Modified gravity and cosmology

Ippocratis D. Saltas

Submitted for the degree of Doctor of Philosophy

University of Sussex

August 2012

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

Signature:

Ippocratis D. Saltas

UNIVERSITY OF SUSSEX

Ippocratis D. Saltas, Doctor of Philosophy

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Summary

Having as a starting point the problem of dark energy described before, this thesis studies modifications of General Relativity (GR), as possible gravitational scenarios for the early and late time Universe, motivated by both classical as well as quantum considerations. In particular, it focuses on modifications of GR of the type $f(R)$ as well as the $f(R, G)$ ones, where R and G is the Ricci scalar and Gauss–Bonnet term respectively. On the same time, a modification of GR based on the Renormalisation Group approach to quantum gravity is considered, as well as its link to $f(R)$ gravity. The main goal of the investigations carried out in this thesis, is to understand the structure, as well as the phenomenological implications of non-linear modifications of GR for cosmology, at both the background as well as the linear perturbation level.

In particular, chapter 2 presents a brief introduction to the dynamics of GR in the presence of a "dark component" at the background, as well as at the linear perturbation level, while chapter 3 is an introduction to the fundamental properties of non–linear modifications of GR, reviewing important results of the relevant literature.

Chapter 4 elaborates with a fundamental property of non–linear gravity models, namely the study of different representations of vacuum actions proportional to $f(R)$ as well as $f(G)$, in view of Legendre transformations, for the case of spacetime manifolds with a boundary. As it is explicitly shown there, although the dynamical equivalence is always true in the bulk, it is not guaranteed on the boundary of the spacetime manifold.

On the other hand, chapter 5 focuses on understanding the role of the effective anisotropic stress present in $f(R, G)$ gravity models, attempting to construct

particular models of the latter type, with a vanishingly small anisotropic stress, so as to agree with current observations. As it turns out, suppression of the effective anisotropic stress in this class of models is very difficult, highlighting the role of the effective anisotropic stress as a smoking gun for testing modified gravity models with current and future observations.

Chapter 6 serves as an introduction to the idea of the Renormalisation Group (RG) and its applications in cosmology, while chapter 7 starts from an RG improved Einstein–Hilbert action and studies its connection with $f(R)$ gravity, as well as its implications for the primordial and the late time acceleration of the Universe. It is shown that the effective $f(R)$ model has some remarkable properties and interesting implications for both early and late time cosmology.

Acknowledgements

During my PhD years, different people helped me along with one or the other way, to a smaller or a larger extent. This thesis would have never been completed without them. I really owe to all of them a sincere thank you.

To begin with, my gratitude goes to my supervisors. It has been a pleasure to know and work with them.

Taking things from the start, I am thankful to Martin Kunz for accepting to work with me as his PhD student in 2008. But much more than that, I owe to thank him for his endless support, encouragement and guidance all throughout my PhD years.

Deciding not to follow Martin to Geneva, my supervision was taken over by Mark Hindmarsh. It has been a great pleasure working with Mark, while his constant help, support and advice have been very important for this thesis.

Though I had never had the chance to work closely with my second supervisor, Andrew Liddle, he has always been encouraging and supporting me with valuable advice at all the crucial times of my PhD path.

Sussex has been a great place to be. The last four years I had the chance of meeting different people for a shorter or a longer time; My thanks go to David Parkinson, Isaac Roseboom, Christoph Rahmede, Mafalda Dias, Kevin Falls, Kostas Nikolakopoulos, Edouard Marchais, Leon Baruah, Nicola Mehrtens, Leonidas Christodoulou, Dimitri Skliros, Donough Regan, Jonny Frazer, as well as all the current and former students and postdocs of the Physics and Astronomy department. What is more, Seb Oliver for his constant support and help with various issues regarding my study at Sussex, as well as Daniel Litim and all the faculty members of the Astronomy and Particle Physics groups. Everybody had its own direct or indirect, smaller or larger contribution, and that is probably hard to describe in words.

During the last year, I have been very glad to have the opportunity to collaborate with Luca Amendola and Ignacy Sawicki from the University of Heidelberg, who I learned a lot from; I am also thankful to them for their warm hospitality when visiting Heidelberg.

The GTA funding, granted to me by the department of Physics and Astronomy at Sussex during my PhD studies, has been a very important support. What is more, as I spent part of my time as a Teaching Assistant, I have much enjoyed working with Lesley Onoura, Barry Garraway, Kathy Romer, Jon Loveday, Jose Verdu Galiana, Jeff Hartnell, Mike Hardiman, Maria Brooks and Jackie Grant.

I am also grateful to the examiners of my thesis defense, Dr. Kazuya Koyama and Dr. Xavier Calmet, for a constructive discussion and very useful feedback on the work presented in this thesis.

Then, of course my sincere thanks go to my close friends for their invaluable help and courage.

Last, but never least, I am thankful to my family for everything they have done up to now; my parents, my sister and my grandmother.

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Chapter 1

Introduction

The 2011 Nobel prize in physics was awarded to Saul Perlmutter, Brian P. Schmidt and Adam G. Riess "for the discovery of the accelerating expansion of the Universe through observations of distant supernovae" [1]. The path for this discovery was the study of the luminosity distant of a set of high redshift Supernovae of Type Ia (SNIa) Riess et al. (1998); Perlmutter et al. (1999). The latter is a rather surprising observation, as one would expect that the large matter concentrations [2] in the Universe gravitationally attract each other, yielding a slowing down the of the Universe's expansion. In the context of Einstein–Hilbert gravity, such an accelerated expansion could be achieved with the introduction of a new component in the equations, with a rather special property: it should have a negative pressure, so that it counteracts the gravitational force between the pressureless matter in the Universe, producing this way an "antigravity" effect leading to the observed accelerated expansion of the Universe. This mysterious yet component was termed as "dark energy".

What makes supernovae special is the fact that they can be assumed as "standardisable candles", in the sense that their absolute magnitude can be correlated with their light curve, with brighter supernovae yielding broader light curves Hamuy et al. (1996) and Amendola and Tsujikawa (2010) and references therein. This allows for an efficient measurement of their luminosity distance, which in turn depends on the energy–momentum content of the Universe. By studying a set of both high-redshift as well as low redshift SN Ia data, Riess et al. (1998) showed that the dark energy

[1] http://www.nobelprize.org/nobel_prizes/physics/laureates/2011/

[2] With "large matter concentrations" here we mean clusters and superclusters of galaxies, as the acceleration of the Universe is observable only at large scales.

component has the form of a cosmological constant Λ at the 99% confidence level. A positive cosmological constant, as we will describe also later on, is able to produce negative pressure, while its density ρ_Λ, which is constant at all times, is related to the pressure as $p_\Lambda = -\rho_\Lambda$. The old cosmological paradigm, the so–called Cold Dark Matter (CDM) model, where the only components in the Universe were baryons, radiation (including relativistic particle species), and the as yet undiscovered pressureless dark matter, had to be extended to account for the mysterious dark energy. In the presence of a cosmological constant, the old CDM paradigm was extended to the so–called ΛCDM .

On the same time, the supernovae data were not the only observations indicating the need for dark energy. Independent observations regarding the age of the Universe, the large scale structure of the Universe, as well as observations of the Cosmic Microwave Background (CMB) were leading towards the presence of a dark energy component of the Universe, which is very close to a cosmological constant Λ, i.e with an equation of state $p_{\rm DE} \simeq -\rho_{\rm DE}$.

It is obvious that the age of the Universe should be larger than the age of any galaxy or star. However, without the assumption of the dark energy, estimating the age of Milky Way's globular clusters Carretta et al. (2000); Jimenez et al. (1996); Hansen et al. (2002) showed that there was a contradiction with the estimated age of the Universe; while the latter was estimated to be about 10 Gyr, globular clusters seemed to be older than 11Gyr. This crucial contradiction was resolved by the assumption of dark energy, since as it turns out, the age of the Universe becomes larger in the presence of a dark energy component.

On the other hand, the power spectrum of the CMB is also dependent on the energy-momentum content of the Universe, and it provides with another independent test of the existence of dark energy. In particular, the observed position of the acoustic peaks, as well as the integrated Sachs–Wolfe (ISW) effect in the CMB power spectrum, require dark energy in the form very close to the cosmological constant.

Finally, observational evidence for dark energy comes from the large scale structure of the Universe, in particular the clustering of galaxies. The quantity describing the strength of clustering is the matter power spectrum, and depends on the scale, i.e the wave number k in Fourier space. A key point here is that the scale corresponding to the peak of the matter power spectrum is related to the wave number that entered the cosmological ("Hubble") horizon at the particular time of the Universe evolution, when the matter and radiation energy densities where equal. What is more, the wave number at matter–radiation equality depends on the relative fraction of the pressureless matter in the Universe today, $\Omega_{\mathrm{m}}^{(0)}$, in particular, decreases (increases) with decreasing (increasing) $\Omega_{\mathrm{m}}^{(0)}$. Since by definition the sum of all the particular fractions corresponding to the different constituents of the Universe should equal one, the presence of dark energy affects the matter fraction $\Omega_{\mathrm{m}}^{(0)}$, and in turn the position of the peak in the matter power spectrum, making the latter another test of dark energy. For a study of the matter power spectrum from Luminous Red Galaxies (LRG), as well as main galaxy data from the SDSS see for example Tegmark et al. (2006).

Above three observational tests provide with independent evidence for the existence of an extra ingredient of the Universe, with the main property an antigravity effect (negative pressure) at large scales, causing distant galaxies to recede away from each other in an accelerating way. What is more, observations indicate that the dark energy accounts for about the 71% of the total energy density of the Universe. As for the remaining components, about 25% consists of the pressureless dark matter, while a 4% of baryons, and a 0.005% corresponds to the observed CMB (black body) radiation.

In order to get an idea about the nature of the problem of dark energy, as well as the different resolutions suggested, one has to introduce the basic concepts of Einstein's General Relativity (GR), which is the fundamental framework for understanding the evolution of the Universe from the very early times of its evolution up to now. For review works on the dark energy and the cosmological problem the reader can refer to Copeland et al. (2006a); Peebles and Ratra (2003a); Frieman et al. (2008); Perivolaropoulos (2006); Padmanabhan (2006); Durrer and Maartens (2008); Sapone (2010); Padmanabhan (2003); Sahni (2002).

1.1 General Relativity and alternatives

General Relativity (GR) is a theory for gravity. Its fundamental principle is the Einstein Equivalence Principle (EPP), which in fact is the foundation of all metric theories of gravity, not only of GR Will (1981). [3] The EPP is an extension of the Weak Equivalence Principle (WEP), the latter stating that: *If an uncharged test body is placed at an initial event in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its inertial structure and composition* Will (1981). The latter statement describes in a more formal language the Newtonian Equivalence Principle that the gravitational mass equals the inertial mass, i.e all masses fall in a gravitational field in the same way.

The great importance of the Einstein Equivalence Principle is that it generalises the Weak one to include all laws of physics, like for example the laws of electrodynamics. In particular, the EEP assumes the WEP to be valid and further states that *any local nongravitational test experiment is independent of the velocity of the freely falling laboratory, as well as from the particular point and time in the Universe, the experiment is carried out.* Independence of experiment of the particular spacetime point translates to general covariance, which forces the equations of motion to be of tensorial character.

Following the argument in Will (1981), if a gravitational theory satisfies the EEP it should also satisfy the three postulates of metric theories of gravity: Spacetime is equipped with a metric $g_{\mu\nu}$, test particles follow the geodesics [4] of the metric $g_{\mu\nu}$, and that in local freely falling frames (local Lorentz frames), Special Relativity is the description of the nongravitational laws of physics. For the explicit argument supporting the latter statement, as well as its implications and experimental evidence the reader is referred to Will (1981); Ortín (2004). Here we will only use above postulates as our starting point to discuss the action and equations of motion of General Relativity.

Einstein's fundamental idea to arrive at General Relativity, was to relate the effects of a gravitational field, with the curvature of spacetime. What is more, the principle of general covariance implies that the equations should be relations

[3] In the following discussion we shall be closely following C. M. Will's book Will (1981).

[4] A geodesic is the curved space analogue of a straight line on the plane.

between tensors. In this description, spacetime is modeled as a Riemanian manifold. A Riemanian manifold is a differentiable manifold equipped with a metric (for a description of these notions and their connection with General Relativity please see for example Wald (1984); Schutz (2009); Hawking and Ellis (1974)). The significance of the metric is that it allows to measure lengths of curves on the spacetime manifold through,

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \equiv \sum_\alpha \sum_\beta g_{\alpha\beta} dx^\alpha dx^\beta. \tag{1.1}$$

The metric is associated with a *covariant derivative* ∇_γ used to parallel transport vectors along the spacetime manifold, and when applied on a tensor $T^{\alpha\beta}$ yields

$$\nabla_\gamma T^{\alpha\beta} = \partial_\gamma T^{\alpha\beta} + \Gamma^\alpha_{\mu\gamma} T^{\mu\beta} + \Gamma^\beta_{\mu\gamma} T^{\alpha\mu}, \tag{1.2}$$

where $\Gamma^\alpha_{\mu\gamma}$ is the connection or Christoffel symbol defined in (1.4). A straightforward generalisation of above formula to tensors with more than two indices (see for example Wald (1984)). Above, and for the rest of this thesis, unless otherwise stated, repeated indices will imply summation, i.e

$$A_\alpha B^\alpha \equiv A_0 B^0 + \ldots + A_k B^k. \tag{1.3}$$

The object $\Gamma^\beta_{\mu\gamma}$ introduced in the expression for the covariant derivative above, is the *Christoffel symbol* or *the connection* defined as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta} \right), \tag{1.4}$$

and is symmetric with respect to its lower indices, i.e $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$. The latter symmetry is true only when the derivative operator ∇_μ is torsion free, i.e when $\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu$, with f any scalar function.

An important property of the metric is that it satisfies the so–called *compatibility condition*,

$$\nabla_\rho g_{\mu\nu} = 0. \tag{1.5}$$

We can use the covariant derivative in order to parallel transport a vector V^μ along a closed loop, in order to describe the intrinsic curvature of the manifold, through the Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$ as

$$[\nabla_\alpha, \nabla_\beta]V_\gamma \equiv \nabla_\alpha \nabla_\beta V_\gamma - \nabla_\beta \nabla_\alpha V^\gamma = R_{\alpha\beta\gamma}{}^{\delta} V_\delta, \tag{1.6}$$

with the Riemann curvature tensor defined as

$$R^{\alpha}{}_{\beta\gamma\delta} \equiv \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}_{\beta\gamma}. \tag{1.7}$$

The Riemann tensor satisfies the following (anti) symmetry relations under commutation of its indices

$$R^{\alpha}{}_{\beta\gamma\delta} = -R_{\beta}{}^{\alpha}{}_{\gamma\delta} = -R^{\alpha}{}_{\beta\delta\gamma} = R_{\beta}{}^{\alpha}{}_{\delta\gamma}. \tag{1.8}$$

We can also define the Ricci tensor and Ricci scalar through contractions of $R^{\alpha}{}_{\beta\gamma\delta}$ with the metric field as

$$R_{\alpha\beta} \equiv g^{\mu\nu} R_{\mu\alpha\nu\beta}, \tag{1.9}$$

$$R \equiv g^{\alpha\beta} R_{\alpha\beta}. \tag{1.10}$$

Equipped with a Riemannian manifold, as well as with the underlying principle of GR, that the gravitational field of matter fields expresses itself as spacetime curvature, Einstein introduced the following set of equations

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta}, \tag{1.11}$$

with $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ known as the Einstein tensor, and $T_{\alpha\beta}$ is the energy–momentum tensor associated with any matter fields present. Λ is the cosmological constant to which we shall come back in a while. Notice that because of the index symmetry of the metric both $G_{\alpha\beta}$ and $T_{\alpha\beta}$ are symmetric tensors.

Equations (1.11) are the *fundamental field equations of GR*. They describe the way curvature (l.h.s) reacts to matter (r.h.s), and vice verse. With "matter" here is meant any sort of matter, e.g baryonic, relativistic or dark energy as well, as we will see later on.

By virtue of the Bianchi identities the covariant derivative of the l.h.s of (1.11) is identically zero, implying energy–momentum conservation of the r.h.s. What is more, the tensorial nature of the equation makes its form independent of the coordinate system. For details on its derivation and its significance, along with historical remarks we refer to Schutz (2009).

The field equations (1.11) can be formally derived through a variational principle, from the *Einstein–Hilbert* action (see for example Carroll (2003)),

$$S \equiv S[g] = \int d^4x \sqrt{-g}\frac{R-2\Lambda}{16\pi G} + S_{\rm m}[g,\psi], \tag{1.12}$$

with R being the Ricci scalar, G and Λ Newton's and cosmological constant respectively, and $S_{\rm m}$ denoting collectively the part of the action corresponding to any matter field content.

Notice that integration in the action integral is assumed along the spacetime manifold. A consistent initial value formulation of GR for a manifold with a boundary requires the introduction of the so–called Gibbons–Hawking terms in the action Gibbons and Hawking (1977). We leave this issue until chapter 4.

Let us get back to the problem of dark energy. As was mentioned also before, observations indicate that there should be an extra "matter" component in the Universe, with the peculiar property of having a negative pressure. In view of the Einstein field equations (1.11), the problem accounts in a missing component in the equations that could account for dark energy (let us assume for a moment that $\Lambda = 0$ in the equations). There are two paths that have been suggested here: Either to modify the l.h.s or the r.h.s of the equations respectively. In the first case, we are dealing with a modification of gravity, while on the second case with the addition of an extra energy–momentum component with the desired properties. However, such a distinction is only a formal one, as one can always move any term from the l.h.s to the r.h.s of the equations, interpreting it as some sort of an effective energy momentum tensor. The equations of motion themselves are not able to make such a distinction.

The simplest scenario for dark energy is that of a cosmological constant Λ in equations (1.11). A cosmological constant has a constant energy density $\rho_{\rm vac}$, where "vac" stands for "vacuum". On the one hand, it can be thought as a purely classical term allowed by the symmetries of GR, on the other hand, from a quantum mechanical point of view, it should receive contributions from the zero-point fluctuations of the different fields present in the Universe. In the latter case, the cosmological constant represents the vacuum energy associated with the zero point fluctuations of some fields, naturally those being the fundamental fields of the Standard Model of Particle Physics. However, this assumption leads to a much larger value for the cosmological constant than the one observed, which is also known as the problem of magnitude.

To be more precise, all fields present in a our description of the Universe, should fluctuate around their vacuum expectation value, first predicted by Casimir Casimir (1948). The fluctuations are associated with a particular amount of energy, which is expected to contribute to the Einstein equations in the semiclassical limit as the expectation value of an energy-momentum tensor on the r.h.s, i.e

$$G_{\alpha\beta} = 8\pi G \langle T_{00} \rangle_{\text{vac}}, \tag{1.13}$$

with

$$< T_{00} >_{\text{vac}} \sim \int_0^{\Lambda_c} \sqrt{k^2 + m^2} k^2 dk. \tag{1.14}$$

Above integral sums up the zero point energy of some field with mass m, up to the cut–off energy scale Λ_c. The latter has to be introduced, otherwise the integral will yield an infinite result, which would not make sense physically. In fact, the cut–off dependent, bare zero point energy (1.14) does not correspond to the observed, renormalised one. The latter should be given as the sum of the bare and suitable counter terms. For the expectation value $< T_{00} >_{\text{vac}}$, it can be seen that it diverges as the fourth power of the cut–off, i.e $\sim \Lambda_c^4$. By assuming a Planck scale cutoff, this yields $< T_{00} >_{\text{vac}} \simeq m_{p^4} c^5/\hbar^2 \sim 10^{76} GeV^2$ [5]. However, the observed value for the cosmological constant is $\sim 10^{-47} GeV^4$, which is 123 orders of magnitude smaller than the bare one. Therefore, the counter terms to be added to the bare energy to yield the renormalised one, should be such that they cancel the very high value of the bare part, which requires an extreme fine tuning. This is the magnitude problem associated with the cosmological constant.

Notice that the discrepancy between the bare and the observed vacuum energy

[5] Notice that here we recover the speed of light c, which we had set equal to one.

can be made smaller by choosing the cut–off at a lower scale, like the QCD scale, but even there the difference between the bare and the observed value can be found to be unacceptably large (~ 40 orders of magnitude, since $\Lambda_{QCD} \sim 10^{-3} GeV^4$).

The second problem associated with the cosmological constant is the so–called coincidence problem, which is related to the fact that the present value of the cosmological constant is of the same order of magnitude. In principle, this is a surprising observation, as a priori one would have not expected these two numbers to be related in such a way, also implying that we are living in a very special time of the Universe evolution. [6] For interesting reviews on these issues the reader is referred to Sahni (2002); Peebles and Ratra (2003b); Padmanabhan (2003); Copeland et al. (2006b), as well as Hollenstein et al. (2012) for a recent and interesting discussion on the renormalisation of zero point fluctuations.

Above two problems associated with the ΛCDM model, together with motivation coming from particle physics, has lead cosmologists to study alternative scenarios to describe the late time acceleration of the Universe. Many alternatives have been suggested in the literature, which can be broadly divided in two main categories: models that modify the r.h.s (energy-momentum part) of the Einstein equations, and those that modify the l.h.s (gravitational part) of the equations respectively.

Typical examples of the first category are minimally coupled to gravity scalar field models with either canonical (quintessence) Ratra and Peebles (1988) or non–canonical (k–essence) Chiba et al. (2000); Armendariz-Picon et al. (2000) kinetic terms. When non–minimally coupled to gravity, above models do not exhibit any imperfections at the linear level, like anisotropic stress or momentum flux (see chapter 2 for a definition of these terms.) Although, the positioning of a particular contribution on either side of the Einstein equations is a matter of convention, we classify these models as a modification of the r.h.s of the field equations in the sense that their interaction with gravity is minimal, i.e they do not "mix" with gravity in any non–trivial way as the models described below.

On the other hand, models where the modification of the equations has a purely gravitational origin (or more formally speaking, those that modify the l.h.s of the equations) include non–linear modifications of GR like $f(R)$ or $f(R, G)$ gravity mod-

[6]For an interesting discussion about other problems related to the ΛCDM model see Perivolaropoulos (2011, 2008).

els, where G is the Gauss–Bonnet term, or scalar–tensor theories Brans and Dicke (1961); Bergmann (1968); Wagoner (1970); Nordtvedt (1970); Amendola (1999); Uzan (1999); Chiba (1999); Bartolo and Pietroni (1999); Perrotta et al. (1999); Fujii and Maeda (2003); Charmousis et al. (2012) the most well known probably being the Brans–Dicke gravity Brans and Dicke (1961) (For a review on $f(R)$ gravity see Sotiriou and Faraoni (2010); De Felice and Tsujikawa (2010), while for more general models Nojiri and Odintsov (2006a); Durrer and Maartens (2008); Capozziello and Francaviglia (2008); Clifton et al. (2012) and references therein). Non–linear modifications of gravity modify the GR action to include non–linear curvature terms, yielding equations of motion of fourth–order for the metric field, while scalar–tensor theories introduce a non–minimal coupling between the scalar field and curvature in the action. The non–minimal coupling to gravity is the cause of the appearance of imperfections at the linear level, like anisotropic stress, in contrast for example to quintessence or k–essence models. A description of the fundamental properties and dynamics of these models, as well as of some more general ones, can be found in chapter 3.

We stress that at the classical level there is a link between non–linear modifications of gravity and scalar–tensor theories, as the two classes of theories are formally related via a Legendre transformation through the introduction of auxiliary fields. We will explicitly discuss this issue in chapter 4.

Furthermore, there are models which combine both of the above formal classes, in the sense that although they possess non–minimal couplings between a scalar field and curvature, their non–minimal coupling to curvature cannot be eliminated through an appropriate transformation like for example in $f(R)$ gravity. Examples of such theories are general scalar–tensor theories described by the Horndenski lagrangian Horndeski (1974), as well as kinetic gravity braidings Pujolas et al. (2011); Deffayet et al. (2010) and galileon models Deffayet et al. (2009a,b); Silva and Koyama (2009).

Another class of modifications of gravity include higher-dimensional models, like for example braneworld models, a characteristic example being the DGP one, according to which, our Universe is confined on a 4-dimensional (3+1) surface (brane), embedded in a 5-dimensional bulk space. Standard model particles are restricted on the brane, but gravity is allowed to propagate in the bulk space. For reviews on braneworld cosmology see for example March-Russell (2000); Langlois (2002); Wands (2002); Brax et al. (2004); Maartens and Koyama (2010) and references therein.

A very interesting approach to the dark energy problem has been suggested in the context of the Renormalisation Group (RG) cosmology, where the cosmological constant is promoted to a dynamical variable running with cosmic time. As it turns out, this scenario can successfully account for dark energy, and will be explicitly studied in chapters 6 and 7. This approach shares many common features with Brans–Dicke and $f(R)$ theories, as we will discuss in chapters 6 and 7.

Let us close this section by referring the interested reader for more details on the theory of General Relativity and its applications in cosmology to the following textbooks Misner et al. (1973); Wald (1984); Carroll (2003); Hawking and Ellis (1974); Will (1981). What is more, studies of the theoretical and observational foundations of modern cosmology can be found in Dodelson (2003); Liddle (1999); Peebles (1993); Peacock (1999); Amendola and Tsujikawa (2010); Mukhanov (2005).

1.2 Notation and conventions

Unless otherwise stated, we will work in units where $c = \hbar = 1$. We will also use the metric signature $(- + ++)$. We shall denote both Newton's constant and the Gauss–Bonnet term with "G", and the distinction will be made explicit wherever there is a danger of confusion. Newton's G is related to the Planck mass as $m_{\mathrm{p}} = G^{-1/2} = 1.2211 \times 10^{19}$ GeV.

Whenever no particular reference is made about the values of indices, those are assumed to represent abstract tensor ones. We also adopt the Einstein convention for indices, i.e repeated indices will imply summation, unless otherwise stated.

Chapter 2

Theory of cosmological perturbations

In this chapter we will introduce the basic background equations of GR in a Friedmann–Lemaitre-Robertson-Walker (FLRW) background, as well as review the theory of (scalar) linear perturbation equations in a component language, that will be helpful for the analysis in some of the next chapters.

The importance of perturbation theory in a cosmological context lies in the fact that it allows us to understand how small inhomogeneities in the matter distribution collapse to form the bound structures as we observe them in the Universe: stars, galaxies, clusters of galaxies, and so on. At the scales of cosmological interest, $k/H \gg 1$, where k is the wave number associated with some typical cosmological scale, the perturbations are well described by the linear approximation, that is ignoring terms of second and higher order in the perturbative expansion of the fields involved.

In principle, the metric field carries tensor, vector and scalar modes. However, in the study of cosmological perturbation theory, only tensor and scalar modes are of interest, since vector modes decay very quickly, unless there are active sources such as defects or primordial magnetic fields. On the other hand, tensor fluctuations in a cosmological context are predicted by the theory of cosmological (primordial) inflation, with their production occurring in the very early Universe, for this reason also called "primordial gravitational waves" (for details see for example Lyth and Liddle (2009)). Their detection poses a big challenge for observational cosmology, and there are prospects of being detected with future experiments Krauss et al. (2010).

The scalar modes are of the greatest interest for late time cosmology, since they are responsible for the formation of bound gravitational structures, i.e their role is to act as a source for the gravitational force that makes matter inhomogeneities collapse. On the same time, as said before, we will be interested in first (linear) order terms in the perturbative expansion. This is justified by the fact that, at large scales in the Universe, gravitational (or matter) fluctuations are assumed to be small, at least small enough to be consistently described at first order.

2.1 Background equations

The action of General Relativity (GR) in the presence of (any sort of) matter fluids is described by

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}R + S_{\rm m} + S_{\rm b} + S_{\rm X}, \tag{2.1}$$

where R is the Ricci scalar and g denotes the determinant of the background metric field, $g \equiv \det g_{\alpha\beta}$. $S_{\rm m}$ and $S_{\rm b}$ respectively stand for the part of the action describing dark and baryonic matter respectively. The usual approach is to describe both as perfect fluids with zero pressure.

On the other hand, $S_{\rm X}$ denotes the dark component which also takes into account a possible modification of GR. In the latter case, it will be a function of the metric and its derivatives, and at the level of the equations of motion it can be thought as contributing an effective energy–momentum tensor on the r.h.s of the equations.

Variation of above action with respect to the metric,

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0, \tag{2.2}$$

gives rise to the background equations of motion

$$G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - \frac{1}{2}g^{\mu}{}_{\nu}R = \frac{\kappa^2}{3}\left(T_{\rm (b)}{}^{\mu}{}_{\nu} + T_{\rm (m)}{}^{\mu}{}_{\nu} + T_{\rm (X)}{}^{\mu}{}_{\nu}\right), \tag{2.3}$$

which we call the *Einstein equations*.

The Einstein equations satisfy the Bianchi identities,

$$\nabla_\mu G^\mu{}_\nu = 0 = \nabla_\mu \left(T_{(\mathrm{b})}{}^\mu{}_\nu + T_{(\mathrm{m})}{}^\mu{}_\nu + T_{(\mathrm{X})}{}^\mu{}_\nu \right) , \tag{2.4}$$

which come as a result of the gauge invariance of the theory, i.e they are gauge identities due to the diffeomorphism invariance of GR. Since the l.h.s is identically zero, the r.h.s should be as well, leading to a set of conservation equations for the (effective) matter fields. Below we will evaluate the latter for an energy–momentum tensor described by a perfect fluid.

Cosmological observations show that our Universe is to very high accuracy isotropic and homogeneous. The line element satisfying these requirement is described by the four-dimensional homogeneous and isotropic spacetime, called Friedmann–Lemaitre-Robertson-Walker (FLRW) spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) dx^i dx^i, \tag{2.5}$$

where t is cosmic time and repeated indices imply summation. We have set the spatial curvature equal to zero, as this is the case that we will consider throughout this thesis. In fact, there is good observational evidence that the Universe is flat to high accuracy coming from the CMB Komatsu et al. (2011).

We can also define the metric element in terms if conformal time defined as

$$\eta \equiv \int a^{-1}(t) dt. \tag{2.6}$$

In the following we might use one or the other definition of the time variable, and that will be made clear in the text.

In the homogeneous and isotropic background described by the FLRW metric, either baryonic or dark matter are modeled as perfect fluids. The energy–momentum tensor for a perfect fluid with energy density $\rho = \rho(t)$, pressure $p = p(t)$ and 4-velocity u^μ reads as

$$T^\mu{}_\nu = (\rho + p) u^\mu u_\nu + \delta^\mu{}_\nu p. \tag{2.7}$$

In comoving coordinates, the the 4-velocity of the fluid is $u^\mu = (-1, 0, 0, 0)$ and satisfies the timelike normalisation relation

$$u^{\mu}u_{\mu} \equiv g_{\mu\nu}u^{\mu}u^{\nu} = -1. \tag{2.8}$$

The Einstein equations according to the FLRW metric, give rise to the so called Friedmann, acceleration equation respectively,

$$H^2 = \frac{\kappa^2}{3}\left(\rho_{\rm b} + \rho_{\rm m} + T_{(\rm X)}{}^{0}{}_{0}\right), \tag{2.9}$$

$$3H^2 + 2\dot{H} = -\kappa^2\left(w_{\rm b}\rho_{\rm b} + w_{\rm m}\rho_{\rm m} + T_{(\rm X)}{}^{j}{}_{j}\right). \tag{2.10}$$

The r.h.s of the Einstein equations has to be covariantly conserved as well,

$$\nabla_{\mu}T_{(\rm X)}{}^{\mu}{}_{\nu} + T_{(\rm m)}{}^{\mu}{}_{\nu} = 0, \tag{2.11}$$

where we neglected the baryons' energy–momentum tensor for simplicity. Above equation allows for a general coupling between dark energy and dark matter as

$$\nabla_{\mu}T_{(\rm X)}{}^{\mu}{}_{\nu} = C(t), \tag{2.12}$$

$$\nabla_{\mu}T_{(\rm m)}{}^{\mu}{}_{\nu} = -C(t), \tag{2.13}$$

and when evaluated on an FLRW background they give

$$\dot{\rho}_{\rm X} + 3H(1 + w_{\rm X})\rho_{\rm X} = C(t), \tag{2.14}$$

$$\dot{\rho}_{m} + 3H(1 + w_{\rm m})\rho_{\rm m} = -C(t), \tag{2.15}$$

with no summation implied in $T^{j}_{(\rm X)j}$ and a dot denoting differentiation with respect to cosmic time. The coupling $C(t)$ takes into account a possible interaction between dark matter (or baryons) and dark energy.

$H = H(t)$ is the *Hubble parameter* defined as

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}, \tag{2.16}$$

and the *barotropic index* w_i with $i = \mathrm{X}, \mathrm{m}, \mathrm{b}$, is defined as

$$w_i(t) \equiv \frac{p_i(t)}{\rho_i(t)}. \tag{2.17}$$

In principle, w_i will be a function of time. Baryonic and dark matter are usually considered to be pressure less, i.e $w_\mathrm{b} = w_\mathrm{m} = 0$. We can also define the Hubble parameter with respect to conformal time as

$$\mathcal{H}(\eta) \equiv \frac{1}{a(\eta)}\frac{da}{d\eta} = H(t)a(t). \tag{2.18}$$

The background behavior of a given cosmological model is completely described by two functions, which could for example be the Hubble parameter $H = H(t)$, and the barotropic index $w_i = w_i(t)$. $H = H(t)$ is determined through the Friedman equation. On the other hand, knowledge of w_i and $H(t)$ allows to solve for the matter density evolution. Mathematically speaking, we have two (first order) equations with two unknown variables.

2.2 Linear perturbations and the choice of gauge

In this section we will describe the formalism needed to describe departures from the smooth, homogeneous and isotropic Universe described in the previous section. More precisely, we will consider small, linear fluctuations around the FLRW background for both gravitational and matter degrees of freedom. As dictated by the Einstein equations, a fluctuation in the l.h.s of the equation will source a fluctuation in the r.h.s and vice versa. This analysis is essential to understand how the primordial, quantum fluctuations generated at the end of inflation in the very early Universe, are amplified to form the bound gravitational structures observed in the late time Universe.

Small inhomogeneities in an expanding Universe collapse gravitationally to form galaxies, clusters of galaxies and so on. According to the theory of inflation, small departures from the smooth cosmological background are generated through quantum fluctuations of the inflaton field, which are then amplified and finally become classical Lyth and Liddle (2009). Their treatment as linear perturbations holds as long as the fluctuations remain small, otherwise the validity of the linear approximation fails.

What is more, different models of dark energy which are degenerate at the background, can give different predictions at the perturbation level. Therefore, cosmological observables related to perturbation variables are extremely important in making predictions about the dark fluid. For example, two very important observables at this level is the matter power spectrum and the weak lensing potential. The first describes how much (dark) matter clusters or in other words how densely matter is distributed in space, while the second one is the gravitational potential that forces light from distant galaxies to bend along the line of sight.

For an explicit presentation of the theory of cosmological perturbations in both GR and non–linear gravity models, the reader is referred to Ma and Bertschinger (1994); Mukhanov et al. (1992); Hwang and Noh (2005).

2.2.1 Perturbed field equations

In this section, we move from cosmic time t, to conformal time η,

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j = a^2(\tau)(d\tau^2 - \gamma_{ij}dx^i dx^j). \tag{2.19}$$

For the shake of simplicity, and unless otherwise stated, we will also denote the total energy-momentum tensor as $T^\mu{}_\nu$, without referring to its particular constituents.

We want to consider small fluctuations around FLRW spacetime in the Einstein equations (2.3), sourced through the field fluctuations as

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \quad \rho_i \to \rho_i + \delta\rho_i. \tag{2.20}$$

Notice that similar relations will hold for any other field variable present in the field equations. In the above notation, the dark component is included in ρ_i.

The perturbed Einstein equations then read as

$$G^\mu{}_\nu + \delta G^\mu{}_\nu = \kappa^2 \left(T^\mu{}_\nu + \delta T^\mu{}_\nu\right), \tag{2.21}$$

and assuming the background equations hold, i.e $G^\mu{}_\nu = \kappa^2 T^\mu{}_\nu$ we arrive at the linear perturbation equation

$$\delta G^{\mu}{}_{\nu} = \kappa^2 \delta T^{\mu}{}_{\nu}. \tag{2.22}$$

Above equation tells us that a fluctuation in the gravitational part of the Einstein equation will generate a fluctuation in the matter part and vice versa, which is the key idea of gravitational instability. In order to make use of it, we have to express both sides in terms of the fields fluctuations. We shall begin with the Einstein tensor perturbation, and later on we will also discuss the explicit form of the energy–momentum tensor perturbation. The perturbed Einstein tensor reads as

$$\delta G^{\mu}{}_{\nu} = \delta R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}\delta R, \tag{2.23}$$

with $\delta^{\mu}{}_{\nu}$ the Kronecker delta, which should not be confused with the variation symbol. The Ricci tensor and scalar variations have to be calculated by varying their explicit expressions in terms if the connection. For example, for the variation of the Ricci scalar one finds,

$$\delta R \equiv \delta\left(g^{\alpha\beta} R_{\alpha\beta}\right) = \delta g^{\alpha\beta} R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}, \tag{2.24}$$

which assumes knowledge of $\delta R_{\alpha\beta}$. Of course, given the metric fluctuation, the first quantity that has to be calculated is the fluctuation of the connection given by

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\delta g^{\alpha\kappa}\left(2g_{\kappa(\beta,\gamma)} - g_{\beta\gamma,\kappa}\right) + \frac{1}{2}g^{\alpha\kappa}\left(2\delta g_{\kappa(\beta,\gamma)} - \delta g_{\beta\gamma,\kappa}\right), \tag{2.25}$$

with $(A, B) \equiv \frac{1}{2}(AB + BA)$.

Having in hand all the expressions for the perturbed curvature tensors, the next step is to choose an explicit expression for the metric fluctuation $\delta g_{\mu\nu}$. The metric, being a rank two tensor field, carries in total 10 degrees of freedom, and one can write down a general decomposition consisting of scalar, vector and tensor modes respectively. The general decomposition of the metric perturbation reads as

$$\delta g_{\mu\nu} = a^2(\eta)\begin{pmatrix} -2\Psi & w_i \\ w_i & 2\Phi\delta_{ij} + h_{ij} \end{pmatrix}. \tag{2.26}$$

ψ and ϕ are spatial scalars, w_i is a 3-vector and h_{ij} is a traceless spatial rank two tensor field, i.e $\delta^{ij}h_{ij} = 0$.

We can further decompose the components of the perturbed metric element (2.26). The vector part w_i can be decomposed into longitudinal (curl free) and transverse (divergence free) component as

$$w_i = w_i^{||} + w_i^{\perp} = \nabla_i w + w_i^{\perp} \equiv w_{,i} + w_i^{\perp}, \tag{2.27}$$

since the longitudinal part, being curl free, i.e $\nabla \times w_i^{||} = 0$, can be expressed as the gradient of a scalar function.[1]The vector part satisfies $\nabla \cdot w_i^{\perp} = 0$.

In a similar fashion, the tensor part of (2.26) can be expressed in terms of pure scalar, vector and tensor part respectively as

$$h_{ij} = h_{ij}^{||} + h_{ij}^{\perp} + h_{ij}^{T}, \tag{2.28}$$

with T standing for transverse. $h_{ij}^{||}$ and $h_{ij}^{\perp}$ can be further decomposed into scalar and vector parts while h_{ij}^{T} is pure transverse and cannot be decomposed any further. We have

$$h_{ij}^{||} = (\partial_i\partial_j - \frac{1}{3}\partial^2)h, \tag{2.29}$$

$$h_{ij}^{\perp} = \frac{1}{2}(h_{i,j} + h_{j,i}), \tag{2.30}$$

with h and h_i denoting the pure scalar and vector parts respectively, and $\partial^2 \equiv \delta^{ij}\partial_i\partial_j$. The following relations hold for the different components

$$\delta^{ij}h_{ij}^{||} = 0, \tag{2.31}$$

$$\delta^{ij}h_{i,j} = 0, \tag{2.32}$$

$$\delta^{ij}h_{ij}^{T} = 0, \quad \delta^{ik}h_{ij,k}^{T} = 0. \tag{2.33}$$

Therefore, the scalar part $h_{ij}^{||}$ is symmetric and traceless, the vector $h_{ij}^{\perp}$ is symmetric and traceless, and h_{ij}^{T} is symmetric, transverse and traceless.

We can sum up all the different modes included in the perturbed element $\delta g_{\mu\nu}$: There are four scalars (ψ, ϕ, w, h), two vectors (w_i, h_i) and a tensor part h_{ij}^{T}. Re-

[1]The comma here denotes partial derivative, i.e $_{,i} \equiv \partial_i$.

member that the total degrees of freedom of the metric field add up to ten. Among these the four scalars contribute four, the 2 vectors are subject to two constraints and contribute two, and the tensor mode satisfies 4 constraints and contributes two degrees of freedom respectively. Adding up, the sum gives ten, as it should. Not all of the ten degrees of freedom are physically relevant, and in order to calculate physically meaningful quantities one has to choose a gauge, a procedure that will describe in the following. All we can say for now is that, choosing a gauge artifact degrees of freedom are removed from the calculation of observables.

An important point is that at the linear perturbation level, which we are interested in, the different modes, scalar, vector and tensor ones, propagate independently, i.e there is no mixing between different modes. At the linear level, each mode sources a different part of the perturbed energy-momentum tensor (r.h.s of the Einstein equations) and vice versa. As mentioned also before the scalar modes are responsible for gravitational collapse. They source perturbations in the energy density, pressure and scalar part of the velocity of the energy-momentum tensor on r.h.s. On the other hand, vector modes source rotational velocity perturbations and since they decay in time, they are not of cosmological interest. Finally, tensor modes represent gravitational waves, and in a cosmological context are suspected to be generated at the end of the inflationary era (primordial gravitational waves), and could be potentially observable in CMB.

2.2.2 Perturbed energy-momentum tensor

We saw in the previous section how to express the perturbed l.h.s of the Einstein equation. In this section we will present how the r.h.s should be perturbed, which corresponds to perturbations in the energy-momentum tensor of the fluid(s).

We will restrict ourselves to the case of the energy-momentum tensor of a perfect fluid which is the most common case in cosmology, at least concerning baryonic and dark matter fluids. The case of perturbations in a general energy-momentum tensor will be described in a following chapter, where we will discuss perturbations for a general dark fluid. Notice also that in the case that the energy-momentum tensor $T^{\alpha}{}_{(X)\beta}$ is of geometrical origin, in that case its explicit perturbed form is derived in the same way as the l.h.s of the Einstein equations. (Remember that by moving an energy momentum tensor of purely gravitational origin to the r.h.s is just a matter of taste, so that we can view the equation as GR on the l.h.s sourced by some effective "fluid" on the r.h.s.)

For a general fluid, the energy momentum-tensor has the form

$$T^{\alpha}{}_{\beta} = (\rho + p)u^{\alpha}u_{\beta} + p\delta^{\mu}{}_{\nu} + (2q_{(\alpha}u_{\beta)} + \sigma_{\alpha\beta}), \tag{2.34}$$

with ρ the energy density, p the pressure, u^{α} the fluid rest frame velocity, q_{α} the momentum flux and $\sigma_{\alpha\beta}$ the anisotropic shear. For a perfect fluid, $q_{\alpha} = \sigma_{\alpha\beta} = 0$. Also notice that even for an imperfect fluid, q_{α} and $\sigma_{\alpha\beta}$ will be zero at the FLRW background, but not at the perturbation level. The fluid velocity is a timelike vector satisfying $u^{\alpha}u_{\alpha} = -1$, and at the FLRW background takes the form

$$u_{\alpha} = \frac{1}{a(\eta)}(-1, 0, 0, 0). \tag{2.35}$$

Perturbing the energy-momentum tensor we have that

$$\delta T^{\alpha}{}_{\beta} = (\delta\rho + \delta p)u^{\alpha}u_{\beta} + (\delta\rho + \delta p)(\delta u^{\alpha}u_{\beta} + u^{\alpha}\delta u_{\beta}) + \delta p\delta^{\mu}{}_{\nu}, \tag{2.36}$$

$$\equiv \rho\left[\delta(1 + c_s^2)u^{\alpha}u_{\beta} + (1 + w)(\delta u^{\alpha}u_{\beta} + u^{\alpha}\delta u_{\beta}) + c_s^2\delta\delta^{\mu}{}_{\nu}\right], \tag{2.37}$$

where the sound speed, relating the pressure with the energy perturbation, is defined as

$$c_s^2 \equiv \frac{\delta p}{\delta\rho}. \tag{2.38}$$

At the linear level, knowledge of the sound speed is essential as it closes the system of equations, by relating the energy with pressure perturbations, in a similar way the barotropic index does for pressure and energy at the background. In particular, for the case of a barotropic fluid where $p = p(\rho)$ the sound speed becomes $c_s^2 \equiv dp/d\rho = \dot{p}(\rho)/\dot{\rho}$. However, in general pressure depends on entropy too, so

$$c_s^2 \equiv \frac{\delta p(\rho, s)}{\delta\rho} = \frac{\partial p}{\partial\rho} + \frac{\partial p}{\partial s}\frac{\partial s}{\partial\rho} \equiv c_{s(a)}^2 + c_{s(na)}^2, \tag{2.39}$$

with $c_{s(a)}^2$ the adiabatic and $c_{s(na)}^2$ the non-adiabatic part of the sound speed. Obviously, for a barotropic fluid, it is $c_{s(na)}^2 = 0$.

The sound speed is associated with a characteristic length, the Jeans length λ_J, which is defined as Padmanabhan (1993)

$$\lambda_J \equiv \sqrt{\pi}\frac{c_\mathrm{s}}{(G\rho)^{1/2}}, \tag{2.40}$$

with c_s is the sound speed of the collapsing component under study, and ρ the energy density of the dominant component in the case of a multicomponent Universe. At a given time, the growth of modes smaller than the Jeans length, $\lambda < \lambda_J$ will be suppressed, while the opposite will be true for modes outside the Jeans length. In the latter case, pressure support of the matter density cannot counterbalance the gravitational attraction and the small inhomogeneity collapses under gravity, as the timescale for gravitational collapse $t_\mathrm{grav} \sim (G\rho_\mathrm{m})^{-1/2}$ is smaller than the one need for pressure re adjustment $t_\mathrm{press} \sim \lambda/c_\mathrm{s}$.

In fact, in a multicomponent Universe, gravitational collapse of the perturbed species can be prevented even for modes outside the Jeans, when the background expansion is fast enough to prevent collapse. In that case, the expansion timescale $t_\mathrm{exp} \sim (G\rho_\mathrm{dominant})^{-1/2}$ is smaller than the timescale for gravitational collapse $t_\mathrm{grav} \sim (G\rho_\mathrm{m})^{-1/2}$. In such situation it is $t_\mathrm{exp} < t_\mathrm{grav} < t_\mathrm{pressure}$ Padmanabhan (1993),

$$(G\rho_\mathrm{dominant})^{-1/2} < (G\rho_\mathrm{m})^{-1/2} < \frac{\lambda}{c_\mathrm{s}}. \tag{2.41}$$

For a detailed presentation of above issue the reader is referred to the textbook Padmanabhan (1993).

Let us now elaborate with perturbing the energy momentum tensor, which we assume to have a perfect fluid form. Considering only scalar perturbations, the components of the perturbed energy-momentum tensor are

$$\delta T^0{}_0 = -\delta\rho, \tag{2.42}$$

$$\delta T^0{}_i = -\delta T^i{}_0 = (1+w)\rho v_i, \tag{2.43}$$

$$\delta T^i{}_j = c_s^2 \delta\rho \delta^i{}_j. \tag{2.44}$$

For the 4-velocity perturbation $u^\alpha \to u^\alpha + \delta u^\alpha$ we have,

$$u^\alpha = \frac{1}{a}\left(1-\Psi, v_i\right), \quad u_\alpha = a\left(-(1+\Psi), v_i - w_i\right), \tag{2.45}$$

with $a \equiv a(\eta)$ and $v^i \equiv dx^i/d\eta \equiv au^i$ is the matter, coordinate peculiar velocity. Notice that the timelike normalisation is preserved.

The set of the perturbed Einstein equations (2.21) is supplemented with the perturbation of the conservation equations for the fluid(s) energy-momentum tensor(s) on the r.h.s. of the equations. This leads to

$$\delta\left(\nabla_\alpha T^\alpha{}_\beta\right) = 0 \equiv \delta\left(T^\alpha{}_{\beta,\alpha} - \Gamma^\kappa_{\beta\lambda}T^\lambda{}_\kappa + \Gamma^\kappa_{\kappa\lambda}T^\lambda{}_\beta\right), \tag{2.46}$$

$$\sim \partial\delta T + \Gamma\delta T + T\delta\Gamma, \tag{2.47}$$

with the last line being a schematic representation of the kind of terms one finds after evaluation of the variation on the first line. We will derive explicit expressions for above equation considering particular cases for fluids in the following.

2.2.3 Choice of gauge

As mentioned before, there is gauge freedom in the perturbed, gravitational field equations. In the context of perturbation theory in gravity, the gauge freedom lies in the different ways one can move from the unperturbed (background) manifold to the perturbed one. The latter corresponds to the "real", observed Universe.

Let us look at this issue a bit more closely. Let us recall the notion of a gauge transformation. The gauge transformation of some field variable, is the change in that variable induced by an infinitesimal diffeomorphism, generated by a vector field ξ^α. (The change in the field induced by such transformation defines the Lie derivative.) In particular, for the metric field a gauge transformation transforms the field as,

$$g_{\alpha\beta} \to g_{\alpha\beta} + 2\nabla_{(\alpha}\xi_{\beta)}, \tag{2.48}$$

with $(A, B) \equiv \frac{1}{2}(AB + BA)$.

In the context of perturbation theory in gravity, by doing a gauge transformation of some field (e.g the metric), one changes the point in the background spacetime corresponding to a point in the physical space Ellis and Bruni (1989). Therefore, even for quantities which are scalars under gauge transformations, the value of the perturbation will not be invariant under the transformation if the quantity is non-zero and position dependent in the background Ellis and Bruni (1989). This results in the Walker-Stewart lemma Stewart and Walker (1974) which simply states that quantities which are constant or zero in the background spacetime will be gauge invariant. The latter is the standard approach in constructing gauge invariant variables, especially in the covariant approach to perturbation theory of Ellis and Bruni (1989); Bruni et al. (1992). Following Ellis and Bruni (1989), for the metric perturbation we have,

$$\delta g_{\alpha\beta} = g_{\alpha\beta} - \bar{g}_{\alpha\beta}, \tag{2.49}$$

with the bar indicating the background metric field. If from observations we were able to fully reconstruct the "real" metric $g_{\alpha\beta}$, *there is no unique way of reconstructing the idealised, background metric* $\bar{g}_{\alpha\beta}$; the gauge freedom allows for different mappings from the background to perturbed Universe. In order for the calculation of observable quantities to be meaningful, spurious degrees of freedom have to be removed, or in other words a particular mapping from the background (homogeneous) to the perturbed (inhomogeneous) spacetime has to be chosen, through the "gauge fixing" procedure. After the gauge is fixed, the different local coordinate transformations of the metric field are uniquely fixed, and the extra gauge degrees of freedom are eliminated.

Let us describe the most commonly used gauge choices for scalar perturbations in cosmology. The condition $B = w = 0$ [2] defines the so–called *Newtonian* or *longitudinal gauge*. The advantage of this gauge can be seen by the fact that in the solar system limit Ψ plays the role of the Newtonian potential. Morover, it is the combinations of Φ and Ψ that comes into Weak Lensing (WL) measurements, making it extremely useful for calculating observables. Another important point is that in this gauge, Φ and Ψ coincide with the gauge invariant potentials $\Phi_{\rm GI}$, $\Psi_{\rm GI}$ Mukhanov et al. (1992). Note also that in this gauge the $0-0$ Einstein equation plays the role of the generalized Poisson equation, well known from its Newtonian gravity analogue.

[2] This w should not be confused with the barotropic index.

The condition $\Psi = w = 0$ defines the *synchronous gauge*. This is the gauge where all observers agree on the notion of time. Although the metric shift is zero, the energy-momentum tensor includes a velocity perturbation v_i in its $0-i$ component. For the metric evolution equations in this gauge see for example Ma and Bertschinger (1994). The coordinates in this gauge are not totally fixed, leading to the appearance of unphysical gauge modes. The latter fact requires one to be cautious when interpreting results calculated in this frame Bednarz (1985); Mukhanov et al. (1992).

One can also work with *gauge invariant* perturbation variables, an approach introduced by Bardeen Bardeen (1980) (see also Mukhanov et al. (1992)). The introduction of gauge invariant variables refers to the construction of expressions relating perturbation variables of different gauges, in such a way that any gauge transformation leaves them invariant.

Finally, another approach is to work in the so–called covariant formalism, which is background independent. In this approach, one works with manifestly, gauge invariant fluid quantities. For details of this approach one can look at Ellis and Bruni (1989); Bruni et al. (1992) and references therein.

In this thesis, we shall work in the Newtonian gauge, which as described before is defined through $w = h = 0$, and therefore characterised by the two scalar potentials Ψ and Φ. We will refer to them as the "Newtonian potentials". In this gauge, the components of the perturbed Einstein equations in the presence of multiple perfect fluids and a "dark" energy-momentum tensor ${T_{(\mathrm{X})}}^{\alpha}{}_{\beta}$ take the following form

$$\frac{a^2 \delta G^0{}_0}{2} \equiv 3\mathcal{H}(\mathcal{H}\Psi - \Phi') + \nabla^2\Phi = -\frac{\kappa^2 a^2}{2}(\delta\rho_i - \delta {T_{(\mathrm{X})}}^0{}_0) \tag{2.50}$$

$$\frac{a^2 \nabla^i \delta G^0{}_i}{2} \equiv \nabla^2(\Phi' - \mathcal{H}\Psi) = \frac{\kappa^2 a^2}{2}\left((1+w_i)\rho_i\theta_i + \delta {T_{(\mathrm{X})}}^0{}_i\right) \tag{2.51}$$

$$\frac{a^2 \delta G^i{}_j}{2} \equiv \Phi'' + 2\mathcal{H}\Phi' - \mathcal{H}\Psi' - (\mathcal{H}^2 + 2\mathcal{H}')\Psi = -\frac{\kappa^2 a^2}{2}\left(c_{si}^2\delta\rho_i + \delta {T_{(\mathrm{X})}}^i{}_j\right) \tag{2.52}$$

$$\left(\delta G^i{}_j - \frac{1}{3}\delta G^k{}_k\right) \equiv \left(\partial^i\partial_j - \frac{1}{3}\partial^k\partial_k\right)(\Phi + \Psi) = -\frac{\kappa^2 a^2}{2}\delta\Pi_{(X)}, \tag{2.53}$$

where $\delta\Pi_{(X)}$ is the scalar anisotropic stress contribution of the dark component X, and is defined through ${\delta\Pi_{(\mathrm{X})}}^i{}_j \equiv \delta {T_{(\mathrm{X})}}^i{}_j - \frac{1}{3}\delta^i{}_j \delta {T_{(\mathrm{X})}}^k{}_k = \left(\partial^i\partial_j - \frac{1}{3}\partial^k\partial_k\right)\delta\Pi_{(X)}$. Furthermore, we have also defined the velocity gradient θ as $\theta_i \equiv \nabla^i v_i$. Notice also that on the r.h.s of the above equations we assume that the perfect fluid components are summed over. i.e $\delta\rho_i \equiv \delta\rho_1 + \delta\rho_2 + \ldots$. For baryonic and dark matter we have $w = c_s^2 = 0$.

The set of perturbation equations is not complete yet. We still have to perturb the fluid(s) conservation equations, as shown in equation (2.46). We will do it for the case of a perfect fluid with barotropic index and sound speed w_i and c_{si} respectively, in the Newtonian gauge, yielding two first order equations, as follows Amendola and Tsujikawa (2010),

$$\delta_i' + 3\mathcal{H}(c_{si}^2 - w_i)\delta_i = -(1+w_i)(\theta_i + 3\Phi'), \tag{2.54}$$

$$\theta_i' + \left(\mathcal{H}(1-3w_i) + \frac{w_i'}{1+w_i}\right)\theta_i = -\nabla^2\left(\frac{c_{si}^2}{1+w_i}\delta_i + \Psi\right). \tag{2.55}$$

Above equations are a result of energy and momentum conservation respectively. One can also derive a single, second order evolution equation with respect to time, for the fraction δ_i, by differentiating the energy equation with respect to time, and then substituting the momentum one to eliminate θ_i and its time derivative.

Notice that for the case of pressure less matter, like for example dark matter, where $w = w' = c_s^2 = 0$, above equations simplify a lot.

The system of perturbation equations is now complete [3]. Notice that only two of the Einstein equations are independent. We have therefore two gravitational equations for two gravitational variables (Φ and Ψ) and two equations for each two fluid variables (δ_i and θ_i). In the sub horizon approximation ($H^2/k^2 \ll 1$), which is the appropriate approximation when studying structure formation in the late time Universe, the two gravitational equations that are used is the Poisson equation, coming from the $0-0$ component of the Einstein equations, and the anisotropy equation. We will use this approximation in the following chapters when we will evaluate the equations for particular models of the dark component $T^{\alpha}{}_{(X)\beta}$.

Relativistic species, like neutrinos, also contribute to the anisotropic stress, however their contribution becomes important at the early stages of the cosmological evolution, and we can neglect them when studying the Universe at later times. Notice that in the absence of any anisotropic stress contribution, we get $\Phi = -\Psi$,

[3] For the transition to Fourier from real space, given a variable f it is: $f = \int d^3k f_k e^{i\mathbf{k}\cdot\mathbf{r}}$, with k the Fourier mode.

which is a signature of GR or scalar field models like quintessence or k-essence (but not true for galileon models). It is no longer true in non-linear gravity models like $f(R)$ gravity, and therefore any departure from $|\Phi/\Psi| \sim 1$, will signal a modification of gravity. We will come back to this issue in a later chapter, when we will discuss the significance of anisotropic stress as a key observable in testing modified gravity theories.

The weak lensing potential in the Newtonian gauge is defined as

$$\phi_{\rm WL} \equiv \Psi - \Phi. \tag{2.56}$$

It is the potential to which light rays respond when passing close to some large concentration of matter, like a cluster of galaxies. Therefore, the potential ϕ_{WL} can be extracted from weak lensing surveys. In fact, the light rays correspond to the scalar potentials and not directly to the matter concentrations. This means that the scalar potentials might acquire modifications compared to their GR value, either due to the presence of some unknown clustering component, or because of a possible modification of gravity itself. This fact has to be taken into account when constructing phenomenological dark energy parametrisations at the linear level (see for example Amendola et al. (2008)).

The approach we used above to discuss linear perturbations is using the language of components. In fact, complicated models of the scalar–tensor type, especially those with second derivatives in the energy–momentum tensor, the simplest of this case being $f(R)$ theories, the component language can become quite complicated when analysing the physics. A more intuitive approach one can use is the covariant language of fluids. In this context, the dynamics of the model can be expressed as the evolution of fluid variables like energy, provided knowledge of appropriate closure relations between energy and pressure.

Our aim is not to discuss this subject in details, and for details we refer the reader to the work in progress Sawicki et al. (2012), which is soon to appear. However, let us present the fundamental equations in the context of a scalar–tensor theory.

Given a particular action, which apart from the metric included also a scalar field [4], the starting point for a covariant fluid description is to start off by defining

[4]We do not make any particular assumption about the form of the kinetic term or the potential

the *scalar frame*, which is the frame of an observer comoving with the fluid, with velocity U^μ,

$$U^\mu = -\frac{\nabla^\mu \phi}{\sqrt{2X}}, \tag{2.57}$$

with $X \equiv -\frac{1}{2}\partial_\kappa \phi \partial^\kappa \phi$, which is positive, $X > 0$, in our metric signature $(-+++)$. Notice that with this definition, the velocity vector U^μ is time like. All other velocity fields, like for example the dark matter one, can be decomposed into perpendicular and parallel components to U^μ.

The covariant derivative of the velocity acquires the decomposition into the kinematical quantities a_μ, θ and $\sigma_{\mu\nu}$ as

$$\nabla_\mu U_\nu = -U_\mu a_\nu + \sigma_{\mu\nu} + \frac{1}{3} \perp_{\mu\nu} \theta, \tag{2.58}$$

with

$$a_\nu \equiv U^\mu \nabla_\mu U_\nu \tag{2.59}$$

the acceleration, $\sigma_{\mu\nu}$ the transverse, traceless symmetric shear, and $\theta \equiv \nabla^\mu U_\mu$ the expansion. In a FLRW background it is $\theta = 3H$.

All tensor quantities can be then decomposed covariantly, into parallel and perpendicular components to U^μ. In particular, the energy–momentum tensor is decomposed into its "irreducible" fluid quantities, namely energy $\mathcal{E}$, pressure $\mathcal{P}$, momentum flux q_μ and shear $\tau_{\mu\nu}$, according to

$$T_{\mu\nu} = \mathcal{E} U_\mu U_\nu + \perp_{\mu\nu} \mathcal{P} + 2U_{(\mu} q_{\nu)} + \tau_{\mu\nu}, \tag{2.60}$$

with the projector perpendicular to the velocity U_μ defined as $\perp_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$, with $g_{\mu\nu}$ the background spacetime metric.

Covariant conservation of energy and momentum are derived by projecting out the covariant derivative of the energy–momentum tensor appropriately,

$$U^\mu \nabla^\nu T_{\mu\nu} = \dot{\mathcal{E}} + (\mathcal{E} + \mathcal{P})\theta - U^\mu \dot{q}_\mu + \nabla_\mu q^\mu + \sigma_{\mu\nu}\tau^{\mu\nu}, \tag{2.61}$$

$$\perp_\lambda{}^\nu \nabla^\mu T_{\mu\nu} = (\mathcal{E} + \mathcal{P}) a_\lambda + \perp_\lambda^\mu \nabla_\mu \mathcal{P} + \frac{4}{3}\theta q_\lambda + \perp_{\mu\lambda} \dot{q}^\mu + \sigma_{\mu\lambda} q^\mu + \perp_{\lambda\nu} \nabla_\mu \tau^{\mu\nu}, \tag{2.62}$$

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of the scalar field here.

while the Einstein equations are obtained in a similar fashion, for example for the 00, $0i$ and ij we have

$$U^\mu U^\nu G_{\mu\nu} = U^\mu U^\nu T_{\mu\nu}, \tag{2.63}$$

$$U^\mu \perp_\lambda {}^\nu G_{\mu\nu} = U^\mu \perp_\lambda {}^\nu T_{\mu\nu}, \tag{2.64}$$

$$\left(\perp^{\alpha\mu}\perp_{\beta\nu} - \frac{1}{3}\perp^{\mu\nu}\perp^{\alpha\beta}\right) G_{\mu\nu} = \left(\perp^{\alpha\mu}\perp_{\beta\nu} - \frac{1}{3}\perp^{\mu\nu}\perp^{\alpha\beta}\right) T_{\mu\nu}, \tag{2.65}$$

with $T_{\mu\nu}$ here just denoting the total energy–momentum tensor for simplicity.

It is important to note that the conservation equations (2.61)–(2.62) are true at all orders. Evaluated at a FLRW background the first leads to the usual energy density conservation, while the second one is identically zero. To study linear perturbations the procedure is then similar to the component approach presented before; one chooses a gauge, and linearises the conservation equations,as well as the gravitational ones. What is more, in the context of scalar perturbations, vector and tensor quantities, like for example the momentum flux q_μ, can be expressed as the spatial gradient of a scalar function as

$$q_\mu = \perp^\kappa{}_\mu \nabla_\kappa q, \tag{2.66}$$

with $q \equiv q(t)$ a background function dependent on cosmic time t. A similar decomposition exists for the shear tensor $\tau_{\mu\nu}$. We will not elaborate more on this issue and we refer the reader to the work soon to appear in Sawicki et al. (2012).

Let us close the chapter with a little summary. We described the background gravitational equations in the presence of multiple perfect fluids and a dark component X for the homogeneous and isotropic FLRW metric. We then discussed how to perturb the background equations to describe small departures from homogeneity and isotropy, a procedure that leads to understanding gravitational instability and therefore formation of structures in the Universe. We described the different gauge choices and then presented the perturbation equations in the so-called Newtonian gauge, a gauge which is very useful when calculating observables, and we will also be using in the following.

Chapter 3

Non-linear gravity

3.1 Introduction

We discussed in chapter 1 that in order to model the dark energy various modifications of GR have been studied. In this chapter, we will focus on the case of non-linear gravity models. The latter, modify the standard GR action through the inclusion of non-linear curvature terms. These can be either non-linear functions of the Ricci scalar, or more general combinations of the Riemann tensor and the metric field. We can formally write down the most general action of this class of models in four dimensions as

$$S = \int d^4x\sqrt{-g}f(g_{\mu\nu}, R_{\alpha\beta\gamma\delta}) + S_{\rm m}(g_{\mu\nu}, \psi_i), \tag{3.1}$$

with $S_{\rm m}$ denoting collectively the action of any sort of matter field present.

From above action, one can construct infinitely many scalar combinations, through the operation of contraction. In principle, the equations of motion resulting through variation with respect to the metric, will be of fourth order (with respect to the metric) for this particular type of action, except for particular non-linear curvature combinations known as the *Lovelock scalars*, which have the property of retaining the second-order character of the equations of motion. The most famous of such a scalar is probably the Gauss–Bonnet term defined as

$$G \equiv R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\kappa\lambda}R_{\mu\nu\kappa\lambda}. \tag{3.2}$$

The Gauss-Bonnet term is the second order term in an infinite sum of curvature combinations introduced by Lovelock Lovelock (1971).

$$S = \int d^d x \sqrt{-g} R^{\gamma_1 \delta_1}_{\alpha_1 \beta_1} \dots R^{\gamma_n \delta_n}_{\alpha_n \beta_n} \delta^{\alpha_1 \beta_1 \dots \alpha_n \beta_n}_{\gamma_1 \delta_1 \dots \gamma_n \delta_n}, \tag{3.3}$$

where $\delta^{\alpha_1 \beta_1 \dots \alpha_n \beta_n}_{\gamma_1 \delta_1 \dots \gamma_n \delta_n}$ is the alternating tensor, which is antisymmetric with respect to the interchange of two neighboring indices, and n an integer number, $n = 0, 1, 2, \dots$, and d the spacetime dimensionality. Notice that the action integral is defined in d dimensions. For a given spacetime dimension d, not all terms of the infinite series in above action contribute to the equations of motion. The non vanishing components in the action are these that satisfy $n \leq d/2$. This is an immediate result of the antisymmetric properties of the alternating tensor. The terms that satisfy the inequality $n < d/2$, are non-zero and contribute to the equations of motion. On the other hand, the term with $n = d/2$ is non-zero, but is a total derivative, only yielding a surface term. The latter term starts to contribute at dimension $d + 1$.

In particular, for $d = 4$, we have $n \leq 2$. The term with $n = 1$ corresponds simply to the Ricci scalar, while the one with $n = 2$ to the Gauss–Bonnet one. One sees that the latter will be a total derivative in $d = 4$, and will only start to contribute in $d = 5$. The third order term, $n = 3$, is non-zero in $d = 6$ (total derivative) and starts contributing to the equations of motion from $d = 7$ and on. Notice that the cosmological constant corresponds to the trivial zeroth order term, $n = 0$, in above expansion.

Although the Gauss–Bonnet term is a total derivative in four dimensions, this is not true anymore if it is coupled to a scalar field, or enters the action in a non-linear form. The latter case will concern us in the following where we will study the so-called $f(R, G)$ models. In fact, as we will see in chapter 4 the two cases are related, i.e one can always re-express an $f(R, G)$ model in a scalar–tensor form. A Gauss–Bonnet term coupled to a scalar appears in the low energy limit of string theory, where the scalar field is the dilaton Zwiebach (1985).

It is interesting to note that action (3.3) can be written as a sum of a bulk and surface term, with a particular relation relating the two terms. This means that knowledge of the bulk term only is enough to determine the surface one Kolekar and Padmanabhan (2010). This is called the holographic property of the action. It is an important feature when studying gravity from a thermodynamical point of view (see Padmanabhan (2011) and references therein).

Let us now go back to the more general case of action (3.1). As mentioned before, this class of actions will give at most fourth order equations of motion for the metric field. This implies that the structure of these theories is going to be richer than that of GR. We will start by discussing the vacuum structure for the actions described by (3.1). Recall that in GR without a cosmological constant, the only vacuum solution is Minkowski spacetime, which is stable. (Anti-) de Sitter spacetime solutions are only possible through the addition of a cosmological constant, and in that case the solution reads as $R = \Lambda/4$. The situation is very different for the case of non-linear models (3.1). In general, their vacuum solutions include Minkowski spacetime as well as (anti-) de Sitter solutions, even in the absence of a cosmological constant. Vacuum solutions are characterised by a constant Ricci-curvature, i.e $R = R_0 = \textit{constant}$, and correspond to maximally symmetric spaces. Therefore, the Riemann tensor is given by

$$R_{\alpha\beta\gamma\delta} \equiv R_{0\alpha\beta\gamma\delta} = \frac{1}{12} R_0 \left(g_{\gamma\alpha} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \right), \tag{3.4}$$

with "0" denoting evaluation on the vacuum solution. Substituting above expression into the function "f", the latter becomes a function of Ricci scalar only, i.e $f = f(R_0$. Then, using the equations of motion resulting form action (3.1) after variation with respect to the metric, one can show that vacuum spacetimes are solutions of the following equation

$$R_0 f'(R_0) - 2f(R_0) = 0, \tag{3.5}$$

with $f'(R_0) \equiv df(R_0)/dR_0$. Similar equations to the above hold for the more particular cases of $f(g_{\mu\nu}, R_{\alpha\beta})$ and $f(R)$. Now, for different forms of the function "f" one can in principle find multiple (anti-) de Sitter solutions. Their stability can be found by performing a linear analysis around the background solution. Obviously, the existence of a stable de Sitter solution is the key in describing the dark energy problem in the context of non–linear gravity models.

The question that arises is, what the criteria are for a particular form of f to be acceptable. In order to answer this question we have first to define what we mean by "acceptable". From a mathematical point of view, it should be obvious that any scalar combination will be acceptable, since it will preserve general covariance (of course, general covariance has deep physical implications too). The restrictions should then come from the physics: a generally viable gravitational action should behave correctly both at classical and quantum level. Let us start with the classical one. There, one has to make sure the dynamics are free from: singularities and any sort of dynamical instabilities that could endanger the phenomenological viability of the theory. Furthermore, the theory should have the correct sequence of cosmological eras and the proper (post) Newtonian limit. A viable cosmological evolution should include an early de Sitter era (inflation), followed by a radiation and matter era respectively, leading to a late time acceleration period. The particular physics of each era pose each own restrictions to the model; during inflation the primordial density perturbations are generated, while during radiation the light elements form, also known as the Big Bang Nucleosynthesis (BBN). On the other hand, during matter domination, the large scale structures form. The late time de Sitter era correspond to the presently observed acceleration of the Universe. Consequently, the radiation and matter eras should be stable enough, to allow for the correct BBN and structure formation to occur. On the other hand, a crucial restriction at the quantum level is that the theory should be ghost-free and renormalisable. In particular, the presence of ghost-like degrees of freedom will violate unitarity of the theory.Furthermore, renormalisability ensures that calculation of observable quantities, like the mass or the charge of a particle, yields finite numbers.

From above discussion it turns out that it is important to understand the field content of non–linear gravity models. In GR, the only propagating degree of freedom is the massless, spin-two particle, the well known graviton. In models described by action (3.1), the situation is more complex. Following Hindawi et al. (1996) we can get an idea about the propagating fields in these theories by looking at the Cauchy problem of the theory. Since the theory is diffeomorphism invariant, the independent components of the metric can be reduced from ten to six, as in GR. However, the metric field satisfies fourth-order equations of motion [1], which means that initial data will require specification of the field itself as well as its first, second and third

[1] In fact, we can re-express the original fourth-order theory, in a dynamical equivalent fashion, as a second-order one through the introduction of auxiliary fields. This will be the subject of chapter 4.

derivative respectively. This makes up in total twelve degrees of freedom for the theory as defined in action (3.1). However, we know that six of these degrees of freedom should correspond to the two helicity states of the graviton. Therefore, the remaining six degrees of freedom should be attributed to some new fields present in the theory.

In order to understand the new propagating fields present in the theory, we can expand action (3.1) up to second order in curvature, around some vacuum solution characterized by $R = R_0 = constant$. The result is Chiba (2005a); Hindawi et al. (1996)

$$S = \frac{\beta}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}R_0 + \frac{1}{6m_0^2}R^2 - \frac{1}{m_2^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right], \tag{3.6}$$

with β, m_0^2, m_2^2 constants. To identify the field content of above action one can perform a linearised analysis to identify the field propagators, as was done in Stelle (1978); Nunez and Solganik (2005a). In any case, as a first step, it is useful to re-express it as a second-order one through the introduction of two new field variables, χ and $\pi^{\mu\nu}$, as was done for example in Hindawi et al. (1996); Chiba (2005a) and after a conformal redefinition of the metric $\tilde{g}_{\mu\nu} = e^{\chi}g_{\mu\nu}$ arrive at the dynamically equivalent action

$$S = \int d^4x\sqrt{-g}\left[\tilde{R}(\tilde{g}) - \frac{3}{2}(\tilde{\nabla}\chi)^2 - \frac{3}{2}m_0^2(1-e^{-\chi})^2 - \frac{1}{2}R_0e^{-2\chi} - \tilde{G}_{\mu\nu}\tilde{\pi}^{\mu\nu} + \frac{1}{4}m_2^2(\tilde{\pi}^{\mu\nu}\tilde{\pi}_{\mu\nu} - \tilde{\pi}^2)\right], \tag{3.7}$$

with $\tilde{G}_{\mu\nu}$ the Einstein tensor. We first notice that the new scalar χ is a canonical, massive scalar field. As for $\tilde{\pi}^{\mu\nu}$, it can be shown from the equations of motion for the metric field, that it satisfies a transverse, traceless condition; therefore it is a spin-two field. As a result, the last term in the action describes the interaction between two spin-two tensor fields, $\tilde{g}^{\mu\nu}$ and $\tilde{\pi}^{\mu\nu}$. The latter interaction violates the no–go theorem Aragone and Deser (1980); Boulanger et al. (2001) which states that the only possible ghost-free interaction between two spin-two fields is the massive Fierz–Pauli one [2]. This way, we find that $\tilde{\pi}^{\mu\nu}$ is a ghost field. From a calculation point of view one could see this explicitly by making a field redefinition in order to

[2] Remember that the mass term in the massive Fierz–Pauli action, in an expansion around flat spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, is $-\frac{1}{4}m^2\left(h^{\mu\nu}h_{\mu\nu} - h^2\right)$. For details see for example Ortín (2004).

diagonalise the kinetic terms of the spin two fields, and show that one of them will have the wrong sign.

We conclude that the field content of general theories described by (3.1) is a massless, spin two field (graviton), a massive spin-two ghost field, and a massive scalar.

However, it has been shown Navarro and Van Acoleyen (2006) that for the particular theories that admit the form

$$S = \int d^4x\sqrt{-g}f(R, R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu}R^{\mu\nu}), \tag{3.8}$$

the spin-two ghost field disappears, i.e $m_2^{-1} = 0$ in the action (3.6). To this special class of theories belong the cases of $f(R)$ and $f(R, G)$ gravities that will be studied in more detail in the following sections. The absence of the ghost spin-two tensor field makes these theories obviously more attractive. Notice however, that the massive, scalar degree of freedom still exists, and one has to make sure that it is not ghost-like or tachyonic, and it is also well "hidden" at solar system scales, where GR is expected to be recovered. The key point here, as was further showed in Navarro and Van Acoleyen (2006), for this type of theories the scalar mass acquires a dependence on the background curvature and therefore its effects are suppressed at high curvature environments like the solar system, while it becomes light at low curvatures, e.g large scales of the order H, and can act as a source of dark energy. The latter property is essentially the chameleon mechanism first suggested in Khoury and Weltman (2004). The idea behind it is that the mass of the scalar is dependent on the matter background density, making it very light at cosmological scales where the matter density is low, while at high density environments like the solar system, the scalar becomes very massive, and therefore effectively unobservable. The chameleon is not the only screening mechanism for scalar fields. The Vainshtein Vainshtein (1972); Deffayet et al. (2002) as well as the symmetron Hinterbichler et al. (2011) mechanism have been also suggested as ways to screen a scalar at high density regions. The former becomes significant in the presence of derivative self couplings of the scalar, which dominate in high density regions, while the latter is based on making the vacuum expectation value of the scalar dependent on the environment mass density.

As mentioned earlier, a viable gravity model should reproduce the correct background cosmological evolution. The analysis in Carroll et al. (2005) is one of the first attempts to study the (late) cosmology of general models of the type

$$f = \frac{\mu^{4n+2}}{(aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda})^n}, \tag{3.9}$$

with a, b, c dimensionless constants and $n > 0$. Notice that "f" acquires a large value at sufficiently small curvatures, which is the reason it makes it interesting for describing the late time acceleration of the Universe. It was found that models of this type poses an unstable late time de Sitter solution, as well as other power law attractors that could account for the observed late time acceleration, when the mass scale $\mu \sim H_0$. Probably the most well studied models in the literature have been the $f(R)$ and $R + f(G)$ ones, that will be described in more detail in the next sections. However, it is worth mentioning here that it has been found that by a suitable choice of the function f, models of this type have the correct sequence of cosmological eras, from scaling solutions to a late time, stable dS attractor Amendola et al. (2007b); Zhou et al. (2009). Furthermore, in a similar context, early and late time acceleration unifying models have been also investigated in detail in Nojiri and Odintsov (2011). In the following, we will mainly focus on $f(R)$ and $f(R, G)$ models, which are the cases that have been widely studied in the literature, as they are free from the unwanted, ghost spin-two field.

The behavior of $f(R)$ and $f(R, G)$ models has been also well studied at the linear perturbation level, where they pose some distinct signatures that distinguish them from ΛCDM or scalar field models like quintessence or k–essence. Both class of models enhance the growth of structure, by modifying the matter power spectrum Gannouji et al. (2009); Bean et al. (2007); Tsujikawa (2008a); Pogosian and Silvestri (2008); De Felice and Suyama (2011a). However, the characteristic signature of non–linear gravity models in general, is the existence of an effective anisotropic stress, which is a key observational quantity that can be extracted from combining galaxy clustering with weak lensing surveys Amendola et al. (2008); Saltas and Kunz (2011). Furthermore, because of the fourth order nature of non–linear gravity, one expects in principle to find superluminal modes at the linear perturbation level, as was shown for general $f(R, G)$ gravity in De Felice and Suyama (2009). The special case of

$R + f(G)$ models has been ruled out, due to a singularity present at the linear level in the presence of matter De Felice et al. (2010b).

Non–linear gravity models suffer in principle from various types of early or late singularities, which are absent in GR. In particular, for $f(R)$ gravity an early time curvature singularity was found in Starobinsky (2007) and explained in Frolov (2008), while other types of singularities have been studied in Abdalla et al. (2005).

In principle, one expects that non–linear modifications of the GR action will yield corrections to the corresponding solar system limit, which is what is found in practice. For the case of the models described by (3.9), it was found in Navarro and Van Acoleyen (2006) that they poses an acceptable Newtonian limit at small distance from the source, but they can have observationally significant effects at galactic scales. In particular, for the models given by (3.8), in Navarro and Van Acoleyen (2006) it was found that the scalar degree of freedom present in the theory acquires a dependence on the environment curvature, and effectively decouples in the vicinity of a matter source like a star so that solar system tests are successfully passed. The latter behavior is the so-called chameleon mechanism, and is also present in the special case of $f(R)$ models. For the case of $R + f(G)$ models it was shown that they can accept a viable solar system limit upon a suitable choice of the function f Davis (2007a); De Felice and Tsujikawa (2009c).

Let us close our review of non–linear gravity by mentioning a special type of non–linear gravity models, the so called conformal gravity. As its name declares, the special thing about this type of gravity is that it is invariant under conformal transformations of the metric. More precisely, invariance here means that both action and equations of motion will be unaffected after conformally redefining the metric field.

The form of a conformally invariant action for gravity depends on the spacetime dimensionality. The very first hint for this is given by the transformation of the action measure under a conformal transformation $g_{\alpha\beta} \to \Omega^2(x^\mu)g_{\alpha\beta}$,

$$\int_V d^n x\sqrt{-g}\mathcal{L} \to \int_V d^n x\sqrt{-\tilde{g}}\Omega^{-n}\tilde{\mathcal{L}}, \tag{3.10}$$

which implies that $\mathcal{L}$ should transform appropriately if the action has to be conformally invariant. In four dimensions, the action for conformal gravity is

$$S = \alpha_g \int_V d^4x \sqrt{-g} C^{\alpha}{}_{\beta\gamma\delta} C_{\alpha}{}^{\beta\gamma\delta}, \tag{3.11}$$

with $C^{\alpha}{}_{\beta\gamma\delta}$ the Weyl tensor, and α_g a dimensionless coupling. The Weyl tensor is the trace part of the Riemann tensor, and is defined as

$$C^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta\gamma\delta} - \frac{2}{n-2}\left(g^{\alpha}{}_{[\gamma} R_{\delta]\beta} - g_{\beta[\gamma} R_{\delta]}{}^{\alpha}\right) + \frac{2}{(n-1)(n-2)} R g^{\alpha}{}_{[\gamma} g_{\delta]\beta}. \tag{3.12}$$

Notice that the Weyl tensor is conformally invariant when in the form with one raised index and the rest being low, i.e $C^{\alpha}{}_{\beta\gamma\delta}$, while it possesses the same symmetries with the Riemann tensor. Expressing the Weyl tensor in terms of the Riemann tensor and its contractions, action (3.11) can be expressed in the equivalent form

$$S = 2\alpha_g \int_V d^4x \sqrt{-g} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2\right), \tag{3.13}$$

where we dropped the Gauss–Bonnet term, as in four dimensions it only contributes a surface term. The second form of the action makes obvious the connection with the non–linear gravity theories described before.

In four dimensions, cosmological and astrophysical aspects of conformal gravity have been studied. In particular, in this context, galactic rotation curves have been successfully fitted, see for example Mannheim and O'Brien (2010); O'Brien and Mannheim (2011); Mannheim (2012) and references therein. On the cosmology side, it has been claimed that the cosmological problem accepts a solution in view of a conformally invariant action proportional to the square of the Weyl term, Mannheim (2012) and references therein, while for a study of the unitarity of the theory one can look at reference Bender and Mannheim (2008).

We can find a generalisation of the conformally invariant gravity action in any dimensions using as our starting point the Lovelock action (3.3). The flow of thinking is as follows: the action that has to be constructed should consist of some product of Weyl tensors. What is more, the Weyl tensor has the same symmetries under index permutations as the Riemann tensor. Therefore, we can try to use the same form with the Lovelock action, but with the Riemann tensor substituted by the Weyl one.

Therefore, we can attempt to write a conformally invariant action in n dimensions as

$$S = \int_V \sqrt{-g}\, \delta^{135\ldots 2n-1}_{246\ldots 2n} C^{24}_{13} C^{68}_{57} \ldots C^{2n-2\;2n}_{2n-3\;2n-1}, \tag{3.14}$$

with $1, 2 \ldots \equiv \mu_1, \mu_2, \ldots$ for simplicity and $\delta^{135\ldots 2n-1}_{246\ldots 2n}$ denotes as in the Lovelock case the antisymmetric tensor. Above action is indeed the correct action in n dimensions, and it has been also used in Deser and Schwimmer (1993) in an analysis of conformal anomalies in (even) arbitrary dimensions.

In this section, we reviewed some fundamental properties of non–linear gravity actions. In the following, we will focus on the special class of actions proportional to $f(R)$, and $f(R, G)$ respectively. For detailed reviews on modified gravity models the reader is referred to Sotiriou and Faraoni (2010); De Felice and Tsujikawa (2010); Clifton et al. (2012); Capozziello and Francaviglia (2008); Durrer and Maartens (2008); Nojiri and Odintsov (2006a).

3.2 $f(R)$ gravity

3.2.1 Equations of motion and dynamics

$f(R)$ gravity is the simplest case of the general action (3.1). It modifies GR by promoting its action to an arbitrary function of the Ricci scalar. One of the first studies of $f(R)$ theory was its application to describe primordial inflation Starobinsky (1980a), and after then it has been also suggested as a candidate for the late time acceleration of the Universe. Different formulations of $f(R)$ gravity have been studied in the literature, namely the metric Buchdahl (1970), the Palatini Buchdahl (1970) and the metric–affine formalism Sotiriou and Liberati (2007a,b). It is important to note that although for an action linear to the curvature R, the metric and Palatini formalism yield the same set of equations of motion, this is not true for the non-linear case of $f(R)$ theories. In this thesis, we will be interested in the metric formalism, where the only independent field variable is the metric $g_{\alpha\beta}$, and the equations of motion are of fourth–order. Below, we will first derive the equations of motion and then we will discuss the background dynamics and reconstruction, as well as the stability conditions for metric $f(R)$ models.

The action for $f(R)$ gravity in four dimensions reads as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) + S_{\rm m}(g_{\mu\nu},\psi_i), \tag{3.15}$$

where $S_{\rm m}$ denotes collectively all matter fields present.

Let us for the sake of illustration derive the equations of motion for above action explicitly by varying it with respect to the metric field. Variation of the matter part will yield the matter energy-momentum tensor,

$$T_{\alpha\beta} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta g^{\alpha\beta}}. \tag{3.16}$$

Varying the gravitational part, omitting for the moment the factor $1/\kappa^2$, we get,

$$\delta S_{f(R)} = \int d^4x\sqrt{-g}\left(-\frac{1}{2}f(R)g_{\alpha\beta}\delta g^{\alpha\beta} + f_R\delta R\right), \tag{3.17}$$

with $f_R(R) \equiv df(R)/dR$. For the second term in the variation we have,

$$\begin{aligned}
&\int d^4x\sqrt{-g}f_R\delta R \\
&= \int d^4x\sqrt{-g}f_R(R)\left[R_{\alpha\beta}\delta g^{\alpha\beta} + \nabla_\rho(g^{\alpha\beta}\delta\Gamma^\rho_{\alpha\beta} - g^{\alpha\rho}\delta\Gamma^\mu_{\alpha\mu})\right] \\
&\equiv C_1 + C_2.
\end{aligned} \tag{3.18}$$

Evaluating B by integration by parts we get,

$$\begin{aligned}
C_2 = &\int_{\partial\Sigma} d^3x\sqrt{-h}f_R(R)\left(g^{\alpha\beta}\delta\Gamma^\rho_{\alpha\beta} - g^{\alpha\rho}\delta\Gamma^\mu_{\alpha\mu}\right)n_\rho \\
&- \int_\Sigma d^4x\sqrt{-g}\left(g^{\alpha\beta}\delta\Gamma^\rho_{\alpha\beta} - g^{\alpha\rho}\delta\Gamma^\mu_{\alpha\mu}\right)\nabla_\rho f_R(R) \\
&\equiv C_3 - C_4,
\end{aligned} \tag{3.19}$$

with $\partial\Sigma$ denoting the boundary of the four-dimensional manifold Σ. The first term (C) is a surface term and won't contribute to the equations of motion. In the case the boundary is taken to be infinity, the surface term C_3 can be assumed to vanish, i.e as $\partial\Sigma \to \infty$, $C_3 \to 0$.

On the other hand, evaluating the Christoffel symbol variation in the bulk term

(C_4), and using integration by parts again we arrive at

$$C_4 = -\int_M dM\sqrt{-g}\left[\left(\nabla^\kappa\nabla^\alpha f_R(R)\right)\delta g_{\kappa\alpha} + \left(\nabla^\kappa\nabla_\kappa f_R(R)\right)g^{\alpha\beta}\delta g_{\alpha\beta}\right]. \tag{3.20}$$

Plugging above relation in the original action variation, while keeping only terms integrated along the bulk Σ and requiring the variation to be zero we arrive at the equations of motion,

$$G_{\alpha\beta} = \frac{\kappa^2}{f_R(R)}T_{\alpha\beta} + \frac{1}{f_R(R)}\left[\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta}\right)\nabla^\mu\nabla^\nu f_R(R) + \frac{1}{2}g_{\alpha\beta}\left(f(R) - Rf_R(R)\right)\right]. \tag{3.21}$$

Let us come back to the surface term (C_3) derived in (3.19). After integrating by parts and using the definition of the extrinsic curvature (A.7) it takes the form

$$C_3 = 2\int_{\partial\Sigma} d^3x\sqrt{-h}Kf_R(R), \tag{3.22}$$

where $K \equiv K^\alpha_\alpha$ is the trace of the extrinsic curvature tensor and $h \equiv h^\kappa{}_\kappa$ is the trace of the metric on the boundary surface $\partial\Sigma$, also known as the "projection operator". It reduces to the surface term encountered in GR, for $f_R = 1$.

For a spacetime with a boundary, the surface term (3.22) has to be cancelled in a formal way, by introducing a suitable counter term in the original action. In GR, the form of the appropriate term, also known as Gibbons–Hawking term Gibbons and Hawking (1977), is that of (3.22) with $f_R = 1$, which after variation with respect to the metric, will cancel the normal derivatives of the metric field on the boundary surface. However, in the case of $f(R)$ gravity, in principle there does not exist such a term. We will come back to this issue in chapter 4 where we will discuss the dynamical equivalence between different action representations, and will see that in contrary to the original representation of $f(R)$ gravity described by (3.15), the corresponding Jordan and Einstein frame representations always possess a Gibbons–Hawking term.

What does the equation of motion (3.21) tell us about $f(R)$ gravity? The first thing to note, in contrast to GR, is that the equation is fourth order with respect to the metric field. More precisely, it includes second derivatives of $f_R(R)$, and the latter in turn hides second derivatives of the metric field. This implies that the range of solutions will be larger than in GR.

Let us take the trace of (3.21),

$$\Box f_R(R) - \frac{dV_{f(R)}}{dR} = \frac{\kappa^2}{3} T, \tag{3.23}$$

with $T \equiv g^{\alpha\beta} T_{\alpha\beta}$, and

$$\frac{dV_{f(R)}}{dR} \equiv \frac{1}{3}\left(2f(R) - R f_R(R)\right). \tag{3.24}$$

Above equation is a Klein–Gordon equation for $f_R(R)$, sourced by a scalar potential of gravitational origin and the trace of the matter energy-momentum tensor. This tells us that in $f(R)$ gravity there is an extra, massive scalar degree of freedom propagating, apart from the massless spin-two tensor field, the well known graviton. In the original $f(R)$ action (3.15) the scalar degree of freedom, described by $f_R(R)$, has pure gravitational origin, however it can exist as an independent, true scalar field in the Jordan or Einstein frame representation, as we will see later. In the literature the scalar degree of freedom of $f(R)$ gravity has been dubbed as "scalaron".

We can define the effective mass of the scalaron as

$$m_{\text{eff}}^2 \equiv \frac{d^2 V_{f(R)}}{dR^2} \equiv \frac{1}{3}\frac{f_R - R f_{RR}}{f_{RR}}. \tag{3.25}$$

Looking at the equation of motion (3.21), we see that we have moved the extra part coming from the modification of the action to the r.h.s, while keeping GR on the l.h.s, interpreting this way the gravitational modification as an effective energy-momentum tensor, which we will denote as ${T_{(\text{eff})}}^{\alpha}{}_{\beta}$. Notice that it corresponds to ${T_{(\text{X})}}^{\alpha}{}_{\beta}$ of the previous chapters.

The degrees of freedom in metric $f(R)$ gravity should satisfy a set of stability conditions if the theory has to be viable. What are they? To answer this question, let us first re-express the original action in the so-callled Jordan frame through the introduction of a scalar field ϕ

$$S_J = \int \sqrt{-g}\left[f_\phi(\phi) R - (\phi f_\phi(\phi) - f(\phi))\right] + S_{\text{m}}(g_{\mu\nu}, \psi), \tag{3.26}$$

with $f_\phi(\phi \equiv df(\phi)/d\phi$, and original action is recovered for $\phi = R$. We will not get into more details on the transition to the Jordan frame action, since a more detailed study on the subject will be made in a next chapter.

Now, looking at (3.26) we see that the Ricci scalar R has the correct sign, i.e graviton is not a ghost, if $f_\phi(\phi) > 0$. In the original representation therefore, graviton is not a ghost for $f_R(R) > 0$. Furthermore, the scalar f_R is not a tachyon for $m^2_{\text{eff}} > 0$. The latter condition is also required for the stability of de Sitter space Faraoni and Nadeau (2005).

3.2.2 Cosmological evolution and reconstruction

$f(R)$ models modify the cosmological equations by introducing new, fourth order terms with respect to the metric. In equation (3.21) we have moved them on the r.h.s, interpreting them this way as an effective fluid, of gravitational origin. The $0-0$ and $i-i$ components of (3.21) evaluated on a flat FLRW background read as

$$H^2 = \frac{\kappa^2}{3F}(\rho_{\text{b}} + \rho_{\text{m}}) + \frac{1}{3F}\left[\frac{1}{2}(FR - f) - 3H\dot{F}\right], \tag{3.27}$$

$$2\dot{H} - 3H^2 = \frac{\kappa^2}{F}(w_{\text{b}}\rho_{\text{b}} + w_{\text{m}}\rho_{\text{m}}) + \frac{1}{F}\left[\ddot{F} + 3H\dot{F} + \frac{1}{2}(f - FR)\right], \tag{3.28}$$

where we denote $f_R \equiv F$, $f(R) \equiv f$ for simplicity, and dots denote derivative with respect to cosmic time t. It is trivial therefore to extract the effective energy density and pressure for $f(R)$ gravity,

$$\rho_{\text{eff}} \equiv \frac{1}{3F}\left[\frac{1}{2}(FR - f) - 3H\dot{F}\right], \tag{3.29}$$

$$p_{\text{eff}} \equiv \frac{1}{F}\left[\ddot{F} + 3H\dot{F} + \frac{1}{2}(f - FR)\right]. \tag{3.30}$$

In the standard way, one can define an index w_{eff} for the effective fluid as

$$w_{\text{eff}} \equiv \frac{p_{\text{eff}}}{\rho_{\text{eff}}}. \tag{3.31}$$

Today, w_{eff} should be close to -1, i.e mimic a cosmological constant at the background. A reconstruction method for constructing $f(R)$ models given a background evolution, the so-called "designer method", will be described explicitly later.

Cosmological evolution of $f(R)$ models by means of a dynamical system analysis has been studied in Amendola et al. (2007b). It has been shown that a variety of cosmological fixed points exist, and upon a suitable choice of the function "f" one can achieve a viable evolution from a radiation/matter domination period to a late time de Sitter attractor.

Let us know describe the so called "designer $f(R)$" method, i.e how to reconstruct, at least numerically, an $f(R)$ model given a background expansion history $H = H(t)$. Because, of the higher order nature of $f(R)$ gravity, there are multiple forms of $f(R)$ able to reproduce a given background expansion. The degeneracy is only uplifted at the perturbation level.

We begin by parametrising the Friedmann equation as

$$H^2 = \frac{\kappa^2}{3}(\rho + \rho_X), \tag{3.32}$$

with ρ_X stands for the energy density of dark energy. Then, we define the following dimensionless quantities as

$$y = \frac{f(R)}{H_0^2},\ E = \frac{H^2}{H_0^2}, \tag{3.33}$$

with H_0 the value of the Hubble parameter today. A simple parametrisation for the quantity E is

$$E = (1 - \Omega_{DE})a^{-3} + \Omega a^{-3(1+w)}, \tag{3.34}$$

neglecting radiation and assuming constant equation of state for the dark energy fluid. This could be generalized to account for $w = w(a)$ or include radiation as well. The evolution of the Ricci scalar is given by

$$R = 6(\dot{H} + 2H^2). \tag{3.35}$$

Using the dimensionless quantity E we can write,

$$\frac{R}{H_0^2} = 3(E' + 4E),\quad \frac{R'}{H_0^2} = 3(E'' + 4E'),\quad \frac{d}{dt} = H\frac{d}{d\ln a}, \tag{3.36}$$

and

$$\frac{\partial f}{\partial R} \equiv F = \frac{\partial f}{\partial \ln a}\frac{\partial \ln a}{\partial R} = \frac{f'}{R'} = \frac{1}{3}\left(\frac{y'}{4E' + E''}\right). \tag{3.37}$$

Here, we use the following notation for derivatives: $' \equiv \frac{d}{\ln a} = a\frac{d}{da}$ and $\dot{} \equiv \frac{d}{dt}$.

Using above relations and after a bit of algebra, we can re-express the Friedmann equation 3.27 as,

$$y'' - \left[1 + \frac{1}{2}\frac{E'}{E} + \frac{4E'' + E^{(3)}}{(4E' + E'')^2}\right] y' + \frac{1}{2}\left(\frac{4E' + E''}{E}\right) y = -\frac{\kappa^2}{H_0^2}\rho_X \left(\frac{4E' + E''}{E}\right). \tag{3.38}$$

The second order nature of above differential equation is related to the fourth order nature of $f(R)$ gravity. Being a second order differential equation, it allows for a family of solutions depending on the initial conditions chosen, i.e there is not a unique $f(R)$ model corresponding to a given expansion.

As it was shown in Hu and Sawicki (2007), if $y_\pm$ correspond to the two solutions of the homogeneous part of (3.38) at the high curvature regime, with $y_\pm \propto a^{p_\pm}$, then

$$p_\pm = \frac{-7 \pm \sqrt{73}}{4}. \tag{3.39}$$

The p_- branch will violate the requirement that the $f(R)$ model will mimic GR at high curvature, and its amplitude is set to zero. Then, at the high curvature regime a particular solution of the full equation was shown to be

$$y_{part} = \frac{6\Omega_{DE}}{6w^2 + 5w - 2} a^{-3(1+w)}, \tag{3.40}$$

with w constant. Therefore, the initial conditions at some initial time a_i at the high curvature era are given by

$$y(\ln a_i) = Ay_+(\ln a_i) + y_{part}(\ln a_i), \tag{3.41}$$

$$y'(\ln a_i) = Ap_+ y_+(\ln a_i) - 3(1+w) y_{part}(\ln a_i). \tag{3.42}$$

Different values of the constant A, will yield different $f(R)$ models with the same expansion history. The degeneracy is only uplifted at the linear perturbation level (see for example Hu and Sawicki (2007)).

At the level of linear, scalar perturbations around FLRW, $f(R)$ models are characterised by two main regimes, namely the regime where $\lambda_{f_R}\ lla^2/k^2$ and $\lambda_{f_R} \gg a^2/k^2$ respectively. λ_{f_R} denotes the Compton wavelength of the scalaron and k is the wave number. The first regime corresponds to modes outside the scalaron range, and in this regime the growth of structure evolves as in GR, i.e $\delta_{\rm m} \propto a \propto t^{2/3}$ during matter domination. In the second regime, the growth of structure is enhanced, and grows like $\delta_{\rm m} \propto t^{(\sqrt{33}-1)/6}$. Since the scalaron Compton radius evolves as $\lambda_{f_R} \propto t^{4(n+1)}$, a mode which initially lies in the first regime can enter in the second one after some time during matter domination. The calculation of observationally relevant quantities like the matter power spectrum and the ISW effect on the CMB have been worked out in Tsujikawa (2008b).

It is important to note that, $f(R)$ models do not modify weak lensing explicitly, i.e the weak lensing equations has the same form as in GR, and the modifications enter implicitly through the different evolution of the Newtonian potentials Φ and Ψ. [3] In the Σ and Q language of Amendola et al. (2008) this means that for $f(R)$ gravity $Q \neq 1$ and $\Sigma = 1$.

3.3 $f(R,G)$ gravity

3.3.1 Equations of motion

In this section we will be concerned with a more general class of non-linear gravity models, these that are a function of both Ricci scalar R and Gauss–Bonnet term G. They are described by the following action

$$S = \int_M d^4x\sqrt{-g}\left[f(R,G) + L_{\rm matter}\right], \tag{3.43}$$

where R and G are the Ricci and Gauss–Bonnet scalar respectively, M denotes the four dimensional spacetime, and $L_{\rm matter}$ is the Lagrangian for any matter fields or fluids present.The form of the function f is constrained by both classical and quantum stability requirements, as well as agreement with large scale and solar

[3] That should be expected, since they are equivalent to Einstein gravity plus a canonical scalar field non–minimally coupled to matter.

system data. (We will revisit this point in Section 5.3.) In the following, we will work in natural units where $G = c = 1$, unless otherwise stated.

The Gauss–Bonnet term was introduced and described in section 3.1, but let us recall its definition for convenience,

$$G \equiv R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\kappa\lambda}R_{\mu\nu\kappa\lambda}. \tag{3.44}$$

The curvature scalars R and G are both functions of the metric and its derivatives, however they enter the function f as independent degrees of freedom, in the sense that the dependence of f on them is in principle arbitrary. Varying action (3.43) with respect to the metric $g^{\mu\nu}$, and using the Bianchi identities, we get the equations of motion Carroll et al. (2005); De Felice and Suyama (2009)

$$FG_{\mu\nu} = T^{(\mathrm{matter})}_{\mu\nu} + T^{(\mathrm{eff})}_{\mu\nu}, \tag{3.45}$$

where $T^{(\mathrm{eff})}_{\mu\nu}$ the effective energy-momentum tensor for $f(R, G)$ gravity defined as

$$\begin{aligned} T^{(\mathrm{eff})}_{\mu\nu} \equiv \Big(& \nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F + 2R\nabla_\mu\nabla_\nu\xi - 2g_{\mu\nu}R\Box\xi - 8R_{(\mu}{}^{\kappa}\nabla_\kappa\nabla_{\nu)}\xi + 4R_{\mu\nu}\Box\xi \\ & + 4g_{\mu\nu}R^{\kappa\lambda}\nabla_\kappa\nabla_\lambda\xi + 4R_{\mu\kappa\lambda\nu}\nabla^\kappa\nabla^\lambda\xi - \frac{1}{2}g_{\mu\nu}V(R, G)\Big), \end{aligned} \tag{3.46}$$

and we used the additional definitions [4]

$$F \equiv f_R \frac{\partial f(R, G)}{\partial R}, \tag{3.47}$$

$$\xi \equiv f_G \equiv \frac{\partial f(R, G)}{\partial G}, \tag{3.48}$$

$$V(R, G) \equiv RF + \xi G - f(R, G). \tag{3.49}$$

Taking the limits $\xi \to 0$ and $F \to 1$ in (3.45), we recover the $f(R)$ and $R + f(G)$ equations of motion respectively. We chose here to bring all the non-GR gravitational contributions to the l.h.s of the equations of motion, and treat them as an effective energy-momentum tensor. However, this choice is rather a matter of convenience.

[4] Here we follow the notation of De Felice and Suyama (2009).

Looking at (3.45) one can see that the equations of motion of $f(R,G)$ gravity will be of fourth-order with respect to the metric field, as was expected. For their trace, we get

$$\left(3g^{\mu\nu}\frac{\partial}{\partial\lambda} - 4G^{\mu\nu}\frac{\partial}{\partial\sigma}\right)\nabla_\mu\nabla_\nu f(\lambda,\sigma) + (FR + 2\xi G - 2f) = T^{\kappa}{}_{\kappa}^{(\mathrm{matter})}. \tag{3.50}$$

For the case where the Ricci and the Gauss–Bonnet scalar enter in a particular combination in f through some function $\Omega \equiv \Omega(R,G)$, and $f(R,G) = f(\Omega)$ then above equation reduces to an evolution equation for the single scalar $\partial f(\Omega)/\partial\Omega$.

In contrast to $f(R)$ gravity, which can be expressed as a scalar–tensor theory through the introduction of an auxiliary scalar, we expect that for the case of $f(R,G)$ models two scalars will be needed, one corresponding to the Ricci and the other to the Gauss–Bonnet scalar respectively. Following De Felice and Suyama (2009) we can introduce two scalar fields λ and σ and re-express the action (3.43) as

$$S = \int_M d^4x\sqrt{-g}\left[RF(\lambda,\sigma) + G\xi(\lambda,\sigma) - V(\lambda,\sigma) + L_{\mathrm{matter}}\right]. \tag{3.51}$$

Varying above action with respect to λ and σ we get the following equations of motion for the scalar fields

$$(R-\lambda)F_\lambda + (G-\sigma)F_\sigma = 0, \tag{3.52}$$

$$(R-\lambda)F_\sigma + (G-\sigma)\xi_\sigma = 0, \tag{3.53}$$

with $F_\lambda \equiv \partial F/\partial\lambda$, and so on. Above system of equations admits the solution

$$\lambda = R, \quad \sigma = G, \tag{3.54}$$

which can be plagued into (3.51) to recover (3.43). The equivalence holds also at the level of the equations of motion through (3.54).

Equations 3.52 and 3.52 are independent from each other only if

$$F_\lambda\xi_\sigma - F_\sigma^2 \neq 0. \tag{3.55}$$

In the case condition (3.55) is satisfied the two scalar degrees of freedom λ and σ

are independent from each other, while in the opposite case they are not. Condition (3.55) plays a key role at the stability of linear perturbations of $f(R,G)$ gravity as we will discuss later in this section. One can notice that for the special cases of $f(R)$ and $R+f(G)$ models condition (3.55) is not satisfied.

3.3.2 Cosmological evolution and stability

We can compute effective fluid quantities for $f(R,G)$ gravity, as we did for the $f(R)$ case. We have to keep in mind in this case too that $T_{\mu\nu}^{(\text{eff,total})}$ although covariantly conserved, is not an energy-momentum tensor in the usual sense, since it is a function of the spacetime geometry and its first and second derivatives.

In the following, as usual we will be interested in homogeneous, isotropic and flat cosmologies, described by the flat, four dimensional FLRW metric

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2, \tag{3.56}$$

with $a(t)$ the scale factor. In this background, the two key quantities, R and G, can be expressed purely as a function of the Hubble parameter $H \equiv H(t)$ and its time derivative,

$$R(t) = 6\left(2H^2 + \dot{H}\right), \tag{3.57}$$

$$G(t) = 24H^2\left(H^2 + \dot{H}\right). \tag{3.58}$$

The t–t component of the $f(R,G)$ equations of motion (3.45) gives a modified version of the usual Friedman equation which reads as

$$3H^2 = \frac{1}{F}T^{(\text{mat})0}{}_0 + T^{(\text{eff})0}{}_0, \tag{3.59}$$

$$2\dot{H} - 3H^2 = \frac{1}{F}T^{(\text{mat})i}{}_j + T^{(\text{eff})i}{}_j. \tag{3.60}$$

with the effective "fluid" components defined as

$$T^{(\text{eff})0}{}_0 \equiv \rho_{\text{eff}} \equiv \frac{1}{F}\left(-H\dot{F} - 4H^3\dot{\xi} + \frac{1}{6}V\right), \tag{3.61}$$

$$T^{(\text{eff})i}{}_j \equiv p_{\text{eff}} \equiv \frac{1}{F}\left(\ddot{F} + 4H^2\ddot{\xi} + 2H\dot{F} + 8H\frac{\ddot{a}}{a}\dot{\xi} - \frac{1}{2}V\right), \tag{3.62}$$

with dots denoting differentiation with respect to cosmic time t. Notice that the above equation is of fourth order with respect to the scale factor, in contrast to the usual Friedman equation.

We saw previously that we can reconstruct an $f(R)$ model, in a way that it reproduces a given background expansion using the "designer method". For consistency, in this subsection we will extend this method for the case of $f(R, G)$ models. Remember that as we noticed for $f(R)$, there was in fact a class of $f(R)$ models, parametrised by a single parameter, that was able to reproduce a given expansion history. We expect the same to be true for the $f(R, G)$ case, and what is more in this case there is an additional freedom due to the different combinations between the curvature scalars R and G.

We can start by proceeding in a similar way as in $f(R)$ gravity, by defining the following dimensionless quantities in terms of the "dimensionless background" $E \equiv E(a)$,

$$\tilde{G} \equiv \frac{G}{H_0^4} = 12(EE' + E^2), \tag{3.63}$$

$$\xi \equiv \frac{\partial f}{\partial G} = \frac{f'}{G'}, \quad F \equiv \frac{\partial f}{\partial R} = \frac{f'}{R'}. \tag{3.64}$$

Then, after some algebra, the Friedmann equation (3.59) can be re-written as

$$E_1 y'' + E_2 y' + \frac{1}{2} y = -\frac{\rho_m}{H_0^2}, \tag{3.65}$$

with the functions E_1 and E_2 defined as

$$E_1 = E\left(\frac{E}{EE'' + (E')^2 + 4EE'} + \frac{1}{E'' + 4E'}\right), \tag{3.66}$$

$$\begin{aligned} E_2 =& \frac{1}{2}\left[\frac{2E}{E'' + 4E'} - \frac{E\left(E' + 2E\right)}{E\left(E'' + 4E'\right) + E'^2} - \frac{E' + 4E}{E'' + 4E'} - \frac{2E\left(E^{(3)} + 4E''\right)}{\left(E'' + 4E'\right)^2}\right. \\ &\left. - \frac{2E^2\left(4E'^2 + E\left(E^{(3)} + 4E''\right) + 3E'E''\right)}{\left(E\left(E'' + 4E'\right) + E'^2\right)^2}\right]. \end{aligned} \tag{3.67}$$

For a given background expansion, the solution $y = y(t)$ provides us with no information on the dependence of the function $f(R,G)$ on R and G respectively, unless we make from the start some ansatz, e.g $f(R,G) = f(R+G/M^2)$. Therefore, in principle, and in contrast to the case of $f(R)$ gravity, in this case there is an extra degeneracy when trying to reconstruct the function f, coming from the different ways R and G can be combined in f.

To set the initial conditions, we work as in $f(R)$ gravity, by choosing a time at the high curvature regime, where the dark energy contribution is negligible and $y \propto a^{p_\pm}$. We find that

$$p_\pm = \frac{-1 \pm 12}{4}, \tag{3.68}$$

with only acceptable solution the positive branch, $p_+ = 11/4$.

The background dynamics of $f(R,G)$ gravity have been studied extensively only for the particular case of $R+f(G)$, first in Li et al. (2007) where it was claimed that these models cannot reproduce arbitrary background expansion histories, because of the change of sign of the Gauss–Bonnet term from positive on negative at a particular time of the cosmological evolution. However, in Zhou et al. (2009) the background evolution for these models was revisited by means of a phase space analysis, and the problem pointed out in Li et al. (2007) was cured by replacing G with its absolute value. It was found that they exhibit rich phase space dynamics, and in particular they possess de Sitter, radiation as well as matter domination cosmological solutions. There, it was also found that upon a suitable choice of the function $f(G)$, a viable cosmological evolution from radiation domination to late time acceleration can be obtained. In De Felice and Tsujikawa (2009a) the conditions for cosmological viability of $f(G)$ models was studied, where it was found that a stability of a radiation/matter domination as well as a de Sitter era require that $d^2 f(G)/d^2 G > 0$.

At the linear level, it was shown in De Felice et al. (2010b), that for $f(G)$ models in the presence of a perfect fluid, the density perturbations of the latter exhibit an UV instability irrespective of the form of the function $f(G)$, rendering effectively these models incompatible with large structure observations. The linear scalar perturbations around an FLRW background for the more general $f(R,G)$ models was studied in De Felice and Suyama (2009) for the vacuum case. There, it was found that for these models a new instability can arise, associated with the group velocity of short-wavelength modes depending linearly on the wave number k, yielding an in principle superluminal propagation for these modes. The latter instability was shown to persist in the presence of a perfect fluid, in De Felice et al. (2010a). It is interesting to note that the particular subclasses of $f(R, G)$ models, the $f(R)$ and $f(G)$ ones, do not share this instability.

Chapter 4

Dynamical equivalence of non-linear gravity models

4.1 Introduction

In this chapter, we shall be closely following the collaborative work with M. Hindmarsh reported in Saltas and Hindmarsh (2011).

The topic we will be elaborating on below regards a fundamental aspect of non–linear gravity actions, namely their dynamical equivalence to different representations, an issue that has been the subject of research and intense debate in the literature from the early days of non–linear gravity.

What the term “dynamical equivalence” means is that a particular gravitational action can be re-expressed as a new one, with a new set of field variables, and that there is an invertible mapping that relates the two sets of field variables, as well as the two actions (or Lagrangians) respectively. The variational principle of the new action will in principle require different boundary conditions and possibly different Gibbons–Hawking (GH) terms as well. The GH terms are required for a well posed variational principle in the case of a manifold with a boundary. Probably the most well known example of such an equivalence is that of the $f(R)$ action to Brans–Dicke and Einstein–Hilbert one, the first through the introduction of an auxiliary scalar field Higgs (1959); Bicknell (1974); Teyssandier and Tourrenc (1983); Whitt (1984); Schmidt (1987); Wands (1994), and the second through a conformal transformation of the metric Higgs (1959); Teyssandier and Tourrenc (1983); Whitt

(1984), Magnano et al. (1987); Ferraris et al. (1988); Jakubiec and Kijowski (1988); Maeda (1988); Barrow and Cotsakis (1988).

The motivation of re-expressing a gravitational action by introducing a new set of field variables might be related mainly to two things. The first is mathematical simplicity and convenience, if the new set of variables is to make the calculations one wants to perform simpler. Furthermore, if one is able to move from a variational principle that will lead to fourth order equations of motion to another that will lead to second order ones, that could be a benefit, since second order equations are in principle easier to handle as well as to interpret physically. The second possible motivation is related to physics. One gets more intuition and understanding of a gravitational theory, by studying its equivalence to other ones (like the equivalence between $f(R)$ and Einstein gravity).

From a physical point of view care must be taken in the interpretation of physical quantities in the two different representations. The question that often arises is which of the two representations is the physical one. For example, in the case of the conformal equivalence between $f(R)$ and Einstein gravity, the inclusion of matter in the action can raise the question of along which of the two metrics (original and conformally transformed one) do matter particles actually fall. For some interesting discussions on the subject one can refer to Brans (1988); Sokolowski (1989a,b); Magnano et al. (1990a); Magnano and Sokołowski (1994); Sokolowski (1995); Magnano (1995); Faraoni and Gunzig (1999); Capozziello et al. (1997); Sotiriou et al. (2008).

Given the equivalence between the bulk parts of two actions, this does not mean that the equivalence holds for the surface parts as well. More precisely, given the GH term of an action in one representation, then the GH term calculated using the equivalence with the other reresentation is possible to be problematic as far as the particular variational principle is concerned. As we will discuss later on, this is the case for $f(R)$ and $R + f(G)$ gravity, when the equivalence between the original and the Jordan frame action is considered. As we will see, the latter equivalence breaks on the boundary surface.

In this section, we will focus on $f(R)$ Nojiri and Odintsov (2006a); Capozziello and Francaviglia (2008); Sotiriou and Faraoni (2010); De Felice and Tsujikawa (2010) and $R + f(G)$ Nojiri and Odintsov (2006a, 2005); Nojiri et al. (2005) models, where

G is the Gauss–Bonnet (GB) term Lovelock (1971), also defined before in a previous section

$$G \equiv R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha}{}_{\beta\gamma\delta}R_{\alpha}{}^{\beta\gamma\delta}, \tag{4.1}$$

"f" being an in principle non linear function of its arguments. We will study the dynamical equivalence of above theories to other representations in vacuum, using as our main tool the rather general approach of Legendre transformation, which for the case of $f(R)$ coincides with the standard procedure of introducing an auxiliary scalar field followed by a conformal transformation, something which is not true for $R+f(G)$ theories, i.e. the latter cannot be conformally transformed to a minimally coupled, scalar-tensor frame. However, for the $R+f(G)$ theory, we will show how using a Legendre transformation we can re-express it as a second order theory, with a new extra rank two tensor field. We will work at the level of the action and we will include in our analysis the relevant Gibbons–Hawking terms Gibbons and Hawking (1977), which are important for the consistency of the initial value formulation of the theory. Furthermore, we will calculate them explicitly wherever necessary.

Let us sketch the structure of this chapter. In Section 4.2 we will briefly describe the approach of Legendre transforming a higher order gravitational action. In view of the latter approach, in Section 4.3 we show the equivalence of the full (including the relevant GH term) $f(R)$ action to the Einstein–Hilbert one. We also discuss the equivalence between the relevant GH terms. Then, in Section 4.5 we consider the $R+f(G)$ action and after calculating the GH term in the Jordan (scalar–tensor) frame, we explicitly study the effect of conformal transformation on the full Jordan frame action. Finally, in Section 4.6 we re-express the original $R+f(G)$ action with one scalar and two extra tensor fields present (apart from the metric), one of them though being independent, and then discuss the classical dynamics of the system. We include various useful formulas and explicit calculations in the Appendix. In this section, we will work in natural units, $c = G = 1$.

4.2 Establishing dynamical equivalence

In this section we will briefly describe the idea and motivation behind Legendre transforming a gravitational action as a tool of moving to a new, dynamically equivalent variational principle of second-order, first applied in Magnano et al. (1987, 1990b).

The higher order nature of non–linear gravitational Lagrangians like the $f(R)$ or $R+f(G)$ ones, comes from the fact that they are non–linear with respect to the second derivatives of the metric, since "f" is an in principle non–linear function with respect to its arguments, and $R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta\gamma\delta}\,(g^2, (\nabla g)^2, \nabla^2 g)$.

However, we can try to re–express higher order gravitational Lagrangians linearly with respect to $\nabla^2 g$, by making them linear with respect to the curvature tensors [1] through the introduction of the appropriate "velocities" and "momenta", in a similar fashion to the ordinary Hamiltonian formalism. The new variational principle will then lead to second order equations of motion for the new set of field variables.

The appropriate identification for the generalised "position" and "momenta" as well as the Legendre transformed Lagrangian will read schematically as

$$q \leftrightarrow g_{\alpha\beta}, \quad \dot{q}_i \leftrightarrow \left(R, R^{\alpha\beta}, R^{\alpha}{}_{\beta\gamma\delta}\right), \tag{4.2}$$

$$\widetilde{L}(-g)^{-1/2} = \dot{q}_i p_i - \left[\dot{q}_i(q,p)p_i - (-g)^{-1/2}L(q,p)\right] \equiv \dot{q}_i p_i - H(q,p), \tag{4.3}$$

assuming the invertibility condition holds, $\partial^2 L/(\partial\dot{q}_i\partial\dot{q}_j) \neq 0$. Quantities entering $\widetilde{L}$ will be in principle tensor objects. Different gravity actions give us different options in defining generalised "velocities" and "momenta". This will be made clear in Sections 4.4 and 4.6, where we will apply the above formalism for the case of $f(R)$ and $R+f(G)$ gravity respectively.

4.3 Dynamical equivalence of $f(R)$ gravity, part I

It is well known in the literature that through the introduction of an auxiliary scalar, the $f(R)$ action can be re-expressed as a non–minimally coupled scalar-tensor one (also called Jordan frame action), and it is the latter that is usually conformally transformed to the so-called Einstein frame action. In this section, we we will focus and attempt to clarify the role of the relevant GH terms in the two representations, original and Jordan frame one. Then, in the next subsection we will demonstrate

[1] This is because curvature tensors are linear with respect to $\nabla^2 g$.

the equivalence of the full action (bulk and surface part) using the general approach of the Legendre transformation.

Starting from the bulk $f(R)$ action on a manifold M,

$$S = \int_M d^n x \sqrt{-g} f(R), \tag{4.4}$$

through the introduction of an auxiliary scalar ψ, one can re-express it in a dynamically equivalent way as

$$S^{\mathrm{J}} = \int_M d^n x \sqrt{-g} \left[\Phi R(g) - V(\Phi) \right], \tag{4.5}$$

with $\Phi \equiv f'(\psi)$, and $V(\Phi) \equiv \Phi f'^{-1}(\Phi) - f\left(f'^{-1}(\Phi)\right)$. For the latter we require that $f''(\psi) \neq 0$, so we are able to solve for $\psi = f'^{-1}(\Phi)$. Action (4.5) is the so-called Jordan frame action.

The transition to the Einstein frame action will be shown explicitly in Section 4.4 by means of a Legendre transformation.

4.3.1 The $f(R)$ Gibbons–Hawking term in the Jordan frame

When considering gravitational actions on manifolds with boundary Σ, the variation gives boundary terms containing normal derivatives of the metric variation $\nabla_n^{(k)} \delta g_{\alpha\beta}$. However, a well defined variational principle requires that only a particular set of dynamical coordinates ($g_{\alpha\beta}$ and possibly its derivatives up to some order depending on the theory) is fixed on the boundary. In order to cancel the extra, unwanted surface terms, one needs to add a so-called Gibbons-Hawking (GH) term in the action Gibbons and Hawking (1977). The appropriate modification of the Einstein-Hilbert action turns out to be

$$S^{EH} = \int_M d^n x \sqrt{-g} R + 2 \int_\Sigma d^{n-1} x \sqrt{-h} K, \tag{4.6}$$

where h is the induced metric on the surface Σ, and K is the trace of its extrinsic curvature. Variation of (4.6) is then performed keeping only the metric $g_{\alpha\beta}$ fixed on the boundary.

In the original $f(R)$ action (4.4) there is no natural GH term which cancels the extra unwanted higher derivative boundary terms, except for the particular case of maximally symmetric spacetimes Madsen and Barrow (1989). The implication of this is that there is no counter term that can be added in the action, in order to cancel the normal derivatives of the metric field after variation of the action with respect to the metric. The non–vanishing of the latter terms can pose the variational principle of the action problematic for the case of an action defined on a manifold with a boundary. However, if one only cares about eh local equations of motion of the theory, it can be assumed that in the action variation, the variation of the metric together with its first, second and third derivatives are zero. For a nice discussion of these issues see Dyer and Hinterbichler (2009).

However, in the Jordan frame of $f(R)$ one can find an appropriate Gibbons–Hawking term in full generality, and then under the assumption that the dynamical equivalence between different representations holds on the boundary surface, one is able to re-express it in the original $f(R)$ representation. Let us examine this more carefully.

First we want to find the GH term in the Jordan frame, and so we vary (4.5) with respect to the bulk metric and after discarding the bulk contributions we get

$$\delta S_{\Sigma}^{\mathrm{J}} = -\int_{\Sigma} d^{n-1}x\,\sqrt{-h}\nabla_{\rho}\left(\Phi\, g^{\alpha\beta}\delta\Gamma^{\rho}_{\alpha\beta}\right) = 2\int_{\Sigma} d^{n-1}x\,\sqrt{-h}\Phi\delta K. \tag{4.7}$$

Therefore, the GH term that should be added in the Jordan frame action (4.9) is

$$S_{\Sigma}^{\mathrm{J}} = -2\int_{\Sigma} d^{n-1}x\,\sqrt{-h}\Phi K, \tag{4.8}$$

and variation should be performed with $(\delta g_{\alpha\beta}, \delta\Phi)$ vanishing on Σ. After using the correspondence between Jordan and original frame, $\Phi \leftrightarrow f'(R)$, we find the GH term in the original frame to be [2]

$$S_{\Sigma}^{\mathrm{J}} = -2\int_{\Sigma} d^{n-1}x\,\sqrt{-h}f'(R)K. \tag{4.9}$$

[2] We obtained the GH term in the original $f(R)$ representation by substituting the equation of motion for Φ, $V'(\Phi) = R(g)$, into equation (4.8). This makes clear that the equivalence is demonstrably valid only on-shell (i.e. at the level of the classical equations of motion).

However, variation of this boundary term generates a new term

$$\delta S_{\Sigma} = -2\int_{\Sigma} d^{n-1}x\,\sqrt{-h}f''(R)K\delta R, \tag{4.10}$$

which vanishes only by requiring that $\delta R = 0$ on the boundary. In Dyer and Hinterbichler (2009) it has been shown that the GH term (4.9) is necessary in order to derive the correct Wald entropy for $f(R)$ gravity. [3]However, keeping R fixed on the boundary surface can be problematic. Since R includes both the first and the second derivatives of the metric, keeping it fixed would in principle require that the second derivatives of the metric are held fixed too, which overconstrains the actual formulation. The only possibility that would prevent the latter from happening would be that the condition $\delta R = 0$ is satisfied through some special configuration of the field variations on the boundary, something that would restrict the generality of our variational principle.

Therefore, we see that equivalence between the two representations breaks down at the boundary, when the consistency of the variational principle is considered. This failure indicates that the two theories cannot be truly equivalent. Furthermore, the two theories are inequivalent at the quantum level as well; considering the path integral defined in the Jordan frame, the integration over Φ, will generate extra terms in the effective action, making the latter inequivalent to the one defined in the original representation.

4.4 Dynamical equivalence of $f(R)$ gravity, part II

We now want to exploit the dynamical equivalence between the full $f(R)$ and Einstein–Hilbert action, using solely the Legendre transformation approach, presented in section 4.2, which is more general and for the $f(R)$ case gives the same result with the conformal transformation. This was done in Magnano et al. (1987) for the bulk Lagrangian, discarding a total derivative. In the following we will show how to cure this by Legendre transforming the GH term, apart from the bulk part, getting

[3]The Wald entropy extends the notion of entropy to black holes, and applies to any metric theory of gravity. In particular, the Wald black hole entropy formula relates the entropy of the black hole with the area of the black hole's horizon, $S = cA$, where A is the horizon area and c a constant. For more details see Wald (1984).

this way the correct Einstein–Hilbert GH term as well. [4]

Let us begin with the action

$$S = \int_M d^n x \sqrt{-g} f(R) + \int_\Sigma d^{n-1} x \sqrt{-h} f'(R) K, \tag{4.11}$$

including the GH term found in the previous section[5]. As we will see below, the inclusion of the latter is indeed a good choice. We will need separate variables for the boundary surface, and the Legendre transfomed action will be of the form

$$\widetilde{S} = \int_M d^n x \sqrt{-g} \left(\dot{q}_B p_B - H_B \right) + \int_\Sigma d^{n-1} x \sqrt{-h} \left(\dot{q}_\Sigma p_\Sigma - H_\Sigma \right), \tag{4.12}$$

with $H_i \equiv H_i(q,p)$. Let us first naively associate for the generalised bulk velocity, $\dot{q} \leftrightarrow R_{\alpha\beta}$. Then we get the bulk conjugate momentum as

$$p_B(q,\dot{q}) \leftrightarrow \tilde{g}^{\alpha\beta} \equiv \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial R_{\alpha\beta}} = f'(R) g^{\alpha\beta}. \tag{4.13}$$

We see that the definition of the conjugate momentum defines a conformal relation between two different metrics. In fact, as we will see below, $\tilde{g}_{\alpha\beta}$ is the metric in the Einstein frame. However, the correct association for $\dot{q}$ is not exactly $R_{\alpha\beta}$, but R, since relation (4.13) cannot be inverted for $R_{\alpha\beta}$.

We proceed by identifying

$$\dot{q}_B \leftrightarrow R, \quad \dot{q}_\Sigma \leftrightarrow K, \tag{4.14}$$

$$p_B(q,\dot{q}) \leftrightarrow \Phi \equiv \tfrac{1}{\sqrt{-g}} \tfrac{\partial L_B}{\partial R(g)} = f'(R), \tag{4.15}$$

$$p_\Sigma(q,\dot{q}) \leftrightarrow \Phi \equiv \tfrac{1}{\sqrt{-h}} \tfrac{\partial L_\Sigma}{\partial K(h)} = f'(R). \tag{4.16}$$

Now, using the intuition gained from (4.13), and using (4.15)-(4.16), we define the

[4]The transition to the Einstein frame by means of a conformal transformation, including the relevant GH terms, has been studied in Dyer and Hinterbichler (2009).

[5]As it was previously discussed, the GH term (4.9) can be in general problematic, however it turns to be neccessary here in order to cancel extra surface terms after Legendre transforming the action.

following relations for the bulk and surface metric respectively

$$\tilde{g}^{\alpha\beta} \equiv \Phi^{\frac{2}{(n-2)}} g^{\alpha\beta} \quad \text{and} \quad \tilde{h}^{\alpha\beta} \equiv \Phi^{\frac{2}{(n-2)}} h^{\alpha\beta}. \tag{4.17}$$

The invertibility condition is not satisfied on the boundary Σ, since $\partial^2 L_\Sigma / \partial K^2 = 0$. However, the surface part of $\widetilde{L}$ can be still defined, with the only difference that the surface Hamiltonian will vanish identically, $H_\Sigma = 0$. The bulk Hamiltonian is calculated after solving one of relations (4.15) for R, $H_B(\Phi) \equiv \Phi f'^{-1}(\Phi) - f(f'^{-1}(\Phi))$. Using this together with (4.14)-(4.16), and substituting in (4.12) we arrive at

$$\widetilde{S} = \int_M d^n x \sqrt{-g} \left[\Phi R(g) - H_B(\Phi)\right] + \int_\Sigma d^{n-1} x \sqrt{-h} \Phi K. \tag{4.18}$$

For the transition to the Einstein frame we will use the general equations (A.26) and (A.28) relating two Ricci (extrinsic curvature) tensors, evaluated for two different metrics $g_{\alpha\beta}$ $(h_{\alpha\beta})$ and $\tilde{g}_{\alpha\beta}$ $(\tilde{h}_{\alpha\beta})$. Defining $\Phi = \exp[\left(1/\sqrt{2\omega}\right)\phi]$ and $\omega(n) \equiv (n-1)/(n-2)$ we get

$$\begin{aligned} \widetilde{S}_B = \; & = \int_M d^n x \sqrt{-\tilde{g}} \left[\widetilde{R}(\tilde{g}) - \tfrac{1}{2}\partial_\kappa \phi \partial^\kappa \phi - e^{\frac{n}{1-n}\sqrt{\frac{\omega(n)}{2}}\phi} H_B(\phi)\right] \\ & \quad + \left(2\omega(n)\right)^{1/2} \int_\Sigma d^{n-1} x \sqrt{-\tilde{h}} (\partial^\kappa \phi) \tilde{n}_\kappa, \end{aligned} \tag{4.19}$$

$$\widetilde{S}_\Sigma = \int_\Sigma d^{n-1} x \sqrt{-\tilde{h}} \left[2\widetilde{K}(\tilde{h}) - \left(2\omega(n)\right)^{1/2} (\partial^\kappa \phi) \tilde{n}_\kappa\right], \tag{4.20}$$

and after summing up we arrive at

$$\widetilde{S} = \int_M d^n x \sqrt{-\tilde{g}} \left[\widetilde{R}(\tilde{g}) - \frac{1}{2}\partial_\kappa \phi \partial^\kappa \phi - e^{\frac{n}{1-n}\sqrt{\frac{\omega(n)}{2}}\phi} H_B(\phi)\right] + 2\int_\Sigma d^{n-1} x \sqrt{-\tilde{h}} \widetilde{K}(\tilde{h}), \tag{4.21}$$

$\tilde{n}_\kappa$ denoting the normal vector to Σ. We see that we arrive at the correct, full Einstein–Hilbert action, following a conceptually different and more fundamental procedure. The conformal relation between the two metrics was revealed naturally through the definitions of the conjugate momenta.

4.5 Dynamical equivalence of $f(G)$ gravity, part I

In this section we will aim to express the Jordan (scalar–tensor) frame of the $R+f(G)$ action as a minimally–coupled theory by means of a conformal transformation. Firstly we will derive the GH term in the Jordan frame, and then find the appropriate one in the original frame, as dictated by the equivalence between frames. Then, in subsection 4.5.2 we will continue with conformally transforming the full, Jordan $R+f(G)$ action.

Our starting point is the $R+f(G)$ action

$$S=\int_M d^n x\,\sqrt{-g}\,[\alpha R+f(G)]\,, \tag{4.22}$$

with G defined in (4.1) and α a dimensionless constant.

Through the introduction of an auxiliary scalar field ψ we get the Jordan frame action as

$$S^{\mathrm{J}}=\int_M d^n x\,\sqrt{-g}\,[\alpha R+\Phi G-V(\Phi)]\,, \tag{4.23}$$

with $\Phi=f'(\psi)$, and $V(\Phi)\equiv[\Phi f'^{-1}(\Phi)-f\left(f'^{-1}(\Phi)\right)]$, assuming that $f''(\psi)\neq 0$.

.

4.5.1 The $f(G)$ Gibbons–Hawking term in the Jordan frame

The motivation of this subsection is the same as in the $f(R)$ case, as explained in Section 4.3.1. We will derive the appropriate GH term in the original action (4.22) as dictated by the equivalence with the Jordan frame, by first calculating the Jordan frame one, presenting the explicit results of the surface parts of the action variation. Some useful variation formulas and definitions used can be found in A.2.

We start from the Jordan frame action (4.23) and vary each of the Gauss–Bonnet terms separately with respect to $g_{\alpha\beta}$ using relations (A.3)– (A.15). We focus on the $f(G)$ term, since the GH term for R is given by (4.9) for $f'(R)=1$. We will again present only the boundary part of the variation, as well as work in Riemann and Gaussian normal coordinates Misner et al. (1973). With the aid of integration by parts, and using equation (A.3), we get

$$\delta S^{\mathrm{J}}_{1\Sigma} = -4\int_\Sigma d^{n-1}x\,\sqrt{-h}\,\Phi R\delta K, \tag{4.24}$$

$$\delta S^{\mathrm{J}}_{2\Sigma} = -4\int_\Sigma d^{n-1}x\,\sqrt{-h}\,\Phi\Big[2n^\beta R^{\alpha\kappa}\nabla_\kappa - n^\lambda R^{\alpha\beta}\nabla_\lambda - n_\lambda h^{\alpha\beta}R^{\kappa\lambda}\nabla_\kappa, \\ -n^\alpha n^\beta n_\kappa R^{\kappa\lambda}\nabla_\lambda\Big]\delta g_{\alpha\beta}, \tag{4.25}$$

$$\delta S^{\mathrm{J}}_{3\Sigma} = \int_\Sigma d^{n-1}x\,\sqrt{-h}\,\Phi\left[n_\lambda R^{\alpha\kappa\lambda\beta}\nabla_\kappa\right]\delta g_{\alpha\beta}. \tag{4.26}$$

The geometric relevance of the above terms becomes evident if we express them in terms of tensor objects defined on the boundary surface using the Gauss–Codacci equations Misner et al. (1973). Doing this, and adding up all three terms together, we arrive at

$$\delta S^{\mathrm{J}}_\Sigma = \int_\Sigma d^{n-1}x\sqrt{-h}\,\Phi\Bigg[2\Bigg(2\widehat{G}^{\beta\gamma}\delta K_{\beta\gamma} + 2K_\mu{}^\beta K^{\mu\gamma}\delta K_{\beta\gamma} \\ -2KK^{\beta\gamma}\delta K_{\beta\gamma} + K^2\delta K - K_{\alpha\mu}K^{\alpha\mu}\delta K\Bigg)\Bigg], \tag{4.27}$$

with $\widehat{G}^{\beta\gamma}$ the Einstein tensor defined on Σ. Since we require that $\delta g_{\alpha\beta} = 0$ on Σ (or $\delta h_{\alpha\beta} = 0$), it follows that $\delta\widehat{G}_{\alpha\beta} = 0$ and $\delta K_{\alpha\beta} = \delta K^{\alpha\beta} = \delta K^\alpha{}_\beta$ on Σ as well. Using those facts, we can go backwards in (4.27) and check that it is the variation of the following quantity

$$S^{\mathrm{J}}_\Sigma = \int_\Sigma d^{n-1}x\,\sqrt{-h}\quad\Phi\left[2\left(2\widehat{G}^{\alpha\beta}K_{\alpha\beta} + J\right)\right], \tag{4.28}$$

with $J \equiv \frac{2}{3}K^\rho{}_\kappa K^{\kappa\lambda}K_{\lambda\rho} - KK_{\kappa\lambda}K^{\kappa\lambda} + \frac{1}{3}K^3$. The appropriate supplement for the initial scalar–tensor action is therefore equation (4.28) with a minus sign instead. The GH term for a simple Gauss–Bonnet action ($L \propto \sqrt{-g}G$) has been derived under more general assumptions in a braneworld context in Davis (2003), as well as in Myers (1987) using the calculus of differential forms.

Now, as in the $f(R)$ case, we can use the equivalence $f'(G) \leftrightarrow \Phi$, to find the GH term in the original $f(G)$ frame if the equivalence is to hold on the boundary,

$$S_\Sigma = -\int_\Sigma d^{n-1}x\,\sqrt{-h}\,f'(G)\left[2\left(2\widehat{G}^{\alpha\beta}K_{\alpha\beta} + J\right)\right]. \tag{4.29}$$

Now, variation of the action requires $\delta g_{\alpha\beta} = 0$ and $\delta G = 0$ on Σ. The latter condition can yield a problematic variational principle for the same reasons discussed in Section 4.3.1. Therefore, for $R + f(G)$ theories as well, true equivalence on the boundary surface is broken.

4.5.2 Conformal transformation of the Jordan frame action

We now want to study if the non–minimally coupled, full Jordan frame action (4.29), can be decoupled from Φ and written in a $G+$ scalar field form, similar to the $f(R)$ case, using a conformal transformation of the metric.

Before we start with the calculations, let us introduce a notation that will make our equations look shorter. So, only for the rest of this section, we shall define: $n_i \equiv (n - i)$ and $r_i \equiv 1/n_i$, where n is the spacetime dimensionality. n_i is not to be confused with the surface normal n_α.

We shall begin with the bulk term. Using the transformation formula (A.23) together with the conformal factor identification

$$\Omega = \Phi^{1/(n-4)} \equiv \Phi^{r_4}, \tag{4.30}$$

and omitting the potential which transforms trivially, the action (4.23) after the redefinition $\Phi = \exp[\phi]$ becomes

$$\begin{aligned} \int_M d^n x \sqrt{-g}\, \Phi G \mapsto \int_M d^n x \sqrt{-\tilde{g}} \Bigg\{ & \widetilde{G} \\ & -8r_4 n_3 \left[\phi^{\tilde{;}\alpha\beta} - r_4 n_5 \phi^{,\alpha}\phi^{,\beta}\right] \widetilde{R}_{\alpha\beta} - 2r_4 n_3 \left[3r_4 n_4 \phi^{,\kappa}\phi_{,\kappa} - 2\tilde{\Box}\phi - a e^{(1-r_4 n_2)\phi}\right] \widetilde{R} \\ & +4r_4^4 n_3 n_2 \left[2n_5 n_3 + \tfrac{n n_1}{4} - 3a r_4^2 n_5 n_1 e^{(1-r_4 n_2)\phi}\right] \left(\phi_{,\kappa}\phi^{,\kappa}\right)^2 \\ & +4r_4^3 n_5 n_3 n_2 \left[\phi_{,\alpha}\phi_{,\beta} + 2\phi_{\tilde{;}\alpha\beta}\right] \left(\phi^{,\alpha}\phi^{,\beta}\right) \\ & +4r_4^2 n_3 n_2 \left[\tilde{\Box}\phi - r_4\left(2n_5 + n_1\right)\phi^{,\kappa}\phi_{,\kappa} + 2a r_4 n_1 e^{(1-r_4 n_2)\phi}\right] \tilde{\Box}\phi \\ & -4r_4^2 n_3 n_2 \phi^{\tilde{;}\alpha\beta}\, \phi_{\tilde{;}\alpha\beta} \Bigg\}, \end{aligned} \tag{4.31}$$

with $\tilde{\Box} \equiv \widetilde{\nabla}^\kappa \widetilde{\nabla}_\kappa$.

Identification (4.30) breaks down for $n = 4$, and in fact it is valid only for $n \geq 5$.

This means that we are unable to decouple the scalar Φ from the Gauss–Bonnet term unless $n \geq 5$. For $n \geq 5$, the GB term is minimally coupled to the scalar $\Phi = \exp[\phi]$, but there are new couplings between the derivatives of ϕ, the Ricci tensor and Ricci scalar. In this case, action (4.31), plus the scalar potential term of (4.23), describes a fourth order, non minimally coupled scalar–tensor theory.

Let us now turn attention to the conformal transformation of the relevant GH term, calculated in A.3. One can see that the variational principle requires that we impose apart from $\delta\tilde{g} = 0$ and $\delta\phi = 0$, the extra conditions $\widetilde{\nabla}\delta\tilde{g} = 0$ and $\widetilde{\nabla}\delta\phi = 0$ on Σ. The $R + f(G)$ action cannot be expressed as a second order, minimally–coupled scalar tensor one, in contrast with $f(R)$ gravity.

4.6 Dynamical equivalence of $f(G)$ gravity, part II

The richer structure of the $R + f(G)$ action gives us more options in identifying generalised velocities, compared to the $f(R)$ one. In this section we want to take advantage of the latter fact, and re-express the original $R + f(G)$ action as a new one with not only a new scalar, but with new tensor fields as well, by means of a Legendre transformation. The new variational principle will be of second order.

Our starting point is the action

$$S = \int_M d^n x \sqrt{-g} \left[\alpha R + f(G)\right]. \tag{4.32}$$

We proceed with defining our conjugate momenta as

$$p_1 \leftrightarrow \Psi \equiv \tfrac{1}{\sqrt{-g}} \tfrac{\partial L}{\partial R} = \alpha + 2Rf'(G), \tag{4.33}$$

$$p_2 \leftrightarrow \tilde{g}^{\alpha\beta} \equiv \tfrac{1}{\sqrt{-g}} \tfrac{\partial L}{\partial R_{\alpha\beta}} = -8f'(G)R^{\alpha\beta}, \tag{4.34}$$

$$p_3 \leftrightarrow \sigma_\alpha{}^{\beta\gamma\delta} \equiv \tfrac{1}{\sqrt{-g}} \tfrac{\partial L}{\partial R^\alpha{}_{\beta\gamma\delta}} = 2f'(G)R_\alpha{}^{\beta\gamma\delta}. \tag{4.35}$$

Defining $\Phi \equiv \Phi(G) \equiv f'(G)$, the inverse of the above relations read

$$R = \tfrac{1}{2\Phi}\left(\Psi - \alpha\right), \tag{4.36}$$

$$R^{\alpha\beta} = -\tfrac{1}{8\Phi}\tilde{g}^{\alpha\beta}, \tag{4.37}$$

$$R_\alpha{}^{\beta\gamma\delta}(\Phi, g, \tilde{g}, \sigma) = \tfrac{1}{2\Phi}\sigma_\alpha{}^{\beta\gamma\delta}, \tag{4.38}$$

with $\tilde{g}^{\kappa}{}_{\kappa} \equiv g^{\kappa\lambda}\tilde{g}_{\kappa\lambda}$ and $f'(G) \neq 0$. In fact, we will use $g_{\alpha\beta}$ to raise and lower indices for the rest of the section.

For the calculation of the Hamiltonian we will need to express the Gauss–Bonnet term in terms of the new fields $(\Psi, \tilde{g}^{\alpha\beta}, \sigma_{\alpha}{}^{\beta\gamma\delta})$. Using the inverse relations (4.36)-(4.38) we get

$$G \equiv R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu}{}_{\nu\rho\sigma}R_{\mu}{}^{\nu\rho\sigma} = \tfrac{1}{4\Phi^2}\Gamma(\Psi, \tilde{g}, \sigma), \tag{4.39}$$

with the function Γ defined as

$$\Gamma(\Psi, \tilde{g}, \sigma) \equiv (\Psi - \alpha)^2 - \frac{\tilde{g}^{\mu\nu}\tilde{g}_{\mu\nu}}{4} + \ \sigma_{\mu}{}^{\nu\rho\sigma}\sigma^{\mu}{}_{\nu\rho\sigma}. \tag{4.40}$$

Furthermore, we assume that we can invert relation (4.39) [6] and express the Gauss–Bonnet term in terms of the function Γ as

$$G = G^{-1}(\Gamma) \equiv J(\Gamma), \tag{4.41}$$

so that

$$f'(G) = f'(J(\Gamma)) \equiv F(\Gamma). \tag{4.42}$$

Using all the above, we can now calculate the Hamiltonian as

$$\begin{aligned} H(\Gamma(\Psi, \tilde{g}, \sigma)) &= \Psi R(\Psi, \tilde{g}, \sigma) + \tilde{g}^{\alpha\beta}R_{\alpha\beta}(\Psi, \tilde{g}, \sigma) + \sigma_{\alpha}{}^{\beta\gamma\delta}R^{\alpha}{}_{\beta\gamma\delta}(\Psi, \tilde{g}, \sigma) - (-g)^{-1/2}L(\Psi, \tilde{g}, \sigma) \\ &= \tfrac{\Gamma}{2F(\Gamma)} - f\left(J(\Gamma)\right). \end{aligned} \tag{4.43}$$

Notice that the fields $(\Psi, \tilde{g}^{\alpha\beta}, \sigma_{\alpha}{}^{\beta\gamma\delta})$ enter implicitly in the Hamiltonian through the function Γ.

The Legendre transformed action then reads

$$\widetilde{S}[\Psi, g, \tilde{g}, \sigma] = \int_M d^n x\, \sqrt{-g} \quad \left[\Psi R(g) + \tilde{g}^{\alpha\beta}R_{\alpha\beta}(g) + \sigma_{\alpha}{}^{\beta\gamma\delta}R^{\alpha}{}_{\beta\gamma\delta}(g) - H(\Gamma(\Psi, \tilde{g}, \sigma))\right]. \tag{4.44}$$

To get the equations of motion we vary the action $\widetilde{S}$ with respect to the four fields

[6] The necessary condition is that $[(f'(G))^2\, G]' \neq 0$, implying that $f(G) \neq C_1\sqrt{G} + C_2$.

$(\Psi, \tilde{g}^{\alpha\beta}, \sigma_\alpha{}^{\beta\gamma\delta}, g_{\alpha\beta})$ to get

$$\tfrac{\delta \widetilde{S}}{\delta \Psi} = R(g) - 2H'\left(\Psi - \alpha\right) = 0, \tag{4.45}$$

$$\tfrac{\delta \widetilde{S}}{\delta \tilde{g}^{\alpha\beta}} = R_{\alpha\beta}(g) + \tfrac{1}{2}H'\tilde{g}_{\alpha\beta} = 0, \tag{4.46}$$

$$\tfrac{\delta \widetilde{S}}{\delta \sigma_\alpha{}^{\beta\gamma\delta}} = R^\alpha{}_{\beta\gamma\delta}(g) - 2H'\sigma^\alpha{}_{\beta\gamma\delta} = 0, \tag{4.47}$$

$$\begin{aligned}\frac{\delta \widetilde{S}}{\delta g_{\alpha\beta}} \quad &= \Psi G^{\alpha\beta} - \nabla^\alpha\nabla^\beta\Psi + g^{\alpha\beta}\nabla^\kappa\nabla_\kappa\Psi - \nabla_\kappa\nabla^{(\alpha}\tilde{g}^{\beta)\kappa} \\ &+\tfrac{1}{2}\nabla^\rho\nabla_\rho\tilde{g}^{\alpha\beta} + \tfrac{1}{2}g^{\alpha\beta}\nabla_\kappa\nabla_\lambda\tilde{g}^{\kappa\lambda} - 2\nabla_\kappa\nabla_\lambda\sigma^{\kappa(\alpha\beta)\lambda} \\ &-\tfrac{1}{2}g^{\alpha\beta}\Big[\tilde{g}^{\kappa\lambda}R_{\kappa\lambda}(g) + \sigma_\kappa{}^{\lambda\mu\nu}R^\kappa{}_{\lambda\mu\nu}(g) - H(\Gamma)\Big] \\ &-\tfrac{1}{2}H'\Big[8\sigma^{\kappa\lambda\mu(\alpha}\sigma^{\beta)}{}_{\mu\lambda\kappa} - \tilde{g}^{\kappa(\alpha}\tilde{g}_\kappa{}^{\beta)}\Big] = 0,\end{aligned} \tag{4.48}$$

with $G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$, $H' \equiv H'(\Gamma) \equiv \partial H/\partial\Gamma$ and covariant derivatives ∇_α defined with respect to $g_{\alpha\beta}$.

Variation with respect to $g_{\alpha\beta}$ yields surface terms $\propto \nabla g_{\alpha\beta}$. We want to keep only the fields fixed on Σ and not their derivatives, so we have to add in action (4.44) the following GH term

$$\widetilde{S}_\Sigma = 2\int_\Sigma d^{n-1}x\,\sqrt{-h}\,\Big(\Psi K + \tilde{g}^{\alpha\beta}\Gamma^\kappa_{\alpha[\kappa}n_{\beta]} + \sigma_\alpha{}^{\beta\gamma\delta}\Gamma^\alpha_{\beta[\gamma}n_{\delta]}\Big), \tag{4.49}$$

with $[A, B] \equiv \frac{1}{2}\left(AB - BA\right)$.

If we now contract equation (4.47) with $g_\alpha{}^\gamma$ and add it to (4.47) we get the relation

$$\sigma^\gamma{}_{\alpha\gamma\beta} \equiv \sigma_{\alpha\beta} = -\frac{1}{4}\tilde{g}_{\alpha\beta}. \tag{4.50}$$

The latter implies that $\sigma_{\alpha\beta\gamma\delta}$ can be expressed as some combination of $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ plus some traceless part, while the trace of that expression should give (4.50). To find the latter expression we can expand the Riemann tensor in terms of he Ricci tensor and scalar according to

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - a_n \left(g_{\alpha[\delta} R_{\gamma]\beta} + g_{\beta[\gamma} R_{\delta]\alpha} \right) - b_n R g_{\alpha[\gamma} g_{\delta]\beta}, \tag{4.51}$$

with $a_n \equiv \frac{2}{n-2}$, $b_n \equiv \frac{2}{(n-1)(n-2)}$ and $C_{\alpha\beta\gamma\delta}$ the Weyl tensor which is traceless in all its indices. After use of equation (4.46), relation (4.51) can be expressed as

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{a_n}{2} H' \left(g_{\alpha[\delta} \tilde{g}_{\gamma]\beta} + g_{\beta[\gamma} \tilde{g}_{\delta]\alpha} \right) + \frac{b_n}{2} H' \tilde{g} g_{\alpha[\gamma} g_{\delta]\beta}, \tag{4.52}$$

and plugging the latter into equation (4.47) to substitute for the Riemann tensor, we get a relation between $\sigma_{\alpha\beta\gamma\delta}$, $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$

$$\sigma_{\alpha\beta\gamma\delta} = \frac{1}{2H'} C_{\alpha\beta\gamma\delta}(g) + \frac{a_n}{4} \left(g_{\alpha[\delta} \tilde{g}_{\gamma]\beta} + g_{\beta[\gamma} \tilde{g}_{\delta]\alpha} \right) + \frac{b_n}{4} \tilde{g} g_{\alpha[\gamma} g_{\delta]\beta}. \tag{4.53}$$

Combining equations (4.45) and (4.46) we can find a similar relation for Ψ

$$\Psi(g, \tilde{g}) = \alpha - \frac{1}{4} \tilde{g}, \tag{4.54}$$

with $\tilde{g} \equiv g^{\alpha\beta} \tilde{g}_{\alpha\beta}$. One would like to be able to solve equation (4.53) for $\sigma_{\alpha\beta\gamma\delta} = \sigma_{\alpha\beta\gamma\delta}(\tilde{g}, g)$. However, this is not in principle possible unless $H' =$ constant (corresponding to the trivial case of $f(G) = G$) or $C_{\alpha\beta\gamma\delta}(g) = 0$. The latter case includes the case of the FLRW spacetime or maximally symmetric spacetimes like the Minkowski one. In that case, all fields can be expressed in terms of $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ and we can get a solution for the latter ones by solving the appropriate system of second order differential equations, which we derive below.

Now, we want to derive a system of evolution equations for the set of fields $(g_{\alpha\beta}, \tilde{g}_{\alpha\beta})$. The first equation we will use results from equation (4.46) after taking its trace once, together with some simple algebra. To get the second equation, we use relations (4.53) and (4.54) together with the $C_{\alpha\beta\gamma\delta}(g) = 0$ ansatz to express the last of the equations of motion, equation (4.48), in terms of $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ alone. This way we arrive at the new system of second order equations for the set of fields $(g_{\alpha\beta}, \tilde{g}_{\alpha\beta})$

$$G_{\alpha\beta} = -\frac{1}{2} H' \left(\tilde{g}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \tilde{g}^{\kappa}{}_{\kappa} \right), \tag{4.55}$$

$$\left(\widehat{P}^{\kappa\lambda}_{\mu\nu}\right)^{(\alpha\beta)} \nabla_\kappa \nabla_\lambda \tilde{g}^{\mu\nu} - H' g^{\alpha\beta}\Big[p_n \tilde{g}^{\kappa\lambda}\tilde{g}_{\kappa\lambda} + q_n \tilde{g}^2 - \alpha\tilde{g} - 2\tfrac{H}{H'}\Big]$$
$$-H'\Big[r_n \tilde{g}^{\kappa(\alpha}\tilde{g}^{\beta)}{}_\kappa + s_n \tilde{g}\tilde{g}^{\alpha\beta} + 2\alpha\tilde{g}^{\alpha\beta}\Big] = 0, \tag{4.56}$$

with $H \equiv H(\Gamma(g,\tilde{g}))$ and the operator $\widehat{P} \equiv \widehat{P}(g)$ defined as

$$\left(\widehat{P}^{\kappa\lambda}_{\mu\nu}\right)^{(\alpha\beta)} \equiv (1-a_n)\Big[c_n g^{\kappa(\alpha}g^{\beta)\lambda}g_{\mu\nu} - c_n g^{\alpha\beta}g^{\kappa\lambda}g_{\mu\nu} - 4g^{\lambda(\alpha}\delta^{\beta)}_\mu\delta^\kappa_\nu + 2g^{\kappa\lambda}\delta^\alpha_\mu\delta^\beta_\nu + 2g^{\alpha\beta}\delta^\kappa_\mu\delta^\lambda_\nu\Big], \tag{4.57}$$

while the constants a_n, b_n, c_n, p_n, q_n, r_n, s_n respectively

$$a_n \equiv \tfrac{2}{n-2}, \quad b_n \equiv \tfrac{2}{(n-1)(n-2)}, \quad c_n \equiv \tfrac{1-2b_n}{1-a_n}, \tag{4.58}$$
$$p_n \equiv a_n - \tfrac{a_n^2}{2b_n}(1+b_n), \quad q_n \equiv \tfrac{1}{4}[1 - a_n(a_n - 2b_n)], \tag{4.59}$$
$$r_n \equiv 2[1 - \tfrac{a_n^2}{4}(n-4)], \quad s_n \equiv \tfrac{1}{2}[2a_n(a_n-2b_n) - 1]. \tag{4.60}$$

To arrive at equation (4.56) we have used the following relations

$$\Gamma \equiv \Gamma(g,\tilde{g}) = \frac{1}{16}\left[\left(1 - \frac{4(n-3)}{(n-2)^2(n-1)}\right)\tilde{g}^2 - 4\left(\frac{n-3}{n-2}\right)\tilde{g}^{\mu\nu}\tilde{g}_{\mu\nu}\right], \tag{4.61}$$

and

$$\sigma^{\alpha\mu\nu\rho}\sigma^\beta{}_{\mu\nu\rho}(g,\tilde{g}) = \frac{1}{8(n-2)^2}\left[g^{\alpha\beta}\tilde{g}^{\rho\mu}\tilde{g}_{\rho\mu} + (n-4)\tilde{g}^{\rho\alpha}\tilde{g}^\beta{}_\rho - 2\left(\frac{n-3}{n-1}\right)\tilde{g}\tilde{g}^{\alpha\beta}\right], \tag{4.62}$$

as well as $\sigma^\alpha{}_{\beta\gamma\delta}(g,\tilde{g})$ given by (4.53) with $C^\alpha{}_{\beta\gamma\delta}(g) = 0$.

A look at the first equation of the new system, equation (4.55), shows that at the level of the equations of motion we can express the dynamics as GR, minimally–coupled to an effective energy–momentum tensor (the r.h.s of the equation) described by the spin two field $\tilde{g}_{\alpha\beta}$.

There is one extra constraint the fields satisfy, that is the Bianchi identities. Since the l.h.s of equation (4.55) is covariantly conserved, as dictated by the Bianchi identities, then the r.h.s should be as well,

$$\nabla^\alpha\Big[H'\Big(\tilde{g}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\tilde{g}\Big)\Big] = 0. \tag{4.63}$$

The latter equation is a condition the set of fields $(g_{\alpha\beta}, \tilde{g}_{\alpha\beta})$ have to satisfy, together with the equations of motion.

We will not seek solutions of the system described by equations (4.55)-(4.56) in this paper, leaving this for a possible future work. However, it is easy to see that Minkowski space is a solution for $(g_{\alpha\beta}, \tilde{g}_{\alpha\beta}) = (\eta_{\alpha\beta}, 0)$.

We see that at the classical level of this representation there are two independent fields, $(g_{\alpha\beta}, \tilde{g}_{\alpha\beta})$, satisfying a system of second order equations together with a second order condition, the Bianchi identity. These equations should be classically equivalent to the original fourth order ones, as $\tilde{g}_{\alpha\beta}$ is related to the second derivatives of $g_{\alpha\beta}$. It should be noted that there is no reason to expect complete equivalence of the quantum equations, as the measure of the path integral in the different representations can introduce new terms.

4.7 Conclusions

Let us close the chapter with a summary and some remarks on the previous analysis. It is clear, that Legendre transformations are a fundamental tool to study the dynamical equivalence between different modified gravity actions, with the aim of understanding better the nature of the theories under study. When working the level of the action, a consistent analysis should take into account the appropriate Gibbons–Hawking (GH) terms (full action). Although in a general context there are no natural GH terms for both $f(R)$ and $R + f(G)$ actions, however one can define them considering the dynamical equivalence between two different representations of the particular action on the boundary surface, as it was done in Section 4.5.1, when we considered the equivalence between the original action and the Jordan frame one. However, the GH terms found through this procedure turn out to render the variational principle inconsistent.

The disagreement between the GH terms in two different frames is associated with the fact that the two representations are not equivalent at the quantum level, as pointed out in Section 4.3.1.

Due to the structural simplicity of the full $f(R)$ action, the Legendre transformation yields in this case the same result as a conformal transformation of the original action. On the other hand, the $R + f(G)$ Jordan (non–minimally coupled) frame action cannot be re-expressed as a second order theory through a conformal transformation, despite the fact that the auxiliary scalar decouples from the Gauss–Bonnet term for $dim \geq 5$. The resulting theory is still of fourth–order, as was calculated explicitly for the full action in Section 4.5.

However, the more complex structure of the $R + f(G)$ action, allows one to re-express it, by means of a Legendre transformation, as a second order theory with extra tensor fields apart from scalars. In the new representation, it turns out that only two fields are the independent ones, the metric $g_{\alpha\beta}$ and the rank two field $\tilde{g}_{\alpha\beta}$. At the level of the equations of motion, we are able to recover GR, sourced by an effective energy–momentum tensor, which is a function of $\tilde{g}_{\alpha\beta}$. Although the two representations are classically equivalent in vacuum without boundary, at the quantum level they differ, as integrating out the extra fields generates new terms in the effective action. We briefly comment on the physical equivalence between different representations in the next section.

4.8 Physical equivalence between different representations

In this section we would very briefly like to review and comment the arguments presented previously in the literature favoring one frame instead of the other. The physical predictions and validity of either frame has been in fact a long controversy in the literature. We shall restrict ourselves to Brans–Dicke gravity, as it has been the most studied theory in terms of the dynamical equivalence, on the same time being one of the simplest cases one can consider. Notice that the case of $f(R)$ gravity corresponds to Brans–Dicke gravity with $\omega = 0$.

Let us start by noting that the analysis in the previous sections was performed for the case of vacuum, without including matter fields, as we were interested purely in the gravitational sector of the theories. However, the introduction of matter fields complicates things; in the original $f(R)$ or Jordan frame representation, the matter fields are minimally coupled to gravity, which is no longer true after performing a conformal transformation to the Einstein frame. This implies that in the original representation matter fields follow the geodesics of the Jordan frame metric, but the similar is not true for the Einstein frame.

To begin with, in Magnano and Sokolowski (1994) it is argued that the "true physical variables are exactly those which describe the equivalent general relativistic model" (in the case of $f(R)$ gravity the latter implies the Einstein frame variables). Of course, one should first define what "physical" means. According to Magnano and Sokolowski (1994) physical is the frame defined through a " a set of field variables which are (at least in principle) measurable and satisfy all general requirements of classical eld theory". What is more, in the same work it is argued that the Weak Energy Condition can be violated in one frame, but not in the other. Although this could be in principle true, in Flanagan (2004) it is argued that there is no physical observable whose predicted value in all conformal frames is the sign of $G_{\mu\nu}u^{\mu}$ for a timelike u^{μ}. In general, all observable quantities should be conformally invariant, and therefore independent of the particular frame used. After all, at the classical level a conformal transformation just accounts for a field re-definition, and of course physical observables should be invariant.

What is more, starting from the fact that two conformally related metrics can interact in a different way with external matter fields (e.g baryons) in the action, in Magnano and Sokolowski (1994) it is argued that the physical metric should be chosen such that its geodesic lines are those followed by external matter test particles. A

In Faraoni and Gunzig (1999) it is explicitly demonstrated that in the Jordan frame of Brans–Dicke theory, wave-like gravitational fields violate the weak energy condition, implying an "infrared catastrophe" for scalar gravitational waves. However, again in this case the question is if there is any observable where this violation manifests itself.

An important point regarding conformal transformations and sometimes neglected in the various studies in the literature, is a point emphasized by Dicke's original paper Dicke (1962); a conformal transformation accounts to a change in units. In particular, the new units run as some particular function of the conformal factor and in turn as a function of the spacetime coordinates. This view is further adopted in Flanagan (2004); Faraoni and Nadeau (2007) pointing out that the latter change of units should be always kept in mind when calculating physical observables, and of course, physics should be independent of the choice of units.

However, it has to be stressed that one expects physical observables to be independent of the choice of variables; this means that one should always ask the question of what is really observed before reaching any conclusions about the viability of one frame against another. An important example is the calculation of primordial inflationary spectra in the context of non-linear gravity theories in both Jordan and Einstein frame, yielding the same answer, Kolb et al. (1990); Kaiser (1995) as well as De Felice and Tsujikawa (2010) and references therein.

It has been also pointed out Faraoni (2009), that different conformal frames give a different effective mass and range for the scalar, and care must be taken when for example confronting scalar–tensor theories with solar system experiments. In particular, one should be careful with the definition of mass in different frames of the same action.

One could ask the question of what happens with the physical equivalence between different frames at the quantum level. From a path integral point of view, different conformal frames will yield physically inequivalent theories, as the transformation on the action as well as the path integral measure will yield additional factors that will affect the path integral in a non-trivial way.

As it is argued in Flanagan (2004) the only case where two conformal frames will yield physically equivalent theories at the quantum level is the semiclassical case, where only the external matter fields are quantised, but not the metric and the scalar, which should be expected as in that case the conformal transformation accounts only for a redefinition of units.

When gravity is treated as a quantum effective field theory, i.e a quantum theory which is accurate only up to a particular cut–off energy scale, an important role is played by the equivalence theorem which states that the S- matrix is invariant under non-linear, local field redefinitions (Chisholm (1961); Kamefuchi et al. (1961), as well as Flanagan (2004); Faraoni and Nadeau (2007) and references therein). Conformal transformations belong to this class of transformations, however the theorem only holds at the regime where the perturbative approach is valid. As a result, calculation of tree level quantities will yield equivalent results in different conformal frames, but this is not in general true beyond the effective description of the theory, i.e for energies beyond the energy cut–off.

Chapter 5

Anisotropic stress and stability in non–linear gravity models

5.1 Introduction

This chapter closely follows the collaborative work together with M. Kunz reported in Saltas and Kunz (2011).

In particular, we will elaborate on an important feature of modified gravity models, namely the effective anisotropic stress, and will try to understand its importance for current and future cosmological observations.

Although strictly speaking cosmological probes in general cannot provide conclusive proof Kunz and Sapone (2007); Kunz et al. (2008); Hu and Sawicki (2007), the presence of a significant anisotropic stress could be a smoking gun for a modification of GR at large scales: canonical scalar fields do not create additional anisotropic stress, while the modified-gravity (MG) models like scalar-tensor theories, brane-world models like the Dvali-Gabadadze-Porrati (DGP) model Dvali et al. (2000) and $f(R, G)$ type theories generically induce a large effective anisotropic stress.

In this chapter, we investigate one specific class of models, $f(R, G)$ type modifications of GR, and ask the question whether it is possible to construct viable models with a vanishing, or arbitrarily small effective anisotropic stress. Or in other words, is it possible to mimic "GR" with these models, at least up to first order in perturbation theory and in the sense that the extra anisotropic stress is small enough? Since $f(R)$ models have many things in common with scalar-tensor theories, we expect that our discussion is also relevant for those models, and as we discuss later, also for DGP and other braneworld models.

The structure of the chapter is as follows: In Section 5.2 we discuss the notion of anisotropic stress in general, and how this plays an important role in modified gravity models and then we investigate the possibility of a vanishing anisotropic stress in the particular cases of $f(R)$ and $f(G)$ models, before we look at the more general $f(R,G)$ case. In Section 5.3 we identify and discuss the link between anisotropic stress and stability in modified gravity models in the context of both homogeneous and inhomogeneous perturbations around de Sitter space. We further derive the relevant stability conditions. We generalize the discussion to arbitrary backgrounds in Section 5.4 and give some results for a matter dominated evolution. In Section 5.5, we apply the above to characteristic toy models, and then discuss our conclusions. Some explicit intermediate calculations and formulas can be found in the Appendix.

The $f(R,G)$ models were introduced in section (3.3), where some of their fundamental properties was discussed. For convenience let us recall some equations that will need in the following. In the flat, FLRW background the two key quantities, R and G, can be expressed purely as a function of the Hubble parameter $H \equiv H(t)$ and its time derivative,

$$R(t) = 6\left(2H^2 + \dot{H}\right), \tag{5.1}$$

$$G(t) = 24H^2\left(H^2 + \dot{H}\right). \tag{5.2}$$

We will also use the notation $F \equiv f_R \equiv \frac{\partial f(R,G)}{\partial R}$ and $\xi \equiv f_G \equiv \frac{\partial f(R,G)}{\partial G}$, and over dots denoting derivative with respect to cosmic time t.

Notice also that in this section we will be using ϕ and ψ to denote the scalar Newtonian potentials, and Φ and Ψ for the gauge invariant ones.

5.2 The effective anisotropic stress in higher order gravity

Let us here introduce the notion of anisotropic stress in gravity. As a starting point, we consider scalar perturbations around a flat FLRW background in the conformal

Newtonian gauge, where the metric is of the form[1]

$$ds^2 = -(1 + 2\psi)dt^2 + a(t)^2 (1 - 2\phi)\, d\mathbf{x}^2, \tag{5.3}$$

and the gravitational potentials $\psi \equiv \psi(\mathbf{x}, t)$ and $\phi \equiv \phi(\mathbf{x}, t)$ are closely related to observations: light deflection is sourced by the lensing potential $\phi + \psi$ and non-relativistic particle motion by ψ alone.

The *scalar anisotropic stress* Π is then defined as the difference in the potentials

$$\phi - \psi \equiv \Pi(\mathbf{x}, t), \tag{5.4}$$

or the difference of the relevant potentials in some other gauge. Equation (5.4) is called the *anisotropy equation*, and can be found by calculating the ij $(i \neq j)$ component of the perturbed equations of motion around the FLRW metric,

$$\begin{aligned} \delta G^i{}_j - \frac{1}{3} g^i_j \delta G^\kappa{}_\kappa &= \delta T^{(\mathrm{eff,total})i}{}_j - \frac{1}{3} g^i_j \delta T^{(\mathrm{eff,total})\kappa}{}_\kappa \\ &\equiv \Pi^{(\mathrm{eff})i}{}_j, \end{aligned} \tag{5.5}$$

from which one then extracts the scalar part as usual to get

$$\phi - \psi = \Pi^{(\mathrm{eff})}. \tag{5.6}$$

We emphasize that this is the anisotropic stress one would infer by assuming GR to hold, not only the anisotropic stress from the matter fields. Indeed, here we are precisely interested in the contribution to $\phi - \psi$ due to a modification of gravity. While relativistic particles do induce an anisotropic stress, it is small at late times and we will neglect the contribution of $T^{(\mathrm{matter})}_{\mu\nu}$ in equation (3.45) to $\phi - \psi$. Notice that because of the nature of $T^{(\mathrm{eff})i}{}_j$ in modified gravity theories, the r.h.s of above equation will in principle have a spacetime dependence, i.e it will be a function of ϕ, ψ as well as their first and second derivatives with respect to time (in Fourier space), in contrast to GR, where the r.h.s is just a function of the matter content. The usefulness of (5.6) is that it has a GR-like l.h.s., allowing to compute predictions for cosmological observations as usual, while all the extra contributions are moved

[1]The general form of the perturbed line element is given in the Appendix.

to the r.h.s. and interpreted as a "modified gravity energy-momentum tensor".

In particular, for GR (and neglecting any relativistic species) we have $\Pi^{(\mathrm{eff})} = 0$ and therefore $\phi = \psi$ *at all times*. Therefore, the inequality of the Newtonian potentials is a "signature" of departures from GR on large scales Kunz and Sapone (2007).

The ratio ϕ/ψ, or variables derived from it, like $\eta(t,k) \equiv \frac{\psi}{\phi} - 1$, can be extracted observationally by combining weak lensing experiments with e.g. galaxy surveys or redshift space distortions, making cosmological observations a powerful test of GR Amendola et al. (2008). In particular, one can extract the weak lensing potential from weak lensing surveys, with the former being equal to the difference between the two scalar potentials, $\phi_{\mathrm{WL}} = \psi + \phi$[2]. On the same time, measurements of the peculiar velocities of galaxies can provide an estimation of the scalar potential ψ through the momentum conservation equation for the pressure less matter fluid (Euler equation). Combining the two observations, i.e weak lensing and peculiar velocities measurements, one can then extract the ratio ψ/ϕ or equivalently the parameter η. Current limits on η are rather weak, with deviations of order unity from $\eta = 0$ still allowed, but future probes will measure the ratio ψ/ϕ with an accuracy of a few percent (e.g. Bean and Tangmatitham (2010); Daniel et al. (2010); Zhao et al. (2010); Song et al. (2011)). For a discussion on a model independent measurement of the parameter η see Amendola et al. (2013).

In this paper, we raise and investigate the following question: Can we construct a viable modified gravity model with $\phi/\psi = 1$, or in other words, is $\phi \neq \psi$ an unavoidable consequence of modifying gravity to explain the dark energy? We will try to answer this question step by step, by investigating the anisotropy equations of $f(R)$, $R + f(G)$ as well as of the more general $f(R, G)$ gravity models.

The equations for general spaces tend to be complex and in general do not admit simple solutions. For this reason in this paper we will first focus on the case of a de Sitter background. On the one hand, solutions that explain the observed accelerated expansion usually tend towards a de Sitter fixed point, and also the observed background expansion requires no deviation from $p = -\rho$ for the inferred

[2]Notice the different form of the lensing potential compared to the one defined in 2.56. The difference here is due to the different sign used for the potential ϕ in the perturbed line element (5.3) compared the one defined before.

dark energy component. On the other hand, the equations simplify significantly in this limit, which allows us to give explicit solutions that we can then discuss in detail. We comment on the behavior for other backgrounds in section 5.4, but leave a fully general study for future work. We nonetheless expect our conclusions to be quite generic for models that try to explain the dark energy.

5.2.1 The anisotropic stress in $f(R)$ models

Let us begin with the special case of $f(R)$ gravity, described by the action

$$S = \int_M d^4x\sqrt{-g}f(R), \tag{5.7}$$

which corresponds to the limit of $\xi \to 0$ of the general $f(R, G)$ models. It is well known that these models are characterized by an extra, dynamical scalar degree of freedom F, which is proportional to the first derivative of $f(R)$, $F \equiv f_R(R) \equiv f'(R)$, and its equation of motion given in (3.23). However, unless the theory is written in the so–called Jordan frame, the latter degree of freedom (also called "scalaron") is still of geometrical origin.

In $f(R)$ gravity, the anisotropic stress equation (5.15), defined in the Newtonian gauge, becomes

$$\phi - \psi = \frac{\delta F}{F} \equiv \Pi_R^{(\mathrm{eff})}, \tag{5.8}$$

which holds for any spacetime, not just de Sitter.

Since $\delta F = f_{RR}(R)\delta R$, the stress contribution is proportional to the derivative of the extra scalar degree of freedom with respect to R, that is, it depends on the evolution of the scalar $F \equiv f'(R)$. Seeking a form for the function $f(R)$ that would make $\Pi^{(\mathrm{eff})}$ vanish at all times corresponds to solving the equation $f_{RR} = 0$ with a general solution $f(R) = R + \Lambda$, i.e. of all $f(R)$ models it is *precisely* GR that satisfies this equation. In other words, the requirement of zero anisotropic stress in $f(R)$ theories is equivalent to suppressing the extra degree of freedom of the theory, leading to the GR limit. (In the PPF framework of Hu and Sawicki (2007), $f_{RR} \to 0$ corresponds to $B \to 0$, B being a parameter introduced to quantify the modification from GR).

Although it is not possible to make $\Pi^{(\mathrm{eff})}$ exactly zero at all times without reverting back to GR, one can try to make it sufficiently small for a given cosmological period, by an appropriate choice of the model parameters. This corresponds to setting f_{RR} sufficiently small for some particular initial conditions and ensuring that it stays small, by an approriate choice of model. This has been done for example in Pogosian and Silvestri (2008). The price one pays is a rapid oscillatory behavior for both the gravitational potentials and the curvature perturbation. What is more, the amplitude of the latter can grow arbitrarily. We will come back to this this later, when we will study the relevant stability conditions and will see that this is a general feature of $f(R,G)$ and other modified gravity models: the existence of anisotropic stress is related to the extra scalar degree of freedom of these models, and an attempt to suppress it causes unstable behavior. In the $f(R)$ case, suppression of the extra scalar corresponds to $f_{RR} \to 0$.

As can be seen from equation (5.8), another way to force $\Pi_R^{(\text{eff})} = 0$ would be to impose the condition $\delta R = 0$. The crucial difference between $\delta R = 0$ and $f_{RR} = 0$, is that the latter is a background requirement, i.e a requirement on the particular form of the $f(R)$ action. On the other hand, the condition $\delta R = 0$ imposes a dynamical condition on the potentials ϕ, ψ and their first and second time derivatives. If we also take into account that in that case the l.h.s implies $\phi = \psi$, we find the equation

$$\ddot{\phi} + 5H\dot{\phi} + 3H^2 \left(\frac{H^2}{\dot{H}} + \frac{k^2}{6H^2 a^2} + 2 \right) \phi = 0, \tag{5.9}$$

which not only in general is unstable, but also fixes the perturbation evolution needed to keep $\delta R = 0$, which is in general incompatible with the desired evolution of the Universe, e.g. structure formation. In other words, the requirement $\delta R = 0$ *imposes* an evolution that is in principle of no phenomenological interest. For this reason, what we seek in this paper is a condition of the first kind, i.e. a condition on model space rather than on the evolution of the perturbations.

5.2.2 The anisotropic stress in $f(G)$ models

Since $f(R)$ models do not allow for a vanishing anisotropic stress, we will instead look at the other limiting case of $f(R,G)$ models, namely those described by the action

$$S = \int_M d^4x\sqrt{-g}\,[R + f(G)]\,. \tag{5.10}$$

These models posses an instability in the presence of a matter fluid, irrespective of the form of the function $f(G)$ De Felice et al. (2010b), which rules them out as realistic scenarios, but here we just want to see whether it is possible to construct $f(G)$ models that contribute no additional effective anisotropic stress.

The first term in the action does not contribute any extra anisotropic stress. In an FLRW background, these models posses an extra scalar degree of freedom, proportional to $\xi \equiv f_G(G) = f'(G)$. The anisotropy equation in a general spacetime in this case reads as

$$\phi - \psi \equiv \Pi_G^{(\mathrm{eff})} = 4H\dot{\xi}\psi - 4\ddot{\xi}\phi + 4\left(H^2 + \dot{H}\right)\delta\xi, \tag{5.11}$$

with $\delta\xi = f_{GG}\delta G$. For a de Sitter background, the equation simplifies to $\phi - \psi = 4H_0^2 f_{GG}\delta G$. One possibility to have no anisotropic stress is to set $f_{GG} = 0$ at all times, leading to the model $f(G) = G + \Lambda$. In four dimensions G is a topological invariant Lovelock (1971), i.e. it is a total derivative and so it has no contribution to the equations of motion, and we are left only with $R + \Lambda$ for the relevant gravitational Lagrangian, which is equivalent to GR. Alternatively we require $\delta G = 0$ which suffers from the same problems as $\delta R = 0$ and does not allow in general for a sensible evolution of the perturbations.

For a general background, the similarity to the case of $f(R)$ is spoiled by the first two terms in the anisotropy equation. In general, the condition on the evolution of ϕ and ψ imposed by those terms will again be difficult to enforce as a function of time. On the other hand, if the background quantities vary only slowly, $\dot{\xi}, \ddot{\xi} \approx 0$, then the anisotropy equation can be simplified as [3]

$$\phi - \psi \equiv \Pi_G^{(\mathrm{eff})} = -4\,(1 + 3w_{\mathrm{eff}})\,\delta\xi, \tag{5.12}$$

where we used the relation

$$\frac{\dot{H}}{H^2} \approx -\frac{3}{2}(1 + w_{\mathrm{eff}}), \tag{5.13}$$

[3] We obtain effectively the same condition on scales that are well inside the horizon, $k \gg aH$, as $\delta\xi$ is in general boosted by factors of $(k/(aH))^2$ relative to ϕ and ψ.

with $w_{\rm eff} \equiv p/\rho$ being the effective equation of state parameter for the background evolution. Now, the situation is again similar to the one encountered for $f(R)$: one has either to require either $f_{GG} = 0$, $\delta G = 0$, or $w_{\rm eff} = -1/3$. As discussed above, the first condition leads to GR (in which case automatically $\dot{\xi} = \ddot{\xi} = 0$ at all times), while the second does not allow for an acceptable evolution of the perturbations. The third condition, which corresponds to the evolution of a Universe dominated by curvature, is also not very relevant given current observational results in cosmology.

5.2.3 The anisotropic stress in $f(R, G)$ models

We saw in the previous sections that the vanishing of the anisotropic stress in $f(R)$ and $f(G)$ models corresponds to either trivial or unphysical situations. We now turn to study the more general case of $f(R, G)$ models. Here, the function $f(R, G)$ has two contributions, coming from the R- and G- part respectively, and from (B.25) the anisotropy equation reads as De Felice and Suyama (2009)

$$\phi - \psi = \frac{1}{F}\left[\delta F + 4H\dot{\xi}\psi - 4\ddot{\xi}\phi + 4\left(H^2 + \dot{H}\right)\delta\xi\right]. \tag{5.14}$$

Unlike the $f(R)$ case, where we simply had to demand that $f_{RR}(R) = 0$, the nature of the anisotropy equation here does again not allow us to write down an explicit condition for the function $f(R, G)$ that would give a zero anisotropic stress contribution in a general spacetime: as in $f(G)$ models, we find extra factors of ϕ, ψ and their time derivatives. The only case for which we can find a simple condition is for the de Sitter spacetime, and therefore we shall restrict ourselves in this case for the time being. Furthermore, for models that try to explain the dark energy, it is at late times that we expect modifications of gravity to become important, and that deviations from GR should appear in observations. For such a late-time accelerating epoch, a de Sitter spacetime is expected to provide a reasonable approximation.

The anisotropy equation in de Sitter space reads as

$$\begin{aligned} \phi - \psi &= \frac{1}{F}\left[\delta F + 4H_0^2\delta\xi\right] \\ &\equiv \Pi_G^{(\rm eff)} + \Pi_R^{(\rm eff)} \equiv \Pi_{\rm tot}^{(\rm eff)}, \end{aligned} \tag{5.15}$$

where, as before, we have defined the contribution coming from the R- and G- part of the action respectively as

$$\Pi_R^{(\mathrm{eff})} \equiv \frac{\delta F}{F} \quad \text{and} \quad \Pi_G^{(\mathrm{eff})} \equiv 4H_0^2 \frac{\delta \xi}{F}. \tag{5.16}$$

Notice that this case is just the sum of the corresponding limiting cases of $f(R)$ and $R + f(G)$ gravity respectively, although now either term depends on both R and G.

We now ask the same question as before: Is it possible in this case to find a class of $f(R, G)$ models that give a zero anisotropic stress $\Pi_{tot}^{(\mathrm{eff})} = 0$, having at the same time a sensible evolution of the perturbations? By inspection of (5.16) one can see that in order for the total scalar anisotropic stress to be zero, we require that at all times

$$\Pi_R^{(\mathrm{eff})} = -\Pi_G^{(\mathrm{eff})}. \tag{5.17}$$

In other words, we require that the particular anisotropic stress contributions have equal magnitude and opposite sign at all times, or at least for the cosmological era of interest.

We can rewrite condition (5.17) using the relations

$$\delta F = F_R(R, G)\delta R + F_G(R, G)\delta G, \tag{5.18}$$

$$\delta \xi = \xi_R(R, G)\delta R + \xi_G(R, G)\delta G. \tag{5.19}$$

In de Sitter space we have additionally

$$G = 4H_0^2 R, \tag{5.20}$$

which implies that $\delta G = 4H_0^2 \delta R$. Using the last relation together with (5.18) and (5.19) (and so limiting ourselves to de Sitter backgrounds) condition (5.17) becomes

$$\left(f_{RR} + 4H_0^2 f_{RG} + 4H_0^2 f_{GR} + 16H_0^4 f_{GG}\right) \delta R = 0. \tag{5.21}$$

If $f(R, G)$ is an analytic function we have $f_{RG} = f_{GR}$, and requiring that the above equation is valid for any variation δR (see discussion in $f(R)$ section) we arrive at

$$f_{RR} + 8H_0^2 f_{RG} + 16H_0^4 f_{GG} = 0. \tag{5.22}$$

The above equation is a second order PDE with constant coefficients, for the class of functions $f \equiv f(R,G)$ that give a vanishing anisotropic stress in de Sitter space. Its general solution is

$$f(R,G) = f_1(\Omega) + R f_2(\Omega), \tag{5.23}$$

with $\Omega \equiv R - G/(4H_0^2)$, and f_1, f_2 arbitrary but analytic functions of Ω.

We specify the function $f(R,G)$ in the action, which is agnostic of quantities like H_0. For this reason it is preferable to consider a more general class of models with

$$\Omega \equiv \left(R - \frac{G}{M^2}\right), \tag{5.24}$$

with M a parameter with mass dimensions, so that $\Pi^{(\text{eff})} \to 0$ corresponds to the special case of a model with a de Sitter expansion rate of $H_0 = M/2$. As we will also discuss later on, the mass parameter M controls which of the two contributions in $f(R,G)$ dominates.

Assuming that the de Sitter point exists and is stable, we see that it is in principle possible to find a non-trivial class of $f(R,G)$ models that give exactly zero anisotropic stress in de Sitter space at all times, by selecting a model in the class (5.23). However, as we will see by studying the stability of de Sitter space below, the case $M \to 2H_0$ corresponds to a singularity for the actual model, and therefore the model cannot be viable. Furthermore, we will see that the anisotropic stress cannot become arbitrarily small, since this will cause unstable behavior for the curvature perturbations.

5.3 Anisotropic stress and stability for a de Sitter background

There are different stability criteria that a gravitational theory aiming to describe the late time acceleration should satisfy, each leading to a different condition for the form of the function $f(R,G)$. At the background level, a viable model should give rise to sufficiently long radiation and matter eras, as well as a transition to a stable de Sitter era Amendola et al. (2007b); Nojiri and Odintsov (2006b); Zhou et al. (2009). Furthermore, avoidance of singularities and of rapid collapse of perturbations (positivity of the sound speed) as well as agreement with local gravity constraints should be ensured Davis (2007b); Amendola et al. (2007a); De Felice and Tsujikawa (2009d). Of great importance is also the absence of ghost like degrees of freedom De Felice et al. (2006); Chiba (2005b); Nunez and Solganik (2005b). For the class of $f(R,G)$ models the latter requirement translates into $f_R(R,G) > 0$.

Modified gravity models of the type $f(R)$ or $R + f(G)$ suffer from a curvature singularity at very early times of the cosmological evolution Starobinsky (2007); Tsujikawa (2008b); Frolov (2008); De Felice and Tsujikawa (2009b); Sotiriou (2007). The latter singular behavior can lead to oscillations of the scalar degree of freedom with infinite amplitude and frequency. As explained in Frolov (2008), the singularity lies at a finite field value and energy level and therefore is easily accessible. We will see in the following that this singularity is a feature of $f(R,G)$ models as well.

In this paper we are interested in the classical stability, and particularly its connection to the effective anisotropic stress. As we will show and discuss below, the attempt of turning off or making sufficiently small the effective anisotropic stress for a de Sitter background leads to serious stability problems that question the actual viability of models with vanishing $\Pi^{(\mathrm{eff})}$.

5.3.1 Existence of a de Sitter point

Since we will specifically study the behaviour near the de Sitter point, it is necessary that this solution exists for the models of interest. De Sitter space is a vacuum, maximally symmetric space described by the conditions

$$H = H_0 = \text{constant} > 0, \quad \dot{R} = \dot{G} = \dot{F} = \dot{\xi} = 0. \tag{5.25}$$

Furthermore, in maximally symmetric spaces any curvature invariant can be expressed as a function of the Ricci scalar, and particularly for the Gauss–Bonnet term we get

$$G = \frac{R^2}{6}. \tag{5.26}$$

We can derive the condition for the existence of the de Sitter point by taking the trace of the equations of motion (3.45) and using relations (5.25), (5.26), to arrive at

$$F(R)R + 2G(R)\xi(R) - 2f(R) = 0, \tag{5.27}$$

where everything is assumed to be expressed in terms of the Ricci scalar and evaluated on de Sitter space. The cases $\xi = 0$ and $F = 1$, give the relevant conditions for $f(R)$ and $R + f(G)$ gravity respectively. Solving the algebraic equation given above, we get the de Sitter point solution, which in general is not unique. Minkowski space corresponds to the special case of $R_0 = H_0 = 0$.

For the models of the type (5.23), we find with the help of equation (5.27) that the de Sitter point is given by solutions of the equation

$$f_1(u) + u f_2(u) = 0. \tag{5.28}$$

and $R = 2u$. The next step in our analysis will be the study of the stability of de Sitter space at both homogeneous and inhomogeneous level.

5.3.2 Homogeneous perturbations

Now we turn to study the stability of the de Sitter solution, first with respect to homogeneous (background) perturbations. As we will see, there is a strong link between effective anisotropic stress and stability in modified gravity models.

Let us consider the time–time component of the Friedman equation (3.45) and perturb it linearly around the de Sitter solution $H = H_0$

$$H(t) = H_0 + \delta H(t). \tag{5.29}$$

Under perturbation (5.29) the perturbed function $f(R, G)$ reads as

$$f = f_0 + F_0 \delta R + \xi_0 \delta G, \tag{5.30}$$

and similar expressions hold for the other quantities of interest. The explicit formulas and calculations for any space can be found in the Appendix.

Now, evaluating relations (B.13)–(B.15) and using conditions (5.25), we can write the linearized perturbed modified Friedman equation (3.59) in the form

$$C_1\delta\ddot{H} + C_2\delta\dot{H} + C_3\delta H = 0, \tag{5.31}$$

with the constants C_1, C_2 and C_3 defined in the Appendix. There is no constant term since we know that de Sitter, $\delta H \equiv 0$, is a solution. This equation then admits an exponential solution of the form

$$\delta H = Ae^{a^+t} + Be^{a^-t}, \tag{5.32}$$

with

$$a^{\pm} \equiv \frac{3}{2}H_0 \pm \sqrt{\frac{9}{4}H_0^2 - \left(\frac{F}{3\omega} - 4H_0^2\right)}, \tag{5.33}$$

and

$$\omega \equiv F_R + 4H_0^2\left(2F_G + 4H_0^2\xi_G\right), \tag{5.34}$$

where we dropped the subscript "0" from F_R e.t.c for simplicity.

From solution (5.33), we can read off the condition for de Sitter space stability with respect to homogeneous perturbations:

$$\frac{F}{3\left[F_R + 4H_0^2\left(2F_G + 4H_0^2\xi_G\right)\right]} - 4H_0^2 \geq 0. \tag{5.35}$$

The latter condition ensures that the de Sitter point is an attractor for the particular $f(R, G)$ model under study, which is important for the viability of a cosmological model of gravity. The limit $\xi \to 0$ in (5.35) gives the corresponding condition for $f(R)$ gravity, that has been derived before in Faraoni (2005),

$$\frac{F}{3F_R} - 4H_0^2 \geq 0, \tag{5.36}$$

while when $F \to 1$ we get a similar condition for the $R + f(G)$ models also derived in De Felice and Tsujikawa (2009b)

$$\frac{1}{48H_0^4 f_{GG}} - 4H_0^2 \geq 0. \tag{5.37}$$

The stability condition (5.35) is general, but now we can check what it tells us for the class of models that give a zero anisotropic stress, described by equation (5.24) as $M^2 \to 4H_0^2$. We can see that in this case necessarily $\omega \to 0$, and so the eigenvalues (5.33) tend to infinity[4]. In particular, when ω is exactly zero, which corresponds to the case of a vanishing anisotropic stress, it is not possible to reach the de Sitter state without triggering a singularity in the model: the quantity $C_1 = 18H\omega$ in equation (5.31) goes to zero as we approach de Sitter, together with $C_2 \to 0$ (see appendix B.1). In general $\delta\ddot{H} \sim (C_3/C_1)\delta H \to \infty$ which requires $\delta\ddot{H}$ to diverge in order to satisfy the evolution equation, except possibly for a lower dimensional and thus infinitely fine-tuned set of trajectories in specific models. We will show and discuss this explicitly in section 5.5 considering examples for particular $f(R, G)$ models.

If the effective anisotropic stress is not exactly zero, but sufficiently small, in that case the rapid and large background oscillations render the linear analysis unreliable, i.e the evolution becomes non linear. We will come back to this again in section 5.5.

Similarly, in the $f(R)$ and $R + f(G)$ cases, where the zero anisotropic stress condition was that $f_{RR}(R) = 0$ and $f_{GG}(G) = 0$ respectively, conditions (5.36) and (5.37), give the obvious result that one gets infinities when trying to suppress the extra degree of freedom. The difference with the more general $f(R, G)$ models is that the singularity appears for a finite value of the mass parameter M of the model, while in $f(R)$ and $R + f(G)$ the same happens for rather trivial cases. We conclude therefore that a $f(R, G)$ type model that has no anisotropic stress in a de Sitter background cannot dynamically reach this background solution.

5.3.3 Inhomogeneous perturbations

In this subsection we will study the behavior of inhomogeneous perturbations in de Sitter space and we will first show that the stability condition coincides with the

[4]Since the no-ghost condition requires that $F > 0$, the question whether the background solution moves towards or away from de Sitter depends on whether $\omega \to 0^+$ or $\omega \to 0^-$.

stability condition derived in the section on homogeneous perturbations. We will then make the relation between anisotropic stress and stability clear by studying the evolution of the perturbations. The full set of perturbation equations together with some useful relations can be found in the Appendix.

We follow De Felice and Suyama (2009) and choose the gauge invariant expression

$$\Phi \equiv \frac{1}{2F}\left[\delta F + 4H_0^2\delta\xi\right]. \tag{5.38}$$

for the gravitational potential, as it reduces to ϕ in the Newtonian gauge and remains well-defined for a de Sitter background. For that background, we find that the potential is just given by

$$\Phi = \frac{(f_{RR} + 8H_0^2 f_{RG} + 16H_0^4 f_{GG})\,\delta R}{2F}, \tag{5.39}$$

where we used the fact that $f_{RG} = f_{GR}$ and that in de Sitter space we have $\delta G = 4H^2\delta R$. From condition (5.21) we see that in de Sitter space and for models that have no anisotropic stress, Φ is necessarily zero. However, let us assume that we are not exactly in this limit. Then by substituting the expression of δR in terms of the gauge invariant Φ, relation (B.31), we arrive at the evolution equation,

$$\ddot{\Phi} + 3H_0\dot{\Phi} + \left(\frac{k^2}{a^2} + m_{\rm eff}^2\right)\Phi = 0, \tag{5.40}$$

with

$$m_{\rm eff}^2 \equiv \frac{F}{3\omega} - 4H_0^2, \tag{5.41}$$

for ω defined in (5.34), and $a(t) \propto \exp(H_0 t)$. $m_{\rm eff}^2$ is the effective mass of the Klein–Gordon type equation for the scalar perturbation in de Sitter space, and has a purely geometrical origin. Equation (5.40) reduces to that of $f(R)$ and $R + f(G)$ for the limits of $\xi \to 0$ and $F \to 1$ respectively.

As $k \to 0$, the requirement for superhorizon stability dictates that the effective mass is positive,

$$m_{\rm eff}^2 > 0\ , \tag{5.42}$$

which leads to the same stability condition as derived before with the homogeneous analysis, equation (5.35). Therefore, the two stability criteria, with respect to homogeneous and inhomogeneous perturbations respectively, lead to the same conditions, as it is the case for $f(R)$ gravity as well Faraoni (2005).

Turning back to the effective anisotropic stress, we can see that considering again the class of models found in (5.23) and requiring $M \to 2H_0$ ($\Pi^{(\mathrm{eff})}_{\mathrm{tot}} \to 0$), will make the denominator of (5.41) go to zero so that

$$\lim_{M \to 2H_0} m^2_{\mathrm{eff}} \equiv \lim_{\omega \to 0} \left(\frac{F}{3\omega} - 4H_0^2 \right) = \pm\infty, \tag{5.43}$$

depending on the sign of ω as it approaches zero. In the case of positive infinity the stability condition is not violated, while the minus infinity will obviously violate the stability condition, as it would make the effective mass negative (tachyonic).

The effective mass going to infinity means that the scalar degree of freedom becomes frozen and so it is effectively suppressed. This is also the case in the special cases of $f(R)$ and $R + f(G)$ gravity, as can be seen by inspection of equations (5.36) and (5.37) for $f_{RR} \to 0$ and $f_{GG} \to 0$ respectively. However, here the singularity appears in a non trivial way, i.e for a critical value of the mass parameter M where the two different contributions, i.e the R- and the G- contribution in (5.17) balance each other. By consequence, in $f(R, G)$ type models, the anisotropic stress is related to the extra scalar degree of freedom of the theory. As the same happens in scalar-tensor models (e.g. equation (43) of Amendola et al. (2008)), and also in DGP where the absence of anisotropic stress requires the crossover scale to diverge, $r_c \to \infty$, which effectively restores GR, we conjecture that this is a quite general feature of modified gravity models. In addition, in the $f(R, G)$ case, turning the anisotropic stress off (or trying to make it sufficiently small) has a direct impact on the stability and time evolution of the model.

To see what happens when the mass diverges, it is possible to study the solution of the evolution equation (5.39) using a WKB approximation for $\ddot{\Phi} \ll 1$. We discuss the procedure in more detail in appendix B.3, where we show that the solution in this regime, and for a sufficiently large effective mass m_{eff}, is approximately given by

$$\Phi(t) \approx \sum_{\pm} C_{\pm} \mathrm{e}^{(-H_0 \,\pm\, 2i m_{\mathrm{eff}})t}, \tag{5.44}$$

with $C_{\pm}$ constants and $H_0 > 0$. From the above solution it can be seen that the frequency of the oscillations is proportional to $m_{\rm eff}$. Suppressing the anisotropic stress leads to a very large effective mass and thus to a very rapid oscillation of Φ. Although we have shown this here only for the de Sitter limit, we expect that the result is more general, and similar oscillations have been seen for example in Pogosian and Silvestri (2008) during matter domination for numerically reconstructed $f(R)$ models which mimic GR at early times.

From relation (5.41) it can be seen that a large effective mass corresponds to a small anisotropic stress and a small potential Φ. However, this is not true for the curvature perturbation δR (or δG) which has an amplitude that is $\propto m_{\rm eff}$,

$$\delta R(t) = 6\left(m_{\rm eff}^2 + 4H_0^2\right)\Phi(t) \tag{5.45}$$

and as $m_{\rm eff}^2 \gg 1$ one can get large curvature perturbations, that grow significantly at earlier times. The latter behavior, occurring while we try to suppress the effective anisotropic stress, is very similar to the one caused by the singularity found in Starobinsky's "disappearing cosmological constant" model, Starobinsky (2007) and Frolov (2008). In that case, the singularity appeared in the high curvature limit of the particular model, while in our case it appears in the model space of different $f(R,G)$ models respectively. The latter oscillatory behavior endangers the stability of the actual model as has been pointed out in Starobinsky (2007) and Frolov (2008), and for an explicit discussion on the subject the reader is referred to Starobinsky (2007); Frolov (2008).

Another interesting aspect of the models of the type $f(\Omega)$ concerns the sound speed. The propagation speed in de Sitter space equals the speed of light ($c_s^2 = 1$). However, using the formula derived in De Felice and Suyama (2009) [5] we find that the sound speed in a general background is given by

$$c_s^2 = 1 + \frac{8\dot{H}}{4H^2 - M^2} \equiv 1 + \left(\frac{2}{1-\gamma}\right)\frac{\dot{H}}{H^2} \tag{5.46}$$

where $\gamma \equiv \frac{M^2}{4H^2}$ is a dimensionless parameter (constant in a de Sitter background). $\gamma \gg 1$ implies that the Ricci scalar part of the $f(R,G)$ contribution to the aniso-

[5] It is relation (6.20) in De Felice and Suyama (2009).

tropic stress dominates, while for $\gamma \ll 1$ the Gauss–Bonnet part is larger. $\gamma = 1$ corresponds to the case where the two contributions in $f(R, G)$ models become equal and cancel.

We can calculate $\dot{H}$ from the equations of motion, and for this particular class of models we get

$$\dot{H} = \frac{(1-\gamma)(H\dot{\xi} - \ddot{\xi})}{8H^2(F + 4H\dot{\xi})}, \tag{5.47}$$

which can then be substituted in (5.46). However, considering an expansion characterized by an effective w_{eff}, equation (5.13), the sound speed takes the form

$$c_s^2 \approx 1 - \frac{3(1 + w_{\text{eff}})}{1 - \gamma}. \tag{5.48}$$

Assuming a background with $w_{\text{eff}} \neq -1$, we immediately see that as $\gamma \to 1$, $c_s^2 \to \infty$. The sound speed becomes negative for $\gamma < 1$ (Gauss–Bonnet part dominates) and positive for $\gamma > 1$ (Ricci scalar part dominates) respectively. The value $\gamma = 1$, which corresponds to the effective anisotropic stress becoming zero is the critical value where the sound speed diverges and changes sign. In other words, if one wishes to enforce $c_s \leq 1$ then one has to ensure that the model lies sufficiently far from the regime where the two contributions balance.

5.4 General and matter-dominated background

In this section we extend the analysis to a general background evolution, and then consider specifically the important case of matter domination. In general we have to consider equation (5.14). In this equation, δF and $\delta\xi$ are functions of δR and δG through Eqs. (5.18) and (5.19). These in turn can be expressed in terms of the metric perturbations, ϕ and ψ, see e.g. De Felice and Suyama (2011b). In the small-scale limit, $k \gg aH$, we find that $\phi = \psi$ implies

$$f_{RR} + 16(H^2 + \dot{H})(H^2 + 2\dot{H})f_{GG} + 4(2H^2 + 3\dot{H})f_{RG} = 0. \tag{5.49}$$

In order to re-transform this condition into one involving only R and G, we can eliminate H and $\dot{H}$ with the help of equations (5.1) and (5.2),

$$\begin{aligned} H^2 &= \frac{1}{12}\left(R+\sqrt{R^2-6G}\right), &(5.50)\\ \dot{H} &= -\frac{1}{6}\sqrt{R^2-6G}. &(5.51)\end{aligned}$$

Using this prescription we find for the general no-anisotropic-stress condition

$$\begin{aligned} 0 &= f_{RR}+\frac{2}{9}\left(-9G+2R\left(R-\sqrt{R^2-6G}\right)\right)f_{GG}\\ &+\frac{2}{3}\left(R-2\sqrt{R^2-6G}\right)f_{RG}. \qquad (5.52)\end{aligned}$$

While it is difficult to find general solutions, we can instead study the case for a background evolving with a given $w_{\rm eff}$, as defined in (5.13). We notice that in this case equation (5.49) can be written as

$$\begin{aligned} 0 &= f_{RR}-2H^2(5+9w_{\rm eff})f_{RG}\\ &\quad +8H^4(2+9w_{\rm eff}(1+w_{\rm eff}))f_{GG}. \qquad (5.53)\end{aligned}$$

For $w_{\rm eff}=-1$ (de Sitter expansion) we recover equation (5.22), while for $w_{\rm eff}=0$ (matter dominated expansion) we find

$$f_{RR}-10H^2f_{RG}+16H^4f_{GG}=0. \qquad (5.54)$$

The Hubble parameter in the latter equation can be eliminated in favor of R and G using equations (5.1) and (5.2) evaluated for a matter background,

$$R=3H^2,\quad G=-12H^4,\quad G=-\frac{4}{3}R^2. \qquad (5.55)$$

We now try to construct an explicit example for a model that has no anisotropic stress during matter domination. For this purpose, we make an ansatz

$$f(R,G)=R+G^n\beta(R) \qquad (5.56)$$

Here we take β as an a-priori general function of R. Inserting this model into equation (5.54) and using (5.55) we can re-express the condition in terms of R only. We find that β needs to satisfy the following differential equation:

$$2n(n-1)\beta + 5nR\beta' + 2R^2\beta'' = 0. \tag{5.57}$$

This equation has clearly a power-law solution, $\beta(R) = cR^m$, with

$$m_{1,2} = \frac{1}{4}\left(2 - 5n \pm \sqrt{4 + n(9n-4)}\right), \tag{5.58}$$

and the general solution is of the form

$$f(R,G) = R + c_1 G^n R^{m_1} + c_2 G^n R^{m_2} \tag{5.59}$$

where $m_i = m_i(n)$ is given by the equations for m_1 and m_2 above.

A successful model with zero anisotropic stress should at the same time satisfy the Friedmann equation as well. During matter domination we can write the latter as

$$\begin{aligned} 0 &= R^2 f_{RR} - 2G^2 f_{GG} + RG f_{RG} - \frac{1}{6} R f_R + \frac{1}{6} G f_G \\ &\quad - \frac{1}{6} f + \frac{3}{4}\rho_0 R. \end{aligned} \tag{5.60}$$

Here we chose R and G so as to correspond to the partial derivatives, since the choice is not unique. The final term is due to $\rho_m(t) \propto t^{-2} \propto R$. Inserting a model of the form (5.59) but for a general exponent m, we find the condition

$$-6m^2 + m(7-6n) + (n-1)(12n-1) = 0. \tag{5.61}$$

A model of this form that satisfies simultaneously (5.58) and (5.61) allows for a matter dominated evolution and contributes no anisotropic stress during that period. This is the case for

$$n = \frac{1}{90}\left(11 \pm \sqrt{41}\right), \quad m = \frac{1}{180}\left(61 \pm 11\sqrt{41}\right) \tag{5.62}$$

where one needs to use either both positive or both negative signs. An additional solution is given by $m = 0$ and $n = 1$, which is just GR.

Therefore, there is at least one model in the context of $f(R, G)$ gravity that is able to give a zero effective anisotropic stress, in the subhorizon limit of a matter background. Numerically we find that the evolution of the model close to the matter point can be stable for a significant amount of time, although the matter point is not an attractor solution (and thus the anisotropic stress does in general not vanish exactly).

Let us now turn attention to homogeneous perturbations around the matter point, keeping the function $f(R, G)$ in its general form for the start. In Appendix B.1 we calculate the evolution of homogeneous perturbations for a general expansion $a(t) \propto t^p$. For the matter case we get for $p = 2/3$

$$\delta\ddot{H} + \left(\frac{\dot{\omega}}{\omega} + 9H\right)\delta\dot{H} + m_{\text{eff}}^2\delta H = \frac{\delta\rho_{\text{m}}}{18H\omega}, \tag{5.63}$$

with the effective mass defined as

$$m_{\text{eff}}^2 \equiv \frac{F}{3\omega} \equiv \frac{F}{3\left[F_R + 4H^2\left(2F_G + 4H^2\xi_G\right)\right]}. \tag{5.64}$$

Equation (5.63) can be solved approximately at the WKB regime using an iterative approach Starobinsky (2007); Tsujikawa (2008b),

$$\delta H = \delta H_{(\text{osc.})} + \delta H_{(\text{ind.})}. \tag{5.65}$$

$\delta H_{(\text{osc.})}$ is the solution describing oscillations of the scalar degree of freedom, obtained setting $\delta\rho_{\text{m}} = 0$. $\delta H_{(\text{ind.})}$ denotes the matter induced part, which is obtained by turning off all the derivatives on the l.h.s of equation (5.63). We assume that $\delta H_{(\text{osc.})} \ll \delta H_{(\text{ind.})}$, so that the deviations from GR are sufficiently small.

Stability in this case requires, apart from the no–ghost condition $F > 0$, that the effective mass is positive,

$$m_{\text{eff}}^2 > 0. \tag{5.66}$$

Let us turn attention to the oscillatory part of the solution (5.65). It can be obtained using the WKB approximation, by assuming the solution is a slowly varying quantity in time,

$$\delta H_{(\mathrm{osc})} \approx A e^{i\theta(t)}, \tag{5.67}$$

with $\ddot{\theta} \ll 1$. Plugging above ansatz into (5.63), and after some algebra, we find that

$$\delta H_{(\mathrm{osc})} \approx \sum_{\pm} A_{\pm} \exp\left[-\frac{1}{2}\int_{t_0}^{t} dt' \left(9H + \frac{\dot{\omega}}{\omega}\right)\right] \times \exp\left[\pm 2i \int_{t_0}^{t} dt' m_{\mathrm{eff}}^2\right]. \tag{5.68}$$

with A a constant. Using the fact that $H \equiv H_{\mathrm{m}}(t) = 2/(3t)$, and performing the integration in the first exponential we arrive at,

$$\delta H_{(\mathrm{osc})} \approx \sum_{\pm} \frac{A_{\pm}}{t^3 \omega^{1/2}} \exp\left[\pm 2i \int_{t_0}^{t} dt' m_{\mathrm{eff}}^2\right]. \tag{5.69}$$

The second integration can be performed after choosing a particular model. From (5.69) one can see that the amplitude of the oscillating solution grows as one goes backwards in time, which is exactly the behavior pointed out for $f(R)$ models in Starobinsky (2007); Tsujikawa (2008b); Frolov (2008), and was due to a curvature singularity as explained in Frolov (2008). Therefore, $f(R,G)$ models suffer from the same problem too.

We also notice that the models of type (5.23) that have no anisotropic stress during a de Sitter phase with specific expansion rate $H_0 = M/2$ will pass through $\omega = 0$ and thus $m_{\mathrm{eff}} \to \infty$ in any background if the expansion rate $H(t)$ crosses this critical value $M/2$.

A different, general way to decrease the anisotropic stress is to move close to GR by decreasing the deviations from the extra $f(R,G)$ contributions, which effectively implies

$$f_{RR}, f_{RG}, f_{GG} \ll 1. \tag{5.70}$$

In this case, we also make ω small while $F \to 1$. Again this will lead to rapid oscillations, and we suspect that this is the reason for those seen in Pogosian and Silvestri (2008). Once the genie of extra degrees of freedom is out of the bottle, it is difficult to push it back in without further complications.

5.5 Toy models

In this section we will study the de Sitter behavior for some characteristic cases of the class of models found in (5.23). For the sake of generality we will consider

$$\Omega = R + \epsilon \frac{G}{M^2}, \tag{5.71}$$

with $\epsilon = \pm 1$. The particular class of models with a vanishing of the anisotropic stress, found in (5.23), correspond to $\epsilon \to -1$ and $M \to 2H_0$.

First note that, for the class of models (5.23), it is possible to parametrize both the de Sitter existence and stability conditions in terms of the parameter γ, which controls the different regimes of the model. For simplicity and illustration let us assume that $f_2 = 0$. Then, the de Sitter condition (5.27) becomes

$$\left(\frac{\gamma + \epsilon}{2\gamma + \epsilon}\right) f_\Omega \Omega_0 - f(\Omega_0) = 0, \tag{5.72}$$

with $f_\Omega \equiv f_\Omega(\Omega_0)$, and

$$\Omega_0 = 6H_0^2 \left(\frac{2\gamma + \epsilon}{\gamma}\right). \tag{5.73}$$

Furthermore, for the de Sitter stability condition (5.42) we get

$$\left(\frac{\gamma^2}{\gamma^2 + 2\epsilon\gamma + 1}\right) \frac{f_{\Omega\Omega}}{f_\Omega} \geq 2 \left(\frac{\gamma}{2\gamma + \epsilon}\right) \Omega_0, \tag{5.74}$$

We will assume that $\Omega_0 > 0$, $\gamma > 0$ and real. The limits $\gamma \to \infty$ and $\gamma \to 0$ correspond to the pure $f(R)$ and $f(G)$ regimes respectively.

In principle, we will assume that through (5.72) we can express Ω_0 in terms of γ and the other possible parameters of the model as $\Omega_0 = \Omega_0(\gamma, c_i)$, and then use (5.74) to get a constraining condition.

5.5.1 $f(\Omega) = \Omega + \Omega \ln\left(\Omega/c^2\right)$

Here, c is a positive constant of mass dimensions. This model is able to re-produce a late-time acceleration, since at late times $\Omega \ll 1$, and the logarithmic term will dominate. In four dimensions the linear term Ω is essentially equivalent to the Ricci scalar R since the Gauss-Bonnet term does not contribute to the equations of motion. The absence of a Minkowski solution makes this model rather unrealistic.

A non trivial de Sitter solution can be found using (5.27)

$$\Omega_0 = c e^{\epsilon/\gamma}, \tag{5.75}$$

and the Hubble parameter is then trivially given by (5.73).

The stability condition (5.74) yields

$$\frac{2\gamma^2 - 1}{2\gamma + \epsilon} \geq 0. \tag{5.76}$$

For both branches, $\epsilon = \pm 1$, de Sitter space is stable when $\gamma > \sqrt{2}/2$.

To illustrate the singularity when trying to reach de Sitter, we set $\gamma = 1$ and for simplicity $c = \sqrt{6e}$ so that the de Sitter solution is given by $H_0 = 1$. Expanding the equation of motion in δH we find to first order,

$$2\left(1 + 2(\delta\dot{H})^2\right)\delta H + O\left((\delta H)^2\right) = 0. \tag{5.77}$$

Only in the second order term a contribution $\delta\ddot{H}(\delta H)^2$ appears. We notice that it is not possible to solve the first term for real δH, so that necessarily $\delta\ddot{H} \propto 1/\delta H$ will diverge when we try to dynamically reach de Sitter. The only exception is $\delta H = 0$, i.e. the solution that is always de Sitter.

5.5.2 $f(\Omega) = \Omega + c\Omega^n$

In the context of $f(R)$ gravity, models of this type were suggested as an explanation for late time acceleration Capozziello et al. (2003); Carroll et al. (2004) with $n < 0$, while models with $n > 0$ can lead to acceleration at early times and explain inflation. Furthermore, it was found that de Sitter space is unstable unless $cn < 0$ Faraoni (2005). Here, we assume that $cn > 0$, otherwise the no–ghost condition $F > 0$ could be violated.

The de Sitter point equation (5.72) gives two solutions, namely $\Omega_0 = 0$ which corresponds to Minkowski spacetime and a non trivial de Sitter one,

$$\Omega_0 = \left[\frac{\gamma}{c(\gamma(n-2) + \epsilon(n-1))}\right]^{1/(n-1)}. \tag{5.78}$$

In order for Ω_0 to be real and positive one has to ensure that the quantity in the denominator in the latter relation is positive. We shall also require that the Hubble parameter, as given implicitly in relation (5.73), will be real and positive too.

For both branches $\epsilon = \pm 1$, de Sitter is always unstable when $n < 0$. For $n > 0$, it is always unstable if $\epsilon = 1$, but for $\epsilon = -1$, $n > 2$, the stability condition (5.74) gives

$$\frac{n-1}{n-2} < \gamma < \frac{n + \sqrt{n/2} - 1}{n-2}, \tag{5.79}$$

with both Ω and H_0 being real and positive.

To avoid a superluminal sound speed, the model should lie in the $f(R)$ regime, characterized by $\gamma > 1$, which is satisfied here, as the right branch of above inequality approaches the value 1^+ as $n \to \infty$. Further, for $n > 2$, Minkowski space is always stable.

To consider the equation of motion close to de Sitter, we set $\epsilon = -1$, $\gamma = 1$ and choose $c = -6^{(1-n)}$, for which $H_0 = 1$. We also assume that $n \neq 1$. We again expand in δH. The lowest order equation becomes now

$$\left(1 + 2n(\delta \dot{H})^2\right) \delta H + O\left((\delta H)^2\right) = 0. \tag{5.80}$$

Again the second derivative of δH appears only at order $(\delta H)^2$. This time we can in principle make the first order term vanish for $n < 0$, which would allow to cross $\delta H = 0$ with a finite second derivative. However, there are two problems: Firstly, we can only cross, not move into and stay on $\delta H = 0$, since locally we need $\delta H \sim (t - t_0)/\sqrt{-2n}$ to avoid triggering the instability, and secondly this requires an infinite amount of fine-tuning in the initial conditions: we need to reach de Sitter at exactly the right speed, else we are either repelled, or a catastrophe engulfs the Universe. So in reality again it is impossible to reach de Sitter dynamically.

5.5.3 $f(\Omega) = \Omega + c_0\lambda\left((1 + \frac{\Omega^2}{c_0^2})^{-n} - 1\right)$

This is a straightforward generalization of Starobinsky's disappearing cosmological constant model Starobinsky (2007). It was proposed in the context of $f(R)$ gravity as a late time acceleration model, that has a vanishing cosmological constant in Minkowski spacetime. It is trivial to check that Minkowski, $f(0) = 0$, is indeed a solution, but unstable since $f_{\Omega\Omega}(0) < 0$.

The model is characterized by three parameters, c_0, λ, $\gamma > 0$.

From the de Sitter point equation, one can find an expression for λ as a function of γ, and $x_1 \equiv \Omega_0/c_0$

$$\lambda = \frac{x_1(g_0 - 1)(1 + x_1^2)^{n+1}}{[x_1^2(2ng_0 + 1) - (1 + x_1^2)^{n+1} + 1]}, \tag{5.81}$$

where $g_0 \equiv (\gamma + \epsilon)/(2\gamma + \epsilon)$. Taking the limit $\gamma \to \infty$ in the above expression one recovers the one given in Starobinsky's paper Starobinsky (2007).

Let us assume that c_0 is of the order of the de Sitter scale, $\Omega_0/c_0 \equiv x_1 = 1$. The de Sitter stability condition then reads

$$2(\gamma + \epsilon)n^2 + (2\gamma + 1)n + (2\gamma + \epsilon)(1 - 2^n) \leq 0, \tag{5.82}$$

For $n = 1$ de Sitter is stable if

$$-\frac{2\gamma}{2\gamma + \epsilon} \leq 0, \tag{5.83}$$

which is never satisfied for both branches $\epsilon = \pm 1$. However, choosing $x_1 = 1/2, n = 1$, we find that de Sitter is stable for $\epsilon = 1$ and $\gamma > 1/2$, as well as for $\epsilon = -1$ and $\gamma > 0.68$.

Stability can be established for a wide range of the model parameters, but that would require a detailed exploration of the parameter space of $\{c_0, \lambda, n\}$, and we are not interested in this here.

For the critical case $\epsilon = -1$, $\gamma = 1$, choosing the λ of (5.81) and in addition $c_0 = 6H_0^2$ for simplicity, we find to first order in δH

$$\left[1 - \frac{n\left(H_0^4 - 2n(\delta\dot{H})^2\right)}{(2^n - 1)H_0^4}\right]\delta H + O\left((\delta H)^2\right) = 0. \tag{5.84}$$

This equation is of the same kind as the one found for the previous toy model, and it leads to the same behaviour. The special case $n = 1$ leads to the equation $(\delta\dot{H})^2\delta H = 0$, which prohibits any crossing of $\delta H = 0$ as otherwise $\delta\ddot{H}$ has to diverge.

5.6 Conclusions

In this chapter we studied the anisotropic stress in $f(R,G)$ type modified gravity models. In this context, we investigated the possibility of finding models that are able to mimic GR at least in the sense that they do not create an additional, effective contribution to the anisotropic stress, i.e $\phi = \psi$ in the Newtonian gauge. For the needs of our analysis, we also derived the necessary background stability conditions. We started by considering the case of a de Sitter background, since this allowed us to find the general class of models with vanishing anisotropic stress. The de Sitter case is in addition interesting as current observations indicate that the Universe is approaching this state. We further considered the general case in the small-scale limit, and in more detail the case of a matter dominated expansion.

We find that for de Sitter expansion, the anisotropic stress is inextricably linked to the presence of an extra scalar degree of freedom. The same is true for a matter expansion in the subhorizon limit. Suppressing the effective, geometric anisotropic stress is equivalent to suppressing the extra degree of freedom, which either requires the model to revert back to GR or else leads to an instability in the background evolution. In addition, it leads problematic effects like rapid oscillations of the gravitational potential and the curvature perturbation (with possible runaway production of scalar particles). The same problems appear when one tries to generally decrease the extra degrees of freedom through a model reconstruction, in order to obtain an evolution similar to GR. We think that this has been observed for numerically reconstructed $f(R)$ models in a matter dominated background Pogosian and Silvestri (2008), indicating that it is more general and not restricted to de Sitter.

Furthermore, our stability analysis reveals that the curvature singularity present in $f(R)$ models Starobinsky (2007); Tsujikawa (2008b); Frolov (2008) appears in the more general $f(R,G)$ case as well. What is more, its unwanted effect on the behavior of curvature perturbation is amplified for all models that try to suppress the anisotropic stress by decreasing f_{RR}, f_{RG} and f_{GG}. In these cases we find rapid curvature oscillations with arbitrarily high amplitude.

In the case of a pure matter dominated background, we were able to construct an explicit model that gives a zero effective anisotropic stress in the subhorizon limit. At late times, when the gravity modifications are expected to appear and the evolution ceases to be matter dominated, this model will no longer give $\phi = \psi$. This could possibly be avoided by constructing such models for a whole expansion history including late-time accelerated expansion. However, such a procedure would necessarily involve significant fine-tuning as changes in the expansion rate would have to coincide with changes in the behavior of the function $f(R, G)$, which would in general depend sensitively on initial conditions. This appears to be rather difficult to construct. In addition, as discussed above, such a model would not be able to reach the de Sitter state without encountering a singularity.

While the link between effective anisotropic stress and the scalar degree of freedom of the theory was studied here in the context of $f(R, G)$ models, it is also present in scalar-tensor and DGP models: If a scalar-tensor model is coupled to the Ricci scalar in the action through $F(\varphi)R$ then the anisotropic stress is proportional to $(F'/F)\delta\varphi$ and the analogy to the $f(R)$ case is obvious. In DGP, the effective anisotropic stress vanishes for $r_c \propto M_4^2/M_5^3 \to \infty$ where M_4 and M_5 are the four- and five-dimensional Planck scales Koyama and Maartens (2006); Lue et al. (2004). In this limit, the 5-dimensional part of the action is suppressed and only the usual 4D Einstein-Hilbert action remains.

We conjecture that suppressing the effective anisotropic stress in modified gravity models is difficult, if not impossible, to achieve in a realistic scenario. In models with a single extra degree of freedom that we looked at ($f(R)$, $f(G)$, scalar-tensor models and DGP) it is not possible at all to have no effective anisotropic stress except in the GR limit. In more complicated cases like $f(R, G)$ it is possible to cancel the contributions to the effective anisotropic stress coming from several extra degrees of freedom, but this appears to be fine tuned and the resulting models tend to develop fatal singularities. This reinforces the role of the anisotropic stress as a key observable for current and future dark energy surveys. While the observation of a strong anisotropic stress would point towards a modification of GR, the absence of anisotropic stress would present a significant challenge for modified gravity models and would require strong fine-tuning, which in turn favors scenarios where the dark energy is a cosmological constant or an extra minimally-coupled field with negative pressure.

Chapter 6

Introducing Renormalisation Group cosmology

In this chapter, we will describe the basic idea behind renormalisation, and then discuss its application as a non-perturbative approach to quantum gravity. We will then be interested to apply it in cosmology, and understand the basic steps needed in order for the latter implementation to be consistent, working at the level of the equations of motion.

Before we discuss particular applications in cosmology, we will first try to provide a brief introduction to the following three questions: What is the idea behind renormalisation? Why do we need renormalisation in physics? How is it applied in the context of quantum gravity? We will try to give a basic, but intuitive description of above questions in the following sections.

6.1 An example from quantum field theory

In this section we shall be closely following Delamotte (2004) and describe an example of renormalisation at the perturbative level, of the kind that is typically appearing in Quantum Field Theory (QFT).

Let us consider an abstract theory with one coupling constant, say g_0, with g_0 being the bare coupling. It is important to stress that this is not the observed coupling, i.e the one that is measured in an experiment. We will come back to this below. Let us also assume that we would like to calculate perturbatively a particular physical quantity which we shall denote as $F(x)$. The quantity $F(x)$ could represent for example the amplitude for a particular scattering process, in which case the variable x would denote particle energy-momentum.

Since we want to use perturbation theory we can start by expanding $F(x)$ in powers of the bare coupling g_0 as

$$F(x) = g_0 + g_0^2 F_1(x) + \ldots + \mathcal{O}(g_0^n). \tag{6.1}$$

In QFT, the functions F_i would correspond to loop integrals over infinite virtual particle states in momentum space, and it is common that they are divergent. A typical example of a divergent integral in QFT would be a logarithmically divergent one,

$$F_1(x) = \int_0^\infty \frac{dt}{t+x}. \tag{6.2}$$

As said above, g_0 is the bare coupling, and it is not a physical quantity. What is measured in an experiment at $x = \mu$ is $F(\mu)$. Since there is only one coupling in this toy theory, one measurement will be needed to fix g_0 so as to reproduce the physical quantity $F(\mu)$ *at a given order in perturbation theory.* In other words, the principle idea is to start with a theory with a particular set of parameters (couplings), calculate physical quantities (e.g scattering amplitudes) in terms of the bare parameters, and then use experimental input to *re-parametrise* physical quantities in a way that they re-produce the experimental measurements. This is done using the *renormalisation prescription*,

$$F(\mu) = g_R, \tag{6.3}$$

with g_R denoting the *renormalised coupling.* The particular complications with this procedure enter when the expansion (6.1) is singular, e.g logarithmically divergent in our particular example.

Let us point out that we can expand the bare coupling as a power series of the renormalised one, i.e

$$g_0 = g_R + \delta_2 g + \ldots, \tag{6.4}$$

with $\delta_n g \sim \mathcal{O}(g_R^n)$. At this point enters the question of *renormalisability* of the theory: If the theory is renormalisable, it should be possible to convert the ill-defined perturbative expansion into a well defined (non-singular) one, by re-parametrising in terms of measured parameters, instead of the bare ones. It should be stressed that, in view of the perturbative expansion of a given quantity, renormalisability should be studied at each order of perturbation theory separately.

The first step for renormalising the perturbative expansion (6.1) is by *regularising* the expansion through the introduction of a cut-off or regulator Λ in the UV diverging integral, and if the theory (at the particular perturbation order) is renormalisable, then by taking the limit $\Lambda \to \infty$, the original expansion should be recovered.

Let us see this more closely. We introduce new, regularised functions $F_i(x)$ as

$$F_i(x) \to F_{i,\Lambda}(x), \tag{6.5}$$

such that the diverging integral(s) are well defined. The expansion (6.1) then takes the form

$$F(x) \to F_\Lambda(x)(x, g_0, \Lambda) \equiv g_0 + g_0^2 F_{1,\Lambda}(x) + \ldots + \mathcal{O}(g_0^n). \tag{6.6}$$

For example, the integral (6.2) can be written as

$$F_1(x) \to F_{1,\Lambda}(x) = \int_0^\Lambda \frac{dt}{t+x}. \tag{6.7}$$

Now, we can use the renormalisation condition (6.3) together with the well-defined expansion (6.6) to re-parametrise the latter in terms of the measured coupling g_R. If the renormalisability hypothesis holds then after this procedure we should be able to safely take the limit $\Lambda \to \infty$, at a given scale $x = \mu$ and a given measurement for the coupling g_R.

As also mentioned before, given a perturbative expansion, the renormalisation procedure should be applied order by order. Here, for illustration we shall restrict ourselves only in the first and second order. At first order in g_0,

$$F_\Lambda(x) = g_0 + \mathcal{O}(g_0^2), \tag{6.8}$$

and using the renormalisation condition (6.3) we have

$$g_R = g_0 + \mathcal{O}(g_R^2). \tag{6.9}$$

In other words, at first order the renormalisation condition just tells us that the bare coupling g_0 should equal the measured one. i.e $g_0 = g_R$.

At second order, things become more interesting. At this order it is,

$$F_\Lambda(x)(x, g_0, \Lambda) \equiv g_0 + g_0^2 F_{1,\Lambda}(x) + \mathcal{O}(g_0^3), \tag{6.10}$$

which after using the coupling expansion (6.4) yields

$$F_\Lambda(x)(x, g_0, \Lambda) \equiv g_R + \delta_2 g + g_R^2 F_{1,\Lambda}(x) + \mathcal{O}(g_0^3). \tag{6.11}$$

Using condition (6.3) we get,

$$\delta_2 g = -g_R^2 F_{1,\Lambda}(\mu), \tag{6.12}$$

which is explicitly equal to

$$\delta_2 g = -\alpha g_R^2 \int_0^\Lambda \frac{dt}{t+\mu} = -\alpha g_R^2 log \frac{\Lambda + \mu}{\mu}, \tag{6.13}$$

which diverges as $\Lambda \to \infty$.

Plugging above expression into (6.11), to eliminate $\delta_2 g$ in favor of the renormalised coupling we find

$$F_\Lambda(x)(x, g_0, \Lambda) \equiv g_R + g_R^2 \left(F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)\right) + \mathcal{O}(g_R^3) \tag{6.14}$$

$$= g_R + \alpha(\mu - x) g_R^2 \int_0^\Lambda \frac{dt}{(t+x)(t+\mu)} + \mathcal{O}(g_R^3), \tag{6.15}$$

which is well defined for $\Lambda \to \infty$, i.e

$$\lim_{\Lambda\to\infty} F_\Lambda(x)(x, g_R) \to \text{finite quantity}. \tag{6.16}$$

Therefore, the theory is *renormalisable up to second order.*

Although we shall not continue to discuss above procedure for higher order perturbation theory, at this level we can still extract a couple of general features of above procedure which are true at all orders in perturbation theory. As we saw before, the divergence in the second order term was cancelled by a term coming from the expansion of the first order bare coupling g_0 in powers if the renormalised one g_R; this behavior is true at all orders n. The second point to be stressed out is that it would not be possible to cancel the divergence in $F_{1,\Lambda}(x)$ using $F_{1,\lambda}(\mu)$, if the former would depend on x; in that case, one would need an extra renormalisation condition for the divergence to be removed. For details one can refer to Delamotte (2004).

In this section we presented a simple, but typical example of renormalisation at the perturbative level. In the next section we will discuss it at the level of the action, using a non-perturbative approach.

6.2 The Functional Renormalisation Group and Asymptotic Safety

In this section we will be interested to discuss renormalisation techniques beyond perturbation theory, i.e in a non-perturbative fashion. The main tool to do this is the so–called Exact Renormalisation Group (ERG) or the Functional Renormalisation Group (FRG). The word "functional" declares that the actual analysis is performed at the level of the action, i.e using functional methods. Therefore, the starting point is the definition of an action which describes the degrees of freedom of our theory as well as all the interactions between them. Then the ERG approach is the mathematical machinery to employ the following, and fundamental statement about physical systems:

The properties of a particular physical system depend on the scale one performs an observation on the actual system.

Above statement says something very deep about physical systems: one expects the properties of a given physical system to be scale dependent, i.e dependent on the energy (or length) scale on performs a particular experiment. This fact has tremendous applications both in physics and mathematics, from fractals to quantum gravity. A simple example of this behavior can be found by looking at a gas: Macroscopically, the gas can be described as a fluid with a temperature, entropy, e.t.c. However, as one "zooms" into the fluid, i.e studying it at smaller scales, discovers that it consists of molecules with a velocity (effectively described by Brownian motion), and even at smaller scales the molecules consist of atoms, which in turn are built up from electrons, neutrons and protons, and so on. It can be seen that at each particular scale, different physics sets in, e.g at large scales fluid mechanics is sufficient to analyse the gas behavior, while at smaller scales one has to take into the quantum properties of the particles the gas consists of.

In above example, it is important to explain the way the term "macroscopic" is used. With the latter term, we mean the scales where the gas can be effectively described ignoring its molecular structure; this would be true for scales much larger than the correlation length of its particular degrees of freedom (e.g molecules), i.e for scales λ that satisfy

$$\lambda \gg \lambda_{\rm sep}, \tag{6.17}$$

with $\lambda_{\rm sep}$ denoting the typical separation between internal degrees of freedom.

It has to be pointed out that in principle, a particular system will exhibit many degrees of freedom per correlation length, that can range from a few to infinity. If the interaction range is r_0, and assuming locality for the interactions, then one can then split the system into different patches of length r_0. The next step is to "coarse grain" the system over patches of characteristic length r_0, and successively apply the coarse graining procedure each time over neighboring patches. The ERG, provides us with an analytic tool to perform such a "coarse graining". Below, we will discuss how this is applied to understand the non–perturbative renormalisability of quantum gravity.

Let us now focus on the significance of the ERG for QFT, following mainly Gies (2006). A great advantage of the ERG is that it does not require for the couplings to be small; this is why it also termed as an non–perturbative approach. The application of the ERG requires a functional formulation of the particular theory. Armed with with these two, i.e a functional formulation of our theory together with the RG, the idea is to study the fluctuations of the system successively at different scales, in a "coarse–grained" fashion. In particular, in this context, one studies the change of the correlation functions of the theory, induced by an infinitesimal momentum shell of fluctuations. Application of this procedure leads to a flow equation, which describes the evolution of the generating functional of correlation functions, under the process of integrating out degrees of freedom. From the flow equation, it is then possible to extract the evolution equations for the couplings of the theory as a function of the cut–off energy scale.

Now, we will briefly sketch the derivation of the flow equation focusing on scalar field theory, but the extension to more complicated theories, like gravity can be found in the work where it was first introduced Wetterich (1993). For more details on the derivation and interpretation one should also refer to Gies (2006); Reuter and Saueressig (2012); Litim (2008a); Reuter and Saueressig (2007a). We start with the notion of the generating functional $Z[J]$ of n-point correlation functions,

$$Z[J] \equiv e^{W[J]} = \int \mathcal{D}\varphi e^{-S[\varphi]+\int J\varphi}. \tag{6.18}$$

$J \equiv J(x)$ is the source and we also defined

$$\int J\varphi \equiv \int d^D x J(x)\varphi(x). \tag{6.19}$$

The n-point correlation functions for a scalar field theory are then derived from $Z[J]$ through functional differentiation [1]. $W[J]$ is the generating functional of the n-point connected correlation functions.

The *effective action* $\Gamma[\phi]$ is defined through a Legendre transformation,

$$\Gamma[\phi] = sup_J \left(\int J\phi - W[J] \right), \tag{6.20}$$

with “sup” standing for “supremum”, and

$$\phi(x) \equiv \langle\varphi(x)\rangle_J = \frac{\delta W_k[J]}{\delta J(x)}. \tag{6.21}$$

In the context of the RG, the effective action can be calculated in a very intuitive and efficient way, by integrating out momentum modes shell by shell. To achieve this we first define a variation of the effective action, the *effective average action* Γ_k,

[1]Please refer to standard QFT textbooks for details, e.g Peskin and Schroeder (1995).

parametrised through the cut–off scale k. By definition the action Γ_k interpolates between two extreme limits, the UV and the IR one, corresponding to $k \to \infty$ and $k \to 0$ respectively.

The action Γ_k can be derived through a modification of the functional $Z[J]$, through an IR regulator ΔS_k as

$$Z_k[J] \equiv e^{W_k[J]} \equiv \int \mathcal{D}\varphi e^{-S[\varphi]-\Delta S_k[\varphi]+\int J\varphi}, \tag{6.22}$$

with the regulator defined as

$$\Delta S_k[\varphi] \equiv \frac{1}{2}\int \frac{d^Dq}{(2\pi)^D}\varphi(-q)R_k(q)\varphi(q). \tag{6.23}$$

The IR regulator term above can be viewed as a momentum-dependent mass term, since it is of quadratic nature with respect to the field φ. The regulator function $R_k(q)$, is a matrix-valued cut-off and its form can in principle be chosen arbitrarily, however it is subject to the following three conditions,

$$\lim_{q^2/k^2\to 0} R_k(q) > 0, \tag{6.24}$$

$$\lim_{k^2/q^2\to 0} R_k(q) = 0, \tag{6.25}$$

$$\lim_{k^2\to\Lambda\to\infty} R_k(q) \to \infty. \tag{6.26}$$

The first condition implements the idea of an infrared regulator, i.e that the regulator suppresses the integration over the IR momentum modes, the second one that the regulator vanishes as $k \to 0$, while the third one that the functional integral is dominated by the stationary point of the action in this limit, justifying this way the saddle-point approximation. The second condition ensures that the usual generating functional is recovered as $k \to 0$.

The effective average action Γ_k can be defined formally through a Legendre transformation as before,

$$\Gamma[\phi] = sup_J\left(\int J\phi - W[J]\right) - \Delta S_k[\phi]. \tag{6.27}$$

One can then derive the flow equation of $\Gamma_k[\phi]$ as a function of the cut–of scale k,

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\partial_t R_k \left(\Gamma_k^{(2)} + R_k\right)^{-1}\right], \tag{6.28}$$

with the inverse propagator defined as

$$\Gamma_k^{(2)} \equiv \frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi}, \tag{6.29}$$

and "Tr" standing for the trace of the r.h.s of the equation, or in other words defining the integral over all momentum modes k. For the case of a field with index structure, like gravity, (6.29) should be modified appropriately.

Above equation is an exact functional equation for Γ_k, having an 1-loop structure. Γ_k can in principle include all possible operators of the field. Equation (6.28) then interpolates between the two extreme cases: the UV case, where the bare action is recovered, $\Gamma_{k\to 0} = \Gamma_{bare}$, and the IR case where $\Gamma_{k\to\infty} = \Gamma_0$. Different solutions of equation (6.28) give rise to different families of effective field theories, $\Gamma_k \equiv \Gamma_k[g_{\mu\nu}]$ with $0 < k < \infty$, defining a Wilsonian RG flow on the theory space. With the term "theory space" one means the space consisting of all diffeomorphism invariant functionals $\Gamma_k[g_{\mu\nu}]$. The solution of (6.28) defines a curve on the "theory space, which is the space characterised by all functionals Γ_k.

The existence of the regulator R_k in the denominator of (6.28), ensures that the IR regulation, i.e that sufficiently low momenta will be suppressed. On the same time, the term $\partial_t R_k$ in the numerator, together with the conditions (6.24)-(6.25) takes care of the UV regulation, restricting the integration between a momentum shell near $p^2 \sim k^2$. This is in pure agreement with the underlying idea of the Wilsonian procedure of integrating out momenta shell by shell.

The regulator can be in principle chosen arbitrarily, however, one has to make sure conditions (6.24)-(6.26) are satisfied. The precise form of the trajectory on the theory space depends on the particular form of the regulator, however the endpoint of the trajectory should not, as implied by relations (6.24)-(6.26). As a consequence the choice of the regulator should not influence the position of a possible non-trivial fixed point in the UV or IR.

Once the field content, $\Phi(x)$, of the theory is defined, the theory space is defined as $\Phi \mapsto \Gamma[\Phi]$, i.e of all action functionals of the particular field(s) which are compatible with the symmetries of the particular theory. We assume that the effective action can be expanded in a basis of operators P_i,

$$\Gamma[\Phi] = \sum_{i=1}^{\infty} g_i(k) P_i[\Phi], \tag{6.30}$$

and we also define the dimensionless couplings $\tilde{g}_i(k) \equiv k^{-d_i} g_i(k)$, with d_i the canonical mass dimension of the dimension full coupling $g_i(k)$. No assumptions about the smallness of couplings are made. One then defines the couplings $g_i(k \to \infty)$ as the "bare" couplings, whereas $g(k \to 0)$ as the "dressed" or the "renormalised" ones.

Differentiating (6.30) with respect to "RG time" $t \equiv \log k$, and taking into account that the operators do not depend on the RG scale, we find,

$$\frac{d\Gamma_k}{dt} = \sum_{i=1}^{\infty} \beta_i(\tilde{g}_j, k) P_i[\Phi], \tag{6.31}$$

with the functions β_i defined as

$$\beta_i(\tilde{g}_j, k) \equiv \frac{d\tilde{g}_i(\tilde{g}_j, k)}{dt} = k \frac{d\tilde{g}_i(\tilde{g}_j, k)}{dk}. \tag{6.32}$$

The functions $\beta_i(\tilde{g}_j, k)$ are called the *beta functions*, and it is important to emphasize on the fact that they are not restricted to small couplings as in the usual perturbation theory. The explicit form of the beta functions can be found by expanding the r.h.s of the RG equation (6.28) in powers of the operators P_i, and comparing with the r.h.s of (6.31). The beta functions define a vector field $\vec{\beta}$ on the theory space, which' integral curves are the effective functionals $k \mapsto \Gamma_k$, running from the UV, $k \to \infty$, to the IR, $k \to 0$. The couplings of the action as a function of scale k is found by solving a system of coupled, first order differential equations,

$$k \partial_k \bar{g}_i = \beta_i(\tilde{g}_j), \tag{6.33}$$

with j in principle running from 1 to ∞. In other words, one ends up with an infinite set of differential equations. However, for practical calculations, the effective action has to be truncated, i.e a particular ansatz with a finite set of operators (and couplings) has to be assumed. The truncation where only the linear and zeroth order curvature terms are kept, the Ricci scalar and the cosmological respectively, is called the Einstein–Hilbert truncation, sometimes also referred as the RG improved Einstein–Hilbert action. In this truncation there are two couplings present, namely Newton's and the cosmological "constant" respectively.

Fixed points under the RG, are the points where the beta functions vanish, i.e $\beta_i(\tilde{g}_j) = 0$. They can be distinguished into two main categories: Trivial fixed points or Gaussian Fixed Points (GFP) are the ones where $g_i = 0$ for all i's, i.e where the theory is "free". It is around a GFP where usual perturbation theory applies. The second class of fixed points are the non-trivial ones, where the couplings acquire non-zero values.

Let us consider the linearised flow around a (non trivial) fixed point,

$$\partial_t \tilde{g}_i = \frac{\partial \beta_i}{\partial \tilde{g}_j}(\tilde{g}_j - \tilde{g}_{j*}) \equiv M^i_j(\tilde{g}_j - \tilde{g}_{j*}), \tag{6.34}$$

$$\tilde{g}_i(k) = \tilde{g}_{i*} + \sum_j C^j_{(i)} \mathbf{B}^j_{(i)} \left(\frac{k}{k_0}\right)^{-\theta_{(i)j}}, \tag{6.35}$$

with $\tilde{g}_*$ the fixed point value, C_i integration constants, $\mathbf{B^j}$ are eigenvectors of the matrix M^i_j, and k_0 some reference energy scale. The quantities $\theta_{(i)} \equiv -\lambda_i$, where λ_i is an eigenvalue of M^i_j are called the critical exponents.

The critical surface (also called unstable manifold) around a (non trivial) fixed point is defined as the collection of all points in the theory space that evolve towards the fixed point with increasing energy k (inverse RG flow). Its dimensionality is defined by the number of attractive directions in theory space. Asymptotic Safety requires that the dimensionality of the critical surface is finite, i.e a finite number of relevant couplings exist in the vicinity of the UV fixed points.

The idea of Asymptotic Safety relies on the existence of a non-trivial fixed point under the RG, in the UV, first proposed by Stephen Weinberg Hawking S. W. (1979); if such a fixed point exists, then the limit $k \to \infty$ can be safely taken, and the theory is UV complete, i.e is well defined at high energies and does not suffer from any UV divergences as $k \to \infty$.

Different investigations have shown that a non-trivial UV fixed point exists in the Einstein-Hilbert truncation Reuter (1998); Souma (1999, 2000); Lauscher and Reuter (2002); Litim (2004); Reuter and Saueressig (2002); Percacci and Perini (2003b,a) (see also Litim (2004) and references therein), as well as in higher truncations Narain and Percacci (2010); Narain and Rahmede (2010); Codello et al. (2009, 2008a); Fischer and Litim (2006), providing strong evidence that quantum gravity is renormalisable in a non-perturbative way. Recently, the existence of a non–trivial RG fixed point in the IR has been also investigated Donkin and Pawlowski (2012); Nagy et al. (2012).

Before we close this section let us refer the reader to Gies (2006); Reuter and Saueressig (2012); Pawlowski (2007); Litim (2008a); Reuter and Saueressig (2007a); Percacci (2007, 2011) for detailed reviews on RG, Asymptotic Safety and its application to gravity.

6.2.1 The flow equation of Einstein–Hilbert gravity

Before we discuss the non–perturbative flow equation for Einstein–Hilbert gravity under the RG, let us first pause and briefly discuss why the standard perturbation approach to quantum Einstein–Hilbert gravity fails. We shall mainly follow the discussions in Zee (2010), Hambert (2010), to which we refer the reader for more details. In the context of perturbation theory, and for the case where the action is described by the Einstein–Hilbert one [2],

$$S = \frac{1}{16\pi G}\int d^n x\sqrt{-g}R[g], \tag{6.36}$$

the first step consists in expanding the spacetime metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sqrt{16\pi G}h_{\mu\nu}, \tag{6.37}$$

with $\bar{g}_{\mu\nu}$ being a classical background metric field, and $h_{\mu\nu}$ represents the quantum fluctuation. What is more, the coupling G is the bare Newton's constant. The standard approach followed in the context of perturbation theory is the so–called dimensional regularisation which accounts to performing calculations in n dimensions and in the end taking the limit of $n \to 4$.

[2]Here, we assume that the cosmological constant is zero.

Substituting above expansion for the metric field in the lagrangian, the latter can then be expanded up to different orders in the fluctuating field, i.e

$$\mathcal{L} \sim \mathcal{O}(h^2) + \mathcal{O}(h^3) + \mathcal{O}(h^4) + \mathcal{L}_{gf} + \mathcal{L}_{ghost}, \tag{6.38}$$

with the last two terms corresponding to the gauge fixing and ghost terms respectively. The introduction of the gauge fixing term makes the graviton propagator well defined, and on the same time gives rise to the ghost part according to the Faddeev–Popov procedure[3]. The quadratic part of the action will give the graviton propagator, while the higher order terms in h the different interaction vertices of the relevant order, and the Feynman rules then follow accordingly.

Einstein–Hilbert quantum gravity, with or without a cosmological constant, is known to be perturbatively non–renormalisable, due to the problem of the UV divergences. One could expect this behavior by looking at the dimensionality of Newton's constant which scales as

$$G \sim E^{2-n}, \tag{6.39}$$

where E is an energy scale, and n the spacetime dimensions. Notice that for $n > 2$ Newton's constant has negative mass dimensions, which is the root of the problem of the UV divergences of Einstein–Hilbert quantum gravity. If with Λ we denote a UV cut–off energy scale, the lowest order loop diagrams will be proportional to the dimensionless product

$$\sim G\Lambda^{n-2}, \tag{6.40}$$

which implies that for $n > 2$ loop corrections of first order will be divergent with increasing cut–off Λ. In particular, for the case of $n = 4$, the 1–loop correction to the graviton-graviton scattering we will require evaluation of the following type of integral

$$\sim G^2 \int^{\Lambda} d^4k(1/k^2) \sim G^2\Lambda^2, \tag{6.41}$$

which is divergent as $\Lambda \to \infty$. On the same time, for the amplitude $\mathcal{M}$ of the

[3] For details see for example Rivers (1987); Peskin and Schroeder (1995); Hambert (2010).

graviton-graviton interaction it will be

$$\mathcal{M} \sim G + G^2\Lambda^2 + \mathcal{O}(G^3), \tag{6.42}$$

which in turn implies that the effective Newton's constant will run as a function of the cut–off energy scale as

$$\frac{G(\Lambda)}{G} \sim 1 + G\Lambda + \mathcal{O}(G^2), \tag{6.43}$$

which is also divergent as $\Lambda \to \infty$.

In general, in n spacetime dimensions the graviton propagator in momentum space will scale as $\sim 1/k^2$, the vertex functions as k^2, while the measure in n dimensions as $d^n k$, yielding for the superficial degree of divergence[4]

$$\mathcal{D} = 2 + (n-2)L, \tag{6.44}$$

where L is the number of the loops involved. From relation (6.44) one can see that for spacetime dimensions greater than two, $n > 2$, the superficial degree of divergence $\mathcal{D}$ will be proportional to the number of loop corrections L.

Using above simple arguments, we see that in the context of perturbation theory Einstein–Hilbert gravity is perturbatively non–renormalisable in $n > 2$, with the root of the problem lying in the negative mass dimensions of Newton's constant (for $n > 2$). The resolution that Asymptotic Safety provides to the latter problem is the assumption that a fixed point exists for Newton's constant G under the RG, in the limit of the cut–off going to infinity. If the UV fixed point exists, this means that the dimensionless product $G\Lambda^2$ in relation (6.43) tends to a finite value as $\Lambda \to \infty$, ensuring on the same time that scattering amplitudes are UV finite. The flow of the coupling(s) to the UV fixed point, if the latter exists, is a non–perturbative effect that in principle cannot be seen with standard perturbation methods. Below, we will discuss how one can go beyond perturbation theory using RG methods for the case of gravity.

Let us now proceed with discussing how the non–perturbative flow for the grav-

[4]For a discussion of the derivation see for example Zee (2010).

itational couplings under the RG is calculated, having as a starting point the exact RG equation defined in (6.28). Before we focus on the particular case of an Einstein–Hilbert action with a cosmological constant, we will discuss the more general case of an action including higher order curvature terms in the so–called $f(R)$ ansatz.

The first thing to point out is that in order to proceed with the evaluation of the flow equation (6.28) for the effective action one has to assume a particular ansatz for the latter. In gravity, general covariance dictates that all scalar curvature combinations compatible with it should be included in the action functional. However, for practical calculations a specific ansatz has to be chosen, as an infinite number of operators will yield an infinite set of beta functions, which is obviously impossible to solve. Furthermore, simple truncations provide with valuable insight of the behavior of metric gravity under the RG, on the same time allowing a set of equations which is relatively easy to cope with.

In the $f(R)$ context, an ansatz widely used in the literature has been the one where the $f(R)$ function is expanded in positive powers of the Ricci scalar R [5],

$$\Gamma_k[g] = \int d^4x\sqrt{-g}f(R) \equiv \int \sqrt{-g}d^4x \left(\frac{R-2\Lambda}{16\pi G} + c_2R^2 + c_3R^3 + \ldots + c_nR^n\right). \tag{6.45}$$

An ansatz with negative power of the curvature R has been studied in Machado and Saueressig (2008). Before we proceed, let us note that in the following, and unless otherwise stated, we will use the letter k to denote the RG cut–off scale, and Λ the dimension full cosmological constant respectively.

We will not get into the details of the derivation of the flow equation for above ansatz, however let us sketch the basic steps involved in the calculation as these were followed for example in Codello et al. (2009), and with the appropriate modifications they apply to any action, not only a (purely) gravitational one:

• Assumption of a particular ansatz for the effective action Γ_k, which should be then supplemented with the suitable gauge fixing and ghost terms. In principle, one can consider all curvature combinations that preserve general covariance. For example, (Codello et al. (2009)), Machado and Saueressig (2008) Codello et al. (2008b) have investigated the asymptotic safety scenario assuming an $f(R)$ form

[5] For a rather complete treatment of the RG application on $f(R)$ gravity see Machado (2010).

for the effective action, while (Narain and Rahmede (2010); Narain and Percacci (2010)) assuming a general scalar–tensor one. However, inclusion of other curvature combinations have been studied, as for example, in Benedetti et al. (2009) for the case of an action supplemented with the square of the Weyl tensor.

After assuming a form for the gravitational part of the action, a choice for the IR regulator function R_k has to be made. One expects that any particular regulator choice that satisfies the general requirements described before, (6.24)–(6.26), will not alter the qualitative features of the RG flow, like for example the fixed point structure. In fact, the position of the fixed point fluctuates with changing the regulator function, as well as new spurious ones are generated. A fixed point solution to be accepted, should persist under regulator or gauge variations.

• The next step is the calculation of the second variation of the effective action with respect to the field variables of the effective action. At this point, the metric fluctuation is decomposed into its irreducible components, as well as the background is fixed. A commonly used choice for the background metric is a Euclidean de Sitter one, which is also used in Codello et al. (2009). Both the decomposition in term of the irreducible components as well as the choice of a spherical background allow for an exact inversion of the kinetic operator. [6]Then, the inverse propagators for the different irreducible modes are computed, and inserted together with the explicit expression of the regulator function into the RG equation (6.28). Evaluation of the trace over momenta on the r.h.s of the RG equation requires the use of a heat kernel expansion. The result yields the flow equation for the particular effective action, which is an in principle non–linear differential equation that involves the couplings and their derivatives. The form of the flow equation can be simplified significantly for particular choices of the gauge.

• After the flow equation has been derived, the beta functions, as well as the fixed points and the corresponding critical exponents of the theory can be calculated, though Taylor expanding both sides of the flow equation and comparing similar terms. That will be made more explicit in the following for the case of a polynomial ansatz for the effective action.

One expects that truncating the action leads to a particular error, as in this way

[6]In a cosmological context, spherically symmetric backgrounds play an important role too, as both the FLRW as well as de Sitter spacetime belong to this class of backgrounds.

the effect of higher order contributions allowed by the symmetries of the theory is neglected. In a non–perturbative context, however, there is not any standard way of allowing such an estimation. However, one can still get an estimation by studying the dependence of universal quantities, like the critical exponents, on the choice of the regulator function.

What is more, the existence of a particular type of a non–trivial fixed point under the RG for a particular truncation, is not itself enough to claim its existence in general. In fact, one has to confirm its existence and stability for different truncations, cut–off schemes, as well as gauge choices. Even a successful outcome of such a procedure might not again provide a conclusive proof for the existence of the fixed point, but it will still be a rather strong evidence.

Let us now present the beta functions for Einstein–Hilbert gravity, as these were derived in Litim (2004, 2008b). They read as

$$\partial_t\lambda = \beta_\lambda(g,\lambda) \equiv -2\lambda - 12g - \tfrac{24g\left(3g+\frac{1}{2}(1-3\lambda)\right)}{2g-\frac{1}{2}(1-2\lambda)^2}, \tag{6.46}$$

$$\partial_t g = \beta_g(g,\lambda) \equiv 2g + \tfrac{24g^2}{4g-(1-2\lambda)^2}, \tag{6.47}$$

where $t \equiv \ln k$, and β_λ, β_g the beta functions, and we also defined the dimensionless couplings

$$g(k) \equiv k^2 G(k)/24\pi, \quad \lambda(k) \equiv \Lambda/k^2. \tag{6.48}$$

In above equations the factor of 24π is included to remove phase space factors. Above beta functions were calculated using the optimised cut–off and a type of harmonic background field gauge introduced in Litim (2004).

There are two fixed points of the above RG flows, a free or Gaussian one, and an interacting one which is attractive in the UV ($k \to \infty$), with

$$(g^*,\lambda^*)_{\rm GFP} = (0,0), \quad (g^*,\lambda^*)_{\rm UV} = (0.015625, 0.25), \tag{6.49}$$

(Refs Litim (2008b)-Reuter and Saueressig (2007b) and references therein). The different types of trajectories arising from above beta functions can be seen in figure 6.1. In particular, the trajectories of type IIa define the "separatrix", which separate the trajectories starting off from the UV fixed point and evolving towards negative and positive values of λ respectively. The observationally accepted trajectories are those of Type $IIIa$, leading to a positive λ at late times, and a classical regime around the Gaussian Fixed Point.

The eigenvalues corresponding to the linearised beta functions around the UV fixed point are complex conjugate with negative real parts, leading to the spiraling behavior of the evolution in that regime. The sign of the real parts of the eigenvalues imply that the UV fixed point is attractive (repelling) for increasing (decreasing) cut–off energy k. However, the complex nature of the eigenvalues is not generally true for higher truncations, see for example Codello et al. (2008b).

A phenomenologically viable trajectory is one that starts at high energies from the UV fixed point and then evolves towards smaller values of g as k is lowered, passes close to the GFP, until it turns to the right towards increasing values of λ. A trajectory passing sufficiently close to the GFP will have a long classical regime, i.e $G \simeq G_0$, $\Lambda \simeq \Lambda_0$, with "0" here denoting the present value. In other words, the closer the trajectory passes to the GFP, the larger amount of "RG time" t it spends close to it. In the vicinity of the GFP the couplings acquire tiny values[7], leading to a large hierarchy between classical and UV ($\sim m_p$) scales. Under the requirement of the existence of a classical regime, a small (and constant) cosmological constant comes for free. From the linearised solutions around the GFP, one can see that the dimensionless couplings in this regime scale canonically, implying constancy for the dimensionfull ones. The classical regime covering many orders of magnitude in scales is required by terrestrial, solar and galactic tests, as well as consistency with cosmological evolution since Big Bang Nucleosynthesis. We shall come back with a more detailed analysis of above beta functions in chapter 7, where we will discuss how classical GR arises, as well as the emerging early and late time cosmology for the RG-improved Einstein–Hilbert action. For a nice presentation of the solar and astrophysical scales limit of the RG improved Einstein–Hilbert action see Reuter and Weyer (2004a).

[7] In chapter 7 we will come back to this issue where we will also discuss the relevant orders of magnitude for these couplings.

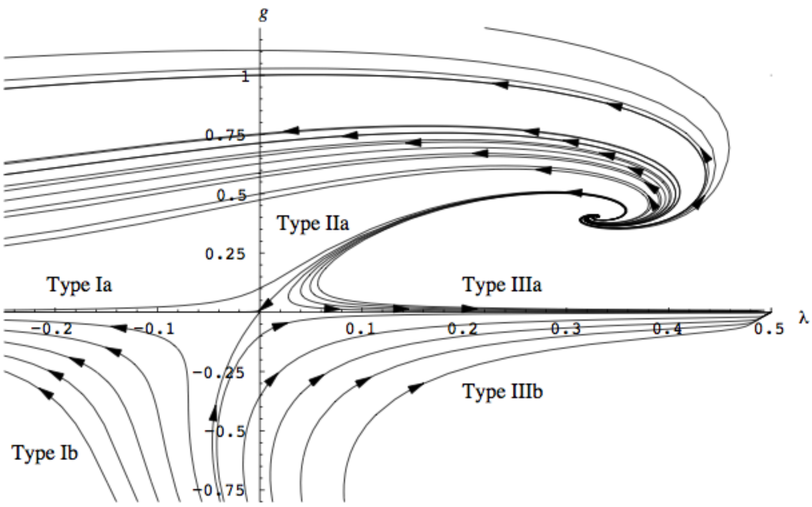

Figure 6.1: The different types of trajectories of the theory space in the Einstein–Hilbert truncation. Plot is taken from Reuter and Saueressig (2002). The observationally accepted trajectories are those of Type $IIIa$, spiraling around the UV fixed point for large values of the cut–off scale k, and evolving towards the Gaussian Fixed Point (GFP) with decreasing k. At some point very close to the GFP, the trajectory turns right and evolves towards larger values of λ in the IR. The classical regime, covering from earth to astrophysical scales, is realised in the vicinity of the GFP, where both the dimension full cosmological and Newton's constant acquire constant values.

6.3 The flow equation in $f(R)$ gravity

After having presented the flow equation of Einstein–HIlbert gravity, and for the sake of completeness, let us now go back to the ansatz (6.45), in order to present the form of its flow equation. We will be following the approach of Codello et al. (2009), [8]

We start by trivially writing (6.45) as

$$\Gamma_k[g] = \int \sqrt{-g} d^4 x f(R) \equiv V \times f(R), \tag{6.50}$$

where V denotes the volume of spacetime. Differentiating above action with respect to RG time $t \equiv \log k$, we get

$$\partial_t \Gamma_k = V \partial_t f(R). \tag{6.51}$$

It is very convenient to work with dimensionless quantities, so let us proceed by defining

$$\tilde{R} \equiv k^{-2} R, \quad \tilde{f}(\tilde{R}, k) \equiv k^{-4} f(\tilde{R}, k) \equiv k^{-4} f_k(\tilde{R}, k), \tag{6.52}$$

Notice that the dependence of $\tilde{f}(\tilde{R}, k)$ on the RG scale k comes through the running couplings $\tilde{g}(k)$, i.e $\tilde{f}(\tilde{R}, k) \equiv \tilde{f}(\tilde{R}, \tilde{g}(k))$. We these definitions, and remembering that

$$\frac{\partial}{\partial t} \equiv \left.\frac{\partial}{\partial_t}\right|_{\tilde{R}} + \frac{\partial \tilde{R}}{\partial t} \left.\frac{\partial}{\partial \tilde{R}}\right|_t, \tag{6.53}$$

with the bar implying that the corresponding variable is to be kept fixed during differentiation, we have the following relations between dimensionless and dimension full quantities,

$$\partial_t f_R = k^2 \left(2\tilde{f}_{\tilde{R}} + 2\tilde{R}\tilde{f}_{\tilde{R}\tilde{R}} + \partial_t \tilde{f}_{\tilde{R}} \right), \tag{6.54}$$

$$\partial_t f = k^4 \left(4\tilde{f} - 2\tilde{R}\tilde{f}_{\tilde{R}} + \partial_k \tilde{f} \right). \tag{6.55}$$

We shall restrict ourselves to (Euclidean) spherical symmetry, as the flow equation itself is evaluated on a (Euclidean) de Sitter background, which means that the

[8] The reader is referred to Codello et al. (2009) for details regarding the choice of the regulator function as well as the choice of the gauge in the derivation of the flow equation presented here.

volume element acquires the following form

$$V = \frac{384\pi^2}{R^2}. \tag{6.56}$$

After following the procedure schetched before, and for the regulator and gauge conventions of Codello et al. (2009), the flow equation for $f(R)$ gravity reads as Codello et al. (2009)

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= \frac{384\pi^2}{30240\tilde{R}^2}\Bigg[-\frac{1008\left(511\tilde{R}^2-360\tilde{R}-1080\right)}{\tilde{R}-3}-\frac{2016\left(607\tilde{R}^3-360\tilde{R}-2160\right)}{\tilde{R}-4}\\
&+20\frac{\left(311\tilde{R}^3-126\tilde{R}^2-22680\tilde{R}+45360\right)\partial_t\tilde{f}_R-252\left(\tilde{R}^2+360\tilde{R}-1080\right)\tilde{f}_R}{3\tilde{f}_R-(\tilde{R}-3)\tilde{f}_R}\\
&+\Big[1008\left(29\tilde{R}^2+273\tilde{R}^2-3240\right)\tilde{f}'+4\left(185\tilde{R}^3+3654\tilde{R}^2+22680\tilde{R}+45360\right)\partial_t\tilde{f}'\\
&-2016\left(29\tilde{R}^3+273\tilde{R}^2-3240\right)\partial_t\tilde{f}_{RR}-9\left(181\tilde{R}^4+3248\tilde{R}^3+15288\tilde{R}^2-90720\right)\Big]\times\\
&\left[\tilde{f}_{RR}(\tilde{R}-3)^2+2\tilde{f}+(3-2\tilde{R})\tilde{f}_R\right]^{-1}\Bigg].\end{aligned} \tag{6.57}$$

Above equation is a non–linear differential equation with partial derivatives, whose complexity makes it in general impossible to be solved analytically unless a particular form for the function $f(R)$ is chosen, such as an expansion in positive powers of the curvature scalar R. Notice that the r.h.s of (6.57) is expressed solely in terms of dimensionless quantities.

If we differentiate the effective action (6.50) with respect to $t \equiv logk$ we get

$$\partial_t\tilde{f} - 2\tilde{R}\tilde{f}_R + 4\tilde{f} = \tilde{V}^{-1}\partial_t\Gamma_k. \tag{6.58}$$

The beta functions can be then derived by expanding both sides with respect to the dimensionless curvature $\tilde{R}$ around $\tilde{R} = 0$,

$$\frac{dg_i}{dt} = \frac{1}{i!}\frac{\partial^i}{\partial\tilde{R}^i}\frac{1}{\tilde{V}}\frac{d\Gamma_k}{dt}\bigg|_{\tilde{R}=0}, \tag{6.59}$$

where it is assumed that everything is expressed in terms of dimensionless quantities.

On the fixed point, the RG derivative of the dimensionless version of f vanishes, i.e $\partial_t\tilde{f}_k = 0$. We can work out the fixed points for a particular $f(R)$ ansatz straight from above flow equation. To do this, we set all the derivatives of dimensionless quantities with respect to t in (6.58), e.g $\partial_t \tilde{f} = 0$, to get

$$-2\tilde{R}\tilde{f}_R + 4\tilde{f} = \tilde{V}^{-1}\partial_t \Gamma_k. \tag{6.60}$$

Expanding both sides around $\tilde{R} = 0$, and comparing equal powers of the expansion, we can arrive at a set of $n \times n$ coupled, algebraic equations, where n is the order of the truncation, i.e the maximum power of R in the effective action.

For the case of the $f(R)$ ansatz in a polynomial expansion in powers of R, a strong evidence for the existence of the UV fixed point would require the investigation of as much of the truncation space as possible, i.e by including even more powers of the Ricci curvature (together with the associated couplings) in the truncated action. In particular, in Codello et al. (2009), the case of a truncation up to the power $i = 8$ was studied, where it was found that the fixed point value for Newton's and cosmological constant couplings, is stable with increasing the truncation order. What is more, in the same work, and for the same effective action ansatz up to order $i = 8$, it was found that the dimensionality of the critical surface is three.

In a following chapter we will present the explicit form of the beta functions for the Einstein–Hilbert truncation, and we will also discuss the fixed points as well as the phenomenologically relevant trajectories in the theory space.

6.4 RG improved Friedmann equations

As also explained before, Einstein gravity without a cosmological constant, although successful at solar and galactic scales, is challenged by cosmological observations of the early and late time Universe. Both early (inflation) and late time acceleration of our Universe require either the introduction of an extra degree of freedom in the action, like a scalar or tensor field, or a modification of gravity itself. On the same time, the trivial extension of GR, i.e the introduction of a cosmological constant, is plagued by the magnitude and coincidence problem described at an earlier chapter. Modifications of the gravitational action like scalar-tensor actions or non–linear extensions of Einstein–Hilbert action, probably the most famous being the Brans–Dicke one Brans and Dicke (1961) and $f(R)$ theories respectively Starobinsky (1980a); Nojiri and Odintsov (2006a); Capozziello and Francaviglia (2008); Sotiriou and Faraoni (2010); De Felice and Tsujikawa (2010); Clifton et al. (2012). A common characteristic among scalar-tensor and modified theories alike is that they lead to a modification of Newton's constant G_N, which acquires a scale dependence, for example in scalar-tensor theories through the coupling of gravity with a scalar field.

GR with a cosmological constant Λ has been very successful in describing the late time acceleration of the Universe from a phenomenological point of view, but it is unable to account for a primordial inflationary era. One of the most challenging problems a cosmological constant faces from a theoretical point of view is the order of magnitude problem, i.e why it has such a tiny value, as well as the coincidence problem, or in other words why it is only at recent times that Λ becomes dynamically relevant. In the context of scalar field or modified gravity models, the vacuum energy is replaced by a dynamically evolving, effective energy-momentum tensor, but this only partly solves the problem, as any effective energy-momentum tensor has to reproduce the tiny value of Λ today.

In this section we will briefly present the standard approach to combine the renormalisation group with cosmology. We will work at the level of the equations of motion, and we will discuss how the usual cosmological equations are modified and how one can solve them. For simplicity and illustration we will focus only on the Einstein–Hilbert truncation, where the only couplings are Newton's $G \equiv G_k(k)$ and the cosmological constant $\Lambda \equiv \Lambda_k(k)$, with k the renormalisation group cut-off scale. The running of G and Λ changes the cosmological dynamics resulting from the action, and has also been suggested as a possible resolution to the coincidence problem Bonanno and Reuter (2002a, 2004); Weinberg (2010); Tye and Xu (2010); Bonanno et al. (2011); Contillo (2011); Reuter and Weyer (2004a); Grande et al. (2011). In particular, Shapiro and Sola (2000, 2002, 2008, 2009) have studied the cosmological consequences of a running Newton's G, while Shapiro et al. (2005); Bauer (2005b,a) studied the case of a running cosmological constant. Comparison with cosmological observations, including supernovae data, has been carried out in Guberina et al. (2003); Shapiro and Sola (2004); Espana-Bonet et al. (2004); Shapiro et al. (2005).

6.4.1 Cut–off identification in cosmology

In a cosmological context, it is attractive to think of the cut-off energy scale k as dynamically evolving with cosmic time Bonanno and Reuter (2002a,b); Reuter and Saueressig (2005). There are different ways to understand this connection in an expanding Universe. Since in the effective action modes with momentum $p^2 \gg k^2$ are integrated out, k defines the energy scale of the theory, i.e. the typical scale at which the couplings in the effective action are evaluated.

This identification can be either performed at the level of the equations of motion or the action. What is more, the identification can be done through an ansatz or in a dynamical way. In the latter case, the identification results naturally through the cosmological equations, by requiring that the Bianchi identities are satisfied. On the other hand, by doing some particular ansatz, $k = k(t)$, it is not ensured that the Bianchi identities will be satisfied. In fact, one has to pick up an ansatz that will not violate the satisfaction of the Bianchi identities. We will make these points more precise in the following.

Let us start by gathering some intuition about the form of the identification $k = k(t)$. The typical energy of particles in an expanding Universe with temperature T at a particular time, is directly linked with the expansion; the Universe starts off from a hot state and cools down as it expands with cosmic time t and in particular, for a homogeneous and isotropic Universe described by the FLRW metric, characterised by the scale factor $a(t)$, the typical energy of relativistic particles scales as $1/a(t)$. One could then think of identifying the cut-off scale k as

$$k \sim k_B T(t) \sim \frac{E_0}{a(t)}, \tag{6.61}$$

where k_B is the Boltzmann constant and E_0 a constant with dimensions of energy.

Alternatively, one can think that the horizon size of the Universe $d_H \sim 1/H(t)$, with $H \equiv \dot{a}(t)/a(t)$ the Hubble parameter, defines the typical scale of correlations between different quantum degrees of freedom, and identify

$$k^{-1} \sim d_H(t) \sim H^{-1}(t). \tag{6.62}$$

Notice that both identifications (6.61) and (6.62) are monotonically decreasing functions of cosmic time, in the context of a Hot Big Bang scenario. However, it is not in principle guaranteed that a particular ansatz made at the level of the equations of motion will prove to satisfy the Bianchi identities. The latter, provide the condition for all consistent identifications Bonanno and Reuter (2002a); Babic et al. (2005); Reuter and Saueressig (2005); Hindmarsh et al. (2011); Reuter and Weyer (2004b), and in the following we shall expand on the following issue in more detail.

On the other hand, the cut-of identification can be also performed at the level of the action, in a covariant way. In that case, the Bianchi identities will be automatically be satisfied. That will be the subject of a later section, where we will identify the cut-off scale with the scalar curvature, i.e $k \sim R$.

6.4.2 Evolution equations at the background

Let us start by stating our conventions for this section: a dot will denote derivative with respect to conformal time unless otherwise stated and primes will denote derivative with respect to the cut–off energy scale k.

Let us begin with the RG improved effective action in the Einstein–Hilbert truncation

$$S[g,\psi] = \int \sqrt{-g}\left[\frac{1}{16\pi G_k}\left(R - 2\Lambda_k\right) + L_{\rm matter}(g_{\alpha\beta},\psi)\right], \tag{6.63}$$

with $G_k \equiv G(k)$, $\Lambda_k \equiv \Lambda(k)$, and $L_{\rm matter}$ collectively denoting all matter fluids present.

The Einstein equations read as

$$G^{\mu}{}_{\nu} = 8\pi G(k) T^{\mu}{}_{\nu} - \Lambda(k)\delta^{\mu}{}_{\nu}. \tag{6.64}$$

The cosmological constant has been moved on the r.h.s and could be interpreted as some sort of effective fluid.

Let us turn attention to the conservation of the r.h.s of the Einstein equations. We have

$$\nabla_{\mu}\left(8\pi G(k) T^{\mu}{}_{\nu} - \Lambda(k)\delta^{\mu}{}_{\nu}\right) = 0. \tag{6.65}$$

In standard cosmology, where Λ and G do not depend on k above relation leads to the usual conservation equation(s) for the matter fluid described by $T^{\mu}{}_{\nu}$. We assume that k is some function of cosmological time t, i.e $k = k(t)$. In addition, we can require that the energy–momentum tensor of matter fluids is separately conserved. In this case we arrive at the following two equations

$$8\pi G(k)\nabla_\mu T^\mu{}_\nu = 0, \tag{6.66}$$

$$8\pi T^\mu{}_\nu \nabla_\mu G(k) - \nabla_\nu \Lambda(k) = 0. \tag{6.67}$$

The second equation will be referred as the "consistency condition".

The first equation will yield the usual matter conservation equations, while the second one a consistency relation between $\Lambda(k)$ and $G(k)$ that will ensure that the Bianchi identities are satisfied. One can see that the second constraint in fact implies that the time evolution of G and Λ should be such that the Bianchi identities are satisfied. The presence of an extra equation in our system, reflects also the fact that there is a new variable, the RG cut–off scale k.

The consistency condition for $\nu = 0$ yields

$$8\pi\dot{G} + \dot{\Lambda} = 0, \tag{6.68}$$

while for $\nu = i$ we get a trivial equation, since all quantities depend only on time in the FLRW background. The conservation of matter fluids yields the usual equation, which we write again here for convenience

$$\dot{\rho} + 3H(1+w)\rho = 0. \tag{6.69}$$

Following the standard approach in cosmology, if we try to solve for $a(t)$ and $\rho(t)$ for some given index w (assuming a single fluid for simplicity), we see that our system is overdetermined since we have three equations, i.e the Friedmann, matter conservation and consistency condition for two variables. In fact, one of the equations should be used to identify the scale k as a function of time, i.e $k = k(t)$. Let us see how this is done below.

From the consistency condition (6.68) one can solve for $\rho = \rho(k(t))$,

$$\rho(t) = -\frac{1}{8\pi}\frac{G'(k(t))}{\Lambda'(k(t))}, \tag{6.70}$$

where we assume that k is some function of cosmic time, that we will find later. From the matter conservation equation we get the usual solution $\rho \propto a^{-3(1+w)}$, which after combining with (6.70) gives an expression for the scale factor

$$a(t) = C\left[-\frac{G'(k(t))}{\Lambda'(k(t))}\right]^{\frac{1}{3(1+w)}}, \tag{6.71}$$

with the constant C given by $C \equiv a_0(8\pi\rho_0)^{\frac{1}{3(1+w)}}$.

The only unused equation so far is the Friedmann equation. We can use it to find an expression for the identification $k = k(t)$. Plugging (6.70) and (6.71) above relations into the Friedman equation we arrive at an equation which' solution will provide $k = k(t)$,

$$\frac{dk}{dt} = \frac{1}{3}\frac{a(k)}{a'(k)}\left[\Lambda(k) + 8\pi G(k)\rho(k)\right]. \tag{6.72}$$

The system of equations (6.70), (6.71) and (6.72) is now closed. For a given trajectory in the RG phase space, $k \to (G(k), \Lambda(k))$, we get a background cosmological evolution described by (6.70), (6.71), (6.72). Background quantities like the matter density parameter $\Omega_{\rm m}$ and the deceleration parameter can be expressed in terms of RG data, i.e the beta functions, the couplings and their derivatives (see for example Reuter and Saueressig (2005)).

It is important to notice that it is not a-priori obvious if a given RG trajectory will give a sensible cosmological evolution. In Reuter and Saueressig (2005) above procedure was followed and was found that one can induce a physically acceptable cosmological evolution, which starts from very high energies (UV) and asymptotes to a de Sitter phase at late times (IR).

Let us close this section by mentioning that other approaches in the RG cosmology have been used in the literature. In Grande et al. (2011), in the context of the Einstein–Hilbert truncation, a particular RG-inspired ansatz was used for the running of $G(k)$ and $\Lambda(k)$, and it was also shown that the identification $k = H$, is consistent with the Bianchi identities. The particular approach used there was an expansion of the form,

$$\rho_\Lambda = \Sigma_n C_n k^{2n}, \tag{6.73}$$

$$G^{-1} = \Sigma_n D_n k^{2n}, \tag{6.74}$$

with n an integer number and $\rho_\Lambda \equiv \Lambda/(16\pi G)$. The coefficients C_n, D_n in above expansions depend on different fields of masses M_i. For $k \ll M_i$ the series are expanded with respect to the small parameter k/M_i, and k is assumed to be of the order of some characteristic energy momentum–scale in an FLRW cosmological context. What is more, in the same work, it was shown that assuming the identification $k = H$ and a quadratic evolution law for ρ_λ,

$$\rho_\Lambda = C_0 + C_1 H^2, \tag{6.75}$$

with C_0, C_1 constants defined as

$$C_0 \equiv \rho_{\Lambda 0} - \frac{3\nu}{8\pi} m_P^2 k_0^2, \tag{6.76}$$

$$C_1 \equiv \frac{3\nu}{8\pi} m_P^2, \tag{6.77}$$

one can then use the differential constraint (6.68) to obtain a suitable evolution for $G = G(k)$,

$$G(H, \nu) = \frac{G_0}{1 + \nu \ln(H^2/H_0^2)}, \tag{6.78}$$

with $G(H_0) \equiv G_0 \equiv 1/m_P^2$. The parameter ν is defined as

$$\nu \equiv \pm \frac{1}{12\pi} \frac{M^2}{m_P^2}, \tag{6.79}$$

with $\pm$ corresponding to whether bosonic or fermionic fields contribute to the mass spectrum below the Planck scale. One can see that for $\nu > 0$ Newton's G decreases logarithmically with H, a slow enough evolution to account for a viable phenomenology.

Let us stress that above ansatzs for ρ_Λ and G^{-1} do not arise from a particular exact RG flow, but they are rather effective relations attempting to "catch" general RG features, and we refer the reader to Grande et al. (2011) for a detailed presentation of the issue. On the other hand, the approach we will use in chapter 7 to study the early and late time cosmological dynamics is based on an RG flow coming from an exact RG equation for gravity. A disadvantage of the latter approach is the fact that in principle it does not exist an analytic solution for the running of the couplings for the complete RG flow, however one can always resort to numerical techniques as well as analytical methods for particular parts of the RG evolution as we will see later.

6.4.3 Equivalence with Brans–Dicke gravity

We discussed before how we can handle a running Newton's G and cosmological constant at the level of the equations of motion. In this section, we will still focus at the level of the equations of motion, but we will present how one can find a link between the RG improved Einstein equations and Brans–Dicke gravity, as was also shown in Reuter and Weyer (2004b); Cai and Easson (2011)

The starting point is the usual RG improved Einstein–Hilbert action (in four dimensions),

$$S = \int d^4x\sqrt{-g}\left(\frac{R-2\Lambda(k)}{16\pi G(k)}\right), \tag{6.80}$$

considering vacuum for simplicity. It is important to remind here that the couplings are functions of the cut-off k but *not* of the metric. This is an important point when it comes to deriving the equations of motion. Let's vary the action with respect to the metric field. Schematically, we get

$$\delta S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}(\delta R - 2\delta\Lambda) - \frac{1}{16\pi G}(R-2\Lambda)\delta G\right] - \int \delta\sqrt{-g}\left(\frac{R-2\Lambda}{16\pi G}\right), \tag{6.81}$$

with the variation symbol here implying variation with respect to the metric, i.e $\delta \equiv \delta_g$.

Since the couplings do not depend on the spacetime metric we have

$$\delta G = \delta\Lambda = 0, \tag{6.82}$$

yielding

$$\delta S = \int d^4x\sqrt{-g}\frac{1}{16\pi G}\delta R - \int \delta\sqrt{-g}\left(\frac{R-2\Lambda}{16\pi G}\right). \tag{6.83}$$

The variation of the Ricci scalar reads schematically as

$$\delta R = \mathcal{O}(g) + \mathcal{O}(\nabla g) + \mathcal{O}(\nabla^2 g). \tag{6.84}$$

After the necessary integrations by parts, we then arrive at the equations of motion which read as,

$$G\widehat{D}_{\mu\nu}G^{-1} + G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{6.85}$$

with $\widehat{D}_{\mu\nu}$ a second order differential operator. Now, one can see that the equivalence with Brans–Dicke is revealed through the identification $\phi = G^{-1}$, i.e it is G^{-1} that plays the role of the Brans–Dicke scalar field. Notice also, that is has mass dimensions two, as we would expect for the Brans–Dicke scalar to have.

As we also discussed before, when performing the identification for the cut–off at the level of the equations of motion, it has to be ensured that the Bianchi identities are satisfied, mainly playing the role of an integrability condition for the Einstein equations. Let's see explicitly how this is achieved for the "Brans–Dicke" equations of motion derived above. The latter, equation (6.85), can be trivially re-written in the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(k)T^m_{\mu\nu} - \Lambda g_{\mu\nu} + G\widehat{D}_{\mu\nu}G^{-1}. \tag{6.86}$$

Covariant differentiation on both sides of above equation then requires that

$$\nabla^\mu G_{\mu\nu} = 0 = 8\pi T_{\mu\nu}\nabla^\mu G - g_{\mu\nu}\nabla^\mu\Lambda + (\nabla^\mu G)\widehat{D}_{\mu\nu}G^{-1} + G\nabla^\mu\widehat{D}_{\mu\nu}G^{-1}, \tag{6.87}$$

since $\nabla^\mu G_{\mu\nu} = 0$ identically and assuming also that the matter fields are conserved separately. Let us focus on the $\nabla^\mu\widehat{D}_{\mu\nu}G^{-1}$ term on the r.h.s of equation (6.87). It yields

$$\begin{aligned} \nabla^\mu\widehat{D}_{\mu\nu}G^{-1} &\equiv \nabla^\mu[(\nabla_\mu\nabla_\nu)G^{-1}] - \nabla_\nu(\Box G^{-1}) & (6.88)\\ &= (\nabla^\kappa\nabla_\nu - \nabla_\nu\nabla^\kappa)\nabla_\kappa G^{-1} & \\ &= R_{\nu\lambda}\nabla^\lambda G^{-1}. & (6.89) \end{aligned}$$

Using the latter relation in equation (6.87) we arrive at

$$\begin{aligned}
&G^{-1}\left[8\pi G T_{\mu\nu} + G\widehat{D}_{\mu\nu}G^{-1}\right]\nabla^{\mu}G - \nabla_{\nu}\Lambda + GR_{\nu\lambda}\nabla^{\lambda}G^{-1} = 0,\\
&G^{-1}\left[G_{\mu\nu} + \Lambda g_{\mu\nu}\right]\nabla^{\mu}G - \nabla_{\nu}\Lambda + GR_{\nu\lambda}\nabla^{\lambda}G^{-1} = 0,\\
&-(R-2\Lambda)\frac{G_{,k}}{G}\nabla_{\nu}k = 2\Lambda_{,k}\nabla_{\nu}k,
\end{aligned} \tag{6.90}$$

with $G_{,k} \equiv \partial G/\partial k$ and similar definitions hold for the other quantities.

Relation (6.90) is a differential constraint equation for the running couplings $G(k)$, $\Lambda(k)$, complementing the equations of motion. As in a cosmological context we expect the RG scale k to be related to cosmological time t, k becomes a dynamical variable at the level of equations of motion. Equation (6.90) then provides us with an extra condition that dictates the consistent dependence of k on time t so that the Bianchi identities are satisfied.

Let us make a point regarding the above way of establishing equivalence between an RG-improved action with Brans–Dicke gravity, i.e by working at the level of the equations of motion. Although it works for the Einstein–Hilbert truncation, in fact, it not very helpful if we would like to see if higher truncations are still equivalent with Brans–Dicke. To see this, let's consider the more general action

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} + c_2(k)R^2 + c_3(k)R^3 + \mathcal{O}(c_n(k)R^n)\right) \equiv \int d^4x\sqrt{-g}f(R), \tag{6.91}$$

Varying the action with respect to the metric field, we get schematically

$$\delta S = \int d^4x\sqrt{-g}f_R(R)\delta R + \int d^4x\delta\sqrt{-g}f(R) \tag{6.92}$$

$$\sim \left[(\nabla\nabla)^{\mu\nu}_R + (\nabla\nabla)^{\mu\nu}_k\right] f_R(R) + f_R G^{\mu\nu} + \dots, \tag{6.93}$$

with $(\nabla\nabla)_R$ and $(\nabla\nabla)_k$ denoting differentiation with constant R and constant k respectively. Now, it can be seen that the simple identification $G^{-1} = \phi$ that was made in the Einstein–Hilbert truncation is not enough to provide us with the equivalence with Brans–Dicke, due to the existence of more couplings other than $G(k)$.

The way out to this, is study the equivalence with BD through a covariant cut–

off identification, like for example $k^2 = \rho R$, arriving at an $f(R)$ model, which can then be expressed in the Jordan frame. This cut–off identification will be performed in a later section for the case of the Einstein–Hilbert truncation.

6.4.4 Evolution at the linear level

We discussed in the previous section how the RG improvement applies for the Einstein–Hilbert action in an FLRW cosmology. Here, for the same theory, we will go beyond the background, and will describe how in this context the linear, scalar perturbation equations around FLRW are derived. As the couplings now run with the energy scale, we will describe how that should be taken into account at the linear level, and in fact will find that it leads us to a non-trivial modification of the usual linearised Einstein equations.

Let us first point out that there is an important change of notation that we will be using below: in particular, we will still represent the Fourier mode with a k, and we will use the symbol k_{RG} for the RG scale. Furthermore, dots and primes will imply differentiation with respect to conformal time η and RG scale respectively.

We will be considering a perturbed FLRW background in the Newtonian gauge, with the line element reading as

$$ds^2 = a^2(\eta)\left[-(1+2\Psi)d\eta^2 + (1+2\Phi)dx^2\right], \tag{6.94}$$

with Φ and Ψ the scalar Newtonian potentials.

We can now proceed with perturbing the Einstein equations. Of course, we have to take into account the running of the coupling constants,

$$G(t,k) \to G(t) + \delta G(t,k), \tag{6.95}$$

$$\Lambda(t,k) \to \Lambda(t) + \delta\Lambda(t,k). \tag{6.96}$$

Notice that the perturbed coupling constants acquire a dependence on space, or on the mode k in Fourier space.

We then arrive at the perturbed Einstein equations which read as

$$\delta G^{\mu}{}_{\nu} = 8\pi G \left(T^{\mu}{}_{\nu}\frac{\delta G}{G} + \delta T^{\mu}{}_{\nu} - \frac{\delta\Lambda}{8\pi G}\delta^{\mu}{}_{\nu} \right). \tag{6.97}$$

Furthermore, we will assume that the energy–momentum tensor of the matter species in the Universe, like baryons or dark matter, acquires the form of a perfect fluid with a barotropic index $w \equiv p_{\rm m}/\rho_{\rm m}$,

$$T^{0}{}_{0} = -\rho_m, \quad T^{i}{}_{j} = w\rho_m\delta^{i}{}_{j}, \tag{6.98}$$

$$\delta T^{0}{}_{0} = -\delta\rho_m, \quad \delta T^{0}{}_{i} = (1+w)\rho_m\partial_i v, \tag{6.99}$$

with v_i the velocity field which we express as the gradient of a scalar, since we are interested in scalar perturbations only. We also define the velocity gradient as $\theta \equiv \nabla_i v^i$.

We will be interested in the evolution of small inhomogeneities at scales well inside the horizon, i.e we will be focusing on those modes that satisfy $k \ll H$. In this sub-horizon regime, the $0-0$ part of the perturbed Einstein equations gives the Poisson equation,

$$k^2\Phi = 4\pi a^2 G\rho_m \left(\delta_G + \delta_m + \frac{\delta\Lambda}{8\pi G\rho_m} \right), \tag{6.100}$$

with the definitions

$$\delta_m \equiv \frac{\delta\rho_m}{\rho_m}, \quad \delta_G \equiv \frac{\delta G}{G}. \tag{6.101}$$

One notices here that the r.h.s of the Poisson equation, acquires a contribution of the perturbed G and Λ apart from the matter perturbation.

On the other hand the off-diagonal part of the $i-j(i \neq j)$ component of the Einstein equations gives the anisotropy equation

$$\Phi + \Psi = 0. \tag{6.102}$$

The r.h.s of above equation is zero, since here we are interested in cosmological periods well after radiation domination, where the anisotropic stress coming from relativistic species is negligible. On the other hand, the running of the coupling constants does not yield any anisotropic stress contribution either.

The Poisson and the anisotropy equation are the main gravitational equations we will need. As described in a previous section, at the linear level of an FLRW background we also have two conservations equations, one corresponding to energy and the another to momentum conservation respectively,

$$\dot{\delta}_m = -\theta_m - 3\dot{\Phi}, \tag{6.103}$$

$$\dot{\theta}_m = -H\theta_m - k^2\Psi. \tag{6.104}$$

Remember that $\theta_m \equiv ik^i u_i$ is the velocity divergence of the pressure less matter velocity field in Fourier space. Using the Poisson equation (6.101), equation (6.103) can be written as

$$\dot{\theta} = -\mathcal{H}\theta + \frac{3}{2}\Omega_m \left(\delta_G + \delta_m\right), \tag{6.105}$$

where we neglected a θ term suppressed by k^2.

The second term on the r.h.s of (6.104) can be neglected in the sub-horizon regime we are interested in. This can be seen as follows. Differentiating the Poisson equation (6.101) with respect to conformal time, and using the perturbed Bianchi conditions (6.107)-(6.108), one can check that the term $\dot{\Phi}$ contributes four terms, proportional to $\delta_m, \dot{\delta}_m, \theta_m, \delta_G$, all being suppressed by H^2/k^2, which is very small in this regime. Therefore, equation 6.103 reduces to

$$\dot{\delta}_m = -\theta_m. \tag{6.106}$$

However, the system of equations is not yet complete. In fact, we should also perturb the integrability condition (6.67), which ensures that the Bianchi identities are satisfied. Remember that in the background these equations had to be satisfied as a result of the Bianchi identities not identically satisfied due to the running of the couplings. Considering also relations (6.95) and (6.96), perturbing (6.67) linearly we arrive at the following equations (for $\nu = 0$ and $\nu = i$ respectively),

$$Y\left(\delta_m + \delta_G\right) + \dot{\delta}_G + \frac{\delta\dot{\Lambda}}{8\pi\dot{G}\rho_m} = 0, \tag{6.107}$$

$$k^2\left(w\rho_m\delta_G - \frac{\delta\Lambda}{8\pi G}\right) = (1+w)Y\theta, \tag{6.108}$$

where we defined the quantity $Y \equiv \dot{G}/G$, which describes the relative variation of Newton's G with respect to conformal time. We shall assume in the following that the relative time variation of G is sufficiently small during matter domination, therefore neglecting second time derivatives of G as well as terms of order Y^2.

Remembering that,

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}, \tag{6.109}$$

relation (6.108) can also be re-expressed in terms of ρ_Λ as

$$k^2\left[\delta_G(w\rho_m + \rho_\Lambda) - \delta\rho_\Lambda\right] = \rho_m(1+w)Y\theta. \tag{6.110}$$

Relation (6.108) tells us something very interesting. Noting that,

$$\delta G(k_{RG}) \equiv \delta G = \frac{dG(k_{RG})}{dk_{RG}}\delta k_{RG}, \quad \delta\Lambda(k_{RG}) \equiv \delta\Lambda = \frac{d\Lambda(k_{RG})}{dk_{RG}}\delta k_{RG}, \tag{6.111}$$

we can express the perturbed Bianchi condition (6.108) as

$$\delta k_{RG} = \frac{(1+w)}{w - \frac{\Lambda'}{\Lambda}\frac{G}{G'}\frac{\Omega_\Lambda}{\Omega_m}}\dot{k}_{RG}\frac{\theta}{k^2}. \tag{6.112}$$

Above relation shows that the perturbation of the cut–off scale k_{RG} is proportional to the velocity gradient of the matter fluid present. A similar result has been found in Contillo et al. (2012a) in the context of scalar field inflation. Notice that it is $\delta k_{RG} = 0$, for the case of a Λ-dominated Universe, i.e $w = -1$, as well as whenever the background cut–off scale stops evolving, $\dot{k}_{RG} = 0$.

Apart from the two scalar potentials Φ and Ψ, there are four more variables at the linear level, namely δ_m, δ_G, δ_Λ(or $\delta\rho_\Lambda$) and θ. Equations (6.103)-(6.104) describe the evolution of δ_m and θ respectively, while equations (6.107)–(6.108) provide us with constrains between different variables. In particular, (6.107) relates the time derivative of δ_G with that of $\delta\Lambda$, while (6.108) relates δ_G and $\delta\Lambda$ with the velocity gradient θ. Therefore, taking also into account the two gravitational equations, we have six equations and six variables.

Differentiating equation (6.108) and with the aid of the following equations

$$\dot{\rho_m} + 3\mathcal{H}(\rho_m + p_m) = 0, \tag{6.113}$$

$$\dot{\delta_G} \equiv \left(\frac{\delta G}{G}\right)^{\cdot} \equiv \dot{\delta_G} = \frac{\delta\dot{G}}{G} - \delta_G Y, \quad \frac{\ddot{G}}{G} = \dot{Y} + Y^2, \tag{6.114}$$

we arrive at a first order evolution equation for δ_G,

$$\dot{\delta_G} = \alpha Y \frac{\theta}{k^2} + \left[\frac{3}{2}\frac{Y\Omega_m(1+w)}{k^2} + \frac{\beta}{1+w} - \frac{Y}{1+w}\right]\delta_G - \frac{Y}{1+w}\delta_m. \tag{6.115}$$

For convenience we have defined

$$\alpha \equiv \frac{\dot{w}}{1+w} + \frac{\dot{Y}}{Y} + Y - (2+3w)\mathcal{H}, \tag{6.116}$$

$$\beta \equiv \frac{3w}{1+w}\mathcal{H} - \dot{w} - wY. \tag{6.117}$$

For consistency, we have derived above equation without fixing w, however in this section we are interested in the case of a pressure less matter dominated Universe, i.e $w = 0$. However, setting $w = 0$, and neglecting second derivatives of G as well as terms of order Y^2 we get

$$\dot{\delta_G} = -2\mathcal{H}Y\frac{\theta}{k^2} - \frac{3}{2}Y\delta_G - Y\delta_m. \tag{6.118}$$

Equations (6.104),(6.106) and (6.118), form a closed system of coupled, first order evolution equations for δ_m, θ and δ_G. Notice that for $Y = \dot{\Lambda} = \delta_G = \delta\Lambda = 0$, the system reduces to the ordinary linear equations with constant G and Λ.

The variation of Newton's G affects the evolution of the system in two ways; By affecting the scalar potential in the Poisson equation through the variation of G, as well as the appearance of the new term $\delta_G \equiv \delta G/G$ on the r.h.s of the equation. What is more, the variation of G implicitly affects the evolution the Hubble parameter $\mathcal{H}$. The running of Λ affects itself the scalar potential in the Poisson equation as well.

We notice that the way the running of the coupling constants here affects the evolution of the linear equations shares some common features with scalar tensor theories, where the scalar field non–minimally coupled to curvature in the action plays the role of the effective (running) Newton's G, while the scalar field perturbation also appears on the r.h.s of the Poisson equation. An important difference in the RG scenario we have studied and scalar-tensor theories, is that in the former, a crucial role is play by the Bianchi identities, which provide the necessary constraint equations to close the system.

It would be worth trying to find solutions for above equations, to understand the way the growth of matter is affected by the evolution of the coupling constants. We shall leave this for future work, as the main goal of this section was mainly to only present the form of linear equations in this scenario. We shall also notice that similar analysis has been performed in Fabris et al. (2007); Grande et al. (2010, 2011).

Chapter 7

Effective $f(R)$ action from running couplings

7.1 Introduction

In this chapter, we will be closely following the work done together with Mark Hindmarsh in Hindmarsh and Saltas (2012). In the previous chapter we introduced the application of the RG to quantum gravity, and we also discussed its application in cosmology. In particular, as we previously discussed, in a cosmological context, one needs to identify the RG scale k with some function of cosmic time, focusing on the case of the Einstein–Hilbert truncation, and working at the level of the equations of motion. As it turned out to be, it is crucial that the identification chosen satisfies the Bianchi identities, a fact that itself introduces significant complications to the actual analysis.

In this chapter, we will consider a different cut–off identification, namely one which is performed at the level of the action, motivated by an analogous procedure which generates the effective potential for a scalar field theory. We will associate k with the scalar curvature, i.e

$$k^2 \sim R, \tag{7.1}$$

through which we can view the RG improved Einstein-Hilbert action as an effective $f(R)$ model. We will then go on to study the properties as well as the resulting cosmology of the particular $f(R)$ model.

The idea of Newton's G running with curvature has been suggested previously Frolov and Guo (2011), although not with the correct beta-function, and here we include the cosmological constant with the full non-perturbative beta functions for both couplings. The resulting $f(R)$ model does not include the renormalisation effects of matter, or any gravitational invariants other than R, and so it should be viewed as a prototype. However, it will turn out that it has some remarkable properties, also allowing us to study the RG improved action in an elegant way. One feature is that the scale identification is performed at the level of the action, in a covariant fashion, so there is no need to add extra dynamical conditions through the Bianchi identities as described above.

In the following, we will be working in a unit system with $c = \hbar = 1$, unless otherwise stated, as well as use $G = m_p^{-2} = 8\pi\kappa^2$. Unless otherwise stated, mass scales will be presented in Planck units.

7.2 RG improved Einstein–Hilbert action

This section will serve as an introduction to the basic concepts and notation that we will use in the following, reminding also briefly about notions introduced in the previous chapter.

Our starting point is the RG improved effective action in the Einstein–Hilbert truncation,

$$\Gamma_k[g,\psi] = \int d^4x\sqrt{-g}\left[\left(\frac{R(g) - 2\Lambda_k}{16\pi G_k}\right) + \mathcal{L}_{\text{matter}}(\psi, g)\right], \tag{7.2}$$

with k is the renormalisation group cut-off scale, which sets the momentum scale above which modes are integrated out. The effective, "coarse-grained" action functional, $\Gamma_k[g,\psi]$, interpolates between the true effective action in the infrared (IR, $k \to 0$) and the bare action defined in the UV at a cut–off scale k_{max}. The interpolation of the effective action as a function of scale is controlled by the exact renormalisation group equation (ERGE) Wetterich (1993). If k_{max} can be taken to infinity the theory is renormalisable, signaled by a UV fixed point in the couplings of the theory.

The quantum corrections can be encoded in the evolution of the coupling constants as a function of energy,[1] whose beta-functions can be extracted from the ERGE. The form of the latter depends on the choice of the cut-off function choice and the gauge. We will follow the conventions of Litim (2000); Litim and Manuel (2001), noting that different choices of cut-off function and gauge do not change the qualitative features of the beta functions.

In the standard approach, one defines the dimensionless Newton's and cosmological constant as

$$g(k) \equiv k^2 G(k)/24\pi, \quad \lambda(k) \equiv \Lambda/k^2, \tag{7.3}$$

and the running of the dimensionless couplings in $d = 4$ is described through the set of first order, coupled differential equations Litim (2004, 2008b),

$$\partial_t \lambda = \beta_\lambda(g, \lambda) \equiv -2\lambda - 12g - \frac{24g\big(3g + \frac{1}{2}(1-3\lambda)\big)}{2g - \frac{1}{2}(1-2\lambda)^2}, \tag{7.4}$$

$$\partial_t g = \beta_g(g, \lambda) \equiv 2g + \frac{24g^2}{4g - (1-2\lambda)^2}, \tag{7.5}$$

where $t \equiv \ln k$, and β_λ, β_g the beta functions. In above equations the factor of 24π is included to remove phase space factors.

There are two fixed points of the above RG flows, a free or Gaussian one, with $(g^*, \lambda^*)_{\mathrm{GFP}} = (0, 0)$, and an interacting one which is attractive in the UV ($k \to \infty$), with $(g^*, \lambda^*)_{\mathrm{UV}} = (0.015625, 0.25)$ (Litim (2008b)-Reuter and Saueressig (2007b) and references therein). The existence of a UV fixed point points to consistent quantum behavior of the system at high energies, realising Weinberg's Asymptotic Safety scenario Hawking S. W. (1979). The Gaussian fixed point ($k \to 0$) describes a free theory.

A phenomenologically viable solution (trajectory) of the system (7.4)-(7.5) on the $g - \lambda$ plane is one that starts at high energies from the UV fixed point and then evolves towards smaller values of g as k is lowered, passes close to the GFP, until it turns to the right towards increasing values of λ. A trajectory passing sufficiently close to the GFP will subsequently have a long classical regime, i.e $G \simeq G_0$, $\Lambda \simeq \Lambda_0$, with "0" here denoting the present value. The classical regime covering many orders of magnitude in scales is required by terrestrial, solar and galactic tests, as well as consistency with cosmological evolution since Big Bang Nucleosynthesis.

[1] For a cautionary note see Anber and Donoghue (2011).

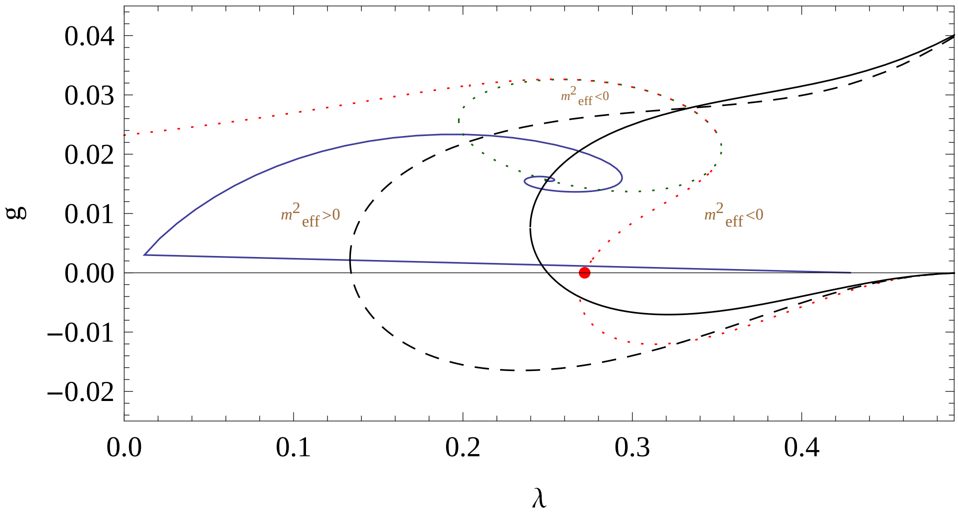

Figure 7.1: A viable RG trajectory (blue, continuous curve) on the $g - \lambda$ plane in the Einstein–Hilbert truncation for the choice $\rho = 1$ and the initial conditions (7.35). It spirals around the UV RG fixed point and evolves towards the IR as curvature R decreases. The intersection of the phase curve with the de Sitter line (black, continouous) corresponds to a *de Sitter point* in the cosmological evolution, while intersection with the dashed (black) one is where for the slow roll parameter $\epsilon_V = 1$. The regions where $m^2_{\text{eff}} > 0$ ($m^2_{\text{eff}} < 0$) are seperated by the dotted lines, with m^2_{eff} the Jordan frame mass squared, defined in (7.14). The dotted curve consists of two separate curves (green and red dots) corresponding to the vanishing of the numerator (denominator) of m^2_{eff}. They join at the upper part of the dotted "ellipsis", where m^2_{eff} remains finite and non-zero. Along the lower part of the "ellipsis" (green) m^2_{eff} vanishes. The dotted curves outside the "ellipsis" (red) correspond to $m^2_{\text{eff}} \to \infty$. Notice that beyond the red dot at $\lambda_* \simeq 0.27$ on the λ-axis, m^2_{eff} becomes negative, and therefore de Sitter space unstable too.

The Einstein-Hilbert truncation has a couple of features which may not be present in all truncations. The eigenvalues of the linearised flow in the vicinity of the UV fixed point are complex conjugate, causing oscillatory behavior of the trajectory around it (see Figure 7.1). Also, the flow (7.4,7.5) has a singularity at $\lambda = 1/2$, which terminates the classical regime. It has been conjectured that this is an artifact of the truncation, and that there may actually be a non-trivial fixed point in the IR Bonanno and Reuter (2002b).

7.3 Scale identification & effective $f(R)$ action

As explained before, the first step in studying the cosmology of an RG improved action is to identify the cut-off scale k with as a function of cosmic time $k = k(t)$.

In this section, we will work at the level of the action and use a particular ansatz that will allow us to view the effects of RG running of the couplings as an effective $f(R)$ model, by identifying

$$k^2 = \rho R, \tag{7.6}$$

where R is the Ricci scalar and ρ is a dimensionless constant. Here, with the particular identification (7.6) the dimensionless couplings are defined as $g(k) \equiv \rho R \times G(R), \quad \lambda(R) \equiv \Lambda(R)/(\rho R)$, and action (7.2) takes the following form

$$\begin{aligned} S_{f(R)} &= \int d^4x\sqrt{-g}\frac{R^2 h(R)}{384\pi^2} + S_{\rm m}(\psi, g) &(7.7)\\ &\equiv \int d^4x\sqrt{-g}\frac{f(R)}{2\tilde{\kappa}^2} + S_{\rm m}(\psi, g), &(7.8)\end{aligned}$$

with $h(R) \equiv \rho(\frac{1-2\rho\lambda}{g})$, and the extra factor of 24π appearing in the first line because of the rescaling of g performed in the beta functions (7.4) and (7.5). We absorb it into the factor $\tilde{\kappa}^2 = 192\pi^2$.

The quantum corrections are now expressed in the non-linear effective action, which takes the form of an $f(R)$ model (7.8). This provides us with a different view of the RG effects on the Einstein–Hilbert action (7.2). What is more, the particular scale identification preserves general covariance of the action.

We can compare this procedure with the RG-improvement of the effective potential in scalar field theory Coleman and Weinberg (1973). There, if one starts with the tree potential $V = \lambda\phi^4/4!$, solves the RG equation for the coupling, and makes the identification $k = \alpha\phi$, one obtains at one loop

$$V = \frac{1}{4!}\frac{\lambda_0}{1 - b(\lambda_0)\ln(\alpha\phi/k_0)}\phi^4, \tag{7.9}$$

where $b(\lambda) = 3\lambda/16\pi^2$, which recovers and improves on the one-loop effective potential calculated by the standard graphical methods. The constants are constrained by a renormalisation condition such as

$$\left.\frac{d^4V}{d\phi^4}\right|_{\phi_0} = \lambda_{\rm r}, \tag{7.10}$$

where $\lambda_{\rm r}$ is the physical coupling as inferred, say, from a scattering experiment with the background field set at ϕ_0. Normally, we can avoid all mention of α by writing $V = \lambda\phi^4/4!$, with $\lambda = \lambda_{\mathrm{r}}/(1 - b(\lambda_{\mathrm{r}})\ln(\phi/\phi_0))$. However, it is still implicit in the relationship between ϕ_0 and the scale k_0.

The renormalisation conditions for the effective Einstein-Hilbert action can be taken as

$$f_R|_{R_0} = \frac{\tilde{\kappa}^2}{8\pi G_0}, \qquad \left.\frac{Rf_R - f}{2f_R}\right|_{R_0} = \Lambda_0, \tag{7.11}$$

where R_0 is the curvature scalar evaluated today.

Some remarks regarding the action (7.8) are in order. Firstly, we can see that on a fixed point, where $h(R)$ is constant, the Lagragian is effectively R^2, which is renormalisable Stelle (1977). Secondly, there is a singularity of the RG flow in the Einstein–Hilbert truncation Litim (2008b); Reuter and Saueressig (2005): the beta functions diverge for $4g = (\lambda - \frac{1}{2})^2$. We will therefore restrict ourselves to cosmological evolution which does not reach the singularity.

Finally, let us comment on the dimensionless parameter ρ, defined through our identification (7.6), relating the RG scale k and the cosmological scale R. We will see in equation (7.33) that it determines the scalaron mass, and so in principle could also be fixed. However, as we do not know the scalaron mass, we will leave ρ free, and investigate what range of values give an acceptable cosmology. As ρ describes to what extent the RG scale k follows the curvature R, we would hope to find that $\rho \sim 1$: it is natural to think of the RG scale as the scale of the important dynamics, which in the cosmological context is given by the curvature. It will in fact turn out $\rho \sim 1$ gives a viable cosmology.

7.4 Stability and the GR limit

7.4.1 Degrees of freedom and stability conditions

As a first step to understand the resulting effective $f(R)$ action from the renormalisation group (7.8), we want to study its stability and its approach to the limiting case of GR. Below we remind ourselves about some basic facts about $f(R)$ gravity that will be necessary for the rest of the chapter Starobinsky (1980a); Nojiri and Odintsov (2006a); Capozziello and Francaviglia (2008); Sotiriou and Faraoni (2010);

De Felice and Tsujikawa (2010); Clifton et al. (2012).

As described also in a previous chapter, it is well known that $f(R)$ models exhibit an extra, massive scalar degree of freedom, dubbed "scalaron". It satisfies a Klein–Gordon type equation, which can be found by varying action (7.8) with respect to the metric and then taking the trace,

$$\Box f_R(R) + \frac{dV_{\text{eff}}(R)}{df_R} = \frac{\tilde{\kappa}^2}{3} T_{(\text{m})}, \tag{7.12}$$

where $\Box$ is the d'Alembertian associated with the metric $g_{\alpha\beta}$, $T_{(\text{m})} \equiv g^{\mu\nu} T_{(\text{m})\mu\nu}$ is the trace of any matter sources present, and

$$\frac{dV_{\text{eff}}(R)}{df_R} \equiv \frac{1}{3} \left[R f_R(R) - 2f(R) \right]. \tag{7.13}$$

From (7.13) we can deduce the scalaron's mass in the frame defined by action (7.8),

$$m_{\text{eff}}^2 \equiv \frac{d^2 V_{\text{eff}}(R)}{df_R^2} = \frac{f_R - R f_{RR}}{3 f_{RR}}. \tag{7.14}$$

Expression (7.14) also appears as the effective mass in a stability analysis around de Sitter spacetime Sotiriou and Faraoni (2010) (and references therein).[2] Therefore, stability of the scalaron propagation (i.e avoidance of tachyonic instability), as well as stability of de Sitter spacetime requires that $m_{\text{eff}}^2 > 0$.

While an unstable scalaron just means that long-wavelength scalar fluctuations will grow, the graviton kinetic term must certainly have the correct sign, in order to avoid ghosts. This means that

$$f_R > 0, \tag{7.15}$$

which at small values of the couplings λ and g, is ensured through condition

$$\frac{\rho R}{g} > 0. \tag{7.16}$$

In order to make the connection with the RG, we will express both stability conditions, for the scalar and for the graviton, in terms of the beta functions, using the explicit form of action (7.2).

[2]Note that there is another definition for the scalaron mass in the Einstein frame, which we will present later.

In the RG-improved Einstein–Hilbert action derivatives of f can be expressed in terms of RG data, as

$$\frac{d}{dR} = \frac{\partial}{\partial R} + \frac{1}{2R}\left(\beta_g \frac{\partial}{\partial g} + \beta_\lambda \frac{\partial}{\partial \lambda}\right), \tag{7.17}$$

For example, for f_R we have,

$$f_R = 2R\left[h - \frac{1}{4g}(h\beta_g + 2\rho^2\beta_\lambda)\right], \tag{7.18}$$

while the second derivative is

$$\begin{aligned} f_{RR} =& 2h + \frac{\beta_g}{4g}\left(-6h + 2h\frac{\beta_g}{g} + 2\rho^2\frac{\beta_\lambda}{g} - h\beta_{g,g} - 2\rho^2\beta_{\lambda,g}\right) \\ & - \frac{\beta_\lambda}{4g}\left(8 + 4\rho^2 - 2\frac{\beta_g}{g} + h\beta_{g,\lambda} + 2\rho^2\beta_{\lambda,\lambda}\right). \end{aligned} \tag{7.19}$$

Plugging above relations into expression (7.14), and using the beta functions (7.4,7.5), the scalar mass m^2_{eff} can be re-expressed as $m^2_{\text{eff}} = m^2_{\text{eff}}(R, g, \lambda)$. The same is in principle true for f_R and other quantities of interest as we will see later.

In particular, from the explicit expression of f_R in terms of the couplings, one can check that the no-ghost condition (7.15) is always satisfied in the domain of interest, $0 < \lambda < 0.5$, $0 < g \lesssim 0.02$ and of course $R > 0$. (see also Figure 7.2 for a plot of f_R and Figure 7.1 for the phase space of a viable RG trajectory.)

7.4.2 The $f(R)$ model in the perturbative regime

Let us now see how GR is recovered in this framework. Let us remind the reader that the phase diagram of the Einstein–HIlbert truncation was presented in section 6.2.1. If an RG trajectory is to be viable, it should have a sufficiently long classical regime, where any quantum corrections are suppressed enough not to be observed in astrophysical or solar system tests, and therefore the coupling constants should be effectively constant, and acquire the values observed at these scales. Therefore, GR is recovered in the sense that under the RG flow Newton's constant acquires its classical value, $G \simeq G_0 = 1/m_p^2$ for a sufficient "RG time", large enough to cover the range of classical scales (earth, solar and galactic). In the classical regime, Λ

has to have a negligible variation too. It has been shown that both requirements are achieved if the viable RG trajectory passes sufficiently close to the GFP at $(g, \lambda) = (0, 0)$ Reuter and Weyer (2004a). It is after the close passage to the GFP when the classical regime starts, and it turns out that the closer the trajectory passes to it, the longer it lasts in RG time, and the greater range of scales the classical regime covers.

We will therefore need to first linearise the system of beta functions (7.4)-(7.5) around the GFP Reuter and Weyer (2004a). To make the analysis more clear, it would be better to first proceed with the linearisation of the equations without assuming any identification for k, i.e keeping k as the independent variable in (7.4)-(7.5). We get,

$$\partial_t \lambda = -2\lambda + 2\alpha g, \tag{7.20}$$

$$\partial_t g = 2g, \tag{7.21}$$

with renormalisation group time $t \equiv \ln(k/k_0)$, and k_0 a reference scale. The parameter α cut-off function dependent, but is always positive and of order 1. For the optimised cut–off, used to derive the beta functions (7.4)-(7.5), $\alpha = 6$.

The solution of the linearised system reads

$$g = c_1 k^2 \equiv g_T \frac{k^2}{k_T^2} \tag{7.22}$$

$$\lambda = \frac{1}{2}\alpha c_1 k^2 + \frac{c_2}{k^2} \equiv \frac{1}{2}\lambda_T \left(\frac{k^2}{k_T^2} + \frac{k_T^2}{k^2} \right), \tag{7.23}$$

with k_T the value of the cut-off scale around the turning point in the vicinity of the GFP, and $g_T \equiv g(k_T), \lambda_T \equiv g(k_T)$. Notice also that is,

$$\lambda_T / g_T = \alpha, \tag{7.24}$$

which implies that $\lambda_T \sim g_T$, since $\alpha \sim O(1)$. From the above linearised relations we get for the dimensionful couplings,

$$\frac{8\pi G}{\tilde{\kappa}^2} = c_1 = \frac{g_T}{k_T^2} = \text{const.} \tag{7.25}$$

$$\Lambda = \frac{1}{2}\alpha c_1 k^4 + c_2 = \frac{1}{2}\lambda_T \frac{k^4}{k_T^2} + \frac{1}{2}\lambda_T k_T^2, \tag{7.26}$$

following the notation of Reuter and Weyer (2004a). Equation (7.25) tells us that in this regime, Newton's G becomes a constant, and we identify $c_1 = 8\pi G_0/\tilde{\kappa}^2$. For scales $k \ll k_T$, Λ is also effectively constant, and we may identify $c_2 = \Lambda_0$. Hence

$$\frac{g_T \lambda_T}{2} = \frac{8\pi G_0 \Lambda_0}{\tilde{\kappa}^2}, \tag{7.27}$$

and

$$g_T = \sqrt{\frac{16\pi G_0 \Lambda_0}{\alpha \tilde{\kappa}^2}}, \quad k_T = \left(\frac{\tilde{\kappa}^2 \Lambda_0}{\alpha 4\pi G_0}\right)^{\frac{1}{4}}. \tag{7.28}$$

From the observed values of Λ_0 and G_0, we have

$$g_T \sim \lambda_T \sim 10^{-60}, \quad k_T \sim 10^{-30} m_{\mathrm{p}}. \tag{7.29}$$

Let us now turn to the solution of the system under the identification $k^2 = \rho R$. The linearised equations are not enough as higher-order terms contribute already at $\mathrm{O}(R^2)$. An efficient way to include the higher-order terms is to substitute into the Talyor expansion around the renormalisation point R_0 (7.11)

$$f(R) = f(R_0) + f_R|_{R_0}(R - R_0) + \frac{1}{2} f_{RR}|_{R_0}(R - R_0)^2. \tag{7.30}$$

We find that for the optimised cut-off where $\alpha = 6$,

$$f(R) \simeq \frac{\tilde{\kappa}^2}{G_0}(R - 2\Lambda_0) + 6(2-\rho)\rho(R - R_0)^2. \tag{7.31}$$

From the small coupling expansion of (7.14) the scalaron mass squared in the classical regime is given by

$$m_{\mathrm{eff},0}^2 = \left.\frac{f_R - R f_{RR}}{3 f_{RR}}\right|_{R_0} \simeq \frac{1}{36(2-\rho)} \frac{R_0}{g}, \tag{7.32}$$

(see also equation 7.49) and we see that it is positive provided $0 < \rho < 2$. Using the renormalisation condition (7.11) we find

$$m^2_{\text{eff},0} \simeq \frac{1}{36(2-\rho)} \frac{\tilde{\kappa}^2}{8\pi G_0}, \tag{7.33}$$

and observe that the scalaron mass is safely at the Planck scale, so large deviations from GR at laboratory, solar and astrophysical scales are avoided.

To get an idea of the realistic values of the couplings in the classical regime we can evaluate them at solar and galactic scales, taking $k^2 \sim R$. With $R_{\text{sol}}^{-1/2} \sim 1\text{AU}$ [3] and $R_{\text{gal}}^{-1/2} \sim 10^{21}\text{m}$ we find that

$$g_{\text{sol}} \simeq R_{\text{sol}} \times G_{\text{sol}} \simeq 10^{-92}, \;\; g_{\text{gal}} \simeq R_{\text{gal}} \times G_{\text{gal}} \simeq 10^{-112}, \tag{7.34}$$

assuming that $G_{\text{gal}} = G_{\text{sol}} \simeq 10^{-70}\ \text{m}^2$. We see that the classical value of the dimensionless coupling g acquires a tiny value. For λ we cannot follow the same analysis, since Λ has been only measured at cosmological scales, $k \sim H_0$, with H_0 the Hubble parameter today. However, the product $g\lambda \sim G_0\Lambda_0$, so $\lambda_{\text{sol}} \ll 1$. Therefore, the values of both g and λ on solar and galactic scales lie extremely close to the GFP. However, it is intriguing to note that by this reasoning, λ evaluated at the Hubble scale is of order 1, where non-perturbative effects in the beta functions are important Reuter and Weyer (2004a).

The form of a phenomenologically viable RG evolution on the $g - \lambda$ plane is given in Figure 7.1, for the choice of $\rho = 1$ and [4]

$$\begin{aligned} &R_{\text{min}} = 8 \times 10^{-5}, \; R_T = 5 \times 10^{-3}, \; R_{\text{max}} = 50, \\ &\lambda(R_T) = 10^{-2}, \; g(R_T) = 10^{-3}, \end{aligned} \tag{7.35}$$

where R_T is the curvature at the turning point close to the GFP, in Planck units. The above initial conditions are not realistic, but they allow for a good numerical illustration. Figure 7.2 shows the evolution of the derivative $f_R(R)$ of the resulting $f(R)$ model, for the above choice of initial conditions.

[3] $1\text{AU} \simeq 1.496 \times 10^{11}$ m.

[4] Numerical solutions in this paper are obtained using Mathematica's differential and algebraic solvers, making use of the stiffness option as well as increasing the maximum step number when appropriate. Plots are also produced with the same software.

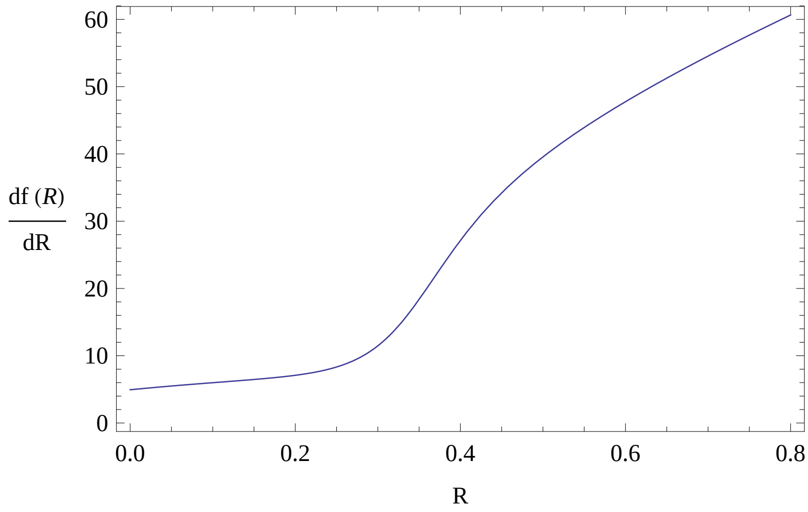

Figure 7.2: The derivative of the $f(R)$ model for the initial conditions (7.35) (in Planck units). For large values of R, the model effectively behaves as R^2 gravity, since $f_R(R) \sim R$. For smaller R, the evolution enters the classical regime near the Gaussian fixed point. The Einstein-Hilbert term dominates, $f_R(R)$ becomes nearly constant, with a small positive slope reflecting the positivity of the scalaron mass-squared. At very small R (not shown) there is a departure from the Einstein-Hilbert action due to the IR divergence in the beta functions. This part of the action is never encountered as the system freezes at the IR de Sitter point.

7.5 Cosmological dynamics

Now, we proceed with studying the cosmological dynamics of the model, i.e the cosmological fixed points, their stability and the transition from one cosmological era to the other, as well as if inflation can be viable in this scenario.

A viable cosmological model, aiming to describe the background evolution of the Universe from early to late times, should have a period of accelerated expansion at early times (inflation), followed by a radiation and matter era respectively, evolving asymptotically towards de Sitter at late times. Each particular period has its own requirements in order to be viable. For example, a UV de Sitter point should be unstable, while an IR one stable, while the matter point should be a saddle with damped oscillation, so that strucutre formation has enough time to take place.

7.5.1 Transition to the Einstein frame

It will be useful for the latter analysis to first calculate the Einstein frame action, as an aid in calculating inflationary quantities, like the slow roll parameters. To do this we will introduce an auxiliary field and then conformally transform the metric appropriately. However, in the context of action (7.8) the latter transformation requires care, since Newton's G is running with curvature.

Let us see this in more detail. Our original action (7.8) is a function of $R, g(R), \lambda(R)$ and implicitly of the metric through the Ricci scalar R. In the standard way, we introduce auxiliary scalars σ and ϕ, and write our original theory in the Jordan frame as

$$S = \frac{1}{2\tilde{\kappa}^2} \int \sqrt{-g} \left[f'(\sigma) R - (f'(\sigma)\sigma - f(\sigma)) \right] \tag{7.36}$$

$$\equiv \frac{1}{2\tilde{\kappa}^2} \int \sqrt{-g} \left[\phi R - V(\phi) \right], \tag{7.37}$$

with $V(\phi) = \phi\sigma(\phi) - f(\sigma(\phi))$, and $\phi = f'(\sigma)$. We require that $f''(\sigma) \neq 0$, so that the function f' can be inverted to find σ as a function of ϕ. Note that the equation of motion for σ gives the constraint which reproduces the original action, i.e $\sigma = R$.

The Jordan frame scalar ϕ, plays the role of the inverse of Newton's constant in front of R. For the transition to the Einstein frame, Newton's constant will have to be re-introduced through the conformal redefinition of the metric, and the question that arises in our scenario, is which Newton's constant should that be, since using a running $G = G(R)$ could lead to ambiguities. We can resolve this issue by using Newton's G today, denoted $G = G_0$.

We can now perform the conformal redefinition of the metric as

$$\tilde{g}_{\alpha\beta} = \frac{8\pi G_0 \phi}{\tilde{\kappa}^2} g_{\alpha\beta}, \tag{7.38}$$

combined with a redefinition on the scalar ϕ

$$\phi = \phi_0 \exp\left(\sqrt{\frac{16\pi G_0}{3}} \Phi \right), \tag{7.39}$$

with ϕ_0 constant. Performing above two field redefinitions in action (7.37), we finally

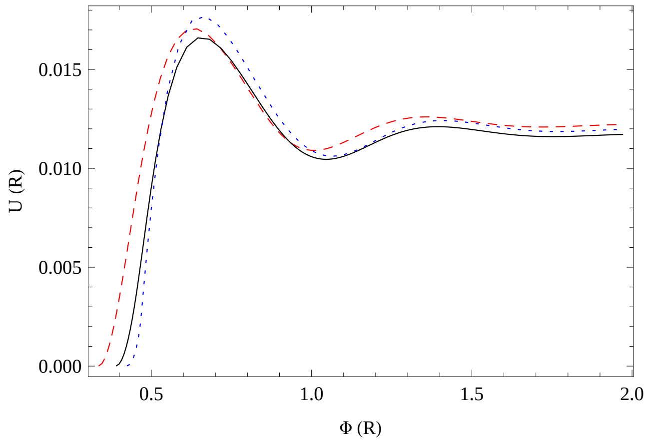

Figure 7.3: The Einstein frame scalar potential (in Planck units), described by relations (7.41)-(7.42), for $\rho = 0.8$ (red, dashed), $\rho = 1$ (back, continuous) and $\rho = 1.15$ (blue, dotted) respectively. Cosmological evolution starts from the maximum of the potential, which corresponds to the unstable UV de Sitter point, and evolves towards smaller values of the field Φ.

end up with the Einstein frame action

$$\widetilde{S} = \int d^4x \sqrt{-\tilde{g}} \left(\frac{1}{16\pi G_0} \widetilde{R} - \frac{1}{2} (\nabla \Phi)^2 - U(\Phi) \right) + \widetilde{S}_m(\tilde{g}, \psi, \Phi). \tag{7.40}$$

The scalaron potential in the Einstein frame is then given in parametric form as,

$$U(R) = \frac{\tilde{\kappa}^2}{2\,(8\pi G_0)^2} \frac{R f_R(R) - f(R)}{f_R(R)^2}, \tag{7.41}$$

$$\Phi(R) = \sqrt{\frac{3}{16\pi G_0}} \ln f_R(R). \tag{7.42}$$

The mass of the scalar Φ is defined in the usual way through the Einstein frame potential as

$$\widetilde{m}^2_{\text{eff}} = \frac{d^2 U}{d\Phi^2}. \tag{7.43}$$

We will use the above two relations later when we will work out the inflationary slow roll parameters. The Einstein frame potential for different values of ρ is plotted in Figure 7.3. The maximum corresponds to the unstable de Sitter point in the UV, with the cosmological evolution occurring "to the left" of it, i.e to smaller field values.

7.5.2 de Sitter solutions

Let us now look for the simplest cosmological solutions, which are the maximally symmetric constant curvature ones. In $f(R)$ gravity, they correspond to the points where the potential in (7.12) has an extremum, i.e solution of the algebraic equation

$$R f_R(R) - 2f(R) = 0. \tag{7.44}$$

One can check that the same condition is also derived in the Einstein frame by requiring that $dU/d\Phi = 0$.

Using relation (7.18), condition (7.44) implies that

$$2\rho^2 g \beta_\lambda + (1 - 2\rho\lambda)\beta_g = 0. \tag{7.45}$$

For β_λ, β_g non-zero, and a given ρ, equation (7.45) defines a family of solutions, described by a curve in the $g-\lambda$ plane, which is the locus of all de Sitter points. Any intersection of it with the RG trajectory will imply a de Sitter era in the particular cosmological evolution. It is interesting to note that any RG fixed point will always satisfy the de Sitter condition (7.44) or (7.45), since there, $\beta_g = \beta_\lambda = 0$. This is an identity for fixed points, as $f(R) \propto R^2$ there, but we can check that they are de Sitter by inspecting the Einstein frame potential. In particular, the UV RG fixed point is a always a de Sitter point, as the potential (7.41) stays finite as $R \to \infty$.

The location of the de Sitter line depends on the value of the parameter ρ, which shifts the scale of both early and late time de Sitter points. As a starting point, we can get an idea of the de Sitter points structure by setting $\rho = 1$ in equation (7.45) and working out the resulting de Sitter line, which is shown in Figure 7.1. The de Sitter line passes through the UV RG fixed point, yielding this way an infinite number of de Sitter points. This can be seen as follows: as pointed out before, the RG UV fixed point is a de Sitter point itself. On the same time, the behavior of the RG evolution in the vicinity of the RG UV fixed point is described by an unstable spiral, which circles the fixed point infinitely many times as $k \to \infty$ (or $R \to \infty$). As a consequence there will be an infinite number of intersections between the de Sitter line and the RG phase curve. The UV RG fixed point is the limiting de Sitter point of the above infinite set of de Sitter points.

Furthermore, as Figure 7.1 shows, there is an "outer" de Sitter point in the UV regime, and another one in the IR. For the case $\rho = 1$, we find

$$(g_{\rm dS}, \lambda_{\rm dS})_{\rm UV} \simeq (0.02, 0.27), \tag{7.46}$$

while it is easy to show that

$$(g_{\rm dS}, \lambda_{\rm dS})_{\rm IR} \simeq (0, 0.25). \tag{7.47}$$

Notice that the "inner" UV de Sitter points cannot be accessed, since they are protected by the outer one. At least in the Einstein–Hilbert truncation, and under the cut–off identification considered here ($k^2 = \rho R$), this seems to be a general behavior: There is always an infinite set of UV de Sitter points, all hidden by the most outer one, and a de Sitter point in the IR. As a consequence, classical cosmological evolution cannot reach the extreme UV regime around the UV RG fixed point, i.e for $k^2 = \rho R \to \infty$.

We now want to understand how the de Sitter line changes as we vary the dimensionless parameter ρ in our cut–off identification. There are two extreme cases leading to two limiting de Sitter lines, one for $\rho \to \infty$ and another for $\rho \to 0$. Solving equation (7.45) for g and taking the limit $\rho \to \infty$ the limiting curve is described by

$$g_{\rho\to\infty}(\lambda) = \frac{1}{96}\left(12\lambda^2 - 4\lambda - 3 + \sqrt{144\lambda^4 + 672\lambda^3 - 824\lambda^2 + 216\lambda + 9}\right), \tag{7.48}$$

which for a realistic RG evolution gives a de Sitter point at the UV, and another one very close to the GFP, i.e $\lambda \sim g \sim 10^{-35}$. Therefore, by tuning the parameter ρ to very large values, both UV and IR de Sitter points are shifted towards the UV along the RG trajectories. Notice that letting $\lambda \to 0$ in (7.48) we get $g_{\rho\to\infty} \to 0$, i.e the curve passes through the GFP at $(\lambda, g) = (0, 0)$.

On the other hand, as $\rho \to 0$, the de Sitter line becomes

$$g_{\rho\to 0}(\lambda) = \frac{1}{16}(1 - 2\lambda)^2,$$

which again gives a de Sitter point in the UV and a second one for $(g, \lambda) \simeq (0, 0.5)$,

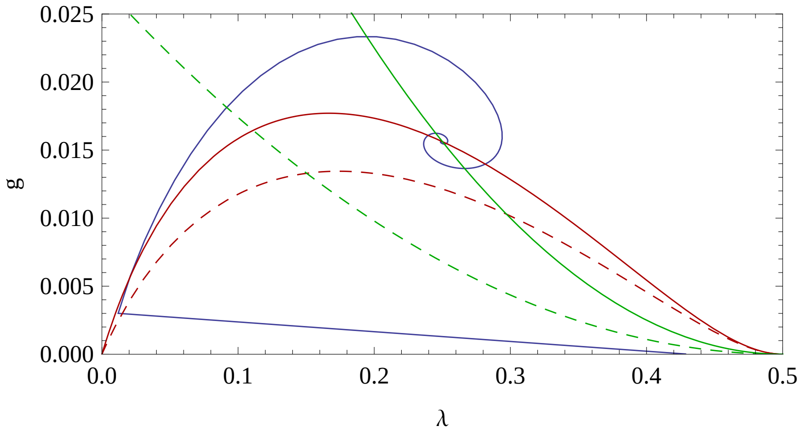

Figure 7.4: The limiting de Sitter (continuous) and slow roll lines (dashed) respectively. Red colour corresponds to $\rho \to \infty$ ("bell" shaped curves), and the green to $\rho \to 0$ respectively. For $\rho \to \infty$ both de Sitter and slow roll line go to zero as $\lambda \to 0$.

as $g \ll 1$ in the IR regime, but now both points are shifted towards the IR. The de Sitter lines corresponding to the extreme cases described above can be seen in Figure 7.4.

To summarise: the general trend is that by making ρ smaller, the position of the UV de Sitter point is shifted towards smaller values of R (i.e moving away from the RG UV fixed point), while the situation is the opposite for increasing ρ.

Let us turn attention to the stability of the de Sitter points. As said before, a de Sitter point is (un)stable if ($m^2_{\text{eff}} < 0$) $m^2_{\text{eff}} > 0$. Therefore, the equation

$$m^2_{\text{eff}}(R, g, \lambda) = 0$$

will in turn define a line on the $g - \lambda$ plane along which the square of the mass becomes zero. Another useful line on the $g-\lambda$ plane is the one along which the square mass diverges, i.e its denominator becoming zero. Both $m^2_{\text{eff}} = 0$, and $m^2_{\text{eff}} \to \infty$ lines will divide the $g - \lambda$ plane into regions of positive and negative mass squared. This can be seen in Figure 7.1 for $\rho = 1$. If the UV de Sitter point should play the role of an inflationary era, it should be an unstable, while the IR one should be stable. We will see later that this can be achieved for a range of values for the parameter ρ.

The general expression for the scalaron mass in terms of the dimensionless couplings is quite complicated, but it simplifies reasonably in the classical and IR regime, where $g \ll 1$. In this case, the mass takes the form

$$m^2_{\mathrm{eff(IR)}} \simeq \frac{R(2\lambda-1)^3}{36g\,(8\lambda^3\rho+4\lambda^2\rho+\lambda(8-6\rho)+\rho-2)}, \tag{7.49}$$

where we neglected terms of order g^2 and higher. The critical points where the denominator of the scalaron mass vanishes will signal a singularity, with the scalaron mass going to infinity.

Let us study the positivity of $m^2_{\mathrm{eff(IR)}}$ by first studying the special case of the GFP regime, where in addition to $g \ll 1$, it is also $\lambda \ll 1$. In this case, relation (7.49) simplifies to

$$m^2_{\mathrm{eff(GFP)}} \simeq \frac{R}{36g\,(2-\rho)}, \tag{7.50}$$

and the sign of it is positive when $\rho < 2$, while for $\rho \to 2$ it blows up. We recall that the renormalisation condition (7.11) fixes $\rho R_0/g_0 \sim m_{\mathrm{P}}^2$, so the scalaron has a Planck-scale mass near the Gaussian fixed point.

In the IR regime, where $\lambda \sim O(1)$, we have to study the full relation (7.49). We distinguish two regimes, one when $\rho < 2$ and another when $\rho > 2$. For $\rho < 2$, the vanishing of the denominator of (7.49) has only one relevant solution $\lambda = \lambda_*(\rho)$ being a function of ρ.

$$0 < \lambda < \lambda_* : m^2_{\mathrm{eff}} > 0, \tag{7.51}$$

$$\lambda_* < \lambda < 0.5 : m^2_{\mathrm{eff}} < 0. \tag{7.52}$$

In the limiting case of $\rho \to 0$, $\lambda_* \to 0$, while as $\rho \to 2^-$, it is $\lambda_* \simeq 0.31$.

On the other hand, for $\rho > 2$, there are two relevant solutions, $\lambda_{*(1)}$ and $\lambda_{*(2)}$. We have the following cases

$$0 < \lambda < \lambda_{*(1)} : m^2_{\mathrm{eff}} < 0, \tag{7.53}$$

$$\lambda_{*(1)} < \lambda < \lambda_{*(2)} : m^2_{\mathrm{eff}} > 0, \tag{7.54}$$

$$\lambda_{*(2)} < \lambda < 0.5 : m^2_{\mathrm{eff}} < 0, \tag{7.55}$$

with both $\lambda_{*(1)}$, $\lambda_{*(2)}$ varying with ρ, i.e $\lambda_{*(1)} \equiv \lambda_{*(1)}(\rho)$, $\lambda_{*(2)} \equiv \lambda_{*(2)}(\rho)$. In particular, we have that as $\rho \to 2^+$ $\lambda_{*(1)} = 0$ and $\lambda_{*(2)} \simeq 0.31$, while for $\rho \to \infty$ $\lambda_{*(1)} \simeq 0.2$ and $\lambda_{*(2)} \simeq 0.5$.

We conclude from the above analysis that the case $\rho > 2$ is rejected, since $m^2_{\rm eff}$ is negative around the GFP. On the other hand, for $\rho < 2$ the mass $m^2_{\rm eff}$ is positive around the GFP ($\lambda \ll 1$) and it stays positive for $\lambda < \lambda_*$ with λ_* approaching $\lambda_* \simeq 0.3$ as $\rho \to 2$.

An important point when $\rho < 2$, concerns the position of the IR de Sitter point. From above, it is understood that both the position of de Sitter points as well as the critical point $\lambda_*(\rho)$, beyond which $m_{\rm eff}$ becomes negative, depend on the parameter ρ. What it turns out to be is that the corresponding position of the IR de Sitter point, will lie ahead of λ_* on the λ-axis for $\rho \lesssim 0.9$, which means that the de Sitter point will be unstable. As a result, all trajectories with $g \ll 1$ and $\rho \lesssim 0.9$, will posses an unstable IR de Sitter point. In other words, the RG trajectory will pass through the mass singularity point λ_*, making $m^2_{\rm eff}$ negative, before the trajectory reaches its actual terminating (de Sitter) point.

From the above stability analysis, we see that the parameter ρ has been constrained to be $0.9 \lesssim \rho < 2$. In the next section, we will further constraint ρ by requiring that the different cosmological periods are connected with each other in a viable way, finding that $\rho \sim 1$.

7.5.3 Dynamical evolution from UV to IR

We saw that in principle we can have de Sitter solutions, and the existence of a classical regime ensures for a standard radiation/matter era respectively. It is important though, that the cosmological eras are connected dynamically in a viable way. This will be the subject of this section. More precisely, we will consider action (7.8) in the presence of a perfect fluid with barotropic index $w \equiv p/\rho$, and study its dynamics by means of a dynamical system analysis, by improving the dynamical system for $f(R)$ gravity, presented in Amendola et al. (2007b), to account for our RG-inspired $f(R)$ model.

We can start by defining the following dimensionless variables,

$$x_1 = \frac{-\dot{f}_R}{Hf_R}, \tag{7.56}$$
$$x_2 = \frac{-f}{6H^2 f_R}, \tag{7.57}$$
$$x_3 = \frac{R}{6H^2}, \tag{7.58}$$
$$x_4 = \frac{\tilde{\kappa}^2 \rho_r}{3H^2 f_R}, \tag{7.59}$$

with an over dot denoting derivative with respect to cosmic time. The Hubble parameter is defined as $H \equiv \dot{a}/a$, with a the Universe scale factor. In the absence of radiation it is $x_4 = 0$.

Then, the background dynamics can be expressed in terms of the dynamical system Amendola et al. (2007b),

$$x_1' = -1 - x_2 - 3x_2 + x_1^2 - x_1 x_3 + x_4, \tag{7.60}$$
$$x_2' = \frac{x_1}{x_3} - x_2(2x_3 - x_1 - 4), \tag{7.61}$$
$$x_3' = \frac{-x_1 x_3}{m} - 2x_3(x_3 - 2), \tag{7.62}$$
$$x_4' = -2x_3 x_4 + x_1 x_4, \tag{7.63}$$

with the constraint

$$\Omega_m \equiv \frac{\tilde{\kappa}^2 \rho_m}{3H^2 f_R} = 1 - x_1 - x_2 - x_3 - x_4, \tag{7.64}$$

and primes here denoting differentiation with respect to $\ln a$.

The quantity $m = m(r)$ is defined as

$$m \equiv \frac{d \ln f_R}{d \ln R} = \frac{R f_{RR}}{f_R}, \tag{7.65}$$
$$r \equiv -\frac{d \ln f}{d \ln R} = -\frac{R f_R}{f} = \frac{x_3}{x_2}. \tag{7.66}$$

$m = m(r)$ characterizes the particular $f(R)$ model, and it needs to be given a priori in order for the dynamical system to close. In principle, given a particular $f(R)$ model, one is able to invert $r = r(R)$ and plug into m to get $m = m(r)$. However, in our case the form of the $f(R)$ model is dictated through the particular running of the couplings $g(R)$, $\lambda(R)$, by solving the system of beta functions. Therefore, in order to close the dynamical system (7.60)-(7.63) we will need to evolve the couplings with time as well.

In addition, the effective equation of state is given by,

$$w_{\text{eff}} = -\frac{1}{3}\left(2x_3 - 1\right). \tag{7.67}$$

For the dimensionless couplings we can write,

$$g' = \frac{\partial g}{\partial R}\frac{dR}{dr}\frac{dr}{dN} = \frac{\beta_g}{2R}\frac{\partial R}{\partial r}\left(\frac{\partial r}{\partial x_2}x_2' + \frac{\partial r}{\partial x_3}x_3'\right), \tag{7.68}$$

$$\lambda' = \frac{\partial \lambda}{dR}\frac{dR}{\partial r}\frac{dr}{dN} = \frac{\beta_\lambda}{2R}\frac{\partial R}{\partial r}\left(\frac{\partial r}{\partial x_2}x_2' + \frac{\partial r}{\partial x_3}x_3'\right). \tag{7.69}$$

After some algebra, we get

$$g' = \frac{\beta_g}{2R}\left(\frac{f^2}{f_R^2 R - f_R f - f_{RR} f R}\right)\left(\frac{x_3' x_2 - x_2' x_3}{x_2^2}\right), \tag{7.70}$$

$$\lambda' = \frac{\beta_\lambda}{2R}\left(\frac{f^2}{f_R^2 R - f_R f - f_{RR} f R}\right)\left(\frac{x_3' x_2 - x_2' x_3}{x_2^2}\right), \tag{7.71}$$

where $x_i' \equiv x_i'(x_i, g, \lambda)$ through the relevant evolution equation. The complete dynamical set of equations is now (7.60)-(7.63) supplemented with (7.70)-(7.71). Notice that any fixed point of (7.60)-(7.63) automatically satisfies (7.70)-(7.71) as well.

One should be reminded here that the derivatives with respect to R, e.g f_R, can be explicitly expressed using (7.8), (7.18) and (7.19). In addition, both r and m are implicit functions of curvature R, through $r \equiv r(\lambda(R), g(R))$ and $g \equiv g(\lambda(R), g(R))$.

The RG improved dynamical system with $x_4 = 0$ has three cosmological fixed points: An early time de Sitter, a matter, and a late time de Sitter point respectively. Of course, we expect that a radiation fixed point will appear by the time we introduce x_4. For a complete analysis and the fixed point structure and their stability one can refer to Ref Amendola et al. (2007b).

The de Sitter point P_1, the matter point P_5 and the radiation point P_7, are given in the general form $P = (x_1, x_2, x_3, x_4)$ as,

$$P_1 = (0, 1, -2, 0), \tag{7.72}$$

$$P_5 = \left(\frac{3m_0}{m_0+1}, -\frac{4m_0+1}{2(m_0+1)^2}, \frac{4m_0+1}{2(m_0+1)}, 0\right), \tag{7.73}$$

$$P_6 = \left(\frac{4m_0}{m_0+1}, -\frac{2m_0}{(m_0+1)^2}, \frac{2m_0}{m_0+1}, \frac{-5m_0^2-2m_0+1}{(m_0+1)^2}\right). \tag{7.74}$$

The de Sitter point P_1 is characterised by $r = -2$, and is stable as long as

$$0 < m|_{r=-2} < 1. \tag{7.75}$$

On the other hand, the points P_5 and P_6 define a family of fixed points parametrized by m, all lying on the line $m = -r - 1$. An acceptable matter era requires that standard GR is recovered, i.e $m \to 0$ ($f_{RR} \simeq 0$), yielding $P_5 = (0, -1/2, 1/2)$, and therefore $r = -1$. For $m \simeq 0$, and in the presence of radiation, a radiation fixed point will also exist in the vicinity of P_5. In particular, the existence of a saddle matter era requires that at the matter point,

$$m|_{r=-2} \simeq +0, \quad \left.\frac{dm(r)}{dr}\right|_{r=-2} > -1. \tag{7.76}$$

The shape of the curve $m = m(r)$ on the $m{-}r$ plane can provide us with sufficient information regarding the asymptotic behavior of the particular $f(R)$ model. In our case, we can work out the $m = m(r)$ curve by integrating the system of beta functions, and then evaluating both $r = r(\lambda, g)$ and $g = g(\lambda, g)$. By choosing a typical RG trajectory for $\rho = 1$ (i.e $k^2 = R$), and initial conditions for the system of beta functions those of (7.35), we get the $m - r$ curve shown in Figure 7.5. We see that cosmological evolution begins from an unstable ($r > 1$) early time de Sitter point, and then evolves towards the (radiation) matter point at $(r, m) \simeq (-1, 0)$. It then leaves the matter point and evolves towards a stable IR de Sitter point at $r = -2$. Notice that the matter point is approached from positive values of m as condition (7.76) requires.

Let us comment on a point regarding the crossing of the $m = -r - 1$ line on the $m - r$ plane, Fig. 7.5. In Amendola et al. (2007b) it is argued that the line

$m = -r - 1$ cannot be crossed, and that cosmological evolution should be restricted between successive roots on this line. In particular, it is straightforward to derive the following equation for the evolution of $r = r(N)$,

$$\frac{dr}{dN} = (m + r + 1)\,\frac{1}{R}\frac{dR}{dN}, \tag{7.77}$$

which implies that evolution of $r = r(N)$ stops whenever $m = -r - 1$, provided that dR/dN does not diverge. However, we would like to show that in our case the derivative dm/dr diverges as the $m(r)$ curve approaches the point $(r, m) = (-1, 0)$. We start by noticing that for r, m expressed in terms of g, λ, the latter point corresponds to $(g, \lambda) = (0, 0)$, and therefore we expect the couplings to be small as the curve approaches that point on the $m - r$ plane. On the $g - \lambda$ plane, this translates into lying close the GFP.

Under the assumption that $g, \lambda \ll 1$, we can neglect higher order terms to find

$$m \simeq 12g + \mathcal{O}(g^2, \lambda^2), \tag{7.78}$$

$$r \simeq -1 - 2\lambda + \mathcal{O}(g^2, \lambda^2). \tag{7.79}$$

What is more, for the derivative dm/dr we find that

$$\frac{dm}{dr} \simeq -\frac{6g}{(4g + 4\lambda - 1)^2\,(6g - \lambda)}, \tag{7.80}$$

For our present analysis the linearised expressions are enough, and we will not present the full expressions of $m(g, \lambda)$, $r(g, \lambda)$, as they are rather complicated. From (7.80) we see that when the couplings are small, the denominator becomes zero when $6g - \lambda = 0$, which when combined with expressions (7.78)-(7.79), gives

$$6g - \lambda = \frac{1}{2}\,(m + r + 1)\,, \tag{7.81}$$

which in turn implies that the denominator will vanish when $m = -r - 1$, i.e when the $m(r)$ curve crosses the latter line. In fact, one can numerically show that the vanishing of the denominator happens along $m = -r - 1$ for the general expressions, i.e not only at the linearised case. Therefore, for $g \neq 0$ the derivative dm/dr will diverge at the point $(r, m) = (-1, 0)$ according to (7.80)

$$\left.\frac{dm}{dr}\right|_{m\to -r-1} \to \infty, \tag{7.82}$$

allowing to bypass condition (7.77). We can also get an analytic expression for the curve $m = m(r)$ around $r \simeq -1$, $m \simeq 0$, by first using (7.78), (7.79) into (7.80) to arrive at the following expression for the derivative

$$\frac{dm}{dr} \simeq -\frac{m}{m+r+1}. \tag{7.83}$$

Notice first that from the linearised expression (7.78) one can see that m is of order g in this regime, i.e much smaller than one, but not exactly zero. Therefore, as the line $m+r+1 = 0$ is crossed the derivative (7.83) diverges. Furthermore, differential equation (7.83) can be solved to give

$$m \simeq -(r+1) \pm \sqrt{(r+1)^2 - (r_{\mathrm{T}}+1)^2}, \tag{7.84}$$

with r_{T} a constant, and $\pm$ denoting the positive and negative branch of the solution, corresponding to that part of $m(r)$ before and after the crossing with $m+r+1 = 0$ respectively. One can check that above solution indeed reproduces the expected behavior, i.e as $r \to -1$, $m \to 0$, and it is worth noting that although the derivative along the line $m = -r-1$ diverges, the curve $m = m(r)$ itself is continuous, as can be also seen in figure 7.5.

Note also that the turn-around on the $m-r$ plane corresponds to a turn-around on the $g-\lambda$ plane. In particular, if we denote $m_{\mathrm{T}} \equiv m(r_{\mathrm{T}})$, with T denoting the value at the turning point, it is straightforward to see that $m_{\mathrm{T}} \simeq 12 g_T$ and $r_{\mathrm{T}} \simeq -1 - 2\lambda_T$, where g_{T} and λ_{T} are defined in (7.22) and (7.23). We also recall that $g_{\mathrm{T}} \sim \lambda_{\mathrm{T}} \sim \sqrt{G_T \Lambda_T}$ from (7.28)–(7.29), allowing one to estimate how close the turning point is to $(m, r) = (0, -1)$. Classical GR-like evolution begins beyond the turning point, on the lower branch. Therefore, on the $g-\lambda$ plane, radiation and matter domination occur around the turning point in the vicinity of the GFP.

For illustrative purposes, Figure 7.6 shows the cosmological evolution from the

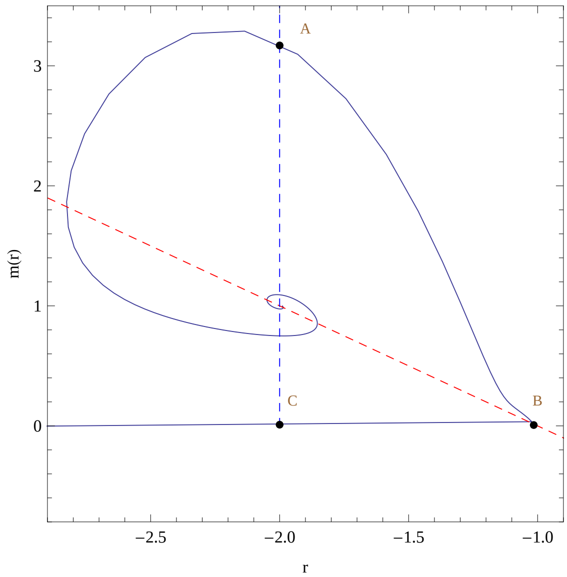

Figure 7.5: The $m-r$ plane for $\rho = 1$ and the set of initial conditions (7.35), with $m(r)$ and r given by relations (7.65) and (7.66) respectively. Point A corresponds to the unstable UV de Sitter point, point B to the saddle matter point, while C to the stable IR de Sitter respectively, as described in section 7.5.3. The dashed lines correspond to $r = -2$ and $m = -r - 1$ respectively.

matter to the IR de Sitter point in the coordinate space, while Figure 7.8 shows the evolution of the effective index and slow roll parameter w_{eff} and ϵ_V respectively, from the UV de Sitter point to the matter one.

As it turns out, under a suitable choice of initial conditions for g, λ and ρ, it is possible to get a cosmology where the UV regime is correctly connected with the IR one. The question that arises is if there are any bounds on the parameter ρ in this direction. In fact, for $\rho \gtrsim 1.1$ the behavior of the evolution on the $m - r$ plane starts becoming unstable, and evolution does not reach the late time de Sitter point, after leaving the matter era. What is more, as ρ increases the matter era happens to be approached from negative values of m, which as explained before is forbidden. Furthermore, as was also explained in the previous section, the positivity of m^2_{eff} in the IR regime (stability of IR de Sitter point) as well as in the GFP regime puts the extra restriction $0.9 \lesssim \rho < 2$.

Therefore, we conclude that the viability of both the classical regime and late

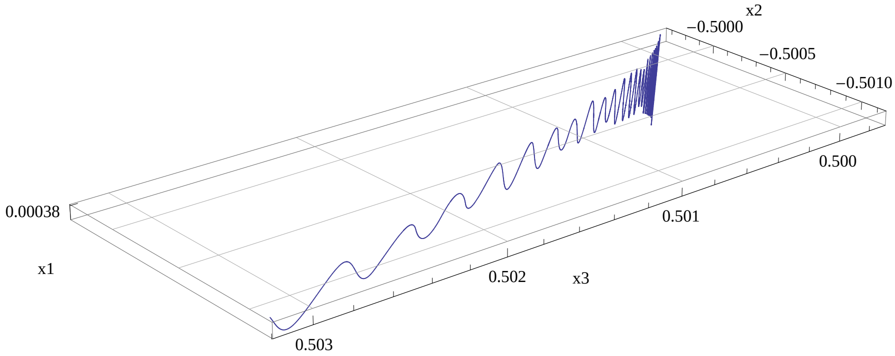

Figure 7.6: The cosmological trajectory described by the dynamical system (7.60) - (7.63), in the space of the coordinates (x_1, x_2, x_3), leaving the matter point and evolving towards the IR de Sitter. In particular, it spirals around the unstable matter point, and then evolves towards the stable de Sitter in the IR. The initial conditions chosen are $(\lambda_0, g_0) = (10^{-2}, 10^{-5})$, and $(x_{10}, x_{20}, x_{30}) = (x_{10(\mathrm{m})} + 10^{-5}, x_{20(\mathrm{m})} - 10^{-6}, x_{30(\mathrm{m})} + 10^{-5})$, with $(x_{10(\mathrm{m})}, x_{20(\mathrm{m})}, x_{30(\mathrm{m})})$ denoting the coordinates of the matter fixed point given in (7.73), and m_0 is evaluated as $m_0(\lambda_0, g_0)$ using (7.65) and (7.18)-(7.19). Above initial conditions give $r_0 \simeq -1.02$, $m_0 \simeq 1.24 \times 10^{-4}$. We also assumed that $x_4 = 0$. The amplitude of the oscillation along the x_1 axis is of the order 10^{-5}.

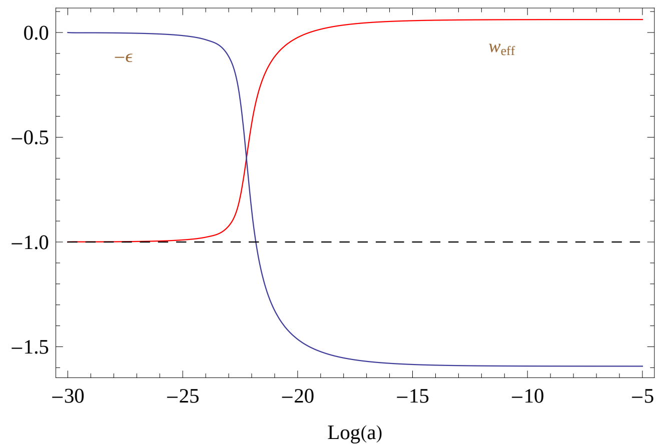

Figure 7.7: The effective index w_{eff} and the slow roll parameter ϵ (relations (7.67) and (7.88) respectively) from the UV de Sitter to matter domination for initial conditions: $(x_{10}, x_{20}, x_{30}, \lambda_0, g_0) = (10^{-2}, -1-10^{-3}, 2-10^{-5}, 0.26, 0.02)$ and $\ln(a_0) = -30$, and ϵ re-expressed as $\epsilon = 2 - x_3$.

time cosmology restricts ρ to lie in the range

$$0.9 \lesssim \rho \lesssim 1.1. \tag{7.85}$$

7.5.4 Inflationary dynamics

We showed that our particular $f(R)$ model exhibits an unstable UV de Sitter point, which can be dynamically connected with the radiation/matter era in a viable way. We would like to understand if the UV de Sitter point, describing a primordial inflationary era, could be observationally viable i.e if the scalar and gravitational fluctuations amplitudes as well as the number of e-foldings are those that are required according to observations. Recall that the only free parameter in our model is the dimensionless parameter ρ.

Below, we will evaluate all inflationary quantities in the Einstein frame, ignoring the non-minimal coupling between matter and the scalar field, since inflation is a (almost) vacuum dominated period.

Let us first revise some standard notions of scalar field inflation. To start with, the slow roll parameters ensure that the scalar field (inflaton) has a small kinetic energy during inflation, compared to the potential energy, so that the latter dominates. The two slow roll parameters are defined as

$$\epsilon \equiv \frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN}, \tag{7.86}$$

$$\eta \equiv \frac{\ddot{\Phi}}{H\dot{\Phi}} = \epsilon - \frac{1}{2\epsilon}\frac{d\epsilon}{dN}, \tag{7.87}$$

with the overdot denoting differentiation with respect to cosmic time. For a scalar field action with a kinetic term and a potential, they can be alternatively (and equivalently to first order in ϵ, η) defined as

$$\epsilon_V = \frac{m_p^2}{16\pi}\left(\frac{U_\Phi}{U}\right)^2, \tag{7.88}$$

$$\eta_V = \frac{m_p^2}{8\pi}\frac{U_{\Phi\Phi}}{U}, \tag{7.89}$$

with the subscript Φ denoting differentiation with respect to the Einstein frame scalar field Φ respectively.

Inflation occurs as long as the slow roll condition is satisfied, i.e

$$\epsilon_V \ll 1, \eta_V \ll 1, \tag{7.90}$$

and ends when $\epsilon_V, \eta_V \sim O(1)$. Smallness of ϵ_V ensures that the spacetime during inflation remains sufficiently close to de Sitter, while smallness of η_V ensures that variation of ϵ_V per e-fold is sufficiently small.

The number of e-folds is given by

$$N \equiv \ln \frac{a_{\mathrm{f}}}{a_{\mathrm{i}}} \approx \int_{\Phi_{\mathrm{i}}}^{\Phi_{\mathrm{f}}} \frac{U}{U_\Phi} d\Phi, \tag{7.91}$$

with a_{i}, a_{f} the scale factor at the start and end of inflation respectively, and the slow roll approximation used in the last approximation. Above integral can be of course evaluated in terms of the couplings and curvature R through,

$$d\Phi = \left(\frac{\partial \Phi}{\partial g} \frac{dg}{dR} + \frac{\partial \Phi}{\partial \lambda} \frac{d\lambda}{dR} + \frac{\partial \Phi}{\partial R} \right) dR, \tag{7.92}$$

and the integral (7.91) can be calculated between two points R_{i} and R_{f} along the RG trajectory. Notice that in the vicinity of a de Sitter point the number of e-folds diverges since there $U_\Phi \to 0$.

Fluctuations of the scalar field during inflation, generate scalar and gravitational perturbations, whose power spectra in the slow roll approximation are given by (see e.g. Lyth and Liddle (2009))

$$\mathcal{P}_s = \frac{128\pi}{3} \left. \frac{U^3}{m_p^6 U_\Phi^2} \right|_{k=aH}, \tag{7.93}$$

$$\mathcal{P}_g = \frac{128}{3} \left. \frac{U}{m_p^4} \right|_{k=aH}, \tag{7.94}$$

assuming evaluation at the horizon crossing of the relevant mode. The scalar power spectrum becomes infinite when evaluated on a de Sitter point, reflecting the standard infra-red divergence. This behavior can be seen in Figure 7.8.

Notice that expressing the derivative of the potential as

$$\frac{\partial U}{\partial \Phi} = \frac{\partial U}{\partial R} \frac{\partial R}{\partial \Phi} = \sqrt{\frac{16\pi G_0}{3}} \frac{f_R}{f_{RR}} \frac{\partial U}{\partial R}, \tag{7.95}$$

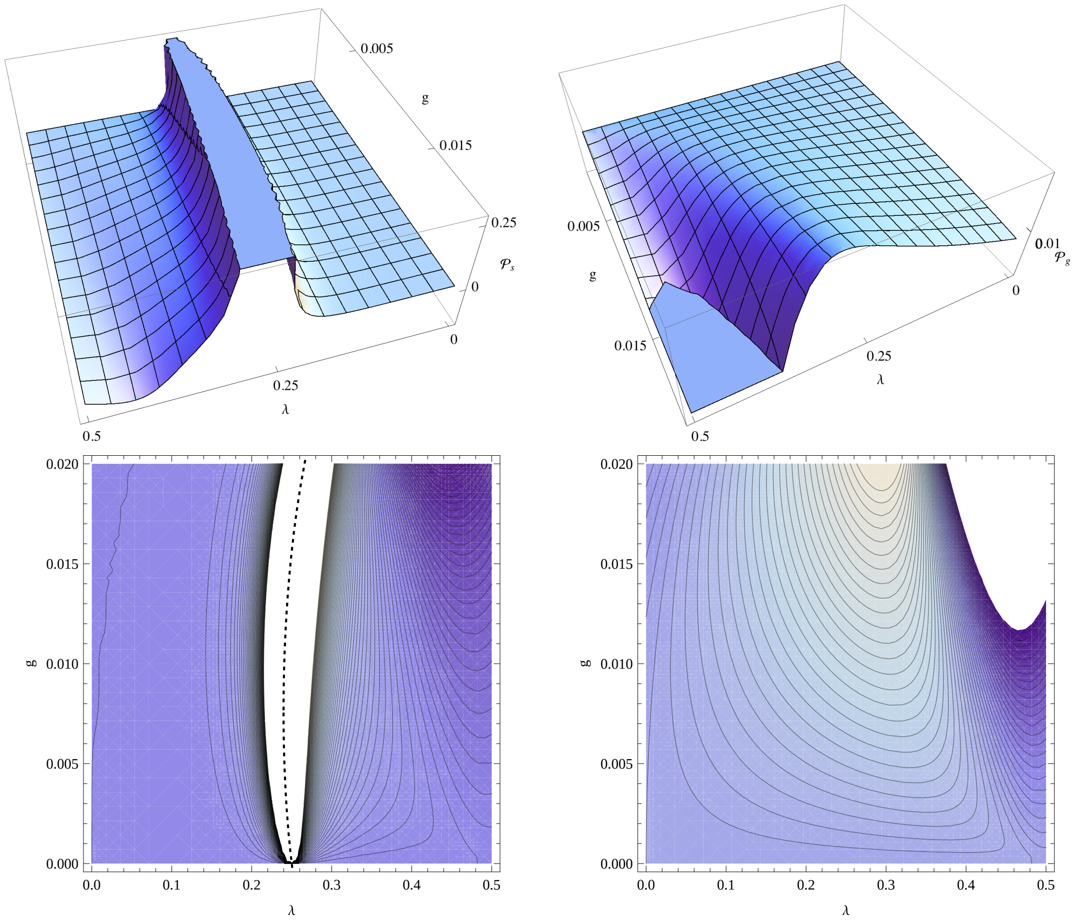

Figure 7.8: Upper row: The scalar (left) and gravitational (right) fluctuation power spectrum, as given by relations (7.98) and (7.99) respectively, as a function of the couplings λ, g, and setting $\rho = 1$. The scalar power spectrum peaks along the de Sitter line, as on a de Sitter point it is $\mathcal{P}_s \to \infty$. Lower row: The corresponding contour plots of the scalar (left) and gravitational (right) spectrum of upper row. In the scalar power spectrum the dotted line corresponds to the de Sitter line, along which the power spectrum diverges. Higher values correspond to lighter shaded areas.

and using a similar expression for the second derivative, the slow roll parameters in the Einstein frame can be calculated to be

$$\epsilon_V(R) = \frac{1}{3}\left(\frac{2f - Rf_R}{f - Rf_R}\right)^2, \tag{7.96}$$

$$\eta_V(R) = \frac{2}{3}\frac{f_R^2 + f_{RR}f_R R - 4f_{RR}f}{f_{RR}(Rf_R - f)}. \tag{7.97}$$

These relations can also be viewed as a function of the scalar $\Phi = \Phi(R)$, through relation (7.42), as well as functions of g, λ, ρ through relations (7.18) and (7.19).

The equation $\epsilon_V(g, \lambda) = 1$ defines a curve in the $g - \lambda$ plane ("slow roll line"), whose intersection with the RG phase curve corresponds to the end of inflation, and is associated with the corresponding de Sitter line for a given ρ. The slow roll line for $\rho = 1$ can be seen in Figure 7.1.

In general, decreasing ρ, the slow roll lines shift away from the UV RG fixed point along the RG evolution, and vice versa as $\rho \to \infty$. The opposite is true for the de Sitter lines, which means that an increasing ρ increases (decreases) the scale where inflation starts (ends), and the opposite is true for decreasing ρ. It is interesting to note that for $\rho \to \infty$, the low energy de Sitter point lies before the point where $\epsilon_V = 1$. The limiting slow roll lines for $\rho \to 0, \infty$ are shown in Figure 7.4.

Let us now move to the power spectra, given in (7.93) and (7.94). In order to match the scalar fluctuation amplitude according to the CMB observations Komatsu et al. (2011), we need $\mathcal{P}_s \simeq 2 \times 10^{-9}$, and $\mathcal{P}_g \lesssim 0.2\mathcal{P}_s$. The precise value of the amplitudes depends on a set of values for (g, λ, ρ) evaluated at the particular scale of interest. It will be useful first to give the explicit expressions of the spectra in terms of g, λ and ρ, for the beta functions (7.4, 7.5). We find

$$\mathcal{P}_s = \frac{128}{3\rho}\frac{A(g,\lambda,\rho)B(g,\lambda,\rho)^3}{C(g,\lambda,\rho)^2 D(g,\lambda,\rho)^2}, \tag{7.98}$$

$$\mathcal{P}_g = \frac{128}{\rho}\frac{A(g,\lambda,\rho)B(g,\lambda,\rho)}{C(g,\lambda,\rho)^2}, \tag{7.99}$$

with the additional definitions

$$
\begin{aligned}
A(g,\lambda,\rho) &\equiv g\left(4g-(1-2\lambda)^2\right),\\
B(g,\lambda,\rho) &\equiv 96g^2\rho + g\left(\left(-24\lambda^2+4\lambda+6\right)\rho-6\right)\\
&\quad -(1-2\lambda)^2\lambda\rho,\\
C(g,\lambda,\rho) &\equiv -192g^2\rho+4g\left(3\left(4\lambda^2-1\right)\rho+2\right)\\
&\quad +(1-2\lambda)^2\\
D(g,\lambda,\rho) &\equiv -192g^2\rho+4g\left(\left(12\lambda^2-4\lambda-3\right)\rho+4\right)\\
&\quad +(1-2\lambda)^2(4\lambda\rho-1).
\end{aligned} \tag{7.100}
$$

We arrived at relations (7.98)-(7.99), using relations (7.95) and (7.18)-(7.19) to re-express the spectra appropriately. Analgous (but more complicated) expressions can be derived for beta functions with other gauges and cut-off functions.

We have seen in the previous sections that stability requirements of the classical regime (GFP regime) as well as of the late time cosmology require that $0.9 \lesssim \rho \lesssim 1.1$. Therefore, the first thing to investigate is inflation can be observationally viable for ρ in this range.

So, let us proceed by studying the case of $\rho = 1$. In this case, we also know the values of the couplings at which inflation starts and ends , $P_{\text{start}} \equiv (g_{\text{start}}, \lambda_{\text{start}}) \simeq (0.02, 0.27)$ and $P_{\text{end}} \equiv (g_{\text{end}}, \lambda_{\text{end}}) \simeq (0.02, 0.22)$, with P_{start} corresponding to the UV de Sitter point, and P_{end} to the point where $\epsilon_V = 1$ (see also Figure 7.1). For the connection with observations one is in principle interested at the value of the power spectra about 60 e-foldings before the end of inflation. Now, for $\rho = 1$, and as can also be seen in Figure 7.5.4, between P_{start} and P_{end} both power spectra are smooth, decreasing functions of g and λ, acquiring their lowest value at P_{end},

$$\mathcal{P}_s \simeq 0.067, \quad \mathcal{P}_g \simeq 0.052. \tag{7.101}$$

One sees that the (lowest) values of the power spectra (7.101), are too large to agree with observations, yielding a non-viable inflationary period for $\rho = 1$. It is not difficult to check that this behavior is true for all values of ρ between $0.9 \lesssim \rho \lesssim 1.1$. Therefore, a viable late time cosmology cannot be combined with a viable primordial inflation.

Having seen that an observationally viable inflationary era is not in agreement with a viable late time cosmology, which requires $\rho \sim 1$, we ask the following question: could inflation be viable on its own for some parameter ρ, away from $\rho \sim 1$? Let us try to understand this by checking the behavior of the power spectra (7.98) and (7.99) for the extreme cases of $\rho \to 0$ and $\rho \to \infty$ respectively. Assuming a (finite) value of g and λ we find that

$$\lim_{\rho\to 0} \mathcal{P}_s, \mathcal{P}_g = \infty, \tag{7.102}$$

which is obviously unacceptable.

On the other extreme, i.e when $\rho \to \infty$, the power spectra go to zero,

$$\lim_{\rho\to \infty} \mathcal{P}_s, \mathcal{P}_g = 0, \tag{7.103}$$

which is potentially viable. For the scalar to tensor ratio we find that

$$\left.\frac{\mathcal{P}_g}{\mathcal{P}_s}\right|_{\rho\to\infty} = \frac{48\left(48g^2 + g\left(-12\lambda^2 + 4\lambda + 3\right) - (1-2\lambda)^2\lambda\right)^2}{\left(96g^2 + g\left(-24\lambda^2 + 4\lambda + 6\right) - (1-2\lambda)^2\lambda\right)^2}. \tag{7.104}$$

Remembering that when $\rho \gg 1$, the end of inflation, which is described on the phase space of $g - \lambda$ by the slow roll line, is shifted towards smaller values of the couplings, as can also be seen in Figure 7.4. Therefore, we can get an estimate of above ratio by assuming that the fluctuations are produced at a point in the linear regime of the RG evolution, where $g \sim \lambda \ll 1$, yielding $\mathcal{P}_g/\mathcal{P}_s \sim O(1)$, which is observationally unacceptable.

Before concluding this section, let us comment on another possibility of understanding inflation in this scenario, that is modeling it as R^2 inflation Starobinsky (1980b) using the $f(R)$ model found in (7.31) at large R:

$$f(R) \simeq \frac{\tilde{\kappa}^2}{G_0}\left(R - 2\Lambda_0\right) + 6(2-\rho)\rho R^2. \tag{7.105}$$

Matching to the perturbation amplitude, R^2 inflation can account for the observations if the coefficient of the R^2 term is of order 10^{11} Starobinsky (1980a). Hence we see in approximate way how tuning ρ to very large values suppresses the perturbations. However, this results in an unacceptable classical limit as well as a non-viable late time cosmology for the reasons explained in previous sections.

To conclude this section, it turns out that primordial inflation in this scenario cannot agree with observations unless ρ is very large, in which case the mass of the scalaron diverges and becomes tachyonic in the subsequent evolution. Hence the observed fluctuations must be generated at a later period of inflation, which requires that more degrees of freedom should be introduced in the action, like for example a scalar field. Scalar field inflation in the Asymptotic Safety scenario, and with scale identification in the equations of motion, has been considered in Contillo et al. (2012b). A more exotic possibility is that the extra degrees of freedom produce a fixed point with a very small fluctuations. One notes that for small g, the tensor power spectrum becomes

$$\mathcal{P}_g \simeq 128 g\lambda \tag{7.106}$$

which is suggestive that a fixed point with small $g\lambda$ could be viable. Note the appearance of the product $g\lambda \sim G\Lambda$, which is the expected scale of tensor fluctuations in Einstein-Hilbert gravity in a de Sitter phase with cosmological constant Λ.

Closing this section we would like to make a comment on reheating after inflation. Any observationally viable inflationary theory should predict a period of reheating after the end of inflation, where the scalar field driving inflation ("inflaton") decays into relativistic matter, and of course the same should apply for asymptotically safe inflation. However, in our analysis we did not consider reheating as inflation turn out to be non–viable due to the large scalar and tensor fluctuations. Therefore, the matter content we introduced to study matter domination earlier in this chapter, was introduced rather by hand with the aim of understanding the occurrence of a viable matter domination in this context. A viable cosmological model describing the Universe evolution from its early to late stages, should provide us with a reheating mechanism generating the matter content in the Universe after the vacuum dominated period of inflation, while on the same time predicting the correct order of primordial fluctuations as well as yielding a viable late time cosmology.

7.6 Discussion and conclusions

We studied the cosmology of an $f(R)$ model generated by the RG improvement of the Einstein–Hilbert action. The transition to $f(R)$ gravity was achieved by identifying the renormalisation group scale to be proportional to scalar curvature,

$$k^2 = \rho R, \tag{7.107}$$

in the non-perturbative beta-functions calculated from the exact renormalisation group equation.

We found that the resulting $f(R)$ model has some remarkable properties. Firstly, it maintains the correct sign for the graviton and scalaron kinetic terms. Very close to a non-trivial RG fixed point it behaves like R^2 gravity, which is scale invariant, while it reduces to GR in the vicinity of the Gaussian fixed point. At solar and galactic scales, the scalaron's mass is of the order of Planck mass, preventing observable departures from GR at these scales. On the other hand, in the vicinity of the UV RG fixed point, the scalaron mass vanishes, reflecting the scale invariance of the action in that regime.

The cosmological solutions of the $f(R)$ model are also interesting. It naturally exhibits an unstable UV de Sitter point which evolves to a stable one in the IR. The effective cosmological constants are exponentially separated when Newton's G and the cosmological constant are matched to their observed values. What is more, there are an infinite set of de Sitter points as the UV RG fixed point is approached ($R \to \infty$). However, classical cosmological evolution starts from the outermost de Sitter point, and therefore the UV RG fixed point is hidden behind it, and cannot be accessed. The Big Bang singularity is avoided, since the de Sitter point is reached at infinite time in the past, i.e as $t \to -\infty$. The model therefore satisfies the requirements of a successful $f(R)$ model itemized in Nojiri and Odintsov (2011).

Introducing matter content to the cosmology, we found that the UV de Sitter point can be connected to the IR de Sitter era through a radiation/matter era, with a stable scalaron, provided

$$0.9 \lesssim \rho \lesssim 1.1. \tag{7.108}$$

Unfortunately, the fluctuations generated during inflation at the outer UV de Sitter point are too large to account for the observations (Section 7.5.4).

Therefore observable inflation requires extra degrees of freedom in the action, for example a scalar field driving inflation at a lower scale. A more remote possibility would be that the extra degrees of freedom move the fixed point to a smaller value of $g\lambda$, which could suppress the fluctuations.

To make the comparison with previous cut-off identifications in the literature, performed at the level of the equations of motion, our constraint for the parameter ρ, i.e $\rho \sim 1$, is broadly consistent with scale identifications made in the equations of motion, rather than the action as here. In particular, in Reuter and Saueressig (2005), it was numerically found that for the identification $k^2 \sim cH^2$, the constant c should be of order one, which is consistent with $\rho \sim 1$ in our identification.

This model can be improved by extending the analysis performed in this paper to higher truncations, i.e by including higher order curvature terms in the action. It is interesting to ask what features are generic. The existence of a UV fixed point seems to be a universal feature of all truncations found so far, so we expect the Einstein frame potential of the scalaron to tend to a constant at large values of the field. However, we do not expect the presence of an infinite number of de Sitter points to be generic, as it arose from the complex eigenvalues of the fixed point, which are not present for the general four-derivative truncation Benedetti et al. (2009). We should also include matter fields in the renormalisation group equations. With these modifications it might turn out that there is a model for which both early and late time cosmology agrees with observations.

Chapter 8

Conclusions and outlook

As this thesis comes to its end, let us try to think about what the current status regarding dark energy is, and if one should be optimistic about any future surprises. Einstein's "biggest blunder", that is the introduction of the cosmological constant in his field equations, is currently the simplest description to the dark energy problem, on the same time supported by most of the current observations. However, its success at the phenomenological level is not shared by theory, where as we discussed before, conceptual problems like the magnitude problem, are a yet unresolved puzzle.

We should stress that until very recently, Einstein's theory of gravity (GR), had been only tested at solar system scales. It is only the last years that cosmologists have access to large scale observations of the Universe. It is not obvious at all that GR should be the theory of gravity at large scales, and it could be possible that this is what the observations tell us. However, if gravity behaves differently at the very large scales, that should have a characteristic imprint on observations at the linear level, which includes the large scale structure of the Universe, as well as weak lensing experiments. Probably one of the most distinct signatures for some modification of gravity, beyond the ΛCDM paradigm would be the existence of a non–zero anisotropic stress, which implies that $|\Phi/\Psi| \neq 1$, with Φ and Ψ the scalar Newtonian potentials. The fact that no anisotropic stress has been yet detected in no way does provide any conclusive proof about these models, as the current observational bounds are weak, with deviations of order unity for the ratio $|\Phi/\Psi| \neq 1$ still allowed. Future observations of higher precision will be needed to put tighter constraints, with an accuracy of a few percent. As we showed in chapter 5, the existence of a non–zero anisotropic stress is an essential feature of non–linear gravity models and stability requirements do not allow it to be arbitrarily small, as its suppression endangers the stability of both background and linear evolution of the non–linear gravity model under study. Therefore, as it turns out, anisotropic stress is a key observable in testing gravity modifications at large scales, as apart from ΛCDM, minimally coupled scalar field models (e.g quintessence, k–essence) predict a zero anisotropic stress contribution.

The recent discovery of the Higgs boson might be the start for possible future discoveries at the Large Hadron Collider (LHC) at CERN, that could open new paths for cosmology [1]. Searching for new particles at the laboratory is equally important as studying the Universe at its largest scales, as it could be the case that the discovery of a new particle with the properties of dark energy, could provide the answer to the problem. What is more, the (non) discovery of the dark matter particles will probably give strong hints about the nature of the dark energy. Furthermore, for the case where the effects of dark matter are explained through a modification of the gravitational law at galactic scales, the dark matter effects could be just a manifestation of a more general gravitational law unifying both dark energy and dark matter under the same framework.

It could be that the answer to the dark energy problem might be lying in a combination of the current theories we have. In particular, Quantum Field Theory predicts that the constants in Nature evolve with scale (or energy), and we know that according to the Standard Hot Big Bang scenario the Universe's size (and temperature) has evolved from a tiny size of the order of Planck length up to the current horizon scale today. Therefore, the fundamental constants of Nature are in principle expected to vary from the very early (hot) up to the recent (cooled) state of the Universe. In this context, it just happens that at the typical Earth and solar system scales, they do acquire an effectively constant value, equal to the one we observe at these scales. [2] A vacuum energy that runs with scale could provide a way out to the magnitude problem associated with the cosmological constant, on the

[1] As far the Higgs boson is concerned, its large mass (125 – 126 GeV) implies that it could not be responsible for the late time acceleration. (Higgs mass result according to http://press.web.cern.ch/press/PressReleases/Releases2012/PR17.12E.html)

[2] A varying Newton's constant G at solar scales could also have a huge impact on the stability of the solar system.

same time retaining the successful predictions of the ΛCDM model. What is more, it could successfully provide a unification of the primordial with the observed late time acceleration of the Universe, under a common framework, which is well motivated from the well known and studied methods of Quantum Field Theory. This was the subject of chapter 7, where the running of Newton's and cosmological constant was suggested by the Renormalisation Group (RG) improvement of the action in the context of the Asymptotic Safety scenario. Our analysis there revealed that the (non–perturbative) running of the coupling constants under the RG describe a successful background cosmology, from matter (and radiation) domination up to recent times. An equally important result was that primordial acceleration cannot be successful in the simplest implementation of such a scenario, due to the large primordial fluctuations produced. The latter outcome is associated with a fundamental property of the theory space of the Einstein–Hilbert truncation, in particular with the position of the UV fixed under the RG. However, that is not the end of the story. A successful inflation in this context would call for the investigation of other truncations, beyond the Einstein–Hilbert one, as well as the introduction of extra degrees of freedom in the action. What is more, it is important to study and understand the evolution of (linear) perturbations in this scenario, and possible distinct signatures either at cosmological or astrophysical scales.

The hunt for the nature of primordial or late time acceleration, requires studying different theories, and testing them against observations. The essence of such a task lies not only in understanding the inflation or dark energy mystery, but also understanding the nature of gravity itself. However, generalisations of GR, either purely gravitational, or through the introduction of new (scalar) fields, possess in principle a high amount of complexity, or dynamics with a whole new range of solutions compared to the so far studied GR ones. Therefore, it is of great importance to develop and investigate tools that allow us either to simplify the calculations involved, or gain a deeper intuition about the structure of the different theories, by comparing them with well known ones. A tool of this kind is the application of Legendre transformations of higher order actions, as presented and studied in chapter 4 for the case of $f(R)$ and $f(G)$, with G the Gauss–Bonnet term, both suggested in the literature as candidates for the description of the late time acceleration of the Universe. The equivalence of vacuum $f(R)$ gravity with Einstein–Hilbert with a minimally coupled canonical scalar field (Einstein frame), through a conformal transformation, has been well known in the literature since while ago. In chapter 4 we explored a similar procedure in the context of models which are an arbitrary function of the Gauss–Bonnet term, the so–called $f(G)$ ones. As we showed there, this class of fourth order theories can be re-expressed as second order ones through the introduction of a new scalar and tensor field variable, leading to a kind of bi–metric scalar tensor theory. The key tool for the transition to the new representation was a Legendre transformation, which when applied in the $f(R)$ context yields the well known transition to the Einstein frame. What is more, as we found in the same chapter, dynamical equivalence can be broken on the boundary spacetime. Therefore, care must be taken when studying the dynamical equivalence between different frames for spacetimes with boundary, as the equivalence does not always hold for the Gibbons–Hawking terms of the two representations.

Any physical theory should have as its upper goal the confrontation with experiment. The development of appropriate (unified) frameworks that allow for concrete observational predictions about different gravitational theories is therefore of great importance. What is more, investigating equivalent, but on the same time more intuitive descriptions of gravitational theories, can reveal properties that went unseen before, or make the prediction of observables easier. Such frameworks are especially useful at the linear level as there one is able to break the background degeneracies among different gravity models. In this context, a very useful tool is the covariant fluid description of linear perturbations for (scalar–tensor) gravity models with second order derivatives in their energy–momentum tensor, the most well known example being probably $f(R)$ or Brans–Dicke gravity. This is to be presented in an upcoming work in Sawicki et al. (2012).

Cosmology is certainly entering a new era. Over the course of the last years, cosmological observations have seen a rapid development, allowing for testing (gravitational) theories of both early and late time Universe, with future missions promising an even higher accuracy of observations.

It is not the first time Einstein's theory of gravity is in doubt. Until the year 1959 the small amount of observational evidence for GR (probably the most celebrated one being the deflection of light by the sun), led a number of physicists to suggest various alternative theories for gravity. However, during the years 1959-1960 new tests together with older ones performed with higher accuracy, enlarged significantly the observational evidence supporting the validity of GR [3], establishing it as the accepted (classical) relativistic theory of gravity. Today, Einstein's theory is again under question. The answer is probably to be revealed by the research to follow in the years ahead.

[3]For a nice historical review see Will (1981).

Appendix A

Dynamical equivalence of non–linear gravity

A.1 Basic geometrical tools and definitions

In this section we will present some useful geometrical tools and definitions from chapter 4.

As an aid in deriving the GH terms of sections 4.3 and 4.5, we combine two special coordinate systems, the so–called Gauss and Riemann normal coordinates. The first one is related with the spacetime splitting in $(n-1)+1$ form, while the second with the coordinate choice around a point P on the $(n-1)$–dimensional hypersurface Σ.

To begin with, let $(M, g_{\alpha\beta})$ be a globally hyperbolic spacetime, foliated by successive $(n-1)$–dimensional spacelike Cauchy hypersurfaces Σ_n, parametrised by the coordinate n. Let also n^μ (or $\mathbf{n} = \partial/\partial n$) be the unit normal vector to the hypersurfaces Σ_n. One can now use the general ADM spacetime splitting to write the metric $g_{\alpha\beta}$ in terms of the lapse function and vector Misner et al. (1973). However, in order to make our calculations simpler, we consider a special spacetime splitting using Gaussian normal coordinates (GNC) in the neighborhood of the $(n-1)$–dimensional surface Σ_n. In this splitting the metric becomes

$$ds^2 = \epsilon^{-1} dn^2 + g_{ij} dx^i dx^j, \qquad \text{(A.1)}$$

with $\epsilon \equiv n^{\kappa}n_{\kappa} = \mathbf{n.n} = \pm 1$ for timelike and spacelike surface respectively. The important property of this special spacetime splitting is that a geodesic which is normal to a spacelike hypersurface, at some value of the parameter n, will intersect normally to the next hypersurface, at $n+dn$ Misner et al. (1973). The n coordinate, in fact, measures lapse of proper time (or length) along the geodesic.

If we now pick a point P_0 on a hypersurface Σ, we can always locally construct an inertial frame, where free particles will move along straight lines (at least locally). Such an inertial frame is described by the Riemann normal coordinates (RNC) system. An important property of this coordinate system is that at the coordinate centre P_0 it is

$$\Gamma^{\alpha}_{\beta\gamma}(P_0) = 0, \tag{A.2}$$

or in other words, the space is locally flat. Using this property in the derivations of the GH terms defined on the hypersurface Σ, and without losing generality, since we are dealing with tensors, we set $\Gamma^{\alpha}{}_{\beta\gamma}(P_0) = 0$.

The bulk metric $g_{\alpha\beta}$ induces an $(n-1)$–dimensional metric $h_{\alpha\beta}$ on the boundary surface Σ as

$$h_{\alpha\beta} = g_{\alpha\beta} \pm n_{\alpha}n_{\beta}, \tag{A.3}$$

for a spacelike $(+)$ and timelike $(-)$ surface Σ respectively. Its determinant, h, is defined as the determinant of h_{ij}, with $i, j = 1 \dots (n-1)$. Furthermore, we associate a covariant derivative with $h_{\alpha\beta}$ denoted as D_{α}.

We can then define a projection operator from the tangent space to the bulk M to the tangent space to the boundary Σ at a point P_0, through the projection operator

$$h^{\alpha}{}_{\beta} = g^{\alpha}{}_{\beta} \pm n^{\alpha}n_{\beta}. \tag{A.4}$$

We will use the same symbol for both the induced metric and the projection operator. The following relations hold

$$h_{\alpha\beta}h_{\delta}{}^{\alpha} = h_{\beta\delta} \quad , \quad g^{\alpha\gamma}h_{\beta\gamma} = h^{\alpha}_{\beta}, \quad h \equiv g^{\alpha\beta}h_{\alpha\beta} = n-1, \tag{A.5}$$

as well as

$$h^{\alpha\beta}n_{\beta} = 0. \tag{A.6}$$

Notice that the projection operator $h_{\alpha\beta}$ sometimes is also denoted as $\perp_{\alpha\beta}$.

The extrinsic curvature $K_{\alpha\beta}$ is an $(n-1)$–dimensional tensor that measures the "bending" of Σ in the bulk spacetime M, and is defined as

$$\begin{aligned} K_{\alpha\beta} &= \frac{1}{2}\pounds_n h_{\alpha\beta} = \nabla_\alpha \xi_\beta = h^\gamma{}_\alpha \nabla_\gamma \xi_\beta \\ &= h_\alpha{}^\gamma \nabla_\gamma n_\beta = h_\alpha{}^\gamma \left(\partial_\gamma n_\beta - \Gamma^\rho_{\gamma\beta} n_\rho\right), \end{aligned} \tag{A.7}$$

where "$\pounds$" is the Lie derivative, ξ^β a unit tangent to the geodesic congruences orthogonal to Σ, n^β any other normal to Σ, and ∇_a defined with respect to the bulk metric $g_{\alpha\beta}$. If we express $K_{\alpha\beta}$ in the special coordinate system of GNC, giving up for a moment the abstract index notation, we get

$$\begin{aligned} K_{ij} &= -\Gamma^0_{ij}\, n_0 \\ &= -\frac{1}{2}\epsilon \frac{\partial}{\partial n} g_{ij} = -\frac{1}{2}\epsilon \pounds_n h_{ij}, \end{aligned} \tag{A.8}$$

with $\epsilon = \pm 1$ for a timelike and spacelike surface Σ respectively. Relation (A.8) shows that $K_{\alpha\beta}$ measures the rate of change of the induced metric h_{ij} along the geodesic congruence orthogonal to Σ.

The bulk curvature tensors are related to the extrinsic curvature of the surface Σ and its derivatives through the Gauss-Codazzi equations Wald (1984); Misner et al. (1973),

$$R^\alpha{}_{\beta\gamma\delta} = \widehat{R}^\alpha{}_{\beta\gamma\delta} + \epsilon\left(K_{\beta\gamma} K_\delta{}^\alpha - K_{\beta\delta} K^\alpha{}_\gamma\right), \tag{A.9}$$

$$R^n{}_{\beta\gamma\delta} = \epsilon R_{n\beta\gamma\delta} = \epsilon\left(D_\delta K_{\beta\gamma} - D_\gamma K_{\beta\delta}\right), \tag{A.10}$$

with the index n in the second equation being fixed and denoting direction along the normal n^α.

A.2 Useful formulas

A.2.1 Variation formulas

The formulas we present in this subsection are used to calculate the GH terms presented in Sections 4.3, 4.5 and 4.6. They are evaluated using the special coordinate systems of GNC and RNC respectively.

For the variation of the Christoffel symbol and Riemann tensor respectively we have

$$\delta\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\delta g_{\nu[\rho,\mu]} + \delta g_{\rho\mu,\nu}\right), \tag{A.11}$$

$$\delta R^{\alpha}{}_{\beta\gamma\delta} = g^{\alpha\kappa}\left(\delta g_{\kappa[\delta;\gamma]\beta} - \delta g_{\beta[\gamma;\delta]\kappa}\right), \tag{A.12}$$

with $[A, B] \equiv \frac{1}{2}(AB - BA)$. The variation of the Ricci tensor and scalar can be found beginning from (A.12) and calculating the variation of the appropriate contractions, for example, $\delta R_{\beta\delta} \equiv \delta(g^{\gamma}{}_{\alpha}R^{\alpha}{}_{\beta\gamma\delta})$.

Variation of the extrinsic curvature, A.8, with respect to $g_{\alpha\beta}$ gives

$$\delta K_{\alpha\beta} = -h_{\alpha}{}^{\gamma}\delta\Gamma^{\delta}_{\beta\gamma}n_{\delta} \tag{A.13}$$

$$= -\frac{1}{2}\,n_{\delta}\,h_{\alpha}{}^{\gamma}\;g^{\delta\rho}\nabla_{[\rho}\delta g_{\beta]\gamma}, \tag{A.14}$$

and for its trace respectively

$$\delta K \equiv \delta K^{\alpha}{}_{\alpha} = \frac{1}{2}n^{\rho}h^{\alpha\gamma}\nabla_{\rho}\delta g_{\alpha\gamma}. \tag{A.15}$$

A.2.2 Conformal transformation formulas

If M is an n–dimensional manifold supplied with a metric g_{ab}, and $\Omega \equiv \Omega(x^{\alpha})$ is a smooth, strictly positive function, then a conformal transformation $(M, g_{\alpha\beta}) \mapsto (M, \tilde{g}_{\alpha\beta})$ is defined as $g_{\alpha\beta} \mapsto \tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$. It follows that,

$$\tilde{g}^{\alpha\beta} = \Omega^{-2}g^{\alpha\beta}, \qquad \tilde{g}^{\alpha\beta}\tilde{g}_{\beta\gamma} = g^{\alpha\beta}g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}, \tag{A.16}$$

$$\tilde{h}_{\alpha\beta} = \Omega^2 h_{\alpha\beta}, \qquad \tilde{h}^{\alpha}{}_{\beta} = h^{\alpha}{}_{\beta}, \tag{A.17}$$

$$\sqrt{-\tilde{g}} = \sqrt{-g}\,\Omega^{n}, \qquad \sqrt{-\tilde{h}} = \sqrt{-h}\,\Omega^{n-1}, \tag{A.18}$$

$$\tilde{n}_{\alpha} = \Omega\, n_{\alpha}, \qquad \tilde{n}^{\alpha} = \Omega^{-1}\, n^{\alpha}, \tag{A.19}$$

since in general, $n_\alpha = (-N, 0, 0, \ldots, 0)$ and $\tilde{N} = \Omega N$, N being the lapse function.

The starting point for the transformation of the curvature objects is the transformation of the Christoffel symbol and using this we get the transformation of the Ricci tensor, Ricci scalar and GB term respectively. We have Wald (1984); Dabrowski et al. (2009),

$$\Gamma^\gamma_{\alpha\beta} = \widetilde{\Gamma}^\gamma_{\alpha\beta} - \Omega^{-1}\left[\delta^\gamma_\alpha \Omega_{,\beta} + \delta^\gamma_b \Omega_{,\alpha} - g_{\alpha\beta}\Omega^{,\gamma}\right], \tag{A.20}$$

$$R_{\alpha\beta} = \widetilde{R}_{\alpha\beta} - \Omega^{-2}(n-1)\tilde{g}_{\alpha\beta}\Omega_{,\kappa}\Omega^{,\kappa} + \Omega^{-1}\left[(n-2)\,\Omega_{\tilde{;}\alpha\beta} + \tilde{g}_{\alpha\beta}\tilde{\Box}\Omega\right], \tag{A.21}$$

$$R = \Omega^2\left[\widetilde{R} + 2(n-1)\Omega^{-1}\tilde{\Box}\Omega - n(n-1)\Omega^{-2}\tilde{g}^{\alpha\beta}\Omega_{,\alpha}\Omega_{,\beta}\right], \tag{A.22}$$

$$\begin{aligned} G =& \Omega^4\Big[\widetilde{G} - 4n_3\Omega^{-1}\left(2\widetilde{R}_{\alpha\beta}\Omega^{\tilde{;}\alpha\beta} - \widetilde{R}\tilde{\Box}\Omega\right) \\ & + 2n_2n_3\Omega^{-2}\left(2(\tilde{\Box}\Omega)^2 - 2\Omega_{\tilde{;}\alpha\beta}\Omega^{\tilde{;}\alpha\beta} - \widetilde{R}\Omega_{,\kappa}\Omega^{,\kappa}\right) \\ & - n_1n_2n_3\Omega^{-3}\left(4(\tilde{\Box}\Omega)\Omega_{,\kappa}\Omega^{,\kappa} - n\Omega^{-1}(\Omega_{,\kappa}\Omega^{,\kappa})^2\right)\Big], \end{aligned} \tag{A.23}$$

where in the last formula we use the convention $n_i \equiv (n-i)$, with n the spacetime dimension.

Beginning from the definition of the extrinsic curvature A.7, and using property A.6, we have

$$K_{\alpha\beta} = \Omega^{-1}\left[\widetilde{K}_{ab} - \Omega^{-1}\tilde{h}_{\alpha\beta}\Omega_{,\kappa}\tilde{n}^\kappa\right]. \tag{A.24}$$

Contracting with $h^{\alpha\beta}$ we find

$$K^\alpha{}_\alpha \equiv K = \Omega\left[\widetilde{K} - (n-1)\Omega^{-1}\Omega_{,\kappa}\tilde{n}^\kappa\right]. \tag{A.25}$$

For two different metrics $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$, defined on the same manifold M, and not necessarily conformally related, we have

$$R_{\alpha\beta\gamma}{}^\delta(g) - \widetilde{R}_{\alpha\beta\gamma}{}^\delta(\tilde{g}) = \widetilde{\nabla}_\beta C^\delta{}_{\alpha\gamma} - \widetilde{\nabla}_\alpha C^\delta{}_{\beta\gamma} + C^\kappa{}_{\alpha\gamma}C^\delta{}_{\beta\kappa} - C^\kappa{}_{\beta\gamma}C^\delta{}_{\alpha\kappa}, \tag{A.26}$$

$$K_{\alpha\beta}(h) - \widetilde{K}_{\alpha\beta}(\tilde{h}) = \frac{1}{2}\pounds_n \tilde{h}_{\alpha\beta} - \frac{1}{2}\pounds_n h_{\alpha\beta} \tag{A.27}$$

$$= \tilde{h}_{\alpha}{}^{\gamma}\widetilde{\nabla}_{\gamma} n_{\beta} - h_{\alpha}{}^{\gamma}\nabla_{\gamma} n_{\beta} = -h_{\alpha}{}^{\gamma} C^{\kappa}{}_{\gamma\beta} n_{\kappa}, \tag{A.28}$$

with $C^{\alpha}{}_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\sigma}(\widetilde{\nabla}_{\beta} g_{\gamma\sigma} + \widetilde{\nabla}_{\gamma} g_{\beta\sigma} - \widetilde{\nabla}_{\sigma} g_{\beta\gamma})$. The first relation results by starting from the definition of the Riemman tensor and evaluating it for two different metrics. Doing the necessary contractions in (A.26), we get similar expressions for the Ricci tensor and scalar as well as the extrinsic curvature trace respectively. As it was the case in Section 4.4, for two metrics conformally related, $\tilde{g}_{\alpha\beta} = \Phi g_{\alpha\beta}$ ($\Omega^2 \equiv \Phi$), beginning from (A.26) we get for the Ricci tensor

$$R(g) = \Phi \left[\widetilde{R}(\tilde{g}) - \frac{1}{4}(n-2)(n-1)\Phi^{-2}\partial_{\kappa}\Phi\partial^{\kappa}\Phi + (n-1)\widetilde{\nabla}^{\kappa}(\Phi^{-1}\widetilde{\nabla}_{\kappa}\Phi) \right], \tag{A.29}$$

and similarly contracting (A.28) with $\tilde{h}^{\alpha\beta}$ for the trace of the extrinsic curvature

$$h^{\alpha\beta} K_{\alpha\beta} = \Phi \left[\tilde{h}^{\alpha\beta}\widetilde{\nabla}_{\alpha} n_{\beta} + \frac{1}{2}\tilde{h}^{\alpha\beta} n^{\kappa}\widetilde{\nabla}_{\kappa} g_{\alpha\beta} \right]$$

$$= \Phi^{1/2} \left[\tilde{h}^{\alpha\beta}\widetilde{\nabla}_{\alpha}\tilde{n}_{\beta} - \frac{1}{2}(n-1)\tilde{n}^{\kappa}\Phi^{-1}\widetilde{\nabla}_{\kappa}\Phi \right], \tag{A.30}$$

with $\tilde{n}_{\alpha} = \Phi^{1/2} n_{\alpha}$.

A.3 Conformal transformation of the Gauss–Bonnet GH term

Here we will present the conformal transformation of the Gauss–Bonnet GH term, (4.28). For the two terms of (4.28) we get respectively

$$\int_{\Sigma} d^{n-1}x\, \sqrt{-h}\; J \mapsto \int_{\Sigma} d^{n-1}x \sqrt{-\tilde{h}}\; \Omega^{4-n}\; \Phi \Bigg\{ \widetilde{J} + n_3 \Omega^{-1} \left[\widetilde{K}^2 - \widetilde{K}_{\alpha\beta}\widetilde{K}^{\alpha\beta} \right] (\Omega_{,\kappa}\tilde{n}^{\kappa})$$

$$- n_3 n_2 \widetilde{K}\; \Omega^{-2} \left(\Omega_{,\kappa}\tilde{n}^{\kappa}\right)^2 + \frac{1}{3} n_3 n_2 n_1 \Omega^{-3} \left(\Omega_{,\kappa}\tilde{n}^{\kappa}\right)^3 \Bigg\}. \tag{A.31}$$

$$\begin{aligned}\int_\Sigma d^{n-1}x\,\sqrt{-h}\;\Phi\widehat{G}^{\alpha\beta}K_{\alpha\beta} \mapsto \int_\Sigma d^{n-1}x\sqrt{-\tilde{h}}\;\Omega^{4-n}\Phi\Bigg\{&\widetilde{\widehat{G}}^{\alpha\beta}\widetilde{K}_{\alpha\beta}\\ + n_3\Omega^{-1}\left[\frac{1}{2}\widetilde{\widehat{R}}(\Omega_{,\kappa}\tilde{n}^\kappa) + \widetilde{K}_{\alpha\beta}\Omega^{\tilde{;}\alpha\beta} - \widetilde{K}\tilde{\Box}\Omega\right]&\\ + n_3\Omega^{-2}\left[\frac{n_2}{2}\widetilde{K}(\Omega_{,\kappa}\Omega^{,\kappa}) - \tilde{\Box}\Omega(\Omega_{,\kappa}\tilde{n}^\kappa) + n_1\tilde{\Box}\Omega(\Omega_{,\kappa}\tilde{n}^\kappa)\right]&\\ - \frac{1}{2}\Omega^{-3}n_3n_2n_1\left(\Omega_{,\kappa}\Omega^{,\kappa}\right)^2\left(\Omega_{,\lambda}\tilde{n}^\lambda\right)\Bigg\}.&\end{aligned} \tag{A.32}$$

Adding up terms (A.31) and (A.32) we get the GB GH term in the conformally transformed frame. However, boundary terms resulting by variation of action (4.31) with respect to $\tilde{g}_{\alpha\beta}$ and Φ, will not be able to cancel with the GH subterms in (A.31) and (A.32), as was the case in $f(R)$. Consequently, we are left with terms proportional to first and second order derivatives of both the metric and scalar field on the boundary surface Σ, which should be held fixed in the initial value formulation, together with $g_{\alpha\beta}$ and ϕ themselves, in order for the GH term to be zero in the total variation.

Appendix B

Anisotropic stress and stability in modified gravity models

B.1 Homogeneous perturbations of $f(R,G)$

Below we present some additional mathematical supplement from chapter 5.

We will first present the stability analysis of any fixed point of the the $f(R,G)$ Friedmann equation, using homogeneous perturbations around the relevant solution. Our starting point is the $t-t$ equation (3.59), which for convenience we reproduce it here again,

$$3H^2F + 3H\dot{F} + 12H^3\dot{\xi} - \frac{1}{2}V - \rho_i = 0. \tag{B.1}$$

If $H \equiv H(t)$ is a solution of above equation then perturbing around it as $H(t) \to H(t) + \delta H(t)$, and keeping up to first order terms we get for the curvature scalars and their first time derivatives respectively

$$R \to R + 6\left(4H\delta H + \delta\dot{H}\right), \tag{B.2}$$

$$G \to G + 24\left[2(2H^3 + \dot{H}H)\delta H + H^2\delta\dot{H}\right]. \tag{B.3}$$

The next step is to perturb the modified Friedman equation (3.59). Particularly, the scalar potential becomes

$$
\begin{aligned}
V &\to V + R\delta F + G\delta\xi \\
&= V + (RF_R + G\xi_R)\delta R + (RF_G + G\xi_G)\delta G \\
&\equiv V + V_{(R)}\delta R + V_{(G)}\delta G,
\end{aligned}
\tag{B.4}
$$

and after evaluating the scalar field perturbations, it takes the form

$$
V = V + 6\left(V_{(R)} + 4H^2 V_{(G)}\right)\delta\dot{H} + 24\left(HV_{(R)} + 4H^3 V_{(G)}\right)\delta H, \tag{B.5}
$$

where subscripts in brackets simply denote indices, while those outside brackets denote derivative with respect to the corresponding variable.

Using relations given above, and after some algebra, the modified Friedman equation becomes

$$
C_{(H)}\delta H + C_{(R)}\delta R + C_{(G)}\delta G + C_{(\dot{R})}\delta\dot{R} + C_{(\dot{G})}\delta\dot{G} = 0, \tag{B.6}
$$

with

$$
\begin{aligned}
C_{(H)} &\equiv 6HF + 3\dot{F} + 36H^2\dot{\xi}, && \text{(B.7)} \\
C_{(R)} &\equiv 3H^2 F_R + 3H\dot{F}_R + 12H^3\dot{\xi}_R - \frac{1}{2}V_{(R)}, && \text{(B.8)} \\
C_{(G)} &\equiv 3H^2 F_G + 3H\dot{F}_G + 12H^3\dot{\xi}_G - \frac{1}{2}V_{(G)}, && \text{(B.9)} \\
C_{(\dot{R})} &\equiv 3HF_R + 12H^3\xi_R, && \text{(B.10)} \\
C_{(\dot{G})} &\equiv 3HF_G + 12H^3\xi_G. && \text{(B.11)}
\end{aligned}
$$

Substituting for the perturbations of δR, δG and their derivatives we arrive at

$$
C_1\delta\ddot{H} + C_2\delta\dot{H} + C_3\delta H - \delta\rho_i = 0, \tag{B.12}
$$

with

$$C_1 \equiv 6\left[C_{(\dot{R})} + 4C_{(\dot{G})}H^2\right], \tag{B.13}$$

$$C_2 \equiv \left[C_{(R)} + 4C_{(R)}H^2 + 4C_{(\dot{R})}H + 16C_{(\dot{G})}(H^3 + \dot{H}H)\right], \tag{B.14}$$

$$C_3 \equiv 6\Big[HF + \frac{1}{2}\dot{F} + 6H^2\dot{\xi}_G + 4C_{(R)}H + 8C_{(G)}(2H^3 + \dot{H}H) + 8C_{(\dot{G})}(6H^2\dot{H} + \dot{H}^2 + \ddot{H}H)\Big]. \tag{B.15}$$

Defining $\omega \equiv F_R + 4H^2(2F_G + 4H^2\xi_G)$, the generalisation of equation (5.34) for arbitrary H, we find that always

$$C_1 = 18H\omega. \tag{B.16}$$

For a polynomial background expansion, described by $a(t) \propto t^p$, the other coefficients become

$$C_2 = \frac{18H}{p}\left[p\dot{\omega} + 8H^3(1+3p)(\xi_R + 2H^2\xi_G) + (1+3p)HF_R\right], \tag{B.17}$$

$$C_3 = \frac{3}{p^2}\Big\{p^2\dot{F} + 2Hp^2F - 12H^2\Big[2p^2(H\omega - \dot{\omega}) + 4pH^2(\dot{\xi}_R + 4H^2\dot{\xi}_G) - 2(10p+4)H^3(\xi_R + 4H^2\xi_G)\Big]\Big\}. \tag{B.18}$$

For a de Sitter expansion, $a(t) \propto \exp[H_0 t]$, and $H = H_0 = const.$, we get

$$C_2 = 3H_0C_1 \tag{B.19}$$

$$C_3 = \left(\frac{F}{3\omega} - 4H_0^2\right)C_1. \tag{B.20}$$

B.2 Inhomogeneous perturbations and de Sitter stability

The general metric element for scalar perturbations around a flat FLRW background reads

$$\begin{aligned} ds^2 = &- (1+2\alpha)dt^2 - 2a(t)\partial_i\beta\, dt\, dx^i \\ &+ a(t)^2 \left(\delta_{ij} - 2\phi\delta_{ij} + 2\partial_i\partial_j\gamma\right) dx^i dx^j . \end{aligned} \tag{B.21}$$

The general form of scalar perturbation equations around FLRW for $f(R,G)$ models can be found in De Felice and Suyama (2009). Here, we shall present the full set of equations for the case of de Sitter space only.

Before we proceed, let us define the gauge invariant variable Φ as

$$\Phi \equiv \Phi(t) \equiv \frac{\delta F + 4H^2\delta\xi}{2F}, \tag{B.22}$$

with $H \equiv H_0$ as well as the rest of the background quantities evaluated on the de Sitter point. The perturbation equations then read as

$$3H^2\psi + \frac{k^2}{a^2}\left(H\chi + \phi\right) + 3H\dot{\phi} = 3H\dot{\Phi} + \left(\frac{k^2}{a^2} - 3H^2\right)\Phi, \tag{B.23}$$

$$H\psi + \dot{\phi} = \dot{\Phi} - H\Phi, \tag{B.24}$$

$$\dot{\chi} + H\chi + \phi - \psi = 2\Phi, \tag{B.25}$$

$$\begin{aligned} \delta R = -2\Big[&12H^2\psi + 3\ddot{\phi} + 12H\dot{\phi} + 3H\dot{\psi} \\ &\frac{k^2}{a^2}\left(\dot{\chi} + 2H\chi + 2\phi - \psi\right)\Big], \end{aligned} \tag{B.26}$$

$$\begin{aligned} \delta G = -8\Big[&12H^4\alpha - 3H^2\ddot{\phi} + 3H^3\dot{\alpha} - 12H^3\dot{\phi} \\ &+ \frac{k^2}{a^2}H^2\left(2H\chi + \chi - 2\phi - \alpha\right)\Big]. \end{aligned} \tag{B.27}$$

Equations (B.23), (B.24) and (B.25) correspond to the 00, the $0i$ and the $ij (i \neq j)$

components respectively. Particularly, equation (B.25) is the anisotropy equation, and the choice of variable Φ is now evident: it is the r.h.s of the latter equation, describing the effective anisotropic stress in de Sitter space, $\Phi = \Pi^{(\mathrm{eff})}$, and therefore is gauge invariant.

In order to re-express above equations in terms of gauge invariant variables only, we need a second gauge invariant variable apart from Φ. Following De Felice and Suyama (2009) we define

$$\Psi \equiv \Phi + \phi - H\chi. \tag{B.28}$$

Now, using equation (B.24) in (B.23) we get

$$\Phi = \phi + H\chi, \tag{B.29}$$

which can be inserted into (B.28) to give

$$\Psi = 0. \tag{B.30}$$

Using equations (B.24), (B.25) as well as (B.29) we can re-express the curvature perturbation in terms of the gauge invariant potential Φ

$$\delta R = -6\left[\ddot{\Phi} + 3H\dot{\Phi} + \left(\frac{k^2}{a^2} - 4H_0^2\right)\right]. \tag{B.31}$$

B.3 Sub-horizon solution for Φ in the WKB approximation

Considering the evolution equation (5.40) in de Sitter space for the gauge invariant potential Φ, we assume a solution of the form

$$\Phi = C\mathrm{e}^{\mathrm{i}\theta(\mathrm{t})}, \tag{B.32}$$

with C a constant, and $\ddot{\theta}(t) \ll 1$. Then, we can calculate that

$$\begin{aligned}
\Phi(t) &\approx \sum_{\pm} C_{\pm} \exp\left[i \int_0^t dt' \dot{\theta}^{\pm}(t')\right] \qquad \text{(B.33)}\\
&\equiv \sum_{\pm} C_{\pm} \exp\left[-\int_0^t dt' \left(\widetilde{A}(t') \pm i\widetilde{B}(t')\right)\right]
\end{aligned}$$

with

$$\widetilde{A} \equiv A + \frac{2B\dot{B}}{C^2}, \qquad \widetilde{B} \equiv B + \frac{A\dot{B}}{C^2}, \qquad \text{(B.34)}$$

$$A \equiv 3H_0, \;\; B \equiv \sqrt{4\left(k^2 \mathrm{e}^{-2H_0 t'} + m_{\text{eff}}^2\right) - 9H_0^2}, \qquad \text{(B.35)}$$

$$C^2 \equiv A^2 + 4B^2. \qquad \text{(B.36)}$$

From solution (B.33) we can calculate the limit when $m_{\text{eff}}^2 \gg 1$, which is the case when $\Pi^{(\text{eff})} \to 0$. In this case we have,

$$\widetilde{A} \approx 3H_0, \qquad \widetilde{B} \approx 2m_{\text{eff}}, \qquad \text{(B.37)}$$

and the solution is approximately given by

$$\Phi(t) \approx \sum_{\pm} C_{\pm} \exp\left[-H_0 t \pm 2im_{\text{eff}} t\right]. \qquad \text{(B.38)}$$

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