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Higher Integrability of the Gradient of Conformal Maps

by

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Declaration

I hereby declare that this thesis has not been submitted in whole or in part to this or any other University for the award of a degree.

Signature:

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UNIVERSITY OF SUSSEX

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Abstract

Let ϕ be a conformal map from $\mathbb{D} \subset \mathbb{C}$ onto a simply connected domain Ω, with its inverse $\psi = \phi^{-1} : \Omega \longrightarrow \mathbb{D}$. *Brennan's conjecture* states that, for all such ϕ,

$$\int\int_{\mathbb{D}} |\phi'|^{2-p} \mathrm{d}x\mathrm{d}y = \int\int_{\Omega} |\psi'|^{p} \mathrm{d}x\mathrm{d}y < \infty, \text{ for } \frac{4}{3} < p < 4. \tag{1}$$

In connection with this conjecture, we generalize the formula $I_{-1}(r, \phi') = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{|\phi'(re^{i\theta})|}$, (cf.[52]) to becomes $I_{-p}(r, \phi') = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{|\phi'(re^{i\theta})|^p} < \infty$ for $-1.1697 < p < \frac{2}{3}$, where ϕ is normalized by two conditions $\phi(0) = 0$, $\phi'(0) = 1$. In this respect, we show that the quantity $I_{-1}(r, \phi')$ diverges, if we consider ϕ be a möbious transformation, where $w(z) = \frac{1}{\sqrt{\phi'(z)}}$ be a solution of the second order of differential equation $w''(z) + q(z)w(z) = 0 \quad \text{for} \quad z \in \mathbb{D}$.

Also, we prove that if ϕ is an univalent function in $\mathbb{D}$ with $|\phi(z)| \leq 1$ for all z and $\phi(a) = b$ for some $a, b \in \mathbb{D}$. Then the range of the pth-power integrable function in (1) can be extended depending on behaviour of self-conformal maps ϕ.

We show a nice expansion of the range of integral mean $I(r, \phi')$ of univalent function under the boundedness condition. Also, we prove that any holomorphic function on unit disk $(|z| < 1)$ with $Re(zF'(z)) > 0$ in $|z| < 1$, generates a starlike function defined on the unit disk.

For conformal self-maps ϕ of the unit disk, we study weighted composition operator $C_{\alpha}^{\frac{t}{2}}$, is defined as a mapping $f \longrightarrow f \circ \phi \cdot (\phi')^b$. We are interested in their boundedness property as operators acting in weighted Bergman spaces A^2_{α},$A^2_{\alpha-1}$. Our approach addresses how to generate two types of an holomorphic functions, one of them defined on the cardioid domain and other belongs to *Hardy space* $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D})$, $n \in \mathbb{N}$, depending on the range of angle θ, $n\pi < \theta \leq (1 + n)\pi$.

Make yourself the measure (for dealing) between you and others.
Thus, you should desire for others what you desire for yourself, and hate for others what do you hate for yourself.
Do not oppress as you do not like to be oppressed, and do good to others as you would like good to be done to you.

Imam Abu Alhassan(peace be upon him).

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There are some people I always go back to them $\cdots$ My family : my parents, my brothers and sisters, you have given me so much, thanks for your faith in me, and your prayer to me in all time, I owe a great debt of gratitude to my wonderful parents for all of the sacrifices that they have made to us, and for teaching me that if I face challenges in my daily lives when people around me do not value my dedication, commitment and pass on negative criticism to belittle my efforts, and if there are people around me who try to pull me down by subjecting me to ridicule, harassment and negative comments, it is bound to affect my self-morale. It is important that I don't surrender, and lose sight of my goals and strive towards achieving the best results. It will take some effort from my side to counter the negativity that people spread around me.

Shatha,

April, 2013.

Dedication

I might not know where the life's road will take me, but walking with God, through this journey has given me strength.

I diedicate this thesis to God Almighty for his mercies to me and my family.

As I lovingly dedicate who supported me each step of the way.

Shatha,

April, 2013.

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Chapter 1

Introduction

Our starting point is the classical theory of conformal mappings which is a transformation $w = \phi(z)$ "preserves angles" if any holomorphic map ϕ defined on non-empty open subset of $\mathbb{C}$ is surjective(that is *Univalent*), then its derivative vanishes nowhere.

The conformal mapping has some elementary properties;

- the inverse is also a conformal mapping.
- A conformal mapping is a homeomorphism, that is; a continuous injective map with continuous inverse.
- conformal maps are locally univalent, that is; the derivative does not vanish and there are only simple poles.
- Angles between curves including their orientation are preserved by conformal amps.
- The conformal image of a (measurable) subset A has the area

$$\text{area } \phi(A) = \int\int_A |\phi'(z)|^2 \mathrm{d}x\mathrm{d}y.$$

Conformal mappings involving best possible pointwise estimates of the derivative, thus supplying a measure of the extremal expansion (contraction) possible for a conformal mapping . It is natural to consider also the integral means of $|\phi'|^p$ along circles $|z| = r$, where ϕ is the conformal mapping in question and p is a real parameter, $0 < r < 1$ if ϕ is defined in the unit disk $\mathbb{D}$, while $1 < r < +\infty$ if ϕ is defined in the exterior disk.

The growth of the integral means as $r \to 1$ in the classical pointwise estimates is by far a very fast however, one can see some interesting results which have been proved by *Clunie, Makarov*, *Pommerenke, Bertilsson*, *Shimorin, J. Brennan, Hedenmalm,* ([52],[5],[58],[9],[34] respectively), and others.

In applications, if $\Omega \subset \mathbb{C}$ is a simply-connected domain, then a conformal mapping ϕ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto Ω exists by the *Riemann* mapping theorem [48](if $\Omega \neq \mathbb{C}$), it seems not easy to compute conformal maps if the geometry of Ω is complicated, therefore it is of interest to find a relation between properties of the Riemann map and the geometry of Ω with general properties of conformal maps. Hence, the field which precisely describes the conformal maps is the main part of *geometric function theory*, and in the beginning of the century, initiated new horizons to this topic through the works of *Koebe* [5].

Theory of conformal mappings was introduced by $Carath\acute{e}odory$ in 1912 for variable regions so that this map may be continuously extended from the boundary of Ω onto unit circle, if Ω is bounded by a*Jordan* curve, cf. *Caratheodory extension theorem* ([19, pp.12]) which was employed by *Montel* in 1917 to study the properties of prime ends under conformal mappings, that's why the theory of conformal mappings has been used more in the study of boundary behaviour of conformal maps, according to the basic idea that uses the conformal maps to transform the boundary- value problem, for the region R into a corresponding one for the unit circle or half plane and then find the solution to solve the given problem by employing the inverse of conformal mapping. [1]

In general, there is a question raised about *Riemann* maps, namely; what is the image of $\phi(z)$, where z is fixed? Surely, we need some normalization on ϕ to specify its image. Hence, let us normalize ϕ under two conditions $\phi(0) = 0$ and $\phi'(0) = 1$, such that the set of conformal maps of the unit disk with this normalization is denoted by S and is called class of *univalent* functions [2] on unit disk $\mathbb{D}$, which have an upper and lower bounds on $|\phi(z)|$ that are independent

[1] We refer the reader to several excellent textbooks [11, 29] and [6].

[2] Here, we refer that the univalent function is called *Schlicht function.*

of $\phi \in S$ as proved by *Koebe* [50, pp.21] . This result implies that S is compact in the topology of locally uniform convergence [65].

There exist bounds on the derivatives $\phi'(z)$, $\phi''(z)$,..., valid for all $\phi \in S$, where we have the distortion estimates

$$\frac{1-r}{(1+r)^3} \leq |\phi'(z)| \leq \frac{1+r}{(1-r)^3}, \quad |z| = r < 1. \tag{1.1}$$

The above estimates are sharp, as it is known by the example of an appropriate rotation of the *Koebe function*

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots, \quad z \in \mathbb{D};$$

which maps the unit disk $\mathbb{D}$ to the complement of the ray $(-\infty, \frac{-1}{4}]$. Short calculations show that

$$k'(z) = \frac{1+z}{(1-z)^3}, \quad z \in \mathbb{D}.$$

Hence, it will be helpful to have a better understanding of the behaviour of the sets in $\mathbb{D}$, where $|\phi'(z)|$ is either large or small. For example, $|k'(z)|$ is large near the boundary of unit disk (that is, near $|z| = 1$), small near a point $z = -1$, and to be an unassuming elsewhere.

Integral means. In the following we consider one way to obtain measure of the overall size of $|\phi|$, Define the *integral means*

$$I_p(r, \phi') = \frac{1}{2\pi} \int_{|z|=r} |\phi'(z)|^p \mathrm{d}\theta \qquad (0 < r < 1), \tag{1.2}$$

where $\mathrm{d}\theta$ is the angular measure $\frac{|\mathrm{d}z|}{r}$. In cases $p > 0$ and $p < 0$, these quantities are measure tools to how much the conformal map expands (compresses, respectively). Therefore, the main problem is to estimate

$$\int_{|z|=r} |\phi'(z)|^p \mathrm{d}\theta \quad \text{as} \quad r \to 1, \tag{1.3}$$

as

$$\lim_{p \to +\infty} \left(\int_{|z|=r} |\phi'(z)|^p \mathrm{d}\theta \right)^{\frac{1}{p}} = \max_{|z|=r} |\phi'(z)|.$$

Hardy space on $\mathbb{D}$. For $1 \leq p \leq \infty$ let H^p be the *Hardy* space on the unit disk $\mathbb{D}$ equipped with its usual norm

$$||\phi||_{H_p} = \sup_{0\leq r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |\phi(re^{i\theta})|^p \mathrm{d}\theta \right)^{\frac{1}{p}} < \infty, \text{ for } 0 < p \leq \infty,$$
$$||\phi||_{H_\infty} = \sup_{|z|<1} |\phi(z)| \text{ for } p = \infty.$$

Denote by $H^p(\mathbb{D})$ the normed vector space defined as the space of all holomorphic functions ϕ in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

While, $h^p(\mathbb{D}) = \{U \in h(\mathbb{D}) : ||U||_p < \infty\}$, for $p \in (0, \infty]$ is the family of all complex harmonic functions U on the open unit disk $\mathbb{D}$ such that for $0 < p \leq \infty$, [3]

$$||U||_p = \sup_{0\leq r<1} ||U_r||_p = \sup_{0\leq r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |U(re^{i\theta})|^p \mathrm{d}\theta \right)^{\frac{1}{p}}, \quad \text{if } p \in (0, \infty)$$
$$||U||_\infty = \sup_{|z|<1} |U(z)| \text{ for } p = \infty.$$

The growth of the integral means of the derivative $\phi'(re^{i\theta})$, where $z = re^{i\theta}$, is related with Hardy spaces in case $\phi(z)$ is an holomorphic function in the unit disc $\{z = re^{i\theta} : |z| < 1\}$, univalent or not, to belong to Hardy space $H^p(\mathbb{D})$, if $\mathrm{I}_p(r, \phi)$ remains bounded as $r \to 1$, because

$$\mathrm{I}_p(r, \phi) < \mathrm{I}_q(r, \phi); \;\; p < q.$$

Hence, each $\phi(z) \in H^p(\mathbb{D})$, where $z = re^{i\theta}$ has the property

$$\int_0^1 \mathrm{I}_\infty(r, \phi)\mathrm{d}r < \infty.$$

Which implies that $\phi(z)$ has radial limit

$$\lim_{r\to 1} \phi(re^{i\theta}) = \phi(e^{i\theta}) \quad \text{in almost every direction.}$$

This fact is due to *Fatou's theorem* which states that; for any holomorphic function $\phi : \mathbb{D} \to \mathbb{C}$, such that

$$\sup_{0<r<1} ||\phi(re^{i\theta})||_{L^p(\mathbb{T})} < \infty, \text{ where } \mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi]\} = \{z = e^{i\theta} \in \mathbb{C} : |z| = 1\},$$

[3] we refer the reader to the book [39].

$\phi(re^{i\theta})$ converges for $r \to 1$ to some function $\phi(e^{i\theta}) \in L^p(\mathbb{T})$ pointwise almost everywhere in $L^p(\mathbb{T})$, that is the limit being taken is along a straight line from the center of unit disk to the boundary of the circle that's why the pointwise limit is a radial limit, cf. [63, 26] and [56].

The boundary function ϕ cannot vanish on any set of positive measure unless ϕ is the zero function, but the converse is not true, because it is possible to find an holomorphic function which has a maximum modulus and increases arbitrary slowly to infinity but fails to have a radial limit on any set of positive measure, that's why there is a very close relation between the mean growth of the $\phi'(re^{i\theta})$ and the smoothness of the boundary function $\phi(e^{i\theta})$ [19]. It is reasonable to expect an holomorphic function to be smooth on the boundary if its derivative grows slowly, and vice versa.

Further- more; each function $\phi(re^{i\theta}) \in H^p(\mathbb{D})$, $(0 < p \leq \infty)$ has a nontangential limit $\phi(e^{i\theta})$ at almost every boundary point and *Littlewood's* theorem [19] shows uniform boundedness of this quantity for $0 < r < 1$, which gives rise to the well- known Hardy spaces $H^p(\mathbb{D})$, that's why *Hardy* spaces is a good area to study the growth of $\mathrm{I}_p(r, \phi)$ and $\mathrm{I}_p(r, \phi')$, cf. [37, 35].

Boundary behavior of conformal maps. It is obvious that with *Riemann* mapping we will have a definite condition for existence of a conformal map of the unit disk $\mathbb{D}$ onto any simply connected domain Ω, but we have to be careful when the boundary $\partial\Omega$ of Ω is irregular, we might expect the mapping and its derivative to have rough behaviours as they approach the boundary of the unit disk. A conformal mapping $\phi : \mathbb{D} \to \Omega$ can be extended continously to $\partial\mathbb{D}$ (unit circle) exactly when $\partial\mathbb{D}$ is locally connected. Unfortunately this does not mean that the extension will be well behaved in any way.

P. L. Duren [19] shows that the radial limit in *Hardy* spaces theory

$$\lim_{r\to 1} \phi(re^{i\theta}) = \phi(e^{i\theta}).$$

of any function $\phi \in S$ exists finitely almost everywhere in θ. On the other hand, *G. Piranian A. J. Lohwater and W. Rudin* [42], prove that the radial limit of a meromorphic function and bounded in $\mathbb{D}$ exists and is finite, for almost all points on $\partial\mathbb{D}$, but the radial limit of the derivative of a meromorphic function and bounded in $\mathbb{D}$ fails to exist and be finite.

Brennan's conjecture. We now study one of the pillars of this thesis, *Brennan's conjecture*, this conjecture is formulated as an estimate for conformal maps $\psi : \Omega \to \mathbb{D}$,

$$\int\int_{\Omega} |\psi'|^p \mathrm{d}x\mathrm{d}y < \infty. \tag{1.4}$$

for $\frac{4}{3} < p < 4$, and $\mathrm{d}x\mathrm{d}y = \mathrm{d}A$ is the area measure on the plane. *Brennan's* idea for studying the L^p- norm of $|\psi'|^p$, had a significant impact in approximation theory. Changing the variables will offer us the possibility to write (1.4) in terms of $\psi^{-1} = \phi$:

$$\int\int_{\mathbb{D}} |\phi'|^{2-p} \mathrm{d}x\mathrm{d}y < \infty. \tag{1.5}$$

Brennan [9], introduces an interesting result about increasing of upper bound to $3 + \tau$ by using a harmonic measure argument of *Carleson*. Pommerenke [52] shows that (1.5) holds for $\frac{3}{4} < p < 3.399$. In [9] *B. Dahlberg and J. Lewis* prove that, if Ω is starlike domain [4]or even close-to-convex domain then (1.4) holds for $\frac{4}{3} < p < 4$.

Weighted composition operator C_ϕ^b on Bergman space A_α^t. Clearly, the inequality (1.1) shows that there exists a positive number β, depending on t, such that

$$I_t(r, \phi') = O\left(\frac{1}{(1-r)^\beta}\right) \quad \text{as} \quad r \to 1^-. \tag{1.6}$$

Let ϕ be univalent on $\mathbb{D}$, the function $\beta_\phi(t)$ is usually defined as the infimum of all β such that (1.6) is valid. This is called the *integral means spectrum* of ϕ, and can be written as follows:

$$\beta_\phi(t) = \inf\left\{\beta : \int |\phi'(re^{i\theta})|^t \mathrm{d}\theta = O(\frac{1}{(1-r)^\beta})\right\} \quad \text{as} \quad r \to 1^-. \tag{1.7}$$

which lead us to define the *universal integral means spectrum* for the class S which is defined by

$$\mathbf{B}_S(t) = \sup\{\beta_\phi(t) : \phi \text{ is univalent}\}. \tag{1.8}$$

In this thesis, we introduce the *weighted composition operator* C_ϕ^b on the *Bergman space* A_α^t of holomorphic functions on $\mathbb{D}$, for positive integers b, then we shall define the *classical composition operator* C_ϕ associated with conformal self- maps ϕ of unit disk which is defined as follows:

$$f \longmapsto f \circ \phi = f(\phi(z)),$$

[4] We refer the reader to Appendix A, definition (A.0.12) for more details on **starlike domain**.

so the *weighted composition operator* is defined as a mapping

$$C_{\phi}^{b} : f \longmapsto f \circ \phi \cdot (\phi')^{b}, \text{ for each } b \in \mathbb{R},$$

where $(\phi')^b$ denotes the angular derivative of ϕ at the boundary of unit disk $\partial\mathbb{D}$.

In fact, the properties of weighted composition operator C_{ϕ}^{b} are associated with estimating the integral means of derivatives of univalent functions. Specially, the boundebness of the operators C_{ϕ}^{b}, with $b \in (-1, 0)$ in the classical *Bergman space* A^2 contributes to reformulating *Brennan's conjecture* for conformal mappings (cf. [58, Theorem 2.9.pp.7]).

Shimorin [58] proved a number of estimates for universal function $\mathcal{A}(t) = \sup\limits_{\phi} \alpha_{\phi}(t)$, which correspond to known estimates for the universal integral means spectrum $\mathbf{B}(t)$, where

$$\alpha_{\phi}(t) = \inf\{\beta > 0 : C_{\phi}^{\frac{t}{2}} \text{ is bounded in } \mathcal{A}_{\beta-1}^{2}\},$$

depending on critical value t_0, in the *Carleson- Makarov* theorem [5], his work is based on the strong result obtained by *D. Bertilsson* [5, Theorem 3.7(e). pp.44]

He proved also that *Brennan's conjecture* is equivalent to the equality $A(t) = |t| - 1$ for $t \leq -2$ (as well- known as a property that $(\phi')^b \in A^2$ for any $b \in (-1, 0)$ and ϕ conformal mappings of $\mathbb{D}$ or the universal integral means spectrum function as $\mathbf{B}(t) = 1$ for $t = -2$).

Concerning unknown values of t, *Shimorin* [58] confirmes that all the estimates of the universal integral means spectrum which is proven in [57] are valid also for the universal function $\mathcal{A}(t)$ with only a slight modification. *Shimorin* [57] found that the multiplier norm of the *Schwarzian derivative* from the *Bergman space* A_{α}^{2} to $A_{\alpha+4}^{2}$ can be estimated accurately by applying the area methods directly rather than going through classical pointwise estimate

$$\left| \frac{\phi'''(z)}{\phi'(z)} - \frac{3}{2} \left[\frac{\phi''(z)}{\phi'(z)} \right]^2 \right| \leq \frac{6}{(1 - |z|^2)^2}, \qquad z \in \mathbb{D},$$

which leads to obtain a good estimate of the universal integral means spectrum function $\mathbf{B}(-1)$ and $\mathbf{B}(-2)$ more than formerly known. *Hedenmalm* and *Shimorin* [34] found a collection of estimates of multiplier norm type, parametrized by a real parameter θ, $0 < \theta \leq 1$, via

[5] *Carleson- Makarov* theorem: There exists a $t_0 \leq -2$ such that $\mathbf{B}_{Sb}(t) = |t| - 1, \quad \text{for } t = t_0$.

Prawitz theorem [6] which generalizes the *Grönwall* area theorem and application of the diagonal restriction operation on the bidisk and the use of sharp constants in norm estimates, also their work includes numerical estimates of $\mathbf{B}_S(\tau)$, for real values of τ. In [62] *Alan Sola* studies the behaviour of univalent functions in the mean and at the boundary and uses the similar methods which applied by [34] to obtain that $\mathbf{B}_S(-1) \leq 0.388$ and $\mathbf{B}_S(-2) \leq 1.206$.

The objective of this thesis is to show how the *Brennan's* conjecture which concerns integrability of the derivative of a conformal unit disk $\mathbb{D}$ onto the simlpy connected domain Ω becomes a useful tool when we have to study the properties of *Riemann* map and geometry of Ω with general properties of conformal maps through constructing several examples.

Next, we shall see later, some interesting results on the problem of estimating the integral means of derivatives of univalent functions for instance :

Generalizing the value of power integrable function which have been addressed by Pommerenke [52].

As we show how to extend the range of the value of the pth- power integrable function of *Brennan*'s conjecture up to become $\frac{4}{3} < p < 5$ depending on behaviour of self-conformal maps.

In connection with the problem of estimating the integral means of derivatives of univalent functions, we show a nice expansion of the range of integral mean of univalent function under the boundedness condition.

Also, we prove that any holomorphic function on unit disk with $Re(zF'(z)) > 0$ in $|z| < 1$, generates a starlike function on unit disk

We introduces a Theorem which comes as a corollary of the *Koebe one-quarter theorem* and *Koebe distortion theorem* which considers a sharp result on the integrability of gradient of Cauchy transform $\hat{g}(z)$ over a non-decreasing sequence $\partial\mathcal{D}_i$ in $\mathcal{D}$, such that

$$\int_{\partial\mathcal{D}_i} \nabla \left(\frac{|\hat{g}(z)|}{\sqrt{|z|}} \right) \mathrm{d}\mu_i \quad \text{exists and is finite on } \partial\mathcal{D}_i$$

if the *Cauchy transform* $\hat{g}(z)$ of $g \in L^q(E, \mathrm{d}A)$ for some $1 < q \leq 2$, is identically zero in $\mathbb{C}\backslash E$ and there exists a non-decreasing sequence $\partial\mathcal{D}_i$ in $\mathcal{D}$, where E is a compact subset of the plane having connected complement, D is a connected domain $\mathcal{D} \subset E$.

[6] For a straight- forward proof of *Prawitz* theorem, cf. Theorem A.0.8, Appendix A

In another respect, we are interested in the boundedness of weighted composition operator $C_\phi^b : f \longmapsto f \circ \phi \cdot (\phi')^b$, for each $b \in \mathbb{R}$ in the classical weighted *Bergman* space $\mathbb{A}_\alpha^2$ of functions $f \in H(\mathbb{D})$, where ϕ is an holomorphic self-map of the unit disk $\mathbb{D} \subset \mathbb{C}$ depending only on function of the form $(1-\overline{\lambda}z)^{\frac{-\gamma}{2}}$ and the convexity property of the function $\alpha_\phi(t)$.

We also show that the existance of cusp on the boundary of *cardioid* domain arising from integrability of conformal maps through one of the polar functions in the general solution of *Laplace* equation by proving that there is an holomorphic function defined on the boundary of cardioid domain when $\phi'(0) = 0$, for $0 < n \leq 1$, $n \in \mathbb{N}$ and another belongs to *Hardy space* $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D})$, $n \in \mathbb{N}$, on the boundary of unit disk.

1.1 Background

This section serves as motivation to the main results of this thesis. Appendices **A, B** are added to give a full support.

1.1.1 Riemann's mapping theorem

The theory of conformal mappings is inseparable from that of complex holomorphic functions. One of the most remarkable results from complex analysis states that, for any simply connected domain Ω (other than the whole complex plane) there is always an holomorphic function $\phi : \Omega \to \mathbb{D}$ that maps the interior of Ω to that of unit disk $\mathbb{D}$. For more details cf. [40].

Theorem 1.1.1. *Let Ω be a simply connected domain (open and connected subset of $\mathbb{C}$) which is not the whole plane and a be a point in Ω. Then there is a unique holomorphic function of Ω onto the unit disk $\mathbb{D}$ having the properties ($\phi(a) = 0; \phi'(a) > 0; \phi$ is one to one and $\phi(\Omega) = \mathbb{D}\,\{z : |z| < 1\}$). In other words, we can say (Ω is analytically isomorphic to the unit disk).*[7]

Proof. :-

- **(Uniqueness)**:- Fix $a \in \Omega$.

[7] We refer the reader to excellent textbook [40].

Now, we have to prove that there is a unique holomorphic function $\phi : \Omega \longrightarrow \mathbb{D}$ with required properties as follows:

Let $g : \Omega \longrightarrow \mathbb{D}$ be holomorphic function and satisfy the same properties of ϕ, that is, g is (one to one), $g'(a) > 0$ and $g(a) = 0$.

So, $g(z)$ has inverse function (by Inverse function theorem), such that $\phi \circ g^{-1} : \mathbb{D} \longrightarrow \mathbb{D}$ will be an holomorphic function which is one to one.

By *Schwartz* theorem, [8] we have $\phi \circ g^{-1}$ is an automorphism of $\mathbb{D}$, that fixes 0, such that $\phi \circ g^{-1}(0) = 0$.

Hence, we can deduce that $\phi \circ g^{-1}$ is a rotation, namely there exist α, a constant, $|\alpha| = 1$ such that $\phi(z) = \alpha g(z)$. This implies that $\phi'(a) = \alpha\ g'(a) \Rightarrow \alpha > 0$, since $\phi'(a) > 0$ and $g'(a) > 0$.

It follows, (as $|\alpha| = 1$ and $\alpha > 0$) or $\phi = g$.

Finally, ϕ is an unique function.

- **(Existence)** :- Let us assume that

$$F = \{\phi : \phi \in H(\Omega)^{9} : \phi \text{ is (one-to- one)}, \phi'(a) > 0, \phi(\Omega) \subseteq \mathbb{D}\}.$$

We claim that F is non-empty.

Let $w_0 \in \mathbb{C} \backslash \Omega$.

Then there exists a holomorphic map $\phi : \Omega \longrightarrow \mathbb{C}$ such that $\phi(z)^2 = z - w_0$.

Let $z_1, z_2 \in \Omega$ such that $\phi(z_1) = \phi(z_2)$. Then $z_1 - w_0 = \phi(z_1)^2 = \phi(z_2)^2 = z_2 - w_0$, hence $z_1 = z_2$. Therefore ϕ is injective.

Also, one sees that

$$\phi(z_1) \neq -\phi(z_2) \tag{1.9}$$

[8] We refer the reader to cf. Appendix B for more details on *Schwartz theorem*.

[9] We refer to define each of the following vector spaces, the space $H(\Omega)$ is the class of all holomorphic function defined on Ω. The space $H^{\infty}(\mathbb{D})$ is defined as the vector space of bounded holomorphic functions on the unit disk, with the norm $||\phi||_{H^{\infty}} = \sup_{|z|<1} |\phi(z)|$.

for each $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$.

By the *Open Mapping* theorem [10] ϕ is open. Thus, $\phi(\Omega)$ contains a disc $\mathbb{D}_r(b)$ with $0 < r < |b|$, for some b. Then equation (1.9) implies

$$\mathbb{D}_1(-b) \cap \phi(\Omega) = \emptyset.$$

Since $\phi(\Omega)$ is open, it follows that

$$\overline{\mathbb{D}_r(-b)} \cap \phi(\Omega) = \emptyset.$$

Hence $|\phi(z) + b| > r$, for all $z \in \Omega$. Therefore the map

$$\phi^{-1} := \frac{r}{\phi + b},$$

is in F. From the previous step we have

$$F = \{\phi : \phi \in H(\Omega) : \phi \text{ is (one-to- one)}, \phi'(a) > 0, \phi(\Omega) \subseteq \mathbb{D}\}.$$

such that it will be shown that the desired mapping is the unique in F whose derivative at z_0 is maximal, and has the required properties in F family.

Suppose first that F is non- empty; then clearly $\overline{F} = F \cup \{0\}$.

Now, since $\phi(\Omega) \subseteq \mathbb{D} \;\Rightarrow\; |F(z)| < 1$, for each $z \in \Omega$, and hence

$$\sup\{\phi : |\phi(z)| \leq 1, \quad z \in \Omega\}.$$

Hence, F is locally bounded; that is, if there exist $r > 0$ such that

$$\sup\{|\phi(z)| : |z - a| < r, \phi \in F\} < \infty,$$

which implies that F is normal set due to *Montel's* theorem [11] , that is, every ϕ_n sequence in F has a subsequence ϕ_{n_k} that converges to ϕ uniformly on compact subset of Ω and then

$$\phi'_{n_k}(a) \longrightarrow \phi'(a).$$

[10] *Open Mapping* theorem [3]: The image of an open set under a non- constant holomorphic mapping is an open set.

[11] *Montel's* Theorem states : $\{$*A family F in $H(\Omega)$ is a locally bounded iff F is normal*$\}$ [12].

So, ϕ is an holomorphic function or a constant function (by *Weierstrass* theorem), this implies that F is closed set, therefore F is a compact set (since, F is normal set iff it's closure is compact).

Therefore, there is $f \in F$ such that, $\phi'(a) \leq f'(a)$ $, \forall \phi \in F$.

Consider the function ϕ in F whose derivative at a maximum.

Let M be the least upper bound of the derivatives $\phi^{'}(a)$ as ϕ ranges over F, then $M > 0$ but it may be the case that $M = \infty$.

Pick a sequence $\phi_n \subset F$ such that $\phi_n^{'}(a)$ approaches M, and we know that F is a normal family, hence ϕ_n contains a subsequence ϕ_{n_k} which converges to a function ϕ uniformly on every compact subset of Ω.

The function ϕ is an holomorphic function, and $\phi_{n_k}^{'}$ converges to $\phi^{'}$ uniformly on compact subsets of Ω due to *Weierstrass Theorem*, this implies that $\phi^{'}(a) = M$ and M is a positive real number, means, M is finite.

Now, we assert that $\phi \in F$ through out, that is ϕ not constant, since $f^{'}(a) = M > 0$ and ϕ is holomorphic, the *Open mapping Theorem* implies $|\phi(z)| < 1$ for all $z \in \Omega$.

We have to show that ϕ is injective.

Let $w \in \Omega$, consider the functions $g_w(z) = g(z) - g(w)$, where $g \in F$, such that $g_w(z)$ are non -zero on $\Omega \setminus \{w\}$ as each g is injective .

Then $\phi(z) - \phi(w)$ is a limit of such functions, we deduce that it is nowhere zero on $\Omega \setminus \{w\}$ due to *Hurwitz's Theorem*, since w is chosen arbitrarily, it follows that ϕ is injective, thus $\phi \in F$, and ϕ is an element of F with maximal derivative at a.

It remains to show that $\phi(\Omega) = \mathbb{D}$.

As known, $\phi(\Omega) \subseteq \mathbb{D}$ holds; that is, we have to prove $\mathbb{D} \subseteq \phi(\Omega)$.

Let $w \in \mathbb{D}$ and $w \notin \phi(\Omega)$, then there is a *Möbius* transformation defined of $\mathbb{D}$ onto itself.

$$\varphi_w : \mathbb{D} \to \mathbb{D} \quad \text{such that} \quad \varphi_w(z) = \frac{z - w}{1 - \overline{w}z}, \quad z \in \mathbb{D}.$$

Hence $\varphi_w \circ \phi : \Omega \longrightarrow \mathbb{D}$ is a nowhere zero holomorphic function, hence $\varphi_w \circ \phi$ has an holomorphic square root (that is, $h^2(z) = \varphi_w \circ \phi : \Omega \longrightarrow \mathbb{D}$).

$$\Rightarrow h^2(z) = \varphi_w(\phi(z)) = \frac{\phi(z) - w}{1 - \overline{w}\phi(z)}, \quad \forall z \in \Omega. \Rightarrow h(z) \subseteq \mathbb{D}.$$

Fixed $h(a)$ in $\mathbb{D}$, and define *Möbius* transformation

$$\varphi_{h(a)} = \frac{z - h(a)}{1 - \overline{h(a)}z} : \mathbb{D} \longrightarrow \mathbb{D},$$

such that $g := \varphi_{h(a)} \circ h : \Omega \longrightarrow \mathbb{D}$ can defined by

$$\varphi_{h(a)}(h(z)) = \frac{h(z) - h(a)}{1 - \overline{h(a)}h(a)} = 0, \text{ for } z \in \Omega,$$

satisfies $g(a) = 0$, as $g(\Omega) \subseteq \mathbb{D}$, and this, in turn, generates contradiction with uniqueness of the function ϕ. Hence, $\phi(\Omega) = \mathbb{D}$.

□

The following theorem establishes the correspondence of the boundaries $\partial\Omega$ of Ω and $\partial\mathbb{D}$ of $\mathbb{D}$ in case $\partial\Omega$ is a Jordan curve.

Theorem 1.1.2. ***(Carathéodory- Osgood's theorem)***

Let Ω be a Jordan domain, that is, a domain bounded by a Jordan curve, and let ϕ be a conformal mapping $\phi : \Omega \to \mathbb{D}$. Then, ϕ can be extended one-to-one continuously to the closure $\overline{\Omega} := \Omega \cup \partial\Omega$ of the domain Ω.

Now, we give a group of theorems and corollaries that constitute the first sharp results in *univalent* functions . For additional information the reader is referred to references [19, 53] and [50].

Theorem 1.1.3. ***(Area theorem)*** *[27]*

If $g \in \Sigma$[12], then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$, with equality if and only if $g \in \tilde{\Sigma}$. [13]

[12] Σ is the class of functions $g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$ holomorphic and univalent in the exterior of the unit disk $\Delta = \{z : |z| > 1\}$, except for a simple pole at infinity with residue 1, and which map Δ onto the complement of a compact connected set E [19].

[13] For a straight-forward proof of Area theorem, cf. Theorem A.0.5, Appendix A

Corollary 1.1.4. *If $g \in \Sigma$, then $|b_1| \leq 1$, and equality holds if and only if $g = z + b_\circ + 2e^{2i\theta} z^{-1}$ where $b_\circ \in \mathbb{C}$, $\theta \in \mathbb{R}$.*

The following lemma gives a basic estimate which leads to the distortion theorem and related results.

Lemma 1.1.5. *For each $\phi \in S$* [14],

$$\left| z\frac{\phi''}{\phi'} - \frac{2|z|^2}{1-|z|^2} \right| \leq \frac{4|z|}{1-|z|^2}.$$

Theorem 1.1.6. ***(Distortion Koebe theorem)*** *For each $\phi \in S$ defined on unit disc $\mathbb{D}$,*

$$\frac{1-r}{(1+r)^3} \leq |\phi'(z)| \leq \frac{1+r}{(1-r)^3}, \quad |z| = r < 1.$$

Equality holds if and only if ϕ is a suitable rotation of the Koebe function [15]

Lemma 1.1.7. ***(Parseval formula)***
let $\phi(z)$ be an holomorphic function in unit disk $\mathbb{D}$ such that it is represented there by Taylor series expansion $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ then [16]

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |\phi(re^{i\theta})|^2 \, \mathrm{d}\theta.$$

1.1.2 Schwarz-Christoffel transformation

The *Schwarz-Christoffel transformation* is a conformal mapping from the upper half of the complex plane to a polygonal domain. It allows many physical problems posed on two-dimensional polygons. In specific case, one method of determining solutions is by taking the polygonal domain in the complex plane and determining a conformal map, which preserves the structure of *Laplace's equation*, that restarts the problem in a simpler domain. For such polygonal domains, a method of determining the specific transform needed is provided by the following theorem:

[14] For a straight-forward proof of this assertion, cf. Theorem A.0.6, Appendix A
[15] For a straight- forward proof of Distortion Koebe theorem, cf. Theorem A.0.7, Appendix A
[16] For a straight-forward proof of *Parseval Formula*, cf. Lemma A.0.9, Appendix A

Given a polygonal curve Γ, its interior P is a simply connected domain. Thus, by the *Riemann mapping* theorem, there exist a function F that conformally maps the upper half plane onto P. The *Schwarz-Christoffel* theorem provides a concrete description of such maps.

Theorem 1.1.8. ***(Schwarz-Christoffel theorem)***
Let P be the interior of a polygon Γ having vertices $w_1, w_2, \ldots, w_n$ and interior angles $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_n\pi$ in counterclockwise order. Let ϕ be any conformal map from the upper half- plane $\mathbb{H}^+$ to P with $\phi(\infty) = w_n$. Then

$$\phi(z) = A + C\int^z \prod_{k=1}^{n-1}(\zeta - z_k)^{\alpha_k - 1}\mathrm{d}\zeta$$

for some complex constants A and C, where $w_k = \phi(z_k)$ for $k = 1, \ldots, n-1$.

Definition 1.1.9. **(Bergman spaces)** For $0 < p < +\infty$ and a given parameter $-1 < \alpha < +\infty$, *Bergman space* $A^p_\alpha(\mathbb{D})$ is a function space of holomorphic functions f in the unit disk $\mathbb{D}$ with the norm property

$$||f||^p_{p,\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha \mathrm{d}A(z) < \infty, \tag{1.10}$$

where $\mathrm{d}A(z) = \frac{\mathrm{d}x\mathrm{d}y}{\pi}$, (z=x+iy) is the normalized *Lebesgue* area measure on $\mathbb{D}$, and the function $w(z) = (1-|z|^2)^\alpha : \mathbb{D} \longrightarrow (0,\infty)$ is a radial weight. [17]

In addition, for $1 \leq p < \infty$ and $\alpha > -1$ fixed, A^p_α is a *Banach space*. For $p = 2$, A^p_α is a *Hilbert space*.

1.1.3 Smirnov domain

Let $\phi(w) = z$ be a conformal mapping *(one-to- one)* from the unit disk $\mathbb{D}$ onto simply connected domain Ω boundary by rectifiable Jordan curve in the complex plane $\mathbb{C}$, then ϕ has a continuous injection extension to closure of $\mathbb{D}$ with no cut points (cf. [53], Theorem 2.6, pp.24) so that the length of the Jordan curve $L(\gamma)$ to be $L(\gamma) < \infty \Leftrightarrow \phi' \in H^1(\Omega)$ (cf. [53], Section 6.3). Hence

[17] A function $w(z) : \mathbb{D} \to (0,\infty)$, integrable over $\mathbb{D}$, is called a *weight function* or simply a *weight*. In addition, it is *radial* if $w(z) = w(|z|)$ for all $z \in \mathbb{D}$.

Ω is called a *Smirnov domain* if for $|w| < 1$ the harmonic function $\log|\phi'(w)|$ can be written as the *Poisson integral* of its non-tangential boundary values $\log|\phi'(e^{i\theta})|$ as follows:

$$\log|\phi'(w)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\ \log|\phi'(e^{i\theta})|\ \mathrm{d}\theta.$$

Since ϕ' as a non-zeros function belongs to $H^1(\Omega)$, and as is known in *Hardy* spaces for $0 < p \leq \infty$, every non-zero function such that $\phi' \in H^1(\Omega)$ can be written as the product $\phi'(w) = M(w)G(w)$, where $M(w)$ is a inner function[18] and $G(w)$ is an outer function [19](cf. [18], Theorem 17.17, [61] and [67]).

Therefore, Ω is called a *Smirnov domain* when $M(w) \equiv 1$.

One thing that we need to mention here is that, $E^p(\Omega)$ is the set of all holomorphic functions in Ω which is bounded by rectifiable *Jordan* curve γ, such that for $0 < p \leq \infty$ there is a sequence of closed rectifiable *Jordan* curves $\gamma_i \subset \Omega$, $i = 1, 2, \cdots$, and satisfy the following:

i. γ_i tends to γ, as $i \to \infty$ in the sense that, if Ω_i is the bounded domain with boundary γ_i, then $\Omega_1 \subset \Omega_2 \subset \cdots \Omega_i \subset \Omega$ and $\bigcup_{i=1}^{\infty} = \Omega$;

ii.

$$\int_{\gamma_i} |f(z)|^p|\mathrm{d}z| \leq const < \infty.$$

1.1.4 Comparison theorem for ODEs

This section involves some comparison theorems for ODEs which play an important role in comparing the properties of the growth of the integral means of derivative of univalent function with properties of differential equations to find differential inequalities which gives good estimates refer to be the growth of the integral means of derivative of univalent function that exist and is finite for some values of pth-power integrable function. For further details cf. [66, 15] and [4].

[18] $M(w)$ is an inner function if and only if $|M(w)| \leq 1$ on the unit disk $\mathbb{D}$ and $\lim_{r\to 1} M(re^{i\theta})$ exists at almost all θ and it's modulus is equal to 1

[19] $G(w)$ is an outer (exterior) function if it takes the form $G(w) = c\exp\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(\phi(e^{i\theta}))\mathrm{d}\theta\right]$ for some complex number c with $|c| = 1$, and some positive measurable function ϕ on the unit circle such that $\log\phi$ is integrable on the circle.

Lemma 1.1.10. *Let p, q be continuous functions in $[a, b)$, such that $p(x) \in \mathbb{R}$, $q(x) > 0$ for $a \leq x < b$, suppose u is twice differentiable and $u'' < pu' + qu$, $v'' = pv' + qv$, if*

$$u(a) < v(a) \text{ and } u'(a) < v'(a), \text{ then } u(x) < v(x) \text{ for } a \leq x < b. \tag{1.11}$$

Proof. Let $\phi = u - v$ such that,

$$u(a) < v(a) \Rightarrow u(a) - v(a) < 0 \Rightarrow \phi(a) < 0$$

and

$$u'(a) < v'(a) \Rightarrow u'(a) - v'(a) < 0 \Rightarrow \phi'(a) < 0$$

such that,

$$\left.\begin{aligned} u'' &< pu' + qu \\ v'' &= pv' + qv \end{aligned}\right\} \Rightarrow \phi'' < p\phi' + q\phi \tag{1.12}$$

Let us assume that,

$$u(x) < v(x); \text{ for } x \in [a, b)$$

is false, this implies that there exists a point $\zeta \in [a, b)$ such that $\phi(\zeta) = 0$, and $\phi(x) < 0$, for all $x \in (a, \zeta)$.

Therefore, for any $x \in (a, \zeta) \ni a < x < \zeta$, we obtain

$$\phi(a) < \phi(x) < \phi(\zeta) \Rightarrow \phi(x) < 0, \text{ for } x \in (a, \zeta).$$

And hence, $\phi(a) < 0$ *and* $\phi'(a) < 0$, which in turn implies that ϕ has a local minimum; assume point $x_{min} \in (a, \zeta)$.

Clearly, $\phi(x_{min}) < 0 \Rightarrow \phi'(x_{min}) = 0$, and then $\phi''(x_{min}) \geq 0$, this contradiction with

$$\phi'' < p\phi' + q\phi,$$

showing that $\phi(x) < 0$, or $u(x) < v(x)$ is true for $x \in [a, b)$. $\square$

Lemma 1.1.11. *Let $q(x)$ be continuous and positive $(q(x) > 0)$ on $[a, b)$, suppose u is four times differentiable and $u^{(4)} < qu$, $v^{(4)} = qv$, $u^{(k)}(a) < v^{(k)}(a)$ for $k = 0, 1, 2, 3$ then $u(x) < v(x)$ for $a \leq x < b$.*

Proof. Let $\phi = u - v$ be such that,

$$u(a) < v(a) \;\Rightarrow\; u(a) - v(a) < 0 \;\Rightarrow\; \phi(a) < 0 \;\text{ for } k = 0. \tag{1.13}$$

and

$$u^{(k)}(a) < v^{(k)}(a) \;\Rightarrow\; u^{(k)}(a) - v^{(k)}(a) < 0 \;\Rightarrow\; \phi^{(k)}(a) < 0. \tag{1.14}$$

Also, we have

$$\left.\begin{aligned} u^{(4)} &< qu \\ v^{(4)} &= qv \end{aligned}\right\} \;\Rightarrow\; \phi^{(4)} < q\phi \tag{1.15}$$

Let us assume that

$$u(x) < v(x); \;\; \text{for } x \in [a, b)$$

is false, this implies that there exist a point $\zeta \in [a, b)$ such that $u(\zeta) = v(\zeta) \;\Rightarrow\; \phi(\zeta) = 0$.

Therefore, for any $x \in (a, \zeta) \;\ni\; a < x < \zeta$, we obtain

$$\phi(a) < \phi(x) < \phi(\zeta) \;\Rightarrow\; \phi(x) < 0,$$

for $x \in (a, \zeta)$, since $\phi(a) < 0$.

Hence, $\phi'(a) < 0$; for $k = 1$ (cf. equation (1.14)).

So it is impossible that, $\phi''(x) < 0$, for $a < x \leq \zeta$, which gives $\phi'' = \psi$ has zero in (a, ζ), such that $\psi(a) = \phi''(a) < 0$ and $\psi'(a) = \phi'''(a) < 0$ by equation (1.14).

$\Rightarrow$ ψ has a local minimum $x_\circ \in (a, \zeta)$.

In particular, if ψ is twice differentiable at stationary point $x_\circ$, that is, $\psi'(x_\circ) = 0$ such that $\psi''(x_\circ) \geq 0$, which in turn implies that $\phi^{(4)}(x_\circ) = \psi'' \geq 0$, which leads to contradiction with equation (1.15), since $q(x) > 0$, $\phi(x) < 0$, such that $u(x) < v(x)$, for $a \leq x < b$ is true. □

1.1.5 Cauchy transform

The Cauchy transform of a positive measure plays an important role in complex analysis, and especially in the approximation problem . We will highlight a bit in this respect and in chapter 3, cf. [55, 36].

Let $g \in \mathcal{C}^{\infty}$ be a complex smooth function defined on $\overline{\Omega}$, let $z_{\circ}$ be a point in the interior of Ω.

Consider; complex 1- form,

$$w = g\,\chi = \frac{g(z)}{(z - z_{\circ})}\,\mathrm{d}z.$$

so that $\chi = \frac{\mathrm{d}z}{z - z_{\circ}}$ is a closed 1-form because $\mathrm{d}\chi = 0$ and w is defined on $\overline{\Omega}$ except at $z_{\circ}$, that is; $z_{\circ}$ is a singularity of

$$w = \frac{g(z)}{(z - z_{\circ})}\,\mathrm{d}z. \tag{1.16}$$

Now, we shall remove a small disk $\mathbb{D}(z_{\circ}, r)$ centered at $z_{\circ}$ and with the radius of so small r that $\mathbb{D}(z_{\circ}, r) \subset \Omega$.

The remaining part of Ω which is an annulus like domain $\mathcal{D}' = \Omega \diagdown \mathbb{D}(z_{\circ}, r)$, such that the boundary of $\mathcal{D}'$ is $\partial\Omega \cup \partial\mathbb{D}(z_{\circ}, r)$.

It is important to mention that the boundary of $\mathbb{D}(z_{\circ}, r)$ is oriented in the opposite way, that is, in the negative oriented (clockwise), such that

$$\partial\mathcal{D}' = \partial\Omega \cup (-\partial\mathbb{D}(z_{\circ}, r)).$$

Moreover, $w = \frac{g(z)}{(z - z_{\circ})}\,\mathrm{d}z$ has no singularity in $\mathcal{D}'$

By appyling *Stokes' Theorem* [20] we obtain,

$$\begin{aligned}\int\int_{\mathcal{D}'} \mathrm{d}w &= \int_{\partial\mathcal{D}'} w \\ &= \int_{\partial\Omega} w - \int_{\partial\mathbb{D}(z_{\circ}, r)} w.\end{aligned} \tag{1.17}$$

$\partial\mathbb{D}(z_{\circ}, r)$ is a circle γ around $z_{\circ}$ such that $\gamma(t) = z_{\circ} + re^{it}$, $0 \leq t \leq 2\pi$, such that $g(\gamma(t)) = g(z_{\circ} + re^{it})$ convergent to $g(z_{\circ})$ as the radius r shrinks to 0.

The last integral in (1.17) gives

$$\begin{aligned}\int_{\partial\mathbb{D}(z_{\circ}, r)} w &= \int_{0}^{2\pi} \frac{g(z_{\circ} + re^{it})}{z_{\circ} + re^{it} - z_{\circ}}\, rie^{it}\mathrm{d}t. \\ &= i\int_{0}^{2\pi} g(z_{\circ} + ie^{it})\mathrm{d}t.\end{aligned}$$

[20]we refer the reader to the in Appendix B, Theorem (B.0.20) for more information on *Stokes' Theorem*.

Hence,

$$i\int_0^{2\pi} g(z_\circ + re^{it})\mathrm{d}t = i\int_0^{2\pi} g(z_\circ) = 2\pi i g(z_\circ).$$

So, equation (1.17) can be written as follows

$$\int\int_{\mathcal{D}'} \mathrm{d}w = \int_{\partial\Omega} w - 2\pi i g(z_\circ) \text{ as r decreases to 0.}$$

Letting $r \to 0$ in equation (1.17), we will notice that the disk $\mathbb{D}(z_\circ, r)$ shrinks and $\mathcal{D}'$ fills up Ω.

$$\Rightarrow \int\int_{\Omega} \mathrm{d}w = \int_{\partial\Omega} w - 2\pi i g(z_\circ). \tag{1.18}$$

Derive equation (1.16) to obtain,

$$\mathrm{d}w = \frac{\frac{\partial g}{\partial \bar{z}}}{z - z_\circ}\mathrm{d}\bar{z} \wedge \mathrm{d}z. \tag{1.19}$$

Substituting equations (1.16) and (1.19) in equation (1.18).

$$\int\int_{\Omega} \frac{\frac{\partial g}{\partial \bar{z}}}{z - z_\circ}\mathrm{d}\bar{z} \wedge \mathrm{d}z = \int_{\partial\Omega} \frac{g(z)}{(z - z_\circ)}\,\mathrm{d}z - 2\pi i g(z_\circ).$$

$$\Rightarrow \frac{1}{2\pi i}\int\int_{\Omega} \frac{\frac{\partial g}{\partial \bar{z}}}{z - z_\circ}\mathrm{d}\bar{z} \wedge \mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{g(z)}{(z - z_\circ)}\,\mathrm{d}z - g(z_\circ).$$

$$\Rightarrow g(z_\circ) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{g(z)}{(z - z_\circ)}\,\mathrm{d}z - \frac{1}{2\pi i}\int\int_{\Omega} \frac{\frac{\partial g}{\partial \bar{z}}}{z - z_\circ}\mathrm{d}\bar{z} \wedge \mathrm{d}z, \tag{1.20}$$

which is called *Cauchy Pompeiu's* formula.

Now, we shall discuss two special cases:-

- when $g(z)$ is an holomorphic function $\Rightarrow$ $\frac{\partial g}{\partial \bar{z}} = 0$.

  Hence , we have

  $$g(z_\circ) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{g(z)}{(z - z_\circ)}\,\mathrm{d}z. \quad (\textit{ Cauchy formula}).$$

  This formula tells us that :

  i. The value of $g(z)$ at $z_\circ$ in Ω is completely determined by the values of $g(z)$ on the boundary.

ii. The formula:

$$\frac{1}{2\pi i}\int_{\gamma} w = \frac{1}{2\pi i}\int_{\gamma}\frac{\mathrm{d}z}{z - z_\circ} = 1,$$

holds for any closed curve $\gamma = z_\circ + re^{it}$ surrounding $z_\circ$, not just for a circle with $z_\circ$ as its center.

- when $g(z)$ vanish on the boundary of Ω, that is $g(z)|_{\partial\Omega} = 0$ then; equation (1.20) becomes,

$$g(z_\circ) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{g(z)}{(z - z_\circ)}\,\mathrm{d}z - \frac{1}{2\pi i}\int\int_{\Omega}\frac{\frac{\partial g}{\partial \overline{z}}}{z - z_\circ}\mathrm{d}\overline{z}\wedge \mathrm{d}z.$$

$$\Rightarrow\ g(\zeta) = 0 - \frac{1}{2\pi i}\int\int_{\Omega}\frac{\frac{\partial g}{\partial \overline{z}}}{-(\zeta - z)}2i\mathrm{d}x\wedge \mathrm{d}y.$$

$$\Rightarrow\ g(\zeta) = \frac{1}{\pi}\int\int_{\Omega}\frac{\frac{\partial g}{\partial \overline{z}}}{\zeta - z}\mathrm{d}x\mathrm{d}y.$$

$$\Rightarrow\ g(\zeta) = \frac{1}{\pi}\int\int_{\Omega}\frac{\frac{\partial g}{\partial \overline{z}}}{-(z - \zeta)}\mathrm{d}x\mathrm{d}y.$$

$$\Rightarrow\ -\pi g(\zeta) = \int\int_{\Omega}\frac{\frac{\partial g}{\partial \overline{z}}}{z - \zeta}\mathrm{d}x\mathrm{d}y = \frac{\partial}{\partial \overline{z}}\int\int_{\Omega}\frac{g}{z - \zeta}\mathrm{d}x\mathrm{d}y = \frac{\partial}{\partial \overline{z}}\hat{g}.$$

Finally,

$$-\pi g(\zeta) = \frac{\partial}{\partial \overline{z}}\hat{g}.$$

where $z_\circ$ is switched to $\zeta,\ \zeta \in \Omega$.

Now, the following question arises:-

Q:- Does *Cauchy* transform $\hat{g}$ vanish identically in the unbounded complement of $\overline{\Omega}$?

Ans:- By going back to the *Cauchy Pompeiu's* Formula (cf. equation (1.20)). We will notice that when g vanishes on the boundary of Ω, that is $g(z)|_{\partial\Omega} = 0$, then

$$\frac{\partial \hat{g}}{\partial \overline{z}} = -\pi g.$$

$$\Rightarrow\ \int \frac{\partial \hat{g}}{\partial \overline{z}}\mathrm{d}\overline{z} = -\int Q\pi g\mathrm{d}\overline{z} = 0,\ \text{for every polynomial}\, Q$$

where $Q = 1$ is a non- zero constant function which is a polynomial function of degree zero. This implies that $\hat{g} = 0$ and on the other hand; it is know that $\frac{\partial \hat{g}}{\partial \overline{z}} = 0$ when $\hat{g}$ is a holomorphic function.

Finally, $\hat{g} = c \;\Rightarrow\; \hat{g} \equiv 0 \;\text{ in }\; (\Omega^{\circ})^{c}$.

We shall indicate that *Cauchy* transform possesses two important properties [45]:

Let μ be a complex regular Borel measure with compact support Ω and define

$$\hat{\mu}(z) = \int_{\Omega} \frac{\mathrm{d}\mu(\zeta)}{\zeta - z}$$

to be the *Cauchy transform* of μ. Then

i. $\hat{g}$ converges almost everywhere in the plane depending on the area. Let $z \in \mathbb{C}\backslash\Omega$, choose X subset of the plane so that $\Omega \subset X$ and let R sufficiently large so that for any $\zeta \in \Omega$, we notice that X is contained in the disk $\mathcal{D}_R(\zeta) = \{z : |\zeta - z| < R\}$, this implies that

$$\int_X \frac{\mathrm{d}A}{|\zeta - z|} \leq \int_0^{2\pi} \int_0^R \frac{1}{r}\, r \mathrm{d}x \mathrm{d}y = 2\pi R < \infty,$$

by *Fubini's* theorem, we obtain

$$\int_X \int_{\Omega} \frac{\mathrm{d}|\mu|(\zeta)}{|\zeta - z|} \mathrm{d}A = \int_{\Omega} \int_X \frac{\mathrm{d}A}{|\zeta - z|} \mathrm{d}|\mu|(\zeta) \leq 2\pi R |\mu|(\Omega).$$

ii. The *Cauchy* transform is also continuous and holomorphic in $\mathbb{C}\backslash\Omega$ choose z_0 in $\mathbb{C}\backslash\Omega$ and z in a neighborhood X of z_0 such that $\overline{X} \cap \Omega = \emptyset$, then we have that

$$\left| \int_{\Omega} \frac{\mathrm{d}\mu(\zeta)}{\zeta - z_0} - \int_{\Omega} \frac{\mathrm{d}\mu(\zeta)}{\zeta - z} \right| \leq |z_0 - z| \int_{\Omega} \frac{\mathrm{d}|\mu|(\zeta)}{|\zeta - z_0||\zeta - z|} \leq C|z_0 - z|,$$

where the constant C depends on the distance between z_0 and Ω.

Choose Γ to be a closed curve that lies in $\mathbb{C}\backslash\Omega$ which does not surround Ω and z_0, since $\frac{1}{\zeta - z_0}$ is holomorphic function then $\int_{\Gamma} \frac{1}{\zeta - z_0} = 0$, this implies that

$$\int_{\Gamma} \left(\int_{\Omega} \frac{\mathrm{d}\mu(\zeta)}{\zeta - z_0} \right) \mathrm{d}z = \int_{\Omega} \left(\int_{\Gamma} \frac{\mathrm{d}z}{\zeta - z_0} \right) \mathrm{d}\mu(\zeta) = 0$$

Since $\hat{\mu}(z)$ is continuous and $\int_{\Gamma} \hat{\mu}(z) = 0$ over any closed curve Γ lies in $\mathbb{C}\backslash\Omega$, hence $\hat{\mu}(z)$ is holomorphic by *Moreras* Theorem [21] [1].

[21] *Moreras* Theorem: If a single - valued function $\phi(z)$ of a complex variable z in a domain Ω is continuous and if its integral over any closed rectifiable contour $\gamma \subset \Omega$ is equal to zero, that is, if

$$\int_{\gamma} \phi(z) \mathrm{d}z = 0, \qquad \gamma \subset \Omega,$$

then $\phi(z)$ is an holomorphic function in Ω. This theorem was obtained by *G. Morera*

1.2 Outline of the Thesis

It is time to outline in detail the plan of the thesis and discussion of the results.

Chapter 2: Brennan's conjecture, higher integrability of gradient of conformal maps. In this chapter we start with Brennan conjecture

$$\int\int_{\mathbb{D}} |\phi'|^{2-p} \mathrm{d}x\mathrm{d}y = \int\int_{\Omega} |\psi'|^{p} \mathrm{d}x\mathrm{d}y < \infty,$$

which $\frac{4}{3} < p < 4$, with $\psi = \phi^{-1} : \Omega \to \mathbb{D}$, as a main tool in this area, as introduced by *Pommerenke* [52]. It was proved that

$$I_{-1}(r, \phi') = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta}{|\phi'(re^{i\theta})|} = O(1-r)^{-0.601}.$$

We generalize the formula above to show that,

$$I_{-p}(r, \phi') = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta}{|\phi'(re^{i\theta})|^p} < \infty \quad \text{for} \quad -1.1697 < p < \frac{2}{3}$$

Also, we prove that the range of the pth-power integrable function can be extended, if ϕ is an univalent function in $\mathbb{D}$ with $|\phi(z)| \leq 1$ for all z and $\phi(a) = b$ for some $a, b \in \mathbb{D}$, as presented in Theorem 2.1.5.

In connection with *Brennan*'s conjecture, we construct some interesting examples (2.2.4), (2.2.5), (2.2.2) and (2.2.3) which show the behaviour of the boundary derivative of conformal maps ϕ from polygon P onto unit disk $\mathbb{D}$ and the behaviour of the boundary derivative of the inverse maps, where polygon P such as rectangular domain and triangular domain or simply connected domains bounded by a circular arc polygon $P = \Omega$ such as crescent domain and lens domain. This requires the usage of upper half-plane $\mathbb{H}^+$ in the construction of a composite function, which in turn help to estimate the derivative of conformal map and the derivative of the inverse maps easily through some quantities that belong to $\mathbb{H}^+$, in order to obtain a good estimation, and to circumvent the difficulty in applying *Schwarz- Christoffel* formula (disk version) directly in some examples, such as (2.2.2) and (2.2.3).

As part of this chapter, we construct several examples on the integrability of powers gradient of conformal mapping over infinite sector W, when $\alpha = \frac{\pi}{n}$, for some integer n, which are presented in the examples (2.2.6),(2.2.7) and (2.2.8).

In addition, we present a full proof of *Hardy* identity, for being a fundamental tool to estimate the integral means of univalent function on unit disk $\mathbb{D}$ as presented in Pommerenke's work [52]. Seemingly, there is no convenient reference, at least to our knowledge.

Chapter 3: Integral means of the derivative of univalent function. In connecton with chapter 2, we show a good comparison between Theorems (3.1.2) and (3.1.3), where we prove that the boundedness condition on the univalent function contributes to the expansion of the range of pth-power integrable function.

In Theorem (3.1.4) we proved that there exists starlike function [22] ϕ in unit disk ($|z| < 1$) such that

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})| \mathrm{d}\theta \leq \int_0^r M(r) \left[\frac{2(1+\rho)^2}{\rho^2(1-\rho)^2} + \frac{4\rho+2\rho^2}{\rho(1-\rho)^2} + \frac{1+\rho}{\rho^2(1-\rho)}\right] \mathrm{d}\rho$$

We also proved that the quantity

$$I_{-1}(r,\phi') = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{|\phi'(re^{i\theta})|}$$

diverges , if we consider $\phi : \mathbb{D} \to \mathbb{D}$ be a m$\ddot{o}$bious transformation, such that $w(z) = \frac{1}{\sqrt{\phi'(z)}}$ be a solution of the second order of differential equation $w''(z) + q(z)w(z) = 0 \quad \text{for} \quad z \in \mathbb{D}$.

Among other things, we assumed that $\phi : \Omega \to \mathbb{D}$, with a great role for each of *Koebe one-quarter* theorem and *Koebe distortion* theorem to prove there is a constant K depending on the modulus of z in Ω such that $1 - |\phi(z)| \leq K\sqrt{|z|}$, for some $z \in \Omega$, see Theorem 3.2.6.

Theorem (3.2.6) generates a sharp result on the integrability of gradient of Cauchy transform $\hat{g}(z)$ over a sequence $\partial\mathcal{D}_i$ which is getting larger in $\mathcal{D}$, such that

$$\int_{\partial\mathcal{D}_i} \nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right) \mathrm{d}\mu_i \quad \text{exists and is finite on } \partial\mathcal{D}_i$$

if the *Cauchy transform* $\hat{g}(z)$ of $g \in L^q(E, \mathrm{d}A)$ for some $1 < q \leq 2$, is identically zero in $\mathbb{C}\backslash E$ and there exist a non-decreasing sequence $\partial\mathcal{D}_i$ in $\mathcal{D}$, where E is a compact subset of the plane having connected complement, D be a connected domain $\mathcal{D} \subset E$, see Theorem 3.2.8.

Also, Theorem 3.2.6 comes as a tool for proving the integrability of the derivative of conformal mappping over bounded domain by a class C^1 Jordan curve exists and is finite for all $p < 2$, as presented in Theorem 3.2.7.

[22] We refer the reader to Appendix A, definition (A.0.13) for more details on starlike function.

Chapter 4: Weighted composition operator $C_\alpha^{\frac{t}{2}}$ in Bergman spaces A_α^2, $A_{\alpha-1}^2$. Throughout this chapter, we assume that ϕ is a conformal self- map of $\mathbb{D}$ to study the boundedness of weighted composition operators $C_\alpha^{\frac{t}{2}}$ in Bergman spaces A_α^2, $A_{\alpha-1}^2$.

The work's idea throughout this chapter comes of *S.Shimorin*'s paper[58], so what is new in the boundedness of weighted composition operator $C_\alpha^{\frac{t}{2}}$? The new thing in this work is prove that, the weighted composition operator $C_\alpha^{\frac{t}{2}}$ is bounded in Bergman spaces A_α^2 without depending on convexity property of the function $\alpha_\phi(t)$ and satisfies.

$$\int_{\mathbb{D}} \frac{|\phi'|^t}{|1-\overline{\lambda}\phi|^\gamma}(1-|u|^2)^\alpha \mathrm{d}A(u) \leq \frac{C_1^{t-s}(\phi,t)\cdot C_2^s(\phi,t)}{(1-|\lambda|^2)^{\gamma-\alpha-2t}}, \quad \text{for some } \gamma > \alpha+2t.$$

While that, the operator $C_\alpha^{\frac{t}{2}}$ is a bounded in $A_{\alpha-1}^2$ depending upon the convexity property of $\alpha_\phi(t)$ and satisfies

$$\int_{\mathbb{D}} \frac{|\phi'|^t}{|1-\overline{\gamma}\phi|^\gamma}(1-|u|^2)^{\alpha-1} \mathrm{d}A(u) \leq \frac{C_8}{(1-|\lambda|^2)^{\lambda-|t|-\epsilon}},$$

is presented in Theorems 4.1.1 and 4.1.2 respectively.

Chapter 5: Integrability over cardioid domain, simply connected domain. In this chapter, we consider $\Omega \subset \mathbb{R}^2$ be a bounded domain for all $1 \leq p < \frac{4}{3}$, such that the boundary of Ω is a smooth *Jordan* curve except the point $(z=0)$, and a harmonic function on Ω which belongs to the *Sobolev* space $\mathbb{W}_0^{1,p}(\Omega)$, where $\mathbb{W}_0^{1,p}(\Omega)$ is the closure of $\mathbb{C}_0^\infty(\Omega)$ in the $\mathbb{W}^{1,p}(\Omega)$-norm, as put forward by *P. Hajlasz* in [28].

P. Hajlasz in ([28, Theorem 1. pp.80]), proved that $u(z) = \Im(z^{-1/2} + \frac{i}{2})$ is a harmonic function in Ω and belong to $\mathbb{W}_0^{1,p}(\Omega)$ for all $1 \leq p < \frac{4}{3}$. Our work in last chapter is in part an elaboration of *P. Hajlasz*'s idea, that's why we consider *Laplace's* equation with *Dirichlet* conditions [23] on a bounded simply connected domain $\Omega \subset \mathbb{R}^2$;

$$\left.\begin{aligned} \Delta u = 0, &\quad x \in \Omega; \\ u = 0, &\quad x \in \partial\Omega. \end{aligned}\right\} \tag{1.21}$$

[23] *Dirichlet* conditions: in the sense, one can be specified only of values along the boundary of Ω, e.g. the function ux takes prescribed values on the boundary $\partial\Omega$, that is, $u(x) = $ constant [46]

Here, $u \equiv 0$ in the closure of Ω. Then, we chose one of the harmonic functions which belong to the general solution of the *Laplace's* equation

$$u(r, \theta) = r^{-\sqrt{\mu}} \sin \sqrt{\mu}\theta, \tag{1.22}$$

where $u(r, \theta)$ vanishes on $\partial\Omega$ under the condition $r = \phi(\theta) = (\sin \sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$, μ is an eigenvalue, to study the existence of cusp on the boundary cardioid domain, we classify it into (*inward cusp*, *outward cusp*) on the boundary cardioid domain via an integrability of conformal maps at the intersection point $(z = 0)$ and its neighborhoods.

Consequently, we prove that the integrability of pth-power of the gradient of $u(r, \theta) = r^{-\sqrt{\mu}} \sin \sqrt{\mu}\theta$ exists and is finite for $p < \frac{2}{\sqrt{\mu}+1}$ when $\mu = \frac{1}{n^2}$ $\quad n \in \mathbb{N}$ as follows:

i. In case $n = 2$, we have $|\nabla u| \in L_p(\Omega)$ for $p < \frac{4}{3}$ and $u(r, \theta)$ vanishes on $\partial\Omega$ except $z = 0$ (discontinuity point).

ii. In case $n \geq 3$, we have $|\nabla u| \in L_p(\Omega)$ for $p < \frac{2}{\sqrt{\mu}+1}$, and $u(r, \theta)$ vanishes on $\partial\Omega$ except at the neighborhoods of $z = 0$.

iii. Also, we show the existence of an outward-pointing cusp on the boundary of Ω when $n \notin \mathbb{N}$; for instance, $n = \sqrt{2} \notin \mathbb{N}$ such that the tangent vector does not equal to zero.

see figures (5.1), (5.2) and table (5.1).

Our approach addresses how to generate two types of an holomorphic functions as follows:

i. $u(r, \theta) = \left(r^{-\sqrt{\mu}} \cos \sqrt{\mu}\theta\right) - 1$, defined on cardioid domain Ω, vanishes on $\partial\Omega$, where $r = (\cos \sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$ which is fully settled in Theorem (5.2.1).

ii. We consider *M. Keldysh, M. Laurentiev* theorem ([18] Sec.10.1) as fundamental tool to generate an holomorphic function belongs to $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D})$, $n \in \mathbb{N}$, and does not have poles on $\partial\mathbb{D} = \mathbb{T}$, depending on the range of angle θ, $n\pi < \theta \leq (1 + n)\pi$, where Ω is a *Smirnov* domain (cardioid type).

   Our work is largely related to some results obtained by *D. Khavinson* [14, 38].

   Theorem (5.2.4) gives the detail.

Chapter 2

Brennan's conjecture, higher integrability of gradient of conformal maps

Introduction

This chapter provides an overview on *Brennan's conjecture* which is associated with estimating the integral means of derivatives of univalent functions via generalizing some cases which have posed by *Pommerenke* [52]. We also show the behaviour of the boundary derivative of conformal maps from polygon domain onto unit disk, and its inverse maps. Moreover, we study the existence and finiteness the integrability of the derivative of conformal maps over an infinite sector $\mathbb{W}$.

We must begin by assuming that Ω be a simply connected domain in complex plane $\mathbb{C}$, and let $\phi : \mathbb{D} \longrightarrow \Omega$ be a conformal mapping.

Brennan's conjecture states that, for all such ϕ,

$$\int\int_{\mathbb{D}} |\phi'|^{2-p} \mathrm{d}x\mathrm{d}y = \int\int_{\Omega} |\psi'|^{p} \mathrm{d}x\mathrm{d}y < \infty. \tag{2.1}$$

holds for $\frac{4}{3} < p < 4$ with $\psi = \phi^{-1} : \Omega \to \mathbb{D}$, $\mathrm{d}x\mathrm{d}y = \mathrm{d}A$ is the area measure on the plane.

This conjecture is justified, in the following respects:

(a) In case $p = 2$, the integral represents the unit disk area and is therefore finite. This can be shown as follows:

Proof. :- Let $\phi^{-1} : \Omega \to \mathbb{D}, \quad \psi = \phi^{-1}$. So, by using the formula below

$$\int\int_{\mathbb{D}} |\phi'|^{2-p}\mathrm{d}x\mathrm{d}y = \int\int_{\Omega} |\psi'|^{p}\mathrm{d}x\mathrm{d}y \tag{2.2}$$

$$\Rightarrow \int\int_{\Omega} |\psi'|^{2}\mathrm{d}x\mathrm{d}y = \int\int_{\mathbb{D}} r\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\int_0^1 r\mathrm{d}r\mathrm{d}\theta = \pi.$$

□

(b) Equation (2.2) holds, for all $\frac{4}{3} < p < 3$. Proof can be given as follows:

Proof. :- Assume that, $2 - p = q$ in (2.2) and then apply second *Green's* identity.

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial|\phi(re^{i\theta})|^q}{\partial r} r\ \mathrm{d}\theta &= \frac{1}{2\pi}\int\int_{\mathbb{D}} \Delta(|\phi|^q)\ \mathrm{d}x\mathrm{d}y \\ &= \frac{q^2}{2\pi r}\int\int_{\{|w|<r\}} |\phi(w)|^{q-2}|\phi^{'}(w)|^2\mathrm{d}x\mathrm{d}y.\end{aligned}$$

We substitute $\phi(w) = z$, since ϕ is holomorphic and univalent.

So, $\phi(w)$ has a maximum modulus $\mathrm{M}_\infty(r, \phi) = \max\limits_{|w|=r} |\phi(w)|$, such that

$$\int_{\mathbb{D}} |\phi'(w)|^2\mathrm{d}x\mathrm{d}y = \int_{\Omega} \mathrm{d}x\mathrm{d}y$$

$$|\phi(w)| \le \max_{|w|=r} |\phi(w)| = \mathrm{M}_\infty(r, \phi)$$

By *Prawitz's* theorem, this implies that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi(re^{i\theta})|^q\mathrm{d}\theta\right) &= \frac{q^2}{2\pi r}\int\int_{\{|w|<r\}}|\phi(w)|^{q-2}|\phi^{'}(w)|^2\mathrm{d}x\mathrm{d}y \\
&= \frac{q^2}{2\pi r}\int\int_{\phi(\{|w|<r\})}|z|^{q-2}\mathrm{d}A(z) \\
&\leq \frac{q^2}{2\pi r}\int\int_{\phi(\{|w|<r\})}\max|z|^{q-2}\mathrm{d}A(z) \\
&= \frac{q^2}{2\pi r}\int_0^{2\pi}\int_0^{\mathrm{M}(r)}t^{q-2}t\mathrm{d}t\mathrm{d}\theta \\
&= \frac{q^2}{r}\int_0^{\mathrm{M}(r)}t^{q-1}\mathrm{d}t \\
&= \frac{q}{r}\mathrm{M}^q(r) \leq \frac{q}{r}\frac{r^q}{(1-r)^{2q}} \qquad \text{since} \quad \mathrm{M}(r) \leq \frac{r}{(1-r)^2}
\end{aligned}$$

Taking integral of both sides over r gives

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}|\phi(re^{i\theta})|^q\mathrm{d}\theta &\leq q\int_0^r\frac{t^q}{t(1-t)^{2q}}\mathrm{d}t \\
&\leq q\int_0^r\frac{t^{q-1}}{(1-t)^{2q}}\mathrm{d}t \\
&\leq q\left[\left(\int_0^r t^{q-1}\mathrm{d}t\right)\left(-\int_0^r -(1-t)^{-2q}\mathrm{d}t\right)\right] \\
&\leq [t^q|_0^r\left[\frac{-(1-t)^{1-2q}}{1-2q}\right|_0^r.
\end{aligned}$$

When $q > 0 \Rightarrow [t^q|_0^r < \infty; \qquad r \leq 1.$

And when, $1-2q > 0$, we have

$$\begin{aligned}
\left[\frac{-(1-t)^{1-2q}}{1-2q}\right|_0^r &= \frac{1}{2q-1}\left[-(1-r)^{1-2q}+1\right] \\
&= \frac{1}{2q-1}\left[1-\frac{1}{(1-r)^{2q-1}}\right] \\
&= \frac{1}{2q-1}\left[\frac{(1-r)^{2q-1}-1}{(1-r)^{2q-1}}\right] \\
&= \left(\frac{(1-r)^{2q-1}-1}{2q-1}\right)\left(\frac{1}{(1-r)^{2q-1}}\right) \\
&\leq \frac{C_q}{(1-r)^{2q-1}} \qquad \text{since} \quad \left(\frac{(1-r)^{2q-1}-1}{2q-1}\right) \leq C_q \quad \text{as} \quad r \to 1.
\end{aligned}$$

Therefore we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi(re^{i\theta})|^q \mathrm{d}\theta \leq \frac{C_q}{(1-r)^{2q-1}}$$

By *Distortion* theorem

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|\phi'(re^{i\theta})|^q \mathrm{d}\theta &\leq \left(\frac{(1+r)^q}{r^q(1-r)^q}\right)\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi(re^{i\theta})|^q \mathrm{d}\theta\right)\\ &\leq \left(\frac{(1+r)^q}{r^q(1-r)^q}\right)\left(\frac{C_q}{(1-r)^{2q-1}}\right)\\ &\leq \frac{C_{q'}}{(1-r)^{3q-1}} \qquad \text{where} \quad \frac{C_q(1+r)^q}{r^q} \leq C_{q'}.\end{aligned}$$

$$\begin{aligned}\Rightarrow \frac{1}{2\pi}\int_0^1\int_0^{2\pi}|\phi'(re^{i\theta})|^q \,\mathrm{d}r\mathrm{d}\theta \leq \int_0^1 \frac{C_{q'}}{(1-r)^{3q-1}} &= -\int_0^1 C_{q'}\mathrm{d}r\int_0^1\frac{-\mathrm{d}r}{(1-r)^{3q-1}}\\ &= \left[-\int_0^1 C_{q'}\mathrm{d}r)(\frac{(1-r)^{2-3q}}{2-3q}\right|_0^1.\end{aligned}$$

so,when $2-3q>0 \Rightarrow 3q<2$. But, we have $q=(2-p) \Rightarrow 3(2-p)<2$ such that $p>\frac{4}{3}$. □

In the following, we present a full proof of hardy identity which is of course well known, but there seems to be no convenient reference, at least to our knowledge.

Hardy Identity

Proposition 2.0.1. (First identity)

Let ϕ be holomorphic in $\mathbb{D}$. If $\phi(z) \neq 0$ in $\mathbb{D}$ then

$$r\,\frac{\partial}{\partial r}\,|\phi|(z) = |\phi(z)|\,\Re\left(z\frac{\phi'(z)}{\phi(z)}\right). \tag{2.3}$$

Proof. :- Let $\phi(z)=u(r,\theta)+iv(r,\theta),\;\; z=re^{i\theta}$. then

$$|\phi|(z)=\sqrt{u^2+v^2}.$$

Differentiating $\phi(z)$ with respect to r, fixing θ then take limit along the ray where the argument is equal to θ.

$$\phi'(re^{i\theta}) = \frac{\partial \phi(re^{i\theta})}{\partial r} = \frac{1}{e^{i\theta}}(u_r + iv_r). \tag{2.4}$$

Now;

$$z\,\frac{\phi'(z)}{\phi(z)} = re^{i\theta}\,\frac{\frac{1}{e^{i\theta}}(u_r + iv_r)}{u + iv} = r\,\frac{uu_r + vv_r}{u^2 + v^2} + ir\,\frac{uv_r - vu_r}{u^2 + v^2}$$

$$\Rightarrow\ |\phi(z)|\,\Re\left(z\,\frac{\phi'(z)}{\phi(z)}\right) = r\frac{uu_r + vv_r}{\sqrt{u^2 + v^2}}.$$

As is known that,

$$\frac{\partial |\phi|(z)}{\partial r} = \frac{\partial \sqrt{u^2 + v^2}}{\partial r} = \frac{uu_r + vv_r}{\sqrt{u^2 + v^2}}.$$

So, this implies that

$$r\,\frac{\partial |\phi|(z)}{\partial r} = |\phi(z)|\,\Re\left(z\,\frac{\phi'(z)}{\phi(z)}\right)$$

□

Proposition 2.0.2. (Second identity)

Let ϕ be holomorphic in $\mathbb{D}$. If $\phi(z) \neq 0$ in $\mathbb{D}$ then

$$\frac{\partial}{\partial \theta}\,|\phi|(z) = -|\phi(z)|\,\Im\left(z\frac{\phi'(z)}{\phi(z)}\right). \tag{2.5}$$

Proof. :-Let $\phi(z) = u(r, \theta) + iv(r, \theta),\ \ z = re^{i\theta}$ then

$$|\phi|(z) = \sqrt{u^2 + v^2}.$$

Differentiating $\phi(z)$ with respect to θ, fixing r then take limit along the circle.

$$\phi'(re^{i\theta}) = \frac{\partial \phi(re^{i\theta})}{\partial \theta} = \frac{1}{re^{i\theta}}(u_\theta + iv_\theta)$$

$$z\,\frac{\phi'(z)}{\phi(z)} = re^{i\theta}\,\frac{\frac{1}{re^{i\theta}}(u_\theta + iv_\theta)}{u+iv} = \frac{v_\theta u - vu_\theta}{u^2+v^2} + i\,\frac{-u_\theta u - vv_\theta}{u^2+v^2}$$

$$\Rightarrow\ |\phi|\,\Im\left(z\,\frac{\phi'(z)}{\phi(z)}\right) = \frac{-u_\theta u - vv_\theta}{\sqrt{u^2+v^2}}.$$

As it is known that,

$$\frac{\partial|\phi|(z)}{\partial\theta} = \frac{\partial\sqrt{u^2+v^2}}{\partial\theta} = \frac{u_\theta u + vv_\theta}{\sqrt{u^2+v^2}}.$$

So,this implies that

$$\frac{\partial|\phi|(z)}{\partial\theta} = -|\phi(z)|\,\Im\left(z\,\frac{\phi'(z)}{\phi(z)}\right)$$

□

Proposition 2.0.3. (**Third identity**)

Let ϕ be holomorphic in $\mathbb{D}$. If $\phi(z)\neq 0$ in $\mathbb{D}$ then

$$r\,\frac{\partial}{\partial r}|\phi(z)|\,\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - \frac{\partial}{\partial\theta}|\phi(z)|\,\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = |\phi(z)|\left|\frac{z\phi'(z)}{\phi(z)}\right|^2 \tag{2.6}$$

Proof. :- As we stated earlier

$$|\phi(z)|\left|\frac{z\phi'(z)}{\phi(z)}\right|^2 = \sqrt{u^2+v^2}\left|re^{i\theta}\,\frac{\frac{1}{re^{i\theta}}\,(v_\theta - iu_\theta)}{u+iv}\right|^2 = \frac{(u_\theta^2+v_\theta^2)}{\sqrt{u^2+v^2}}.$$

In this case we have to prove that

$$r\,\frac{\partial}{\partial r}|\phi(z)|\,\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - \frac{\partial}{\partial\theta}|\phi(z)|\,\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = \frac{(u_\theta^2+v_\theta^2)}{\sqrt{u^2+v^2}}.$$

As follows

$$r\,\frac{\partial}{\partial r}|\phi(z)|\,\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) = r\,\frac{\partial}{\partial r}\sqrt{u^2+v^2}\ r\,\frac{uu_r+vv_r}{u^2+v^2} = r\,\frac{\partial}{\partial r}\ (r\,\frac{uu_r+vv_r}{\sqrt{u^2+v^2}})$$

A short calculation gives:

$$r\,\frac{\partial}{\partial r}|\phi(z)|\,\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) = \frac{u^2v_\theta^2 + r^2u^3u_{rr} + u^2u_\theta^2 + r^2u^2vv_{rr} + v^2v_\theta^2 + r^2uv^2u_{rr} + v^2u_\theta^2 + r^2v^3v_{rr}}{\sqrt{u^2+v^2}(u^2+v^2)}$$
$$+\,\frac{-u^2v_\theta^2 + 2uvu_\theta v_\theta - v^2u_\theta^2 + u^3v_\theta - vu^2u_\theta + uv^2v_\theta - v^3u_\theta}{\sqrt{u^2+v^2}(u^2+v^2)}$$

Next, to calculate $-\frac{\partial}{\partial\theta}|\phi(z)|\,\Im\left(\frac{z\phi'(z)}{\phi(z)}\right)$,

$$-\frac{\partial}{\partial\theta}|\phi(z)|\,\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = -\frac{\partial}{\partial\theta}\sqrt{u^2+v^2}\left[\frac{-uu_\theta - vv_\theta}{(u^2+v^2)}\right]\frac{\partial}{\partial\theta}\left[\frac{uu_\theta + vv_\theta}{(u^2+v^2)}\right].$$

So,

$$r\,\frac{\partial}{\partial r}|\phi(z)|\,\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - \frac{\partial}{\partial\theta}|\phi(z)|\,\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = \frac{(u_\theta^2+v_\theta^2)}{\sqrt{u^2+v^2}}.$$

This implies that:

$$r\,\frac{\partial}{\partial r}|\phi(z)|\,\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - \frac{\partial}{\partial\theta}|\phi(z)|\,\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = |\phi(z)|\,\left|\frac{z\phi'(z)}{\phi(z)}\right|^2$$

□

Theorem 2.0.4. ***(Hardy Identity for*** $I(r,\phi)$***)***

Let $\Phi(t)$ *be a twice continuously differentiable function,* $\Psi(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\left[t\Phi'(t)\right], 0 \le t < \infty$*, let* $\phi(z)$ *be holomorphic in unit disk* $\mathbb{D}$ *and*

$$I(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(|\phi(re^{i\theta})|)\mathrm{d}\theta,\quad z = re^{i\theta};\quad 0 \le r < 1.$$

which is the integral mean of the modulus of $\phi(z)$*.*

If $\phi(z) \neq 0$ *for* $|z| = r$ *then*

$$r\,\frac{\partial}{\partial r}\left[r\;I'\right] = \frac{1}{2\pi}\int_0^{2\pi}\Psi(|\phi(z)|)\left|\frac{z\phi'}{\phi}\right|^2\mathrm{d}\theta.$$

Proof. :- first we have to choose

$$\Phi(t) = t \;\Rightarrow\; \Phi'(t) = 1 \Rightarrow\; \Psi(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\left[t\right] = t.$$

Let $t = |\phi(z)|$, to obtain

$$r\,\frac{\partial}{\partial r}\;|\phi|(z) = |\phi(z)|\,\Re\left(z\frac{\phi'(z)}{\phi(z)}\right).\quad \text{(see equation (2.3))}$$

$$\frac{\partial}{\partial\theta}\;|\phi|(z) = -|\phi(z)|\,\Im\left(z\frac{\phi'(z)}{\phi(z)}\right).\quad \text{(see equation (2.5))}$$

Multiplying both identities above by $(r\frac{\partial}{\partial r})$, $(\frac{\partial}{\partial\theta})$ respectively. we obtain,

$$(r\,\frac{\partial}{\partial r})^2\,|\phi|(z) = r\,\frac{\partial}{\partial r}\,|\phi(z)|\,\Re\left(z\frac{\phi'(z)}{\phi(z)}\right). \tag{2.7}$$

$$(\frac{\partial}{\partial\theta})^2\,|\phi|(z) = -\frac{\partial}{\partial\theta}\,|\phi(z)|\,\Im\left(z\frac{\phi'(z)}{\phi(z)}\right). \tag{2.8}$$

Adding equation (2.8) to (2.7) we obtain

$$(r\,\frac{\partial}{\partial r})^2\,|\phi|(z) + (\frac{\partial}{\partial\theta})^2\,|\phi|(z) = r\,\frac{\partial}{\partial r}\,|\phi(z)|\,\Re\left(z\frac{\phi'(z)}{\phi(z)}\right) - \frac{\partial}{\partial\theta}\,|\phi(z)|\,\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) \tag{2.9}$$

Integrating (2.9) with respect to θ, when $0 \leq \theta \leq 2\pi$ and using identity (2.6).

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}(r\frac{\partial}{\partial r})^2|\phi|(z)\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi}(\frac{\partial}{\partial\theta})^2|\phi|(z)\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial}{\partial r}|\phi(z)|\Re\left(\frac{z\phi'(z)}{\phi(z)}\right)\mathrm{d}\theta \\ - \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\theta}|\phi(z)|\Im\left(\frac{z\phi'(z)}{\phi(z)}\right)\mathrm{d}\theta.\end{aligned}$$

$$\Rightarrow\; r\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi(z)|\mathrm{d}\theta\right)\right] = \frac{1}{2\pi}\int_0^{2\pi}|\phi(z)|\left|\frac{z\phi'}{\phi}\right|^2\mathrm{d}\theta.$$

□

In the attempt to prove *Hardy* identity with respect to the mean value of the modulus of ϕ on the circle $|z| = r$, we have been led to prove a good deal more. In particular, for the function $I_p(r,\phi)$,

$$I_p(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi}|\phi(re^{i\theta})|^p\mathrm{d}\theta,\quad z = re^{i\theta};\;\; 0 \leq r < 1.$$

where p is any positive number.

To this end, we will try to prove some more identities related to $|\phi|^p$.

Proposition 2.0.5. (Fourth identity)

Let ϕ be holomorphic in $\mathbb{D}$. If $\phi(z) \neq 0$ in $\mathbb{D}$ then

$$r\,\frac{\partial}{\partial r}\,|\phi|^p(z) = p|\phi(z)|^p\,\Re\left(z\frac{\phi'(z)}{\phi(z)}\right). \tag{2.10}$$

Proof. :- Let $\phi(z) = u(r,\theta) + iv(r,\theta), \;\; z = re^{i\theta}$ then

$$|\phi(z)| = \sqrt{u^2+v^2} \;\Rightarrow\; |\phi(z)|^p = (u^2+v^2)^{\frac{p}{2}}.$$

Differentiating $\phi(z)$ with respect to r, fixing θ then taking limit along the ray where the argument is equal to θ.

$$\phi'(re^{i\theta}) = \frac{\partial \phi(re^{i\theta})}{\partial r} = \frac{1}{e^{i\theta}}(u_r + iv_r). \tag{2.11}$$

Now;

$$z\,\frac{\phi'(z)}{\phi(z)} = re^{i\theta}\,\frac{\frac{1}{e^{i\theta}}(u_r+iv_r)}{u+iv} = r\,\frac{uu_r+vv_r}{u^2+v^2} + ir\,\frac{uv_r - vu_r}{u^2+v^2}.$$

$$\Rightarrow\; p|\phi(z)|^p\,\Re\left(z\,\frac{\phi'(z)}{\phi(z)}\right) = r(u^2+v^2)^{\frac{p}{2}-1}(uu_r+vv_r).$$

As we stated earlier,

$$r\frac{\partial|\phi|^p(z)}{\partial r} = r\frac{\partial}{\partial r}(u^2+v^2)^{\frac{p}{2}} = rp\,(u^2+v^2)^{\frac{p}{2}-1}\,(uu_r+vv_r).$$

This implies that

$$r\,\frac{\partial|\phi|^p(z)}{\partial r} = p\,|\phi(z)|^p\,\Re\left(z\,\frac{\phi'(z)}{\phi(z)}\right)$$

□

Proposition 2.0.6. (Fifth identity)

Let ϕ be holomorphic in $\mathbb{D}$, If $\phi(z) \neq 0$ in $\mathbb{D}$ then

$$\frac{\partial}{\partial\theta}\,|\phi|^p(z) = -p|\phi(z)|^p\,\Im\left(z\frac{\phi'(z)}{\phi(z)}\right). \tag{2.12}$$

Proof. :- Let $\phi(z) = u(r,\theta) + iv(r,\theta), \;\; z = re^{i\theta}$ then

$$|\phi(z)| = \sqrt{u^2+v^2} \;\Rightarrow\; |\phi(z)|^p = (u^2+v^2)^{\frac{p}{2}}.$$

Differentiating $\phi(z)$ with respect to θ, fixing r then taking limit along the circle.

$$\phi'(re^{i\theta}) = \frac{\partial\phi(re^{i\theta})}{\partial\theta} = \frac{1}{re^{i\theta}}(u_\theta + iv_\theta)$$

$$z\,\frac{\phi'(z)}{\phi(z)} = re^{i\theta}\,\frac{\frac{1}{re^{i\theta}}(u_\theta + iv_\theta)}{u+iv} = \frac{v_\theta u - v u_\theta}{u^2+v^2} + i\,\frac{-u_\theta u - v v_\theta}{u^2+v^2}$$

$$\Rightarrow\; p\,|\phi|^p(z)\,\Im\left(z\,\frac{\phi'}{\phi}\right) = -p\,(u^2+v^2)^{\frac{p}{2}-1}\,(u_\theta u + v v_\theta).$$

As we stated earlier,

$$\frac{\partial|\phi|^p(z)}{\partial\theta} = \frac{\partial}{\partial\theta}\,(u^2+v^2)^{\frac{p}{2}} = p\,(u^2+v^2)^{\frac{p}{2}-1}\,(u_\theta u + v v_\theta).$$

It leads to

$$\frac{\partial|\phi|(z)}{\partial\theta} = -p\,|\phi|^p(z)\,\Im\left(z\,\frac{\phi'(z)}{\phi(z)}\right)$$

□

Proposition 2.0.7. (Sixth identity)

Let ϕ be holomorphic in $\mathbb{D}$, If $\phi(z)\neq 0$ in $\mathbb{D}$ then

$$r\,\frac{\partial}{\partial r}p\,|\phi(z)|^p\,\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - \frac{\partial}{\partial\theta}p\,|\phi(z)|^p\,\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = p^2|\phi(z)|^p\left|\frac{z\phi'(z)}{\phi(z)}\right|^2. \quad (2.13)$$

Proof.

$$p^2\,|\phi(z)|^p\,\left|\frac{z\phi'(z)}{\phi(z)}\right|^2 = p^2\,(u^2+v^2)^{\frac{p}{2}}\,\left|re^{i\theta}\,\frac{\frac{1}{re^{i\theta}}\,(v_\theta - iu_\theta)}{u+iv}\right|^2 = p^2\,(u^2+v^2)^{\frac{p}{2}-1}\,(u_\theta^2+v_\theta^2).$$

Let us prove that

$$r\,\frac{\partial}{\partial r}\,p\,|\phi(z)|^p\,\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - \frac{\partial}{\partial\theta}\,p\,|\phi(z)|^p\,\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = p^2\,(u^2+v^2)^{\frac{p}{2}-1}\,(u_\theta^2+v_\theta^2).$$

As follows:

$$r\,\frac{\partial}{\partial r}\,p\,|\phi(z)|^p\,\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) = r\,\frac{\partial}{\partial r}\,(u^2+v^2)^{\frac{p}{2}}\;r\,\frac{uu_r+vv_r}{u^2+v^2} = p\,r\,\left[\frac{\partial}{\partial r}\,r\,(u^2+v^2)^{\frac{p}{2}-1}\,(uu_r+vv_r)\right]$$

A short calculation gives

$$r\frac{\partial}{\partial r}p|\phi(z)|^p\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) = r^2p(u^2+v^2)^{\frac{p}{2}-1}({u_r}^2 + uu_{rr} + {v_r}^2 + vv_{rr})$$
$$+\,rp(uu_r+vv_r)(u^2+v^2)^{\frac{p}{2}-1} + r^2\,p\,(\frac{p-2}{2})\,(uu_r+vv_r)(u^2+v^2)^{\frac{p}{2}-2}\,(2uu_r+2vv_r).$$

$$r\frac{\partial}{\partial r}\ p\ |\phi(z)|^p\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) = p\ (u^2+v^2)^{\frac{p}{2}-1}(v_\theta{}^2 + ruv_{r\theta} - uv_\theta + u_\theta{}^2 - rvu_{r\theta} + vu_\theta)$$

$$+p(uv_\theta - vu_\theta)(u^2+v^2)^{\frac{p}{2}-1} + p^2(u^2+v^2)^{\frac{p}{2}-2}(uv_\theta - vu_\theta)^2 - 2p(u^2+v^2)^{\frac{p}{2}-2}(uv_\theta - vu_\theta)^2.$$

Now, we calculate $-p\ \frac{\partial}{\partial\theta}\ |\phi(z)|^p\ \Im\left(\frac{z\phi'(z)}{\phi(z)}\right)$ as follows:

$$-p\ \frac{\partial}{\partial\theta}\ |\phi|^p(z)\ \Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = p\ \frac{\partial}{\partial\theta}\ (u^2+v^2)^{\frac{p}{2}-1}\ \ (uu_\theta + vv_\theta)\,.$$

This will produce:

$$\Rightarrow\ -p\frac{\partial}{\partial\theta}|\phi(z)|^p\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = p(u^2+v^2)^{\frac{p}{2}-1}(u_\theta{}^2 - ruv_{r\theta} + v_\theta{}^2 + rvv_r\theta)$$
$$- 2p(u^2+v^2)^{\frac{p}{2}-2}(uu_\theta + vv_\theta)^2 + p^2(u^2+v^2)^{\frac{p}{2}-2}(uu_\theta + vv_\theta)^2.$$

So,

$$r\ p\frac{\partial}{\partial r}|\phi(z)|\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - p\frac{\partial}{\partial\theta}|\phi(z)|\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = 2p(u^2+v^2)^{\frac{p}{2}-1}(v_\theta{}^2 + u_\theta{}^2)$$
$$+ p^2(u^2+v^2)^{\frac{p}{2}-1}(v_\theta{}^2 + u_\theta{}^2) - 2p\ (u^2+v^2)^{\frac{p}{2}-1}(v_\theta{}^2 + u_\theta{}^2).$$

It implies that

$$rp\frac{\partial}{\partial r}|\phi(z)|\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - p\frac{\partial}{\partial\theta}|\phi(z)|\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = p^2(u^2+v^2)^{\frac{p}{2}-1}(v_\theta{}^2 + u_\theta{}^2)$$

$$\Rightarrow\ rp\frac{\partial}{\partial r}|\phi(z)|\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - p\frac{\partial}{\partial\theta}|\phi(z)|\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) = p^2|\phi(z)|^p\left|\frac{z\phi'(z)}{\phi(z)}\right|^2.$$

□

Theorem 2.0.8. ***(Hardy Identity for*** $I_p(r,\phi)$***)***

Let $\Phi(t)$ *be a twice continuously differentiable function and* $\Psi(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\left[t\Phi'(t)\right],\ (0 \le t < \infty)$, *let* $\phi(z)$ *be holomorphic in unit disk* $\mathbb{D}$ *and*

$$I_p(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(|\phi(re^{i\theta})|^p)\mathrm{d}\theta,\ \ z = re^{i\theta};\ \ 0 \le r < 1.$$

which is the integral mean of the $|\phi|^p(z)$*, where* p *is any positive number.*

If $\phi(z) \ne 0$ *for* $|z| = r$ *then*

$$r\ \frac{\partial}{\partial r}\ [r\ I'] = \frac{1}{2\pi}\ \int_0^{2\pi}\ \Psi(|\phi(z)|^p)\ \left|\frac{z\phi'}{\phi}\right|^2\mathrm{d}\theta.$$

Proof. We can choose

$$\Phi(t) = t^p \;\Rightarrow\; \Phi'(t) = pt^{p-1}.$$

$$\Rightarrow\; \Psi(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\left[p\,t\,t^{p-1}\right] = p^2\,t^p.$$

Let $t = |\phi(z)| \Rightarrow\; t^p = |\phi(z)|^p$, then:

$$r\,\frac{\partial}{\partial r}\,|\phi|^p(z) = p|\phi(z)|^p\,\Re\left(z\frac{\phi'(z)}{\phi(z)}\right) \quad \text{(see equation(2.10))}.$$

$$\frac{\partial}{\partial \theta}\,|\phi|^p(z) = -p|\phi(z)|^p\,\Im\left(z\frac{\phi'(z)}{\phi(z)}\right) \quad \text{(see equation (2.12))}.$$

Multiplying both identities above by $(r\frac{\partial}{\partial r})$, $(\frac{\partial}{\partial \theta})$ respectively. we get,

$$(r\,\frac{\partial}{\partial r})^2\,|\phi|^p(z) = p\,r\,\frac{\partial}{\partial r}\,|\phi(z)|^p\,\Re\left(z\frac{\phi'(z)}{\phi(z)}\right). \tag{2.14}$$

$$(\frac{\partial}{\partial \theta})^2\,|\phi|^p(z) = -p\,\frac{\partial}{\partial \theta}\,|\phi(z)|^p\,\Im\left(z\frac{\phi'(z)}{\phi(z)}\right). \tag{2.15}$$

Adding equation (2.15) to (2.14) produces:

$$(r\,\frac{\partial}{\partial r})^2|\phi|^p(z) + (\frac{\partial}{\partial \theta})^2|\phi|^p(z) = p\,r\frac{\partial}{\partial r}\,|\phi(z)|^p\Re\left(z\frac{\phi'(z)}{\phi(z)}\right) - p\,\frac{\partial}{\partial \theta}\,|\phi(z)|^p\Im\left(\frac{z\phi'(z)}{\phi(z)}\right) \tag{2.16}$$

Integrating (2.16) with respect to θ when $0 \le \theta \le 2\pi$, and using equation (2.13).

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}(r\frac{\partial}{\partial r})^2|\phi|^p(z)\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi}(\frac{\partial}{\partial \theta})^2|\phi|^p(z)\mathrm{d}\theta &= \frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial}{\partial r}|\phi(z)|^p\Re\left(\frac{z\phi'}{\phi}\right)\mathrm{d}\theta \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial \theta}|\phi(z)|^p\Im\left(\frac{z\phi'(z)}{\phi(z)}\right)\mathrm{d}\theta.\end{aligned}$$

$$\Rightarrow\; r\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi|^p(z)\mathrm{d}\theta\right)\right] = \frac{p^2}{2\pi}\int_0^{2\pi}|\phi(z)|^p\left|\frac{z\phi'}{\phi}\right|^2\mathrm{d}\theta.$$

□

2.1 On the integral means of the derivative of an univalent function

The aim of this section is to show some interesting results on the integral means of the derivative of a univalent function by generalizing the following Theorem (2.1.1), according to the source [52].

Theorem 2.1.1. *[52] If ϕ is univalent function in unit disk $\mathbb{D}$ then, as $r \to 1-0$,*

$$I_{-1}(r,\phi') = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{|\phi'(re^{i\theta})|} = O((1-r)^{-0.601}). \tag{2.17}$$

Proof. :- Given that ϕ is univalent function in $\mathbb{D}$. Therefore, ϕ is holomorphic function and (one-to-one) such that $\phi'(z) \neq 0$ this implies that $(\phi^{-1}(z))' = \frac{1}{\phi'(z)}$, is holomorphic function $\forall z \in \mathbb{D}$, except at a possible pole z_0 of ϕ where $\frac{1}{\phi'(z_0)}$ has a double zeros, which can be written as follows

$w(z) = \sqrt{\frac{1}{\phi'(z)}} \Rightarrow\ w^2(z) = (\phi'(z))^{-1}$, is holomorphic function.

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^{-1}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} |w(re^{i\theta})|^2\mathrm{d}\theta\ ;\quad z = re^{i\theta}$$

Let $I_{-1}(r,\phi') = I(r)$. So,

$$I_{-1}(r,\phi') = I(r) = \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^{-1}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} |w(re^{i\theta})|^2\mathrm{d}\theta\ ;\quad z = re^{i\theta} \tag{2.18}$$

where $w(z)$ has a *Taylor* series $w(z) = \sum_{n=0}^{\infty} a_n z^n\ ;\quad z \in \mathbb{D}$.

By *Parseval* Formula (cf. Lemma 1.1.7).

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |w(re^{i\theta})|^2\mathrm{d}\theta \tag{2.19}$$

$$\Rightarrow\ I(r) = \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^{-1}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} |w(re^{i\theta})|^2\mathrm{d}\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \tag{2.20}$$

Differentiating equation (2.20) gives,

$$\begin{aligned}
I'(r) &= \sum_{n=1}^{\infty} 2n|a_n|^2 r^{2n-1}\\
I''(r) &= \sum_{n=1}^{\infty} 2n(2n-1)|a_n|^2 r^{2n-2}\\
I^{(3)}(r) &= \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)|a_n|^2 r^{2n-3}\\
I^{(4)}(r) &= \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4}
\end{aligned}$$

And

$$
\begin{aligned}
w'(z) &= \sum_{n=1}^{\infty} n a_n z^{n-1} \\
w''(z) &= \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} \\
\Rightarrow \ |w''(z)|^2 &= \sum_{n=2}^{\infty} n^2(n-1)^2 |a_n|^2 |z|^{2n-4} \\
\Rightarrow \ I^{(4)}(r) &\leq K |w''(re^{i\theta})|^2 \quad \text{by lemma } (1.1.11)
\end{aligned}
$$

Let us find K by comparing the coefficients between $I^{(4)}(r)$ and $|F''(re^{i\theta})|^2$ to find K as follows:

$$
\begin{aligned}
2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4} &\leq K n^2 (n-1)^2 |a_n|^2 r^{2n-4} \\
4(2n-1)(n-1)(2n-3) &\leq K n (n-1)^2 \\
4(2n-1)(2n-3) &\leq K n(n-1) \\
4(2-\frac{1}{n})(2-\frac{1}{n-1}) &\leq K
\end{aligned}
$$

So, $K = 16$ is smallest such constant as $n \to \infty$.

$$
I^{(4)}(r) = \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4} \leq 16 \sum_{n=2}^{\infty} n^2(n-1)^2 |a_n|^2 |z|^{2n-4}
$$

$$
\Rightarrow \ I^{(4)}(r) \leq \frac{16}{2\pi} \int_0^{2\pi} |w''(re^{i\theta})|^2 \mathrm{d}\theta \tag{2.21}
$$

Differentiating $w(z) = (\phi')^{\frac{-1}{2}}$ twice we get $w''(re^{i\theta})$ as follows:-

$$
\begin{aligned}
w'(z) &= \frac{-\phi''}{2\phi'\sqrt{\phi'}} \\
w''(z) &= -\frac{1}{2} \ \left[\left(\frac{\phi'''\phi' + \phi''^2}{\phi'^2}\right) - \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)^2 \right] \\
w''(z) &= -\frac{1}{2} \ (\phi')^{-\frac{1}{2}} \ \left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi''}{\phi'}\right) - \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)^2 \right] \\
w''(z) &= -\frac{1}{2} \ w(z) \ \left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi''}{\phi'}\right) - \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)^2 \right] \quad ; \quad (|z| < 1)
\end{aligned}
$$

where,

$$\{S_\phi\} = \left[\frac{\mathrm{d}}{\mathrm{d}z}(\frac{\phi''}{\phi'}) - \frac{1}{2}(\frac{\phi''}{\phi'})^2\right] \quad \text{is so called } \textbf{Schwarzian derivative}.$$
$$w''(z) = -\frac{1}{2}\, w(z)\, \{S_\phi\}; \text{ it is known that } \{S_\phi\} \leq \frac{6}{(1-|z|^2)^2}.$$

We return to inequality (2.20) to find the final formula for the differential inequality of $I_p(r)$.

$$\begin{aligned} I^{(4)}(r) \leq \frac{16}{2\pi}\int_0^{2\pi} |w''(re^{i\theta})|^2 \mathrm{d}\theta &= \frac{16}{2\pi}\int_0^{2\pi} \left|-\frac{1}{2}\, w(z)\, \{S_\phi\}\right|^2 \mathrm{d}\theta \\ &= 4\,|\{S_\phi\}|^2 \left(\frac{1}{2\pi}\int_0^{2\pi} |w(z)|^2 \mathrm{d}\theta\right). \\ &\leq 4\left(\frac{36}{(1-|z|^2)^4}\right)\left(\frac{1}{2\pi}\int_0^{2\pi} |w(z)|^2 \mathrm{d}\theta\right). \\ &= 144(1-r^2)^{-4} I_p(r) \qquad \text{by equation (3.21)} \end{aligned}$$
$$\Rightarrow I^{(4)}(r) \leq 144(1-r^2)^{-4} I_p(r); \text{ for } a \leq r < 1 \text{ and suitable } a < 1. \tag{2.22}$$

Now, we need to check if the inequality (2.22) bounded or not, that is,

$$I^{(4)}(r) \leq 144(1-r^2)^{-4} I_p(r) < C(1-r)^{-4} I_p(r)$$

such that,

$$\Rightarrow 144(1+r)^{-4} < C; \quad a \leq r < 1$$
$$\frac{144}{(1+r)^4} < C.$$

If $r = 1$ then $C > 9$.

So, the value of C is slightly more than 9.(that is; $C = 9.01$ will do).

$$\Rightarrow I^{(4)}(r) \leq 144(1-r^2)^{-4} I_p(r) < 9.01(1-r)^{-4} I_p(r)$$

This differential Inequality is corresponding to the differential equation below by lemma (1.1.11):

$$\nu^{(4)}(r) = 9.01(1-r)^{-4}\, \nu(r) \tag{2.23}$$

Which has a solution

$$\nu(r) = A(1-r)^{-\beta}; \text{where} A \text{is a constant and} \beta > 0.$$

We attempt to find the value of β as follows:

$$\begin{aligned}
\nu'(r) &= A\beta(1-r)^{-\beta-1} \\
\nu''(r) &= A\beta(\beta+1)(1-r)^{-\beta-2} \\
\nu^{(3)}(r) &= A\beta(\beta+1)(\beta+2)(1-r)^{-\beta-3} \\
\nu^{(4)}(r) &= A\beta(\beta+1)(\beta+2)(\beta+3)(1-r)^{-\beta-4}
\end{aligned}$$

Substituting $\nu(r)$, $\nu^{(4)}(r)$ in equation (2.22) we get,

$$\begin{aligned}
A\beta(\beta+1)(\beta+2)(\beta+3)(1-r)^{-\beta-4} &= 9.01\,(1-r)^{-4}\,(A(1-r)^{-\beta}) \\
\Rightarrow\ \beta(\beta+1)(\beta+2)(\beta+3) &= 9.01.
\end{aligned}$$

Is a quartic polynomial with the roots:

$$\beta_1 = -3.6009,\ \beta_2 = 0.6009,\ \beta_3 = -1.5 - 1.3834i,\ \beta_4 = -1.5 + 1.3834i \qquad \text{for} j = 1, 2, 3, 4.$$

Hence, when j goes from $1 \to 4$, $Re(\beta_j)$ will be less than 0.6009, such that $Re\beta < 0.601$. Therefore,

$$\nu(r) = A(1-r)^{-0.601}$$

is the solution of (2.23).

Choosing a sufficiently large constant A in front of the $(1-r)^{-0.601}$- term, allow us to apply Lemma (1.1.11) so that $I^{(4)}(r_0) > \nu^{(4)}(r_0)$ for $k = 0, 1, 2, 3.$ at $r_0 \leq r < 1$.

$$\begin{aligned}
I(r) < \nu(r)\ ,\ \forall r,\ \ r_0 \leq r < 1 \\
\Rightarrow I(r) < \nu(r) = A(1-r)^{-0.601} = O(1-r)^{-0.601}.
\end{aligned}$$

□

Theorem 2.1.2. *[52] If ϕ is holomorphic and univalent in unit disk $\mathbb{D}$, then*

$$I_p(r, \phi') = O((1-r)^{-\beta}) \quad \textit{as } r \to 1-0 \quad \textit{for} \quad p \in \mathbb{R} \tag{2.24}$$

and

$$\beta > -\frac{1}{2} + p + \sqrt{\frac{1}{4} - p + 4p^2} \tag{2.25}$$

Proof. :- We have

$$I_p(r, \phi') = \frac{1}{2\pi} \int_0^{2\pi} |\phi'(re^{i\theta})|^p \mathrm{d}\theta \tag{2.26}$$

Multiplying both sides of equation (2.51) by the $(r^2(\frac{\partial}{\partial r})^2)$ using Theorem (2.0.8) and Lemma (1.1.5), and with a bit of calculations, we get differential inequality of $I_p(r, \phi')$.

$$r^2(\frac{\partial}{\partial r})^2 I_p = r^2(\frac{\partial}{\partial r})^2(\frac{1}{2\pi} \int_0^{2\pi} |\phi'(re^{i\theta})|^p \mathrm{d}\theta)$$
$$r^2(\frac{\partial}{\partial r})^2 I_p = r\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^p \mathrm{d}\theta\right)\right].$$
$$r\frac{\partial}{\partial r}(rI_p') = \frac{p^2}{2\pi}\int_0^{2\pi} |\phi'|^p |z\frac{\phi''}{\phi'}|^2 \mathrm{d}\theta$$
$$r^2 I_p''(r) + rI_p'(r) = \frac{p^2}{2\pi}\int_0^{2\pi} |\phi'|^p |z\frac{\phi''}{\phi'}|^2 \mathrm{d}\theta$$

$$r^2 I_p''(r) = \left(\frac{p^2}{2\pi}\int_0^{2\pi} |\phi'|^p \left|z\frac{\phi''}{\phi'}\right|^2 \mathrm{d}\theta\right) - rI_p'(r). \tag{2.27}$$

Since $I_p'(r) \geq 0 \Rightarrow -I_p'(r) \leq 0$.

$$\left(\frac{p^2}{2\pi}\int_0^{2\pi} |\phi'|^p |z\frac{\phi''}{\phi'}|^2 \mathrm{d}\theta\right) - rI_p'(r) \leq \frac{p^2}{2\pi}\int_0^{2\pi} |\phi'|^p \left|z\frac{\phi''}{\phi'}\right|^2 \mathrm{d}\theta$$

By inequality (2.27);

$$r^2 I_p''(r) \leq \frac{p^2}{2\pi}\int_0^{2\pi} |\phi'|^p \left|z\frac{\phi''}{\phi'}\right|^2 \mathrm{d}\theta \tag{2.28}$$

Now, since ϕ is univalent, by lemma (1.1.5), we can write:

$$\left|z\frac{\phi''}{\phi'}\right| = \left|z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} + \frac{2r^2}{1-r^2}\right|,$$

and we obtain that

$$r^2 I_p''(r) \leq \frac{p^2}{2\pi} \int_0^{2\pi} |\phi'|^p \left| \left(z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} \right) + \frac{2r^2}{1-r^2} \right|^2 \mathrm{d}\theta \tag{2.29}$$

Simplifying the inequality (2.28) gives:

$$\left| z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} + \frac{2r^2}{1-r^2} \right|^2 \leq \left(\left| z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} \right| + \left| \frac{2r^2}{1-r^2} \right| \right)^2 = \left| z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} \right|^2 + \left| \frac{2r^2}{1-r^2} \right|^2 + 2\left| z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} \right| \left| \frac{2r^2}{1-r^2} \right| \tag{2.30}$$

Using lemma (1.1.5), we have

$$\left| z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} \right|^2 \leq \frac{16r^2}{(1-r^2)^2}.$$

Also,

$$\begin{aligned} \left| \frac{2r^2}{1-r^2} \right|^2 &= \frac{4r^4}{(1-r^2)^2} \\ 2\left| z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} \right| \left| \frac{2r^2}{1-r^2} \right| &= \left| z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} \right| \left| \frac{4r^2}{1-r^2} \right| \\ &= \left[\mathrm{Re}(z\frac{\phi''}{\phi'}) - \frac{2r^2}{1-r^2} \right] (\frac{4r^2}{1-r^2}); \qquad 0 \leq r < 1 \end{aligned}$$

Equation (2.30) becomes

$$\left| z\frac{\phi''}{\phi'} - \frac{2r^2}{1-r^2} + \frac{2r^2}{1-r^2} \right|^2 \leq \frac{16r^2}{(1-r^2)^2} + \frac{4r^4}{(1-r^2)^2} + \left[\mathrm{Re}(z\frac{\phi''}{\phi'}) - \frac{2r^2}{1-r^2} \right] (\frac{4r^2}{1-r^2})$$

Hence, inequality (2.29) becomes

$$r^2 I_p''(r) \leq \frac{p^2}{2\pi} \int_0^{2\pi} |\phi'|^p \left(\frac{16r^2}{(1-r^2)^2} + \frac{4r^4}{(1-r^2)^2} \right) \mathrm{d}\theta + \frac{p^2}{2\pi} \int_0^{2\pi} |\phi'|^p \left[\mathrm{Re}(z\frac{\phi''}{\phi'}) \right] (\frac{4r^2}{1-r^2}) \mathrm{d}\theta$$

$$- \frac{p^2}{2\pi} \int_0^{2\pi} |\phi'|^p \left(\frac{8r^4}{(1-r^2)^2} \right) \mathrm{d}\theta \tag{2.31}$$

Note that

$$I_p(r) = \frac{1}{2\pi} \int_0^{2\pi} |\phi'|^p \mathrm{d}\theta \tag{2.32}$$

$$\text{Let}\quad \phi(z) = u + iv \;\text{ then } \phi' = \frac{\partial \phi}{\partial r} = \frac{1}{e^{i\theta}}(u_r + iv_r); \qquad \text{(see equation (2.11))}$$
$$\Rightarrow\ |\phi'| = \sqrt{u_r^2 + v_r^2}$$

When $|\phi'|$ is raised to the power p, then

$$\Rightarrow\ |\phi'|^p = (u_r^2 + v_r^2)^{\frac{p}{2}}$$

$$\begin{aligned}
\Rightarrow\ \frac{\partial |\phi'|^p}{\partial r} &= \frac{p}{2}\,(u_r^2 + v_r^2)^{\frac{p}{2}-1}\,(2u_r u_{rr} + 2v_r v_{rr}).\\
&= p\,(u_r^2 + v_r^2)^{\frac{p}{2}}\,\frac{(u_r u_{rr} + v_r v_{rr})}{(u_r^2 + v_r^2)}\\
&= p\,|\phi'|^p\,\Re(\frac{\phi''}{\phi'}).
\end{aligned}$$

and,

$$\begin{aligned}
\frac{\partial}{\partial r} I_p(r) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial r}|\phi'|^p \mathrm{d}\theta\\
I'_p(r) &= \frac{p}{2\pi}\int_0^{2\pi} |\phi'|^p \Re(\frac{\phi''}{\phi'})\mathrm{d}\theta
\end{aligned}$$

Multiplying both sides by r, we obtain

$$rI'_p(r) = \frac{p}{2\pi}\int_0^{2\pi} |\phi'|^p \mathrm{Re}(z\frac{\phi''}{\phi'})\mathrm{d}\theta \tag{2.33}$$

Substituting equations (2.32) & (2.33) in equation (2.31) we get

$$\begin{aligned}
r^2 I_p''(r) &\le \left(\frac{16p^2r^2}{(1-r^2)^2} + \frac{4p^2r^4}{1-r^2}\right) I_p(r) + \left(\frac{4pr^2}{1-r^2}\right) rI'_p(r) - \left(\frac{8p^2r^4}{(1-r^2)^2}\right) I_p(r)\\
r^2 I_p''(r) &\le \left(\frac{16p^2r^2}{(1-r^2)^2} + \frac{4p^2r^4}{1-r^2} - \frac{8r^4}{(1-r^2)^2}\right) I_p(r) + \left(\frac{4pr^3}{1-r^2}\right) I'_p(r)
\end{aligned}$$

$$r^2 I_p''(r) \le \left(\frac{16p^2r^2 - 4p^2r^4}{(1-r^2)^2}\right) I_p(r) + \left(\frac{4pr^3}{1-r^2}\right) I'_p(r) \tag{2.34}$$

Equation (2.34) is a differential inequality. So, we have to compare (2.34) with the differential equation (1.11) in lemma (1.1.10).

To apply lemma (1.1.10) we have to find r_0 lies between 0 and r as we already have $0 \le r < 1$.

Given $\epsilon > 0$ we find $r_0 = r_0 < 1$ such that, $0 < r_0 \leq r < 1$ as follows:

Dividing equation (2.34) by r^2 leads to

$$I_p''(r) \leq \left(\frac{4pr^3}{r^2(1-r^2)}\right) I'_p(r) + \left(\frac{16p^2r^2 - 4p^2r^4}{r^2(1-r^2)^2}\right) I_p(r) \tag{2.35}$$

We show that each term in equation (2.33) is bounded by ϵ.

$$\frac{4pr^3}{r^2(1-r^2)} = \frac{4pr^3}{r^2(1-r)(1+r)} = \frac{2p}{(1-r)} \leq \frac{2p+\epsilon}{(1-r)}\,; \qquad \text{when } r \to 1.$$

And the same thing with respect to the second term.

$$\frac{16p^2r^2 - 4p^2r^4}{r^2(1-r^2)^2} = \frac{4p^2r^2(4-r^2)}{r^2(1-r)^2(1+r)^2} = \frac{12p^2}{4(1-r)^2} \leq \frac{3p^2+\epsilon}{(1-r)^2}\,; \qquad \text{when } r \to 1.$$

$$\Rightarrow \; I_p''(r) \leq \frac{2p+\epsilon}{1-r} I_p'(r) + \frac{3p^2+\epsilon}{(1-r)^2} I_p(r) \tag{2.36}$$

Now, we compare equation (2.36) with DE below.

$$\nu''(r) = \frac{2p+\epsilon}{1-r}\nu'(r) + \frac{3p^2+\epsilon}{(1-r)^2}\nu(r), \tag{2.37}$$

which is $Cauchy - EulerEquation$, the solution of this equation is defined as follows:

$$\nu(r) = c(1-r)^{-\alpha}$$
$$\nu'(r) = \alpha c(1-r)^{-\alpha-1}$$
$$\nu''(r) = \alpha(\alpha+1)c(1-r)^{-\alpha-2}$$

We can rewrite equation (2.35) as follows:

$$(1-r)^2\nu''(r) = (2p+\epsilon)(1-r)\nu'(r) + (3p^2+\epsilon)\nu(r)$$
$$(1-r)^2[\alpha(\alpha+1)c(1-r)^{-\alpha-2}] = (2p+\epsilon)(1-r)[\alpha c(1-r)^{-\alpha-1}] + (3p^2+\epsilon)c(1-r)^{-\alpha}.$$

$$\Rightarrow \; [\alpha(\alpha+1) - \alpha(2p+\epsilon) - (3p^2+\epsilon)](1-r)^{-\alpha} = 0$$

Knowning that $(1-r)^{-\alpha} > 0$, it is a positive solution to the equation (2.37)

$$\alpha(\alpha+1) - \alpha(2p+\epsilon) - (3p^2+\epsilon) = 0$$
$$\alpha^2 + (1-2p-\epsilon)\alpha - (3p^2+\epsilon) = 0.$$

Let's look for its limit when $\epsilon \longrightarrow 0$.

$$\lim_{\epsilon \to 0}[\alpha^2 + (1-2p-\epsilon)\alpha - (3p^2+\epsilon)] = 0$$
$$\alpha^2 + (1-2p)\alpha - 3p^2 = 0$$

$$\Rightarrow \ \alpha = \frac{-(1-2p) \pm \sqrt{(1-2p)^2 + 12p^2}}{2} = -\frac{1}{2} + p + \sqrt{4p^2 - p + \frac{1}{4}}$$

$\Rightarrow$ α tends to the quantity on the right - hand side of equation (2.25).

Choose the constant c in $\nu(r) = c(1-r)^{-\alpha}$ so large enough to make

$$\nu(r_0) > I(r_0) \ \ and \ then \ \ \nu'(r_0) = \alpha c(1-r_0)^{-\alpha-1} > I'(r_0)$$

Then by lemma (1.1.10), we obtain

$$I(r) < \nu(r) \ \ for \ \ r_0 \leq r < 1$$
$$I(r) < \frac{c}{(1-r)^{\alpha}}$$
$$\Rightarrow \ I(r) = O((1-r)^{-\beta}).$$

□

Lemma 2.1.3. *[41] Let ϕ be holomorphic on the unit disk $\mathbb{D}$, and assume that $|\phi(z)| \leq 1$ for all z, and $\phi(a) = b$ for some a, b in $\mathbb{D}$, then*

$$|\phi'(a)| \leq \frac{1-|b|^2}{1-|a|^2}.$$

Main results:

According to the theorem (2.1.2) above, there are questions to raise; Does the integral means of the derivative of univalent function

$$I_{-p}(r, \phi') = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta}{|\phi'(re^{i\theta})|^p}, \tag{2.38}$$

exist for some values of p? Theorem below is an answer to the questions raised, and improving to what was proved by Pommerenke in [52].

Theorem 2.1.4. *If ϕ is an univalent function in unit disk $\mathbb{D}$ then, as $r \to 1-0$,*

$$I_{-p}(r, \phi') = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{|\phi'(re^{i\theta})|^p} < \infty \quad \textit{for} \quad -1.1697 < p < \frac{2}{3} \tag{2.39}$$

Proof. Given

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{|\phi'(re^{i\theta})|^p} = \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^{-p}\mathrm{d}\theta$$

Suppose that, $p = -1-\alpha$. It is worth to mention that this hypothesis was adopted by Brennan [9], when $0 \le \alpha < 0.399$ to prove that

$$\int_0^{2\pi} |\phi'(re^{i\theta})|^p \mathrm{d}\theta < \infty \quad \text{for} \quad -1-\alpha < p < \frac{2}{3}$$

.

Hence, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^{-p}\mathrm{d}\theta &= \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^{1+\alpha}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|\,(|\phi'(re^{i\theta})|)^{\alpha}\mathrm{d}\theta \\
&= O(1-r)^{-3\alpha}\ \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta \quad \textit{(Distortion theorem)}
\end{aligned}$$

Now, we have to calculate the integral means

$$\mathrm{M}_p(r, \phi') = \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta$$

Applying *Hölder inequality*, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta \le \left(\int_0^{2\pi} |\phi'(re^{i\theta})|^p\mathrm{d}\theta\right)^{\frac{1}{p}} \left(\int_0^{2\pi} |\phi'(re^{i\theta})|^q\mathrm{d}\theta\right)^{\frac{1}{q}}$$

So, when $\frac{1}{p} = \frac{1-\delta}{2}$, as known that $\frac{1}{p}+\frac{1}{q} = 1 \Rightarrow q = \frac{2}{1+\delta}$. This implies to

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta \le \left(\int_0^{2\pi} |\phi'(re^{i\theta})|^2\mathrm{d}\theta\right)^{\frac{1-\delta}{2}} \left(\int_0^{2\pi} |\phi'(re^{i\theta})|^{\frac{2\delta}{1+\delta}}\mathrm{d}\theta\right)^{\frac{1+\delta}{2}} \tag{2.40}$$

We have to estimate two integrals on the right-hand side inequality (2.40) as follows:

By lemma (3.1.1) we have

$$\left(\int_0^{2\pi} |\phi'(re^{i\theta})|^2 \mathrm{d}\theta\right)^{\frac{1-\delta}{2}} = O((1-r)^{-1})^{\frac{1-\delta}{2}}.$$

By using theorem (1) in [52] when $p = \frac{2\delta}{1+\delta}$.

$$\left(\int_0^{2\pi} |\phi'(re^{i\theta})|^{\frac{2\delta}{1+\delta}} \mathrm{d}\theta\right)^{\frac{1+\delta}{2}} = O((1-r)^{-\beta})^{\frac{1+\delta}{2}} = O(1-r)^{\frac{-\beta(1+\delta)}{2}}$$

Finally,

$$\int_0^{2\pi} |\phi'(re^{i\theta})| \mathrm{d}\theta = O(1-r)^{\frac{-\beta(1+\delta)}{2}} O((1-r))^{\frac{-(1-\delta)}{2}}$$
$$\int_0^{2\pi} |\phi'(re^{i\theta})| \mathrm{d}\theta = O(1-r)^{\frac{-1}{2}(1+\delta)\beta - \frac{1}{2} + \frac{\delta}{2}}$$

for every β satisfies;

$$\beta > -\frac{1}{2} + p + \sqrt{(\frac{1}{4} - p + 4p^2)} \qquad \text{see Theorem 2.1.2.}$$

by choosing $\delta = 0.0364;\ (0 < \delta < 1)$, then the value of β as follows:

$$\beta > -\frac{1}{2} + p + \sqrt{(\frac{1}{4} - p + 4p^2)} = -\frac{1}{2} + \frac{2\delta}{1+\delta} + \sqrt{\frac{1}{4} - p + 4(\frac{2\delta}{1+\delta}^{2})}$$

It immediately follows that $\beta > 0.0168898$.

Suppose $\beta = 0.0169$

$$\begin{aligned}
\Rightarrow \int_0^{2\pi} |\phi'(re^{i\theta})| \mathrm{d}\theta &= O(1-r)^{\frac{-1}{2}(1+\delta)\beta - \frac{1}{2} + \frac{\delta}{2}} \\
&= O(1-r)^{\frac{-1}{2}(1.0364)(0.0169) - 0.5 + 0.0182} \\
&= O(1-r)^{-0.00875758 - 0.4818} \\
&= O(1-r)^{-0.49055758} \\
&= O(1-r)^{-0.491}.
\end{aligned}$$

Hence,

$$\int_0^{2\pi} |\phi'(re^{i\theta})|^{-p} \mathrm{d}\theta = O(1-r)^{-0.491}\, O(1-r)^{-3\alpha}$$
$$\int_0^1 \int_0^{2\pi} |\phi'(re^{i\theta})|^{-p} \mathrm{d}\theta \mathrm{d}r = \int_0^1 O(1-r)^{-3\alpha-0.491} \mathrm{d}r$$

This implies that $-3\alpha - 0.491 + 1 > 0 \Rightarrow \alpha < 0.1697$.

By assumption, we have $p = -1 - \alpha \Rightarrow p > -1.1697$, this implies to $-1.1697 < p < \frac{2}{3}$, depending on *Brennan's* result.

$\square$

Theorem 2.1.5. *If ϕ is an univalent function in $\mathbb{D}$ with $|\phi(z)| \leq 1$ for all z and $\phi(a) = b$ for some $a, b \in \mathbb{D}$ then, as $r \to 1 - 0$,*

(a) The range of the pth-power integrable function in Brennan's conjecture has increased to be $\frac{4}{3} < p < 5$ depending on behaviour of self-conformal maps

(b) $|a_n| \leq e^1$, where $e^1 = 2.71828$.

Proof. (a) The integral

$$\int\int_{\mathbb{D}} |\phi'|^p \mathrm{d}x\mathrm{d}y < \infty, \text{ for } -1-\alpha < \lambda < \frac{2}{3}, \quad \text{(cf.[9])}.$$

Let $\lambda = -1 - \alpha$

$$
\begin{aligned}
\int_0^1 \int_0^{2\pi} |\phi'|^{\lambda} \mathrm{d}\theta &= \int_0^1 \int_0^{2\pi} |\phi'|^{-1-\alpha} r\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^1 \int_0^{2\pi} |\phi'|^{-\alpha} |\phi'|^{-1} r\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^1 \int_0^{2\pi} O\left((1-r)^{-1}\right)^{\alpha} \; |\phi'|^{-1} r\mathrm{d}r\mathrm{d}\theta; \quad \textit{Distortion}\text{ theorem } (1.1.6) \\
&\leq \int_0^1 \int_0^{2\pi} O(1-r)^{-\alpha} \left(\frac{1-|\phi(z)|^2}{1-|z|^2}\right)^{-1} r\mathrm{d}r\mathrm{d}\theta; \quad \textit{(Schwarz-Pick lemma (2.1.3))} \\
&\leq \int_0^1 \int_0^{2\pi} O(1-r)^{-\alpha} \left(\frac{1}{1-|z|}\right)^{-1} r\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^1 \int_0^{2\pi} O(1-r)^{-\alpha} \frac{1}{(1-r)^{-1}} r\mathrm{d}r\mathrm{d}\theta \\
&= C \int_0^1 \frac{r\mathrm{d}r}{(1-r)^{\alpha-1}} \\
&\leq C \int_0^1 \frac{\mathrm{d}r}{(1-r)^{\alpha-1}} \\
&= \int_0^1 (1-r)^{-\alpha+1} \mathrm{d}r = \left[\frac{(1-r)^{-\alpha+2}}{-\alpha+2}\right|_0^1 .
\end{aligned}
$$

when $-\alpha + 2 > 0 \Rightarrow \lambda = -1 - \alpha > -3 \Rightarrow -3 < \lambda < \frac{2}{3}$.

Brennan's conjecture stated that,

$$
\int\int_{\mathbb{D}} |\phi'|^{2-p} \mathrm{d}x\mathrm{d}y = \int\int_{\Omega} |\psi'|^{p} \mathrm{d}x\mathrm{d}y < \infty; \quad \text{whenever } \frac{4}{3} < p < 4.
$$

Hence, Let $2 - p = \lambda \Rightarrow p = 2 - \lambda \Rightarrow \frac{4}{3} < p < 5$.

We notice that the range of the pth-power integrable function in *Brennan's conjecture* has increased depending on behaviour of self-conformal maps. □

Proof. (b) Given $\phi(z)$ is bounded and univalent function in $\mathbb{D}$, that is; $\phi(z)$ is a holomorphic function in $\mathbb{D}$. Our aim is to estimate the coefficients of ϕ which are denoted $|a_n|$

as follows:

$$\phi(z) = a_0 + a_1 z + a_2 z^2 \ldots + a_n z^n + \ldots$$
$$\phi'(z) = a_1 + 2a_2 z \ldots + n a_n z^n + \ldots$$
$$n a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\phi'(z)}{z^n} \mathrm{d}z$$

Let $z = re^{i\theta}$ for $0 \le r < 1$.

$$|n a_n| = |\frac{1}{2\pi i}\int_0^{2\pi} \frac{\phi'(re^{i\theta})}{r^n e^{in\theta}} re^{i\theta}\mathrm{d}\theta| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|\phi'(re^{i\theta})|}{r^{n-1}}\mathrm{d}\theta$$

$$r^n |n a_n| \le \frac{r}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta \tag{2.41}$$

As well-known that

$$l(r) = r\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta; \quad (0 \le r < 1), \tag{2.42}$$

is the length of the curve $\{\phi(z) : |z| = r\}$. and let $r = 1 - \frac{1}{n+1} = \frac{n}{n+1}$, Since

$$\begin{aligned}
r^n |n a_n| &\le \frac{r}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta. \\
|n a_n| &\le \frac{e^1}{2\pi}(\frac{n}{n+1})\int_0^{2\pi}\left(\frac{1-|\phi(z)|^2}{1-|z|^2}\right)\mathrm{d}\theta; \quad \textit{(Schwarz-Pick lemma (2.1.3))} \\
|n a_n| &\le \frac{e^1}{n+1}\left(\frac{1}{1-|z|}\right) \\
|a_n| &\le \frac{e^1}{n+1}\left(\frac{1}{1-r}\right). \\
|a_n| &\le e^1, \qquad \text{where } \frac{1}{1-r} = n+1
\end{aligned}$$

□

2.2 Boundary behaviour for modulus of the derivative of conformal mapping of polygon region to unit disk and its inverse

We start this section with basic definition of the polygonal domain and we present some typical examples to examine the boundary behaviour of the derivatives of conformal mapping of polygon to unit disk, as well as the boundary behaviour of the derivatives of the inverse maps.

Definition 2.2.1 (Polygon). A polygon P is usually defined as a collection of n vertices $v_1, v_2, ..., v_n$ and n edges $v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1$ such that no pair of nonconsecutive edges share a point. We deviate from the usual practice by defining a polygon as the closed finite connected region of the plane bounded by these vertices and edges. The collection of vertices and edges will be referred to as the boundary of P, denoted by ∂P,[1] a polygon of n vertices will sometimes be called an n-gon.

Riemann mapping theorem guarantees the existence of a conformal map ϕ from polygonal domain $P \subset \mathbb{C}$ conformally onto the unit disk $\mathbb{D}$ $(|w| < 1)$, which can be extended continuously to the boundary (cf. *Carathéodory's theorem 1.1.2*). Worth of a mention that is not yet possible write down a simple formula for the conformal map from one region to another. Hence, in case of a map from the (upper half-plane $(\Im z > 0)$ or unit disk $\mathbb{D}$) to a polygon, then *Schwarz- Christoffel* formula [17]

allows to compute the conformal map ϕ defined as follows:

Consider $\phi : P \longrightarrow \mathbb{D}$, be a conformal mapping, where ∂P be a circular arc or straight line segment γ, normalized by the conditions $\phi(z_0) = 0$ and $\phi'(z_0) > 0$ (where z_0 is

[1]The term *"polygon"* is often modified by *"simple"* to distinguish it from polygons that cross themselves, *Simple polygons* are also called *Jordan polygons*, because such a polygon divides the plane into two regions. The boundary of a polygon is a *"Jordan curve"* [2]: it separates the plane into two disjoint regions, the interior and the exterior of the polygon.

some point in P), and observe that ϕ maps γ onto an arc $\hat{\gamma}$ of the unit circle $|w| = 1$ with $\psi(w) = \phi^{-1}(z) : \mathbb{D} \longrightarrow P$.

$$\phi(z) = A + C \int^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} \mathrm{d}\zeta$$

where A and C are suitably chosen constants (cf. Theorem 1.1.8).

So, we deliberately construct ϕ as the composition of one *Schwarz-Christoffel* map from P into upper half-plane (by applying *Schwarz- Christoffel transformation*) and another map of the upper half-plane to unit disk , in the examples (2.2.2), (2.2.3) where ϕ maps the P to unit disk.

This technique could help to study the behavior of conformal mapping by estimating some quantities which belong to interior upper half-plane, which in turn will help to analyze the boundary behaviour of the derivative of this map.

In the following some examples describe of the above:

Example 2.2.2. The derivative of the conformal mapping defined on Rectangular domain to the Unit disk is bounded but the derivative of the inverse maps is unbounded.

Solution. Map the interior of the rectangular domain with vertices at points $z_1 = 1$, $z_2 = -1$, $z_3 = -a$, and $z_4 = a$ with $a > 1$ into upper half plane needs to define *Schwarz-Christoffel transformtion* which maps $\mathbb{H}^+$ into rectangle as follows:

$$z = A + B \int_{\zeta_o}^{\zeta} \frac{\mathrm{d}s}{(s-1)^{\frac{1}{2}}(s+1)^{\frac{1}{2}}(s-a)^{\frac{1}{2}}(s+a)^{\frac{1}{2}}}$$
$$z = A + B \int_{\zeta_o}^{\zeta} \frac{\mathrm{d}s}{(s^2-1)^{\frac{1}{2}}(s^2-a^2)^{\frac{1}{2}}}$$

Suppose $A = 0$ and $B = 1$ for convenience.

$$z = \int_{\zeta_o}^{\zeta} \frac{\mathrm{d}s}{\sqrt{(s^2-1)(s^2-a^2)}} \tag{2.43}$$

Suppose $\zeta_o = 0$ and let $a = \frac{1}{k}$, then the integral (2.52) is transformed to

$$z = \int_{\zeta_o}^{\zeta} \frac{\mathrm{d}s}{\sqrt{(s^2-1)(s^2-(\frac{1}{k})^2)}} = k \int_{0}^{\zeta} \frac{\mathrm{d}s}{\sqrt{(s^2-1)(k^2s^2-1)}}$$

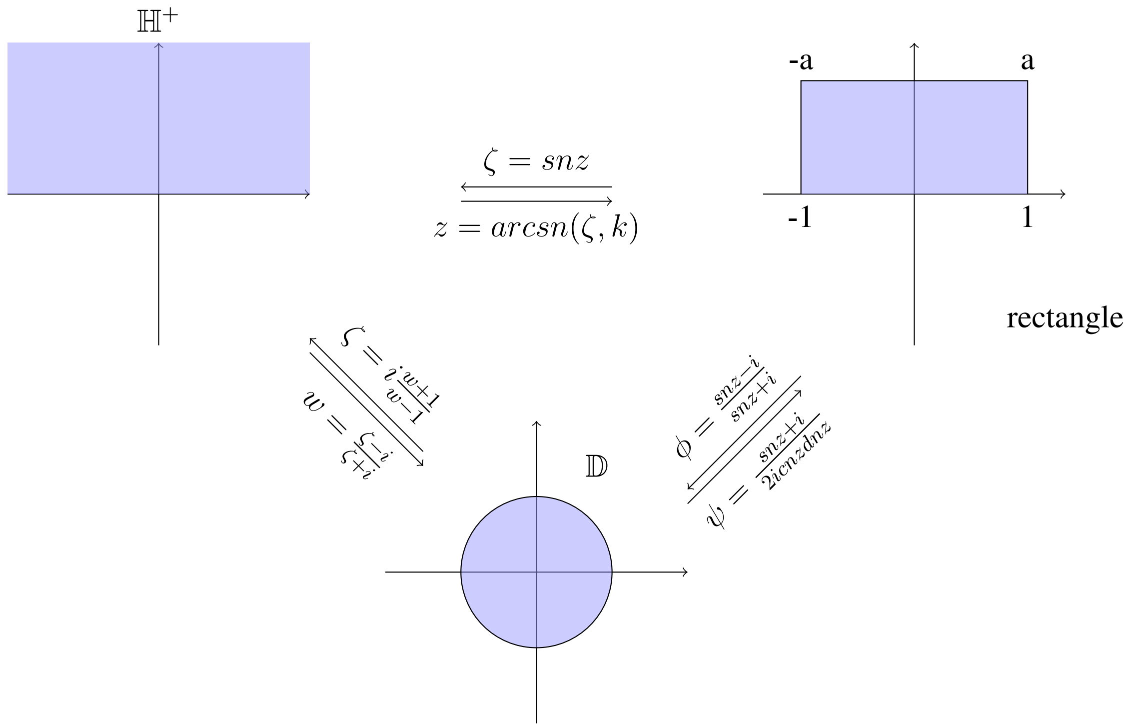


Figure 2.1: Conformal mappings from rectangular domain onto unit disk

$$z = k\int_0^\zeta \frac{\mathrm{d}s}{\sqrt{(1-s^2)(1-k^2s^2)}} \tag{2.44}$$

The integral (2.44) is called an elliptic integral of first kind and k is a modulus of the elliptic integral with $0 < k < 1$, denote by

$$z = sn^{-1}(\zeta, k). \tag{2.45}$$

The inverse mapping of integral (2.44) is known as the *Jacobi elliptic* function denoted by

$$\begin{aligned}
&\zeta = sn(z,k). \\
\Rightarrow\ &\phi = \frac{\zeta - i}{\zeta + i} = \frac{snz - i}{snz + i} : \text{Rectangular domain} \longrightarrow \mathbb{D} \\
&\phi' = \frac{(snz+i)(cnz\ dnz) - (snz-i)(\ dnz)}{(snz+i)^2} = \frac{2i\ cnz\ dnz}{(snz+i)^2}
\end{aligned}$$

such that; $dnz = \sqrt{1-k^2sn^2z}$, $cnz = \sqrt{1-sn^2z}$.

$$|\phi'| = |\frac{2i\ cnz\ dnz}{(snz+i)^2}| = \frac{2|cnz\ dnz|}{|snz+i|^2}$$

$$snz \in H^+ \Rightarrow\ snz+i \in H^+ \text{ that is; } |snz+i| \geq 1\ \Rightarrow\ \frac{1}{|snz+i|} \leq 1$$

$$|cnz\ dnz| = |\sqrt{1-k^2sn^2z}|\ |\sqrt{1-sn^2z}|$$

$$|1-k^2sn^2z| \leq 1+|snz|^2$$

$$|1-sn^2z| \leq 1+|k\ snz|^2$$

$$\Rightarrow\ |\phi'| = \frac{2|cnz\ dnz|}{|snz+i|^2} \leq 2(1+|snz|^2)(1+|k\ snz|^2)$$

$\Rightarrow |\phi'|$ is bounded.

It remains to show that the inverse of the derivative of such function $\phi(z)$ is unbounded as follows:

$$\psi = \phi^{-1} : \mathbb{D} \longrightarrow \text{rectangular domain}$$

$$\psi' = \frac{1}{\phi'} = \frac{(snz+i)^2}{2i\ cnz\ dnz}$$

$$\Rightarrow\ \psi = \frac{(snz+i)^2}{2i\ cnz\ dnz}$$

$$|\psi'| = \frac{|snz+i|^2}{2|cnz\ dnz|}$$

$snz \in H^+ \Rightarrow (snz+i) \in H^+ \Rightarrow\ |snz+i|^2 \geq 1.$

$$|\psi'| = \frac{|snz+i|^2}{2|cnz\ dnz|} \geq \frac{1}{2|cnz\ dnz|}$$

$\Rightarrow |\psi'|$ is unbounded as $|z| \longrightarrow 0$.

Example 2.2.3. The derivative of the conformal mapping defined on Triangular domain $\triangle$ to the unit disk is bounded but the derivative of the inverse maps is unbounded.

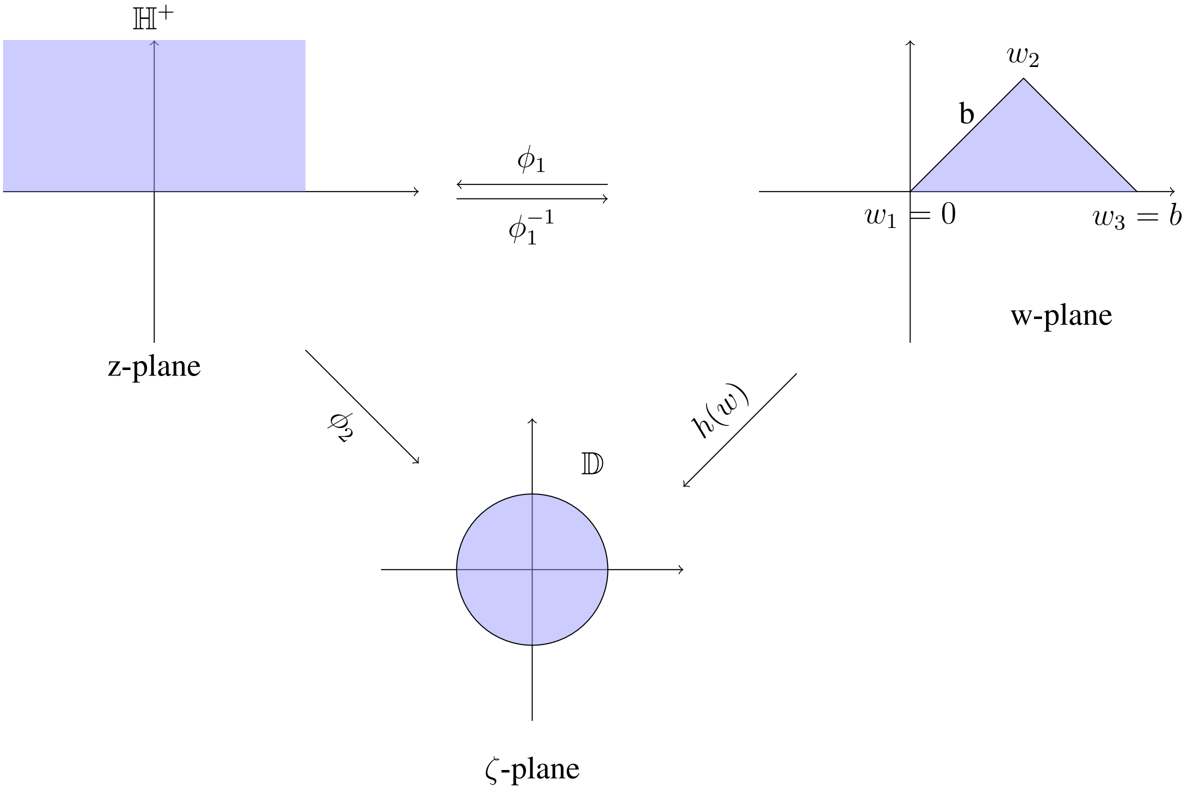


Figure 2.2: Conformal mappings from triangular domain onto unit disk

Solution. To construct conformal mapping defined on triangle domain to unit disk $\mathbb{D}$. we have to define conformal mapping on triangular domain to upper half plane H^+ and then define another mapping from $\mathbb{H}^+$ to unit disk $\mathbb{D}$ to achieve our aim.

To do so, First : we have to establish conformal mapping that maps upper half- plane $\mathbb{H}^+$ onto triangular domain $\triangle$ by *Schwarz-Christoffel* transformation as follows:

Let

$$\phi_1 = A \int_{z_o}^{z} (s - x_1)^{-k_1} \, (s - x_2)^{-k_2} \mathrm{d}s + B.$$

be such that $-k_i = \frac{\alpha_i}{\pi} - 1$; $\forall i = 1, 2$ to be *Schwars-Christoffel* transformation that maps $\mathbb{H}^+$ into the interior of the equilateral triangle $\triangle$ such that $\alpha_i = \frac{\pi}{3}$; $\forall i = 1, 2, 3$.

Now, by assisstance that $z_o = 1$, $A = 1$ and $B = 0$, we obtain *Schwars-Christoffel* transformation defined as follows:

$$\phi_1 = \int_1^z (s+1)^{\frac{-2}{3}} (s-1)^{\frac{-2}{3}} \mathrm{d}s. \tag{2.46}$$

Which maps $x_1 = -1, x_2 = 1$ and $x_3 = \infty$ into $\triangle\ w_1 w_2 w_3$ as follows:

i. In case $z = 1 \Rightarrow\ \phi_1(1) = 0$; that is , $w_1 = 0$ in H^+.

ii. In case $z = -1$, we have

$$\phi_1(-1) = \int_1^{-1} (s+1)^{\frac{-2}{3}} (s-1)^{\frac{-2}{3}} \mathrm{d}s = w_2. \tag{2.47}$$

iii. In case $z = \pm\infty$, we have

$$\lim_{z\to\pm\infty} \phi_1 = \int_1^{\pm\infty} (s+1)^{\frac{-2}{3}} (s-1)^{\frac{-2}{3}} \mathrm{d}s = w_3. \tag{2.48}$$

To solve these integrals, let us consider first the equation (2.47) by choosing a path of the integration $z = x$ along the real axis in the positive sense, that is; by writing

$$s - 1 = |s-1|\, e^{i\theta_1}$$
$$s + 1 = |s+1|\, e^{i\theta_2}.$$

The argument $(\theta_1 + \theta_2)$ remains constant throughout integration from -1 to 1 since $(s+1)$ stays positive with zero argument, and $(s-1)$ has constant argument π. Therefore equation (2.47) yields

$$w_2 = \phi_1(-1) = -\int_{-1}^{1} (x+1)^{\frac{-2}{3}} (1-x)^{\frac{-2}{3}} (-e^{-\frac{2\pi i}{3}}) \mathrm{d}s.$$

$$w_2 = \phi_1(-1) = e^{\frac{\pi i}{3}} \int_0^1 \frac{2\mathrm{d}x}{(1-x^2)^{\frac{-2}{3}}}. \tag{2.49}$$

By letting $x = \sqrt{t}$ in equation (2.49), we obtain *Beta function* $\mathcal{B}(\frac{1}{2}, \frac{1}{3})$ and

$$w_2 = \phi_1(-1) = e^{\frac{\pi i}{3}}\ \mathcal{B}(\frac{1}{2}, \frac{1}{3}).$$

$$\Rightarrow\quad w_2 = \mathbf{b}\, e^{\frac{\pi i}{3}}.$$

where b is the value of $\mathcal{B}(\frac{1}{2},\frac{1}{3})$ Now, the vertex w_3 lies on the positive u- axis.

So, w_3 must be represented by the boundary integral 1 to ∞ as follows:

$$w_3 = \phi_1(\infty) = \int_1^{\infty} (x+1)^{\frac{-2}{3}}\ (1-x)^{\frac{-2}{3}} \mathrm{d}x$$

$$w_3 = \phi_1(\infty) = \int_1^{\infty} \frac{\mathrm{d}x}{(x^2-1)^{\frac{-2}{3}}}. \tag{2.50}$$

But w_3 is also represented by integral (2.55) when $z = -\infty$ along the negative real axis.

So,

$$\begin{aligned}
w_3 = \phi_1(-\infty) &= \int_1^{-\infty} (x+1)^{\frac{-2}{3}}\ (1-x)^{\frac{-2}{3}} \mathrm{d}x \\
&= \int_1^{-1} (x+1)^{\frac{-2}{3}}\ (1-x)^{\frac{-2}{3}}\ e^{\frac{-2\pi i}{3}} \mathrm{d}x\ +\ \int_{-1}^{-\infty} (x+1)^{\frac{-2}{3}}\ (1-x)^{\frac{-2}{3}}\ e^{\frac{-4\pi i}{3}} \mathrm{d}x. \\
w_3 &= w_1 + e^{\frac{-\pi i}{3}} \int_1^{\infty} \frac{\mathrm{d}x}{(x^2-1)^{\frac{2}{3}}} \\
\Rightarrow\quad w_3 &= w_1 + e^{\frac{-\pi i}{3}}\ w_3.
\end{aligned} \tag{2.51}$$

Solving (2.51) for w_3 we obtain:

$$\begin{aligned}
\Rightarrow\quad & w_3 - e^{\frac{-\pi i}{3}}\ w_3 = w_1. \\
& w_3\ (1 - e^{\frac{-\pi i}{3}}) = b\, e^{\frac{-\pi i}{3}} \\
& w_3 = b\ ; \qquad \text{since } (1 - e^{\frac{-\pi i}{3}}) = e^{\frac{-\pi i}{3}}.
\end{aligned}$$

In the end, we found the conformal mapping that maps upper half- plane $\mathbb{H}^+$ onto triangular domain $\triangle$.

It is known that $\phi_2 = \frac{z-i}{z+i}$ maps upper half- plane $\mathbb{H}^+$ onto unit disk $\mathbb{D}$.

Hence, we have

$$\begin{aligned}
\phi_1 &= \int_1^z (s^2-1)^{\frac{-2}{3}} \mathrm{d}s :\ \text{upper half plane} \mathbb{H}^+ \longrightarrow \text{triangular domain } \triangle \\
\text{and}\quad \phi_2 &= \frac{z-i}{z+i} :\ \text{upper half- plane}\mathbb{H}^+ \to \text{unit disk}\mathbb{D}.
\end{aligned}$$

Let

$$h(w): \text{ triangular domain } \triangle \longrightarrow \text{ unit disk } \mathbb{D}.$$

defined as follows:

$$\begin{aligned}
h(w) &= (\phi_2 \circ \phi_1^{-1})(w) = \phi_2(\phi_1^{-1})(w) \\
h'(w) &= \phi_2'(\phi_1^{-1})(\phi_1'^{-1}) \\
&= (\frac{\phi_1^{-1} - i}{\phi_1^{-1} + i})' \, (\frac{1}{\phi_1'}) \\
&= \frac{(\phi_1^{-1} + i)(\phi_1'^{-1}) - (\phi_1^{-1} - i)(\phi_1'^{-1})}{(\phi_1^{-1} + i)^2} \, \frac{1}{\phi_1'}. \\
\Rightarrow \; h'(w) &= \frac{2i(z^2 - 1)^{\frac{2}{3}}}{(\phi_1^{-1} + i)^2} \, (z^2 - 1)^{\frac{2}{3}} \\
&= \frac{2i(z^2 - 1)^{\frac{4}{3}}}{(\phi_1^{-1} + i)^2}. \\
|h'(w)| &= |\frac{2i(z^2 - 1)^{\frac{4}{3}}}{(\phi_1^{-1} + i)^2}| = \frac{2i|z^2 - 1|^{\frac{4}{3}}}{|\phi_1^{-1} + i|^2},
\end{aligned}$$

where $(\phi_1^{-1})' = \frac{1}{\phi_1'} = (z^2 - 1)^{\frac{2}{3}}$ and $\phi_1^{-1} \in \mathbb{H}^+$, so that $(\phi_1^{-1} + i) \in \mathbb{H}^+$.

This implies to

$$\begin{aligned}
&|\phi_1^{-1} + i|^2 \geq 1 \;\Rightarrow\; \frac{1}{|\phi_1^{-1} + i|^2} \leq 1 \\
&\Rightarrow\; |h'(w)| \leq 2(|z|^2 + 1)^{\frac{4}{3}}.
\end{aligned}$$

$|h'(w)|$ is bounded.

What remains is to prove that the inverse of the derivative of $h(w)$ is unbounded.

Note that; $\phi_2^{-1} = -i\frac{\zeta+1}{\zeta-1}$ is the *Möbius transformation*, it maps the unit disk $\mathbb{D}$ to upper half-plane $\mathbb{H}^+$, thus we have

$$(\phi_2^{-1})' = \frac{(\zeta - 1)(-i) - (-i\zeta - i)}{(\zeta - 1)^2} = \frac{2i}{(\zeta - 1)^2}.$$

We use *Schwarz-Christoffel* transformation for mapping $\mathbb{H}^+$ into triangular domain $\triangle$.

$$\phi_1 = \int_1^z (s^2 - 1)^{\frac{-2}{3}} \mathrm{d}s.$$

Therefore,

$$
\begin{aligned}
h^{-1}(\zeta) &= (\phi_1 \circ \phi_2^{-1})(\zeta) : \mathbb{D} \longrightarrow \triangle \ . \\
(h^{-1}(\zeta))' &= \phi_1'(\phi_2^{-1}(\zeta))\ (\phi_2^{-1}(\zeta))' \\
&= [(\phi_2^{-1}(\zeta))^2 - 1]^{\frac{-2}{3}}\ \frac{2i}{(\zeta-1)^2}. \\
&= \frac{2i\left[-(\frac{\zeta+1}{\zeta-1})^2 - 1\right]^{\frac{-2}{3}}}{(\zeta-1)^2} \\
&= \frac{2i\left[\frac{-2\zeta^2-2}{(\zeta-1)^2}\right]^{\frac{-2}{3}}}{(\zeta-1)^2}. \\
&= \frac{2i}{(\zeta-1)^2\ \left[\frac{-2\zeta^2-2}{(\zeta-1)^2}\right]^{\frac{-2}{3}}} \\
&= \frac{2i}{(\zeta-1)^2\ (-2\zeta^2-2)^{\frac{2}{3}}(\zeta-1)^{\frac{-4}{3}}} \\
&= \frac{2i}{(\zeta-1)^{\frac{2}{3}}\ (-2\zeta^2-2)^{\frac{2}{3}}}.
\end{aligned}
$$

$$
\begin{aligned}
|(h^{-1}(\zeta))'| &= \left|\frac{2i}{(\zeta-1)^{\frac{2}{3}}\ (-2\zeta^2-2)^{\frac{2}{3}}}\right|. \\
|(h^{-1}(\zeta))'| &= \left|\frac{2}{(\zeta-1)^{\frac{2}{3}}(2\zeta^2+2)^{\frac{2}{3}}}\right| \longrightarrow \infty \ \text{ as } \ \zeta \longrightarrow 1.
\end{aligned}
$$

Example 2.2.4. The derivative of the conformal mapping defined on crescent domain to unit disk is bounded but the derivative of the inverse maps is unbounded.

Solution. Compute the conformal mapping of a crescent domain onto unit disk by setting a sequence of functions as follows:

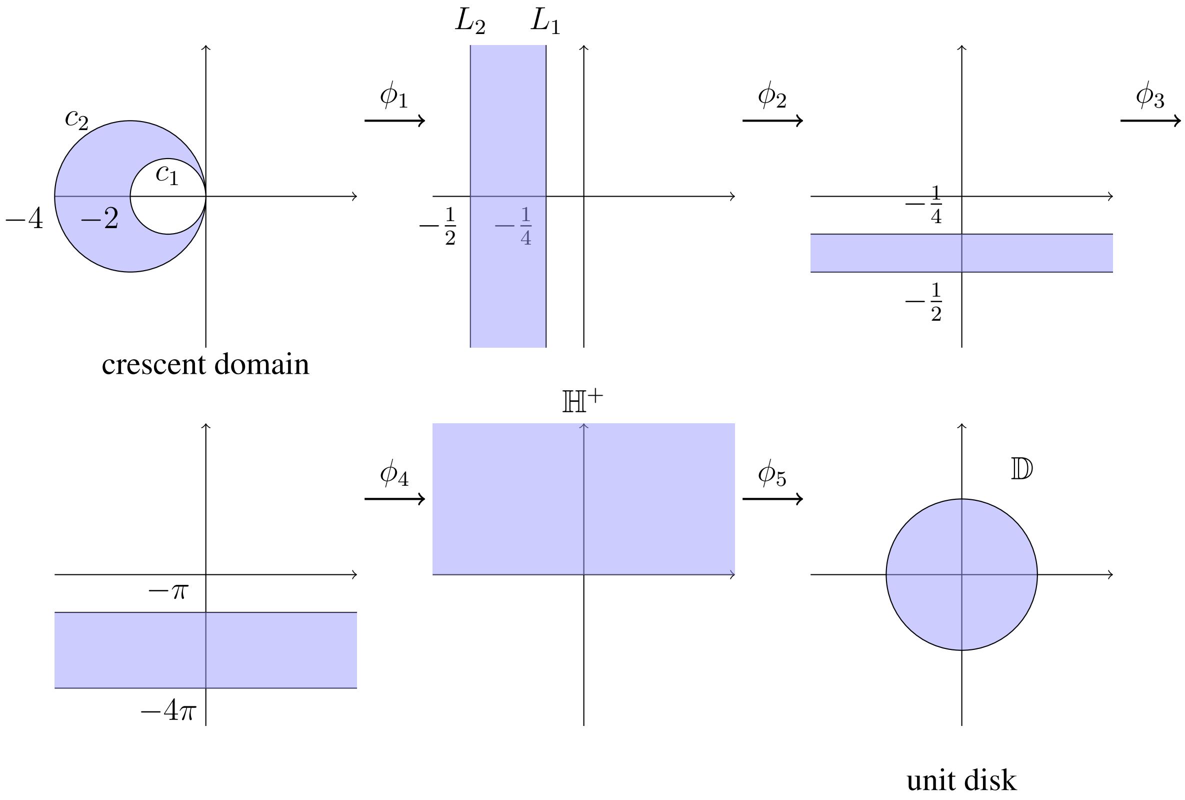


Figure 2.3: Conformal mappings from crescent domain onto unit disk

Let

$\phi_1 = \dfrac{1}{z}$ maps $C_1 \to L_1$ and $C_2 \to L_2$.

$\phi_2 = \dfrac{i}{z}$ rotates the stripe in the left plane onto stripe in the lower half plane.

$\phi_3 = \dfrac{4\pi i}{z}$ extends the stripe in the lower half plane $\mathbb{H}^-$ between $-\pi, -2\pi$

$\phi_4 = e^{\frac{4i\pi}{z}}$ maps the stripe in the lower half plane $\mathbb{H}^-$into $\mathbb{H}^+$.

$\phi_5 = \dfrac{e^{\frac{4i\pi}{z}} - i}{e^{\frac{4i\pi}{z}} + i}$ maps the $\mathbb{H}^+$ onto unit disk $\mathbb{D}$.

So;

$$\phi(z) = \frac{e^{\frac{4i\pi}{z}} - i}{e^{\frac{4i\pi}{z}} + i} : \text{crescent domain} \longrightarrow \text{unit disk } \mathbb{D}.$$

Using short calculation we obtain

$$\phi'(z) = \frac{(e^{\frac{4i\pi}{z}} + i)(\frac{-4i\pi}{z^2} e^{\frac{4i\pi}{z}}) - (e^{\frac{4i\pi}{z}} - i)(\frac{-4i\pi}{z^2} e^{\frac{4i\pi}{z}})}{(e^{\frac{4i\pi}{z}} + i)^2}.$$

$$\Rightarrow \ \phi'(z) = \frac{\frac{8\pi}{z^2} e^{\frac{4i\pi}{z}}}{(e^{\frac{4i\pi}{z}} + i)^2} = \frac{8\pi e^{\frac{4i\pi}{z}}}{z^2(e^{\frac{4i\pi}{z}} + i)^2}.$$

$$|\phi'(z)| = \frac{8\pi|e^{\frac{4i\pi}{z}}|}{|z|^2|e^{\frac{4i\pi}{z}} + i|^2}.$$

Now;

$$|e^{\frac{4i\pi}{z}}| = 1 \ \Rightarrow \ e^{\frac{4i\pi}{z}} \in H^+ \ \Rightarrow \ (e^{\frac{4i\pi}{z}} + i) \in H^+.$$

$$\Rightarrow \ |e^{\frac{4i\pi}{z}} + i| \geq 1 \ \Rightarrow \ |e^{\frac{4i\pi}{z}} + i|^2 \geq 1$$

$$\Rightarrow \ \frac{1}{\left|e^{\frac{4i\pi}{z}} + i\right|^2} \leq 1.$$

$$\Rightarrow \ |\phi'(z)| = \frac{8\pi|e^{\frac{4i\pi}{z}}|}{|z|^2|e^{\frac{4i\pi}{z}} + i|^2}.$$

$$\leq \frac{8\pi}{|z|^2}$$

Hence, $|\phi'(z)|$ is bounded. We show that the inverse of the derivative of such function $\phi(z)$ is unbounded as follows:

$$\phi^{-1}(w) = \psi(w) : \mathbb{D} \longrightarrow \text{crescent domain.}$$

$$\psi = \phi^{-1} = \frac{1}{\phi} = \frac{e^{\frac{4\pi i}{z}} + i}{e^{\frac{4\pi i}{z}} - i}.$$

$$\Rightarrow \ \psi' = \frac{1}{\phi'}$$

$$\Rightarrow \ \psi' = \frac{z^2(e^{\frac{4i\pi}{z}} + i)}{8\pi e^{\frac{4i\pi}{z}}}$$

$$\Rightarrow \ |\psi'| = \frac{|z|^2|e^{\frac{4i\pi}{z}} + i|}{8\pi|e^{\frac{4i\pi}{z}}|}.$$

It is known, $e^{\frac{4i\pi}{z}} \in H^+ \Rightarrow (e^{\frac{4i\pi}{z}} + i) \in H^+$. Hence

$$|e^{\frac{4i\pi}{z}} + i| \geq 1 \ \& \ |e^{\frac{4i\pi}{z}}| = 1.$$

$$\Rightarrow \ |\psi'| = \frac{|z|^2|e^{\frac{4i\pi}{z}} + i|}{8\pi|e^{\frac{4i\pi}{z}}|} \geq \frac{|z|^2}{8\pi}.$$

In the end, we obtain $|\psi'|$ is unbounded.

Example 2.2.5. The derivative of the conformal mapping defined on Lens- shaped domain to the unit disk is bounded but the derivative of the inverse maps is unbounded.

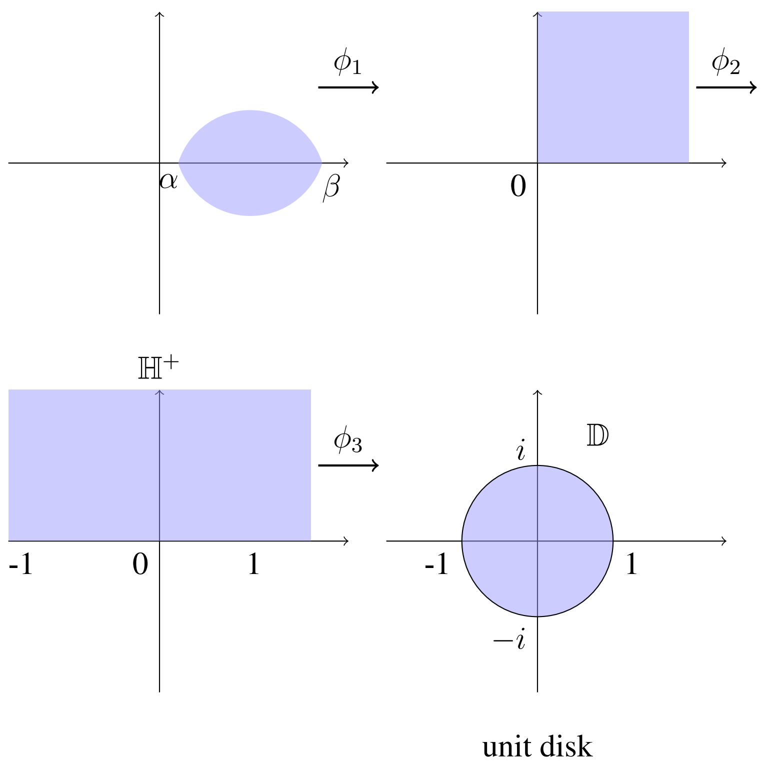


Figure 2.4: Conformal mappings from lens domain onto unit disk

Solution. Compute the conformal mapping a lens domain onto unit disk by setting a sequence of functions as follows:

Let

$$\phi_1 = \frac{z-\alpha}{z-\beta} \text{ maps Lens-shaped domain to the first quarter plane}$$

$$\phi_2 = z^2 \text{ maps the first quarter plane to the upper half- plane.}$$

$$\phi_3 = \frac{z^2-i}{z^2+i} \text{ maps upper half plane } \mathbb{H}^+ \text{ to the unit disk } \mathbb{D}.$$

So,

$$\phi(z) = \frac{(\frac{z-\alpha}{z-\beta})^2 - i}{(\frac{z-\alpha}{z-\beta})^2 + i} : \text{Lens-shaped domain} \longrightarrow \text{unit disk } \mathbb{D}.$$
$$\Rightarrow \phi(z) = \frac{(z-\alpha)^2 - i(z-\beta)^2}{(z-\alpha)^2 + i(z-\beta)^2}.$$

Therefore when

$$z = \alpha \Rightarrow \phi(z) = -1 \text{ in } \mathbb{D}$$
$$z = \beta \Rightarrow \phi(z) = 1 \text{ in } \mathbb{D}.$$

When

$$(z-\alpha)^2 = -(z-\beta)^2$$
$$\Rightarrow \phi(z) = \frac{-(z-\beta)^2 - i(z-\beta)^2}{-(z-\beta)^2 + i(z-\beta)^2} = i \text{ in } \mathbb{D}.$$

In the end, if

$$(z-\alpha)^2 = (z-\beta)^2$$
$$\Rightarrow \phi(z) = \frac{(z-\beta)^2 - i(z-\beta)^2}{(z-\beta)^2 + i(z-\beta)^2} = -i \text{ in } \mathbb{D}.$$

$$\Rightarrow \phi'(z) = \frac{4i(z-\beta)^2(z-\alpha) - 4i(z-\beta)(z-\alpha)^2}{[(z-\alpha)^2 + i(z-\beta)^2]^2}$$
$$\Rightarrow |\phi'(z)| = \frac{4|\alpha-\beta||z-\beta||z-\alpha|}{(z-\alpha)^4 + (z-\beta)^4}.$$

where, $4|\alpha - \beta| = c$; c is a constant; $(\alpha < \beta)$.

Also,

$$|z-\beta| \leq |z| + |\beta| = M_1$$
$$\& \ |z-\alpha| \leq |z| + |\alpha| = M_2$$
$$\Rightarrow |\phi'(z)| \leq \frac{M}{(z-\alpha)^4 + (z-\beta)^4}$$

$\Rightarrow |\phi'(z)|$ is a bounded for every z in *Lens-shaped domain*. Again, we can show that the inverse of the derivative of such function $\phi(z)$ is unbounded as follows:

$$\begin{aligned}\psi' = \frac{1}{\phi'} &= \frac{[(z-\alpha)^2 + i(z-\beta)^2]^2}{4i(z-\beta)^2(z-\alpha) - 4i(z-\beta)(z-\alpha)^2} \\ \Rightarrow\ |\psi'| &= \frac{|(z-\alpha)^2 + i(z-\beta)^2|^2}{4|(z-\beta)(z-\alpha)(\alpha-\beta)|} \\ \Rightarrow\ |\psi'| &= \frac{(z-\alpha)^4 + (z-\beta)^4}{4|z-\beta||z-\alpha||\alpha-\beta|}.\end{aligned}$$

If $z \longrightarrow \alpha$ or $z \longrightarrow \beta$, then $|\psi'| \longrightarrow \infty$, so $|\psi'|$ is unbounded.

The following examples show the integrability of the derivative of conformal maps on infinite sector $\mathbb{W}$ exists and is finite for some pth-power integrable function ϕ when $\alpha = \frac{\pi}{n}$ is a number for some integer n. Further details can be found in the books of Di Francesco [16] and of M. Stein [64].

Example 2.2.6. Let $\phi(z)$ be a conformal mapping defined on infinite sector W for the angle $\alpha = \frac{\pi}{2}$ onto unit disk $\mathbb{D}$ as follows:

$$\phi(z) = (\phi_2 \circ \phi_1)(z) = \frac{z^2 - i}{z^2 + i} : W \to \mathbb{D} \quad \text{so that} \quad \phi_1(z) = z^2.$$

If maps the infinite sector onto upper half plane $\mathbb{H}^+$ and $\phi_2(w) = \frac{w-i}{w+i}$ maps the upper half- plane $\mathbb{H}^+$ onto unit disk $\mathbb{D}$ (see Figure 2.5). Then the integrability of the derivative of conformal mapping ϕ, is as follows:

$$\int\int_W |\phi'(z)|^p \,\mathrm{d}x\mathrm{d}y < \infty; \quad \text{for each } p > \frac{2}{3}.$$

Solution. Given

$$\begin{aligned}\phi(z) &= \frac{z^2 - i}{z^2 + i} : W \to \mathbb{D} \\ \Rightarrow\ \phi'(z) &= \frac{(z^2+i)(2z) - (z^2-i)(2z)}{(z^2+i)^2} \\ &= \frac{i4z}{(z^2+i)^2} \\ \Rightarrow\ |\phi'(z)| &= \frac{4|z|}{|z^2+i|^2}.\end{aligned}$$

Now, $\mathbb{W}$-plane is an infinite sector. that is; $r = |z| \to 0 - \infty$.

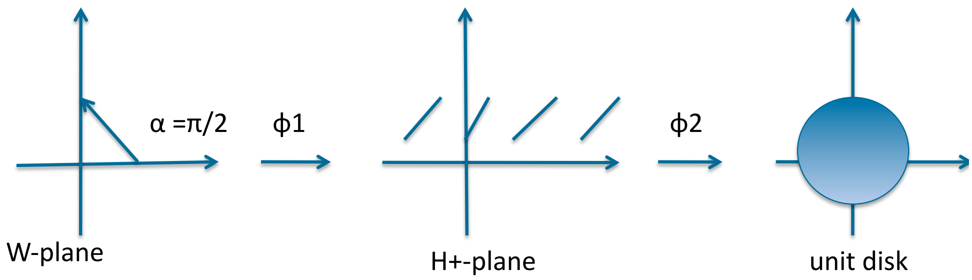


Figure 2.5: Infinite sector W for the angle $\alpha = \frac{\pi}{2}$

- so, $r = |z| \to \infty$ (i.e; $|z|$ is large). We know that

$$\begin{aligned} |z^2 + i| &\geq |z|^2 - 1 \geq \frac{1}{2}\,|z|^2. \\ \Rightarrow \frac{1}{|z^2+i|} &\leq \frac{2}{|z|^2}. \\ \Rightarrow \frac{1}{|z^2+i|^2} &\leq \frac{4}{|z|^4}, \end{aligned}$$

refer to the behaviour of $|z^2 + i|$ at ∞ with respect to the region.

$$\Rightarrow\ |\phi'(z)| = \frac{4|z|}{|z^2+i|^2} \leq \frac{16|z|}{|z|^4} = 16|z|^{-3}.$$

- and $\mathbf{r = |z| \sim 0}$

$$\Rightarrow\ |\phi'(z)| = \frac{4|z|}{|z^2+i|^2}\ \Rightarrow\ |\phi'(z)| = \frac{4|z|}{|i|^2} = 4|z|.$$

$$\Rightarrow\ |\phi'(z)| \leq \begin{cases} 16|z|^{-3} & ;\ |z| \ \text{ is large} \\ 4|z| & ; |z|\ \sim 0 \end{cases}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\infty} |\phi'|^p \,\mathrm{d}x\mathrm{d}y \leq 4\int_0^{\frac{\pi}{2}}\int_0^{1} |z|^p\, r\,\mathrm{d}r\mathrm{d}\theta + 16\int_0^{\frac{\pi}{2}}\int_1^{\infty} |z|^{-3p}\, r\,\mathrm{d}r\mathrm{d}\theta.$$
$$\int_0^{\frac{\pi}{2}}\int_0^{\infty} |\phi'|^p \,\mathrm{d}x\mathrm{d}y = 2\pi\left[\int_0^{1} r^{p+1}\,\mathrm{d}r\right] + 8\pi\left[\int_1^{\infty} r^{-3p+1}\,\mathrm{d}r\right].$$

$$\Rightarrow \int_0^{\frac{\pi}{2}}\int_0^{\infty} |\phi'|^p \,\mathrm{d}x\mathrm{d}y \leq 2\pi\left[\frac{r^{p+2}}{p+2}\right|_0^1 + 2\pi\left[\frac{r^{-3p+2}}{-3p+2}\right|_1^{\infty} \tag{2.52}$$

There are two definite integrals on the right-hand side of inequality (2.52).

The first one is clearly finite, and the second one is:

$$-3p + 2 < \ 0 \Rightarrow \ -3p < -2 \Rightarrow \ p > \frac{2}{3}.$$

Example 2.2.7. Let $\phi(z)$ be a conformal mapping defined on infinite sector $\mathbb{W}$ onto unit disk $\mathbb{D}$ as follows:

$$\phi(z) = (\phi_2 \circ \phi_1)(z) = \frac{z^4 - i}{z^4 + i} : W \to \mathbb{D}$$

such that $\phi_1(z) = z^4$ maps the infinite sector onto upper half plane $\mathbb{H}^+$ and $\phi_2(w) = \frac{w-i}{w+i}$ maps the upper half plane $\mathbb{H}^+$ onto unit disk $\mathbb{D}$ (see Figure 2.6). then the integrability of the derivative of conformal mapping is:

$$\int\int_W |\phi'(z)|^p \, \mathrm{d}x\mathrm{d}y < \infty \ \ ; \text{for each } p > \frac{2}{5}.$$

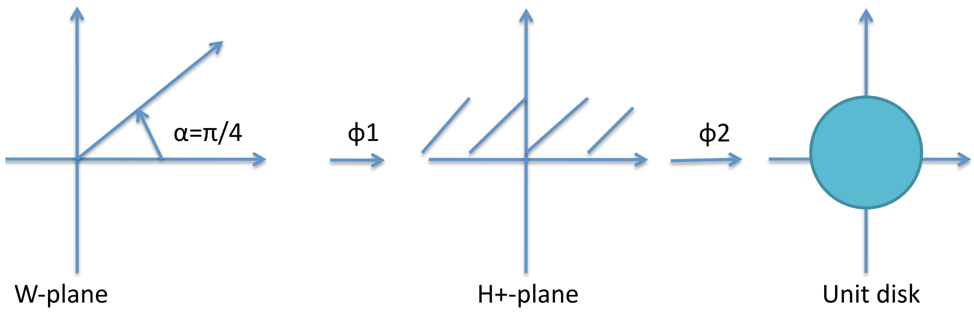


Figure 2.6: Infinite sector W for the angle $\alpha = \frac{\pi}{4}$

Solution. Given

$$\begin{aligned} \phi(z) &= \frac{z^4 - i}{z^4 + i} : W \to \mathbb{D} \\ \Rightarrow \ \phi'(z) &= \frac{(z^4 + i)(4z^3) - (z^4 - i)(4z^3)}{(z^4 + i)^2} \\ &= \frac{i8z^3}{(z^4 + i)^2} \\ \Rightarrow \ |\phi'(z)| &= \frac{8|z|^3}{|z^4 + i|^2}. \end{aligned}$$

Now, W-plane is an infinite sector. that is; $r = |z| \to 0 - \infty$.

- so,when $r = |z| \to \infty$ (that is; $|z|$ be large). We know that

$$|z^4 + i| \geq |z|^4 - 1 \geq \frac{1}{2}\,|z|^4.$$
$$\Rightarrow \frac{1}{|z^4 + i|} \leq \frac{2}{|z|^4}.$$
$$\Rightarrow \frac{1}{|z^4 + i|^2} \leq \frac{4}{|z|^8}.$$

This is referring to the behaviour of $|z^4 + i|$ at ∞ with respect to the region.

$$\Rightarrow |\phi'(z)| = \frac{8|z|^3}{|z^4 + i|^2} \leq \frac{8|z|^3}{|z|^8} = 32|z|^{-5}.$$

- and when $\mathbf{r} = |\mathbf{z}| \sim \mathbf{0}$

$$\Rightarrow |\phi'(z)| = \frac{8|z|^3}{|z^4 + i|^2} \Rightarrow |\phi'(z)| = \frac{8|z|^3}{|i|^2} = 8|z|^3.$$

$$\Rightarrow |\phi'(z)| \leq \begin{cases} 8|z|^{-5} & ; \ |z| \ \text{ is large} \\ 8|z|^3 & ; |z| \ \sim 0 \end{cases}$$

$$\int_0^{\frac{\pi}{4}} \int_0^{\infty} |\phi'|^p \, \mathrm{d}x\mathrm{d}y \leq 8 \int_0^{\frac{\pi}{4}} \int_0^1 r^{3p} r \, \mathrm{d}r\mathrm{d}\theta + 8 \int_0^{\frac{\pi}{4}} \int_1^{\infty} r^{-5p} r \, \mathrm{d}r\mathrm{d}\theta$$
$$= 2\pi \left[\int_0^1 r^{3p+1} \, \mathrm{d}r\right] + 2\pi \left[\int_1^{\infty} r^{-5p+1} \, \mathrm{d}r\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \int_0^{\infty} |\phi'|^p \, \mathrm{d}x\mathrm{d}y \leq 2\pi \, [\frac{r^{3p+2}}{3p+2}|_0^1 + 2\pi \, [\frac{r^{-5p+2}}{-5p+2}|_1^{\infty} \tag{2.53}$$

The first term on the right -hand- side of (2.53) is finite, and the second one is:

$$-5p + 2 < \ 0 \Rightarrow \ -5p < -2 \Rightarrow \ p > \frac{2}{5}.$$

Example 2.2.8. Let $\phi(z)$ be a conformal mapping defined on infinite sector $\mathbb{W}$ onto unit disk $\mathbb{D}$ as follows:

$$\phi(z) = (\phi_2 \circ \phi_1)(z) = \frac{z^n - i}{z^n + i} : W \to \mathbb{D}$$

such that $\phi_1(z) = z^n$ maps the infinite sector onto upper half- plane $\mathbb{H}^+$ where α is of the form $\alpha = \frac{\pi}{n}$ for some integer n and $\phi_2(w) = \frac{w-i}{w+i}$ maps the upper half- plane $\mathbb{H}^+$ onto unit disk $\mathbb{D}$ (see Figure 2.7). Then the integrability of the derivative of conformal mapping is as follows:

$$\int\int_W |\phi'(z)|^p \,\mathrm{d}x\mathrm{d}y < \infty \;\; ; \text{ for each } \;\; p > \frac{2}{\frac{3\pi}{\alpha}+1}.$$

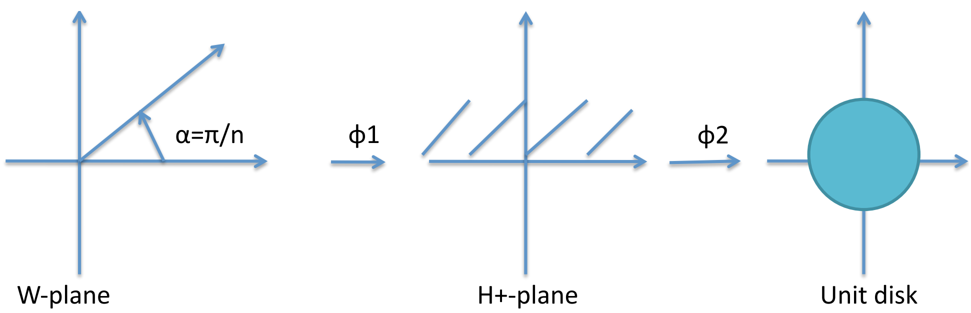


Figure 2.7: Infinite sector W for the angle $\alpha = \frac{\pi}{n}$

Solution. Let $\alpha = \frac{\pi}{n}$ be the angle of the infinite sector W which is mapped by $\phi_1 = z^n$ onto upper half plane $\mathbb{H}^+$.

One can write $\phi_1 = z^{\frac{\pi}{\alpha}}$. We define the power function $\phi_1 = z^{\frac{\pi}{\alpha}}$ to be the multivalued function

$$z^{\frac{\pi}{\alpha}} = e^{\frac{\pi}{\alpha}\log z} \;\; ; \;\; z \neq 0.$$

$$\begin{aligned}\Rightarrow \; z^{\frac{\pi}{\alpha}} &= e^{\frac{\pi}{\alpha}\log|z| + i arg z} \\ &= r^{\frac{\pi}{\alpha}}\, e^{i\frac{\pi}{\alpha}\theta}\, e^{\pm i\frac{2\pi^2}{\alpha}k}\end{aligned}$$

Various values of $z^{\frac{\pi}{\alpha}}$ are obtained from the principal value $e^{\frac{\pi}{\alpha}\log z}$ by multiplying by the integral power $(e^{i\frac{2\pi^2}{\alpha}})^k$ of $e^{i\frac{2\pi^2}{\alpha}}$.

Let $\alpha = \frac{\pi}{n}$ is a number for some integer n, then the integral powers $e^{i\frac{2\pi^2}{\alpha}k}$ of $e^{i\frac{2\pi^2}{\alpha}}$ are exactly the nth roots of unity, and the values of $z^{\frac{\pi}{\alpha}}$ are the n nth roots of z.

$$\begin{aligned}\Rightarrow \phi(z) &= \frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i} \\ &= \frac{r^{\frac{\pi}{\alpha}}\, e^{i\frac{\pi}{\alpha}} - i}{r^{\frac{\pi}{\alpha}}\, e^{i\frac{\pi}{\alpha}} + i}\end{aligned}$$

$$\phi(z) = \frac{r^{\frac{\pi}{\alpha}}\ \cos\frac{\pi}{\alpha}\theta + ir^{\frac{\pi}{\alpha}}\ \sin\frac{\pi}{\alpha}\theta - i}{r^{\frac{\pi}{\alpha}}\ \cos\frac{\pi}{\alpha}\theta + ir^{\frac{\pi}{\alpha}}\ \sin\frac{\pi}{\alpha}\theta + i}. \tag{2.54}$$

Simplify last equation (2.54) we get:

$$\begin{aligned}\phi(z) &= \frac{r^{\frac{2\pi}{\alpha}} - 2ir^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}\theta - 1}{r^{\frac{2\pi}{\alpha}} + 2r^{\frac{\pi}{\alpha}}\sin\frac{\pi}{\alpha}\theta + 1} \\ &= \frac{r^{\frac{2\pi}{\alpha}} - 1}{r^{\frac{2\pi}{\alpha}} + 2r^{\frac{\pi}{\alpha}}\sin\frac{\pi}{\alpha}\theta + 1} + i\frac{-2r^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}\theta}{r^{\frac{2\pi}{\alpha}} + 2r^{\frac{\pi}{\alpha}}\sin\frac{\pi}{\alpha}\theta + 1}\end{aligned}$$

When $\theta = 0$ or $\theta = 2\pi$, it implies that:

$$\begin{aligned}\phi(z) &= \frac{r^{\frac{2\pi}{\alpha}} - 1}{r^{\frac{2\pi}{\alpha}} + 1} + i\frac{-2r^{\frac{\pi}{\alpha}}}{r^{\frac{2\pi}{\alpha}} + 1} \\ &= \frac{r^{\frac{2\pi}{\alpha}} - 1 - 2ir^{\frac{\pi}{\alpha}}}{r^{\frac{2\pi}{\alpha}} + 1}\end{aligned}$$

The derivative of $\phi(z)$ can be calculated:

$$\begin{aligned}\phi'(z) &= \frac{(r^{\frac{2\pi}{\alpha}} + 1)[\frac{2\pi}{\alpha}\, r^{\frac{2\pi}{\alpha}-1} - 2i\frac{\pi}{\alpha}\, r^{\frac{\pi}{\alpha}-1}] - (r^{\frac{2\pi}{\alpha}} - 1 - 2ir^{\frac{\pi}{\alpha}})[\frac{2\pi}{\alpha}r^{\frac{2\pi}{\alpha}-1}]}{(r^{\frac{2\pi}{\alpha}} + 1)^2} \\ &= \frac{2i\frac{\pi}{\alpha}r^{\frac{3\pi}{\alpha}-1} + \frac{4\pi}{\alpha}r^{\frac{4\pi}{\alpha}-1} - 2i\frac{\pi}{\alpha}r^{\frac{\pi}{\alpha}-1}}{(r^{\frac{2\pi}{\alpha}} + 1)^2} \\ &= \frac{\frac{4\pi}{\alpha}r^{\frac{4\pi}{\alpha}-1} + i\frac{2\pi}{\alpha}(r^{\frac{3\pi}{\alpha}-1} - r^{\frac{\pi}{\alpha}-1})}{(r^{\frac{2\pi}{\alpha}} + 1)^2} \\ |\phi'(z)| &= \frac{\left|\frac{4\pi}{\alpha}r^{\frac{4\pi}{\alpha}-1} + i\frac{2\pi}{\alpha}(r^{\frac{3\pi}{\alpha}-1} - r^{\frac{\pi}{\alpha}-1})\right|}{(r^{\frac{2\pi}{\alpha}} + 1)^2} \\ |\phi'(z)| &= \frac{\sqrt{\frac{16\pi^2}{\alpha^2}\, r^{\frac{8\pi}{\alpha}-2} + \frac{4\pi^2}{\alpha^2}(r^{\frac{3\pi}{\alpha}-1} - r^{\frac{\pi}{\alpha}-1})^2}}{(r^{\frac{2\pi}{\alpha}} + 1)^2}\end{aligned}$$

Again, W-plane is an infinite sector. that is; $|z| \to 0 - \infty$.

i. In case $r = |z| \to \infty$ (that is; $|z|$ be large).

$$|\phi'(z)| = \frac{\sqrt{\frac{16\pi^2}{\alpha^2}\ |z|^{\frac{8\pi}{\alpha}-2} + \frac{4\pi^2}{\alpha^2}(|z|^{\frac{3\pi}{\alpha}-1} - |z|^{\frac{\pi}{\alpha}-1})^2}}{(|z|^{\frac{2\pi}{\alpha}}+1)^2}.$$

We know that,

$$|z|^{\frac{2\pi}{\alpha}} + 1 \geq |z^{\frac{2\pi}{\alpha}} + 1| \geq |z|^{\frac{2\pi}{\alpha}} - 1 \geq \frac{1}{2}|z|^{\frac{2\pi}{\alpha}}$$
$$\Rightarrow\ |z|^{\frac{2\pi}{\alpha}} + 1 \geq \frac{1}{2}|z|^{\frac{2\pi}{\alpha}}$$
$$\Rightarrow\ \frac{1}{(|z|^{\frac{2\pi}{\alpha}}+1)^2} \leq \frac{4}{|z|^{\frac{4\pi}{\alpha}}}.$$

Hence,

$$|\phi'(z)| = \frac{\sqrt{\frac{16\pi^2}{\alpha^2}\ |z|^{\frac{8\pi}{\alpha}-2} + \frac{4\pi^2}{\alpha^2}(|z|^{\frac{3\pi}{\alpha}-1} - |z|^{\frac{\pi}{\alpha}-1})^2}}{(|z|^{\frac{2\pi}{\alpha}}+1)^2} \leq \frac{4\sqrt{\frac{16\pi^2}{\alpha^2}\ |z|^{\frac{8\pi}{\alpha}-2} + \frac{4\pi^2}{\alpha^2}(|z|^{\frac{3\pi}{\alpha}-1} - |z|^{\frac{\pi}{\alpha}-1})^2}}{|z|^{\frac{4\pi}{\alpha}}}.$$

$$\begin{aligned} |\phi'(z)| \leq \frac{4\sqrt{\frac{16\pi^2}{\alpha^2}\ r^{\frac{8\pi}{\alpha}-2} + \frac{4\pi^2}{\alpha^2}(r^{\frac{3\pi}{\alpha}-1} - r^{\frac{\pi}{\alpha}-1})^2}}{r^{\frac{4\pi}{\alpha}}} &= \frac{4\sqrt{\frac{16\pi^2}{\alpha^2}\ r^{\frac{8\pi}{\alpha}-2} + \frac{4\pi^2}{\alpha^2} r^{\frac{2\pi}{\alpha}-2}(r^{\frac{2\pi}{\alpha}} - 1)^2}}{r^{\frac{4\pi}{\alpha}}} \\ &= \frac{4\sqrt{\frac{4\pi^2}{\alpha^2}\ r^{\frac{2\pi}{\alpha}-2}\left[4r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}} - 1)^2\right]}}{r^{\frac{4\pi}{\alpha}}} \end{aligned}$$

$$\begin{aligned} |\phi'(z)| \leq \frac{4\sqrt{\frac{4\pi^2}{\alpha^2}\ r^{\frac{2\pi}{\alpha}-2}\left[4r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}} - 1)^2\right]}}{r^{\frac{4\pi}{\alpha}}} &= \frac{\frac{8\pi}{\alpha}\ r^{\frac{\pi}{\alpha}-1}\sqrt{4\, r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}} - 1)^2}}{r^{\frac{4\pi}{\alpha}}} \\ &= \frac{8\pi}{\alpha}\ r^{\frac{-3\pi}{\alpha}-1}\sqrt{4\, r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}} - 1)^2} \end{aligned}$$

ii. In case $\mathbf{r} = |\mathbf{z}| \sim \mathbf{0}$

$$\begin{aligned} |\phi'(z)| &= \sqrt{\frac{16\pi^2}{\alpha^2}\ |z|^{\frac{8\pi}{\alpha}-2} + \frac{4\pi^2}{\alpha^2}(|z|^{\frac{3\pi}{\alpha}-1} - |z|^{\frac{\pi}{\alpha}-1})^2}. \\ \Rightarrow\ |\phi'(z)| &= \sqrt{\frac{16\pi^2}{\alpha^2}\ r^{\frac{8\pi}{\alpha}-2} + \frac{4\pi^2}{\alpha^2}(r^{\frac{3\pi}{\alpha}-1} - r^{\frac{\pi}{\alpha}-1})^2} \\ &= \sqrt{\frac{16\pi^2}{\alpha^2}\ r^{\frac{8\pi}{\alpha}-2} + \frac{4\pi^2}{\alpha^2} r^{\frac{2\pi}{\alpha}-2}(r^{\frac{2\pi}{\alpha}} - 1)^2} \\ &= \sqrt{\frac{4\pi^2}{\alpha^2}\ r^{\frac{2\pi}{\alpha}-2}\left[4r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}} - 1)^2\right]} \\ &= \frac{2\pi}{\alpha}\ r^{\frac{\pi}{\alpha}-1}\sqrt{4\, r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}} - 1)^2}. \end{aligned}$$

In the end,

$$\Rightarrow\ |\phi'(z)| \leq \begin{cases} \frac{8\pi}{\alpha}\, r^{\frac{-3\pi}{\alpha}-1}\sqrt{4\, r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}}-1)^2} & ;\ |z| \ \text{ is large} \\ \\ \frac{2\pi}{\alpha}\, r^{\frac{\pi}{\alpha}-1}\sqrt{4\, r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}}-1)^2}. & ;|z|\ \sim 0 \end{cases}$$

$$\int_0^{\frac{\pi}{n}}\int_0^{\infty} |\phi'(z)|^p \mathrm{d}x\mathrm{d}y \leq \int_0^{\frac{\pi}{n}}\int_0^{1} (\frac{2\pi}{\alpha})^p\, r^{(\frac{\pi}{\alpha}-1)p}\left[4\, r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}}-1)^2\right]^{\frac{p}{2}} .r\mathrm{d}r\mathrm{d}\theta$$
$$+\int_0^{\frac{\pi}{n}}\int_1^{\infty} (\frac{8\pi}{\alpha})^p\, r^{(\frac{-3\pi}{\alpha}-1)p}\left[4\, r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}}-1)^2\right]^{\frac{p}{2}} r\mathrm{d}r\mathrm{d}\theta.$$

$$\int_0^{\frac{\pi}{n}}\int_0^{\infty} |\phi'(z)|^p \mathrm{d}x\mathrm{d}y \leq (\frac{2\pi}{\alpha})^p(\frac{\pi}{n})\int_0^{1} r^{(\frac{\pi}{\alpha}-1)p+1}\mathrm{d}r\int_0^{1}\left[4\, r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}}-1)^2\right]^{\frac{p}{2}} .r\mathrm{d}r$$

$$+(\frac{8\pi}{\alpha})^p(\frac{\pi}{n})\int_1^{\infty} r^{(\frac{-3\pi}{\alpha}-1)p+1}\mathrm{d}r\int_1^{\infty}\left[4\, r^{\frac{6\pi}{\alpha}} + (r^{\frac{2\pi}{\alpha}}-1)^2\right]^{\frac{p}{2}}\mathrm{d}r. \quad (2.55)$$

such that, we have four terms . The first, second and fourth terms on the right -hand-side of (2.55) becomes finite only when

$$\begin{aligned} &(\frac{-3\pi}{\alpha}-1)p+2 \leq 0 \Rightarrow\ (\frac{-3\pi}{\alpha}-1)p \leq -2 \\ &\Rightarrow\ -(\frac{3\pi}{\alpha}+1)p \leq -2 \\ &\quad\Rightarrow\ (\frac{3\pi}{\alpha}+1)p \geq 2 \\ &\Rightarrow\ p > \frac{2}{\frac{3\pi}{\alpha}+1}. \end{aligned}$$

Chapter 3

Integral means of the derivative of univalent function

This chapter is divided into two sections; the first section is connected with chapter 2, through some results concerning the integral means of univalent function for some pth-power integrable function, where $1 < p < \infty$.

Theorem 3.1.3 comes as a nice extension of Theorem 3.1.2 but with a stronger condition, boundedness, which contributes to the expansion of the range of integrability.

Pommerenke in Theorem 2.1.1 proved that

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi'(re^{i\theta})|^{-1}\mathrm{d}\theta = O(1-r)^{-0.601}\text{as } r \to 1-0.$$

In this regard, we prove that the integral above diverges, if an holomorphic function

$$w(z) = \frac{1}{\sqrt{\phi'(z)}} \in H^2(\mathbb{D})$$

be a solution of the *Cauchy-Euler* differential equation $w''(z) + q(z)w(z) = 0$, for $z \in \mathbb{D}$, where ϕ is m$\ddot{o}$bious transformation of unit disk D onto itself. Finally, we prove that any holomorphic function on unit disk with $Re(zF'(z)) > 0$ in $|z| < 1$, generates a starlike function on unit disk such that

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi'(re^{i\theta})|\mathrm{d}\theta \le \int_0^r M(r)\left[\frac{2(1+\rho)^2}{\rho^2(1-\rho)^2} + \frac{4\rho+2\rho^2}{\rho(1-\rho)^2} + \frac{1+\rho}{\rho^2(1-\rho)}\right]\mathrm{d}\rho.$$

Which appears clearly in Theorem (3.1.4).

For detail about most of the results in this section, cf. [51, 52, 9].

Second section introduces Theorem (3.2.6) which comes as a corollary of the *Koebe one-quarter theorem* and *Koebe distortion theorem*. This result together with Theorem (3.2.7) establish the existence and finiteness of the integrability of the derivative of conformal mappping for all $p < 2$. Further more Theorem (3.2.6) considers a sharp result on the integrability of gradient of Cauchy transform $\hat{g}(z)$ over a non-decreasing sequence $\partial\mathcal{D}_i$ in $\mathcal{D}$, such that

$$\int_{\partial\mathcal{D}_i} \nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right) \mathrm{d}\mu_i \quad \text{exists and is finite on } \partial\mathcal{D}_i$$

if the *Cauchy transform* $\hat{g}(z)$ of $g \in L^q(E, \mathrm{d}A)$ for some $1 < q \leq 2$, is an identically zero in $\mathbb{C}\backslash E$ and there exists a non-decreasing sequence $\partial\mathcal{D}_i$ in $\mathcal{D}$, where E is a compact subset of the plane having connected complement, D is a connected domain $\mathcal{D} \subset E$, to produce Theorem (3.2.8).

3.1 Some results on the integral means of derivative of univalent function

Lemma 3.1.1. *[50]*

Let T be an open subset of $[0, 2\pi]$ and $0 \leq r < 1$. If $\phi \in S$,then

$$\int_T |\phi'(re^{i\theta})|^2 |\phi(re^{i\theta})|^{p-2} \mathrm{d}\theta \leq \begin{cases} K(p)(1-r)^{-1}(\mathrm{M}_T(r))^p & \text{if } p > 0, \\ K(p)(1-r)^{-1} & \text{if } p < 0. \end{cases}$$

such that

$$\mathrm{M}_T(r) = \max_{\theta \in T} |\phi(re^{i\theta})| \; ; \;\; 0 \leq r < 1 \; ; \;\; T \subset [0, 2\pi]$$

At this stage, we show that the boundedness condition in Theorem (3.1.3) contributes to the expansion of the range of integrability, which in turn provides a good comparison between Theorems (3.1.2) and (3.1.3).

Theorem 3.1.2. *If ϕ is holomorphic and univalent in unit disk $\mathbb{D}$, then*

$$I(r,\phi') = O((1-r)^{-2.914}) \quad \textit{as } r \to 1-0 \tag{3.1}$$

Proof. :- Let $\phi'(z) = (\sqrt{\phi'(z)})^2$ such that $(\phi'(z))^{\frac{1}{2}} = F(z)$ be holomorphic function in unit disk since ϕ is univalent on unit disk such that $F(z) = \sum\limits_{n=0}^{\infty} a_n z^n$.

We notice that $|\phi'(z)| = |F(z)|^2$.

$$I(r,\phi') = \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2\mathrm{d}\theta\ ; \quad z = re^{i\theta} \tag{3.2}$$

So, $F(z)$ has a Taylor series such that $F(z) = \sum_{n=0}^{\infty} a_n z^n\ ; \quad z \in \mathbb{D}$.

By *Parseval formula* (1.1.7).

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2\mathrm{d}\theta \tag{3.3}$$

Hence,

$$I(r) = \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2\mathrm{d}\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \tag{3.4}$$

By differentiating equation (3.4) we get,

$$\begin{aligned}
I'(r) &= \sum_{n=1}^{\infty} 2n|a_n|^2 r^{2n-1} \\
I''(r) &= \sum_{n=1}^{\infty} 2n(2n-1)|a_n|^2 r^{2n-2} \\
I^{(3)}(r) &= \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)|a_n|^2 r^{2n-3} \\
I^{(4)}(r) &= \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4}
\end{aligned}$$

And

$$
\begin{aligned}
F'(z) &= \sum_{n=1}^{\infty} n a_n z^{n-1} \\
F''(z) &= \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} \\
|F''(z)|^2 &= \sum_{n=2}^{\infty} n^2(n-1)^2 |a_n|^2 |z|^{2n-4} \\
I^{(4)}(r) &\leq K|F''(re^{i\theta})|^2 \quad \text{by lemma 1.1.11and } \textit{Parseval} \text{ formula 1.1.7}
\end{aligned}
$$

Let us find K by comparing the coefficients between $I^{(4)}(r)$ and $|F''(re^{i\theta})|^2$ to find K as follows:

$$
\begin{aligned}
2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4} &\leq Kn^2(n-1)^2|a_n|^2 r^{2n-4} \\
4(2n-1)(n-1)(2n-3) &\leq Kn(n-1)^2 \\
4(2n-1)(2n-3) &\leq Kn(n-1) \\
4(2-\frac{1}{n})(2-\frac{1}{n-1}) &\leq K
\end{aligned}
$$

So, $K = 16$ is smallest such constant as $n \to \infty$.

$$
I^{(4)}(r) = \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4} \leq 16 \sum_{n=2}^{\infty} n^2(n-1)^2 |a_n|^2 |z|^{2n-4}
$$

$$
I^{(4)}(r) = \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4} \leq \frac{16}{2\pi} \int_0^{2\pi} |F''(re^{i\theta})|^2 \mathrm{d}\theta \quad (3.5)
$$

$$
\therefore I^{(4)}(r) \leq \frac{16}{2\pi} \int_0^{2\pi} |F''(re^{i\theta})|^2 \mathrm{d}\theta \quad (3.6)
$$

Differentiate $F(z)=(\phi')^{\frac{1}{2}}$ twice to get $F''(re^{i\theta})$ as follows:-

$$\begin{aligned}
F'(z) &= \frac{1}{2}(\phi')^{\frac{1}{2}-1}\phi'' \\
F''(z) &= \left(\frac{1}{2}(\phi')^{\frac{1}{2}-1}\right)\phi''' + \phi''\left(\frac{1}{2}(\frac{1}{2}-1)(\phi')^{\frac{1}{2}-2}\phi''\right) \\
&= \frac{1}{2}(\phi')^{\frac{1}{2}}\left[\frac{1}{2}(\frac{\phi''}{\phi'})^2 + \frac{\mathrm{d}}{\mathrm{d}z}(\frac{\phi''}{\phi'})\right]. \\
&= \frac{1}{2}(\phi')^{\frac{1}{2}}\left[(\frac{\phi''}{\phi'})^2 + \left\{\frac{\mathrm{d}}{\mathrm{d}z}(\frac{\phi''}{\phi'}) - \frac{1}{2}(\frac{\phi''}{\phi'})^2\right\}\right]. \\
&= \frac{1}{2}(\phi')^{\frac{1}{2}}\left[(\frac{\phi''}{\phi'})^2 + \{S_\phi\}\right].
\end{aligned}$$

where $\{S_\phi\} = \left[\frac{\mathrm{d}}{\mathrm{d}z}(\frac{\phi''}{\phi'}) - \frac{1}{2}(\frac{\phi''}{\phi'})^2\right]$ is called *Schwarzian derivative*

$$\begin{aligned}
|F''(z)| &\leq \frac{1}{2}|\phi'|^{\frac{1}{2}}\left|(\frac{\phi''}{\phi'})^2 + \{S_\phi\}\right| \leq \frac{1}{2}|\phi'|^{\frac{1}{2}}\left[|\frac{\phi''}{\phi'}|^2 + |S_\phi|\right]. \\
|F''(z)|^2 &\leq \frac{1}{4}|\phi'|\left[|\frac{\phi''}{\phi'}|^2 + |S_\phi|\right]^2 \\
&= \frac{1}{4}|\phi'|\left[|\frac{\phi''}{\phi'}|^4 + \frac{12}{(1-r^2)^2}|\frac{\phi''}{\phi'}|^2 + \frac{36}{(1-r^2)^4}\right],
\end{aligned}$$

where $\{S_\phi\} \leq \frac{6}{(1-|z|^2)^2}$. (cf. [47], [19, pp.261-263]) [1].

$$\Rightarrow \quad |F''(z)|^2 \leq \frac{1}{4}|\phi'|\left[\frac{1325.25}{(1-r)^4}\right] = |\phi'|\,\frac{331.3125}{(1-r)^4}.$$

$$\Rightarrow \int_0^{2\pi} |F''(z)|^2 \mathrm{d}\theta \leq \frac{331.3125}{(1-r)^4}\int_0^{2\pi} |\phi'(re^{i\theta})|\mathrm{d}\theta.$$

And hence,

$$I^{(4)}(r) \leq\ 331.3125(1-r)^{-4}I(r)$$

The differential Inequality corresponds to differential equation below by lemma (1.1.11):

$$\nu^{(4)}(r) = 331.3125(1-r)^{-4}\,\nu(r) \tag{3.7}$$

[1] We refer the reader to Appendix A, (Theorem (A.0.14) and Remark (A.0.15)) for more details on *Schwarzian* derivative

Which has a solution

$$\nu(r) = A(1-r)^{-\beta}; \text{where A is a constant and } \beta > 0$$

Now, we find the value of β as follows:

$$\nu'(r) = A\beta(1-r)^{-\beta-1}$$
$$\nu''(r) = A\beta(\beta+1)(1-r)^{-\beta-2}$$
$$\nu^{(3)}(r) = A\beta(\beta+1)(\beta+2)(1-r)^{-\beta-3}$$
$$\nu^{(4)}(r) = A\beta(\beta+1)(\beta+2)(\beta+3)(1-r)^{-\beta-4}$$

Substitute $\nu(r)$, $\nu^{(4)}(r)$ in equation (2.22) to get,

$$A\beta(\beta+1)(\beta+2)(\beta+3)(1-r)^{-\beta-4} = \ 331.3125\ (1-r)^{-4}\ (A(1-r)^{-\beta})$$

$$\text{then } \beta(\beta+1)(\beta+2)(\beta+3) = 331.3125\ ; \ \ \text{(Quartic Polynomial)},$$

and the quartic roots for β are.

$$\beta_1 = -5.9136$$
$$\beta_2 = -1.5000 + 4.1206i$$
$$\beta_3 = -1.5000 + 4.1206i$$
$$\beta_4 = 2.9136$$

So, we notice that $Re(\beta_j)$ when j goes from $1 \to 4$ will be less than 2.9136. $\Rightarrow \beta <$ 2.914. and then,

$$\nu(r) = A(1-r)^{-2.914}$$

is the solution of the equation (3.7). Therefor if we choose A sufficiently large, we will be able to use lemma (1.1.11) such that $I^{(4)}(r_0) > \nu^{(4)}(r_0)$ for $k = 0, 1, 2, 3.$ at $r_0 \leq r < 1.$

$$I(r) < \nu(r)\ , \ \forall r, \ \ r_0 \leq r < 1$$
$$I(r) < \nu(r) = A(1-r)^{-2.914}$$
$$I(r) = O(1-r)^{-2.914}.$$

□

Theorem 3.1.3. *If $\phi(z)$ is bounded and univalent in $\mathbb{D}$ then,*

$$\int\int_D |\phi'(z)|\mathrm{d}x\mathrm{d}y = O(1-r)^{-0.497}. \tag{3.8}$$

Proof. :- Given $\phi \in S$ a bounded function.

Hence, if $\delta > 0$, then from *Cauchy- Schwarz* inequality we obtain the bound

$$\int_0^{2\pi} |\phi'|^{1+\delta}\mathrm{d}\theta \leq \underbrace{\left(\int_0^{2\pi} |\phi'|^{2\delta}\mathrm{d}\theta\right)^{\frac{1}{2}}}_{I_1} \underbrace{\left(\int_0^{2\pi} |\phi'|^{2}\mathrm{d}\theta\right)^{\frac{1}{2}}}_{I_2}$$

Now, we have to estimate two integrals in the right-hand side as follows: Define,

$$I_2(r) = \int_0^{2\pi} |\phi'|^2\mathrm{d}\theta = O(1-r)^{-1} \quad \text{by lemma(3.1.1)}$$

And then define,

$$I_1(r) = \int_0^{2\pi} |\phi'|^{2\delta}\mathrm{d}\theta.$$

Suppose $[\phi']^\delta = F(z)$, where $F(z) = \sum_{n=0}^{\infty} c_n z^n$ is holomorphic in unit disk since ϕ is univalent in unit disk.

$$I_1(r) = \int_0^{2\pi} |F(z)|^2\mathrm{d}\theta.$$

By *Parseval formula* (1.1.7)

$$I_1(r) = \int_0^{2\pi} |F(z)|^2\mathrm{d}\theta = 2\pi\sum_{n=0}^{\infty} |c_n|^2 r^{2n}$$

It follows that,

$$I_1'(r) = 2\pi\sum_{n=1}^{\infty} 2n|c_n|^2 r^{2n-1}.$$

$$I_1''(r) = 2\pi\sum_{n=1}^{\infty} 2n(2n-1)|c_n|^2 r^{2n-2}.$$

$$I_1''(r) = 8\pi\sum_{n=1}^{\infty} n^2|c_n|^2r^{2n-2} - 4\pi\sum_{n=1}^{\infty} n|c_n|^2r^{2n-2} \leq 8\pi\sum_{n=1}^{\infty} n^2|c_n|^2r^{2n-2}$$

Hence,

$$\Rightarrow \quad I_1''(r) \leq 8\pi\sum_{n=1}^{\infty} n^2|c_n|^2r^{2n-2} \tag{3.9}$$

By lemma (1.1.5) we know that for any $\phi \in S$, $|\frac{\phi''}{\phi'}| \leq \frac{6}{1-r}$; $|z| = re^{i\theta}$. This implies to

$$\begin{aligned} 2\pi\sum_{n=1}^{\infty} n^2|c_n|^2r^{2n-2} &= \int_0^{2\pi} |F'(re^{i\theta})|^2 \mathrm{d}\theta \\ &= \delta^2\int_0^{2\pi} |\frac{\phi''}{\phi'}|^2|\phi'|^{2\delta}\mathrm{d}\theta \end{aligned}$$

$$2\pi\sum_{n=1}^{\infty} n^2|c_n|^2r^{2n-2} \leq \frac{36\ \delta^2}{(1-r)^2}\int_0^{2\pi} |\phi'|^{2\delta}\mathrm{d}\theta \tag{3.10}$$

If we combine inequalities (3.9) and (3.10); we obtain

$$\begin{aligned} I_1''(r) \leq 8\pi\sum_{n=1}^{\infty} n^2|c_n|^2r^{2n-2} &\leq \frac{144\ \delta^2}{(1-r)^2}I_1(r) \\ \frac{I_1''(r)}{I_1(r)} &\leq \frac{144\ \delta^2}{1-r)^2} \end{aligned}$$

$$(\log I_1(r))'' = \frac{I_1''(r)}{I_1(r)} - (\frac{I_1'(r)}{I_1(r)})^2 \leq \frac{I_1''(r)}{I_1(r)} \leq \frac{144\ \delta^2}{1-r)^2} \tag{3.11}$$

Integrating twice yields;

$$\begin{aligned} &\log I_1(r)_0^r \leq -144\ \delta^2\log(1-r). \\ \Rightarrow \quad &\log I_1(r) \leq \log\frac{2\pi}{(1-r)^{144\ \delta^2}} \\ \Rightarrow \quad &I_1(r) = O(1-r)^{-144\ \delta^2}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \int_0^{2\pi} |\phi'|^{1+\delta}\mathrm{d}\theta &= O(1-r)^{-\frac{1}{2}}\ O(1-r)^{-72\ \delta^2} \\ &= O(1-r)^{-\frac{1}{2}-72\ \delta^2} \qquad \text{as } r \to 1-0. \end{aligned}$$

At this stage, we shall suppose that there is $0 < \alpha < 1$ with define the set

$$A = \{\theta : |\phi'| > (1-r)^{-\alpha}\}$$

then

$$|\phi'(z)| = |\phi'(z)|^{1+\delta-\delta} = |\phi'(z)|^{1+\delta}\, |\phi'(z)|^{-\delta} \leq |\phi'(z)|^{1+\delta}\, (1-r)^{\alpha\delta}.$$

Therefore $\displaystyle\int_0^{2\pi} |\phi'(z)| \mathrm{d}\theta \leq (1-r)^{-\frac{1}{2}-72\delta^2+\alpha\delta}$

If we set $\delta = \frac{\alpha}{144} \Rightarrow -\frac{1}{2} + 0.00347\alpha^2 > -\alpha$ and then $\alpha = 0.49915 \backsimeq 0.4992$. Finally,

$$\int_0^{2\pi} |\phi'(re^{i\theta})| \mathrm{d}\theta = O(1-r)^{-0.497}$$

$\square$

Theorem 3.1.4. *If F be an holomorphic function on unit disk $\mathbb{D}$, and $Re(zF'(z)) > 0$ in $|z| < 1$, then there exist a starlike function ϕ in $|z| < 1$ such that*

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})| \mathrm{d}\theta \leq \int_0^r M(r)\left[\frac{2(1+\rho)^2}{\rho^2(1-\rho)^2} + \frac{4\rho+2\rho^2}{\rho(1-\rho)^2} + \frac{1+\rho}{\rho^2(1-\rho)}\right] \mathrm{d}\rho$$

Proof. Let us assume $F(z) = \log\phi$ with ϕ be an univalent function in $|z| < 1$, then $F'(z) = (\log\phi(z))' = \frac{\phi'(z)}{\phi(z)}$ so that $zF'(z) = z\frac{\phi'(z)}{\phi(z)}$.

Since $Re(zF'(z)) > 0$ in $|z| < 1$, we obtain $Re(z\frac{\phi'(z)}{\phi(z)}) > 0$, this implies ϕ is a starlike function in $|z| < 1$.

Let $h(z) = zF'(z) \Rightarrow h(z) = \frac{z\phi'}{\phi}$, and suppose that $G(z) = z\phi'$ then $G(z) = \phi(z)h(z)$, to obtain

$$\begin{aligned}
G'(z) &= \phi'(z)h(z) + \phi(z)h'(z)\\
&= \frac{\phi(z)h(z)}{z}h(z) + \phi(z)h'(z)\\
&= \frac{\phi}{z}h^2(z) + \phi(z)h'(z)
\end{aligned}$$

$$\begin{aligned}
|G'(z)| = \left|\frac{\phi}{z}h^2(z) + \phi(z)h'(z)\right| &\leq r^{-1}|\phi(z)||h(z)|^2 + |\phi(z)||h'(z)|\\
&\leq r^{-1}\max_{|z|=r}|\phi(z)||h(z)|^2 + \max_{|z|=r}|\phi(z)||h'(z)|\\
&= r^{-1}M(r)|h(z)|^2 + M(r)|h'(z)|
\end{aligned}$$

where $M(r) = \max_{|z|=r} |\phi(z)|$, is the maximum modulus of $\phi(z)$, and hence we obtain,

$$\frac{1}{2\pi}\int_0^{2\pi} |G'(re^{i\theta})|\mathrm{d}\theta \le \underbrace{r^{-1}M(r)\frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|^2\mathrm{d}\theta}_{I_1} + \underbrace{M(r)\frac{1}{2\pi}\int_0^{2\pi} |h'(re^{i\theta})|\mathrm{d}\theta}_{I_2}. \tag{3.12}$$

Now, we have to estimate two integrals in inequality (3.12), but before that, we need apply estimates of length such that we can write

$$|G(z)| \le \int_0^r |G'(\rho e^{i\theta})|\mathrm{d}\rho.$$

$$L(r) = \frac{1}{2\pi}\int_0^{2\pi} |G(re^{i\theta})| \le \frac{1}{2\pi}\int_0^{2\pi}\left[\int_0^r |G'(\rho e^{i\theta})|\mathrm{d}\rho\right]\mathrm{d}\theta. \tag{3.13}$$

Estimate the integrals I_1, I_2 and set $h(z) = zF'(z)$ to obtain

$$\begin{aligned} I_1 &= r^{-1}M(r)\frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|^2\mathrm{d}\theta. \\ &= r^{-1}M(r)\frac{1}{2\pi}\int_0^{2\pi} |zF'(re^{i\theta})|^2\mathrm{d}\theta. \end{aligned}$$

$$I_1 \le r^{-1}M(r)\left(\frac{1+r}{1-r}\right)^2 \qquad \text{(cf.[50, lemma1.3 and equation(1.2.13)])} \tag{3.14}$$

and

$$\begin{aligned} I_2 &= \frac{M(r)}{2\pi}\int_0^{2\pi} |h'(re^{i\theta})|\mathrm{d}\theta. \\ &= \frac{M(r)}{2\pi}\int_0^{2\pi} |zF''(re^{i\theta}) + F'(re^{i\theta})|\mathrm{d}\theta. \\ &\le \frac{rM(r)}{2\pi}\int_0^{2\pi} |F''(re^{i\theta})|\mathrm{d}\theta + \frac{M(r)}{2\pi}\int_0^{2\pi} F'(re^{i\theta})|\mathrm{d}\theta. \end{aligned}$$

$$I_2 \le \frac{rM(r)}{2\pi}\int_0^{2\pi} |F''(re^{i\theta})|\mathrm{d}\theta + M(r)\frac{1+r}{r(1-r)} \qquad \text{(cf.[50, lemma1.3 and equation(1.2.13)]).} \tag{3.15}$$

Now, we have

$$|F''(z)| = \left|\frac{\phi''}{\phi} - (\frac{\phi'}{\phi})^2\right| \leq \left|\frac{\phi''}{\phi}\right| + \left|\frac{\phi'}{\phi}\right|^2 = \left|\frac{\phi''}{\phi}\right| + |F'(z)|^2. \tag{3.16}$$

Hence, $\left|\frac{\phi''}{\phi}\right|$ can be estimated and substitute it in equation (3.16) as follows:

$$\left|\frac{\phi''}{\phi}\right| = \left|\frac{\phi''}{\phi'}\cdot\frac{\phi'}{\phi}\right| = \left|\frac{\phi''}{\phi'}\right|\left|\frac{\phi'}{\phi}\right| \leq \left(\frac{4+2r}{1-r^2}\right)\left(\frac{1+r}{r(1-r)}\right) \text{ (cf.[50, lemma(1.3)and equation(1.2.13)]),}$$

to obtain

$$|F''(z)| = \left|\frac{\phi''}{\phi}\right| + |F'(z)|^2 \leq \left(\frac{4+2r}{1-r^2}\right)\left(\frac{1+r}{r(1-r)}\right) + \frac{(1+r)^2}{r^2(1-r)^2}. \tag{3.17}$$

In this stage, we shall substitute equation (3.17) in equation (3.15) to obtain

$$I_2 \leq rM(r)\left[\left(\frac{4+2r}{1-r^2}\right)\left(\frac{1+r}{r(1-r)}\right) + \frac{(1+r)^2}{r^2(1-r)^2}\right] + M(r)\frac{1+r}{r(1-r)}.$$

$$I_2 \leq M(r)\left[(\frac{4r+2r^2}{1-r^2})(\frac{1+r}{(1-r)}) + \frac{(1+r)^2}{r(1-r)^2} + \frac{1+r}{r(1-r)}\right]. \tag{3.18}$$

Substitute equations (3.18) and (3.14) in (3.12) we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|G'(re^{i\theta})|\mathrm{d}\theta &\leq r^{-1}M(r)\frac{1}{2\pi}\int_0^{2\pi}|h(re^{i\theta})|^2\mathrm{d}\theta + M(r)\frac{1}{2\pi}\int_0^{2\pi}|h'(re^{i\theta})|\mathrm{d}\theta.\\ &\leq M(r)\frac{(1+r)^2}{r(1-r)^2} + M(r)\left[(\frac{4r+2r^2}{1-r^2})(\frac{1+r}{(1-r)}) + \frac{(1+r)^2}{r(1-r)^2} + \frac{1+r}{r(1-r)}\right].\end{aligned}$$

Integral tha last equation with respect to $\rho \to 0-r$.

$$\int_0^r\left[\frac{1}{2\pi}\int_0^{2\pi}|G'(re^{i\theta})|\mathrm{d}\theta\right]\mathrm{d}\rho \leq \int_0^r M(r)\left[\frac{2(1+\rho)^2}{\rho(1-\rho)^2} + \frac{4\rho+2\rho^2}{(1-\rho)^2} + \frac{1+\rho}{\rho(1-\rho)}\right]\mathrm{d}\rho.$$

From equation (3.13) can be concluded that

$$\frac{1}{2\pi}\int_0^{2\pi}|G(re^{i\theta})|\mathrm{d}\theta \leq \int_0^r M(r)\left[\frac{2(1+\rho)^2}{\rho(1-\rho)^2} + \frac{4\rho+2\rho^2}{(1-\rho)^2} + \frac{1+\rho}{\rho(1-\rho)}\right]\mathrm{d}\rho$$
$$\frac{1}{2\pi}\int_0^{2\pi}|z\phi'(re^{i\theta})|\mathrm{d}\theta \leq \int_0^r M(r)\left[\frac{2(1+\rho)^2}{\rho(1-\rho)^2} + \frac{4\rho+2\rho^2}{(1-\rho)^2} + \frac{1+\rho}{\rho(1-\rho)}\right]\mathrm{d}\rho.$$

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi'(re^{i\theta})|\mathrm{d}\theta \leq \int_0^r M(r)\left[\frac{2(1+\rho)^2}{\rho^2(1-\rho)^2} + \frac{4\rho+2\rho^2}{\rho(1-\rho)^2} + \frac{1+\rho}{\rho^2(1-\rho)}\right]\mathrm{d}\rho.$$

The proof is complete. $\square$

Theorem 3.1.5. *Let ϕ be a möbious transformation of unit disk $\mathbb{D}$ onto itself, and let*

$$w''(z) + q(z)w(z) = 0 \quad \textit{for} \quad z \in \mathbb{D} \tag{3.19}$$

be the differential equation whose solution $w(z) = \frac{1}{\sqrt{\phi'(z)}} \in H^2(\mathbb{D})$ then

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^{-1} \mathrm{d}\theta \geq \frac{(1-r^2)^3}{k\alpha} \sum_{n=2}^{\infty} n(2n-2)(2n-3) \quad \textit{for} \quad z = re^{i\theta}. \tag{3.20}$$

Proof. Assume $w(z)$ is continuous in the closed unit disk $\mathbb{D}$ such that $w(z) = \sum_{n=0}^{\infty} a_n z^n$.

Given $w(z) = \frac{1}{\sqrt{\phi'(z)}}$, we have $|w(z)|^2 = |\phi'(z)|^{-1}$.

And hence,

$$I_{-1}(r, \phi') = I(r) = \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^{-1} \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} |w(re^{i\theta})|^2 \mathrm{d}\theta, \tag{3.21}$$

where $z = re^{i\theta}$

By *Parseval* formula (1.1.7),

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |w(re^{i\theta})|^2 \mathrm{d}\theta \tag{3.22}$$

$$\Rightarrow \quad I(r) = \frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^{-1} \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} |w(re^{i\theta})|^2 \mathrm{d}\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \tag{3.23}$$

By differentiation equation (3.23) we obtain,

$$\begin{aligned}
I'(r) &= \sum_{n=1}^{\infty} 2n|a_n|^2 r^{2n-1} \\
I''(r) &= \sum_{n=1}^{\infty} 2n(2n-1)|a_n|^2 r^{2n-2} \\
I^{(3)}(r) &= \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)|a_n|^2 r^{2n-3} \\
I^{(4)}(r) &= \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4}
\end{aligned}$$

And

$$\begin{aligned}
w'(z) &= \sum_{n=1}^{\infty} n a_n z^{n-1} \\
w''(z) &= \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} \\
\Rightarrow \ |w''(z)|^2 &= \sum_{n=2}^{\infty} n^2(n-1)^2 |a_n|^2 |z|^{2n-4} \\
I^{(4)}(r) &\leq K|F''(re^{i\theta})|^2 \quad \text{by lemma (1.1.11) and } \textit{Parseval} \text{ formula (1.1.7)}
\end{aligned}$$

Let us find K by comparing the coefficients between $I^{(4)}(r)$ and $|F''(re^{i\theta})|^2$ to find K as follows:

$$\begin{aligned}
2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4} &\leq Kn^2(n-1)^2|a_n|^2 r^{2n-4} \\
4(2n-1)(n-1)(2n-3) &\leq Kn(n-1)^2 \\
4(2n-1)(2n-3) &\leq Kn(n-1) \\
4(2-\frac{1}{n})(2-\frac{1}{n-1}) &\leq K
\end{aligned}$$

So, $K = 16$ is smallest such constant as $n \to \infty$.

$$I^{(4)}(r) = \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4} \leq 16 \sum_{n=2}^{\infty} n^2(n-1)^2|a_n|^2|z|^{2n-4}$$

$$\Rightarrow \ I^{(4)}(r) \leq \frac{16}{2\pi} \int_0^{2\pi} |w''(re^{i\theta})|^2 \mathrm{d}\theta \tag{3.24}$$

Integrate the last inequality with respect to r as $r \to 0-1$.

Multiplying both sides of an equation 3.24 by r.

$$\begin{aligned}
\int_0^1 I^{(4)}(r) r \mathrm{d}r &\leq \frac{16}{2\pi} \int_0^1 \int_0^{2\pi} |w''(re^{i\theta})|^2 r \mathrm{d}r \mathrm{d}\theta \\
&= \frac{16}{2\pi} \int_0^1 \int_0^{2\pi} |q(z)w(z)|^2 r \mathrm{d}r \mathrm{d}\theta \\
&= \frac{16}{2\pi} \int \int_{\mathbb{D}} |q(z)|^2 |w(z)|^2 \mathrm{d}A_r
\end{aligned}$$

There is an absolute constant $\alpha > 0$ such that

$$\int_0^1 I^{(4)}(r) r \mathrm{d}r \leq \frac{k\alpha}{(1-r^2)^3} \sum_{n=0}^{\infty} |a_n|^2 \quad \text{(cf. Theorem 2, pp.26,31 in [51]).} \tag{3.25}$$

Multiply the last inequality by r^{2n}

$$\int_0^1 I^{(4)}(r) r^{2n+1} \mathrm{d}r \leq \frac{k\alpha}{(1-r^2)^3} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{k\alpha}{(1-r^2)^3} \left(\frac{1}{2\pi} \int_0^{2\pi} |\phi'(re^{i\theta})|^{-1} \mathrm{d}\theta \right) \tag{3.26}$$

This implies to

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |\phi'(re^{i\theta})|^{-1} \mathrm{d}\theta &\geq \frac{(1-r^2)^3}{k\alpha} \int_0^1 \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4} r^{2n+1} \mathrm{d}r \\ &= \frac{(1-r^2)^3}{k\alpha} \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 \int_0^1 r^{4n-3} \mathrm{d}r. \end{aligned}$$

Finally,

$$\frac{1}{2\pi} \int_0^{2\pi} |\phi'(re^{i\theta})|^{-1} \mathrm{d}\theta \geq \frac{(1-r^2)^3}{k\alpha} \sum_{n=2}^{\infty} n(2n-2)(2n-3) \quad \text{holds.}$$

□

3.2 Integrability of gradient of conformal mapping

The main result of this section is Theorem (3.2.6) which comes as a corollary of the *Koebe distortion theorem*. Theorem (3.2.6) discusses existence of positive constant K, depending only on the modulus of z in Ω, such that $(1 - |\phi(z)|) \leq K\sqrt{|z|}$, for some $z \in \Omega$, has been used this result as an essential tool to prove Theorems (3.2.7), (3.2.8).

Theorem (3.2.6) related closely with similar lemma [2] which is well-known in polynomial approximation and plays important role to find conditions on $\Omega \subset \mathbb{C}$ (be a bounded simply domain) and w be a positive measurable function defined on Ω which imply that

[2] For interested reader, we refer to [33] as a more convenient reference

$H^p(\Omega, \mathrm{d}A) = L^p_a(\Omega, \mathrm{d}A)$ [3]. for each p, $1 \leq p < \infty$ in turn, helps to prove that polynomials Q in $L^p(\Omega, \mathrm{d}A)$ are said to be complete in $L^p_a(\Omega, \mathrm{d}A)$, we refer to ([31, 30] and [32]) for more details.

Definition 3.2.1. (**Harmonic measure**) Let Ω be a bounded, open domain in n-dimentional Euclidean space $\mathbb{R}^n$, $n \geq 2$, and let $\partial\Omega$ denote the boundary of Ω. Any finite real- valued continuous function f on $\partial\Omega$, $f : \partial\Omega \longrightarrow \mathbb{R}$ corresponds to a unique function $u(x)$ on the closure $\overline{\Omega}$ of the region, is called a solution of the *Dirichlet problem*, if

i. u is a continuous on $\overline{\Omega}$.

ii. u is a harmonic in Ω, that is $\Delta u \equiv 0$ in Ω.

iii. $u|_{\partial\Omega} = f$.

A solution of the *Dirichlet* problem u corresponding to the continuous boundary function f, is called a harmonic extension of f, let us call it, $u_f = u(f)$.

If the point $x \in \Omega$ is assumed to be fixed, then by *Riesz representation* theorem [4] and *the maximum principle*, for $u(f)$ defined on the compact set $C_c(\Omega)$ there exists a unique Borel measure $\mu(x)$ at the point x on Ω, define $u_\mu \in C_c(\Omega)^\star$ by

$$u_\mu(f) = \int f \mathrm{d}\mu(x, \Omega),$$

for all f in $C_c(\Omega)$, and the measure $\mu(x, \Omega)$ is called the *Harmonic measure*.

In particular, if the solution of the *Dirichlet* problem u corresponding to a Boral measurable $E \subset \partial\Omega$ with the boundary data

$$f = \chi_E = \begin{cases} 1 & \text{if } \zeta \in E, \\ 0 & \text{if } \zeta \notin E. \end{cases}$$

[3] $H^p(\Omega, \mathrm{d}A)$- space consists of all the functions, that can be approximated arbitrarily in the $L^p(\Omega, \mathrm{d}A)$-norm by a sequence of polynomials; $L^p_a(\Omega, \mathrm{d}A)$ denotes the class of all the functions in $L^p(\Omega, \mathrm{d}A)$ which are holomorphic in Ω

[4] A straight-forward proof of *Riesz representation* theorem in Appendix B, Theorem (B.0.18)

such that $f = 1_E$ that is, f takes the value 1 on the part E of $\partial\Omega$ and $f = 0$ on the remaining $\partial\Omega$, such that, this solution is called the harmonic measure of E, and is denoted by $\mu(x, E; \Omega)$.

For much extra information about harmonic measure and other topic which are related to it, we refer to [20, 60, 25] and [2].

Theorem 3.2.2. ***(Koebe One-Quarter theorem)***[5]

The range of every function of class S[6] contains the disk $\{w : |w| < \frac{1}{4}\}$.

Lemma 3.2.3. *[8]*

Let E be a compact subset of the plane having connected complement and let $g \in L(E, \mathrm{d}A)$ for some $q > 1$. If $\hat{g} = 0$ identically in $\mathbb{C}\backslash E$ then $\hat{g}(z_0) = 0$ at every point $z_0 \in \partial E$, where

$$\int_E \frac{|g(z)|^q}{|z - z_0|}\mathrm{d}A < \infty.$$

Lemma 3.2.4. *[8, 7]*

Let E be a compact set with connected complement and let $g \in L(E, \mathrm{d}A)$ for some $1 < q \leq 2$. If $\hat{g} = 0$ identically in $\mathbb{C}\backslash E$ and ζ is a point of E° (the interior of E) at a distance $\delta(z) = \frac{1}{e}$ from ∂E then

$$|\hat{g}(\zeta)| \leq C\left\{g^\star(\zeta_0)\delta\log(\frac{1}{\delta}) + \left[\Gamma_q(\delta)\int_{|z-\zeta_0|\leq 4\delta}|g(z)|^q\mathrm{d}A\right]^{\frac{1}{q}}\right\}$$

where $g^\star(\zeta_0) = \sup_r(\pi r^2)^{-1}\int_{|z-\zeta|<r}|g(z)|\mathrm{d}A$ is the Hardy- Littlewood maximal function, $\Gamma_q(\delta)$ is equal to $\log(\frac{1}{\delta})$ or $\delta^q - 2$ according to whether $q = 2$ or $q < 2$ and, C is a constant depending only on q and the diameter of E.

[5] A straight-forward proof of theorem *Koebe One-Quarter Theorem* in Appendix A.

[6] class S of functions ϕ holomorphic and univalent in the unit disk $\mathbb{D} = \{z : |z| < 1\}$, normalized by the conditions $\phi(0) = 0$ and $\phi'(0) = 1$.

Lemma 3.2.5. [7] *There exist positive constants K_1 and K_2, depending only on $\delta\left(\phi^{-1}(0)\right)$, such that*

$$K_1 \frac{1-|\phi(z)|}{\delta(z)} \leq |\phi'(z)| \leq K_2 \frac{1-|\phi(z)|}{\delta(z)}.$$

where $\phi^{-1}(0) = z$, and ϕ maps a simply connected domain Ω conformally onto unit disk $\mathbb{D}$.

Now, our starting point in this section will be with Theorem (3.2.6) and to indicate how this theorem can be used to produce other results.

Main results:

Theorem 3.2.6. *Let ϕ be a conformal mapping of a simply connected domain Ω onto the unit disk $\mathbb{D}$, then there is a constant K depending only on the modulus of z in Ω such that*

$$1 - |\phi(z)| \leq K\sqrt{|z|}, \quad \textit{for some} \quad z \in \Omega.$$

Proof. Let Ω be a bounded simply connected domain, and ϕ be a conformal map defined as follows:

$$\phi : \Omega \longrightarrow \mathbb{D}\ (|w| < 1)$$

So, the inverse function

$$\psi = \phi^{-1} : \mathbb{D} \longrightarrow \Omega$$

Apply *Distortion* theorem (1.1.6) to the inverse function $\psi = \phi^{-1}(w)$, to obtain that $\phi^{-1}(w) \in S$, normalized by the conditions $\phi^{-1}(0) = 0$ and $(\phi^{-1})'(0) = 1$.

Then, for $w \in \mathbb{D}$;

$$\frac{|w|}{(1+|w|)^2} \leq |\phi^{-1}(w)| \leq \frac{|w|}{(1-|w|)^2}.$$

Fix $w_0 \in \mathbb{D}$

$$\Rightarrow\ |\phi^{-1}(w_0)| \leq \frac{|w_0|}{(1-|w_0|)^2} \tag{3.27}$$

[7] A proof of Lemma (3.2.5) can be found in [50, pp.21-22]

Fix $z_0 \in \Omega$. Then apply *Koebe one-quarter* theorem (3.2.2) in order to show that the range of the function $\phi^{-1}(w) \in S$ contains the disk $\{\phi^{-1}(w) : |\phi^{-1}(w)| < \frac{1}{4}\} \subset \Omega^{\circ}$, such that $z_0 \notin \{\phi^{-1}(w) : |\phi^{-1}(w)| < \frac{1}{4}\}$ which implies

$$|z_0| \geq \frac{1}{4} \quad \text{in} \;\; \Omega \tag{3.28}$$

Equation (3.28) can therefore be written as follows:

$$|z_0| \geq \frac{1}{4}|\,(\phi^{-1})'(0)| \quad \text{in} \;\; \Omega \qquad \text{by condition} \;\; (\phi^{-1})'(0) = 1.$$

We can assume that $w_0 \in \mathbb{D}$ is the image of -1, that is ; $\phi(-1) = w_0$.

By taking the inverse of both sides , we obtain

$$|\phi^{-1}(w_0)| = 1 \tag{3.29}$$

Distortion theorem which represents by equation (3.27) will now be applied to obtain

$$\begin{aligned} 1 = |\phi^{-1}(w_0)| &\leq \frac{|w_0|}{(1-|w_0|)^2} = \frac{|w_0|\;|(\phi^{-1})'(0)|}{(1-|w_0|)^2} \\ 1 \leq \frac{|w_0|\;|(\phi^{-1})'(0)|}{(1-|w_0|)^2} &\leq \frac{4|z_0|\;|w_0|}{(1-|w_0|)^2} \qquad \text{by equation (3.28).} \end{aligned}$$

Since, w_0 and z_0 are arbitrary points, this implies that

$$1 \leq \frac{4|z|\;|w|}{(1-|w|)^2}.$$

Hence

$$\begin{aligned} 1 \leq \frac{4|z|\;|w|}{(1-|w|)^2} &< \frac{C\;|z|}{(1-|w|)^2} = \frac{C\;|z|}{(1-|\phi(z)|)^2}. \\ \Rightarrow \;\; (1-|\phi(z)|)^2 &\leq C\;|z|. \end{aligned}$$

Finally, we obtain

$$(1-|\phi(z)|) \leq K\sqrt{|z|}, \qquad \text{for all} \quad z \in \Omega.$$

The proof is complete. □

Theorem 3.2.7. *Let Ω be a bounded simply connected domain, whose boundary is a class $\mathcal{C}^1$ Jordan curve. If ϕ is a conformal map of Ω to the unit disk $\mathbb{D}(|w| < 1)$, then*

$$\int\int_{\Omega} |\phi'|^p \, \mathrm{d}x\mathrm{d}y < \infty, \quad \textit{for all} \quad p < 2.$$

Proof. We shall assume that $z_0 \in \Omega$ and $\phi(z_0) = 0$. It can then be inferred from the co-area formula.[8].

Let ϕ be a conformal mapping of Ω (simply connected domain) onto the open unit disk D ($|z| < 1$), that is; $\phi : \Omega \to \mathbb{D}$ with A be a measurable subset of $\mathbb{D}$.

Let $g(z) : \ B = \{z \in \Omega : |\phi(z)| = r\} \subset \Omega \to \ [0, \infty)$ be a measurable function defined on the measurable set B in Ω, as the following

$$g(z) = \frac{\chi_{B(z)}}{|\phi'(z)|} : \Omega \to [0, \infty)$$

where $\chi_B(z) : B \subset \Omega \to \{0, 1\}$ is the characteristic function.

$$\chi_B = \begin{cases} 1 & \text{if } z \in B \\ 0 & \text{if } z \notin B \end{cases}$$

Because we have to calculate the integral over $B = \{z \in \Omega : |\phi(z)| = r\}$ and we know that $0 < r \leq 1 \Rightarrow \chi_B = 1; \ z \in B$ we have then

$$\int\int_{\Omega} \frac{\chi_{\mathcal{B}}}{|\phi'|} |\phi'| \, \mathrm{d}x\mathrm{d}y = \int_0^1 \left(\int_{B=\{z\in\Omega:|\phi(z)|=r\}} \frac{\chi_{B(z)}}{|\phi'|} \, r\mathrm{d}\theta \right) \mathrm{d}r.$$

$$\Rightarrow \int\int_{\Omega} \mathrm{d}x\mathrm{d}y = \int_0^1 \left(\int_{B(z)=\{z\in\Omega:|\phi(z)|=r\}} \frac{1}{|\phi'|} \, \mathrm{d}s \right) \mathrm{d}r.$$

Now, we have

$$\begin{aligned} \int\int_{\Omega} |\phi'(z)|^p \mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_{B=\{z\in\Omega:|\phi(z)|=r\}} |\phi'(z)|^p \frac{1}{|\phi'|} \, \mathrm{d}s \right) \mathrm{d}r. \\ &= \int_0^1 \left(\int_{B=\{z\in\Omega:|\phi(z)|=r\}} |\phi'(z)|^{p-1} \, \mathrm{d}s \right) \mathrm{d}r. \\ &= \int_0^1 |\phi'|^{p-1} \left(\int_0^{2\pi r} \mathrm{d}s \right) \mathrm{d}r \\ &= \int_0^1 2\pi r |\phi'|^{p-1} \, \mathrm{d}r \end{aligned}$$

[8]A straight-forward proof of this assertion is in Definition (B.0.16), Appendix B

By Lemma (3.2.5) we have $|\phi'| \leq C\frac{1-|\phi|}{\delta(z)}$, so let us assume that $\delta(z) = |z|$ such that

$$\int\int_{\Omega} |\phi'(z)|^p \mathrm{d}x\mathrm{d}y = \int_0^1 2\pi r|\phi'|^{p-1}\,\mathrm{d}r.$$
$$\leq \int_0^1 2\pi r\frac{(1-r)^{p-1}}{|z|^{p-1}}\mathrm{d}r.$$

In Lemma (3.2.6) we deduced that $(1-|\phi|) \leq k_1\sqrt{|z|} \implies \frac{k_2}{|z|^{p-1}} \leq \frac{1}{(1-r)^{2(p-1)}}$.

Hence,

$$\int_0^1 2\pi r\frac{(1-r)^{p-1}}{|z|^{p-1}}\mathrm{d}r \leq \int_0^1 2\pi r\frac{(1-r)^{p-1}}{(1-r)^{2(p-1)}}\mathrm{d}r = \int_0^1 2\pi r(1-r)^{-(p-1)}\mathrm{d}r.$$
$$= 2\pi r\left[\frac{(1-r)^{-(p-1)+1}}{-(p-1)+1}\right|_0^1$$

when $-(p-1)+1 > 0 \Rightarrow p < 2$. □

Theorem 3.2.8. *Let E be a compact subset of the plane having connected complement, $\mathcal{D} \subset E$ be a connected domain, let $\hat{g}$ be a cauchy transform of a function g, where $g \in L^q(E, \mathrm{d}A)$ for some $1 < q \leq 2$, if $\hat{g}(z)$ is an identically zero in $\mathbb{C}\backslash E$ and there exist a non-decreasing sequence $\partial\mathcal{D}_i$ in $\mathcal{D}$ then*

$$\int_{\partial\mathcal{D}_i} \nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right)\mathrm{d}\mu_i \quad \textit{exists and is finite on } \partial\mathcal{D}_i$$

Proof. Let $\mathcal{D}$ be a connected domain $\mathcal{D} \subset E$. Fix arbitrary point $\zeta \in \mathcal{D}$ and assume μ be the harmonic measure on $\partial\mathcal{D}$ representing ζ.

Choose $\mathcal{D}_i$ be a non-decreacing sequence such that $\zeta \in \mathcal{D}_i$ for all i and $\bigcup \mathcal{D}_i$ fill up $\mathcal{D}$.

Given $g \in L^q(E)$, and assume the property $\int Qg\mathrm{d}A = 0$ for every polynomial $Q \in H^p(E, \mathrm{d}A)$, then $\hat{g}$ vanishes identically[9] in the unbounded complementary component of E.

We obtain $|g(\hat{\zeta})|$ is bounded in E at Euclidean distance $\delta(z) = dis(z, \partial E) < \frac{1}{e}$ (see lemma 3.2.4), that is; $|g(\hat{\zeta})|$ is bounded in $\mathcal{D}$.

[9] we refer the reader to ([8, pp.119])

Hence, multiply and divide $|\hat{g}(z)|$ by $\sqrt{\delta(z)}$ where $\delta(z) = dis(z, \partial\mathcal{D}_i) = |z|$ this yield the identity below:

$$\log \frac{\sqrt{|z|}\,|\hat{g}(z)|}{\sqrt{|z|}} = \log \sqrt{|z|} + \log \frac{|\hat{g}(z)|}{\sqrt{|z|}}$$
$$\Rightarrow \quad \log |\hat{g}(z)| = \log \sqrt{|z|} + \log \frac{|\hat{g}(z)|}{\sqrt{|z|}}$$

Integrate the last quantity over $\partial\mathcal{D}_i$ with respect to harmonic measure $\mathrm{d}\mu_i$ this will imply,

$$\int_{\partial\mathcal{D}_i} \log |\hat{g}(z)| \mathrm{d}\mu_i = \int_{\partial\mathcal{D}_i} \log \sqrt{|z|} \mathrm{d}\mu_i + \int_{\partial\mathcal{D}_i} \log \frac{|\hat{g}(z)|}{\sqrt{|z|}} \mathrm{d}\mu_i.$$

Here, we will pay particular attention to the second integral.

As known in [8, pp. 145] that,

$$\int_{\partial\mathcal{D}_i} \log \frac{|\hat{g}(z)|}{\sqrt{|z|}} \mathrm{d}\mu_i < \int_{\partial\mathcal{D}_i} \frac{|\hat{g}(z)|}{\sqrt{|z|}} \mathrm{d}\mu_i \tag{3.30}$$

It is known in measure theory [10]

$$\sum_{j=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x_j} \int_{\partial\mathcal{D}_i} \frac{|\hat{g}(z)|}{\sqrt{|z|}} \mathrm{d}\mu_i = \int_{\partial\mathcal{D}_i} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{|\hat{g}(z)|}{\sqrt{|z|}} \right) \mathrm{d}\mu_i$$
$$= \int_{\partial\mathcal{D}_i} \nabla \left(\frac{|\hat{g}(z)|}{\sqrt{|z|}} \right) \mathrm{d}\mu_i$$

Hence, we have

$$\int_{\partial\mathcal{D}_i} \nabla \left(\frac{|\hat{g}(z)|}{\sqrt{|z|}} \right) \mathrm{d}\mu_i = \int_{\partial\mathcal{D}_i} |\hat{g}(z)| \nabla \left(\frac{1}{\sqrt{|z|}} \right) \mathrm{d}\mu_i + \int_{\partial\mathcal{D}_i} \frac{1}{\sqrt{|z|}} \nabla |\hat{g}(z)| \, \mathrm{d}\mu_i. \tag{3.31}$$

This implies that

$$\left| \int_{\partial\mathcal{D}_i} \nabla \left(\frac{|\hat{g}(z)|}{\sqrt{|z|}} \right) \mathrm{d}\mu_i \right| \leq \underbrace{\int_{\partial\mathcal{D}_i} \left| |\hat{g}(z)| \nabla \left(\frac{1}{\sqrt{|z|}} \right) \right| |\mathrm{d}\mu_i|}_{I_1} + \underbrace{\int_{\partial\mathcal{D}_i} \left| \frac{\nabla |\hat{g}(z)|}{\sqrt{|z|}} \right| |\mathrm{d}\mu_i|}_{I_2}. \tag{3.32}$$

[10] For interested reader we refer to [23] and definition (B.0.17) in Appendix B

So we may direct out efforts toward finding a bounds for I_1 and I_2. In that endeavor we have to define a *Green function* which is a harmonic function on $\mathcal{D}_i$ and it is defined on $\mathcal{D}'_i$ as well. In this case should be define the harmonic function $\mathrm{d}\mu_i = \frac{\partial G_i}{\partial n}|\mathrm{d}z|$.

i. Remove a small disk $|z-\zeta| \le r_\circ$ from $\mathcal{D}_i$ we obtain $\mathcal{D}'_i$ such that $|z-\zeta| \le r_\circ$ is contained in every $\mathcal{D}_i$, and its boundary is smooth, this lead to, any continuous function on $\partial\mathcal{D}_i$ to $\mathbb{R}$ will generate harmonic function on $\mathcal{D}_i$ with singularity (pole) at ζ.

ii. Let ϕ_i be a conformal map of $\mathcal{D}_i$ onto unit disk $\mathbb{D} = \{w : |w| < 1\}$ with $\phi_i(\zeta) = 0$; and as clear $\psi_i = \phi_i^{-1}$, which satisfies the following:

a. $|\nabla G_i| \le C|\phi'_i|$ on $\mathcal{D}'_i$.

b. $|\psi'_i(w)| \ge C(1-|w|)$.

An estimation on I_1 can be obtained by applying *Hölder inequality* with short calculation as follows:

$$
\begin{aligned}
I_1 &= \int_{\partial\mathcal{D}_i} \left| |\hat{g}(z)| \nabla \left(\frac{1}{\sqrt{|z|}}\right) \right| |\mathrm{d}\mu_i| \\
&= \int_{\partial\mathcal{D}'_i} |\hat{g}(z)| \nabla \left(\frac{1}{\sqrt{|z|}}\right) \nabla G_i \mathrm{d}A \\
&\le \left(\int_{\partial\mathcal{D}'_i} |\hat{g}|^q \mathrm{d}A\right)^{\frac{1}{q}} \left(\int_{\partial\mathcal{D}'_i} \left|\nabla \left(\frac{1}{\sqrt{|z|}}\right) \nabla G_i\right|^p \mathrm{d}A\right)^{\frac{1}{p}} \\
&= ||\hat{g}||_q \left(\int_{\partial\mathcal{D}'_i} \left|\nabla \left(\frac{1}{\sqrt{|z|}}\right) \nabla G_i\right|^p \mathrm{d}A\right)^{\frac{1}{p}}.
\end{aligned}
$$

And hence we obtain

$$\begin{aligned}
{I_1}^p &\leq C_4 \int_{\partial\mathcal{D}_i'} \left| \nabla \left(\frac{1}{\sqrt{|z|}} \right) \nabla G_i \right|^p \mathrm{d}A \\
&= C_4 \int_{\partial\mathcal{D}_i'} \left| \nabla \left(\frac{1}{\sqrt{|z|}} \right) \right|^p |\nabla G_i|^p \mathrm{d}A \\
&\leq C_4 \int_{\partial\mathcal{D}_i} |\phi_i'|^{p-2} \left| \nabla \left(\frac{1}{1-|\phi_i|} \right) \right|^p |\phi_i'|^2 \mathrm{d}A \\
&= C_4 \int_{|w|<1} \frac{1}{|\psi_i'|^{p-2}} \left| \nabla \left(\frac{1}{1-r} \right) \right|^p \mathrm{d}A \\
&= C_4 \int_{|w|<1} \frac{\mathrm{d}A}{(1-r)^{3p-2}} \\
&= C_4 \left| \frac{(1-r)^{-3p+3}}{-3p+3} \right]_0^1
\end{aligned}$$

such that when $-3p + 3 > 0$ this implies to $p < 1$.

It is a consequence of *Hölder inequality*[11] and *Calderón-Zygmund* theorem on the continuity of singular integral operators, (cf.[32] &, [10, pp.564]), that

$$\begin{aligned}
I_2 &= \int_{\partial\mathcal{D}_i} \left| \frac{\nabla|\hat{g}(z)|}{\sqrt{|z|}} \right| |\mathrm{d}\mu_i| \\
&= \int_{\partial\mathcal{D}_i'} |\nabla|\hat{g}(z)|| \left| \frac{|\nabla G_i|}{\sqrt{|z|}} \right| \mathrm{d}A \\
&\leq C||g||_q \left(\int_{\partial\mathcal{D}_i'} \left| \frac{|\nabla G_i|}{\sqrt{|z|}} \right|^p \mathrm{d}A \right)^{\frac{1}{p}} \\
&\leq C_1 \left(\int_{\partial\mathcal{D}_i'} \left| \frac{\nabla G_i}{\sqrt{|z|}} \right|^p \mathrm{d}A \right) \\
&\leq C_1 \int_{\mathcal{D}_i} \frac{|\phi_i'|^{p-2}}{1-|\phi_i|} |\phi_i'|^2 \mathrm{d}A \qquad \text{by inequality } (a) \\
&= C_1 \int_{|w|<1} \frac{1}{|\psi_i'|^{p-2}} \frac{1}{(1-|w|)} \mathrm{d}A \qquad \text{by inequality } (b) \\
&= C_2 \int_{|w|<1} \frac{\mathrm{d}A}{(1-|w|)^{p-1}} \\
&= C_3 \left| \frac{(1-r)^{-p+2}}{-p+2} \right]_0^1,
\end{aligned}$$

[11] For further information about the role of *Hölder inequality*

when $-p+2>0$ this implies to $p<2$.

Finally, the quantity

$$\int_{\partial \mathcal{D}_i} \nabla \left(\frac{|\hat{g}(z)|}{\sqrt{|z|}} \right) \mathrm{d}\mu_i \quad \text{exists and is finite on } \partial \mathcal{D}_i.$$

$\square$

Chapter 4

Weighted composition operator associated with holomorphic self -map

The aim of this chapter is to deal with the boundedness of weighted composition operator C_ϕ^b for univalent function ϕ on classical weighted *Bergman space* A_α^2 and $A_{\alpha-1}^2$ depending only on functions of the form $(1-\overline{\lambda}z)^{-\gamma/2}$ and the convexity property of the function $\alpha_\phi(t)$.

We shall show the opertators C_ϕ^b as acting in classical weighted *Bergman space* A_α^p. Let ϕ be an holomorphic self- map of the unit disk $\mathbb{D} \subset \mathbb{C}$. The classical composition operator C_ϕ is defined as a mapping $f \mapsto f \circ \phi$. One can define a weighted composition operator C_ϕ^b as a mapping $C_\phi^b : f \mapsto f \circ \phi \cdot (\phi')^b$ for each $b \in \mathbb{R}$ in the classical weighted *Bergman* spaces A_α^p.[1]

The space A_α^p is equivalent to a finite, positive borel measure μ in $\mathbb{D}$ which is called an A_α^p- **Carleson measure** if the inequality

$$\int_{\mathbb{D}} |f(z)|^p \mathrm{d}\mu(z) \leq C \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z),$$

holds for $f \in A_\alpha^p(\mathbb{D})$.

[1] The interested reader is referred to chapter 1, definition (1.1.9) for further information on the classical weighted *Bergman* spaces A_α^p.

In addition, if we assume that $D(z, r)$ is a disk of radius r centered at z, and for any $\lambda \in \mathbb{D}$, then a finite positive measure μ in $\mathbb{D}$ is an A^p_α **-Carleson measure** if and only if

$$\mu\left(D(\lambda, \frac{1}{2}(1-|\lambda|))\right) \leq C(\mu)(1-|\lambda|)^{\alpha+2}. \tag{4.1}$$

One more equivalent condition is

$$\int_{\mathbb{D}} \frac{\mathrm{d}\mu(z)}{|1-\overline{\lambda}z|^\gamma} \leq \frac{C(\mu,\gamma)}{(1-|\lambda|^2)^{\gamma-\alpha-2}} \tag{4.2}$$

for some $\gamma > \alpha + 2$ and any $\lambda \in \mathbb{D}$.

The property of a measure μ to be an A^p_α **-Carleson measure** depends only on α and the product pb, which implies that C^b_ϕ to be bounded in A^p_α, as shown in the lemma below.

Lemma 4.0.9. *Let ϕ be a conformal self-map of $\mathbb{D}$. Then C^b_ϕ is bounded in A^p_α if and only if the measure $\mathrm{d}\mu$ defined as*

$$\mu(E) := \int_{\phi^{-1}(E)} |\phi'(z)|^{pb}(1-|z|^2)^\alpha \mathrm{d}A(z), \tag{4.3}$$

is an A^p_α -Carleson measure.

That's why *Shimorin* [58] considered only *Hilbert space* case $p = 2$, to study the boundedness of operators C^b_ϕ in A^2_α, by applying the *Carleson* measure condition (4.2) to the measure μ in (4.1) as follows.

Lemma 4.0.10. *[58] The operator C^b_ϕ is bounded in A^2_α if and only if for some $\gamma > \alpha + 2$*

$$\int_{\mathbb{D}} \frac{|\phi'(u)|^{2b}}{|1-\overline{\lambda}|\phi(u)|^\gamma} (1-|u|^2)^\alpha \mathrm{d}A(u) \leq \frac{C(\phi,\gamma)}{(1-|\lambda|^2)^{\gamma-\alpha-2}} \quad \forall \lambda \in \mathbb{D}. \tag{4.4}$$

Lemma above shows that it is enough to consider the function of the form $(1-\overline{\lambda}z)^{-\gamma/2}$ with $\gamma > \alpha + 2$, to prove boundedness of C^b_ϕ. This function contributes to the reproducing kernel for the space A^2_α (cf. [49, pp.131]).

Furthermore the operator C^b_ϕ, is bounded for fixed p and b in case α is sufficiently big but not necessarily true for all α and p. That's why he introduced two functions as follows:

$$\alpha_\phi(t) = \inf\{\beta > 0 : C^{\frac{t}{2}}_\phi \text{ is bounded in } \mathcal{A}^2_{\beta-1}\} \tag{4.5}$$

$$\mathcal{A}(t) := \sup_\phi \alpha_\phi(t) \tag{4.6}$$

with application the convexity property of the function $\alpha_\phi(t)$

$$|\alpha_\phi(t_1) - \alpha_\phi(t_2)| \leq |t_1 - t_2|. \tag{4.7}$$

which is equivalent to two estimates

$$\alpha_\phi(t+\epsilon) \leq \alpha_\phi(t) + \epsilon \ \text{ and } \ \alpha_\phi(t-\epsilon) \leq \alpha_\phi(t) + \epsilon, \qquad \text{valid for } \epsilon \geq 0. \tag{4.8}$$

Hence, this technique is enough to prove that $\mathcal{A} \leq |t| - 1$ for $t \leq t_0$, which is based on the amazing result obtained by *Bertilsson*

Theorem 4.0.11. ***(Bertilsson**[5]) For $t \leq t_0$, there exists a constant $C = C(t)$ such that for any $\phi \in S$*

$$\int_{\partial\mathbb{D}=\mathbb{T}} \left| r^2 \frac{\phi'(r\zeta)}{\phi^2(r\zeta)} \right|^t \mathrm{d}m(\zeta) \leq \frac{C}{(1-r)^{|t|-1}}. \tag{4.9}$$

Corollary 4.0.12. *[58] Let ϕ be a conformal self-map of $\mathbb{D}$ such that $\phi(0) = 0$. The function*

$$f(z) = \frac{\phi'(0)^{-1}\phi(z)}{(1-\overline{\lambda}\phi(z))^2} \qquad \textit{for some } \ \lambda \in \mathbb{D},$$

is in the class S with

$$\frac{z^2 f'(z)}{f^2(z)} = \phi'(0)\left(\frac{z}{\phi(z)}\right)^2 \phi'(z)(1+\overline{\lambda}\phi(z))(1-\overline{\lambda}\phi(z)).$$

Then, for $t \leq t_0$, with inequality (4.9)

$$\int_{\mathbb{T}} \frac{|\phi'(r\zeta)|^t}{|1-\overline{\lambda}\phi(r\zeta)|^{|t|}} \mathrm{d}m(\zeta) \leq C(\phi,t) \int_{\partial\mathbb{D}=\mathbb{T}} \left| r^2 \frac{\phi'(r\zeta)}{\phi^2(r\zeta)} \right|^2 \mathrm{d}m(\zeta) \leq \frac{C_1(\phi,t)}{(1-r)^{|t|-1}}. \tag{4.10}$$

Lemma 4.0.13. *[58, pp.6] If ϕ is a conformal self-map of $\mathbb{D}$ satisfying $\phi(0) = 0$, then*

i. $|1-\overline{\lambda}\phi(z)| \geq \frac{(1-|\phi(z)|^2)}{2}$, *for any* $\lambda \in \mathbb{D}$.

ii. $\int_{\mathbb{D}} \frac{|\phi'|^{t-\epsilon}}{|1-\overline{\lambda}\phi|^{\gamma}} |\phi'|^{\epsilon}(1-|z|^2)^{\alpha+\epsilon-1} \mathrm{d}A(z) \leq \left(\frac{2}{c^2}\right)^{\epsilon} \int_{\mathbb{D}} \frac{|\phi'|^t}{|1-\overline{\lambda}\phi|^{\gamma-\epsilon}} (1-|z|^2)^{\alpha-1} \mathrm{d}A(z)$.

Our point of work, started from *Shimorin*'s paper and specifically from the convexity property of the function $\alpha_\phi(t)$ which plays a main role in our work to prove the boundedness of the operator $C_\phi^{\frac{t}{2}}$, in weighted *Bergman space* A_α^2 if

$$\int_{\mathbb{D}} \frac{|\phi'|^t}{|1-\overline{\lambda}\phi|^\gamma}(1-|z|^2)^\alpha \mathrm{d}A(z) \le \frac{C_1^{t-s}(\phi,t)\cdot C_2^s(\phi,t)}{(1-|\lambda|^2)^{\gamma-\alpha-2t}}, \text{ for some } \gamma > \alpha+2t, \quad (4.11)$$

without depending on the convexity of $\alpha_\phi(t)$. But in case, when $\alpha_\phi(t)$ is convex then the operator $C_\phi^{\frac{t}{2}}$ is bounded in $A_{\alpha-1}^2$ if

$$\int_{\mathbb{D}} \frac{|\phi'|^t}{|1-\overline{\lambda}\phi|^\gamma}(1-|u|^2)^{\alpha-1} \mathrm{d}A(u) \le \frac{C_1(\phi,t)(C_8}{(1-|\lambda|^2)^{\gamma-|t|-1}}.$$

These results are settled in theorems (4.1.1), (4.1.2) respectively.

4.1 Main results

Theorem 4.1.1. *Assume that ϕ is a conformal self- map of $\mathbb{D}$, then $C_\phi^{\frac{t}{2}}$ is bounded in A_α^2 if*

$$\int_{\mathbb{D}} \frac{|\phi'|^t}{|1-\overline{\lambda}\phi|^\gamma}(1-|u|^2)^\alpha \mathrm{d}A(u) \le \frac{C_1^{t-s}(\phi,t)\cdot C_2^s(\phi,t)}{(1-|\lambda|^2)^{\gamma-\alpha-2t}}, \quad \textit{for some } \gamma > \alpha+2t. \quad (4.12)$$

Proof. Let $s \in (0,1)$ and assume $t = t-s+s = (t-s)+s$ and $\alpha = \alpha-1+1 = (\alpha-1)+1$, then applying the *Hölder* inequality with $\frac{1}{a} = t-s$, and $\frac{1}{b} = s$, one obtains

$$\int_{\mathbb{D}} \frac{|\phi'|^t}{|1-\overline{\lambda}\phi|^\gamma}(1-|u|^2)^\alpha \mathrm{d}A(u) \le \left(\int_{\mathbb{D}} \frac{|\phi'|^{\frac{t}{t-s}}}{|1-\overline{\lambda}\phi|^{\frac{\lambda}{2(t-s)}}}(1-|u|^2)^{\frac{\alpha-1}{t-s}} \mathrm{d}A(u)\right)^{t-s} \cdot$$
$$\cdot \left(\int_{\mathbb{D}} \frac{|\phi'|^{\frac{t}{s}}}{|1-\overline{\lambda}\phi|^{\frac{\lambda}{2s}}}(1-|u|^2)^{\frac{1}{s}} \mathrm{d}A(u)\right)^{s}$$

By Lemma (4.0.10)we obtain,

$$\begin{aligned}
&\le \left(\frac{C_1(\phi,t)}{(1-|\lambda|^2)^{\frac{\gamma}{2(t-s)}-\frac{\alpha-1}{t-s}-2}}\right)^{t-s} \cdot \left(\frac{C_2(\phi,t)}{(1-|\lambda|^2)^{\frac{\gamma}{2s}-\frac{1}{s}-2}}\right)^{s} \\
&= \frac{C_1^{t-s}(\phi,t)}{(1-|\lambda|^2)^{\frac{\gamma}{2}-\alpha+1-2t+2s}} \cdot \frac{C_2^s(\phi,t)}{(1-|\lambda|^2)^{\frac{\gamma}{2}-1-2s}} \\
&= \frac{C_1^{t-s}(\phi,t)\cdot C_2^s(\phi,t)}{(1-|\lambda|^2)^{\gamma-\alpha-2t}}.
\end{aligned}$$

Finally, when $\gamma - \alpha - 2t > 0 \Rightarrow \gamma > \alpha + 2t$. $\square$

Theorem 4.1.2. *Let ϕ be a conformal self-map of $\mathbb{D}$ satisfying $\phi(0) = 0$, and let $\alpha_\phi(t)$ be convex. Then the operator $C_\phi^{\frac{t}{2}}$ is bounded in $A^2_{\alpha-1}$ if*

$$\int_{\mathbb{D}} \frac{|\phi'|^t}{|1-\overline{\lambda}\phi|^\gamma}(1-|u|^2)^{\alpha-1}\mathrm{d}A(u) \leq \frac{C_1(\phi,t)}{(1-|\lambda|^2)^{\gamma-|t|-1}},$$

where $C_1(\phi, t) > 0$ is a constant depends on ϕ and t .

Proof. Fix $\lambda \in \mathbb{D}$, and pick some $\epsilon \in (0, 1)$

$$\begin{aligned}\int_{\mathbb{D}} \frac{|\phi'(u)|^t}{|1-\overline{\lambda}\phi(u)|^\gamma}(1-|u|^2)^{\alpha-1}\mathrm{d}A(u) &= \int_{\mathbb{D}} \frac{|\phi'(u)|^{t-\epsilon+\epsilon}}{|1-\overline{\lambda}\phi|^\gamma}(1-|u|^2)^{\alpha-1+\epsilon-\epsilon}\,\mathrm{d}A(u)\\ &= \int_{\mathbb{D}} \frac{|\phi'(u)|^{t-\epsilon}}{|1-\overline{\lambda}\phi(u)|^\gamma}|\phi'(u)|^\epsilon\,(1-|u|^2)^{\alpha+\epsilon-1}\\ &\quad\cdot(1-|u|^2)^{-\epsilon}\mathrm{d}A(u)\end{aligned}$$

by applying inequality (4.0.13(i)), and inequality (4.0.13(ii))

$$\begin{aligned}&\leq \left(\frac{2}{c^2}\right)^\epsilon \int_{\mathbb{D}} \frac{|\phi'(u)|^t}{|1-\overline{\lambda}\phi(u)|^{\gamma-\epsilon}}\frac{(1-|\phi(u)|^2)^\epsilon}{1-|u|^2)^\epsilon}(1-|u|^2)^{\alpha-1-\epsilon}\mathrm{d}A(u)\\ &= \left(\frac{2}{c^2}\right)^\epsilon \int_{\mathbb{D}} \frac{|\phi'(u)|^t}{|1-\overline{\lambda}\phi(u)|^{\gamma}}\cdot\frac{(1-|u|^2)^{\alpha-1-2\epsilon}}{|1-\overline{\lambda}\phi(u)|^{-\epsilon}}(1-|\phi(u)|^2)^\epsilon\mathrm{d}A(u)\\ &\leq \left(\frac{4}{c^2}\right)^\epsilon \int_{\mathbb{D}} \frac{|\phi'(u)|^t}{|1-\overline{\lambda}\phi(u)|^{\gamma}}\cdot\frac{(1-|u|^2)^{\alpha-1-2\epsilon}}{|1-\overline{\lambda}\phi(u)|^{-\epsilon}}|1-\overline{\lambda}\phi(u)|^\epsilon\mathrm{d}A(u)\\ &= \left(\frac{4}{c^2}\right)^\epsilon \int_{\mathbb{D}} \frac{|\phi'(u)|^t}{|1-\overline{\lambda}\phi(u)|^{\gamma}}\cdot\frac{(1-|u|^2)^{\alpha-1-2\epsilon}}{|1-\overline{\lambda}\phi(u)|^{-2\epsilon}}\mathrm{d}A(u)\end{aligned}$$

Let $\gamma = \gamma - |t| + |t|$, and then apply the corollary (4.0.12), to obtain.

$$\begin{aligned}&\left(\frac{4}{c^2}\right)^\epsilon \int_{\mathbb{D}} \frac{|\phi'(u)|^t}{|1-\overline{\lambda}\phi(u)|^{\gamma}}\cdot\frac{(1-|u|^2)^{\alpha-1-2\epsilon}}{|1-\overline{\lambda}\phi(u)|^{-2\epsilon}}\mathrm{d}A(u) =\\ &= \left(\frac{4}{c^2}\right)^\epsilon \int_0^1\int_{\partial\mathbb{D}} \frac{|\phi'(r\zeta)|^t}{|1-\overline{\lambda}\phi(r\zeta)|^{\gamma+|t|-|t|}}\cdot\frac{(1-r^2)^{\alpha-1-2\epsilon}}{|1-\overline{\lambda}\phi(r\zeta)|^{-2\epsilon}}\mathrm{d}m(\zeta)2r\mathrm{d}r\\ &= \left(\frac{4}{c^2}\right)^\epsilon \int_0^1\int_{\partial\mathbb{D}} \frac{|\phi'(r\zeta)|^t}{|1-\overline{\lambda}\phi(r\zeta)|^{|t|}}\cdot\frac{(1-r^2)^{\alpha-1-2\epsilon}}{|1-\overline{\lambda}\phi(r\zeta)|^{\gamma-|t|-2\epsilon}}\mathrm{d}m(\zeta)2r\mathrm{d}r\\ &\leq C_3\left(\frac{4}{c^2}\right)^\epsilon \int_0^1 \frac{C_1(\phi,t)}{(1-r)^{|t|-1}}\cdot\frac{(1-r^2)^{\alpha-1-2\epsilon}}{(1-r|\lambda|)^{\gamma-|t|-2\epsilon}}\mathrm{d}r\end{aligned}$$

For any ϕ, a trivial estimate of $\alpha_\phi(t)$ is $\alpha_\phi(t) \leq |t|$.

$$\begin{aligned}
&\leq C_3\left(\frac{4}{c^2}\right)^{\epsilon}\int_0^1 \frac{C_1(\phi,t)}{(1-r)^{|t|-1}}\cdot\frac{(1-r^2)^{|t|-1-2\epsilon}}{(1-r|\lambda|)^{\gamma-|t|-2\epsilon}}\mathrm{d}r \\
&\leq C_4\left(\frac{4}{c^2}\right)^{\epsilon}\int_0^1 \frac{C_1(\phi,t)(1-r)^{-2\epsilon}}{(1-r|\lambda|)^{\gamma-|t|-2\epsilon}}\mathrm{d}r \\
&= C_5C_1(\phi,t)\left(\underbrace{\int_0^{|\lambda|}\frac{(1-r)^{-2\epsilon}}{(1-r|\lambda|)^{\gamma-|t|-2\epsilon}}\mathrm{d}r}_{I_1}+\underbrace{\int_{|\lambda|}^1\frac{(1-r)^{-2\epsilon}}{(1-r|\lambda|)^{\gamma-|t|-2\epsilon}}\mathrm{d}r}_{I_2}\right)
\end{aligned}$$

In this,

$$\begin{aligned}
I_1 = \int_0^{|\lambda|}\frac{(1-r)^{-2\epsilon}}{(1-r|\lambda|)^{\gamma-|t|-2\epsilon}}\mathrm{d}r &\leq (1-|\lambda|)^{-2\epsilon}\int_0^{|\lambda|}\frac{\mathrm{d}r}{(1-r|\lambda|)^{\gamma-|t|-2\epsilon}} \\
&\leq \frac{C_6C_1(\phi,t)}{(1-|\lambda|)^{\gamma-|t|-1}}
\end{aligned}$$

And then,

$$\begin{aligned}
I_2 = \int_{|\lambda|}^1\frac{(1-r)^{-2\epsilon}}{(1-r|\lambda|)^{\gamma-|t|-2\epsilon}}\mathrm{d}r &\leq \frac{1}{(1-|\lambda|)^{\gamma-|t|-2\epsilon}}\int_{|\lambda|}^1(1-r)^{-2\epsilon}\mathrm{d}r \\
&= \frac{C_7C_1(\phi,t)}{(1-|\lambda|)^{\gamma-|t|-1}}.
\end{aligned}$$

Finally

$$\int_{\mathbb{D}}\frac{|\phi'|^t}{|1-\overline{\lambda}\phi|^\gamma}(1-|u|^2)^{\alpha-1}\mathrm{d}A(u) \leq \frac{C_1(\phi,t)}{(1-|\lambda|^2)^{\gamma-|t|-1}},$$

where $C_1(\phi,t) > 0$ is a constant depends on ϕ and t. □

Chapter 5

Integrability over cardioid domain, and simply connected domain

In this chapter, we begin with *P. Hajlasz* [28] in the case Ω is the image of the two dimensional disk $\mathbb{D} = \{z : |z - i| < 1\}$ under the mapping $z \to z^2$, and the boundary of Ω is a smooth curve except at the point $z = 0$, where we have an inward cusp, then

$$u(r, \theta) = \frac{1}{2} - \frac{1}{\sqrt{r}} \sin \frac{\theta}{2}$$

is harmonic in Ω and it vanishes on $\partial\Omega$, except at the discontinuity point $z = 0$ and hence u belongs to $\mathbb{W}_0^{1,p}(\Omega)$ for all $1 \leq p < \frac{4}{3}$.

This motivates us to consider *Laplace's equation* on a bounded simply connected domain $\Omega \subset \mathbb{R}^2$, with *Dirichlet boundary conditions*;

$$\left.\begin{aligned} \Delta u = 0, \qquad & x \in \Omega; \\ u = 0, \qquad & x \in \partial\Omega. \end{aligned}\right\} \tag{5.1}$$

It is well known that, $u \equiv 0$ in the closure of Ω, if u belongs to the sobolev space $\mathbb{W}_0^{1,p}(\Omega)$, [1] $p \geq 2$ (or $p > 1$ if $\partial\Omega$ is smooth). For non-smooth domain this is no longer true.

[1] $\mathbb{W}_0^{1,p}(\Omega)$ is the closure of $\mathbb{C}_0^\infty(\Omega)$ in the $\mathbb{W}^{1,p}(\Omega)$- norm, cf. ([13, 54] and [59]).

Let

$$u(r,\theta) = r^{-\sqrt{\mu}} \sin \sqrt{\mu}\theta, \tag{5.2}$$

one of the polar functions of the general solution of the *Laplace* equation under polar coordinates (r, θ)

$$U(r,\theta) = \underbrace{A_1 r^{\sqrt{\mu}} cos\sqrt{\mu}\theta}_{T_1} + \underbrace{A_2 r^{\sqrt{\mu}} \sin\sqrt{\mu}\theta}_{T_2} + \underbrace{A_3 r^{-\sqrt{\mu}} cos\sqrt{\mu}\theta}_{T_3} + \underbrace{A_4 r^{-\sqrt{\mu}} \sin\sqrt{\mu}\theta}_{T_4}, \tag{5.3}$$

in the circular sector $\{(r,\theta) : \ 0 < \theta < \alpha, \ 0 < r < a\} = \Omega$, where $0 \leq \theta \leq \pi$, and $\mu = \left(\frac{n\pi}{a}\right)^2, n = 0, 1, 2, \cdots$ is an eigenvalue, and $r = \phi(\theta) = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$ that makes the equation (5.2) vanish on the boundary of Ω, in order to study the existance of cusp on the boundary of cardioid domain, with classification into inward cusp, outward cusp via an integrability of conformal maps at $z = 0$, and its neighborhoods.

In Section (5.2), we construct an holomorphic function on the cardioid domain Ω, where $\phi'(0) = 0$, for $0 < n \leq 1$ in Theorem (5.2.1), and another belongs to *Hardy space* $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D})$, $n \in \mathbb{N}$, on the unit disk in Theorem (5.2.4).

For more information regarding such kind of work, cf. *D. Khavinson* papers [14] ,[38] and the books ([18] pp.168 -169), [22] are also excellent reference sources on this subject.

5.1 Existence of a cusp on the boundary of cardioid domain

Definition 5.1.1. [1],[44] Let $\Omega \subset \mathbb{R}^n$ be a simply connected domain. Fix point $z = 0$ in Ω and let $\partial\Omega$ be the boundary of Ω, let γ in Ω be defined as a simple Jordan arc which divides Ω into two subdomains.

Let $K = (\gamma_n)_{n=1}^{\infty}$, a sequence of γ_n in the given domain Ω, be called a chain, if it satisfies all the following conditions :

i. The diameter of γ_n tends to zero as $n \to \infty$.

ii. for each n the intersection $\gamma_n \cap \gamma_{n+1}$ is empty.

iii. any path connecting $z = 0$ in Ω with arc γ_n for all $n > 1$ intersects with arc γ_{n-1}.

Moreover, any two chains $K = (\gamma_n)$ and $K' = (\gamma_n')$ in Ω are equivalent if the arc γ_n separates the point $z = 0$ from all arcs γ_n' except for a finite number of them. An equivalence class of chains in Ω is called a prime end.

Remark 5.1.2. [53] Let ϕ map unit disk $\mathbb{D}$ conformally onto simply connected domain $\Omega \subset \mathbb{C}$ with locally connected boundary $\partial\Omega$. Let $\zeta = e^{i\theta} \in \partial\mathbb{D}$. Then $\partial\Omega$ has a corner of opening $\pi\alpha(0 \leq \alpha \leq 2)$ at $\phi(\zeta) \neq \infty$ if

$$arg[\phi(e^{it}) - \phi(e^{i\theta})] \to \begin{cases} \gamma & \text{as } t \to \theta_+, \\ \gamma + \pi\alpha & \text{as } t \to \theta_-. \end{cases} \tag{5.4}$$

Hence, if ϕ maps the unit disk onto the domain Ω, this will induce a one-to-one mapping between the points on the unit circle and the prime ends of Ω. That is, there may exist another point $\zeta' \in \partial\mathbb{D}$ with $\phi(\zeta') = \phi(\zeta)$ where there may be a corner of opening $\pi\alpha'$ or none at all. Also, if $\alpha = 1$ then we obtain a tangent of direction angle γ. If $\alpha = 0$, then we will obtain an outward-pointing cusp, and if $\alpha = 2$, we will get an inward-pointing cusp.

Theorem 5.1.3. *[53]*
Let ϕ maps $\mathbb{D}$ conformally onto the bounded domain $\Omega \subset \mathbb{C}$. Then the following four conditions are equivalent:

i. ϕ has a continuous extension to $\overline{\mathbb{D}}$;

ii. $\partial\Omega$ is a curve, that is $\partial\Omega$=$\{\varphi(\zeta) : \zeta \in \partial\mathbb{D}\}$;

iii. $\partial\Omega$ is locally connected;

iv. $\mathbb{C}\backslash\Omega$ is locally connected.

The main result of this section reads as follows:

Consider the two-dimentional Laplace equation in polar coordinates (r, θ)

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0, \tag{5.5}$$

in the circular sector $\{(r, \theta) : \ 0 < \theta < \alpha, \ 0 < r < a\} = \Omega$.

Let us start with the general solution of the *Laplace* equation under polar coordinates (r, θ) [54] as follows:

$$U(r, \theta) = \underbrace{A_1 r^{\sqrt{\mu}} cos\sqrt{\mu}\theta}_{T_1} + \underbrace{A_2 r^{\sqrt{\mu}} \sin\sqrt{\mu}\theta}_{T_2} + \underbrace{A_3 r^{-\sqrt{\mu}} cos\sqrt{\mu}\theta}_{T_3} + \underbrace{A_4 r^{-\sqrt{\mu}} \sin\sqrt{\mu}\theta}_{T_4}. \tag{5.6}$$

Choose $u(r, \theta) = r^{-\sqrt{\mu}} \sin\sqrt{\mu}\theta$, a harmonic function on Ω, such that

$$u(r, \theta) = r^{-\sqrt{\mu}} \sin\sqrt{\mu}\theta - 1 \tag{5.7}$$

to be zero on the boundary of Ω, where $r = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$, then the integrability $\int\int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}y$ depends on the local behaviour at a point $e^{i\theta} \in \partial\Omega$ as follows:

$$|\nabla u|^2 = \left[(\frac{\partial u}{\partial r})^2 + (\frac{1}{r}\frac{\partial u}{\partial \theta})^2\right]$$

$$|\nabla u|^2 = \mu r^{2(-\sqrt{\mu}-1)}\sin^2\sqrt{\mu}\theta + \mu r^{2(-\sqrt{\mu}-1)}cos^2\sqrt{\mu}\theta$$

$$|\nabla u|^p = \mu^{\frac{p}{2}} r^{(-\sqrt{\mu}-1)p}$$

Then,

$$\int\int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}y = \int\int_\Omega \mu^{\frac{p}{2}} r^{(-\sqrt{\mu}-1)p+1}\mathrm{d}r\mathrm{d}\theta \tag{5.8}$$

$$= \mu^{\frac{p}{2}} \int_0^{n\pi} \mathrm{d}\theta \left[\frac{r^{(-\sqrt{\mu}-1)p+2}}{(-\sqrt{\mu}-1)p+2}\right|_0^a \tag{5.9}$$

such that $(-\sqrt{\mu}-1)p + 2 > 0 \ \Rightarrow p < \frac{2}{\sqrt{\mu}+1}$.

Hence,

$$\int\int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}y < \infty \quad for \ \ p < \frac{2}{\sqrt{\mu}+1}. \tag{5.10}$$

It is clear that all the values of p depend on μ, which in turn depend on the condition

$$r = \phi(\theta) = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}, \tag{5.11}$$

such that if $0 \leq \theta \leq \pi$ then $\phi(\theta) = 0$ when $\theta = \frac{\pi}{\sqrt{\mu}}$.

Calculate the tangent vector for the function $r = \phi(\theta)$ in equation (5.11), by using the formula below:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\phi(\theta)\cos\theta + \sin\theta\phi'(\theta)}{-\phi(\theta)\sin\theta + \cos\theta\phi'(\theta)}$$

such that when $\theta = \frac{\pi}{\sqrt{\mu}}$ this implies that $\phi(\theta = \frac{\pi}{\sqrt{\mu}}) = 0$

Hence we will get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\theta\phi'(\theta)}{\cos\theta\phi'(\theta)} = \tan\theta \quad \text{at} \quad \theta = \frac{\pi}{\sqrt{\mu}}.$$

Suppose, $\tan\theta = 0$ at $\theta = \frac{\pi}{\sqrt{\mu}}$, then $\frac{\pi}{\sqrt{\mu}} = 0, \pi, 2\pi, \ldots = n\pi$, $n \in \mathbb{N}$, and it gives that

$$\mu = \frac{1}{n^2}, \quad n \in \mathbb{N}. \tag{5.12}$$

Consequently, we deduce that

$$\int\int_{\Omega} |\nabla u|^p \mathrm{d}x\mathrm{d}y < \infty \qquad for\ \ p < \frac{2}{\sqrt{\mu}+1}, \qquad \text{where} \quad \mu = \frac{1}{n^2}, \quad n \in \mathbb{N}.$$

Hence, we can classify this result, depending on n as follows:-

i. In case, $n = 2$ then $|\nabla u| \in L_p(\Omega)$ for all $p < \frac{4}{3}$ and $u(r,\theta)$ vanishes on $\partial\Omega$ except the discontinuity point $z = 0$.

ii. In case, $n \geq 3$ then $|\nabla u| \in L_p(\Omega)$ for all $p < \frac{2}{\sqrt{\mu}+1}$, and u vanishes on $\partial\Omega$ except some inward cusps at the neighborhoods of $z = 0$.

iii. According to the above, the harmonic function $u = r^{-\sqrt{\mu}}\sin\sqrt{\mu}\theta - 1$ in Ω, belongs to $\mathbb{W}^{1,p}(\Omega)$ for all $p < \frac{2}{\sqrt{\mu}+1}$, where $\mu = \frac{1}{n^2}$, $n \in \mathbb{N}$, that is, $u = 0$ on $\partial\Omega$ for $r = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$ and we deduce that when we approach the vertex of the cusp

$z = 0$ along the boundary of Ω we will have zero limit, whereas if we properly approach the vertex of the cusp $z = 0$ from the interior of Ω we will have

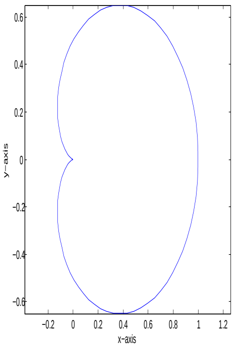
$$\lim_{z \to 0} u = r^{-\sqrt{\mu}} \sin \sqrt{\mu}\theta - 1 = -\infty.$$

For this we can say that $u = 0$ on the boundary of a given domain except at discontinuity point $z = 0$.

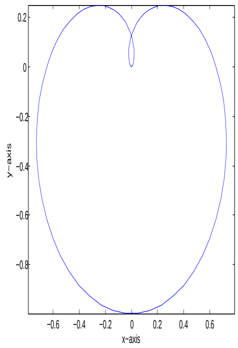
Likewise, it can be carried out along the same lines for another harmonic function in the general solution of *Laplace equation* which is:

$$u(r,\theta) = r^{-\sqrt{\mu}} \cos \sqrt{\mu}\theta - 1, \quad \text{where} \quad r = (\cos \sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}, \quad \text{and} \quad \mu \quad \text{is an eigenvalue.}$$

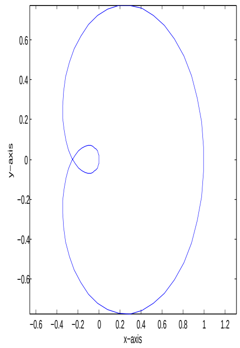
In order to derive more information about the existence of *inward-pointing cusp* on the boundary of Ω at the point $z = 0$ and its neighborhoods, see figure (5.1) and table (5.1) below.



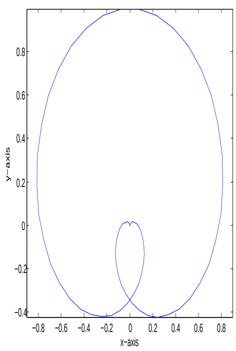
(a) Figure for $r = (\sin\theta/2)^2$



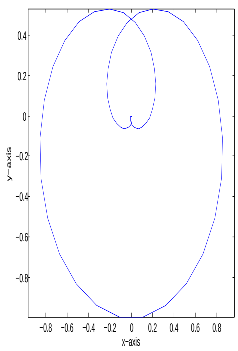
(b) Figure for $r = (\sin\theta/3)^3$



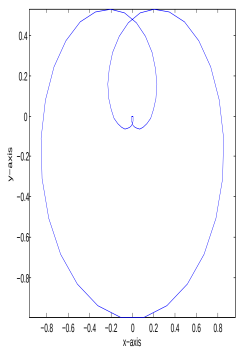
(c) Figure for $r = (\sin\theta/4)^4$



(d) Figure for $r = (\sin\theta/5)^5$



(e) Figure for $r = (\sin\theta/6)^6$



(f) Figure for $r = (\sin\theta/7)^7$

Figure 5.1: Existence of inward cusp on the boundary Ω where $r = (sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$.

Table 5.1: Self intersection points for the polar function $r = \phi(\theta) = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$.

n	$\mu = \frac{1}{n^2}$	$r = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$	(θ_1, θ_2)	inward-pointing cusp at the neighborhood of $z = (r, \theta) = 0$
3	$\frac{1}{9}$	$\sin^3\frac{\theta}{3}$	$(\frac{\pi}{2}, \frac{5\pi}{2})$	(0.125,$\frac{\pi}{2}$)
				(0.125,$\frac{5\pi}{2}$)
4	$\frac{1}{16}$	$\sin^4\frac{\theta}{4}$	$(\pi, 3\pi)$	(0.2500,π)
				(0.2500,3π)
5	$\frac{1}{25}$	$\sin^5\frac{\theta}{5}$	$(\frac{3\pi}{2}, \frac{7\pi}{2})$	(0.0028,$\frac{\pi}{2}$)
			$(\frac{\pi}{2}, \frac{9\pi}{2})$	(0.0028,$\frac{9\pi}{2}$)
6	$\frac{1}{36}$	$\sin^6\frac{\theta}{6}$	(2π,4π)	(0.0156,π)
			$(\pi, 5\pi)$	(0.0156,5π)
7	$\frac{1}{49}$	$\sin^7\frac{\theta}{7}$	$(\frac{5\pi}{2}, \frac{9\pi}{2})$	(0.4819,$\frac{5\pi}{2}$)
			$(\frac{3\pi}{2}, \frac{11\pi}{2})$	(0.4819,$\frac{9\pi}{2}$)
			$(\frac{\pi}{2}, \frac{13\pi}{2})$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

At this stage, we consider $n \notin \mathbb{N}$ for example, then

$$\int\int_{\Omega} |\nabla u|^p \mathrm{d}x\mathrm{d}y \nless \infty \quad \text{for } p < \frac{2}{\sqrt{\mu}+1}, \quad \text{where} \quad \mu = \frac{1}{n^2}.$$

However there is no inward-pointing cusp on the boundary of Ω, that is, we have outward-pointing cusp on the boundary of Ω.

For instance, let $n = \sqrt{2} \notin \mathbb{N}$, then $\mu = \frac{1}{2} \Rightarrow \sqrt{\mu} = \frac{1}{\sqrt{2}}$, such that

$$u(r,\theta) = \left(r^{\frac{-1}{\sqrt{2}}} \sin \frac{\theta}{\sqrt{2}} \right) - 1, \quad 0 \leq \theta \leq \sqrt{2}\pi.$$

and $u = 0$ on $\partial\Omega$ where $r = \phi(\theta) = (\sin \frac{\theta}{\sqrt{2}})^{\sqrt{2}}$.

Calculating the tangent vector for the function $r = \phi(\theta)$ as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\phi(\theta)\cos\theta + \sin\theta\phi'(\theta)}{-\phi(\theta)\sin\theta + \cos\theta\phi'(\theta)}$$

such that in case, $\theta = 0 \Rightarrow \phi(\theta = 0) = 0$, which implies to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\theta\phi'(\theta)}{\cos\theta\phi'(\theta)} = \tan\theta$$

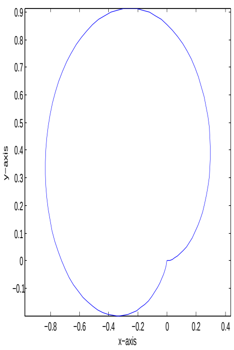
and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, however in case $\theta = \sqrt{2}\pi \Rightarrow f(\theta = \sqrt{2}\pi) = 0$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan(\sqrt{2}\pi) = 3.6202$$

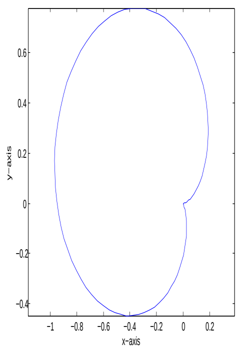
is a straight-line equation

$$y = 3.6202\, x + c, \qquad c \text{ is a constant.}$$

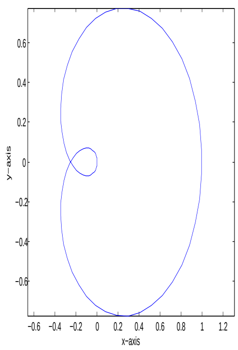
Some figures in (5.2) which plotted in Matlab program help to locate more information about the existence of *outward-pointing cusp* on the boundary of Ω at the point $z = 0$ and its neighborhoods for the function $r = \phi(\theta) = (sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$.



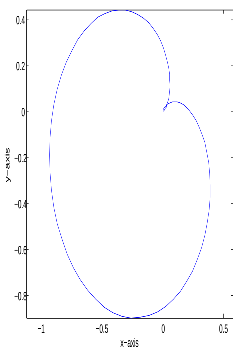
(a) Figure for $r = (\sin\theta/\sqrt{2})^{\sqrt{2}}$



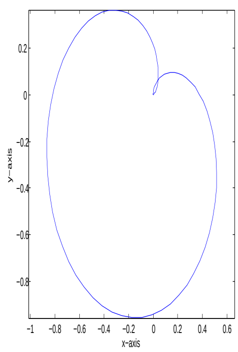
(b) Figure for $r = (\sin\theta/\sqrt{3})^{\sqrt{3}}$



(c) Figure for $r = (\sin\theta/\sqrt{5})^{\sqrt{5}}$



(d) Figure for $r = (\sin\theta/\sqrt{6})^{\sqrt{6}}$



(e) Figure for $r = (\sin\theta/\sqrt{7})^{\sqrt{7}}$

Figure 5.2: Existence of outward cusp on the boundary Ω where $r = (sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$.

5.2 Generating an holomorphic function on cardioid domain, unit disk

This section is of great interest to illustrate how to generate an holomorphic function on the cardioid domain Ω by a harmonic function defined on Ω, and vanishes on the boundary of Ω, and another on the unit disk by holomorphic function belongs to *Smirnov domain (cardioid type)*, which is settled in Theorems 5.2.1 and 5.2.4.

Theorem 5.2.1. *Let* $u(r,\theta) = \left(r^{-\sqrt{\mu}} \cos \sqrt{\mu}\theta\right) - 1$ *be a harmonic function on cardioid domain* Ω *and* $u(r,\theta) = 0$ *on* $\partial\Omega$ *where* $r = (\cos \sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$, *where* μ *is an eigenvalue. Then the polar function* $r = (\cos \sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$ *generates holomorphic function on* Ω *for all* $0 < n \leq 1$.

Proof. Let

$$z = \psi(\zeta) = c(1+\zeta)^n \tag{5.13}$$

be a conformal mapping defined on the simply connected domain Ω onto unit disk $\{\zeta : |\zeta| < 1\}$. To derive polar function we shall define $\zeta = e^{i\alpha}$ on the boundary of the unit circle such that

$$z = \psi(\zeta) = c(1+e^{i\alpha})^n = c(1+\cos\alpha + i\sin\alpha)^n \tag{5.14}$$

Since $z = re^{i\theta}$ is a point on the curve C in the interior of Ω then equation (5.14) becomes

$$\begin{aligned} z &= c(1+\cos\alpha + i\sin\alpha)^n \\ re^{i\theta} &= c\left[2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]^n \\ &= c\left[2\cos\frac{\alpha}{2}(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2})\right]^n. \\ re^{i\theta} &= c\left[2^n\cos^n\frac{n\alpha}{2}\right]e^{\frac{in\alpha}{2}} \end{aligned}$$

Therefore $\theta = \frac{n\alpha}{2}$ and

$$r = c\left[2\cos\frac{\alpha}{2}\right]^n \tag{5.15}$$

Substituting $\alpha = \frac{2\theta}{n}$ into equation (5.15) we obtain

$$r = c_1 \left[\cos\frac{\theta}{2}\right]^n \quad \text{where} \quad c_1 = 2^n c \quad \text{is a constant} \tag{5.16}$$

This polar function will generate a cardioid domain when $n = 2$. Apply *Fourier* expansion to the function

$$f(\theta) = \cos^{2n}\frac{\theta}{n}, \quad (-L \leq \theta \leq L) \quad \text{where} \quad L = \frac{n\pi}{2}. \tag{5.17}$$

We obtain that

$$f(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi\theta}{L}.$$

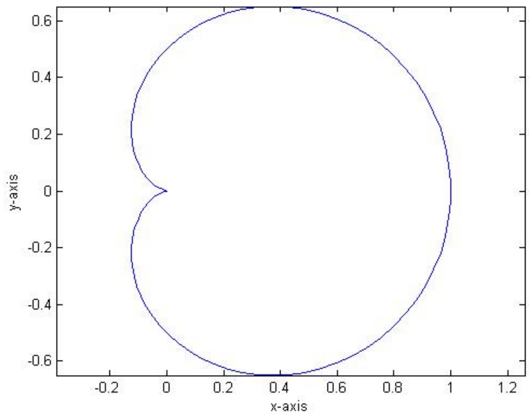


Figure 5.3: Cardioid domain, $r = \left[\cos\frac{\theta}{2}\right]^2$

We know that $f(\theta)$ being even implies $b_k = 0$. Substituting equation (5.17) and $L = \frac{n\pi}{2}$ into a_k-Formula, $a_k = \frac{2}{L}\int_0^L f(\theta)\cos\frac{k\pi\theta}{L}\mathrm{d}\theta$, we obtain

$$a_k = \frac{4}{n\pi}\int_0^{\frac{n\pi}{2}} \cos^{2n}\frac{\theta}{n}\cos\frac{2k\theta}{n}\mathrm{d}\theta. \tag{5.18}$$

Assuming $\frac{\theta}{n} = \theta_1$ in equation (5.18), for convenience, gives

$$a_k = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n}\theta_1 \cos(2k\theta_1)\mathrm{d}\theta. \tag{5.19}$$

Now, we need to apply *Cauchy's* formula which is related to *Gamma* function [2]

Put $\rho = 2n$ and $\beta = 2k$, to obtain

$$a_k = \frac{n}{2^{2n-2}} \frac{\Gamma(2n)}{\Gamma(1+n+k)\Gamma(1+n-k)}, \quad \text{for } \; k = 0, 1, 2, \dots \tag{5.20}$$

Substituting equation (5.20) into the *Fourier* series expansion as follows

$$f(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi\theta}{L}.$$

$$\cos^{2n}\frac{\theta}{n} = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \frac{n}{2^{2n-2}} \frac{\Gamma(2n)}{\Gamma(1+n+k)\Gamma(1+n-k)} \cos\frac{2k\theta}{n}.$$

$$\cos^{2n}\frac{\theta}{n} = \frac{n\Gamma(2n)}{2^{2n-1}} \left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty} \frac{\cos\frac{2k\theta}{n}}{\Gamma(1+n+k)\Gamma(1+n-k)} \right] \tag{5.21}$$

We notice that $\cos^{2n}\frac{\theta}{n} = r^2$ by equation (5.16) and

$$\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty} \frac{\cos\frac{2k\theta}{n}}{\Gamma(1+n+k)\Gamma(1+n-k)} \right] = Re \left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty} \frac{e^{ik\alpha}}{\Gamma(1+n+k)\Gamma(1+n-k)} \right] \tag{5.22}$$

where $\alpha = \frac{2\theta}{n}$. And hence, there exists function defined on the boundary of unit disk $\partial\mathbb{D}$ in ζ-plane, which is

$$\Phi(\zeta) = \frac{n\Gamma(2n)}{2^{2n-1}} \left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty} \frac{e^{ik\alpha}}{\Gamma(1+n+k)\Gamma(1+n-k)} \right] \tag{5.23}$$

such that $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha) = \eta + i\xi = \zeta.$, that is

$$\Phi(\zeta) = \frac{n\Gamma(2n)}{2^{2n-1}} \left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty} \frac{\zeta^k}{\Gamma(1+n+k)\Gamma(1+n-k)} \right] \tag{5.24}$$

By equation (5.13) we obtain $\zeta = (\frac{z}{c})^{\frac{1}{n}} - 1$.

[2] we refer to *Cauchy*'s formula which is related with *Gamma* function

$$\int_0^{\frac{\pi}{2}} (\cos t)^{\rho} \cos(\beta t) \mathrm{d}t = \frac{\pi\Gamma(1+\rho)2^{-\rho-1}}{\Gamma(1+\frac{1}{2}\rho+\frac{1}{2}\beta)\Gamma(1+\frac{1}{2}\rho-\frac{1}{2}\beta)}.$$

Consequently, we deduce

$$\begin{aligned} &\phi(z) = \phi_1(x,y) + i\phi_2(x,y) \\ \Rightarrow\quad &\phi(z) = \phi(\psi(\zeta)) = \phi_1(\eta,\xi) + i\phi_2(\eta,\xi) = \Phi(\zeta), \quad \text{where } z = \psi(\zeta) \end{aligned}$$

such that,

$$\phi(z) = \frac{n\Gamma(2n)}{2^{2n-1}}\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{(\frac{z}{c})^{\frac{1}{n}} - 1}{\Gamma(1+n+k)\Gamma(1+n-k)}\right]. \tag{5.25}$$

It is clear that ϕ is holomorphic function at $z = 0$ whose derivative exists and is continuous at $z = 0$, such that $\phi'(0) = 0$ for $0 < n \leq 1$. However on the other hand, $\phi(z)$ is not holomorphic at $z = 0$ for $1 < n \leq 2$, because $\phi'(z)$ at $z = 0$ does not exist.

□

Before moving on to a new theorem, we need just a little more background about *Smirnov* classes and *M.Keldysh, M. Laurentiev* Theorem.

Definition 5.2.2. **(Smirnov classes)** [18]

Any holomorphic function defined on Ω is said to be of class $E^p(\Omega)$ for $0 < p \leq \infty$ if there exists a sequence of rectifiable *Jordan* curves $\gamma_1, \gamma_2, \ldots$ in Ω, approaching the boundary Ω (in the sense of γ) such that

$$\int_{\gamma_i} |f(z)|^p |\mathrm{d}z| \leq const < \infty.$$

Theorem 5.2.3. ***(M.Keldysh, M. Laurentiev)***

$f(z) \in E^p(\Omega)$ if and only if $F(w) = f(\phi(w))[\phi'(w)]^{1/p} \in H^p(\mathbb{D})$ for some conformal mapping $\phi(w)$ of the unit disk onto Ω.

Theorem 5.2.4. *Let Ω be a domain bounded by a curve that is real holomorphic except at the point z_0 where it has a corner with interior angle θ. If $n\pi < \theta \leq (n+1)\pi$, $n \in \mathbb{N}$, then for all $p \geq 2 - \frac{\theta}{n\pi}$, $n \in \mathbb{N}$ every $f(z) \in E^p(\Omega)$, generates an holomorphic function on unit disk $\mathbb{D}$ does not have poles on $\partial\mathbb{D} = \mathbb{T}$ such that $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}) \subseteq E^p(\Omega)$.*

Proof. Consider a simply connected domain $\Omega \subset \mathbb{C}$, bounded by a Jordan rectifiable curve γ, let $\phi(w) = (1-w)^2 : \mathbb{D} \to \Omega$ be a conformal mapping of the unit disk $\mathbb{D}$ onto Ω (cardioid type). Assume that there is a function $F(w) = i\frac{1-w}{1+w}$ which maps unit disk onto the upper- half plane H^+.

So, there is a function $f(\phi(w)) = F(\phi^{-1}(z))$ which maps Ω(cardioid type) onto upper-half plane H^+, where $\phi^{-1}(z) = w$.

Given $n\pi < \theta \leq (n+1)\pi$, that is, $1 < \frac{\theta}{n\pi} \leq 1 + \frac{1}{n}$ where $n \in \mathbb{N}$.

We will start to show that in such Ω there exists a function $f(z) \in E^p(\Omega)$ with real boundary values for some values of p.

For this purpose, consider $F(w) = i\frac{1-w}{1+w} : \mathbb{D} \to \mathbb{H}^+$ and according to *M.Keldysh, M. Laurentiev* Theorem 5.2.3 and ([53], Theorem 3.9 pp.52), we obtain

$$\begin{aligned} f(\phi(w))[\phi'(w)]^{\frac{1}{p}} &= f(\phi(w))\left[(1-w)^{\frac{\theta}{n\pi}-1}g(w)\right]^{\frac{1}{p}} \\ &= F(w)\left[(1-w)^{\frac{\theta}{n\pi p}-\frac{1}{p}}g(w)^{\frac{1}{p}}\right]; \text{ since } f(\phi(w)) = F(w). \\ &= i\frac{(1-w)^{1+\frac{\theta}{n\pi p}-\frac{1}{p}}}{(1+w)}g(w)^{\frac{1}{p}} \end{aligned}$$

So, $f(\phi(w))[\phi'(w)]^{\frac{1}{p}} \in H^p(\mathbb{D})$ for $1 < p < 1+\frac{1}{n}$, since $p(1+\frac{\theta}{n\pi p}-\frac{1}{p}) = p+\frac{\theta}{n\pi}-1 > 1$, which follows from the fact that, $\frac{\theta}{n\pi} < p+\frac{\theta}{n\pi}-1 < \frac{1}{n}+\frac{\theta}{n\pi}$, and in addition; $\frac{\theta}{n\pi} > 1$ implies $p \geq 2-\frac{\theta}{n\pi}$, $n \in \mathbb{N}$.

Hence, there exists a function such $f(z) \in E^p(\Omega)$, for $p \geq 2-\frac{\theta}{n\pi}$.

At this stage we shall apply Theorem 5.2.3 and ([53],Theorem 3.9, pp.52) once again to prove that, there exists an holomorphic function with pole at $w=1$ of order greater than 1 on $\partial\mathbb{D} = \mathbb{T}$ as follows:

$$f(\phi(w))[\phi'(w)]^{\frac{1}{2n\pi-\theta/n\pi}} = f(\phi(w))\left[(1-w)^{\frac{\theta}{n\pi}-1}g(w)\right]^{\frac{n\pi}{2n\pi-\theta}}. \tag{5.26}$$

$$= f(\phi(w))\left[(1-w)^{\frac{\theta-n\pi}{2n\pi-\theta}}g(w)^{\frac{n\pi}{2n\pi-\theta}}\right] \in H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}). \tag{5.27}$$

Let

$$G(w) = f(\phi(w))[\phi'(w)]^{\frac{n\pi}{2n\pi-\theta}}.$$

$$\Rightarrow \; G(w) = f(\phi(w))(1-w)^{1-\frac{3n\pi-2\theta}{2n\pi-\theta}} g(w)^{\frac{n\pi}{2n\pi-\theta}} \tag{5.28}$$

such that

$$\Rightarrow \; \frac{G(w)(1-w)^{\frac{3n\pi-2\theta}{2n\pi-\theta}}}{g(w)^{\frac{n\pi}{2n\pi-\theta}}} = f(\phi(w))(1-w) \in H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}).$$

Set,

$$G^*(w) = \frac{G(w)(1-w)^{\frac{3n\pi-2\theta}{2n\pi-\theta}}}{g(w)^{\frac{n\pi}{2n\pi-\theta}}} = f(\phi(w))(1-w).$$

The last equation can be written as follows:

$$G^*(w)(1-\overline{w}) = f(\phi(w))(1-w)(1-\overline{w}).$$

$$\Rightarrow \; G^*(w)(1-\overline{w}) = f(\phi(w))|1-w|^2.$$

Set again,

$$K(w) = G^*(w)(1-\overline{w})$$

such that

$$K(w) = f(\phi(w))|1-w|^2 \in H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}), \qquad n \in \mathbb{N} \tag{5.29}$$

As we obtained $f(z) \in E^p(\Omega)$, for $p \geq 2 - \frac{\theta}{n\pi}$, it can be set,

$$f(z) = u(x,y) + iv(x,y)$$

Now, we need to rewrite equation (5.29) as follows:

$$K(w) = f(\phi(w))\frac{w-\alpha_1}{1-w}\,(1-\overline{\alpha_2}w)(w-\alpha_2), \quad \text{where} \quad \alpha_1 \in \partial\mathbb{D} = \mathbb{T} \;\&\; \alpha_2 \in \overline{\mathbb{D}}. \tag{5.30}$$

Hence, $\frac{w-\alpha_1}{1-w} \in \mathbb{R}$ on $\partial\mathbb{D} = \mathbb{T}$.

Let us assume $w = x + iy$, $\alpha_1 = a + ib$ and $(1 - \overline{\alpha_2}w)(w - \alpha_2) = t$, where $t \in \mathbb{R}$, since it gives real values. Then

$$\begin{aligned} K(w) &= f(\phi(w))\frac{w - \alpha_1}{1 - w}\,(1 - \overline{\alpha_2}w)(w - \alpha_2) \\ &= t\,[u(x, y) + iv(x, y)]\,\frac{(x - a) + i(y - b)}{(1 - x) - iy} \\ &= t\,[u(x, y) + iv(x, y)]\,\frac{[(x - a)(1 - x) - y(y - b)] + i[(y - b)(1 - x) + y(x - a)]}{(1 - x)^2 + y^2}. \end{aligned}$$

Since $\frac{w-\alpha_1}{1-w}$ is a real value on $\partial\mathbb{D} = \mathbb{T}$ when $a = 1, b = 0$, then $\alpha_1 = 1$, so that

$$K(w) = t\,[u(x, y) + iv(x, y)]\frac{w - 1}{1 - w} = tf(\phi(w)), \tag{5.31}$$

that is, $K(w)$ does not have poles on $\partial\mathbb{D} = \mathbb{T}$.

Hence, by equations (5.29), (5.31) we obtain that, $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}) \subseteq E^p(\Omega)$. $\square$

Appendix A

Background material on univalent functions

Theorem A.0.5. ***(Area theorem)***
If $g \in \Sigma$, then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$, with equality if and only if $g \in \tilde{\Sigma}$.

Proof. Given E be a compact connected set such that E be a set omitted by g. Define,

$$g(z) : \triangle(|z| > 1) \longrightarrow E^c$$

hence, for $r > 0$, let C_r be the image of boundary $\partial \mathbb{D}_1 = |z| = r$ under the function g, such that C_r be a boundary of the domain E_r.
Let $w = x + iy \;\Rightarrow\; \mathrm{d}w = \mathrm{d}x + i\mathrm{d}y$ and $\overline{w} = (x - iy)$ we get,

$$\overline{w}\mathrm{d}w = (x - iy))\,(\mathrm{d}x + i\mathrm{d}y).$$
$$\overline{w}\mathrm{d}w = (x\mathrm{d}x + y\mathrm{d}y) + i(x\mathrm{d}y - y\mathrm{d}x)$$

By applying *Green theorem*

$$\begin{aligned}\int_{C_r} (x\mathrm{d}x + y\mathrm{d}y) + i(x\mathrm{d}y - y\mathrm{d}x) &= \int_{E_r} 2i\mathrm{d}x\mathrm{d}y.\\ &= 2i\; A_r.\end{aligned}$$

Hence,

$$2i\ A_r = \int_{C_r} \overline{w}\mathrm{d}w.$$
$$\Rightarrow\ A_r = \frac{1}{2i}\int_{C_r} \overline{w}\mathrm{d}w.$$

Let $w = g(z) \Rightarrow \overline{w} = \overline{g(z)}$, $\mathrm{d}w = g'(z)\mathrm{d}z$.

$$\begin{aligned}
A_r = \frac{1}{2i}\int_{C_r} \overline{w}\mathrm{d}w &= \frac{1}{2i}\int_{|z|=r} \overline{g(z)}\ g'(z)\mathrm{d}z. \\
&= \frac{1}{2}\int_0^{2\pi} \left[re^{-i\theta} + \sum_{n=0}^{\infty} \overline{b}_n r^{-n} e^{in\theta}\right] \left[1 - \sum_{n=0}^{\infty} n\bar{b}_n r^{-n-1} e^{-i(n+1)\theta}\right] re^{i\theta}\mathrm{d}\theta \\
&= \frac{1}{2}\left[2\pi r^2 - 2\pi \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right]. \\
&= \pi\left[r^2 - 2\pi \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right]. \quad (r > 1).
\end{aligned}$$

Letting r decreasing. that is, $r \to 1$.

$$\Rightarrow\ A_r = \pi\left[1 - \sum_{n=1}^{\infty} n|b_n|^2\right].$$
$$= m(E_r) \quad \text{is outer measure of} \quad E$$

Since $m(E_r) \geq 0$. Then

$$\pi\left[1 - \sum_{n=1}^{\infty} n|b_n|^2\right] \geq 0.$$
$$\Rightarrow\ 1 - \sum_{n=1}^{\infty} n|b_n|^2 \geq 0$$
$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

□

Lemma A.0.6. *For each $\phi \in S$,*

$$\left| z\frac{\phi''}{\phi'} - \frac{2|z|^2}{1-|z|^2} \right| \leq \frac{4|z|}{1-|z|^2}.$$

Proof. Given $\phi \in S$, fix $\zeta \in \mathbb{D}$.

Now, we can define function

$$F_\zeta(z) = \frac{\phi(\frac{z+\zeta}{1-\bar{\zeta}z}) - \phi(\zeta)}{(1-|\zeta|^2)\phi'(\zeta)} \text{ is disk automorphism belong to } S.$$

$$\Rightarrow \; F(z) = z + b_2 z^2 + b_3 z^3 + ...$$

$$\text{Since } \; F \in \mathcal{S} \; \Rightarrow \; F_\zeta(0) = \frac{\phi(\zeta) - \phi(\zeta)}{(1-|\zeta|^2)\phi'(\zeta)} = 0.$$

$$\text{and } F'_\zeta(0) = \frac{[(1-|\zeta|^2)\phi'(\zeta)] \; [(1-|\zeta|^2)\phi'(\zeta)]}{[(1-|\zeta|^2)\phi'(\zeta)]^2} = 1$$

Now, b_2 is a coefficient of z^2 for the function $F(z)$ such that $F(z)$ has Tayler expansion this means

$$b_2 = \frac{F''(z=0)}{2!}$$

$$F''(z) = \frac{[(1-|\zeta|^2)\phi'(\zeta)] \left[\phi' \frac{-2\bar{\zeta}(1-|\zeta|^2)(1-\bar{\zeta}z)}{(1-\bar{\zeta}z)^4} + \frac{1-|\zeta|^2}{(1-\bar{\zeta}z)^2}\phi'' \frac{1-|\zeta|^2}{(1-\bar{\zeta}z)^2}\right]}{[(1-|\zeta|^2)\phi'(\zeta)]^2}$$

$$F''(z) = \frac{\phi' \; \frac{-2\bar{\zeta}(1-|\zeta|^2)}{(1-\bar{\zeta}z)^3} + \phi'' \frac{(1-|\zeta|^2)^2}{(1-\bar{\zeta}z)^2}}{(1-|\zeta|^2)\phi'(\zeta)}$$

$$F''(0) = \frac{\phi''}{\phi'}(1-|\zeta|^2) - 2\bar{\zeta}$$

$$\Rightarrow \; b_2 = \frac{1}{2}\left[\frac{\phi''}{\phi'}(1-|\zeta|^2) - 2\bar{\zeta}\right]$$

By Biebarback's theorem, $|b_2| \leq 2$.

$$\Rightarrow \; \frac{1}{2}\left|\frac{\phi''}{\phi'}(1-|\zeta|^2) - 2\bar{\zeta}\right| \leq 2$$

$$\Rightarrow \; \left|\frac{\phi''}{\phi'}(1-|\zeta|^2) - 2\bar{\zeta}\right| \leq 4$$

$$\Rightarrow \; \left|\frac{\phi''}{\phi'} - \frac{2\bar{\zeta}}{(1-|\zeta|^2)}\right| \leq \frac{4}{1-|\zeta|^2}$$

Multiply both of sides by $|\zeta|$.

$$\begin{aligned}
&\left|\zeta\,\frac{\phi''}{\phi'}-\frac{2\bar{\zeta}\zeta}{(1-|\zeta|^2)}\right|\le\frac{4|\zeta|}{1-|\zeta|^2}\\
\Rightarrow&\left|z\,\frac{\phi''}{\phi'}-\frac{2|z|^2}{(1-|z|^2)}\right|\le\frac{4|z|}{1-|z|^2}\\
\Rightarrow&\left|z\,\frac{\phi''}{\phi'}-\frac{2r^2}{(1-r^2)}\right|\le\frac{4r}{1-r^2}.
\end{aligned}$$

□

Theorem A.0.7. ***(Distortion Koebe theorem)*** *: For each $\phi\in S$ defined on unit disc $\mathbb{D}$,*

$$\frac{1-r}{(1+r)^3}\le|\phi'(z)|\le\frac{1+r}{(1-r)^3},\quad |z|=r<1.$$

For each $z\in\mathbb{D}$,$z\neq 0$, equality occurs if and only if ϕ is a suitable rotation of the koebe function $\phi(z)=\frac{z}{(1-z)^2}$.

Proof. :- Given $\phi(z)\in S$, by lemma (1.1.5)

$$\begin{aligned}
&\frac{-4r}{1-r^2}\le\frac{z\phi''}{\phi'}-\frac{2r^2}{1-r^2}\le\frac{4r}{1-r^2}\\
\Rightarrow&\frac{-4r}{1-r^2}\le\frac{z\phi''}{\phi'}-\frac{2r^2}{1-r^2}\le\frac{4r}{1-r^2}\\
\Rightarrow&\frac{2r^2-4r}{1-r^2}\le Re(\frac{z\phi''}{\phi'})\le\frac{4r+2r^2}{1-r^2}
\end{aligned}$$

Now $\phi'(z)$ is not vanish in $\mathbb{D}$ when $z=0$, that is, $\phi'(0)\neq 0$. so, there is an holomorphic branch $\log\phi'(z)$ will be vanish in $\mathbb{D}$ when $z=0$ as $r\to 0$

$$\begin{aligned}
\frac{\partial}{\partial r}(\log\phi'(z))&=\frac{\partial}{\partial z}(\log\phi')\,\frac{\partial z}{\partial r}+\frac{\partial}{\partial\bar{z}}(\log\phi')\,\frac{\partial\bar{z}}{\partial r}\\
&=\frac{\phi''}{\phi'}\,e^{i\theta}+0\\
&=\frac{\phi''}{\phi'}\,e^{i\theta}\\
Re(\frac{\partial}{\partial r}(\log\phi'(z)))&=Re(\frac{\phi''}{\phi'}\,e^{i\theta})
\end{aligned}$$

and we know,

$$
\begin{aligned}
Re(\frac{\partial}{\partial r}(\log\phi'(z))) &= \frac{\partial}{\partial r}Re(\log\phi'(z))\\
\Rightarrow \frac{\partial}{\partial r}Re(\log\phi'(z)) &= Re(\frac{\phi''}{\phi'}\,e^{i\theta})\\
\Rightarrow r\frac{\partial}{\partial r}Re(\log\phi'(z)) &= Re(\frac{z\phi''}{\phi'})\\
\Rightarrow \frac{2r^2-4r}{1-r^2} \leq r\frac{\partial}{\partial r}&Re(\log\phi'(z)) \leq \frac{4r+2r^2}{1-r^2}
\end{aligned}
$$

By dividing both of sides on r and integrate each term of inequality with respect to r such that $0 < r < R$, we obtain

$$
\begin{aligned}
\frac{2r-4}{1-r^2} &\leq \frac{\partial}{\partial r}Re(\log\phi'(z)) \leq \frac{4+2r}{1-r^2}\\
\int_0^R \frac{2r-4}{1-r^2} &\leq \int_0^R \frac{\partial}{\partial r}Re(\log\phi'(z)) \leq \int_0^R \frac{4+2r}{1-r^2}\\
\int_0^R \frac{2r-4}{1-r^2} &\leq \int_0^R \frac{\partial}{\partial r}\log|\phi'(z)| \leq \int_0^R \frac{4+2r}{1-r^2}\\
\int_0^R \frac{2r-4}{1-r^2} &\leq \log|\phi'(Re^{i\theta})| \leq \int_0^R \frac{4+2r}{1-r^2}
\end{aligned}
$$

such that

$$
\begin{aligned}
&\int_0^R \frac{2r-4}{1-r^2} = \log(1-R) - \log(1+R)^3 = \log\frac{1-R}{(1+R)^3}\\
\text{and}\quad &\int_0^R \frac{2r+4}{1-r^2} = \log(1+R) - \log(1-R)^3 = \log\frac{1+R}{(1-R)^3}\\
\Rightarrow\quad &\log\frac{1-R}{(1+R)^3} \leq \log|\phi'(Re^{i\theta})| \leq \log\frac{1+R}{(1-R)^3}.
\end{aligned}
$$

By exponentiation

$$\frac{1-R}{(1+R)^3} \leq |\phi'(Re^{i\theta})| \leq \frac{1+R}{(1-R)^3}.$$

□

Theorem A.0.8. ***(Prawitz's theorem)*** *: If $\psi \in S$ such that $\psi : \mathcal{D} \to \Omega$, then for $0 < p < \infty$,*

$$\mathcal{M}_p^p(r,\psi) \leq p\int_0^r \frac{1}{t}\mathcal{M}_\infty^p(t,\psi)\mathrm{d}t, \quad 0 < r < 1.$$

Proof. :- Let ϕ^{-1}:$\mathcal{D} \to \Omega$, such that $\psi = \phi^{-1}$, Given $\mathcal{M}_p^p(r,\psi)$ be a growth of the integral means of the function $\psi(w)$ which define in unit disk.

$$\mathcal{M}_p^p(r,\psi) = \frac{1}{2\pi}\int_0^{2\pi} |\psi(re^{i\theta})|^p \,\mathrm{d}\theta;\ (z = re^{i\theta}, 0 \le r < 1)$$

Multiply both of sides by $(r\frac{\partial}{\partial r})$ such that

$$r\frac{\partial}{\partial r}(\mathcal{M}_p^p(r,\psi)) = r\frac{\partial}{\partial r}[\frac{1}{2\pi}\int_0^{2\pi} |\psi(re^{i\theta})|^p \,\mathrm{d}\theta] = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial r}|\psi(re^{i\theta})|^p \,r\mathrm{d}\theta$$

Now, by second *Green*'s identity,

$$\int_{\partial D} \frac{\partial\psi}{\partial n}\mathrm{d}S = \int_D \Delta\psi\mathrm{d}V$$

We have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial|\psi(re^{i\theta})|^p}{\partial r} r\mathrm{d}\theta &= \frac{1}{2\pi}\int\int_D \Delta(|\psi|^p)\mathrm{d}x\mathrm{d}y \\
\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial|\psi(re^{i\theta})|^p}{\partial r} \mathrm{d}\theta &= \frac{1}{2\pi r}\int\int_D \Delta(|\psi|^p)\mathrm{d}x\mathrm{d}y \\
&= \frac{p^2}{2\pi r}\int\int_{\{|w|<r\}} |\psi(w)|^{p-2}|\psi'(w)|^2\mathrm{d}x\mathrm{d}y
\end{aligned}$$

we can substitute $\psi(w) = z$ and because ψ is holomorphic functions and one-to-one.
$\Rightarrow\psi(w)$ is an univalent function.
So, $\psi(w)$ has a maximum modulus

$$\mathrm{M}_\infty(r,\psi) = \max_{|w|=r}|\psi(w)|.$$

such that

$$|\psi(w)| \le \max_{|w|=r}|\psi(w)| = \mathrm{M}_\infty(r,\psi)$$

and it's known that

$$\int_D |\psi'(w)|^2\mathrm{d}x\mathrm{d}y = \int_\Omega \mathrm{d}x\mathrm{d}y$$

This implies to

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{1}{2\pi}\int_0^{2\pi}|\psi(re^{i\theta})|^p\mathrm{d}\theta\Big) = \frac{p^2}{2\pi r}\int\int_{\{|w|<r\}}|\psi(w)|^{p-2}|\psi'(w)|^2\mathrm{d}x\mathrm{d}y$$
$$= \frac{p^2}{2\pi r}\int\int_{\psi(\{|w|<r\})}|z|^{p-2}\mathrm{d}A(z)$$
$$\leq \frac{p^2}{2\pi r}\int\int_{\psi(\{|w|<r\})}\max|z|^{p-2}\mathrm{d}A(z) = \frac{p^2}{2\pi r}\int_0^{2\pi}\int_0^{\mathrm{M}_\infty(r)} t^{p-2}\ t\ \mathrm{d}t\ \mathrm{d}\theta$$

So, where

$$\frac{p^2}{2\pi r}\int_0^{2\pi}\int_0^{\mathrm{M}_\infty(r)} t^{p-2}\ t\ \mathrm{d}t\ \mathrm{d}\theta = \frac{p^2}{r}\int_0^{\mathrm{M}_\infty(r)} t^{p-1}\mathrm{d}t$$
$$= \frac{p}{r}\mathrm{M}_\infty^p(r,\psi)$$

We will get,

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{1}{2\pi}\int_0^{2\pi}|\psi(re^{i\theta})|^p\mathrm{d}\theta\Big) \leq \frac{p}{r}\mathrm{M}_\infty^p(r,\psi)$$

Integrate both of sides with respect to r as follows:

$$\mathcal{M}_p^p(r,\psi) = \frac{1}{2\pi}\int_0^{2\pi}|\psi(re^{i\theta})|^p\mathrm{d}\theta \leq p\int_0^r \frac{1}{t}\mathrm{M}_\infty^p(t,\psi)\mathrm{d}t.$$

□

Lemma A.0.9. ***(Parseval formula)***

If $\phi(z)$ be holomorphic function in unit disk $\mathbb{D}$ such that can be represent it by Taylor series expansion $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\sum_{n=0}^{\infty}|a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}|\phi(re^{i\theta})|^2\ \mathrm{d}\theta.$$

Proof. Given $\phi(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \cdots$

So, by Cauchy Integral Formula

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\phi(z)}{z^{n+1}}\ \mathrm{d}z$$

$$
\begin{aligned}
\Rightarrow \int_0^{2\pi} |\phi(re^{i\theta})|^2 \mathrm{d}\theta &= \int_0^{2\pi} \phi(re^{i\theta})\overline{\phi(re^{i\theta})}\, \mathrm{d}\theta \\
\int_0^{2\pi} |\phi(re^{i\theta})|^2 \mathrm{d}\theta &= \int_0^{2\pi} \phi(re^{i\theta})\ (\sum_{n=0}^{\infty} \overline{a_n (re^{i\theta})^n})\, \mathrm{d}\theta \\
&= \sum_{n=0}^{\infty} \int_0^{2\pi} \frac{\phi(re^{i\theta})\overline{a_n} r^n}{e^{ni\theta}}\, \mathrm{d}\theta \\
&= \sum_{n=0}^{\infty} 2\pi \overline{a_n} r^{2n} (\frac{1}{2\pi i} \int_0^{2\pi} \frac{\phi(re^{i\theta})}{(re^{i\theta})^{n+1}}\, ire^{i\theta}\, \mathrm{d}\theta) \\
\Rightarrow \int_0^{2\pi} |\phi(re^{i\theta})|^2 \mathrm{d}\theta &= \sum_{n=0}^{\infty} (2\pi \overline{a_n} r^{2n}) a_n \\
&= 2\pi \sum_{n=0}^{\infty} \overline{a_n} a_n r^{2n} \\
&= 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.
\end{aligned}
$$

This is implies to

$$
\frac{1}{2\pi} \int_0^{2\pi} |\phi(re^{i\theta})|^2 \mathrm{d}\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.
$$

□

Theorem A.0.10. ***(Bieberbach's theorem)***
If $\phi \in S$, then $|b_2| \leq 2$, with equality if and only if ϕ is a rotation of the Koebe function.

Proof. :- Given $\phi(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in S, Since $\phi(z) = 0$ only at the origin. then a single valued branch of the square root can be chosen as follow

$$
g(z) = \sqrt{\phi(z^2)}
$$

which is a square- root transformation. such that

$$
\begin{aligned}
g(z) = \phi(z^2)^{\frac{1}{2}}. &= \{z^2 + a_2 z^4 + a_3 z^6 + \cdots\}^{\frac{1}{2}}. \\
&= z\{1 + a_2 z^2 + a_3 z^4 + \cdots\}^{\frac{1}{2}}. \\
&= z\{1 + \frac{a_2}{2} z^2 + (\frac{a_3}{2} - \frac{a_2^2}{8}) z^4 + \cdots\}. \\
&= z + (\frac{a_2}{2}) z^3 + (\frac{a_3}{2} - \frac{a_2^2}{8}) z^5 + \cdots .
\end{aligned}
$$

Now, we have to apply inversion transformation on the function g such that we get another function h in $\Sigma = \{z : |z| > 1\}$ which is defined as follows:

$$h(z) = \frac{1}{g(z)}.$$

such that

$$h(z) = \frac{1}{g(z)} = \frac{1}{z} - (\frac{a_2}{2})z + \cdots = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n.$$

Then $h(z)$ is univalent in $0 < |z| < 1$, and so the image of $|z| = r$ by $h(z)$ is a simple closed curve for $0 < r < 1$ this implies to the function $h(z)$ of class Σ.
For this by corollary of the area theorem.

$$|\frac{-a_2}{2}| \leq 1 \;\Rightarrow\; |\frac{a_2}{2}| \leq 1 \;\Rightarrow\; |a_2| \leq 2.$$

and equality is possible only if $b_n = 0$ $(n > 1)$, and in this case

$$h(z) = \frac{1}{g(z)} = \frac{1}{z} - ze^{i\theta} = \frac{1 - z^2 e^{i\theta}}{z}.$$

And then,

$$g(z) = \frac{1}{h(z)} = \frac{z}{1 - e^{i\theta} z^2}.$$

Finally, we have

$$\begin{aligned} g(z) &= \phi(z^2)^{\frac{1}{2}}. \\ \Rightarrow \frac{z}{1 - e^{i\theta} z^2} &= \phi(z^2)^{\frac{1}{2}}. \\ \Rightarrow \phi(z^2) &= \frac{z^2}{(1 - e^{i\theta} z^2)^2}. \\ \phi(z) &= \frac{z}{(1 - e^{i\theta} z)^2}. \end{aligned}$$

□

Theorem A.0.11. ***(Koebe One-Quarter theorem)****[19]. The range of every function of class S contains the disk $\{w : |w| < \frac{1}{4}\}$.*

Proof. Given $\phi = z + a_2 z^2 + a_3 z^3 + \cdots$ belongs to class S and it's range omits value $w \in \mathbb{C}$, then by properties of univalent function (cf. property (vi) in [19]).

$$h(z) = \frac{w\,\phi(z)}{w - \phi(z)} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \cdots$$

is holomorphic and univalent function in $\mathbb{D}$.

Using *Bieberbach theorem* (A.0.10) gives

$$\left| a_2 + \frac{1}{w} \right| \leq 2$$

Combined with the inequality $|a_2| \leq 2$. we get $|\frac{1}{w}| \leq 4$ which implies to $|w| \geq \frac{1}{4}$.

So, every value omits of the range ϕ must lie outside the disk $\{w : |w| < \frac{1}{4}\}$. □

Definition A.0.12. **(starlike set)** If $\mathcal{A} \subset \mathbb{C}$, then we could say that $\mathcal{A}$ is a *starlik with respect to the point* $z_0 \in \mathcal{A}$, if the line segment joining z_0 to any point of $\mathcal{A}$ is contained in $\mathcal{A}$.

Definition A.0.13. **(starlike function)** A function ϕ is said to be *starlike*, if it maps the unit disk $\mathbb{D}(|z| < 1)$ conformally onto a set that is starlike with respect to the origin, or if $Re\left(\frac{z\phi'(z)}{\phi(z)}\right) \geq 0$, $|z| < 1$. The class of starlike functions is denoted by $S^{\star}$ and contained in class S.

Theorem A.0.14. *(**Nehari's Theorem**) Let ϕ be a regular function (holomorphic and single valued function (1-1)) in $\mathbb{D}$ and suppose its Schwarzian derivative satisfies*

$$|S_\phi| \leq 2(1 - |z|^2)^2, \quad |z| < 1. \tag{A.1}$$

Then ϕ is univalent in $\mathbb{D}$.

Remark A.0.15. Inequality (A.1) is a necessary condition for univalent function by inversion of the function

$$F(z) = \frac{\phi\left(\frac{z+\zeta}{1+\bar{\zeta}z} - \phi(\zeta)\right)}{(1 - |\zeta|^2)\phi'(\zeta)} = z + A_2 z^2 + A_3 z^3 + \cdots$$

to replace a constant 2 by 6. This result was rediscovered by *Nehari* [47].

Appendix B

Some basic notions of analysis

Co-area formula

The following, termed to co-area formula which has proved to be essential tool in analysis with a wide range of applications. For smooth functions the formula considers as a result in multivariate calculus and it follows from a simple change of variables, and has established by Herbert Federer (1959) and Fleming & Rishel (1960) for Lipschitz functions,Sobolev functions respectively, cf. [21], [24] and [43]. We will briefly review the definition of Co-area formula as following.

Definition B.0.16. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be Lipschtiz continuous assume that for $a.e.\ r \in \mathbb{R}$, assume that the level set $\{x \in \mathbb{R}^n | \phi(x) = r, r \in \mathbb{R}\}$ is a smooth and $(n-1)$-dimentional hypersurface in $\mathbb{R}^n$. Also, suppose that there is $g : \mathbb{R}^n \to \mathbb{R}$ is continuous and summable. Then

$$\int_{\mathbb{R}^n} g|\nabla u| \mathrm{d}x = \int_{-\infty}^{\infty} \left(\int_{\{\phi=r\}} g \mathrm{d}s \right) \mathrm{d}r.$$

Theorem B.0.17. *Let X be an open subset of $\mathbb{R}$, and $\mathcal{M}$ be a measure space.*

Suppose $f : X \times \mathcal{M} \longrightarrow \mathbb{R}$ satisfies the following condations:

i. $f(x, w)$ is a lebesgue- integrable function of w for each $x \in X$

ii. For almost all $w \in \Omega$, the derivative $\frac{\partial f(x,w)}{\partial x}$ exist for all $x \in X$

iii. there is an integrable function $\Theta : \mathcal{M} \longrightarrow \mathbb{R}$ such that $|\frac{\partial f(x,w)}{\partial x}| \leq \Theta(w)$ for all $x \in X$

Then $\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathcal{M}} f(x,w)\mathrm{d}w = \int_{\mathcal{M}} \frac{\partial}{\partial x} f(x,w)\mathrm{d}w.$ for all $x \in X$.

Theorem B.0.18. *(**Riesz representation theorem**) ([68], pp.5)*

Let Ω is a compact metric space, let $M(\Omega)$ denote the space of probability measures on Ω, that is, measure with $\mu(\Omega) = 1$. The map $M(\Omega) \longrightarrow C(\Omega)^\star$, $\mu \longrightarrow \phi_\mu$, is a bijection of $M(\Omega)$.

For $\mu \in \mu(\Omega)$, define $\phi_\mu \in C(\Omega)^\star = \{\phi \in C(\Omega)^\star | \phi(f) \geq 0, \quad \text{for all} \quad f \geq 0, \phi(1) = 1\}$ by $\phi_\mu(f) = \int f \mathrm{d}\mu$.

Theorem B.0.19. *(**Schwartz theorem**) $\{$if $\phi \in H^\infty(\mathbb{D})$, $\|\phi\| \leq 1$, and $\phi(0) = 0$. Then*

$$|\phi(z)| \leq |z|, \qquad (z \in \Omega = \mathbb{D}), \tag{B.1}$$

$$|\phi'(0)| \leq 1; \tag{B.2}$$

if equality holds in (B.1) for one $z \in \Omega - \{0\}$, or if equality holds in (B.2), then $\phi(z) = \alpha z$, where $\alpha \in \mathbb{C}$ is a constant , $|\alpha| = 1\}$.

Theorem B.0.20. *(**Stokes' theorem**) For w a differential $(k-1)$-form with compact support on an oriented k-dimensional manifold with boundary $\partial\Omega$,*

$$\int_\Omega \mathrm{d}w = \int_{\partial\Omega} w,$$

wher $\mathrm{d}w$ is the exterior derivative of the differential form w.

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