagrange equations associated with $\mathbb{F}_p$ over $\mathcal{A}_p(\Omega)$ we are *confronted* with the task of obtaining *necessary* and *sufficient* conditions on the *vector* field

$$\begin{aligned}[\nabla u]^t \Delta_p u = \nabla \mathbf{s} + \Big\{ & r\mathbf{s}\mathbf{A}^2 - r^2\mathbf{s}\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle \mathbf{I}_n + \\ & \frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) + \frac{1}{r^{n-1}}\frac{d}{dr}(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2)\mathbf{I}_n \Big\}\theta,\end{aligned}$$

with $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ and $\mathbf{s} = (n+r^2|\dot{\mathbf{Q}}\theta|)^{\frac{p-2}{2}}$ for it to be a *gradient*, specifically, to coincide with $\nabla\mathfrak{p}$. This *analysis* occupies a major part of this chapter and is *fully* settled in Theorem 3.4.2 and Theorem 3.4.3.

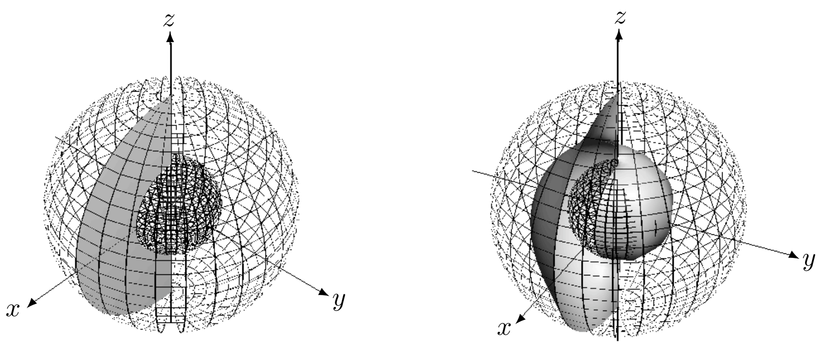

FIGURE 1.3: A schematic of how a generalised twist deforms a vertical plane in a three dimensional annulus.

The conclusion that the above analysis bares on to the *original* Euler-Lagrange equations turns to be a surprising discrepancy between *even* and *odd* dimensions. Indeed it follows that in *even* dimensions the latter system of equations admits *infinitely* many *smooth* solutions, modulo isometries, in the form of *generalised* twists whilst in *odd* dimensions this number *severely* reduces to *one*.

In Chapter 4, we consider $\Omega \subset \mathbb{R}^n$ to be a bounded *starshaped* domain and the energy functional (1.1) over the space of *admissible* maps (1.2) when $p \in [1, \infty[$ and while $\varphi = \bar{\xi}x$ and $\bar{\xi} \in \mathbb{M}_{n\times n}$ with $\det \bar{\xi} = 1$. Here the integrand $\mathbf{F} : \mathbb{M}_{n\times n} \to \mathbb{R}$ is of class $\mathbf{C}^1$ and for future reference we associate with it the set of hypotheses [**H1**], [**H2**], $[\mathbf{H3}]_\xi$ and

$[\mathbf{H4}]_\xi$ (*Rank-one* convexity at ξ) For *fixed* $\xi \in \mathbb{M}_{n\times n}$ and all *rank-one* $\zeta \in \mathbb{M}_{n\times n}$ the function

$$\mathbb{R} \ni t \mapsto \mathbf{F}(\xi + t\zeta) \in \mathbb{R},$$

is *convex* at $t = 0$. If the subscript ξ is *omitted* $\mathbf{F}$ is taken rank-one convex everywhere. [5]

Here we are primarily concerned with the question of *uniqueness* for solutions to the system of Euler-Lagrange equations (1.3), associated with the energy functional (1.1) over the space of (1.2), as well as that for its *strong* local minimizers (*see* Definition 2.2.1).

Indeed, the *former*, under the stated $\mathbf{C}^1$ regularity assumption on $\mathbf{F}$, the question of uniqueness of solutions to the associated system of Euler-Lagrange equations [subject to *linear* boundary conditions] was established in a seminal paper of Knops & Stuart (*see* [44]). There it is shown that subject to $\mathbf{F}$ being of class $\mathbf{C}^2$, *rank-one* convex everywhere and *strictly* quasiconvex at $\bar{\xi}$ any smooth solution u in a *starshaped* domain satisfying $\det \nabla u = 1$ in Ω and $u = \bar{\xi}x$ on $\partial\Omega$ satisfies $u = \bar{\xi}x$ on $\bar{\Omega}$.

In this short chapter we give a *new* proof of the *aforementioned* uniqueness result of Knops & Stuart [44]. This is based on *firstly* removing the measure preserving condition $\det \nabla u = 1$ and considering instead a suitable *unconstrained* functional [with the aid of the Lagrange multiplier $\mathfrak{p}$] and *secondly* utilising the so-called *stationarity* condition followed by comparison with *homogeneous* degree-one extensions as introduced in [71] by Taheri. This approach has the advantage of *extending* the uniqueness result to all *weak* solutions u of class $\mathbf{C}^1$ satisfying the *weak* form of the stationarity condition (*see* (4.4) below).

Finally we prove a new uniqueness result for *strong* local minimizers of $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$ to the effect that subject to [**H1**], $[\mathbf{H3}]_{\bar{\xi}}$ alone *any* such $u \in \mathcal{A}_p(\Omega)$ satisfies $\mathbb{F}[u, \Omega] = \mathbb{F}[\bar{\xi}x, \Omega]$ and therefore subject to the additional *strictly* quasiconvexity of $\mathbf{F}$ at $\bar{\xi}$ it must be that $u = \bar{\xi}x$ on $\bar{\Omega}$! We note that in this chapter for technical reasons one needs to restrict to $p \in [n, \infty[$ for the multiplicity result relating to *strong* local minimizers and to $p \in]1, \infty[$ for the one relating to smooth solutions.

[5] For a comprehensive treatment of the *convexity* notions [**H3**], [**H4**] and their significance in the *Calculus of Variations* we refer the interested reader to the books [11], [17], [22] and [30]-[32].

In the *final* Chapter, we return to the domain Ω in Chapter 3, i.e., generalised *annulus*, and consider the integral functionals $\mathbb{F}$ given by

$$\mathbb{F}[u,\Omega] := \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \phi(\det \nabla u)\right] dx, \tag{1.6}$$

over the space of admissible maps

$$\mathcal{A}(\Omega) = \left\{ u \in W^{1,2}_\varphi(\Omega, \mathbb{R}^n) : \det \nabla u > 0 \ a.e. \text{ in } \Omega \right\}, \tag{1.7}$$

where

$$W^{1,2}_\varphi(\Omega) := \left\{ u \in W^{1,2}(\Omega; \mathbb{R}^n) : u|_{\partial\Omega} = \varphi \right\},$$

and where φ is the *identity* map.

Regarding the function ϕ appearing in the energy functional $\mathbb{F}$ we make the following set of *hypotheses.*

[**h1**] $\phi :]0,\infty[\to [0,\infty[$,

[**h2**] ϕ is *convex*,

[**h3**] $\phi \in \mathbf{C}^2(]0,\infty[)$,

[**h4**] ϕ has the two *limiting* behaviours

$$\lim_{s\downarrow 0} \phi(s) = \lim_{s\uparrow\infty} \frac{\phi(s)}{s} = \infty,$$

[**h5**] there exists $\beta > 0$ and $\delta > 0$ such that for all $s \in]0,\infty[$ and $\alpha > 0$ satisfying $|\alpha - 1| < \delta$ we have that

$$|s\phi'(\alpha s)| \leq \beta[\phi(s) + 1]. \tag{1.8}$$

We are primarily concerned with the task of *extremising* the energy functional $\mathbb{F}$ over the space $\mathcal{A}(\Omega)$ and *examining* a special class of maps as solutions to the corresponding system of Euler-Lagrange equations which can *formally* be written as

$$\begin{cases} \Delta u + \nabla\left[\phi'(\det \nabla u) \ \mathrm{cof}\ \nabla u\right] = 0 & \text{in } \Omega, \\ \det \nabla u > 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Again we proceed by introducing a class of maps, referred to as *generalised* twists, characterised and defined by

$$u(x) = \mathbf{G}(r)\theta,$$

with

$$\mathbf{G}(r) = f(r)\mathbf{Q}(r),$$

where $r = |x|$, $\theta = x/|x|$, $\mathbf{Q} \in \mathbf{C}([a,b], \mathbf{SO}(n))$ and $f \in \mathbf{C}[a,b]$. In addition, to ensure *admissibility*, i.e., $u \in \mathcal{A}(\Omega)$ it suffices to impose a *further* L^2-summability on $\dot{f} := df/dr$ and $\dot{\mathbf{Q}} := d\mathbf{Q}/dr$ along with $\dot{f} > 0$ $\mathcal{L}^1$-a.e. on $]a,b[$ while $f(a) = a$, $f(b) = b$ and $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$.

Next by restricting the energy functional $\mathbb{F}$ to the space of such *twists* we can write

$$\begin{aligned} \mathbb{E}[\mathbf{Q}, f] := \frac{2}{\omega_n}\mathbb{F}[u,\Omega] = \int_a^b \Big\{ f^2 \Big[n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2 \Big] + n\dot{f}^2 + \\ 2n\phi(\dot{f}(\frac{f}{r})^{n-1}) \Big\} r^{n-1}\, dr, \end{aligned}$$

and the corresponding Euler-Lagrange equations can again be *formally* shown to be the *second* order system of ordinary *differential* equations

$$\begin{cases} \frac{d}{dr}\Big[r^{n-1} f^2 \mathbf{Q}^t \frac{d}{dr}\mathbf{Q} \Big] = 0, \\ \\ \frac{d}{dr}\Big[r^{n-1}\dot{f} + f^{n-1}\phi' \Big] = (n-1)[r^{n-3} f + \dot{f} f^{n-2}\phi'] + \\ \qquad\qquad \frac{1}{n} r^{n-1} f |\dot{\mathbf{Q}}|^2, \end{cases}$$

on $]a,b[$ where $\phi' = \phi'(\dot{f}(\frac{f}{r})^{n-1})$.

Now in order to characterise among solutions to the above system *all* those which grant a solution to the Euler-Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}(\Omega)$ we are *confronted* with the of task of verifying the *necessary* and *sufficient* condition $\dot{\mathbf{Q}}(r) \in \mathbb{R}\mathbf{SO}(n)$ on $]a,b[$. This *analysis* occupies a major part of the chapter and is *fully* settled in Theorem 5.5.3 and Theorem 5.6.3.

Again the conclusion that this analysis bares on to the *original* Euler-Lagrange equations turns to be a similar type of discrepancy between *even* and *odd* dimensions as that arose from the model in Chapter 3. Indeed it follows that in *even* dimensions the latter system of equations admit *infinitely* many *smooth* solutions, modulo isometries, in the form of *generalised* twists whilst in *odd* dimensions this number *severely* reduces to *one*.

We end by noting following in dealing with the *polyconvexity* in the *last* chapter.

[**1**] It is convenient to extend ϕ to the *entire* real line by setting $\phi(s) = \infty$ for $s \in]-\infty, 0]$. Evidently with this convention for any $u \in W^{1,2}_{\varphi}(\Omega, \mathbb{R}^n)$ we have that

$$\mathbb{F}[u, \Omega] < \infty \implies \det \nabla u > 0 \ \mathcal{L}^n\text{-}a.e. \text{ in } \Omega.$$

However notice that the *reverse* implication, in general, need *not* be true.

[**2**] The system of Euler-Lagrange equations associated with $\mathbb{F}$ for any solution of class $\mathbf{C}^2$ can alternatively be expressed as (*see* Definition 5.6.2)

$$\begin{cases} [\nabla u]^t \Delta u + \det \nabla u \, \nabla \Big[\phi'(\det \nabla u) \Big] = 0 & \text{in } \Omega, \\ \det \nabla u > 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

This being a consequence of the so-called *Piola's* identity (*see*, e.g., Morrey [48] pp. 122) and the pointwise *invertibility* of the gradient matrix.

Chapter 2

Continuous self-maps of annuli

The aim of this chapter is to describe the topology of the space of orientation preserving Sobolev maps on n-dimensional annuli that are required for the development of the thesis. At the heart of this investigation lies the profound problem of *enumerating* the path-connected components of its associated space of self-map. The material in this chapter is taken from Shahrokhi-Dehkordi & Taheri [60] and Taheri [68].

Assume to begin that $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ with $0 < a < b < \infty$ and that the boundary data φ in (1.2) is taken $\varphi = x$, the *identity* map. Then it can be shown that the space of Sobolev maps

$$\mathcal{A}_p(\Omega) := \left\{ u \in W^{1,p}(\Omega, \mathbb{R}^n) : \det \nabla u > 0 \ a.e. \text{ in } \Omega, u|_{\partial\Omega} = x \right\},$$

with $p \geq n$ embeds continuously and compactly into the space of self-maps of Ω, that is

$$\mathfrak{A}(\Omega) := \left\{ \phi \in \mathbf{C}(\bar{\Omega}, \bar{\Omega}) : \phi(x) = x \text{ for } x \in \partial\Omega \right\}. \tag{2.1}$$

Here $\mathfrak{A} := \mathfrak{A}(\Omega)$ is equipped with the topology of *uniform* convergence. The reader is referred to Morrey [48] for this last statement or Taheri [67], [70] and [73] for further details and proofs.

2.1 Degree of continuous self-maps on annuli

Definition 2.1.1. (**Homotopy**)

A pair of maps $\phi_0, \phi_1 \in \mathfrak{A}$ are referred to as *homotopic* if and only if there exists a continuous map $h : [0, 1] \times \bar{\Omega} \to \bar{\Omega}$ such that

[1] $h(0, x) = \phi_0(x)$ for all $x \in \bar{\Omega}$,

[2] $h(1, x) = \phi_1(x)$ for all $x \in \bar{\Omega}$,

[3] $h(t, x) = x$ for all $t \in [0, 1]$ and $x \in \partial\Omega$.

The collection of *all* maps homotopic to $\phi \in \mathfrak{A}$ is referred to as the homotopy *class* of ϕ and denoted by $[\phi]$. In order to give a characterisation of the homotopy classes $\{[\phi] : \phi \in \mathfrak{A}\}$, below, we consider the cases $n = 2$ and $n \geq 3$ separately.

The case ($n = 2$). Fix $\phi \in \mathfrak{A}$. Then, using polar coordinates, for $\theta \in [0, 2\pi]$ (*fixed*) the $\mathbb{S}^1$-valued *curve*

$$\gamma^\theta(r) = \frac{\phi}{|\phi|}(r, \theta) : [a, b] \to \mathbb{S}^1,$$

has a well-defined *index* or *winding* number about the *origin*. Furthermore, in view of continuity of ϕ, this is independent of the particular choice of $\theta \in [0, 2\pi]$. The latter correspondence will be denoted by

$$\phi \mapsto \mathbf{deg}(\frac{\phi}{|\phi|}).$$

Note that this integer also agrees with the Brouwer *degree* of the map resulting from identifying $\mathbb{S}^1 \cong [a, b]/\{a, b\}$, justified as a result of $\gamma_\theta(a) = \gamma_\theta(b)$. On the other hand for a differentiable curve (taking advantage of $\mathbb{S}^1 \subset \mathbb{C}$) we specifically have the formula

$$\mathbf{deg}(\frac{\phi}{|\phi|}) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}.$$

Theorem 2.1.2. *Let $n = 2$. Then, the map*

$$\mathbf{deg}(\cdot) : \Big\{[\phi] : \phi \in \mathfrak{A}\Big\} \to \mathbb{Z},$$

is a bijection. Moreover, for a pair of maps $\phi_0, \phi_1 \in \mathfrak{A}$, we have that

$$[\phi_0] = [\phi_1] \iff \mathbf{deg}(\frac{\phi_0}{|\phi_0|}) = \mathbf{deg}(\frac{\phi_1}{|\phi_1|}).$$

The case ($n \geq 3$). Fix $\phi \in \mathfrak{A}$. Then, using the identification $\bar{\Omega} \cong [a, b] \times \mathbb{S}^m$, with $m = n - 1$, it is plain that the map

$$\omega[r](\cdot) = \frac{\phi}{|\phi|}(r, \cdot) : [a, b] \to \mathbf{C}_\varphi(\mathbb{S}^m, \mathbb{S}^m),$$

uniquely defines an element of the group $\pi_1[\mathbf{C}_\varphi(\mathbb{S}^m, \mathbb{S}^m)]$. Where φ denoted as *identity* map of the m-sphere and $\mathbf{C}_\varphi(\mathbb{S}^m, \mathbb{S}^m)$ is the path-connected component of $\mathbf{C}(\mathbb{S}^m, \mathbb{S}^m)$ containing φ. By considering the action of $\mathbf{SO}(n)$ on $\mathbb{S}^m$ (viewed as its group of orientation *preserving* isometries, i.e., through

$$\Phi[\cdot] : \mathbf{SO}(n) \mapsto \mathbf{C}(\mathbb{S}^m, \mathbb{S}^m),$$

where $\Phi[\xi](x) = \xi x$ for $x \in \mathbb{S}^m$) it can be shown that the latter induces an *isomorphism* between $\pi_1[\mathbf{SO}(n)] \cong \mathbb{Z}_2$ and $\pi_1[\mathbf{C}_\varphi(\mathbb{S}^m, \mathbb{S}^m)]$. Thus, we are naturally lead to the correspondence

$$\phi \mapsto \mathbf{deg}_2(\frac{\phi}{|\phi|}) \in \mathbb{Z}_2.$$

Theorem 2.1.3. *Let $n = 3$. Then, the map*

$$\mathbf{deg}_2(\cdot) : \Big\{[\phi] : \phi \in \mathfrak{A}\Big\} \to \mathbb{Z}_2,$$

is a bijection. Moreover, for a pair of maps $\phi_0, \phi_1 \in \mathfrak{A}$, we have that

$$[\phi_0] = [\phi_1] \iff \mathbf{deg}_2(\frac{\phi_0}{|\phi_0|}) = \mathbf{deg}_2(\frac{\phi_1}{|\phi_1|}).$$

Remark 2.1.4. In the case of a *punctured* ball, say, $\Omega = \mathbb{B}\backslash\{0\}$ for any pair of maps $\phi_0, \phi_1 \in \mathfrak{A} := \{\phi \in C(\bar{\Omega}, \bar{\Omega}) : \phi = \varphi \text{ on } \partial\Omega = \{0\} \cup \partial\mathbb{B}\}$ the *continuous* path $[0,1] \ni t \mapsto \phi_t := (1-t)\phi_0 + t\phi_1$ lies within $\mathfrak{A}$ and joins ϕ_0 to ϕ_1. Therefore, here, $\mathfrak{A}$ consists of a *single* component only!

2.2 Homotopy characterisation and strong local minimizers

Let Ω as before be an n-dimensional annulus and $\mathcal{A}_p(\Omega)$ the space defined in (1.2). When $p \in [n, \infty[$ by taking advantage of the embedding $\mathcal{A}_p(\Omega) \subset \mathfrak{A}(\Omega)$ it follows that every $u \in \mathcal{A}_p := \mathcal{A}_p(\Omega)$ has a representative (again, denoted u) in $\mathfrak{A}$. Hence, we can set,

[**1**] ($n = 2$) for each $m \in \mathbb{Z}$,

$$\mathfrak{c}_m[\mathcal{A}_p] := \Big\{u \in \mathcal{A}_p : \mathbf{deg}(\frac{u}{|u|}) = m\Big\}. \tag{2.2}$$

As a result the latter are *pairwise* disjoint and that

$$\mathcal{A}_p = \bigcup_{m \in \mathbb{Z}} \mathfrak{c}_m[\mathcal{A}_p].$$

[**2**] ($n \geq 3$) for each $\alpha \in \mathbb{Z}_2 = \{0, 1\}$,

$$\mathfrak{c}_\alpha[\mathcal{A}_p] := \Big\{u \in \mathcal{A}_p : \mathbf{deg}_2(\frac{u}{|u|}) = \alpha\Big\}. \tag{2.3}$$

As a result, again, the latter are *pairwise* disjoint and that

$$\mathcal{A}_p = \bigcup_{\alpha \in \mathbb{Z}_2} \mathfrak{c}_\alpha[\mathcal{A}_p].$$

Definition 2.2.1. (**Strong local minimizer**)
A map $\bar{u} \in \mathcal{A}_p(\Omega)$ is a *strong* local minimizer of the functional $\mathbb{F}$, given by (1.1), if and only if there exists $\delta = \delta(\bar{u}) > 0$ such that $\mathbb{F}[\bar{u}, \Omega] \leq \mathbb{F}[u, \Omega]$ for all $u \in \mathcal{A}_p(\Omega)$ satisfying $\|\bar{u} - u\|_{L^1} < \delta$.

Over the next two propositions, we will show that the homotopy classes of $\mathfrak{c}_\star[\mathcal{A}_p]$ are *sequentially* weakly closed, hence by minimizing $\mathbb{F}$ on each homotopy class we arrive at a *strong* local minimizer. Note that when $p \in [1, n[$ this argument encounters *two* serious obstacles, *firstly*, there is *no* embedding of $\mathcal{A}_p(\Omega)$ into $\mathfrak{A}(\Omega)$, and *secondly*, the *determinant* function *fails* to be sequentially *weakly* continuous.

Proposition 2.2.2. *Let* $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ *be a generalised annulus and* $\mathbf{F}$ *an integrand satisfying* [**H2**]. *Fix* $p \in [n, \infty[$, *and consider the classes* $\mathfrak{c}_\star[\mathcal{A}_p]$ *as defined either by* (2.2) *or* (2.3). *Then,*
[**1**] $\mathfrak{c}_\star[\mathcal{A}_p]$ *is* $W^{1,p}$*-sequentially weakly closed,*
[**2**] *for* $u \in \mathfrak{c}_\star[\mathcal{A}_p]$ *and* $s > 0$ *there exists* $\delta = \delta(u, s) > 0$ *such that*

$$\left\{ \begin{array}{l} v \in \mathcal{A}_p, \\ \|v - u\|_{L^1} < \delta, \\ \mathbb{F}[v, \Omega] < s, \end{array} \right\} \implies v \in \mathfrak{c}_\star[\mathcal{A}_p].$$

Proof. [**1**] Let $(u_j)_{j \in \mathbb{N}} \subset \mathfrak{c}_\star[\mathcal{A}_p]$ and $u_j \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^n)$. Then, as a result of $p \geq n$, by passing to a subsequence (not re-labaled) we have

$$\det \nabla u_j \overset{*}{\rightharpoonup} \det \nabla u,$$

in $\mathcal{M}(\Omega)$ and so $u \in \mathcal{A}_p$. Moreover, in view of $u_j \to u$ uniformly on $\bar{\Omega}$, an application of Theorem 2.1.2 or Theorem 2.1.3 gives $u \in \mathfrak{c}_\star[\mathcal{A}_p]$. This justifies [**1**].
[**2**] Assume the contrary. Then, there exists $u \in \mathfrak{c}_\star[\mathcal{A}_p]$, $s > 0$ and $(v_j)_{j \in \mathbb{N}}$ such that

$$\left\{ \begin{array}{l} v_j \in \mathcal{A}_p, \\ \|v_j - u\|_{L^1} \to 0, \\ \mathbb{F}[v_j, \Omega] < s, \end{array} \right.$$

while $v_j \notin \mathfrak{c}_\star[\mathcal{A}_p]$. However the above imply that by passing to a subsequence (not re-labeled) $v_j \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^n)$ and as in [**1**], $v_j \to u$ uniformly on $\bar{\Omega}$. Hence, again either by Theorem 2.1.2 or Theorem 2.1.3,

$$\mathbf{deg}_\star(\frac{v_j}{|v_j|}) \to \mathbf{deg}_\star(\frac{u}{|u|}).$$

As the above quantities are *integers* (and the one on the *right* being a constant) it follows that for j large enough $v_j \in \mathfrak{c}_\star[\mathcal{A}_p]$ which is a contradiction. This completes the proof. □

In view of the sequential weak lower semicontinuity of the energy functional $\mathbb{F}$, an application of the direct methods of the calculus of variations leads us to the following conclusion.

Proposition 2.2.3. *Let* $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ *be a generalised annulus and* $\mathbf{F}$ *an integrand satisfying* [**H1**], [**H2**] *and* [**H3**]. *Fix* $p \in [n, \infty[$ *and consider the classes* $\mathfrak{c}_\star[\mathcal{A}_p]$ *as defined either by* (2.2) *or* (2.3). *Then, there exists*

$$\bar{u} = \bar{u}(x; a, b, \star) \in \mathfrak{c}_\star[\mathcal{A}_p],$$

such that

$$\mathbb{F}[\bar{u}, \Omega] = \inf_{\mathfrak{c}_\star[\mathcal{A}_p]} \mathbb{F}[\cdot, \Omega]. \tag{2.4}$$

In addition, for each such $\bar{u}$ *there exists* $\delta = \delta(\bar{u}) > 0$ *such that*

$$\mathbb{F}[\bar{u}, \Omega] \leq \mathbb{F}[v, \Omega], \tag{2.5}$$

for all $v \in \mathcal{A}$ *satisfying* $\|\bar{u} - v\|_{L^1} < \delta$.

Proof. Let $(v_j)_{j\in\mathbb{N}} \subset \mathfrak{c}_\star[\mathcal{A}_p]$ be an *infimizing* sequence, $\mathbb{F}[v_j, \Omega] \downarrow \alpha := \inf_{\mathfrak{c}_\star[\mathcal{A}_p]} \mathbb{F}[\cdot, \Omega]$. Then as $\alpha < \infty$ it follows that by passing to a subsequence (not re-labeled) $v_j \rightharpoonup \bar{u}$ in $W^{1,p}(\Omega, \mathbb{R}^n)$ where by [**1**] in Proposition 2.2.2, $\bar{u} \in \mathfrak{c}_\star[\mathcal{A}_p]$. As a result we can write

$$\alpha \leq \mathbb{F}[\bar{u}, \Omega] \leq \liminf_{j\uparrow\infty} \mathbb{F}[v_j, \Omega] \leq \alpha,$$

and so $\bar{u}$ is a minimizer as required.

To establish the final assertion, fix $\star$ and $\bar{u}$ as above and with $s = 1 + \mathbb{F}[\bar{u}, \Omega]$ pick $\delta > 0$ as [**2**] in Proposition 2.2.2. Then, any $v \in \mathcal{A}_p$ satisfying $\|\bar{u} - v\|_{L^1} < \delta$ also satisfies (2.5). Otherwise $\mathbb{F}[v, \Omega] < \mathbb{F}[\bar{u}, \Omega] < s$ implying that $v \in \mathfrak{c}_\star[\mathcal{A}_p]$ and hence in view of $\bar{u}$ being a minimizer, $\mathbb{F}[v, \Omega] \geq \mathbb{F}[\bar{u}, \Omega]$ which is a contradiction. □

Chapter 3

Measure-preserving maps and generalised twists

In this chapter we introduce a class of maps referred to as *generalised* twists and *examine* them in connection with the Euler-Lagrange equations associated with the p-Dirichlet energy

$$\mathbb{F}_p[u,\Omega] := p^{-1} \int_\Omega |\nabla u(x)|^p \, dx, \tag{3.1}$$

with $p \in]1,\infty[$, over the space of measure preserving maps (1.2). The main result is an interesting discrepancy between *even* and *odd* dimensions. Here we show that in even dimensions the latter system of equations admits *infinitely* many smooth solutions, modulo isometries, amongst such maps. In odd dimensions this number reduces to *one*. The result relies on a careful analysis of the *full* versus the *restricted* Euler-Lagrange equations where a key ingredient is a *necessary* and *sufficient* condition for an associated vector field to be a *gradient*. The material in this chapter is taken from Shahrokhi-Dehkordi & Taheri [59], [60] and partly [69] by Taheri.

3.1 Generalised twists

We begin this section by introducing a class of maps, referred to as *generalised* twists and then proceed to study some properties of these maps.

Definition 3.1.1. (**Generalised twists**)

Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$. A map $u \in \mathbf{C}(\bar{\Omega}, \bar{\Omega})$ is referred to as a *generalised* twists if and only if it can be expressed as

$$u(x) = \mathbf{G}(r)\theta, \tag{3.2}$$

with

$$\mathbf{G}(r) = f(r)\mathbf{Q}(r),$$

where $r = |x|$, $\theta = x/|x|$, $\mathbf{Q} \in \mathbf{C}([a,b], \mathbf{SO}(n))$ and $f \in \mathbf{C}[a,b]$.

When $n = 2$ and $f \equiv r$ a *generalised* twist can be shown to take, in *polar* coordinates, the alternative form

$$(r, \theta) \mapsto (r, \theta + g(r)), \tag{3.3}$$

for a suitable $g \in \mathbf{C}[a,b]$. Maps of the type (3.3) frequently arise in the study of *mapping* class groups of *surfaces* and are better known as *Dehn-twists*, e.g., see Dehn [23]. In *higher* dimensions, by contrast, no such simple representation of (3.2) is feasible in generalised *spherical* coordinates, however, the terminology here is suggested by analogy with (3.3) when $n = 2$. The continuous function $\mathbf{G}$ in the above definition will be referred to as the twist *path*. When additionally $\mathbf{G}(a) = \mathbf{G}(b)$ we refer to $\mathbf{G}$ as the twists *loop*.

Notice that as a result of the basic requirement $\det \nabla u = 1$ *a.e.* in Ω built into the definition of a *generalised* twists it follows in particular we assume $f \equiv r$ in $[a,b]$, *see* equation (5.2) in Proposition 5.1.1. Therefore along this chapter we assume always $f(r) = r$ on $[a,b]$.

Proposition 3.1.2. *Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$. A generalised twist u lies in $\mathcal{A}_p = \mathcal{A}_p(\Omega)$ with $p \in [1, \infty[$ provided that the following hold.*
[**1**] $\mathbf{Q} \in W^{1,p}([a,b], \mathbf{SO}(n))$,
[**2**] $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$.

Thus, in particular, when a *generalised* twist u lies in $\mathcal{A}_p$ its corresponding twist path forms a *loop* in the pointed space $(\mathbf{SO}(n), \mathbf{I}_n)$.

Proof. Assume that u is a *generalised* twist. Then $u \in \mathcal{A}_p(\Omega)$ if and only if the following hold.
(i) $u = x$ on $\partial\Omega$,
(ii) $\det \nabla u = 1$ in Ω, and,
(iii) $\|u\|_{W^{1,p}(\Omega)} < \infty$.
Evidently [**2**] gives (i). Moreover, a straight-forward calculation gives

$$\nabla u = \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta$$

$$= \mathbf{Q}(\mathbf{I}_n + \mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta), \tag{3.4}$$

where $r = |x|$, $\theta = x/|x|$ and $\dot{\mathbf{Q}} := d\mathbf{Q}/dr$. Hence in view of $\det \mathbf{Q} = 1$ we can write

$$\begin{aligned}
\det \nabla u &= \det(\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta) \\
&= \det(\mathbf{I}_n + r\mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta) \\
&= 1 + \langle \mathbf{Q}^t\dot{\mathbf{Q}}\theta, \theta\rangle \\
&= 1 + \langle \dot{\mathbf{Q}}\theta, \mathbf{Q}\theta\rangle = 1,
\end{aligned}$$

where in the last identity we have used the fact that $\langle \mathbf{Q}\theta, \mathbf{Q}\theta\rangle = |\theta|^2 = 1$ for all $\theta \in \mathbb{S}^{n-1}$ and so as a result

$$\frac{d}{dr}\langle \mathbf{Q}\theta, \mathbf{Q}\theta\rangle = \langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta\rangle + \langle \dot{\mathbf{Q}}\theta, \mathbf{Q}\theta\rangle = 0.$$

This therefore gives (ii). Finally, to justify (iii) we *first* note that

$$\begin{aligned}
|\nabla u|^2 &= tr\Big\{[\nabla u][\nabla u]^t\Big\} \\
&= tr\Big\{(\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta)(\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta)\Big\} \\
&= tr\Big\{\mathbf{I}_n + r\mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta + r\dot{\mathbf{Q}} \otimes \mathbf{Q}\theta + r^2\dot{\mathbf{Q}}\theta \otimes \dot{\mathbf{Q}}\theta\Big\} \\
&= n + 2r\langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta\rangle + r^2\langle \dot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\rangle.
\end{aligned}$$

Therefore as a result of $\langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta\rangle = 0$ for any $p \in [1, \infty[$ we have that

$$|\nabla u|^p = (n + r^2|\dot{\mathbf{Q}}\theta|^2)^{\frac{p}{2}}. \tag{3.5}$$

Hence in view of $|u| = r\sqrt{\langle \mathbf{Q}\theta, \mathbf{Q}\theta\rangle} = r$ we can write

$$\int_\Omega |u|^p + |\nabla u|^p = \int_a^b \int_{\mathbb{S}^{n-1}} \Big\{r^p + (n + r^2|\dot{\mathbf{Q}}\theta|^2)^{\frac{p}{2}}\Big\} r^{n-1}\, d\mathcal{H}^{n-1}(\theta) dr,$$

and so referring to [**1**] the conclusion follows. □

Proposition 3.1.3. *Suppose that u is a generalised twist with the associated twist path $\mathbf{Q} \in \mathbf{C}^2(]a, b[, \mathbf{SO}(n))$. Then for $p \in [1, \infty[$ we have that*

$$\begin{aligned}
\Delta_p u :=& \mathrm{div}(|\nabla u|^{p-2})\nabla u \\
=& \mathbf{Q}\Big[\nabla \mathbf{s} \otimes \theta + \frac{1}{r^n}\frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big) + r\mathbf{s}\mathbf{A}^2\Big]\theta,
\end{aligned}$$

where $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ *and* $\mathbf{s} = \mathbf{s}(r,\theta) := (n + r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}}$.

Proof. [**1**] ($p = 2$) Referring to Definition 3.1.1 and using the notation $u = (u_1, u_2, \dots, u_n)$ we can write with the aid of (3.4) in Proposition 3.1.2 that

$$
\begin{aligned}
\Delta u_i =& \sum_{j=1}^n \frac{\partial}{\partial x_j}\left\{\mathbf{Q}_{ij} + r\sum_{k=1}^n \dot{\mathbf{Q}}_{ik}\theta_k\theta_j\right\} \\
=& \sum_{j=1}^n \left\{\dot{\mathbf{Q}}_{ij}\theta_j + \theta_j\sum_{k=1}^n \dot{\mathbf{Q}}_{ik}\theta_k\theta_j + r\sum_{k=1}^n \ddot{\mathbf{Q}}_{ik}\theta_j\theta_k\theta_j + \right. \\
& \left. \sum_{k=1}^n \dot{\mathbf{Q}}_{ik}(\delta_{kj} - \theta_j\theta_k)\theta_j + \sum_{k=1}^n \dot{\mathbf{Q}}_{ik}\theta_k(1 - \theta_j\theta_j)\right\} \\
=& 2\sum_{j=1}^n \dot{\mathbf{Q}}_{ij}\theta_j + r\sum_{j=1}^n \ddot{\mathbf{Q}}_{ij}\theta_j + (n-1)\sum_{j=1}^n \dot{\mathbf{Q}}_{ij}\theta_j \\
=& (n+1)\sum_{j=1}^n \dot{\mathbf{Q}}_{ik}\theta_k + r\sum_{j=1}^n \ddot{\mathbf{Q}}_{ij}\theta_j.
\end{aligned}
$$

As this is true for $1 \le i \le n$ going back to the original *vector* notation and using the substitutions $\dot{\mathbf{Q}} = \mathbf{Q}\mathbf{A}$ and $\ddot{\mathbf{Q}} = \mathbf{Q}[\dot{\mathbf{A}} + \mathbf{A}^2]$ we have that,

$$
\begin{aligned}
\Delta u &= [(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}]\theta \\
&= \mathbf{Q}[(n+1)\mathbf{A} + r\dot{\mathbf{A}} + r\mathbf{A}^2]\theta \\
&= \mathbf{Q}\left[\frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{A}) + r\mathbf{A}^2\right]\theta,
\end{aligned}
$$

which is the required result for $p = 2$. [Note that in this case $\mathbf{s} = \mathbf{s}(r,\theta) \equiv 1$.]

[**2**] ($p \in [1,\infty[$) According to definition we have that

$$
\begin{aligned}
\Delta_p u &= \operatorname{div}(|\nabla u|^{p-2}\nabla u) \\
&= \operatorname{div}(\mathbf{s}\nabla u) = \nabla u\nabla\mathbf{s} + \mathbf{s}\Delta u.
\end{aligned}
$$

Now a straight-forward *differentiation* gives

$$
\begin{aligned}
\nabla\mathbf{s} &= \nabla(n + r^2|\dot{\mathbf{Q}}\theta|^2)^{\frac{p-2}{2}} \\
&= \nabla(n + r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}} \\
&= \beta\left[r\mathbf{A}^t\mathbf{A} + r^2\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\mathbf{I}_n\right]\theta, \qquad (3.6)
\end{aligned}
$$

where $\beta = \beta(r,\theta,p) := (p-2)(n+r^2|\mathbf{A}\theta|^2)^{\frac{p-4}{2}}$. Thus we can write

$$\begin{aligned}\Delta_p u =& \nabla u \nabla \mathbf{s} + \mathbf{s}\Delta u \\ =& \mathbf{Q}\Big[\mathbf{I}_n + r\mathbf{A}\theta\otimes\theta\Big]\nabla\mathbf{s} + \mathbf{s}\mathbf{Q}\Big[(n+1)\mathbf{A} + r\dot{\mathbf{A}} + r\mathbf{A}^2\Big]\theta \\ =& \mathbf{Q}\nabla\mathbf{s} + r\beta\mathbf{Q}\Big[\mathbf{A}\theta\otimes\theta\Big]\Big[r\mathbf{A}^t\mathbf{A} + r^2\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle\mathbf{I}_n\Big]\theta + \\ & \mathbf{s}\mathbf{Q}\Big[(n+1)\mathbf{A} + r\dot{\mathbf{A}} + r\mathbf{A}^2\Big]\theta.\end{aligned}$$

In order to further simplify the *second* term on the *right* in the last identity we *first* notice that

$$\begin{aligned}\mathbf{s}_r := \frac{\partial\mathbf{s}}{\partial r} &= \frac{\partial}{\partial r}(n+r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}} \\ &= \beta\Big[r|\mathbf{A}\theta|^2 + r^2\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle\Big],\end{aligned}$$

and consequently

$$\begin{aligned}r\mathbf{s}_r\mathbf{Q}\mathbf{A}\theta &= r\beta\mathbf{Q}\Big[r|\mathbf{A}\theta|^2 + r^2\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle\Big]\mathbf{A}\theta \\ &= r\beta\mathbf{Q}\Big[\mathbf{A}\theta\otimes\theta\Big]\times\Big[r\mathbf{A}^t\mathbf{A} + r^2\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle\mathbf{I}_n\Big]\theta.\end{aligned}$$

Therefore substituting back gives

$$\begin{aligned}\Delta_p u =& \mathbf{Q}\Big[\nabla\mathbf{s}\otimes\theta + r\mathbf{s}_r\mathbf{A} + (n+1)\mathbf{s}\mathbf{A} + r\mathbf{s}\dot{\mathbf{A}} + r\mathbf{s}\mathbf{A}^2\Big]\theta \\ =& \mathbf{Q}\Big[\nabla\mathbf{s}\otimes\theta + \frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) + r\mathbf{s}\mathbf{A}^2\Big]\theta,\end{aligned}$$

which is the required conclusion. □

Proposition 3.1.4. *Suppose that u is a generalised twist with the associated twist path $\mathbf{Q} \in \mathbf{C}^2(]a,b[,\mathbf{SO}(n))$. Then for $p \in [1,\infty[$ we have that*

$$\begin{aligned}[\nabla u]^t\Delta_p u = \nabla\mathbf{s} + \Big\{& r\mathbf{s}\mathbf{A}^2 - r^2\mathbf{s}\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle\mathbf{I}_n + \\ & \frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) + \frac{1}{r^{n-1}}\frac{d}{dr}(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2)\mathbf{I}_n\Big\}\theta,\end{aligned} \tag{3.7}$$

where $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ and $\mathbf{s} = \mathbf{s}(r,\theta) = (n+r^2|\mathbf{A}\theta|)^{\frac{p-2}{2}}$.

Proof. In view of (3.4) we have that

$$[\nabla u]^t = [\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta]^t = [\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta] = [\mathbf{I}_n + r\theta \otimes \mathbf{A}\theta]\mathbf{Q}^t.$$

Therefore by substituting for $[\nabla u]^t$ and $\Delta_p u$ (from the previous proposition) we arrive at

$$\begin{aligned}
[\nabla u]^t \Delta_p u =& \Big[\mathbf{I}_n + r\theta \otimes \mathbf{A}\theta\Big] \times \Big[\nabla \mathbf{s} \otimes \theta + \frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) + r\mathbf{s}\mathbf{A}^2\Big]\theta \\
=& \Big[\nabla \mathbf{s} \otimes \theta + \frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) + r\mathbf{s}\mathbf{A}^2\Big]\theta + \\
& \Big[r\langle \nabla \mathbf{s}, \mathbf{A}\theta\rangle + \frac{1}{r^{n-1}}\langle \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}), \mathbf{A}\theta\rangle + r^2\mathbf{s}\langle \mathbf{A}^2\theta, \mathbf{A}\theta\rangle\Big]\theta.
\end{aligned}$$

However, in view of $\mathbf{A}$ being *skew*-symmetric it can be easily verified that $\langle \mathbf{A}^2\theta, \mathbf{A}\theta\rangle = 0$ and in a similar way referring to (3.6)

$$\begin{aligned}
\langle \nabla \mathbf{s}, \mathbf{A}\theta\rangle &= \langle \beta\Big[r\mathbf{A}^t\mathbf{A} + r^2\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle \mathbf{I}_n\Big]\theta, \mathbf{A}\theta\rangle \\
&= \beta r\Big\{\langle \mathbf{A}^3\theta, \theta\rangle + r\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\langle \mathbf{A}\theta, \theta\rangle\Big\} = 0.
\end{aligned}$$

Thus *summarising*, we have that

$$\begin{aligned}
[\nabla u]^t \Delta_p u &= \nabla \mathbf{s} + \Big\{r\mathbf{s}\mathbf{A}^2 + \frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) + \\
&\qquad \frac{1}{r^{n-1}}\langle \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}), \mathbf{A}\theta\rangle \mathbf{I}_n\Big\}\theta \\
&= \nabla \mathbf{s} + \Big\{r\mathbf{s}\mathbf{A}^2 - r^2\mathbf{s}\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle \mathbf{I}_n + \frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) + \\
&\qquad \frac{1}{r^{n-1}}\frac{d}{dr}(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2)\mathbf{I}_n\Big\}\theta.
\end{aligned}$$

The proof is thus complete. □

3.2 The p-energy restricted to the loop space

For a *generalised* twist u referring to (3.5) we have for any $p \in [1, \infty[$ that

$$\int_\Omega |\nabla u|^p = \int_a^b \int_{\mathbb{S}^{n-1}} (n + r^2|\dot{\mathbf{Q}}\theta|^2)^{\frac{p}{2}} r^{n-1}\, d\mathcal{H}^{n-1}(\theta) dr.$$

Motivated by the above representation in this section we introduce the energy functional

$$\mathbb{E}_p[\mathbf{Q}] := \int_a^b \mathbf{E}(r, \dot{\mathbf{Q}}) r^{n-1}\, dr$$

where the *integrand* itself is given through the integral

$$\mathbf{E}(r, \xi) = \int_{\mathbb{S}^{n-1}} (n + r^2|\xi\theta|^2)^{\frac{p}{2}}\, d\mathcal{H}^{n-1}(\theta).$$

Associated with the energy functional $\mathbb{E}_p$ and in line with Proposition 3.1.2 we introduce the space of *admissible* loops

$$\mathcal{E}_p := \left\{ \mathbf{Q} = \mathbf{Q}(r) : \begin{array}{l} \mathbf{Q} \in W^{1,p}([a,b], \mathbf{SO}(n)), \\ \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n. \end{array} \right\}$$

Our primary objective here is to obtain the Euler-Lagrange equation associated with the energy functional $\mathbb{E}_p$ over the space of loops $\mathcal{E}_p$. In doing so the following observation will prove useful.

Proposition 3.2.1. *Let* $\mathbf{Q} \in \mathbf{SO}(n)$ *and* $\mathbf{R} \in \mathbb{M}_{n\times n}$. *Then the following are equivalent:*

[**1**] $\mathbf{R}\mathbf{Q}^t + \mathbf{Q}\mathbf{R}^t = \mathbf{0}$,

[**2**] $\mathbf{R} = (\mathbf{F} - \mathbf{F}^t)\mathbf{Q}$ *for some* $\mathbf{F} \in \mathbb{M}_{n\times n}$.

Moreover, $\mathbf{F}$ *in* [**2**] *is unique if it is assumed skew-symmetric, i.e.,* $\mathbf{F}^t = -\mathbf{F}$.

Proof. The implication [**2**] $\implies$ [**1**] follows from a direct verification. For the reverse implication it suffices to assume $\mathbf{F}^t + \mathbf{F} = \mathbf{0}$ and then take $2\mathbf{F} = \mathbf{R}\mathbf{Q}^t$. □

Proposition 3.2.2. *Let* $p \in [1, \infty[$. *Then the Euler-Lagrange equation associated with* $\mathbb{E}_p$ *over* $\mathcal{E}_p$ *takes the form*

$$\frac{d}{dr}\left\{ r^{n-1} \Big[\mathbf{E}_\xi(r, \dot{\mathbf{Q}})\mathbf{Q}^t - \mathbf{Q}\mathbf{E}^t_\xi(r, \dot{\mathbf{Q}}) \Big] \right\} = 0. \tag{3.8}$$

Proof. Fix $\mathbf{Q} \in W^{1,p}([a,b], \mathbf{SO}(n))$ and pick a *variation* $\mathbf{H} \in \mathbf{C}_0^\infty([a,b], \mathbb{M}_{n\times n})$. For $\varepsilon \in \mathbb{R}$ put $\mathbf{Q}_\varepsilon = \mathbf{Q} + \varepsilon\mathbf{H}$. Then,

$$\begin{aligned} \mathbf{Q}_\varepsilon \mathbf{Q}_\varepsilon^t &= [\mathbf{Q} + \varepsilon\mathbf{H}][\mathbf{Q} + \varepsilon\mathbf{H}]^t \\ &= \mathbf{I}_n + \varepsilon[\mathbf{H}\mathbf{Q}^t + \mathbf{Q}\mathbf{H}^t] + \varepsilon^2\mathbf{H}\mathbf{H}^t. \end{aligned}$$

Hence for $\mathbf{Q}_\varepsilon$ to take values on $\mathbf{SO}(n)$ to the *first* order it suffices to have

$$\mathbf{H}\mathbf{Q}^t + \mathbf{Q}\mathbf{H}^t = \mathbf{0},$$

on $[a, b]$. In view of Proposition 3.2.1 this is equivalent to assuming that for some $\mathbf{F} \in \mathbf{C}_0^\infty([a,b], \mathbb{M}_{n\times n})$ the *variation* $\mathbf{H}$ has the form

$$\mathbf{H} = (\mathbf{F} - \mathbf{F}^t)\mathbf{Q}.$$

With this assumption in place we examine the vanishing of the *first* derivative of the *energy*, i.e., that indeed

$$\begin{aligned}
0 &= \frac{d}{d\epsilon}\mathbb{E}_p[\mathbf{Q}_\varepsilon]\bigg|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_a^b \mathbf{E}(r, \dot{\mathbf{Q}}_\varepsilon) r^{n-1}\, dr\bigg|_{\varepsilon=0} \\
&= \int_a^b \left\{ \frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}}_\varepsilon) : \frac{d}{d\varepsilon}\dot{\mathbf{Q}}_\varepsilon \right\} r^{n-1}\, dr\bigg|_{\varepsilon=0} \\
&= \int_a^b \left\{ \frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}}) : \Big[(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t)\mathbf{Q} + (\mathbf{F} - \mathbf{F}^t)\dot{\mathbf{Q}} \Big] \right\} r^{n-1}\, dr \\
&=: \mathbf{I} + \mathbf{II}.
\end{aligned}$$

We now proceed by evaluating each term separately. Indeed, with regards to the *first* term we have that

$$\begin{aligned}
\mathbf{I} &= \int_a^b \left\{ \frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}}) : (\dot{\mathbf{F}} - \dot{\mathbf{F}}^t)\mathbf{Q} \right\} r^{n-1}\, dr \\
&= \int_a^b \left\{ \frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t : (\dot{\mathbf{F}} - \dot{\mathbf{F}}^t) \right\} r^{n-1}\, dr \\
&= \int_a^b \left\{ -\frac{d}{dr}\left[r^{n-1}\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t \right] : (\mathbf{F} - \mathbf{F}^t) \right\} dr.
\end{aligned}$$

Note that in the *third* line we have used *integration* by *parts* which together with the *boundary* conditions $\mathbf{F}(a) = \mathbf{F}(b) = \mathbf{0}$ gives

$$\begin{aligned}
0 =& r^{n-1}\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t : (\mathbf{F} - \mathbf{F}^t)\bigg|_a^b \\
=& \int_a^b r^{n-1}\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t : (\dot{\mathbf{F}} - \dot{\mathbf{F}}^t)\, dr + \\
& \int_a^b \frac{d}{dr}\left[r^{n-1}\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t \right] : (\mathbf{F} - \mathbf{F}^t)\, dr.
\end{aligned}$$

On the other hand for the *second* term a direct verification reveals that

$$\begin{aligned}
\mathbf{II} &= \int_a^b \left\{ \frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}}) : (\mathbf{F} - \mathbf{F}^t)\dot{\mathbf{Q}} \right\} r^{n-1}\, dr \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} p(n + r^2|\dot{\mathbf{Q}}\theta|)^{\frac{p-2}{2}} \langle \dot{\mathbf{Q}}\theta, (\mathbf{F} - \mathbf{F}^t)\dot{\mathbf{Q}}\theta \rangle\, r^{n+1}\, dr = 0,
\end{aligned}$$

as a result of the *pointwise* identity $\langle \dot{\mathbf{Q}}\theta, (\mathbf{F}-\mathbf{F}^t)\dot{\mathbf{Q}}\theta\rangle = 0$. Thus, *summarising*, we have that

$$\frac{d}{d\varepsilon}\mathbb{E}_p[\mathbf{Q}_\varepsilon]\bigg|_{\varepsilon=0} = \int_a^b \left\{ -\frac{d}{dr}\left[r^{n-1}\frac{\partial \mathbf{E}}{\partial \xi}(r,\dot{\mathbf{Q}})\mathbf{Q}^t \right] : (\mathbf{F}-\mathbf{F}^t) \right\} dr = 0.$$

As this is true for every $\mathbf{F} \in \mathbf{C}_0^\infty([a,b], \mathbb{M}_{n\times n})$ it follows that the *skew*-symmetric part of the *tensor* field in the brackets in the equation above is zero. This gives the required conclusion. $\square$

Proposition 3.2.3. *The Euler-Lagrange equation associated with $\mathbb{E}_p$ over $\mathcal{E}_p$ can be alternatively expressed as*

$$\int_a^b \int_{\mathbb{S}^{n-1}} \left\langle \left\{ \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) \right\}\theta, (\mathbf{F}-\mathbf{F}^t)\theta \right\rangle \, d\mathcal{H}^{n-1}(\theta) dr = 0,$$

for all $\mathbf{F} \in \mathbf{C}_0^\infty(]a,b[, \mathbb{M}_{n\times n})$ where $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ and $\mathbf{s} = (n + r^2|\mathbf{A}\theta|)^{\frac{p-2}{2}}$.

Proof. Referring to the proof of Proposition 3.2.2 and making the substitutions described above for $\mathbf{A}$ and $\mathbf{s}$ we can write

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\mathbb{E}_p[\mathbf{Q}_\varepsilon]\bigg|_{\varepsilon=0} =: \mathbf{I} \\
&= \int_a^b \left\{ \frac{\partial \mathbf{E}}{\partial \xi}(r,\dot{\mathbf{Q}}) : (\dot{\mathbf{F}}-\dot{\mathbf{F}}^t)\mathbf{Q} \right\} r^{n-1}\, dr \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} p\langle r^{n+1}\mathbf{s}\mathbf{A}\theta, (\dot{\mathbf{F}}-\dot{\mathbf{F}}^t)\theta\rangle \, d\mathcal{H}^{n-1}(\theta) dr \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} -p\left\langle \left\{ \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) \right\}\theta, (\mathbf{F}-\mathbf{F}^t)\theta \right\rangle \, d\mathcal{H}^{n-1}(\theta) dr,
\end{aligned}$$

which is the required conclusion. $\square$

Any twist *loop* forming a solution to the Euler-Lagrange equation associated with $\mathbb{E}_p$ over $\mathcal{E}_p$ (as described in the above proposition) will be referred to as a *p-stationary* loop.

Remark 3.2.4. In view of Proposition 3.2.3 a *sufficient* condition for an admissible loop $\mathbf{Q} \in \mathcal{E}_p$ to be *p-stationary* is the *stronger* condition

$$\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) = 0. \tag{3.9}$$

Interestingly for $p = 2$ the latter is *equivalent* to the Euler-Lagrange equation described in Proposition 3.2.3 (*see* [60]). However, in general, i.e., for $p \neq 2$, this need not be the case as in the *original* Euler-Lagrange equation the function $\mathbf{s}$ depends on both r and θ. In fact, *if*, $\mathbf{s}$ were to be *independent* of θ then the Euler-Lagrange equation described in Proposition 3.2.3 could be easily shown to be *equivalent* to (3.9).

3.3 Minimizing p-stationary loops

Consider as in the previous section for $p \in [1, \infty[$ the energy functional

$$\mathbb{E}_p[\mathbf{Q}] = \int_a^b \mathbf{E}(r, \dot{\mathbf{Q}}) r^{n-1} \, dr,$$

with the *integrand*

$$\mathbf{E}(r, \xi) = \int_{\mathbb{S}^{n-1}} (n + r^2 |\xi\theta|^2)^{\frac{p}{2}} \, d\mathcal{H}^{n-1}(\theta),$$

over the space of *admissible* loops

$$\mathcal{E}_p := \left\{ \mathbf{Q} = \mathbf{Q}(r) : \begin{array}{l} \mathbf{Q} \in W^{1,p}([a,b], \mathbf{SO}(n)), \\ \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n. \end{array} \right\}$$

According to an elementary version of *Sobolev* embedding theorem any $\mathbf{Q} \in \mathcal{E}_p$ has a *continuous* representative (again denoted $\mathbf{Q}$). Thus each such $\mathbf{Q}$ represents an element of the fundamental group $\pi_1[\mathbf{SO}(n)]$ which is denoted by $]\mathbf{Q}[$. As is well-known (*see*, e.g., Bredon [16])

$$\pi_1[\mathbf{SO}(n)] \cong \begin{cases} \mathbb{Z} & \text{when } n = 2, \\ \mathbb{Z}_2 & \text{when } n \geq 3, \end{cases}$$

and so these facts combined enables one to introduce the following *partitioning* of the loop space $\mathcal{E}_p$.
[**1**] ($n = 2$) for each $m \in \mathbb{Z}$ put

$$\mathfrak{c}_m[\mathcal{E}_p] := \Big\{ \mathbf{Q} \in \mathcal{E}_p :]\mathbf{Q}[= m \Big\}.$$

As a result the latter are *pairwise* disjoint and that

$$\mathcal{E}_p = \bigcup_{m \in \mathbb{Z}} \mathfrak{c}_m[\mathcal{E}_p].$$

[**2**] ($n \geq 3$) for each $\alpha \in \mathbb{Z}_2 = \{0, 1\}$ put

$$\mathfrak{c}_\alpha[\mathcal{E}_p] := \Big\{ \mathbf{Q} \in \mathcal{E}_p :]\mathbf{Q}[= \alpha \Big\}.$$

As a result, again, the latter are *pairwise* disjoint and that

$$\mathcal{E}_p = \bigcup_{\alpha \in \mathbb{Z}_2} \mathfrak{c}_\alpha[\mathcal{E}_p].$$

When $p > 1$ an application of the direct methods of the *calculus of variations* to the energy functional $\mathbb{E}_p$ together with the observation that the homotopy classes $\mathfrak{c}_\star[\mathcal{E}_p] \subset \mathcal{E}_p$ are *sequentially* weakly closed gives the existence of [multiple] *minimizing* p-stationary loops. Note that the *sequential* weak closedness of the homotopy classes $\mathfrak{c}_\star[\mathcal{E}_p]$ is a result of $\mathbf{SO}(n)$ having a *tubular* neighbourhood that projects back onto itself and this in turn follows from $\mathbf{SO}(n)$ being a smooth *compact* manifold. The only missing ingredient in this regard is the following statement implying the *coercivity* of $\mathbb{E}_p$ over $\mathcal{E}_p$.

Proposition 3.3.1. *Let $p \in [1, \infty[$. Then there exists $c = c(n, p) > 0$ such that*

$$\fint_{\mathbb{S}^{n-1}} |\mathbf{F}\theta|^p \, d\mathcal{H}^{n-1}(\theta) \geq c|\mathbf{F}|^p,$$

for every $\mathbf{F} \in \mathbb{M}_{n\times n}$.

Proof. Fix $\mathbf{F} \in \mathbb{M}_{n\times n}$. Then the non-negative *symmetric* matrix $\mathbf{F}^t\mathbf{F}$ is orthogonally *diagonalisable*, that is, $\mathbf{F}^t\mathbf{F} = \mathbf{P}^t\mathbf{D}\mathbf{P}$ where $\mathbf{D} = diag(\lambda_1[\mathbf{F}^t\mathbf{F}], \dots, \lambda_n[\mathbf{F}^t\mathbf{F}])$ and $\mathbf{P} \in \mathbf{O}(n)$. As a result for $\theta \in \mathbb{S}^{n-1}$ we can write

$$|\mathbf{F}\theta| = |\langle \mathbf{F}\theta, \mathbf{F}\theta\rangle|^{\frac{1}{2}} = |\langle \mathbf{F}^t\mathbf{F}\theta, \theta\rangle|^{\frac{1}{2}} = |\langle \mathbf{P}^t\mathbf{D}\mathbf{P}\theta, \theta\rangle|^{\frac{1}{2}} = |\langle \mathbf{D}\mathbf{P}\theta, \mathbf{P}\theta\rangle|^{\frac{1}{2}}.$$

Setting $w := \mathbf{P}\theta$ and noting that $\mathbf{O}(n)$ acts as the group of *isometries* on $\mathbb{S}^{n-1}$, an application of Jensen's inequality followed by Hölder's inequality on *finite* sequences (*see*, e.g., [53] or [25]) gives

$$\begin{aligned}
\left\{\fint_{\mathbb{S}^{n-1}} |\mathbf{F}\theta|^p \, d\mathcal{H}^{n-1}(\theta)\right\}^{\frac{1}{p}} &\geq \fint_{\mathbb{S}^{n-1}} |\mathbf{F}\theta| \, d\mathcal{H}^{n-1}(\theta) \\
&\geq \fint_{\mathbb{S}^{n-1}} \left\{\sum_{j=1}^n \lambda_j[\mathbf{F}^t\mathbf{F}] w_j^2(\theta)\right\}^{\frac{1}{2}} d\mathcal{H}^{n-1}(\theta) \\
&\geq \frac{1}{\sqrt{n}} \sum_{j=1}^n \lambda_j^{\frac{1}{2}}[\mathbf{F}^t\mathbf{F}] \fint_{\mathbb{S}^{n-1}} |w_j(\theta)| \, d\mathcal{H}^{n-1}(\theta) \\
&\geq \frac{\alpha_n}{\sqrt{n}} \left\{\sum_{j=1}^n \lambda_j[\mathbf{F}^t\mathbf{F}]\right\}^{\frac{1}{2}} = \frac{\alpha_n}{\sqrt{n}} |\mathbf{F}|.
\end{aligned}$$

Hence the conclusion follows with the choice of

$$c = \alpha_n^p n^{1-\frac{p}{2}} \omega_n = \min_{\substack{1 \leq j \leq n \\ \theta_j \neq 0}} \left\{\fint_{\mathbb{S}^{n-1}} |\theta_j| \, d\mathcal{H}^{n-1}(\theta)\right\}^p n^{1-\frac{p}{2}} \omega_n > 0.$$

□

Proposition 3.3.2. *Let* $p \in [1, \infty[$. *Then there exists* $d = d(n, p, \Omega) > 0$ *such that*

$$\mathbb{E}_p[\mathbf{Q}] \geq d \|\mathbf{Q}\|^p_{W^{1,p}},$$

for all $\mathbf{Q} \in \mathcal{E}_p$.

Proof. In view of Proposition 3.3.1 it is enough to note that for $\mathbf{Q} \in \mathcal{E}_p$ we can write

$$\begin{aligned}\mathbb{E}_p[\mathbf{Q}] &= \int_a^b \int_{\mathbb{S}^{n-1}} (n + r^2|\dot{\mathbf{Q}}\theta|^2)^{\frac{p}{2}} r^{n-1} \, d\mathcal{H}^{n-1}(\theta) dr \\ &\geq \int_a^b \int_{\mathbb{S}^{n-1}} r^{p+n-1} |\dot{\mathbf{Q}}\theta|^p \, d\mathcal{H}^{n-1}(\theta) dr \\ &\geq c \int_a^b r^{p+n-1} |\dot{\mathbf{Q}}|^p \, dr,\end{aligned}$$

and so the conclusion follows by an application of Poincaré inequality. □

Theorem 3.3.3. *Let* $p \in]1, \infty[$. *Then the following hold.*

[**1**] $(n = 2)$ *for each* $m \in \mathbb{Z}$ *there exists* $\mathbf{Q}_m \in \mathfrak{c}_m[\mathcal{E}_p]$ *such that*

$$\mathbb{E}_p[\mathbf{Q}_m] = \inf_{\mathfrak{c}_m[\mathcal{E}_p]} \mathbb{E}_p,$$

[**2**] $(n \geq 3)$ *for each* $\alpha \in \mathbb{Z}_2$ *there exists* $\mathbf{Q}_\alpha \in \mathfrak{c}_\alpha[\mathcal{E}_p]$ *such that*

$$\mathbb{E}_p[\mathbf{Q}_\alpha] = \inf_{\mathfrak{c}_\alpha[\mathcal{E}_p]} \mathbb{E}_p.$$

In either case the resulting minimizers satisfy the corresponding Euler-Lagrange equations (3.8).

We return to the question of existence of *multiple* p-stationary loops having *specific* relevance to the *original* energy functional $\mathbb{F}_p$ over the space $\mathcal{A}_p$ towards the end of the paper. Before this, however, we pause to discuss in detail the implications that the *original* Euler-Lagrange equations [*see* Definition 3.4.1 below] will exert upon the *twist* loop associated with a *generalised* twist.

3.4 Generalised twists as classical solutions

The aim of this section is to give a complete *characterisation* of *all* those p-stationary loops $\mathbf{Q} \in \mathcal{E}_p$ whose resulting *generalised* twist

$$u = \mathbf{Q}(r)x,$$

furnishes a solution to the Euler-Lagrange equations associated with the energy functional $\mathbb{F}_p$ over the space $\mathcal{A}_p$. To this end we begin by *clarifying* the notion of a [*classical*] solution.

Definition 3.4.1. (**Classical solution**)

A pair $(u, \mathfrak{p})$ is said to be a *classical* solution to the Euler-Lagrange equations associated with the energy functional (3.1) and subject to the constraint (1.2) if and only if

[**1**] $u \in \mathbf{C}^2(\Omega, \mathbb{R}^n) \cap \mathbf{C}(\bar{\Omega}, \mathbb{R}^n)$,

[**2**] $\mathfrak{p} \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\bar{\Omega})$, and

[**3**] $(u, \mathfrak{p})$ satisfy the *system* of equations [1]

$$\begin{cases} [\text{cof } \nabla u(x)]^{-1} \Delta_p u(x) = \nabla \mathfrak{p}(x) & x \in \Omega, \\ \det \nabla u(x) = 1 & x \in \Omega, \\ u(x) = x & x \in \partial\Omega. \end{cases}$$

In view of Proposition 3.1.4 the task outlined at the start of this section amounts to verifying that under what *additional* conditions would the *vector* field described by the expression on the *right* in (3.7) be a gradient. The answer to this question is given by the following *two* theorems.

Theorem 3.4.2. *Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ and consider the vector field $\mathbf{v} \in \mathbf{C}^1(\Omega, \mathbb{R}^n)$ defined in spherical coordinates through*

$$\mathbf{v} = \left\{ r\mathbf{s}\mathbf{A}^2 - r^2\mathbf{s}\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta \rangle \mathbf{I}_n + \frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) + \frac{1}{r^{n-1}}\frac{d}{dr}(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2)\mathbf{I}_n \right\}\theta,$$

where $r \in]a, b[$, $\theta \in \mathbb{S}^{n-1}$, $\mathbf{A} = \mathbf{A}(r) \in \mathbf{C}^1(]a, b[, \mathbb{M}_{n\times n})$ is skew-symmetric and

$$\begin{aligned} \mathbf{s} &= \mathbf{s}(r, \theta) \\ &=: (n + r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}}, \end{aligned} \tag{3.10}$$

with $p \in [1, \infty[$. Then the following are equivalent.

[**1**] $\mathbf{v}$ *is a gradient,*

[**2**] $\mathbf{A}^2 = -\sigma \mathbf{I}_n$ *for some $\sigma \in \mathbf{C}^1]a, b[$ with $\sigma \geq 0$ and*

$$\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) = 0. \tag{3.11}$$

[1]Note that $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$.

Proof. [**2**] $\Longrightarrow$ [**1**]

Assuming $\mathbf{A}$ to be *skew*-symmetric and $\mathbf{A}^2 = -\sigma \mathbf{I}_n$ it follows that

$$\begin{aligned}\mathbf{s} &= (n + r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}} \\ &= (n - r^2\langle \mathbf{A}^2\theta, \theta\rangle)^{\frac{p-2}{2}} \\ &= (n + \sigma r^2)^{\frac{p-2}{2}},\end{aligned}$$

and so in particular $\mathbf{s} = \mathbf{s}(r)$. Now referring to (3.11) we can write

$$\begin{aligned}0 &= \frac{1}{r^n}\langle \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A})\theta, \mathbf{A}\theta\rangle \\ &= (n+1)\mathbf{s}|\mathbf{A}\theta|^2 + r\mathbf{s}_r|\mathbf{A}\theta|^2 + r\mathbf{s}\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle \\ &= \frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2) - r\mathbf{s}\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle. \end{aligned} \tag{3.12}$$

As a result the vector field $\mathbf{v}$ can be simplified and hence *re-written* in the form

$$\mathbf{v} = r\mathbf{s}\mathbf{A}^2\theta = -(n + \sigma r^2)^{\frac{p-2}{2}}\sigma\theta.$$

Denoting now by F a suitable *primitive* of $f(r) := -(n + \sigma r^2)^{\frac{p-2}{2}}\sigma$ it is evident that

$$\mathbf{v} = \nabla F,$$

and so $\mathbf{v}$ is a gradient. This gives [**1**].

[**1**] $\Longrightarrow$ [**2**]

For the sake of clarity and convenience we break this part into *two* steps. In the *first* step we establish (3.11) and in the *second* one the particular *diagonal* form of $\mathbf{A}^2$. Thus it is important to note that in the first *two* steps the function $\mathbf{s}$ depends on both r and θ!

Step 1. [Justification of (3.11)] We begin by *extracting* a gradient out of $\mathbf{v}$ and hence *rewriting* it in the form

$$\mathbf{v} = \nabla\mathbf{t} + \left\{\frac{1}{r^n}\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) + \frac{1}{r^{n-1}}\frac{d}{dr}(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2)\mathbf{I}_n\right\}\theta,$$

where $\mathbf{t} = -p^{-1}(n + r^2|\mathbf{A}\theta|^2)^{\frac{p}{2}}$.

To the vector field $\mathbf{v} = (v_1, \dots, v_n)$ we now assign the differential 1-*form* $\omega = v_1 dx_1 + \dots + v_n dx_n$. Then in view of $\mathbf{v}$ being a *gradient*, for any *closed* path $\gamma \in \mathbf{C}^1([0, 2\pi], \mathbb{S}^{n-1})$ it *must* be that

$$\begin{aligned}
0 =& \int_{r\gamma} \omega \\
=& \int_0^{2\pi} \langle \mathbf{v}(r\gamma(t)), r\gamma'(t) \rangle \, dt \\
=& \frac{1}{r^n} \int_0^{2\pi} \langle \frac{d}{dr} \left[r^{n+1} \mathbf{s}(r, \gamma(t)) \mathbf{A} \right] \gamma(t), r\gamma'(t) \rangle \, dt + \\
& \frac{1}{r^{n-1}} \int_0^{2\pi} \langle \frac{d}{dr} \left[r^{n+1} \mathbf{s}(r, \gamma(t)) |\mathbf{A}\gamma(t)|^2 \right] \gamma(t), r\gamma'(t) \rangle \, dt \\
=& \frac{1}{r^n} \int_0^{2\pi} \langle \frac{d}{dr} \left[r^{n+1} \mathbf{s}(r, \gamma(t)) \mathbf{A} \right] \gamma(t), r\gamma'(t) \rangle \, dt,
\end{aligned} \tag{3.13}$$

where in concluding the *last* line we have used the *pointwise* identity $\langle \gamma, \gamma' \rangle = 0$ which holds as a result of γ taking values on $\mathbb{S}^{n-1}$ and consequently implying that

$$\begin{aligned}
0 =& \int_0^{2\pi} \langle \frac{d}{dr} \left[r^{n+1} \mathbf{s}(r, \gamma(t)) |\mathbf{A}\gamma(t)|^2 \right] \gamma(t), r\gamma'(t) \rangle \, dt \\
=& \int_0^{2\pi} \frac{d}{dr} \left[r^{n+1} \mathbf{s}(r, \gamma(t)) |\mathbf{A}\gamma(t)|^2 \right] r \langle \gamma(t), \gamma'(t) \rangle \, dt.
\end{aligned}$$

Anticipating on (3.11) we *first* note that in view of $\mathbf{A}$ being *skew*-symmetric it can be *orthogonally* diagonalised, i.e., [2]

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^t, \tag{3.14}$$

where $\mathbf{P} = \mathbf{P}(r) \in \mathbf{SO}(n)$ and $\mathbf{D} = \mathbf{D}(r) \in \mathbb{M}_{n \times n}$ is in *special* block diagonal form, i.e.,

[**1**] $(n = 2k)$

$$\mathbf{D} = diag(d_1 \mathbf{J}_2, d_2 \mathbf{J}_2, \dots, d_k \mathbf{J}_2),$$

[**2**] $(n = 2k + 1)$

$$\mathbf{D} = diag(d_1 \mathbf{J}_2, d_2 \mathbf{J}_2, \dots, d_k \mathbf{J}_2, 0),$$

with $\{\pm d_1 i, \pm d_2 i, \dots, \pm d_k i\}$ or $\{\pm d_1 i, \pm d_2 i, \dots, \pm d_k i, 0\}$ denoting the *eigen*-values of the *skew-symmetric* matrix $\mathbf{A}$ [as well as $\mathbf{D}$] respectively. We emphasize that *nowhere* in this proof have we assumed *continuity* or *differentiability* of $\mathbf{P} = \mathbf{P}(r)$ or $\mathbf{D} = \mathbf{D}(r)$ with respect to r. These in general need not even be true! [*see*, e.g., [43], Chapter five.]

[2] At this stage the reader is encouraged to consult the *Appendix* at the end of the thesis where some *notation* as well as basic properties related to the matrix *exponential* as a mapping between the space of *skew-symmetric* matrices and the *special* orthogonal group is discussed.

With the aid of (3.14) and for the sake of convenience we now introduce the *skew-symmetric* matrix

$$\mathbf{F} = \mathbf{F}(r,\theta) := \mathbf{P}^t \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A})\mathbf{P}. \tag{3.15}$$

Then a straight-forward *differentiation* shows that

$$\begin{aligned}
\mathbf{F} &= \mathbf{P}^t \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A})\mathbf{P} \\
&= \mathbf{P}^t \left\{ r^n[(n+1)\mathbf{s} + r\mathbf{s}_r]\mathbf{A} + r^{n+1}\mathbf{s}\dot{\mathbf{A}} \right\}\mathbf{P} \\
&= \mathbf{P}^t \left\{ r^n[(n+1)\mathbf{s} + r\mathbf{s}_r]\mathbf{P}\mathbf{D}\mathbf{P}^t + r^{n+1}\mathbf{s}\dot{\mathbf{A}} \right\}\mathbf{P} \\
&= r^n[(n+1)\mathbf{s} + r\mathbf{s}_r]\mathbf{D} + r^{n+1}\mathbf{s}\mathbf{P}^t\dot{\mathbf{A}}\mathbf{P}.
\end{aligned} \tag{3.16}$$

Evidently establishing (3.11) is *equivalent* to showing that

$$\mathbf{F}(r,\theta) = 0, \tag{3.17}$$

for *all* $r \in]a,b[$ and *all* $\theta \in \mathbb{S}^{n-1}$.

On the other hand for each *fixed* $r \in]a,b[$ setting $\omega := \mathbf{P}^t\gamma$ [also a *closed* path in $\mathbf{C}^1([0,2\pi],\mathbb{S}^{n-1})$] in (3.13) we have that expressed as

$$\begin{aligned}
0 &= \int_0^{2\pi} \langle \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A})\gamma, \gamma' \rangle \, dt \\
&= \int_0^{2\pi} \langle \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A})\mathbf{P}\omega, \mathbf{P}\omega' \rangle \, dt \\
&= \int_0^{2\pi} \langle \mathbf{P}^t\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A})\mathbf{P}\omega, \omega' \rangle \, dt \\
&= \int_0^{2\pi} \langle \mathbf{F}\omega, \omega' \rangle \, dt,
\end{aligned}$$

where in the above $\mathbf{s} = \mathbf{s}(r,\mathbf{P}\omega)$ and $\mathbf{F} = \mathbf{F}(r,\mathbf{P}\omega)$. Thus the *necessary* condition (3.13) can be equivalently expressed as

$$\int_0^{2\pi} \langle \mathbf{F}(r,\mathbf{P}\omega)\omega, \omega' \rangle \, dt = 0, \tag{3.18}$$

for every *closed* path $\omega \in \mathbf{C}^1([0,2\pi],\mathbb{S}^{n-1})$.

With this introduction the conclusion in **step 1** now amounts to proving the implication (3.18) $\Longrightarrow$ (3.17). This will be established below in a *componentwise* fashion. Note that in view of the *skew-symmetry* of $\mathbf{F}$ it suffices to justify the latter in the form $\mathbf{F}_{pq}(r,\theta) = 0$ *only* when $1 \le p < q \le n$.

Indeed consider a *parameterised* family of *closed* paths $\rho \in \mathbf{C}^\infty([0,2\pi],\mathbb{S}^{n-1})$ given by

$$\rho : [0,2\pi] \ni t \mapsto \rho(t) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n, \tag{3.19}$$

with

$$\begin{cases} \rho_1 = \sin t \sin\phi_2 \sin\phi_3 \dots \sin\phi_{n-1}, \\ \rho_2 = \cos t \sin\phi_2 \sin\phi_3 \dots \sin\phi_{n-1}, \\ \rho_3 = \cos\phi_2 \sin\phi_3 \dots \sin\phi_{n-1}, \\ \vdots \\ \rho_{n-1} = \cos\phi_{n-2} \sin\phi_{n-1}, \\ \rho_n = \cos\phi_{n-1}, \end{cases}$$

where $\phi_j \in [0,\pi]$ for all $2 \le j \le n-1$. For *fixed* $1 \le p < q \le n$ we introduce the matrix Γ^{pq} as that obtained by *simultaneously* interchanging the *first* and *p-th* and the *second* and *q-th* rows of $\mathbf{I}_n$, i.e.,

$$\Gamma^{pq} e_j = \begin{cases} e_p & \text{if } j = 1, \\ e_1 & \text{if } j = p, \\ e_q & \text{if } j = 2, \\ e_2 & \text{if } j = q, \\ e_j & \textit{otherwise}, \end{cases}$$

where $\{e_1, e_2, \dots, e_n\}$ denotes the *standard* basis of $\mathbb{R}^n$. In view of $\Gamma^{pq} \in \mathbf{O}(n)$ setting $\omega = \Gamma^{pq}\rho$ it is clear that ω is a *closed* path in $\mathbf{C}^\infty([0,2\pi],\mathbb{S}^{n-1})$.

Claim 1. For any *skew-symmetric* matrix $\mathbf{F} \in \mathbb{M}_{n\times n}$ and $\omega = \Gamma^{pq}\rho$ as above we have that

$$\int_0^{2\pi} \langle \mathbf{F}\omega(t), \omega'(t)\rangle \, dt = 2\pi(\rho_1^2 + \rho_2^2)\mathbf{F}_{pq}.$$

The proof of this claim follows by direct *verification* noting that here

$$\omega'(t) = \Gamma^{pq}\rho'(t) = \Gamma^{pq}(\rho_2, -\rho_1, 0, \dots, 0).$$

We now proceed by substituting ω as described above into (3.18) and then considering the following two *distinct* cases.

[**1**] ($p = 2j-1, q = 2j$ for some $1 \le j \le k = \lfloor n/2 \rfloor$) In this case by utilising the *special* block

diagonal form of $\mathbf{D}$ a straight-forward calculation shows that

$$\begin{aligned}\mathbf{s} &= \mathbf{s}(r, \mathbf{P}\omega(t)) \\ &= (n - r^2\langle \mathbf{D}^2\omega(t), \omega(t)\rangle)^{\frac{p-2}{2}} \\ &= (n - r^2\langle \mathbf{D}^2\Gamma^{pq}\rho(t), \Gamma^{pq}\rho(t)\rangle)^{\frac{p-2}{2}} \\ &= (n + r^2[d_1^2\rho_p^2 + d_1^2\rho_q^2 + \cdots + d_j^2(\rho_1^2 + \rho_2^2) + \cdots])^{\frac{p-2}{2}},\end{aligned}$$

is indeed *independent* of the t *variable* [as $\rho_1^2 + \rho_2^2$ does *not* depend on t]. Hence the same is true of $\mathbf{F}(r, \mathbf{P}\omega)$ and so referring to (3.18) and utilising **claim 1** we can write

$$\begin{aligned}0 &= \int_0^{2\pi} \langle \mathbf{F}(r, \mathbf{P}\omega)\omega, \omega'\rangle \, dt \\ &= \int_0^{2\pi} \langle \mathbf{F}(r, \mathbf{P}\Gamma^{pq}\rho(t))\Gamma^{pq}\rho(t), \Gamma^{pq}\rho'(t)\rangle \, dt \\ &= 2\pi(\rho_1^2 + \rho_2^2)\mathbf{F}_{pq}(r, \mathbf{P}\omega),\end{aligned}$$

which in turn for $\rho_1^2 + \rho_2^2 \neq 0$ gives [3]

$$\mathbf{F}_{pq}(r, \mathbf{P}\omega) = 0. \tag{3.20}$$

Now to get (3.17) for the latter choice of p, q pick $\theta \in \mathbb{S}^{n-1}$ and set $\alpha = [\Gamma^{pq}]^t\mathbf{P}^t\theta$. Then $\alpha \in \mathbb{S}^{n-1}$ and thus can be written in *generalised* spherical coordinates as

$$\begin{cases} \alpha_1 = \sin\phi_1 \sin\phi_2 \sin\phi_3 \ldots \sin\phi_{n-1}, \\ \alpha_2 = \cos\phi_1 \sin\phi_2 \sin\phi_3 \ldots \sin\phi_{n-1}, \\ \alpha_3 = \cos\phi_2 \sin\phi_3 \ldots \sin\phi_{n-1}, \\ \vdots \\ \alpha_{n-1} = \cos\phi_{n-2} \sin\phi_{n-1}, \\ \alpha_n = \cos\phi_{n-1}, \end{cases}$$

where $\phi_1 \in [0, 2\pi]$ and $\phi_j \in [0, \pi]$ for all $2 \le j \le n-1$. Considering now the closed *path* ρ in (3.19) for the latter choice of *parameters* $\phi_2, \phi_3, \ldots, \phi_{n-1}$ a straight-forward calculation gives

$$\begin{aligned}\mathbf{s}(r, \theta) &= (n + r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}} \\ &= (n + r^2|\mathbf{D}\Gamma^{pq}\alpha|^2)^{\frac{p-2}{2}} \\ &= (n + r^2|\mathbf{D}\Gamma^{pq}\rho|^2)^{\frac{p-2}{2}}\end{aligned}$$

[3]Note that $(\rho_1^2 + \rho_2^2) = \prod_{2 \le j \le n-1} \sin^2\phi_j$ and so $\rho_1^2 + \rho_2^2 = 0 \iff \sum_{3 \le j \le n} \rho_j^2 = 1 \iff \phi_j \in \{0, \pi\}$ for some $2 \le j \le n-1$. This set is a copy of $\mathbb{S}^{n-3}$ lying in $\mathbb{S}^{n-1}$.

$$= (n + r^2|\mathbf{AP}\omega|^2)^{\frac{p-2}{2}}$$
$$= \mathbf{s}(r, \mathbf{P}\omega),$$

and so referring to (3.20) for $\rho_1^2 + \rho_2^2 \neq 0$ we obtain

$$\mathbf{F}_{pq}(r, \theta) = \mathbf{F}_{pq}(r, \mathbf{P}\omega) = 0,$$

as required.

[**2**] $(p, q$ not as in [**1**]) Unlike the case with [**1**] here $\mathbf{s}$ depends *explicitly* on the t *variable* [yet in a *specific* manner (*see* below)] whilst $\mathbf{D}_{pq} = 0$ as can be verified by inspecting its block diagonal representation.

Now referring, again, to (3.18) and noting that the *p-th* and *q-th* components of ω' are given by $\omega_p' = \rho_1' = \rho_2$ and $\omega_q' = \rho_2' = -\rho_1$ [with *all* the remaining derivatives vanishing] we can write using $\mathbf{F} = \mathbf{F}(r, \mathbf{P}\omega)$

$$\begin{aligned} 0 &= \int_0^{2\pi} \langle \mathbf{F}\omega, \omega' \rangle \, dt \\ &= \int_0^{2\pi} \left\{ \sum_{j=1}^n \mathbf{F}_{pj}\omega_j\omega_p' + \sum_{j=1}^n \mathbf{F}_{qj}\omega_j\omega_q' \right\} dt \\ &= \int_0^{2\pi} \left\{ (\mathbf{F}_{pq}\rho_2^2 - \mathbf{F}_{qp}\rho_1^2) + \rho_2 \sum_{\substack{j=1 \\ j\neq q}}^n \mathbf{F}_{pj}\omega_j - \rho_1 \sum_{\substack{j=1 \\ j\neq p}}^n \mathbf{F}_{qj}\omega_j \right\} dt \\ &= \mathbf{I} + \mathbf{II} - \mathbf{III}. \end{aligned} \tag{3.21}$$

In order to evaluate the above terms we *first* observe that here $\mathbf{s}$ takes the form

$$\begin{aligned} \mathbf{s} &= \mathbf{s}(r, \mathbf{P}\omega(t)) \\ &= (n - r^2\langle \mathbf{D}^2\omega(t), \omega(t)\rangle)^{\frac{p-2}{2}} \\ &= (n - r^2\langle \mathbf{D}^2\Gamma^{pq}\rho(t), \Gamma^{pq}\rho(t)\rangle)^{\frac{p-2}{2}} \\ &= (n + r^2[d_1^2\rho_p^2 + d_2^2\rho_q^2 + \cdots + d_\xi^2\rho_1^2 + \cdots + d_\zeta^2\rho_2^2 + \cdots])^{\frac{p-2}{2}} \\ &=: \mathfrak{s}(\sin^2 t, \cos^2 t). \end{aligned} \tag{3.22}$$

Returning to (3.21) we have that

$$\mathbf{II} = \int_0^{2\pi} \rho_2 \sum_{\substack{j=1 \\ j\neq q}}^n \mathbf{F}_{pj}\omega_j \, dt$$

$$
\begin{aligned}
&= \int_0^{2\pi} \rho_2 \sum_{\substack{j=1\\ j\neq q}}^{n} \Big[\mathbf{P}^t \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A})\mathbf{P}\Big]_{pj} \omega_j \, dt \\
&= \sum_{\substack{j=1\\ j\neq q}}^{n} \Big[\mathbf{P}^t \frac{d}{dr}\Big(r^{n+1}\Big\{\int_0^{2\pi} \rho_2 \mathbf{s}\, dt\Big\}\mathbf{A}\Big)\mathbf{P}\Big]_{pj} \omega_j,
\end{aligned}
$$

and in a similar way

$$
\begin{aligned}
\mathbf{III} &= \int_0^{2\pi} \rho_1 \sum_{\substack{j=1\\ j\neq p}}^{n} \mathbf{F}_{qj}\omega_j \, dt \\
&= \int_0^{2\pi} \rho_1 \sum_{\substack{j=1\\ j\neq p}}^{n} \Big[\mathbf{P}^t \frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A})\mathbf{P}\Big]_{qj} \omega_j \, dt \\
&= \sum_{\substack{j=1\\ j\neq p}}^{n} \Big[\mathbf{P}^t \frac{d}{dr}\Big(r^{n+1}\Big\{\int_0^{2\pi} \rho_1 \mathbf{s}\, dt\Big\}\mathbf{A}\Big)\mathbf{P}\Big]_{qj} \omega_j,
\end{aligned}
$$

where in concluding the *last* line in both equalities we have used the fact that the only components of ω depending explicitly on the t *variable* are $\omega_p = \rho_1$ and $\omega_q = \rho_2$ where in each case *one* is excluded from the *summation* sign and the other has a *zero* coefficient in view of the *skew-symmetry* of the matrix preceding it.

However in view of the specific manner in which $\mathbf{s}$ depends on t [*see* (3.22)] it follows that both integrals *vanish* and so as a result $\mathbf{II} = \mathbf{III} = 0$. [4] Hence returning to (3.21) and utilising the *skew*-symmetry on $\mathbf{F}$ and (3.16) we can write

$$
\begin{aligned}
\mathbf{I} &= \int_0^{2\pi} (\mathbf{F}_{pq}\rho_2^2 - \mathbf{F}_{qp}\rho_1^2)\, dt \\
&= \int_0^{2\pi} (\rho_1^2 + \rho_2^2)\mathbf{F}_{pq}\, dt \\
&= \int_0^{2\pi} r^{n+1}(\rho_1^2 + \rho_2^2)\mathbf{s}[\mathbf{P}^t\dot{\mathbf{A}}\mathbf{P}]_{pq}\, dt \\
&= r^{n+1}(\rho_1^2 + \rho_2^2)\Big\{\int_0^{2\pi} \mathbf{s}\, dt\Big\}[\mathbf{P}^t\dot{\mathbf{A}}\mathbf{P}]_{pq} = 0.
\end{aligned}
$$

Thus as $\mathbf{s} > 0$ for $\rho_1^2 + \rho_2^2 \neq 0$ it follows that $[\mathbf{P}^t\dot{\mathbf{A}}\mathbf{P}]_{pq} = 0$. Since for the latter range of p, q we have that $\mathbf{D}_{pq} = 0$ referring to (3.16) it immediately follows that $\mathbf{F}_{pq} = 0$.

[4] It can be easily shown that as a result of periodicity the following indentities hold:

$$
\begin{aligned}
&\int_0^{2\pi} \mathfrak{s}(\sin^2 t, \cos^2 t)\sin t\, dt = 0, \\
&\int_0^{2\pi} \mathfrak{s}(\sin^2 t, \cos^2 t)\cos t\, dt = 0.
\end{aligned}
$$

Hence *summarising* we have shown that in both cases [**1**] and [**2**] for *fixed* $r \in]a,b[$ we have $\mathbf{F}_{pq}(r,\cdot) = 0$ outside a copy of $\mathbb{S}^{n-3}$. By *continuity* of $\mathbf{F}_{pq}(r,\cdot)$ on $\mathbb{S}^{n-1}$ this gives (3.17) and as a result (3.11). The proof of **step 1** is therefore complete.

Step 2. $[\mathbf{A}^2 = -\sigma \mathbf{I}_n]$ Here we establish the *remaining* part of [**2**] namely that $\mathbf{A}^2 = -\sigma \mathbf{I}_n$ for some $\sigma \in \mathbf{C}^1]a,b[$ with $\sigma \geq 0$. To this end, we *first* observe that by utilising (3.11) the vector field $\mathbf{v}$ can be considerably simplified and *re-written* in the form [as in (3.12)]

$$\mathbf{v} = r\mathbf{s}\mathbf{A}^2\theta.$$

Now for $\mathbf{v} = (v_1, v_2, \dots, v_n)$ to be a *gradient* it is necessary that the differential 1-*form* $\omega = v_1 dx_1 + \dots + v_n dx_n$ be *closed*. In other words $d\omega = 0$ which in turn amounts to

$$\frac{\partial v_q}{\partial x_p} - \frac{\partial v_p}{\partial x_q} = 0,$$

for all $1 \leq p, q \leq n$. Setting $\mathbf{F} = \mathbf{A}^2$ we have that

$$\frac{\partial v_q}{\partial x_p} = r\frac{\partial \mathbf{s}}{\partial x_p}[\mathbf{F}\theta]_q + r\mathbf{s}[\dot{\mathbf{F}}\theta]_q\theta_p + \mathbf{s}\mathbf{F}_{qp},$$

and in a similar way

$$\frac{\partial v_p}{\partial x_q} = r\frac{\partial \mathbf{s}}{\partial x_q}[\mathbf{F}\theta]_p + r\mathbf{s}[\dot{\mathbf{F}}\theta]_p\theta_q + \mathbf{s}\mathbf{F}_{pq}.$$

Thus in view of the *symmetry* of $\mathbf{F}$ for the latter range of p, q we have that

$$\begin{aligned} 0 =& \frac{\partial v_q}{\partial x_p} - \frac{\partial v_p}{\partial x_q} \\ =& r\frac{\partial \mathbf{s}}{\partial x_p}[\mathbf{F}\theta]_q - r\frac{\partial \mathbf{s}}{\partial x_q}[\mathbf{F}\theta]_p + r\mathbf{s}\Big\{[\dot{\mathbf{F}}\theta \otimes \theta]_{qp} - [\dot{\mathbf{F}}\theta \otimes \theta]_{pq}\Big\}. \end{aligned}$$

Alternatively using *tensor* notation this can be *simplified* in the form

$$\begin{aligned} 0 =& \nabla\mathbf{s} \otimes \mathbf{F}\theta - \mathbf{F}\theta \otimes \nabla\mathbf{s} + \\ & \mathbf{s}(\theta \otimes \dot{\mathbf{F}}\theta - \dot{\mathbf{F}}\theta \otimes \theta) \\ =& \frac{1}{2}\beta r^2 \langle \dot{\mathbf{F}}\theta, \theta\rangle (\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta) + \\ & \mathbf{s}(\theta \otimes \dot{\mathbf{F}}\theta - \dot{\mathbf{F}}\theta \otimes \theta), \end{aligned} \tag{3.23}$$

where in concluding the *second* identity we have used

$$
\begin{aligned}
\nabla \mathbf{s} &= \nabla (n + r^2 |\mathbf{A}\theta|^2)^{\frac{p-2}{2}} \\
&= \nabla (n - r^2 \langle \mathbf{F}\theta, \theta \rangle)^{\frac{p-2}{2}} \\
&= -\beta \left[\frac{1}{2} r^2 \langle \dot{\mathbf{F}}\theta, \theta \rangle \mathbf{I}_n + r\mathbf{F} \right] \theta,
\end{aligned}
$$

with $\beta = \beta(r, \theta, p) := (p-2)(n - r^2 \langle \mathbf{F}\theta, \theta \rangle)^{\frac{p-4}{2}}$. Next a straight-forward calculation using (3.11) gives

$$
\dot{\mathbf{F}} = -2 \Big(\frac{n+1}{r} + \frac{\mathbf{s}_r}{\mathbf{s}} \Big) \mathbf{F}. \tag{3.24}
$$

Therefore substituting this into (3.23) results in

$$
\begin{aligned}
0 =& \frac{1}{2} \beta r^2 \langle \dot{\mathbf{F}}\theta, \theta \rangle (\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta) - \\
& \mathbf{s}(\dot{\mathbf{F}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{F}}\theta) \\
=& \left\{ 2 \Big(\frac{n+1}{r} + \frac{\mathbf{s}_r}{\mathbf{s}} \Big) \Big(\mathbf{s} - \frac{1}{2} \beta r^2 \langle \mathbf{F}\theta, \theta \rangle \Big) \right\} \Big(\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta \Big) \\
=& \gamma \times (\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta),
\end{aligned} \tag{3.25}
$$

where for the sake of convenience we have introduced

$$
\begin{aligned}
\gamma &= \gamma(r, \theta, p) \\
&=: 2 \Big(\frac{n+1}{r} + \frac{\mathbf{s}_r}{\mathbf{s}} \Big) \Big(\mathbf{s} - \frac{1}{2} \beta r^2 \langle \mathbf{F}\theta, \theta \rangle \Big).
\end{aligned} \tag{3.26}
$$

Claim 2. Let $p \in [1, \infty[$. Then $\gamma = \gamma(r, \theta, p) > 0$ for *all* $r \in]a, b[$ and $\theta \in \mathbb{S}^{n-1}$.

The proof of this claim follows by *direct* verification. Indeed here a straight-forward *differentiation* gives

$$
\begin{aligned}
\mathbf{s}_r = \frac{\partial \mathbf{s}}{\partial r} &= \frac{\partial}{\partial r} (n + r^2 |\mathbf{A}\theta|^2)^{\frac{p-2}{2}} \\
&= \frac{\partial}{\partial r} (n - r^2 \langle \mathbf{F}\theta, \theta \rangle)^{\frac{p-2}{2}} \\
&= -\beta \left[r \langle \mathbf{F}\theta, \theta \rangle + \frac{1}{2} r^2 \langle \dot{\mathbf{F}}\theta, \theta \rangle \right].
\end{aligned}
$$

Now *eliminating* the term $\langle \dot{\mathbf{F}}\theta, \theta\rangle$ in the above expression with the aid of (3.24) results in

$$\mathbf{s}_r = \frac{nr\beta\mathbf{s}\langle \mathbf{F}\theta, \theta\rangle}{\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle}.$$

(*See* below for a justification that $\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle \neq 0$.) Hence referring to (3.26) we can write

$$\begin{aligned}\gamma &= 2\Big(\frac{n+1}{r} + \frac{\mathbf{s}_r}{\mathbf{s}}\Big)\Big(\mathbf{s} - \frac{1}{2}\beta r^2\langle \mathbf{F}\theta, \theta\rangle\Big)\\ &= \frac{(n+1)\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle}{r(\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle)}\Big(2\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle\Big)\\ &=: \frac{\mathbf{I}}{\mathbf{II}} \times \mathbf{III}.\end{aligned}$$

We now proceed by evaluating each term *separately*. Indeed with regards to the *first* term we have that

$$\begin{aligned}\mathbf{I} &= (n+1)\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle\\ &= (n - r^2\langle \mathbf{F}\theta, \theta\rangle)^{\frac{p-4}{2}}\Big[n(n+1) - (n+p-1)r^2\langle \mathbf{F}\theta, \theta\rangle\Big],\end{aligned}$$

and in a *similar* way

$$\begin{aligned}\mathbf{II} &= r(\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle)\\ &= r(n - r^2\langle \mathbf{F}\theta, \theta\rangle)^{\frac{p-4}{2}}\Big[n - (p-1)r^2\langle \mathbf{F}\theta, \theta\rangle\Big],\end{aligned}$$

and

$$\begin{aligned}\mathbf{III} &= (2\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle)\\ &= (n - r^2\langle \mathbf{F}\theta, \theta\rangle)^{\frac{p-4}{2}}\Big[2n - pr^2\langle \mathbf{F}\theta, \theta\rangle\Big].\end{aligned}$$

Now in view of $-\langle \mathbf{F}\theta, \theta\rangle = \langle \mathbf{A}^t\mathbf{A}\theta, \theta\rangle = |\mathbf{A}\theta|^2 \geq 0$ for all $r \in]a, b[$ and $\theta \in \mathbb{S}^{n-1}$ along with $p \in [1, \infty[$ it follows that *all* the terms $\mathbf{I}, \mathbf{II}$ and $\mathbf{III}$ are *strictly* positive. As a result

$$\gamma > 0, \tag{3.27}$$

and so the claim is justified.

Now returning to the identity (3.25) it follows as a result of (3.27) that necessarily

$$\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta = 0, \tag{3.28}$$

for all $r \in]a, b[$ and $\theta \in \mathbb{S}^{n-1}$. The conclusion in **step 2** is now an immediate result of the following statement.

Claim 3. Let $\mathbf{F} \in \mathbb{M}_{n\times n}$. Then (3.28) holds for all $\theta \in \mathbb{S}^{n-1}$ if and only if there exists $-\sigma \in \mathbb{R}$ such that $\mathbf{F} = -\sigma \mathbf{I}_n$.

For a *proof* of **claim 3** we refer the interested reader to Proposition B.0.6 in Appendix B. Finally $\sigma \in \mathbf{C}^1]a, b[$ and $\sigma \geq 0$ are consequences of the representation above and the hypothesis of the theorem. With this the *proof* of Theorem 3.4.2 is complete. □

Theorem 3.4.3. *Let* $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ *and consider the vector field* $\mathbf{v}$ *as defined in Theorem* 3.4.2. *Then the following are equivalent.*
[**1**] $\mathbf{v}$ *is a gradient,*
[**2**] $\mathbf{A} = \mu \mathbf{J}$ *for some* $\mu \in \mathbf{C}^1]a, b[$ *with* $\mu \geq 0$, $\mathbf{J} \in \mathbb{M}_{n\times n}$ *skew-symmetric with* $\mathbf{J}^2 = -\mathbf{I}_n$ *and*

$$\frac{d}{dr}(r^{n+1}\mathbf{s}\mu) = 0, \tag{3.29}$$

in $]a, b[$. *Here* $\mathbf{s} = (n + r^2\mu^2)^{\frac{p-2}{2}}$.

Proof. [**2**] $\Longrightarrow$ [**1**] The argument here is similar to that in Theorem 3.4.2 and so will be *abbreviated.* [**1**] $\Longrightarrow$ [**2**] Let $\mathbf{v}$ be a *gradient.* Then according to [**2**] in Theorem 3.4.2, $\mathbf{A}^2 = -\sigma \mathbf{I}_n$ for some $\sigma \in \mathbf{C}^1(]a, b[)$ with $\sigma \geq 0$ and so $\mathbf{A} = \sqrt{\sigma}\mathbf{J}$ where $\mathbf{J} = \mathbf{J}(r)$ and $\mathbf{J}^2 = -\mathbf{I}_n$. The aim is to show that $\mathbf{J}$ is *independent* of r. Note that in general there is *no* uniqueness or even *finiteness* associated with the choice of a *square* root of a matrix! Thus an argument *purely* based on continuity would not yield the *aforementioned* claim and it is *crucial* to additionally utilise (3.11). To this end we proceed as follows. Indeed according to [**2**] in Theorem 3.4.2,

$$\frac{d}{dr}(r^{n+1}\mathbf{s}\mathbf{A}) = 0.$$

Integrating the above equation gives $r^{n+1}\mathbf{s}\mathbf{A} = \xi$ for some *constant* $\xi \in \mathbb{M}_{n\times n}$. Moreover,

$$-(r^{n+1}\mathbf{s})^2\sigma \mathbf{I}_n = (r^{n+1}\mathbf{s}\mathbf{A})^2 = \xi^2, \tag{3.30}$$

giving $(r^{n+1}\mathbf{s})^2\sigma \equiv c$ for some *non-negative* constant c. Thus *either* $\sigma \equiv 0$ in which case $\mathbf{A} \equiv 0$ on $]a, b[$ and so the choice $\mu \equiv 0$ gives the conclusion *or* else $\sigma > 0$ on $]a, b[$ and so setting

$$\mathbf{J} := \frac{1}{\sqrt{c}}\xi,$$

we have as a result of (3.30) that $\mathbf{J}^2 = -\mathbf{I}_n$. Furthermore setting

$$\mu := \frac{1}{\sqrt{c}} r^{n+1} \mathbf{s}\sigma,$$

it follows that $\mu \in \mathbf{C}^1]a,b[$, $\mu^2 = \sigma$ and by substitution $\mathbf{A} = \mu\mathbf{J}$. As a result μ also satisfies (3.29). The proof of the theorem is thus complete. $\square$

Remark 3.4.4. Referring to the above *proof* it follows from $r^{n+1}\mathbf{s}\mu = c$ on $]a,b[$ that when $p > 1$ the function μ remains *bounded* on $]a,b[$.

Theorem 3.4.5. *Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ and $u \in \mathcal{A}_p$ with $p \in]1,\infty[$ be a generalised twist whose corresponding twist loop $\mathbf{Q} \in \mathbf{C}^2(]a,b[, \mathbf{SO}(n))$. Then the following are equivalent.*

[**1**] *u is a classical solution to the Euler-Lagrange equations associated with $\mathbb{F}_p$ over $\mathcal{A}_p$,*

[**2**] *depending on whether n is even or odd we have that*

[**2a**] *($n = 2k$) there exist $g = g(r) \in \mathbf{C}[a,b] \cap \mathbf{C}^2]a,b[$ with $g(a), g(b) \in 2\pi\mathbb{Z}$ and $\mathbf{P} \in \mathbf{O}(n)$ such that*

$$\mathbf{Q} = \mathbf{P} diag(\mathfrak{R}(g), \mathfrak{R}(g), \ldots, \mathfrak{R}(g))\mathbf{P}^t,$$

whilst g is a solution on $]a,b[$ to

$$\frac{d}{dr}\left\{ r^{n+1}(n + r^2 g'^2)^{\frac{p-2}{2}} g' \right\} = 0, \tag{3.31}$$

or

[**2b**] *($n = 2k+1$) necessarily $u = x$ on $\bar{\Omega}$.*

Proof. [**1**] $\implies$ [**2**] Let $u = \mathbf{Q}(r)x$ be a *classical* solution to the stated Euler-Lagrange equations. Then setting $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ an application of Proposition 3.1.4 in conjunction with Theorem 3.4.3 gives

$$\frac{d}{dr}\mathbf{Q} = \mu\mathbf{Q}\mathbf{J}, \tag{3.32}$$

where $\mu \in \mathbf{C}^1]a,b[$ satisfies (3.29) and $\mathbf{J}^2 = -\mathbf{I}_n$. Moreover *either* $\mu \equiv 0$ or else $\mu > 0$ and bounded on $]a,b[$. (*See* Remark 3.4.4.) We now consider the cases [**2a**] and [**2b**] *separately.*

[**2a**] ($n = 2k$) Let $g \in \mathbf{C}[a,b] \cap \mathbf{C}^2]a,b[$ be a *primitive* of μ satisfying $g(a) \in 2\pi\mathbb{Z}$. (The *continuity* of g on $[a,b]$ follows from g being *monotone* and $g' = \mu$ being bounded on $]a,b[$.) Next, a straight-forward calculation gives

$$\begin{aligned} \mathbf{s} &= (n + r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}} \\ &= (n + r^2 g'^2 |\mathbf{J}\theta|^2)^{\frac{p-2}{2}} \end{aligned}$$

$$= (n + r^2 g'^2)^{\frac{p-2}{2}}.$$

Thus in view of (3.29) g satisfies (3.31) on $]a, b[$. An application of Tonelli and Hilbert-Weierstrass *differentiability* theorems (*see*, e.g., [18] pp. 57-61) now gives $g \in \mathbf{C}^2[a, b]$ and so in particular $\mu \in \mathbf{C}^1[a, b]$. As will be seen in the next section (3.31) is the Euler-Lagrange equation corresponding to the energy functional $\mathbb{G}_p$ over the space $\mathcal{G}_p^m$ [*see* (3.33), (3.35)]. In particular it follows that $g \in \mathbf{C}^\infty[a, b]$.

With this introduction now put $\mathbf{C} = g\mathbf{J}$. Then $\mathbf{A} = g'\mathbf{J} = \mu\mathbf{J}$. In particular $\mathbf{A}$ and $\mathbf{C}$ *commute* and so we have that

$$\frac{d}{dr} e^{\mathbf{C}} = e^{\mathbf{C}}\mathbf{A} = g' e^{\mathbf{C}}\mathbf{J} = \mu e^{\mathbf{C}}\mathbf{J}.$$

Thus $e^{\mathbf{C}}$ is a solution to (3.32). Moreover by bringing $\mathbf{C}$ into a block diagonal form we can write $\mathbf{C} = g\mathbf{P}\mathbf{J}_n\mathbf{P}^t$ where $\mathbf{P} \in \mathbf{O}(n)$ and $\mathbf{J}_n = diag(\mathbf{J}_2, \mathbf{J}_2, \ldots, \mathbf{J}_2)$. As a result

$$\begin{aligned} e^{\mathbf{C}} &= e^{g\mathbf{P}\mathbf{J}_n\mathbf{P}^t} \\ &= \mathbf{P} e^{g\mathbf{J}_n} \mathbf{P}^t \\ &= \mathbf{P} diag(\mathfrak{R}(g), \mathfrak{R}(g), \ldots, \mathfrak{R}(g))\mathbf{P}^t. \end{aligned}$$

Since $g(a) \in 2\pi\mathbb{Z}$ the above shows that $e^{\mathbf{C}}|_{r=a} = \mathbf{Q}(a) = \mathbf{I}_n$ and so by *uniqueness* of solutions to initial value problems $\mathbf{Q} = e^{\mathbf{C}}$ on $[a, b]$. Since $\mathbf{Q}(b) = \mathbf{I}_n$ it follows in a similar way that $g(b) \in 2\pi\mathbb{Z}$.
[**2b**] $(n = 2k + 1)$ Here in view of the *skew-symmetry* of $\mathbf{Q}^t\dot{\mathbf{Q}}$, pre-multiplying (3.32) by $\mathbf{Q}^t$ and then taking *determinants* from both sides, $\mu \equiv 0$ and so $\dot{\mathbf{Q}} \equiv 0$ on $]a, b[$. As $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ this gives $\mathbf{Q} \equiv \mathbf{I}_n$ on $[a, b]$ and so $u = x$ on $\bar{\Omega}$.
[**2**] $\Longrightarrow$ [**1**] For the case [**2b**] this is *trivial* and for [**2a**] it is enough to note that for such u, (3.31) is equivalent to (3.11). □

3.5 A characterisation of all twist solutions

In section 3.3 we proved the existence of *multiple* p-stationary loops by directly *minimizing* the energy functional $\mathbb{E}_p$ over the homotopy classes $\mathfrak{c}_\star[\mathcal{E}_p]$ of the *loop* space $\mathcal{E}_p$. By contrast in this section we focus on the Euler-Lagrange equation itself and present a class of p-stationary loops that in turn will prove fruitful in discussing the *existence* of multiple solutions to the Euler-Lagrange equations associated with the energy functional $\mathbb{F}_p$ over the space $\mathcal{A}_p$.

To this end we consider the case of *even* dimensions ($n = 2k$) and for $p \in [1, \infty[$ and $m \in \mathbb{N}$ set

$$\mathcal{G}_p^m = \mathcal{G}_p^m(a, b) := \Big\{ g = g(r) \in W^{1,p}(a, b) : g(a) = 0, g(b) = 2\pi m \Big\}. \tag{3.33}$$

Now for $g \in \mathcal{G}_p^m$ and $\mathbf{P} \in \mathbf{O}(n)$ set

$$\mathbf{Q} = \mathbf{P} diag(\mathfrak{R}(g), \mathfrak{R}(g), \ldots, \mathfrak{R}(g))\mathbf{P}^t. \tag{3.34}$$

It is then evident that the path $\mathbf{Q}$ so defined forms an *admissible* loop, i.e., lies in $\mathcal{E}_p$. It is thus natural to set

$$\begin{aligned} \mathbb{G}_p[g] := \mathbb{E}_p[\mathbf{Q}] &= \int_a^b \int_{\mathbb{S}^{n-1}} (n + r^2 |\dot{\mathbf{Q}}\theta|^2)^{\frac{p}{2}} r^{n-1} \, d\mathcal{H}^{n-1}(\theta) dr \\ &= n\omega_n \int_a^b (n + r^2 g'^2)^{\frac{p}{2}} r^{n-1} \, dr. \end{aligned} \tag{3.35}$$

An application of the direct methods of the calculus of variations and standard *regularity* theory (*see*, e.g., [18] pp. 57-61) leads us to the following statement.

Theorem 3.5.1. *Let $p \in]1, \infty[$ and consider the energy functional $\mathbb{G}_p$ over the space $\mathcal{G}_p^m$. Then for each $m \in \mathbb{N}$ there exists a unique $g = g(r; m, a, b) \in \mathcal{G}_p^m$ such that*

$$\mathbb{G}_p[g] = \inf_{\mathcal{G}_p^m} \mathbb{G}_p.$$

Moreover $g(r; m, a, b)$ satisfies the corresponding Euler-Lagrange equation

$$\frac{d}{dr} \Big\{ r^{n+1} (n + r^2 g'^2)^{\frac{p-2}{2}} g' \Big\} = 0, \tag{3.36}$$

on $]a, b[$. Additionally $g \in \mathbf{C}^\infty[a, b]$.

Remark 3.5.2. The Euler-Lagrange equation (3.36) for g is *equivalent* to equation (3.9) for the *twist* loop $\mathbf{Q}$ defined through (3.34) and imply the Euler-Lagrange equation (3.9) [or alternatively that given in Proposition 3.2.3 for $\mathbf{A} = \mathbf{Q}^t \dot{\mathbf{Q}}$]. Hence for every $\mathbf{P} \in \mathbf{O}(n)$ and every $m \in \mathbb{Z}$ the corresponding $\mathbf{Q}$ given by (3.34) with $g = g(r; m, a, b)$ is a p-stationary *loop.*

Theorem 3.5.3. *Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$. Consider the energy functional $\mathbb{F}_p$ with $p \in]1, \infty[$ over the space $\mathcal{A}_p$. Then the set $\mathfrak{S}$ of all generalised twist solutions to the corresponding Euler-Lagrange equations can be characterised as follows.*

[**1**] $(n = 2k)$ $\mathfrak{S}$ *is infinite and any generalised twist* $u \in \mathfrak{S}$ *can be described as*

$$
\begin{aligned}
u &= r\mathbf{Q}(r; a, b, m)\theta \\
&= r\mathbf{P} diag(\mathfrak{R}(g), \mathfrak{R}(g), \dots, \mathfrak{R}(g))(r)\mathbf{P}^t\theta,
\end{aligned}
$$

where $\mathbf{P} \in \mathbf{O}(n)$ *and* $g \in \mathbf{C}^\infty[a, b]$ *satisfies*

$$\frac{d}{dr}\left\{ r^{n+1}(n + r^2 g'^2)^{\frac{p-2}{2}} g' \right\} = 0,$$

with $g(a), g(b) \in 2\pi\mathbb{Z}$*,*

[**2**] $(n = 2k + 1)$ $\mathfrak{S}$ *consists of the single map* $u = x$*.*

Proof. This is an immediate consequence of Theorem 3.4.5 and Theorem 3.5.1. □

Remark 3.5.4. Is it possible to consider *generalised* twists u whose twist *loop* lies in other spaces [than $\mathbf{SO}(n)$ already considered] with the hope of finding *new* classes of classical solutions to the Euler-Lagrange equations associated with the energy functional $\mathbb{F}_p$ over $\mathcal{A}_p$?

Motivated by the requirement $\det \nabla u = 1$ on such maps the choice of loops in $\mathbf{SL}(n) \supset \mathbf{SO}(n)$ seems a natural one. [5] However it turns out that the choice $\mathbf{SO}(n)$ is *no* less general than $\mathbf{SL}(n)$!

Claim. Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$. For $p \in [1, \infty[$ consider the map $u \in \mathbf{C}(\bar{\Omega}, \bar{\Omega})$ defined via

$$u = \mathbf{F}(r)x,$$

where $r = |x|$ and $\mathbf{F} \in W^{1,p}([a, b], \mathbf{SL}(n))$. Then

$$u \in \mathcal{A}_p(\Omega) \implies \mathbf{F} \in W^{1,p}([a, b], \mathbf{SO}(n)).$$

Proof. A straight-forward calculation as in the proof of Proposition 3.1.2 gives

$$
\begin{aligned}
\nabla u &= \mathbf{F} + r\dot{\mathbf{F}}\theta \otimes \theta \\
&= \mathbf{F}(\mathbf{I}_n + r\mathbf{F}^{-1}\dot{\mathbf{F}}\theta \otimes \theta).
\end{aligned}
$$

[5] Recall that for every non-negative integer n we have that

$$\mathbf{SL}(n) = \mathbf{SL}(\mathbb{R}, n) := \left\{ \mathbf{F} \in \mathbb{M}_{n \times n}(\mathbb{R}) : \det \mathbf{F} = 1 \right\}.$$

Hence in view of $\det \mathbf{F} = 1$ we can write

$$\begin{aligned}\det \nabla u &= \det(\mathbf{F} + r\dot{\mathbf{F}}\theta \otimes \theta)\\ &= \det(\mathbf{I}_n + r\mathbf{F}^{-1}\dot{\mathbf{F}}\theta \otimes \theta)\\ &= 1 + r\langle \mathbf{F}^{-1}\dot{\mathbf{F}}\theta, \theta\rangle.\end{aligned}$$

Evidently $u \in \mathcal{A}_p(\Omega)$ provided that

(i) $u = x$ on $\partial\Omega$,

(ii) $\det \nabla u = 1$ in Ω, and,

(iii) $\|u\|_{W^{1,p}(\Omega)} < \infty$.

Now again referring to the proof of Proposition 3.1.2 we have that

$(i) \iff \mathbf{F}(a) = \mathbf{F}(b) = \mathbf{I}_n$,

whilst

$(ii) \iff \langle \mathbf{F}^{-1}\dot{\mathbf{F}}\theta, \theta\rangle = 0$ for all $\theta \in \mathbb{S}^{n-1} \iff \mathbf{F}^{-1}\dot{\mathbf{F}} + \dot{\mathbf{F}}^t\mathbf{F}^{-t} = 0$.

However, anticipating on the latter, we can write

$$\begin{aligned}\mathbf{F}^{-1}\dot{\mathbf{F}} + \dot{\mathbf{F}}^t\mathbf{F}^{-t} = 0 &\iff \dot{\mathbf{F}} + \mathbf{F}\dot{\mathbf{F}}^t\mathbf{F}^{-t} = 0,\\ &\iff \dot{\mathbf{F}}\mathbf{F}^t + \mathbf{F}\dot{\mathbf{F}}^t = 0\\ &\iff \frac{d}{dr}(\mathbf{F}\mathbf{F}^t) = 0.\end{aligned}$$

This together with (i) and the *continuity* of $\mathbf{F}$ on $[a, b]$ gives $\mathbf{F}\mathbf{F}^t = \mathbf{I}_n$ and so the conclusion follows.

3.6 Limiting behaviour of the generalised twists as the inner hole shrinks to a point

In this section we consider the case where $b = 1$ and $a = \varepsilon > 0$ with the aim of discussing the *limiting* properties of the *generalised* twists from Theorem 3.5.3 as $\varepsilon \downarrow 0$. This is particularly interesting since in the limit (the *punctured* ball), by Remark 2.1.4, *all* components of the function space collapse to a single one and so it is important to have a clear understanding as to how the twist solutions and their energies [for each *fixed* integer m] behave.

To this end, let $\Omega_\varepsilon := \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\}$ where $n = 2k$ and for each $m \in \mathbb{Z}$ let $u_\varepsilon \in \mathcal{A}_p$ denote the *generalised* twist from **[1]** in Theorem 3.5.3, that is, with the notation $x = r\theta$,

$$\begin{aligned} u_\varepsilon &= r\mathbf{Q}(r;\varepsilon,1,m)\theta \\ &= r\mathbf{P}_\varepsilon diag(\mathfrak{R}(g_\varepsilon),\mathfrak{R}(g_\varepsilon),\ldots,\mathfrak{R}(g_\varepsilon))\mathbf{P}_\varepsilon^t\theta, \end{aligned}$$

where $\mathbf{P}_\varepsilon \in \mathbf{O}(n)$ and $g_\varepsilon(r) = g(r;\varepsilon,1,m)$.

In order to make the study of the *limiting* properties of u_ε more tractable, we fix the domain to be the unit ball and extend each map by *identity* off Ω_ε. [In what follows, unless otherwise stated, we speak of u_ε in this extended sense.] Thus, here, we have that

$$u_\varepsilon : (r,\theta) \mapsto \big(r, \mathbf{G}_\varepsilon(r)\theta\big), \tag{3.37}$$

where

$$\mathbf{G}_\varepsilon(r) = \mathbf{P}_\varepsilon diag(\mathfrak{R}(g_\varepsilon),\mathfrak{R}(g_\varepsilon),\ldots,\mathfrak{R}(g_\varepsilon))\mathbf{P}_\varepsilon^t,$$

and

$$g_\varepsilon(r) = \begin{cases} 0 & r \le \varepsilon, \\ g(r;\varepsilon,1,m) & \varepsilon \le r \le 1. \end{cases}$$

In discussing the *limiting* properties of u_ε it is convenient to introduce a so-called *comparison* map. Indeed, *fix* $m \in \mathbb{Z}$ and consider the *generalised* twist

$$v_\varepsilon : (r,\theta) \mapsto (r, \mathbf{H}_\varepsilon(r)\theta). \tag{3.38}$$

where

$$\mathbf{H}_\varepsilon(r) = \mathbf{P}_\varepsilon diag(\mathfrak{R}(h_\varepsilon),\mathfrak{R}(h_\varepsilon),\ldots,\mathfrak{R}(h_\varepsilon))\mathbf{P}_\varepsilon^t,$$

and

$$h_\varepsilon(r) := \begin{cases} 0 & r \in (0,\varepsilon), \\ 2m\pi(\frac{r}{\varepsilon}-1) & r \in (\varepsilon, 2\varepsilon), \\ 2m\pi & r \in (2\varepsilon, 1). \end{cases}$$

Proposition 3.6.1. *Let $p \in]1,\infty[$. The family of generalised twists (v_ε) enjoys the following properties.*

[1] *$v_\varepsilon \to x$ in $W^{1,p}(\mathbb{B},\mathbb{R}^n)$,*

[2] *$v_\varepsilon \to x$ uniformly on $\bar{\mathbb{B}}$.*

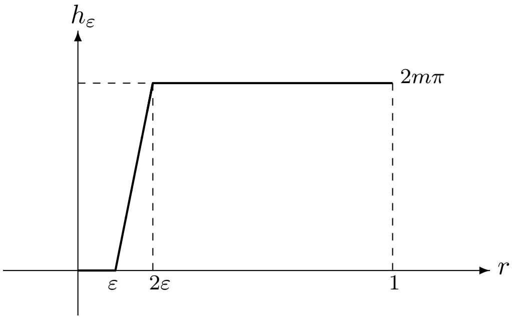

FIGURE 3.1: The function h_ε associated with the *extended* twist loop $\mathbf{H}_\varepsilon$.

Proof. [**1**] Using (3.38) and a straight-forward calculation we have that,

$$\begin{aligned}\|v_\varepsilon - x\|^p_{W^{1,p}_0} &= \int_{\mathbb{B}} |\nabla v_\varepsilon - \mathbf{I}_n|^p \\ &= \int_{\mathbb{B}_{2\varepsilon}\setminus\mathbb{B}_\varepsilon} |\nabla v_\varepsilon - \mathbf{I}_n|^p \le 2^{p-1} \int_{\mathbb{B}_{2\varepsilon}\setminus\mathbb{B}_\varepsilon} |\nabla v_\varepsilon|^p + |\mathbf{I}_n|^p.\end{aligned}$$

Furthermore, referring to Proposition 3.1.2 [*see* (3.5)] we can write

$$\begin{aligned}\int_{\mathbb{B}_{2\varepsilon}\setminus\mathbb{B}_\varepsilon} |\nabla v_\varepsilon|^p &= \int_\varepsilon^{2\varepsilon} \int_{\mathbb{S}^{n-1}} (n + r^2|\dot{\mathbf{H}}_\varepsilon\theta|^2)^{\frac{p}{2}} r^{n-1}\, d\mathcal{H}^{n-1}(\theta) dr \\ &= n\omega_n \int_\varepsilon^{2\varepsilon} (n + r^2 h_\varepsilon'^2)^{\frac{p}{2}} r^{n-1}\, dr \\ &\le \omega_n (2^n - 1)\varepsilon^n [n + 4(2m\pi)^2]^{\frac{p}{2}}. \end{aligned} \tag{3.39}$$

The above estimates when combined give [**1**] as a result of Poincaré inequality.

[**2**] By direct verification we have that

$$\begin{aligned}|v_\varepsilon - x|^2 &= |r\mathbf{H}_\varepsilon(r)\theta - r\theta|^2 \\ &= r^2\Big|\mathbf{P}_\varepsilon diag(\mathfrak{R}(h_\varepsilon), \dots, \mathfrak{R}(h_\varepsilon))\mathbf{P}^t_\varepsilon\theta - \theta\Big|^2 \\ &= r^2\Big|\mathbf{P}_\varepsilon \big[diag(\mathfrak{R}(h_\varepsilon), \dots, \mathfrak{R}(h_\varepsilon)) - \mathbf{I}_n\big]\mathbf{P}^t_\varepsilon\theta\Big|^2 \\ &= r^2\Big|\big[diag(\mathfrak{R}(h_\varepsilon), \dots, \mathfrak{R}(h_\varepsilon)) - \mathbf{I}_n\big]\omega_\varepsilon\Big|^2 \qquad \big(\omega_\varepsilon := \mathbf{P}^t_\varepsilon\theta\big) \\ &= \frac{1}{2} r^2 |\mathfrak{R}(h_\varepsilon) - \mathbf{I}_2|^2. \end{aligned} \tag{3.40}$$

However a straight-forward calculation gives

$$|\mathfrak{R}(h_\varepsilon) - \mathbf{I}_2|^2 = 4(1 - \cos h_\varepsilon) = 8 \sin^2 \frac{h_\varepsilon}{2}.$$

Thus combining the above and referring to the definition of h_ε we arrive at the bound

$$\sup_{\mathbb{B}} |v_\varepsilon - x| = \sup_{[\varepsilon, 2\varepsilon]} 2r \left| \sin \frac{h_\varepsilon}{2} \right| \leq 4\varepsilon,$$

which gives the required conclusion. $\square$

Let $p \in]1, \infty[$ and *fix* $m \in \mathbb{Z}$. Then $g_\varepsilon, h_\varepsilon \in \mathcal{G}_p^m(\varepsilon, 1)$ [*see* (3.33)] and so according to the minimizing property of g_ε we have that

$$\mathbb{F}_p[u_\varepsilon, \mathbb{B}] = \frac{1}{p}\mathbb{E}_p[\mathbf{G}_\varepsilon] = \frac{1}{p}\mathbb{G}_p[g_\varepsilon] \leq \frac{1}{p}\mathbb{G}_p[h_\varepsilon] = \frac{1}{p}\mathbb{E}_p[\mathbf{H}_\varepsilon] = \mathbb{F}_p[v_\varepsilon, \mathbb{B}]. \tag{3.41}$$

This in conjunction with [**1**] in Proposition 3.6.1 implies the boundedness of (u_ε) in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$ and so as a result (u_ε) admits a *weakly* convergent subsequence. Indeed more is true!

Theorem 3.6.2. *Let* $\Omega_\varepsilon := \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\}$. *For* $p \in]1, \infty[$ *and* $m \in \mathbb{Z}$ *let* $(u_\varepsilon)_{\varepsilon > 0}$ *denote the family of generalised twists as in* (3.37). *Then,*
[**1**] $u_\varepsilon \to x$ *in* $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$,
[**2**] $u_\varepsilon \to x$ *uniformly in* $\bar{\mathbb{B}}$.

Before proof we note that here both convergences are in reference to the *entire* sequence and not merely a subsequence as was implied in discussing the *weak* convergence prior to the proposition. The argument is standard and will be abbreviated.

Proof. [**1**] Fix $m \in \mathbb{Z}$ and let v_ε be as in (3.38). Then referring to (3.41) it follows that by passing to a subsequence (not re-labeled) $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$. Appealing to the *sequential* weak lower semicontintuity of $\mathbb{F}_p$ and [**1**] in Proposition 3.6.1 we can write

$$\begin{aligned} \mathbb{F}_p[x, \mathbb{B}] \leq \mathbb{F}_p[u, \mathbb{B}] &\leq \liminf_{\varepsilon \downarrow 0} \mathbb{F}_p[u_\varepsilon, \mathbb{B}] \\ &\leq \limsup_{\varepsilon \downarrow 0} \mathbb{F}_p[u_\varepsilon, \mathbb{B}] \\ &\leq \lim_{\varepsilon \downarrow 0} \mathbb{F}_p[v_\varepsilon, \mathbb{B}] = \mathbb{F}_p[x, \mathbb{B}]. \end{aligned}$$

This in view of the *strict* convexity of $\mathbb{F}_p$ (on $W^{1,p}$) gives $u = x$. As a result of the *uniform* convexity of the p-norm ($p > 1$) the *aforementioned* weak convergence can now be improved to *strong* convergence and this gives [**1**].
[**2**] By [**1**] we can assume without loss of generality that $u_\varepsilon \to x$ $\mathcal{L}^n$-a.e. in $\mathbb{B}$. To justify the *uniform* convergence in [**2**] let g_ε be as that described in (3.37) and *fix* $\sigma \in (0, 1)$. Then we claim that $g_\varepsilon \to 2m\pi$ uniformly on $[\sigma, 1]$. Indeed, (u_ε) bounded in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$ gives (u_ε) bounded in

$W^{1,p}(\mathbb{B}\backslash\bar{\mathbb{B}}_\sigma, \mathbb{R}^n)$ and so referring to (3.5) and using a calculation similar to that in (3.39) we have (g_ε) bounded in $W^{1,p}(\sigma,1)$. Hence, there exists $f = f_\sigma \in W^{1,p}(\sigma,1)$ so that passing to a subsequence (not relabeled)

$$\begin{cases} g_\varepsilon \rightharpoonup f & \text{in } W^{1,p}(\sigma,1), \\ g_\varepsilon \to f & \text{in } L^\infty[\sigma,1], \\ f(1) = 2m\pi. \end{cases}$$

In addition referring again to (3.37) we can assume in view of $\mathbf{O}(n)$ being *compact*, that by passing to a *further* subsequence (again, not relabeled) $\mathbf{P}_\varepsilon \to \mathbf{P}$ for some $\mathbf{P} \in \mathbf{O}(n)$. Hence for $\mathcal{L}^n$-a.e. $x \in \mathbb{B}\backslash\bar{\mathbb{B}}_\sigma$ we can write

$$\begin{aligned} x = r\theta &= \lim_{\varepsilon\downarrow 0} u_\varepsilon(x) \\ &= \lim_{\varepsilon\downarrow 0} r\mathbf{G}_\varepsilon(r)\theta \\ &= \lim_{\varepsilon\downarrow 0} r\mathbf{P}_\varepsilon diag(\Re(g_\varepsilon),\ldots,\Re(g_\varepsilon))\mathbf{P}_\varepsilon^t\theta \\ &= r\mathbf{P} diag(\Re(f),\ldots,\Re(f))\mathbf{P}^t\theta, \end{aligned}$$

giving $\Re(f) = \mathbf{I}_2$ and in turn that $f = 2\pi n(r)$ for some $n(r) \in \mathbb{Z}$. The continuity of f along with $f(1) = 2m\pi$ now gives $f = 2m\pi$ on $[\sigma,1]$ justifying the assertion. Next, arguing as in (3.40) we can write

$$\begin{aligned} |u_\varepsilon - x|^2 &= \left|r\mathbf{G}_\varepsilon(r)\theta - r\theta\right|^2 \\ &= 2r^2(1-\cos g_\varepsilon) \\ &= 4r^2\sin^2\frac{g_\varepsilon}{2}. \end{aligned}$$

Thus, to conclude [**2**] fix $\delta > 0$ and *first* take $\sigma \in (0, 2^{-1}\delta]$ and then ε_0 such that $|\sin(2^{-1}g_\varepsilon)| \le 2^{-1}\delta$ on $[\sigma,1]$ for $\varepsilon < \varepsilon_0$. Then $\sup_{\mathbb{B}} |u_\varepsilon - x| \le \max(2\sigma,\delta) = \delta$. □

The *uniform* convergence in [**2**] above looks at *first* counter-intuitive, as, how can u_ε and x be *uniformly* close when u_ε twists m times while the *limit* x none? Indeed a *careful* consideration reveals that the latter twists occur at a distance ε from the origin and within a layer of thickness $O(\varepsilon)$ and this is in no conflict with the stated *uniform* convergence!

Chapter 4

Quasiconvexity and uniqueness of stationary points

Throughout this chapter we assume $\Omega \subset \mathbb{R}^n$ to be a bounded *starshaped* domain and consider the energy functional

$$\mathbb{F}[u, \Omega] := \int_\Omega \mathbf{F}(\nabla u(x))\, dx,$$

over the space of measure-preserving maps

$$\mathcal{A}_p(\Omega) := \Big\{ u \in \bar{\xi} x + W_0^{1,p}(\Omega, \mathbb{R}^n) : \det \nabla u = 1 \ a.e. \text{ in } \Omega \Big\},$$

with $p \in [1, \infty[$, $\bar{\xi} \in \mathbb{M}_{n\times n}$ satisfying $\det \bar{\xi} = 1$. The hypotheses [**H1**]-[**H3**] on the integrand $\mathbf{F}$ here refer to those outline in Chapter 1. We address the question of *uniqueness* for solutions of the corresponding system of Euler-Lagrange equations. In particular we give a short and new proof of the celebrated result of Knops & Sturat [44] using the method based on *comparison* with the homogeneous degree-one extension maps. The material in this chapter is taken from Shahrokhi-Dehkordi & Taheri [58].

4.1 Quasiconvexity and uniqueness in starshaped domain

Let $\Omega \subset \mathbb{R}^n$ be a $\mathbf{C}^1$ bounded *starshaped* domain (with respect to the origin). [1] Without loss of generality we assume in the sequel that there exists a *strictly* positive function $d : \mathbb{S}^{n-1} \to \mathbb{R}$ of class

[1] Recall that a set $\mathcal{S} \subset \mathbb{R}^n$ is said to be *starshaped* with respect to the point $x_0 \in \mathcal{S}$ if and only if whenever the point x belongs to $\mathcal{S}$, the straight line segment joining x_0 to x also lies in $\mathcal{S}$.

$\mathbf{C}^1$ such that

$$\partial\Omega = \left\{ \omega \neq 0 : |\omega| = d(\frac{\omega}{|\omega|}) \right\}.$$

It is then clear that $\Omega = \{0\} \cup \{x \neq 0 : |x| < d(x/|x|)\}$. Moreover the *unit* outward normal to the boundary at a point $\omega \in \partial\Omega$ is given by

$$\nu = \frac{1}{\alpha(\theta)} \left[\theta - (\mathbf{I}_n - \theta \otimes \theta) \frac{\nabla d(\theta)}{d(\theta)} \right],$$

where $\alpha(\theta) = d(\theta)^{-1} \sqrt{d(\theta)^2 + |\nabla d(\theta)|^2 - \langle \theta, \nabla d(\theta) \rangle^2}$ and $\theta = \omega/|\omega|$.

Definition 4.1.1. (**Classical solution**)

A pair $(u, \mathfrak{p})$ is said to be a *classical* solution to the Euler-Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$ if and only if the following hold.

[**1**] $u \in \mathbf{C}^2(\Omega, \mathbb{R}^n) \cap \mathbf{C}^1(\bar{\Omega}, \mathbb{R}^n)$,

[**2**] $\mathfrak{p} \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\bar{\Omega})$,

[**3**] $(u, \mathfrak{p})$ satisfy the *system* of equations

$$\begin{cases} \operatorname{div} \{\mathbf{F}_\xi(\nabla u) - \mathfrak{p}[\operatorname{cof} \nabla u]\} = 0 & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega. \end{cases}$$

Although we are primarily concerned with the case $v = \bar{\xi}x$, for reasons that will become clear later, we allow $v \in \mathbf{C}^1(\bar{\Omega}, \mathbb{R}^n)$ to be arbitrary. Now suppose that $(u, \mathfrak{p})$ is a classical solution as described in Definition 4.1.1. We set

$$\mathbf{G}(x, z, \xi) = \mathbf{G}(x, z, \xi; \mathfrak{p}) := \mathbf{F}(\xi) - \mathfrak{p}(x)[\det \xi - 1], \tag{4.1}$$

for all $x \in \Omega$, $z \in \mathbb{R}^n$ and $\xi \in \mathbb{M}_{n\times n}$. Next with the aid of $\mathbf{G}$ we introduce the *Hamilton* [or the *energy-momentum*] tensor

$$\mathbf{T}_\alpha^\beta(x, z, \xi) := \xi_\alpha^i \mathbf{G}_{\xi_\beta^i}(x, z, \xi) - \delta_\alpha^\beta \mathbf{G}(x, z, \xi). \tag{4.2}$$

Theorem 4.1.2. *Let $(u, \mathfrak{p})$ be a classical solution to the Euler-Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$. Let $\mathbf{F}$ be of class $\mathbf{C}^2$. Then with $\mathbf{G}$ and $\mathbf{T}$ as in* (4.1) *and* (4.2) *we have that*

$$\operatorname{div}\{\mathbf{T}(x, u, \nabla u)\} + \mathbf{G}_x(x, u, \nabla u) = 0, \tag{4.3}$$

in Ω.

Proof. (By direct *verification*) Indeed expanding the above identity *componentwise* we have that

$$\begin{aligned}
\mathbf{L}_\alpha :=& \Big[\operatorname{div}\{\mathbf{T}(x,u,\nabla u)\} + \mathbf{G}_x(x,u,\nabla u)\Big]_\alpha \\
=& \frac{\partial \mathbf{T}^\beta_\alpha}{\partial x_\beta}(x,u,\nabla u) + \mathbf{G}_{x_\alpha}(x,u,\nabla u) \\
=& \frac{\partial}{\partial x_\beta}\left\{u^i_{,\alpha}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof}\nabla u]_{i\beta}\right)\right\} - \\
& \frac{\partial}{\partial x_\alpha}\Big\{\mathbf{F} - \mathfrak{p}(x)[\det \nabla u - 1]\Big\} - \frac{\partial \mathfrak{p}}{\partial x_\alpha}(x)[\det\nabla u - 1].
\end{aligned}$$

Therefore taking advantage of $\det \nabla u = 1$ and by direct *differentiation* we can write

$$\begin{aligned}
\mathbf{L}_\alpha =& u^i_{,\alpha\beta}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof}\nabla u]_{i\beta}\right) + \\
& u^i_{,\alpha}\frac{\partial}{\partial x_\beta}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof}\nabla u]_{i\beta}\right) - \mathbf{F}_{\xi^i_\beta} u^i_{,\alpha\beta} \\
=& -\mathfrak{p}(x)\frac{\partial}{\partial x_\alpha}\det\nabla u + u^i_{,\alpha}\frac{\partial}{\partial x_\beta}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof}\nabla u]_{i\beta}\right) \\
=& u^i_{,\alpha}\frac{\partial}{\partial x_\beta}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof}\nabla u]_{i\beta}\right) = 0,
\end{aligned}$$

which is the required conclusion. □

We note that the equation (4.3) is the so-called *stationarity* condition in its *strong* form as opposed to its *weak* form given by (4.4) below. For the sake of future reference we next introduce the *unconstrained* energy functional

$$\begin{aligned}
\mathbb{G}[u,\mathfrak{p};\Omega] :=& \int_\Omega \mathbf{G}(x,u,\nabla u)\,dx \\
=& \int_\Omega \Big(\mathbf{F}(\nabla u) - \mathfrak{p}(x)[\det\nabla u - 1]\Big)\,dx.
\end{aligned}$$

Then setting $u_\varepsilon(x) := u(x+\varepsilon\varphi)$ with $\varphi \in \mathbf{C}^\infty_c(\Omega,\mathbb{R}^n)$ an application of Theorem 4.1.2 and the *divergence* theorem along with a straight-forward calculation gives

$$\begin{aligned}
\frac{d}{d\varepsilon}\mathbb{G}[u_\varepsilon,\mathfrak{p};\Omega]\Big|_{\varepsilon=0} =& \int_\Omega \Big(\mathbf{T}^\beta_\alpha \varphi^\alpha_{,\beta} - \mathbf{G}_{x_\alpha}\varphi^\alpha\Big)\,dx \\
=& \int_\Omega \Big(u^i_{,\alpha}\mathbf{G}_{\xi^i_\beta}\varphi^\alpha_{,\beta} - \delta^\beta_\alpha \mathbf{G}\varphi^\alpha_{,\beta} - \mathbf{G}_{x_\alpha}\varphi^\alpha\Big)\,dx = 0.
\end{aligned} \tag{4.4}$$

In the course of the proof of next theorem we make repeated use of the following integration formula.

Proposition 4.1.3. *For every $f \in L^1(\Omega)$ we have that*

$$\int_\Omega f(x)\,dx = \int_0^1 \int_{\partial\Omega} \rho^{n-1}\frac{d(\theta)}{\alpha(\theta)} f(\rho\omega)\,d\mathcal{H}^{n-1}(\omega)d\rho.$$

Proof. As d and α are bounded away from zero a straight-forward proof of this assertion follows from *co-area* formula (*see* e.g., [28], Theorem 3.2.12, pp. 249) with the particular choice of $f(x) = |x|/d(x/|x|)$ there. □

Theorem 4.1.4. *Let $(u, \mathfrak{p})$ be a classical solution to the Euler-Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$. Assume that*

[**1**] $\mathbf{F}$ *is of class* $\mathbf{C}^2$*,*

[**2**] $\mathbf{F}$ *satisfies* $[\mathbf{H4}]_\xi$ *for all $\xi \in \{\nabla u(\omega) : \omega \in \partial\Omega\}$.*

Then with $\mathbf{G}$ *and* $\mathbf{T}$ *as in* (4.1) *and* (4.2) *we have that*

$$\mathbb{G}[u, \mathfrak{p}; \Omega] \le \mathbb{G}[\bar{u}, \bar{\mathfrak{p}}; \Omega], \tag{4.5}$$

where $\bar{u}, \bar{\mathfrak{p}}$ denote the homogeneous degree-one and degree-zero extensions of $u, \mathfrak{p}$ to Ω respectively, that is,

$$\bar{u}(x) := \frac{r}{d(\theta)} u(\theta d(\theta)),$$

and

$$\bar{\mathfrak{p}}(x) := \mathfrak{p}(\theta d(\theta)),$$

for $x \in \bar{\Omega}$ where $r = |x|$ and $\theta = x/|x|$.

Proof. For the sake of clarity and convenience we present this in the following *two* steps.

Step 1. ($\mathbb{G}[u, \mathfrak{p}; \Omega]$ as a *boundary* integral) For $t \in [0, 1]$ and $\varepsilon > 0$ put

$$\mathbf{s}_\varepsilon(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1 - \varepsilon, \\ 1 - \frac{t-(1-\varepsilon)}{\varepsilon} & \text{for } 1 - \varepsilon \le t \le 1, \end{cases}$$

and set

$$\varphi(x) = \mathbf{s}_\varepsilon\Big(\frac{|x|}{d(\theta)}\Big)x. \tag{4.6}$$

Then one can easily verify that

$$\begin{aligned}\nabla\varphi(x) &= \mathbf{s}_\varepsilon\Big(\frac{|x|}{d(\theta)}\Big)\mathbf{I}_n + |x|\frac{1}{d(\theta)}\mathbf{s}'_\varepsilon\Big(\frac{|x|}{d(\theta)}\Big)\theta \otimes \left(\theta - (\mathbf{I}_n - \theta\otimes\theta)\frac{\nabla d(\theta)}{d(\theta)}\right) \\ &= \mathbf{s}_\varepsilon\Big(\frac{|x|}{d(\theta)}\Big)\mathbf{I}_n + |x|\frac{\alpha(\theta)}{d(\theta)}\mathbf{s}'_\varepsilon\Big(\frac{|x|}{d(\theta)}\Big)\theta \otimes \nu,\end{aligned}$$

where $\theta = x/|x|$ and $\nu = \nu(\theta d(\theta))$ is the unit *outward* normal to $\partial\Omega$. Moreover it is evident that

$$\mathbf{1}_\Omega = \lim_{\varepsilon \downarrow 0} \mathbf{s}_\varepsilon(\frac{|x|}{d(\theta)}), \tag{4.7}$$

where the *limit* is being understood both as $\mathcal{L}^n$-a.e. in Ω and strongly in $L^1(\Omega)$. Now upon substituting φ as given by (4.6) into (4.4) and re-arranging terms it follows after taking into account (4.7) that

$$\begin{aligned} n\mathbb{G}[u, \mathfrak{p}; \Omega] =& \lim_{\varepsilon \downarrow 0} \int_\Omega n\mathbf{s}_\varepsilon(\frac{|x|}{d(\theta)})\mathbf{G}(x, u, \nabla u)\, dx \\ =& \lim_{\varepsilon \downarrow 0} \int_\Omega \Big\{ -\frac{\alpha(\theta)}{d(\theta)}|x|(\theta \cdot \nu)\mathbf{s}'_\varepsilon(\frac{|x|}{d(\theta)})\mathbf{G}(x, u, \nabla u) + \\ & \mathbf{s}_\varepsilon(\frac{|x|}{d(\theta)})\langle \mathbf{G}_\xi(x, u, \nabla u), \nabla u\rangle + \\ & \frac{\alpha(\theta)}{d(\theta)}|x|\mathbf{s}'_\varepsilon(\frac{|x|}{d(\theta)})\langle \mathbf{G}_\xi(x, u, \nabla u), \nabla u\, \theta \otimes \nu\rangle \Big\}\, dx \\ =& \lim_{\varepsilon \downarrow 0} \Big\{ \mathbf{I} + \mathbf{II} + \mathbf{III} \Big\}. \end{aligned} \tag{4.8}$$

We now proceed by considering each term separately. Indeed, with regards to the *first* term we have that

$$\begin{aligned} \mathbf{I} = \mathbf{I}(\varepsilon) =& \int_\Omega -\frac{\alpha(\theta)}{d(\theta)}|x|(\theta \cdot \nu)\mathbf{s}'_\varepsilon(\frac{|x|}{d(\theta)})\mathbf{G}(x, u, \nabla u)\, dx \\ =& \int_\Omega -\frac{1}{d(\theta)}|x|\mathbf{s}'_\varepsilon(\frac{|x|}{d(\theta)})\mathbf{F}(\nabla u(x))\, dx \\ =& \int_{1-\varepsilon}^1 \int_{\partial\Omega} \frac{1}{\varepsilon}\rho^n \frac{d(\theta)}{\alpha(\theta)}\mathbf{F}(\nabla u(\rho\omega))\, d\mathcal{H}^{n-1}(\omega) d\rho. \end{aligned}$$

Thus by passing to the limit $\varepsilon \downarrow 0$ we have that

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbf{I} =& \lim_{\varepsilon \downarrow 0} \int_{1-\varepsilon}^1 \int_{\partial\Omega} \frac{1}{\varepsilon}\rho^n \frac{d(\theta)}{\alpha(\theta)}\mathbf{F}(\nabla u(\rho\omega))\, d\mathcal{H}^{n-1}(\omega) d\rho \\ =& \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)}\mathbf{F}(\nabla u(\omega))\, d\mathcal{H}^{n-1}(\omega). \end{aligned}$$

In a similar way with regards to the *second* term we have that

$$\begin{aligned} \mathbf{II} = \mathbf{II}(\varepsilon) =& \int_\Omega \mathbf{s}_\varepsilon(\frac{|x|}{d(\theta)})\langle \mathbf{G}_\xi(x, u, \nabla u), \nabla u\rangle\, dx \\ =& \int_\Omega \mathbf{s}_\varepsilon(\frac{|x|}{d(\theta)})\langle \mathbf{F}_\xi(\nabla u) - \mathfrak{p}(x)[\mathrm{cof}\ \nabla u], \nabla u\rangle\, dx. \end{aligned}$$

Utilising (4.7) and Lebesgue's theorem on dominated converge, passing to the limit $\varepsilon \downarrow 0$ yields

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \mathbf{II} &= \lim_{\varepsilon\downarrow 0} \int_\Omega \mathbf{s}_\varepsilon(\frac{|x|}{d(\theta)})\langle \mathbf{F}_\xi(\nabla u) - \mathfrak{p}(x)[\mathrm{cof}\ \nabla u], \nabla u\rangle\, dx \\
&= \int_\Omega \langle \mathbf{F}_\xi(\nabla u) - \mathfrak{p}(x)[\mathrm{cof}\ \nabla u], \nabla u\rangle\, dx \\
&= \int_{\partial\Omega} \langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\ \nabla u(\omega)], u(\omega)\otimes\nu\rangle\, d\mathcal{H}^{n-1}(\omega),
\end{aligned}$$

where in the *second* identity we have appealed to the *divergence* theorem along with the fact that $(u, \mathfrak{p})$ is a solution to the Euler-Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}_p$.

Finally with regards to the *third* term we can write

$$\begin{aligned}
\mathbf{III} = \mathbf{III}(\varepsilon) &= \int_\Omega \frac{\alpha(\theta)}{d(\theta)}|x|\mathbf{s}'_\varepsilon(\frac{|x|}{d(\theta)})\langle \mathbf{G}_\xi(x, u, \nabla u), \nabla u\, \theta\otimes\nu\rangle\, dx \\
&= \int_\Omega \frac{\alpha(\theta)}{d(\theta)}|x|\mathbf{s}'_\varepsilon(\frac{|x|}{d(\theta)})\langle \mathbf{F}_\xi(\nabla u) - \mathfrak{p}(x)[\mathrm{cof}\ \nabla u], \nabla u\, \theta\otimes\nu\rangle\, dx \\
&= \int_{1-\varepsilon}^1 \int_{\partial\Omega} -\frac{1}{\varepsilon}\rho^n d(\theta)\times \\
&\quad \left\{\langle \mathbf{F}_\xi(\nabla u(\rho\omega)) - \mathfrak{p}(\rho\omega)[\mathrm{cof}\ \nabla u(\rho\omega)], \nabla u(\rho\omega)\, \theta\otimes\nu\rangle\right\} d\mathcal{H}^{n-1}(\omega) d\rho.
\end{aligned}$$

Thus by passing to the limit $\varepsilon \downarrow 0$ we have that

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \mathbf{III} &= \lim_{\varepsilon\downarrow 0} \int_{1-\varepsilon}^1 \int_{\partial\Omega} -\frac{1}{\varepsilon}\rho^n d(\theta)\times \\
&\quad \left\{\langle \mathbf{F}_\xi(\nabla u(\rho\omega)) - \mathfrak{p}(\rho\omega)[\mathrm{cof}\ \nabla u(\rho\omega)], \nabla u(\rho\omega)\, \theta\otimes\nu\rangle\right\} d\mathcal{H}^{n-1}(\omega) d\rho \\
&= \int_{\partial\Omega} -d(\theta)\langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\ \nabla u(\omega)], \nabla u(\omega)\, \theta\otimes\nu\rangle\, d\mathcal{H}^{n-1}(\omega).
\end{aligned}$$

Hence referring to (4.8) and *summarising* the above conclusions we have that

$$\begin{aligned}
n\mathbb{G}[u, \mathfrak{p}; \Omega] &= \int_\Omega n\mathbf{G}(x, u, \nabla u)\, dx \\
&= \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)}\mathbf{F}(\nabla u(\omega))\, d\mathcal{H}^{n-1}(\omega) + \\
&\quad \int_{\partial\Omega} \langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\ \nabla u(\omega)], u(\omega)\otimes\nu\rangle\, d\mathcal{H}^{n-1}(\omega) - \\
&\quad \int_{\partial\Omega} d(\theta)\langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\ \nabla u(\omega)], \nabla u(\omega)\theta\otimes\nu\rangle\, d\mathcal{H}^{n-1}(\omega).
\end{aligned} \tag{4.9}$$

Step 2. (A *lower* bound on $\mathbb{G}[\bar{u}, \bar{\mathfrak{p}}; \Omega]$) Recall that the *homogeneous* degree-one extension of u to Ω is given by

$$\bar{u}(x) = \frac{|x|}{d(\theta)}u(\theta d(\theta)),$$

for $x \in \bar{\Omega}$ with $\theta = x/|x|$. It can therefore be easily checked that

$$\begin{aligned}\nabla \bar{u}(x) =& \nabla u(\theta d(\theta)) + \Big\{ \Big(\frac{u(\theta d(\theta))}{d(\theta)} - \nabla u(\theta d(\theta))\theta \Big) \otimes \Big(\theta - (\mathbf{I}_n - \theta \otimes \theta) \frac{\nabla d(\theta)}{d(\theta)} \Big) \Big\} \\ =& \nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)} \Big\{ [u(\omega) - d(\theta) \nabla u(\omega)\theta] \otimes \nu \Big\},\end{aligned} \tag{4.10}$$

for $x \in \bar{\Omega}$ where $\omega = \theta d(\theta) \in \partial\Omega$. In particular we have that

$$\begin{aligned}\det \nabla \bar{u}(x) =& \det \nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)} \langle [\nabla u(\omega)]^{-1} [u(\omega) - d(\theta) \nabla u(\omega)\theta], \nu \rangle \\ =& 1 + \frac{\alpha(\theta)}{d(\theta)} \langle [\mathrm{cof}\ \nabla u(\omega)]^t [u(\omega) - d(\theta) \nabla u(\omega)\theta], \nu \rangle.\end{aligned} \tag{4.11}$$

Thus we can write

$$\begin{aligned}n\mathbb{G}[\bar{u}, \bar{\mathfrak{p}}; \Omega] =& n \int_\Omega \mathbf{G}(x, \bar{u}, \nabla \bar{u}; \bar{\mathfrak{p}})\, dx \\ =& n \int_\Omega \mathbf{F}(\nabla \bar{u}) - \bar{\mathfrak{p}}(x)[\det \nabla \bar{u} - 1]\, dx \\ =& n \int_0^1 \int_{\partial\Omega} \rho^{n-1} \frac{d(\theta)}{\alpha(\theta)} \times \\ & \Big\{ \mathbf{F}(\nabla \bar{u}(\rho\omega)) - \bar{\mathfrak{p}}(\rho\omega)[\det \nabla \bar{u}(\rho\omega) - 1] \Big\}\, d\mathcal{H}^{n-1}(\omega) d\rho \\ =& \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)} \Big\{ \mathbf{F}(\nabla \bar{u}(\omega)) - \bar{\mathfrak{p}}(\omega)[\det \nabla \bar{u}(\omega) - 1] \Big\}\, d\mathcal{H}^{n-1}(\omega),\end{aligned} \tag{4.12}$$

where in concluding the *last* line we have used the identities $\nabla \bar{u}(\rho\omega) = \nabla \bar{u}(\omega)$ and $\bar{\mathfrak{p}}(\rho\omega) = \bar{\mathfrak{p}}(\omega)$ for $\rho \in [0, 1]$ and $\omega \in \partial\Omega$ as a consequence of *homogeneity.*

Now anticipating on the integral on the *right* in (4.12) we *first* note that in view of the *rank-one* convexity of $\mathbf{F}$ at the points $\nabla u(\omega)$ using (4.10) [with $x = \omega$] we have that

$$\begin{aligned}\mathbf{F}(\nabla \bar{u}(\omega)) =& \mathbf{F}\Big(\nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)} [u(\omega) - d(\theta) \nabla u(\omega)\theta] \otimes \nu \Big) \\ \geq& \mathbf{F}\big(\nabla u(\omega)\big) + \frac{\alpha(\theta)}{d(\theta)} \langle \mathbf{F}_\xi(\nabla u(\omega)), [u(\omega) - d(\theta) \nabla u(\omega)\theta] \otimes \nu \rangle.\end{aligned} \tag{4.13}$$

Hence substituting from (4.11) and (4.13) into (4.12) and making note of the inequality $d/\alpha > 0$ we can write

$$\begin{aligned}n\mathbb{G}[\bar{u}, \bar{\mathfrak{p}}; \Omega] =& \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)} \Big\{ \mathbf{F}(\nabla \bar{u}(\omega)) - \bar{\mathfrak{p}}(\omega)[\det \nabla \bar{u}(\omega) - 1] \Big\}\, d\mathcal{H}^{n-1}(\omega) \\ \geq& \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)} \mathbf{F}(\nabla u(\omega))\, d\mathcal{H}^{n-1}(\omega) +\end{aligned}$$

$$\int_{\partial\Omega} \langle \mathbf{F}_\xi(\nabla u(\omega)), [u(\omega) - d(\theta)\nabla u(\omega)\theta] \otimes \nu \rangle \, d\mathcal{H}^{n-1}(\omega) -$$
$$\int_{\partial\Omega} \mathfrak{p}(\omega) \langle [\text{cof } \nabla u(\omega)]^t [u(\omega) - d(\theta)\nabla u(\omega)\theta], \nu \rangle \, d\mathcal{H}^{n-1}(\omega).$$

Finally, re-arranging terms and comparing the expression on the *right* in the above with (4.9) immediately yields

$$\begin{aligned} n\mathbb{G}[\bar{u}, \bar{\mathfrak{p}}; \Omega] \geq & \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)} \mathbf{F}(\nabla u(\omega)) \, d\mathcal{H}^{n-1}(\omega) + \\ & \int_{\partial\Omega} \langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\text{cof } \nabla u(\omega)], u(\omega) \otimes \nu \rangle \, d\mathcal{H}^{n-1}(\omega) - \\ & \int_{\partial\Omega} d(\theta) \langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\text{cof } \nabla u(\omega)], \nabla u(\omega)\theta \otimes \nu \rangle \, d\mathcal{H}^{n-1}(\omega) \\ \geq & n\mathbb{G}[u, \mathfrak{p}; \Omega], \end{aligned}$$

which is the required conclusion. □

4.2 Uniqueness theorems on starshaped domain

Theorem 4.2.1. (*Uniqueness* **I**) *Let $\Omega \subset \mathbb{R}^n$ be a $\mathbf{C}^1$ bounded starshaped domain and consider the energy functional $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$. Assume that*

[**1**] $\mathbf{F}$ *is of class* $\mathbf{C}^2$,

[**2**] $\mathbf{F}$ *satisfies* [**H1**] *and* $[\mathbf{H3}]_{\bar{\xi}}$,

[**3**] $(u, \mathfrak{p})$ *is a classical solution* (*see Definition* 4.1.1),

[**4**] $\mathbf{F}$ *satisfies* $[\mathbf{H4}]_\xi$ *for all* $\xi \in \{\nabla u(\omega) : \omega \in \partial\Omega\}$.

Then,

$$\mathbb{F}[u, \Omega] = \mathbb{F}[\bar{\xi}x, \Omega] = \inf_{\mathcal{A}_p(\Omega)} \mathbb{F}[\cdot, \Omega].$$

If, additionally, $\mathbf{F}$ *is strictly quasiconvex at* $\bar{\xi}$ *then* $u = \bar{\xi}x$ *on* $\bar{\Omega}$.

Proof. Evidently $\bar{u} = \bar{\xi}x$ and therefore $\det \nabla \bar{u} = 1$ in Ω. It should note that in general $\det \nabla \bar{u} = 1$ is *false*! [*See* (4.11)] However, interestingly, subject to $u = \bar{\xi}x$ on $\partial\Omega$ as described the latter identity holds throughout Ω. Hence referring to the estimate (4.5) in Theorem 4.1.4 and the *quasiconvexity* of $\mathbf{F}$ at $\bar{\xi}$ we can write

$$\mathbb{F}[\bar{u}, \Omega] \leq \mathbb{F}[u, \Omega] = \mathbb{G}[u, \mathfrak{p}; \Omega] \leq \mathbb{G}[\bar{u}, \bar{\mathfrak{p}}; \Omega] = \mathbb{F}[\bar{u}, \Omega].$$

The remaining assertion is now a *trivial* consequence of the latter and the *strict* quasiconvexity of $\mathbf{F}$ at $\bar{\xi}$. □

Remark 4.2.2. The proof of Theorem 4.1.4 and Theorem 4.2.1 remain unchanged if $\mathbf{F}$ is of class $\mathbf{C}^1$ and in Definition (4.1.1), [**1**] is replaced by $u \in \mathbf{C}^1(\bar{\Omega}, \mathbb{R}^n)$, [**2**] by $\mathfrak{p} \in \mathbf{C}(\bar{\Omega})$ and [**3**] by $(u, \mathfrak{p})$ being a *weak* solution to the corresponding system of Euler-Lagrange equation provided that *additionally* (4.4) holds.

Theorem 4.2.3. (*Uniqueness* **II**) *Let $\Omega \subset \mathbb{R}^n$ be a bounded starshaped domain and consider the energy functional $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$. Assume that*

[**1**] $\mathbf{F}$ *is of class* $\mathbf{C}$,

[**2**] $\mathbf{F}$ *satisfies* [**H1**] *and* $[\mathbf{H3}]_{\bar{\xi}}$,

[**3**] $u \in \mathcal{A}_p(\Omega)$ *is a strong local minimizer of $\mathbb{F}$, i.e., that there exists $\rho = \rho(u) > 0$ such that $\mathbb{F}[u, \Omega] \leq \mathbb{F}[w, \Omega]$ for all $w \in \mathcal{A}_p(\Omega)$ with $\|u - w\|_{L^1} \leq \rho$.*

Then,

$$\mathbb{F}[u, \Omega] = \mathbb{F}[\bar{\xi}x, \Omega] = \inf_{\mathcal{A}_p(\Omega)} \mathbb{F}[\cdot, \Omega]. \tag{4.14}$$

If, additionally, $\mathbf{F}$ is strictly quasiconvex at $\bar{\xi}$ then $u = \bar{\xi}x$ on $\bar{\Omega}$.

Proof. The *second* identity in (4.14) is a result of [**1**], [**2**] and a straight-forward *approximation* and so it suffices to justify only the *first* equality. Indeed for the sake of a contradiction assume $\mathbb{F}[u, \Omega] > \mathbb{F}[\bar{\xi}x, \Omega]$ and for $\delta \in (0, 1]$ and $x \in \Omega$ set

$$u_\delta(x) := \begin{cases} \delta u(\frac{x}{\delta}) & x \in \bar{\Omega}_\delta, \\ \bar{\xi}x & x \in \Omega \backslash \bar{\Omega}_\delta, \end{cases}$$

where $\Omega_\delta = \delta\Omega$. Then $\det \nabla u_\delta = 1$ $\mathcal{L}^n$-a.e. in Ω and so $u_\delta \in \mathcal{A}_p(\Omega)$. Moreover, a straight-forward calculation gives

$$\begin{aligned} \mathbb{F}[u_\delta, \Omega] &= \mathbb{F}[u, \Omega] + (1 - \delta^n)\Big\{\mathbb{F}[\bar{\xi}x, \Omega] - \mathbb{F}[u, \Omega]\Big\} \\ &< \mathbb{F}[u, \Omega], \end{aligned}$$

whilst $u_\delta \to u$ in $W^{1,p}$ as $\delta \uparrow 1$. This contradicts [**3**] and so the assertion is justified. The final part is now a *trivial* consequence of the latter and the *strict* quasiconvexity of $\mathbf{F}$ at $\bar{\xi}$. □

Chapter 5

Polyconvexity and generalised twists

In this chapter we consider the energy functional $\mathbb{F}$ as given by (1.6) over the space of orientation preserving maps $\mathcal{A}(\Omega)$ as defined by (1.7) in the first chapter and discuss the question of existence of multiple *strong* local minimizers for $\mathbb{F}$. Motivated by their signification in topology and the study of mapping class groups, we consider a class of maps, referred to as *generalised* twists as defined in Chapter 3, and examine them in connection with the corresponding Euler-Lagrange equation and we show that in even dimensions the latter system of equations admits *infinitely* many smooth solutions, modulo isometries, amongst such maps. In odd dimensions this number reduces to *one*. The material in this chapter is taken from Shahrokhi-Dehkordi & Taheri [57].

5.1 Generalised twists and the space of orientation preserving maps

We start this section by recalling the definition of a generalised twist. Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$. A map $u \in \mathbf{C}(\bar{\Omega}, \bar{\Omega})$ is referred to as a *generalised* twist if and only if it can be expressed as

$$u(x) = \mathbf{G}(r)\theta,$$

with

$$\mathbf{G}(r) = f(r)\mathbf{Q}(r),$$

where $r = |x|$, $\theta = x/|x|$, $\mathbf{Q} \in \mathbf{C}([a, b], \mathbf{SO}(n))$ and $f \in \mathbf{C}[a, b]$.

Notice that as a result of the basic requirement $u \in \mathbf{C}(\bar{\Omega}, \bar{\Omega})$ built into the definition of a *generalised* twist it follows in particular that $a \le |f| \le b$ on $[a, b]$ *see* (5.5). The continuous function $\mathbf{G}$ in the above definition will be referred to as the twist *path*. When additionally $\mathbf{G}(a) = \mathbf{G}(b)$ we refer to $\mathbf{G}$ as the twist *loop*.

Proposition 5.1.1. *Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$. A generalised twist u lies in $\mathcal{A}(\Omega)$ provided that the following set of conditions hold.*

[**1**] *Conditions on* $\mathbf{Q}$:

[**1a**] $\mathbf{Q} \in W^{1,2}([a, b], \mathbf{SO}(n))$,

[**1b**] $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$,

[**2**] *Conditions on* f :

[**2a**] $f \in W^{1,2}(a, b)$,

[**2b**] $\dot{f} > 0$ $\mathcal{L}^1$*-a.e. on* (a, b),

[**2c**] $f(a) = a$ *and* $f(b) = b$.

Proof. Let u be a *generalised* twists as in Definition 3.1.1. Then u lies in $\mathcal{A}(\Omega)$ if and only if the following conditions hold.

(i) $u = x$ on $\partial\Omega$,

(ii) $\det \nabla u > 0$ $\mathcal{L}^n$-a.e. in Ω, and,

(iii) $\|u\|_{W^{1,2}(\Omega)} < \infty$.

Evidently [**1b**] and [**2c**] together give (i). In addition a straight-forward *differentiation* reveals that

$$\nabla u = \frac{f}{r}\mathbf{Q} + (\dot{f} - \frac{f}{r})\mathbf{Q}\theta \otimes \theta + f\dot{\mathbf{Q}}\theta \otimes \theta. \tag{5.1}$$

Here we have denoted $\dot{f} := \frac{d}{dr}f$ and in a similar way $\dot{\mathbf{Q}} := \frac{d}{dr}\mathbf{Q}$. Therefore using the latter we can write

$$\begin{aligned}
\det \nabla u &= \det\left[\frac{f}{r}\mathbf{Q} + (\dot{f} - \frac{f}{r})\mathbf{Q}\theta \otimes \theta + f\dot{\mathbf{Q}}\theta \otimes \theta\right] \\
&= \det\left[\frac{f}{r}\mathbf{Q}\right] \det\left[\mathbf{I}_n + (\frac{r\dot{f}}{f} - 1)\mathbf{Q}\theta \otimes \mathbf{Q}\theta + r\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta\right] \\
&= (\frac{f}{r})^n\left[1 + (\frac{r\dot{f}}{f} - 1)\langle\mathbf{Q}\theta, \mathbf{Q}\theta\rangle + r\langle\dot{\mathbf{Q}}\theta, \mathbf{Q}\theta\rangle\right] \\
&= \dot{f}(\frac{f}{r})^{n-1},
\end{aligned} \tag{5.2}$$

where in concluding the *last* identity we have used the fact that $\langle \mathbf{Q}\theta, \mathbf{Q}\theta \rangle = |\theta|^2 = 1$ for all $\theta \in \mathbb{S}^{n-1}$ and so as a result

$$\begin{aligned} \langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta \rangle = \langle \dot{\mathbf{Q}}\theta, \mathbf{Q}\theta \rangle &= \frac{1}{2}\frac{d}{dr}\langle \mathbf{Q}\theta, \mathbf{Q}\theta \rangle \\ &= \frac{1}{2}\Big[\langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta \rangle + \langle \dot{\mathbf{Q}}\theta, \mathbf{Q}\theta \rangle\Big] = 0. \end{aligned} \tag{5.3}$$

Since as a result of [**2a**], [**2b**] and [**2c**] we have that $f \in \mathbf{C}[a,b]$ and $f(r) \in [a,b]$ for *all* $r \in [a,b]$ this immediately gives (ii). Now to justify (iii) we begin by *first* noting that

$$\begin{aligned} |\nabla u|^2 &= tr\Big\{[\nabla u][\nabla u]^t\Big\} \\ &= tr\Big\{\Big[\frac{f}{r}\mathbf{Q} + (\dot{f} - \frac{f}{r})\mathbf{Q}\theta \otimes \theta + f\dot{\mathbf{Q}}\theta \otimes \theta\Big] \times \\ &\qquad \Big[\frac{f}{r}\mathbf{Q}^t + (\dot{f} - \frac{f}{r})\theta \otimes \mathbf{Q}\theta + f\theta \otimes \dot{\mathbf{Q}}\theta\Big]\Big\} \\ &= tr\Big\{\frac{f}{r}\Big[\frac{f}{r}\mathbf{I}_n + (\dot{f} - \frac{f}{r})\mathbf{Q}\theta \otimes \mathbf{Q}\theta + f\mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta\Big] + \\ &\qquad (\dot{f} - \frac{f}{r})\Big[\frac{f}{r}\mathbf{Q}\theta \otimes \mathbf{Q}\theta + (\dot{f} - \frac{f}{r})\mathbf{Q}\theta \otimes \mathbf{Q}\theta + f\mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta\Big] + \\ &\qquad f\Big[\frac{f}{r}\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta + (\dot{f} - \frac{f}{r})\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta + f\dot{\mathbf{Q}}\theta \otimes \dot{\mathbf{Q}}\theta\Big]\Big\} \\ &= n(\frac{f}{r})^2 + (\dot{f}^2 - (\frac{f}{r})^2)\langle \mathbf{Q}\theta, \mathbf{Q}\theta \rangle + 2f\dot{f}\langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta \rangle + f^2\langle \dot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta \rangle \\ &= (n-1)(\frac{f}{r})^2 + \dot{f}^2 + f^2|\dot{\mathbf{Q}}\theta|^2. \end{aligned} \tag{5.4}$$

Next it is evident that

$$|u|^2 = \langle \mathbf{G}(r)\theta, \mathbf{G}(r)\theta \rangle = \langle f(r)\theta, f(r)\theta \rangle = |f|^2. \tag{5.5}$$

Hence by combining the latter we can write [1]

$$\begin{aligned} \|u\|^2_{W^{1,2}(\Omega)} = &\int_a^b \int_{\mathbb{S}^{n-1}} \Big\{f^2 + (n-1)(\frac{f}{r})^2 + \dot{f}^2 + \\ &\qquad f^2|\dot{\mathbf{Q}}\theta|^2\Big\} r^{n-1}\, d\mathcal{H}^{n-1}(\theta) dr \\ =&\omega_n \int_a^b \Big\{f^2\Big[n + n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2\Big] + n\dot{f}^2\Big\} r^{n-1}\, dr, \end{aligned} \tag{5.6}$$

[1]Here we are taking advantage of the *identity*

$$\int_{\mathbb{S}^{n-1}} \langle \mathbf{F}\theta, \theta \rangle \, d\mathcal{H}^{n-1}(\theta) = \omega_n tr\mathbf{F},$$

that holds for *any* given $\mathbf{F} \in \mathbb{M}_{n\times n}$. A straight-forward proof of this assertion is in Proposition B.0.8, Appendix B.

and so referring again to [**1a**] and [**2a**] the conclusion follows. □

Proposition 5.1.2. *Suppose that* u *is a generalised twist as in Definition* 3.1.1*. Then subject to* $\mathbf{Q} \in \mathbf{C}^2(]a,b[, \mathbf{SO}(n))$ *and* $f \in \mathbf{C}^2(]a,b[)$ *we have that*

$$\Delta u = [\alpha \mathbf{Q} + \beta \dot{\mathbf{Q}} + f \ddot{\mathbf{Q}}]\theta, \tag{5.7}$$

where

$$\alpha := \ddot{f} + \frac{n-1}{r}(\dot{f} - \frac{f}{r}),$$

and

$$\beta := 2\dot{f} + \frac{n-1}{r} f.$$

Proof. Referring to Definition 3.1.1 and using the notation $u = (u_1, u_2, \dots, u_n)$ we can write with the aid of (5.1) in Proposition 5.1.1 that

$$\begin{aligned}
\Delta u_i = & \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left\{ \frac{f}{r} \mathbf{Q}_{ij} + (\dot{f} - \frac{f}{r}) \sum_{k=1}^{n} \mathbf{Q}_{ik} \theta_k \theta_j + f \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k \theta_j \right\} \\
= & \sum_{j=1}^{n} \left\{ \frac{r\dot{f} - f}{r^2} \mathbf{Q}_{ij} \theta_j + \frac{f}{r} \dot{\mathbf{Q}}_{ij} \theta_j + (\ddot{f} - \frac{r\dot{f} - f}{r^2}) \sum_{k=1}^{n} \mathbf{Q}_{ik} \theta_k \theta_j^2 + \right. \\
& (\dot{f} - \frac{f}{r}) \left[\sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k \theta_j^2 + \frac{1}{r} \sum_{k=1}^{n} \mathbf{Q}_{ik} (\delta_{kj} - \theta_k \theta_j) \theta_j + \right. \\
& \left. \frac{1}{r} \sum_{k=1}^{n} \mathbf{Q}_{ik} \theta_k (1 - \theta_j^2) \right] + \dot{f} \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k \theta_j^2 + f \sum_{k=1}^{n} \ddot{\mathbf{Q}}_{ik} \theta_k \theta_j^2 + \\
& \left. \frac{f}{r} \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} (\delta_{kj} - \theta_k \theta_j) \theta_j + \frac{f}{r} \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k (1 - \theta_j^2) \right\}.
\end{aligned}$$

Hence we have that

$$\begin{aligned}
\Delta u_i = & \ddot{f} \sum_{k=1}^{n} \mathbf{Q}_{ik} \theta_k + \frac{(n-1)}{r} (\dot{f} - \frac{f}{r}) \sum_{k=1}^{n} \mathbf{Q}_{ik} \theta_k + \\
& 2\dot{f} \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k + f \sum_{k=1}^{n} \ddot{\mathbf{Q}}_{ik} \theta_k + \frac{(n-1)}{r} f \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k \\
= & \left[\ddot{f} + \frac{(n-1)}{r} (\dot{f} - \frac{f}{r}) \right] \sum_{k=1}^{n} \mathbf{Q}_{ik} \theta_k + \\
& \left[2\dot{f} + \frac{(n-1)}{r} f \right] \sum_{k=1}^{n} \dot{\mathbf{Q}}_{ik} \theta_k + f \sum_{k=1}^{n} \ddot{\mathbf{Q}}_{ik} \theta_k.
\end{aligned}$$

As this is true for every $1 \leq i \leq n$ using *vector* notation we can write

$$\Delta u = \left\{ \left[\ddot{f} + \frac{(n-1)}{r}(\dot{f} - \frac{f}{r}) \right] \mathbf{Q} + \left[2\dot{f} + \frac{(n-1)}{r} f \right] \dot{\mathbf{Q}} + f \ddot{\mathbf{Q}} \right\} \theta,$$

which is the required identity. □

5.2 The energy restricted to the space of twists

For a *generalised* twist u as in Definition 3.1.1 using (5.1) and (5.2) in Proposition 5.1.1 we can write

$$\begin{aligned}
\mathbb{F}[u, \Omega] &= \int_\Omega \left[\frac{1}{2} |\nabla u|^2 + \phi(\det \nabla u) \right] dx \\
&= \frac{1}{2} \int_a^b \int_{\mathbb{S}^{n-1}} \Big\{ (n-1)(\frac{f}{r})^2 + \dot{f}^2 + f^2 |\dot{\mathbf{Q}} \theta|^2 + \\
&\qquad 2\phi(\dot{f}(\frac{f}{r})^{n-1}) \Big\} r^{n-1} \, d\mathcal{H}^{n-1}(\theta) dr \\
&= \frac{\omega_n}{2} \int_a^b \Big\{ f^2 \left[n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2 \right] + n\dot{f}^2 + \\
&\qquad 2n\phi(\dot{f}(\frac{f}{r})^{n-1}) \Big\} r^{n-1} \, dr.
\end{aligned}$$

Motivated by the above representation in what follows we introduce the energy functional

$$\begin{aligned}
\mathbb{E}[\mathbf{Q}, f] := \int_a^b \Big\{ f^2 \left[n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2 \right] + n\dot{f}^2 + \\
2n\phi(\dot{f}(\frac{f}{r})^{n-1}) \Big\} r^{n-1} \, dr,
\end{aligned}$$

over the space of *admissible* maps

$$\mathcal{E} := \left\{ (\mathbf{Q}, f) : \begin{array}{l} \mathbf{Q} \in W^{1,2}([a,b], \mathbf{SO}(n)), \\ \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n, \\ f \in W^{1,2}[a,b], \\ \dot{f} > 0 \ \mathcal{L}^1\text{-}a.e. \text{ on } (a,b), \\ f(a) = a, f(b) = b. \end{array} \right\}$$

Our primary objective here is to obtain the Euler-Lagrange equations associated with the energy functional $\mathbb{E}$ over the space $\mathcal{E}$. Before that, we recall the Proposition 3.2.1 which in effect gives a characterisation of the tangent space to the *orthogonal* group at an arbitrary matrix $\mathbf{Q} \in \mathbf{SO}(n)$. This will turn useful in computing the *aforementioned* Euler-Lagrnage equations.

Proposition 5.2.1. *Let* $(\mathbf{Q}, f) \in \mathcal{E}$ *with* $\mathbf{Q} \in \mathbf{C}^2(]a,b[, \mathbf{SO}(n))$, $f \in \mathbf{C}^2(]a,b[)$ *and* $\dot{f} > 0$ *on* $]a,b[$. *Then assuming* $\mathbb{E}[\mathbf{Q}, f] < \infty$ *the Euler-Lagrange equations associated with* $\mathbb{E}$ *over* $\mathcal{E}$ *at* $(\mathbf{Q}, f)$ *take the form*

$$\mathbb{EL}[(\mathbf{Q}, f)] = 0,$$

that is,

$$\begin{cases} (i)\ \frac{d}{dr}\left[r^{n-1} f^2 \mathbf{Q}^t \frac{d}{dr}\mathbf{Q}\right] = 0, \\ \\ (ii)\ \frac{d}{dr}\left[r^{n-1}\dot{f} + f^{n-1}\phi'\right] = (n-1)[r^{n-3} f + \dot{f} f^{n-2}\phi'] + \frac{1}{n} r^{n-1} f |\dot{\mathbf{Q}}|^2, \end{cases}$$

on $]a,b[$ *where* $\phi' = \phi'(\dot{f}(\frac{f}{r})^{n-1})$.

Proof. First *fix* $\mathbf{Q}$ as described and for $\varepsilon \in \mathbb{R}$ put $\mathbf{Q}_\varepsilon = \mathbf{Q} + \varepsilon \mathbf{Q}(\mathbf{F} - \mathbf{F}^t)$ where $\mathbf{F} \in \mathbf{C}_0^\infty(]a,b[, \mathbb{M}_{n\times n})$. Then by utilising Proposition 3.2.1 we can write

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\mathbb{E}[\mathbf{Q}_\varepsilon, f]\bigg|_{\varepsilon=0} \\ 0 &= \frac{d}{d\varepsilon}\bigg[\int_a^b \bigg\{ f^2\bigg[n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}_\varepsilon|^2\bigg] + n\dot{f}^2 + \\ &\qquad\qquad 2n\phi(\dot{f}(\frac{f}{r})^{n-1})\bigg\} r^{n-1}\,dr\bigg]\bigg|_{\varepsilon=0} \\ &= \int_a^b r^{n-1} f^2 \langle \mathbf{Q}^t\dot{\mathbf{Q}}, (\dot{\mathbf{F}} - \dot{\mathbf{F}}^t)\rangle\,dr \\ &= -\int_a^b \langle \frac{d}{dr}[r^{n-1} f^2 \mathbf{Q}^t \dot{\mathbf{Q}}], (\mathbf{F} - \mathbf{F}^t)\rangle\,dr. \end{aligned}$$

Note that in concluding the *last* line we have used integration by *parts* together with the *boundary* conditions $\mathbf{F}(a) = \mathbf{F}(b) = 0$. Now in view of $\mathbf{Q}^t\dot{\mathbf{Q}}$ being *skew-symmetric* it follows that

$$\frac{d}{dr}\left[r^{n-1} f^2 \mathbf{Q}^t \frac{d}{dr}\mathbf{Q}\right] = 0,$$

which is the *first* equation in the system.

Next *fix* f as described and for $\varepsilon \in \mathbb{R}$ put $f_\varepsilon = f + \varepsilon g$ where $g \in \mathbf{C}_0^\infty(]a,b[)$. As $\dot{f} \in \mathbf{C}(]a,b[)$ and $\mathbf{K} := \text{supp } g \subset]a,b[$ is *compact* it follows that $\dot{f} \geq c > 0$ on $\mathbf{K}$. Thus for $|\varepsilon|$ sufficiently small ($|\varepsilon| \times \sup_{[a,b]} |\dot{g}| < c$) we have $\dot{f}_\varepsilon > 0$ on $]a,b[$ and so $(\mathbf{Q}, f_\varepsilon) \in \mathcal{E}$.

In addition by choosing ε smaller we have $\mathbb{E}[\mathbf{Q}, f_\varepsilon] < \infty$. The latter follows from the observation that $\dot{f}_\varepsilon = \dot{f}$ on $]a, b[\backslash\mathbf{K}$, the assumption $\mathbb{E}[\mathbf{Q}, f] < \infty$ and the *lower* and *upper* bounds

$$\frac{c}{2} \le \dot{f}_\varepsilon = \dot{f} + \varepsilon\dot{g} \le \sup_{\mathbf{K}} \dot{f} + |\varepsilon| \sup_{]a,b[} |\dot{g}|,$$

on $\mathbf{K}$ provided that $c \ge 2|\varepsilon| \sup_{]a,b[} |\dot{g}|$.

We can now proceed by considering the *variations* of $\mathbb{E}$ along the path $(\mathbf{Q}, f_\varepsilon)$ and as a result we can write

$$\begin{aligned}
0 =& \frac{d}{d\varepsilon}\mathbb{E}[\mathbf{Q}, f_\varepsilon]\Big|_{\varepsilon=0} \\
0 =& \frac{d}{d\varepsilon}\Bigg[\int_a^b \Bigg\{ f_\varepsilon^2 \Big[n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2\Big] + n\dot{f}_\varepsilon^2 + \\
& \qquad 2n\phi(\dot{f}_\varepsilon(\frac{f_\varepsilon}{r})^{n-1})\Bigg\} r^{n-1}\,dr\Bigg]\Bigg|_{\varepsilon=0} \\
=& \int_a^b \Bigg\{ 2fg\Big[n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2\Big] + 2n\dot{f}\dot{g} + \\
& \qquad 2n\Big[(\frac{f}{r})^{n-1}\dot{g} + (n-1)\frac{1}{r^{n-1}}\dot{f}f^{n-2}g\Big]\phi'\Bigg\} r^{n-1}\,dr \\
=& \int_a^b \Big[(n-1)[r^{n-3}f + \dot{f}f^{n-2}\phi'] + \frac{1}{n}r^{n-1}f|\dot{\mathbf{Q}}|^2\Big] g\,dr + \\
& \int_a^b \Big[r^{n-1}\dot{f} + f^{n-1}\phi'\Big]\dot{g}\,dr.
\end{aligned}$$

Now using integration by *parts* on the *second* term on the *right* together with the fact that $g(a) = g(b) = 0$ we obtain

$$\begin{aligned}
0 =& \int_a^b \Big[(n-1)[r^{n-3}f + \dot{f}f^{n-2}\phi'] + \frac{1}{n}r^{n-1}f|\dot{\mathbf{Q}}|^2\Big] g\,dr - \\
& \int_a^b \frac{d}{dr}\Big[r^{n-1}\dot{f} + f^{n-1}\phi'\Big] g\,dr.
\end{aligned}$$

As the latter is true for *all* $g \in \mathbf{C}_0^\infty(]a, b[)$ it follows that

$$\frac{d}{dr}\Big[r^{n-1}\dot{f} + f^{n-1}\phi'\Big] = (n-1)[r^{n-3}f + \dot{f}f^{n-2}\phi'] + \frac{1}{n}r^{n-1}f|\dot{\mathbf{Q}}|^2,$$

on $]a, b[$ which is the *second* equation in the system. This completes the proof. □

Any twist *loop* $\mathbf{G} = f\mathbf{Q}$ forming a solution to the Euler-Lagrange equations associated with $\mathbb{E}$ over $\mathcal{E}$ [i.e., whose corresponding f, $\mathbf{Q}$ satisfy (i), (ii) above] will be referred to as a *stationary* loop.

5.3 Energy minimizing loops in homotopy classes

Consider as in the previous section the energy functional

$$\mathbb{E}[\mathbf{Q}, f] := \int_a^b \Big\{ f^2 \Big[n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2 \Big] + n\dot{f}^2 + 2n\phi(\dot{f}(\frac{f}{r})^{n-1}) \Big\} r^{n-1}\, dr,$$

over the space of admissible maps

$$\mathcal{E} := \left\{ (\mathbf{Q}, f) : \begin{array}{l} \mathbf{Q} \in W^{1,2}([a,b], \mathbf{SO}(n)), \\ \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n, \\ f \in W^{1,2}[a,b], \\ \dot{f} > 0 \ \mathcal{L}^1\text{-}a.e. \text{ on } (a,b), \\ f(a) = a, f(b) = b. \end{array} \right\}$$

According to an elementary version of the *Sobolev* embedding theorem any pair $(\mathbf{Q}, f) \in \mathcal{E}$ has a *continuous* representative [again denoted $(\mathbf{Q}, f)$]. In particular each such $\mathbf{Q}$ represents an element of the fundamental group $\pi_1[\mathbf{SO}(n)]$ denoted $]\mathbf{Q}[$. As is well-known

$$\pi_1[\mathbf{SO}(n)] \cong \begin{cases} \mathbb{Z} & \text{when } n = 2, \\ \mathbb{Z}_2 & \text{when } n \geq 3, \end{cases}$$

and these facts together enables one to introduce the following *partitioning* of the Sobolev space $\mathcal{E}$.

[**1**] $(n = 2)$ for each $m \in \mathbb{Z}$ put

$$\mathfrak{c}_m[\mathcal{E}] := \Big\{ (\mathbf{Q}, f) \in \mathcal{E} :]\mathbf{Q}[= m \Big\}.$$

As a result the latter are *pairwise* disjoint and that

$$\mathcal{E} = \bigcup_{m \in \mathbb{Z}} \mathfrak{c}_m[\mathcal{E}].$$

[**2**] $(n \geq 3)$ for each $\alpha \in \mathbb{Z}_2 = \{0, 1\}$ put

$$\mathfrak{c}_\alpha[\mathcal{E}] := \Big\{ (\mathbf{Q}, f) \in \mathcal{E} :]\mathbf{Q}[= \alpha \Big\}.$$

As a result, again, the latter are *pairwise* disjoint and that

$$\mathcal{E} = \bigcup_{\alpha \in \mathbb{Z}_2} \mathfrak{c}_\alpha[\mathcal{E}].$$

An application of the direct methods of the *calculus of variations* to the energy functional $\mathbb{E}$ together with the observation that the homotopy classes $\mathfrak{c}_\star[\mathcal{E}] \subset \mathcal{E}$ are *sequentially* weakly closed results in the existence of [multiple] *minimizing* stationary loops (*See* Theorem 5.3.2.). We note that the *sequential* weak closedness of the homotopy classes $\mathfrak{c}_\star[\mathcal{E}]$ is a result of $\mathbf{SO}(n)$ having a *tubular* neighbourhood that projects back onto it and this in turn follows from $\mathbf{SO}(n)$ being a smooth *compact* manifold.

We begin by *first* establishing the following straight-forward lower bound on $\mathbb{E}$ amounting to it being coercive on $\mathcal{E}$.

Proposition 5.3.1. (Coercivity). *There exists $d = d(n,a,b) > 0$ such that*

$$\mathbb{E}[\mathbf{Q}, f] \geq d(\|\mathbf{Q}\|^2_{W^{1,2}} + \|f\|^2_{W^{1,2}}),$$

for all $(\mathbf{Q}, f) \in \mathcal{E}$.

Proof. Since for all $(\mathbf{Q}, f) \in \mathcal{E}$ we have that $a \leq f \leq b$ on $[a,b]$ by taking into account that $\phi \geq 0$ we can write

$$\begin{aligned}\mathbb{E}[\mathbf{Q}, f] = &\int_a^b \left\{ f^2 \left[n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2 \right] + n\dot{f}^2 + \right.\\ &\left. \qquad 2n\phi(\dot{f}(\frac{f}{r})^{n-1}) \right\} r^{n-1}\, dr\\ \geq &\int_a^b \left\{ n\left[(n-1)\frac{1}{r^2} f^2 + \dot{f}^2 \right] + f^2|\dot{\mathbf{Q}}|^2 \right\} r^{n-1}\, dr.\end{aligned}$$

The conclusion now follows by utilising the Poincaré inequality. $\square$

Theorem 5.3.2. (Existence of energy minimizing loops).

Consider the energy functional $\mathbb{E}$ over the space of admissible maps $\mathcal{E}$. Then,

[**1**] $(n = 2)$ *for each $m \in \mathbb{Z}$ there exists $(\mathbf{Q}_m, f_m) \in \mathfrak{c}_m[\mathcal{E}]$ such that*

$$\mathbb{E}[\mathbf{Q}_m, f_m] = \inf_{\mathfrak{c}_m[\mathcal{E}]} \mathbb{E},$$

[**2**] $(n \geq 3)$ *for each $\alpha \in \mathbb{Z}_2$ there exists $(\mathbf{Q}_\alpha, f_\alpha) \in \mathfrak{c}_\alpha[\mathcal{E}]$ such that*

$$\mathbb{E}[\mathbf{Q}_\alpha, f_\alpha] = \inf_{\mathfrak{c}_\alpha[\mathcal{E}]} \mathbb{E}.$$

Proof. First of all we note that each homotopy class $\mathfrak{c}_\star[\mathcal{E}] \subset \mathcal{E}$ admits a pair $(\mathbf{Q}, f)$ for which $\mathbb{E}[\mathbf{Q}, f] < \infty$ follows by taking, e.g., $f = r$ and $\mathbf{Q}$ a *smooth* loop representing the corresponding element of $\pi_1[\mathbf{SO}(n)]$. Let $(\mathbf{Q}_j, f_j) \subset c_\star[\mathcal{E}]$ denote an *infimizing* sequence for $\mathbb{E}$ over $\mathfrak{c}_\star[\mathcal{E}]$. Then appealing to Proposition 5.3.1 it follows that by passing to a subsequence (not re-labeled) we have that

$$\begin{cases} \mathbf{Q}_j \to \mathbf{Q} & \text{in } \mathbf{C}([a,b], \mathbf{SO}(n)), \\ \mathbf{Q}_j \rightharpoonup \mathbf{Q} & \text{in } W^{1,2}([a,b], \mathbf{SO}(n)), \\ f_j \to f & \text{in } \mathbf{C}[a,b], \\ f_j \rightharpoonup f & \text{in } W^{1,2}(a,b). \end{cases}$$

As a result we conclude in particular that $a \le f \le b$ on $[a, b]$ and additionally

$$\begin{aligned} &f_j \dot{\mathbf{Q}}_j \rightharpoonup f \dot{\mathbf{Q}} \\ &\dot{f}_j (\frac{f_j}{r})^{n-1} \rightharpoonup \dot{f} (\frac{f}{r})^{n-1}, \end{aligned}$$

where both convergences are interpreted as *weakly* in L^2. Therefore by standard lower *semicontinuity* results (*see*, e.g., [18]) we have that

$$\begin{aligned} \inf_{\mathfrak{c}_\star[\mathcal{E}]} \mathbb{E} &\le \mathbb{E}[\mathbf{Q}, f] \\ &\le \liminf_{j \uparrow \infty} \mathbb{E}[\mathbf{Q}_j, f_j] \\ &\le \inf_{\mathfrak{c}_\star[\mathcal{E}]} \mathbb{E} < \infty. \end{aligned} \tag{5.8}$$

The above *firstly* implies that $\dot{f} > 0$ $\mathcal{L}^1$-a.e. on $]a, b[$ and as a result $(\mathbf{Q}, f) \in \mathcal{E}$. This in view of the *closedness* of the homotopy classes $\mathfrak{c}_\star[\mathcal{E}] \subset \mathcal{E}$ with respect to the topology of *uniform* convergence gives

$$]\mathbf{Q}[=]\mathbf{Q}_j[,$$

and therefore $(\mathbf{Q}, f) \in \mathfrak{c}_\star[\mathcal{E}]$. A *second* appeal to (5.8) now implies $(\mathbf{Q}, f)$ to be the required minimizer on $\mathfrak{c}_\star[\mathcal{E}]$. □

Remark 5.3.3. It can be shown that in [**1**] and [**2**] above the resulting minimizers satisfy the corresponding Euler-Lagrnage equations described in Proposition 5.2.1. The argument here will follow closely that given in detail in the proof of Theorem 5.4.3 and hence will be abbreviated.

5.4 Alternative construction of multiple stationary loops

In section 5.3 we proved the existence of *multiple* stationary loops by directly *minimizing* the energy functional $\mathbb{E}$ over the homotopy classes $\mathfrak{c}_\star[\mathcal{E}]$ of the *loop* space $\mathcal{E}$. By contrast in this section we focus on the Euler-Lagrange equation itself and present a class of stationary loops that in turn will prove fruitful in discussing the *existence* of multiple solutions to the Euler-Lagrange equations associated with the energy functional $\mathbb{F}$ over the space $\mathcal{A}(\Omega)$.

Indeed here we establish the existence of multiple (*infinitely* many) *stationary* loops $\mathbf{G} = f\mathbf{Q}$ where the pair $(\mathbf{Q}, f) \in \mathcal{E}$, depending on whether the dimension n is *even* or *odd*, has one of the following *specific* forms.

[**1**] $(n = 2k)$

$$\mathbf{Q} = \mathbf{Q}[\mathbf{g}] := \mathbf{P}diag(\mathfrak{R}(g_1), \mathfrak{R}(g_2), \dots, \mathfrak{R}(g_k))\mathbf{P}^t,$$

[**2**] $(n = 2k + 1)$

$$\mathbf{Q} = \mathbf{Q}[\mathbf{g}] := \mathbf{P}diag(\mathfrak{R}(g_1), \mathfrak{R}(g_2), \dots, \mathfrak{R}(g_k), 1)\mathbf{P}^t,$$

where $\mathbf{P} \in \mathbf{O}(n)$ is *fixed*, $\mathbf{g} = (g_1, g_2, \dots, g_k) \in \mathcal{J}_\mathbf{m}$ (*see* below) and the matrix $\mathfrak{R} \in \mathbb{M}_{2\times 2}$ is given by

$$\mathfrak{R}(s) := \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}.$$

Indeed for $\mathbf{m} = (m_1, m_2, \dots, m_k) \in \mathbb{Z}^k$ we put

$$\mathcal{J}_\mathbf{m} = \mathcal{J}_\mathbf{m}(a,b) := \left\{ (\mathbf{g}, f) : \begin{array}{l} \mathbf{g} \in [W^{1,2}(a,b)]^k, \\ \mathbf{g}(a) = 0, \mathbf{g}(b) = 2\pi\mathbf{m}, \\ f \in W^{1,2}(a,b), \\ \dot{f} > 0 \ \mathcal{L}^1\text{-}a.e. \text{ on } (a,b), \\ f(a) = a, f(b) = b. \end{array} \right\}$$

It is thus evident that for each such $\mathbf{m}$ and *fixed* $\mathbf{P} \in \mathbf{O}(n)$ and with $\mathbf{Q} = \mathbf{Q}[\mathbf{g}]$ we have that

$$(\mathbf{g}, f) \in \mathcal{J}_\mathbf{m} \iff (\mathbf{Q}, f) \in \mathcal{E}.$$

Next for $(\mathbf{g}, f) \in \mathscr{J}_{\mathbf{m}}$ as described above and *fixed* $\mathbf{P} \in \mathbf{O}(n)$ denoting again $\mathbf{Q} = \mathbf{Q}[\mathbf{g}]$ we introduce

$$\begin{aligned}\mathbb{J}[\mathbf{g}, f] :=& \mathbb{E}[\mathbf{Q}, f] \\ =& \int_a^b \left\{ f^2 \left[n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2 \right] + n\dot{f}^2 + 2n\phi(\dot{f}(\frac{f}{r})^{n-1}) \right\} r^{n-1}\, dr \\ =& \int_a^b \left\{ f^2 \left[n(n-1)\frac{1}{r^2} + 2|\dot{\mathbf{g}}|^2 \right] + n\dot{f}^2 + 2n\phi(\dot{f}(\frac{f}{r})^{n-1}) \right\} r^{n-1}\, dr \\ =:& \int_a^b \mathbf{J}(r, \dot{\mathbf{g}}, f, \dot{f})\, dr, \end{aligned} \tag{5.9}$$

where we have set $|\dot{\mathbf{g}}|^2 = \sum_{j=1}^k \dot{g}_j^2$ whilst

$$\mathbf{J}(r, \mathbf{s}, z, p) := n\left\{ z^2 \left[(n-1)\frac{1}{r^2} + \frac{2}{n}|\mathbf{s}|^2 \right] + p^2 + 2\phi(p(\frac{z}{r})^{n-1}) \right\} r^{n-1}.$$

Proposition 5.4.1. *There exists $L > 0$ and $\sigma > 0$ so that for all $\alpha > 0$ satisfying $|\alpha - 1| < \sigma$ we have that*

$$\left| z\mathbf{J}_z(r, \mathbf{s}, \alpha z, p) \right| \leq L \left[\mathbf{J}(r, \mathbf{s}, z, p) + 1 \right],$$

for all $r \in [a, b]$, $\mathbf{s} \in \mathbb{R}^k$, $z \in]0, \infty[$ and $p \in]0, \infty[$.

Proof. This follows by *direct* verification and use of [**h5**]. □

Proposition 5.4.2. *For fixed $r \in [a, b]$, $\mathbf{s} \in \mathbb{R}^k$ and $z \in]0, \infty[$ the function $\mathbf{J}_p$ has the following limiting behaviours*

$$\lim_{p \downarrow 0} \mathbf{J}_p(r, \mathbf{s}, z, p) = -\infty,$$

$$\lim_{p \uparrow \infty} \mathbf{J}_p(r, \mathbf{s}, z, p) = \infty$$

Proof. This is an immediate consequence of [**h2**]-[**h4**]. □

We are now in a position to state the main result of this section on existence of *infinitely* many stationary loops $(\mathbf{Q}, f) \in \mathcal{E}$ for which $\mathbf{Q} = \mathbf{Q}[\mathbf{g}]$.

Theorem 5.4.3. (Existence and regularity of multiple stationary loops).
Consider the energy functional $\mathbb{J}$ over the space $\mathscr{J}_{\mathbf{m}}$. Then for each $\mathbf{m} \in \mathbb{Z}^k$ there exists $(\mathbf{g}, f) \in \mathscr{J}_{\mathbf{m}}$ such that

$$\mathbb{J}[\mathbf{g}, f] = \inf_{\mathscr{J}_{\mathbf{m}}} \mathbb{J}[\cdot].$$

In addition the pair $(\mathbf{g}, f)$ *satisfies the corresponding Euler-Lagrange equations*

$$\mathbb{EL}[\mathbf{g}, f] = 0,$$

that is,

$$\begin{cases} \frac{d}{dr}\Bigg[r^{n-1}f^2\dot{\mathbf{g}}\Bigg] = 0, \\ \\ \frac{d}{dr}\Bigg[r^{n-1}\dot{f} + f^{n-1}\phi'\Bigg] = (n-1)[r^{n-3}f + \dot{f}f^{n-2}\phi'] + \frac{2}{n}r^{n-1}f|\dot{\mathbf{g}}|^2, \end{cases}$$

on $]a, b[$ *where* $\phi' = \phi'(\dot{f}(\frac{f}{r})^{n-1})$ *whilst* $(\mathbf{g}, f) \in \mathbf{C}^2[a, b] \times \cdots \times \mathbf{C}^2[a, b]$ *and* $\dot{f} > 0$ *on* $[a, b]$.

Note that the above Euler-Lagrange equations will be shown to be satisfied by the pair $(\mathbf{g}, f)$ as a result of its *minimizing* property. One can then verify that the latter equations result from those in Proposition 5.2.1 upon making the substitution $(\mathbf{Q}, f) = (\mathbf{Q}[\mathbf{g}], f)$. Thus any such $(\mathbf{g}, f)$ gives rise to an associated *stationary* loop!

Proof. (*Existence*) Let $(\mathbf{g}_j, f_j) \subset \mathcal{J}_{\mathbf{m}}$ denote an *infimizing* sequence for $\mathbb{J}$ over $\mathcal{J}_{\mathbf{m}}$. An application of Proposition 5.3.1 (with $\mathbf{Q}_j := \mathbf{Q}[\mathbf{g}_j]$) gives

$$\begin{aligned} \infty > \mathbb{J}[\mathbf{g}_j, f_j] = \mathbb{E}[\mathbf{Q}_j, f_j] &\geq d(\|\mathbf{Q}_j\|^2_{W^{1,2}} + \|f_j\|^2_{W^{1,2}}) \\ &\geq d\Bigg[n(b-a) + 2\|\dot{\mathbf{g}}_j\|^2_{L^2} + \|f_j\|^2_{W^{1,2}}\Bigg], \end{aligned}$$

and so as a result $(\mathbf{g}_j, f_j) \subset \mathcal{J}_{\mathbf{m}}$ is bounded. It thus follows that by passing to a subsequence (not re-labeled) we have that

$$\begin{cases} \mathbf{g}_j \to \mathbf{g} & \text{in } \mathbf{C}[a, b], \\ \mathbf{g}_j \rightharpoonup \mathbf{g} & \text{in } W^{1,2}(a, b), \\ f_j \to f & \text{in } \mathbf{C}[a, b], \\ f_j \rightharpoonup f & \text{in } W^{1,2}(a, b). \end{cases}$$

Hence in particular $a \leq f \leq b$ on $[a, b]$ and that

$$\begin{aligned} &f_j\dot{\mathbf{g}}_j \rightharpoonup f\dot{\mathbf{g}} \\ &\dot{f}_j(\frac{f_j}{r})^{n-1} \rightharpoonup \dot{f}(\frac{f}{r})^{n-1}, \end{aligned}$$

where both convergences are interpreted as *weakly* in L^2. Therefore, again, by standard lower *semicontinuity* results we have that

$$\begin{aligned}\inf_{\mathcal{J}_{\mathbf{m}}} \mathbb{J} &\le \mathbb{J}[\mathbf{g}, f] \\ &\le \liminf_{j\uparrow\infty} \mathbb{J}[\mathbf{g}_j, f_j] \\ &\le \inf_{\mathcal{J}_{\mathbf{m}}} \mathbb{E} < \infty. \end{aligned} \tag{5.10}$$

The above *firstly* implies that $\dot{f} > 0$ $\mathcal{L}^1$-*a.e.* on $]a, b[$ which gives $(\mathbf{g}, f) \in \mathcal{J}_{\mathbf{m}}$ and *secondly* that $(\mathbf{g}, f)$ is the required *minimizer*.

(*Regularity*) Let $(\mathbf{g}, f) \in \mathcal{J}_{\mathbf{m}}$ be the *minimizer* from the previous part. For the sake of clarity and convenience we present the proof of this in the following *three* steps.

Step 1 ($\mathbf{g} \in \mathbf{C}^1[a, b]$) Evidently $f \in \mathbf{C}[a, b]$ and $a \le f \le b$ on $[a, b]$. Hence the assertion follows immediately in view of $\mathbf{g}$ *minimizing* the integral

$$\mathbf{h} \mapsto \int_a^b |\dot{\mathbf{h}}|^2 f^2 r^{n-1}\, dr,$$

among all $\mathbf{h}$ with $(\mathbf{h}, f) \in \mathcal{J}_{\mathbf{m}}$.

Step 2 ($f \in \mathbf{C}^1[a, b]$) The argument here is based upon *suitably* modifying a well-known technique from [4], Theorem 7.3. To this end for $j \in \mathbb{N}$ put

$$E_j := \left\{ r \in]a, b[: j^{-1} \le \dot{f}(r) \le j \right\}.$$

Then (E_j) is *monotone* increasing and $\mathcal{L}^1(]a, b[\setminus \cup_{j=1}^{\infty} E_j) = 0$. Now *fix* j and pick $w \in L^\infty(a, b)$ such that

$$\int_{E_j} w = \int_a^b w \mathbf{1}_{E_j} = 0. \tag{5.11}$$

For $\varepsilon \in \mathbb{R}$ put

$$f_\varepsilon(r) := f(r) + \varepsilon \int_a^r w \mathbf{1}_{E_j}.$$

Then we have that

[**1**] $f_\varepsilon(a) = f(a) = a$,

[**2**] $f_\varepsilon(b) = f(b) = b$,

[**3**] $\dot{f}_\varepsilon(r) = \dot{f}(r)$ for $\mathcal{L}^1$-a.e. $r \notin E_j$,

[**4**] $\dot{f}_\varepsilon(r) > 0$ $\mathcal{L}^1$-a.e. on $]a, b[$, provided that $|\varepsilon| \times ||w||_{L^\infty(a,b)} < j^{-1}$.

The aim is now to derive the Euler-Lagrnage equation associated with f as a result of differentiating the energy functional $\mathbb{J}$ along f_ε at $\varepsilon = 0$. To this end consider *first* the *difference* quotient

$$
\begin{aligned}
\left| \frac{\mathbf{J}(r, \dot{\mathbf{g}}, f_\varepsilon, \dot{f}_\varepsilon) - \mathbf{J}(r, \dot{\mathbf{g}}, f, \dot{f})}{\varepsilon} \right| \le & \left| \frac{\mathbf{J}(r, \dot{\mathbf{g}}, f_\varepsilon, \dot{f}_\varepsilon) - \mathbf{J}(r, \dot{\mathbf{g}}, f_\varepsilon, \dot{f})}{\varepsilon} \right| + \\
& \left| \frac{\mathbf{J}(r, \dot{\mathbf{g}}, f_\varepsilon, \dot{f}) - \mathbf{J}(r, \dot{\mathbf{g}}, f, \dot{f})}{\varepsilon} \right| = \mathbf{I} + \mathbf{II}.
\end{aligned}
$$

Then an application of the *mean* value theorem gives

$$
\mathbf{I} = \left| \frac{\mathbf{J}(r, \dot{\mathbf{g}}, f_\varepsilon, \dot{f}_\varepsilon) - \mathbf{J}(r, \dot{\mathbf{g}}, f_\varepsilon, \dot{f})}{\varepsilon} \right| \le c,
$$

where $c = c(j) > 0$ is independent of ε. [Notice that indeed $\mathbf{I} = 0$ for $\mathcal{L}^1$-a.e. $r \notin E_j$.] In a similar way we have that

$$
\begin{aligned}
\mathbf{II} &= \left| \frac{\mathbf{J}(r, \dot{\mathbf{g}}, f_\varepsilon, \dot{f}) - \mathbf{J}(r, \dot{\mathbf{g}}, f, \dot{f})}{\varepsilon} \right| \\
&= \left| \mathbf{J}_z(r, \dot{\mathbf{g}}, f + \theta[f_\varepsilon - f], \dot{f}) \right| \left| \frac{f_\varepsilon - f}{\varepsilon} \right|,
\end{aligned}
$$

where $\theta = \theta(\varepsilon, r) \in [0, 1]$. However, since

$$
\begin{aligned}
f + \theta[f_\varepsilon - f] &= f \left[1 + \theta \frac{f_\varepsilon - f}{f} \right] \\
&= f \left[1 + \varepsilon\theta \frac{1}{f} \int_a^r w \mathbf{1}_{E_j} \right],
\end{aligned}
$$

it follows from Proposition 5.4.1 that upon choosing ε sufficiently small we can write

$$
\begin{aligned}
\mathbf{II} &= \left| \frac{\mathbf{J}(r, \dot{\mathbf{g}}, f_\varepsilon, \dot{f}) - \mathbf{J}(r, \dot{\mathbf{g}}, f, \dot{f})}{\varepsilon} \right| \\
&\le L \left[\mathbf{J}(r, \dot{\mathbf{g}}, f, \dot{f}) + 1 \right] \int_a^r w \mathbf{1}_{E_j} =: F(r),
\end{aligned}
$$

where $F \in L^1(a, b)$ [note that $\mathbb{J}[\mathbf{g}, f] < \infty$]. Hence an application of Lebesgue's theorem on *dominated* convergence gives

$$
\begin{aligned}
0 &= \frac{d}{d\varepsilon} \mathbb{J}[\mathbf{g}, f_\varepsilon] \Big|_{\varepsilon = 0} \\
&= \lim_{\varepsilon \to 0} \int_a^b \frac{\mathbf{J}(r, \dot{\mathbf{g}}, f_\varepsilon, \dot{f}_\varepsilon) - \mathbf{J}(r, \dot{\mathbf{g}}, f, \dot{f})}{\varepsilon} \, dr \\
&= \int_a^b \left[\mathbf{J}_p(r, \dot{\mathbf{g}}, f, \dot{f}) w \mathbf{1}_{E_j} + \mathbf{J}_z(r, \dot{\mathbf{g}}, f, \dot{f}) \int_a^r w \mathbf{1}_{E_j} \right] dr
\end{aligned}
$$

$$= \int_a^b w\mathbf{1}_{E_j}\left[\mathbf{J}_p(r,\dot{\mathbf{g}},f,\dot{f}) - \int_a^r \mathbf{J}_z(\rho,\dot{\mathbf{g}},f,\dot{f})\,d\rho\right] dr,$$

where in concluding the *last* line we have used a convenient form of integration by parts. Therefore recalling (5.11) it follows from the above (note that arguing as in estimating **II** above [*see* Proposition 5.4.1] and taking into account $\mathbb{J}[\mathbf{g},f] < \infty$ and $a \le f \le b$ it follows that $\mathbf{J}_z(r,\dot{\mathbf{g}},f,\dot{f})$ is *summable* on $]a,b[$)

$$\mathbf{J}_p(r,\dot{\mathbf{g}},f,\dot{f}) - \int_a^r \mathbf{J}_z(\rho,\dot{\mathbf{g}},f,\dot{f})\,d\rho = c_j,$$

for $\mathcal{L}^1$-a.e. $r \in E_j$. [Here c_j is an *arbitrary* constant.] Now in view of (E_j) being *monotone* increasing it follows that c_j is independent of j and in view of $\cup_{j=1}^{\infty} E_j$ having *full* measure in $]a,b[$ that

$$\mathbf{J}_p(r,\dot{\mathbf{g}},f,\dot{f}) = c + \int_a^r \mathbf{J}_z(\rho,\dot{\mathbf{g}},f,\dot{f})\,d\rho, \tag{5.12}$$

for $\mathcal{L}^1$-a.e. $r \in]a,b[$. As the term on the *right* is absolutely continuous on $[a,b]$, using [**h2**] and Proposition 5.4.2, it follows that by modifying $\dot{f}$ on an $\mathcal{L}^1$-null set, we have $\dot{f} > 0$ and *equality* in (5.12) holds *everywhere* on $[a,b]$ (hence $\mathbf{J}_p(r,\dot{\mathbf{g}},f,\dot{f})$ is continuous on $[a,b]$). Standard arguments (*see*, e.g., [4] pp. 584 or [18] pp. 57-61) now give the continuity of $\dot{f}$ on $[a,b]$. A close inspection of the proof of Theorem 2.6(ii) in [18] reveals that having (5.12) the *same* conclusion holds if the assumption $\mathbf{J}$ being of class $\mathbf{C}^1$ is replaced by $\mathbf{J}_{\dot{f}}$ being of class $\mathbf{C}$. In a similar way the conclusion of Theorem 2.6(iii) holds if the assumption $\mathbf{J}$ being of class $\mathbf{C}^2$ is replaced by $\mathbf{J}_{\dot{f}}$ being of class $\mathbf{C}^1$.
Step 3 ($\mathbf{g} \in \mathbf{C}^2[a,b]$, $f \in \mathbf{C}^2[a,b]$) The required regularity of g follows using the conclusion in **step 2** in **step 1** and that of f from the conclusion in **step 2** and the Hilbert-Weierstrass differentiability theorem (*see* [18]). □

5.5 The restricted versus the full Euler-Lagrange equations

In this section we discuss in detail the implications that the Euler-Lagrange equations associated with the energy functional $\mathbb{F}$ will exert upon the *twist* loop $\mathbf{G} = f\mathbf{Q}$ of a *generalised* twist $u \in \mathcal{A}(\Omega)$.

Theorem 5.5.1. *Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ and let u be a generalised twist as in Definition 3.1.1. Assume in addition that the twist loop $\mathbf{G} = f\mathbf{Q}$ satisfies the following assumptions.*

[**1**] $\mathbf{Q} \in \mathbf{C}^2(]a,b[, \mathbf{SO}(n))$,

[**2**] $f \in \mathbf{C}^2(]a,b[)$,

[**3**] $\dot{f} > 0$ *on* $]a,b[$,

[**4**] $(\mathbf{Q}, f) \in \mathcal{E}$.

Then the following implication holds.

$$\mathbb{EL}[(\mathbf{Q}, f)] = 0 \implies \mathbb{EL}[u] := [\nabla u]^t \Delta u + (\det \nabla u)\nabla \Big[\phi'(\det \nabla u)\Big] = f\left[\frac{1}{n}\dot{f}|\dot{\mathbf{Q}}|^2 \mathbf{I}_n - \frac{f}{r}\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} - (\dot{f} - \frac{f}{r})|\dot{\mathbf{Q}}\theta|^2 \mathbf{I}_n\right]\theta.$$

Proof. We proceed by evaluating each of the expressions in $\mathbb{EL}[u]$ separately. Indeed with regards to the *first* term using (5.1) in Proposition 5.1.1 in conjunction with (5.7) in Proposition 5.1.2 we can write

$$\begin{aligned}
[\nabla u]^t \Delta u &= \left[\frac{f}{r}\mathbf{Q}^t + (\dot{f} - \frac{f}{r})\theta \otimes \mathbf{Q}\theta + f\theta \otimes \dot{\mathbf{Q}}\theta\right]\left[\alpha \mathbf{Q} + \beta \dot{\mathbf{Q}} + f\ddot{\mathbf{Q}}\right]\theta \\
&= \Bigg\{\alpha \frac{f}{r}\mathbf{I}_n + \beta \frac{f}{r}\mathbf{Q}^t\dot{\mathbf{Q}} + \frac{f}{r}^2 \mathbf{Q}^t\ddot{\mathbf{Q}} + \\
&\qquad \left[\alpha\langle \mathbf{Q}\theta, \mathbf{Q}\theta\rangle + \beta\langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta\rangle + f\langle \mathbf{Q}\theta, \ddot{\mathbf{Q}}\theta\rangle\right](\dot{f} - \frac{f}{r})\mathbf{I}_n + \\
&\qquad \left[\alpha\langle \dot{\mathbf{Q}}\theta, \mathbf{Q}\theta\rangle + \beta\langle \dot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\rangle + f\langle \dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle\right]f\mathbf{I}_n\Bigg\}\theta \\
&= \Bigg\{\frac{f}{r}\left[\beta \mathbf{Q}^t\dot{\mathbf{Q}} + f\mathbf{Q}^t\ddot{\mathbf{Q}}\right] + f(\dot{f} - \frac{f}{r})\langle \mathbf{Q}\theta, \ddot{\mathbf{Q}}\theta\rangle \mathbf{I}_n + \alpha \dot{f}\mathbf{I}_n + \\
&\qquad \left[\beta|\dot{\mathbf{Q}}\theta|^2 + f\langle \dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle\right]f\mathbf{I}_n\Bigg\}\theta,
\end{aligned} \tag{5.13}$$

where in concluding the *last* equation we have made repeated *use* of the identity (5.3).

Now referring to the Euler-Lagrange equations in Proposition 5.2.1 it follows upon expansion of (i) that

$$\begin{aligned}
\frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}f^2\mathbf{Q}^t\dot{\mathbf{Q}}\right] &= (n-1)\frac{f^2}{r}\mathbf{Q}^t\dot{\mathbf{Q}} + 2f\dot{f}\mathbf{Q}^t\dot{\mathbf{Q}} + f^2\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} + f^2\mathbf{Q}^t\ddot{\mathbf{Q}} \\
&= f\left\{\left[(n-1)\frac{f}{r} + 2\dot{f}\right]\mathbf{Q}^t\dot{\mathbf{Q}} + f\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} + f\mathbf{Q}^t\ddot{\mathbf{Q}}\right\} \\
&= f\left[\beta \mathbf{Q}^t\dot{\mathbf{Q}} + f\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} + f\mathbf{Q}^t\ddot{\mathbf{Q}}\right] = 0.
\end{aligned} \tag{5.14}$$

By *pre*-multiplying (5.14) with $\dot{\mathbf{Q}}^t\mathbf{Q}$ and ignoring the *non-zero* factor f we can thus conclude that

$$\begin{aligned}
0 &= \dot{\mathbf{Q}}^t\mathbf{Q}\left[\beta \mathbf{Q}^t\dot{\mathbf{Q}} + f\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} + f\mathbf{Q}^t\ddot{\mathbf{Q}}\right] \\
&= \beta\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} + f\dot{\mathbf{Q}}^t\mathbf{Q}\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} + f\dot{\mathbf{Q}}^t\ddot{\mathbf{Q}}.
\end{aligned}$$

However in view of the *second* term in the *last* line being *skew-symmetric* the above equation in particular implies that for *all* $\theta \in \mathbb{S}^{n-1}$ we have

$$\begin{aligned} \langle [\beta \dot{\mathbf{Q}}^t \dot{\mathbf{Q}} + f \dot{\mathbf{Q}}^t \mathbf{Q} \dot{\mathbf{Q}}^t \dot{\mathbf{Q}} + f \dot{\mathbf{Q}}^t \ddot{\mathbf{Q}}]\theta, \theta \rangle =& \beta |\dot{\mathbf{Q}}\theta|^2 + f \langle \dot{\mathbf{Q}}\theta, \theta \ddot{\mathbf{Q}} \rangle \\ =& \beta |\dot{\mathbf{Q}}\theta|^2 + f \langle \dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta \rangle \\ =& 0. \end{aligned} \tag{5.15}$$

In a similar way referring to (5.3) we have that

$$\begin{aligned} \frac{d}{dr} \langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta \rangle =& \langle \dot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta \rangle + \langle \mathbf{Q}\theta, \ddot{\mathbf{Q}}\theta \rangle \\ =& |\dot{\mathbf{Q}}\theta|^2 + \langle \mathbf{Q}\theta, \ddot{\mathbf{Q}}\theta \rangle \\ =& 0. \end{aligned} \tag{5.16}$$

Therefore by substituting (5.14), (5.15) and (5.16) into (5.13) respectively we arrive at the identity

$$\begin{aligned} [\nabla u]^t \Delta u = & \Big\{ \frac{f}{r} \Big[\beta \mathbf{Q}^t \dot{\mathbf{Q}} + f \mathbf{Q}^t \ddot{\mathbf{Q}} \Big] + f(\dot{f} - \frac{f}{r}) \langle \mathbf{Q}\theta, \ddot{\mathbf{Q}}\theta \rangle \mathbf{I}_n + \alpha \dot{f} \mathbf{I}_n + \\ & \Big[\beta |\dot{\mathbf{Q}}\theta|^2 + f \langle \dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta \rangle \Big] f \mathbf{I}_n \Big\} \theta, \\ = & \Big[-\frac{f}{r}^2 \dot{\mathbf{Q}}^t \dot{\mathbf{Q}} - f(\dot{f} - \frac{f}{r}) |\dot{\mathbf{Q}}\theta|^2 \mathbf{I}_n + \alpha \dot{f} \mathbf{I}_n \Big] \theta. \end{aligned} \tag{5.17}$$

Next referring again to the Euler-Lagrange equation in Proposition 5.2.1 it follows upon expansion of (ii) that

$$\begin{aligned} (\det \nabla u) \nabla \Big[\phi'(\det \nabla u) \Big] =& \dot{f} (\frac{f}{r})^{n-1} \frac{d}{dr} \Big[\phi'(\dot{f} (\frac{f}{r})^{n-1}) \Big] \theta \\ =& - \Big[\frac{(n-1)}{r} \dot{f}^2 + \dot{f} \ddot{f} - \frac{(n-1)}{r^2} f \dot{f} - \frac{1}{n} f \dot{f} |\dot{\mathbf{Q}}|^2 \Big] \theta \\ =& - \Big[\alpha \dot{f} - \frac{1}{n} f \dot{f} |\dot{\mathbf{Q}}|^2 \Big] \theta. \end{aligned} \tag{5.18}$$

Therefore, by combining (5.17) and (5.18), we arrive at

$$\begin{aligned} \mathbb{EL}[u] =& [\nabla u]^t \Delta u + (\det \nabla u) \nabla \Big[\phi'(\det \nabla u) \Big] \\ =& [\nabla u]^t \Delta u + (\det \nabla u) \phi''(\det \nabla u) \nabla \Big[\det \nabla u \Big] \\ =& \Big[-\frac{f}{r}^2 \dot{\mathbf{Q}}^t \dot{\mathbf{Q}} - f(\dot{f} - \frac{f}{r}) |\dot{\mathbf{Q}}\theta|^2 \mathbf{I}_n + \alpha \dot{f} \mathbf{I}_n + \end{aligned}$$

$$\frac{1}{n}f\dot{f}|\dot{\mathbf{Q}}|^2\mathbf{I}_n - \alpha\dot{f}\mathbf{I}_n\Big]\theta$$
$$=\Big[\frac{1}{n}f\dot{f}|\dot{\mathbf{Q}}|^2\mathbf{I}_n - \frac{f}{r}^2\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} - f(\dot{f}-\frac{f}{r})|\dot{\mathbf{Q}}\theta|^2\mathbf{I}_n\Big]\theta,$$

which is the required conclusion. □

Theorem 5.5.2. *Let* $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ *and let* u *be a generalised twist as in Definition 3.1.1. Assume* $\mathbb{F}[u,\Omega] < \infty$ *and that the twist loop* $\mathbf{G} = f\mathbf{Q}$ *satisfies the following assumptions.*

[**1**] $\mathbf{Q} \in \mathbf{C}^2(]a,b[, \mathbf{SO}(n))$,

[**2**] $f \in \mathbf{C}^2(]a,b[)$,

[**3**] $\dot{f} > 0$ *on* $]a,b[$,

[**4**] $(\mathbf{Q}, f) \in \mathcal{E}$.

Then the following equivalence between the full and the restricted Euler-Lagrange equations holds. [2]

$$\mathbb{EL}[u] = 0 \iff \left\{\begin{array}{l} (i)\ \mathbb{EL}[(\mathbf{Q},f)] = 0, \\ (ii)\ \dot{\mathbf{Q}}(r) \in \mathbb{R}\mathbf{SO}(n) \text{ for all } r \in]a,b[. \end{array}\right\}$$

Proof. Let u be a *generalised* twist and let $\mathbf{G} = f\mathbf{Q}$ denote its twist *loop*. Then in view of [**1**]-[**4**] above an application of Theorem 5.5.1 gives

$$\begin{aligned}\mathbb{EL}[u] = 0 &\iff [\nabla u]^t\Delta u + (\det\nabla u)\nabla\Big[\phi'(\det\nabla u)\Big] = 0 \\ &\Longrightarrow \Big[\frac{1}{n}\dot{f}|\mathbf{F}|^2\mathbf{I}_n - (\dot{f}-\frac{f}{r})|\mathbf{F}\theta|^2\mathbf{I}_n - \frac{f}{r}\mathbf{F}^t\mathbf{F}\Big]\theta = 0, \end{aligned} \tag{5.19}$$

with $\mathbf{F} = \dot{\mathbf{Q}}(r)$. Moreover, we have that

$$\mathbb{EL}[u] = 0 \implies \mathbb{EL}[(\mathbf{Q},f)] = 0. \tag{5.20}$$

[This follows, e.g., by arguing as in Proposition 5.2.1 and noting that the equation on the *left* results from taking a larger class of variations in $\mathbb{F}$ than that on the *right*.]

With the aid of the *equivalence* and the implications in (5.19) and (5.20) we now proceed by establishing the *two* implications in the conclusion of the theorem separately.

(*Sufficiency* "$\Longleftarrow$") Fix $r \in]a,b[$ and assume that $\mathbf{F} := \dot{\mathbf{Q}}(r) \in \mathbb{R}\mathbf{SO}(n)$. Then by definition

[2]Recall that for every *non-negative* integer n we have that

$$\mathbb{R}\mathbf{SO}(n) := \Big\{\mathbf{F} : \mathbf{F} = \rho\mathbf{Q} \text{ where } \rho \in \mathbb{R}, \mathbf{Q} \in \mathbf{SO}(n)\Big\}.$$

there exists $\rho = \rho(r) \in \mathbb{R}$ and $\mathbf{R} = \mathbf{R}(r) \in \mathbf{SO}(n)$ such that

$$\mathbf{F} = \rho\mathbf{R}.$$

A straight-forward calculation now gives

$$\begin{aligned} 0 =& \left[\dot{f}\rho^2 - (\dot{f} - \frac{f}{r})\rho^2 - \frac{f}{r}\rho^2\right]\mathbf{I}_n \\ =& \left[\frac{1}{n}\dot{f}|\mathbf{F}|^2\mathbf{I}_n - (\dot{f} - \frac{f}{r})|\mathbf{F}\theta|^2\mathbf{I}_n - \frac{f}{r}\mathbf{F}^t\mathbf{F}\right] \\ =& \left[\frac{1}{n}\dot{f}|\dot{\mathbf{Q}}|^2\mathbf{I}_n - (\dot{f} - \frac{f}{r})|\dot{\mathbf{Q}}\theta|^2\mathbf{I}_n - \frac{f}{r}\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\right]. \end{aligned}$$

Therefore if $\mathbb{EL}[(\mathbf{Q}, f)] = 0$ an application of Theorem 5.5.1 immediately gives $\mathbb{EL}[u] = 0$.

(*Necessity* "$\Longrightarrow$") Assume that $\mathbb{EL}[u] = 0$. Fix $r \in]a, b[$ and put $\mathbf{Q} := \mathbf{Q}(r)$ and $\mathbf{F} := \dot{\mathbf{Q}}(r)$. Then referring to (5.19) for every $\theta \in \mathbb{S}^{n-1}$ we have that

$$\begin{aligned} 0 =& \langle\left[\frac{1}{n}\dot{f}|\mathbf{F}|^2\mathbf{I}_n - (\dot{f} - \frac{f}{r})|\mathbf{F}\theta|^2\mathbf{I}_n - \frac{f}{r}\mathbf{F}^t\mathbf{F}\right]\theta, \theta\rangle \\ =& \frac{1}{n}\dot{f}|\mathbf{F}|^2 - (\dot{f} - \frac{f}{r})|\mathbf{F}\theta|^2 - \frac{f}{r}|\mathbf{F}\theta|^2 \\ =& \dot{f}\left[\frac{1}{n}|\mathbf{F}|^2 - |\mathbf{F}\theta|^2\right]. \end{aligned}$$

In view of the latter being true for *all* $\theta \in \mathbb{S}^{n-1}$ [and that $\dot{f}(r) \neq 0$] it follows that $\mathbf{F} \in \mathbb{R}\mathbf{O}(n)$. Indeed *fix* $\mathbf{F} \in \mathbb{M}_{n\times n}$ and put $\mathbf{A} := \mathbf{F}^t\mathbf{F}$. Then it is evident that

$$\frac{1}{n}|\mathbf{F}|^2 = \langle\mathbf{F}\theta, \mathbf{F}\theta\rangle \iff \frac{1}{n}tr\mathbf{A} = \langle\mathbf{A}\theta, \theta\rangle,$$

[for *all* $\theta \in \mathbb{S}^{n-1}$]. Since $\mathbf{A}$ is *symmetric* and *non-negative* its *eigen-values* are *real* and satisfy $0 \leq \lambda_1[\mathbf{A}] \leq \cdots \leq \lambda_n[\mathbf{A}]$. Testing the above *identity* in turn with corresponding *eigen-vectors* gives at once $\lambda_1[\mathbf{A}] = \cdots = \lambda_n[\mathbf{A}] := \lambda$ and so $\mathbf{A} = \lambda\mathbf{I}_n$. This can now easily be seen to give $\mathbf{F} \in \mathbb{R}\mathbf{O}(n)$. However as $\mathbf{QF}^t$ is *skew-symmetric* it follows from Proposition A.0.5 that $\mathbf{QF}^t \in \mathbb{R}\mathbf{SO}(n)$ and so $\mathbf{F} \in \mathbb{R}\mathbf{SO}(n)$. This together with (5.20) completes the proof. □

Theorem 5.5.3. *Let* $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ *and let* u *be a generalised twist as in Definition* 3.1.1. *Assume* $\mathbb{F}[u, \Omega] < \infty$ *and that the twist loop* $\mathbf{G} = f\mathbf{Q}$ *satisfies the following assumptions.*

[1] $\mathbf{Q} \in \mathbf{C}^2(]a, b[, \mathbf{SO}(n))$,

[2] $f \in \mathbf{C}^2(]a, b[)$,

[3] $\dot{f} > 0$ *on* $]a, b[$,

[**4**] $(\mathbf{Q}, f) \in \mathcal{E}$.

Then the following equivalence between the full and the restricted Euler-Lagrange equations hold.

$$
\mathbb{EL}[u] = 0 \iff \left\{ \begin{array}{ll} [\mathbf{a}]\ (n = 2k) & \textit{there exist } g = g(r) \in \mathbf{C}^2[a,b] \textit{ with} \\ & g(a), g(b) \in 2\pi\mathbb{Z} \textit{ and } \mathbf{P} \in \mathbf{O}(n) \textit{ so} \\ & \textit{that} \\ & (i)\ \mathbf{Q} = \mathbf{P} diag(\mathfrak{R}(g), ..., \mathfrak{R}(g))\mathbf{P}^t, \\ & (ii)\ \mathbb{EL}[g, f] = 0. \\ & \textit{Moreover we have that} \\ & (iii)\ f \in \mathbf{C}^2[a,b]. \\ [\mathbf{b}]\ (n = 2k+1) & \\ & (i)\ \mathbf{Q} = \mathbf{I}_n, \\ & (ii)\ \mathbb{EL}[0, f] = 0, \\ & \textit{Moreover we have that} \\ & (iii)\ f \in \mathbf{C}^2[a,b]. \end{array} \right\}
$$

Note that in $[\mathbf{a}](ii)$ *and* $[\mathbf{b}](ii)$ *above we have denoted*

$$
\mathbb{EL}[g, f] = 0,
$$

as an abbreviation for the second order system

$$
\left\{ \begin{array}{l} \frac{d}{dr}\left[r^{n-1} f^2 \dot{g} \right] = 0, \\ \\ \frac{d}{dr}\left[r^{n-1}\dot{f} + f^{n-1}\phi' \right] = (n-1)[r^{n-3} f + \dot{f} f^{n-2}\phi'] + r^{n-1} f \dot{g}^2, \end{array} \right.
$$

where $\phi' = \phi'(\dot{f}(\frac{f}{r})^{n-1})$ *on* $]a, b[$.

Proof. We establish each of the *two* implications in the conclusion of the theorem separately.

(*Sufficiency* "$\Longleftarrow$") We restrict to the case [**a**] only as for [**b**] the conclusion is *trivially* true. Indeed let g, $\mathbf{P}$ and $\mathbf{Q}$ be as described. Then a straight-forward *differentiation* gives

$$
\begin{aligned}
\dot{\mathbf{Q}}^t \dot{\mathbf{Q}} =& [\mathbf{P} diag(\dot{\mathfrak{R}}(g), \dots, \dot{\mathfrak{R}}(g))\mathbf{P}^t]^t \times \\
& [\mathbf{P} diag(\dot{\mathfrak{R}}(g), \dots, \dot{\mathfrak{R}}(g))\mathbf{P}^t] \\
=& \dot{g}^2 \mathbf{P} \mathbf{I}_n \mathbf{P}^t = \dot{g}^2 \mathbf{I}_n,
\end{aligned}
$$

while $diag(\dot{\mathfrak{R}}(g), \dots, \dot{\mathfrak{R}}(g)) \in \mathbb{R}\mathbf{SO}(n)$. Hence $\dot{\mathbf{Q}}(r) \in \mathbb{R}\mathbf{SO}(n)$ for *all* $r \in]a, b[$. Next, using the same expression for $\mathbf{Q}$ we can verify that

$$\begin{aligned}\mathbf{Q}^t\dot{\mathbf{Q}} =& [\mathbf{P}diag(\mathfrak{R}(g), \dots, \mathfrak{R}(g))\mathbf{P}^t]^t \times \\ & [\mathbf{P}diag(\dot{\mathfrak{R}}(g), \dots, \dot{\mathfrak{R}}(g))\mathbf{P}^t] \\ =& \dot{g}\mathbf{P}diag(\mathbf{J}_2, \dots, \mathbf{J}_2)\mathbf{P}^t,\end{aligned}$$

and in a similar way that

$$|\dot{\mathbf{Q}}|^2 = tr[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}] = tr[\dot{g}^2\mathbf{I}_n] = n\dot{g}^2.$$

Therefore referring to Proposition 5.2.1 it follows that

$$\mathbb{EL}[(\mathbf{Q}, f)] = \mathbb{EL}[g, f] = 0,$$

where in concluding the *second* equality we have appealed to [**a**](ii) above. The assertion is now easily seen to follow from Theorem 5.5.2.

(*Necessity* "$\Longrightarrow$") Assume that $\mathbb{EL}[u] = 0$. Then according to Theorem 5.5.2 we have that

$$\begin{cases} (i)\ \mathbb{EL}[(\mathbf{Q}, f)] = 0, \\ \\ (ii)\ \dot{\mathbf{Q}}(r) \in \mathbb{R}\mathbf{SO}(n) \text{ for all } r \in]a, b[. \end{cases}$$

Now referring to (i) above by integrating the *first* equation in the corresponding system (*see* Proposition 5.2.1) we can write

$$r^{n-1}f^2\mathbf{Q}^t\frac{d}{dr}\mathbf{Q} = \mathbf{A}, \tag{5.21}$$

where $\mathbf{A} \in \mathbb{M}_{n\times n}$ is *skew-symmetric* and by (ii) above $\mathbf{A} \in \mathbb{R}\mathbf{SO}(n)$. We now consider the cases [**a**] and [**b**] separately.

[**a**] ($n = 2k$) By utilising Proposition A.0.5 there exist $\alpha \in \mathbb{R}$, $\mathbf{P} \in \mathbf{O}(n)$ such that we can re-write the above equation in the more convenient form

$$\begin{aligned}\frac{d}{dr}\mathbf{Q} &= \alpha\frac{1}{r^{n-1}f^2}\mathbf{Q}\mathbf{P}diag(\mathbf{J}_2, \mathbf{J}_2, \dots, \mathbf{J}_2)\mathbf{P}^t \\ &=: \mu\mathbf{Q}\mathbf{J}.\end{aligned} \tag{5.22}$$

Let $g \in \mathbf{C}^1[a,b] \cap \mathbf{C}^2]a,b[$ be a primitive of μ satisfying $g(a) \in 2\pi\mathbb{Z}$ and then *fix* α so that $g(b) \in 2\pi\mathbb{Z}$. Then a straight-forward *differentiation* gives

$$\begin{aligned}\frac{d}{dr}e^{g\mathbf{J}} &= \dot{g}e^{g\mathbf{J}}\mathbf{J}\\ &= \mu e^{g\mathbf{J}}\mathbf{J},\end{aligned}$$

whilst

$$\begin{aligned}e^{g\mathbf{J}} &= e^{g\mathbf{P}diag(\mathbf{J}_2,\ldots,\mathbf{J}_2)\mathbf{P}^t}\\ &= \mathbf{P}e^{g[diag(\mathbf{J}_2,\ldots,\mathbf{J}_2)]}\mathbf{P}^t\\ &= \mathbf{P}diag(\mathfrak{R}(g),\ldots,\mathfrak{R}(g))\mathbf{P}^t.\end{aligned}$$

Hence by the *uniqueness* of solutions to initial value problems [applied to (5.22)] it follows that $\mathbf{Q} = e^{g\mathbf{J}}$ on $[a,b]$. This gives [**a**](i). Using the latter conclusion [**a**](ii) follows as in the proof of the *sufficiency* part using (i) above. Finally that $g, f \in \mathbf{C}^2[a,b]$ follows by using an adaptation of the argument in Theorem 5.4.3 along with the well-known Hilbert-Weierstrass *differentiability* theorem (*See* [4] pp. 584 and [18] pp. 57-61). As will be seen in the next section $\mathbb{EL}[g,f] = 0$ is a *genuine* Euler-Lagrange equation [in fact corresponding to the energy functional $\mathfrak{J}$ over the space $(2\pi m_a, 0) + \mathcal{J}_{m_b - m_a}$ (*see* Section 5.6)].
[**b**] $(n = 2k+1)$ An application of Proposition A.0.5 gives $\mathbf{A} = 0$. Hence referring to (5.21) together with the *boundary* conditions $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ it follows that $\mathbf{Q} = \mathbf{I}_n$ on $[a,b]$. This gives [**b**](i). Finally according to (i) above we have that

$$\mathbb{EL}[0,f] = \mathbb{EL}[(\mathbf{I}_n, f)] = 0$$

which gives [**b**](ii). The proof is thus complete. □

5.6 A characterisation of all twist solutions

In the previous section we discussed the implications that the Euler-Lagrange equations associated with the energy functional $\mathbb{F}$ exerted upon the *twist* loop $\mathbf{G} = f\mathbf{Q}$ corresponding to a *generalised* twist $u \in \mathcal{A}(\Omega)$ in order for the latter to furnish a solution to these equations. In this section we show how this analysis enables one to give a *complete* characterisation of all such *twist* solutions. (*See* Definition 5.6.2.)

We begin by considering the case of *even* dimensions ($n = 2k$). Here for each *fixed* $m \in \mathbb{Z}$ we set

$$\mathcal{J}_m := \mathcal{J}_m(a,b) := \left\{ (g,f) : \begin{array}{l} g \in W^{1,2}(a,b), \\ g(a) = 0, g(b) = 2\pi m, \\ f \in W^{1,2}(a,b), \\ \dot{f} > 0 \ \mathcal{L}^1\text{-}a.e. \text{ on } (a,b) \\ f(a) = a, f(b) = b, \end{array} \right\},$$

and

$$\begin{aligned} \mathfrak{J}[g,f] :=& \mathbb{J}[\mathbf{g},f] \\ =& \int_a^b \left\{ f^2 \left[n(n-1)\frac{1}{r^2} + n\dot{g}^2 \right] + n\dot{f}^2 + 2n\phi(\dot{f}(\frac{f}{r})^{n-1}) \right\} r^{n-1}\, dr, \end{aligned}$$

where $\mathbf{g} = (g, g, \ldots, g)$. With the aid of this notations we have the following statement.

Theorem 5.6.1. (Existence and regularity of special stationary loops).
Consider the energy functional $\mathfrak{J}$ over the space $\mathcal{J}_m$. Then for each $m \in \mathbb{Z}$ there exist $g = g(r;a,b,m)$ and $f = f(r;a,b,m)$ with $(g,f) \in \mathcal{J}_m$ such that

$$\mathfrak{J}[g,f] = \inf_{\mathcal{J}_m} \mathfrak{J}[\cdot].$$

Moreover the pair (g,f) satisfies the corresponding Euler-Lagrange equations

$$\mathbb{EL}[g,f] = 0,$$

that is,

$$\begin{cases} \frac{d}{dr}\left[r^{n-1} f^2 \dot{g} \right] = 0, \\ \\ \frac{d}{dr}\left[r^{n-1}\dot{f} + f^{n-1}\phi' \right] = (n-1)[r^{n-3}f + \dot{f}f^{n-2}\phi'] + r^{n-1}f\dot{g}^2, \end{cases}$$

on $]a,b[$ where $\phi' = \phi'(\dot{f}(\frac{f}{r})^{n-1})$. Additionally $(g,f) \in \mathbf{C}^2[a,b] \times \mathbf{C}^2[a,b]$ and $\dot{f} > 0$ on $[a,b]$.

Proof.
The argument here is similar to that used in Theorem 5.4.3 and hence will be abbreviated. □

We now return to the energy functional $\mathbb{F}$ defined over the space of admissible maps $\mathcal{A}(\Omega)$. For the sake of clarity and future reference we proceed with the following *definition*.

Definition 5.6.2. (**Classical solution**)

An admissible map $u \in \mathcal{A}(\Omega)$ is referred to as a *classical* solution to the Euler-Lagrange equations associated with the energy functional (1.6) over the space (1.7) if and only if the following hold:

[**1**] $\mathbb{F}[u, \Omega] < \infty$,

[**2**] $u \in \mathbf{C}^2(\Omega, \mathbb{R}^n) \cap \mathbf{C}(\bar{\Omega}, \mathbb{R}^n)$,

[**3**] u satisfies the *system* of equations

$$\begin{cases} [\nabla u]^t \Delta u + \det \nabla u \nabla \Big[\phi'(\det \nabla u)\Big] = 0 & \text{in } \Omega, \\ \det \nabla u > 0 & \text{in } \Omega, \\ u = x & \text{on } \partial\Omega. \end{cases}$$

Note that when speaking of a *classical* solution in the form of a *generalised* twist [i.e., $u(x) = f(r)\mathbf{Q}(r)\theta$] in connection with [**2**] above we *implicitly* assume the pair $(\mathbf{Q}, f)$ to be of class $\mathbf{C}^2$, i.e., that $f \in \mathbf{C}^2]a, b[$ and $\mathbf{Q} \in \mathbf{C}^2(]a, b[, \mathbf{SO}(n))$. Moreover, in connection with [**3**] we have $\det \nabla u > 0$ in $\Omega \iff \dot{f} > 0$ in $]a, b[$. [*See* (5.2).]

We are now in a position to present the main result of this chapter which is a complete characterisation of all *twist* solutions to the Euler-Lagrange equations associated with the energy functional $\mathbb{F}$.

Theorem 5.6.3. (Characterisation of all twist solutions).

Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ and consider the energy functional $\mathbb{F}$ over the space $\mathcal{A}(\Omega)$. Then the set $\mathfrak{S}$ of all classical solutions in the form of generalised twists to the corresponding Euler-Lagrange equations can be characterised as follows.

[**1**] $(n = 2k)$ $\mathfrak{S}$ *is infinite and any generalised twist $u \in \mathfrak{S}$ can be described as*

$$\begin{aligned} u =& \mathbf{G}(r)\theta \\ =& f(r)\mathbf{P} diag(\mathfrak{R}(g), \dots, \mathfrak{R}(g))\mathbf{P}^t\theta, \end{aligned}$$

where $\mathbf{P} \in \mathbf{O}(n)$ and $f, g \in \mathbf{C}^2[a, b]$ satisfy the second order system [notation as in Theorem 5.6.1]

$$\mathbb{EL}[g, f] = 0.$$

[**2**] $(n = 2k + 1)$ $\mathfrak{S}$ *consist of the single map $u = x$.*

Proof. [**1**] That $\mathfrak{S}$ is *infinite* follows from Theorem 5.6.1. The remaining assertions follow from [**a**] in Theorem 5.5.3.

[**2**] Assume $u \in \mathfrak{S}$. Then referring to [**b**] in Theorem 5.5.3 it follows that $\mathbf{Q} = \mathbf{I}_n$ while $f \in \mathbf{C}^2[a,b]$ and $\mathbb{EL}[0,f] = 0$. Evidently $f = r$ is a solution to the latter. An application of the phase-plane argument in [55] (*see* pp. 111-117) shows that the latter is indeed the *only* solution. □

5.7 The limiting behaviour of twists when the inner boundary converges to a point

In this section we consider the case where $b = 1$ and $a = \varepsilon > 0$ with the aim of discussing the *limiting* properties of the *generalised* twists from Theorem 5.6.3 as $\varepsilon \downarrow 0$. This is particularly interesting since in the limit (the *punctured* ball) *all* components of the function space collapse to a single one and so it is important to have a clear understanding as to how the twist solutions and their energies [for each *fixed* integer m] behave.

To this end, let $\Omega_\varepsilon := \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\}$ where $n = 2k$ and for each $m \in \mathbb{Z}$ let $u_\varepsilon \in \mathcal{A}(\Omega)$ denote the *generalised* twist from [**1**] in Theorem 5.6.3, that is, with the notation $x = r\theta$, set

$$\begin{aligned} u_\varepsilon &= \mathbf{G}(r;\varepsilon,1,m)\theta \\ &= f_\varepsilon(r)\mathbf{P}_\varepsilon diag(\mathfrak{R}(g_\varepsilon),\dots,\mathfrak{R}(g_\varepsilon))\mathbf{P}_\varepsilon^t\theta, \end{aligned}$$

where $\mathbf{P}_\varepsilon \in \mathbf{O}(n)$, $f_\varepsilon(r) = f(r;\varepsilon,1,m)$ and $g_\varepsilon(r) = g(r;\varepsilon,1,m)$.

In order to make the study of the *limiting* properties of u_ε more tractable, we *fix* the domain to be the unit ball and extend each map by *identity* off Ω_ε. [In what follows, unless otherwise stated, we speak of u_ε in this extended sense.] Thus, here, we have that

$$u_\varepsilon : (r,\theta) \mapsto (f_\varepsilon(r), \mathbf{Q}_\varepsilon(r)\theta), \tag{5.23}$$

where

$$\mathbf{Q}_\varepsilon(r) = \mathbf{P}_\varepsilon diag(\mathfrak{R}(g_\varepsilon),\dots,\mathfrak{R}(g_\varepsilon))\mathbf{P}_\varepsilon^t,$$

and

$$g_\varepsilon(r) = \begin{cases} 0 & r \le \varepsilon, \\ g(r;\varepsilon,1,m) & \varepsilon \le r \le 1, \end{cases}$$

while

$$f_\varepsilon(r) = \begin{cases} r & r \le \varepsilon, \\ f(r;\varepsilon,1,m) & \varepsilon \le r \le 1. \end{cases}$$

In discussing the *limiting* properties of u_ε it is convenient to introduce a so-called *comparison* map. Indeed, *fix* $m \in \mathbb{Z}$ and consider the *generalised* twist

$$\begin{aligned} v_\varepsilon &= \mathbf{H}_\varepsilon(r)\theta \\ &= r\mathbf{P}_\varepsilon diag(\Re(h_\varepsilon), \ldots, \Re(h_\varepsilon))\mathbf{P}_\varepsilon^t\theta, \end{aligned}$$

where $\mathbf{P}_\varepsilon \in \mathbf{O}(n)$ is as above and

$$h_\varepsilon(r) := \begin{cases} 0 & r \in (0, \varepsilon), \\ 2m\pi(\frac{r}{\varepsilon} - 1) & r \in (\varepsilon, 2\varepsilon), \\ 2m\pi & r \in (2\varepsilon, 1). \end{cases}$$

Thus in particular we can write

$$v_\varepsilon : (r, \theta) \mapsto (r, \mathbf{R}_\varepsilon(r)\theta), \tag{5.24}$$

where

$$\mathbf{R}_\varepsilon(r) = \mathbf{P}_\varepsilon diag(\Re(h_\varepsilon), \ldots, \Re(h_\varepsilon))\mathbf{P}_\varepsilon^t.$$

The following proposition describes some of the basic properties of the *family* of comparison maps (v_ε).

Proposition 5.7.1. *The family of comparison maps* (v_ε) *enjoys the following properties.*
[**1**] $\det \nabla v_\varepsilon = 1$ *in* $\mathbb{B}$,
[**2**] $v_\varepsilon \to x$ *in* $W^{1,2}(\mathbb{B}, \mathbb{R}^n)$,
[**3**] $v_\varepsilon \to x$ *uniformly on* $\bar{\mathbb{B}}$.

Proof. [**1**] Evidently v_ε is a *generalised* twist with the corresponding twist *loop*

$$\begin{aligned} \mathbf{H}_\varepsilon(r) :=& r\mathbf{R}_\varepsilon(r) \\ =& r\mathbf{P}_\varepsilon diag(\Re(h_\varepsilon), \ldots, \Re(h_\varepsilon))\mathbf{P}_\varepsilon^t. \end{aligned}$$

An application of (5.2) in Proposition 5.1.1 [with the choice $f = r$] immediately gives [**1**].
[**2**] Indeed referring to the definition of v_ε we can write

$$\begin{aligned} \|v_\varepsilon - x\|^2_{W_0^{1,2}} &= \int_{\mathbb{B}} |\nabla v_\varepsilon - \mathbf{I}_n|^2 \, dx \\ &= \int_{\mathbb{B}_{2\varepsilon} \setminus \mathbb{B}_\varepsilon} |\nabla v_\varepsilon - \mathbf{I}_n|^2 \, dx \le 2 \int_{\mathbb{B}_{2\varepsilon} \setminus \mathbb{B}_\varepsilon} \left(|\nabla v_\varepsilon|^2 + |\mathbf{I}_n|^2 \right) dx. \end{aligned}$$

However in view of (5.4) in Proposition 5.1.1 [again with the choice $f = r$] we have that

$$|\nabla v_\varepsilon|^2 = n + r^2 |\dot{\mathbf{R}}_\varepsilon \theta|^2.$$

Therefore we can write

$$\begin{aligned}\int_{\mathbb{B}_{2\varepsilon}\setminus\mathbb{B}_\varepsilon} |\nabla v_\varepsilon|^2\, dx &= \int_\varepsilon^{2\varepsilon} \int_{\mathbb{S}^{n-1}} (n + r^2|\dot{\mathbf{R}}_\varepsilon\theta|^2) r^{n-1}\, d\mathcal{H}^{n-1}(\theta) dr \\ &= n\omega_n \int_\varepsilon^{2\varepsilon} (n + r^2 \dot{h}_\varepsilon^2) r^{n-1}\, dr \\ &\le \omega_n (2^n - 1)[n + 4(2m\pi)^2]\varepsilon^n.\end{aligned}$$

The above estimates when combined give [**2**] as a result of Poincaré *inequality.*

[**3**] Again by direct verification we have that

$$\begin{aligned}|v_\varepsilon - x|^2 &= |\mathbf{H}_\varepsilon(r)\theta - r\theta|^2 \\ &= r^2 \Big|\mathbf{P}_\varepsilon diag(\Re(h_\varepsilon), \dots, \Re(h_\varepsilon))\mathbf{P}_\varepsilon^t \theta - \theta\Big|^2 \\ &= r^2 \Big|\mathbf{P}_\varepsilon \big[diag(\Re(h_\varepsilon), \dots, \Re(h_\varepsilon)) - \mathbf{I}_n\big] \mathbf{P}_\varepsilon^t\theta\Big|^2 \\ &= r^2 \Big|\big[diag(\Re(h_\varepsilon), \dots, \Re(h_\varepsilon)) - \mathbf{I}_n\big]\omega_\varepsilon\Big|^2 \qquad (\omega_\varepsilon := \mathbf{P}_\varepsilon^t\theta) \\ &= \frac{1}{2} r^2 |\Re(h_\varepsilon) - \mathbf{I}_2|^2. \end{aligned} \tag{5.25}$$

However a straight-forward calculation gives

$$|\Re(h_\varepsilon) - \mathbf{I}_2|^2 = 8 \sin^2 \frac{h_\varepsilon}{2}.$$

Thus by substitution and referring to the definition of h_ε we immediately arrive at the bound

$$\sup_{\mathbb{B}} |v_\varepsilon - x| = \sup_{[\varepsilon, 2\varepsilon]} 2r |\sin \frac{h_\varepsilon}{2}| \le 4\varepsilon,$$

which is the required conclusion. This complete the proof. □

Fix $m \in \mathbb{Z}$ and let $\mathbf{g}_\varepsilon := (g_\varepsilon, \dots, g_\varepsilon)$, $\mathbf{h}_\varepsilon := (h_\varepsilon, \dots, h_\varepsilon)$. It is evident that the pairs $(g_\varepsilon, f_\varepsilon), (h_\varepsilon, r) \in \mathcal{J}_m(\varepsilon, 1)$ and so according to the minimizing property of $(g_\varepsilon, f_\varepsilon)$ we have that

$$\begin{aligned}\frac{2}{\omega_n}\mathbb{F}[u_\varepsilon, \mathbb{B}] &= \mathbb{E}[\mathbf{Q}_\varepsilon, f_\varepsilon] = \mathfrak{I}[g_\varepsilon, f_\varepsilon] \\ &\le \mathfrak{I}[h_\varepsilon, r] = \mathbb{E}[\mathbf{R}_\varepsilon, r] = \frac{2}{\omega_n}\mathbb{F}[v_\varepsilon, \mathbb{B}].\end{aligned} \tag{5.26}$$

This in conjunction with [**1**], [**2**] in Proposition 5.7.1 implies the boundedness of (u_ε) in $W^{1,2}(\mathbb{B},\mathbb{R}^n)$ and as a result (u_ε) admits a *weakly* convergent subsequence. Indeed more is true!

Theorem 5.7.2. (Limiting behaviour of twists).
Let $\Omega_\varepsilon := \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\}$. For fixed $m \in \mathbb{Z}$ let (u_ε) denote the family of generalised twists as in (5.23)*. Then we have the following convergences.*
[**1**] *$u_\varepsilon \to x$ in $W^{1,2}(\mathbb{B},\mathbb{R}^n)$,*
[**2**] *$u_\varepsilon \to x$ uniformly on $\bar{\mathbb{B}}$.*

Proof. [**1**] Fix $m \in \mathbb{Z}$ and let v_ε be as in (5.24). Then referring to (5.26) it follows that by passing to a subsequence (not re-labeled) we have that

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } W^{1,2}(\mathbb{B},\mathbb{R}^n), \\ u_\varepsilon \to u & \mathcal{L}^n\text{-}a.e. \text{ in } \mathbb{B}. \end{cases}$$

In addition we can write

$$\begin{aligned} \mathbb{F}[x,\mathbb{B}] &\le \liminf_{\varepsilon\downarrow 0} \mathbb{F}[u_\varepsilon,\mathbb{B}] \\ &\le \limsup_{\varepsilon\downarrow 0} \mathbb{F}[u_\varepsilon,\mathbb{B}] \\ &\le \lim_{\varepsilon\downarrow 0} \mathbb{F}[v_\varepsilon,\mathbb{B}] = \mathbb{F}[x,\mathbb{B}]. \end{aligned} \tag{5.27}$$

Now *fix* $\sigma \in (0,1)$ and recall the pair $(g_\varepsilon, f_\varepsilon)$ used in expressing (u_ε). Then (u_ε) bounded in $W^{1,2}(\mathbb{B},\mathbb{R}^n)$ gives (u_ε) bounded in $W^{1,2}(\mathbb{B}\backslash\bar{\mathbb{B}}_\sigma,\mathbb{R}^n)$ and so as a result $(g_\varepsilon, f_\varepsilon)$ is *bounded* in $[W^{1,2}(\sigma,1)]^2$. In particular there exist $(g,f) \in [W^{1,2}(\sigma,1)]^2$ such that passing to a subsequence (not re-labeled) we have that

$$\begin{cases} g_\varepsilon \rightharpoonup g & \text{in } W^{1,2}(\sigma,1), \\ g_\varepsilon \to g & \text{in } \mathbf{C}[\sigma,1], \\ f_\varepsilon \rightharpoonup f & \text{in } W^{1,2}(\sigma,1), \\ f_\varepsilon \to f & \text{in } \mathbf{C}[\sigma,1], \\ g(1) = 2m\pi, & \\ f(1) = 1. & \end{cases}$$

As a consequence we have in particular that

$$\begin{aligned} &f_\varepsilon \dot{g}_\varepsilon \rightharpoonup f\dot{g}, \\ &\dot{f}_\varepsilon (\frac{f_\varepsilon}{r})^{n-1} \rightharpoonup \dot{f}(\frac{f}{r})^{n-1}, \end{aligned}$$

where both convergences are interpreted as *weakly* in $L^2(\sigma, 1)$. Therefore [using the *same* notation as in (5.9)] standard lower *semicontinuity* results (*see*, e.g., [18]) give

$$\int_\sigma^1 \mathbf{J}(r, \dot{\mathbf{g}}, f, \dot{f})\, dr \le \liminf_{\varepsilon \downarrow 0} \int_\sigma^1 \mathbf{J}(r, \dot{\mathbf{g}}_\varepsilon, f_\varepsilon, \dot{f}_\varepsilon)\, dr. \tag{5.28}$$

Now referring to (5.23) we can assume that $\mathbf{P}_\varepsilon \to \mathbf{P}$ as a result of $\mathbf{O}(n)$ being a *compact* manifold. Hence with the aid of the above we can write

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} u_\varepsilon(x) &= \lim_{\varepsilon \downarrow 0} \mathbf{G}_\varepsilon(r)\theta \\
&= \lim_{\varepsilon \downarrow 0} f_\varepsilon \mathbf{P}_\varepsilon diag(\mathfrak{R}(g_\varepsilon), \dots, \mathfrak{R}(g_\varepsilon))\mathbf{P}_\varepsilon^t \theta \\
&= f\mathbf{P} diag(\mathfrak{R}(g), \dots, \mathfrak{R}(g))\mathbf{P}^t\theta := w,
\end{aligned} \tag{5.29}$$

where the convergence is interpreted as *uniformly* on $\mathbb{B}\backslash\bar{\mathbb{B}}_\sigma$. Recalling the *pointwise* convergence of (u_ε) we thus conclude that $u = w$ $\mathcal{L}^n$-*a.e.* in $\mathbb{B}\backslash\bar{\mathbb{B}}_\sigma$. Hence by combining (5.27) and (5.28) we have that

$$\begin{aligned}
\mathbb{F}[u, \mathbb{B}\backslash\bar{\mathbb{B}}_\sigma] &= \mathbb{F}[w, \mathbb{B}\backslash\bar{\mathbb{B}}_\sigma] \\
&\le \liminf_{\varepsilon \downarrow 0} \mathbb{F}[u_\varepsilon, \mathbb{B}\backslash\bar{\mathbb{B}}_\sigma] \\
&\le \liminf_{\varepsilon \downarrow 0} \mathbb{F}[u_\varepsilon, \mathbb{B}] \\
&\le \limsup_{\varepsilon \downarrow 0} \mathbb{F}[u_\varepsilon, \mathbb{B}] \le \mathbb{F}[x, \mathbb{B}].
\end{aligned}$$

Note that the energy functional $\mathbb{F}[\cdot, \Omega]$ is *not* sequentially *weakly* lower semicontinuous on $\mathcal{A}(\Omega)$. However (5.28) demonstrates that the same is true if one *restricts* to *generalised* twists! An application of Lebesgue's theorem on *monotone* convergence now gives

$$\mathbb{F}[x, \mathbb{B}] \le \mathbb{F}[u, \mathbb{B}] = \lim_{\sigma \downarrow 0} \mathbb{F}[u, \mathbb{B}\backslash\bar{\mathbb{B}}_\sigma] \le \mathbb{F}[x, \mathbb{B}].$$

Hence $\mathbb{F}[x, \mathbb{B}] = \mathbb{F}[u, \mathbb{B}]$ and this in turn together with the *strict* quasiconvexity of $\mathbb{F}$ gives $u = x$ in $\bar{\mathbb{B}}$. Finally referring again to (5.27) we have that

$$\left\{ \begin{array}{l} u_\varepsilon \rightharpoonup u, \\ u = x, \\ \mathbb{F}[u_\varepsilon, \mathbb{B}] \to \mathbb{F}[x, \mathbb{B}], \end{array} \right\} \implies u_\varepsilon \to x,$$

which is the required conclusion in [**1**].

[**2**] In view of the assertion in [**1**] and the characterisation of the *pointwise* limit of the family (u_ε) in (5.29) we have that

$$\begin{aligned} w &= f(r)\mathbf{P}diag(\mathfrak{R}(g), \dots, \mathfrak{R}(g))\mathbf{P}^t \\ &= r\theta = x, \end{aligned}$$

$\mathcal{L}^n$-*a.e.* in $\mathbb{B}$. Therefore according to f, g both being *continuous* on the interval $]0,1] = \cup_{\sigma\in]0,1[}[\sigma,1]$ it follows that

$$f\mathfrak{R}(g) = r\mathbf{I}_2$$

on $]0,1]$. This gives $f = r$ (e.g., by taking the *norm* of both sides and noting that f is *non-negative*) and $\mathfrak{R}(g) = \mathbf{I}_2$ which in turn gives $g(r) = 2\pi n(r)$ for some $n(r) \in \mathbb{Z}$. Referring to $g(1) = 2\pi m$ it follows again by appealing to the continuity of g that $g(r) = 2\pi m$ on $]0,1]$. Next, arguing as in (5.25) we can write

$$\begin{aligned} |u_\varepsilon - x|^2 &= |\mathbf{G}_\varepsilon(r)\theta - r\theta|^2 \\ &= 2r^2(1 - \cos g_\varepsilon) \\ &= 4r^2 \sin^2 \frac{g_\varepsilon}{2}. \end{aligned}$$

Thus, to conclude [**2**] fix $\delta > 0$ and *first* take $\sigma \in (0, 2^{-1}\delta]$ and then ε_0 such that $|\sin(2^{-1}g_\varepsilon)| \le 2^{-1}\delta$ on $[\sigma, 1]$ for $\varepsilon < \varepsilon_0$. Then $\sup_{\mathbb{B}} |u_\varepsilon - x| \le \max(2\sigma, \delta) = \delta$. $\square$

Appendix A

Skew-symmetric matrices and the orthogonal group

Recall from linear algebra that *all* eigen-values of a [real] *skew*-symmetric matrix have zero *real* parts. Hence they *either* appear as *purely* imaginary conjugate pairs *or* zero. In particular when n is *odd* there is necessarily a zero eigen-value. Thus distinguishing between the cases when n is *even* and *odd* respectively we can bring every *skew-symmetric* matrix to a *block* diagonal form. In what follows we set

$$\mathbf{J}_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Proposition A.0.3. *Let* $\mathbf{A} \in \mathbb{M}_{n\times n}$ *be skew-symmetric. Then there exist* $(\lambda_j)_{j=1}^k \subset \mathbb{R}$ *and* $\mathbf{P} \in \mathbf{SO}(n)$ *such that*

[**1**] $(n = 2k)$

$$\mathbf{A} = \mathbf{P} diag(\lambda_1\mathbf{J}_2, \lambda_2\mathbf{J}_2, \dots, \lambda_k\mathbf{J}_2)\mathbf{P}^t,$$

[**2**] $(n = 2k+1)$

$$\mathbf{A} = \mathbf{P} diag(\lambda_1\mathbf{J}_2, \lambda_2\mathbf{J}_2, \dots, \lambda_k\mathbf{J}_2, 0)\mathbf{P}^t.$$

Proof. Indeed, here, $\mathbf{A}$ is *normal* [i.e., it commutes with its *transpose* $\mathbf{A}^t = -\mathbf{A}$] and so the conclusion follows from the the well-known *spectral* theorem. □

We note that by allowing $\mathbf{P} \in \mathbf{O}(n)$ we can additionally arrange for the *sequence* $(\lambda_j)_{j=1}^k$ to be *non-negative*. On the other hand the choices of $\mathbf{P}$ and $(\lambda_j)_{j=1}^k$ are in general *non-unique*. Indeed it is a *trivial* matter to see that by suitably adjusting $\mathbf{P}$ one can replace *any* λ_j with $-\lambda_j$.

In what follows we set

$$\mathfrak{R}(s) := \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}.$$

Proposition A.0.4. *Let* $\mathbf{A} \in \mathbb{M}_{n\times n}$ *be skew-symmetric. Then using the notation in Proposition A.0.3 we have that*

[**1**] $(n = 2k)$

$$e^{s\mathbf{A}} = \mathbf{P}diag(\mathfrak{R}(s\lambda_1), \mathfrak{R}(s\lambda_2), \ldots, \mathfrak{R}(s\lambda_k))\mathbf{P}^t,$$

[**2**] $(n = 2k+1)$

$$e^{s\mathbf{A}} = \mathbf{P}diag(\mathfrak{R}(s\lambda_1), \mathfrak{R}(s\lambda_2), \ldots, \mathfrak{R}(s\lambda_k), 1)\mathbf{P}^t.$$

Proof. A straight-forward calculation gives

$$e^{s\mathbf{J}} = \sum_{n=0}^{\infty} \frac{1}{n!} s^n \mathbf{J}_2^n = \mathfrak{R}(s).$$

The conclusion now follows by noting that $e^{\mathbf{A}} = e^{\mathbf{PDP}^t} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^t$. □

Proposition A.0.5. *Let* $\mathbf{A} \in \mathbb{M}_{n\times n}$ *be skew-symmetric. Assume in addition that* $\mathbf{A} \in \mathbb{R}\mathbf{O}(n)$. *Then the following hold.*

[**1**] $(n = 2k)$ *there exists* $\alpha \in \mathbb{R}$ *and* $\mathbf{P} \in \mathbf{O}(n)$ *such that*

$$\mathbf{A} = \alpha\mathbf{P}diag(\mathbf{J}_2, \mathbf{J}_2, \ldots, \mathbf{J}_2)\mathbf{P}^t,$$

[**2**] $(n = 2k+1)$ *necessarily* $\mathbf{A} = 0$.

Therefore it follows that indeed $\mathbf{A} \in \mathbb{R}\mathbf{SO}(n)$.

Proof. In view of $\mathbf{A} \in \mathbb{R}\mathbf{O}(n)$ there exists $\alpha \in \mathbb{R}$ such that $\mathbf{A}^t\mathbf{A} = \mathbf{A}\mathbf{A}^t = \alpha^2\mathbf{I}_n$. In what follows we proceed by considering each of the cases $n = 2k$ and $n = 2k+1$ separately.

[**1**] Since $\mathbf{A}$ is *skew-symmetric* it follows from [**1**] in Proposition A.0.3 that there exist $(\lambda_j)_{j=1}^k$ and $\mathbf{R} \in \mathbf{O}(n)$ such that $\mathbf{A} = \mathbf{R}\, diag(\lambda_1\mathbf{J}_2, \lambda_2\mathbf{J}_2, \ldots, \lambda_k\mathbf{J}_2)\mathbf{R}^t$. Hence

$$\begin{aligned}
\mathbf{A}^t\mathbf{A} =& [\mathbf{R}diag(\lambda_1\mathbf{J}_2, \lambda_2\mathbf{J}_2, \ldots, \lambda_k\mathbf{J}_2)\mathbf{R}^t]^t \times \\
& [\mathbf{R}diag(\lambda_1\mathbf{J}_2, \lambda_2\mathbf{J}_2, \ldots, \lambda_k\mathbf{J}_2)\mathbf{R}^t] \\
=& \mathbf{R}diag(\lambda_1^2\mathbf{I}_2, \lambda_2^2\mathbf{I}_2, \ldots, \lambda_k^2\mathbf{I}_2)\mathbf{R}^t \\
=& \alpha^2\mathbf{I}_n
\end{aligned}$$

and so $\lambda_1^2 = \lambda_2^2 = \ldots = \lambda_k^2 = \alpha^2$. In particular there exists $(\beta_j)_{j=1}^k \subset \{\pm 1\}$ such that $\mathbf{A} = \alpha \mathbf{R}\, diag(\beta_1 \mathbf{J}_2, \beta_2 \mathbf{J}_2, \ldots, \beta_k \mathbf{J}_2)\mathbf{R}^t$. The conclusion now follows by *post*-multiplying $\mathbf{R}$ with suitable *orthogonal* matrices through an application of the following *trivial* identity relating $-\mathbf{J}_2$ to $\mathbf{J}_2$,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

[**2**] This is an immediate consequence of $\det \mathbf{A} = 0$. $\square$

For more details and basics properties related to the matrix exponential as a mapping between the spaces of *skew-symmetric* matrices and the special orthogonal groups, we refer the interested reader to the books [21], [39] and [54].

Appendix B

Symmetric matrices and vector fields

Proposition B.0.6. *Let $\mathbf{F} \in \mathbb{M}_{n\times n}$ be fixed and consider for $\theta \in \mathbb{S}^{n-1}$ the identity*

$$\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta = \mathbf{0}. \tag{B.1}$$

Then (B.1) *holds for all $\theta \in \mathbb{S}^{n-1}$ if and only if there exists $\sigma \in \mathbb{R}$ such that $\mathbf{F} = \sigma \mathbf{I}_n$.*

Proof. (*Sufficiency*) If $\mathbf{F} = \sigma \mathbf{I}_n$ for some $\sigma \in \mathbb{R}$ then (B.1) is *trivially* true for all $\theta \in \mathbb{S}^{n-1}$.
(*Necessity*) Assume that (B.1) holds for all $\theta \in \mathbb{S}^{n-1}$. To justify the assertion it suffices to consider the following steps.
[**1**] By substituting the choices $\theta \in \{e_1, e_2, \dots, e_n\}$ (the *standard* basis) it follows that $\mathbf{F}$ must be *diagonal*.
[**2**] Assume now that $\mathbf{F} = diag(d_1, d_2, \dots, d_n)$. Then (B.1) is equivalent to the set of equations

$$\theta_i \theta_j (d_i - d_j) = 0,$$

for $1 \leq i, j \leq n$ where $\theta = (\theta_1, \theta_2, \dots, \theta_n)$. It thus follows that $d_1 = d_2 = \cdots = d_n$ and so denoting the common value as σ gives the conclusion. $\square$

Proposition B.0.7. *Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ and consider the vector field $\mathbf{v} = \mathbf{A}(r)x$ in Ω where $\mathbf{A} \in \mathbf{C}^1(]a, b[, \mathbb{M}_{n\times n})$ is symmetric. Then the following are equivalent.*
[**1**] $\mathbf{v}$ *is a gradient,*
[**2**] $\mathbf{A} = s\mathbf{I}_n + \mathbf{K}$ *for some $s \in \mathbf{C}^1]a, b[$ and constant symmetric matrix $\mathbf{K} \in \mathbb{M}_{n\times n}$.*

Proof. ([**1**] $\Longrightarrow$ [**2**])

If $\mathbf{v} = (v_1, v_2, \dots, v_n)$ is a *gradient* field in Ω then it is necessary that for all $1 \le p, q \le n$,

$$\frac{\partial v_q}{\partial x_p} - \frac{\partial v_p}{\partial x_q} = 0.$$

Substituting for $\mathbf{v}$ and denoting $r = |x|$ this means that

$$\begin{aligned} 0 =& \frac{\partial}{\partial x_p} \sum_{j=1}^n \mathbf{A}_{qj} x_j - \frac{\partial}{\partial x_q} \sum_{j=1}^n \mathbf{A}_{pj} x_j \\ =& \left\{ r \sum_{j=1}^n \dot{\mathbf{A}}_{qj} \theta_j \theta_p + \sum_{j=1}^n \mathbf{A}_{qj} \delta_{jp} \right\} - \left\{ r \sum_{j=1}^n \dot{\mathbf{A}}_{pj} \theta_j \theta_q + \sum_{j=1}^n \mathbf{A}_{pj} \delta_{jq} \right\}, \end{aligned}$$

or in view of $\mathbf{A}$ being *symmetric* that

$$\begin{aligned} 0 &= \sum_{j=1}^n \left\{ \dot{\mathbf{A}}_{qj}(r) \theta_j \theta_p - \dot{\mathbf{A}}_{pj}(r) \theta_j \theta_q \right\} \\ &= \left[\dot{\mathbf{A}}(r)\theta \otimes \theta - \theta \otimes \dot{\mathbf{A}}(r)\theta \right]_{qp}, \end{aligned}$$

for $r \in (a, b)$ and $\theta \in \mathbb{S}^{n-1}$. An application of Proposition B.0.6 [with $\mathbf{F} = \dot{\mathbf{A}}(r)$] now gives $\dot{\mathbf{A}}(r) = \sigma(r)\mathbf{I}_n$ where $\sigma \in \mathbf{C}]a, b[$. Consequently by integration we arrive at

$$\mathbf{A} = s\mathbf{I}_n + \mathbf{K},$$

on $]a, b[$ where $s \in \mathbf{C}^1]a, b[$ is a suitable *primitive* for σ and $\mathbf{K} \in \mathbb{M}_{n\times n}$ is *constant* and *symmetric*. This gives [**2**].

([**2**] $\Longrightarrow$ [**1**])

Assume now $\mathbf{A}(r) = s(r)\mathbf{I}_n + \mathbf{K}$ then clearly $\mathbf{v} = s(r)x + \mathbf{K}x$ in Ω. To show that $\mathbf{v}$ is a *gradient* it suffices to consider $f(x) := \rho(r) + \frac{1}{2}\langle \mathbf{K}x, x\rangle$ for some $\rho \in \mathbf{C}^2]a, b[$ to be determined. Then as $\mathbf{K}$ being a *symmetric* matrix we have

$$\nabla f(x) = \dot{\rho}(r)\frac{x}{r} + \mathbf{K}x,$$

which in turn gives $\dot{\rho}(r) = rs(r)$. An integration now leads to ρ and so here $\mathbf{v} = \nabla f$. □

Proposition B.0.8. *Let* $\mathbf{F} \in \mathbb{M}_{n\times n}$. *Then*

$$\int_{\mathbb{S}^{n-1}} \langle \mathbf{F}\theta, \theta\rangle \, d\mathcal{H}^{n-1}(\theta) = \omega_n tr\mathbf{F},$$

where $\omega_n = \mathcal{L}^n(\mathbb{B})$.

Proof. Consider the *vector* field $\mathbf{v} := \mathbf{F}x$ for $x \in \bar{\mathbb{B}}$. Then an application of the *divergence* theorem gives

$$\begin{aligned}
\int_{\partial\mathbb{B}} \langle \mathbf{F}\theta, \theta\rangle \, d\mathcal{H}^{n-1}(\theta) &= \int_{\partial\mathbb{B}} \langle \mathbf{v}(\theta), \theta\rangle \, d\mathcal{H}^{n-1}(\theta) \\
&= \int_{\mathbb{B}} \operatorname{div} \mathbf{v}(x) \, dx \\
&= \int_{\mathbb{B}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\mathbf{F}_{ij} x_j) = \omega_n tr \mathbf{F}.
\end{aligned}$$

$\square$

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