On Blow-up Solutions of Parabolic Problems

by

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To The Memory of My Father,
Abdul-kadhim

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another university for the award of any other degree.

Signature:

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Abstract

This thesis is concerned with the study of the *Blow-up phenomena* for parabolic problems, which can be defined in a basic way as the inability to continue the solutions up to or after a finite time, the so called blow-up time. Namely, we consider the blow-up location in space and its rate estimates, for special cases of the following types of problems:

(i) Dirichlet problems for semilinear equations,

(ii) Neumann problems for heat equations,

(iii) Neumann problems for semilinear equations,

(iv) Dirichlet (Cauchy) problems for semilinear equations with gradient terms.

For problems of type (i), (ii), we extend some known blow-up results of parabolic problems with power and exponential type nonlinearities to problems with nonlinear terms, which grow faster than these types of functions for large values of solutions. Moreover, under certain conditions, some blow-up results of the single semilinear heat equation are extended to the coupled systems of two semilinear heat equations.

For problems of type (iii), we study how the reaction terms and the nonlinear boundary terms affect the blow-up properties of the blow-up solutions of these problems.

The noninfluence of the gradient terms on the blow-up bounds is showed for problems of type (iv).

The study of blow-up is considerably more difficult and interesting when the equations involved are PDEs, and indeed, it has become both a kind of industry and an art.

Prof. Juan Luis Vazquez, [64]

Contents

Chapter 1

Introduction

Many physical and engineering problems can be modeled mathematically in the form of evolution equations (partial differential equations depending on time). We cannot obtain a well-defined solution for these equations without adding suitable additional conditions (initial and boundary conditions). Since the last century, many authors have studied the existence and uniqueness for the linear types of these problems.

Nonlinear partial differential equations are more complicated and have more properties than linear equations, these properties are related to important features of the real world phenomena, on the other hand, these properties are connected with the difficulties of the mathematical treatment.

In the last decades, partial differential equations became one of the most active areas of mathematics research because it helped mathematicians to find answers and explanations to many phenomena of the nonlinear world.

It is known that singularities occur in the solution of linear problems when the problem has singular coefficients or singular data, the so called fixed singularities. One of the most important properties of nonlinear partial differential equations is the possibility of eventual occurrence of singularities starting from smooth data (coefficient and initial or boundary conditions), the so called well posedness in the small, meaning the existence and uniqueness and continuity of the classical solutions can be established for small time.

Singularities of nonlinear problems may come from the effects of nonlinear terms, which occur in the partial differential equations or in the boundary conditions, usually they depend on the time and the location, the so called moving singularities.

One of the most remarkable type of these singularities is what we call the *Blow-up phenomena*. Basically, in a nonlinear problem, blow-up is a form of the spontaneous singularities appear when one or more of the depending variables go to infinity as time goes to a certain finite time.

In this thesis we consider the blow-up phenomena for parabolic problems, which we will describe in more detail in the next section.

1.1 Background

Blow-up phenomena occur in an elementary form in the theory of ordinary differential equations, and it is equivalent to global nonexistence (see [49]), for instance, the problem of reaction equation with positive constant initial value, namely

$$u_t = f(u), \quad t > 0, \quad u(0) = a > 0,$$

where f is positive and continuous. It is well known that, for any solution of this problem, the condition

$$\int_U^\infty \frac{du}{f(u)} < \infty, \quad U \geq 1 \tag{1.1}$$

is the necessary and sufficient condition for the occurrence of blow-up in finite time, see [31]. For the special case (the power type problem), namely

$$\left.\begin{aligned} u_t &= u^p, & t &> 0, \\ u(t) &= a, & t &= 0, \end{aligned}\right\} \tag{1.2}$$

where $p > 1, a > 0$, it is easy to see that the unique solution to this problem takes the form

$$u(t) = \frac{C}{(T-t)^{\frac{1}{p-1}}}, \quad T = \frac{1}{a^{p-1}(p-1)}, \quad C = \frac{1}{(p-1)^{\frac{1}{p-1}}}. \tag{1.3}$$

It is clear that this solution is nonsingular if $0 < t < T$, and $u(t)$ goes to infinity as $t \to T^-$. We say that the solution of this problem blows up at $t = T$. Clearly, the number $\frac{1}{p-1}$ is the (algebraic) blow-up rate for this solution. On the other hand, for the Cauchy problem for the heat equation, namely

$$\left.\begin{array}{ll} u_t = \Delta u, & x \in R^n, t > 0, \\ u(x,0) = u_0(x), & x \in R^n, \end{array}\right\} \tag{1.4}$$

it is known that the fundamental solution of this problem takes the form

$$u(x,t) = \frac{1}{(4\pi t)^{(n/2)}} \int_{R^n} u_0(y) \exp[-\frac{|x-y|^2}{4t}]dy, \tag{1.5}$$

which means, it decays like $t^{-\frac{n}{2}}$.

Fujita [25] has considered the initial value problem of a semilinear equation, which is a combination of the two problems (1.2), (1.4), namely

$$\left.\begin{array}{ll} u_t = \Delta u + u^p, & x \in R^n,\ t > 0, \\ u(x,0) = u_0, & x \in R^n. \end{array}\right\} \tag{1.6}$$

He proved that there are no global, nontrivial solutions of (1.6) whenever $\frac{1}{p-1} \geq \frac{n}{2}$, while there are both global, nontrivial solutions and blow-up solutions, if the blow-up rate is smaller than the decay rate. Therefore, the study of ordinary differential equations supplies basic ideas for the theory of blow-up and singularities.

Starting from these examples above, for partial differential equations defined in a domain Ω with some $t > 0$, the concept of blow-up means the solution cannot be continued globally in time at some or many points in $\overline{\Omega}$, because of the infinite growth of some variables of the problem describing the evolution process. In other words, blow-up occurs if the solution becomes infinite at some or many points in $\overline{\Omega}$ in finite time.

In general, blow-up can be discussed in any normed space, however in this thesis we deal with only *Blow-up in $L^\infty-norm$*, which can be defined as follows

Definition 1.1.1. For any parabolic equation, we say that the classical solution u blows up in $L^\infty -$ norm or blows up (for short), if there exists $T < \infty$, called the blow-up time, such that u is well defined for all $0 < t < T$, while it becomes unbounded in $L^{\infty} -$ norm, when t approach to T, that is

$$\sup_{x\in\Omega} |u(x,t)| \to \infty \quad \text{as } t \to T^{-}.$$

For a system of two coupled semilinear parabolic equations, namely

$$u_t = \Delta u + F(u,v), \quad v_t = \Delta v + G(u,v), \quad (x,t) \in \Omega \times (0,T),$$

we say that a solution (u,v) blows up in finite time, if there exist $T < \infty$ such that either u or v blows up at $t = T$, this means

$$\sup_{x\in\Omega} |u(x,t)| \to \infty, \quad \text{or} \quad \sup_{x\in\Omega} |v(x,t)| \to \infty, \quad \text{as } t \to T^{-},$$

while

$$\sup_{x\in\Omega}\{|u(x,t)| + |v(x,t)|\} \leq C < \infty, \quad t < T.$$

Moreover, we say that u, v *blow up simultaneously,* if both of u, v blow up at T, see for instance [44].

Remark 1.1.2. It is well known that for some problems, see [35, 56], the solution stays bounded, while a space derivative may blow up in a finite time, the so called *gradient blow-up* (**GBU**). For some other problems, the time derivative becomes unbounded (blows up) when the solution reaches a certain finite level in finite time, the so called *guenching phenomena*, see [12]. Clearly, in these two cases, blow-up and global nonexistence are nonequivalent.

Blow-up solutions of partial differential equations have been investigated by many authors specially after the important results by Kaplan [37], Fujita [25], Friedman and McLeod [24] and some other authors. There is a very extensive literature on the blow-up phenomena, however, there is as yet no complete theory for many problems.

To study the blow-up phenomena for parabolic problems defined on a domain Ω, $t > 0$, with the initial function u_0, it is natural to ask some important questions, which have been discussed by many authors (see [31, 64]), such as blow-up location and its behavior in space and time. In fact, as we will see, the qualitative properties of blow-up solutions are controlled by three criteria: the size of the initial data u_0, the geometry of the domain Ω and the type on the nonlinearity of the function of solutions, which appears in the equation as a reaction term or appears in the problem as a boundary condition term.

We can briefly, summarize these questions as follows:

1- Does blow-up occur?

It is known that the existence and uniqueness can be discussed in different function spaces, and since blow-up is the inability to continue the solutions in that function space up to or after a finite time, blow-up may occur in a function space but not in another one, for instance, blow-up may occur for classical solutions, while there exists a global weak solution L^1 (see [20, 49]).

To study the blow-up for classical solutions, the above question can be split into two questions:

i-Which problems do have finite time blow-up solutions?

The answer depends on the form of the problem (the coefficients and the nonlinear terms which appear in the equation or more generally its structural conditions) and the form of the initial date. For example, consider the Dirichlet problem for the semilinear heat equation defined in a bounded domain Ω, with smooth boundary and nonnegative initial condition, namely

$$\left.\begin{array}{ll} u_t = \Delta u + f(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0, & x \in \Omega, \end{array}\right\} \tag{1.7}$$

where $f \in C^1$, positive for $u > 0$, convex function and satisfies the condition (1.1). It has been proved in [37] that if u_0 is nonnegative and large enough, then the nontrival solution to this problem blows up in finite time. For some other problems, blow-up may occur due to the effect of the boundary conditions even in case of the equation is linear and has smooth coefficients, for instance, the problem of heat equation with nonlinear boundary condition, namely

$$\left.\begin{array}{ll} u_t = \Delta u, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = f(u), & x \in \partial B_R, \\ u(x,0) = u_0, & x \in B_R. \end{array}\right\} \tag{1.8}$$

It has been shown in [34], that if $f \in C^1$, positive nondecreasing for $u > 0$ and satisfies the condition (1.1), then there is a finite blow-up time for any positive initial data u_0.

ii- Which solutions do blow up in finite time?

In case of the problem has a blow-up solution one may ask whether each solution blows up in finite time. Problems may have both of global and blow-up solutions for different initial data. For instance, we recall the results of Fujita [25], which we have discussed before, we see that each nontrivial solution of problem (1.6) starting from nonzero initial data, blows up in finite time, if $1 < p \leq 1 + 2/n$, while the problem may have global or blow-up solutions, if $p > 1 + 2/n$, depending on the size of u_0.

2- When does blow-up happen?

The solutions of parabolic problems can be classified into four cases depending on the location of the blow-up time T, as follows:

(i) **Bounded global solution:** the solution stays uniformly bounded in time.

(ii) **Unbounded global solution:** the solution blows up (becomes unbounded) when time goes to infinity ($T = \infty$).

(iii) **Blow-up solution in finite time:** the solution becomes unbounded in finite time ($T < \infty$).

(iv) **Instantaneous blow-up solution:** the solution is unbounded at any arbitrary small time, $t > 0$ ($T = 0$).

It is known that for problem (1.7), where $f(u) = \lambda u, \lambda > 1$ blow-up in infinite time occurs, while if f is a superlinear function, then in this case we may have bounded or unbounded global solutions (see [20] and the examples therein). In general, by global solution we mean case (i) or (ii). The case (iii) is called the standard blow-up case. For an example of (iv), consider problem (1.7), where $f(u) = \lambda e^u,\ \lambda > 0,\ n \geq 10$, with a singular initial data $u_0(x) \geq S(x) = -2\ln|x|$, $u_0 \not\equiv S$, it has been showed in [31] that $u(x,t) = \infty$ for any arbitrary small $t > 0$, which means, the nonexistence of a locally in time, nontrivial solution of this problem.

Remark 1.1.3. We may ask whether any estimate for the finite blow-up time can be found. In general for many problems, it is not easy task, however, the blow-up time estimates have been shown in the literature for some special cases (see for instance [51]).

3- Where does blow-up happen?

The blow-up set $B(u_0)$ is a closed subset of $\overline{\Omega}$ and it is a function of the initial condition u_0, it can be defined as follows

Definition 1.1.4. Let u blows up in finite time $T > 0$. Then $x_0 \in \overline{\Omega}$ is a *blow-up point* if $u(x_n, t_n) \to \infty$ for some $\{x_n, t_n\}_{n=1}^{\infty} \subset \Omega \times (0, T)$ such that $(x_n, t_n) \to (x_0, T)$ as $n \to \infty$. *The blow-up set* $B(u_0)$ is the set of all blow-up points.

The blow-up set can be only one of the following three cases

(i) **Finite blow-up point:** where $B(u_0)$ has only one point (single blow-up point) or a finite number of points.

(ii) **Regional blow-up:** in this case $B(u_0) \subsetneq \overline{\Omega}$ and the measure of $B(u_0)$ is finite and positive.

(iii) **Global blow-up:** where $B(u_0) = \overline{\Omega}$.

It was shown in [24] that for problem (1.7), where f is of power or exponential type, if $\Omega = B_R$ and u_0 is nonzero radially decreasing function, then the blow-up occurs only at $x = 0$, therefore, in this case we have a single blow-up point, while if $f(u) = u^p$, $\Omega = (-1, 1)$, it was shown in [20], that for given any integer k and $-1 < x_1 < \cdots < x_k < 1$, there is u_0 such that u blows up at $t = T < \infty$ and $B(u_0) = \{x_1, \ldots, x_k\}$. In [29], it has been shown that in case of $f(u) = (1 + u)\log^2(1 + u)$, $\Omega = R$, u_0 is radially nonincreasing and satisfies some additional assumptions, the blow-up set is exactly $[-\pi, \pi]$. Therefore, in this case we have a reginal blow-up. Another example of regional blow-up is the problem (1.8), it was shown in [34] that if $f \in C^2$ and convex function, then the blow-up in this problem occurs only on the boundary (∂B_R). Global blow-up may occur in some problems, for instance, in the problem of semilinear heat equations with a gradient term (see the subsection 5.2.3).

Remark 1.1.5. In first and second cases the blow-up solutions are called localized blow-up.

4-How does blow-up occur?

In order to understand the space-time behavior of blow-up solutions near the blow-up points as t approaches the blow-up time, we need to study two aspects:

Blow-up rate estimate: It is the rate at which each blow-up solution $u(x,t)$ diverges as t approaches the blow-up time T and x approaches a blow-up point.

For Dirichlet and Cauchy problems for semilinear parabolic equations, blow-up is said to be of *type I*, if the solutions blow up with the same rate as the solutions of the corresponding ordinary differential equation, otherwise blow-up is said to be of *type II* (see [55, 56]). For instance, it was shown in [24] for problem (1.7), where $f(u) = u^p, p > 1$, Ω is a ball or convex domain, that there exist two constants $C, c > 0$ such that the upper (lower) blow-up rate estimate to the positive blow-up solution take the following form

$$c(T-t)^{-1/(p-1)} \leq \max_{x \in \overline{\Omega}} u(x,t) \leq C(T-t)^{-1/(p-1)}, \quad t \in (0,T).$$

It is clear that the above upper (lower) blow-up rate is the same rate as of the solutions (1.3) of the corresponding ordinary differential equation (1.2), therefore, the blow-up of this problem is of type I.

In the literature there are some different techniques used to derive the lower (upper) blow-up rate estimates, some of these techniques depend on the rescaling arguments, which means one rescales only space or both space and time variables, the limiting equation obtained is either elliptic or parabolic. The solutions of these new equations are bounded not only at the non blowing points, but rather at the blow-up set, see for instance [7, 33].The other common technique relying on maximum principle arguments (applied to some suitable auxiliary functions), see for instance, [24]. For the problems of parabolic equations with nonlinear boundary conditions, many authors have used the integral equation methods to find the blow-up rate estimates, see for instance [36, 44].

Blow-up profile: It is the asymptotic behavior of each blow-up solution u, as limits of $u(x,t)$ when $t \to T^-$ near and at the blow-up point. More generally, the ultimate goal being to describe the blow-up behavior of u at the final time T, for x close to the blow-up point, the so called the final blow-up profile.

As already pointed out by Giga and Kohn [32, 33], the blow-up rate estimate is crucial in studying the asymptotic behavior to problem (1.7), where Ω is a bounded, convex domain or R^n, $f(u) = u|u|^{p-1}$, and p is in the subcritical Sobolov parameter range, namely

$$\left.\begin{array}{ll} 1 < p < \frac{n+2}{n-2} & \text{if } n \geq 3, \\ 1 < p < \infty & \text{if } n = 1, 2. \end{array}\right\} \tag{1.9}$$

They have used the similarity variables and the asymptotic expansion to prove that

$$\lim_{t \to T} (T-t)^{1/(p-1)} u(x_0 + y\sqrt{T-t}, t) = 0 \text{ or } k,$$

uniformly for $|y| < C$, where $k = (p-1)^{-1/(p-1)}$. This means, if we restrict the spatial domain to the (time-dependent) domain $|x - x_0| < C\sqrt{T-t}$, then the self similar blow-up profile is given by

$$u(x,t) \approx \frac{k}{(T-t)^{1/(p-1)}}, \quad \text{as} \quad t \to T.$$

Clearly, if x_0 is a blow-up point, then the limit above cannot be zero. This limit describes the asymptotic behavior of u in space-time domain prior to (x_0, T), for any $x_0 \in \Omega$. Furthermore, for $\Omega = B_R$, u is positive, radially decreasing solution, it is known [24, 55] that u blows up at only $x = 0$, moreover, the final pointwise blow-up profile is given by

$$u(x,T) \approx \frac{C}{|x|^{\frac{2}{p-1}}}, \quad \text{as} \quad |x| \to 0.$$

5-What does happen after blow-up occurs?

It is desirable task to study the possibility of continuation of the classical blow-up solution in some weaker sence after the blow-up time. In general blow-up of classical solutions of any problem has to be one of the following three cases:

(i) **Complete blow-up:** In this case the solution cannot be continued again after blow-up occurs. For instance, Baras and Cohen [2] have considered the blow-up solution to problem (1.7), where $f(u) = u^p$, and p is in the subcritical Sobolov parameter range (1.9). They proved that a continuation in any sense is not possible because it leads to the conclusion that

$$u(x,t) = \infty, \quad x \in \Omega, t > T.$$

(ii) **Incomplete blow-up:** In this case the solution can be continued in weak sense in some subset of $\overline{\Omega}$, with some $t > T$. For instance, for problem (1.7), where $f(u) = u^p$, $p \geq \frac{n+2}{n-2}$, $n > 2$, and Ω is convex, it has been shown in [20] under some restricted assumptions on u_0, that the problem has an unbounded global weak solution.

(iii) **Transient blow-up:** In this case the solution becomes bounded immediately after T. For instance, in [31] it was discussed a type of problems that has a radial solution, which blows up at a momentary single blow-up point peak at $t = T$ and then evolves immediately into a classical bounded solution for the rest of time $t > T$, such blow-up solution called the peaking solution.

1.2 Outline of the Thesis

The aim of this thesis is to extend the known blow-up results to several parabolic problems, and further, to address some of the standard blow-up questions, which have been discussed in the last section, namely, we consider the blow-up sets and the blow-up rate estimates for these problems. Each of the forthcoming chapters is devoted to study a specific type of parabolic problems.

Chapter 2 considers the problems of semilinear parabolic equations with zero Dirichlet boundary conditions, defined in a ball. Two spatial cases are studied. Firstly, the heat equation with the exponential of a power type function. Secondly, coupled systems of two semilinear heat equations. Finally, the ignition model system is studied as a special case of those systems. For these problems, we extended the known blow-up results by Friedman and McLeod in [24], showing that the blow-up occurs at only a single point, as a consequence of deriving the pointwise estimates of their classical solutions. Moreover,by using the maximum principle arguments (applied to some suitable auxiliary functions) we derive the upper blow-up rate estimates for these problems.

Chapter 3 is devoted to study the problems for the heat equation (system) with the exponential of power type functions as Neumann boundary conditions, defined in a ball. For the scalar problem, we use the maximum principle arguments to derive the blow-up rate estimate, while the integral equation method is used to find the blow-up rate estimates for the system problem. Moreover, depending on these upper blow-up rate estimates, as in the other studied cases (see [44, 46]), we show that the blow-up occurs only on the boundary for the system problem.

Chapter 4 considers the problems of a semilinear equation (system) with nonlinear boundary conditions, defined in a ball. We consider the spatial case, where the reaction terms and the boundary conditions are of exponential type functions. The integral equation method is used to derive the lower and the upper blow-up rate estimate for the scalar problem and the system problem, respectively, while the maximum principle arguments is used to derive upper (lower) and lower blow-up rate estimate for the scalar problem and the system problem, respectively. We show that the reaction terms have an important effect on the upper blow-up rate estimates which become more singular than those known for the cases where the reaction terms are absent, while under certain assumptions the lower blow-up rate estimates take the same forms as those known for the problem where the reaction terms are absent (see [46]). Under some restricted assumptions on these problems, we prove that the blow-up can only occur on the boundary.

Chapter 5 is devoted to study the problems of semilinear parabolic equations with gradient terms. Two spatial cases are studied. Firstly, we consider the zero Dirichlet problem for heat equation with the exponential function of solutions and a negative sign gradient term function, defined in a ball. Secondly, we consider the Cauchy and Dirichlet problems for the system of heat equations with power type functions of solutions and positive sign gradient terms functions, defined in a ball or R^n. For the first problem, we derive the upper pointwise and the blow-up rate estimates using the maximum principle arguments, while for the system problem, we use a technique that depends on rescaling arguments, to derive the upper rate estimates for the blow-up solutions and their gradients functions. These blow-up bounds take the same forms as those known for the cases where the gradient terms are absent (see [24, 61]). This shows that under certain assumptions these gradient terms have no effect on the blow-up bounds.

In *Chapter 6*, we briefly summarize our main results and conclusions and discuss some possible areas of further research.

This thesis contains two appendices:

In *Appendix A*, we introduce the domain notation and symbols, which have been used throughout the thesis, furthermore, we review the standard function spaces and the definitions of superlinear functions, radial functions, uniformly parabolic equations, classical and weak solutions.

In *Appendix B*, we recall same maximum and comparison principles, which we frequently use in this thesis.

Chapter 2

Dirichlet Problems for Semilinear Parabolic Equations

Introduction

It is well known that semilinear parabolic equations arise in many physical situations, where diffusive phenomena and source terms have to be modeled. In [39] Lacy presents a number of physical situations including chemical reactions and electrical heating, where blow-up has physical significance.

The purpose of this chapter is to study the blow-up rate estimates and the blow-up set for a semilinear parabolic equation (system) with zero Dirichlet boundary conditions defined in a ball. In section one we consider the problem of heat equation with a special reaction term, which is the exponential of a power type function.The second section is devoted to study a general form of systems of semilinear heat equations, and then we study the special case where the reaction terms are of exponential type functions, as an example of our results.

2.1 The Semilinear Heat Equation

This section is concerned with the problem of the semilinear heat equation with zero Dirichlet boundary condition:

$$\left.\begin{array}{ll} u_t = \Delta u + f(u), & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R, \end{array}\right\} \tag{2.1}$$

where $f \in C^1(R) \cap C^2(R \backslash \{0\})$ is positive and increasing function in $(0, \infty)$, $u_0 \in C^2(\overline{B}_R)$ is nonzero, nonnegative, radially nonincreasing function, vanishing on ∂B_R. That is, it satisfies the following conditions

$$\left.\begin{array}{ll} u_0(x) = u_0(|x|), & x \in B_R, \\ u_0(x) = 0, & x \in \partial B_R, \\ u_{0r}(|x|) \leq 0, & x \in B_R. \end{array}\right\} \tag{2.2}$$

Moreover, it satisfies

$$\Delta u_0(x) + f(u_0(x)) \geq 0, \quad x \in B_R. \tag{2.3}$$

Blow-up phenomena for reaction-diffusion problems in bounded domain have been studied for the first time in [37] by Kaplan, he showed that, if the convex source terms $f = f(u)$ satisfying the condition

$$\int_U^\infty \frac{du}{f(u)} < \infty, \quad U \geq 1, \tag{2.4}$$

then diffusion cannot prevent blow-up when the initial state is large enough. In fact, the dynamics of equation (2.1) can be understood as a competition between the Laplacian term and nonlinear reaction term.

The problem of semilinear parabolic equation defined in a ball has been introduced in [24, 47, 56, 66]. For instance, in [24] Friedman and McLeod have studied problem (2.1) with (2.2), under fairly general assumptions on u_0, f, they proved that the solutions of this problem are positive, radially decreasing and blow up in finite time at only a single point $x = 0$. They have considered problem (2.1) with two special cases of f, namely, the power type ($f(u) =$

$u|u|^{p-1},\ p > 1$. Here $u^p \equiv u|u|^{p-1}$), and the exponential type ($f(u) = e^u$). For the power type, they showed that for any $\alpha \geq 2/(p-1)$, the upper pointwise estimate takes the following form

$$u(x,t) \leq C|x|^{-\alpha}, \quad x \in B_R \setminus \{0\} \times (0,T),$$

which shows that the only possible blow-up point is $x = 0$. Moreover, under an additional assumption of monotonicity in time (2.3), the corresponding lower estimate on the blow-up profile can be established (see [56]) as follows

$$u(x,T) \geq C|x|^{-2/(p-1)}, \quad x \in B_{R^*} \setminus \{0\},$$

for some $R^* \leq R$, $C > 0$. On the other hand, it has been shown in [24] that the upper (lower) blow-up rate estimates take the following form

$$c(T-t)^{-1/(p-1)} \leq u(0,t) \leq C(T-t)^{-1/(p-1)}, \quad t \in (0,T).$$

For the second case (the exponential type), Friedman and McLeod showed similar results, they proved that the point $x = 0$ is the only blow-up point due to the upper pointwise estimate, which takes the following form

$$u(x,t) \leq \log C + \frac{2}{\alpha}\log(\frac{1}{|x|}), \quad (x,t) \in B_R \setminus \{0\} \times (0,T),$$

where $0 < \alpha < 1, C > 0$. Moreover, the upper (lower) blow-up rate estimate takes the following form

$$\log c - \log(T-t) \leq u(0,t) \leq \log C - \log(T-t), \quad t \in (0,T). \tag{2.5}$$

The aim of this section is to show that the results of Friedman and McLeod hold true for problem (2.1), where f takes the special case $f(u) = e^{u|u|^{p-1}}, p > 1$, namely

$$\left.\begin{array}{ll} u_t = \Delta u + e^{u|u|^{p-1}}, & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R. \end{array}\right\} \tag{2.6}$$

In other words, we prove that $x = 0$ the only possible blow-up point for this problem. Furthermore, we show that the upper blow-up rate estimate takes the following form

$$u(0,t) \leq \log C - \frac{1}{p}\log(T-t), \quad t \in (0,T).$$

2.1.1 Preliminaries

Since f is a C^1 function, the existence and uniqueness of local classical solutions to problem (2.1) are well known, see (Ch. 7, Th. 6, [22]). Moreover, for the regularity results ($u \in C^{2,1}(\overline{B}_R \times [0,T))$, see [56]. On the other hand, there are various conditions in the literature which ensure that $T < \infty$, for instance, it has been shown in [37] that if f is convex, then the codition (2.4) is the necessary and sufficient condition on f to achieve a blow-up solution.Therefore, the solution of problem (2.1) with conditions (2.2) may blow up in finite time for large initial data.

The next lemma shows some properties of the solutions of problem (2.1) with conditions (2.2). We denote for simplicity $u(r,t) = u(x,t)$.

Lemma 2.1.1. *Let u be a classical solution of (2.1) with (2.2). Then*

(i) *$u(x,t)$ is positive and radial, $u_r \leq 0$ in $[0,R) \times (0,T)$. Moreover, $u_r < 0$ in $(0,R] \times (0,T)$.*

(ii) $u_t > 0, \quad (x,t) \in B_R \times (0,T)$.

(iii) *For $f(u) = e^{u^p}$, u blows up in finite time for large initial data and the blow-up set contains $x = 0$.*

Proof of (i):

The proof that u is positive in $B_R \times (0,T)$ is followed directly by Proposition B.1.1.

Next, the aim is to show that the solution of problem (2.1) is radial.

Define the function v as follows:

$$v(x,t) = u((|x|, 0, \ldots, 0), t), \quad (x,t) \in B_R \times (0,T).$$

Clearly, v is a solution to problem (2.1) with the initial function

$$v_0(|x|) = u_0((|x|, 0, \ldots, 0), t).$$

Since u_0 is radial, it follows that

$$u_0(x) = v_0(x), \quad x \in B_R.$$

Therefore, $v(x,t)$ is a solution to problem (2.1) with u_0 as well.

Since it is known that for any initial function u_0 the problem (2.1) has a unique solution in $B_R \times (0,T)$, thus

$$u(x,t) \equiv v(x,t), \quad (x,t) \in B_R \times (0,T).$$

By using Lemma A.2.5, it follows that u is radial.

The final aim is to show that $u_r < 0$, for $(x,t) \in B_R \times (0,T) \cap \{r > 0\}$.

Set $z = r^{n-1}u_r$. Since $f \in C^1([0,\infty)) \cap C^2(0,\infty)$ and $u > 0$ in $\Omega \times (0,T)$, by parabolic regularity results (see Ch. 3, Theorem 13,[22]), we obtain

$$u \in C^{4,2}(B_R \times (0,T)) \cap C^{2,1}(\overline{B}_R \times [0,T)). \tag{2.7}$$

The first equation in (2.1) can be written as follows:

$$u_t - \frac{1}{r^{n-1}} z_r = f, \quad (x,t) \in B_R \times (0,T) \cap \{r > 0\}.$$

Differentiating with respect to r

$$z_t + \frac{n-1}{r} z_r - z_{rr} - f^{'}(u)z = 0, \quad (x,t) \in B_R \times (0,T) \cap \{r > 0\}. \tag{2.8}$$

From the zero Dirichlet boundary condition in (2.1) and since $u > 0$ in $B_R \times (0,T)$, it follows that

$$z(x,t) = R^{n-1}u_r(R,t) < 0, \quad (x,t) \in \partial B_R \times (0,T).$$

Moreover, from (2.2), we have

$$\begin{aligned} &z(x,0) = r^{n-1}u_{0r}(x) \le 0, && x \in B_R \cap \{r > 0\}, \\ &z(0,t) = 0, && t \in (0,T). \end{aligned}$$

Also, by (2.7)

$$u_r \in C^{2,1}((0,R) \times (0,T)) \cap C([0,R] \times [0,T)).$$

Since f' is continuous, $f'(u)$ is bounded in $[0,R]\times[0,t]$, for $t<T$.

From above, it follows by the maximum principle B.1.1 that

$$u_r<0, \text{ for } (x,t)\in \overline{B}_R\times(0,T)\cap\{r>0\}.$$

Proof of (ii):

Set $v=u_t$, by (2.7)

$$v\in C^{2,1}(B_R\times(0,T))\cap C(\overline{B}_R\times[0,T)).$$

Clearly, v satisfies

$$\left.\begin{array}{ll} v_t=\Delta v+f'(u)v, & (x,t)\in B_R\times(0,T) \\ v(x,t)=0, & (x,t)\in\partial B_R\times(0,T), \\ v(x,0)=\Delta u_0+f(u_0)\geq 0, & x\in B_R. \end{array}\right\}$$

From Proposition B.1.1, it follows that

$$v>0, \quad (x,t)\in B_R\times(0,T).$$

Moreover, by Proposition B.1.5, it follows that

$$\frac{\partial v}{\partial\eta}<0, \quad \text{on} \quad \partial B_R\times(0,T).$$

Proof of (iii):

Since the function $f(u)=e^{u^p}$ is convex on $(0,\infty)$ $(f''(u)>0, \forall u>0)$ and satisfies the condition (2.4), the solutions of problem (2.6) blow up in finite time for large initial function. On the other hand, from the comparison principle B.1.2, it is easy to see that if u^*, u are classical solutions (starting from u_0) to problems (2.6) and (2.1), where $f(u)=u^p, p>1$, respectively, then

$$u^*\geq u, \quad \text{in} \quad B_R\times(0,T).$$

It is well known that $x=0$ is the only blow-up point to problem (2.1), (2.2), where $f(u)=u^p$. Therefore, the blow-up solutions of problem (2.6) with (2.2), blow up at $x=0$. Thus (iii) holds.

Remark 2.1.2. Since $u\geq 0$ in $\overline{B}_R\times(0,T)$, we have

$$e^{u^p}\equiv e^{u|u|^{p-1}}, \quad \text{in} \quad \overline{B}_R\times(0,T).$$

2.1.2 Pointwise Estimates

This subsection considers the pointwise estimate of the solutions of problem (2.6) with (2.2), which shows that the blow-up cannot occur if x is not equal to zero. In order to prove that, we need first to recall the following lemma, which has been proved by Friedman and McLeod in (p. 428, [24]).

Lemma 2.1.3. *Let u be a blow-up solution of problem (2.1) with (2.2). Also suppose that*

$$u_{0r}(r) \leq -\delta r, \qquad for\ 0 < r \leq R, \quad where\ \delta > 0. \tag{2.9}$$

Consider $F \in C^2(0,\infty) \cap C^1([0,\infty))$, such that F is positive in $(0,\infty)$ and satisfies

$$F^{'}, F^{''} \geq 0 \quad in\ (0,\infty). \tag{2.10}$$

Also if it satisfies with f the following condition,

$$f^{'}F - fF^{'} \geq 2\varepsilon FF^{'} \quad in \quad (0,\infty). \tag{2.11}$$

Then the function $J = r^{n-1}u_r + \varepsilon r^n F(u)$ is nonpositive in $B_R \times (0,T)$ for some $\varepsilon > 0$.

Proof. Set $z = r^{n-1}u_r, c(r) = \varepsilon r^n$.

Since

$$u_r \in C^{2,1}((0,R) \times (0,T)) \cap C([0,R] \times [0,T))$$

and $F \in C^2(0,\infty) \cap C^1([0,\infty))$, it follows

$$J \in C^{2,1}((0,R) \times (0,T)) \cap C([0,R] \times [0,T)).$$

By using (2.8), a direct calculation shows

$$\begin{aligned} J_t + \frac{n-1}{r}J_r - J_{rr} &= f^{'}(u)z + cF^{'}f + \frac{2(n-1)}{r}cF^{'}u_r \\ &+ \frac{n-1}{r}c^{'}F - cF^{''}u_r^2 - 2c^{'}F^{'}u_r - c^{''}F \equiv B. \end{aligned}$$

Using $u_r = z/r^{n-1}$ and $z = -cF + J$, it follows that

$$\begin{aligned} B &= bJ - c(f^{'}F - fF^{'}) - \frac{c^3}{r^{2n-2}}F^{''}F^2 + \frac{2cc^{'}}{r^{n-1}}F^{'}F \\ &\quad + \frac{n-1}{r}c^{'}F - \frac{2(n-1)c^2}{r^n}F^{'}F - c^{''}F, \end{aligned}$$

where

$$b = f^{'} + \frac{2(n-1)}{r^n}cF^{'} - \frac{2c^{'}F^{'}}{r^{(n-1)}} = f^{'} - 2\varepsilon F^{'}$$

when $c = \varepsilon r^n$.

Clearly, b is a bounded function for $0 < r < R$, $0 < t \leq T^* < T$.

Thus

$$J_t + \frac{n-1}{r}J_r - J_{rr} - bJ \leq 0, \quad (r,t) \in (0,R) \times (0,T)$$

provided

$$f^{'}F - fF^{'} - \frac{2c^{'}}{r^{n-1}}F^{'}F + \frac{2(n-1)}{r^n}cF^{'}F + (c^{''} - \frac{n-1}{r}c^{'})\frac{F}{c} \geq 0.$$

Since $c = \varepsilon r^n$, and with choosing ε small enough the last inequality becomes

$$f^{'}(u)F(u) - f(u)F^{'}(u) \geq 2\varepsilon F(u)F^{'}(u), \quad \text{in } (0,R) \times (0,T).$$

From (2.11), it is clear that the last inequality holds.

Since $u_t > 0$ in $(0,R)\times(0,T)$ and from the zero Dirichlet boundary condition, it clear that

$$u_r(R,t) < u_{0r}(R), \quad t \in (0,T).$$

Thus

$$\begin{aligned} &J(R,t) \leq R^{n-1}[u_{0r}(R) + \varepsilon RF(0)] \leq R^n[-\delta + \varepsilon F(0)] \leq 0, \quad t \in (0,T), \\ &J(r,0) = r^{n-1}[u_{0r}(r) + \varepsilon rF(u_0(r))] \leq r^n[-\delta + \varepsilon F(u_0(r))] \leq 0, \end{aligned}$$

provided

$$\varepsilon \leq \frac{\delta}{\max_{(0,R]} F(u_0)}.$$

Morevor, $J(0,\cdot) = 0$.

From above and Proposition B.1.3, it follows that

$$J \leq 0, \quad \text{in} \quad [0,R] \times (0,T).$$

Hence

$$J \leq 0, \quad \text{in} \quad \overline{B}_R \times (0,T).$$

□

The last lemma has been used in [24] for the cases where the reaction term is of power or exponential type functions, to prove that the blow-up can only occur at a single point. The next theorem extends these results to the problem (2.6) with (2.2).

Theorem 2.1.4. *Let u be a blow-up solution of problem (2.6) with (2.2). Also suppose that u_0 satisfies (2.9). Then $x = 0$ is the only blow-up point.*

Proof. Let

$$F(u) = e^{\delta u^p}, \quad 0 < \delta < 1.$$

It is clear that F satisfies (2.10). The next aim is to show that the inequality (2.11) holds.

A direct calculation shows

$$\begin{aligned} f^{'}(u)F(u) - f(u)F^{'}(u) &= pu^{p-1}e^{(1+\delta)u^p} - \delta pu^{p-1}e^{(1+\delta)u^p} \\ &= pu^{p-1}e^{(1+\delta)u^p}[1-\delta]. \end{aligned} \tag{2.12}$$

On the other hand,

$$2\varepsilon F(u)F^{'}(u) = 2\varepsilon\delta pu^{p-1}e^{2\delta u^p}. \tag{2.13}$$

From (2.12), (2.13) it is clear that (2.11) holds true provided ε, δ are small enough.

Thus, by Lemma 2.1.3

$$J = r^{n-1}u_r + \varepsilon r^n e^{\delta u^p} \leq 0, \quad (r,t) \in (0,R) \times (0,T),$$

or

$$-\frac{u_r}{e^{\delta u^p}} \geq \varepsilon r. \tag{2.14}$$

Let $G(s) = \int_s^\infty \frac{du}{e^{\delta u^p}}$.

It is clear that

$$\frac{d}{dr}G(u(r,t)) = \frac{d}{dr}\int_u^\infty \frac{du}{e^{\delta u^p}} = -\frac{d}{dr}\int_\infty^u \frac{du}{e^{\delta u^p}} = -\frac{d}{du}\int_\infty^u \frac{u_r}{e^{\delta u^p}}du = -\frac{u_r}{e^{\delta u^p}}.$$

Thus, by (2.14), we obtain

$$G(u(r,t))_r \geq \varepsilon r.$$

Now, integrate the last equation from 0 to r

$$G(u(r,t)) - G(u(0,t)) \geq \frac{1}{2}\varepsilon r^2.$$

It follows

$$G(u(r,t)) \geq \frac{1}{2}\varepsilon r^2. \tag{2.15}$$

If for some $r > 0$, $u(r,t) \to \infty$, as $t \to T$, then $G(u(r,t)) \to 0$, as $t \to T$, a contradiction to (2.15). □

Remark 2.1.5. Under the assumptions of Theorem 2.1.4, it follows from (2.15) that the upper pointwise estimate for problem (2.6) with (2.2) takes the following form

$$u(x,t) \leq \log C + \frac{2}{\delta}\log(\frac{1}{|x|}), \quad (x,t) \in B_{R_0} \setminus \{0\} \times (t_0, T),$$

where $R_0 \in (0,R)$ and $t_0 \in [0,T)$ such that $u(R_0,t_0) \geq 1$.

2.1.3 Blow-up Rate Estimates

The following theorem considers the upper bound of the blow-up rate for problem (2.6) with (2.2), following the procedure used in [24].

Theorem 2.1.6. *Let u be a solution of (2.6) with (2.2) and (2.9), which blows up in finite time T. Then there exists a positive constant C such that*

$$u(0,t) \leq \log C - \frac{1}{p}\log(T-t), \quad t \in (0,T). \tag{2.16}$$

Proof. Define the function F as follows,

$$F(x,t) = u_t - \alpha f(u), \quad (x,t) \in B_R \times (0,T),$$

where $f(u) = e^{u^p}$, $\alpha > 0$.

Since $F \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T))$, a direct calculation shows

$$\begin{aligned} F_t - \Delta F &= u_{tt} - \alpha f^{'} u_t - \Delta u_t + \alpha \Delta f(u), \\ &= u_{tt} - \Delta u_t - \alpha f^{'}[u_t - \Delta u] + \alpha |\nabla u|^2 f^{''}, \\ &= f^{'} u_t - \alpha f^{'} f(u) + \alpha |\nabla u|^2 f^{''}. \end{aligned}$$

Thus

$$F_t - \Delta F - f^{'}(u) F = \alpha |\nabla u|^2 f^{''} \geq 0, \quad (x,t) \in B_R \times (0,T), \tag{2.17}$$

due to $f^{''}(u) > 0$, for u in $(0, \infty)$.

Since $f^{'}$ is continuous, $f^{'}(u)$ is bounded in $\overline{B}_R \times [0,t]$, for $t < T$.

By Lemma 2.1.1, $u_t(x,t) > 0$, in $B_R \times (0,T)$, and since u blows up at $x = 0$, there exist $k > 0$, $\varepsilon \in (0,R)$, $\tau \in (0,T)$ such that

$$u_t(x,t) \geq k, \quad (x,t) \in \overline{B}_\varepsilon \times [\tau, T).$$

Also, we can find $\alpha > 0$ such that $u_t(x,\tau) \geq \alpha f(u(x,\tau))$, for $x \in B_\varepsilon$. Thus

$$F(x,\tau) \geq 0 \quad \text{for } x \in B_\varepsilon. \tag{2.18}$$

On the other hand, because of u blows up at only $x = 0$, there exists $C_0 > 0$ such that

$$f(u(x,t)) \leq C_0 < \infty, \quad \text{in} \quad \partial B_\varepsilon \times (0,T),$$

If we choose α is small enough such that $k \geq \alpha C_0$, then we get

$$F(x,t) \geq 0, \quad (x,t) \in \partial B_\varepsilon \times [\tau, T), \tag{2.19}$$

By (2.17), (2.18), (2.19) and Proposition B.1.1 (starting from τ instead of 0), it follows that

$$F(x,t) \geq 0, \quad (x,t) \in \overline{B}_\varepsilon \times (\tau, T).$$

Thus

$$u_t(0,t) \geq \alpha e^{u^p(0,t)}, \quad \text{for} \quad \tau \leq t < T. \tag{2.20}$$

Since u is increasing in time and blows at T, there exist $\tau^* \leq \tau$ such that

$$u(0,t) \geq p^{\frac{1}{(p-1)}} \quad \text{for} \quad \tau^* \leq t < T,$$

provided τ is close enough to T, which leads to

$$e^{u^p(0,t)} \geq e^{pu(0,t)}, \quad \tau^* \leq t < T. \tag{2.21}$$

From (2.20), (2.21), it follows that

$$u_t(0,t) \geq \alpha e^{pu(0,t)}, \quad \text{for} \quad \tau \leq t < T. \tag{2.22}$$

Integrate (2.22) from t to T

$$\int_t^T u_t(0,t) e^{-pu(0,t)} \geq \alpha(T-t).$$

Thus

$$-\frac{1}{p} e^{-pu(0,t)}|_t^T \geq \alpha(T-t). \tag{2.23}$$

Since

$$u(0,t) \to \infty, \quad e^{-pu(0,t)} \to 0, \quad \text{as} \quad t \to T,$$

the inequality (2.23) becomes

$$\frac{1}{e^{pu(0,t)}} \geq p\alpha(T-t).$$

Thus

$$e^{pu(0,t)}(T-t) \leq C^*, \quad C^* = 1/(p\alpha), \quad t \in [\tau, T)$$

Therefore, there exist a positive constant C such that

$$u(0,t) \leq \log C - \frac{1}{p}\log(T-t), \quad t \in (0,T).$$

□

Remark 2.1.7. Depending on the size of the initial data, at a large time enough, the solution of problem (2.6) is larger than or equal to the solution of problem (2.1), where $f(u) = e^{pu}$, and this can be shown by the comparison principle B.1.2. However, from Theorem 2.1.6, we observe that the two problems have the same upper blow-up rate estimate (2.16).

2.2 Coupled Systems of Reaction Diffusion Equations

In this section, we consider the system of two semilinear heat equations with zero Dirichlet boundary conditions defined in a ball:

$$\left.\begin{array}{lll} u_t = \Delta u + f(v), & v_t = \Delta v + g(u), & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & v(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in B_R, \end{array}\right\} \tag{2.24}$$

where $f, g \in C^1(R) \cap C^2(R \setminus \{0\})$, are positive and increasing superlinear functions on $(0, \infty)$, $1/f, 1/g$ are integrable at infinity, moreover, the functions $f^{'}, g^{'}, f^{''}$ and $g^{''}$ are positive in $(0, \infty)$, u_0 and v_0 are smooth, nonnegative, radially nonincreasing functions, vanishing on ∂B_R, this means they satisfy the following conditions:

$$\left.\begin{array}{lll} u_0(x) = u_0(|x|), & v_0(x) = v_0(|x|), & x \in B_R, \\ u_0(x) = 0, & v_0(x) = 0, & x \in \partial B_R, \\ u_{0r}(|x|) \leq 0, & v_{0r}(|x|) \leq 0, & x \in B_R. \end{array}\right\} \tag{2.25}$$

Moreover, we assume that they satisfy the following conditions

$$\Delta u_0 + f(v_0) \geq 0, \quad \Delta v_0 + g(u_0) \geq 0, \quad \forall x \in B_R. \tag{2.26}$$

According to [11], the problem (2.24) has been formulated from physical models arising in various fields of applied sciences, for example, in the chemical reaction process, the chemical concentration and the temperature are governed by a coupled system of reaction diffusion equations in the form of (2.24).

The problem of a semilinear parabolic system defined in a ball was introduced in [11, 23, 43, 61]. For instance, in [23] Friedman and Giga have studied the blow-up solution to the system (2.24) in one dimensional space, namely

$$u_t = u_{xx} + f(v), \qquad v_t = v_{xx} + g(u), \quad (x,t) \in (-R, R) \times (0,T),$$

where f, g are positive, increasing and superlinear functions, and u_0, v_0 are defined as in (2.25) and suitably large. It was proved (under some assumptions on f, g), that the blow-up to this problem occurs at only a single point. They have studied as examples of their results, two special cases of f, g; firstly, the power model

$$f(v) = v|v|^{p-1}, \quad g(u) = u|u|^{q-1}, \quad p = q, \tag{2.27}$$

secondly, the exponential model

$$f(u) = Ae^{u}, \quad g(u) = Be^{u} \quad A, B > 0. \tag{2.28}$$

Recently, in [61], it has been considered the positive solutions to problem (2.24) in general dimensional space, where f, g are of power type functions, namely

$$u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q, \quad (x,t) \in B_R \times (0,T) \quad p, q > 1. \tag{2.29}$$

For this problem, it was proved single point blow-up for the radially decreasing solutions. Moreover, it was shown that the lower pointwise estimates for the final blow-up profiles take the forms

$$u(x,T) \geq c_1|x|^{-2\alpha}, \qquad v(x,T) \geq c_2|x|^{-2\beta},$$

where

$$\alpha = \frac{p+1}{pq-1}, \quad \beta = \frac{q+1}{pq-1}.$$

On the other hand, the blow-up rate estimates for this problem have been considered by many authors (see for instance [11]), it was shown that if the condition (2.26) is satisfied, then the upper (lower) blow-up rate estimates take the following forms:

$$c_1(T-t)^{-\alpha} \leq u(0,t) \leq c_2(T-t)^{-\alpha}, \quad t \in (0,T),$$

$$c_3(T-t)^{-\beta} \leq u(0,t) \leq c_4(T-t)^{-\beta}, \quad t \in (0,T).$$

Similar results were obtained for the second special case of problem (2.24), where f, g are of exponential type , namely

$$u_t = \Delta u + e^{pv}, \qquad v_t = \Delta v + e^{qu}, \quad (x,t) \in B_R \times (0,T), \quad p, q > 0. \tag{2.30}$$

For this problem, it has been shown in [11] that the only blow-up point is $x = 0$ and the blow-up rate estimates take the following forms:

$$\log c - \log[q(T-t)] \leq qu(0,t) \leq \log C - \log[q(T-t)], \quad t \in (0,T),$$

$$\log c - \log[p(T-t)] \leq pv(0,t) \leq \log C - \log[p(T-t)], \quad t \in (0,T).$$

The aim of this section is to study some conditions under which the blow-up in problem (2.24) occurs only at a single point, furthermore, to derive a formula for the upper (lower) blow-up rate estimates under some restricted assumptions on f, g. Finally, the special case, where f, g take the forms as in (2.28) (the so called *Ignition system*, [71]), will be studied in general dimensional space as an example of the results of this section.

2.2.1 Preliminaries

Since f, g are C^1 functions, which means they are locally Lipschitz functions, the local existence of the unique classical solutions to problem (2.24) is guaranteed (see [40]). On the other hand, it is well known [27, 28] that $T < \infty$ for a large class of functions f, g, when the initial data (u_0, v_0) are suitably large. Moreover, since (2.24) is coupled system, only simultaneous blow-up can occur.

The next lemma shows some properties of the classical solutions of problem (2.24) with (2.25). We denote for simplicity $u(r,t) = u(x,t),\ v(r,t) = v(x,t)$.

Lemma 2.2.1. *Let (u, v) be a classical solution to the problem (2.24), (2.25). Then*

(i) *u and v are positive and radial, $u_r \leq 0,\ v_r \leq 0$ in $[0,R) \times (0,T)$. Moreover, $u_r < 0,\ v_r < 0$ in $(0,R] \times (0,T)$.*

(ii) $u_t > 0,\ v_t > 0, \quad (x,t) \in B_R \times (0,T)$.

(iii) *If (u, v) is a blow-up solution, then $x = 0$ is a blow-up point.*

Proof. The proofs of (i) and (ii) are similar to the proof of Lemma 2.1.1, with using some of maximum principles from Appendix B, for parabolic systems.

From (i) we conclude that the blow-up sets for u and v coincide with some intervals $[-a, a]$ and $[-b, b]$ respectively, where $a, b < R$. This means, the blow-up set for (u, v) contains $r = 0$. Thus (iii) holds. □

2.2.2 Blow-up Set

In this subsection we show under some assumptions that the only possible blow-up point to problem (2.24), (2.25) is $x = 0$.

Theorem 2.2.2. *Let (u, v) be a blow-up solution of problem (2.24) with (2.25). Suppose that*

$$u_{0r}(r) \leq -\delta_1 r,\ v_{0r}(r) \leq -\delta_2 r \quad \text{for } 0 < r \leq R, \quad \text{where } \delta_1, \delta_2 > 0. \tag{2.31}$$

If there exist two functions $F, G \in C^2([0, \infty))$ such that F, G are positive in $(0, \infty)$ and their first and second derivatives are nonnegative in $(0, \infty)$, moreover, they satisfy with f, g the following conditions

$$\int_s^\infty \frac{dv}{F(v)} < \infty, \quad \int_s^\infty \frac{du}{G(u)} < \infty, \quad \text{for} \quad s > 0,$$

$$\left.\begin{array}{ll} f'(v)F(v) - f(v)G'(u) \geq 2\varepsilon G(u)G'(u), & \text{in} \quad (0, R) \times (0, T), \\ g'(u)G(u) - g(u)F'(v) \geq 2\varepsilon F(v)F'(v), & \text{in} \quad (0, R) \times (0, T), \end{array}\right\} \tag{2.32}$$

for some $\varepsilon \in (0, 1)$, then the blow-up set has only one point $x = 0$.

Proof. We follow the procedures of Friedman and McLeod used in [24] for the scalar problem (2.1).

Since both u and v are radial, we denote for simplicity

$$u(r, t) = u(x, t), \quad v(r, t) = v(x, t).$$

Clearly, the system (2.24) can be written as follows:

$$\left.\begin{array}{ll} u_t = u_{rr} + \frac{n-1}{r} u_r + f(v), & (r, t) \in (0, R) \times (0, T), \\ v_t = v_{rr} + \frac{n-1}{r} v_r + g(u), & (r, t) \in (0, R) \times (0, T). \end{array}\right\} \tag{2.33}$$

Set

$$J_1 = r^{n-1}u_r + \varepsilon r^n G(u), \quad J_2 = r^{n-1}v_r + \varepsilon r^n F(v).$$

By parabolic regularity results

$$u_r, v_r \in C^{2,1}((0,R)\times(0,T)) \cap C([0,R]\times[0,T)).$$

Since $F, G \in C^2([0,\infty))$,

$$J_1, J_2 \in C^{2,1}((0,R)\times(0,T)) \cap C([0,R]\times[0,T)).$$

Denote for convenience

$$w_1 = r^{n-1}u_r, \quad w_2 = r^{n-1}v_r, \quad , c(r) = \varepsilon r^n.$$

Thus

$$J_1 = w_1 + c(r)G(u), \quad J_2 = w_2 + c(r)F(v).$$

A direct calculation shows

$$\begin{aligned} w_{1t} =& r^{n-1}u_{rt}, \\ w_{1r} =& r^{n-1}u_{rr} + (n-1)r^{n-2}u_r, \\ w_{1rr} =& r^{n-1}u_{rrr} + (n-1)r^{n-2}u_{rr} + (n-1)(n-2)r^{n-3}u_r \\ &+ (n-1)r^{n-2}u_{rr}. \end{aligned}$$

This leads to

$$\begin{aligned} w_{1t} + \frac{n-1}{r}w_{1r} - w_{1rr} \quad = \quad & r^{n-1}u_{rt} + (n-1)r^{n-2}u_{rr} + (n-1)^2 r^{n-3}u_r \\ & - r^{n-1}u_{rrr} - (n-1)r^{n-2}u_{rr} \\ & - (n-1)(n-2)r^{n-3}u_r - (n-1)r^{n-2}u_{rr}. \end{aligned}$$

From (2.33), it follows that

$$u_{rrr} = u_{tr} - \frac{n-1}{r}u_{rr} + \frac{n-1}{r^2}u_r - f' v_r.$$

Thus

$$w_{1t} + \frac{n-1}{r}w_{1r} - w_{1rr} = w_2 f'(v).$$

In the same way we can show

$$w_{2t} + \frac{n-1}{r} w_{2r} - w_{2rr} = w_1 g^{'}(u).$$

Also it is clear

$$\begin{aligned}
[c(r)G(u)]_t &= c(r)G^{'}(u)u_t = \varepsilon r^n G^{'}(u)(u_{rr} + \frac{n-1}{r}u_r + f(v)), \\
[c(r)G(u)]_r &= \varepsilon r^n G^{'}(u)u_r + \varepsilon n G(u) r^{n-1}, \\
\frac{(n-1)}{r}[c(r)G(u)]_r &= \varepsilon(n-1)r^{n-1}G^{'}(u)u_r + \varepsilon n(n-1)G(u)r^{n-2}, \\
[c(r)G(u)]_{rr} &= \varepsilon r^n (G^{'}(u)u_{rr} + u_r^2 G^{''}(u)) + \varepsilon G^{'}(u)u_r n r^{n-1} \\
&\quad + \varepsilon n G(u)(n-1)r^{n-2} + \varepsilon n r^{n-1} G^{'}(u)u_r.
\end{aligned}$$

From above, it follows that

$$\begin{aligned}
J_{1t} + \frac{n-1}{r}J_{1r} - J_{1rr} &= f^{'}(v)[J_2 - \varepsilon r^n F(v)] + \varepsilon r^n G^{'}(u) f(v) \\
&\quad - 2\varepsilon G^{'}(u)[r^{n-1}u_r] - \varepsilon r^n G^{''}(u) u_r^2.
\end{aligned}$$

Using the relation $r^{n-1}u_r = w_1 = J_1 - \varepsilon r^n G(u)$, we obtain

$$\begin{aligned}
J_{1t} + \frac{n-1}{r}J_{1r} - J_{1rr} &\leq f^{'}(v)[J_2 - \varepsilon r^n F(v)] \\
&\quad + \varepsilon r^n G^{'}(u) f(v) - 2\varepsilon G^{'}(u)[J_1 - \varepsilon r^n G(u)]
\end{aligned}$$

Thus

$$J_{1t} + \frac{n-1}{r}J_{1r} - J_{1rr} - bJ_1 - cJ_2 \leq -\varepsilon r^n H, \tag{2.34}$$

where

$$H = F(v)f^{'}(v) - f(v)G^{'}(u) - 2\varepsilon G(u)G^{'}(u).$$

From our assumption (2.32), it follows that $H \geq 0$ in $(0, R) \times (0, T)$.

Thus

$$J_{1t} + \frac{n-1}{r}J_{1r} - J_{1rr} - bJ_1 - cJ_2 \leq 0, \quad (x,t) \in (0,R) \times (0,T).$$

where, $b = -2\varepsilon G^{'}(u), c = f^{'}(v)$.

In the same way we can show that

$$J_{2t} + \frac{n-1}{r} J_{2r} - J_{2rr} - dJ_2 - hJ_1 \leq 0, \quad (x,t) \in (0,R) \times (0,T),$$

where, $d = -2\varepsilon F^{'}(v), h = g^{'}(u)$.

Clearly, c, h, d and b are bounded functions on $(0,R) \times [0,t]$ for any fixed $t \in (0,T)$, moreover, $c, h \geq 0$.

Also,

$$J_1(0,t) = J_2(0,t) = 0, \quad t \in [0,T).$$

By (2.31), we obtain

$$\begin{aligned} J_1(r,0) &= r^{n-1}[u_{0r}(r) + \varepsilon r G(u_0(r))] \leq r^n[-\delta_1 + \varepsilon G(u_0(r))], \\ J_2(r,0) &= r^{n-1}[v_{0r}(r) + \varepsilon r F(v_0(r))] \leq r^n[-\delta_2 + \varepsilon F(v_0(r))]. \end{aligned}$$

Since u, v are increasing in time in the domain $B_R \times (0,T)$, it follows that

$$u > u_0, v > v_0, \quad (x,t) \in B_R \times (0,T),$$

and from the zero Dirichlet boundary conditions, it is easy to see that

$$u_r(R,t) < u_{0r}(R) < 0, \quad v_r(R,t) < v_{0r}(R) < 0, \quad t \in (0,T).$$

Thus

$$\begin{aligned} J_1(R,t) &\leq R^{n-1}[u_{0r}(R) + \varepsilon r G(0)] \leq R^n[-\delta_1 + \varepsilon G(0)], \quad t \in (0,T), \\ J_2(R,t) &\leq R^{n-1}[v_{0r}(R) + \varepsilon r F(0)] \leq R^n[-\delta_2 + \varepsilon F(0)], \quad t \in (0,T). \end{aligned}$$

Therefore, each of the functions $J_1(r,0)$, $J_2(r,0)$, $J_1(R,t)$, $J_2(R,t)$, are non-positive, for $r \in (0,R)$, $t \in (0,T)$, provided

$$\varepsilon \leq \min\{\frac{\delta_1}{\max_{(0,R]} G(u_0)}, \frac{\delta_2}{\max_{(0,R]} F(v_0)}\}.$$

From above and Proposition B.2.1, it follows that

$$J_1, J_2 \leq 0, \quad (x,t) \in B_R \times (0,T). \tag{2.35}$$

Define

$$G^*(s) = \int_s^\infty \frac{du}{G(u)}, \quad F^*(s) = \int_s^\infty \frac{dv}{F(v)}.$$

From (2.35), it follows that

$$\frac{-u_r}{G(u)} \geq \varepsilon r$$

Clearly,

$$\frac{d}{dr}G^*(u(r,t)) = \frac{d}{dr}\int_u^\infty \frac{du}{G(u)} = -\frac{d}{dr}\int_\infty^u \frac{du}{G(u)} = -\frac{d}{du}\int_\infty^u \frac{u_r}{G(u)}du = -\frac{u_r}{G(u)}.$$

Thus

$$G^*(u(r,t))_r \geq \varepsilon r.$$

Now, integrate the last equation from 0 to r

$$G^*(u(r,t)) - G^*(u(0,t)) \geq \frac{1}{2}\varepsilon r^2.$$

It follows that

$$G^*(u(r,t)) \geq \frac{1}{2}\varepsilon r^2. \tag{2.36}$$

In the same way we can show that

$$F^*(v(r,t)) \geq \frac{1}{2}\varepsilon r^2. \tag{2.37}$$

If for some $r > 0$ $u(r,t) \to \infty$ or $v(r,t) \to \infty$ as $t \to T$, then $G^*(u(r,t)) \to 0$ or $F^*(v(r,t)) \to 0$ as $t \to T$, a contradiction to (2.36), (2.37). □

Remark 2.2.3. Theorem 2.2.2 implies that any point $x \neq 0$ does not belong to the blow-up set. Therefore, under the assumption of Theorem 2.2.2, the blow-up set of the problem (2.24), (2.25) has only a single point $x = 0$.

In section 2.2.4, we study an example for the assumptions assumed in the last theorem.

2.2.3 Blow-up Rate Estimates

The following theorem considers the lower (upper) bounds on the blow-up rate estimates for problem (2.24), (2.25) with some restricted assumptions on f, g.

Theorem 2.2.4. *Let (u, v) be a solution to (2.24), (2.25), which blows up at only one point ($x = 0$). Assume there exsits $\gamma > 1$ such that*

$$g(u) \leq \gamma f(v), \quad f(v) \leq \gamma g(u), \quad (x,t) \in B_R \times (0,T). \tag{2.38}$$

Then there exist positive constants c_1, c_2, c_3 and c_4 such that

$$G_1^{-1}(c_1(T-t)) \leq u(0,t) \leq G_1^{-1}(c_2(T-t)), \quad t \in (0,T), \tag{2.39}$$

$$F_1^{-1}(c_3(T-t)) \leq v(0,t) \leq F_1^{-1}(c_4(T-t)), \quad t \in (0,T), \tag{2.40}$$

where

$$G_1(s) = \int_s^\infty \frac{du}{g(u)}, \quad F_1(s) = \int_s^\infty \frac{dv}{f(v)}. \tag{2.41}$$

Proof. We first consider the lower bounds.

Set

$$U(t) = u(0,t), \quad V(t) = v(0,t), \quad t \in [0,T).$$

Since (u, v) attains its maximum at $x = 0$, we obtain

$$\Delta U(t) \leq 0, \quad \Delta V(t) \leq 0, \quad 0 \leq t < T.$$

From (2.24) it follows that

$$U_t(t) \leq f(V(t)), \quad V_t(t) \leq g(U(t)), \quad 0 < t < T. \tag{2.42}$$

From (2.38) and (2.42), it follows that

$$U_t(t) \leq \gamma g(U(t)), \quad V_t(t) \leq \gamma f(V(t)), \quad 0 < t < T.$$

Thus

$$\frac{U_t(t)}{g(U(t))} \leq \gamma, \quad \frac{V_t(t)}{f(V(t))} \leq \gamma, \quad 0 < t < T. \tag{2.43}$$

Clearly,

$$-\frac{dG_1(u(0,t))}{dt} = -\frac{d}{dt}\int_{u(0,t)}^\infty \frac{du}{g(u(0,t))} = -\frac{d}{dt}\int_t^T \frac{(du/dt)}{g(u(0,t))}dt = \frac{d}{dt}\int_T^t \frac{u_t}{g(u(0,t))}dt,$$

which leads to

$$-\frac{dG_1(u(0,t))}{dt} = \frac{u_t(0,t)}{g(u(0,t))},$$

where G_1 defined as in (2.41). From the last equation and equation (2.43), it follows that

$$-\frac{dG_1(u)}{dt} \leq \gamma, \quad 0 < t < T. \tag{2.44}$$

Integrate (2.44) from t to T

$$G_1(u(0,t)) - G_1(u(0,T)) \leq \gamma(T-t).$$

Clearly, $G_1(u(0,T)) = 0$.

Thus

$$G_1(u(0,t)) \leq \gamma(T-t), \quad 0 < t < T.$$

Since G_1 is decreasing, by the last equation

$$u(0,t) \geq G_1^{-1}(\gamma(T-t)), \quad 0 < t < T.$$

For v in the same way we can show that

$$v(0,t) \geq F_1^{-1}(\gamma(T-t)), \quad 0 < t < T.$$

Next, we consider the upper bounds.

As in the proof of Theorem 2.1.6, define the functions Q, H as follows

$$Q(x,t) = u_t - \theta f(v), \quad H(x,t) = v_t - \theta g(u), \quad (x,t) \in B_R \times (0,T),$$

where $\theta > 0$. By parabolic regularity, we have

$$u_t, v_t \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T)),$$

and since $f, g \in C^2(0,\infty) \cap C([0,\infty))$, it follows that

$$F, G \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T)).$$

A direct calculation shows

$$\begin{aligned} Q_t - \Delta Q &= u_{tt} - \theta f' v_t - \Delta u_t + \theta \Delta f(v), \\ &= u_{tt} - \Delta u_t - \theta f'[v_t - \Delta v] + \theta |\nabla v|^2 f'', \\ &= f' v_t - \theta f' g(u) + \theta |\nabla v|^2 f''. \end{aligned}$$

Thus

$$Q_t - \Delta Q - f^{'}(v)H = \theta|\nabla v|^2 f^{''} \geq 0, \quad \text{in} \quad B_R \times (0,T),$$

due to $f^{''}$ is a positive function in $(0,\infty)$. In the same we can show that

$$H_t - \Delta H - g^{'}(u)Q = \theta|\nabla u|^2 g^{''} \geq 0, \quad \text{in} \quad B_R \times (0,T).$$

Since $f^{'}, g^{'}$ are continuous functions, $f^{'}(v), g^{'}(u)$ are bounded in $\overline{B}_R \times [0,t]$ for $t < T$.

By Lemma 2.2.1, $u_t, v_t > 0$, in $B_R \times (0,T)$, and since u, v blow up at $x = 0$, there exist $k_1 > 0, k_2 > 0$, $\varepsilon \in (0,R)$, $\tau \in (0,T)$ such that

$$u_t(x,t) \geq k_1, \quad v_t(x,t) \geq k_2, \quad (x,t) \in \overline{B}_\varepsilon \times [\tau, T).$$

Also, we can find $\theta > 0$ such that

$$u_t(x,\tau) \geq \theta f(v(x,\tau)),\ v_t(x,\tau) \geq \theta g(u(x,\tau)), \quad \text{for } x \in B_\varepsilon.$$

Thus

$$F(x,\tau) \geq 0,\ G(x,\tau) \geq 0 \quad \text{for } x \in B_\varepsilon.$$

Since, u, v blow up at only $x = 0$, there exists $C_1, C_2 > 0$ such that

$$f(v(x,t)) \leq C_1 < \infty,\ g(u(x,t)) \leq C_2 < \infty, \quad \text{in} \quad \partial B_\varepsilon \times (0,T),$$

If we choose θ is small enough such that

$$\theta \leq \min\{\frac{k_1}{C_1}, \frac{k_2}{C_2}\},$$

then, we can get

$$F(x,t) \geq 0,\ G(x,t) \geq 0 \quad (x,t) \in \partial B_\varepsilon \times [\tau, T),$$

From above and by Proposition B.2.1 (starting from τ instead of 0), it follows that

$$F(x,t) \geq 0,\ G(x,t) \geq 0 \quad (x,t) \in \overline{B}_\varepsilon \times (\tau, T).$$

This leads to

$$u_t(0,t) \geq \theta f(v(0,t)), \quad v_t \geq \theta g(u(0,t)), \quad \text{for} \quad \tau \leq t < T. \tag{2.45}$$

By (2.38), we obtain

$$u_t(0,t) \geq \frac{\theta}{\gamma} g(u(0,t)), \quad v_t \geq \frac{\theta}{\gamma} f(v(0,t)), \quad \tau \leq t < T. \tag{2.46}$$

Since

$$-\frac{dG_1(u(0,t))}{dt} = \frac{u_t(0,t)}{g(u(0,t))}.$$

From (2.46) and the last equation, it follows that

$$-\frac{dG_1(u(0,t))}{dt} \geq \frac{\theta}{\gamma}, \quad \tau \leq t < T.$$

Integrating the last inequality from t to T

$$\int_t^T -dG_1(u(0,t)) = G_1(u(0,t)) - G_1(u(0,T)) \geq \frac{\theta}{\gamma}(T-t).$$

Thus

$$G_1(u(0,t)) \geq \frac{\theta}{\gamma}(T-t), \quad \tau \leq t < T. \tag{2.47}$$

Since G_1 is decreasing, from (2.47), it follows that

$$u(0,t) \leq G_1^{-1}(\frac{\theta}{\gamma}(T-t)), \quad \tau \leq t < T.$$

Thus, there exist $c_2 > 0$ such that

$$u(0,t) \leq G_1^{-1}(c_2(T-t)), \quad 0 < t < T.$$

Similarly, we can find $c_4 > 0$ such that

$$v(0,t) \leq F_1^{-1}(c_4(T-t)), \quad 0 < t < T.$$

$\square$

In next section, we study an example for the assumptions assumed in the last theorem.

2.2.4 The Ignition System

Theorems 2.2.2 and 2.2.4 can be applied to a large class of functions f, g, including the following forms

$$f(v) = Ae^v, \quad g(u) = Be^u, \tag{2.48}$$

where A, B are positive constants. To show that the condition (2.38) holds for such type of system, we prove the following lemma, which has been proved in [23] for one dimensional space.

Lemma 2.2.5. *Let (u, v) be a nontrivial solution to problem (2.24), (2.25), where f, g take the forms of (2.48).Then there exist $M > 1$ such that*

$$e^v \leq Me^u, \quad e^u \leq Me^v, \quad (x,t) \in B_R \times (0,T). \tag{2.49}$$

Proof. Define

$$J = Me^u - e^v, \quad (x,t) \in B_R \times (0,T).$$

Clearly, $J \in C^{2,1}(\overline{\Omega} \times [0,T))$. A direct calculation shows

$$\begin{aligned} J_t &= Me^u u_t - e^v v_t, \\ \nabla J &= Me^u \nabla u - \nabla v e^v, \\ \Delta J &= Me^u \Delta u + Me^u |\nabla u|^2 - e^v \Delta v - e^v |\nabla v|^2. \end{aligned} \tag{2.50}$$

Thus

$$\begin{aligned} J_t - \Delta J &= Me^u[u_t - \Delta u] - e^v[v_t - \Delta v] + e^v|\nabla v|^2 - Me^u|\nabla u|^2 \\ &= (MA - B)e^{u+v} + e^v|\nabla v|^2 - Me^u|\nabla u|^2. \end{aligned} \tag{2.51}$$

From (2.50), we obtain

$$\nabla u = \frac{1}{Me^u}[\nabla v e^v + \nabla J].$$

This leads to

$$|\nabla u|^2 = \frac{1}{M^2 e^{2u}}[e^{2v}|\nabla v|^2 + 2e^v \nabla v \cdot \nabla J + |\nabla J|^2].$$

Therefore, (2.51) becomes

$$\begin{aligned} J_t - \Delta J &= (MA - B)e^{u+v} + [e^v - \frac{e^{2v}}{Me^u}]|\nabla v|^2 \\ &\quad -[\frac{2e^v}{Me^u}\nabla v + \frac{1}{Me^u}\nabla J] \cdot \nabla J. \end{aligned}$$

Since

$$e^v - \frac{e^{2v}}{Me^u} = e^v \frac{J}{Me^u},$$

the last equation can be rewritten as follows:

$$J_t - \Delta J - b \cdot \nabla J - cJ = (MA - B)e^{u+v} \geq 0, \quad (x,t) \in B_R \times (0,T)$$

provided $M \geq B/A$, where

$$b = -[\frac{2e^v}{Me^u}\nabla v + \frac{1}{Me^u}\nabla J], \; c = \frac{e^v}{Me^u}|\nabla v|^2.$$

It is clear that, c is bounded in $B_R \times (0,t]$, for $t < T$.

Moreover, $J(R,\cdot) = M - 1 > 0$ and $J(\cdot,0) = Me^{u_0} - e^{v_0} \geq 0$, provided M is large enough.

From above and Proposition B.1.3, we deduce that

$$J \geq 0, \quad (x,t) \in B_R \times (0,T).$$

Similarly, we can show that the function $H = Me^v - e^u$ is nonnegative in $B_R \times (0,T)$. □

The next theorem shows that Theorem 2.2.2 can be applied to the ignition system (problem (2.24), (2.25) with (2.48)) with appropriate choice for F, G.

Theorem 2.2.6. *Let (u,v) be a blow-up solution to problem (2.24), (2.25), where f, g are given as in (2.48), (u_0, v_0) satisfies (2.31). Then there exist only a single blow-up point. Moreover, the pointwise estimates take the following forms:*

$$u \leq \log C - \frac{2}{\alpha}\log(r), \quad v \leq \log C - \frac{2}{\alpha}\log(r), \quad (r,t) \in (0,R] \times (0,T).$$

Proof. Let

$$F(v) = e^{\alpha v},\ G(u) = e^{\alpha u}, \quad \alpha \in (0,1). \tag{2.52}$$

In order to make use of Theorem 2.2.2, we need to show that F, G satisfy the condition (2.32).

A direct calculation shows

$$f^{'}F - fG^{'} = Ae^{v}[e^{\alpha v} - \alpha e^{\alpha u}]. \tag{2.53}$$

By (2.49)

$$e^{v} \geq \frac{1}{M}e^{u}, \quad (x,t) \in B_R \times (0,T).$$

Thus (2.53) becomes

$$\begin{aligned} f^{'}F - fG^{'} &\geq \frac{A}{M}e^{u}[\frac{1}{M^{\alpha}}e^{\alpha u} - \alpha e^{\alpha u}] \\ &\geq \frac{A}{M}[\frac{1}{M^{\alpha}} - \alpha]e^{2\alpha u} \geq 2\varepsilon\alpha e^{2\alpha u} = 2\varepsilon GG^{'} \end{aligned}$$

provided $\alpha < \frac{1}{M}$, ε is small enough such that

$$\varepsilon \leq \frac{A}{2M}[\frac{1}{\alpha M^{\alpha}} - 1].$$

In the same way we can show that

$$g^{'}G - gF^{'} \geq \frac{B}{M}[\frac{1}{M^{\alpha}} - \alpha]e^{2\alpha v} \geq 2\varepsilon\alpha e^{2\alpha v} = 2\varepsilon FF^{'}, \tag{2.54}$$

provided

$$\varepsilon \leq \frac{B}{2M}[\frac{1}{\alpha M^{\alpha}} - 1].$$

Thus the condition (2.32) is met. Therefore, according to Theorem 2.2.2, we conclude that $x = 0$ is the only blow-up point.

The next aim is to derive the pointwise estimates. As in Theorem 2.2.2, define the functions G^{*}, F^{*} as follows

$$G^{*}(s) = \int_{s}^{\infty} \frac{du}{G(u)}, \quad F^{*}(s) = \int_{s}^{\infty} \frac{dv}{F(v)}, \quad s \geq 0.$$

From (2.52), we get

$$G^{*}(s) = F^{*}(s) = \frac{1}{\alpha e^{\alpha s}}, \quad s > 0.$$

Therefore, (2.36), (2.37) become

$$\frac{1}{\alpha e^{\alpha u}} \geq \varepsilon \frac{r^2}{2}, \quad \frac{1}{\alpha e^{\alpha v}} \geq \varepsilon \frac{r^2}{2}.$$

Thus

$$e^{\alpha u} \leq \frac{2}{\alpha \varepsilon r^2}, \quad e^{\alpha v} \leq \frac{2}{\alpha \varepsilon r^2}, \quad (r,t) \in (0,R] \times (0,T)$$

or

$$u \leq \log C - \frac{2}{\alpha} \log(r), \quad v \leq \log C - \frac{2}{\alpha} \log(r), \quad (r,t) \in (0,R] \times (0,T).$$

□

The next theorem considers the blow-up rate estimates for problem (2.24), (2.25), where f, g take the forms of (2.48).

Theorem 2.2.7. *Let (u, v) be a blow-up solution to ignition system (problem (2.24), (2.25), where f, g take the forms of (2.48)). Moreover, assume that (u_0, v_0) satisfies (2.31). Then the upper (lower) blow-up rate estimates take the following forms*

$$\log C_1 - \log(T-t) \leq u(0,t) \leq \log C_2 - \log(T-t), \quad t \in (0,T),$$

$$\log C_3 - \log(T-t) \leq v(0,t) \leq \log C_4 - \log(T-t), \quad t \in (0,T),$$

where C_1, C_2, C_3 and C_4 are positive constants.

Proof. Define the functions G_1, F_1 as in Theorem 2.2.4 as follows:

$$G_1(s) = \int_s^\infty \frac{du}{Be^u}, \quad F_1(s) = \int_s^\infty \frac{dv}{Ae^v} ds.$$

It is obviously that

$$G_1(s) = \frac{1}{Be^s}, \quad F_1(s) = \frac{1}{Ae^s}, \quad s \geq 0.$$

Moreover,

$$G_1^{-1}(s) = -\log(Bs), \quad F_1^{-1}(s) = -\log(As), \quad s > 0.$$

Therefore, from (2.39) it follows that

$$-\log(Bc_1(T-t)) \leq u(0,t) \leq -\log(Bc_2(T-t)), \quad t \in (0,T).$$

Thus, there exist $C_1, C_1 > 0$ such that

$$\log C_1 - \log(T-t) \leq u(0,t) \leq \log C_2 - \log(T-t), \quad t \in (0,T).$$

In the same way, depending on (2.40), there exist $C_3, C_4 > 0$ such that

$$\log C_3 - \log(T-t) \leq v(0,t) \leq \log C_4 - \log(T-t), \quad t \in (0,T).$$

$\square$

Chapter 3

Neumann Problems for Heat Equations

The problems of heat equation with nonlinear boundary conditions have been formulated from many physical models arising in various fields of the applied sciences, for example, in the chemical reactions, heat transfer and population dynamics. Also the problem of two heat equations coupling the nonlinear Neumann boundary values, describes some cross boundary flux. See [50] and the references therein.

The main objective here is to establish estimates on the blow-up rates and find the blow-up set for such type of problems defined in a ball. In the first section of this chapter we consider the scalar problem, while the problem for the system is studied in section two. The nonlinear boundary conditions, which we consider in this chapter are the exponential of power type functions.

3.1 The Heat Equation with a Nonlinear Boundary Condition

In this section we consider the problem of the heat equation with a nonlinear boundary condition, namely

$$\left.\begin{array}{lll} u_t = \Delta u, & & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = f(u), & & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & & x \in B_R, \end{array}\right\} \tag{3.1}$$

where $f \in C^1(R)$ and is positive increasing function in $(0,\infty)$, u_0 is smooth, nonnegative, radial function and satisfying the compatibility condition

$$\frac{\partial u_0}{\partial \eta} = f(u_0), \quad x \in \partial B_R. \tag{3.2}$$

Moreover, it satisfies

$$\Delta u_0 \geq 0, \quad u_{0r}(|x|) \geq 0, \quad x \in \overline{B}_R. \tag{3.3}$$

The problem of heat equation with nonlinear Neumann boundary conditions defined in a ball, has been introduced in [9, 16, 18, 34]. For instance, in [34] it has been shown that if f is nondecreasing and $1/f$ is integrable at infinity for $u > 0$, then the blow-up occurs in finite time for any positive initial data u_0 (not necessarily radial), moreover, if f is $C^2(0,\infty)$, increasing and convex in $(0,\infty)$, then blow-up occurs only on the boundary.

For the special case of problem (3.1), where $f(u) = u|u|^{p-1}$, it has been proved in [16] that for any u_0, the finite time blow-up occurs, where $p > 1$, and it occurs only on the boundary. Moreover, it has been shown in [18, 36] that the upper (lower) blow-up rate estimate take the following form

$$C_1(T-t)^{\frac{-1}{2(p-1)}} \leq \max_{x\in\overline{B}_R} u(x,t) \leq C_2(T-t)^{\frac{-1}{2(p-1)}}, \quad t \in (0,T).$$

In [9], it has been considered the second special case of problem (3.1), where $f(u) = e^u$, in one dimensional space defined in the domain $(0,1) \times (0,T)$, it has been proved that every positive solution blows up in finite time and the blow-up occurs only on the boundary ($x = 1$). Also, the upper (lower) blow-up rate estimates take the following forms

$$C_1(T-t)^{-1/2} \leq e^{u(1,t)} \leq C_2(T-t)^{-1/2}, \quad 0 < t < T.$$

This section is concerned with the blow-up solutions of problem (3.1), where $f(u) = e^{u|u|^{p-1}}, p > 1$, namely

$$\left.\begin{array}{ll} u_t = \Delta u, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = e^{u|u|^{p-1}}, & (x,t) \in S_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R. \end{array}\right\} \tag{3.4}$$

We prove that the upper blow-up rate estimate takes the following form

$$\max_{\overline{B}_R} u(x,t) \leq \log C - \frac{1}{2p}\log(T-t), \quad 0 < t < T.$$

3.1.1 Preliminaries

The local existence of the unique classical solutions to problem (3.1) is well known from the next theorem, which has been proved in [34].

Theorem 3.1.1. *Let $f \in C^1$ and let $u_0 \in C^{2+\alpha}(\overline{B}_R)$, where $\alpha \in (0,1)$, satisfies the condition (3.2).Then there exists $T^* > 0$ such that problem (3.1) admits a solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{B}_R \times [0,T^*])$. Moreover, there is a unique maximal solution. If f is bounded in $C^1(R)$, there exists a solution for any $T > 0$.*

The following lemma shows some properties of the solution of problem (3.1). We denote for simplicity $u(r,t) = u(x,t)$.

Lemma 3.1.2. *Let u be a classical unique solution to problem (3.1).Then*

(i) *$u > 0$, radial on $\overline{B}_R \times (0,T)$. Moreover, $u_r \geq 0$, in $[0,R] \times [0,T)$.*

(ii) *$u_t > 0$ in $\overline{B}_R \times (0,T)$. Moreover, if $\Delta u_0 \geq a > 0$, in $\overline{B}_R$, then $u_t \geq a$, in $\overline{B}_R \times [0,T)$.*

(iii) *For $f(u) = e^{u^p}$, u blows up in a finite time and the blow-up occurs only on the boundary.*

Proof of (i):

We can show that $u > 0$ in $\overline{B}_R \times (0,T)$, by just applying Proposition B.1.5 to problem (3.1), or alternatively, by the following proof, which has been given in [34].

Let $\delta > 0$, and

$$||u||_{L^\infty(\overline{B}_R \times [0,T-\delta))} \leq M.$$

Let $f^* \in C^1$ be a bounded function such that

$$f^*(u) = f(u), \quad 0 \leq u \leq M+1$$

and $f^* > 0$ in $R^1 \setminus \{0\}$. Let v be the global solution to problem (3.1) with f replaced by f^*.

We claim that $v \geq 0$. In fact suppose v attains a negative minimum in $\overline{B}_R \times [0, T-\delta)$. As it is a solution of the heat equation it should be attained at the parabolic boundary point. As $u_0 \geq 0$ there should exist $x_0 \in \partial B_R, t_0 > 0$ such that

$$v(x_0,t_0) = \min_{\overline{B}_R \times [0,T-\delta)} v.$$

Thus

$$0 \geq \frac{\partial v}{\partial \eta}(x_0,t_0) = f(v(x_0,t_0)) > 0.$$

This a contradiction and thus $v \geq 0$.

Now, we claim that

$$v = u, \quad \text{in} \quad \overline{B}_R \times [0, T-\delta),$$

if not, let $0 \leq \tau_0 < T - \delta$ be such that

$$v = u \quad \text{in} \quad \overline{B}_R \times [0, \tau_0).$$

By continuity,

$$v \leq M+1 \quad \text{in} \quad B_R \times (0, \tau_0 + \varepsilon).$$

Thus

$$f^* = f \quad \text{in} \quad \overline{B}_R \times [0, \tau_0 + \varepsilon).$$

By uniqueness $v = u$ in $\overline{B}_R \times [0, \tau_0 + \varepsilon)$. Therefore, $u = v$ in $\overline{B}_R \times [0, T - \delta)$. Thus

$$u \geq 0, \quad \text{in} \quad \overline{B}_R \times [0, T - \delta).$$

We claim that

$$u > 0, \quad \text{in} \quad \overline{B}_R \times (0, T - \delta).$$

If $u = 0$ at $(x_0, t_0) \in \partial B_R \times (0, T - \delta)$ it would be a minimum of u on $\overline{B}_R \times (0, T - \delta)$. Thus

$$\frac{\partial v}{\partial \eta}(x_0, t_0) < 0.$$

On the other hand

$$\frac{\partial v}{\partial \eta}(x_0, t_0) = f(u(x_0, t_0)) > 0.$$

Thus $u > 0$ on $\partial B_R \times (0, T - \delta)$. As u is a solution of the heat equation it cannot attain interior minimum without being constant. Therefore,

$$u > 0, \quad \text{in} \quad \overline{B}_R \times (0, T - \delta).$$

As δ is arbitrary, thus

$$u > 0, \quad \text{in} \quad \overline{B}_R \times (0, T).$$

Now, we prove that u is radial. Let $x \in \overline{B}_R$, and $x^{'}$ be a rotation of x given by $x^{'} = Ax$, where $A = (a_{ij})$ is an orthogonal matrix, that is

$$AA^T = A^T A = I.$$

It is well known that the solution of heat equation is invariant under rotations (see [34]), therefore, each of $u(x, t), u(x^{'}, t)$ is a solution of problem (3.1) at the point x with the initial conditions $u_0(x), u_0(x^{'})$ respectively. Since $|x| = |x^{'}|$ (from the properties of orthogonal matrix) and u_0 is radial, we obtain

$$u_0(x^{'}) = u_0(x).$$

But, for any u_0, the problem has a unique solution, therefore

$$u(x, t) = u(x^{'}, t),$$

which means that u is radial.

The next aim is to show that u_r is nonnegative.

Set $z(x,t) = u_r(r,t)$. Clearly, z is a solution of

$$\left.\begin{array}{ll} z_t - \Delta z + \frac{n-1}{r^2} z = 0, & (x,t) \in B_R \times (0,T), \\ z(x,t) = f(u), & (x,t) \in \partial B_R \times [0,T), \\ z(x,0) = u_{0r}(r), & x \in B_R. \end{array}\right\}$$

Suppose that $z(x,t) < 0$ for some points in $\overline{B}_R \times [0,T)$. Let z attends its negative minimum in $\overline{B}_R \times [0,T)$ at the point (x_0,t_0).

Since $z(x,0) \geq 0$, $t_0 > 0$. If $x_0 \in B_R$, then

$$\Delta z(x_0,t_0) \geq 0, \quad z_t(x_0,t_0) \leq 0.$$

Thus

$$0 \geq z_t(x_0,t_0) - \Delta z(x_0,t_0) = -\frac{(n-1)}{r^2} z(x_0,t_0) > 0.$$

If $x_0 \in \partial B_R$, then

$$0 > z(x_0,t_0) = f(u(x_0,t_0)) > 0.$$

Clearly, in each of both cases above it follows a contradiction.

Therefore,

$$z(x,t) \geq 0, \quad \text{in} \quad \overline{B}_R \times [0,T).$$

Proof of (ii):

Set $w = u_t$. Clearly, w is the solution of the following problem

$$\left.\begin{array}{ll} w_t = \Delta w, & (x,t) \in B_R \times (0,T) \\ \frac{\partial w}{\partial \eta} - f'(u) w = 0, & (x,t) \in \partial B_R \times (0,T), \\ w(x,0) = \Delta u_0 \geq 0, & x \in \overline{B}_R. \end{array}\right\}$$

It is well known that this problem has a unique nonnegative solution (see [50], Theorem 2.1). Moreover, by Proposition B.1.5, it follows that

$$w > 0, \quad \text{in} \quad \overline{B}_R \times (0,T).$$

Next, we consider the case, when $\Delta u_0 \geq a > 0 \quad \text{in} \quad \overline{B}_R$. We follow the proof of (Proposition 1.3, [34]).

Since u is a classical solution, $\Delta u \in C(\overline{B}_R \times [0, T))$.

Let $0 < b < a$, there exist $\varepsilon_0 > 0$ such that $\varepsilon < \varepsilon_0$ implies

$$u_t(x, \varepsilon) = \Delta u(x, \varepsilon) > b, \quad x \in \overline{B}_R,$$

which leads to

$$u(x, \varepsilon) > u(x, 0) + b\varepsilon.$$

Let

$$u_\varepsilon = u(x, t + \varepsilon) - b\varepsilon$$

It is clear that u_ε is a solution of the heat equation in $B_R \times (0, T - \varepsilon)$,

$$u_\varepsilon(x, 0) > u(x, 0) \quad \text{in} \quad \overline{B}_R$$

and

$$\frac{\partial u_\varepsilon}{\partial \eta}(x, t) = \frac{\partial u}{\partial \eta}(x, t + \varepsilon) = f(u(x, t + \varepsilon)) = f(u_\varepsilon(x, t) + b\varepsilon) \geq f(u_\varepsilon(x, t)).$$

From Proposition B.1.6, it follows that

$$u_\varepsilon(x, t) > u(x, t), \quad \text{in} \quad \overline{B}_R \times (0, T - \varepsilon).$$

This implies that

$$u_t \geq b \quad \text{in} \quad \overline{B}_R \times [0, T).$$

As $b < a$ is arbitrary

$$u_t \geq a \quad \text{in} \quad \overline{B}_R \times [0, T).$$

Proof of (iii):

Clearly, $f(u) = e^{u^p}$ is $C^2(0, \infty)$, increasing, positive function in $(0, \infty)$ and $1/f$ is integrable at infinity for $u > 0$, moreover f is convex $(f''(u) > 0, \forall u > 0)$. Therefore, according to the result of [34], it follows that (iii) holds.

3.1.2 Blow-up Rate Estimates

The following theorem considers the upper blow-up rate estimate for problem (3.4).

Theorem 3.1.3. *Let u be a blow-up solution to (3.4), where $\Delta u_0 \geq a > 0$ in $\overline{B}_R$, T is the blow-up time. Then there exists a positive constant C such that*

$$\max_{\overline{B}_R} u(x,t) \leq \log C - \frac{1}{2p}\log(T-t), \quad 0 < t < T. \tag{3.5}$$

Proof. We follow the idea of [9], consider the function

$$F(x,t) = u_t(r,t) - \varepsilon u_r^2(r,t), \quad (x,t) \in B_R \times (0,T).$$

Clearly, $F \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T))$.

By a straightforward calculation

$$F_t - \Delta F = 2\varepsilon(\frac{n-1}{r^2}u_r^2 + u_{rr}^2) \geq 0.$$

Since $\Delta u_0 \geq a > 0$, and $u_{0r} \in C(\overline{B}_R)$,

$$F(x,0) = \Delta u_0(r) - \varepsilon u_{0r}^2(r) \geq 0, \quad x \in B_R.$$

provided ε is small enough.

Moreover,

$$\begin{aligned}
\frac{\partial F}{\partial \eta}|_{x\in S_R} &= u_{rt}(R,t) - 2\varepsilon u_r(R,t)u_{rr}(R,t) \\
&= (e^{u^p(R,t)})_t - 2\varepsilon e^{u^p(R,t)}(u_t(R,t) - \frac{n-1}{r}u_r(R,t)) \\
&\geq (p[u(R,t)]^{p-1} - 2\varepsilon)e^{u^p(R,t)}u_t(R,t).
\end{aligned}$$

Since

$$u_t > 0, \quad \text{on} \quad \overline{B}_R \times (0,T).$$

Thus

$$\frac{\partial F}{\partial \eta}|_{x\in S_R} \geq 0, \quad t \in (0,T)$$

provided

$$\varepsilon \leq \frac{p[u_0(R)]^{p-1}}{2}.$$

From the comparison principle B.1.1, it follows that

$$F(x,t) \geq 0, \quad \text{in} \quad \overline{B}_R \times (0,T),$$

in particular $F(x,t) \geq 0$, for $|x| = R$, that is

$$u_t(R,t) \geq \varepsilon u_r^2(R,t) = \varepsilon e^{2u^p(R,t)}, \quad t \in (0,T).$$

Since u is increasing in time and blows at T, there exist $\tau < T$ such that

$$u(R,t) \geq p^{\frac{1}{(p-1)}} \quad \text{for} \quad \tau \leq t < T,$$

which leads to

$$u_t(R,t) \geq \varepsilon e^{2pu(R,t)}, \quad t \in [\tau, T).$$

By integration the above inequality from t to T, it follows that

$$\int_t^T u_t e^{-2pu(R,t)} \geq \varepsilon(T-t).$$

So

$$-\frac{1}{2p} e^{-2pu(R,t)}|_t^T \geq \varepsilon(T-t). \tag{3.6}$$

Since

$$u(R,t) \to \infty, \quad e^{-pu(R,t)} \to 0 \quad \text{as} \quad t \to T,$$

the inequality (3.6) becomes

$$\frac{1}{e^{pu(R,t)}} \geq (2p\varepsilon(T-t))^{1/2},$$

which means

$$(T-t)^{1/2} e^{pu(R,t)} \leq \frac{1}{\sqrt{2p\varepsilon}},$$

Therefore, there exist a positive constant C such that

$$\max_{\overline{B}_R} u(x,t) \leq \log C - \frac{1}{2p} \log(T-t), \quad 0 < t < T.$$

□

Remark 3.1.4. Depending on the size of the initial data, at a large time enough, the solution of problem (3.4) is larger than or equal to the solution of problem (3.1), where $f(u) = e^{pu}$, and this can be shown by the comparison principle B.1.2. However, from Theorem 3.1.3, we observe that the two problems have the same upper blow-up rate estimate (3.5).

3.2 Systems of Heat Equations with Nonlinear Coupled Boundary Conditions

In this section we consider the system of two heat equations with coupled nonlinear Neumann boundary conditions, namely

$$\left.\begin{array}{lll} u_t = \Delta u, & v_t = \Delta v, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = f(v), & \frac{\partial v}{\partial \eta} = g(u), & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in B_R, \end{array}\right\} \tag{3.7}$$

where $f, g \in C^1(R) \cap C^2(R \setminus \{0\})$, increasing functions such that $f, g > 0$ in $(0, \infty)$, u_0, v_0 are smooth, radial, nonzero, nonnegative functions and satisfying the compatibility condition

$$\frac{\partial u_0}{\partial \eta} = f(v_0), \quad \frac{\partial v_0}{\partial \eta} = g(u_0), \quad x \in \partial B_R, \tag{3.8}$$

Moreover, they satisfy

$$\Delta u_0, \Delta u_0 \geq 0, \quad u_{0r}(|x|), v_{0r}(|x|) \geq 0, \quad x \in \overline{B}_R. \tag{3.9}$$

The problem of the system of two heat equations with nonlinear Neumann boundary conditions defined in a ball was introduced in [10, 11, 44, 46]. For instance, in [10] it was studied the blow-up solutions to a special case of the system (3.7), where

$$f(v) = v|v|^{p-1}, \quad g(u) = u|u|^{q-1}, \quad p, q > 1. \tag{3.10}$$

It was proved that for any nonzero, nonnegative initial data (u_0, v_0), the finite time blow-up can only occur on the boundary, moreover, it was shown in [44] that the blow-up rate estimates take the following form

$$c \leq \max_{x\in\overline{\Omega}} u(x,t)(T-t)^{\frac{p+1}{2(pq-1)}} \leq C, \quad t \in (0,T),$$

$$c \leq \max_{x\in\overline{\Omega}} v(x,t)(T-t)^{\frac{q+1}{2(pq-1)}} \leq C, \quad t \in (0,T).$$

In [11, 46], it was considered the solutions of the system (3.7) with exponential Neumann boundary conditions model, namely

$$f(v) = e^{pv}, \quad g(u) = e^{qu}, \quad p,q > 0. \tag{3.11}$$

Also it was showed that for any nonzero, nonnegative initial data, (u_0, v_0), the solution blows up in finite time and the blow-up occurs only on the boundary, moreover, the blow-up rate estimates take the following forms

$$C_1 \leq e^{qu(R,t)}(T-t)^{1/2} \leq C_2, \quad C_3 \leq e^{pv(R,t)}(T-t)^{1/2} \leq C_4.$$

In this section, we study the blow-up solutions of problem (3.7), where f, g are the exponential of power type functions, namely

$$\left.\begin{array}{lll} u_t = \Delta u, & v_t = \Delta v, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = e^{v|v|^{p-1}}, & \frac{\partial v}{\partial \eta} = e^{u|u|^{q-1}}, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in B_R, \end{array}\right\} \tag{3.12}$$

for $p, q > 1$.

We prove that the upper blow-up rate estimates for problem (3.12) take the following form

$$\max_{\overline{B}_R} u(x,t) \leq \log C_1 - \frac{\alpha}{2}\log(T-t), \quad 0 < t < T,$$

$$\max_{\overline{B}_R} v(x,t) \leq \log C_2 - \frac{\beta}{2}\log(T-t), \quad 0 < t < T,$$

where $\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}$.

Moreover, we show that the blow-up occurs only on the boundary.

3.2.1 Preliminaries

The local existence and uniqueness of classical solutions to problem (3.7) are well known by the standard parabolic theory [40]. In case of the compatibility conditions (3.8) being not satisfied, the local existence and uniqueness also hold by the following theorem, which has been proved in [58].

Theorem 3.2.1. *Let $u_0, v_0 \in C^2(\overline{B}_R)$ and nonnegative in B_R, and positive on $\partial\Omega$. Let f and g be strictly positive in $R \setminus \{0\}$, nondecreasing in $(0, \infty)$ with f', g' locally Lipschitz continuous in $R \setminus \{0\}$. There exists a unique maximal classical solution to problem (3.7), (u, v). Let $T^* = T_{\max}(u_0, v_0)$ be the time of existence of the maximal solution. Then*

$$\lim_{t \to T^*} ||u(\cdot, t)||_{L^\infty(\overline{B}_R)} = \infty,$$

$$\lim_{t \to T^*} ||v(\cdot, t)||_{L^\infty(\overline{B}_R)} = \infty.$$

Remark 3.2.2. Theorem 3.2.1 shows that if T^* is finite, then (u, v) blows up simultaneously.

In the following lemma we study some properties of the classical solutions of problem (3.12). We denote for simplicity $u(r, t) = u(x, t),\ v(r, t) = v(x, t)$.

Lemma 3.2.3. *Let (u, v) be a classical unique solution of (3.12). Then*

(i) *(u, v) blows up in finite time and the blow-up set contains ∂B_R.*

(ii) *u, v are positive, radial. Moreover, $u_r, v_r \geq 0$ in $[0, R] \times (0, T)$.*

(iii) *$u_t, v_t > 0$ in $\overline{B}_R \times (0, T)$. Moreover, if $\Delta u_0 \geq a > 0, \Delta v_0 \geq b > 0$ in $\overline{B}_R$, then $u_t \geq a, v_t \geq b$, in $\overline{B}_R \times [0, T)$.*

Proof. The proof of (i) follows from the comparison principle (Proposition B.2.2) and the known blow-up results of problem (3.7) with (3.10).

The proofs of (ii), (iii) are similar to the proof of Lemma 3.1.2, so they are omitted here. $\square$

Remark 3.2.4. When $u_0(x) \equiv v_0(x)$, $p = q$, the problem (3.12) can be reduced to a scalar problem discussed in the first section of this chapter.

3.2.2 Rate Estimates

In order to study the upper blow-up rate estimates for problem (3.7), we need to recall some results from [22, 44].

Lemma 3.2.5. ([44]) *Let $A(t)$ and $B(t)$ be positive C^1 functions in $[0,T)$ and satisfy*

$$A^{'}(t) \geq c\frac{B^p(t)}{\sqrt{T-t}}, \quad B^{'}(t) \geq c\frac{A^q(t)}{\sqrt{T-t}} \quad \text{for} \quad t \in [0,T),$$
$$A(t) \longrightarrow +\infty \quad \text{or} \quad B(t) \longrightarrow +\infty \quad \text{as} \quad t \longrightarrow T^-,$$

where $p,q > 0, c > 0$ and $pq > 1$. Then there exists $C > 0$ such that

$$A(t) \leq C(T-t)^{-\alpha/2}, \quad B(t) \leq C(T-t)^{-\beta/2}, \quad t \in [0,T),$$

where $\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}$.

Lemma 3.2.6. ([22]) *Let $x \in \overline{B}_R$. If $0 \leq a < n-1$. Then there exist $C > 0$ such that*

$$\int_{S_R} \frac{ds_y}{|x-y|^a} \leq C.$$

Theorem 3.2.7. (Jump relation, [22]) *Let $\Gamma(x,t)$ be the fundamental solution of heat equation, namely*

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{(n/2)}} \exp[-\frac{|x|^2}{4t}]. \tag{3.13}$$

Let φ be a continuous function on $S_R \times [0,T]$. Then for any $x \in B_R, x^0 \in S_R, 0 < t_1 < t_2 \leq T$, for some $T > 0$, the function

$$U(x,t) = \int_{t_1}^{t_2} \int_{S_R} \Gamma(x-y, t-z)\varphi(y,z)ds_y d\tau$$

satisfies the jump relation

$$\frac{\partial}{\partial \eta}U(x,t) \rightarrow -\frac{1}{2}\varphi(x^0,t) + \frac{\partial}{\partial \eta}U(x^0,t), \quad \text{as} \quad x \rightarrow x^0.$$

Theorem 3.2.8. *Let (u,v) be a solution of (3.12) which blows up in finite time T. Then there exist positive constants C_1, C_2 such that*

$$\max_{\overline{B}_R} u(x,t) \leq \log C_1 - \frac{\alpha}{2}\log(T-t), \quad 0 < t < T,$$
$$\max_{\overline{B}_R} v(x,t) \leq \log C_2 - \frac{\beta}{2}\log(T-t), \quad 0 < t < T.$$

Proof. We follow the idea of [44], define the functions M and M_b as follows

$$M(t) = \max_{\overline{B}_R} u(x,t), \quad \text{and} \quad M_b(t) = \max_{S_R} u(x,t).$$

Similarly,

$$N(t) = \max_{\overline{B}_R} v(x,t), \quad \text{and} \quad N_b(t) = \max_{S_R} v(x,t).$$

Due to Lemma 3.2.3, both of M, M_b are monotone increasing functions, and since u is a solution of heat equation, it cannot attain interior maximum without being constant, therefore,

$$M(t) = M_b(t). \quad \text{Similarly} \quad N(t) = N_b(t).$$

Moreover, since u, v blow up simultaneously, we have

$$M(t) \longrightarrow +\infty, \quad N(t) \longrightarrow +\infty \quad \text{as} \quad t \longrightarrow T^-. \tag{3.14}$$

As in [36, 44], for $0 < z_1 < t < T$ and $x \in B_R$, depending on the second Green's identity with assuming the Green function:

$$G(x, y; z_1, t) = \Gamma(x - y, t - z_1),$$

where Γ is defined in (3.13), the integral equation to problem (3.12) with respect to u can be written as follows

$$\begin{aligned} u(x,t) &= \int_{B_R} \Gamma(x-y, t-z_1) u(y, z_1) dy + \int_{z_1}^{t} \int_{S_R} e^{v^p(y,\tau)} \Gamma(x-y, t-\tau) ds_y d_\tau \\ &\quad - \int_{z_1}^{t} \int_{S_R} u(y,\tau) \frac{\partial \Gamma}{\partial \eta_y}(x-y, t-\tau) ds_y d\tau, \end{aligned}$$

As in [36], letting $x \to S_R$ and using the jump relation (Theorem 3.2.7) for the third term on the right hand side of the last equation, it follows that

$$\begin{aligned} \frac{1}{2} u(x,t) &= \int_{B_R} \Gamma(x-y, t-z_1) u(y, z_1) dy + \int_{z_1}^{t} \int_{S_R} e^{v^p(y,\tau)} \Gamma(x-y, t-\tau) ds_y d_\tau \\ &\quad - \int_{z_1}^{t} \int_{S_R} u(y,\tau) \frac{\partial \Gamma}{\partial \eta_y}(x-y, t-\tau) ds_y d\tau, \end{aligned}$$

for $x \in S_R, 0 < z_1 < t < T$.

Depending on Lemma 3.2.3 we notice that u, v are positive and radial.Thus

$$\int_{B_R} \Gamma(x-y,t-z_1)u(y,z_1)dy > 0,$$
$$\int_{z_1}^{t}\int_{S_R} e^{v^p(y,\tau)}\Gamma(x-y,t-\tau)ds_y d_\tau = \int_{z_1}^{t} e^{v^p(R,\tau)}[\int_{S_R}\Gamma(x-y,t-\tau)ds_y]d\tau.$$

This leads to

$$\begin{aligned}\frac{1}{2}M(t) \quad &\geq \quad \int_{z_1}^{t} e^{N^p(\tau)}[\int_{S_R}\Gamma(x-y,t-\tau)ds_y]d\tau \\ &\quad - \int_{z_1}^{t} M(\tau)[\int_{S_R}|\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)|ds_y]d\tau, \quad x\in S_R, 0<z_1<t<T.\end{aligned}$$

It is known that (see Ch.5, Lemma 1,[22]) for some $\sigma \in (0,1)$ and for any $1-\frac{\sigma}{2} < \mu < 1$,there exist $C_0 > 0$, such that Γ satisfies

$$|\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)| \leq \frac{C_0}{(t-\tau)^\mu}\cdot\frac{1}{|x-y|^{(n+1-2\mu-\sigma)}}, \quad x,y\in S_R.$$

From Lemma 3.2.6, there exist $C^* > 0$ such that

$$\int_{S_R}\frac{ds_y}{|x-y|^{(n+1-2\mu-\sigma)}} < C^*.$$

Moreover, for $0 < t_1 < t_2$ and t_1 is close to t_2, there exists $c > 0$, such that

$$\int_{S_R}\Gamma(x-y,t_2-t_1)ds_y \geq \frac{c}{\sqrt{t_2-t_1}},$$

Thus

$$\frac{1}{2}M(t) \geq c\int_{z_1}^{t}\frac{e^{N^p(\tau)}}{\sqrt{t-\tau}}d\tau - C\int_{z_1}^{t}\frac{M(\tau)}{|t-\tau|^\mu}d\tau.$$

Since

$$\begin{aligned}C\int_{z_1}^{t}\frac{M(\tau)}{|t-\tau|^\mu}d\tau \leq CM(t)\int_{z_1}^{t}\frac{d\tau}{|t-\tau|^\mu} \quad &= \quad \frac{C}{1-\mu}M(t)|t-z_1|^{1-\mu} \\ &\leq \quad \frac{C}{1-\mu}M(t)|T-z_1|^{1-\mu},\end{aligned}$$

it follows that the last equation becomes

$$\frac{1}{2}M(t) \geq c\int_{z_1}^{t}\frac{e^{N^p(\tau)}}{\sqrt{T-\tau}}d\tau - C_1^*M(t)|T-z_1|^{1-\mu}.$$

Similarly, for $0 < z_2 < t < T$, we have

$$\frac{1}{2}N(t) \geq c\int_{z_2}^{t} \frac{e^{M^q(\tau)}}{\sqrt{T-\tau}}d\tau - C_2^* N(t)|T-z_2|^{1-\mu}.$$

Taking z_1, z_2 so that

$$C_1^*|T-z_1|^{1-\mu} \leq 1/2, \quad C_2^*|T-z_2|^{1-\mu} \leq 1/2,$$

it follows

$$M(t) \geq c\int_{z_1}^{t} \frac{e^{N^p(\tau)}}{\sqrt{T-\tau}}d\tau, \quad N(t) \geq c\int_{z_2}^{t} \frac{e^{M^q(\tau)}}{\sqrt{T-\tau}}d\tau. \tag{3.15}$$

Since both of M, N increasing functions and from (3.14), we can find T^* in $(0,T)$ such that

$$M(t) \geq q^{\frac{1}{(q-1)}}, \quad N(t) \geq p^{\frac{1}{(p-1)}}, \quad \text{for} \quad T^* \leq t < T,$$

which leads to

$$e^{M^q(t)} \geq e^{qM(t)}, \quad e^{N^p(t)} \geq e^{pN(t)}, \quad T^* \leq t < T.$$

Therefore, if we choose z_1, z_2 in (T^*, T), then (3.15) becomes

$$e^{M(t)} \geq c\int_{z_1}^{t} \frac{e^{pN(\tau)}}{\sqrt{T-\tau}}d\tau \equiv I_1(t), \quad e^{N(t)} \geq c\int_{z_2}^{t} \frac{e^{qM(\tau)}}{\sqrt{T-\tau}}d\tau \equiv I_2(t).$$

Clearly,

$$I_1^{'}(t) = c\frac{e^{pN(t)}}{\sqrt{T-t}} \geq \frac{cI_2^p}{\sqrt{T-t}}, \quad I_2^{'}(t) = c\frac{e^{qM(t)}}{\sqrt{T-t}} \geq \frac{cI_1^q}{\sqrt{T-t}}.$$

By Lemma 3.2.5, it follows that

$$I_1(t) \leq \frac{C}{(T-t)^{\frac{\alpha}{2}}}, \quad I_2(t) \leq \frac{C}{(T-t)^{\frac{\beta}{2}}}, \quad t \in (\max\{z_1, z_2\}, T). \tag{3.16}$$

On the other hand, for $t^* = 2t - T$ (assuming that t is close to T such that $t \in (\max\{z_1, z_2\}, T)$), we have

$$I_1(t) \geq c\int_{t^*}^{t} \frac{e^{pN(\tau)}}{\sqrt{T-\tau}}d\tau \geq ce^{pN(t^*)}\int_{2t-T}^{t} \frac{d\tau}{\sqrt{T-\tau}} = 2c(\sqrt{2}-1)\sqrt{T-t}e^{pN(t^*)}.$$

Combining the last inquality with (3.16) yields

$$e^{N(t^*)} \leq \frac{C}{2c(\sqrt{2}-1)(T-t)^{\frac{p+1}{2p(pq-1)}+\frac{1}{2p}}} = \frac{2^{\frac{q+1}{2(pq-1)}}C}{2c(\sqrt{2}-1)(T-t^*)^{\frac{q+1}{2(pq-1)}}}.$$

Thus, there exists a constant $c_1 > 0$ such that

$$e^{N(t^*)}(T-t^*)^{\frac{q+1}{2(pq-1)}} \leq c_1.$$

In the same way we can show

$$e^{M(t^*)}(T-t^*)^{\frac{p+1}{2(pq-1)}} \leq c_2.$$

This leads to that there exist $C_1, C_2 > 0$ such that

$$\max_{\overline{B}_R} u(x,t) \leq \log C_1 - \frac{\alpha}{2}\log(T-t), \quad 0 < t < T, \tag{3.17}$$

$$\max_{\overline{B}_R} v(x,t) \leq \log C_2 - \frac{\beta}{2}\log(T-t), \quad 0 < t < T. \tag{3.18}$$

□

Remark 3.2.9. It is clear that $q\alpha, p\beta > 1$, which leads to

$$\frac{\alpha}{2} > \frac{1}{2q}, \quad \frac{\beta}{2} > \frac{1}{2p}.$$

Therefore, the blow up rate estimates, which have been shown in Theorem 3.2.8, are greater (more singular) than those known for problem (3.7), where f, g take the forms of (3.11), while they are less (less singular) than those known for problem (3.7), where f, g take the forms of (3.10).

3.2.3 Blow-up Set

In order to show that the blow-up in problem (3.12) occurs only on the boundary, we need to recall the following lemma from [46].

Lemma 3.2.10. *Let $w \in C^{2,1}(\overline{B}_R \times [0,T))$ and satisfies*

$$\left.\begin{array}{lll} w_t = \Delta w, & (x,t) \in B_R \times (0,T), & \\ w(x,t) \leq \frac{C}{(T-t)^m}, & (x,t) \in S_R \times (0,T), & m > 0. \end{array}\right\}$$

Then for any $0 < a < R$,

$$\sup\{w(x,t) : 0 \leq |x| \leq a, \ 0 \leq t < T\} < \infty.$$

Proof. Set

$$h(x) = (R^2 - r^2)^2, \; r = |x|,$$
$$z(x,t) = \frac{C_1}{[h(x) + C_2(T-t)]^m}.$$

Clearly,

$$z \in C^{2,1}(\overline{B}_R \times [0,T)).$$

Also, we can show that:

$$\begin{aligned}
\Delta h - \frac{(m+1)|\nabla h|^2}{h} &= 8r^2 - 4n(R^2 - r^2) - (m+1)16r^2 \\
&\geq -4nR^2 - 16R^2(m+1), \\
z_t - \Delta z &= \frac{C_1 m}{[h(x) + C_2(T-t)]^{m+1}}\left(C_2 + \Delta h - \frac{(m+1)|\nabla h|^2}{h + C_2(T-t)}\right) \\
&\geq \frac{C_1 m}{[h(x) + C_2(T-t)]^{m+1}}(C_2 - 4nR^2 - 16R^2(m+1)).
\end{aligned}$$

Let

$$C_2 = 4nR^2 + 16R^2(m+1) + 1$$

and take C_1 to be large such that

$$z(x,0) \geq w(x,0), \quad x \in B_R.$$

Let $C_1 \geq C(C_2)^m$, which implies that

$$z(x,t) \geq w(x,t) \quad \text{on} \quad S_R \times [0,T).$$

Then from Proposition B.1.2, it follows that

$$z(x,t) \geq w(x,t), \quad (x,t) \in \overline{B}_R \times (0,T)$$

and hence

$$\sup\{w(x,t) : 0 \leq |x| \leq a, 0 \leq t < T\} \leq C_1(R^2 - a^2)^{-2m} < \infty, \quad 0 \leq a < R.$$

□

Theorem 3.2.11. *Let the assumptions of Theorem 3.2.8 be in force. Then (u,v) blows up only on the boundary.*

Proof. Using equations (3.17), (3.18)

$$u(R,t) \leq \frac{c_1}{(T-t)^{\frac{\alpha}{2}}}, \quad v(R,t) \leq \frac{c_2}{(T-t)^{\frac{\beta}{2}}}, \quad t \in (0,T).$$

From Lemma 3.2.10, it follows that

$$\sup\{u(x,t) : (x,t) \in \overline{B}_a \times [0,T)\} \leq C_1(R^2 - a^2)^{-\alpha} < \infty,$$

$$\sup\{v(x,t) : (x,t) \in \overline{B}_a \times [0,T)\} \leq C_1(R^2 - a^2)^{-\beta} < \infty,$$

for $a < R$.

Therefore, for (u,v), the blow-up occurs simultaneously only on the boundary. $\square$

Chapter 4

Neumann Problems for Semilinear Parabolic Equations

The present chapter is motivated by the similarities between the problem (4.1) and the two problems (2.1), (3.1), which we have studied in Chapter 2 and Chapter 3, respectively, for special cases of f. We know the effect of f as the reaction term in (2.1) and as the boundary term in (3.1) on blow-up properties of solutions in a finite time. In this chapter we study how the boundary term and the reaction term affect the blow-up rate estimates and the blow-up sets for the problems of reaction diffusion equations with nonlinear boundary conditions, defined in a ball. We show that the reactions terms induce important effects on the upper blow-up rate estimates which become more singular than those known for the case where the reaction terms are absent.

In section one we consider the problem of heat equation with two exponential terms; one appears in the equation as a reaction term and the another one appears on the boundary. A semilinear reaction-diffusion system coupled in both equations and boundary conditions is considered in section two.

4.1 The Semilinear Heat Equation with a Nonlinear Boundary Condition

In this section, we consider the initial-boundary value problem:

$$\left.\begin{array}{ll} u_t = \Delta u + \lambda f(u), & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = g(u), & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{array}\right\} \tag{4.1}$$

where Ω is a bounded domain, $\lambda \in R$, $f, g \in C^1([0,\infty)) \cap C^2(0,\infty)$ are nonnegative functions, u_0 is nonnegative, radial, smooth function satisfying

$$\frac{\partial u_0}{\partial \eta} = g(u_0), \qquad x \in \partial\Omega, \tag{4.2}$$

$$\Delta u_0 + \lambda f(u_0) \geq 0, \quad u_{0r}(|x|) \geq 0, \quad x \in \overline{\Omega}_R. \tag{4.3}$$

This problem has been studied by many authors (see for example [5, 70] in the case of $\lambda < 0$, and [45] in the case of $\lambda > 0$). The crucial point of these works was the question whether the reaction term in the semilinear equation in (4.1) can prevent (affect) blow-up. For instance, in [5] it has been studied the blow-up solutions of problem (4.1), where $\lambda < 0$ and

$$f(u) = u^p, \quad g(u) = u^q, \quad p, q > 1, \tag{4.4}$$

for $n = 1$ or $\Omega = B_R$. Particularly, it was shown that the exponent $p = 2q - 1$ is critical for blow-up in the following sense:

(i) If $p < 2q-1$ (or $p = 2q-1$ and $-\lambda < q$), then there exist solutions, which blow up in finite time and the blow-up occurs only on the boundary.

(ii) If $p > 2q-1$ (or $p = 2q-1$ and $-\lambda > q$), then all solutions exist globally and are globally bounded.

In [57] J. D. Rossi has proved for the case (i), where $n = 1$, $\Omega = [0,1]$, that there exist positive constants C, c such that the upper (lower) blow-up rate estimates take the following forms

$$c \leq \max_{[0,1]} u(\cdot,t)(T-t)^{\frac{1}{2(q-1)}} \leq C, \quad 0 < t < T.$$

In [45] it has been studied another special case of problem (4.1), where $\lambda = 1$, f, g as in (4.4), $\Omega = [0, 1]$ or it is a bounded domain with C^2 boundary. It was proved that the solutions of this problem exist globally, if and only if $\max\{p, q\} \leq 1$, otherwise, every solution has to blow up in finite time. Moreover, the blow-up occurs only on the boundary. The blow-up rate estimate for this case has been studied in [45, 57]. For $n = 1, \Omega = [0, 1]$, it has been shown that there exist positive constants c, C such that

$$c \leq \max_{[0,1]} u(\cdot, t)(T - t)^{\alpha} \leq C, \quad 0 < t < T,$$

where $\alpha = 1/(p - 1)$ if $p \geq 2q - 1$, and $\alpha = 1/[2(q - 1)]$ if $p < 2q - 1$.

We observe that if $p < 2q - 1$, then the nonlinear term at the boundary determines and gives the blow-up rate, while if $p > 2q - 1$, then the reaction term in the semilinear equation dominates and gives the blow-up rate.

Later, in [70] it was considered a second special case of (4.1), where $\lambda = -a, a > 0$, f, g are of exponential forms, namely

$$\left.\begin{array}{ll} u_t = \Delta u - a e^{pu}, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = e^{qu}, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{array}\right\} \tag{4.5}$$

where $p, q > 0$, u_0 satisfies (4.2), (4.3).

It has been shown that in case where Ω is a bounded domain with smooth boundary, the critical exponent can be given as follows

(i) If $2q < p$, the solutions of problem (4.5) are globally bounded.

(ii) If $2q > p$, the solutions of problem (4.5) blow up in finite time for large initial data.

(iii) If $2q = p$, the solutions may blow up in finite time for large initial data.

Moreover, in case where $\Omega = B_R$, the blow-up occurs only on the boundary and there exist positive constants c, C such that the upper (lower) blow-up rate estimate take the following form

$$\log C_1 - \frac{1}{2q}\log(T - t) \leq \max_{\overline{B}} u(\cdot, t) \leq \log C_2 - \frac{1}{2q}\log(T - t), \quad 0 < t < T.$$

We note that, the blow-up properties (blow-up location and bounds) of problem (4.5) are the same as those of problem (4.5), where $a = 0$, see Chapter 3.

In this section we study the blow-up solutions of problem (4.1), where f, g take the exponential forms as in problem (4.5), $\Omega = B_R$, namely

$$\left.\begin{array}{lll} u_t = \Delta u + \lambda e^{pu}, & & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = e^{qu}, & & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & & x \in B_R, \end{array}\right\} \tag{4.6}$$

where $p, q, \lambda > 0$. We consider the blow-up rate estimates and the blow-up set for this problem.

4.1.1 Preliminaries

Since $f(u) = \lambda e^{pu}$, $g(u) = e^{qu}$ are smooth functions and problem (4.6) is uniformly parabolic, also u_0 satisfies the compatibility condition (4.2), it follows that the existence and uniqueness of local classical solutions to problem (4.6) are known by [1, 50] or by the standard existence theory [40]. On the other hand, since $f \in C^\infty(R)$, by regularity results (see [22]), we obtain that the solutions of this problem are smooth in $B_R \times (0,T)$.

In this section we denote for simplicity $u(x,t) = u(r,t)$.

The following lemma shows some properties of the classical solutions to problem (4.6).

Lemma 4.1.1. *Let u be a classical solution to problem (4.6), where u_0 satisfies the assumptions (4.2), (4.3). Then*

(i) $u > 0$, *radial in* $\overline{B}_R \times (0,T)$.

(ii) $u_r \geq 0$, *in* $[0,R] \times [0,T)$.

(iii) $u_t > 0$ *in* $\overline{B}_R \times (0,T)$.

(iv) *u blows up in finite time and the blow-up set contains* ∂B_R.

Proof. Set $f(u) = \lambda e^{pu},\ g(u) = e^{qu}$. As in the proof of Lemma 2.1.1, the result that u is radial follows from the assumptions (u_0 is radial) and the uniqueness of u, while the proof of u is positive in $\overline{B}_R \times (0,T)$ follows from the positivity of f and g, and that by applying Proposition B.1.1 to problem (4.6).

To prove (ii), set $z(x,t) = u_r(r,t)$. It is clear that z is the solution of the following problem

$$\left.\begin{array}{ll} z_t - \Delta z = [-\frac{n-1}{r^2} + f^{'}(u)]z, & \text{in } B_R \times (0,T), \\ z(x,t) = g(u), & \text{on } \partial B_R \times (0,T), \\ z_0(x) = u_{0r} \geq 0, & \text{in } B_R. \end{array}\right\}$$

By Proposition B.1.3, we obtain that

$$z \geq 0, \quad \text{in } \overline{B}_R \times (0,T).$$

Next, we aim to prove (iii). Set $w = u_t$. By differentiating (4.6) with respect to time, it follows that

$$\left.\begin{array}{ll} w_t - \Delta w - f^{'}(u)w = 0, & \text{in } B_R \times (0,T), \\ \frac{\partial w}{\partial \eta} - g^{'}(u)w = 0, & \text{on } \partial B_R \times (0,T), \\ w_0 = \Delta u_0 + f(u_0) \geq 0, & \text{in } \overline{B}_R. \end{array}\right\} \tag{4.7}$$

It is well known that problem (4.7) has a unique nonnegative solution (see [50]). Moreover, by applying Proposition B.1.5 to problem (4.7) yields that

$$w > 0, \quad \text{in} \quad \overline{B}_R \times (0,T).$$

To prove (iv), consider problem (3.1), where $f(u) = e^{qu}$. Clearly, in this case, u is a supersolution to (3.1) (starting with the same initial condition).

Assume that v is the solution of problem (3.1), thus by the comparison principle B.1.2, we obtain

$$v(\cdot,t) \leq u(\cdot,t), \quad 0 < t < T.$$

It is well known that v has to blow up in finite time and the blow-up occurs only on the boundary (see Chapter 3), so, u blows up in finite time and ∂B_R is a subset of the blow-up set of problem (4.6). $\square$

4.1.2 Blow-up Rate Estimates

Since $u_r \geq 0$, in $[0, R] \times (0, T)$, it follows that

$$\max_{\overline{B}_R} u(\cdot, t) = u(R, t), \quad 0 < t < T.$$

Therefore, it is sufficient to derive the lower (upper) bounds of the blow-up rate for $u(R, t)$.

Theorem 4.1.2. *Let u be a solution to problem (4.6), where u_0 satisfies the assumptions (4.2), (4.3), T is the blow-up time.Then there is a positive constant c such that*

$$\log c - \frac{1}{2\alpha} \log(T - t) \leq u(R, t), \quad t \in (0, T),$$

where $\alpha = \max\{p, q\}$.

Proof. Define

$$M(t) = \max_{\overline{B}_R} u(\cdot, t) = u(R, t), \quad \text{for} \quad t \in [0, T).$$

Clearly, $M(t)$ is increasing in $(0, T)$ (due to $u_t > 0$, for $t \in (0, T),\ x \in \overline{B}_R$). As in [70], for $0 < z < t < T, x \in B_R$, the integral equation of problem (4.6) with respect to u, can be written as follows

$$\begin{aligned} u(x,t) \quad = \quad & \int_{B_R} \Gamma(x-y, t-z) u(y,z) dy + \lambda \int_z^t \int_{B_R} \Gamma(x-y, t-\tau) e^{pu(y,\tau)} dy d\tau \\ & + \int_z^t \int_{S_R} \Gamma(x-y, t-\tau) e^{qu(y,\tau)} ds_y d\tau \\ & - \int_z^t \int_{S_R} u(y,\tau) \frac{\partial \Gamma}{\partial \eta_y}(x-y, t-\tau) ds_y d\tau, \end{aligned} \tag{4.8}$$

where Γ is the fundamental solution of the heat equation, which was defined in (3.13).

Since $u(y, t) \leq u(R, t)$ for $y \in \overline{B}_R$, the last equation becomes

$$\begin{aligned} u(x,t) \quad \leq \quad & u(R,z) \int_{B_R} \Gamma(x-y, t-z) dy + \lambda \int_z^t e^{pu(R,\tau)} \int_{B_R} \Gamma(x-y, t-\tau) dy d\tau. \\ & + \int_z^t e^{qu(R,\tau)} \int_{S_R} \Gamma(x-y, t-\tau) ds_y d\tau \\ & + \int_z^t u(R,\tau) \int_{S_R} \mid \frac{\partial \Gamma}{\partial \eta_y}(x-y, t-\tau) \mid ds_y d\tau. \end{aligned}$$

Since u is a continuous function on $\overline{B}_R$, the last inequality leads to

$$\begin{aligned} M(t) \leq\ & M(z)\int_{B_R}\Gamma(x-y,t-z)dy+\lambda e^{pM(t)}\int_z^t\int_{B_R}\Gamma(x-y,t-\tau)dyd\tau \\ & +e^{qM(t)}\int_z^t\int_{S_R}\Gamma(x-y,t-\tau)ds_y d\tau \\ & +M(t)\int_z^t\int_{S_R}\mid\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)\mid ds_y d\tau. \end{aligned} \tag{4.9}$$

It is known from (Ch. 5, [22]) and (Ch. 2, [50]) that for $0<t_1<t_2,\ x,y\in R^n$, Γ satisfies

$$\int_{B_R}\Gamma(x-y,t_2-t_1)dy\leq 1.$$

Moreover, there exist positive constants k_1,k_2 such that

$$\begin{aligned} &\Gamma(x-y,t_2-t_1)\leq\frac{k_1}{(t_2-t_1)^{\mu_0}}\cdot\frac{1}{|x-y|^{n-2+\mu_0}},\quad 0<\mu_0<1,\\ &\mid\frac{\partial\Gamma}{\partial\eta_y}(x-y,t_2-t_1)\mid\leq\frac{k_2}{(t_2-t_1)^{\mu}}\cdot\frac{1}{|x-y|^{n+1-2\mu-\sigma}}, \end{aligned}$$

for some $\sigma\in(0,1)$, and for any $\mu\in(1-\frac{\sigma}{2},1)$.

If we choose $\mu_0=1/2$, then by Lemma 3.2.6, we deduce that there exist positive constants d_1,d_2 such that

$$\int_{S_R}\frac{ds_y}{|x-y|^{n-2+\mu_0}}\leq d_1,\qquad \int_{S_R}\frac{ds_y}{|x-y|^{n+1-2\mu-\sigma}}\leq d_2.$$

From above, it follows that there exist $C_1,C_2>0$ such that the inequality (4.9) becomes

$$M(t)\leq M(z)+\lambda e^{pM(t)}(t-z)+C_1e^{qM(t)}\sqrt{t-z}+C_2M(t)(t-z)^{1-\mu}.$$

Since $t-z\leq T-z$, it follows that

$$M(t)\leq M(z)+\lambda e^{pM(t)}\sqrt{T-z}+C_1e^{qM(t)}\sqrt{T-z}+C_2M(t)(T-z)^{1-\mu}, \tag{4.10}$$

provided $(T-z)\leq 1$.

Clearly,

$$\frac{M(t)}{e^{\alpha M(t)}}\longrightarrow 0,\quad\text{when}\quad t\to T.$$

Thus
$$\frac{M(t)}{e^{\alpha M(t)}} \leq (T-z)^{\frac{1}{2}-(1-\mu)}, \text{ for } t \text{ close to } T.$$

Therefore, the inequality (4.10) becomes

$$M(t) \leq M(z) + \lambda e^{pM(t)}\sqrt{T-z} + C_1 e^{qM(t)}\sqrt{T-z} + C_2 e^{\alpha M(t)}\sqrt{T-z},$$

thus there is a constant C^* such that

$$M(t) \leq M(z) + C^* e^{\alpha M(t)}\sqrt{T-z}, \quad z < t < T, \ t \text{ close to } T.$$

For any z close to T, we can choose $z < t < T$ such that

$$M(t) - M(z) = C_0 > 0,$$

which implies
$$C_0 \leq C^* e^{\alpha M(z) + \alpha C_0}\sqrt{T-z}.$$

Thus
$$\frac{C_0}{C^* e^{(\alpha C_0)}\sqrt{T-z}} \leq e^{\alpha u(R,z)}.$$

Therefore, there exist a positive constant c such that
$$\log c - \frac{1}{2\alpha}\log(T-t) \leq u(R,t), \quad t \in (0,T).$$

□

The next theorem shows similar results to Theorem 4.1.2 with adding more restricted assumptions on q and u_0. However, the proof relies on the maximum principle rather than the integral equation and it is simpler than the proof of Theorem 4.1.2.

Theorem 4.1.3. *Let u be a solution to problem (4.6), where $q \geq 1$, T is the blow-up time, u_0 satisfies the assumptions (4.2), (4.3), moreover, it satisfies the following condition*

$$u_{0r}(r) - \frac{r}{R}e^{u_0(r)} \geq 0, \quad r \in [0,R]. \tag{4.11}$$

Then there is a positive constant c such that

$$\log c - \frac{1}{2\alpha}\log(T-t) \leq u(R,t), \quad t \in (0,T),$$

where $\alpha = \max\{p,q\}$.

Proof. Define the functions J in $C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T))$, as follows:

$$J(x,t) = u_r(r,t) - \frac{r}{R}e^{u(r,t)}, \quad x \in B_R \times (0,T).$$

A direct calculation shows

$$\begin{aligned} J_t &= u_{rt} - \frac{r}{R}e^u[u_{rr} + \frac{n-1}{r}u_r + \lambda p e^{pu}], \\ J_r &= u_{rr} - \frac{r}{R}e^u u_r - \frac{1}{R}e^u, \\ J_{rr} &= [u_{rt} - \frac{n-1}{r}u_{rr} + \frac{n-1}{r^2}u_r - \lambda p e^{pu}u_r] \\ &\quad -\frac{r}{R}[e^u u_{rr} + e^u u_r^2] - \frac{2}{R}e^u u_r. \end{aligned}$$

From above, it follows that

$$J_t - J_{rr} - \frac{n-1}{r}J_r = -\frac{n-1}{r^2}[u_r - \frac{r}{R}e^u] + \lambda p e^{pu}[u_r - \frac{r}{R}e^u] + \frac{r}{R}e^u u_r^2 + \frac{2}{R}e^u u_r.$$

Thus

$$J_t - \Delta J - bJ = \frac{r}{R}e^u u_r^2 + \frac{2}{R}e^u u_r \geq 0,$$

for $(x,t) \in B_R \times (0,T) \cap \{r > 0\}$, where $b = [\lambda p e^{pu} - \frac{n-1}{r^2}]$.

Clearly, from (4.11), it follows that

$$J(x,0) \geq 0, \quad x \in B_R,$$

and

$$J(0,t) = u_r(0,t) \geq 0, \; J(R,t) = 0 \quad t \in (0,T).$$

Since

$$\sup_{(0,R)\times(0,t]} b < \infty, \quad \text{for} \quad t < T,$$

from above and maximum principle B.1.3, it follows that

$$J \geq 0, \quad (x,t) \in B_R \times (0,T).$$

Moreover,

$$\frac{\partial J}{\partial \eta}|_{\partial B_R} \leq 0.$$

This means

$$(u_{rr} - \frac{r}{R}e^u u_r - \frac{1}{R}e^u)|_{\partial B_R} \leq 0.$$

Thus

$$u_t \leq (\frac{n-1}{r}u_r + \lambda p e^{pu} + e^u u_r + \frac{1}{R}e^u)|_{\partial B_R},$$

which implies that

$$u_t(R,t) \leq \frac{n-1}{R}e^{qu(R,t)} + \lambda p e^{pu(R,t)} + e^{(1+q)u(R,t)} + \frac{2}{R}e^{u(R,t)}, \quad t \in (0,T).$$

Thus, there exist a constant C such that

$$u_t(R,t) \leq Ce^{2\alpha u(R,t)}, \quad t \in (0,T).$$

Integrate this inequality from t to T and since u blows up at R, it follows

$$\frac{c}{(T-t)^{\frac{1}{2}}} \leq e^{\alpha u(R,t)}, \quad t \in (0,T)$$

or

$$\log c - \frac{1}{2\alpha}\log(T-t) \leq u(R,t), \quad t \in (0,T).$$

□

Remark 4.1.4. From Theorems 4.1.2 and 4.1.3 we observe that when $q > p$ the boundary term plays the dominating role and the lower blow-up rate takes the form:

$$\log c - \frac{1}{2q}\log(T-t) \leq u(R,t), \quad t \in (0,T),$$

moreover, this estimate is coincident with the lower blow-up rate estimate known for problem (4.6), where $\lambda = 0$ (see Chapter 3), while when $p > q$ the reaction term is dominated and gives the lower blow-up rate as follows

$$\log c - \frac{1}{2p}\log(T-t) \leq u(R,t), \quad t \in (0,T).$$

We next consider the upper bound.

Theorem 4.1.5. *Let u be a solution of problem (4.6), where T is the blow-up time, u_0 satisfies the assumptions (4.2), (4.3) moreover, assume that*

$$\Delta u_0 + f(u_0) \geq a > 0, \quad in\ \overline{B}_R. \tag{4.12}$$

Then there is a positive constant C such that

$$u(R,t) \leq \log C - \frac{1}{q}\log(T-t), \quad t \in (0,T). \tag{4.13}$$

Proof. Define the function J as follows

$$J(x,t) = u_t(r,t) - \varepsilon u_r(r,t), \quad (x,t) \in B_R \times (0,T).$$

Clearly,

$$J \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T)).$$

Since u_{0r} is bounded in $\overline{B}_R$, and by (4.12), for some $\varepsilon > 0$, we have

$$J(x,0) = \Delta u_0(r) + f(u_0(r)) - \varepsilon u_{0r}(r) \geq 0, \quad x \in \overline{B}_R.$$

A simple computation shows

$$\begin{aligned}
J_t &= u_{rrt} + \frac{n-1}{r} u_{rt} + \lambda p e^{pu} u_t - \varepsilon u_{rt}, \\
J_r &= u_{tr} - \varepsilon u_{rr}, \\
J_{rr} &= u_{trr} - \varepsilon u_{tr} + \varepsilon \frac{n-1}{r} u_{rr} - \varepsilon \frac{(n-1)}{r^2} u_r + \varepsilon \lambda p e^{pu} u_r.
\end{aligned}$$

From above, it follows that

$$J_t - J_{rr} - \frac{n-1}{r} J_r - \lambda p e^{pu} J = \varepsilon \frac{(n-1)}{r^2} u_r \geq 0,$$

i.e.

$$J_t - \Delta J - \lambda p e^{pu} J \geq 0, \quad (x,t) \in B_R \times (0,T).$$

Moreover,

$$\begin{aligned}
\frac{\partial J}{\partial \eta}|_{x \in \partial B_R} &= u_{rt}(R,t) - \varepsilon u_{rr}(R,t) \\
&= q e^{qu(R,t)} u_t - \varepsilon [u_t(R,t) - \frac{n-1}{r} u_r(R,t) - \lambda e^{pu(R,t)}] \\
&\geq [q e^{qu(R,t)} - \varepsilon] u_t(R,t)
\end{aligned}$$

Since $u_t > 0$ in $\overline{B}_R \times (0,T)$, we obtain

$$\frac{\partial J}{\partial \eta} \geq 0, \quad \text{on} \quad \partial B_R \times (0,T),$$

provided $\varepsilon \leq q e^{\{q u_0(R)\}}$.

Since e^{pu} is bounded on $B_R \times (0,t]$ for $t < T$, from maximum principle B.1.1 and above, we have

$$J \geq 0, \quad (x,t) \in \overline{B}_R \times (0,T).$$

In particular, $J(x,t) \geq 0$ for $x \in \partial B_R$, that is

$$u_t(R,t) \geq \varepsilon u_r(R,t) = \varepsilon e^{qu(R,t)}, \quad t \in (0,T).$$

Upon integration the above inequality from t to T and since u blows up at R, it follows that

$$e^{qu(R,t)} \leq \frac{1}{q\varepsilon(T-t)}, \quad t \in (0,T),$$

or

$$u(R,t) \leq \log C - \frac{1}{q}\log(T-t), \quad t \in (0,T).$$

□

Remark 4.1.6. Theorem 4.1.5 can be proved without condition (4.12), and that by using a different technique depending on the integral equation (4.8) (see the proof of Theorem 4.2.4 in the next section).

Remark 4.1.7. The upper blow-up rate estimate for problem (4.6), which has been derived in Theorem 4.1.5, is governed by the boundary term even in case $p > q$. On the other hand, as we have mentioned in Chapter 3, the upper blow-up bound for problem (4.6), where $\lambda = 0$ takes the form:

$$u(R,t) \leq \log \frac{C}{(T-t)^{\frac{1}{2q}}}.$$

This shows that the presence of the reaction term has an important effect on the upper blow-up rate estimate.

4.1.3 Blow-up Set

We shall prove in this subsection that the blow-up in problem (4.6) occurs only on the boundary, restricting ourselves to the special case $p = q = 1$ with some certain assumptions on λ.

Theorem 4.1.8. *Suppose that the function $u(x,t)$ is $C^{2,1}(\overline{B}_R \times [0,T))$, and satisfies*

$$\left.\begin{array}{ll} u_t = \Delta u + \lambda e^u, & (x,t) \in B_R \times (0,T), \\ u(x,t) \leq \log \frac{C}{(T-t)}, & (x,t) \in \overline{B}_R \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{array}\right\}$$

where $C < \infty$ *and*

$$\lambda[4R^2(n+1)+1] \leq \min\left\{\frac{1}{C}, \frac{4(n+1)}{[R^2+4(n+1)T]}e^{-||u_0||_\infty}\right\}. \tag{4.14}$$

Then for any $0 \leq a < R$, *there exist a positive constant* A *such that*

$$u(x,t) \leq \log[\frac{1}{A(R^2-r^2)^2}] \quad \text{for} \quad 0 \leq |x| \leq a < R, 0 < t < T.$$

Proof. Let

$$\begin{aligned} v(x) &= A(R^2-r^2)^2, \quad r=|x|, \quad 0 \leq r \leq R, \\ z(x,t) = z(r,t) &= \log\frac{1}{[v(x)+B(T-t)]}, \quad \text{in} \quad \overline{B}_R \times (0,T), \end{aligned}$$

where $B > 0, A \geq \lambda$.

Since $z \in C^{2,1}(\overline{B}_R \times [0,T))$, a direct calculation shows that

$$\begin{aligned} z_t &= \frac{B}{[v(x)+B(T-t)]}, \\ z_r &= \frac{4rA(R^2-r^2)}{[v(x)+B(T-t)]}, \\ z_{rr} &= \frac{[v(x)+B(T-t)][4A(R^2-3r^2)] + 16A^2r^2(R^2-r^2)^2}{[v(x)+B(T-t)]^2}. \end{aligned}$$

Thus

$$\begin{aligned} & z_t - z_{rr} - \tfrac{n-1}{r}z_r - \lambda e^z \\ &= \frac{[B-4A(n-1)(R^2-r^2)-\lambda][v(x)+B(T-t)]}{[v(x)+B(T-t)]^2} \\ &\quad - \frac{[4A(R^2-3r^2)][v(x)+B(T-t)] + 16Ar^2v(x)}{[v(x)+B(T-t)]^2} \\ &\geq \frac{[B-4A(n-1)(R^2-r^2)-\lambda-4A(R^2-3r^2)-16Ar^2]v(x)}{[v(x)+B(T-t)]^2} \\ &\geq \frac{[B-4AR^2n-4AR^2-\lambda]v(x)}{[v(x)+B(T-t)]^2} \\ &\geq \frac{[B-4AR^2n-4AR^2-A]v(x)}{[v(x)+B(T-t)]^2} \geq 0 \end{aligned}$$

provided

$$B \geq A[4R^2(n+1)+1].$$

i.e.

$$z_t - \Delta z - \lambda e^z \geq 0, \quad \text{in} \quad B_R \times (0,T)$$

Moreover,

$$\begin{array}{llll} z(x,0) = \log \frac{1}{[v(x)+BT]} & \geq \log \frac{1}{[AR^4+BT]} & \geq u(x,0), & x \in B_R, \\ z(R,t) = \log \frac{1}{B(T-t)} & \geq \log \frac{C}{(T-t)} & \geq u(R,t), & t \in (0,T) \end{array}$$

provided

$$B \leq \min \left\{ \frac{1}{C}, \frac{4(n+1)}{R^2 + 4(n+1)T} e^{-||u_0||_\infty} \right\}.$$

From above and the comparison principle B.1.2, we obtain

$$z(x,t) \geq u(x,t) \quad \text{in} \quad B_R \times (0,T).$$

Thus

$$u(x,t) \leq \log[\frac{1}{A(R^2-r^2)^2}] < \infty \quad \text{for} \quad 0 \leq |x| \leq a < R, 0 < t < T.$$

□

Remark 4.1.9. For problem (4.6), where $p = q = 1$ and λ satisfies (4.14), it follows from Theorem 4.1.8 and the upper blow-up rate estimate (4.13) that the blow-up occurs only on the boundary. Moreover, we note that, in this case if λ is small enough, then the blow-up set is the same as that of (4.6), where $\lambda = 0$ (see Chapter 3).

4.2 Reaction Diffusion Systems Coupled in both Equations and Boundary Conditions

In this section, we consider the following initial-boundary value problem

$$\left. \begin{array}{lll} u_t = \Delta u + \lambda_1 e^v, & v_t = \Delta v + \lambda_2 e^u, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = e^v, & \frac{\partial v}{\partial \eta} = e^u, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in B_R, \end{array} \right\} \tag{4.15}$$

where $\lambda_1, \lambda_2 > 0$, u_0, v_0 are nonnegative, radial, smooth functions satisfying

$$\left.\begin{array}{lll} \frac{\partial u_0}{\partial \eta} = e^{v_0}, & \frac{\partial u_0}{\partial \eta} = e^{u_0}, & x \in \partial B_R, \\ \Delta u_0 + e^{v_0} \geq 0, & \Delta v_0 + e^{u_0} \geq 0, & x \in \overline{B}_R, \\ u_{0r}(|x|) \geq 0, & v_{0r}(|x|) \geq 0, & x \in \overline{B}_R. \end{array}\right\} \tag{4.16}$$

The problems of semilinear systems coupled in both equations and boundary conditions have been studied very extensively over past years in case the reaction terms and boundary conditions are of power type functions. For instance the following system which has been considered in [26]:

$$\left.\begin{array}{lll} u_t = u_{xx} + v^{p_1}, & v_t = v_{xx} + u^{p_2}, & (x,t) \in (0,1) \times (0,T), \\ u_x(1,t) = v^{q_1}, & v_x(1,t) = u^{q_2}, & t \in (0,T), \\ u_x(0,t) = 0, & v_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in [0,1], \end{array}\right\} \tag{4.17}$$

where $p_1, p_2, q_1, q_2 > 0$, and u_0, v_0 are radially nondecreasing, positive smooth functions, satisfying the conditions

$$u_{0x}(0) = v_{0x}(0) = 0, \quad u_{0x}(1) = v_0^{q_1}(1), \quad v_{0x}(1) = u_0^{q_2}(1).$$

It was shown that if

$$\max\{p_1p_2, p_1q_2, p_2q_1, q_1q_2\} \leq 1,$$

then the solutions of problem (4.17) exist globally, otherwise every solution blows up in finite time. Moreover, the blow-up occurs only at $x = 1$ and the blow-up rate estimates take the following form

$$C_1(T-t)^{-\alpha} \leq u(1,t) \leq C_2(T-t)^{-\alpha}, \quad t \in (0,T),$$

$$C_3(T-t)^{-\beta} \leq v(1,t) \leq C_4(T-t)^{-\beta}, \quad t \in (0,T),$$

where

$$\alpha = \alpha(p_1, p_2, q_1, q_2), \; \beta = \beta(p_1, p_2, q_1, q_2).$$

In [69], it was considered the critical exponents for a system of heat equations with inner absorption reaction terms and coupled boundary conditions of exponential type, namely

$$\left.\begin{array}{lll} u_t = \Delta u - a_1 e^{p_1 u}, & v_t = \Delta v - a_2 e^{p_2 v}, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = e^{q_1 v}, & \frac{\partial v}{\partial \eta} = e^{q_2 u}, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \Omega, \end{array}\right\} \quad (4.18)$$

where Ω is a bounded domain with smooth boundary, $p_1, p_2 \geq 0, q_i, a_i > 0, i = 1, 2,$ u_0, v_0 are nonnegative functions, satisfying the conditions:

$$\frac{\partial u_0}{\partial \eta} = e^{q_1 v_0}, \quad \frac{\partial u_0}{\partial \eta} = e^{q_2 u_0}, \quad x \in \partial B_R.$$

It was shown that if

$$1/\tau_1 > 0, \text{ or } 1/\tau_2 > 0,$$

where

$$\tau_1 = \frac{q_1 + \frac{1}{2}p_2}{q_1 q_2 - \frac{1}{4}p_1 p_2}, \quad \tau_2 = \frac{q_2 + \frac{1}{2}p_1}{q_1 q_2 - \frac{1}{4}p_1 p_2},$$

then the solutions of problem (4.18) with large initial data blow up in finite time.

The main purpose of this section is to derive formulas to the upper and lower blow-up rate estimates for problem (4.15) and to study the blow-up set under some restricted assumptions.

4.2.1 Preliminaries

Since the system (4.15) is uniformly parabolic, also the reaction and the boundary conditions terms are smooth functions and the initial data satisfy the compatibility conditions, it follows that the local existence and uniqueness of the classical solutions of problem (4.15) are known by standard parabolic theory (see [1, 40]). On the other hand, for any initial data (u_0, v_0), the solution of this system blows up simultaneously in finite time and the blow-up set contains the boundary (∂B_R). This can be shown by the comparison principle B.2.2 and the known blow-up results of problem (3.7) with (3.11) discussed in Chapter 3.

In next lemma we denote for simplicity $u(r,t) = u(x,t),\ v(r,t) = v(x,t)$.

Lemma 4.2.1. *Let (u, v) be a classical solution to problem (4.15). Then*

(i) *(u,v) is radial and* $u, v > 0$ *in* $\overline{B}_R \times (0, T)$.

(ii) $u_r, v_r \geq 0$ *in* $[0, R] \times (0, T)$.

(iii) $u_t, v_t > 0$, *in* $\overline{B}_R \times (0, T)$.

This lemma can be proved in similar way to the proof of Lemma 4.1.1 with some modification and by using some comparison principles for parabolic systems from Appendix B.

Next, we prove the following lemma which shows the relation between u and v.

Lemma 4.2.2. *Let* (u, v) *be a solution to problem (4.15), there exist* $\mu > 1$ *such that*

$$e^v \leq \mu e^u, \quad e^u \leq \mu e^v, \quad (x, t) \in \overline{B}_R \times [0, T). \tag{4.19}$$

Proof. Let

$$J(x, t) = \mu e^{u(r,t)} - e^{v(r,t)}, \quad (x, t) \in B_R \times (0, T), \quad r = |x|.$$

Since $J \in C^{2,1}(\overline{B}_R \times [0, T))$, a direct calculation shows

$$\begin{aligned} J_t &= \mu e^u u_t - e^v v_t, \\ J_r &= \mu e^u u_r - e^v v_r, \\ J_{rr} &= \mu e^u u_{rr} + \mu e^u u_r^2 - e^v v_{rr} - e^v v_r^2. \end{aligned} \tag{4.20}$$

Thus

$$\begin{aligned} J_t - J_{rr} - \frac{n-1}{r} J_r &= \mu e^u u_t - e^v v_t - \mu e^u u_{rr} - \mu e^u u_r^2 + e^v v_{rr} + e^v v_r^2 \\ &\quad - \frac{n-1}{r} \mu e^u u_r + \frac{n-1}{r} e^v v_r \\ &= \mu e^u [u_t - u_{rr} - \frac{n-1}{r} u_r] - e^v [v_t - v_{rr} - \frac{n-1}{r} v_r] \\ &\quad - \mu e^u u_r^2 + e^v v_r^2 \\ &= \mu e^u [\lambda_1 e^v] - e^v [\lambda_2 e^u] - \mu e^u u_r^2 + e^v v_r^2. \end{aligned}$$

From (4.20), it follows that

$$\begin{aligned} u_r &= \frac{1}{\mu e^u}[v_r e^v + J_r], \\ u_r^2 &= \frac{1}{\mu^2 e^{2u}}[v_r^2 e^{2v} + 2e^v v_r J_r + J_r^2]. \end{aligned}$$

Therefore,

$$J_t - \Delta J = (\lambda_1 \mu - \lambda_2)e^{u+v} + [e^v - \frac{e^{2v}}{\mu e^u}]v_r^2 - [\frac{2e^v}{\mu e^u}v_r + \frac{1}{\mu e^u}J_r]J_r.$$

Clearly,

$$e^v - \frac{e^{2v}}{\mu e^u} = e^v \frac{J}{\mu e^u}.$$

Therefore, the last equation can be rewritten as follows:

$$J_t - \Delta J - bJ_r - cJ = (\lambda_1 \mu - \lambda_2)e^{u+v} \geq 0, \quad (x,t) \in B_R \times (0,T)$$

provided $\mu > \lambda_2/\lambda_1$, where,

$$b = -[\frac{2e^v}{\mu e^u}v_r + \frac{1}{\mu e^u}J_r], \; c = \frac{e^v}{\mu e^u}v_r^2.$$

It is clear that, b, c are continuous functions and c is bounded in $B_R \times (0, T^*)$, for $T^* < T$.

Moreover,

$$\begin{aligned} \frac{\partial J}{\partial \eta}|_{x \in \partial B_R} &= [\mu e^u u_r - e^v v_r] \\ &= \mu e^{u+v} - e^{u+v} = [\mu - 1]e^{u+v} > 0, \end{aligned}$$

and

$$J(x,0) = \mu e^{u_0} - e^{v_0} \geq 0, \quad x \in \overline{B}_R$$

provided μ is large enough.

From above and Proposition B.1.1, it follows that

$$J \geq 0, \quad \text{in} \quad \overline{B}_R \times [0,T).$$

Similarly, we can show that the function $H = \mu e^v - e^u$ is nonnegative in $\overline{B}_R \times [0,T)$. □

4.2.2 Blow-up Rate Estimates

In this subsection we consider the upper and lower blow-up rate estimates for the solutions of problem (4.15) with (4.16).

Theorem 4.2.3. *Let u be a blow-up solution of problem (4.15) with (4.16), $\lambda_1 = \lambda_2 = \lambda$, T is the blow-up time. Assume that u_0, v_0 satisfy*

$$u_{0r}(r) - \frac{r}{R} e^{v_0(r)} \geq 0, \quad v_{0r}(r) - \frac{r}{R} e^{u_0(r)} \geq 0, \quad r \in [0, R]. \tag{4.21}$$

Then there is a positive constant c such that

$$\log c - \frac{1}{2}\log(T-t) \leq u(R,t), \quad \log c - \frac{1}{2}\log(T-t) \leq v(R,t), \quad t \in (0,T).$$

Proof. Define the functions $J_1, J_2 \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T))$, as follows:

$$J_1(x,t) = u_r(r,t) - \frac{r}{R} e^{v(r,t)}, \quad J_2(x,t) = v_r(r,t) - \frac{r}{R} e^{u(r,t)}.$$

A direct calculation shows

$$\begin{aligned} J_{1t} &= u_{rt} - \frac{r}{R} e^{v} [v_{rr} + \frac{n-1}{r} v_r + \lambda e^{u}], \\ J_{1r} &= u_{rr} - \frac{r}{R} e^{v} v_r - \frac{1}{R} e^{v}, \\ J_{1rr} &= [u_{rt} - \frac{n-1}{r} u_{rr} + \frac{n-1}{r^2} u_r - \lambda e^{v} v_r] \\ &\quad - \frac{r}{R}[e^{v} v_{rr} + e^{v} v_r^2] - \frac{2}{R} e^{v} v_r. \end{aligned}$$

From above, it follows that

$$J_{1t} - J_{1rr} - \frac{n-1}{r} J_{1r} = -\frac{n-1}{r^2}[u_r - \frac{r}{R} e^{v}] + \lambda e^{v}[v_r - \frac{r}{R} e^{u}] + \frac{r}{R} e^{v} v_r^2 + \frac{2}{R} e^{v} v_r.$$

Since from Lemma 4.2.1, we have $v_r \geq 0$, it follows that

$$J_{1t} - \Delta J_1 + \frac{n-1}{r^2} J_1 - \lambda e^{v} J_2 = \frac{r}{R} e^{v} v_r^2 + \frac{2}{R} e^{v} v_r \geq 0,$$

for $(x,t) \in B_R \times (0,T) \cap \{r > 0\}$.

In the same way we can show that

$$J_{2t} - \Delta J_2 + \frac{n-1}{r^2} J_2 - \lambda e^u J_1 \geq 0, \quad (x,t) \in B_R \times (0,T) \cap \{r > 0\}.$$

Clearly, from (4.21) it follows that

$$J_1(x,0),\ J_2(x,0) \geq 0 \quad x \in B_R.$$

And

$$\begin{aligned} &J_1(0,t) = u_r(0,t) \geq 0, J_2(0,t) = v_r(0,t) \geq 0, \\ &J_1(R,t) = J_2(R,t) = 0, \quad t \in (0,T). \end{aligned}$$

Since in the domain $B_R \times (0,t]$ for $t < T$ the suprema of the functions $\lambda e^u, \lambda e^v$ and $\frac{1-n}{r^2}$ are finite, from above and by the maximum principle B.2.1, it follows that

$$J_1, J_2 \geq 0, \quad (x,t) \in B_R \times (0,T).$$

Moreover,

$$\frac{\partial J_1}{\partial \eta}|_{\partial B_R} \leq 0.$$

This means

$$(u_{rr} - \frac{r}{R} e^v v_r - \frac{1}{R} e^v)|_{\partial B_R} \leq 0.$$

Thus

$$u_t \leq (\frac{n-1}{r} u_r + \lambda e^v + \frac{r}{R} e^v v_r + \frac{1}{R} e^v)|_{\partial B_R},$$

which implies that

$$u_t(R,t) \leq \frac{n-1}{R} e^{v(R,t)} + \lambda e^{v(R,t)} + e^{v(R,t)+u(R,t)} + \frac{1}{R} e^{v(R,t)}, \quad t \in (0,T).$$

From the last inequality and Lemma 4.2.2, it follows

$$u_t(R,t) \leq \frac{n-1}{R} \mu e^{u(R,t)} + \lambda \mu e^{u(R,t)} + \mu e^{2u(R,t)} + \frac{\mu}{R} e^{u(R,t)}, \quad t \in (0,T).$$

Thus, there exist a constant C such that

$$u_t(R,t) \leq C e^{2u(R,t)}, \quad t \in (0,T).$$

Integrate this inequality from t to T and since u blows up at R, it follows

$$\frac{c}{(T-t)^{\frac{1}{2}}} \leq e^{u(R,t)}, \quad t \in (0,T)$$

or

$$\log c - \frac{1}{2}\log(T-t) \leq u(R,t), \quad t \in (0,T).$$

We can show in a similar way that

$$\log c - \frac{1}{2}\log(T-t) \leq v(R,t), \quad t \in (0,T).$$

□

Next, we consider the upper bounds

Theorem 4.2.4. *Let u be a blow-up solution of problem (4.15), (4.16), T is the blow-up time. Then there is a positive constant C such that*

$$u(R,t) \leq \log C - \log(T-t), \quad v(R,t) \leq \log C - \log(T-t), \quad t \in (0,T).$$

Proof. Define

$$M(t) = \max_{\overline{B}_R} u(x,t), \quad N(t) = \max_{\overline{B}_R} v(x,t).$$

Clearly, $M(t), N(t)$ are increasing in $(0,T)$ due to the

$$u_t, v_t > 0, \quad (x,t) \in \overline{B}_R \times (0,T).$$

As in Theorem 4.1.2, for $0 < z < t < T, x \in B_R$, the integral equation for problem (4.15) with respect to u can be written as follows

$$\begin{aligned} u(x,t) \quad = \quad & \int_{B_R} \Gamma(x-y,t-z)u(y,z)dy + \lambda_1 \int_z^t \int_{B_R} \Gamma(x-y,t-\tau)e^{v(y,\tau)}dyd\tau \\ & + \int_z^t \int_{S_R} \Gamma(x-y,t-\tau)e^{v(y,\tau)}ds_y d\tau \\ & - \int_z^t \int_{S_R} u(y,\tau)\frac{\partial \Gamma}{\partial \eta_y}(x-y,t-\tau)ds_y d\tau, \end{aligned}$$

where Γ is the fundamental solution of the heat equation, which was defined in (3.13).

Letting $x \to \partial B_R$ and using the jump relation (Theorem 3.2.7) for the fourth term on the right hand side of the last equation, we obtain

$$\begin{aligned}\frac{1}{2}u(x,t) &= \int_{B_R}\Gamma(x-y,t-z)u(y,z)dy + \lambda_1\int_z^t\int_{B_R}\Gamma(x-y,t-\tau)e^{v(y,\tau)}dyd\tau\\ &+\int_z^t\int_{S_R}\Gamma(x-y,t-\tau)e^{v(y,\tau)}ds_yd\tau\\ &-\int_z^t\int_{S_R}u(y,\tau)\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)ds_yd\tau,\end{aligned}$$

for $x \in \partial B_R, 0<z<t<T$.

Since u, v are positive and radial, it follows that

$$\int_{B_R}\Gamma(x-y,t-z)u(y,z)dy > 0,$$
$$\int_z^t\int_{S_R}e^{v(y,\tau)}\Gamma(x-y,t-\tau)ds_yd_\tau \geq \int_z^t e^{v(R,\tau)}[\int_{S_R}\Gamma(x-y,t-\tau)ds_y]d\tau.$$

Thus

$$\begin{aligned}\frac{1}{2}M(t) &\geq \int_z^t e^{N(\tau)}[\int_{S_R}\Gamma(x-y,t-\tau)ds_y]d\tau\\ &-\int_z^t M(\tau)[\int_{S_R}|\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)|ds_y]d\tau, \quad x\in S_R, 0<z<t<T.\end{aligned}$$

It is known that (see [22]) for $0<t_2<t_2$, there is $C^*>0$ such that

$$|\frac{\partial\Gamma}{\partial\eta_y}(x-y,t_2-t_1)| \leq \frac{C^*}{(t_2-t_1)^{\mu_0}}\cdot\frac{1}{|x-y|^{(n+1-2\mu_0-\sigma)}}, \quad x,y\in S_R, \sigma\in(0,1).$$

Choose $1-\frac{\sigma}{2}<\mu_0<1$, by Lemma 3.2.6 there exist $C_1>0$ such that

$$\int_{S_R}\frac{ds_y}{|x-y|^{(n+1-2\mu_0-\sigma)}} < C_1.$$

Also, if t_1 close to t_2, then there exist a constant c such that

$$\int_{S_R}\Gamma(x-y,t_2-t_1)ds_y \geq \frac{c}{\sqrt{t_2-t_1}}.$$

Thus

$$\frac{1}{2}M(t) \geq c\int_z^t\frac{e^{N(\tau)}}{\sqrt{t-\tau}}d\tau - C\int_z^t\frac{M(\tau)}{|t-\tau|^{\mu_0}}d\tau.$$

Since

$$C\int_z^t \frac{M(\tau)}{|t-\tau|^{\mu_0}}d\tau \le CM(t)\int_z^t \frac{d\tau}{|t-\tau|^{\mu_0}} = \frac{C}{1-\mu_0}M(t)|t-z|^{1-\mu_0} \le \frac{C}{1-\mu_0}M(t)|T-z|^{1-\mu_0},$$

it follows that there exist $C_1^* > 0$, such that

$$\frac{1}{2}M(t) \ge c\int_z^t \frac{e^{N(\tau)}}{\sqrt{T-\tau}}d\tau - C_1^* M(t)|T-z|^{1-\mu_0}. \tag{4.22}$$

Taking z so that $C_1^*|T-z|^{1-\mu_0} = 1/2$, it follows

$$M(t) \ge c\int_z^t \frac{e^{N(\tau)}}{\sqrt{T-\tau}}d\tau \equiv A(t). \tag{4.23}$$

Clearly,

$$A'(t) = c\frac{e^{N(t)}}{\sqrt{T-t}}.$$

From Lemma 4.2.2, there exist a constant $\mu > 1$ such that the last equation becomes

$$A'(t) \ge \frac{c}{\mu}\frac{e^{M(t)}}{\sqrt{T-t}} \ge \frac{c}{\mu}\frac{e^{A(t)}}{\sqrt{T-t}},$$

which leads to

$$\int_t^T \frac{dA}{e^A} \ge \int_t^T \frac{c}{\mu}\frac{d\tau}{\sqrt{T-\tau}}.$$

Thus

$$\frac{1}{e^{A(t)}} - \frac{1}{e^{A(T)}} \ge \int_t^T \frac{c}{\mu}\frac{d\tau}{\sqrt{T-\tau}}, \tag{4.24}$$

where $A(T) = \lim_{t\to T} A(t) \le \lim_{t\to T} 2e^{N(t)}(\sqrt{T-z} - \sqrt{T-t}) \le \infty$.

Since A is positive function, we obtain

$$\frac{1}{e^{A(t)}} \ge \frac{1}{e^{A(t)}} - \frac{1}{e^{A(T)}}$$

Thus (4.24) becomes

$$\frac{1}{e^{A(t)}} \ge \frac{2c}{\mu}\sqrt{T-t}.$$

Therefore, there exist a constant $C_0 > 0$ such that

$$e^{A(t)} \le \frac{C_0}{\sqrt{T-t}}, \quad z < t < T. \tag{4.25}$$

On the other hand, for $t_0 = 2t - T$ (assuming that t is close to T), we have

$$A(t) \geq c \int_{t_0}^{t} \frac{e^{N(\tau)}}{\sqrt{T-\tau}} d\tau \geq c e^{N(t_0)} \int_{2t-T}^{t} \frac{d\tau}{\sqrt{T-\tau}} = e^{N(t_0)} 2c(\sqrt{2}-1)\sqrt{T-t}.$$

Combining the last inequality with (4.25), yields

$$\frac{C_0}{\sqrt{T-t}} \geq e^{N(t_0)} 2c(\sqrt{2}-1)\sqrt{T-t},$$

which leads to

$$e^{N(t_0)} \leq \frac{C_0}{c(\sqrt{2}-1)(T-t_0)}.$$

Thus there exist a constant C such that

$$e^{N(t)} \leq \frac{C}{(T-t)}, \quad 0 < t < T$$

or

$$v(R,t) \leq \log C - \log(T-t), \quad t \in (0,T).$$

In the same way we can show

$$u(R,t) \leq \log C - \log(T-t), \quad t \in (0,T).$$

$\square$

Remark 4.2.5. From Theorems 4.2.3 and 4.2.4, we observe that the upper blow-up rate estimates of problem (4.15) are coincident with the upper blow-up rate estimates of the Dirichlet problem for the semilinear system in (4.15) considered in Chapter 2, while the lower blow-up rate estimates of problems (4.15) are coincident with the lower blow-up rate estimates of problem (4.15), where $\lambda_1 = \lambda_2 = 0$ (see Chapter 3).

4.2.3 Blow-up Set

We consider next the blow-up set for problem (4.15), under some certain assumptions on λ_1, λ_2.

Theorem 4.2.6. *Let (u,v) be a blow-up solution to problem (4.15). Assume that the following condition is satisfied*

$$\lambda[4R^2(n+1)+1] \leq \min\left\{\frac{1}{C}, \frac{4(n+1)}{[R^2+4(n+1)T]}e^{-||u_0||_\infty}, \frac{4(n+1)}{[R^2+4(n+1)T]}e^{-||v_0||_\infty}\right\}, \tag{4.26}$$

where T is the blow-up time, C is given in Theorem 4.2.4, $\lambda = \max\{\lambda_1, \lambda_2\}$. Then there exist a positive constant A such that

$$u(x,t) \leq \log[\frac{1}{A(R^2-r^2)^2}], \quad v(x,t) \leq \log[\frac{1}{A(R^2-r^2)^2}],$$

for $(x,t) \in B_R \times (0,T)$.

Proof. Define the functions z_1, z_2 as follows

$$z_1(x,t) = z_2(x,t) = \log\tfrac{1}{[Av(x)+B(T-t)]}, \qquad (x,t) \in \overline{B}_R \times (0,T), \tag{4.27}$$

where $v(x) = (R^2-r^2)^2, \quad r = |x|, \; B > 0, \; A \geq \lambda.$

Since $z_1, z_1 \in C^{2,1}(\overline{B}_R \times [0,T))$, a similar calculation to that in the proof of Theorem 4.1.8 shows that

$$\left.\begin{aligned} z_{1t} - \Delta z_1 - \lambda_1 e^{z_2} \geq z_{1t} - \Delta z_1 - Ae^{z_2} \geq 0, \quad \text{in } B_R \times (0,T), \\ z_{2t} - \Delta z_2 - \lambda_2 e^{z_1} \geq z_{2t} - \Delta z_2 - Ae^{z_1} \geq 0, \quad \text{in } B_R \times (0,T) \end{aligned}\right\} \tag{4.28}$$

provided

$$B \geq A[4R^2(n+1)+1].$$

Moreover,

$$\left.\begin{aligned} z_1(x,0) = \log\tfrac{1}{[Av(x)+BT]} \geq \log\tfrac{1}{[AR^4+BT]} \geq u(x,0), \quad x \in B_R, \\ z_2(x,0) = \log\tfrac{1}{[Av(x)+BT]} \geq \log\tfrac{1}{[AR^4+BT]} \geq v(x,0), \quad x \in B_R \end{aligned}\right\} \tag{4.29}$$

and

$$z_1(R,t) = z_2(R,t) = \log\tfrac{1}{B(T-t)} \geq \log\tfrac{C}{(T-t)}, \quad t \in (0,T) \tag{4.30}$$

provided

$$B \leq \min\left\{\frac{1}{C}, \frac{4(n+1)}{R^2+4(n+1)T}e^{-||u_0||_\infty}, \frac{4(n+1)}{R^2+4(n+1)T}e^{-||v_0||_\infty}\right\},$$

From (4.29), (4.30) and Theorem 4.2.4, it follows that

$$\left.\begin{array}{lll} z_1(R,t) \geq u(R,t), & z_2(R,t) \geq v(R,t), & t \in (0,T), \\ z_1(x,0) \geq u(x,0), & z_2(x,0) \geq v(x,0), & x \in B_R. \end{array}\right\} \tag{4.31}$$

From (4.28), (4.31) and Proposition B.2.3, it follows that

$$z_1(x,t) \geq u(x,t), \quad z_2(x,t) \geq v(x,t), \quad (x,t) \in B_R \times (0,T).$$

Moreover, from (4.27)

$$u(x,t) \leq \log[\frac{1}{A(R^2-r^2)^2}], \quad v(x,t) \leq \log[\frac{1}{A(R^2-r^2)^2}], \tag{4.32}$$

for $(x,t) \in B_R \times (0,T)$. □

Remark 4.2.7. For problem (4.15) with (4.26), from (4.32) we observe that any point $x \in B_R$ cannot be a blow-up point, therefore, the blow-up occurs only at the boundary. This means, if λ_1, λ_2 are small enough, then the blow-up set is the same as that of (3.7) with (3.11), see Chapter 3.

Chapter 5

Semilinear Parabolic Problems with Gradient Terms

Chipot and Weissler introduced in [6] the interesting parabolic equation, which is a semilinear heat equations with gradient term.Their motivation for studying this equation came from earlier work of Levine [41] for the simpler equation in which the gradient term was absent, and more particularly from their interest in extending Levine's work to the semilinear equation, which has a power function of the solutions and a gradient term.

The main purpose of this chapter is to understand whether the gradient terms affect the blow-up bounds. In the first section of this chapter we complete the results of J. Bebernes and D. Eberly [3], considering the pointwise estimate and the blow-up rate estimates for the problem of heat equation with the exponential function of the solutions and a negative sign (dissipative) gradient term, defined in a ball. Next we shall study in section two the blow-up rate estimates for a system of semilinear heat equations with positive sign gradient terms defined in a ball or the whole space.

5.1 The Semilinear Heat Equation with a Gradient Term

Consider the following initial-boundary value problem

$$\left.\begin{array}{ll} u_t = \Delta u - h(|\nabla u|) + f(u), & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R, \end{array}\right\} \tag{5.1}$$

where $f \in C^1(R)$, $h \in C^1([0,\infty))$, $f, h > 0$, $h^{'} \geq 0$ in $(0,\infty)$, $f(0) \geq 0, h(0) = h^{'}(0) = 0$,

$$|h(\xi)| \leq O(|\xi|^2), \tag{5.2}$$

$$sh^{'}(s) - h(s) \leq Ks^q, \quad \text{for} \quad s > 0,\ 0 \leq K < \infty,\ q > 1, \tag{5.3}$$

$u_0 \geq 0$ is smooth, radially nonincreasing function, vanishing on ∂B_R, this means it satisfies the following conditions

$$\left.\begin{array}{ll} u(x) = u_0(|x|), & x \in B_R, \\ u_0(x) = 0, & x \in \partial B_R, \\ u_{0r}(|x|) \leq 0, & x \in B_R. \end{array}\right\} \tag{5.4}$$

Moreover, we assume that

$$\Delta u_0 + f(u_0) - h(|\nabla u_0|) \geq 0, \quad x \in B_R. \tag{5.5}$$

The special case

$$u_t = \Delta u - |\nabla u|^q + u|u|^{p-1},\ p, q > 1 \tag{5.6}$$

was introduced in [6] and it was studied and discussed later by many authors (see for instance [17, 62]). The main issue in those works was to determine for which p and q blow-up in finite time (in the L^∞-norm) may occur. It is well known that it occurs if and only if $p > q$ (see [17]). Therefore, there is a competition between the reaction term, which may cause blow-up as in the problem (2.1), and the gradient term, which fights against blow-up. Equation (5.6) in R^n was considered from similar point of view, in this case blow-up in finite time is also known to occur when $p > q$, but unbounded global solutions always exist (see [62]). For bounded domains, it has been shown in [8] for equation (5.6) with general convex domain Ω that, the blow-up set is compact. Moreover if $\Omega = B_R$, then $x = 0$ is the only possible blow-up point and the upper pointwise estimate takes the following form

$$u \leq c|x|^{-\alpha}, \quad (x,t) \in B_R \setminus \{0\} \times [0,T),$$

for any $\alpha > 2/(p-1)$ if $q \in (1, 2p/(p+1))$, and for $\alpha > q/(p-q)$ if $q \in [2p/(p+1), p)$. We observe that $q/(p-q) > 2/(p-1)$ for $q > 2p/(p+1)$, therefore, the final blow-up profile of the solutions of equation (5.6) is similar to that of $u_t = \Delta u + u^p$ as long as $q < 2p/(p+1)$ (see Chapter 2), whereas for q grater than this critical value, the gradient term induces an important effect on the profile, which becomes more singular.

On the other hand, it was proved in [7, 8, 19, 63] that the upper (lower) blow-up rate estimate in terms of the blow-up time T in the case $q < 2p/(p+1)$ and $u \geq 0$, takes the following form

$$c(T-t)^{-1/(p-1)} \leq u(x,t) \leq C(T-t)^{-1/(p-1)}.$$

J. Bebernes and D. Eberly have considered in [3] a second special case of (5.1), where $f(s) = e^s, h(\xi) = \xi^2$, namely

$$\left.\begin{array}{ll} u_t = \Delta u - |\nabla u|^2 + e^u, & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R. \end{array}\right\} \tag{5.7}$$

The semilinear equation in (5.7) can be viewed as the limiting case of the critical splitting as $p \to \infty$ in the equation (5.6). It has been proved that, the solution of the above problem with u_0 satisfies (5.4) may blow up in finite time and the only possible blow-up point is $x = 0$. Moreover, if we consider the problem in any general bounded domain Ω such that $\partial\Omega$ is analytic, then the blow-up set is a compact set. On the other hand, they proved that, if x_0 is a blow-up point for problem (5.7) with the finite blow-up time T; then

$$\lim_{t \to T^-} [u(x_0,t) + m \log(T-t)] = k,$$

for some $m \in Z^+$ and for some $k \in R$. The analysis therein is based on the observation that the transformation $v = 1 - e^{-u}$ changes the first equation in problem (5.7) into the linear equation $v_t = \Delta v + 1$, moreover, x_0 is a blow-up point for (5.7) with blow-up time T if and only if $v(x_0, T) = 1$.

In this section we consider problem (5.7) with (5.4), our aim is to derive the upper pointwise estimate for the classical solutions of this problem and to find a formula for the upper (lower) blow-up rate estimate.

5.1.1 Preliminaries

Set

$$F(u, \nabla u) = f(u) - h(|\nabla u|). \tag{5.8}$$

Since $F \in C^1(R \times R^n)$, the local existence and uniqueness of classical solutions to problem (5.1), (5.4) is well known by [22, 40], and the regularity of these solutions is guaranteed by [56]. Moreover, the gradient function ∇u is bounded as long as the solution u is bounded due to (5.2) (see also [56]).

In order to show some properties of the classical solutions of problem (5.1) with (5.4), we recall the following lemma, which has been proved in [56]. We may denote for simplicity $u(r, t) = u(x, t)$.

Lemma 5.1.1. *Let u be a classical solution to the problem*

$$\left.\begin{array}{ll} u_t = \Delta u + H(u, \nabla u), & x \in B_R,\ t > 0, \\ u(x,t) = 0, & x \in \partial B_R, t > 0, \\ u(x,0) = u_0(x), & x \in B_R. \end{array}\right\}$$

where

$$H = H(s, \xi) : R \times R^n \to R$$

such that $H \in C^1(R \times R^n)$, $H(s, \xi) = H^*(s, |\xi|)$ and $H(0,0) \geq 0$. Assume $u_0 \geq 0$, such that $u_0 \in C^2(\overline{B}_R)$ is nonnegative and satisfies (5.4) and moreover,

$$\Delta u_0 + H(u_0, \nabla u_0) \geq 0, \quad x \in B_R.$$

Then

(i) $u \geq 0$ *and it is radially nonincreasing in* $[0, R] \times (0, T)$. *Moreover if* $u_0 \not\equiv 0$, *then* $u_r < 0$ *in* $(0, R] \times (0, T)$.

(ii) $u_t \geq 0$ *in* $\overline{B}_R \times [0, T)$.

Remark 5.1.2. It is clear that F (defined in (5.8)) satisfies all the assumptions of H in Lemma 5.1.1, therefore, the classical solutions of problem (5.1) with (5.4) and (5.5) satisfy (i) and (ii). Furthermore, by using Proposition B.1.5, it follows directly that

$$u > 0, \quad \text{in} \quad B_R \times (0, T).$$

Depending on above, the problem (5.1) with (5.4) can be rewritten as follows

$$\left.\begin{array}{ll} u_t = u_{rr} + \frac{n-1}{r} u_r - h(-u_r) + f(u), & (r, t) \in (0, R) \times (0, T), \\ u_r(0, t) = 0, \quad u(R, t) = 0, & t \in [0, T), \\ u(r, 0) = u_0(r), & r \in [0, R], \\ u_r(r, t) < 0, & (r, t) \in (0, R] \times (0, T). \end{array}\right\} \tag{5.9}$$

5.1.2 Pointwise Estimates

In order to derive a formula to the pointwise estimate for problem (5.9), we need first to recall the following theorem, which has been proved in [8].

Theorem 5.1.3. *Assume that, there exist two functions* $F \in C^2([0, \infty))$ *and* $c_\varepsilon \in C^2((0, R]), \varepsilon > 0$, *such that*

$$c_\varepsilon(0) = 0 \textit{ and } c_\varepsilon > 0 \textit{ otherwise}, \; c_\varepsilon{}', c_\varepsilon{}'' \geq 0, \tag{5.10}$$

$$F > 0, F', F'' \geq 0, \quad \textit{in} \quad (0, \infty), \tag{5.11}$$

$$f' F - f F' - 2c_\varepsilon' F' F + c_\varepsilon^2 F'' F^2 - 2^{q-1} K c_\varepsilon^q F^q F' + AF \geq 0, \quad u > 0, 0 < r < R, \tag{5.12}$$

where

$$A = \frac{c_\varepsilon''}{c_\varepsilon} + \frac{n-1}{r} \frac{c_\varepsilon'}{c_\varepsilon} - \frac{n-1}{r^2},$$

$\frac{c_\varepsilon(r)}{r} \to 0$ *uniformly on* $[0, R]$ *as* $\varepsilon \to 0$, *and*

$$G(s) = \int_s^\infty \frac{du}{F(u)} < \infty, \quad s > 0.$$

Let u is a blow-up solution to problem (5.9), where u_0 satisfies

$$u_{0r} \leq -\delta r, \quad r \in (0, R], \quad \delta > 0. \tag{5.13}$$

Suppose that, T is the blow-up time. Then the point $r = 0$ is the only blow-up point, and there is $\varepsilon_1 > 0$ such that

$$u(r,t) \leq G^{-1}(\int_0^r c_{\varepsilon_1}(z)dz), \quad (r,t) \in (0,R] \times (0,T). \tag{5.14}$$

Proof. Since the function F in (5.8) is $C^1(R \times R^n)$, by parabolic regularity results (see [22]), we have

$$u_r \in C^{2,1}((0,R) \times (0,T)) \cap C([0,R] \times [0,T)).$$

Set $w = u_r$.

Differentiating the first equation in (5.9) with respect to r, it follows

$$w_t - \frac{n-1}{r} w_r - w_{rr} = \frac{1-n}{r^2} w + f^{'}(u)w + h^{'}(-u_r)u_{rr}.$$

Define the function

$$J = w + c_\varepsilon(r)F(u)$$

Since $F \in C^2([0,\infty))$, we have $J \in C^{2,1}((0,R) \times (0,T)) \cap C([0,R] \times [0,T))$.

Our aim is to show that $J \leq 0$ in $[0,R] \times [0,T)$.

We compute now the equation for J :

$$\begin{aligned} J_t - \frac{n-1}{r} J_r - J_{rr} &= \frac{1-n}{r^2} w + f^{'}(u)w + h^{'}(-u_r)u_{rr} + c_\varepsilon F^{'}[f(u) - h(-u_r)] \\ &\quad -2wc_\varepsilon^{'}F^{'} + F[\frac{1-n}{r}c_\varepsilon^{'} - c_\varepsilon^{''}] - c_\varepsilon w^2 F^{''}. \end{aligned}$$

Using the relations

$$u_r = w = J - c_\varepsilon F, \; w^2 = c_\varepsilon^2 F^2 + (J - 2c_\varepsilon F)J$$

and

$$u_{rr} = J_r - c_\varepsilon^{'} F - c_\varepsilon F^{'} u_r.$$

A direct calculation shows

$$
\begin{aligned}
J_t &- (\frac{n-1}{r} + h^{'}(-u_r))J_r - J_{rr} - b_0 J \\
&= c_\varepsilon[F(-f^{'} - \frac{c_\varepsilon^{'}}{c_\varepsilon}h^{'}(-u_r) + \frac{n-1}{r^2} - \frac{c_\varepsilon^{''}}{c_\varepsilon} - \frac{n-1}{r}\frac{c_\varepsilon^{'}}{c_\varepsilon}) \\
&\quad +F^{'}(f - u_r h^{'}(-u_r) - h(-u_r)) + 2c_\varepsilon^{'}F^{'}F - c_\varepsilon^2 F^{''}F^2],
\end{aligned}
$$

where

$$b_0 = f^{'} - \frac{n-1}{r^2} - 2c_\varepsilon^{'}F^{'} - c_\varepsilon F^{''}(J - 2c_\varepsilon F).$$

From (5.3) it follows that

$$-u_r h^{'}(-u_r) - h(-u_r) \le K(-u_r)^q = K(c_\varepsilon F - J)^q \le 2^{q-1}K(c_\varepsilon^q F^q + |J|^q).$$

From above and (5.12) and $h^{'}c_\varepsilon^{'} \ge 0$, it follows that

$$J_t - (\frac{n-1}{r} + h^{'}(-u_r))J_r - J_{rr} - bJ \le 0, \quad (r,t) \in (0,R) \times (0,T),$$

where

$$b = b_0 + 2^{q-1}Kc_\varepsilon F^{'}|J|^{q-2}J.$$

Since $u_t \ge 0$ in $(0,R)\times(0,T)$ and from the zero Dirichlet boundary condition, it follows that

$$u_r(R,t) \le u_{0r}(R).$$

Thus, by (5.13), we obtain

$$
\begin{aligned}
&J(R,t) \le u_{0r}(R) + c_\varepsilon(R)F(0) \le -\delta R + c_\varepsilon(R)F(0) \le 0, \quad t \in (0,T), \\
&J(r,0) = u_{0r}(r) + c_\varepsilon(r)F(u_0(r)) \le -\delta r + c_\varepsilon(r)F(u_0(r)) \le 0,
\end{aligned}
$$

provided

$$\frac{c_\varepsilon(r)}{r} \le \frac{\delta}{\max_{(0,R]} F(u_0)}, \quad r \in [0,R].$$

Obviously, $J(0,t) = 0, \quad t \in [0,T)$.

Since b is bounded above in $((0,R)\times(0,t]) \cap \{(r,t) \mid J > 0\}$ for any $t < T$, from above and Proposition B.1.3 with Remark B.1.4, it follows that

$$J \le 0, \quad \text{in} \quad [0,R] \times (0,T).$$

Thus

$$-\frac{u_r}{F(u)} \geq c_\varepsilon(r).$$

Since

$$\frac{d}{dr}G(u) = -\frac{u_r}{F(u)},$$

we get

$$\frac{d}{dr}G(u) \geq c_\varepsilon(r).$$

Integrating this inequality from 0 to r, we obtain

$$G(u(r,t)) \geq G(u(r,t)) - G(u(0,t)) \geq \int_0^r c_\varepsilon(z)dz. \tag{5.15}$$

If for some $r > 0$, $u(r,t) \to \infty$, as $t \to T$, then $G(u(r,t)) \to 0$, as $t \to T$, a contradiction to (5.15).

Since G is nonincreasing, it follows that

$$u(r,t) \leq G^{-1}(\int_0^r c_\varepsilon(z)dz), \quad (r,t) \in (0,R] \times (0,T).$$

□

We are ready now to derive a formula to the pointwise estimate for the blow-up solutions of problem (5.7) with (5.4).

Theorem 5.1.4. *Let u be a blow-up solution to problem (5.7), assume that u_0 satisfies (5.4) and (5.13).Then the upper pointwise estimate takes the following form*

$$u(r,t) \leq \frac{1}{2\alpha}[\log C - m\log(r)], \quad (r,t) \in (0,R] \times (0,T),$$

where $\alpha \in (0, 1/2], C > 0, m > 2$.

Proof. Let $c_\varepsilon = \varepsilon r^{1+\delta}$, where $\delta \in (0,\infty)$.

Clearly, the inequality (5.12) becomes

$$\begin{aligned} &f^{'}F - fF^{'} - 2\varepsilon(1+\delta)r^\delta F^{'}F + \varepsilon^2 r^{2+2\delta}F^{''}F^2 \\ &-2^{q-1}K\varepsilon^q r^{q+\delta q}F^q F^{'} + \frac{\delta(n+\delta)}{r^2}F \geq 0, \;\; u > 0, 0 < r < R. \end{aligned} \tag{5.16}$$

Note that for the semilinear equation in (5.7), $K \geq 1$, and $q = 2$.

To make use of Theorem 5.1.3 for problem (5.7), assume that

$$F(u) = e^{2\alpha u}, \quad \alpha \in (0, 1/2].$$

It is clear that F, and c_ε satisfy the assumptions (5.10) and (5.11), respectively.

With this choice of F the inequality (5.16) takes the form

$$(1 - 2\alpha)e^{(1+2\alpha)u} + 4\alpha^2\varepsilon^2 r^{2(1+\delta)} e^{6\alpha u} + \frac{\delta(n+\delta)}{r^2} e^{2\alpha u} \geq$$
$$4\alpha\varepsilon(1+\delta)r^\delta e^{4\alpha u} + 4\alpha\varepsilon^2 r^{2(1+\delta)} e^{6\alpha u}, \quad u \geq 0, 0 < r \leq R$$

provided $\alpha \leq \frac{1}{2+4\varepsilon R^\delta(1+\delta)}$.

Define the function G as in Theorem 5.1.3 as follows

$$G(s) = \int_s^\infty \frac{du}{e^{2\alpha u}} = \frac{1}{2\alpha e^{2\alpha s}}, \quad s > 0.$$

Clearly,

$$G^{-1}(s) = -\frac{1}{2\alpha}\log(2\alpha s), \quad s > 0.$$

Thus (5.14) becomes

$$u(r,t) \leq \frac{1}{2\alpha}[\log C - m\log(r)], \quad (r,t) \in (0, R] \times (0, T),$$

where $C = \frac{2+\delta}{2\varepsilon\alpha}$, $\quad m = 2 + \delta$. □

Remark 5.1.5. Theorem 5.1.4 shows that, with choosing $\alpha = 1/2$, the upper pointwise estimate of problem (5.7) is the same as that of problem (2.1), where $f(u) = e^u$, which has been considered in [24] (see Chapter 2). Therefore, the gradient term in problem (5.7) has no effect on the pointwise estimates.

5.1.3 Blow-up Rate Estimates

Since under the assumptions of Theorem 5.1.4, $x = 0$ is the only blow-up point for the problem (5.7), in order to estimate the blow-up solution it is sufficient to estimate only $u(0, t)$. The next theorem considers the upper blow-up rate estimate for the general problem (5.1).

Theorem 5.1.6. *Let u be a blow-up solution to problem (5.1), where $u_0 \in C^2(\overline{B}_R)$ and satisfies (5.4), (5.5). Assume that $\frac{h^{'}(s)}{s}$ is continuous function in R. Let T is the blow-up time and $x = 0$ is the only possible blow-up point. If there exist a function, $F \in C^2([0,\infty))$ such that $F > 0$ and $F^{'}, F^{''} \geq 0$ in $(0,\infty)$, moreover,*

$$f^{'}F - F^{'}f + F^{''}|\nabla u|^2 - F^{'}[h^{'}(|\nabla u|)|\nabla u| - h(|\nabla u|)] \geq 0, \ \textit{in}\ B_R \times (0,T), \quad (5.17)$$

then the upper blow rate estimate takes the from

$$u(0,t) \leq G^{-1}(\delta(T-t)), \quad t \in (\tau, T),$$

where $\delta, \tau > 0$, $G(s) = \int_s^\infty \frac{du}{F(u)}$.

Proof. To prove this theorem, we follow the procedures, which have been used in [8].

Introduce the function

$$J = u_t - \delta F(u), \quad (x,t) \in B_R \times (0,T), \quad \text{where } \delta > 0.$$

Since $f^{'}(s)$ and $h^{'}(s)/s$ are continuous functions, it can be shown that $u_t \in C^{2,1}(B_R \times (0,T))$ (see the regularity results in [56]).

Moreover, since $F \in C^2([0,\infty))$,

$$J \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T)).$$

A direct calculation shows

$$\begin{aligned} J_t - \Delta J &= u_{tt} - \Delta u_t - \delta F^{'}[u_t - \Delta u] + \delta F^{''}|\nabla u|^2 \\ &= f^{'}[J + \delta F] - \frac{h^{'}(|\nabla u|)}{|\nabla u|}\nabla u \cdot (\nabla J + \delta F^{'}\nabla u) \\ &\quad -\delta F^{'}[f - h(|\nabla u|)] + \delta F^{''}|\nabla u|^2 \\ &= f^{'}J - \frac{h^{'}(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla J + \delta f^{'}F \\ &\quad -\delta F^{'}[h^{'}(|\nabla u|)|\nabla u| - h(|\nabla u|)] - \delta F^{'}f + \delta F^{''}|\nabla u|^2. \end{aligned}$$

Thus

$$J_t - \Delta J - cJ - b \cdot \nabla J = \delta D,$$

where $c = f'(u), b = -\frac{h'(|\nabla u|)}{|\nabla u|}\nabla u$,

$$D = f'F - F'f + F''|\nabla u|^2 - F'[h'(|\nabla u|)|\nabla u| - h(|\nabla u|)].$$

By (5.17), we have $D \geq 0$ in $B_R \times (0, T)$.

It is clear that c is bounded function on $B_R \times (0, t]$, for any $t < T$.

By Remark 5.1.2, $u_t \geq 0$ in $B_R \times (0, T)$, and since u blows up in finite time at $x = 0$, there exists $\tau > 0, k_0 > 0$ such that

$$u_t(0, t) > k_0 > 0, \quad t \in [\tau, T).$$

In fact, for small $\varepsilon > 0$, we have

$$u_t(r, t) > k > 0, \quad (r, t) \in [0, \varepsilon] \times [\tau, T), \quad k < k_0. \tag{5.18}$$

Also, since F is locally bounded function in $B_R \times (0, T)$, we can find $\delta > 0$ such that

$$k \geq \delta F(u(x, \tau)), \quad x \in B_\varepsilon.$$

Thus

$$J(x, \tau) \geq 0, \quad x \in B_\varepsilon,$$

provided δ is small enough.

Clearly, $F(u)$ is bounded in $\partial B_\varepsilon \times (0, T)$, there exists C_0 such that

$$F(u(x, t)) \leq C_0 < \infty, \quad \text{in} \quad \partial B_\varepsilon \times [\tau, T). \tag{5.19}$$

Thus, by (5.18) and (5.19), it follows that

$$J(x, t) \geq 0, \quad (x, t) \in \partial B_\varepsilon \times (\tau, T), \tag{5.20}$$

provided δ is small enough.

Applying Proposition B.1.3 (starting from τ instead of 0), we obtain

$$J \geq 0, \quad (x, t) \in B_\varepsilon \times (\tau, T),$$

which leads to

$$u_t(0,t) \geq \delta F(u(0,t)), \quad \text{for} \quad \tau < t < T. \tag{5.21}$$

Clearly, (5.21) implies that

$$-\frac{dG(u)}{dt} = \frac{u_t}{F(u)} \geq \delta.$$

By integration,

$$G(u(0,t)) - G(u(0,T)) \geq \delta(T-t).$$

It follows

$$G(u(0,t)) \geq \delta(T-t).$$

Since G is nonincreasing, we obtain

$$u(0,t) \leq G^{-1}(\delta(T-t)), \quad t \in (\tau, T). \tag{5.22}$$

□

For problem (5.7), if one could choose a suitable function F that satisfies the conditions, which have been stated in Theorem 5.1.6, then the upper blow-up rate estimate for this problem would be held.

Theorem 5.1.7. *Let u be a blow-up solution to problem (5.7), where $u_0 \in C^2(\overline{B}_R)$ and satisfies (5.4), (5.13) and the monotonicity assumption*

$$\Delta u_0 + e^{u_0} - |\nabla u_0|^2 \geq 0, \quad x \in B_R,$$

suppose that T is the blow-up time.Then there exist $C > 0$ such that the upper blow-up rate estimate takes the following form

$$u(0,t) \leq \frac{1}{\alpha}[\log C - \log(T-t)], \quad 0 < t < T, \ \alpha \in (0,1].$$

Proof. Let

$$F(u) = e^{\alpha u}, \quad \alpha \in (0,1].$$

It is clear that the inequality (5.17) holds because

$$(1-\alpha)e^{(1+\alpha)u} + \alpha^2 e^{\alpha u}|\nabla u|^2 - \alpha e^{\alpha u}|\nabla u|^2 \geq 0,$$

Set

$$G(s) = \int_s^\infty \frac{du}{e^{\alpha u}} = \frac{1}{\alpha e^{\alpha s}}, \quad s > 0.$$

Clearly,

$$G^{-1}(s) = -\frac{1}{\alpha}\log(\alpha s), \quad s > 0.$$

By Theorem 5.1.6, there is $\delta > 0$ such that

$$u(0,t) \le \frac{1}{\alpha}[\log(\frac{1}{\alpha\delta}) - \log(T-t)], \quad \tau < t < T.$$

Therefore, there exists a positive constant, C such that

$$u(0,t) \le \frac{1}{\alpha}[\log C - \log(T-t)], \quad 0 < t < T.$$

□

Next, we consider the lower blow-up rate for problem (5.7), which is much easier than the upper bound.

Theorem 5.1.8. *Let u be a blow-up solution to problem (5.7), where u_0 satisfies (5.4) and (5.13). Suppose that T is the blow-up time.Then there exist $c > 0$ such that the lower blow-up rate estimate takes the following form*

$$\log c - \log(T-t) \le u(0,t), \quad 0 < t < T.$$

Proof. Define

$$U(t) = u(0,t), \quad t \in [0,T).$$

Since u attains its maximum at $x = 0$,

$$\Delta U(t) \le 0, \quad 0 \le t < T.$$

From the semilinear equation in (5.7) and above, it follows that

$$U_t(t) \le e^{U(t)} \le \lambda e^{U(t)}, \quad 0 < t < T, \tag{5.23}$$

for $\lambda \ge 1$. Integrate (5.23) from t to T, we obtain

$$\frac{1}{\lambda(T-t)} \le e^{u(0,t)}, \quad 0 < t < T.$$

It follows that

$$\log c - \log(T-t) \le u(0,t), \quad 0 < t < T,$$

where $c = 1/\lambda$. □

Remark 5.1.9. Theorem 5.1.8 (Theorem 5.1.7, where $\alpha = 1$) show that, the lower (upper) blow-up rate estimate of problem (5.7) is the same as that of (2.1), where $f(u) = e^u$, which has been considered in [24] (see Chapter 2). Therefore, the gradient term in problem (5.7) has no effect on the blow-up rate estimates.

5.2 Reaction Diffusion Coupled Systems with Gradient Terms

In this section, we consider the Cauchy (Dirichlet) parabolic problem:

$$\left.\begin{array}{llll} u_t = \Delta u + |\nabla u|^{q_1} + v^{p_1}, & v_t = \Delta v + |\nabla v|^{q_2} + u^{p_2} & \text{in} & \Omega \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & \text{in} & \Omega, \end{array}\right\} \tag{5.24}$$

where $p_1, p_2, \in (1,\infty), q_1, q_2 \in (1,2]$, $u_0, v_0 \geq 0$ are nonzero, smooth and bounded functions on Ω (not necessarily radial), $\Omega = R^n$ or B_R. Moreover, in case of $\Omega = B_R$, u, v are further required to satisfy the condition:

$$u(x,t) = 0, \qquad v(x,t) = 0, \quad \text{on} \quad \partial\Omega \times [0,T). \tag{5.25}$$

The problems of semilinear parabolic equations have been studied by many authors, for instance, consider the Cauchy (Dirichlet) problem for the semilinear heat equation:

$$u_t = \Delta u + u^p, \quad \text{in} \quad \Omega \times (0,T), \tag{5.26}$$

where $p > 1$, $\Omega = R^n$ or B_R. For this problem, it is well known that every positive solution blows up in finite time, if the initial data are nonnegative and suitably large [25, 37]. Moreover, it was proved in [24, 67] that the blow-up rate estimate for (5.26) takes the following form

$$u(x,t) \leq c(T-t)^{-\frac{1}{p-1}}, \quad (x,t) \in \Omega \times (0,T).$$

Later, in [38] it has been shown that if we add a positive gradient term to the equation (5.26), namely

$$u_t = \Delta u + |\nabla u|^q + u^p, \quad p, q > 1, \tag{5.27}$$

then that enhancing blow-up, and the influence of the gradient term becoming more important as the value of p decreases. In the case $q = 2$ for radial positive solutions in R^n, it was shown in [29, 30] that blow-up solutions behave asymptotically like the nonconstant self-similar blow-up solution of the first-order *Hamilton-Jacobi equation* without diffusion ($u_t = |\nabla u|^2 + u^p$), which takes the form

$$u(x,t) = (T-t)^{\frac{-1}{p-1}} w(\frac{|x|}{(T-t)^m}), \quad m = (p-2)/2(p-1),$$

where $w \in C^2(R)$ is a positive radial function, vanishing at a finite point or at infinity. Clearly, m describes the range $(-\infty, 1/2)$ for $p \in (1, \infty)$. In particular, this means the blow-up solutions of problem (5.27) blow up with a rate $O((T-t)^{\frac{-1}{p-1}})$, which is the same as that of problem (5.26). However, unlike to problem (5.26) (see [62]), this kind of self similar profile is singular for any $x \in R^n$, where $1 < p < 2$. On the other hand, the existence of nonnegative global solutions to (5.27) is shown in [60] for small initial data.

In [13, 14], it was considered, the Cauchy (Dirichlet) problem for the following semilinear system:

$$u_t = \Delta u + v^{p_1}, \quad v_t = \Delta v + u^{p_2}, \quad (x,t) \in \Omega \times (0,T), \tag{5.28}$$

where $p_1, p_2 > 1$, $\Omega = B_R$ or R^n, with nonzero initial data $u_0, v_0 \geq 0$, it was shown that any positive solution of this problem blows up in finite time, if the initial data are large enough. Moreover, for the Cauchy problem for (5.28), it is well known [13] that every nontrival positive solution blows up in finite time, if

$$\max\{\alpha, \beta\} \geq \frac{n}{2}, \tag{5.29}$$

where

$$\alpha = \frac{p_1 + 1}{p_1 p_2 - 1}, \quad \beta = \frac{p_2 + 1}{p_1 p_2 - 1}. \tag{5.30}$$

The blow-up rate estimates for this system was studied in [7, 11], it was proved that there exist a positive constant C such that

$$u(x,t) \leq C(T-t)^{-\alpha}, \quad (x,t) \in \Omega \times (0,T),$$

$$v(x,t) \leq C(T-t)^{-\beta}, \quad (x,t) \in \Omega \times (0,T).$$

In this section, for problem (5.24), under some certain assumptions, we prove that the upper blow-up rate estimates of the positive solutions and their gradient terms, take the following forms:

$$u(x,t) + |\nabla u(x,t)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \leq C_1(T-t)^{-\alpha}, \quad (x,t) \in \Omega \times (0,T),$$
$$v(x,t) + |\nabla v(x,t)|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}} \leq C_2(T-t)^{-\beta}, \quad (x,t) \in \Omega \times (0,T),$$

where $C_1, C_2 > 0$.

5.2.1 Preliminaries

Set

$$F_1(v, \nabla u) = |\nabla u|^{q_1} + v^{p_1}, \quad F_2(u, \nabla v) = |\nabla v|^{q_2} + u^{p_2}.$$

Since the system (5.24) is uniformly parabolic and its equations have the same principle parts and F_1, F_2 are $C^1([0,\infty) \times R^n)$, moreover, the growth of the nonlinearities F_1 and F_1 with respect to the gradient is sub-quadratic, it follows that the local existence of the unique nonnegative classical solutions to the Dirichlet problem for (5.24) is guaranteed by the standard parabolic theory [40] (see also [52]). Futhermore, in case of $\Omega = R^n$, assuming that the initial data u_0, v_0 are smooth and bounded functions, according to [40] these existence and uniqueness results can also be extended to the Cauchy problem for this system. On the other hand, the following lemma shows that the positive solutions of problem (5.24) may blow up in finite time.

Lemma 5.2.1. *Let (u^*, v^*) and (u, v) are positive solutions of problems (5.24) and (5.28) respectively, where both of them start with $u_0, v_0 \geq 0$. If (u, v) blows up in finite time T, then (u^*, v^*) blows up in finite time T^*, where $T \geq T^*$.*

Proof. set

$$f_1(s_1, s_2) = s_2|s_2|^{p_1-1}, \; f_2(s_1, s_2) = s_1|s_1|^{p_2-1}.$$

Since $p, q > 1$, it follows that f_1, f_2 are C^1.

Clearly, $s_2^{p_1} \equiv s_2|s_2|^{p_1-1}$, $s_1^{p_2} \equiv s_1|s_1|^{p_2-1}$ for $s_1, s_2 \geq 0$.

Thus

$$\left.\begin{array}{ll} u_t - \Delta u - f_1(u,v) = 0 \le |\nabla u^*|^{q_1} = u_t^* - \Delta u^* - f_1(u^*,v^*) & \text{in} \quad \Omega \times (0,T), \\ v_t - \Delta v - f_2(u,v) = 0 \le |\nabla v^*|^{q_2} = v_t^* - \Delta v^* - f_2(u^*,v^*) & \text{in} \quad \Omega \times (0,T). \end{array}\right\}$$

By Proposition B.2.3 (which can also apply without changes to the case of Cauchy problems), it follows that

$$u \le u^*, \ v \le v^* \quad \text{in} \quad \Omega \times (0,T).$$

□

Remark 5.2.2. Since the growth of the nonlinear terms in problem (5.24) with respect to the gradients is sub-quadratic, the gradient functions $\nabla u, \nabla v$ are bounded as long as the solution (u,v) is bounded (see [52]).

5.2.2 Blow-up Rate Estimates

In the next theorem, we establish the upper blow-up rate estimates for the problem (5.24). Furthermore, without comparing the blow-up solutions of this problem with those of problem (5.28), we show that the blow-up can only occur simultaneously.

Theorem 5.2.3. *If p_1, p_2, q_1 and q_2 satisfy the following conditions*

(1) $\max\{\alpha, \beta\} \ge \frac{n}{2}$,

(2) $1 < q_1 < \frac{2\alpha+2}{2\alpha+1}, \quad 1 < q_2 < \frac{2\beta+2}{2\beta+1}$,

where α, β are given in (5.30), then for any positive blow-up solution (u,v) of problem (5.24) there exist positive constants C_1, C_2 such that

$$u(x,t) + |\nabla u(x,t)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \le C_1(T-t)^{-\alpha}, \tag{5.31}$$

$$v(x,t) + |\nabla v(x,t)|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}} \le C_2(T-t)^{-\beta}, \tag{5.32}$$

in $\Omega \times (0,T)$, where $T < \infty$ is the blow-up time.

Proof. Let

$$M_u(t) = \sup_{\Omega\times(0,t]} [u(x,t) + |\nabla u(x,t)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}}],$$

$$M_v(t) = \sup_{\Omega\times(0,t]} [v(x,t) + |\nabla v(x,t)|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}}],$$

for $t \in (0,T)$.

Clearly, M_u, M_v are positive, continuous and nondecreasing functions on $(0,T)$. At least one of them diverges as $t \to T$, due to (u,v) blows up at time T.

We show later that there is $\delta \in (0,1)$ such that

$$\delta \le M_u^{-\frac{1}{2\alpha}}(t) M_v^{\frac{1}{2\beta}}(t) \le \frac{1}{\delta}, \quad t \in (T/2, T). \tag{5.33}$$

So that, consequently, both M_u, M_v have to diverge as $t \to T$.

To establish the blow-up rate estimates, we use a scaling argument similar as in [7].The proof is divided into several steps.

Step 1: Scaling

If M_u diverges as $t \to T$, the following procedure can be applied.

Given $t_0 \in (0,T)$, choose $(x^*, t^*) \in \Omega \times (0, t_0]$ such that

$$u(x^*,t^*) + |\nabla u(x^*,t^*)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \ge \frac{1}{2} M_u(t_0). \tag{5.34}$$

Let $\gamma = \gamma(t_0) = M_u^{-\frac{1}{2\alpha}}(t_0)$ be a scaling factor. Define the rescaled functions

$$\varphi_1^\gamma(y,s) = \gamma^{2\alpha} u(\gamma y + x^*, \gamma^2 s + t^*), \tag{5.35}$$

$$\varphi_2^\gamma(y,s) = \gamma^{2\beta} v(\gamma y + x^*, \gamma^2 s + t^*), \tag{5.36}$$

for $(y,s) \in \Omega_\gamma \times (-\gamma^{-2}t^*, \gamma^{-2}(T - t^*))$, where

$$\Omega_\gamma = \{y \in R^n : \gamma y + x^* \in \Omega\}.$$

Clearly,

$$\Omega_\gamma := \begin{cases} R^n & \text{if} \quad \Omega = R^n, \\ B_{\frac{R}{\gamma}} & \text{if} \quad \Omega = B_R. \end{cases}$$

Next, we aim to show that $(\varphi_1^\gamma, \varphi_2^\gamma)$ is a solution of the following system

$$\left.\begin{array}{l} \varphi_{1s}^\gamma - \Delta\varphi_1^\gamma = \gamma^{\mu_1}|\nabla\varphi_1^\gamma|^{q_1} + (\varphi_2^\gamma)^{p_1}, \\ \varphi_{2s}^\gamma - \Delta\varphi_2^\gamma = \gamma^{\mu_2}|\nabla\varphi_2^\gamma|^{q_2} + (\varphi_1^\gamma)^{p_2}, \end{array}\right\} \tag{5.37}$$

where $\mu_1 = 2\alpha + 2 - (2\alpha+1)q_1,\ \mu_2 = 2\beta + 2 - (2\beta+1)q_2$.

From the assumption (2), it follows that $\mu_1, \mu_2 > 0$.

Clearly,

$$\varphi_{1s}^\gamma = \gamma^{2\alpha+2}u, \quad \nabla\varphi_1^\gamma = \gamma^{2\alpha+1}\nabla u, \quad \Delta\varphi_1^\gamma = \gamma^{2\alpha+2}\Delta u. \tag{5.38}$$

From (5.24), (5.38), it follows

$$\frac{1}{\gamma^{(2\alpha+2)}}\varphi_{1s}^\gamma = \frac{1}{\gamma^{(2\alpha+2)}}\Delta\varphi_1^\gamma + \frac{1}{\gamma^{q_1(2\alpha+1)}}|\nabla\varphi_1^\gamma|^{q_1} + \frac{1}{\gamma^{2p_1\beta}}(\varphi_2^\gamma)^{p_1}.$$

Multiply the last equation by $\gamma^{(2\alpha+2)}$, we get the first equation of (5.37). In the same way we can show that φ_2^γ satisfies the second equation in system (5.37).

Restrict s to $s \in (-\gamma^{-2}t^*, 0]$, our aim now is to show that

$$\varphi_1^\gamma(y,s) + |\nabla\varphi_1^\gamma(y,s)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \le 1, \tag{5.39}$$

for $(y,s) \in \Omega_\gamma \times (-\gamma^{-2}t^*, 0]$.

From (5.38), we obtain

$$\begin{aligned} |\nabla\varphi_1^\gamma(y,s)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} &= \gamma^{[\frac{2(p_1+1)}{p_1p_2-1}+1][\frac{2(p_1+1)}{p_1p_2+2p_1+1}]}|\nabla u|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}}, \\ &= \gamma^{2\alpha}|\nabla u|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}}. \end{aligned} \tag{5.40}$$

Clearly,

$$u(x,t) + |\nabla u(x,t)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \le M_u(t_0), \quad (x,t) \in \Omega \times (0, t^*]. \tag{5.41}$$

From (5.35), (5.40) and (5.41), we get (5.39).

Moreover,

$$\varphi_2^\gamma + |\nabla\varphi_2^\gamma|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}} \le M_u^{-\frac{\beta}{\alpha}}(t_0)M_v(t_0), \tag{5.42}$$

for $(y,s) \in \Omega_\gamma \times (-\gamma^{-2}t^*, 0]$.

On the other hand, from (5.34), we obtain

$$\varphi_1^\gamma(0,0) + |\nabla\varphi_1^\gamma(0,0)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \geq \frac{1}{2}. \tag{5.43}$$

If M_v diverges as $t \to T$ we can proceed in the same way by changing the role of u and v.

Step 2: Schauder's estimates

We need interior Schauder's estimates of the functions φ_1, φ_2 on the sets

$$S_K = \{y \in \Omega_\gamma, |y| \leq K\} \times [-K, KL], \quad K > 0,\ L = 0, 1.$$

Assume that φ_1, φ_2 satisfy in S_{2K} the condition

$$0 \leq \varphi_1^\gamma + |\nabla\varphi_1^\gamma|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \leq B, \quad 0 \leq \varphi_2^\gamma + |\nabla\varphi_2^\gamma|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}} \leq B. \tag{5.44}$$

We claim that for any $K > 0, B > 0$ and $\sigma > 0$ small enough, there is a constant $C = C(K, B, \sigma)$ such that

$$||\varphi_1^\gamma||_{C^{2+\sigma,1+\frac{\sigma}{2}}(S_K)} \leq C, \quad ||\varphi_2^\gamma||_{C^{2+\sigma,1+\frac{\sigma}{2}}(S_K)} \leq C. \tag{5.45}$$

From (5.44) we deduce that each of $\varphi_1^\gamma, \varphi_2^\gamma, \nabla\varphi_1^\gamma, \nabla\varphi_2^\gamma$, is uniformly bounded function in S_{2K}. Therefore, the functions $(\varphi_1^\gamma)^{p_1}, (\varphi_2^\gamma)^{p_2}, |\nabla\varphi_1^\gamma|^{q_1}, |\nabla\varphi_2^\gamma|^{q_2}$ are uniformly bounded in S_{2K}. So, the right hand sides of the two equations in (5.37) are uniformly bounded functions in S_{2K}, applying the interior requlàrity theory (see [40]), we obtain (locally) uniform estimates in $C^{1+\sigma,\frac{1+\sigma}{2}}$-norms. Consequently, by Lemma A.2.2, we obtian (locally) uniform estimates in Hölder norms $C^{\sigma,\frac{\sigma}{2}}$ on the right hand sides of the both equations in (5.37).Thus the parabolic interior Schauder's estimates imply (5.45) (see [22, 40]).

Step 3: The proof of (5.33)

Suppose that this lower bound were false.Then there exist a sequence $t_j \to T$ such that

$$M_u^{-\frac{1}{2\alpha}}(t_j) M_v^{\frac{1}{2\beta}}(t_j) \longrightarrow 0, \quad \text{as } j \to \infty. \tag{5.46}$$

Then clearly M_u diverges as $t_j \to T$. For each t_j in the role of t_0 from Step 1, we scale about the correspoinding point (x_j^*, t_j^*) for all j such that $t_j^* \leq t_j$, with the scaling factor

$$\gamma_j = \gamma(t_j) = M_u^{-\frac{1}{2\alpha}}(t_j).$$

We obtain the corresponding rescaled solution $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$,

$$\varphi_1^{\gamma_j}(y,s) = \gamma_j^{2\alpha} u(\gamma_j y + x_j^*, \gamma_j^2 s + t_j^*), \tag{5.47}$$

$$\varphi_2^{\gamma_j}(y,s) = \gamma_j^{2\beta} v(\gamma_j y + x_j^*, \gamma_j^2 s + t_j^*). \tag{5.48}$$

Clearly, $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$ satisfies (as in Step 1) the following problem

$$\left.\begin{aligned} \varphi_{1s}^{\gamma_j} - \Delta\varphi_1^{\gamma_j} &= \gamma_j^{\mu_1}|\nabla\varphi_1^{\gamma_j}|^{q_1} + (\varphi_2^{\gamma_j})^{p_1}, \\ \varphi_{2s}^{\gamma_j} - \Delta\varphi_2^{\gamma_j} &= \gamma_j^{\mu_2}|\nabla\varphi_2^{\gamma_j}|^{q_2} + (\varphi_1^{\gamma_j})^{p_2}, \end{aligned}\right\} \tag{5.49}$$

with

$$\left.\begin{aligned} &\varphi_1^{\gamma_j}(0,0) + |\nabla\varphi_1^{\gamma_j}(0,0)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \geq 1/2, \\ &0 \leq \varphi_1^{\gamma_j} + |\nabla\varphi_1^{\gamma_j}|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \leq 1, \\ &\varphi_2^{\gamma_j} + |\nabla\varphi_2^{\gamma_j}|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}} \leq M_u^{-\frac{\beta}{\alpha}}(t_j)M_v(t_j), \end{aligned}\right\} \tag{5.50}$$

for $(y,s) \in \Omega_{\gamma_j} \times (-\gamma_j^{-2}t_j^*, 0]$, where

$$\Omega_{\gamma_j} := \begin{cases} R^n & \text{if } \ \Omega = R^n, \\ B_{\frac{R}{\gamma_j}} & \text{if } \ \Omega = B_R. \end{cases}$$

Clearly,

$$\Omega_{\gamma_j} \longrightarrow R^n, \quad \text{as } j \to \infty.$$

From (5.46), (5.50), we see that

$$\varphi_2^{\gamma_j} + |\nabla\varphi_2^{\gamma_j}|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}} \longrightarrow 0, \quad \text{as } j \to \infty.$$

Thus $\varphi_2^{\gamma_j}, \nabla\varphi_2^{\gamma_j}$ are bounded in $\Omega_{\gamma_j} \times (-\gamma_j^{-2}t_j^*, 0]$, $\forall j$.

Using the uniform Schauder's estimate derived in Step 2 to $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$

$$||\varphi_1^{\gamma_j}||_{C^{2+\sigma,1+\frac{\sigma}{2}}(\{y\in\Omega_{\gamma_j},|y|\leq K\}\times[-K,0])} \leq C_K,$$

$$||\varphi_2^{\gamma_j}||_{C^{2+\sigma,1+\frac{\sigma}{2}}(\{y\in\Omega_{\gamma_j},|y|\leq K\}\times[-K,0])} \leq C_K,$$

where C_K is independent of j.

Since $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$ is defined on a compact set, by the Arzela-Ascoli theorem, there exist a convergent subsequance, we still denote it by $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$.

Since $\mu_1, \mu_2 > 0$ and $\nabla\varphi_1^{\gamma_j}, \nabla\varphi_2^{\gamma_j}$ are bounded, it follows that, the limit point (φ_1, φ_2) is a solution of the following system

$$\varphi_{1s} = \Delta\varphi_1 + \varphi_2^{p_1}, \quad \varphi_{2s} = \Delta\varphi_2 + \varphi_1^{p_2}, \quad \text{in} \quad R^n \times (-\infty, 0]. \tag{5.51}$$

Since $\varphi_2^{\gamma_j} \to 0,$ as $j \to \infty$, it follows that $\varphi_2 \equiv 0,$ in $R^n \times (-\infty, 0]$.

Consequently, from the second equation in (5.51), we obtain that

$$\varphi_1 \equiv 0, \quad \text{in} \quad R^n \times (-\infty, 0].$$

This means

$$\varphi_1(0,0) + |\nabla\varphi_1(0,0)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} = 0,$$

which contradicts with (5.50). Thus, the lower bound is held.

To prove the upper bound of (5.33) we proceed similarly as in the proof of lower bound with changing the role of u and v.

Step 4: Estimate on doubling of M_u

As M_u is continuous and diverges as $t \to T$, for any $t_0 \in (0, T)$ we define t_0^+ by

$$t_0^+ = \max\{t \in (t_0, T) : M_u(t) = 2M_u(t_0)\}.$$

Clearly,

$$u(x,t) + |\nabla u(x,t)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \leq 2M_u(t_0), \quad (x,t) \in \Omega \times (0, t_0^+]. \tag{5.52}$$

Take $\gamma = \gamma(t_0) = M_u^{-\frac{1}{2\alpha}}(t_0)$.

We claim that

$$\gamma^{-2}(t_0)(t_0^+ - t_0) \leq A, \quad t_0 \in (\frac{T}{2}, T),$$

where the constant $A \in (0, \infty)$ is independent of t_0. Suppose that this estimate were false, then there would exist a sequence $t_j \to T$ such that

$$\gamma_j^{-2}(t_j)(t_j^+ - t_j) \to \infty,$$

where

$$t_j^+ = \max\{t \in (t_j, T) : M_u(t) = 2M_u(t_j)\}. \tag{5.53}$$

For each t_j we scale about the corresponding point (x_j^*, t_j^*) such that

$$0 < t_j^* \leq t_j, \quad \frac{T}{2} < t_j < t_j^+ < T, \quad \forall j$$

with the scaling factor

$$\gamma_j = \gamma(t_j) = M_u^{-\frac{1}{2\alpha}}(t_j).$$

As in Step 3, we obtain the corresponding rescaled functions $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$, which satisfies (5.49) with the following conditions

$$\left.\begin{array}{l} \varphi_1^{\gamma_j}(0,0) + |\nabla\varphi_1^{\gamma_j}(0,0)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \geq 1/2, \\ 0 \leq \varphi_1^{\gamma_j} + |\nabla\varphi_1^{\gamma_j}|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \leq 2, \\ \varphi_2^{\gamma_j} + |\nabla\varphi_2^{\gamma_j}|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}} \leq M_u^{-\frac{\beta}{\alpha}}(t_j)M_v(t_j^+), \end{array}\right\} \tag{5.54}$$

for $(y,s) \in \Omega_{\gamma_j} \times (-\gamma_j^{-2}t^*, \gamma_j^{-2}(t_j^+ - t_j^*)]$.

From (5.53) and (5.54), it follows that

$$\varphi_2^{\gamma_j} + |\nabla\varphi_2^{\gamma_j}|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}} \leq 2^{\frac{\beta}{\alpha}} M_u^{-\frac{\beta}{\alpha}}(t_j^+)M_v(t_j^+). \tag{5.55}$$

From (5.33), we have

$$M_v(t) \leq \delta^{-2\beta} M_u^{\frac{\beta}{\alpha}}(t), \quad t \in (\frac{T}{2}, T).$$

Therefore, (5.55) becomes

$$\varphi_2^{\gamma_j} + |\nabla\varphi_2^{\gamma_j}|^{\frac{2(p_2+1)}{p_1p_2+2p_2+2}} \leq \frac{2^{\frac{\beta}{\alpha}}}{\delta^{2\beta}}.$$

By using the Schauder's estimates derived in Step 2 for $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$ we get a convergent subseguence in $C_{loc}^{2+\sigma,1+\sigma/2}(R^n \times R)$ to the solution of system (5.51) in $R^n \times R$. This is a contradiction because all the nontrival positive solutions of system (5.51), under the assumption (1), blow up in finite time (see [13]).

Thus, there is $A > 0$ such that

$$\gamma^{-2}(t_0)(t_0^+ - t_0) \leq A, \quad t_0 \in (\frac{T}{2}, T). \tag{5.56}$$

Step 5: Rate estimates

As in Step 4, for any $t_0 \in (T/2, T)$ we define

$$t_1 = t_0^+ \in (t_0, T) \quad \text{such that} \quad M_u(t_1) = 2M_u(t_0).$$

Due to (5.56),

$$(t_1 - t_0) \le AM_u^{-\frac{1}{\alpha}}(t_0).$$

We can use t_1 as a new t_0 and obtain $t_2 \in (t, T)$ such that

$$M_u(t_2) = 2M_u(t_1) = 4M_u(t_0),$$
$$(t_2 - t_1) \le AM_u^{-\frac{1}{\alpha}}(t_1) = 2^{-\frac{1}{\alpha}} AM_u^{-\frac{1}{\alpha}}(t_0).$$

Continuing this process we obtain a sequence $t_j \to T$ such that

$$(t_{j+1} - t_j) \le 2^{-\frac{j}{\alpha}} AM_u^{-\frac{1}{\alpha}}(t_0), \quad j = 0, 1, 2, \ldots$$

If we add these inequalities we get

$$(T - t_0) \le \sum_{j \ge 0} 2^{-\frac{j}{\alpha}} AM_u^{-\frac{1}{\alpha}}(t_0).$$

Thus

$$(T - t_0) \le (1 - 2^{-\frac{1}{\alpha}})^{-1} AM_u^{-\frac{1}{\alpha}}(t_0)$$

From using (5.33) we obtain

$$M_v(t_0) \le \delta^{-2\beta} M_u^{\frac{\beta}{\alpha}}(t_0), \; t_0 \in (T/2, T).$$

Thus

$$M_v(t_0) \le \delta^{-2\beta}(1 - 2^{-\frac{1}{\alpha}})^{-\beta} A^{\beta}(T - t_0)^{-\beta}, \quad t_0 \in (T/2, T).$$

From above there exist two constants C_1^*, C_2^* such that

$$M_u(t_0) \le C_1^*(T - t_0)^{-\alpha}, \quad t_0 \in (\frac{T}{2}, T),$$
$$M_v(t_0) \le C_2^*(T - t_0)^{-\beta}, \quad t_0 \in (\frac{T}{2}, T).$$

From the last two equations and the definitions of M_u, M_v, it follows that there exist constants C_1, C_2 such that

$$u(x,t) + |\nabla u(x,t)|^{\frac{2(p_1+1)}{p_1p_2+2p_1+1}} \le C_1(T - t)^{-\alpha},$$
$$v(x,t) + |\nabla v(x,t)|^{\frac{2(p_2+1)}{p_1p_2+2p_2+1}} \le C_2(T - t)^{-\beta},$$

for $(x,t) \in \Omega \times (0, T)$. $\square$

Remark 5.2.4. If $u_0 \equiv v_0$, $p = p_1 = p_2$, $q = q_1 = q_2$, then problem (5.24) can be reduced to a scalar Dirichlet (Cauchy) problem for (5.27). Moreover, if

$$1 < p \leq 1 + \frac{2}{n}, \quad 1 < q < \frac{2p}{1+p}, \tag{5.57}$$

then in a similar way to the proof of Theorem 5.2.3, we can show that, for a nontrivial positive blow-up solution u, there exist $C > 0$ such that

$$u(x,t) + |\nabla u(x,t)|^{\frac{2}{p+1}} \leq C(T-t)^{\frac{1}{p-1}}, \quad \text{in} \quad \Omega \times (0,T), \tag{5.58}$$

i.e.

$$u(x,t) \leq C(T-t)^{\frac{1}{p-1}}, \quad \text{in} \quad \Omega \times (0,T). \tag{5.59}$$

A similar estimate to (5.58) has been shown in [53, 56] to a large class of semilinear heat equations with gradient terms including (5.6) and (5.27). As we have mentioned before, the rate estimate (5.59) is also known for the blow-up solutions of equations (5.6) and (5.26). Therefore, if p, q satisfy (5.57), then the positive and negative gradient terms which appear in equation (5.27) and (5.6), respectively, do not affect the blow-up rate estimates of these problems. A similar observation holds for problem (5.24) by Theorem 5.2.3, which shows that the upper rate estimates of the Cauchy or Dirichlet problem for system (5.24) are the same as those known for the system (5.28). Therefore, under the assumptions of Theorem 5.2.3, the gradient terms in system (5.24) have no effect on the blow-up rate estimates.

5.2.3 Blow-up Set

It is well known that for the semilinear system (5.28) defined in a ball, under some restricted assumptions on u_0, v_0 (nonnegative, radially decreasing functions), that the only blow-up point is the centre of that ball (see [61]), while it is unknown whether this holds for the system (5.24). However, for the radial solutions of the single equation (5.27) defined in Ω, in case $q = 2$, there is global blow-up, if $1 < p < 2$, $\Omega = B_R$ or R^n, and regional blow-up, if $p = 2$, $\Omega = R^n$, while a single blow-up point, if $p > 2$, $\Omega = B_R$ (see [56, 62] and the references therein).The proof relies on the transformation $v = e^u - 1$, which converts

(5.27) into the semilinear heat equation $v_t = \Delta v + (1+v)\log^p(1+v)$. We note that, these results are much different from those known for equation (5.26) (see [56]), because for any $p > 1$, $\Omega = B_R$ or R^n, only a single blow-up point is known to occur for that problem, where the initial date are nonnegative, radially nonincreasing and bounded function.

Chapter 6

Conclusions and Further Studies

In this thesis, we studied the blow-up properties of second order parabolic problems defined in a ball. Namely, we consider the nature of blow-up set and the rate of blow-up for some problems of the following types:

1. Dirichlet problems for semilinear heat equations,
2. Neumann problems for heat (semilinear heat) equations,
3. Dirichlet (Cauchy) problems for semilinear heat equations with gradient terms.

From this work, we can conclude the following points

- For the Dirichlet problem for the semilinear heat equation (2.1), with nonnegative radially nonincreasing initial data, where the reaction term is the power or the exponential function, it has been shown in [24] that the only possible blow-up point is the centre of the ball. This can be extended to the case where the reaction term grows faster than these types of functions for large values of solutions. Namely, where the reaction term is the exponential of a power type function. Moreover, for this case the upper blow-up rate estimate obtained in Chapter 1, is the same as that known for problem (2.1) where the reaction term is the exponential function of solutions. Similarly, the last conclusion holds for the problem of the heat equation with a nonlinear boundary condition (3.1), while for the system of two heat equations with coupled nonlinear boundary conditions (3.7), in case of the boundary terms are of this type of nonlinearity, the upper blow-up rate estimates obtained in Chapter 3 are greater (more singular) than those known for problem (3.7) where the boundary terms are of exponential type functions of the solutions, but they are less (less singular) than those known for problem (3.7) where the boundary terms are of power type functions. Furthermore, for this case, as in the previous studied cases, the blow-up occurs only on the boundary.

- For the Neumann problem for the semilinear heat equation (4.6), we showed that the presence of the reaction term has an important effect on the upper (lower) blow-up rate estimates in case of the power p of the exponential function that appears in the reaction term is larger than the power q of that appears in the boundary term, otherwise the effect occurs only on the upper bound. Moreover, for the special case $p = q = 1$, and for small enough values of λ, that appears in the reaction term, the blow-up can occur only on the boundary, this means in this case, the reaction term has no effect on the blow-up set. In fact the last conclusion can be extended to the system (4.15), which is coupled in both equations and boundary conditions. Moreover, we conclude that the upper blow-up rate estimates for system (4.15) take the same forms as those considered in Chapter 2 for the Dirichlet problem for this system, while the lower estimates are the same as those known for the problem where the reaction terms are absent.

- For the Dirichlet problem for the semilinear heat equation with negative sign quadratic gradient term (5.7), and nonnegative radially nonincreasing initial data, we showed that the gradient term has no effect neither on the poistwise estimate nor on the blow-up rate estimates for this problem. In other words, these bounds depend only on the exponential function, that appears in the semilinear equation (5.7). A similar conclusion also holds for the Dirichlet (Cauchy) problem for the coupled system of two semilinear heat equations with positive sign gradient terms (5.24). Under certain assumptions on the powers of the nonlinear functions which appear in the equations of the system (5.24), we showed that the upper blow-up rate estimates are the same as those known for the problem where the gradient terms are absent.

We now outline possible directions for future studies

- One may try to find formulas to the blow-up rate estimates and study the blow-up set for the coupled system (2.24), where the reaction terms take the forms as in the scalar problem (2.6).
- It would be interesting to investigate whether, for large values of the parameter λ, or for any $p, q > 0$, which appear in problem (4.6), the blow-up can occur only on the boundary. A similar question can be asked for the system (4.15).
- The blow-up rate estimates (5.31) and (5.32) have been derived under restricted assumptions. We may try to study the blow-up rate estimates for problem (5.24), in case of one or both of the assumptions (1) and (2) of Theorem 5.2.3 are not satisfied.
- It is well known for the Dirichlet problem for the system (5.24) with radially nonincreasing initial data, where the gradient terms are absent, that the blow-up set has only a single point (see [61]). Therefore, it would be really interesting to investigate whether this can be extended to this problem, where the gradient terms are present.

Appendix A

Notation and Definitions

In this appendix we introduce the domains notation and symbols used in this thesis. Furthermore, we review the standard function spaces and the definitions of radial and superlinear functions. Moreover, the definition of uniformly parabolic equations is given in this appendix. Finally, we recall the meaning of maximal classical and weak solutions of parabolic problems.

A.1 Notation for Domains

Let $\Omega \subseteq R^n$, we say that Ω is a *domain*, if it is a nonempty, connected, open set, we refer to the boundary of Ω by $\partial\Omega$ and to its closure by $\overline{\Omega}$. The unit outward normal vector on $\partial\Omega$ at the point $x \in \partial\Omega$ is denoted by $\eta = \eta(x)$, and the outer normal derivative by $\frac{\partial}{\partial\eta}$.

Definition A.1.1. We say that $\partial\Omega$ is C^k, if for each point $x_0 \in \partial\Omega$ there exist $r > 0$ and a C^k function $\gamma : R^{n-1} \to R$ such that

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) \mid x_n > \gamma(x_1, \ldots, x_{n-1})\},$$

where $B(x_0, r)$ is a ball in R^n with centre x_0 and radius r.

Likewise, $\partial\Omega$ is smooth, if the function γ is smooth (infinitely differentiable).

We denote by B_R the open ball in R^n with centre zero and radius R, namely

$$B_R := \{x \in R^n : |x| < R\},$$

where

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Moreover, we refer to the boundary of B_R by ∂B_R or S_R, which is defined as follows

$$\partial B_R := \{x \in R^n : |x| = R\}.$$

The surface measure on ∂B_R will be denoted by ds.

Definition A.1.2. We say that a domain Ω is *symmetric*, if either $\Omega = R^n$, or $\Omega = B_R$, or $\Omega = \{x \in R^n : R_1 < |x| < R_2\}$, where $0 < R_1 < R_2 \leq \infty$.

A.2 Notation for Functions

Throughout this section we introduce the notation for the functions, which defined in the domain, $\Omega \times I \subset R^{n+1}$, where Ω be a bounded domain in R^n, $I \subset R$. Similarly, we can define the same notations for the functions, which are defined in the domain Ω, so they are omitted here.

A.2.1 Function Spaces

Let $\alpha \in (0, 1)$, $k \in Z^+$, we define some classical and parabolic function spaces:

Uniform space:

$$L^\infty(\Omega \times I) := \{u : \Omega \times I \to R \mid ||u||_\infty < \infty\},$$

where

$$||u||_\infty = \sup_{(x,t)\in\Omega\times I} |u(x,t)|.$$

L$^{\mathbf{p}}$-spaces:

For $1 \leq p < \infty$, define

$$L^p(\Omega \times I) := \{u : \Omega \times I \to R \mid u \text{ is measurable and } ||u||_p < \infty\},$$

where

$$||u||_p = (\int_I \int_\Omega |u|^p dxdt)^{\frac{1}{p}}.$$

Hölder spaces:

$$C^{\alpha,\frac{\alpha}{2}}(\Omega \times I) := \{u \in C(\Omega \times I) \mid [u]_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times I)} < \infty\},$$

where

$$[u]_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times I)} := \sup_{x,y\in\Omega,x\neq y;t,s\in I,t\neq s} \frac{|u(x,t)-u(y,s)|}{|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}}}.$$

Moreover, the space $C^{\alpha,\frac{\alpha}{2}}((\overline{\Omega} \times \overline{I}))$ can be equipped with the norm

$$||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times\overline{I})} := ||u||_\infty + [u]_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times\overline{I})}.$$

Lemma A.2.1. *Let $u \in C^{\beta,\frac{\beta}{2}}(\Omega \times I)$, where $\alpha < \beta \leq 1$. Then*

$$u \in C^{\alpha,\frac{\alpha}{2}}(\Omega \times I).$$

Lemma A.2.2. *Let $u \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times\overline{I})$, where $\Omega \subset R^n$ be an open convex bounded set and $p > 1$. Then*

$$u^p \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times \overline{I}).$$

$\mathbf{C^{k,\frac{k}{2}}}$-spaces:

The two spaces $C^{1,\frac{1}{2}}, C^{2,1}$ are defined as follows

$C^{1,\frac{1}{2}}(\Omega\times I) := \{u : \Omega\times I \to R \mid u \text{ is } C^{\frac{1}{2}} \text{ in } t,\ u_{x_i}\text{exist and continuous}, i = 1, 2, ...n\},$

$C^{2,1}(\Omega\times I) := \{u : \Omega\times I \to R \mid u_{x_i}, u_{x_i x_j}, \text{and } u_t \text{ exist and continuous}, \ i, j = 1, 2, ...n\},$

Assuming that u and its partial derivatives are continuous on $\overline{\Omega}\times\overline{I}$, the spaces $C^{1,\frac{1}{2}}(\overline{\Omega} \times \overline{I}), C^{2,1}(\overline{\Omega} \times \overline{I})$ can be equipped with the norms

$$||u||_{C^{1,\frac{1}{2}}(\overline{\Omega}\times\overline{I})} := ||u||_\infty + \sum_{i=1}^n ||u_{x_i}||_\infty,$$

$$||u||_{C^{2,1}(\overline{\Omega}\times\overline{I})} := ||u||_\infty + ||u_t||_\infty + \sum_{i=1}^n ||u_{x_i}||_\infty + \sum_{i,j=1}^n ||u_{x_i x_j}||_\infty,$$

respectively.

$\mathbf{C^{k+\alpha,\frac{k+\alpha}{2}}}$**-spaces:**

The spaces $C^{1+\alpha,\frac{1+\alpha}{2}}$, $C^{2+\alpha,1+\frac{\alpha}{2}}$ can be defined as follows:

$$C^{1+\alpha,\frac{1+\alpha}{2}}(\Omega\times I) =: \{u \mid u \text{ is } C^{\frac{1+\alpha}{2}} \text{ in } t, u_{x_i} \in C^{\alpha,\frac{\alpha}{2}}(\Omega\times I),\ i=1,2,...n\},$$
$$C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times I) =: \{u \in C^{2,1}(\Omega\times I) \mid u_{x_ix_j}, u_t \in C^{\alpha,\frac{\alpha}{2}}(\Omega\times I),\ i,j=1,2,...n\}.$$

Moreover, the spaces $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega}\times\overline{I})$, $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times\overline{I})$ are equipped with the norms

$$||u||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega}\times\overline{I})} := ||u||_{C^{1,\frac{1}{2}}(\overline{\Omega}\times\overline{I})} + \sum_{i=1}^{n}[u_{x_i}]_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times\overline{I})},$$
$$||u||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times\overline{I})} := ||u||_{C^{2,1}(\overline{\Omega}\times\overline{I})} + [u_t]_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times\overline{I})} + \sum_{i,j=1}^{n}[u_{x_ix_j}]_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times\overline{I})}.$$

$\mathbf{C(I,X(\Omega))}$**-space:**

Define

$$C(I,X(\Omega)) := \{u : I \to X(\Omega) \mid u \text{ is continuous}\},$$

where X is a Banach space of functions defined in Ω, such as: L^p, L^∞, C^k.

A.2.2 Superlinear Functions

Definition A.2.3. Let $f = f(u)$, where $f : [0,\infty) \to R$. f is said to be *superlinear*, if it is non dissipative and grow larger than linearly for large values of u. That is, there exist $\varepsilon, A > 0$ such that

$$uf(u) \geq (2+\varepsilon)\int_0^u f(v)dv > 0, \quad \forall\ u \geq A.$$

A.2.3 Radial Functions

Definition A.2.4. Let Ω be a symmetric domain. The function $u : \Omega\times I \to R$, is called *radially symmetric* or simply *radial*, if it satisfies, for each $(x,t) \in \Omega\times I$,

$$u(x,t) = u(x^{'},t), \quad \forall x^{'} \in \Omega, \quad \text{such that} \quad |x^{'}| = |x|.$$

Moreover, it is called *radially nonincreasing* if u is radial and nonincreasing as a function of $r = |x|$.

Lemma A.2.5. *u is radial if and only if*

$$u(x,t) = u((|x|,0,0,\ldots,0),t), \quad \forall (x,t) \in \Omega \times I.$$

A.3 Uniformly Parabolic Equations

Consider the differential equation

$$u_t = \sum_{i,j=1}^{n} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + f(x,t,u,\nabla u), \quad (x,t) \in \Omega \times (0,T), \tag{A.1}$$

where $a_{i,j}, i,j = 1,2,\ldots n$ are defined functions in $\Omega \times (0,T)$, f is defined function in $\Omega \times (0,T) \times R \times R^n$. If the matrix $(a_{i,j})$ is positive definite in $\Omega \times (0,T)$; that is, for every vector $\xi = (\xi_1,\ldots,\xi_n) \in R^n, \xi \neq 0$,

$$\sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j > 0, \quad (x,t) \in \Omega \times (0,T),$$

then we say that (A.1) is of *parabolic type* in $\Omega \times (0,T)$. Moreover, if there exist positive constants λ_1, λ_2 such that, for every vector $\xi = (\xi_1,\ldots,\xi_n) \in R^n$,

$$\lambda_1|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \leq \lambda_2|\xi|^2, \quad (x,t) \in \Omega \times (0,T),$$

then we say that (A.1) is *uniformly parabolic* in $\Omega \times (0,T)$. Similarly, we can define the uniformly elliptic equations.

A.4 Classical and Weak Solutions

For any second order parabolic problem defined in $\{x \in \Omega, t > 0\}$, with the initial function $u_0 \in C^2(\overline{\Omega})$ and for given $T \in (0,\infty]$, by $u \in C([0,T), C^2(\overline{\Omega}))$ is a *classical solution* or a *solution* (for short) in $[0,T)$, we mean that u satisfies the problem for $t \in (0,T)$, $u(\cdot,0) = u_0$ and

$$u \in C^{2,1}(\overline{\Omega} \times [0,T)).$$

If Ω is unbounded, then we also require that $u, \nabla u, \Delta u$ and u_t are bounded on $\overline{\Omega} \times [0, t]$, for every $t < T$.

Moreover, we say that the problem is *well-posed* in $C^2(\overline{\Omega})$ if, for given $u_0 \in C^2(\overline{\Omega})$, there exist $T > 0$ and a unique classical solution in $[0, T]$.

A.4.1 Maximal Solutions

Definition A.4.1. Suppose that we have a parabolic problem such that for each $u_0 \in C^2(\overline{\Omega})$, there exist a unique classical solution u on the interval $[0, T]$, where $T = T(||u_0||_{C^2(\overline{\Omega})})$. If there exist $T_{\max} = T_{\max}(u_0) \in (T, \infty]$ with the following properties:

(i) The solution u can be continued (in a unique way) to a classical solution on the interval $[0, T_{\max})$,

(ii) If $T_{\max} < \infty$, then u cannot be continued to a classical solution on $[0, \tau)$ for any $\tau > T_{\max}$,

(iii) Either $T_{\max} = \infty$ or $\lim_{t \to T_{\max}} ||u(x, t)||_{C^2(\overline{\Omega})} = \infty$,

then we call u the *maximal classical solution* starting from u_0 and $T_{\max}$ its *maximal existence time*.

A.4.2 Weak Solutions

By *weak solutions* of parabolic problems, we mean the functions, which may not be continuously differentiable or even continuous, but they satisfy the problem in weak sense. For example, consider the following problem

$$\left.\begin{array}{ll} u_t = \Delta u + f(u), & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{array}\right\} \tag{A.2}$$

where $0 \leq u_0 \in L^\infty(\Omega)$, Ω is a bounded domain.

Definition A.4.2. The function u is a *weak solution* of (A.2) on $[0,T]$ if

(i) $u \in C([0,T], L^1(\Omega))$,

(ii) $f(u) \in L^1(\Omega \times (0,T))$,

(iii) $$\int_\Omega u(x,t_2)\phi(x,t_2)dx - \int_\Omega u(x,t_1)\phi(x,t_1)dx - \int_{t_1}^{t_2}\int_\Omega u\phi_t dxdt$$
$$= \int_{t_1}^{t_2}\int_\Omega (u\Delta\phi + f\phi)dxdt$$

for every $\phi \in C^{2,1}(\overline{\Omega} \times [0,T])$ with $\phi = 0$ on $\partial\Omega,\ 0 \leq t_1 \leq t_2 \leq T$.

The function u is a *global weak solution*, if it is a weak solution on $[0,T]$ for every $T > 0$.

Appendix B

Maximum and Comparison Principles

Maximum and comparison principles are considered a very useful tool in the study of parabolic problems of scalar equations and systems. Many of the arguments applied in this thesis rely on application of the maximum principles for parabolic equations. In this appendix we recall from ([26, 34, 50, 54, 56, 58]) some maximum and comparison principles, which we frequently use in this thesis.

B.1 Maximum and Comparison Principles for Parabolic Equations

We start with the following maximum principle, which is applicable to the classical solutions of the problems of Dirichlet, Neumann and mixed-type boundary type conditions.

Proposition B.1.1. Let $u \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T))$ be such that

$$\left.\begin{array}{ll} u_t - Lu + cu \geq 0, & (x,t) \in \Omega \times (0,T), \\ \alpha \frac{\partial u}{\partial \eta} + \beta u \geq 0, & (x,t) \in \partial\Omega, \\ u(x,0) \geq 0, & x \in \Omega, \end{array}\right\}$$

where Ω is a bounded domain, L is a uniformly elliptic operator given by

$$Lu \equiv \sum_{i,j=1}^{n} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j},$$

where $a_{i,j}, b_j, i,j = 1,2,\ldots n$, and c are continuos functions in $\Omega \times (0,T)$, moreover, c is bounded in $\Omega \times (0,t]$ for any $t < T$, and α, β are nonnegative continuous functions, such that $\alpha + \beta > 0$ on $\partial\Omega \times (0,T)$. Then

$$u(x,t) \geq 0, \quad (x,t) \in \overline{\Omega} \times (0,T).$$

Moreover

$$u(x,t) > 0, \quad (x,t) \in \Omega \times (0,T) \quad \text{unless} \quad u \equiv 0.$$

As an application of Proposition B.1.1 we have the following comparison principle between the classical solutions u, v of the respective parabolic initial-boundary value problems

$$\left.\begin{array}{lll} u_t - Lu = f_1(x,t,u), & v_t - Lv = f_2(x,t,v), & (x,t) \in \Omega \times (0,T), \\ \alpha\frac{\partial u}{\partial \eta} + \beta u = h_1(x,t), & \alpha\frac{\partial v}{\partial \eta} + \beta v = h_2(x,t), & (x,t) \in \partial\Omega, \\ u(x,0) = u_0, & v(x,0) = v_0, & x \in \Omega, \end{array}\right\} \tag{B.1}$$

where f_1, f_2 are continuous functions in $\Omega \times (0,T)$, u_0, v_0 are smooth function, α, β and L are defined as in Proposition B.1.1.

Proposition B.1.2. Assume that either $\frac{\partial f_1(x,t,s)}{\partial s}$ or $\frac{\partial f_2(x,t,s)}{\partial s}$ is continuous in $s \in R$ and that

$$\left.\begin{array}{ll} f_1(x,t,s) \leq f_2(x,t,s), & (x,t) \in \Omega \times (0,T), \\ h_1(x,t) \leq h_2(x,t), & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) \leq v(x,0), & x \in \Omega. \end{array}\right\}$$

If u, v are the respective solution of (B.1), then

$$u \leq v, \quad (x,t) \in \overline{\Omega} \times (0,T).$$

Moreover, either

$$u = v, \quad \text{or} \quad u < v, \quad (x,t) \in \Omega \times (0,T).$$

The following proposition is a basic maximum principle for classical solutions.

Proposition B.1.3. Let Ω be an arbitrary bounded domain in R^n, $T > 0$,

$$b : \Omega \times (0, T) \to R^n, \ c : \Omega \times (0, T) \to R,$$

$$\sup_{\Omega\times(0,t)} c < \infty, \quad \text{for any } t < T.$$

Assume that

$$v \in C^{2,1}(\Omega \times (0, T)) \cap C(\overline{\Omega} \times [0, T)),$$

and

$$\left.\begin{aligned} v_t - \Delta v \leq b \cdot \nabla v + cv, &\qquad (x, t) \in \Omega \times (0, T),\\ v(x, t) \leq 0, &\qquad (x, t) \in \partial\Omega \times (0, T),\\ v(x, 0) \leq 0, &\qquad x \in \Omega. \end{aligned}\right\} \tag{B.2}$$

Then

$$v(x, t) \leq 0, \quad (x, t) \in \Omega \times (0, T).$$

Remark B.1.4. In Proposition B.1.3 it is sufficient to assume that the first inequality in (B.2) holds in the set $\{(x, t) \in \Omega \times (0, T) \mid v(x, t) > 0\}$.

The following proposition is a version of the strong comparison principle for general semilinear parabolic equations.

Proposition B.1.5. Let Ω be a bounded domain in R^n of class C^2. And

$$u, v \in C^{2,1}(\Omega \times (0, T)) \cap C(\overline{\Omega} \times [0, T)),$$

for some $T > 0$. Assume that

$$u_t - \Delta u - F(x, t, u, \nabla u) \leq v_t - \Delta v - F(x, t, v, \nabla v), \quad (x, t) \in \Omega \times (0, T),$$

where $F = F(x, t, s, \xi) : \overline{\Omega} \times [0, T) \times R \times R^n \to R$ is continuous in x, t and C^1 in s, ξ. Moreover, if F depends on ξ, assume also that

$$\nabla u, \nabla v \in L^\infty(\Omega \times (0, t)), \quad \text{for any } t < T.$$

Let

$$u(x, 0) \leq v(x, 0), \quad x \in \Omega \quad (u(\cdot, 0) \not\equiv v(\cdot, 0)),$$

and

$$u(x,t) \leq v(x,t), \quad (x,t) \in \partial\Omega \times (0,T),$$

or

$$\frac{\partial u}{\partial \eta} + bu \leq \frac{\partial v}{\partial \eta} + bv, \quad \text{on } \partial\Omega \times (0,T), \tag{B.3}$$

where $b \in C^1(\partial\Omega)$. Then

$$u < v \quad \text{in} \quad \Omega \times (0,T).$$

In addition, if $u(x_0,t_0) = v(x_0,t_0)$ for some $x_0 \in \partial\Omega$ and $t_0 \in (0,T)$, then

$$\frac{\partial u(x_0,t_0)}{\partial \eta} > \frac{\partial v(x_0,t_0)}{\partial \eta}.$$

If (B.3) is true, then $u < v$ in $\overline{\Omega} \times (0,T)$.

Finally, we state the following comparison principle for (3.1) (the problem of the heat equation with a nonlinear boundary condition).

Proposition B.1.6. Let $u_i \in C^{2,1}(\overline{B}_R \times [0,T_i))$, $i = 1,2$ be solutions of problem (3.1) with initial data u_1^0, u_2^0 and boundary condition given by the functions f_i. Suppose that

$$f_1 \geq f_2, \quad \text{and} \quad u_1^0 > u_2^0, \quad x \in \overline{B}_R.$$

If f_1 or f_2 are strictly increasing, then

$$u_1 > u_2, \quad \text{in} \quad \overline{B}_R \times [0, \min\{T_1, T_2\}).$$

B.2 Maximum and Comparison Principles for Parabolic Systems

We first give extensions of the previous maximum principle (Proposition B.1.3), to systems of cooperative type.

Proposition B.2.1. Let $0 < T < \infty$, Ω be an bounded domain in R^n,

$$a_{ij} : \Omega \times (0,T) \longrightarrow R, \quad \text{for} \quad i,j \in \{1,2\}, \text{ such that } a_{12}, a_{21} \geq 0,$$

$$\sup_{\Omega\times(0,t)} a_{ij} < \infty, \quad \text{for any} \quad t < T, \quad i,j \in \{1,2\},$$

$b_1, b_2 : \Omega \times (0,T) \longrightarrow R^n$. Assume that for $i = 1, 2$, the function v_i satisfies

$$v_i \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T)),$$

$$\left.\begin{array}{ll} v_{1t} - \Delta v_1 + b_1 \cdot \nabla v_1 \leq a_{11}v_1 + a_{12}v_2, & \text{in} \quad \Omega \times (0,T), \\ v_{2t} - \Delta v_2 + b_2 \cdot \nabla v_2 \leq a_{21}v_1 + a_{22}v_2, & \text{in} \quad \Omega \times (0,T), \end{array}\right\}$$

such that

$$\left.\begin{array}{lll} v_1(x,t) \leq 0, & v_2(x,t) \leq 0, & (x,t) \in \partial\Omega \times (0,T), \\ v_1(x,0) \leq 0, & v_2(x,0) \leq 0, & x \in \Omega. \end{array}\right\}$$

Then

$$v_1(x,t), v_2(x,t) \leq 0 \quad (x,t) \in \Omega \times (0,T).$$

Next, we state a comparison principle to the system of heat equations with Neumann boundary conditions.

Proposition B.2.2. Let (u, v) be a nonnegative *supersolution* to problem (3.7), where $u, v \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T))$. This means (u, v) satisfies the following problem

$$\left.\begin{array}{lll} u_t \geq \Delta u, & v_t \geq \Delta v, & B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} \geq f(v), & \frac{\partial v}{\partial \eta} \geq g(u), & (x,t) \in S_R \times (0,T), \\ u(x,0) \geq u_0(x), & v(x,0) \geq v_0(x), & x \in B_R. \end{array}\right\}$$

If (u^*, v^*) is a nonnegative solution of problem (3.7), starting with the same initial data (u_0, v_0), then

$$u^* \leq u, \quad v^* \leq v, \quad (x,t) \in \overline{B}_R \times [0,T).$$

Let (u_1, v_1) be a nonnegative solution of problem (3.7), starting with (u_{10}, v_{10}), where

$$u_{10} < u_0, \quad v_{10} < v_0, \quad x \in \overline{B}_R.$$

Then

$$u_1 < u, \quad v_1 < v, \quad (x,t) \in \overline{B}_R \times [0,T).$$

The following proposition is a comparison principle for cooperative systems of the form

$$u_t = \Delta u + f_1(u, v), \quad v_t = \Delta v + f_2(u, v). \tag{B.4}$$

Proposition B.2.3. Let $0 < T < \infty$, Ω be an arbitrary domain in R^n, and let $f_i = f_i(s_1, s_2) : R^2 \to R, i = 1, 2,$ be C^1 functions such that

$$\frac{\partial f_1}{\partial s_2} \geq 0, \quad \frac{\partial f_2}{\partial s_1} \geq 0.$$

Let

$$u, v, u^*, v^* \in C^{2,1}(\Omega \times (0, T)) \cap C(\overline{\Omega} \times [0, T)),$$

and

$$u \leq u^*, v \leq v^* \quad \text{in} \quad \Omega \times \{0\},\ \partial\Omega \times (0, T),$$

moreover,

$$\left.\begin{array}{ll} u_t - \Delta u - f_1(u, v) \leq u^*_t - \Delta u^* - f_1(u^*, v^*) & \text{in} \quad \Omega \times (0, T), \\ v_t - \Delta v - f_2(u, v) \leq v^*_t - \Delta v^* - f_2(u^*, v^*) & \text{in} \quad \Omega \times (0, T). \end{array}\right\}$$

i.e. (u^*, v^*) is a supersolution to the system (B.4). Then

$$u \leq u^*, v \leq v^* \quad \text{in} \quad \Omega \times (0, T).$$

Finally, we state the following maximum principle for rection diffusion systems coupled in both equations and boundary conditions.

Proposition B.2.4. Let

$$w, z \in C^{2,1}(\Omega \times (0, T)) \cap C(\overline{\Omega} \times [0, T)),$$

where, $\Omega = (0, R)$, $R > 0$ and $T > 0$, such that

$$\left.\begin{array}{lll} w_t - w_{rr} - \frac{n-1}{r} w_r \geq az, & z_t - z_{rr} - \frac{n-1}{r} z_r \geq bw & (r, t) \in \Omega \times (0, T), \\ w_r(0, t) \leq 0, & z_r(0, t) \leq 0, & 0 < t < T, \\ w_r(R, t) \geq c(R, t) z(R, t), & z_r(R, t) \geq d(R, t) w(R, t), & 0 < t < T, \\ w(r, 0) \geq 0, & z(r, 0) \geq 0, & r \in \overline{\Omega}, \end{array}\right\}$$

where, a, b, c and d are bounded functions in $[0, R] \times [0, t]$, for any $t < T$, we assume also that a, b are nonnegative functions. Then

$$w, z \geq 0, \quad \text{in} \quad \overline{\Omega} \times [0, T).$$

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