Vertex Operators for Cosmic Strings

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Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

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VERTEX OPERATORS FOR COSMIC STRINGS

Summary

Superstring theory posits that as complicated as nature may seem to the naive observer, the variety of observed phenomena may be explained by postulating that at the fundamental scale, matter is composed of lines of energy, namely strings. These oscillating lines would be elementary and would hence have no substructure. They are expected to be incredibly tiny, their line-like structure would become noticeable at scales close to the string scale (which may lie anywhere from the TeV scale all the way up to the Planck scale) and would appear to be point-like to the macroscopic observer. Internal consistency then also requires the presence of higher dimensional objects, namely D-branes, all of which conspire and combine in such a way so as to give rise to the observable Universe. Advances in cosmology suggest the early universe was much hotter and denser than is the Universe at present, that the Universe has expanded and continues to expand (exponentially in fact) at present. This in turn has led a number of theorists to point out the remarkable possibility that some of these strings or D-branes were also stretched with the expansion. The resulting macroscopic strings, the so-called cosmic strings, would potentially stretch across the entire Universe.

Cosmic strings make their presence manifest by oscillating, scattering off other structures, by decaying, producing gravitational waves and so on, and this in turn hints at the available handles that may be used to observe them. Before we can hope to observe cosmic strings however, the first step is then clearly to understand these properties which determine their evolution. A number of approximate (classical) descriptions of cosmic strings have been constructed to date, but approximations break down, especially when potentially interesting things happen (e.g. close to cusps, i.e. points on the string that reach the speed of light) and can obscure the physics. Thankfully, one can go beyond these approximations: all properties of cosmic strings can be concisely and accurately contained or encoded in a single object, the so-called fundamental cosmic string vertex operator. In the present thesis I construct precisely this, covariant vertex operators for general cosmic strings and this is the first such construction.

Cosmic strings, being macroscopic, are likely to exhibit classical behaviour in which case they would most accurately be described by a string theory analogue of the well known harmonic oscillator coherent states. By minimally extending the standard definition of coherent states, so as to include the string theory requirements, I go on to construct both open and closed covariant coherent state vertex operators. The naive construction of the latter requires the existence of a lightlike compactification of spacetime. When the lightlike winding states in the underlying Hilbert space are projected out, the resulting vertex operators have a classical interpretation and can consistently propagate in non-compact spacetime. Using the DDF map I identify explicitly the corresponding general

lightcone gauge classical solutions around which the exact macroscopic quantum states are fluctuating. We go on to show that both the covariant gauge coherent vertex operators, the corresponding lightcone gauge coherent states and the classical solutions all share the same mass and angular momenta, which leads us to conjecture that the covariant and lightcone gauge states are different manifestations of the same state and share identical interactions. Apart from the coherent state vertices I also present a complete set of covariant mass eigenstate vertex operators and these may also be relevant in cosmic string evolution. Finally, I also present the first amplitude computation with the coherent states, the graviton emission amplitude (including the effects of gravitational backreaction) for a simple class of cosmic string loops. As a byproduct of the above, I find that the fundamental building blocks of arbitrarily massive covariant string states are given by elementary Schur polynomials (equivalently complete Bell polynomials). This construction enables one to address the aforementioned questions concerning the properties of cosmic strings, their cosmological signatures, and may lead to the first observations of such objects in the sky. This in turn would be a remarkable way of verifying Superstring theory as the framework underlying the structure of our Universe.

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‘ΠΟΛΥΝΟΪΗΝ ΟΥ ΠΟΛΥΜΑΘΙΗΝ ΑΣΚΕΕΙΝ ΧΡΗ’

ΔΗΜΟΚΡΙΤΟΣ

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Chapter 1

Introduction

The search for a complete physical theory or framework which can account for all observable phenomena is at least 2500 thousand years old, some of the first written accounts of such an attempt being attributed to natural philosophers such as, Pythagoras of Samos (582-496 BCE), Heraclitus of Ephesus (535-475 BCE), Parmenides of Elea (510-440 BCE), and Democritus of Abdera (460-370 BCE). Our understanding of nature has increased immensely since then, and it seems that it is indeed possible that only a handful of concepts or principles (such as the relevance of symmetries, consistency and possibly also elegance) are required to account for observable phenomena. A very rich and profound such framework or proposal which has been used to assemble these ingredients or principles is the so-called Superstring Theory. At least from a perturbative perspective, Superstring Theory posits that as complicated as nature may seem to a naive observer, the variety of observed phenomena may potentially be explained by postulating that at the fundamental scale matter is composed of line-like structures of pure energy with no substructure, oscillating lines of energy, strings. The traditional view is that these strings would be incredibly tiny, that their line-like structure would become noticeable at scales of order 10^{-34} metres. Advances in cosmology however that have been taking place since the late seventies early eighties suggest the early universe was much hotter and denser than is the Universe at present, that the Universe has expanded and continues to expand (exponentially in fact) at present. This in turn has given rise to the remarkable possibility that some of these strings that populate the Universe were also stretched with the expansion, leading effectively to macroscopic strings that would potentially stretch across the entire Universe – cosmic strings.

Although the possibility that superstrings of cosmological extent was initially discarded [1] as a viable way of observing strings in nature, the discovery of dualities [2] and D-branes [3, 4] in the mid-1990's (this epoch is referred to as the second superstring revolution) opened up a huge range of possibilities and the issue of cosmic strings had to be reexamined. The study of cosmic strings, see e.g. [5, 6, 7, 8, 9, 10], subsequently flourished. In the post second superstring revolution era, it was discovered [11] (but see also [12]) [13, 14, 15] that such objects may be produced in string models of the early universe, thus providing an observational signature for Superstring theory [16, 17, 18].

For an overview of cosmic strings in the pre- and post-"Second Superstring Revolution" era see [5, 6] and [7, 8, 19, 20, 21, 22] respectively, and for an excellent review which also contains many of the computational details associated to the latter see [23].

1.1 Brane Inflation

These new developments opened many new avenues for model building [24] and string cosmology, such as the brane inflation scenario [25, 26, 27, 28, 29, 13] in the context of large extra dimensions [30, 31, 32], where macroscopic strings have been found to be produced [11, 14, 33, 17, 12] with string tensions in the range,

$$10^{-12} \leq G\mu \leq 10^{-6}.$$

Here G is the 4-dimensional Newton's constant, μ the string tension.[1] In these brane inflation models it is difficult to obtain a sufficient amount of inflation [34, 35] and in [35] this problem is evaded by considering instead a warped compactification [36, 37], a concrete example of which is the well known $\mathbb{KLMT}$ scenario [38, 35], where all moduli are stabilized [24]. It has since been realized [16] that it is possible in these theories to construct macroscopic non-BPS as well as BPS strings which are stable [39] and potentially observable.

Unfortunately, no completely satisfactory string model of the early universe exists yet: although all moduli are stabilized in the $\mathbb{KLMT}$ brane inflation scenario [38, 35], it suffers from a reheating problem where all the reheating energy arising from the D3/$\overline{\text{D3}}$-annihilation goes into a massless U(1) gauge field that lives on the stabilizing $\overline{\text{D3}}$-brane instead of going into the standard model fields, for an overview see e.g. [21]. Furthermore, in the context of large extra dimensions there is no known mechanism to stabilize the moduli. Nevertheless, these drawbacks may be specific to the models considered to date and it is plausible that in more general constructions these problematic features are absent.

A rough picture of a general D/$\overline{\text{D}}$-brane inflation scenario (without referring to explicit details of any particular model) is as follows. Cosmological inflation is driven by the attractive interaction potential associated to two stacks of parallel D- and $\overline{\text{D}}$-branes which approach each other in the higher dimensional bulk space. These two stacks eventually collide and annihilate via tachyon condensation, see e.g. [40]. Due to the Kibble mechanism [41] any gauge theory with a U(1) gauge symmetry that becomes broken during the evolution of the universe will produce cosmic strings. The crucial observation of [14] was that the low energy string dynamics at the end of brane inflation is described by U(1) symmetry breaking in the tachyon field, and therefore one expects the formation of defects (lower dimensional branes) which are seen as cosmic strings by observers on the (or one

[1] When it is ambiguous, we will explicitly write G_4 or G_d for the 4- or d-dimensional Newton's constant respectively instead of G. Note that $G_4\mu$ is dimensionless and that $[G_d] = L^{d-2}$. In string theory, $\mu = 1/(2\pi\alpha')$.

of the) remaining higher dimensional branes. It has been argued that the production of other defects such as monopoles and domain walls is suppressed [13]. These defects are identified [42, 43] with D1-branes, which follows from computing the conserved charges. Both D-strings and F-strings are expected to arise [44, 17, 16] in this process, even though the standard language of string creation associated to a spontaneous breaking of a U(1) symmetry is not appropriate for F-strings (unless $g_s \gg 1$). The standard model particles of strong and weak interactions correspond to open string modes confined to a remaining D-brane with 3 large non-compact dimensions, and the closed string modes associated to the graviton, radions and massive excitations all correspond to bulk modes.

The presence of cosmic strings is likely to be a fairly generic feature of any string model of the early universe and in the current document I shall assume that such a model can be found and focus instead on the cosmic strings themselves. I will focus in particular on the fundamental cosmic strings which have an exact perturbative (in the string coupling $g_c = e^{\phi}$ and the fundamental string length squared α') description in terms of vertex operators.

1.2 Cosmic String Evolution

The basic properties which collectively determine the evolution are string inter-commutations and reconnections [45, 46, 47, 48, 49], quantum or classical string decay [50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60] and the presence of junctions [61, 62, 63, 64, 65, 66], and possible instabilities [1, 53, 16]. Collectively, these properties and cosmological considerations (such as the expansion rate of the universe, density inhomogeneities, and so on) determine the various observational signatures from cosmic strings.

An initial distribution of long strings is formed via the Kibble mechanism, the shape of any one such string resembling a random walk. The expansion of the universe stretches these strings which intercommute and reconnect producing kinks (i.e. points on the string at which the spacetime embedding tangent vectors associated to left and right-movers are discontinuous). Any one of these kinks then separates into two kinks running along the string in opposite directions. When left- and right-moving modes meet on any given section of a string gravitational radiation is produced. There will also be long strings that self-intercommute and produce loops which subsequently are expected to decay into smaller loops via gravitational radiation.

There is general consensus on the large scale evolution of cosmic strings. Here the string network evolves towards a scaling regime, a regime in which the characteristic length scale of the configuration evolves towards a constant relative to the horizon size [5, 6]. Recently, there has also been some progress in understanding the small scale structure [67, 68, 69, 70, 21, 71, 72]. Here one of the most important questions is: what is the typical size at which loops are produced from long string. There has been large disagreement in the literature with estimates differing by over fifty orders of magnitude [73]. This is an important question and further investigation is required. Another very important question which is also related to the previous one is: what is the importance of gravitational backreaction on the evolution of cosmic strings, see also below.

1.3 Observational Signatures

Signals from cosmic strings have to date not yet been detected. There is a wide range of constraints from gravitational waves [74, 75, 76, 77, 78, 79, 80, 81] (classical gravitational wave emission from loops and infinite strings has been computed in [82, 83] and [84, 85, 86, 87, 71] respectively and from strings with junctions in [88]), strong and weak lensing from strings without [89, 90, 91, 92, 93, 94] (but see also [95]) and with [95, 96, 97] junctions, and the CMB [98, 99, 100, 101, 102, 103, 104, 81, 105]. Future missions searching for a polarization B-mode in the CMB will provide even stronger constraints [106, 107, 108, 109, 71]. Signals from cosmic strings may also show up in ultrahigh-energy cosmic rays [110, 111], radio wave bursts [112], and also diffuse X- and γ-ray backgrounds [111]. There is also the potential to obtain constraints on the underlying compactifications [113]. Even though cosmic strings can only account for a small contribution to the CMB power spectrum, they could instead be the main source of its non-Gaussianities and are expected to dominate over inflationary perturbations at small angular scales, see [22] and references therein.

1.4 Vertex Operators as Cosmic Strings

Given the inherently quantum-mechanical nature of fundamental cosmic strings, the only available handle on such macroscopic objects at present that is capable of accounting for the evolution on the smallest as well as largest scales is given in terms of *vertex operators* [114, 2] which completely characterize the string under consideration. For example, a vertex operator description would be required for cosmic string configurations involving a string theory analogue of cusps (i.e. points on the string that reach the speed of light at discrete instants during the loop's motion) and kinks, as presumably the effective field theory or classical description would break down close to these points.

With a vertex operator construction of cosmic strings one can address various questions, such as what is the decay rate of a given cosmic string configuration, the intercommutation and reconnection probabilities, junction decay rates, emission of massless and massive radiation and so on. The already existing quantum decay rate computations carried out in [51, 52, 54, 55, 56, 57, 58, 59, 60] for instance make use of mass eigenstate vertex operators (with only first harmonics excited) and it is not known at this point whether these are appropriate for the description of cosmic strings. In [58] for instance it was concluded that the spectrum of a particular mass eigenstate does not reproduce the classical gravitational wave spectrum, and one might expect this to be the case also for general mass eigenstates.

It is likely that cosmic strings being macroscopic and massive should have a classical interpretation. If this is the case the appropriate vertex operators are expected (from our experience with standard harmonic oscillator coherent states) to have coherent state-like properties, and so we should be searching for coherent state vertex operators, which from the standard coherent state properties would be expected to have a classical interpretation. The analogous computations to the ones described above with coherent states instead of mass eigenstates would be more desirable and would probably represent a much more realistic description of cosmic strings (pure quantum states are rarely if ever found in nature).

A quantum-mechanical approach to computing the decay process for macroscopic and realistic cosmic string loops is highly desirable as one must also check the usual assumption that the process is classical. Furthermore, the classical computation is not well understood, as calculations based on field theory and the Nambu-Goto approximation differ (a nice discussion of this issue it given in [70]), and gravitational back-reaction is not taken into account which is included very naturally in perturbative string theory.

Finally let us mention that it is very important to find tests which distinguish fundamental strings from solitonic strings; a major difference is the quantum nature of F-strings which leads to a reduced probability for the reconnection of intersecting strings [47] (see also [115] for an alternative approach).

In the current thesis I construct a complete set of covariant vertex operators, i.e. vertices for arbitrarily massive (closed and open) strings, for both mass eigenstates and open and closed string coherent states. We also discuss the corresponding lightcone gauge realization and provide an explicit map from these to general classical (lightcone gauge) solutions.

1.5 Gravitational Radiation and Backreaction

Cusps are generic in loops [116] and are expected to lead to very strong gravitational wave signals [75, 76], although the presence of extra dimensions is likely to weaken the detected signal. In [117, 118] the effect of extra dimensions in cusp formation was studied, as well as the corresponding gravitational waves produced. It was found that the effect of the extra dimensions is to effectively round off cusps, thus decreasing the emission amplitude. It will be interesting to study more carefully the effect of the finite size of the extra dimensions. Cusps on strings with junctions have also been argued to be generic in [64]. Recent evidence [119] suggests that kinks on strings with junctions also provide a very strong gravitational wave signal – the signal from kinks on closed strings *with* junctions is found to be stronger than the signal due to cusps. Note however, that creating a loop with a junction is a higher order process that creating one without a junction, and so the number of such loops in the observable universe may be small. It is very important to test the robustness of all these computations to gravitational backreaction effects. In fact, it is likely that gravitational backreaction can be important for even order of magnitude estimates [120], and developing the necessary tools that enable one to study this problem systematically has been one of the main purposes of the present thesis – in perturbative string theory backreaction effects can be taken into account very naturally.

1.6 Massive Radiation

Apart from the possibility of gravitational backreaction playing a significant role in string evolution, a string theory computation is also required when there is the possibility of massive closed string states being emitted – this might be expected to occur close to cusps and kinks and this massive radiation would presumably be invisible or difficult to calculate in the effective field theory.[2] That massive radiation may dominate over gravitational radiation was suggested in [121, 122], and this was motivated by the observation that loops seemed to be produced at the smallest scales, see also [123, 124], namely at the numerical simulation cutoff scale which is identified with the string width, although their conclusions relied on extrapolation of numerical results beyond the region of validity. Whether a significant amount of massive radiation is emitted is still an open question – this can be addressed in the vertex operator construction of the current document which is expected to give a definite answer to this question. If one is interested in the emission of arbitrarily massive radiation one may proceed along the lines of [51, 52, 54, 55, 56, 57, 58, 59, 60].

1.7 Classicality of Cosmic Strings

Let us now say a few words concerning the classicality of quantum-mechanical string vertex operators. Consider first mass eigenstates. These are specified by certain quantum numbers, the relevant one here being the level number N, and a necessary (but not sufficient) condition for classicality is that these take *large* values. This dates back to Niels Bohr who used this argument when he postulated that any quantum-mechanical system should satisfy the correspondence principle. Typically the quantum numbers of interest in a given quantum system appear in the combination $(N\hbar)$ thus showing that the classical limit $\hbar \to 0$ is related to the large quantum number limit $N \to \infty$ with the combination $N\hbar$ held fixed. For example, this can be seen in the energy spectrum of the hydrogen atom, $E_N \sim \text{const.}/(N\hbar)^2$, the harmonic oscillator, $E_N \sim \text{const.}(N\hbar)$, and also the string spectrum,[3] $E_N \sim \text{const.}\sqrt{(N\hbar)}$. Vertex operators present in the large quantum number limit may in some sense therefore be referred to as being quasi-classical. Mass eigenstates however are nevertheless not truly classical in the sense that they are not expected to have classical expectations values with small uncertainties [125], and one does

[2] I would like to acknowledge an important discussion with Andrew Strominger concerning the relevance of a vertex operator formulation of cosmic strings as opposed to an effective low energy description.

[3] $\hbar$ is usually set equal to 1 but can be re-introduced by examining the path integral $\int \mathcal{D}X e^{\frac{i}{\hbar}S}$, with $S = \frac{1}{2\pi\alpha'}\int d^2z\, \partial X \cdot \bar{\partial} X$. Taking z to be dimensionless, ℓ_s the string length and $[X] = L$ it is seen that

$$2\pi\alpha' = \ell_s^2/\hbar.$$

not expect the spectrum of gravitational radiation to match the classical computation [58] – this is an important issue and deserves further attention.

Coherent states on the other hand, see e.g. [126] and references therein, are expected to possess classical expectation values with small uncertainties, e.g. $\langle J^{\mu\nu}\rangle = J^{\mu\nu}_{\rm cl}$, $\langle X^{\mu}\rangle = X^{\mu}_{\rm cl}$, (with $J^{\mu\nu}$ the spacetime angular momentum and X^{μ} the target space map of the worldsheet into spacetime) and it is likely that these should be identified with fundamental cosmic strings. There are subtleties however concerning the naive classicality requirement $\langle X^{\mu}\rangle = X^{\mu}_{\rm cl}$ (with $X^{\mu}_{\rm cl}$ non-trivially obeying the classical equations of motion, $\partial\bar{\partial}X^{\mu}_{\rm cl} = 0$) and it turns out [125] that this requirement (in the closed string case) is not compatible with the Virasoro constraints (when states are invariant under spacelike worldsheet rigid translations). Suffice it to say here that this is a gauge problem and says nothing about the classicality of the underlying states. We elaborate on this in detail later where we also propose a solution: an alternative to the $\langle X^{\mu}\rangle = X^{\mu}_{\rm cl}$ classicality condition which is compatible with the string symmetries. We will also see that it is possible for closed string (coherent) states to satisfy $\langle X^{\mu}\rangle = X^{\mu}_{\rm cl}$ in lightcone gauge when the underlying spacetime manifold is compactified in a lightlike direction, $X^- \sim X^- + 2\pi R^-$, with X^+ non-compact, because this compactification breaks the invariance under spacelike worldsheet shifts.

1.8 Vertex Operator Constructions

Various prescriptions have been given for the construction of covariant vertex operators, e.g. the construction due to Del Giudice, Di Vecchia and Fubini (DDF) [127, 128, 129, 130] but see also [131], the path integral construction based on symmetry [132, 133, 134] and factorization [135, 136, 137] and operator constructions [138, 139] among others. A powerful method which applies in general backgrounds is given in [140], (although explicit results for high mass states are notoriously difficult to obtain in more general backgrounds). To carry out the map from classical solutions to covariant vertex operators we shall make use of the DDF construction. The power of the DDF construction lies in the following: it generates the entire physical Fock space, and it can be used to translate light-cone gauge states into the corresponding covariant vertex operators, where the standard technology for amplitude computations [114, 141] can be used. This is clearly very useful indeed given that in the construction of vertex operators for cosmic strings we would like to know what the corresponding classical state is, but explicit general classical solutions are best understood in lightcone (not covariant) gauge – the DDF construction provides the appropriate bridge between classical lightcone gauge string solutions and covariant vertex operators.

We also give some explicit results for a number of physical covariant quasi-classical vertex operators (i.e. with large quantum numbers) which lie beyond the leading Regge trajectory without making use of the DDF formalism. Explicit results for high mass vertices are sparse, some notable exceptions being Weinberg's vertex operator construction [132] and also the approach of Sato [133]. In [132] one can find explicit results concerning *monomial* massive vertices and in the present document we derive an explicit representation for general polarization tensors which are appropriate for these vertices. In [133] one can find more general, in particular *polynomial*, vertex operators where constraints on the polarization tensor and other physical state conditions are derived explicitly. A subclass of these vertices has been obtained by Aldazabal et al. [135] by considering the factorization of the tachyon-tachyon amplitude. We here discuss the construction of monomial vertex operators, vertex operators produced in tachyon-tachyon, tachyon-massless[4] and massless-massless scattering and give explicit representations for all polarization tensors. The DDF construction which can be used to generate a complete set of states is essentially identical to a certain factorization of a scattering amplitude with an arbitrary number of massless vertex operators inserted and a tachyonic vacuum.

1.9 Thesis Outline

In Sec. 2 we present a brief overview of the field of cosmic strings. We start with an overview of topological defects in classical field theory. We then go on to discuss type IIA/IIB superstring theory, concentrating in particular on tachyon condensation and the most common topological defects found in string theory. We then describe a cosmological scenario where such objects may be produced, the so called $\mathbb{KLMT}$ scenario. This then leads to a discussion of classical string evolution in Sec. 2.3. We next focus on the problem of taking gravitational backreaction into account (Sec. 2.4) and the associated implications, while emphasizing the importance of doing so. In Sec. 2.5 we discuss flat background evolution and discuss phenomenologically interesting features on cosmic strings, such as cusps. In Sec. 2.6 and 2.7 we present conventions that will be used in the main sections of the text associated to closed and open string mode expansions. This is followed by a section (which is central for cosmic string evolution) on the scaling solution of cosmic strings and a discussion of energy loss mechanisms, namely Sec. 2.8. Finally, in Sec. 2.9 we present an overview of a classical gravitational radiation computation for a string with cusps, that will set the scene for the chapters on the perturbatively exact vertex operator descriptions of cosmic strings and the corresponding graviton emission amplitude.

In Sec. 3 we introduce the necessary material (perturbative string theory) in order to discuss the vertex operator construction of cosmic strings. In particular, in Sec. 3.1, we define conformal field theories, the corresponding Virasoro algebra Sec. 3.2 and representations in Sec. 3.3. We then go on to discuss string amplitudes in Sec. 3.4, and two-point functions in Sec. 3.5. Many details which are omitted from the main text have been included in the Appendices.

In Sec. 4 we discuss the general construction of mass eigenstate vertex operators in bosonic string theory. In Sec. 4.1 in particular, we discuss an explicit construction of monomial covariant vertex operators which lie beyond the leading Regge trajectory and give explicit representations for the polarization tensors for general states. The construc-

[4] Here massless means graviton, dilaton or antisymmetric tensor vertices.

tion is based on the considerations of Weinberg [132] and de Alwis in [142], see also [143, 144, 134, 141]. We find a limitation which arises in this approach given that here polarization tensors have the symmetries of Young tableaux: when the polarization tensor transforms under real representations of SO(25) there is a maximum order of harmonics that can appear in the state (because the order of worldsheet derivatives appearing is always associated to particular a row of the tableau in this approach).

In Sec. 4.2, we expand on the work of Aldazabal et al. [135] and Sato [133] and discuss the construction of vertex operators via factorization of tachyon-tachyon, tachyon-massless and massless-massless scattering processes. Here we emphasize the importance of elementary Schur polynomials (equivalently Bell polynomials) which can be used to write down these vertices very concisely. The importance of Schur polynomials in the construction of vertex operators was identified in [145], as very briefly mentioned in [146].

In Sec. 4.4 we discuss the construction of a complete set of normal ordered mass eigenstate covariant vertex operators using the DDF formalism, which can be used to translate light-cone gauge states into fully covariant vertex operators. The Virasoro constraints are solved completely and the resulting vertex operators are physical for arbitrary polarization tensors that correspond to irreducible representations of SO(25). In the process we show that all covariant vertex operators can naturally be written in terms of elementary Schur polynomials.

In Sec. 5 we show that the construction of physical covariant coherent states becomes clear in the DDF formalism. We construct both open and closed coherent states. These fundamental string states are macroscopic and have a classical interpretation, in the sense that expectation values are non-trivially consistent with the classical equations of motion and constraints. We present an explicit map which relates three classically equivalent descriptions: arbitrary solutions to the equations of motion, the corresponding lightcone gauge coherent states, the corresponding covariant coherent states. We gain further evidence supporting this equivalence by showing that all spacetime components of the angular momenta in all three descriptions are identical. We suggest that these quantum states should be identified with fundamental cosmic strings.

In Sec. 6 we discuss the graviton emission amplitude for a coherent state. The particular coherent state that we will be interested in is also the simplest: a closed string coherent state with first harmonics excited. This computation includes the effects of gravitational backreaction which is always neglected in the classical computations, and which is also believed to be the missing link in understanding the small scale structure of cosmic strings. Depending on the choice of polarization tensor, this vertex operator can for instance represent a collapsed rotating double line (a folded rotating string), but also other configurations. Nevertheless, this computation will be somewhat incomplete because we do not compare the findings with the corresponding classical computation where backreaction is neglected [83], and this will have to await a future publication.

Finally, the Appendices contain extensive overviews of numerous computations, the knowledge of which is taken for granted in the main text. The purpose of these has been to make the thesis self-contained and to set the conventions that are used in the main text.

We restrict my attention to bosonic string theory and it is likely that all results generalize to the superstring. As long as one is able to isolate the tachyonic contribution this should not be too much of a drawback. We always have in mind the superstring when carrying out computations.

The majority of the new work that is presented in this thesis is in Sec. 4.4, 5 and 6.

Chapter 2

Cosmic Strings

In this chapter we discuss cosmic strings, to establish conventions and provide some necessary background. It has been known for many years now [147] that there exist classical field theories which allow for the formation of vortex lines whose equations of motion are governed approximately by the equations of motion of the Nambu-Goto string, see e.g. [5, 6]. The cosmological relevance of such objects was hypothesized just a few years later [41]. As discussed in the Introduction, the study of cosmic strings developed in parallel to the development of string theory, but there was initially only minimal interaction [1] between the two fields. Until the second superstring revolution twenty years later, it was thought that macroscopic fundamental strings would either not be realized in nature, or they would be unobservable [1]. Nevertheless, it has since become clear that macroscopic fundamental strings can be realized, in the sense that they appear to be produced quite naturally in string models with large extra dimensions and warped compactifications. Such objects lead to a number of observational signatures, and may hence provide a phenomenological handle on string theory.

Before giving an overview of these developments, we discuss how cosmic strings arise as topological defects in a simple field theory realization, in order to motivate and facilitate the discussion of the corresponding string theory realization of cosmic strings. We will then show how the field theory defects are related to classical string evolution, and discuss among other things the production of classical gravitational radiation. In the process we will discuss phenomenologically relevant features in the classical evolution of strings, such as cusps and kinks, which lead to very strong gravitational wave signals. The classical evolution however is expected to break down at small scales, where inherently stringy physics is expected to become relevant. Furthermore, in these classical computations the effects of gravitational backreaction are almost always neglected and this can be important for even order of magnitude estimates [120]. This will then motivate the following chapters, and will lead to the main theme of the current document, namely a perturbative string theory description of cosmic strings, the construction of the cosmic string vertex operators, that may possibly be thought of as a perturbatively (in the string coupling, $g_s = e^{\phi}$, and string length $\sqrt{\alpha'}$) exact description of cosmic strings.

2.1 Topological Defects in Classical Field Theory

Topological defects arise as topologically stable solutions to classical field equations in a variety of models with spontaneous symmetry breaking.

Consider a theory with a gauge symmetry group G, under which a multiplet of scalar fields ϕ transforms, $\phi(x) \to g(x)\phi(x)$ with $g(x) \in G$. We define the vacuum manifold, $\mathcal{M}$, by all possible vacuum expectation values, $\langle \phi \rangle = \phi_0$, where the associated potential $V(\phi)$ is minimized. When the symmetry is broken, $\phi \to \phi_0 \in \mathcal{M}$, there will generically remain an unbroken subgroup, H, namely the little group of G,

$$H = \{h \in G \, : \, h\phi_0 = \phi_0\},$$

and so the vacuum manifold is identified with the left coset of H in G, $\mathcal{M} = G/H$.

The type of defects that can exist in the broken phase depends on the topology of $\mathcal{M}$ and in particular on the homotopy groups $\pi_n(\mathcal{M})$, which arise from homotopically equivalent classes of mappings from the n-sphere into the vacuum manifold $\mathcal{M}$. Specifically, in a spacetime with D spatial dimensions, a non-trivial $(D-p-1)^{\text{th}}$ homotopy group,

$$\pi_{D-p-1}(\mathcal{M}) \neq 1,$$

is necessary for defects of dimension p to exist – that is, when $(D-p-1)$-spheres cannot be contracted to a point. Notice that it is the codimension of the defect that determines which homotopy group is the relevant one. However, this does not guarantee that the defects will be stable; one must also ensure that the energy functional has a minimum under scale transformations, $x \to \lambda x$ [148]. For instance, in $D = 3$ dimensions, domain walls ($p = 2$) can exist if $\pi_0(\mathcal{M}) \neq 1$. This will be the case when $\mathcal{M}$ is associated to the breaking of a discrete symmetry, that is if $\mathcal{M}$ has disconnected components. Similarly, strings ($p = 1$) can exist if $\pi_1(\mathcal{M}) \neq 1$. This will be the case if there exist loops, S^1, in the vacuum manifold that are not contractible to a point. In turn, monopoles ($p = 0$) can exist when $\pi_2(\mathcal{M}) \neq 1$, that is if there exist surfaces, S^2, that cannot be contracted to a point. The elements of $\pi_{D-p-1}(\mathcal{M})$, or more precisely the *conjugacy classes* of $\pi_{D-p-1}(\mathcal{M})$, in turn classify the admissible types of dimension-p defects [6].

Let us concentrate on linear topological defects in particular, namely strings. For example, if $\mathcal{M} \simeq S^1$, then the admissible types of strings are characterized by an integer, n, (an example is given below) given that $\pi_1(S^1) = \mathbb{Z}$. Strings can also form in models with a sequence of phase transitions,

$$G \to H_1 \to H_2,$$

where defects can exist if $\pi_1(G/H_1)$ and $\pi_1(H_1/H_2)$ are non-trivial, but will be topologically unstable for instance when $\pi_1(G/H_2) = 1$ [53].

In a cosmological setup, symmetries are expected to dynamically become broken during the expansion of the universe, in both string theory and field theory models of the early universe. In field theory language, as the universe expands, the temperature decreases and this leads to phase transitions. The higher symmetry phase is often associated to the high temperature phase of the model, and the symmetry breaking is identified with a phase transition. The production of topological defects is then described by the *Kibble mechanism* [41]. The universe starts off in the symmetric phase and as it expands and cools below some critical temperature, T_c, the symmetry is broken and the associated scalar field ϕ rolls to a point in the degenerate vacua of the theory. For example, in a scalar field theory with potential $\frac{1}{4}(|\phi|^2-\eta^2)^2$ the vacuum manifold is $\mathcal{M}\simeq S^1$ where $\phi=\eta e^{i\chi}$. Here $G=U(1)$ and $H=1$. The phase χ will be chosen randomly and in different regions of physical space it will (if the two regions are not causally related) take a different value. In a universe with Hubble parameter $H\sim 1/t$, causal processes can only occur within a sphere of radius H^{-1}. In other words, at largely separated distances the field will in general roll to different vacua, characterized by different phases, χ, so the symmetry breaking will be frustrated. Given that $\pi_1(S^1)\neq 1$, values of ϕ around some loop in space will generically form an incontractable loop in $\mathcal{M}$. Therefore, ϕ must leave the vacuum value ϕ_0 in the interior of the loop, and so one or more cosmic strings must have formed in the symmetry breaking.

In order to discuss the production of topological defects, we will consider the simplest scalar field theory with spontaneous symmetry breaking and a local U(1) symmetry, namely the Abelian-Higgs model, characterized by the following spacetime action [147],

$$S[\phi,A]=\int d^4x\sqrt{-G}\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+(\mathcal{D}_\mu\phi)^\dagger(\mathcal{D}^\mu\phi)-V(|\phi|)\Big\}, \tag{2.1}$$

where, focusing on the flat spacetime case, $G_{\mu\nu}=\eta_{\mu\nu}$, the covariant derivative reads $\mathcal{D}_\mu=\partial_\mu+ieA_\mu$ and the field strength $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$.[1] The quantity $\phi(x)$ is a complex scalar field and A_μ the gauge field. The local U(1) symmetry acts on the fields according to,

$$\phi(x)\to e^{i\Lambda(x)}\phi(x),\qquad A_\mu\to A_\mu-\frac{1}{e}\partial_\mu\Lambda(x), \tag{2.2}$$

with $\Lambda(x)$ a real single valued function. $V(|\phi|)$ is the so-called "Mexican hat" potential,

$$V(|\phi|)=\frac{1}{4}\lambda\big(|\phi|^2-\eta^2\big)^2, \tag{2.3}$$

with λ, η constants, related to the mass of the scalar field, and the mass of the vector boson (which is dynamically generated by the Higgs mechanism after spontaneous symmetry breaking). That vortices may be produced in the Abelian-Higgs model follows from the indirect fact that the vacuum manifold, $\mathcal{M}\simeq S^1=\{\phi\,:\,|\phi|^2=\eta^2\}$ is not simply connected, the associated non-trivial fundamental group being,

$$\pi_1(S^1)=\mathbb{Z}.$$

Nevertheless, let us also see this more directly.

[1] Of course, one should not confuse $G\equiv\det G_{\mu\nu}$ with the Newton's constant. When this is ambiguous we shall write G_d for the d-dimensional Newton's constant.

The equations of motion are,

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= 2e\mathrm{Im}\big[\phi(\mathcal{D}^\nu\phi)^\dagger\big],\\ \mathcal{D}^2\phi + \frac{1}{2}\lambda\big(|\phi|^2-\eta^2\big)\phi &= 0,\end{aligned}\tag{2.4}$$

and these define a conserved current $j^\nu \equiv 2e\mathrm{Im}\big[\phi(\mathcal{D}^\nu\phi)^\dagger\big]$.

The presence of strings can be detected by encircling them with a closed loop. The total spacetime flux through an area C that is bounded by a loop ∂C is given by,

$$\Phi = \int_C F = \oint_{\partial C} A,$$

with $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$ and $A = A_\mu dx^\mu$. Let us write $\phi = |\phi|e^{i\chi}$. From the equation of motion for A^μ it then follows that [147],

$$A^\nu = -\frac{1}{2e^2}\frac{j_\nu}{|\phi|^2} - \frac{1}{e}\partial^\nu\chi.$$

Let us consider the case when the boundary ∂C is at asymptotic infinity. At infinity we require the field ϕ to vanish and therefore, $j = 0$, along ∂C. Substituting A^ν into the above, and requiring that ϕ be single-valued as one traverses ∂C, leads to,

$$\Phi = -\frac{2\pi n}{e},$$

with $n \in \mathbb{Z}$. This reflects the fact that an integer number of strings can pass through the area C, either one of which is composed of an integer number of flux quanta, $2\pi/e$. In traversing the path in physical space, it is possible for the scalar field to wrap once around the circle of minima and develop nontrivial winding, $\Delta\chi = 2\pi$. If one now attempts to shrink the loop ∂C, the location of the vortices may be determined more accurately because the loop cannot be shrunk to a point if it contains vortices. At the vortex cores however, the phase is no longer well defined and the phase jump can only be resolved continuously if ϕ rises to the top of the potential where it vanishes, $\phi = 0$. The top of the potential however must be associated with a non-zero energy density, $\sim \frac{1}{4}\lambda\eta^4$, and this is in turn identified with the energy density of strings.

The stability of such a vortex with n flux quanta depends on the ratio

$$\beta \equiv (m_v^{-1}/m_s^{-1})^2,$$

(with m_v and m_s the masses of the vector and Higgs scalar bosons respectively) and in particular on the range of the two associated forces. If lines of magnetic flux approach to within the associated Compton wavelength, $m_v^{-1} \sim (\sqrt{\lambda}\eta)^{-1}$, they will repel each other. On the other hand, the scalar field produces an attractive force that becomes relevant at the associated Compton wavelength, $m_s^{-1} \sim (e\eta)^{-1}$ (because it is energetically favorable to minimize the area over which the energy density is non-zero). It is therefore clear that if $\beta > 1$ the n flux quanta will repel each other and the vortices will only be stable if they carry a single quantum of flux, $2\pi/e$. When on the other hand $\beta < 1$, vortices with arbitrary flux $2\pi n/e$ are stable. The integer n classifies the various possibilities, $n \in \pi_1(S^1) = \mathbb{Z}$.

To identify the above topological defects with strings, one needs to show that they are governed by a string action, which at least in a certain limit should coincide with the Nambu-Goto action,

$$S_N[X] = -\mu \int_\Sigma d^2\sigma \sqrt{-h},$$

with h the determinant of the induced metric, the metric induced by the embedding of the string into spacetime, see below. That is, the effective string action should be proportional to the area of the 2-dimensional worldsheet, Σ, swept out by the string motion. That this is approximately the correct action that reproduces the cosmic string dynamics associated to the Abelian-Higgs model can be seen as follows [6].

If the curvature radius, R, of the string is large, compared to the width of the string, δ, then an appropriate starting point is a Lorentz boosted version of static cylindrically symmetric solution to the equations of motion (2.4). With the gauge choice, $A_0 = 0$, they will be of the form [147],

$$\phi(\mathbf{r}) = e^{in\theta} f(r), \qquad \mathbf{A}(\mathbf{r}) = \frac{\mathbf{r} \times \hat{e}_z}{r} \alpha(r), \tag{2.5}$$

with $\hat{e}_z$ a unit vector along the z-direction, $\mathbf{r}$ a 3-vector, r the radial coordinate, and $|\mathbf{A}(\mathbf{r})| = \alpha(r)$. The appropriate boundary conditions (for a string of finite energy and regular at the origin) are $f(\infty) = \alpha(\infty) = \eta$ and $f(0) = \alpha(0) = 0$. The string thus lies along the z-axis and ϕ, A_i depend only on the polar coordinates (r, θ), with $r^2 = x^2 + y^2$. Suppose we parametrize the worldsheet by the coordinates (τ, σ), so that at a given point (τ, σ) there will exist two orthogonal vectors, $\partial_\alpha X^\mu$ with $\sigma^\alpha = (\tau, \sigma)$, that are tangent to the worldsheet; here $\partial_\alpha = \partial/\partial\sigma^\alpha$ and $\alpha = 0, 1$. In 4 spacetime directions there will then exist two vectors n^A_μ with $A = 1, 2$, that are orthogonal to the worldsheet, $n^A \cdot \partial_\alpha X = 0$, while satisfying $G^{\mu\nu} n^A_\mu n^B_\nu = \delta^{AB}$. Then, a point close to the worldsheet can be mapped into spacetime by,

$$x^\mu(\sigma^\alpha, \rho^A) = X^\mu(\sigma^\alpha) + \rho^A n^\mu_A(\sigma^\alpha),$$

with the second term measuring the transverse distance from the worldsheet, $r^2 = \rho \cdot \rho$.

We next rewrite the Abelian-Higgs model in terms of the new coordinates $x'^\mu = (\sigma^\alpha, \rho^A)$. To do so, we will require that these are single valued and well defined, which amounts to requiring that $\delta \ll |x - X| \ll R$. The Nambu-Goto approximation assumes that the string width, $\delta \to 0$. The measure, $d^4x\sqrt{-G}$ is invariant under diffeomorphisms,

$$\begin{aligned} d^4x\sqrt{-G(x)} &= d^2\sigma d^2\rho \sqrt{-G'(x')} \\ &= d^2\sigma d^2\rho \sqrt{-\det\Big(\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} G_{\rho\sigma}(x)\Big)} \\ &= d^2\sigma d^2\rho \sqrt{-\det\Big(\frac{\partial X^\rho}{\partial \sigma^\alpha}\frac{\partial X^\sigma}{\partial \sigma^\beta} G_{\rho\sigma}(X) + \mathcal{O}\left(\frac{r}{R}\right)\Big)}. \end{aligned} \tag{2.6}$$

We may then substitute the measure (2.6) and solutions (2.5) into the action $S[\phi, A]$. When $r/R \ll 1$ the integrand will only depend on the coordinates ρ^A, with the σ^α coordinate dependence being only through the above determinant. We may then integrate out ρ and arrive at the effective description,

$$S_{\text{effective}}[\phi, A] \simeq S_N[X].$$

The integral over ρ fixes the overall normalization of the Nambu-Goto action, which in turn determines the string tension μ in terms of the underlying field theory quantities. The metric,

$$h_{\alpha\beta}(\sigma) = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X),$$

that appears in the Nambu-Goto action is called the induced metric. It is via the Nambu-Goto action that the correspondence with string theory is normally made. This is because, as explained below, the action that defines string theory, namely the Polyakov action, and the Nambu-Goto action are equivalent classically; although their quantum equivalence is obscured by the highly non-linear dependence of $S_N[X]$ on X, which makes it difficult to quantize.

2.2 Branes, Tachyon Condensation, and Cosmic Strings

Up until the mid 1990's, it was thought that superstring theory is a theory of maps from a two-dimensional worldsheet into spacetime. The worldsheet can be thought of as a string, namely a fundamental string or F-string, that sweeps out a two-dimensional surface as it propagates in time. Five consistent such theories were known, all of which require the number of spacetime dimensions to equal ten – namely, the type I superstring, a theory of open and closed strings; and four closed string theories, namely the type IIA and IIB superstring, and the two heterotic string theories, differing in their gauge groups: $E_8 \times E_8$ and $SO(32)$.

The possibility that superstrings of cosmological size may have been produced in the early universe was first contemplated by Witten [1] who (based on current knowledge of the time) concluded that had they been produced they would either (i) not be observable (they would be produced before inflation and diluted away by the cosmological expansion), (ii) they would be unstable (they would disintegrate into smaller strings long before reaching cosmological scales in the case of Type I strings, or in the case of Heterotic String theory would arise as boundaries of domain walls whose tension would cause the strings to collapse), and in any case (iii) they would nevertheless be excluded by experimental constraints, requiring string tensions, $G\mu \sim 10^{-3}$, while it was clear that strings with $G\mu > 10^{-5}$ had already been ruled out.[2]

During the mid-1990's, a period that is referred to as the second superstring revolution, it was realized that these five seemingly different string theories were different manifestations of a single much larger theory, referred to as M-theory [2]. This conclusion was based

[2] G and μ are are defined on p. 2.

on the fact that various dualities were found to relate these five theories.[3] Furthermore, in addition to the fundamental strings, a number of higher dimensional objects were discovered, and in particular Dirichlet branes or simply D-branes [3, 4], as well as a number of additional string-like objects: namely, D-strings and higher dimensional D-, NS- and M-branes partially wrapped on compact cycles. Therefore, although fundamental strings of cosmological size were discarded as a plausible possibility [1], the second superstring revolution opened up a large window of opportunity for model building, and the possibility that cosmic superstrings may exist was revived. In what follows we shall attempt to offer a glimpse into these developments, while making contact with the results of the previous chapter that was based on a field theory realization of cosmic strings.

String theory, being a first quantized theory, is more akin to quantum mechanics than it is to quantum field theory. In particular, the spectrum of one-particle states arises from quantizing the vibrational modes of a single string. If we consider energy eigenstates, every such state is characterized by its energy and momentum, among other quantum numbers. Carrying out this quantization for the bosonic string leads to the realization that the theory contains a negative mass-squared state, namely a tachyon, as well as massless and an infinite tower of massive states. The presence of a tachyon was seen as one of the main motivations for going beyond the bosonic string, to consider the superstring. A negative mass square indicates an instability in the theory, and in particular of the vacuum around which the system has been quantized perturbatively. In the corresponding low energy field theory, this tachyonic excitation gets promoted to a field, say $T(x)$ with x the zero mode of the target space map, $X:\Sigma\to\mathcal{M}$, of the string into spacetime, and the interactions and couplings in the field theory are determined by computing n-point functions in the string theory from which the field theory descended. Alternatively, one may consider a sigma model associated to the graviton, $G_{\mu\nu}(X)$, dilaton, $\Phi(X)$ and tachyon, $T(X)$, among other fields,

$$\begin{aligned} S[X] = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\partial_z X^\mu \partial_{\bar z} X^\nu G_{\mu\nu}(X) \\ + \frac{1}{8\pi}\int_\Sigma d^2 z R_{(2)}\Phi(X) + \frac{1}{\pi\alpha'}\int_\Sigma d^2 z T(X), \end{aligned} \tag{2.7}$$

with $R_{(2)}$ the worldsheet Ricci scalar and $z,\bar z$ complex worldsheet coordinates, see Appendix B. The normalization of the bosonic fields X is such that the coefficient of the first term is identified with the string tension, $1/(2\pi\alpha')$, when the correspondence with the Nambu-Goto action is made, which is proportional to the area of the worldsheet, $S=\frac{1}{2\pi\alpha'}\text{Area}$. The normalization of the dilaton, $\Phi(X)$, is such that when $\Phi(X)=\Phi_0=\text{const}$, the coefficient is an integer and equal to the Euler characteristic, $\chi(\Sigma)=\frac{1}{8\pi}\int d^2zR_{(2)}$, of the Riemann surface. This choice in turn enables one to identify the quantity, $g_s=e^{\Phi_0}$,

[3]For example, if one compactifies the type IIA on $S^1\times\mathbb{R}^9$ it can be seen that type IIA at radius R is equivalent to type IIB compactified on $S^1\times\mathbb{R}^9$ but with radius α'/R (with $\sqrt{\alpha'}$ the string length) – such an equivalence under spatial inversion of radii is referred to as T-duality. There is also an S-duality which acts instead on the string coupling constant, $g_s\leftrightarrow 1/g_s$, and interchanges the type I superstring with heterotic SO(32) string theory. It was found that all five theories are related by T- and S-dualities of this sort.

with a coupling constant, the significance of which is described below (3.22). The normalization of the tachyon term is conventional. Here the requirement of conformal invariance[4] amounts to requiring that the β-functions vanish, see e.g. [149, 150],

$$\beta_G = \beta_\Phi = \beta_T = 0.$$

These conditions can be interpreted as equations of motion for the zero modes $G_{\mu\nu}(x)$, $\Phi(x)$ and $T(x)$ and so computing the effective action amounts to computing the action (which may not be unique) that gives rise to these equations of motion. In this manner, the tachyon effective action can in principle be computed, as can be the corresponding tachyon potential, say $V(T)$. Clearly, there will be a maximum at $T = 0$, where $V''(0) < 0$. This is somewhat akin to the maximum of the "Mexican hat" potential (2.3) that we encountered in the abelian Higgs model above. One might then wonder whether there exists a stable minimum in the vicinity of $T = 0$, just like in the abelian Higgs potential. Unfortunately, the answer to this question is not known; the tachyon couples to the infinite tower of string modes making it difficult to study the tachyon in isolation, and the $|\text{mass}^2|$ of the tachyon is of the same order of magnitude as that of the massive string modes, and so one cannot integrate out all the heavy fields and work with a *low energy* effective action in the usual manner (one would have to also include massive modes in the effective action). Many take the viewpoint that bosonic string theory is inconsistent because of the tachyon. Thankfully, there exist a number of closed superstring theories which are free of tachyonic excitations where perturbation theory can be carried out consistently.

In addition to closed strings, there also exist superstring theories with open strings, where the string endpoints are defined to live on hypersurfaces referred to as D(irichlet) p-branes. These are p-dimensional extended soliton-like objects, and in a relativistic theory one would hope that they are also dynamical. This is indeed the case [151] with the open string vertex operators, for instance,

$$\oint \partial_n X^\mu A_\mu(X),$$

being interpreted as fluctuations in the shape of the D-brane (n is normal to the worldsheet boundary, the integral is over the woldsheet boundary, and $A_\mu(X)$ is a gauge field). In the presence of N branes of possibly different dimensionality, the gauge field $A_\mu(X)$ can take different values on each of these surfaces. Open strings are oriented and so there can thus exist N^2 different types of open string states, corresponding to the number of ways of attaching the two open string endpoints to N surfaces.

Although superstring vacua (around which a perturbative theory exists) are tachyon-free, there are certain cases when the spectrum of open strings does contain a tachyon, in superstring as well as bosonic string theory. In bosonic string theory there is always a tachyon in the spectrum for any p-dimensional D-brane, whereas in type IIA/IIB superstring theory a tachyonic excitation exists only for odd/even dimensionality p. On the other hand,

[4]Conformal invariance is of paramount importance in string theory. Conformal symmetry is carefully discussed in Appendix D and conformal field theories are discussed in Sec. 3.1.

Dp-branes with $p = 0, 2, 4, 6, 8$ in the type IIA superstring and $p = -1, 1, 3, 5, 7, 9$ type IIB superstring break half of the 32 supersymmetries – they are therefore BPS states [2] and carry conserved Ramond-Ramond (RR) charges [3].[5] That is, the worldvolume of a Dp-brane naturally couples to a $(p+1)$-form RR potential,

$$\mu_p \int C_{p+1},$$

(the integral is over the Dp-brane worldvolume) where μ_p is the conserved Dp-brane electric charge, $\int_{S^{10-p-2}} *F_{p+2}$, and $F_{p+2} = dC_{(p+1)}$ is the associated field strength.[6] The presence of a conserved charge ensures that Dp-branes are stable with only positive or vanishing $(\text{mass})^2$ states appearing in the open string spectrum. The BPS D-branes of either type IIA/IIB theory have mass per unit p-volume, or tension, given by [2],

$$\mu_{D_p} = (2\pi)^{-p} \alpha'^{-(1+p)/2} g_s^{-1}, \tag{2.8}$$

when the RR scalar, C, is set to zero; g_s the closed string coupling constant and $\sqrt{\alpha'}$ the single dimensionful constant of the theory, the string length. D-branes therefore become very heavy in the perturbative limit where $g_s \to 0$, in which case they can be treated as rigid objects, and one need only consider the dynamics of the fundamental strings perturbatively. These D-branes are oriented, and a BPS Dp-brane of opposite orientation is referred to as an anti-Dp-brane, or $\overline{\mathrm{D}}p$-brane. That is, the open string endpoints carry a multiplet of charges transforming under representations Λ, $\bar{\Lambda}$, of a compact Lie group G; these are non-dynamical and are known as Chan-Paton degrees of freedom [2]. When $\Lambda \otimes \bar{\Lambda}$ is in the adjoint representation of G, the lowest energy open string excitations are precisely those of Yang-Mills gauge theory. For N oriented coincident Dp-branes, $\Lambda^* = \bar{\Lambda}$, and the gauge group is $G = \mathrm{U}(N)$.[7]

Although a BPS Dp-brane does not have a tachyonic excitation, a string connecting a Dp-brane–$\overline{\mathrm{D}}p$-brane does have a tachyonic mode when the branes are coincident [152], when the length of the open strings is zero. The tachyon is projected out of the ground state in the Neveu-Schwarz (NS) sector [2] by the GSO [153] projection when both open string endpoints lie on either a Dp-brane or a $\overline{\mathrm{D}}p$-brane, in which case the lowest mass excitations are the massless states.[8] However, for a coincident Dp-brane–$\overline{\mathrm{D}}p$-brane system, the GSO projection is opposite and so the NS ground state remains in the spectrum, giving rise to a tachyonic mode with,

$$(\text{mass})^2 = -1/(2\alpha').$$

[5]The case $p = -1$ is special and corresponds to a D-instanton. These may be thought of as point-like defects that appear at an instant in time.

[6]Note that $n = D - p - 2$ is the dimensionality of the n-sphere that is required to completely surround a Dp-brane in a D-dimensional spacetime.

[7]There also exist un-oriented strings for which $\bar{\Lambda} = \Lambda$ in which case $G = \mathrm{SO}(N)$ or $\mathrm{Sp}(N)$.

[8]The GSO projection in the path integral language corresponds to a sum over all spin structures [154] with certain weights that are determined from modular invariance (invariance under large diffeomorphisms on the worldsheet).

According to the above counting, for oriented open strings there will in fact be two tachyonic modes (corresponding to the two orientations of the open string), which can be combined into a single complex field $T(x)$. There was a $\mathrm{U}(N)$ symmetry that was associated to a stack of N BPS Dp-branes; in the case of $N+M$ coincident Dp-branes and N coincident $\overline{\mathrm{D}}p$-branes the $\mathrm{U}(N)$ gets promoted to a $\mathrm{U}(N+M)\times\mathrm{U}(N)$ gauge symmetry. The tachyon field T lives in the $(N+M,\overline{N})$ (bi-fundamental) representation of the gauge group.[9]

Given that there exists an instability associated to a stack $N+M$ BPS Dp-branes coincident with a stack of N BPS $\overline{\mathrm{D}}p$-branes, and that this instability is associated to a complex tachyon field with a potential in the effective theory such that $V^{''}(0)<0$ at $T=0$, one may wonder whether there exists a stable minimum away from $T=0$. Let us take the simplest case where $N=1$ and $M=0$, which is relevant for a Dp-$\overline{\mathrm{D}}p$-brane pair. Here the massless degrees of freedom are comprised of two $\mathrm{U}(1)$ gauge fields and $2(9-p)$ transverse scalars which are associated to the transverse coordinates of the branes. We consider the effective action, $S_{\text{eff}}(T,\dots)$, which is formally obtained by integrating out the positive (mass)^2 fields, and "..." denote the massless bosonic fields; when the massless fields have been set to zero, the proposed action is of the form [40],

$$S_{\text{eff}}(T)=-\int d^{p+1}xV(T)\sqrt{1+\eta^{\mu\nu}\partial_\mu T\partial_\nu T}$$

The potential $V(T)$ will have a maximum at $T=0$ as appropriate for a tachyonic mode. Let us choose the additive constant in $V(T)$ such that $V(0)$=0. $V(T)$ is found that the effective action has a phase symmetry, $T\to e^{i\alpha}T$, and a family of global minima [40],

$$T=T_0\,e^{i\alpha}.$$

At these minima the sum of the tensions of the original Dp-brane–$\overline{\mathrm{D}}p$-brane pair is exactly cancelled by the negative contribution of the potential [40],

$$V(T_0)+2\mu_{D_p}=0.$$

This implies that the total energy density at the minimum of the potential vanishes; and since the Dp-brane–$\overline{\mathrm{D}}p$-brane pair does not carry any RR charge this should be identified with the vacuum, where there is no D-brane and hence no physical open string excitations. Nevertheless, the equations of motion derived from the effective action do have non-trivial

[9] Dp-branes with $p=-1,1,3,5,7,9$ in the type IIA superstring and $p=0,2,4,6,8$ type IIB superstring also exist, but they have a tachyonic mode and are therefore unstable. These are non-BPS Dp-branes, and can be constructed [155] from an orbifold of either type IIA/IIB theory, the discrete group being generated by the spacetime fermion number associated to left-moving degrees of freedom on the string, $(-)^{F_L}$.[10] Non-BPS Dp-branes have a mass per unit p-volume given by, $\tilde{\mu}_{D_p}=\sqrt{2}(2\pi)^{-p}\alpha'^{-(1+p)/2}g_s^{-1}$, and are unoriented; that is, there is only one way of connecting an open string to a non-BPS Dp-brane and $\Lambda=\bar{\Lambda}$. Therefore, the corresponding tachyon field, $T(x)$, is now real, and can be identified with a particular linear combination of the two tachyons of the Dp-brane–$\overline{\mathrm{D}}p$-brane system that survived the orbifold projection. There is still an infinite tower of massive open string states. Note furthermore that non-BPS Dp-branes are uncharged under the RR $(p+1)$-form potential C_{p+1}.

time-independent classical solutions, and it has been conjectured [42], and subsequently verified [156], that these should be identified with D-branes of lower dimension, and in particular a codimension-two soliton; that is,

$$\text{BPS D}p\text{−brane} - \text{BPS }\overline{\text{D}}p\text{−brane} \quad \rightarrow \quad \text{BPS D}(p-2) \text{ or } \overline{\text{D}}(p-2)\text{−branes.} \tag{2.9}$$

Note that one would expect $\mathrm{D}(p-2)$ and $\overline{\mathrm{D}}(p-2)$ branes to be produced in approximately equal numbers (given that one differs from the other by an overall rotation of 180^o). The same remark holds for the more general cases below. Such processes are referred to as *tachyon condensation* [40]. Since the tachyon is a complex field, it can wind around a codimension-two locus of the potential and non-zero winding will lead to lower dimensional D-branes, just as we described for the abelian Higgs model in Sec. 2.1. A simple example of a solution to the equations of motion is a vortex solution, where T is only allowed to depend on x^{p-1} and x^p say,

$$T(\rho,\theta) = T_0 f(\rho)\, e^{i\theta},$$

where, say $\rho^2 = (x^{p-1})^2 + (x^p)^2$, and $\theta = \arctan(x^p/x^{p-1})$, and the $f(\rho)$ is such that

$$f(\infty) = 1, \qquad \text{and} \qquad f(0) = 0.$$

The potential energy vanishes at infinity, $\rho \rightarrow \infty$, and the accompanying gauge field enforces the covariant derivative of the tachyon to decrease sufficiently rapidly that most of the energy density is concentrated around the $\rho = 0$ region. Clearly, when $p = 3$ the D3-$\overline{\mathrm{D}}$3-brane pair annihilates into a D1-brane, or equivalently a D-string. This is just a simple example of a more general principle, that D-brane annihilation can give rise to lower dimensional D-branes, and in particular macroscopic D-strings.

More generally, recall that on N coincident BPS Dp-branes and N coincident BPS $\overline{\mathrm{D}}p$-branes there is a $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge symmetry. This system annihilates into a stack of BPS $\mathrm{D}(p-2k)$-branes (with $k \in \mathbb{Z}^+$) (or a BPS $\overline{\mathrm{D}}(p-2k)$-brane),

$$N \ \text{D}p - \overline{\text{D}}p\text{−brane pairs} \quad \rightarrow \quad \text{BPS D}(p-2k)\text{−brane}, \qquad \mathrm{k} \leq \mathrm{N} \tag{2.10}$$

The resulting $\mathrm{D}(p-2k)$-branes will carry a $\mathrm{U}(N)$ gauge symmetry, and so in particular when the branes annihilate the $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge symmetry becomes broken,

$$\mathrm{U}(N) \times \mathrm{U}(N) \rightarrow \mathrm{U}(N),$$

and the tachyon develops an expectation value. The vacuum manifold of the resulting configuration is thus,

$$\mathcal{M} = \frac{\mathrm{U}(N) \times \mathrm{U}(N)}{\mathrm{U}(N)} \simeq \mathrm{U}(N).$$

Therefore, according to the general discussion of Sec. 2.1, defects of codimension $2k$ are supported, as advertised in (2.10),

$$\pi_{p-(p-2k)-1}(\mathrm{U}(N)) = \mathbb{Z}, \qquad \text{for} \qquad k \leq N.$$

The first subscript in the homotopy group indicates the dimension in which the condensation takes place, namely p, and the second indicates the dimensionality of the defect that is supported, namely $p - 2k$.

As an example, consider D5-$\overline{\text{D}}$5-brane annihilation where $N = 1$, with the D5 and $\overline{\text{D}}$5-brane wrapping a 2-cycle and the remaining 3 dimensions large and non-compact. According to the above this will lead to the formation of a D3-brane. This may either wrap a 2-cycle and be extended in one non-compact dimension (in which case it should be identified with a macroscopic string); it may wrap a 1-cycle inside the 2-cycle (in which case it would be identified with a domain wall); or, it may be extended in all three non-compact directions.

We would like to identify extended one-dimensional objects with cosmic strings and to do so one needs to take a number of steps. Primarily, one needs to incorporate D-brane annihilation into a cosmological model. Secondly, it must be checked that the Kibble mechanism applies and that cosmic strings can be produced in the early universe. One must also make sure that other defects, such as domain walls or monopoles are either suppressed or not produced, as these would over-close the universe. One then needs to determine whether the resulting strings are stable [16], and what the corresponding cosmological signatures are [21]. Finally, one should check that their presence has not been excluded by experimental constraints.

Thankfully, there exists a very natural implementation of D-brane-$\overline{\text{D}}$-brane annihilation in such a context, and this is referred to as *brane inflation* [25, 26, 27, 28, 29, 13]. Brane inflation models came out of attempts to embed inflation into string theory. Inflation in turn is a desirable feature of a fundamental theory, because it offers a natural explanation for the homogeneity, the isotropy of the universe, and the observed spectrum of density perturbations, see e.g. [157].

A rather natural initial condition for the early universe is to start off with a multitude, or a gas, of D-branes of various dimensionalities. The branes of higher dimensionality will annihilate first and produce lower dimensional branes and branes that are present today. In the most concrete (almost viable) scenario, namely the $\mathbb{KLMT}$ scenario [35], one studies the relative motion of a remaining D3-brane and $\overline{D}$3-brane, which are initially separated by a distance r in the transverse space separating the branes in a throat of a Calabi-Yau (CY) three-fold.[11] The picture we have in mind is that we have compactified type IIB superstring theory on a CY manifold in the presence of flux. The key point here is that the flux induces an inflationary warped throat where the motion of the aforementioned D-branes occurs. The attraction of the two branes will inflate away any other remaining lower dimensional branes, such as domain walls or monopoles. Furthermore, the standard model region on the CY where the standard model particles live, is sufficiently separated from the inflationary throat, so that the D3-$\overline{D}$3-brane annihilation does not interfere with the standard model processes (although the manner in which the moduli stabilization

[11] This is a compactification of the 6 extra dimensions which has certain desirable features, such as $N = 1$ supersymmetry and a potentially realistic gauge group and fermion spectrum.

and standard model fields are introduced does affect the nature/existence of the resulting cosmic strings [16]). All moduli are stabilized. The spacetime metric is of the form,

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}g_{mn}dy^m dy^n,$$

with y^i the coordinates on the compactification manifold with g_{mn} the associated CY metric, $\eta_{\mu\nu}$ a Minkowski metric. That is, the four-dimensional spacetime is scaled by a factor depending on the position in the internal space. The warping, $e^{2A(y)}$, is induced by the three-form fluxes F_3, H_3 [35]. For the D3-$\overline{D}$3 system, supersymmetry is broken and there is a net attractive force due to gravity and RR fields. If the D3-brane is initially located at r_1 and the $\overline{\mathrm{D}}$3-brane is located at r_0, then annihilation will occur in the region of large gravitational redshift where $\min e^{2A(y)} = e^{2A(r_0)} \ll 1$, if $e^{2A(y)}$ is normalized to be $\mathcal{O}(1)$ in the bulk of the CY. This large gravitational redshift has the effect of bringing both the string scale and the inflationary scale, as measured by a ten-dimensional observer, close to the four-dimensional Planck scale, whereas the energy scales as measured by a four-dimensional observer are suppressed by a factor of $e^{A(r_0)}$ [21].

The D3-$\overline{\mathrm{D}}$3-brane pair will eventually annihilate when the distance r becomes of order the string scale and according to the above discussion a tachyonic instability will develop, and defects or D-branes and in particular D-strings will be produced.[12] In particular, the tachyon will acquire a vacuum expectation value, T_0, and since the vacuum manifold is non-trivial it will randomly take different values at different regions in space, especially when these regions are separated by distances greater than the Hubble horizon, H^{-1}. Therefore, the Kibble mechanism applies and defects will be produced. It is important to notice that the Kibble mechanism cannot operate in compactified directions, and in particular the codimension must lie in the uncompactified dimensions. Although we examined the case of D3-$\overline{\mathrm{D}}$3-brane annihilation, there are clearly also more general possibilities. From the above we know that the codimension in type IIB string theory is always even, $2k$, (otherwise the defects are unstable and decay rapidly). Therefore, given that there must be three remaining large non-compact directions (which are to be associated with our perceived spacetime dimensions), only D-branes of codimension-2, i.e. cosmic strings, will be produced by the Kibble mechanism [13, 14]. In particular, the production of monopoles, or domain walls will be heavily suppressed.

During tachyon condensation the open string F-states on the annihilating D-branes are expected to become resonantly excited, and so become large and macroscopic [44]. These can subsequently decay into closed strings or get squeezed into a network of flux tubes, and so we might also expect to find a network of cosmic F-strings as well as a network of D-strings at the end of the phase transition [44, 17]. Furthermore, although F-strings do not have a classical description, they are related by $\mathrm{SL}(2,\mathbb{Z})$ duality to D-strings, and so in the dual picture correspond to topological defects and so must be produced in the same manner [16]. It has also been argued [16] that although only one of the two pictures

[12] This is somewhat akin to the familiar hybrid inflationary models where inflation ends when an instability develops where a mode becomes tachyonic [158].

can apply at any one time, the Kibble argument depends only on causality and so should be valid for both D- and F-strings.

Given the existence of both D- and F-strings, there will also exist (p,q)-strings [2], namely bound states of p F-strings and q D-strings [159], with an associated mass per unit length, or tension,

$$\mu_{p,q} = \frac{1}{2\pi\alpha'}\sqrt{(p-Cq)^2 + g_s^{-2}q^2},$$

where C is the RR scalar and $g_s = e^{\Phi}$ the closed string coupling evaluated at the location of the string [16]. As one would expect, the above result reduces to $\mu_{F_1} = 1/(2\pi\alpha')$ and μ_{D_1} in (2.8) when respectively $(p,q) = (1,0)$ and $(p,q) = (0,1)$ and we set $C = 0$. When strings of type (p,q) and (p',q') meet, they can form a new string of type $(p \pm p', q \pm q')$. Furthermore, from (2.8), the triangle inequality, and the fact that the F-string tension is $\mu_{F_1} = 1/(2\pi\alpha')$, we learn that,

$$\mu_{p,q} \leq p\mu_{F_1} + q\mu_{D_1},$$

as one would expect for a bound state. (p,q)-strings can also form junctions, at which there is the charge conservation condition, $\sum p_i = 0$ and $\sum q_i = 0$, giving rise to a wide range of possibilities for cosmic superstring networks [61, 62, 63, 64, 65, 66].

2.3 Classical String Evolution

The fundamental or F-strings that define the string theory perturbatively are inherently quantum-mechanical and do not have a classical definition. There is a heuristic definition however, and in a later chapter we shall develop and propose a path integral definition of classicality. The heuristic definition of classicality is to study the action and its associated equations of motion and constraints without reverting to path integral computations. There are a number of terms in the string theory action that are not defined in this sense, because they break conformal invariance, which is of course restored at the quantum level. In order to make progress in this section we will therefore have to drop these terms.

Let Σ be a Riemann surface of genus h, with b boundaries and n points at which local functionals (equivalently operators/insertions) are inserted, $V(z_i, \bar{z}_i)$ with $i = 1, \ldots, n$, which are to represent asymptotic states. The theory is defined perturbatively by a path integral, a sum over inequivalent embeddings of Riemann surfaces, Σ, into spacetime, $\mathcal{M}$,

$$X : \Sigma \to \mathcal{M},$$

and a sum over all metrics on Σ. Denote the space of inequivalent metrics (the moduli space) that leave the vertex insertion points invariant by $\mathcal{M}_{h,b}$ and the space of embeddings of the worldsheet into spacetime by $\mathcal{E}_{h,b}$. Schematically, a definition of string theory is then provided by a path integral or scattering amplitude of the form [141], $\langle V^{(1)} \ldots V^{(n)}\rangle = \sum_{h,b=0}^{\infty} \int_{\mathcal{M}_{h,b}\otimes\mathcal{E}_{h,b}} \mathcal{D}g\,\mathcal{D}X\, e^{\frac{i}{\hbar}S[X,g]} V^{(1)} \ldots V^{(n)}$. The finite dimensional measure associated to integrals over the location of the n vertex insertions, is implicitly contained in $V^{(1)} \ldots V^{(n)}$. That all information about asymptotic states can be shrunk to local points on the worldsheet, which in turn allows for the definition of the local functionals $V(z,\bar{z})$, is a manifestation of conformal invariance. The sum over h, b reflects the perturbative nature of the above definition; in Sec. 3.4 and Appendix F we will make these statements more precise.

In the case of bosonic string theory, the action $S[X,g]$ is the unique Poincaré and diffeomorphism invariant action (on the worldsheet and in spacetime) with a conformal symmetry on the worldsheet. There are a number of terms that comprise the action, see Appendix A, depending on the particular set-up of interest.[13] The term that will be of prime interest throughout is the Polyakov action,

$$S_G[X,g] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-g}\,\nabla_\alpha X_\mu(\sigma)\nabla^\alpha X^\mu(\sigma). \tag{2.11}$$

Two metrics appear here – the indices $\alpha = 0,1$, associated to the coordinates[14] $\sigma^\alpha = (\tau_{\rm M},\sigma)$, are contracted with the worldsheet metric $g_{\alpha\beta}$, and the indices $\mu = 0,\dots,26$, associated to the spacetime coordinates $X^\mu(\sigma)$, are contracted using the spacetime metric $G_{\mu\nu}(X)$. Note that $d^2\sigma \equiv d\sigma\wedge d\tau_{\rm M}$. The $X^\mu(\sigma)$ are scalars from the worldsheet point of view and so the covariant derivative $\nabla_\alpha X = \partial_\alpha X$.

A worldsheet and spacetime interval between two neighboring points respectively read,

$$ds_g^2 = g_{\alpha\beta}(\sigma)d\sigma^\alpha d\sigma^\beta, \qquad \text{and} \qquad ds_G^2 = G_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu d\sigma^\alpha d\sigma^\beta,$$

which leads us to identify an induced metric, $h_{\alpha\beta}$,

$$h_{\alpha\beta}(\sigma) = \partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X),$$

i.e. the metric induced by the embedding of the string into spacetime. This can be used to construct the diffeomorphism invariant (in both a worldsheet and spacetime sense) quantity $\sqrt{-h}\,d^2\sigma$, with $h = \det h_{\alpha\beta}$. It is also conformally invariant with respect to $g_{\alpha\beta}$, and so it satisfies all symmetries of bosonic string theory (see Appendix A); of course, it is not conformally invariant with respect to either $h_{\alpha\beta}$ or $G_{\mu\nu}$. Physically, the quantity $\sqrt{-h}\,d^2\sigma$ has the interpretation of an invariant area from the spacetime point of view. This suggests an action principle,

$$S_N[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}, \tag{2.12}$$

whose classical trajectories, $\delta S_N = 0$, (with appropriate boundary conditions) characterize oscillations or perturbations around a surface of minimum area. This 2-dimensional surface represents a string that sweeps out a 2-dimensional surface in spacetime. The quantity S_N is known as the Nambu-Goto action. There are two notable differences between S_G

[13] We will ultimately make the choice of background (A.10). As described in the Appendix both S_Φ and S_μ break conformal invariance at the classical level and so (given that we wish to study the theory at the classical level in this section) we will have to drop these two contributions and concentrate on the Polyakov contribution, S_G, in the current chapter.

[14] The subscript on $\tau_{\rm M}$ is meant to signify that we are on a Minkowski worldsheet, the corresponding Euclidean worldsheet being reached by the replacement, $\tau \equiv \tau_{\rm E} = i\tau_{\rm M}$.

and S_N: (a) S_G is a functional of X and g whereas S_N is constructed solely from X; (b) S_G is quadratic in X, whereas S_N is highly non-linear in X. The two actions are nevertheless equivalent classically but their quantum equivalence (if at all present) has remained obscure, due to the highly non-linear dependence of S_N on X. That they are equivalent classically follows from the constraint associated to the variation of S_G with respect to $g_{\alpha\beta}$ which gives Einstein's equations. Since the Einstein tensor vanishes in 2 dimensions,

$$T_{\alpha\beta} = -\frac{1}{\alpha'}\Big(\partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}\,g_{\alpha\beta}\partial_\gamma X\cdot\partial^\gamma X\Big) = 0, \tag{2.13}$$

with (the dimensionless quantity) $T_{\alpha\beta} \equiv -\frac{4\pi}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}}$ the energy momentum tensor on the worldsheet. $T_{\alpha\beta}$ is covariantly conserved and traceless in 2 dimensions,

$$\nabla^\alpha T_{\alpha\beta} = 0, \qquad \text{and} \qquad T^\alpha_{\ \ \alpha} = 0. \tag{2.14}$$

From $T_{\alpha\beta} = 0$ it follows that $h_{\alpha\beta} = \frac{1}{2}\,g_{\alpha\beta}g^{\gamma\delta}h_{\gamma\delta}$. Taking the determinant of both sides and solving for $g^{\gamma\delta}h_{\gamma\delta}$ enables us to rewrite this constraint as, $\frac{h_{\alpha\beta}}{\sqrt{-h}} = \frac{g_{\alpha\beta}}{\sqrt{-g}}$. Substituting this into S_G gives S_N, thus proving that they are classically equivalent. We did not use the equations of motion for X to prove this equivalence; these may be derived from either action and are given by,

$$\Delta_g X^\sigma - \Gamma^\sigma_{\ \nu\rho}g^{\alpha\beta}\partial_\alpha X^\nu\partial_\beta X^\rho = 0, \tag{2.15}$$

with $\Gamma^\sigma_{\ \nu\rho}$ the Christoffel symbols associated to $G_{\mu\nu}$ and $\Delta_g X \equiv -\frac{1}{\sqrt{-g}}\partial_\alpha\big(\sqrt{-g}g^{\alpha\beta}\partial_\beta X\big)$. Note that $\Delta_g = \Delta_{(0)}$, see Appendix B, where the subscript indicates that it acts on worldsheet scalars. Note that we could just as well have written,

$$\Delta_h X^\sigma - \Gamma^\sigma_{\ \nu\rho}h^{\alpha\beta}\partial_\alpha X^\nu\partial_\beta X^\rho = 0, \tag{2.16}$$

with $\Delta_h X \equiv -\frac{1}{\sqrt{-h}}\partial_\alpha\big(\sqrt{-h}h^{\alpha\beta}\partial_\beta X\big)$, given that $h_{\alpha\beta}$ and $g_{\alpha\beta}$ differ by a conformal factor which drops out of the equations of motion – classically, the induced and worldsheet metrics are indistinguishable. This equivalence is broken by quantum effects, the phenomenological implications of which (in the context of cosmic strings) are not yet understood.

2.4 Gravitational Backreaction

In this section we present a brief overview of the set of equations that need to be solved in order to determine the evolution of a macroscopic string when the effect of gravitational backreaction is included, following [120]. Here the crucial observation is that the string itself is a source of the gravitational background in which it propagates, and which in turn determines its motion. More generally, gravitational (and more general) backreaction is the effect due to which the cosmic string decay products (radiated gravitons or massive states) in turn affect the motion of (or *backreact* on) the radiating cosmic string. It is due to backreaction that a radiating cosmic string can never truly exhibit periodic motion. Even though backreaction is neglected in classical computations, in quantum computations the amplitudes associated to cosmic string decay vanish unless backreaction is taken into account. That is, the amplitude for the process

$$V_{\rm cs} \to V'_{\rm cs} + \text{decay products},$$

with $V_{\rm cs}$ and $V'_{\rm cs}$ the initial and final cosmic sting states respectively, vanishes unless:

$$V_{\rm cs} \neq V'_{\rm cs}.$$

We define backreaction to be the effect due to which $V_{\rm cs} \neq V'_{\rm cs}$.

It is possible that gravitational backreaction can significantly affect the evolution of string, especially close to singular points such as cusps and kinks. In particular, although cusps seem to survive gravitational backreaction, they are weakened. It would be very interesting to determine what happens in string theory when quantum fluctuations are of order the string width and loops may be chopped off (this is further discussed in Sec. 2.8). Non-selfintersecting trajectories with only a few modes excited remain so. Furthermore, and most importantly, it has been suggested [160, 120, 85] that gravitational backreaction sets the scale for the smallest relevant structures in cosmic string evolution, as well as the long sought-after loop production scale – see Sec. 2.8 for further discussion of this important topic. It is therefore of vital importance to understand gravitational backreaction and develop the necessary tools where such questions can be addressed most naturally. In the present section we look at this problem from a classical viewpoint, and it will become clear that this is computationally a highly non-trivial task. This in turn is the reason as to why it is always neglected in the computations of gravitational radiation from cosmic strings [82, 83, 84, 85, 75, 76, 86, 87, 71, 88]. We will argue that in the corresponding quantum computation the effects of gravitational backreaction can be included very naturally, and possibly more easily. This is already seen in a number of massive string decay computations [161, 54, 55, 56, 162, 58, 59, 60] which have been carried out, although it is likely that the vertex operators that characterize the states in these references do not resemble macroscopic classical strings and so their identification with cosmic strings is obscure.

The spacetime energy-momentum tensor associated to the Nambu-Goto action (2.12) is,

$$\begin{aligned} T^{\mu\nu}(x) = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-h}\,\delta^d(x^\mu - X^\mu(\sigma,\tau))\Big[\big(\partial_\sigma X\big)^2\,\partial_\tau X^\mu\partial_\tau X^\nu \\ + \big(\partial_\tau X\big)^2\,\partial_\sigma X^\mu\partial_\sigma X^\nu - \big(\partial_\tau X\cdot\partial_\sigma X\big)\big(\partial_\tau X^\mu\partial_\sigma X^\nu + \partial_\sigma X^\mu\partial_\tau X^\nu\big)\Big], \end{aligned}$$

the contractions being taken with respect to the spacetime metric $G_{\mu\nu}(x)$. Define the lightcone coordinates, $u = \tau + \sigma$, $v = \tau - \sigma$, in terms of which the constraints read, $\big(\partial_u X\big)^2 = 0$, and $\big(\partial_v X\big)^2 = 0$. In worldsheet lightcone coordinates the energy-momentum tensor reduces to,

$$T^{\mu\nu}(x) = \frac{1}{2\pi\alpha'}\int dudv\,\delta^d(x^\mu - X^\mu(u,v))\big(\partial_u X^\mu\partial_v X^\nu + \partial_v X^\mu\partial_u X^\nu\big), \tag{2.17}$$

whereas the equations of motion (2.16) take the form,

$$\partial_u\partial_v X^\mu + \Gamma^\mu_{\nu\rho}\partial_u X^\nu\partial_v X^\rho = 0; \tag{2.18}$$

these preserve [120] the above gauge choice, $\partial_v(\partial_u X)^2 = \partial_u(\partial_v X)^2 = 0$.

The first simplification is to study the backreaction problem perturbatively. That is, we expand the background metric around a flat Minkowski manifold as $G_{\mu\nu}(x) \simeq \eta_{\mu\nu} + h_{\mu\nu}(x)$, such that $|h_{\mu\nu}(x)| \ll 1$. Then, the equations of motion reduce to linear order in $h_{\mu\nu}(x)$ to,

$$\partial_u\partial_v X^\mu + \frac{1}{2}\eta^{\mu\rho}\big(\partial_\sigma h_{\nu\rho} + \partial_\nu h_{\sigma\rho} - \partial_\rho h_{\nu\sigma}\big)(x)\partial_u X^\nu \partial_v X^\rho = 0, \tag{2.19}$$

and the Einstein equations determine the dynamics of $h_{\mu\nu}(x)$, which in the linearized approximation read:

$$\partial^2\Big(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\Big)(x) = -16\pi G_d\, T_{\mu\nu}(x). \tag{2.20}$$

We have chosen a harmonic gauge where, $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h^\mu{}_\mu$ and G_d is the d-dimensional Newton's constant, so that $S = \frac{1}{16\pi G_d}\int d^d x\sqrt{G}R + \dots$. One can invert the Einstein equations using the retarded Green's function technique. This leads to solutions of the form,

$$\begin{aligned} h_{\mu\nu}(x) = \frac{8G_d}{2\pi\alpha'}\int dudv\,\theta(x^0 - X^0)\delta^{d-1}\big((\mathbf{x}-\mathbf{X})^2\big) \\ \times\big(\partial_u X_\mu \partial_v X_\nu + \partial_v X_\mu \partial_u X_\nu - \eta_{\mu\nu}\partial_u X^\rho \partial_v X_\rho\big). \end{aligned} \tag{2.21}$$

If we assume an appropriate compactification of $d-4$ of the spacetime dimensions with volume of compactification V_{d-4}, then (from the associated dimensional reduction of the low energy effective theory) it follows that $G_d = G_4 V_{d-4}$ (and note that $[G_d] = L^{d-2}$). The parameter $G_4\mu \equiv \frac{G_4}{2\pi\alpha'}$ is dimensionless and is therefore the appropriate small parameter that one Taylor expands in. As noted in the introduction typical string models imply that the relevant range is $10^{-12} \lesssim G_4\mu \lesssim 10^{-6}$.

The backreaction problem is then to plug this solution for $h_{\mu\nu}(x)$ into the equations of motion (2.19), and to determine the solutions $X^\mu(u,v)$ which describe the evolution of string, in the presence of a background metric that is itself produced by the string. In [120] these equations are solved iteratively for various initial string trajectories, and the results were compared with computations where the corresponding backreaction effects were neglected.

Let us now discuss how to compute the gravitational radiation from a cosmic string. To make contact with the standard approach [82, 83], let us consider the change in the d-momentum vector during a single period of oscillation; call this quantity $\Delta P^\mu(t)$. The total d-momentum of the loop is related to the energy momentum tensor (2.17) by,

$$P^\mu(t) = \int d^{d-1}\mathbf{x}\, T^{\mu 0}(t,\mathbf{x}), \tag{2.22}$$

and the corresponding change during a period is [120],

$$\Delta P^\mu(t) = \frac{1}{\pi\alpha'}\int dudv\, \partial_u\partial_v X^\mu. \tag{2.23}$$

Here the integral is over a single period (in worldsheet time τ) of oscillation; that is, u,v range from 0 to 1, which is inherited from the intervals $\sigma = [0,1]$ and $\tau = [0,1/2]$. Note that τ and t cannot be set equal to each other (as is done in temporal gauge in flat space) given that the backreaction effect is to shrink a period in real time t with the worldsheet period remaining constant (and equal to 1/2).

Clearly, in the absence of gravitational backreaction both energy and momentum are conserved, given that in this case the free wave equation governs the evolution, $\partial_u\partial_v X^\mu = 0$, and the right-hand side of (2.23) vanishes. Let us consider the timelike component, ΔP^0, the non-vanishing of which implies that energy is not conserved.[15] This non-conservation of energy is precisely due to the fact the an oscillating string radiates gravitational waves which in turn carry away the missing energy. In particular, the total energy radiated by an extended moving object with energy-momentum tensor $T^{\mu\nu}(x)$ is given by the standard expression [163],

$$\Delta E = 2G_d \int_0^\infty \omega^{d-2} d\omega d\Omega_{d-1}\Big(T^*_{\mu\nu}(k)T^{\mu\nu}(k) - \frac{1}{2}|T^\mu_{\ \mu}(k)|^2\Big), \tag{2.24}$$

where $T^{\mu\nu}(k) = \frac{1}{T}\int_0^T dt \int d^{d-1}\mathbf{x}\, T^{\mu\nu}(x)e^{-ik\cdot x}$, and $k^\mu = \omega(1,\mathbf{n})$. Using the equation of motion (2.19), the solution for the perturbation (2.21), and the fact that $\int d^dk\delta(k^2)2k^0\epsilon(k^0) = \int d^{d-1}\mathbf{k} = 2\int_0^\infty \omega^{d-2}d\omega d\Omega_{d-2}$, it can be shown [120] that,

$$\Delta P^0 = -\Delta E,$$

and this proves that energy lost by the string is radiated into space. From the above, one may compute the power in gravitational waves of frequency $\omega_n = 4\pi n/L$ per unit solid angle in a direction $\hat{\mathbf{k}}$, due to a source $T^{\mu\nu}(k)$,

$$\frac{dP_n}{d\Omega_{d-2}} = \frac{G_d\omega_n^{d-2}}{\pi}\Big(T^*_{\mu\nu}(k)T^{\mu\nu}(k) - \frac{1}{2}|T^\mu_{\ \mu}(k)|^2\Big), \tag{2.25}$$

where, writing $P = dE/dt$, the total power in gravitational waves is given by,

$$P = \sum_n P_n. \tag{2.26}$$

L is the dimensionful invariant length of the closed loop.

Given an arbitrary solution to the equation of motion, $X^\mu(u,v)$, the above expression gives the corresponding power radiated in gravitational waves. If the solution includes the effects of gravitational backreaction, so will the gravitational wave computation. Clearly, this is a complicated problem in general and non-trivial explicit solutions that accommodate backreaction effects have only been obtained numerically [120]. However, if it is the gravitational radiation that one is interested in, and *not* the final state of the cosmic string then the above approach may not offer the most efficient approach. In string theory, given an initial string state one can ask what the probability is for this to emit a graviton, or any other string state for that matter, *without* knowledge of the final state of the string.

[15] The corresponding spatial components of (2.23) can be used to study the rocket effect, whereby the center of mass momentum of the loop changes due to the emission of gravitational waves. We will not consider this here.

This can be achieved by means of the optical theorem where the imaginary part of the forward scattering amplitude is linearly related to the cross section for the given initial state to emit a graviton and go into anything. Therefore, the corresponding string theory calculation may also be more tractable than the above classical approach. In particular, analytic results can be obtained as we demonstrate in the final chapter of this thesis, where we compute the graviton emission amplitude of a macroscopic cosmic string loop with first harmonics excited, including the effects of gravitational backreaction. This is one of the many motivations for studying cosmic strings quantum-mechanically. That perturbative string theory can address the gravitational backreaction problem of the classical cosmic string theory, was first pointed out in [161].

2.5 Flat Background Evolution

Let us next consider the case of flat Minkowski background, where ${\Gamma^{\rho}}_{\mu\nu} = 0$, neglecting the effects of gravitational backreaction, and present some standard material on flat space string evolution.

We may use our freedom of choosing two functions, $v^0(\sigma)$ and $v^1(\sigma)$, in order to eliminate 2 components of the metric, $g_{\alpha\beta}$. Given that $g_{\alpha\beta}$ is symmetric, a convenient choice is conformal gauge, where $g_{\alpha\beta}(\sigma) = e^{2\phi(\sigma)}\eta_{\alpha\beta}$. Invariance under the conformal rescaling (A.4) further implies that we can locally set $\phi(\sigma) = 0$; in this section, where we consider only the classical theory, we do not worry about global obstructions to choosing such a gauge.[16]

The $(\sigma, \tau_{\rm M})$-coordinate system

Here the gauge $g_{\alpha\beta}(\sigma) = \eta_{\alpha\beta}$ and the parameterization $\sigma^{\alpha} = (\tau_{\rm M}, \sigma)$ is convenient. The Polyakov action takes the form, $S_G[X, \eta] = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma(-\dot{X}^2 + \acute{X}^2)$, the constraints (2.13) and equation of motion (2.15) respectively reduce to,

$$\begin{gathered} \dot{X}\cdot\acute{X} = 0, \qquad \dot{X}^2 + \acute{X}^2 = 0, \\ \ddot{X}^{\mu} - \acute{\acute{X}}^{\mu} = 0, \end{gathered} \tag{2.27}$$

with $\dot{X} \equiv \partial_{\tau_{\rm M}} X$ and $\acute{X} \equiv \partial_{\sigma} X$. The string therefore evolves according to the free wave equation subject to the above constraints. The first of the constraints implies that we should choose a parameterization such that lines of constant σ are perpendicular to lines of constant $\tau_{\rm M}$.

[16] For closed strings on a worldsheet of *constant* curvature, suffice it to say that we may choose a globally flat worldsheet, $R_{(2)} = 0$, only when it has the topology of a torus, $h = 1$; for worldsheets with the topology of the sphere, $h = 0$, or higher genus surfaces, $h > 1$, the globally well defined choices are a metric of positive constant curvature, $R_{(2)} = +1$, and a metric of constant negative curvature, $R_{(2)} = -1$, respectively.

The (u, v)-coordinate system

Although the above parameterization is common in the cosmic string literature, it is often more convenient to work instead in the lightcone coordinates, $u = \tau_{\rm M} + \sigma$, $v = \tau_{\rm M} - \sigma$, and by convention $\partial_u = \frac{1}{2}(\partial_{\tau_{\rm M}} + \partial_\sigma)$, $\partial_v = \frac{1}{2}(\partial_{\tau_{\rm M}} - \partial_\sigma)$. The Polyakov action here reads, $S_G[X, \eta] = -\frac{1}{2\pi\alpha'}\int \partial_u X \cdot \partial_v X$ (with $\int = \int du \wedge dv$), while the constraints and equations of motion (2.27) respectively now read,

$$(\partial_u X)^2 = (\partial_v X)^2 = 0, \qquad \text{and} \qquad \partial_u \partial_v X^\mu = 0. \tag{2.28}$$

The general solution is,[17]

$$X^\mu(u, v) = X^\mu(u) + X^\mu(v).$$

From the above equations it is clear that we are still free to perform transformations of the form $u \to f(u)$ and $v \to g(v)$. We can fix this remaining gauge invariance by lining up the worldsheet and spacetime timelike components; this is referred to as *temporal gauge*. Writing $X^\mu = (X^0, \mathbf{X})$, the general solution and constraints read respectively,

$$X^0(u, v) = (u + v)L, \qquad \mathbf{X}(u, v) = \mathbf{X}(u) + \mathbf{X}(v),$$

and

$$(\partial_u \mathbf{X})^2 = L^2 = (\partial_v \mathbf{X})^2. \tag{2.29}$$

For agreement with the standard closed string mode expansion, see Sec. 2.6, one is to take $L = \alpha' p^0/2$.

The $(w, \bar{w})$-coordinate system

In the corresponding *Euclidean coordinates* (where $\tau = \tau_{\rm E} = i\tau_{\rm M}$), $w = \sigma + i\tau$, $\bar{w} = \sigma - i\tau$, the worldsheet corresponds to the complex w-plane, with worldsheet time flowing along the imaginary axis and the spacelike distance σ flowing along the real axis. Here, $\partial_w = \frac{1}{2}(\partial_\sigma - i\partial_\tau)$, $\partial_{\bar{w}} = \frac{1}{2}(\partial_\sigma + i\partial_\tau)$ and $d^2w = idw \wedge d\bar{w}$, the Polyakov action reads $S_G[X, \delta] = \frac{1}{2\pi\alpha'}\int d^2w \partial_w X \cdot \partial_{\bar{w}} X$, and the constraints and equations of motion respectively,

$$(\partial_w X)^2 = (\partial_{\bar{w}} X)^2 = 0, \qquad \text{and} \qquad \partial_w \partial_{\bar{w}} X^\mu = 0. \tag{2.30}$$

The $(z, \bar{z})$-coordinate system

This is the coordinate system mostly used in the thesis. It is the Euclidean worldsheet coordinate system on the complex plane. The parameterization $(z, \bar{z})$ is conformally related to the $(w, \bar{w})$ coordinate system by $z = e^{-iw}$ and $\bar{z} = e^{i\bar{w}}$ (in Appendix D we discuss

[17] I am using the conventions of Polchinski ch. 2 [114] where the same letter is used to denote the left- and right-moving modes. In the following chapters we rotate to Euclidean space and the corresponding quantities $X(z)$ and $X(\bar{z})$ will refer to the left- and right-moving modes respectively, so that $X_{\rm L}(z) \equiv X(z)$ and $X_{\rm R}(\bar{z}) = X(\bar{z})$, with $z = \sigma + i\tau_{\rm E}$, $\bar{z} = \sigma - i\tau_{\rm E}$ and $\tau_{\rm E} = i\tau_{\rm M}$. Furthermore, $\partial_z X \equiv \partial_z X(z, \bar{z})$ and similarly for the antiholomorphic sector, $\partial_{\bar{z}} X \equiv \partial_{\bar{z}} X(z, \bar{z})$. For onshell statements of course, $\partial_z X(z, \bar{z}) = \partial_z X(z)$, on account of the equations of motion.

conformal symmetry in more detail). The action,

$$S_G[X,\delta] = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\partial_z X\cdot\partial_{\bar z}X,$$

with $d^2z = idz\wedge d\bar z$, constraints and equations of motion, respectively,

$$(\partial_z X)^2 = (\partial_{\bar z}X)^2 = 0, \qquad \text{and} \qquad \partial_z\partial_{\bar z}X^\mu = 0, \tag{2.31}$$

take the same form as above. The worldsheet metric, $ds^2 = 2g_{z\bar z}dzd\bar z$, and see also Appendix B for other related conventions and definitions. Infinite past, $\tau = -\infty$ is mapped to the origin, $z = \bar z = 0$ and spacelike curves are characterized by $|z|^2 = \text{const}$, see Fig. 3.1. Open strings live in the upper half plane under the identification $z\sim\bar z$, with their endpoints or the worldsheet boundary identified with the fixed point of the identification, $z = \bar z$. Closed strings live in the full complex z-plane. The equation of motion has the general solution,

$$X^\mu(z,\bar z) = X^\mu(z) + X^\mu(\bar z).$$

For easy reference we note that $i\partial_\sigma \equiv z\partial_z - \bar z\partial_{\bar z}$, $\partial_\tau \equiv z\partial_z + \bar z\partial_{\bar z}$ and $2\tau = \ln|z|^2$.

We next discuss these solutions in terms of mode expansions, first for the closed string and subsequently for the open string. The expressions will also apply in the quantum theory where the quantities appearing with a hat are to be interpreted as operators in the canonical approach. We will then consider a particularly interesting class of non-self intersecting loops that was classified by Burden [83], which exhibits cusps at discrete instants during the loops' motion.

Cusps and the Kibble-Turok Sphere

Consider the (u,v) coordinate system, as discussed above in temporal gauge. The vectors $\partial_u\mathbf{X}$, $\partial_v\mathbf{X}$ describe curves on a sphere of radius L and are periodic under $\sigma\to\sigma+2\pi$ (for closed strings),

$$(\partial_u\mathbf{X})^2 = L^2 = (\partial_v\mathbf{X})^2, \tag{2.32}$$

with the $\mathbf{X} = (X^1,\dots,X^D)$ all spacelike, with $D = 25$ for the bosonic string and 10 for the superstring. L can be thought of as representing the size of the string loop. The constraints (2.32) therefore lead to the notion of a Kibble-Turok sphere S^{n+2} [116]. The three-dimensional version S^2 is shown in Fig. 2.5, for the particular case when the two vectors traverse great circles.

The velocity at a given point on the worldsheet is $d\mathbf{X}/dX^0 = (\partial_u\mathbf{X}+\partial_v\mathbf{X})/(2L)$, and therefore if two curves on the sphere intersect, that is if,

$$\partial_u\mathbf{X}(u_0) = \partial_v\mathbf{X}(v_0), \tag{2.33}$$

at a given instant (u_0,v_0) then $(d\mathbf{X}/dX^0)^2 = 1$, which corresponds to a point on the string moving at the speed of light, a *cusp*.

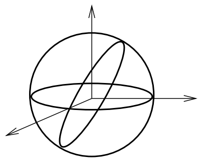

Figure 2.1: The S^{n+2} Kibble-Turok sphere, in the particular case when there are no extra dimensions, $n = 0$. The two great circles are parametrized by the three-dimensional vectors $\partial\mathbf{X}(z)$ and $\bar\partial\mathbf{X}(\bar z)$ and intersect at two points. The angle between the two circles is set by ψ.

The prototypical example of string solutions with cusps is a class of non-selfintersecting classical closed string solutions, that was identified by Burden [83], building on the work of Kibble and Turok [116]. These solutions exhibit cusps at discrete instants during the loops' motion. We will neglect the motion in the extra dimensions which will be a good approximation when the extra dimensions are sufficiently small and fluctuations in the extra dimensions are negligible.

In this case only three components of the vectors $\partial_u\mathbf{X}$, $\partial_v\mathbf{X}$ will be non-vanishing, the case of interest when the string moves in three large spatial dimensions. The solutions of interest in particular correspond to the case when these describe great circles on the Kibble-Turok sphere.[18] In the $(w, \bar w)$ coordinate system (see above), this leads to the following class of solutions which was identified by Burden [83],

$$\begin{aligned}\partial\mathbf{X}(w) &= L\big(\cos nw\,\hat{\mathbf{x}} + \sin nw\,\hat{\mathbf{y}}\big)\\ \bar\partial\mathbf{X}(\bar w) &= L\big[\cos m\bar w\,\hat{\mathbf{x}} + \sin m\bar w\big(\cos\psi\,\hat{\mathbf{y}} + \sin\psi\,\hat{\mathbf{z}}\big)\big].\end{aligned} \tag{2.34}$$

Here n and m are relatively prime. In the $(z, \bar z)$ coordinates, where we conformally transform from the cylinder to the plane $w \to z = e^{-iw}$, $\bar w \to \bar z = e^{i\bar w}$, the $\partial\mathbf{X}(w)$ and $\bar\partial\mathbf{X}(\bar w)$ have conformal dimension $(1, 0)$ and $(0, 1)$ respectively, and so the Burden solutions take the form,

$$\begin{aligned}\partial\mathbf{X}(z) &= \frac{iL}{\sqrt{2}}\big(\xi\,z^{n-1} + \xi^* z^{-n-1}\big)\\ \bar\partial\mathbf{X}(\bar z) &= \frac{iL}{\sqrt{2}}\big(\bar\xi\,\bar z^{m-1} + \bar\xi^* \bar z^{-m-1}\big),\end{aligned} \tag{2.35}$$

with polarization tensors, ξ, $\bar\xi$ (and their complex conjugates), defined as

$$\xi = \frac{1}{\sqrt{2}}\big(\hat{\mathbf{x}} + i\hat{\mathbf{y}}\big), \qquad \bar\xi = \frac{1}{\sqrt{2}}\big(-\hat{\mathbf{x}} + i\cos\psi\hat{\mathbf{y}} + i\sin\psi\hat{\mathbf{z}}\big), \tag{2.36}$$

with the properties $\xi^2 = \bar\xi^2 = 0$, $\xi\cdot\xi^* = 1$, $\bar\xi\cdot\bar\xi^* = 1$, $\xi\cdot\bar\xi = -\frac{1}{2}(1+\cos\psi)$, $\xi\cdot\bar\xi^* = -\frac{1}{2}(1-\cos\psi)$. The simplest solution exhibiting a non-degenerate cusp corresponds to $(n, m) = (2, 1)$ and this is exhibited in Fig. 2.5 for the case $\psi = \pi/4$. Also the case $(m, n) = (1, 1)$ is of interest although here the cusp is degenerate; the string can here for example take the form of a rotating double line which classically is expected to produce

[18] More general solutions of (2.32) which may also be of interest can be found in [164].

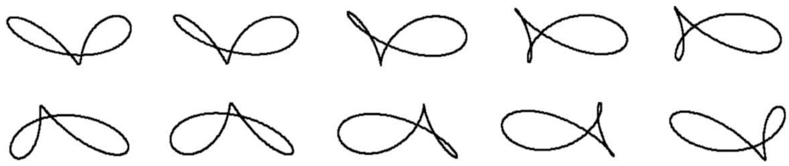

Figure 2.2: Simulation of a Burden loop solution with 2^{nd} harmonics on the left-movers and 1^{st} harmonics on the right-moving modes. Notice the formation of two cusps at discrete instants during the loop's motion.

infinite radiation [83]. The corresponding quantum calculation is expected to give a finite answer and can be determined from the amplitude calculated in Chapter 6.

Gravitational radiation from cusps has been shown [75, 76] to contribute significantly to the gravitational emission from cosmic strings and their detectability prospects. The analysis was done in 3 large dimensions, and the effects of the extra dimensions was neglected. An interesting generalization of these results has more recently appeared in the literature [117, 118] where cusps in the presence of extra dimensions were discussed. It was found that the presence of the extra dimensions significantly damps the gravitational wave signal and has the effect of rounding off cusps. In order to reach these conclusions it was assumed that there are no preferred loop configurations on the KT S^{n+2} sphere and that all configurations are equally likely. It was also pointed out that when the string width is close to the size of the extra dimensions then the motion in the extra dimensions should be irrelevant. This is an important issue and it would be interesting to study the effects of the finite size of the extra dimensions more carefully.

2.6 Closed String Mode Expansion

Consider a worldsheet cylinder with coordinates $0 \leq \sigma \leq 2\pi$ and $-\infty < \tau < \infty$, and the identification $\sigma \sim \sigma + 2\pi$. In the coordinates on the complex plane, $z = e^{-i(\sigma+i\tau)}$ and $\bar{z} = e^{i(\sigma-i\tau)}$, where the string at asymptotic infinity $\tau = -\infty$ is mapped to a point at the origin, the closed string mode expansion for the position operator then reads,

$$X^\mu(z,\bar{z}) = \hat{x}^\mu - i\frac{\alpha'}{2}\,\hat{p}^\mu_{\mathrm{L}}\ln z - i\frac{\alpha'}{2}\,\hat{p}^\mu_{\mathrm{R}}\ln\bar{z} + i\Big(\frac{\alpha'}{2}\Big)^{1/2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha^\mu_n\,z^{-n} + \tilde{\alpha}^\mu_n\,\bar{z}^{-n}\right), \tag{2.37}$$

with $\hat{x}^\mu = \hat{x}^\mu_{\mathrm{L}} + \hat{x}^\mu_{\mathrm{R}}$, total momentum $\hat{p}^\mu = \frac{1}{2}\big(\hat{p}^\mu_{\mathrm{L}} + \hat{p}^\mu_{\mathrm{R}}\big)$, and winding vector $\hat{w}^\mu = \frac{1}{2}\big(\hat{p}^\mu_{\mathrm{L}} - \hat{p}^\mu_{\mathrm{R}}\big)$. If we define $\bar{d}z = dz/(2\pi)$, $\alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}}\hat{p}^\mu_{\mathrm{L}}$ and $\tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}}\hat{p}^\mu_{\mathrm{R}}$, the dimensionless mode expansion operators are given by [114],

$$\alpha^\mu_n = \sqrt{\frac{2}{\alpha'}}\oint \bar{d}z\,\partial X^\mu\,z^n, \qquad \text{and} \qquad \tilde{\alpha}^\mu_n = -\sqrt{\frac{2}{\alpha'}}\oint \bar{d}\bar{z}\,\bar{\partial}X^\mu\,\bar{z}^n,$$

with $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$ and the zero modes are given by [137],

$$\hat{x}^\mu = \oint\Big(\frac{dz}{2\pi i z} - \frac{d\bar{z}}{2\pi i\bar{z}}\Big)X^\mu(z,\bar{z}), \qquad \text{and} \qquad \hat{p}^\mu = \frac{1}{\alpha'}\oint\Big(\bar{d}z\,\partial X^\mu - \bar{d}\bar{z}\,\bar{\partial}X^\mu\Big).$$

The angular momentum operator reads,

$$\hat{J}^{\mu\nu} = \frac{2}{\alpha'}\oint \Big(\bar{d}z X^{[\mu}\partial X^{\nu]} - \bar{d}\bar{z} X^{[\mu}\bar{\partial} X^{\nu]}\Big),$$

the integrals being along a spacelike curve, e.g. $|z|^2 = 1$, and $a^{[\mu\nu]} = \frac{1}{2}(a^{\mu\nu} - a^{\mu\nu})$. These are equivalent to the quantities defined in a general coordinate system in (A.9) in the case of closed strings. All the above can either be interpreted classically as well as quantum-mechanically. We have placed hats on the various operators to make this manifest.

In the quantum theory the solution to the equation of motion, i.e. the factorization of the position operator, $X^\mu(z,\bar{z}) = X^\mu(z) + X^\mu(\bar{z})$, can be carried out formally, but needs to be handled with care due to the presence of zero modes. On account of the commutator interpretation (3.12) discussed in Sec. 3.1, one can show that the standard commutation relations arise,

$$\big[\alpha_n^\mu, \alpha_m^\nu\big] = n\eta^{\mu\nu}\delta_{n+m,0}, \qquad \big[X^\mu(z), \partial_\tau X^\nu(z')\big] = \eta^{\mu\nu}\delta(\sigma - \sigma'), \qquad \text{and} \qquad [x^\mu, p^\nu] = i\eta^{\mu\nu}, \tag{2.38}$$

and similarly for the corresponding antiholomorphic quantities.

2.7 Open String Mode Expansion

We label the spacetime directions tangent to the Dp-brane by lower case latin letters from the beginning of the alphabet, X^a, with $a = 0, \dots, p$, and directions transverse to the brane by upper case latin letters from the middle of the alphabet, X^I, with $I = p+1, \dots 25$. In lightcone *coordinates* and assuming the associated lightcone directions satisfy Neumann boundary conditions we may define,

$$X^\pm = \tfrac{1}{\sqrt{2}}\big(X^0 \pm X^p\big).$$

This is necessary [165] in order to establish the correspondence between covariant and lightcone gauge: recall that in lightcone gauge $X^+ = 2\alpha' p^+ \tau_{\rm M}$ (with $\tau \equiv \tau_{\rm Euclidean} = i\tau_{\rm Minkowski} \equiv i\tau_{\rm M}$), which is compatible with Neumann and not Dirichlet boundary conditions, see (2.39). A general spacetime direction is as always labelled by Greek lower case letters, X^μ. In summary,

$$\begin{aligned} X^a &= \{X^\pm, X^A\}, &\qquad &\text{with} &\qquad &A = 1, \dots, p-1,\\ X^i &= \{X^A, X^I\}, &\qquad &\text{with} &\qquad &I = p+1, \dots, 25,\\ X^\mu &= \{X^\pm, X^i\}, \end{aligned}$$

with the scalar product of two general vectors in components being, $U^\mu V_\mu = -U^- V^+ - U^+ V^- + U^A V^A + U^I V^I$. The directions, X^A, therefore satisfy Neumann boundary conditions, whereas directions transverse to the brane, X^I, satisfy Dirichlet boundary conditions. In the Euclidean worldsheet coordinate[19] $z = e^{-i(\sigma + i\tau)}$, $\bar{z} = e^{i(\sigma - i\tau)}$ with $\sigma \in [0, \pi]$

[19] Our conventions are mostly in agreement with Polchinski [114].

and $\tau \in (-\infty, \infty)$, (considering only the case of NN and DD strings) Neumann and Dirichlet boundary conditions read respectively,

$$\partial_\sigma X^a|_{\partial\Sigma_{1,2}} = 0 \quad \text{(N)} \qquad \text{and} \qquad \partial_\tau X^I|_{\partial\Sigma_{1,2}} = 0 \quad \text{(D)} \tag{2.39}$$

It is useful to note furthermore that, $\partial_\sigma = i(\bar{z}\bar{\partial} - z\partial)$ and $\partial_\tau = \bar{z}\bar{\partial} + z\partial$. In the $(z, \bar{z})$ coordinates the open string physical worldsheet, Σ, is conformally mapped to the upper half plane with the identification, $z \sim \bar{z}$. The fixed point of this identification (the real line, $z = \bar{z}$) defines the open string boundaries,

$$\partial\Sigma_1 \equiv \{z\,|\,z = e^\tau, -\infty < \tau < \infty\}, \qquad \text{and} \qquad \partial\Sigma_2 \equiv \{z\,|\,z = -e^\tau, -\infty < \tau < \infty\}.$$

In the open string conventions, the general solution to the equations of motion, $\partial\bar{\partial}X^\mu = 0$, is given by $X^\mu(z, \bar{z}) = X^\mu(z) + X^\mu(\bar{z})$, with

$$\begin{aligned} X^\mu(z) &= x^\mu_{\rm L} - i\alpha' p^\mu_{\rm L} \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \frac{\alpha^\mu_n}{z^n}, \\ X^\mu(\bar{z}) &= x^\mu_{\rm R} - i\alpha' p^\mu_{\rm R} \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \frac{\tilde{\alpha}^\mu_n}{\bar{z}^n}, \end{aligned}$$

and the momentum is half that of the closed string, $\alpha^\mu_0 = \sqrt{2\alpha'}\hat{p}^\mu_{\rm L}$, $\tilde{\alpha}^\mu_0 = \sqrt{2\alpha'}\hat{p}^\mu_{\rm R}$. If we define the total momentum and winding vectors respectively by,

$$p^\mu = \frac{1}{2}(p_{\rm L} + p^\mu_{\rm R}) \qquad \text{and} \qquad w^\mu = \frac{1}{2}(p^\mu_{\rm L} - p^\mu_{\rm R}), \tag{2.40}$$

it follows that the boundary conditions (2.39) require,

$$w^a = 0, \qquad \alpha^a_n + \tilde{\alpha}^a_n = 0, \qquad \text{and} \qquad p^I = 0, \qquad \alpha^a_n - \tilde{\alpha}^a_n = 0, \tag{2.41}$$

reflecting the fact that open strings cannot wind in the Neumann directions and that the centre of mass momentum in the transverse directions vanishes. Therefore, the string mode expansions take the form,

$$\begin{aligned} N: &\qquad X^\pm(z, \bar{z}) = x^\pm - i\alpha' p^\pm \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha^\pm_n}{n}\Big(\frac{1}{z^n} + \frac{1}{\bar{z}^n}\Big), \\ N: &\qquad X^A(z, \bar{z}) = x^A - i\alpha' p^A \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha^A_n}{n}\Big(\frac{1}{z^n} + \frac{1}{\bar{z}^n}\Big), \\ D: &\qquad X^I(z, \bar{z}) = x^I - i\alpha' w^I \ln\frac{z}{\bar{z}} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha^I_n}{n}\Big(\frac{1}{z^n} - \frac{1}{\bar{z}^n}\Big), \end{aligned} \tag{2.42}$$

with the two string endpoints located respectively at (switching back to a Minkowski worldsheet, $\tau = \tau_{\rm E} = i\tau_{\rm M}$),

$$X^a(z, \bar{z})|_{\partial\Sigma_1} = x^a + (2\alpha')p^a\tau_{\rm M} + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha^a_n}{n} e^{-in\tau_{\rm M}}, \qquad X^I(z, \bar{z})|_{\partial\Sigma_1} = x^I,$$

and

$$X^a(z,\bar{z})|_{\partial\Sigma_2} = x^a + (2\alpha')p^a\tau_{\rm M} + i\sqrt{2\alpha'}\sum_{n\neq 0}(-1)^n\frac{\alpha_n^a}{n}e^{-in\tau_{\rm M}}, \qquad X^I(z,\bar{z})|_{\partial\Sigma_2} = x^I - (2\alpha')w^I\pi.$$

With the definition $\text{đ}z = dz/(2\pi)$, the dimensionless mode expansion operators are as in the closed string [114],

$$\alpha_n^\mu = \sqrt{\frac{2}{\alpha'}}\oint \text{đ}z\,\partial X^\mu\,z^n, \qquad \text{and} \qquad \tilde{\alpha}_n^\mu = -\sqrt{\frac{2}{\alpha'}}\oint \text{đ}\bar{z}\,\bar{\partial}X^\mu\,\bar{z}^n,$$

with $(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$, and using the open string constraints (2.41) one may work with the holomorphic quantity, α_n^μ, only. The zero modes and angular momentum are given by [166],

$$\hat{x}^\mu = \oint\Big(\frac{dz}{2\pi i z} - \frac{d\bar{z}}{2\pi i\bar{z}}\Big)X^\mu(z,\bar{z}), \quad \hat{p}^\mu = \frac{1}{\alpha'}\oint \text{đ}z\,\partial X^\mu, \quad \text{and} \quad \hat{J}^{\mu\nu} = \frac{2}{\alpha'}\oint \text{đ}z X^{[\mu}\partial X^{\nu]},$$

and we have used the doubling trick [114] so that the integrals are along a spacelike curve, e.g. $|z|^2 = 1$, and $a^{[\mu\nu]} = \frac{1}{2}(a^{\mu\nu} - a^{\mu\nu})$. The physical worldsheet is in the upper half plane – one identifies antiholomorphic quantities in the upper half plane with holomorphic quantities in the lower half plane and therefore one may just as well work with holomorphic quantities only in the full complex plane. For example, $\hat{p}^\mu = \frac{1}{2\alpha'}\int_{C_+}\big(\text{đ}z\,\partial X^\mu - \text{đ}\bar{z}\,\bar{\partial}X^\mu\big) = \frac{1}{2\alpha'}\big(\int_{C_+} + \int_{C_-}\big)\text{đ}z\,\partial X^\mu$ and $\int_{C_+} + \int_{C_-} = \oint$, so that C_+ represents an open spacelike contour in the upper half (stretching from $\sigma = 0$ to π), C_- represents the corresponding quantity in the lower half plane (stretching from $\sigma = \pi$ to 2π), and C represents a closed contour, $C = C_- \cup C_+$.

2.8 Scaling and Energy Loss Mechanisms

An important quantity to consider during the evolution of a string network is the typical length scale or correlation length that characterizes the network, ξ. This will be such that in a randomly chosen box of size ξ^3 the average total length of string will be ξ. Therefore, if μ is the energy per unit length, the total density of string is,

$$\rho_s \sim \mu/\xi^2.$$

During the initial stages of cosmic string evolution, the energy density of ordinary matter will initially be only slightly less than the energy density inside a string, and so the string motion will be heavily damped. The regions on the string with high extrinsic curvature will tend to straighten out and, were it not for the expansion of the universe, the total length of string would decrease. The expansion of the universe will however stretch the strings and increase their length. During this *friction-dominated* stage, the typical length scales that characterize the string network, ξ, will grow slightly faster than the Hubble horizon, $H^{-1} \sim t$, and in particular, $\xi \sim t^{5/4}$ [167].

Given that regions in space cannot be correlated over distances greater than the horizon length, H^{-1}, there is an upper bound, $\xi \lesssim t$, and so ξ cannot grow faster than the H^{-1}

indefinitely. Indeed, when ξ/t becomes of order one, the friction-dominated stage will come to an end and strings will thereafter evolve freely. Analytical [168, 169] and numerical [160, 170, 171, 124] work then suggests that the string network will evolve into a scaling regime, where the typical length scales that characterize the network grow linearly in Hubble time, $\xi \sim t$. Here the energy density of the string network, ρ_s, tends to a constant fraction of the radiation or matter density, and correlation lengths of long strings scale with cosmic time t. In particular, let us write $\xi = \gamma t$, with $\gamma \lesssim 1$. The total density of the universe, which is assumed to be flat[20], is $\rho = \frac{3H^2}{8\pi G}$, and therefore in the scaling regime,

$$\frac{\rho_s}{\rho} = \frac{8\pi G\mu}{3\nu^2\gamma^2}, \tag{2.43}$$

with ν equal to 2/3 or 1/2 during the matter or radiation dominated epochs respectively, $H = \nu/t$. Clearly therefore the string tension, and in particular the dimensionless combination $G\mu$, is a phenomenologically important quantity. That the network evolves towards a scaling regime where $\rho_s/\rho = \text{const}$ can be understood as follows. If the density of string becomes large, then strings will intercommute and reconnect more often, leading to the production of loops which in turn decay. Therefore, a surplus of energy in the network will be removed in this manner. If, on the other hand, the density becomes too low, then strings will not meet often enough to produce loops and their density will grow, thus leading asymptotically to the constant fraction (2.43).

Notice that the comoving volume of the universe increases as $V \sim a^3 \sim t^{3\nu}$, and so if the long string was just being stretched by the expansion then the correlation length would increase as $\xi \sim a \sim t^\nu$ and strings would come to dominate the energy density of the universe,

$$\frac{\rho_s}{\rho} \sim \frac{8\pi G\mu}{3\nu^2} t^{2-2\nu}.$$

Therefore, in order for scaling to persist, energy must be continuously removed from the network and it is thought that loop production, and/or massive radiation [121, 122, 70], is responsible for this, more about which will be said below.

A very important question that has remained open for many years now is, what is the scale at which loops are produced [67, 69, 70, 21, 71, 72]. This scale in turn determines the frequency and amplitude of the subsequently emitted gravitational wave signal, as well as (at least in the case of solitonic strings) the fraction of energy going into ultra high energy cosmic rays. As a network of string evolves, a typical long string has the shape of a random walk with step length of order the Hubble radius. Portions of the string network will then intersect with other string, thereby creating kinks, and self-intersections will produce loops. If the loops are large they will oscillate and carry away much of the energy via gravitational radiation. Although their angular momentum will generically tend to prevent a loop from shrinking, the loop's tension and the corresponding loss in energy due to its coupling to gravity should carry away some of the loop's energy, thereby leading to

[20] We are neglecting the cosmological constant which should become important only at later stages of the evolution.

a decrease in its size. The frequency of emitted gravitational radiation is expected to be of order $f \sim L^{-1}$ for a loop of size L and so the frequency will increase as the loop shrinks. The lifetime of a loop can be estimated to be of order

$$\tau \sim \frac{L}{G\mu},$$

and the corresponding power of gravitational radiation can be estimated from the quadrupole formula [82], $P_g \sim GM^2L^4f^6$,

$$P_g \sim G\mu^2, \tag{2.44}$$

with $\mu = M/L$ the string mass per unit length, the string tension. String loops however generically have cusps, points on the string that reach the speed of light. At cusps the string overlaps onto itself and the overlapping region can therefore annihilate and leave behind its energy in (possibly massive) radiation or, in the case of fundamental strings, tiny loops. These tiny loops may either be massive or, in the limit of zero radius, correspond to massless states. If the standard model particles are identified with open string modes associated to strings confined to a "standard model brane", then the tiniest such loops will correspond to bulk modes, namely gravitons, dilatons (if they have not acquired mass), antisymmetric tensor modes $B_{(2)}$ and RR fields. In the simplest scenarios the tiniest loops produced in this manner should therefore be identified with gravitons.

An Abelian-Higgs field theory simulation suggests the size of the chopped off piece of string is of order [172, 173] $\sqrt{\delta L}$, with δ the string width, and this result also takes Lorentz contraction into account. The power associated to cusp annihilation is therefore expected to be of order,

$$P_c \sim \mu\sqrt{\frac{\delta}{L}}. \tag{2.45}$$

Cusp annihilation modifies the behaviour of cusps and changes their shape, producing many daughter cusps but of smaller magnitude [173]. The string overlap will occur when the Lorentz γ-factor reaches the value [87] $\gamma \sim \sqrt{L/\delta}$. In [87] small scale structure was introduced onto the strings and the corresponding effect on cusps was studied. It was found that under most circumstances the presence of small-scale structure close to cusps leads to the formation of loops at the size of the smallest scales. A parameter ϵ was introduced and defined as the ratio of the characteristic wavelength of small-scale structure to the corresponding amplitude of oscillation. It was shown that backreaction is likely to become significant and change the form of the cusp if,

$$\epsilon \gtrsim \sqrt{\frac{\delta}{L}}.$$

As discussed above the tiniest loops are, in string theory, identified with gravitons or $B_{(2)}$ modes. An alternative scenario has been suggested in the corresponding field theory process [121, 122, 70]. When a loop has radiated away its energy and shrunk to a size of order the effective string width, it has been suggested that it may also give up its remaining energy in a burst of particle emission, that may show up as cosmic rays in experiments. There is however disagreement in the literature as to whether the typical initial loop size is large enough for most of the energy to be emitted during the initial stage of decay, namely the gravitational radiation stage. In [121, 122] it was found that the typical scale of loop production is of order the string width, in which case (it was argued) the dominant energy loss mechanism would be particle production, although this conclusion relied on extrapolation of numerical results beyond their range of validity. In the string theory context, it is conceivable that loop production at the tiniest scales, as discussed above, would still be identified with gravitational radiation, and massive modes need not be produced. Close to cusps, it may be that massive loops are produced but these are expected to decay rapidly into gravitons, unless there is a conserved quantity (such as angular momentum) that forbids this from occurring. Nevertheless these are all very important issues and need to be studied in more detail.

Another very important issue, where there is a lot of work that remains to be done, is on the effect of extra dimensions. The effect of extra dimensions on cusp formation has been recently studied in [117, 118]. It was found that the presence of extra dimensions has the effect of rounding off cusps, thus significantly reducing the corresponding gravitational wave amplitude. Here it would be interesting to study in greater detail this effect as a function of the size, among other things, of the extra dimensions. I suspect that it will also be important to take into account the fact that most of the string will be in the large dimensions.

With the vertex operators that I present in the current document, all these questions can be addressed analytically, and definite answers are within reach.

Another pressing open question of interest is whether [174] loops scale with cosmic time [170] or not [160, 123, 124], as well as how or whether this is related to the long sought-after backreaction scale [160]. Here it is harder to consider the corresponding quantum computation because vertex operators in curved backgrounds are highly non-trivial [140]. It may however be possible to proceed with the flat space vertex operators, and take into account the expansion of the universe by constructing an appropriate phenomenological model.

2.9 Gravitational Radiation

A very important energy loss mechanism from cosmic strings is the emission of gravitational radiation. Both long string and small chopped-off loops are expected to radiate gravitationally, and various features such as cusps and kinks on string may lead to a very strong non-Gaussian contribution [75, 76] to the gravitational wave background. In this section we provide a brief overview of some of these computations that have been carried out in this direction, emphasizing in particular some shortcomings and the ways in which these calculations can be improved using the tools developed in the current document.

In the current section we neglect gravitational backreaction, see Sec. 2.4, as is standard in the classical calculations [82, 83, 84, 85, 75, 76, 86, 87, 71, 88]. It has been argued [160, 120] that backreaction should play an important role on the scale of loop production from long strings. On the other hand, the relation of the quantum computations [161, 54, 55, 56, 162, 58, 59, 60] to cosmic strings has remained obscure until the present day, given that the wavefunctions or vertex operators that are required as input into the quantum calculation were not available. We now believe that we have identified the vertex operators that should be identified with cosmic strings and so this should open the door for many new calculations. In the final chapter we present the first of these computations, a backreaction computation for a cosmic string loop, using the coherent state vertex operators constructed in Chapter 5.

Let us first look at the classical approach. Let us consider the case when a gravitational wave is emitted by an arbitrary source and detected by an observer at a distance r from the source, such that

$$\lambda \ll r \ll H^{-1},$$

with H^{-1} the Hubble radius and λ the typical wavelength associated to the perturbation. A gravitational wave is here characterized by the Einstein equations, and is determined by the energy momentum tensor of the source. In a linearized approximation, and when the background geometry is flat, let us can expand the spacetime metric as follows, $G_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ the perturbation, $|h_{\mu\nu}| \ll 1$. In harmonic gauge, $G^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0$, where the Ricci tensor $R_{\mu\nu} \simeq \frac{1}{2}\partial^2 h_{\mu\nu}$, the Einstein equation, $R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = -8\pi G T_{\mu\nu}$ (with G *4-dimensional* Newton's constant here), reduces to a wave equation with a source,

$$\partial^2 \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}, \tag{2.46}$$

with $\bar{h}_{\mu\nu}$ the trace-reversed metric perturbation, $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. This equation can be solved using the Green's function method to find the inverse of the operator ∂^2. Writing, $k^\mu = (\omega, \mathbf{k}) \equiv \omega(1, \mathbf{n})$ and $x = (t, \mathbf{x})$ this leads to [76],

$$\begin{aligned} \bar{h}_{\mu\nu}(x) &= \frac{4G}{r}\sum_{\omega} e^{-i\omega(t-r)}T_{\mu\nu}(k) + \mathcal{O}(1/r^2) \\ &\simeq \frac{2G\ell}{r}\int \frac{d\omega}{2\pi}\, e^{-i\omega(t-r)}T_{\mu\nu}(k), \end{aligned} \tag{2.47}$$

with $r = |\mathbf{x}|$ the distance from an observer at $\mathbf{x}$ to the source at $\mathbf{x}' = 0$, and $\mathbf{n}$ a unit vector in the direction of emission. In the second line we have taken a large frequency limit; ℓ is the characteristic length scale of the radiating object and will be identified with the length of the string. It will be more convenient to work instead with the logarithmic Fourier transform (with respect to retarded time),

$$\begin{aligned} h_{\mu\nu}(k) &\simeq |\omega|\int d(t-r)\, e^{i\omega(t-r)}\,\frac{2G\ell}{r}\int\frac{d\omega'}{2\pi}\, e^{-i\omega'(t-r)}T_{\mu\nu}(k'), \\ &= |\omega|\frac{2G\ell}{r}\,T_{\mu\nu}(k) \end{aligned} \tag{2.48}$$

$T_{\mu\nu}(k)$ is in turn the Fourier transform of the energy momentum tensor (2.17). For a periodic source with fundamental period of oscillation $T = 2\pi/\omega = \ell/2$,

$$\begin{aligned} T_{\mu\nu}(k) &= \frac{1}{T}\int_0^T dt \int d^3\mathbf{x}\, e^{-ik\cdot x} T_{\mu\nu}(x) \\ &= \frac{\mu}{T}\int dudv \big(\partial_u X^\mu \partial_v X^\nu + \partial_v X^\nu \partial_u X^\nu\big)\, e^{-ik\cdot X(u,v)} \\ &= \frac{\mu}{T}\int_0^\ell du\, \partial_u X^\mu e^{-ik\cdot X(u)} \int_0^\ell dv\, \partial_v X^\nu e^{-ik\cdot X(v)} + \mu \leftrightarrow \nu \end{aligned} \tag{2.49}$$

where in the last line we have used the solution to the equation of motion $X^{\mu\nu}(u,v) = X^\mu(u) + X^\mu(v)$, and the worldsheet integral is over a sheet of worldsheet that is swept out in one period. For closed loops $\partial_u X(u+\ell) = \partial_u X(u)$ and $\partial_v X(v+\ell) = \partial_v X(v)$ and the period is $\ell/2$ because $X(\sigma+\ell/2, \tau+\ell/2) = X(\sigma,\tau)$ [82]. It is therefore convenient to define the integrals

$$I_u^\mu(k) = \int_0^\ell du\, \partial_u X^\mu e^{-ik\cdot X(u)}, \qquad \text{and} \qquad I_v^\mu(k) = \int_0^\ell dv\, \partial_v X^\nu e^{-ik\cdot X(v)}. \tag{2.50}$$

Let us focus on the gravitational emission from string loops with cusps. Recall from (2.33) that in temporal gauge, where $X^0(u,v) = (u+v)L$, the condition for a cusp is $\partial_u \mathbf{X}(u_0) = \partial_v \mathbf{X}(v_0)$. Therefore, a cusp will form if there exists a vector ℓ^μ such that $\ell^\mu = \partial_u X^\mu(u_0) = \partial_v X^\mu(v_0)$ with $\ell^2 = 0$. The integrals I_u, I_v will decrease exponentially unless there exists a saddle point [76] such that $\partial_u(k \cdot X(u)) = 0$ and similarly $\partial_v(k \cdot X(v)) = 0$. Given that both k^μ, $\partial_u X^\mu(u)$ and ℓ are null, there will exist a saddle point if $k^\mu \propto \partial_u X^\mu = \ell^\mu$. Given that $\partial_u X^\mu(u) = (L, \mathbf{X}(u))$ and $k^\mu = \omega(1,\mathbf{n})$ with $(\partial_u \mathbf{X})^2 = L^2$ and $\mathbf{n}^2 = 1$ it follows that for $k^\mu = \omega \ell^\mu / L$ there will exist a saddle point. Let us then shift the worldsheet coordinates such that the cusp is located at $(u_0, v_0) = 0$ and $X^\mu(0,0) = 0$, and perform a Taylor expansion around this point,

$$X^\mu(u) = \ell^\mu u + \frac{1}{2}\partial_u^2 X^\mu u^2 + \frac{1}{3!}\partial_u^3 X^\mu u^3 + \ldots, \tag{2.51}$$

with similar expressions for the right-moving sector where $u \leftrightarrow v$. From the constraints, $(\partial_u X)^2 = 0$, and $\partial_u X^\mu(0) = \ell^\mu$ it then follows that

$$k \cdot X(u) = -\frac{\omega}{6L}(\partial_u^2 X(0))^2 u^3.$$

Plugging these results into (2.50) it follows that the physical (i.e. non-gauge) contributions are [76],

$$I_u^\mu(k) = \partial_u^2 X^\mu \int_0^\ell du\, u \exp\left(i\frac{\omega}{6L}(\partial_u^2 X)^2 u^3\right), \tag{2.52}$$

with a similar expression for $I_v^\mu(k)$ and $\partial_u^2 X^\mu = \partial_u^2 X^\mu(0)$. Given that most of the contribution to the integral comes from around the saddle point, $u = 0$, we can extend the limits of integration of u to $\pm\infty$. Then, one finds,

$$\begin{aligned} I_u^\mu(k) &= \partial_u^2 X^\mu \left(\tfrac{|\omega|}{6L}(\partial_u^2 X)^2\right)^{-2/3} \int_{-\infty}^{\infty} dy\, y \exp\left(i\,\mathrm{sign}(\omega) y^3\right) \\ &= \partial_u^2 X^\mu \big(\tfrac{|\omega|}{6L}(\partial_u^2 X)^2\big)^{-2/3} \frac{2\pi i\,\mathrm{sign}(\omega)}{3\Gamma(1/3)} \end{aligned} \tag{2.53}$$

The energy-momentum tensor, $T^{\mu\nu}(k) \sim \frac{\mu}{\ell} I_u^\mu I_v^\nu$, is then of order,

$$T^{\mu\nu}(k) \sim \mu(\ell L)^{2/3}|\omega|^{-4/3},$$

where we have estimated $\partial_u^2 X \sim L/\ell$, which is consistent with the constraints $(\partial_u \mathbf{X})^2 = L^2$ and the mode expansion for $X(u)$, and have dropped constants of order unity. Plugging this into (2.48) and dropping dimensionless constants (such as ℓ) we find that the amplitude reads,

$$h_{\mu\nu}(\omega) \sim \frac{G\mu L^{2/3}}{r}\,|\omega|^{-1/3}\,, \tag{2.54}$$

where the fall-off of the amplitude with frequency, $h(\omega) \sim |\omega|^{-1/3}$, is characteristic of emission from string loops with cusps.

It can be shown [76] that the effect of the expansion of the universe is, roughly speaking, to take $r \to a(t)r$ and $\omega \to (1+z)\omega$, with $a(t)$ the scale factor of the universe and z the redshift at which the cusp event takes place. If $a(t_0)$ is the scale factor today then furthermore, $a(t_0)r \sim t_0 z/(1+z)$, which eliminates the r dependence in favor of the experimentally measurable quantity z.

One then needs to make certain assumptions about the loop size, L, and in particular about whether it scales with the expansion of the universe or not. If it does, that is if $L = \alpha t$, then one would need to determine the z-dependence of t and the parameter α. This parameter is not known, but it is often assumed that $\alpha = \epsilon\Gamma G\mu$, with $\Gamma \sim 50$ and ϵ a dimensionless number that quantifies the uncertainty. As discussed in Sec. 2.4 it is believed that α, which may be identified with the scale of the smallest structures on cosmic strings, the scale of loop production, is set by the backreaction scale [160, 120, 85]; recall that the effect of backreaction is to smooth the strings. A wiggly string would correspond to $\epsilon \sim 10^{-10}$ and for a smooth string, $\epsilon \sim 1$. Given that this uncertainty in the loop production scale will propagate through to the experimentally measurable quantities, a much better understanding of this issue is required to make sound predictions.

In order to complete the calculation, one needs to take into account the number density of string loops in a redshift interval, $(z + dz) - z$, the cusp burst rate and the isotropy of emission. One also needs to make certain assumptions about the probability of intersection and reconnection of cosmic strings [45, 46, 47, 48, 49]. Typical values for the intercommutation probability for superstrings lie in the range $10^{-5} \lesssim p \lesssim 1$, whereas for field theory defects, the probability is essentially $p = 1$ [45]. Many of the above details are explained in the original paper of Damour and Vilenkin [76].

Chapter 3

Perturbative String Theory

String theory is a theory of maps from a 2-dimensional open or closed Riemann surface, the so-called worldsheet Σ, into a 10-dimensional (26 for the bosonic string) manifold $\mathcal{M}$, which is identified with spacetime. The theory is conformal on the worldsheet (but not conformal in spacetime) and so the language of conformal field theory in string theory computations is natural. In the following section we introduce conformal field theories, discuss the spectrum of states, how operators transform under conformal symmetries, and finally what the relation to string theory states is. In the section following this, we discuss string theory and in particular its definition via the string path integral. These overviews will provide the appropriate grounding necessary for the chapters following, where we construct complete sets of string vertex operators for both mass eigenstates and coherent states. This is then followed by an amplitude computation involving coherent states, in particular the graviton emission amplitude for a coherent state with cusps.

3.1 Conformal Field Theory

In this section we give a brief review of conformal field theory, with particular focus on the computational techniques that are unique to $d = 2$ CFT's and the *constraints* it places on the spectrum of the theory and the associated vertex operators.

We start with a definition of $d = 2$ conformal field theory (CFT):

1. There exists a set of fields $\{A_i\}$ which is in general infinite and contains all derivatives of all A_i.

2. There exists a subset of fields $\{\phi_j(z,\bar{z})\} \in \{A_i\}$, termed *primary*, which transform under local conformal transformations (D.20) as components of complex tensors, see also (B.1),
$$\phi(z,\bar{z})(dz)^h(d\bar{z})^{\bar{h}} = \phi'(z',\bar{z}')(dz')^h(d\bar{z}')^{\bar{h}} \tag{3.1}$$
where $h,\bar{h}$ is the *conformal weight* of $\phi(z,\bar{z})$. In general $h,\bar{h}$ need not be integers and the spin of the field ϕ must satisfy $h - \bar{h} = \frac{1}{2}\mathbb{Z}$. The theory is *covariant* in the

sense that the correlation functions preserve the tensorial structure,

$$\begin{aligned}&\langle\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle(dz_1)^{h_1}(d\bar{z}_1)^{\bar{h}_1}\dots(dz_n)^{h_n}(d\bar{z}_n)^{\bar{h}_n}\\&\qquad=\langle\phi_1'(z_1',\bar{z}_1')\dots\phi_n'(z_n',\bar{z}_n')\rangle(dz_1')^{h_1}(d\bar{z}_1')^{\bar{h}_1}\dots(dz_n')^{h_n}(d\bar{z}_n')^{\bar{h}_n}\end{aligned}\tag{3.2}$$

Fields invariant also under the *global* conformal group are called *quasi-primary* or $SL(2,\mathbb{C})$ primaries. Correlation functions are in turn defined in terms of path integrals,

$$\langle\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle=\int\mathcal{D}X e^{-S[X,g]}\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n),$$

where it is to be understood that $\phi(z,\bar{z})$ can be expressed in terms of the canonical fields $X(z,\bar{z})$. We do not normalize the path integral by dividing by the partition function $\langle 1\rangle$.

3. The remaining fields in $\{A_i\}$ can be expressed as linear combinations of primary fields $\{\phi_j\}$ and their derivatives.

4. There is a vacuum $|0\rangle$ invariant under the global conformal group.

In string theory the complex plane, $\mathbb{C}$, we have been considering is identified with the string worldsheet, Σ. Consider a closed string sweeping out a 2-dimensional surface, thus forming a cylinder parametrized by the coordinates $\sigma^\alpha=(\tau,\sigma)$, see Fig. 3.1, and compactify according to $\sigma\sim\sigma+2\pi$. Next define $w=\sigma+i\tau$, $\bar{w}=\sigma-i\tau$, and conformally map to the plane $z=e^{-iw}$, $\bar{z}=e^{i\bar{w}}$ (corresponding to the diagram on the right in Fig. 3.1). Therefore, the slice $\tau=$ const corresponds to surfaces of equal time, infinite past[1] $\tau=-\infty$ is mapped to the origin, $z=0$, and infinite future, $\tau=+\infty$, to $|z|=\infty$. Therefore, dilatations, $z\to\lambda z$ generate time translations, which as shown below (3.15) are generated by $L_0+\bar{L}_0$, which can therefore be regarded as the Hamiltonian of the theory. To build the quantum operator and more generally the quantum theory of conformal fields on the z-plane, we need to realize operators that implement conformal mappings on the plane. We will consider the corresponding path integral quantization in the following section, and here we take a canonical viewpoint – these two descriptions are complementary.

We use the Noether prescription to construct the generators. Recall that in $d+1$ dimensions an exact symmetry has an associated conserved current, j^μ, with $\partial\cdot j=0$. The corresponding conserved charge is constructed by integrating over a fixed timeslice,

$$Q=\int d^dx\,j^0(x),$$

which generates the corresponding infinitesimal symmetry variation of a field A,

$$\delta_\epsilon A=\epsilon[Q,A].\tag{3.3}$$

Now, local coordinate transformations are generated by charges constructed from the energy momentum tensor of the theory, $T^{\mu\nu}$, a symmetric and divergence free tensor. In

[1] The analogous Minkowski space process is reached by taking $\tau\to i\tau$.

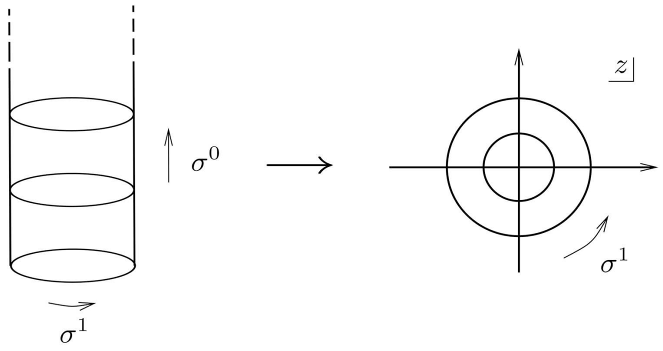

Figure 3.1: Radial quantization

CFT $T^{\mu\nu}$ is also traceless, $T^{\mu}_{\ \mu} = 0$, as can be deduced from requiring that $\partial \cdot j = 0$ when j_μ is identified with the dilatation current, $j_\mu = T_{\mu\nu}x^\nu$ associated to $x \rightarrow \lambda x$. For more general conformal transformations, $x \rightarrow x + \epsilon(x)$, the corresponding current is $j^\mu(x) = T^{\mu}_{\ \nu}\epsilon^\nu(x)$, and when $\epsilon(x)$ is a solution of the conformal Killing equation (D.3) it is conserved, $\partial \cdot j = 0$. In two dimensions the remnant of this energy momentum tensor is, T_{zz}, $T_{\bar{z}\bar{z}}$, with $T_{z\bar{z}} = T_{\bar{z}z} = 0$, from the tracelessness condition

$$T^{\mu}_{\ \mu} = 0, \tag{3.4}$$

where we have assumed the absence of a conformal anomaly – such anomalies always cancel in critical string theory [141]. These are related to the original object by the coordinate transformation, $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. On account also of the conservation law $\partial_\mu T^{\mu}_{\ \nu} = 0$, see (2.14), it follows that the two components are holomorphic and antiholomorphic respectively:

$$T(z) \equiv T_{zz}(z), \qquad \text{and} \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}).$$

Given that in d-dimensions $\partial \cdot j = 0$ leads to the conserved charge $Q = \int d^dx j^0(x)$, with the integral over a timeslice surface, one expects that in 2 dimensions, where $j_z = T(z)\epsilon(z)$, and $j_{\bar{z}} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$, the corresponding conserved charge is obtained (according to the above discussion on constant time-slices on the z-plane) by the integral,

$$Q = \frac{1}{2\pi i}\oint_C \Big(dzT(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\Big). \tag{3.5}$$

This is however formal and cannot be evaluated until the operators in the interior of C are specified. From the above and (3.3) one learns that the symmetry variation of a (primary) field is,

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_C \Big[dzT(z)\epsilon(z), \phi(w,\bar{w})\Big] + \Big[d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \phi(w,\bar{w})\Big]. \tag{3.6}$$

Recall that in the operator formalism one always considers time-ordered products of operators, and in Euclidean space this translates into radial ordering. Define the *radial ordering* operator, R,

$$R\big(A(z)B(w)\big) = \begin{cases} A(z)B(w) & \text{if} \quad |z| > |w| \\ B(w)A(z) & \text{if} \quad |w| < |z| \end{cases} \tag{3.7}$$

with a minus sign for fermionic operators. This leads to the following interpretation of (3.6),

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\Big(\oint_{|z|>|w|} - \oint_{|z|<|w|}\Big)\Big(dz\epsilon(z)R\big(T(z)\phi(w,\bar{w})\big) + d\bar{z}\bar{\epsilon}(\bar{z})R\big(\bar{T}(\bar{z})\phi(w,\bar{w})\big)\Big).$$

and therefore one is to deform the contour around the point $(w,\bar{w})$ where the operator $\phi(w,\bar{w})$ is inserted,

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_w dz\epsilon(z)R\big(T(z)\phi(w,\bar{w})\big) + \frac{1}{2\pi i}\oint_{\bar{w}} d\bar{z}\bar{\epsilon}(\bar{z})R\big(\bar{T}(\bar{z})\phi(w,\bar{w})\big). \tag{3.8}$$

If the integrands are meromorphic then one requires the behavior of the integrands in the limits $z \to w$ and $\bar{z} \to \bar{w}$. Now, $\phi(w,\bar{w})$ is a primary operator and so under $w \to w + \epsilon(w)$, $\bar{w} \to \bar{w} + \bar{\epsilon}(\bar{w})$, according to (3.1),

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) \simeq \big(h\partial\epsilon + \epsilon\partial\big)\phi(w,\bar{w}) + \big(\bar{h}\bar{\partial} + \bar{\epsilon}\bar{\partial}\big)\phi(w,\bar{w}). \tag{3.9}$$

From (3.8), (3.9) and Cauchy's formula we obtain an explicit representation of the radial ordering operator, which is none other than a short distance operator product expansion (OPE),

$$\begin{aligned} R\big(T(z)\phi(w,\bar{w})\big) &= \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + \dots \\ R\big(\bar{T}(\bar{z})\phi(w,\bar{w})\big) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) + \dots \end{aligned} \tag{3.10}$$

The dots denote non-singular terms that do not contribute. These two expressions can be taken to *define* the *quantum* stress energy tensor in $d = 2$ dimensions. This can also be taken to *define* the notion of a primary field and encodes its transformations properties. In what follows we will denote OPE's or radially ordered operator as follows,

$$R\big(A(z)B(w)\big) \equiv A(z)\cdot B(w), \qquad R\big(\bar{A}(\bar{z})\bar{B}(\bar{w})\big) \equiv \bar{A}(\bar{z})\cdot\bar{B}(\bar{w}).$$

In practice, such operator products are computed using the Wick contraction rules with the appropriate propagator.

As an example, consider a single free boson with action $S[X] = \frac{i}{2\pi\alpha'}\int\partial X\wedge\bar{\partial}X$, energy momentum $T(z) = -\frac{1}{\alpha'} : \partial X(z)\partial X(z) :$ and propagator $\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\alpha'}{2}\ln|z-w|^2$. Carrying out the contractions and Taylor expanding one learns that

$$T(z)\cdot\partial X(w,\bar{w}) \cong \frac{1}{(z-w)^2}\partial X(w,\bar{w}) + \frac{1}{z-w}\partial^2 X(w,\bar{w}) + \dots$$

The corresponding contractions in $\bar{T}(\bar{z})\cdot\partial X(w,\bar{w})$ vanish. The symbol $::$ indicates that one should not include self-contractions, whereas the symbol $\cong$ indicates equivalence in an operator product expansion sense. We thus learn that $\partial X(z,\bar{z})$ is a primary conformal field of weight $(h,\bar{h}) = (1,0)$. With this information one may immediately write down the variation of ∂X, under a general conformal transformation $z \to z + \epsilon(z)$. For example,

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\partial X(0) &= \frac{1}{2\pi i}\oint_0 dz\,\epsilon(z)\Big(\frac{1}{z^2}\partial X(0) + \frac{1}{z}\partial^2 X(0)\Big) \\ &= \epsilon_1\partial X(0) + \epsilon_0\partial^2 X(0), \end{aligned} \tag{3.11}$$

where we have expanded $\epsilon(z) = \sum_n \epsilon_n z^n$. Therefore, ∂X transforms nontrivially under $z \to z + \epsilon_0 + (\mathrm{Re}\,\epsilon_1)z + i(\mathrm{Im}\,\epsilon_1)z$ (that is translations, dilatations and rotations), and is invariant under special conformal transformations, $z \to z + \epsilon_2 z^2$, see comments below (3.15). The same comments of course hold for any conformal primary of weight $(1,0)$.

We have shown how to interpret commutation relations of the form (3.6), when the charge Q of the theory is given in terms of a contour integral which encircles one or more insertions. In a similar manner one can show that this procedure generalizes for arbitrary operators of the form,

$$A = \oint dz\, a(z), \qquad \text{and} \qquad B = \oint dw\, b(w),$$

and that there exists the interpretation, see e.g. [175],

$$\begin{aligned} [A,B] &\cong A \cdot B = \oint_0 dw \oint_w dz\, a(z) \cdot b(w), \\ [A, b(w)] &\cong A \cdot b(w) = \oint_w dz\, a(z) \cdot b(w), \end{aligned} \tag{3.12}$$

where now the meromorphy of the integrand has been made manifest, and the dot again denotes operator product expansion (OPE). We make extensive use of (3.12) throughout. The second relation has been proven above. In the first one needs to interpret the quantity

$$[A,B] = \Big(\oint dz \int dw - \oint dw \oint dz\Big) a(z) b(w),$$

and proceed as follows. We fix w and deform the difference between the two z integrations into a contour encircling the single point w. We then perform the z contour deformation using the above radial ordering prescription, and are then free to perform the remaining w integration. This leads to the first relation in (3.12).

3.2 Virasoro Algebra

In (D.21) we Laurent expanded $\epsilon(z)$ in order to show that the $d = 2$ conformal group consists of an infinite number of generators. Similarly, in the quantum theory we can Laurent expand the quantum energy momentum tensor,

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \qquad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n,$$

in which case the quantities L_n are themselves to be interpreted as operators. The exponents of z, $\bar{z}$ have been chosen so that L_n, $\bar{L}_n$ have scaling dimension n; under $z \to \lambda z$, we have $L_n \to \lambda^n L_n$ because $T(z) \to \lambda^{-2} T(\lambda z)$. Inverting this expansion, we can solve for the generators,

$$L_n = \frac{1}{2\pi i} \oint dz\, z^{n+1} T(z), \qquad \text{and} \qquad \bar{L}_n = -\frac{1}{2\pi i} \oint d\bar{z}\, \bar{z}^{n+1} \bar{T}(\bar{z}). \tag{3.13}$$

These expressions are to be interpreted as operator equations in the sense that they can only be evaluated when the insertions inside the contours are specified.

The energy-momentum tensor itself is not a conformal primary, as can be seen in the free boson example above, where $T(z) = -\frac{1}{\alpha'} : \partial X(z) \partial X(z) :$. Computing the OPE's leads to,

$$
\begin{aligned}
T(z) \cdot T(w) &\cong \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w), \\
\bar{T}(\bar{z}) \cdot \bar{T}(\bar{w}) &\cong \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2} \bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}} \bar{\partial} \bar{T}(\bar{w}),
\end{aligned}
\tag{3.14}
$$

For the free boson theory the *central charge* $c = 1$ but for more general CFT's the only constraint is $c \geq 0$ (from requiring $\langle T(z)T(0) \rangle = (c/2)/z^4 \geq 0$) and $c - \bar{c} = 0 \mod 24$. For the matter part of the bosonic string there are $d = 26$ free bosons, $c = 26$. When ghosts are added to the system (by exponentiating the Fadeev-Popov determinants in the path integral measure associated to the space of Riemann surfaces) the central charge of the total energy momentum tensor, $T(z, \bar{z}) + T_{\rm gh}(z, \bar{z})$, vanishes, $c = \bar{c} = d - 26 = 0$ when $d = 26$. We find it more convenient to work in the OCQ (old covariant quantization) [166] formalism where the ghost contribution manifests itself as a restriction on the physical Fock space of states corresponding to the Virasoro conditions (see below). Applying the general expression (3.12) for the charges $L_n, \bar{L}_n$ on account of (3.14) ones learns that they satisfy the algebra,

$$
\begin{aligned}
[L_n, L_m] &= (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \\
[\bar{L}_n, \bar{L}_m] &= (n-m) \bar{L}_{n+m} + \frac{\bar{c}}{12}(n^3 - n)\delta_{n+m,0},
\end{aligned}
\tag{3.15}
$$

which is referred to as the *Virasoro algebra*, is closely related to the Witt algebra (D.24) described above. The difference lies in the central charge term, a quantum-mechanical anomaly. Recall that a conformal transformation is not just a reparametrization of the coordinates, although there is an non-trivial overlap, see Appendix F, $\mathrm{Weyl}(\Sigma) \ltimes \mathrm{Diff}(\Sigma)$, which is what allowed us to derive the conformal algebra in the first place; any sensible theory should be diffeomorphism invariant. Conformal invariance is a diffeomorphism followed by a compensating Weyl rescaling, and the central charge can be thought of as being due to this Weyl rescaling [176]. The line element that is invariant under diffeomorphisms gets rescaled under conformal transformations, $ds^2 \to \Omega ds^2$. The two algebras coincide on the global $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ subgroup spanned by the generators, $L_\pm, L_0$ and $\bar{L}_\pm, \bar{L}_0$, see (D.25).

Clearly, the energy-momentum tensor is not a primary field as it does not satisfy the defining relation (3.10). If c were zero one sees from (3.14) that $T(z)$ would transform as a (2,0) conformal primary. Instead, under conformal transformations it transforms according to,

$$
T'(z')(dz')^2 = \Big(T(z) + \frac{c}{12} S(z', z) \Big) (dz)^2,
$$

with the *Schwarzian* defined as $S(z', z) = (\partial^3 z')(\partial z')^{-1} - \frac{3}{2}(\partial^2 z')^2 (\partial z')^{-2}$. An example is the relation between the energy-momentum tensor in the w and $z = e^{-iw}$ coordinate system,

$$T_{\text{cylinder}}(w) = -z^2 T_{\text{plane}}(z) + \frac{c}{12}.$$

In (2.14) it was seen that the classical constraints enforce the tracelessness of the energy-momentum tensor. At the quantum level, conformal invariance is broken by the presence of the central charge,

$$\langle T^{\alpha}{}_{\alpha} \rangle = -\frac{c}{12} R_{(2)}.$$

However, this need not worry us because when the Fadeev-Popov determinants are properly taken into account in the critical dimensions (d =26 or 10) conformal invariance is always restored [141].

3.3 Representations of the Virasoro Algebra

Building on the above considerations, let us now study representations of the Virasoro algebra, and in particular the constraints placed on the corresponding spectrum. The operator that generates translations in time, namely the Hamiltonian, is constructed from the Hermitian combination $L_0 + \bar{L}_0$, see Appendix D; that is, the charge associated to dilatations on the complex plane. If one is interested in energy eigenstates one should in particular be searching for eigenstates of $L_0 + \bar{L}_0$. Because the Virasoro algebra factorizes, such states will be eigenstates of both L_0 and $\bar{L}_0$. Suppose $|\phi\rangle$ is such a state,

$$L_0|\phi\rangle = h|\phi\rangle, \qquad \bar{L}_0|\phi\rangle = \bar{h}|\phi\rangle. \tag{3.16}$$

From the Virasoro algebra it follows that,

$$[L_0, L_n] = -nL_n,$$

and so one may construct other eigenstates of L_0 by acting with L_n, $L_0\big(L_n|\phi\rangle\big) = (h - n)\big(L_n|\phi\rangle\big)$. Therefore, generators with $n < 0$ raise the energy eigenvalue h by n units and generators with $n > 0$ decrease the energy eigenvalue h by n units. If the energy of the spectrum is bounded from below there must exist states that are annihilated by all $L_{n>0}$, $\bar{L}_{n>0}$, and such states are called *primary* (the highest weight states of the algebra),

$$L_n|\phi\rangle = \bar{L}_n|\phi\rangle = 0 \qquad \text{for all} \qquad n > 0. \tag{3.17}$$

Representations of the Virasoro algebra are then built by acting on primary fields with the raising operators L_{-n}, $\bar{L}_{-n}$ with $n > 0$, and this generates a set of states called the *Verma* module. These correspond to irreducible representations of the Virasoro algebra. If the spectrum of primary states is known, so will be the spectrum of the entire theory. The vacuum of the theory, $|0\rangle$, is invariant under as many symmetries as possible. That is, it has $h = 0$ and obeys

$$L_0|0\rangle = L_{n>0}|0\rangle = 0.$$

Requiring that also $L_{n<0}|0\rangle = 0$ would be inconsistent with (the central charge term in) the Virasoro algebra. States constructed in this manner are not independent, there exist null states with vanishing norm, $\langle\phi_{\rm null}|\phi_{\rm null}\rangle = 0$, and one should be able to identify and isolate such states. Happily, this is accomplished in a very elegant fashion in the DDF construction of Sec. 4.4 as we will show.

So far, we have presented the properties that representations of the Virasoro algebra should satisfy, while focusing in particular on energy eigenstates. Now we would like to relate these concepts to representations of the Virasoro algebra that may be identified with spatially extended (in the background spacetime sense) objects, namely strings. To discuss this link we will first elaborate on the *state-operator* isomorphism that will be crucial in the following developments. The expression (3.13) for the Virasoro generators suggests that for every canonical primary state $|\phi\rangle$, there exists a local insertion on the worldsheet, $\phi(z,\bar{z})$. That is, (3.13) suggests the isomorphism,[2]

$$\begin{aligned} L_n|\phi\rangle &\cong \frac{1}{2\pi i}\oint_C dz z^{n+1}T(z)\cdot\phi(w,\bar{w}), \\ &\cong L_n\cdot\phi(z,\bar{z}), \end{aligned} \tag{3.18}$$

the contour C being taken around the insertion $\phi(w,\bar{w})$, and as discussed above the dot denotes OPE's.

Using our physical intuition let us consider a string worldsheet that is embedded into spacetime and evolving in some unspecified manner without interacting with other objects. (We will become more specific about how to analytically consider such a setup in the next chapter when we discuss path integrals.) This object will generically be extended in the spacetime sense, it will be non-local, and its state at any one point in time will be specified by a wavefunction of the form $|\Psi(X(z,\bar{z}))\rangle$, with X the embedding of the string into spacetime, $X:\Sigma\to\mathcal{M}$. Then, let us suppose that one may consider the initial state of this string, and that this is a well defined possibility.[3] The conformal invariance on the *worldsheet* then suggests that one can rescale any point on this worldsheet without affecting any spacetime observables. Let us therefore rescale all points on the worldsheet, such that this initial string state is shrunk to a point $z,\bar{z}$. This state must be invariant under the string theory symmetries; that is, it must be invariant under such worldsheet rescallings and it must be diffeomorphism invariant (or covariant), in both the spacetime and worldsheet sense. Therefore, given that the initial string state has been shrunk to a point, it follows that there *must* exist a local worldsheet insertion, call it $V(z,\bar{z})$, that has precisely these symmetries which is in one-to-one correspondence with the extended string in spacetime – we call this local insertion, $V(z,\bar{z})$, a *string vertex operator*. We take this one step further and suggest that *any* non-trivial local insertion that has the string theory symmetries and is composed of the string theory fields can be identified with a string vertex operator. No single point on the worldsheet should be distinguished, and therefore vertex operators $V(z,\bar{z})$ should be integrated over,

$$V = \int d^2z V(z,\bar{z}).$$

[2]See Polchinski [114] for a nice discussion of the state-operator map.

[3]This may not be possible in curved spacetimes (e.g. de Sitter space) where there are no asymptotically free states.

Given that d^2z is a conformal primary operator of weight $(-1,-1)$ it follows that vertex operators $V(z,\bar{z})$ must be conformal primary operators of weight $(1,1)$, so that the combination V is conformally invariant, i.e. so that V is a conformal primary with conformal weight (0,0). The state operator map (3.18) and the physical state conditions (3.16) and (3.17) then suggest the following definition of a string theory vertex operator:

$$L_{n>0}\cdot V(z,\bar{z})\cong 0,\qquad L_0\cdot V(z,\bar{z})\cong V(z,\bar{z}),\tag{3.19}$$

and similarly for the antiholomorphic sector,

$$\bar{L}_{n>0}\cdot V(z,\bar{z})\cong 0,\qquad \bar{L}_0\cdot V(z,\bar{z})\cong V(z,\bar{z}).\tag{3.20}$$

In Sec. 4.4 we will explain how to solve these constraints completely and hence identify a complete set of mass eigenstates. In the chapter following that we will construct the corresponding coherent state vertex operators that will be identified with cosmic strings.

We have discussed CFT's, we specified how operators transform under conformal rescallings, and defined the spectrum of states and their relation to string states. In the next section we present the path integral definition of string theory. We will only be able to scratch the surface as this is a vast subject area. Many details can be found in the Appendices and there are also many very good texts on the subject, e.g. [141, 150, 114, 2] are extremely well written and focus on the quantum mechanical perturbative definition of string theory that will be of interest to us (but see also [166, 176]) to name just a handful of these with an approach closest to ours.

3.4 String Amplitudes

In this section we give a systematic overview of string path integrals and scattering amplitudes to set the scene for the vertex operators and scattering amplitudes that we construct and discuss in the following sections. We restrict our attention to the simplest case of bosonic string path integrals for closed strings, although most results carry over also to the corresponding open string construction [177, 178], as well as to the corresponding superstring [141].

Let us consider the scattering of N (in general distinct) vertices of the generic form (4.1). We will be working at arbitrary genus for the main part of the computation until we finally specialize to tree and one-loop perturbation theory where the measure associated to metrics and the moduli space is better understood than the corresponding quantities for higher loop amplitudes.

In the Polyakov approach to string theory [179, 180], one is instructed to integrate over distinct worldsheet metrics, g, associated to a Riemann surface Σ, and target space embeddings of the worldsheet into spacetime,

$$X:\Sigma\to\mathcal{M}.$$

Throughout we will focus on the case of a flat Minkowski background, with an appropriate Wick rotation to Euclidean space $\mathcal{M} = \mathbb{R}^{26}$. S-matrix elements then correspond to path integrals, see e.g. [141], of the form,

$$\big\langle V^{(1)}\dots V^{(N)}\big\rangle \;=\; \sum_{h=0}^{\infty}\int_{\mathcal{E}\times\mathcal{M}_h}\mathcal{D}g\mathcal{D}X e^{-S[g,X]}V^{(1)}\dots V^{(N)}, \tag{3.21}$$

where $S[g,X]$ represents the action for the bosonic string [114]. This identification of the path integral with S-matrix elements presupposes that vertex operators are normalized to 'one string in volume V_{25}' when we truncate space so that $V_{25}\equiv(2\pi)^{25}\delta^{25}(\mathbf{p}'-\mathbf{p})$. Working (in a locally flat Euclidean worldsheet metric) in conformal gauge,[4] $ds^2 = g_{z\bar{z}}(dz\otimes d\bar{z}+d\bar{z}\otimes dz)$, in flat Euclidean spacetime[5], $G_{\mu\nu}(X)\to\delta_{\mu\nu}$, and assuming the dilaton has acquired a constant vacuum expectation value, $\Phi(X)\to\langle\Phi\rangle$, the terms that will be relevant for the thesis take the form,

$$S[g,X] = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\partial_z X\cdot\partial_{\bar{z}}X + \mu^2\int_\Sigma d^2z\sqrt{g} + \langle\Phi\rangle\chi(\Sigma)+\dots, \tag{3.22}$$

where $\chi(\Sigma) = 2-2h$ is the Euler characteristic of the genus h Riemann surface Σ. The first term is the Polyakov action, $S_G[g,X]$, the Euclidean flat space version of (2.11). The second term vanishes classically, $\mu^2 = 0$, but is required for the renormalizability of the theory [181, 144]. The dots denote terms, see Appendix A, that are not relevant for what follows.

The sum appearing in the path integral is over topologies which in the case of closed strings is parametrized by the genus h of Σ. We shall be interested only in closed string scattering and hence only need consider closed Riemann surfaces. One is then instructed to sum over surfaces (worldsheets) with the topology of the sphere, the torus, etc., thus generating a perturbative expansion in $e^{\langle\Phi\rangle}$, which is therefore identified with [114] the string coupling constant,

$$g_s = e^{\langle\Phi\rangle}.$$

In Fig. 3.2 this perturbative expansion is exhibited for the case of n arbitrary asymptotically free (i.e. onshell) string states; we shall refer to these as *vertex operators*. The asymptotic states we shall consider carry arbitrarily large quantum numbers, and will correspond to arbitrarily excited states: the main theme of the current thesis is precisely the construction of these vertex operators – vertex operators that we identify with *fundamental cosmic strings*.

By confining oneself to a flat Minkowski (or Euclidean) (but also a more general *constant*) background (which is what we shall do here), one is in turn effectively considering the tree level term in a perturbative expansion in the (inverse) string tension α'. To see this, one should write the bosonic string action (3.22) in terms of dimensionless fields

[4] We will throughout be using complex coordinates for the worldsheet where holomorphy when present becomes manifest, see Appendix B.

[5] We assume that one may analytically continue back to Minkowski space at the end of the calculation in a consistent manner.

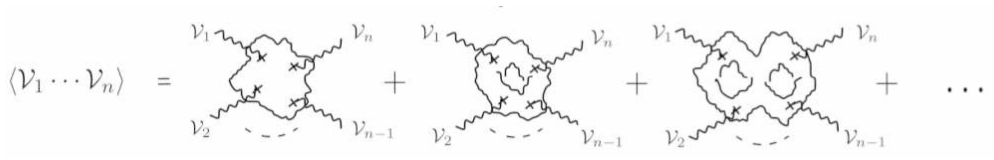

Figure 3.2: A diagrammatic representation of a perturbative expansion of an n-string interaction amplitude in the string coupling g_c. We have conformally transformed the worldsheet in order to shrink the asymptotic string states to local points. This enables us to define local functionals, $\mathcal{V}_i(z_i, \bar{z}_i)$, with $V_i = \int_\Sigma d^2 z_i \sqrt{g} \mathcal{V}_i(z_i, \bar{z}_i)$, which are to represent the corresponding asymptotically free string states. As an example, the one loop diagram represents the amplitude for, say, 2 strings to merge, create a single string which in turn breaks into two and subsequently decays into $n-2$ specified asymptotic states.

X and expand the spacetime metric $G_{\mu\nu}(X)$ (in such a way that the symmetries are preserved, e.g. in normal coordinates) inside the path integral (3.21) around a fixed background, which can be chosen to be Minkowski space $\eta_{\mu\nu}$. The resulting expansion can then be seen to be a perturbative expansion in the string coupling α'.

We was careful above to mention that the integral is to be taken over *distinct* configurations. It turns out [182] that to achieve this we may integrate over *all* embeddings, the space of embeddings being denoted by $\mathcal{E}$, and over *moduli space*, $\mathcal{M}_h$, (see Appendix F)

$$\mathcal{M}_h \equiv \mathrm{Met}(\Sigma)/\mathrm{Weyl}(\Sigma) \ltimes \mathrm{Diff}(\Sigma).$$

One is in essence integrating over deformations of the worldsheet metric that cannot be reached by a symmetry transformation, i.e. by diffeomorphisms and Weyl transformations – these are assumed to be true symmetries of the theory at the quantum as well as the classical level. This restriction to $\mathcal{M}_h$ is thus to ensure no over-counting and is crucial in the path integral approach to string theory. Using the Fadeev-Popov procedure, one decomposes the measure associate to gauge and moduli (physical) deformations of metric, $\mathcal{D}g = J\mathcal{D}(\text{gauge})d(\text{moduli})$, with J the associated Jacobian for the change of coordinates. Notice that the measure associate to moduli deformations is finite dimensional. One then determines the Jacobian, the coordinates on moduli space and drop the measure associated to gauge variations. The resulting object $Jd(\text{moduli})$ is to replace the path integral measure $\mathcal{D}g$ in (3.21). Schematically, a gauge slice associated to the moduli integration integral is shown in Fig. 3.3. This procedure is standard [183, 184, 182, 144, 185, 181, 186, 141, 187, 114] and a brief overview has been included in Appendix F, leading to the gauge fixed form (F.33) of the path integral (3.21),

$$\begin{aligned}&\big\langle V^{(1)} \dots V^{(N)} \big\rangle = \\ &= \sum_{h=0}^{\infty} g_s^{-\chi(\Sigma)} \int_{\mathcal{M}_h} d(\mathrm{WP}) \, \det{}' \Delta_{-1}^{-} \frac{\mathrm{vol(CKV)}^{-1}}{|\det(\psi^z, \psi^z)|} \left(\frac{4\pi^2 \alpha'}{\int_\Sigma d^2 z \sqrt{g}} \det{}' \Delta_{(0)} \right)^{-d/2} \big\langle\!\big\langle V^{(1)} \dots V^{(N)} \big\rangle\!\big\rangle\end{aligned} \tag{3.23}$$

with $d(\mathrm{WP})$ denoting the Weil-Peterson measure on moduli space. Arbitrary smooth metrics on Σ are (due to the uniformization theorem) conformally equivalent to constant curvature Riemann surfaces. We may therefore, if one so pleases, choose a gauge slice

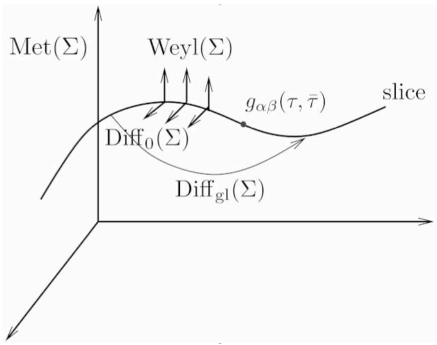

Figure 3.3: A gauge slice in the space of metrics paramterized by the moduli τ, $\bar\tau$. Notice the presence of both global diffeomorphisms of the worldsheet Σ, $\mathrm{Diff}_{\mathrm{gl}}(\Sigma)$, and diffeomorphisms connected to the identity, $\mathrm{Diff}_0(\Sigma)$.

such that we only integrate over constant curvature Riemann surfaces, and this constant curvature gauge slice defines the Weil-Peterson measure; it is defined more explicitly in (F.32). The quantities ψ^z and $\psi^{\bar z}$ form a basis for conformal Killing vectors and vol(CKV) is identified with the volume of the conformal Killing group.[6]. When the surface admits no CKV's, i.e. when $h>1$, one is to set $\mathrm{vol(CKV)}^{-1}=|\det(\psi^z,\psi^z)|$, see (F.30). We have found it convenient to define a correlation function associated to embeddings at *fixed* metric g, $\langle\!\langle V^{(1)}\dots V^{(N)}\rangle\!\rangle$, by the expression (the case of interest being $d=26$)

$$\langle\!\langle V^{(1)}\dots V^{(N)}\rangle\!\rangle\equiv\left(\frac{4\pi^2\alpha'}{\int_\Sigma d^2z\sqrt{g}}\,\det{}'\Delta_{(0)}\right)^{d/2}\int_{\mathcal{E}}\mathcal{D}Xe^{-S_G[g,X]}\,V^{(1)}\dots V^{(N)}. \tag{3.24}$$

We would next like to evaluate the path integral over embeddings, X, (with g fixed) and discuss the various complications that arise when vertex operator insertions are included. We would like to proceed without specifying the exact form of the insertions, and to accomplish this we make the basic observation that mass eigenstate vertex operators can always be cast in the form

$$V^{(\alpha)}=\frac{g_c}{\sqrt{2k^0V_{d-1}}}\int d^2z\sqrt{g}\,P_\alpha(\partial X,\partial^2X,\dots;\bar\partial X,\bar\partial^2X,\dots)e^{ik_{(\alpha)}\cdot X(z,\bar z)},$$

with $P_\alpha(\partial X,\partial^2X,\dots;\bar\partial X,\bar\partial^2X,\dots)$ a polynomial in the arguments, with associated polarization tensors appearing linearly. The couplings g_s and g_c are *a priori* different coupling constants. The former is dimensionless by definition, whereas the latter is determined by requiring that there be 'one string in volume V_{d-1}', which leads to a unitary S-matrix and is such that vertex operators are dimensionless; vertex operator normalization will be discussed in detail later. Coherent states are in turn given by linear combinations of such mass eigenstates, but because coherent states are not eigenstates of energy, the kinematic factor will be different for these. These polynomials are given explicitly in Sec. 4.4 where

[6] In particular, CKV's are globally defined vectors of the form $v^z_{\rm CKV}=a^s\psi^z_s$ with $v^z_{\rm CKV}\in\ker\nabla^{(-1)}_{\bar z}$ (and $v^{\bar z}_{\rm CKV}=\bar a^s\psi^{\bar z}_s$ with $v^{\bar z}_{\rm CKV}\in\ker\nabla^{(1)}_{z}$). The range $s\in\{0,\dots,\dim_{\mathbb{C}}\ker\nabla^{(-1)}_{\bar z}\}$, i.e. the number of CKV's admitted by the surface, depends on its genus h, see (F.13), and on the number of boundaries and crosscaps for open string worldsheets. Furthermore, $\mathrm{vol(CKV)}=\int d^ka\,d^k\bar a$.

we show that they may always be represented in terms of elementary Schur polynomials. Alternatively, they may also reduce to single monomials as shown in Sec. 4.1, when the polarization tensors have the symmetries of Young tableaux.

The task is then to represent the product of vertex operators appearing in a string amplitude,[7] $V^{(1)}\dots V^{(N)}$, in terms of an embedding independent operator, $\mathcal{G}_A$ (which is to include the product of worldsheet integrals, polarization tensors and normalization constants). Omitting for notational simplicity the overall factor $\frac{g_c}{\sqrt{2k_1^0 V_{d-1}}}\dots\frac{g_c}{\sqrt{2k_N^0 V_{d-1}}}$ from the 'one string in volume V_{d-1}' normalization of vertex operators, we can write:

$$V^{(1)}\dots V^{(N)} \equiv \sum_A \mathcal{G}_A \prod_{l\in\mathcal{I}_A^n}\Big(D_l\frac{\delta}{\delta J_{\mu_l}(z_l,\bar z_l)}\Big)e^{i\int d^2z J(z,\bar z)\cdot X(z,\bar z)}, \tag{3.25}$$

provided we take $J(z,\bar z)$ to have some specific form after we have integrated out X. For example, for mass eigenstate vertex insertions, $J(z,\bar z)=\sum_{\alpha=1}^N\delta^2(z-z_\alpha)k_{(\alpha)}$ with the vertex operators $V^{(\alpha)}(z_\alpha,\bar z_\alpha)$ inserted at $z_\alpha,\bar z_\alpha$. We define the operator $\mathcal{G}_A$ and the index set $\mathcal{I}_A^n$ by this expression – the explicit form of $\mathcal{I}_A^n$ and $\mathcal{G}_A$ can be determined once a choice of vertex insertions has been made. From the results of Sec. 4.4 or 5, extracting $\mathcal{G}_A$ for arbitrary vertices is just a matter of algebra. The sum over (possibly multiple indices) A is associated to series expanding the product of polynomials,

$$P_1\dots P_N \equiv \sum_A \mathcal{G}_A\cdot\prod_{l\in\mathcal{I}_A^n}\big(D_lX(z_l,\bar z_l)\big),$$

with the dot denoting spacetime index contractions. We also define the index set $\mathcal{I}_A^n$ by this expression. The differential operators D_l appearing may represent arbitrary worldsheet derivatives,

$$D_l=\frac{\partial^\#}{\partial z_l^\#},$$

and are by definition completely determined by the index $l\in\mathcal{I}_A^N$. The total number of terms in the index set $\mathcal{I}_A^N$, denoted by

$$\mathcal{I}\equiv|\mathcal{I}_A^N|,$$

is equal to the rank of the product of polarization tensors appearing in $\mathcal{G}_A$. The above enables us to represent the path integral over X for arbitrary vertex insertions by,

$$\begin{aligned}\Big\langle\!\Big\langle V^{(1)}\dots V^{(N)}\Big\rangle\!\Big\rangle &= \sum_A\mathcal{G}_A\Big\langle\!\Big\langle D_1X^{\mu_1}(z_1,\bar z_1)\dots D_{\mathcal{I}}X^{\mu_{\mathcal{I}}}(z_{\mathcal{I}},\bar z_{\mathcal{I}})\ e^{i\int d^2zJ(z,\bar z)\cdot X(z,\bar z)}\Big\rangle\!\Big\rangle\\ &=\sum_A(-i)^{\mathcal{I}}\mathcal{G}_A\prod_{l\in\mathcal{I}_A^n}\Big(D_l\frac{\delta}{\delta J_{\mu_l}(z_l,\bar z_l)}\Big)\Big\langle\!\Big\langle e^{i\int d^2zJ(z,\bar z)\cdot X(z,\bar z)}\Big\rangle\!\Big\rangle\end{aligned} \tag{3.26}$$

For the above construction to become possible, we use worldsheet *point splitting* [188]. An un-integrated vertex operator, denoted by $V^{(\alpha)}(z_\alpha,\bar z_\alpha)$, corresponds to a local insertion

[7] Of course, certain vertex operators will be integrated over, others will be fixed if there are CKV's present – this is of no concern at this point.

on the worldsheet at $z_\alpha, \bar{z}_\alpha$. In the $A^{\rm th}$ term of the above series expansion, one splits the n vertex insertion points into $\mathcal{I}$ distinct points on the worldsheet,

$$\{z_1, \bar{z}_1, \dots, z_N, \bar{z}_N\} \to \{z_1, \bar{z}_1, \dots, z_{\mathcal{I}}, \bar{z}_{\mathcal{I}}\},$$

where of course $\mathcal{I} \geq N$. Then perform various computations of interest (e.g. integrate out X) and subsequently *point-merge* back to the original configuration of N vertex insertions,

$$\{z_1, \bar{z}_1, \dots, z_{\mathcal{I}}, \bar{z}_{\mathcal{I}}\} \to \{z_1, \bar{z}_1, \dots, z_N, \bar{z}_N\},$$

before integrating them over the worldsheet (or fixing them in the presence of CKV's). Any singular contributions that arise due to point merging are to be subtracted and this is equivalent to requiring that the original vertex operators are normal ordered [114]. If the original vertex operators are onshell and normal ordered, the metric g will not appear in the vertex operators.

We next integrate out X in the presence of arbitrary vertex operator insertions. From the second line in (3.26) and

$$\Big\langle\!\Big\langle e^{i\int d^2z J(z,\bar{z})\cdot X(z,\bar{z})}\Big\rangle\!\Big\rangle = i(2\pi)^d\delta^d(J_0)e^{-\frac{1}{2}\int d^2z\int d^2z' J(z,\bar{z})\cdot J(z',\bar{z}')G'(z,z')},$$

see Appendix E, it can be seen that we need to compute a Gaussian derivative of arbitrary order. This is computed in the Appendix with the result (E.14). From this follows the general expression,

$$\begin{aligned}
&\Big\langle\!\!\Big\langle D_1X^{\mu_1}(z_1,\bar{z}_1)\dots D_{\mathcal{I}}X^{\mu_{\mathcal{I}}}(z_{\mathcal{I}},\bar{z}_{\mathcal{I}})\ e^{i\int d^2zJ(z,\bar{z})\cdot X(z,\bar{z})}\Big\rangle\!\!\Big\rangle\\
&= i(2\pi)^d\delta^d(J_0)\sum_{k=0}^{\lfloor\mathcal{I}/2\rfloor}\sum_{\pi\in S_{\mathcal{I}}/\sim}\prod_{l=1}^{k}\Big\{\eta^{\mu_{\pi(2l-1)}\mu_{\pi(2l)}}D_{\pi(2l-1)}D_{\pi(2l)}G(z_{\pi(2l-1)},z_{\pi(2l)})\Big\}\\
&\quad\times\prod_{q=2k+1}^{\mathcal{I}}\Big\{i\int d^2zJ^{\mu_{\pi(q)}}(z,\bar{z})D_{\pi(q)}G(z_{\pi(q)},z)\Big\}\ e^{-\frac{1}{2}\int d^2z\int d^2z' J(z,\bar{z})\cdot J(z',\bar{z}')G(z,z')},
\end{aligned}\tag{3.27}$$

from which (after point merging) all scattering amplitudes can be derived. There is a sum over permutations, $\pi \in S_{\mathcal{I}}/\sim$ with $S_{\mathcal{I}}$ the *symmetric group* of degree $\mathcal{I}$ [189], with the equivalence relation defined such that $\pi_i \sim \pi_j$ with $\pi_i, \pi_j \in S_{\mathcal{I}}$ when they define the same element in (3.27). The notation $\lfloor\mathcal{I}/2\rfloor$ in the sum over k indicates that the maximum value of k saturates the inequality $k \leq \mathcal{I}/2$.

$G(z,w)$ is the regularized scalar Green's function, which on multi-loop compact Riemann surfaces [190, 141, 150] reads (for $z \neq w$),

$$G(z,w) = -\frac{\alpha'}{2}\ln|E(z,w)|^2 + \pi\alpha'{\rm Im}\int_z^w \omega_I\,({\rm Im}\Omega)^{-1}_{IJ}\,{\rm Im}\int_z^w\omega_J + \dots \tag{3.28}$$

Here $E(z,w)$ is the *prime form*, the ω_I are *Abelian holomorphic differentials* and Ω_{IJ} is the *period matrix* of the genus h closed Riemann surface. The definitions and a brief overview of the properties of these objects is given in Appendix C. Suffice it to note here that $E(z,w)$ is the natural generalization of a geodesic $z-w$ on $\mathbb{C}$, to a geodesic on an arbitrary genus Riemann surface, and that it has a representation in terms of theta functions (C.16). From the Riemann-Roch-Atiyah-Singer index theorem (F.12), it follows that there are precisely h Abelian holomorphic differentials ω_I, and so $I=\{1,\dots,h\}$. These form a basis for the first cohomology group, $H^1(\Sigma,\mathbb{C})$, and are defined by their natural pairing with the A_I, B_I homology cycles of the Riemann surface,

$$\oint_{A_I}\omega_J=\delta_{IJ},\qquad \text{and}\qquad \oint_{B_I}\omega_J=\Omega_{IJ}.$$

Finally, the $h\times h$ period matrix, Ω_{IJ}, characterizes the complex structure of the Riemann surface, it is symmetric and has a positive imaginary part,

$$\Omega_{IJ}=\Omega_{JI},\qquad \mathrm{Im}\,\Omega_{IJ}>0,$$

and reduces to the familiar complex modulus $\tau=\tau_1+i\tau_2$ at genus $h=1$. See Appendix C for further elaboration and references given therein.

If the vertex operators inserted into the path integral are not normal ordered, self-contractions would have to be included, potentially leading to non-regular contributions to the amplitude from $\lim_{z\to w}E(z,w)$. These divergences can be absorbed by a renormalization of the string coupling, g_c – introduce a UV cut off, $|\epsilon|$, and in amplitudes replace the Green's function at coincident points by the regularized Green's function,

$$G_{\mathrm{R}}(z,z)=\frac{\alpha'}{2}\left(\ln g_{z\bar z}-\ln|\epsilon|^2\right)+\dots \tag{3.29}$$

The regularization has been carried out in a diffeomorphism invariant manner: the invariant distance on the worldsheet, $ds^2=2g_{z\bar z}dzd\bar z$, leads to the natural definition of a UV cut off, $\lim_{z\to w}g_{z\bar z}|E(z,w)|^2=|\epsilon|^2$, where we have used the fact that $E(z,w)\simeq z-w$ for $z\sim w$. Both $G(z,w)$ and $G_{\mathrm{R}}(z,z)$ are derived in Appendix G. The cutoff $|\epsilon|$ can be absorbed by coupling constant renormalization, $g_c\to g_c'$, and the explicit metric dependence in the first term in $G_{\mathrm{R}}(z,z)$ drops out for onshell external vertex operators.

The final expression for a general scattering amplitude follows from substituting (3.27) into (3.26) (with $d=26$ for the bosonic string),

$$\begin{aligned}\langle\!\langle V^{(1)}\dots V^{(N)}\rangle\!\rangle&=i(2\pi)^d\delta^d(J_0)\frac{g_c}{\sqrt{2k_1^0V_{d-1}}}\dots\frac{g_c}{\sqrt{2k_N^0V_{d-1}}}\\ &\sum_A\mathcal{G}_A\cdot\sum_{k=0}^{\lfloor\mathcal{I}/2\rfloor}\sum_{\pi\in S_{\mathcal{I}}/\sim}\prod_{l=1}^{k}\Big\{\eta^{\mu_{\pi(2l-1)}\mu_{\pi(2l)}}D_{\pi(2l-1)}D_{\pi(2l)}G(z_{\pi(2l-1)},z_{\pi(2l)})\Big\}\\ &\times\prod_{q=2k+1}^{\mathcal{I}}\Big\{i\int d^2zJ^{\mu_{\pi(q)}}(z,\bar z)D_{\pi(q)}G(z_{\pi(q)},z)\Big\}\,e^{-\frac12\int d^2z\int d^2z'J(z,\bar z)\cdot J(z',\bar z')G(z,z')},\end{aligned} \tag{3.30}$$

and we have now re-inserted the kinematic factors and coupling constant associated to the 'one string in volume V_{d-1}' normalization of vertex operators. When the vertex operators are normal ordered one is to subtract self-contractions in the point merging procedure –

otherwise the regularized Green's function given above is required. In the particular case when $J(z,\bar{z}) = \sum_{\alpha=1}^{N} \delta^2(z - z_\alpha)k_{(\alpha)}$ with $k^\mu_{(\alpha)}$ the momentum associated to the vertex operator $V^{(\alpha)}$, the exponential factor in (3.30) reduces to,

$$\exp\left\{-\sum_{\alpha<\beta} k_{(\alpha)} \cdot k_{(\beta)}\, \pi\alpha' \mathrm{Im} \int_{z_\alpha}^{z_\beta} \omega_I\, (\mathrm{Im}\Omega)^{-1}_{IJ}\, \mathrm{Im} \int_{z_\alpha}^{z_\beta} \omega_J \right\} \prod_{\alpha<\beta} |E(z_\alpha, z_\beta)|^{\alpha' k_{(\alpha)}\cdot k_{(\beta)}} .$$

The prime form and moduli dependence of the amplitude (3.30) can become manifest by use of a generalization of the binomial theorem,

$$\prod_{i=1}^{n}(A_i + B_i) = \sum_{s=0}^{n} \sum_{\pi\in S_n/\sim} \prod_{i=1}^{n-s} A_{\pi(i)} \prod_{j=n-s+1}^{n} B_{\pi(j)}, \tag{3.31}$$

in the first and second braces in (3.30) with the explicit form for the Green's function (3.28). This holds for commuting objects A_i, B_i. The symmetric group S_n and the equivalence relation are as defined above.

3.5 Two-Point Functions

As an example, let us consider the two-point function. The imaginary part of the two-point function for a cosmic string vertex operator will in a certain factorization limit yield information about its decay rate and decay products [161, 54, 55, 56, 162, 58, 59, 60]. According to the optical theorem the total cross-section for the production of closed string states in the bulk from an initial closed string vertex $V(z,\bar{z})$ of mass m reads,

$$\sigma_{\mathrm{Total}}(m^2) = \frac{1}{2m} \mathrm{Im} \int_\Sigma d^2z \big\langle V^\dagger(z,\bar{z}) V(z',\bar{z}')\big\rangle, \tag{3.32}$$

where, see Appendix F, e.g. at one-loop the two-point function is given by the dimensionless expression,

$$\big\langle V^\dagger(z,\bar{z}) V(z',\bar{z}')\big\rangle = \int_{\mathcal{M}_1} \frac{d\tau d\bar{\tau}}{4\tau_2}\, (4\pi^2\alpha'\tau_2)^{-13} |\eta(\tau)|^{-48} \big\langle\!\big\langle V^\dagger(z,\bar{z}) V(z',\bar{z}')\big\rangle\!\big\rangle. \tag{3.33}$$

Due to the presence of one complex CKV the point $z', \bar{z}'$ can be chosen at will. Suppose that the relevant vertex operator is a certain mass eigenstate vertex operator. In fact, cosmic string vertex operators as we will see in the following chapters turn out to be linear superposition of mass eigenstates. Nevertheless, let us consider this simpler case given that the cosmic string case will correspond to a linear superposition of the mass eigenstate amplitudes. The result can be obtained from (3.30). We may directly take $J(z,\bar{z}) = \delta^2(z - z_1)k - \delta^2(z - z_2)k$ with $J_0 = k_1 + k_2$ and the momentum conserving delta function has enabled us to write $k = k_1 = -k_2$. Suppose furthermore, that one may choose the momenta of this mass eigenstate to be transverse to its polarization tensors.

The amplitude (3.30) reduces to,

$$\begin{aligned}\big\langle\!\big\langle V^\dagger(z,\bar z)V(z',\bar z')\big\rangle\!\big\rangle &= i(2\pi)\delta(k^{0\prime}-k^0)\frac{g_c^2}{2k^0}\sum_A \mathcal{G}_A\cdot\sum_{s=0}^{|\mathcal{I}_A|}\sum_{\pi,\pi'\in S_{|\mathcal{I}_A|}/\sim}\\ \prod_{r=1}^{|\mathcal{I}_A|-s}\eta D_{\pi(r)}D_{\pi'(r')}\ln|E(z_{\pi(r)},z_{\pi'(r')})|^2 &\prod_{j=|\mathcal{I}_A|-s+1}^{|\mathcal{I}_A|}-\eta\,\pi\varepsilon_I\big(\pi(j)\big)\,(\mathrm{Im}\Omega)^{-1}_{IJ}\,\varepsilon_J\big(\pi'(j')\big) \\ \times\exp\left\{k^2\,\pi\alpha'\mathrm{Im}\int_z^{z'}\omega_I\,(\mathrm{Im}\Omega)^{-1}_{IJ}\,\mathrm{Im}\int_z^{z'}\omega_J\right\}&\big|E(z,z')\big|^{-\alpha'k^2},\end{aligned}\tag{3.34}$$

where by $V^\dagger$ we mean: take the complex conjugate of the polarization tensors in V and reverse the momenta – this is the correct prescription in Euclidean signature and corresponds to Hermitian conjugation in Minkowski signature. See also (4.17). We have defined $\varepsilon_I(j)$ by,

$$\begin{aligned}\varepsilon_I(j)&\equiv D_j\left[\int_p^{z_j}\omega_I-\int_p^{\bar z_j}\bar\omega_I\right]\\ &=2i\,D_j\mathrm{Im}\int_p^{z_j}\omega_I,\end{aligned}\tag{3.35}$$

with $2|\mathcal{I}_A|\equiv|\mathcal{I}_A^2|$ and η the Minkowski metric which is contracted with the polarization tensors in $\mathcal{G}_A$ [8]. When for instance the asymptotic states are monomial vertex operators, such as those of Sec. 4.1,

$$\mathcal{G}_0=Z^*_{\mu_1\ldots\mu_{|\mathcal{I}|}}Z_{\mu_{1'}\ldots\mu_{|\mathcal{I}|'}}\lim_{\substack{z_i,\bar z_i\to z,\bar z\\ \forall i\in\mathcal{I}^{(1)}}}\ \lim_{\substack{z_i,\bar z_i\to z',\bar z'\\ \forall i\in\mathcal{I}^{(2)}}},\qquad\text{and}\qquad \mathcal{G}_{A>0}=0.$$

At one-loop one integrates over a single vertex insertion because there is one conformal Killing vector, see Appendix F. For any given two-point function, in the point merging procedure, the resulting expression will contain contact terms,[9] e.g. of the form $\partial_z\partial_{\bar z'}\ln|E(z,z')|^2=-2\pi\delta^2(z-z')$, which in turn seem to lead to a non-analytic contribution to the amplitude (3.34). However, in view of the analyticity of the amplitude in the external momenta k and the fact that the amplitude always contains a factor $|E(z,z')|^{-\alpha'k^2}$, it follows that such terms vanish identically even after the vertex insertion positions have been integrated over [191].[10] We will therefore discard contact terms.

[8]The sum over permutations also permutes the spacetime indices and so in the first and second product we could also have written $\eta^{\mu_{\pi(r)}\nu_{\pi'(r')}}$ and $\eta^{\mu_{\pi(j)}\nu_{\pi'(j')}}$, in which case we would also write $(\mathcal{G}_A)_{\mu_1\mu_2\ldots\nu_1\nu_2\ldots}$.

[9]By contact term we mean a term which only contributes when two or more vertex functionals are inserted at the same point on the worldsheet.

[10]The exact argument is as follows: notice that the exponent of $|E(z,z')|$ in (3.34) can always be made positive by analytic continuation and that when two vertex insertion points come close together the prime form to leading order always has the form $E(z,z')\sim z-z'$. Therefore, given that $\int d^2z|w-z|^{-\alpha'k^2}\delta^2(w-z)$=0 when $-\alpha'k^2>0$ it follows that the amplitude will vanish identically in this region of parameter space of k. It then follows from a famous theorem of complex analysis that the entire expression will vanish everywhere. A similar argument holds for multiple delta functions, e.g. $\int d^2z|z-z'|^{-\alpha'k^2}\delta^2(z-z')\delta^2(z-z')=0$. To see this write this expression as $\lim_{\epsilon\to0}\int d^2z|z-z'+\epsilon|^{-\alpha'k^2}\delta^2(z-z')\delta^2(z-z'+\epsilon)$.

Taking the above considerations into account, it is seen that after point merging the basic building blocks that appear in the first product in the amplitude (3.34) are

$$\omega(z,z') \equiv \partial_z\partial_{z'} \ln E(z,z'), \qquad \bar{\omega}(\bar{z},\bar{z}') \equiv \partial_{\bar{z}}\partial_{\bar{z}'} \ln \bar{E}(\bar{z},\bar{z}'). \tag{3.36}$$

All possible combinations of holomorphic (anti-holomorphic) derivatives of $\omega(z,z')$ ($\bar{\omega}(\bar{z},\bar{z}')$) contribute, the maximum order of derivatives appearing being constrained by the maximum order of derivatives appearing in the original vertex operator. In the literature, see e.g. [192, 141], $\omega(z,z')dz$ and $\bar{\omega}(\bar{z},\bar{z}')d\bar{z}$ are known as differentials of the second kind. These are meromorphic 1-forms with no residues, a double pole at $z = z'$ and zero A_I periods, $\oint_{A_I} \omega(z,z')dz = 0$.

Similarly, the basic building blocks that appear in the second product in (3.34) after point-merging on account of (3.35) are

$$\begin{aligned} K(z,z') &\equiv -\pi\omega_I(z)\,(\mathrm{Im}\Omega)^{-1}_{IJ}\,\omega_J(z'), \quad K(\bar{z},\bar{z}') \equiv -\pi\bar{\omega}_I(\bar{z})\,(\mathrm{Im}\Omega)^{-1}_{IJ}\,\bar{\omega}_J(\bar{z}'), \\ K(z,\bar{z}') &\equiv \pi\omega_I(z)\,(\mathrm{Im}\Omega)^{-1}_{IJ}\,\bar{\omega}_J(\bar{z}'), \quad K(\bar{z},z') \equiv \pi\bar{\omega}_I(\bar{z})\,(\mathrm{Im}\Omega)^{-1}_{IJ}\,\omega_J(z'). \end{aligned} \tag{3.37}$$

The factors that appear contain all combinations of derivatives of K. At genus one, $h = 1$, all K's appear non-differentiated in the amplitude as the Abelian differentials, $\omega_I(z)$, $\bar{\omega}_I(\bar{z})$, are in this case constant (and equal to one) and $\mathrm{Im}\Omega_{IJ} = \tau_2$.

The above two-point function contains all the information about the decay products of the initial vertex operator. When however, the dominant decay channel is massless radiation, which has been demonstrated to be the case for a particular set of vertex operators with first harmonics only in [56, 162], it turns out to be more efficient to instead carry out a forward scattering tree level computation. This is what we do in Sec. 6, where we consider the graviton emission amplitude for a closed string coherent state with first harmonics excited. It is more efficient in the sense that analytic results can be obtained and it is not necessary to resort to numerical simulations. Note however, that massless emission may not always correspond to the dominant decay channel, in which case the analysis of the two-point function becomes more appropriate. This will be the case for vertex operators whose classical analogues self-intersect during the loop's motion and it is conceivable that this is also the case for strings with cusps – that is, points on the string where the determinant of the embedding metric vanishes.

Then, performing the z integration leads to $\lim_{\epsilon\to 0} |\epsilon|^{-\alpha' k^2}\delta^2(\epsilon)$. This in turn vanishes for the following two reasons: the integral $\int d^2\epsilon |\epsilon|^{-\alpha' k^2}\delta^2(\epsilon) = 0$ and the corresponding integrand is non-negative – therefore, the integrand must vanish. Such terms therefore do not contribute and will be set equal to zero in the following.

Chapter 4

Mass Eigenstate Vertex Operators

4.1 Vertex Operators from Symmetry

Working in the functional formalism, the states we consider in this section are due to Weinberg [132], see also de Alwis [142] and [181]. Particular emphasis will be placed on states beyond the leading Regge trajectory.

Working in conformal gauge with worldsheet metric $ds^2 = g_{z\bar{z}}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$, we construct mass eigenstate vertex insertions for onshell physical states of given momentum k^μ from the following symmetry and covariance requirements [132, 142, 181, 134, 141]: a) The vertex V_k should transform like a one-particle state under spacetime translations. b) It should transform like a one-particle state under Lorentz transformations. c) It should be diffeomorphism invariant on the worldsheet. d) It should be conformally invariant on the worldsheet. Focusing on monomial closed string vertex operators,[1] these lead us to consider the following expression,

$$\begin{aligned} V_k =& \frac{g_c}{\sqrt{2k^0 V_{d-1}}} \, C \, Z \cdot \Big(\frac{2}{\alpha'}\Big)^{|\mathcal{I}|/2} \int_\Sigma d^2z \sqrt{g}\; g_{z\bar{z}}^{-N} \\ &\times \underbrace{\partial_z X \dots \partial_z X}_{|I_1|} \dots \underbrace{\partial_z^m X \dots \partial_z^m X}_{|I_m|} \underbrace{\partial_{\bar{z}} X \dots \partial_{\bar{z}} X}_{|\bar{I}_1|} \dots \underbrace{\partial_{\bar{z}}^{\bar{m}} X \dots \partial_{\bar{z}}^{\bar{m}} X}_{|\bar{I}_{\bar{m}}|} \, e^{ik\cdot X(z,\bar{z})}. \end{aligned} \tag{4.1}$$

The factor $\frac{g_c}{\sqrt{2k^0 V_{d-1}}}$ as we will see in Sec. 4.3 is required in order that the vertex operator be normalized to 'one string in volume V_{d-1}' as required in order to give rise to unitary S-matrix elements. The dot product appearing is defined such that every index of the polarization tensor, Z, is contracted with a target spacetime index and the order of the indices is to be respected[2]. By $|I_\ell|$ we mean the number of ℓ^{th} order holomorphic derivatives and likewise for the anti-holomorphic sector, so that the total number of holomorphic derivatives is,[3] $N = |I_1| + 2|I_2| + \dots + m|I_m|$. The quantity C is a combinatorial factor

[1] More general vertices would be polynomials of the form $\int P(\partial X, \partial^2 X, \dots)\bar{P}(\bar{\partial} X, \bar{\partial}^2 X, \dots)e^{ik\cdot X}$. We shall construct these explicitly in Sec. 4.4.

[2] The first index appearing in the polarization tensor is contracted with the first index appearing in the integrand of (4.1), the second with the second and so on.

[3] We use the modulus sign here because by $|I_\ell|$ we actually mean the number of elements of an index

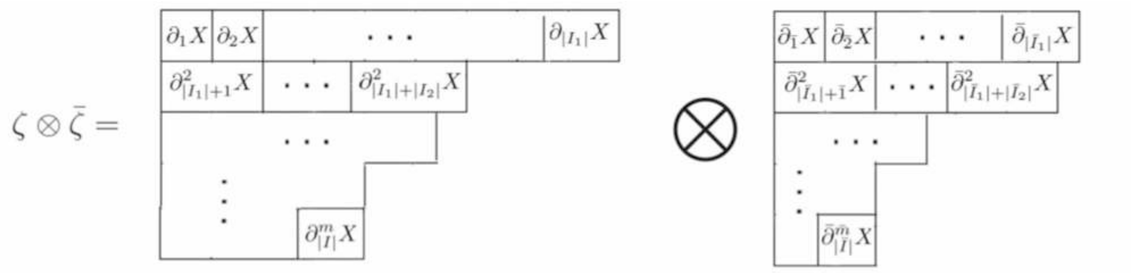

Figure 4.1: The structure of the (anti-)holomorphic part of the physical polarization tensor ζ ($\bar{\zeta}$) can be exhibited by means of Young tableaux, whereby indices of ζ ($\bar{\zeta}$) corresponding to elements in the $\ell^{\rm th}$ row are to be contracted with the spacetime indices of the $\ell^{\rm th}$ derivative terms, $\partial^\ell X$ ($\bar{\partial}^\ell X$), in the vertex insertion $V_k[g,X]$. The total number of boxes in a given row equals the total number of derivatives of a given type, whereas the total number of boxes equals the rank of $\zeta\otimes\bar{\zeta}$, $|\mathcal{I}|=|I|+|\bar{I}|={\rm Rank}(\zeta\otimes\bar{\zeta})$. Note that every box in the above diagram has a corresponding spacetime index associated to it. The notation is such that a box in the diagram containing, say, $\partial_2 X$, indicates that one is to contract the second spacetime index μ_2 in $\zeta_{\mu_1\mu_2\dots}$ with the second term in (4.1), namely ∂X^{μ_2}. Similar remarks hold for any j in $\partial_j^{\#}X$.

that has been determined by Weinberg [132],

$$C=\Big(\prod_{\ell=1}^{m}|I_\ell|!\,(\ell!(\ell-1)!)^{|I_\ell|}\Big)^{-\frac{1}{2}}\times\Big(\prod_{\ell=1}^{\bar{m}}|\bar{I}_\ell|!\,(\ell!(\ell-1)!)^{|\bar{I}_\ell|}\Big)^{-\frac{1}{2}}. \tag{4.2}$$

With this normalization the polarization tensor Z is in turn normalized as

$$Z_{\mu_1\dots\mu_{|\mathcal{I}|}}Z^{*\mu_1\dots\mu_{|\mathcal{I}|}}=1, \tag{4.3}$$

with the definition $|\mathcal{I}|=|I_1|+\dots+|I_m|+|\bar{I}_1|+\dots+|\bar{I}_{\bar{m}}|$. Writing the polarization tensor as $Z=\zeta\otimes\bar{\zeta}$, the ζ and $\bar{\zeta}$ have the symmetries of Young tableaux, see Fig. 4.1.

The vertex (4.1) will be *physical* provided the polarization tensor Z and the number of $\ell^{\rm th}$ order (anti-)holomorphic derivatives, $|I_\ell|$ (and $|\bar{I}_\ell|$) for $1\leq\ell\leq m\,(\bar{m})$, satisfy the following properties [132, 142]:

1. The mass shell constraint (from conformal invariance), $M^2=-k^2=\frac{4}{\alpha'}(N-1)$, and level-matching (from worldsheet translation invariance), $N-\bar{N}=0$ where, $N=\sum_{\ell=1}^m\ell|I_\ell|\in\{0,1,\dots\}$ and $\bar{N}=\sum_{\ell=1}^{\bar{m}}\ell|\bar{I}_\ell|\in\{0,1,\dots\}$.

2. The polarization tensor $Z_{\mu_1\dots\mu_{|\mathcal{I}|}}$ transforms under Lorentz transformations according to a real representation of the little group for k^μ, namely SO(25).

3. The polarization tensor $Z_{\mu_1\dots\mu_{|\mathcal{I}|}}$ (with $Z\equiv\zeta\otimes\bar{\zeta}$) is traceless and transverse with respect to $\eta_{\mu_i\mu_j}$ and k^{μ_i} respectively,

$$\eta^{\mu_i\mu_j}Z_{\mu_1\dots\mu_i\dots\mu_j\dots\mu_{|\mathcal{I}|}}=0,\quad k^{\mu_i}Z_{\mu_1\dots\mu_i\dots\mu_{|\mathcal{I}|}}=0,\quad\forall i,j\in\mathcal{I}. \tag{4.4}$$

set I_ℓ. The notion of an index set becomes indispensable when one considers amplitudes with high spin monomial vertices.

For more general, in particular polynomial, vertex operators this condition for Z is sufficient but not necessary. These two conditions ensure that conformal anomalies associated to certain self-contractions vanish.

4. The holomorphic and anti-holomorphic sector ζ and $\bar{\zeta}$ have the symmetries of *Young tableaux*:

$$(|I_1|, |I_2|, \dots, |I_m|), \quad \text{and} \quad (|\bar{I}_1|, |\bar{I}_2|, \dots, |\bar{I}_{\bar{m}}|), \tag{4.5}$$

respectively. The quantities ℓ and $|I_\ell|$ label respectively the row and number of boxes in that row of the tableau. Clearly then the number of ℓ^{th} order holomorphic derivatives, $|I_\ell|$, is greater or equal to the number of $(\ell+1)^{\text{th}}$ order holomorphic derivatives, $|I_{\ell+1}|$: $|I_1| \geq |I_2| \geq \dots \geq |I_m|$, with $|I_1| + |I_2| + \dots + |I_m| = |I|$. Similar remarks hold for the anti-holomorphic sector.

The physical state conditions above are more restrictive than the conditions laid down by Weinberg [132] and this is so as to avoid the trace anomaly that was subsequently identified by de Alwis [142]. See also [181]. If we did not require a vanishing "cross-term" trace, i.e. a trace with respect to one holomorphic and one anti-holomorphic index of the state polarization tensor, the Ricci scalar $R_{(2)}$ would also appear in vertex operators in such a way so as to absorb the corresponding trace anomalies. Note that in the above vertex operators a dependence on $g_{z\bar{z}}$ appears which naively seems to break conformal invariance, but when the mass shell constraint is enforced (point 1 above) the dependence on $g_{z\bar{z}}$ drops out of path integral computations. Furthermore, we could just as well have written covariant worldsheet derivatives, ∇_z, with a connection associated to a metric $g_{z\bar{z}}$ but one can convince oneself that this would reduce to ∂_z when point 4 is taken into account.

The above considerations are due to Weinberg and de Alwis, and we make the following observation. Given that the polarization tensor is traceless and corresponds to an irreducible representation of SO(25), the sum of the lengths of the first two columns of the Young tableau must be smaller than or equal to 25 (see e.g. [189] p. 394). The structure of the Young tableau therefore puts an upper bound on the number of harmonics (namely m or $\bar{m}$) that can be present in a monomial physical string state in the covariant formalism,[4]

$$(\#\text{boxes in column 1}) + (\#\text{boxes in column 2}) \leq 25, \tag{4.6}$$

which holds for both the holomorphic and anti-holomorphic sector. In terms of the conjugate quantities $|J_\ell|$, $|\bar{J}_\ell|$, see the paragraph containing (I.1), this can be written as $|J_1| + |J_2| \leq 25$ and $|\bar{J}_1| + |\bar{J}_2| \leq 25$.

In Appendix I we show that an explicit representation for Z which satisfies the above physical state conditions is as follows

$$\begin{aligned} Z &= \zeta \otimes \bar{\zeta} \\ &= \big(C_{|J_1|} \otimes C_{|J_2|} \otimes \dots \otimes C_{|J_q|}\big) \otimes \big(\bar{C}_{|\bar{J}_1|} \otimes \bar{C}_{|\bar{J}_2|} \otimes \dots \otimes \bar{C}_{|\bar{J}_{\bar{q}}|}\big), \end{aligned}$$

[4] I am grateful to Steven Weinberg for correspondence concerning this issue.

with C_p and $\bar{C}_p$ being certain completely anti-symmetric spacetime tensors with components

$$C_p^{\mu_1\dots\mu_p} = \frac{1}{p!}\varepsilon_{A_1\dots A_p} e^{\mu_p}_{A_1}\dots e^{\mu_p}_{A_p}, \qquad \bar{C}_p^{\mu_1\dots\mu_p} = \frac{1}{p!}\varepsilon_{A_1\dots A_p} \bar{e}^{\mu_1}_{A_1}\dots \bar{e}^{\mu_p}_{A_p}. \tag{4.7}$$

These specify the spacetime directions in which a given mode (or harmonic) is fluctuating. In particular, the basis vectors appearing, e^μ_A, and $\bar{e}^\mu_A$, permit an expansion of the form[5] $e^\mu_A = N_{AB}\hat{e}^\mu_B$, $\bar{e}^\mu_A = \bar{N}_{AB}\hat{e}^\mu_B$, with

$$k = \begin{pmatrix} k^0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \hat{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \hat{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \dots, \qquad \hat{e}_{12} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ i \\ 0 \end{pmatrix}. \tag{4.8}$$

The subscripts A on $\hat{e}_A$ label the harmonics excited and we restrict our attention to the case where the maximum harmonics appearing satisfy $m, \bar{m} \le 12$ due to the observation we made in (4.6); this is elaborated on below (I.7). The resulting vertex (4.1) is physical provided the independent matrices N_{AB}, $\bar{N}_{AB}$ are elements of SO(12). We are thus free to fix $12(12-1)/2 = 66$ parameters in either N_{AB} or $\bar{N}_{AB}$ and every such choice leads to physically distinct polarization tensors. It is possible to also verify that Z is normalized according to (4.3).

There is also the important relation, $\bar{e}_A = (\bar{N}N^{\mathrm{T}})_{AC}e_C$, and so the choice of matrices N_{AB} and $\bar{N}_{AB}$ also determines the asymmetry between left- and right-movers. We are implicitly considering states with potentially (but not necessarily) asymmetric left-right excitations.[6] An example for a polarization tensor with a left-right excitation asymmetry is the following,

$$\begin{aligned} Z &= \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & & & & & & & & \\ \hline \end{array} \\ &= \big(C_2 \otimes C_2 \otimes C_2\big) \otimes \big(\underbrace{\bar{C}_1 \otimes \bar{C}_1 \otimes \dots \otimes \bar{C}_1}_{9}\big) \end{aligned}$$

The state corresponding to this polarization tensor carries 2^{nd} *and* 1^{st} harmonics on the left-movers and 1^{st} harmonics only on the right-movers. It is not possible with the monomial vertex operators to have only higher harmonics present without having all lower ones as well. This can be traced back to point 4 on p. 64.

In the next section we describe the construction of vertex operators produced in string scattering where the restriction of a maximum number of harmonics is not present as in

[5] Using the Lorentz invariance of the theory we here set the matrices $M^\mu{}_\nu = \delta^\mu{}_\nu$.

[6] An example where left-right asymmetry can become important is in the context of cosmic strings. In massive string states left-right asymmetry is generic and (when the quantum states have a classical interpretation) such an asymmetry seems to be responsible for the presence of cusp-like features in the corresponding classical evolution of strings [83].

the monomial vertices discussed in the present section. The next section serves to set the scene for the general vertex operator construction of Sec. 4.4 but can also be skipped without loss of continuity.

4.2 Vertex Operators from Factorization

In the present section we briefly discuss a standard [193] but alternative formalism which can be used to extract vertex operators. This will serve to introduce the basic ideas that will be necessary in the general vertex operator construction of the next section whereby with some minor modifications a complete set of covariant normal ordered mass eigenstate vertex operators will be constructed.

We here study the tachyon-tachyon, tachyon-massless and massless-massless operator product expansions[7] and extract the resulting vertex operators produced at these three point vertex interactions. The new ingredient here is that (as we shall see) the onshell produced vertex operators are described naturally in terms of elementary Schur polynomials (or equivalently complete Bell polynomials). The vertices produced in tachyon-tachyon scattering are the simplest vertices where the structure and importance of elementary Schur polynomials becomes manifest and the present section will serve to pave the way for the developments of the next sections to come. The approach we adopt is similar in spirit to that of Aldazabal et al. [135] where factorization was carried out by bringing together two (or more) tachyonic vertices in the multi-tachyon amplitude of arbitrary genus and extracting the residues associated to poles arising from the internal vertices going onshell. This procedure is simplified by the use of conformal field theory techniques however where factorization is carried out by examining the limit in which two or more external states approach on the worldsheet, see e.g. Friedan et al. [193]. The internal vertices are read off from the residue of the resulting object. The polarization tensors associated to these states are written naturally in terms of the external momenta and polarization tensors of the original objects. More general polarization tensors can be obtained by examining the limit where more than two vertices approach on the worldsheet [135], whereas explicit constraints for more general polarization tensors appropriate for these vertices has been derived by Sato [133]. The procedure of the next section is more efficient however and the present section is intended to be viewed as a warm-up for the more general construction of the next section.

[7] Massless refers to either of the three particles (graviton, dilaton or antisymmetric tensor) in the massless multiplet for bosonic strings.

Vertices produced in tachyon-tachyon string scattering

Let us start with the vertices produced in tachyon-tachyon scattering. The operator product for the two closed string tachyon process reads [114],

$$
\begin{aligned}
:e^{ip\cdot X(z,\bar{z})}::e^{ip'\cdot X(0,0)}: &= |z|^{\alpha' p\cdot p'} :e^{ip\cdot X(z,\bar{z})}e^{ip'\cdot X(0,0)}: \\
&= |z|^{\alpha' p\cdot p'} :\Big(e^{-ip\cdot X(0,0)} \sum_{m,n=0}^{\infty} \frac{z^m \bar{z}^n}{m!n!} \partial^m \bar{\partial}^n e^{ip\cdot X(0,0)}\Big) e^{i(p+p')\cdot X(0,0)}:
\end{aligned}
\tag{4.9}
$$

where $p^2 = p'^2 = 4/\alpha'$. Conformal invariance of the intermediate states implies that to obtain these we should put the internal momentum $k = p + p'$ onshell, $(p+p')^2 = 4(1-N)/\alpha'$. One then extracts the propagating states from the residue in the above expression,[8]

$$
\oint_0 đz \oint_0 đ\bar{z} \; :e^{ip\cdot X(z,\bar{z})}::e^{ip'\cdot X(0,0)}: \;=\; :\Big(e^{-ip\cdot X(0,0)} \tfrac{\partial^N}{N!} \tfrac{\bar{\partial}^N}{N!} e^{ip\cdot X(0,0)}\Big) e^{ik\cdot X(0,0)}: \tag{4.10}
$$

The right-hand-side of the above expression, call it $V(0,0)$, is a linear combination of all covariant physical vertex operators (i.e. of conformal dimension (1,1)) which can be produced in tachyon-tachyon scattering at mass level N and is thus itself a physical vertex operator. Making use of $\partial\bar{\partial}X = 0$ and shifting from the origin to $z, \bar{z}$, the derivatives in the parenthesis may be evaluated explicitly via Faà di Bruno's formula[9]. The resulting vertex operator reads (we drop the normal ordering symbols $::$ when there is no ambiguity),

$$
\begin{aligned}
V(z,\bar{z}) = \frac{g_c}{\sqrt{2k^0 V_{d-1}}} &\Big[\sum \prod_{\ell=1}^{N} \frac{1}{n_\ell !} \Big(\frac{ip\cdot \partial^\ell X(z,\bar{z})}{\ell !}\Big)^{n_\ell}\Big] \\
&\times \Big[\sum \prod_{r=1}^{N} \frac{1}{\bar{n}_r !} \Big(\frac{ip\cdot \bar{\partial}^r X(z,\bar{z})}{r!}\Big)^{\bar{n}_r}\Big] e^{ik\cdot X(z,\bar{z})},
\end{aligned}
\tag{4.11}
$$

the sum being over the set of positive integers $\{n_\ell\}$ such that, $n_1 + 2n_2 + \cdots + Nn_N = N$, and similarly for the antiholomorphic sector with $n_\ell \to \bar{n}_\ell$. Vertex operator normalization will be discussed in detail in Sec. 4.3. This expression has been obtained by factorizing the explicit multi-loop tachyon amplitude in the path integral formulation by Aldazabal et al. [135]. Furthermore, (4.11) is a special case of the more general vertices considered by Sato [133] when the polarization tensors $\zeta^{\mu_1\mu_2\cdots}$ (there) are taken to equal the symmetric product of vectors $p^{\mu_1}p^{\mu_2}\ldots$. The observation we make here is that the result is most naturally expressed in terms of *elementary Schur polynomials*, $S_m(a_1,\ldots,a_m)$, with the identification $a_\ell = \frac{1}{\ell!}\, ip\cdot\partial^\ell X$, see Appendix J,

$$
\begin{aligned}
V(z,\bar{z}) = \frac{g_c}{\sqrt{2k^0 V_{d-1}}} &S_N\big(ip\cdot\partial X, \ldots, \tfrac{1}{N!}\, ip\cdot\partial^N X\big) \\
&\times S_N\big(ip\cdot\bar{\partial} X, \ldots, \tfrac{1}{N!}\, ip\cdot\bar{\partial}^N X\big)\, e^{ik\cdot X(z,\bar{z})},
\end{aligned}
\tag{4.12}
$$

[8] For notational convenience we write $đz = \frac{dz}{2\pi}$.

[9] This is derived for example in [194]

Notice that level matching is automatically satisfied in (4.12) given that the subscript on S_N denotes the number of worldsheet derivatives appearing in any given term of the polynomial expansion.

Using the integral representation of S_N leads to the following equivalent expression for $V(z,\bar{z})$,

$$\begin{aligned} V(z,\bar{z}) =& \frac{g_c}{\sqrt{2k^0 V_{d-1}}} \oint_0 \text{đ}w \oint_0 \text{đ}\bar{w}\, |w|^{-2N-2} \\ & \times \exp\Big(\sum_{n=1}^{\infty} \frac{w^n}{n!}\, ip\cdot\partial^n X(z,\bar{z}) + \frac{\bar{w}^n}{n!}\, ip\cdot\bar{\partial}^n X(z,\bar{z})\Big) e^{ik\cdot X(z,\bar{z})}. \end{aligned} \tag{4.13}$$

This object is a covariant physical vertex operator of momentum k and level number N.[10] It has been suggested [195] that for a particular initial kinematical configuration with compact external momenta and large radii of compactification this state can acquire the interpretation of a superposition of macroscopic kinked states.

Note that (4.13) is conformally invariant only after the contour integrals have been performed which enforce level matching and worldsheet reparametrization invariance. Given that the original expression, namely (4.9), from which $V(z_i,\bar{z}_i)$ was derived is normal ordered, so is the representation (4.13). Therefore, the appropriate path integral insertion up to normalization reads: $V = \int d^2z V(z,\bar{z})$. Reparametrization invariance is not manifest but becomes so if we insert $\sqrt{g}g_{z\bar{z}}^{-N}$ in the integrand,

$$V = \int d^2z \sqrt{g} g_{z\bar{z}}^{-N} V(z,\bar{z}), \tag{4.14}$$

with $\sqrt{g}$ providing a density for a covariant measure, $g_{z\bar{z}}^{-N}$ ensuring all worldsheet indices are properly contracted and covariant worldsheet derivatives, $\nabla_z^{(n)} = (\partial_z - n\Gamma^z{}_{zz})$, replacing ∂_z (the index n corresponds to the rank of the object on which these derivatives act and $\Gamma^z{}_{zz}$ the connection associated to the metric $g_{z\bar{z}}$). The extra factor $\sqrt{g}g_{z\bar{z}}^{-N}$ has been absent from the outset due to the normal ordering prescription and $\Gamma^z{}_{zz}$ always drops out due to conformal invariance. In the path integral language this extra term $\sqrt{g}g_{z\bar{z}}^{-N}$ combines with self-contractions inside the exponential $e^{-\frac{1}{2}k^2 G_{\rm R}(z,z)}$ to enforce the mass-shell constraints, see e.g. [141]. For some further details on the relation between path integral and CFT vertices see Polchinski [196]. The vertex (4.14) is in agreement with that found in [135] when the identification $g_{z\bar{z}} \simeq \omega(z)\bar{\omega}(\bar{z})$ is made, with $\omega(z)$ a linear combination [135] of the holomorphic Abelian differentials, $\omega_I(z)$, associated to the cycles of the Riemann surface, see Appendix C and e.g. [186, 141].

Vertices produced in tachyon-massless string scattering

Let us next consider the states produced in tachyon-massless string scattering. This is an important interaction because massless states couple universally to all string states and the tachyon is of course the vacuum on which a complete set of states is constructed. This

[10] One may truncate the sum over n at N as terms with $n > N$ do not contribute.

is an example of the process on which the DDF formalism of the next section relies upon in order to construct a complete set of states (in the case of open strings) as we shall see. Having given an explicit computation in Sec. 4.2 for vertices produced in tachyon-tachyon scattering, we omit details of the computation in this and the following subsection which are very similar to the tachyon-tachyon case.

In direct analogy to (4.9) one examines the operator product expansion

$$: e^{ip\cdot X(z,\bar{z})} :: \zeta_{\mu,\nu}\partial X^\mu(w)\bar{\partial}X^\nu(\bar{w})e^{ip'\cdot X(w,\bar{w})} :,$$

and extract the vertex operators from the residue of the resulting expression as in (4.10). This procedure can be seen to lead to the following mass level N vertex operators (up to an overall normalization),

$$\begin{aligned}V(z,\bar{z}) = \frac{g_c}{\sqrt{2k^0V_{d-1}}}\,\zeta_{\mu,\nu}\Big(ip^\mu S_N(-p;z) + \partial X^\mu(z)S_{N-1}(-p;z)\Big)&\\ \times\Big(ip^\nu \bar{S}_N(-p;\bar{z}) + \bar{\partial}X^\nu(\bar{z})\bar{S}_{N-1}(-p;\bar{z})\Big)e^{ik\cdot X(z,\bar{z})}&,\end{aligned} \tag{4.15}$$

with the elementary Schur polynomials defined in Appendix J, with the constraints $p^2 = 4/\alpha'$, $p'^2 = 0$, and we have written $k = p + p'$ so that $k^2 = 4(1-N)/\alpha'$. Vertices produced in tachyon-graviton, tachyon-dilaton or tachyon-antisymmetric tensor scattering are obtained by setting $\zeta_{\mu,\nu}$ equal to $\zeta^{(\mathrm{tg})}_{\mu,\nu} = \frac{1}{2}(\zeta_{\mu,\nu}+\zeta_{\nu,\mu}) - \frac{1}{d}\eta_{\mu\nu}\eta^{\rho\sigma}\zeta_{\rho,\sigma}$, $\zeta^{(\mathrm{td})}_{\mu,\nu} = \frac{1}{d}\eta_{\mu\nu}\eta^{\rho\sigma}\zeta_{\rho,\sigma}$ or $\zeta^{(\mathrm{ta})}_{\mu,\nu} = \frac{1}{2}(\zeta_{\mu,\nu} - \zeta_{\nu,\mu})$ respectively.

Vertices produced in massless-massless string scattering

Finally, the linear combination of vertices produced in massless-massless scattering, again in direct analogy to the above, follow from the residue of the following operator product expansion

$$: \zeta_{\mu,\nu}\partial X^\mu(z)\bar{\partial}X^\nu(\bar{z})e^{ip\cdot X(z,\bar{z})} :: \zeta'_{\rho,\sigma}\partial X^\rho(w)\bar{\partial}X^\sigma(\bar{w})e^{ip'\cdot X(w,\bar{w})} :,$$

with $p^2 = p'^2 = 0$. Taking again $k = p + p'$ with $k^2 = 4(1-N)/\alpha'$ we find for the chiral half, $U(z)$, of the vertex $V(z,\bar{z}) = \frac{g_c}{\sqrt{2k^0V_{d-1}}}U(z)\bar{U}(\bar{z})$,

$$\begin{aligned}U(z) = \zeta_{\mu\rho}\Big\{&\big(p^\mu p^\rho - \eta^{\mu\rho}\big)S_N(-p;z) - i\big(k^\mu - p^\mu\big)\partial X^\rho(z)S_{N-1}(-p;z)\\ &+\sum_{m=1}^{\infty}\frac{1}{(m-1)!}\partial^m X^\mu\Big[\partial X^\rho(z)S_{N-1-m}(-p;z) + ip^\rho S_{N-m}(-p;z)\Big]\Big\}e^{ik\cdot X(z)},\end{aligned} \tag{4.16}$$

where we have formally factorized the polarization tensor as follows, $\zeta_{\mu,\nu}\zeta'_{\rho,\sigma} \equiv \zeta_{\mu\rho,\nu\sigma} = \zeta_{\mu\rho}\tilde{\zeta}_{\nu\sigma}$. For the anti-holomorphic sector one is to replace z, $\zeta_{\mu\rho}$ and $S_N(-p;z)$ by $\bar{z}$, $\tilde{\zeta}_{\nu\sigma}$ and $\bar{S}_N(-p;\bar{z})$ respectively. It seems at this point that the higher mass vertices would become more and more complicated but in fact there is a pattern which we identify in the next section, and this in turn enables one to write down the general result for vertices with arbitrary spin. The DDF approach that we use for this purpose is tailor-made for the construction of a complete set of covariant vertex operators that are in one-to-one correspondence with the lightcone gauge states. Before discussing the DDF construction it will be useful to review general results on the normalization of vertex operators.

4.3 Vertex Operator Normalization and S-Matrix Unitarity

Before moving on the discuss the general DDF construction of vertex operators it will be useful to elaborate on the precise connection of vertex operators to the string S-matrix, as this will in turn enable us to normalize vertex operators correctly, i.e. in such a way that the resulting S-matrix elements are unitary. We will follow the general approach of [132, 114, 197] although the reasoning here will be mostly independent of these references. We will concentrate on mass eigenstates, although these results will go through essentially untouched in the case of coherent states (Sec. 5) as well.

String S-Matrix

Our objective is to use a normalization for vertex operators that is appropriate for scattering amplitude computations, and so we first discuss the precise relation between the string path integral and the S-matrix.

The proper way of constructing a scattering experiment is to first construct vertex operator wave packets for the external string states of interest and then normalize each one of them to "one string in the universe", in direct analogy to the corresponding field theory prescription. Rather than use wavepackets, we may also use momentum eigenstates instead, in which case (due to the uncertainty principle, the infinite spacetime spread of momentum eigenstates) we need to truncate the volume of spacetime at, say, V_{d-1}, the case of interest for the bosonic string being $d = 26$ and for the superstring $d = 10$. According to standard practice [198], we hence identify momentum delta-functions with volume elements and energy delta functions with the time, T, during which the interaction is "turned on",

$$(2\pi)^{d-1}\delta^{d-1}(\mathbf{p}' - \mathbf{p}) \equiv V_{d-1}, \qquad \text{and} \qquad (2\pi)\delta(E' - E) \equiv T. \tag{4.17}$$

By putting the system in a box of size V_{d-1}, the vertex operator normalization condition is changed from "one string in the universe" to "one string in volume V_{d-1}" [199]. Of course, physical observables (cross sections, decay rates, etc...) should not depend on V_{d-1}, although we formally think of taking $V_{d-1} \to \infty$ at the end of the computation.

The "one string in volume V_{d-1}" normalization prescription leads to an S-matrix such that if an initial state of a system is denoted by $|i\rangle$, the final state will be a superposition, $\sum_f |f\rangle\langle f|S|i\rangle$. Therefore, $|S_{fi}|^2$ is interpreted as a transition probability associated to going from $|i\rangle$ to $|f\rangle$,

$$\mathrm{Prob}(f \leftarrow i) = |S_{fi}|^2, \qquad \text{with} \qquad S_{fi} \equiv \langle f|S|i\rangle. \tag{4.18}$$

Conservation of probability, equivalently S-matrix unitarity, requires that,

$$S^\dagger S = \mathbb{1}.$$

In particular, in terms of S_{fi}, unitarity corresponds to the statement:

$$\sum_n S^\dagger_{nf} S_{ni} = \delta_{fi}, \qquad \text{or} \qquad \sum_n S_{fn} S^\dagger_{in} = \delta_{fi}, \tag{4.19}$$

with δ_{fi} a Kronecker delta; working in the Heisenberg picture, $\delta_{fi} \equiv \langle f|i\rangle$. Setting $f = i$ it is seen that unitarity enforces conservation of probability, $\sum_f |S_{fi}|^2 = 1$.

To make the connection with the string path integral, it is conventional and convenient to define a T-matrix which contains the non-trivial contribution to the S-matrix, $S = \mathbb{1} + iT$. Taking matrix elements of both sides and extracting the momentum and energy conserving delta functions leads to,

$$S_{fi} = \delta_{fi} + i(2\pi)^d \delta^d(P_f - P_i) T_{fi}. \tag{4.20}$$

In terms of T_{fi} the unitarity constraint (4.19) reads,

$$T_{fi} - T^\dagger_{if} = i \sum_n (2\pi)^d \delta^d(P_n - P_i) T^\dagger_{nf} T_{ni} \tag{4.21}$$

with P_i or P_f the total momentum associated to the in or out states respectively. With these conventions, the S-matrix is given directly by the string path integral, see Sec. 3.4,

$$\begin{aligned} \langle f|(S - \mathbb{1})|i\rangle &= \sum_{h=0}^{\infty} \int_{\mathcal{E}\times\mathcal{M}_h} \mathcal{D}g\mathcal{D}X e^{-S[g,X]} V^{(1)} \dots V^{(N)} \\ &= i(2\pi)^d \delta^d(P_f - P_i) T_{fi}. \end{aligned} \tag{4.22}$$

where we sum over the genus h of Riemann surfaces. It is to be understood that the integrals are over a single gauge slice, i.e. over all worldsheet embeddings, $\mathcal{E}$, into spacetime and over all worldsheet metrics (or moduli space $\mathcal{M}_h$), such that no two configurations in the integration domain are related by a symmetry. Appropriate integrations over worldsheet insertions are also implicitly included, as are the corresponding Fadeev-Popov determinants.

To interpret the sum over final states in (4.19) or (4.21), note that the number of "one string in volume V_{d-1}" states in a momentum space volume element, $d^{d-1}\mathbf{p}$, is:

$$V_{d-1} \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}}, \tag{4.23}$$

because this is the number of sets $\{n_1, n_2, \dots, n_{d-1}\}$ (with $n_j \in \mathbb{Z}$) for which the momentum

$$\mathbf{p} = \frac{2\pi}{L}(n_1, n_2, \dots, n_{d-1}), \qquad \text{with} \qquad V_{d-1} \equiv L^{d-1},$$

lies in the momentum space volume $d^{d-1}\mathbf{p}$ around $\mathbf{p}$. If there are additional discrete/continuous quantum numbers that label the states under consideration, we would have to sum/integrate over these. For example, in the case of coherent states we would have to include a (dimensionless) integral over polarization tensors.[11] In particular, there will in general be a number of kinematically allowed channels and so we should also include a sum over a complete set of states – we use the compact notation, $\sum\!\!\!\!\!\!\int$, to denote a sum over states and the associated quantum numbers, so that the sum over one-particle states in the final state will be denoted by:

$$\sum_f = \sum\!\!\!\!\!\!\!\int V_{d-1} \int \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}}. \tag{4.24}$$

Both sides of this equation are dimensionless. In relativistic scattering experiments there is also the possibility that the number of strings in the initial and final states is different. Thus, we require the corresponding phase space of multi-particle free states, which will be a sum over products of the single string phase space,

$$\sum_f = \sum_{N_f=0}^{\infty} \prod_{a=1}^{N_f} \Big(\sum\!\!\!\!\!\!\!\int_a V_{d-1} \int \frac{d^{d-1}\mathbf{p}_a}{(2\pi)^{d-1}} \Big), \tag{4.25}$$

with a labeling the string whose phase space we are summing/integrating over, and d is the dimensionality of spacetime in which the strings are allowed to propagate in ($d \leq 26$ or 10 for the bosonic or superstring theory). The phase space sums (4.24) or (4.25) are not Lorentz invariant, but of course Lorentz invariance will be restored in physically observable quantities. This is the price of wanting to construct dimensionless S-matrix elements, S_{fi}, that can be directly interpreted as probabilities.

Vertex Operator Normalization

The normalization of the path integral (or S-matrix) and the normalization of vertex operators is completely determined in terms of the normalization of a single vertex operator by the unitarity constraint (4.21) and the identification (4.22). The normalization of this single vertex operator can in turn be fixed by the "one string in the universe" normalization condition, by making contact with the corresponding field theory, and we describe this next.

Working in the flat Minkowski background,

$$G_{\mu\nu}(X) = \eta_{\mu\nu}, \qquad B_{\mu\nu}(X) = 0, \qquad \text{and} \qquad \Phi(X) = \langle\Phi\rangle,$$

with $\langle\Phi\rangle$ a constant, let us consider the tachyon vertex operator,

$$V(z,\bar{z}) = \mathcal{N} e^{ip\cdot X(z,\bar{z})}. \tag{4.26}$$

The tachyon vertex operator is a very useful quantity to consider in bosonic string theory because it is the basic building block of higher mass vertex operators. We shall eventually relate the normalization of the tachyon to the normalization of all other vertex operators.

[11] In the case of coherent states it is simplest to use lightcone coordinates, see (4.39), because coherent states are (as we will see) eigenstates of p^+, p^i but not of p^-.

To compute the normalization constant $\mathcal{N}$, we notice that V satisfies the equation of motion,

$$\Big(\nabla^2+\frac{4}{\alpha'}\Big)V=0,$$

with the derivative taken with respect to the zero mode x^μ. The low energy field theory corresponding to the tachyon field will therefore be that of a scalar field with mass $m^2=-4/\alpha'$ [140],

$$S[V]=-\frac{1}{(\alpha')^{\frac{d-2}{2}}}\int d^dx\,e^{-2\langle\Phi\rangle}\Big(\frac{1}{2}(\nabla V)^2+\frac{1}{2}m^2V^2+\dots\Big), \tag{4.27}$$

where we have taken into account the fact that the dilaton (even if it is constant in this case) couples universally as shown [140], and we ignore all interaction terms because we are interested in the case when the string under consideration is asymptotically free and onshell, as required by conformal invariance [132]. We have found it convenient to include an appropriate power of α' (with $[\alpha']=L^2$) such that V is dimensionless, $[V]=1$. (This will ensure that the S-matrix is dimensionless independently of the number of vertex operators.) Furthermore, an overall *dimensionless* constant in $S[V]$ is immaterial because it can be absorbed into a shift in $\langle\Phi\rangle$.

As discussed above, the overall normalization of the S-matrix and of all vertex operators *other* than, say, the tachyon are fixed by unitarity. Unitarity will thus relate the normalization of all vertex operators to that of the tachyon. It is convenient to define:

$$g_c\equiv e^{\langle\Phi\rangle}(\alpha')^{\frac{d-2}{4}},\qquad \text{and}\qquad g_s\equiv e^{\langle\Phi\rangle}. \tag{4.28}$$

Now, the "one string in V_{d-1}" constraint can be solved by requiring that the total energy, H, in volume V_{d-1} is that of a single string, $p^0=\sqrt{\mathbf{p}^2+m^2}$ (with $m^2=-4/\alpha'$). We plug the plane wave solution, $V(x)=\mathcal{N}e^{ip\cdot x}+\mathcal{N}^*e^{-ip\cdot x}$, into the Hamiltonian associated to (4.27), which is given by $H(t)=\int_{V_{d-1}}d^{d-1}x[(\partial_0V)\frac{\partial\mathscr{L}}{\partial(\partial_0V)}-\mathscr{L}]$ (with $S[V]=\int dt\mathscr{L}$), and make the link with the string theory vertex operator by identifying $\mathcal{N}$ here with the $\mathcal{N}$ in (4.26). It follows that, $H(t)=|\mathcal{N}|^2 2(p^0)^2V_{d-1}g_c^{-2}$, implying that there will be one string in volume V_{d-1} if:

$$\frac{H(t)}{p^0}=1,\qquad \text{or, equivalently,}\qquad \mathcal{N}=\frac{g_c}{\sqrt{2p^0V_{d-1}}}. \tag{4.29}$$

That is, the "one string in volume V_{d-1}"-normalized tachyon vertex operator is,

$$V(z,\bar z)=\frac{g_c}{\sqrt{2p^0V_{d-1}}}\,e^{ip\cdot X(z,\bar z)}, \tag{4.30}$$

with $E=\sqrt{\mathbf{p}^2+m^2}$ (and $m^2=-4/\alpha'$). Although we will not prove this here, it is not too hard to show that this is precisely the normalization required by: (i) Lorentz invariance of the unitarity constraint of the S-matrix; (ii) Lorentz invariance of the scattering cross section; (iii) the requirement that S-matrix elements, S_{fi}, be dimensionless, so as to interpret $|S_{fi}|^2$ as a probability, as in (4.18).

Notice now that the normalization of the tachyon vertex is such that the most singular term in the operator product expansion is,

$$V(z,\bar{z})\cdot V(0,0)\cong\left(\frac{g_c^2}{2EV_{d-1}}\right)\frac{1}{|z|^4}+\dots. \tag{4.31}$$

This suggests that we may be able to normalize *arbitrarily* massive bosonic string vertex operators by requiring that (4.31) is satisfied. This is indeed the case, and it can be shown (although we shall not do so here) that this statement is compatible with unitarity (4.21). Notice that the normalization condition (4.31) ensures that vertex operators are dimensionless.

S-Matrix Unitarity and Factorization

It is often more convenient to work with vertex operators normalized according to,[12]

$$V(z,\bar{z})\cdot V(0,0)\cong\frac{g_c^2}{|z|^4}+\dots, \tag{4.32}$$

instead of (4.31). Starting from the original normalization (4.31), we extract the factors of $1/\sqrt{2EV_{d-1}}$ out of every vertex operator and, for N asymptotic states in total, define $\mathcal{M}(1,\dots,N)$ according to,

$$T_{fi}\equiv T(1,\dots,N)\equiv\frac{\mathcal{M}(1,\dots,N)}{\sqrt{2E_1V_{d-1}}\dots\sqrt{2E_NV_{d-1}}}, \tag{4.33}$$

with T_{fi} defined in (4.20). When vertex operators are normalized according to (4.32), the path integral yields instead,

$$i(2\pi)^d\delta^d(P_f-P_i)\mathcal{M}(1,\dots,N)=\sum_{h=0}^{\infty}\int_{\mathcal{E}\times\mathcal{M}_h}\mathcal{D}g\mathcal{D}Xe^{-S[g,X]}V^{(1)}\dots V^{(N)}, \tag{4.34}$$

and so according to (4.22) and (4.33) we need to divide (4.34) by the factors $\sqrt{2E_1V_{d-1}}\dots$ to get an S-matrix element,[13]

$$S(1,\dots,N)=\delta_{fi}+i(2\pi)^d\delta^d(P_f-P_i)\frac{\mathcal{M}(1,\dots,N)}{\sqrt{2E_1V_{d-1}}\dots\sqrt{2E_NV_{d-1}}}, \tag{4.35}$$

with $S_{fi}\equiv S(1,\dots,N)$. In terms of $\mathcal{M}(1,\dots,N)$, the unitarity constraint (4.21) in the case where the intermediate strings in the sum over states are *single* string states then reads:

$$\begin{aligned}\mathcal{M}(1,\dots,N)-\mathcal{M}^*(1,\dots,N)&=\\ &=i\sum\!\!\!\!\!\!\int_a\frac{d^{d-1}\mathbf{p}_a}{(2\pi)^{d-1}}\frac{1}{2E_a}(2\pi)^d\delta^d(p_a-P_i)\mathcal{M}(1,\dots,a)\mathcal{M}^*(-a,\dots,N),\end{aligned} \tag{4.36}$$

[12] This is in agreement with the conventions of Polchinski [114], $g_c^{\rm here}\equiv g_c^{\rm Polchinski}$, where it is shown that the relation to the gravitational coupling is $\kappa=2\pi g_c$ with $\kappa^2=8\pi G_{(d)}$ and $G_{(d)}$ the d-dimensional Newton's constant.

[13] Note that the factors of $1/\sqrt{2EV_{d-1}}$ are absent in the S-matrix elements defined in [114].

with the sum/integral being over a complete set of states, written symbolically as a, and their associated quantum numbers. There is an obvious generalization for multi-string intermediate states. (Because of worldsheet duality it is also necessary to sum over both (say) s- and t-channel contributions in the case of $N = 4$, and their natural generalizations for $N > 4$.) It is thus clear that the volume factors have cancelled out and the factors of $\sqrt{2E_i}$ have combined to make the unitarity constraint (4.36) Lorentz invariant. Thus, the factors $\sqrt{2E_i}$ in the vertex operator normalizations are required for Lorentz invariance when the corresponding quantities $\mathcal{M}(1,\dots,N)$ are Lorentz invariant, which is indeed the case in string theory; recall that $\frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}}\frac{1}{2E_\mathbf{p}}$ is the Lorentz invariant phase space, with $E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$. Using $\int \frac{d^dp}{(2\pi)^d}(2\pi)\delta(p^2+m^2)\theta(p^0) = \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}}\frac{1}{2E}$ and $2\pi i\,\delta(x) = \frac{1}{x-i0} - \frac{1}{x+i0}$, it is not too hard to show that tree level unitarity (4.36) is guaranteed if the following factorization formula holds true,

$$i\mathcal{M}(1,\dots,N) = \sum\!\!\!\!\!\!\int_a i\mathcal{M}(1,\dots,a)\cdot\frac{-i\theta(k_a^0)}{k_a^2+m_a^2-i0}\cdot i\mathcal{M}^*(-a,\dots,N), \tag{4.37}$$

and

$$\mathcal{M}(1,\dots,a)\mathcal{M}^*(-a,\dots,N) = \big[\mathcal{M}(1,\dots,a)\mathcal{M}^*(-a,\dots,N)\big]^*.$$

Notice that $\frac{-i\theta(k^0)}{k^2+m^2-i0}$, is the propagator (in the $(-++\dots)$ signature) for a scalar particle of mass m^2 with the correct analytic continuation for a Minkowski process. Given the normalization of the tachyon, the formula (4.37) can be used to derive the normalization of the tree level S-matrix and of all other vertex operators.

Vertex Operator Normalization in Lightcone Coordinates

It is sometimes more convenient (especially in the case of coherent states) to use lightcone coordinates, $\{p^\pm, p^i\}$ with $i = 1,\dots,d-2$ and $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^{d-1})$. In lightcone coordinates, the statement (4.17) is replaced by:

$$(2\pi)\delta(p^{\pm\prime} - p^\pm) \equiv V_\mp, \qquad \text{and} \qquad (2\pi)^{d-2}\delta^{d-2}(\mathbf{p}' - \mathbf{p}) \equiv V_{d-2}. \tag{4.38}$$

The momentum phase space analogous to (4.23) is:

$$\mathcal{V}_{d-1}\frac{d^{d-2}\mathbf{p}}{(2\pi)^{d-1}}\frac{dp^+}{2\pi}, \qquad \text{with} \qquad \mathcal{V}_{d-1} \equiv V_{d-2}V_-. \tag{4.39}$$

For the sum over single string states (4.24) we thus have,

$$\sum_f = \sum\!\!\!\!\!\!\int \mathcal{V}_{d-1}\int_{\mathbb{R}^{d-2}}\frac{d^{d-2}\mathbf{p}}{(2\pi)^{d-1}}\int_0^\infty\frac{dp^+}{2\pi}, \tag{4.40}$$

and similarly for the multi-string case (4.25). We next need the statements analogous to (4.30) and more generally (4.31) in the case of lightcone gauge coordinates.

In direct analogy with the procedure described in the paragraph containing (4.30), we compute the *lightcone coordinate* Hamiltonian associated to the action (4.27), which is given by $H(x^+) = \int d^{d-2}x dx^- [(\partial_+ V)\frac{\partial \mathscr{L}}{\partial(\partial_+ V)} - \mathscr{L}]$ (with $S[V] = \int dx^+ \mathscr{L}$), and enforce the "one string in volume $\mathcal{V}_{d-1}$" constraint by truncating the region of integration in $H(x^+)$ to $\mathcal{V}_{d-1}$ and requiring that $H(x^+)/p^- = 1$. Here $p^- = \frac{1}{2p^+}(\mathbf{p}^2 + m^2)$, is the tachyon onshell condition which yields the lightcone energy associated to a single tachyon (here $m^2 = -4/\alpha'$). Plugging the plane wave solution, $V(x) = \mathcal{N} e^{ip\cdot x} + \mathcal{N}^* e^{-ip\cdot x}$, into the Hamiltonian $H(x^+)$ and requiring that there is one string in volume $\mathcal{V}_{d-1}$, i.e. $H(x^+)/p^- = 1$, thus determines $\mathcal{N}$,

$$\mathcal{N} = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}. \tag{4.41}$$

We make the link with the string theory vertex operator by identifying this $\mathcal{N}$ with that found in (4.26), so that the "one string in volume $\mathcal{V}_{d-1}$"-normalized tachyon vertex operator in lightcone coordinates is,

$$V(z,\bar{z}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\, e^{ip\cdot X(z,\bar{z})}. \tag{4.42}$$

This normalization is such that the most singular term in the operator product expansion is,

$$V(z,\bar{z})\cdot V(0,0) \cong \left(\frac{g_c^2}{2p^+\mathcal{V}_{d-1}}\right)\frac{1}{|z|^4} + \dots, \tag{4.43}$$

and, in direct analogy to the above, this normalization can be used for arbitrarily massive bosonic vertex operators.[14]

Again, as discussed above, see (4.32), it is sometimes more convenient to work with vertex operators normalized according to,

$$V(z,\bar{z})\cdot V(0,0) \cong \frac{g_c^2}{|z|^4} + \dots, \tag{4.44}$$

instead of (4.43). From (4.43), this implies that we should extract the factors of $1/\sqrt{2p^+\mathcal{V}_{d-1}}$ out of every vertex operator and, as in (4.33), for N asymptotic states in total define:

$$T_{fi} \equiv T(1,\dots,N) \equiv \frac{\mathcal{M}(1,\dots,N)}{\sqrt{2p_1^+\mathcal{V}_{d-1}}\dots\sqrt{2p_N^+\mathcal{V}_{d-1}}}. \tag{4.45}$$

As in (4.34), when vertex operators are normalized according to (4.44), the path integral yields,

$$i(2\pi)^d\delta^d(P_f - P_i)\mathcal{M}(1,\dots,N) = \sum_{h=0}^{\infty}\int_{\mathcal{E}\times\mathcal{M}_h}\mathcal{D}g\mathcal{D}X e^{-S[g,X]}V^{(1)}\dots V^{(N)}, \tag{4.46}$$

but now we need to divide (4.34) by the factors $\sqrt{2p_1^+\mathcal{V}_{d-1}}\dots\sqrt{2p_N^+\mathcal{V}_{d-1}}$ to get an S-matrix element, and in particular,

$$S_{fi} = \delta_{fi} + i(2\pi)^d\delta^d(P_f - P_i)\frac{\mathcal{M}(1,\dots,N)}{\sqrt{2p_1^+\mathcal{V}_{d-1}}\dots\sqrt{2p_N^+\mathcal{V}_{d-1}}}. \tag{4.47}$$

[14] The reason as to why lightcone coordinates are useful in the case of coherent states (as mentioned above) is that they are eigenstates of $\hat{p}^+$ and $\hat{\mathbf{p}}$, but not of $\hat{p}^-$, and so it is not possible to factor out $1/\sqrt{2p^0}$, but it is possible to factor out $1/\sqrt{2p^+}$.

The unitarity statement analogous to (4.36) in lightcone coordinates can be derived directly from (4.36) since (4.36) is Lorentz invariant, or it can be derived from (4.21) and (4.45). It reads,

$$\begin{aligned}&\mathcal{M}(1,\dots,N)-\mathcal{M}^*(1,\dots,N)=\\&\quad=i\sum\!\!\!\!\!\!\!\int_a\int_{\mathbb{R}^{d-2}}\frac{d^{d-2}\mathbf{p}_a}{(2\pi)^{d-1}}\int_0^\infty\frac{dp^+}{2\pi}\frac{1}{2p_a^+}(2\pi)^d\delta^d(p_a-P_i)\mathcal{M}(1,\dots,a)\mathcal{M}^*(-a,\dots,N),\end{aligned}\tag{4.48}$$

and the result is (as above) independent of the volume $\mathcal{V}_{d-1}$. To see this let us consider the relativistic phase space integral, $\int\frac{d^dk}{(2\pi)^d}(2\pi)\delta(k^2+m^2)\theta(k^0)$ (which as mentioned above is equivalent to $\int\frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}}\frac{1}{2E_\mathbf{k}}$) with[15] $m^2=2N-2$. In lightcone coordinates (where $dk^-\wedge dk^+=dk^0\wedge dk^{d-1}$), let us redefine the integration variable:

$$k^-=p^-+\frac{N}{p^+},\qquad k^+=p^+,\qquad k^i=p^i,\qquad i=1,\dots,24.\tag{4.49}$$

This removes the N-dependence from the δ-function, $\delta(k^2+2N-2)=\delta(p^2-2)$, and $dk^-\wedge dk^+=dp^-\wedge dp^+$. Ignoring the tachyon, so that $\theta(k^0)=\theta(p^+)$, the Lorentz invariant phase space now reads,

$$\begin{aligned}\int\frac{d^dk}{(2\pi)^d}(2\pi)\delta(k^2+2N-2)\theta(k^0)&=\int\frac{d^dp}{(2\pi)^d}(2\pi)\delta(p^2-2)\theta(p^+)\\&=\int_{\mathbb{R}^{d-2}}\frac{d^{d-2}\mathbf{p}}{(2\pi)^{d-2}}\int_0^\infty\frac{dp^+}{2\pi}\frac{1}{2p^+},\end{aligned}\tag{4.50}$$

where we have integrated out p^-, so that $p^-=\frac{1}{2p^+}(\mathbf{p}^2-2)$, the tachyon onshell condition. Therefore,

$$\int_{\mathbb{R}^{d-1}}\frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}}\frac{1}{2E_\mathbf{k}}=\int_{\mathbb{R}^{d-2}}\frac{d^{d-2}\mathbf{p}}{(2\pi)^{d-2}}\int_0^\infty\frac{dp^+}{2\pi}\frac{1}{2p^+},$$

where it is understood that the integrands are taken onshell; the aforementioned unitarity statement (4.48) is proven.

Tree Level Operator Statements

It is sometimes desirable to compute expectation values of various operators, such as the angular momentum $J^{\mu\nu}$,

$$\langle J^{\mu\nu}\rangle\equiv\langle V|J^{\mu\nu}|V\rangle\equiv J^{\mu\nu}_{\rm cl},\tag{4.51}$$

as this enables one to associate classically computed quantities, such as $J^{\mu\nu}_{\rm cl}$ that is in one-to-one correspondence with solutions of $\partial_z\partial_{\bar z}X^\mu=0$, to quantum-mechanical vertex operators that exhibit these classical characteristics (in the expectation value sense). It is convenient to work in the operator formalism here[16] and absorb the α' and $e^{\langle\Phi\rangle}$ dependence

[15] It is implied here that $\alpha'=2$ in the case of closed strings or $\alpha'=1/2$ in the case of open strings.

[16] The usual path integral definition is not useful here because the path integral associated to two vertex operator insertions vanishes (unless the state under consideration is unstable), because the volume of the CKG is infinite and two vertices are not sufficient to saturate this infinity. This is because the path integral yields only the *non*-trivial contribution to the S-matrix, whereas in (4.51) it is the trivial or non-interacting part that is relevant.

of $V(z,\bar{z})$ into $|0,0;p\rangle$, recall that $g_c = e^{\langle\Phi\rangle}\alpha'^{\frac{d-2}{4}}$, and in particular,

$$|0,0;p\rangle \simeq g_c\, e^{ip\cdot X(z,\bar{z})}. \tag{4.52}$$

At tree level, the factors of $e^{\langle\Phi\rangle}$ (in g_c in each of the two vertex operators in e.g. $\langle V|J^{\mu\nu}|V\rangle$ and the Euler characteristic $e^{-\chi(\Sigma)\langle\Phi\rangle} = e^{-2\langle\Phi\rangle}$) cancel. If we then normalize the state and expectation values in a relativistically invariant manner,

$$|V\rangle = \frac{1}{\sqrt{2E_{\mathbf{p}}V_{d-1}}}|0,0;p\rangle, \qquad \langle 0,0;p'|0,0;p\rangle = 2E_{\mathbf{p}}(2\pi)^{d-1}\delta^{d-1}(\mathbf{p}'-\mathbf{p}), \tag{4.53}$$

then, according to (4.17), such states have unit norm,

$$\langle V|V\rangle = 1.$$

The dimensionality of g_c is precisely that required to make the relativistic normalization shown possible. In lightcone coordinates we have similarly the following relativistic normalization,

$$|V\rangle = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}|0,0;p\rangle, \qquad \langle 0,0;p'|0,0;p\rangle = 2p^+(2\pi)\delta(p^{+\prime}-p^+)(2\pi)^{d-2}\delta^{d-2}(\mathbf{p}'-\mathbf{p}). \tag{4.54}$$

In the following section we will see that higher mass (mass eigen-)states with unit norm can be constructed by acting on the tachyon vertex with DDF operators, A^i_n and $\bar{A}^i_n$, which satisfy $[A^i_n, A^j_m] = n\delta^{ij}\delta_{n+m,0}$:

$$|V\rangle = \frac{1}{\sqrt{2E_{\mathbf{p}}V_{d-1}}}\,C\xi_{ij\dots,kl\dots}A^i_{-n_1}A^j_{-n_2}\dots\bar{A}^k_{-\bar{n}_1}\bar{A}^l_{-\bar{n}_2}\dots|0,0;p\rangle,$$

The combinatorial constant C, defined in (4.62), is chosen such that

$$\langle V|V\rangle = 1,$$

remains true for arbitrarily massive states. There is a similar result in lightcone coordinates with $2p^+\mathcal{V}_{d-1}$ replacing $2EV_{d-1}$, with the corresponding normalization of the tachyonic lightcone vacuum implied as shown above. Furthermore, the corresponding lightcone *gauge* quantities can be obtained by replacing A^i_n and $\bar{A}^i_n$ by α^i_n and $\tilde{\alpha}^i_n$ respectively. Similarly, we will see that the closed string covariant coherent states are of the form,

$$|V\rangle = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,\mathcal{C}_{\lambda\bar{\lambda}}\int_0^{2\pi}\!\! d\!\!\!{}^-\! s\,\exp\Big\{\sum_{n=1}^{\infty}\frac{1}{n}e^{ins}\lambda_n\cdot A_{-n}\Big\}\exp\Big\{\sum_{m=1}^{\infty}\frac{1}{m}e^{-ims}\bar{\lambda}_m\cdot\bar{A}_{-m}\Big\}|0,0;p\rangle, \tag{4.55}$$

see (5.49), which again has unit norm,

$$\langle V|V\rangle = 1,$$

as do the mass eigenstates. Notice that, as mentioned above, for coherent states lightcone coordinates are more convenient.

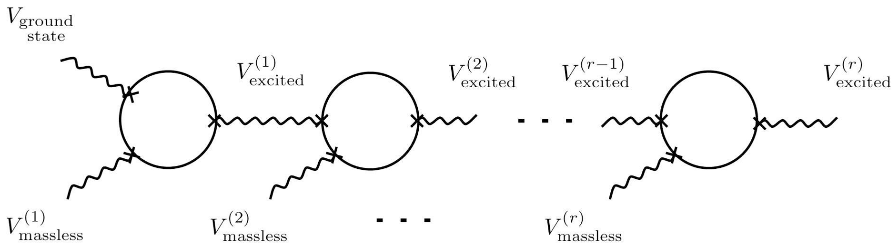

Figure 4.2: The DDF construction of open string physical vertex operators.

Similar results hold for open strings, with g_o and $|0;p\rangle$ replacing g_c and $|0,0;p\rangle$, both vacua being normalized in the same manner, as in (4.53) or (4.54) depending on the choice of coordinates. In addition, in the case of open strings left- and right-movers are related and hence one can construct states using only, say, the holomorphic quantities A^i_n or α^i_n. The closed and open string couplings, g_c and g_o, are related by unitarity [114], e.g. by factorizing the annulus diagram on a closed string pole; in $d = 26$, $g_o^2 = 2^{18}\pi^{25/2}\alpha'^6 g_c$, and in our conventions, see (4.28), where $g_c = e^{\langle\Phi\rangle}\alpha'^6$,

$$g_o = 8\pi^{\frac{1}{4}}(2\pi\alpha')^6 e^{\langle\Phi\rangle/2}. \tag{4.56}$$

Note that the dimensionality of both g_c and g_o is the same. Below we will consider both open and closed string vertex operators in detail.

4.4 Arbitrarily Massive Vertex Operators

In the present section we describe the construction of general covariant vertex operators for the bosonic string. We base our approach on the general (yet practical) approach of Del Giudice, Di Vecchia and Fubini [127, 200, 128] (DDF), see also [129, 130, 166, 131], although we will adopt a somewhat more modern viewpoint.

The geometrical string picture underlying the DDF vertex operator construction is as follows. Arbitrary vertex operators can be extracted from a certain factorization of an N-point scattering amplitude. The setup we have in mind is the following: an initial vacuum state absorbs some number of massless string vertices resulting in an excited state – the resulting excited vertex operator is what we wish to extract. The first non-trivial statement is that a complete set of vertex operators can be obtained from the factorization of a diagram with an arbitrary number of massless open string vertex operator insertions and a vacuum insertion. When the vertex operator we wish to extract is an open string state the appropriate factorization is shown in Fig. 4.2.

As we show below, a complete set of states can be obtained if the i^{th} massless photon vertex operator has momentum $k^\mu_{(i)} = -n_i q^\mu$ and polarization tensor $\xi^j_{(i)}$ with $q^2 = 0$ and n_i a positive integer. All photons therefore approach the vacuum string state from the same angle of incidence with momenta that are only allowed to differ by some integer multiple of a so far arbitrary null vector q^μ. Conformal invariance then enforces the vector q^μ to be transverse to all photon polarization tensors, $\xi^j_{(i)}$, and this leads to spacetime gauge invariance [166]. The vacuum vertex operator, $e^{ip\cdot X}$, which absorbs these photons has momentum p^μ and is tachyonic in the bosonic string, $p^2 = 1/\alpha'$. In addition, to ensure that the internal strings (see Fig. 4.2) are onshell one must require that $(p-Nq)^2 = (1-N)/\alpha'$ for $N = \sum_i n_i$, and therefore: $p\cdot q = 1/(2\alpha')$. The choice of integers $\{n_1, n_2, \dots, n_r\}$ determines the mass level N of the vertex operator we wish to extract and $(p-Nq)^\mu$ is the momentum of the excited state.

In the spirit of the discussion of the previous section this procedure is to be thought of in a step-wize sense: first consider a single photon absorbed by an open string vacuum state. Vertices produced in this process are then given by the residue of the OPE as these two initial states approach on the boundary of the worldsheet,

$$V^{(1)}_{\text{excited}}(w) \cong \oint_w \text{đ}z_1\, V^{(1)}_{\text{massless}}(z_1)\cdot V_{\substack{\text{ground}\\ \text{state}}}(w).$$

The resulting state, $V^{(1)}_{\text{excited}}(w)$ has momentum $(p-n_1q)^\mu$ with n_1 a positive integer of our choice. $V^{(1)}_{\text{excited}}(w)$ is then brought close to an additional photon, $V^{(2)}_{\text{massless}}(z)$, the residue of this OPE now giving rise to a new state,

$$V^{(2)}_{\text{excited}}(w) \cong \oint_w \text{đ}z_2\, V^{(2)}_{\text{massless}}(z_2)\cdot V^{(1)}_{\text{excited}}(w),$$

with momentum $(p-n_1q-n_2q)^\mu$ and so on. Carrying this out r times gives rise to a general vertex operator,

$$V^{(r)}_{\text{excited}}(w) \cong \oint_w \text{đ}z_r\, V^{(r)}_{\text{massless}}(z_r)\dots \oint_w \text{đ}z_2\, V^{(2)}_{\text{massless}}(z_2)\cdot \oint_w \text{đ}z_1\, V^{(1)}_{\text{massless}}(z_1)\cdot V_{\substack{\text{ground}\\ \text{state}}}(w),$$

where it is to be understood that the rightmost integrals are carried out first so as to respect the order with which the photons are absorbed by the vacuum. Defining $A^i_n = \sqrt{\frac{2}{\alpha'}}\oint \text{đ}z\, \partial_z X^i(z) e^{inq\cdot X(z)}$, the above state can be equivalently written as,

$$V^{(r)}_{\text{excited}}(w) \cong \frac{g_o}{\sqrt{2p^+\mathcal{V}_{d-1}}}\, C\xi_{i\dots j}A^i_{-n_1}\dots A^j_{-n_r}\cdot e^{ip\cdot X(w)}, \tag{4.57}$$

with C a to-be-determined normalization constant and $\xi^{ij\cdots} = \xi^i_{(1)}\xi^j_{(2)}\dots$. We have included the factor of $\frac{g_o}{\sqrt{2p^+\mathcal{V}_{d-1}}}$ that we computed (by the 'one string in volume $\mathcal{V}_{d-1}$' condition) in Sec. 4.3 that ensures that S-matrix elements transform correctly under Lorentz transformations. Recall from above, see (4.56), that we denote the open string coupling by g_o.

The A^i_n are the so-called DDF operators [127, 128]. After carrying out the contour integrals the resulting vertex operator, $V(w)\equiv V^{(r)}_{\text{excited}}(w)$, will be composed of a linear superposition of normal ordered terms of the form $\zeta_{\mu\nu\dots}\partial^\# X^\mu \partial^\# X^\nu\dots$ with an overall factor of $e^{i(p-Nq)\cdot X(z)}$ (we shall compute these explicitly). The polarization tensors $\zeta_{\mu\nu\dots}$ will be composed of the quantities, $\xi^{ij\cdots}$, p^μ, and q^μ. There is clearly a one-to-one correspondence between vertex operators $V(w)$ and lightcone gauge states,

$$|V\rangle_{\text{lc}} = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\, C'\xi_{i\dots j}\alpha^i_{-n_1}\dots\alpha^j_{-n_r}|0;p^+,p^i\rangle,$$

with C' an *a priori* different normalization constant to C. It is determined by the condition $\langle V|V\rangle_{\rm lc}=1$ and

$$\langle 0;p^{+\prime},p^{i\prime}|0;p^+,p^i\rangle = 2p^+(2\pi)\delta(p^{+\prime}-p^+)(2\pi)^{d-2}\delta^{d-2}(\mathbf{p}'-\mathbf{p}).$$

Therefore, we reach the important conclusion that covariant vertex operators extracted via factorization of a scattering amplitude with photons and a ground state tachyon form a complete set. A rather non-trivial statement is that $V(w)$ has the same mass and angular momenta as $|V\rangle_{\rm lc}$, and we take this correspondence further and *conjecture* that $V(w)$ and $|V\rangle_{\rm lc}$ also share identical interactions.

Note the DDF vertex operator construction *is* covariant [131], although not manifestly so: even though the $\xi^j_{(i)}$ do not contain any timelike directions (as is also the case for the lightcone gauge states) the resulting polarization tensors $\zeta^{\mu\nu\cdots}$ potentially have all components non-vanishing. We have not enforced any constraint, e.g. $X^+\propto\tau$, on the target space coordinates in the vertex operator $V(w)$, and so the path integral with vertex insertions $V(w)$ includes a measure $\int_{\mathcal{E}}\mathcal{D}X^0\mathcal{D}X^1\dots\mathcal{D}X^{25}e^{\frac{i}{\hbar}S[X]}$. Manifest covariance can be restored as we show with particular examples although this is of course not required in order to plug such vertices into covariant path integrals. The correspondence with the lightcone gauge states suggests also the following: the quantity $\xi^{ij\cdots}$ that appears in the covariant vertex operators are to be identified with tensors corresponding to irreducible representations of SO(25), the little group of SO(25,1) for massive states: that is, $\xi^{ij\cdots}$ have the symmetries of Young tableaux [189].

A good consistency check is the following. Given that the DDF operators are integrals of photon vertex operators, i.e. integrals of (1,0) conformal primary operators, they must be gauge invariant: $[L_n,A^i_m]=0$. Therefore, $V(w)$ must satisfy the Virasoro constraints: the operator $L_{n>0}$ will commute through to hit the vacuum, $e^{ip\cdot X}$, which will be annihilated if it is physical, i.e. if $p^2=1/\alpha'$. The L_0 operator similarly commutes through to hit the vacuum and given that $L_0\cdot e^{ip\cdot X}\cong e^{ip\cdot X}$, the full vertex operator $V(w)$ satisfies the Virasoro constraints automatically:

$$L_0\cdot V(w)\cong V(w),\qquad \text{and}\qquad L_{n>0}\cdot V(w)\cong 0.$$

In direct analogy to the lightcone gauge states the vertices $V(w)$ are transverse to null states [129] as one would expect given the underlying geometrical string picture on which the construction is based.

For the construction of closed string vertex operators it turns out that the naive expression, namely,

$$V(z,\bar z)=\frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,C\xi_{ij\dots,kl\dots}A^i_{-n_1}A^j_{-n_2}\dots\bar A^k_{-\bar n_1}\bar A^l_{-\bar n_2}\dots e^{ip\cdot X(z,\bar z)},\tag{4.58}$$

with the DDF operators $A^i_n=\sqrt{\frac{2}{\alpha'}}\oint d\!\!{}^-z\,\partial_zX^i(z)e^{inq\cdot X(z)}$ and $\bar A^i_n=\sqrt{\frac{2}{\alpha'}}\oint d\!\!{}^-\bar z\,\partial_{\bar z}X^i(\bar z)e^{inq\cdot X(\bar z)}$, is also the correct expression, normalized to 'one string in volume $\mathcal{V}_{d-1}$' as required by unitarity, see Sec. 4.3. The lightcone gauge realization of this state is the expression [129, 130],

$$|V\rangle_{\rm lc} = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,C\xi_{ij\dots,kl\dots}\,\alpha^i_{-n_1}\alpha^j_{-n_2}\dots\tilde{\alpha}^k_{-\bar{n}_1}\tilde{\alpha}^l_{-\bar{n}_2}\dots|0,0;p\rangle. \qquad (4.59)$$

We as usual need to introduce the constraint, $N=\bar{N}$ by hand.[17] The closed string constraints analogous to the open string case are $p^2=4/\alpha'$, $p\cdot q=2/\alpha'$, $q^2=0$ and $q\cdot\xi=0$. The DDF operators commute with the Virasoro generators and so (4.58) again satisfies the Virasoro constraints. The normalization of the vacuum is:

$$\langle 0,0;p'|0,0;p\rangle = 2p^+(2\pi)\delta(p^{+'}-p^+)(2\pi)^{d-2}\delta^{d-2}(\mathbf{p}'-\mathbf{p}),$$

in which case $\langle V|V\rangle_{\rm lc}=1$, see Sec. 4.3.

Caution however is needed in interpreting this expression as a vertex arising in a scattering experiment of massless states and a vacuum (as we did above for the open string). If for example, the vacuum and the corresponding massless states in a string scattering experiment are all bulk vertex operators then a complete set of states would not be generated: e.g., vertices with an asymmetry corresponding to the lightcone gauge states $\alpha^i_{-1}\alpha^j_{-1}\tilde{\alpha}^k_{-2}|p^i,p^+;0,0\rangle$ could not be generated in a closed string scattering experiment of massless vertices and a tachyon. It is likely that rather vertex operators (4.58) can instead be created in an open string scattering experiment: factorization of a (one loop open string) scattering amplitude involving photons and a closed string tachyon should give rise to an arbitrary closed string vertex operator of the form (4.58). It might be worth mentioning that a closed string scattering experiment in a lightlike compactified spacetime, $X^-\sim X^-+2\pi R^-$ with $R^-=\frac{\alpha'}{2}q^-$, of massless vertex operators (with lightlike winding) and a tachyon (without lightlike winding) would generate a complete set of vertex operators of the form (4.58), without the need of introducing open string interactions.

Crucially, the above prescription for extracting vertex operators results in explicit polarization tensors for which there are no additional constraints to be solved, which is a common drawback of many other approaches to vertex operator constructions, see e.g. [132, 193, 196, 141, 133, 201, 139] among others.

Momentum Phase Space

We now examine a subtlety related to the fact that the operators A^i_n depend on the momenta q^μ. The question we want to address here is: when we compute expectation values, can different vertex operators be labelled by different null vectors q^μ? DDF operators satisfy an oscillator algebra, $[A^i_n,A^j_m]=n\delta^{ij}\delta_{n+m,0}$, which is identical to the algebra associated to the α^i_n operators, $[\alpha^i_n,\alpha^j_m]=n\delta^{ij}\delta_{n+m,0}$. In general, one might expect however

[17] Vertex operators (4.58) or (4.59) that do not satisfy the constraint $N=\bar{N}$ still satisfy the Virasoro constraints, $L_0=\bar{L}_0$, but require the presence of a lightlike compactified background. We will discuss vertex operators in lightlike compactified backgrounds in detail when we construct closed string coherent states.

that different vertex operators should be constructed out of DDF operators which in turn are defined with different q^μ – different choices of q^μ for different vertices corresponds to different choices of momentum, $k^\mu = p^\mu - Nq^\mu$. It would then seem that the relevant commutator is $[A_n^i, A_m^{j\,\prime}]$ rather than $[A_n^i, A_m^j]$ with $A_n^{i\,\prime}$ a DDF operator constructed out of q'. To examine this possibility further, let us analyze the constraints and momentum phase space.

Consider the case of open strings with both ends attached to a single Dp-brane, and take $p = 25$; we then generalize the results to arbitrary $p \geq 1$. In this case, we can write down results that hold for both open strings and closed strings when the choice $\alpha' = 1/2$ and $\alpha' = 2$ is made respectively. As discussed above, in the DDF formalism, the momentum of a level N mass eigenstate is:

$$k^\mu = p^\mu - Nq^\mu.$$

Two 26-dimensional vectors p^μ, q^μ are therefore needed to specify the momentum of the state, but there are only 3 constraint equations: $p^2 = 2$, $p \cdot q = 1$, and $q^2 = 0$, so that there remain, $2 \times 26 - 3 = 49$ free parameters. Given that k^μ has only 26 parameters, one of them being eliminated by making use of the mass shell condition, it follows that only 25 of the 49 free parameters are needed in order to completely specify the momentum of a state. Therefore, we can fix $49 - 25 = 24$ of the 2×26 parameters in p^μ, q^μ while still spanning the *full* the phase space. Use this freedom to set

$$q^i = 0, \qquad \text{for} \qquad i = 1, \ldots, 24,$$

for *all* states constructed by DDF operators. Substituting this into the constraint equations (4.76), leads to the positive energy solution,[18]

$$\begin{aligned} p^\mu &= \Big(\frac{c}{2}\big(\mathbf{p}^2 - 2\big) + \frac{1}{2c}, \mathbf{p}, -\frac{c}{2}\big(\mathbf{p}^2 - 2\big) + \frac{1}{2c}\Big), \\ q^\mu &= \big(-c, 0, \ldots, 0, c\big). \end{aligned} \tag{4.60}$$

As required, this choice satisfies $-(p - Nq)^2 \equiv m^2 = 2N - 2$ for any p^i, c. In terms of p^+ we have $c = 1/(\sqrt{2}p^+)$, and $k^- = \frac{1}{2p^+}(\mathbf{p}^2 + 2N - 2)$. The positive energy condition requires $c > 0$ (for non-tachyonic states, $N \geq 1$), and the full phase space (neglecting the tachyon) is:

$$-\infty \leq \mathbf{p} \leq \infty \qquad \text{and} \qquad p^+ > 0,$$

with $p^+ = -1/q^-$.[19] We reach the important conclusion that different vertex operators may indeed be labelled by different q^μ when their momenta differ, but that *all* vertices may be taken to have $q^i = q^+ = 0$ while spanning the *full* phase space. For instance, when

[18] Here for notational simplicity $\alpha' = 1/2$, or $\alpha' = 2$ for the open or closed string case respectively. Also, $\mathbf{p} = (p^1, \ldots, p^{24})$ and as usual $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^{25})$, or in the case of open strings attached to a Dp-brane, $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^p)$.

[19] As an example, if we boost to the rest frame where the $k^i = 0$ and $k^0 = \sqrt{2N-2}$, the vectors p^μ and q^μ are determined completely, and $c^{-1} = \sqrt{2N-2}$.

we compute the inner product of two covariant vertex operators of the form (4.57), we may take one vertex to be constructed out of DDF operators with $q'^- \neq 0$, $q'^i = q'^+ = 0$ and a vacuum with momentum p'^μ and the other to be constructed from DDF operators with $q^- \neq 0$, $q^i = q^+ = 0$ and a vacuum with momentum p^μ. The important point is now that

$$q \cdot q' = 0,$$

and it is due to this fact that $[A'^i_n, A^j_m] = n\delta^{ij}\delta_{n+m,0}$, with A'^i_n and A^i_n the DDF operators constructed out of q' and q respectively. Therefore, different vertex operators can be constructed out of different q^μ provided $q \cdot q' = 0$, which in the coordinate system shown above is equivalent to saying that different vertex operators can be labelled by $\{\mathbf{p}, p^+\}$, which can be taken to be independent for every vertex operator, as required.

In the next two sections we summarize what we have learnt and fill in the details on some of the finer points. We first discuss the closed string and then the modifications required for the open string.

4.4.1 Closed String

One of the virtues of the DDF formalism is that as mentioned above, it provides a dictionary which relates every light-cone gauge state to the corresponding covariant gauge vertex operator. Writing $N = \sum_j n_j$ and $\bar{N} = \sum_j \bar{n}_j$ with $N = \bar{N}$, a general light-cone gauge mass eigenstate state is of the form

$$|V\rangle_{\rm lc} = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\, C\xi_{ij\dots,kl\dots}\, \alpha^i_{-n_1}\alpha^j_{-n_2}\dots\tilde{\alpha}^k_{-\bar{n}_1}\tilde{\alpha}^l_{-\bar{n}_2}\dots|0,0;p^+,p^i\rangle, \tag{4.61}$$

with $|0,0;p^+,p^i\rangle$ an eigenstate of p^+, p^i and annihilated by the (dimensionless) lowering operators $\alpha^i_{n>0}$, $\tilde{\alpha}^i_{n>0}$, normalized according to (4.54). If the polarization tensor $\xi_{ij\dots,kl\dots}$ is normalized to unity $\xi_{ij\dots,kl\dots}\xi^{ij\dots,kl\dots} = 1$, then the combinatorial normalization constant, C, contains [114] a factor of $\frac{1}{\sqrt{n}}$ for every α^i_{-n} that appears and factors of $\frac{1}{\sqrt{\mu_{n,i}!}}$, with $\mu_{n,i}$ the multiplicity of α^i_n in the above product. Similar factors are required for the anti-holomorphic sector; in total[20],

$$C \equiv \frac{1}{\sqrt{\prod_r n_r \prod_{n,i}\mu_{n,i}!}} \times \frac{1}{\sqrt{\prod_s \bar{n}_s \prod_{\bar{n},i}\bar{\mu}_{\bar{n},i}!}}. \tag{4.62}$$

The DDF formalism states that to every light-cone gauge state (4.61) there corresponds [131] the correctly normalized covariant vertex operator of momentum k,

$$V(z,\bar{z}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\, C\xi_{ij\dots,kl\dots}A^i_{-n_1}A^j_{-n_2}\dots\bar{A}^k_{-\bar{n}_1}\bar{A}^l_{-\bar{n}_2}\dots e^{ip\cdot X(z,\bar{z})}, \tag{4.63}$$

[20] The constant C should not be confused with that obtained in the previous sections. Throughout the rest of the section C will be defined according to (4.62). For coherent states (in later sections) C will again be different.

with the (dimensionless) DDF operators, A_n^i, $\bar{A}_n^i$, defined by,[21]

$$A_n^i = \sqrt{\frac{2}{\alpha'}}\oint d\!\!^{-}\!z\,\partial_z X^i(z)e^{inq\cdot X(z)}, \quad \text{and} \quad \bar{A}_n^i = \sqrt{\frac{2}{\alpha'}}\oint d\!\!^{-}\!\bar{z}\,\partial_{\bar{z}} X^i(\bar{z})e^{inq\cdot X(\bar{z})}, \tag{4.64}$$

The indices i are understood to be transverse to q^μ. In accordance with the above considerations the null spacetime vector q^μ and the (tachyonic) vacuum momentum p^μ are such that,

$$p^2 = \frac{4}{\alpha'}, \qquad p\cdot q = \frac{2}{\alpha'}, \qquad \text{and} \qquad q^2 = 0. \tag{4.65}$$

The quantity $k^\mu = p^\mu - Nq^\mu$, as discussed above is identified with the momentum of the vertex operator (4.63): from the definitions of p and q one may confirm that the mass shell condition is automatically satisfied if N is identified with the level number, $N = \sum_i n_i$,[22]

$$k^\mu = p^\mu - Nq^\mu, \qquad \text{and} \qquad k^2 = \frac{4}{\alpha'}(1-N). \tag{4.67}$$

The vertex (4.63) is not yet normal ordered and can be brought into a manifestly normal ordered form by bringing the operators in the integrands close to the vacuum, summing over all Wick contractions using the standard sphere two-point function for scalars,

$$\big\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\big\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2, \tag{4.68}$$

and evaluating the resulting contour integrals so as to extract the residues which correspond to the physical states. The contour integrals in (4.63) are to contain the ground-state vacuum. We are to bring the rightmost operators close to the vacuum first so as to respect the order with which these hit the vacuum. When the right-most DDF operator is brought close to the vacuum one evaluates the associated contour integral (with all other insertions placed outside the contour). Then bring the next DDF operator close to the resulting object, evaluate the operator products and the associated contour integral and so on. The procedure is analogous to the usual procedure of extracting vertex operators from Fock space states [201].

Using the operator product interpretation of the commutators (3.12) it is seen that the DDF operators satisfy an oscillator algebra and annihilate the vacuum when $n>0$ in direct analogy with the corresponding oscillators α_n and $\tilde{\alpha}_n$,

$$\big[A_n^i, A_m^j\big] \cong n\delta^{ij}\delta_{n+m,0}, \quad \text{and} \quad A^i_{n>0}\cdot e^{ip\cdot X(z,\bar{z})} \cong 0, \tag{4.69}$$

In addition, they commute with the Virasoro generators,[23] $L_m\cdot A_n \cong \bar{L}_m\cdot\bar{A}_n \cong \bar{L}_m\cdot A_n \cong L_m\cdot\bar{A}_n \cong 0$, for all $m,n\in\mathbb{Z}$ and the (tachyonic) vacuum on which the DDF operators

[21] Recall also that $d\!\!^{-}\!z = dz/(2\pi)$ which simplifies many formulas.

[22] It is also useful to note that one can always Lorentz boost to a frame where,

$$p = \big(c - 1/(2c), 0, \dots, 0, c + 1/(2c)\big), \qquad q = \big(c, 0, \dots, 0, c\big), \tag{4.66}$$

given that these satisfy $p^2=2$, $p\cdot q = 1$ and $q^2=0$ as required for any c, see Sec. 4.4. As an example, let us boost to the rest frame where the $k^i=0$ and $k^0=\sqrt{2N-2}$. p and q are determined completely, with $c^{-1} = -\sqrt{2N-2}$.

[23] Recall that the Virasoro generators read,

$$L_n = \oint \frac{dz}{2\pi i}z^{n+1}\big(-\frac{1}{\alpha'}\partial X\cdot\partial X\big), \quad \text{and} \quad \bar{L}_n = \oint\frac{d\bar{z}}{2\pi i}\bar{z}^{n+1}\big(-\frac{1}{\alpha'}\bar{\partial}X\cdot\bar{\partial}X\big).$$

act has conformal dimension $(1,1)$ and is therefore an L_0, $\bar{L}_0$ eigenstate, $L_0 \cdot e^{ip\cdot X(z,\bar{z})} \cong \bar{L}_0 \cdot e^{ip\cdot X(z,\bar{z})} \cong e^{ip\cdot X(z,\bar{z})}$. It follows that $V(z,\bar{z})$ is a physical vertex operator given that, $(L_0-1)\cdot V(z,\bar{z}) \cong 0$, $L_{m>0}\cdot V(z,\bar{z}) \cong 0$, and $(\bar{L}_0-1)\cdot V(z,\bar{z}) \cong 0$, $\bar{L}_{m>0}\cdot V(z,\bar{z}) \cong 0$.

An important point that can be mentioned here is that level matching, $(L_0-\bar{L}_0)\cdot V(z,\bar{z}) \cong 0$, is satisfied even for states with asymmetrically excited left- and right-movers, one such state being e.g. $V(z,\bar{z}) = \xi_{i,j}A^i_{-n}\bar{A}^j_{-m}e^{ip\cdot X(z,\bar{z})}$ with $n \neq m$ and positive. In fact, when we normal order this expression it will be seen that the presence of such states requires a lightlike compactification of spacetime – we will have more to say about this later on when we discuss covariant coherent states for closed strings.

We suggest that the states (4.61) and (4.63) are different descriptions of the same state. This is supported from various points of view: (a) there is a one-to-one correspondence between (4.61) and (4.63), and the lightcone gauge states (4.61) describe the complete set of states of the bosonic string; (b) the lightcone and covariant expressions have the same mass and angular momenta; (c) the first mass level states are identical. We *conjecture* and work on the assumption that the lightcone and covariant states share identical correlation functions (provided these are gauge invariant).

As discussed above, that (4.63) is covariant is not manifest due to the explicit presence of transverse indices. However, when the operator products and contour integrals are carried out the resulting object can be given a manifestly covariant form [131] – we will show this explicitly with a couple of examples.

In the next section we fill in the details for the open string covariant vertex operator construction before discussing the normal ordered expression of the closed string vertex operators.

4.4.2 Open String

The open string vertex operator construction proceeds in a similar manner, but there are certain differences that we mention here. First of all note that our open string conventions are presented in Sec. 2.7. We restrict our attention to open strings with both ends attached to a single Dp-brane (with $p \geq 1$ [165]), although such vertex operators are also relevant in scattering amplitude computations involving open string vertices stretched between parallel Dp-branes, the so-called p-p strings. The construction may be generalized to p-p' string vertex operators that stretch between a Dp- and a Dp'-brane along the lines of [202] by making use of the notion of a twist field.

Consider the case of p-p vertex operators where a string worldsheet is attached to two parallel Dp-branes. In a direction transverse to the brane the string satisfies Dirichlet boundary conditions [4],

$$X|_{\partial\Sigma} = x(s),$$

with $x(s)$ parametrizing the boundary of the worldsheet, Σ, which is fixed to the brane. For a worldsheet conformally transformed to the upper half plane with the boundary on the real axis, an example would be a vertex inserted on the real axis at $\mathrm{Im}\, z = 0$ and $\mathrm{Re}\, z = y$, in which case the Dirichlet boundary conditions become,

$$\begin{aligned} X &= 0 \qquad \text{for} \qquad \mathrm{Im}\, z = 0 \ \ \mathrm{Re}\, z < y, \\ X &= L \qquad \text{for} \qquad \mathrm{Im}\, z = 0 \ \ \mathrm{Re}\, z > y, \end{aligned}$$

for the two parallel branes separated by a distance L. A useful formula has been given in [165] for the functional integral,

$$\begin{aligned} &\int_{X|_{\partial\Sigma}=x(z)} \mathcal{D}X e^{-S} \dots \\ &\qquad = \int_{X|_{\partial\Sigma}=0} \mathcal{D}X e^{-S} \exp\left\{ \frac{1}{(2\pi\alpha')^2} \oint_{\partial\Sigma} ds \oint_{\partial\Sigma} ds' x(s) x(s') \partial_\perp \partial'_\perp G_{\mathrm{D}}(z,z') \right\} \dots, \end{aligned} \tag{4.70}$$

with S the Polyakov action, the normal derivatives $\partial_\perp$ acting on the Green's function with Dirichlet boundary conditions, $G_{\mathrm{D}}(z,z') = \langle X(z,\bar{z}) X(z',\bar{z}')\rangle$ with the normalization convention $\partial_z \partial_{\bar{z}} G(z,w) = -\pi\alpha' \delta^2(z-w) + \frac{\pi\alpha' g_{z\bar{z}}}{\int_\Sigma d^2 z \sqrt{g}}$ and $G_{\mathrm{D}}(z,z')|_{z\in\partial\Sigma} = 0$, and the dots "..." denoting vertex operator insertions. This expression shows [165] that we may restrict our attention to the construction of vertex operators with both ends attached to a single brane, say at $X^i|_{\partial\Sigma} = 0$, keeping in mind that one is to include the above exponential factor as appropriate for p-p strings stretching between parallel branes in the various scattering amplitude computations.

Spacetime directions tangent to the Dp-brane are labelled by lower case latin letters from the beginning of the alphabet, X^a, with $a = 0, \dots, p$, and directions transverse to the brane by upper case latin letters from the middle of the alphabet, X^I, with $I = p+1, \dots 25$. It is sometimes useful to work in lightcone coordinates in both covariant and lightcone gauge as this enables us to make the correspondence between the two gauges explicit. Assuming the associated lightcone directions satisfy Neumann boundary conditions we may define,

$$X^\pm = \tfrac{1}{\sqrt{2}}(X^0 \pm X^p).$$

Note that it is necessary [165] for the $X^\pm$ directions to lie in the Neumann directions in order to make the correspondence with lightcone gauge for which $X^+ = (2\alpha')p^+\tau_{\mathrm{M}}$, with $\tau = i\tau_{\mathrm{M}}$, as this is not compatible with Dirichlet boundary conditions, see (4.72). To place the lightcone directions in the Dirichlet directions one needs to instead reformulate lightcone gauge quantization with $X^+ = (2\alpha')p^+\sigma$. A general spacetime direction is as always labelled by Greek lower case letters, X^μ. To summarize,

$$\begin{aligned} X^a &= \{X^\pm, X^A\}, \qquad \text{with} \qquad A = 1, \dots, p-1, \\ X^i &= \{X^A, X^I\}, \qquad \text{with} \qquad I = p+1, \dots, 25, \\ X^\mu &= \{X^\pm, X^i\}. \end{aligned} \tag{4.71}$$

and so the directions X^A satisfy Neumann boundary conditions, whereas directions X^I satisfy Dirichlet boundary conditions. In the Euclidean worldsheet coordinates, $z = e^{-i(\sigma+i\tau)}$,

$\bar{z} = e^{i(\sigma - i\tau)}$ with $\sigma \in [0, \pi]$ and $\tau \in (-\infty, \infty)$, (considering only the case of NN and DD strings) Neumann and Dirichlet boundary conditions read respectively,

$$\partial_\sigma X^a|_{\partial\Sigma_{1,2}} = 0 \quad \text{(N)} \qquad \text{and} \qquad \partial_\tau X^I|_{\partial\Sigma_{1,2}} = 0 \quad \text{(D)} \tag{4.72}$$

Note that, $\partial_\sigma = i(\bar{z}\bar{\partial} - z\partial)$ and $\partial_\tau = \bar{z}\bar{\partial} + z\partial$. In the $(z, \bar{z})$ coordinates the open string physical worldsheet, Σ, is conformally mapped to the upper half plane with the identification, $z \sim \bar{z}$. The associated fixed point, the real line $z = \bar{z}$, defines the open string boundaries.

Using the doubling trick we can as usual write the various expressions needed in terms of holomorphic quantities only [114]: one identifies antiholomorphic quantities in the upper half plane with holomorphic quantities in the lower half plane and therefore one may just as well work with holomorphic quantities only provided one works in the full complex plane. The open string vertex operators are inserted on the real axis. We assume that both ends of the string satisfy the same boundary conditions for any given direction, we thus consider the cases of NN and DD directions only and do not consider mixed boundary conditions. The analogous construction for strings with mixed boundary conditions, i.e. ND and DN, may be constructed along the lines of [202], by introducing the notion of twist operators.

The relevant DDF operators now read,

$$A_n^A = \sqrt{\frac{2}{\alpha'}} \oint \bar{d}z \, \partial X^A(z) e^{inq\cdot X(z)}, \quad \text{and} \quad A_n^I = \sqrt{\frac{2}{\alpha'}} \oint \bar{d}z \, \partial X^I(z) e^{inq\cdot X(z)}, \tag{4.73}$$

for oscillators parallel or transverse to the brane respectively and the closed contour integrals are to contain the operators they act on, which are on the real axis. In a Minkowski signature worldsheet the integrals are along the boundary of the worldsheet which is coincident with the Dp-brane. The null vectors q^μ are restricted to lie within the D-brane worldvolume and are transverse to the DDF operators:

$$q^A = q^I = 0.$$

In direct analogy to the closed string case we create open string vertex operators with fluctuations in the X^A or X^I directions by acting on the vacuum with DDF operators see also Appendix K),

$$V(z, \bar{z}) = \frac{g_o}{\sqrt{2p^+\mathcal{V}_{d-1}}} C\xi_{ij\dots} A_{-n_1}^i A_{-n_2}^j \dots \, e^{ip\cdot X(z)}, \tag{4.74}$$

the vacuum, $e^{ip\cdot X(z)}$ being restricted to the worldsheet boundary (e.g. the real axis in the complex z-plane) and the combinatorial normalization constant C,

$$C \equiv \frac{1}{\sqrt{\prod_r n_r \prod_{n,i} \mu_{n,i}!}}. \tag{4.75}$$

The vertex operators (4.74) are mass level $N = \sum_i n_i$ states with momenta $k^\mu = p^\mu - Nq^\mu$, the onshell constraints now reading,

$$p^2 = \frac{1}{\alpha'}, \qquad p \cdot q = \frac{1}{2\alpha'}, \qquad \text{and} \qquad q^2 = 0, \tag{4.76}$$

so as to ensure that $m^2 = -(p - Nq)^2 = (N-1)/\alpha'$ as appropriate for open strings. The contractions appearing in (4.76) are with respect to all spacetime indices μ. The boundary conditions require in addition, $p^I = 0$, see Sec. 2.7.

Normal ordered vertex operators are obtained from (4.74) by bringing the operators in the integrands close to the vacuum, summing over all Wick contractions using e.g. the upper half plane two-point function for scalars (given in (K.4) in Appendix K) for Neumann (N) or Dirichlet (D) directions, and evaluating the resulting contour integrals so as to extract the residues which correspond to the physical states. In evaluating the operator products one restricts the integrands of the DDF operators to the real axis. Only after the operator products have been computed is one to analytically continue in the variable of integration so as to circle the tachyonic vacuum in order to extract the residue. This is best understood by realizing that the vertex operator (4.74) can be thought of as being created in a sequence of open string scattering events as explained in the introduction and depicted in Fig. 4.2.

The massless states, $V^{(i)}_{\text{massless}}$, that are absorbed by the ground state string, $V_{\text{ground state}} = e^{ip\cdot X(z)}$, are the integrands of the DDF operators polarized in some direction, ξ^i, of our choice, and the final excited state $V^{(r)}_{\text{excited}}$ is given by the vertex operator (4.74) after normal ordering when a sequence of r DDF operators have acted on the vacuum. In what follows we compute this normal ordered expression for a complete set of such open string covariant vertex operators. We give explicit results for the closed string and consider the open string explicitly when we construct coherent states. Open string vertices constructed from the A^A_n operators are related by T-duality to vertices constructed out the A^I_n [203, 204, 4]. The latter are interpreted as ripples in the D-brane worldvolume. The remaining possibility is vertex operators with excitations associated to both transverse and tangent directions to the D-brane, and these may be interpreted as the usual Neumann boundary condition vertices with excitations within the D-brane worldvolume which in addition generate ripples of the D-brane. In the open string coherent state section we will consider vertices constructed from the A^A_n.

As in the closed string case there is a one-to-one correspondence with the lightcone gauge states,

$$|V\rangle_{\text{lc}} = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\, C\xi_{ij\dots}\,\alpha^i_{-n_1}\alpha^j_{-n_2}\dots|0;p^+,p^i\rangle, \tag{4.77}$$

with $|0;p^+,p^i\rangle$ an eigenstate of p^+, p^i and annihilated by the (dimensionless) lowering operators, $\alpha^i_{n>0}$, where

$$\alpha^\mu_n = \sqrt{\frac{2}{\alpha'}}\oint đz\,\partial X^\mu(z)\,z^n,$$

and defined so that (4.54) holds true.

The fact that the covariant gauge vertex operators (4.74) are in one- to one correspondence with the lightcone gauge states (4.77) proves that the former comprise a complete set. We conjecture and work on the assumption that the states $|V\rangle_{\text{lc}}$ and $V(z,\bar z)$ are identical states in the sense that they share identical masses, angular momenta and interactions. We shall obtain evidence supporting this conjecture in what follows.

We next discuss the correspondence between lightcone gauge states and covariant gauge vertex operators, and consider the issue of normal ordering in detail. We start from the graviton and subsequently move on to arbitrarily excited vertex operators.

4.4.3 Covariant equivalent of $\xi_{i,j}\,\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0,0;p^+,p^i\rangle$

We wish to obtain the covariant equivalent of the lightcone gauge graviton (or other massless) state,

$$|V\rangle_{\rm lc} = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,\xi_{i,j}\,\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0,0;p^+,p^i\rangle.$$

Here $m^2=0$, and so from (4.67), $k^\mu=p^\mu-q^\mu$. We see from (4.61) and (4.63), see also (4.52), that the light-cone to covariant vertex map is realized by:

$$\xi_{i,j}\,\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0,0;p^+,p^i\rangle \;\to\; g_c\,\xi_{i,j}\,A^i_{-1}\bar{A}^j_{-1}e^{ip\cdot X(z,\bar{z})}, \tag{4.78}$$

with $\xi\cdot q\equiv 0$. To bring this into a manifestly covariant form, substitute into the right hand side the definitions (4.63). Using the operator products we bring the integrands close to the vacuum and evaluate the resulting contour integrals as explained below (4.64). For the graviton, this procedure can be seen to lead to [131]:[24]

$$\begin{aligned}\xi_{i,j}\,A^i_{-1}\bar{A}^j_{-1}e^{ip\cdot X(z,\bar{z})} &= \frac{2}{\alpha'}\,\xi_{i,j}\oint_z đw\,\partial_w X^i(w)e^{-iq\cdot X(w)}\oint_{\bar{z}} đ\bar{w}\;\partial_{\bar{w}}X^j(\bar{w})e^{-iq\cdot X(\bar{w})}\;e^{ip\cdot X(z,\bar{z})}\\ &\cong \frac{2}{\alpha'}\,\xi_{i,j}\Big(\delta^i_\mu-\frac{\alpha'}{2}\,p^iq_\mu\Big)\Big(\delta^j_\nu-\frac{\alpha'}{2}\,p^jq_\nu\Big)\,\partial X^\mu(z)\bar{\partial}X^\nu(\bar{z})e^{i(p-q)\cdot X(z,\bar{z})}.\end{aligned} \tag{4.79}$$

With the identification $\zeta_{\mu,\nu}=\xi_{i,j}(\delta^i_\mu-\frac{\alpha'}{2}p^iq_\mu)(\delta^j_\nu-\frac{\alpha'}{2}p^jq_\nu)$, we find the manifestly covariant and normal-ordered expression for the graviton vertex,

$$V(z,\bar{z}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,\frac{2}{\alpha'}\,\zeta_{\mu,\nu}\,\partial X^\mu(z)\bar{\partial}X^\nu(\bar{z})e^{ik\cdot X(z,\bar{z})}, \tag{4.80}$$

which has been derived from the corresponding light-cone gauge graviton via the DDF formalism. Note that we could just as well have written $\frac{g_c}{\sqrt{2E_{\mathbf{k}}V_{d-1}}}$ (with $E_{\mathbf{k}}=|\mathbf{k}|$) instead of $\frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}$, provided the momentum phase space in S-matrix elements is taken to be (4.23) instead of (4.39), as discussed in Sec. 4.3. This remark applies also to the other mass eigenstate vertex operators given below as well, but does not apply in the case of coherent states (see later).

The polarization tensor $\zeta_{\mu,\nu}$ is transverse to the graviton momentum k^μ as can be explicitly verified.[25] Notice that depending on our choice of ξ, p and q all entries of the covariant polarization tensor, $\zeta_{\mu,\nu}$, may be non-vanishing in general. Whether or not the corresponding polarization tensor is traceless depends on our choice of $\xi_{i,j}$.

The above procedure generalizes to arbitrarily massive vertices and given that the DDF operators generate the complete set of physical states [129, 130] it is clear that all

[24] We use the convention $X(z,\bar{z})=X(z)+X(\bar{z})$ which can be used inside correlation functions in the absence of sources [196].

[25] Recall that $\xi_{i,j}$ is transverse to q^μ.

arbitrarily massive vertices in covariant gauge may be extracted via this method. The fact that the physical content of the light-cone gauge states (where there are no ghost excitations) is clearer than covariant gauge vertex operators has been one of the great virtues of the light-cone gauge approach – it is seen that this virtue is also present in the covariant gauge if one makes use of the DDF formalism.

4.4.4 Covariant equivalent of $\xi_{i,j}\,\alpha^i_{-N}\tilde{\alpha}^j_{-N}|0,0;p^+,p^i\rangle$

Consider now a not so obvious example which in fact, as will become apparent in the next subsection, is the basic building block of all vertex operators whose polarization tensors are traceless. In this subsection we derive the normal ordered covariant vertex operator corresponding to the lightcone state

$$|V\rangle_{\rm lc} = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,\xi_{i,j}\,\alpha^i_{-N}\tilde{\alpha}^j_{-N}|0,0;p^+,p^i\rangle. \tag{4.81}$$

Here the mass, $m^2 = 4(N-1)/\alpha'$, and so from (4.67), $k^\mu = p^\mu - Nq^\mu$. Following the DDF prescription, we consider the state

$$V(z,\bar{z}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,\xi_{i,j}\,A^i_{-N}\bar{A}^j_{-N}e^{ip\cdot X(z,\bar{z})}. \tag{4.82}$$

As in the graviton example, we use the definitions of the DDF operators and carry out the relevant operator products. Let us consider the holomorphic sector and shift the vertex to $z=0$. This leads us to consider,

$$\begin{aligned}
A^i_{-N}\cdot e^{ip\cdot X(0)} &= \sqrt{\frac{2}{\alpha'}}\oint_0 đw\,\partial X^i(w)e^{-iNq\cdot X(w)}\cdot e^{ip\cdot X(0)}\\
&\cong \sqrt{\frac{2}{\alpha'}}\oint_0\frac{đw}{iw}\Big(p^i\,w^{-N}+\sum_{r=1}^{\infty}\frac{i}{(r-1)!}\,\partial^r X^i(0)\,w^{r-N}\Big)\sum_{m=0}^{\infty}w^m S_m(Nq;0)\,e^{i(p-Nq)\cdot X(0)}\\
&= \sqrt{\frac{2}{\alpha'}}\Big(\frac{\alpha'}{2}p^i S_N(Nq;0)+\sum_{m=1}^{N}\frac{i}{(m-1)!}\,\partial^m X^i(0)S_{N-m}(Nq;0)\Big)e^{i(p-Nq)\cdot X(0)},
\end{aligned} \tag{4.83}$$

with $S_m(Nq;0)$ elementary Schur (or complete Bell) polynomials, see Appendix J,

$$\begin{aligned}
S_m(nq;z) &= \oint_0\frac{dw}{2\pi iw}\,w^{-m}\exp\Big(-inq\cdot\sum_{s=1}^{m}\frac{w^s}{s!}\partial_z^s X(z)\Big),\\
\bar{S}_m(nq;\bar{z}) &= -\oint_0\frac{d\bar{w}}{2\pi i\bar{w}}\,\bar{w}^{-m}\exp\Big(-inq\cdot\sum_{s=1}^{m}\frac{\bar{w}^s}{s!}\partial_{\bar{z}}^s X(\bar{z})\Big),
\end{aligned} \tag{4.84}$$

with $\oint_0\frac{dw}{2\pi iw} = -\oint_0\frac{d\bar{w}}{2\pi i\bar{w}} = 1$, and we have made use of the standard correlator on the complex plane (4.68), as well as the onshell constraints (4.67). The elementary Schur polynomials arise from the Taylor expansion (inside the normal ordering) of $e^{-iNq\cdot X(z)} = \sum_{m=0}^{\infty}z^m S_m(Nq;0)e^{-iNq\cdot X(0)}$ which can be derived from Faà di Bruno's formula [194] for the $m^{\rm th}$ derivative of the exponential, $(e^{iNq\cdot X(z)}\partial^m e^{-iNq\cdot X(z)})_{z=0}$. As a preliminary consistency check note that the subscript N on $S_N(Nq)$ denotes the total number of derivatives and so the level number on both sides of the equation is the same. We have noted also the corresponding expression, $\bar{S}_m(nq;\bar{z})$, for the anti-holomorphic sector. Shifting the insertion back to $z,\bar{z}$ we conclude that the level N lightcone state (4.81) has the covariant manifestation:

$$V(z,\bar{z}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}} : \frac{1}{N}\,\xi_{i,j}H^i_N(z)\bar{H}^j_N(\bar{z})e^{i(p-Nq)\cdot X(z,\bar{z})} : \tag{4.85}$$

with the normalization $C = 1/N$, see (4.62). We have found it convenient to define the polynomials $H^i_N(z)$, $\bar{H}^i_N(\bar{z})$, in $\partial^{\#}X$ and $\bar{\partial}^{\#}X$ respectively,

$$H^i_N(z) \equiv \sqrt{\frac{\alpha'}{2}}p^iS_N(Nq;z) + P^i_N(z), \tag{4.86a}$$

$$\bar{H}^i_N(\bar{z}) \equiv \sqrt{\frac{\alpha'}{2}}p^i\bar{S}_N(Nq;\bar{z}) + \bar{P}^i_N(\bar{z}), \tag{4.86b}$$

with $P^i_N(z)$, $\bar{P}^i_N(\bar{z})$ in turn defined by,

$$P^i_N(z) = \sqrt{\frac{2}{\alpha'}}\sum_{m=1}^{N}\frac{i}{(m-1)!}\,\partial^mX^i(z)S_{N-m}(Nq;z), \tag{4.87a}$$

$$\bar{P}^i_N(\bar{z}) = \sqrt{\frac{2}{\alpha'}}\sum_{m=1}^{N}\frac{i}{(m-1)!}\,\bar{\partial}^mX^i(\bar{z})\bar{S}_{N-m}(Nq;\bar{z}). \tag{4.87b}$$

These polynomials are the fundamental building blocks of normal ordered covariant vertex operators when these correspond in lightcone gauge to a traceless state as we shall see.[26] In the rest frame one is to replace, $H^i_N(z)$, $\bar{H}^i_N(\bar{z})$ with, $P^i_N(z)$, $\bar{P}^i_N(\bar{z})$, respectively as in this case the momenta, $k^\mu = p^\mu - Nq^\mu$, are transverse to the polarization tensors and consequently $\xi_{\dots i\dots}p^i = 0$. Some examples for $N = 0, 1$ and 2 have been given in Appendix J. We next give an explicit example for $m^2 = 4/\alpha'$, mass levels, where $N = 2$, to illustrate that the vertices generated in this manner are the standard covariant vertex operators [193], see also [132, 201, 143, 139], with polarization tensors that range over the entire range of spacetime indices. The difference to the traditional approach (taken in the above cited papers) is that here physical polarization tensors are automatically generated – there are no additional constraints to be solved. First of all note that for $N = 1$ we recover the graviton (or in general the massless) vertex operator(s).[27] For $N = 2$, we have $k^\mu = p^\mu - 2q^\mu$. The covariant vertex operator which is equivalent to the lightcone state $\frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,\frac{1}{2}\xi_{i,j}\,\alpha^i_{-2}\tilde{\alpha}^j_{-2}|0,0;p^+,p^i\rangle$ follows as a corollary of (4.85),

$$|V\rangle = \frac{1}{\sqrt{2E_{\mathbf{k}}V_{d-1}}}\,\frac{1}{2}\big(\chi_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + \zeta_\mu\alpha^\mu_{-2}\big)\big(\bar{\chi}_{\rho\sigma}\tilde{\alpha}^\rho_{-1}\tilde{\alpha}^\sigma_{-1} + \bar{\zeta}_\rho\tilde{\alpha}^\rho_{-2}\big)|0,0;k^\mu\rangle, \tag{4.88}$$

[26] For vertices that correspond to lightcone states whose trace is non-vanishing there is an additional polynomial, $\mathbb{S}_{n,m}(z)$, see below. All these polynomials however are ultimately composed of elementary Schur polynomials, $S_m(nq;z)$.

[27] Recall that in the CFT language there is no Ricci scalar in the dilaton vertex, see Polchinski [196].

where we have made use of the operator-state correspondence, $\alpha^{\mu}_{-n} \simeq \sqrt{\frac{2}{\alpha'}}\frac{i}{(n-1)!}\partial^n X^{\mu}(z)$, $|0,0;k^{\mu}\rangle \simeq g_c\, e^{ik\cdot X(z,\bar{z})}$, and have written $|V\rangle \simeq V(z,\bar{z})$, in order to make manifest the differences to the equivalent lightcone gauge state. We have chosen to write (4.88) in the more conventional coordinates used in covariant gauge, where the vacuum is normalized according to (4.53) and $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}$. From (4.85) one can derive (by expanding out the various polynomials for $N=2$) the manifestly covariant polarization tensors,

$$\zeta_{\mu} = \xi_i\big(\delta^i_{\ \mu} - \tfrac{\alpha'}{2}p^i q_{\mu}\big) \quad \text{and} \quad \chi_{\mu\nu} = \sqrt{\tfrac{\alpha'}{2}}\xi_i\big(\alpha' p^i q_{\mu}q_{\nu} - \delta^i_{\ \mu}q_{\nu} - \delta^i_{\ \nu}q_{\mu}\big), \tag{4.89}$$

with the properties $|\zeta|^2 = |\xi|^2$ (with $|\xi|^2 = 1$ so that the lightcone state is correctly normalized), $\chi_{\mu\nu} = \chi_{\nu\mu}$, $|\chi|^2 = \zeta\cdot k = \chi^{\mu}_{\ \mu} = \chi_{\mu\nu}k^{\mu}k^{\nu} = 0$. As a consistency check note that these polarization tensors solve the physical state conditions, $2\zeta_{\mu} + k^{\nu}\chi_{\mu\nu} = 0$, $2k_{\mu}\zeta^{\mu} + \eta^{\mu\nu}\chi_{\mu\nu} = 0$, which were derived by completely different methods in [139]. There are similar expressions for $\bar{\zeta}_{\mu}$, $\bar{\chi}_{\mu\nu}$ with $\bar{\xi}_i$ replacing ξ_i. One thing to notice is that all components of these polarization tensors may be non-vanishing in general so that the resulting states really are covariant in the usual sense even though the state (4.82) from which (4.88) was derived seems to break spacetime covariance by the explicit choice of transverse indices.

There has been some confusion concerning a state of the form (4.88) in the literature [201, 139] where it is concluded that such a state may satisfy the Virasoro constraints but has zero norm. We disagree in that we find that the state $|V\rangle$ has positive norm,[28] $\langle V|V\rangle = 1$, while satisfying all the Virasoro constraints, $L_{n>0}|V\rangle = 0$, $L_0|V\rangle = |V\rangle$ and is hence physical. In fact, all covariant states generated by the DDF formalism are positive norm physical states. The reason as to why there is disagreement with [201, 139] is because the constraints on the polarization tensors ζ_{μ}, $\chi_{\mu\nu}$ obtained there do not have a unique solution; the solution identified there corresponds to a zero norm state but there is the additional solution, namely (4.89), which gives rise to the positive norm state (4.88).

What we learn from the above exercises is that the DDF vertex operators (4.63) are fully covariant, they all have a lightcone gauge equivalent which can be identified explicitly, and last but not least they generate a complete set of physical states (given that they are in one-to-one correspondence with the light-cone gauge states).

4.4.5 Covariant equivalent of $\xi_{ij\ldots,kl\ldots}\,\alpha^i_{-n}\alpha^j_{-m}\ldots\tilde{\alpha}^k_{-\bar{n}}\tilde{\alpha}^l_{-\bar{m}}\ldots|0,0;p^+,p^i\rangle$

We next generalize the result of the previous subsection and discuss the covariant manifestation of a general lightcone gauge state,

$$|V\rangle_{\rm lc} = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,C\xi_{ij\ldots,kl\ldots}\,\alpha^i_{-n_1}\alpha^j_{-n_2}\ldots\tilde{\alpha}^k_{-\bar{n}_1}\tilde{\alpha}^l_{-\bar{n}_2}\ldots|0,0;p^+,p^i\rangle, \tag{4.90}$$

which according the DDF prescription is given by,

$$V(z,\bar{z}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,C\xi_{ij\ldots,kl\ldots}\,A^i_{-n_1}A^j_{-n_2}\ldots\tilde{A}^k_{-\bar{n}_1}\tilde{A}^l_{-\bar{n}_2}\ldots e^{ip\cdot X(z,\bar{z})}. \tag{4.91}$$

[28]Here we have included the 'one string in volume V_{d-1}' normalizing factor $\frac{1}{\sqrt{2E_{\mathbf{k}}V_{d-1}}}$ and use the relativistic normalization $\langle 0,0;k'|0,0;k\rangle = 2E_{\mathbf{k}}(2\pi)^{d-1}\delta^{d-1}(\mathbf{k}'-\mathbf{k})$.

Here the relevant level numbers associated to left- and right-moving modes is $N = \sum_\ell n_\ell$ and $\bar{N} = \sum_r \bar{n}_r$ and for non-compact spacetimes we are to enforce[29] $N = \bar{N}$. The associated momentum is then, $k^\mu = p^\mu - Nq^\mu$, and the mass shell constraint, $k^2 = 4(1-N)/\alpha'$.

Writing formally $\xi_{ij\ldots,kl\ldots} = \xi_{ij\ldots}\bar{\xi}_{kl\ldots}$ we first consider the case when the polarization tensors ξ and $\bar{\xi}$ are *traceless*,

$$\xi_{\ldots i\ldots j\ldots}\eta^{ij} = \bar{\xi}_{\ldots i\ldots j\ldots}\eta^{ij} = 0,$$

but with $\xi_{\ldots j\ldots}k^j$, $\bar{\xi}_{\ldots j\ldots}k^j$ non-vanishing in general. The normal ordered vertex operator corresponds to a straightforward generalization of (4.85), $\prod_r A^{i_r}_{-n_r} e^{ip\cdot X(z)} \cong \prod_r H^{i_r}_{n_r} e^{i(p-Nq)\cdot X(z)}$ for the holomorphic sector. Therefore, the covariant normal ordered vertex operator associated to a general traceless lightcone state (4.90) is,

$$V(z,\bar{z}) \cong \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}} : C\xi_{ij\ldots,kl\ldots}H^i_{n_1}(z)H^j_{n_2}(z)\ldots\bar{H}^k_{\bar{n}_1}(\bar{z})\bar{H}^l_{\bar{n}_2}(\bar{z})\ldots\, e^{i(p-Nq)\cdot X(z,\bar{z})} : \tag{4.92}$$

with C as given in (4.62). Without referring explicitly to the lightcone state it is seen that C contains a factor of $\frac{1}{\sqrt{n}}$ for every H^i_n that appears and factors of $\frac{1}{\sqrt{\mu_{n,i}!}}$, with $\mu_{n,i}$ the multiplicity of H^i_n.

We can always boost to a frame where $\xi_{\ldots i\ldots}k^i = 0$ (e.g. the rest frame) given that there are no timelike directions in the lightcone gauge polarization tensor, ξ, in which case the above vertex simplifies to,

$$V(z,\bar{z}) \cong \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}} : C\xi_{ij\ldots,kl\ldots}P^i_{n_1}(z)P^j_{n_2}(z)\ldots\bar{P}^k_{\bar{n}_1}(\bar{z})\bar{P}^l_{\bar{n}_2}(\bar{z})\ldots\, e^{i(p-Nq)\cdot X(z,\bar{z})} :$$

We therefore find that when the polarization tensor of a given light-cone state is traceless one may build the corresponding normal ordered covariant vertex operator by making the following identifications,

$$\alpha^i_{-n} \sim H^i_n(z), \qquad \tilde{\alpha}^i_{-\bar{n}} \sim \bar{H}^i_{\bar{n}}(\bar{z}), \qquad \text{and} \qquad |0,0;p^+,p^i\rangle \sim g_c\, e^{i(p-Nq)^\mu X_\mu(z,\bar{z})} \tag{4.93}$$

with an overall combinatorial normalization constant C given in (4.62). If the lightcone states in addition to $\xi_{\ldots i\ldots j\ldots}\eta^{ij} = 0$ satisfy $\xi_{\ldots j\ldots}k^j = 0$ (and similarly for the anti-holomorphic sector), the above identification simplifies to, $\alpha^i_{-n} \sim P^i_n(z)$ and $\tilde{\alpha}^i_{-n} \sim \bar{P}^i_n(\bar{z})$. The resulting covariant vertex operator formed in this way is normal ordered. Note that the normalization of the lightcone state carries over to the covariant vertex unaltered because the normalization for the DDF states is set by the DDF commutation relations (4.69) which are identical to those of the usual creation and annihilation operators.

[29]The $L_0 - \bar{L}_0$ Virasoro constraint is satisfied without the requirement $N = \bar{N}$ but as we discuss later this is only possible in a spacetime with lightlike compactification given that for $N \neq \bar{N}$ we have $k_{\rm L} - k_{\rm R} = -(N-\bar{N})q$ with $q^2 = 0$.

We next construct covariant normal ordered vertex operators in the case when the polarization tensors of the corresponding lightcone gauge states are arbitrary, for which in general,

$$\xi_{\dots i\dots j\dots}\eta^{ij}, \qquad \bar{\xi}_{\dots i\dots j\dots}\eta^{ij}, \qquad \text{and} \qquad \xi_{\dots i\dots}k^i, \qquad \bar{\xi}_{\dots i\dots}k^i,$$

need not vanish. We start from the simplest non-trivial case and then move on to more general cases. Proceeding by induction we then obtain the general result.

For this purpose we'll be needing the following local dimensionless polynomial functionals of $q\cdot\partial^{\#}X(z)$, and $q\cdot\bar{\partial}^{\#}X(\bar{z})$ respectively,

$$\mathbb{S}_{m,n}(z) \equiv \sum_{r=1}^{n} rS_{m+r}(mq;z)S_{n-r}(nq;z), \tag{4.94a}$$

$$\bar{\mathbb{S}}_{m,n}(\bar{z}) \equiv \sum_{r=1}^{n} r\bar{S}_{m+r}(mq;\bar{z})\bar{S}_{n-r}(nq;\bar{z}), \tag{4.94b}$$

with the elementary Schur polynomials, $S_m(nq;z)$, $\bar{S}_m(nq;\bar{z})$, defined in Appendix J. In (4.83) we showed that normal ordering of $A^i_{-n}\cdot e^{ip\cdot X(z)}$ leads to,

$$A^k_{-n}\cdot e^{ip\cdot X(z)} \cong H^k_n(z)\, e^{i(p-nq)\cdot X(z)}. \tag{4.95}$$

Let us apply an additional DDF operator from the left to this expression and normal order the resulting object. We find,

$$A^j_{-m}A^k_{-n}\cdot e^{ip\cdot X(z)} \cong \Big[H^j_m H^k_n + \delta^{jk}\,\mathbb{S}_{m,n}\Big](z)\, e^{i[p-(m+n)q]\cdot X(z)}. \tag{4.96}$$

Proceeding in a similar manner we apply another DDF operator to the resulting expression and normal order the right-hand-side. An important point to note now is that $\mathbb{S}_{m,n}(z)$ commutes with the DDF operators, A^i_ℓ, because $\mathbb{S}_{m,n}(z)$ is a functional of $q\cdot\partial^{\#}X$ and $[A^i_n, q\cdot\partial^{\#}X]=0$. We find,

$$\begin{aligned} &A^i_{-\ell}A^j_{-m}A^k_{-n}\cdot e^{ip\cdot X(z)} \cong \\ &\qquad \cong \Big[H^i_\ell H^j_m H^k_n + \delta^{ij}\,\mathbb{S}_{\ell,m}H^k_n + \delta^{ik}\,\mathbb{S}_{\ell,n}H^j_m + \delta^{jk}\,\mathbb{S}_{m,n}H^i_\ell\Big](z)\, e^{i[p-(\ell+m+n)q]\cdot X(z)} \end{aligned} \tag{4.97}$$

By induction it follows from the above that the general normal ordered expression reads,

$$\begin{aligned} &A^{i_1}_{-n_1}\dots A^{i_g}_{-n_g}\cdot e^{ip\cdot X(z)} \cong \\ &\qquad \cong \sum_{a=0}^{\lfloor g/2\rfloor}\sum_{\pi\in S_g/\sim}\prod_{\ell=1}^{a}\delta^{i_{\pi(2\ell-1)}i_{\pi(2\ell)}}\,\mathbb{S}_{n_{\pi(2\ell-1)},n_{\pi(2\ell)}}(z)\prod_{q=2a+1}^{g}H^{i_{\pi(q)}}_{n_{\pi(q)}}(z)\, e^{i(p-\sum_r n_r q)\cdot X(z)}, \end{aligned} \tag{4.98}$$

with S_g the permutation group of g elements and the equivalence relation $\sim$ being such that $\pi_i\sim\pi_j$ with $\pi_i,\pi_j\in S_g$ when they define indistinguishable terms in (4.98). In all terms where $\mathbb{S}_{n_i,n_j}$ appears we are to only include permutations which preserve the inequality $i\le j$. Furthermore, the notation $\lfloor\cdot\rfloor$ in the summation indicates that the upper limit saturates the inequality $a \leq g/2$. The number of terms in the sum over permutations at fixed a is

$$\frac{2^{-a} g!}{a!(g-2a)!}.$$

For every lightcone gauge state (4.90), with C is as given in (4.62), there exists a covariant normal ordered vertex operator

$$V(z,\bar{z}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}} CU(z)\bar{U}(\bar{z})e^{ip\cdot X(z,\bar{z})}, \tag{4.99}$$

with the normalization being that required by unitarity of the S-matrix, see Sec. 4.3. The normal ordered chiral half $U(z)e^{ip\cdot X(z)}$ is equal to the right hand side of (4.98) when contracted with the lightcone gauge polarization tensor, $\xi_{i_1\dots i_g}$, which corresponds to an arbitrary irreducible representation of SO(25) (or SO(24) for massless states),

$$U(z) = \xi_{i_1\dots i_g}\sum_{a=0}^{\lfloor g/2\rfloor}\sum_{\pi\in S_g/\sim}\prod_{\ell=1}^{a}\delta^{i_{\pi(2\ell-1)}i_{\pi(2\ell)}}\,\mathbb{S}_{n_{\pi(2\ell-1)},n_{\pi(2\ell)}}(z)\prod_{q=2a+1}^{g}H^{i_{\pi(q)}}_{n_{\pi(q)}}(z)\,e^{-i(\sum_{r=1}^{g}n_r)q\cdot X(z)}.$$

There is a similar expression for $\bar{U}(\bar{z})$ with $\bar{\xi}_{ij\dots}$, $\bar{\mathbb{S}}_{n,m}(\bar{z})$, $\bar{H}^i_{\bar{n}}(\bar{z})$ and $e^{-i(\sum_r\bar{n}_r)q\cdot X(\bar{z})}$ replacing $\xi_{ij\dots}$, $\mathbb{S}_{n,m}(z)$, $H^i_n(z)$ and $e^{-i(\sum_r n_r)q\cdot X(z)}$ respectively. If the underlying spacetime manifold is not compactified in a lightlike direction we are to enforce in addition:

$$\sum_r n_r = \sum_r \bar{n}_r;$$

we elaborate on this in the closed string coherent state section. It is curious that it is not string theory symmetries that place this constraint on the level numbers of left- and right-movers, but that it is a phenomenological constraint: our universe does not seem to be lightlike compactified, as such a compactification would (at least globally) break 4-dimensional Lorentz invariance, thus singling out a preferred frame of reference. We therefore *choose* $\sum_r n_r = \sum_r \bar{n}_r$ in order not to break 4-dimensional Lorentz invariance.

When the polarization tensor is traceless, $\xi_{\dots i\dots j\dots}\delta^{ij} = 0$, $U(z)$ reduces to the result obtained in (4.92), the chiral half of which reads, $\xi_{i_1\dots i_s}H^{i_1}_{n_1}\dots H^{i_s}_{n_s}e^{i(p-\sum_r n_r q)\cdot X(z)}$. In the rest frame, $\xi_{\dots i\dots}p^i = 0$, all the $H^i_n(z)$ in $U(z)$ reduce to $P^i_n(z)$.

When all the n_i are equal the sum over permutations may be carried out explicitly. A particularly interesting case is the symmetric representation $\xi^{ij\cdots} = \lambda^i\lambda^j\dots$, in which case (4.98) reduces to,

$$\frac{1}{g!}\big(\lambda\cdot A_{-n}\big)^g\, e^{ip\cdot X(z)} \cong \sum_{a=0}^{\lfloor g/2\rfloor}\frac{1}{a!(g-2a)!}\Big(\frac{1}{2}\lambda\cdot\lambda\,\mathbb{S}_{n,n}\Big)^a\big(\lambda\cdot H_n\big)^{g-2a}\, e^{i(p-gnq)\cdot X(z)}. \tag{4.100}$$

When we sum over g (from 0 to ∞) and multiply by the appropriate kinematic factor, such a object has an interpretation of the chiral half of a closed string coherent state or an open string coherent state as we shall demonstrate in Sec. 5, where we discuss string coherent states in great detail. The corresponding lightcone gauge state is $\frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\exp(\frac{1}{n}\lambda_n\cdot$

$\alpha_{-n})|0;p^+,p^i\rangle$, with $\lambda^i = \frac{1}{n}\lambda_n^i$, which is an eigenstate of $\alpha_{n>0}^i$ with eigenvalue λ_n^i and $\lambda_n^* = \lambda_{-n}$. The covariant gauge expression is not an eigenstate of $\alpha_{n>0}^\mu$ but nevertheless satisfies the definition of a coherent state (see later).

Note finally that the general lightcone state (4.90) is normalized such that:

$$\langle V(p')|V(p)\rangle_{\rm lc} = \delta_{p',p},$$

with $\delta_{p',p}$ a *Kronecker* delta which reduces to unity when $p^{+'} = p^+$ and $\mathbf{p}' = \mathbf{p}$ and vanishes otherwise. The associated vertex operator (4.91) or (4.99), is normalized by the most singular term in the operator product expansion (4.43):

$$V^\dagger(z,\bar{z})\cdot V(0,0) \cong \Big(\frac{g_c^2}{2p^+\mathcal{V}_{d-1}}\Big)\frac{1}{|z|^4} + \dots,$$

the dimensionless coefficient having been fixed by Lorentz covariance and unitarity of the S-matrix. We have made use of the relation between operator product expansions and commutators (3.12), although an operator product expansion can also be used instead. With this normalization the string path integral yields the S-matrix directly, see (4.22).

Chapter 5

String Coherent States

It is possible that cosmic strings being macroscopic and massive should have a classical interpretation. If this is the case, one may suspect that the appropriate vertex operators for the description of cosmic superstrings (from our experience with standard harmonic oscillator coherent states) would have coherent state-like properties. With this motivation in mind we will be searching for coherent state vertex operators, which from the standard coherent state properties would be expected to have a classical interpretation.

The states we have considered in the previous sections are mass eigenstates. The dictionary described above, which identifies the states (4.61) and (4.63), is tailor-made for light-cone to covariant mass eigenstate maps. Coherent states however are not mass eigenstates in general.[1] In the construction of string coherent states one normally proceeds in direct analogy with the construction of coherent states in the harmonic oscillator, whereby coherent states are constructed by exponentiation of the creation operator, $e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$, with $a|0\rangle = 0$ and $[a, a^\dagger] = 1$. In the string case there is an infinite number of creation operators and the vacuum depends on the center of mass momentum. The usual approach is to proceed in lightcone gauge where the constraints are solved automatically and the open string construction is trivial, see e.g. [205]. Rather than drop spacetime covariance we shall make use of the spectrum generating DDF operators which can be used to generate covariant physical states.

In what follows we construct covariant and lightcone gauge open and closed coherent states and show that these states have a classical interpretation by associating them to general classical solutions. We will see that these states are macroscopic and this suggests that they be identified with fundamental cosmic strings. We will primarily define what we mean by a quantum state with a *classical interpretation*:

- *String states with a classical interpretation* should possess classical expectation values (with small uncertainties modulo zero mode contributions) provided these are compatible with the symmetries of string theory. These classical expectation values should be non-trivially consistent with the classical equations of motion and

[1] The coherent states constructed here are eigenstates of momentum however in the spacetime directions transverse to q^a as we shall see.

constraints.

Starting with the *open string* we shall define a string coherent state and, using DDF operators, will proceed by analogy to the harmonic oscillator to construct string coherent states. The definition of a coherent state that we adopt is very general but standard [126] which we minimally extend to include the string theory requirements.[2] After establishing that the coherent state properties are satisfied for the states under consideration we go on to show that the covariant and lightcone gauge states share identical angular momenta and present the explicit map to general classical solutions. We show that these coherent states indeed possess classical expectation values, thus proving that the above definition of classicality is satisfied.

We then go on to discuss the construction of *closed string* coherent states. Here the naive construction leads to the requirement of a lightlike compactification of spacetime, $X^- \sim X^- + 2\pi R^-$. We show that all states considered are indeed physical and single-valued under translations around the compact direction, X^-.

We are then, according to the above definition of classicality, led to search for classical expectation values. In the closed string case the string symmetries forbid [125] the naive expectation that $\langle X^\mu(z,\bar{z})\rangle = X^\mu_{\rm cl}(z,\bar{z})$[3] should be satisfied by a state with a classical interpretation. We elaborate on this and discuss various definitions of classicality and their range of applicability. Here we provide a new classicality requirement (in accordance with the above definition) that applies in all the usual gauges of interest (e.g. lightcone and covariant, but not in static gauge for instance) where the vertices are invariant under spacelike worldsheet shifts where the naive definition $\langle X^\mu\rangle = X^\mu_{\rm cl}$ does not apply.

Finally, we construct coherent closed string states in fully non-compact spacetimes by projecting out the lightlike winding states in the underlying Hilbert space and go on to show that all the coherent state properties are satisfied by the projected states as well, and therefore that the projected states have a classical interpretation. We also compute the angular momenta of the projected states in both lightcone and covariant gauge and show that they are both identical to the angular momentum associated to the corresponding classical solutions which we identify explicitly.

For a good overview of coherent states (but not explicitly in the context of string theory) see Klauder and Skagerstam's book [126] and the excellent review article by Zhang, Feng and Gilmore [206].

5.1 Open String

We here construct covariant coherent string states which according to the above discussion are likely to be good candidates for the description of general cosmic strings.

[2] The naive definition, that a coherent state should be an eigenstate of the annihilation operators is not in general compatible with the string theory symmetries.

[3] Here $X^\mu_{\rm cl}(z,\bar{z})$ is an arbitrary non-trivial solution of the wave equation, $\partial\bar{\partial}X^\mu_{\rm cl}(z,\bar{z}) = 0$.

Starting with the open string, we primarily define an *open string coherent state*, $V(\lambda,\dots)\cong|V(\lambda,\dots)\rangle$, to be a state that:

(a) is specified by a set of *continuous* labels $\lambda=\{\lambda_n^i\}$;

(b) there must exist a resolution of unity,

$$\mathbb{1}=\sum\!\!\!\!\!\!\int\int d\lambda\big|V(\lambda,\dots)\big\rangle\big\langle V(\lambda,\dots)\big|; \tag{5.1}$$

(c) it must transform correctly under all symmetries of bosonic (or super-) string theory.

We also allow for the possibility that the state depends on other discrete or continuous quantum numbers (such as momentum), denoted by "…", which are to be summed or integrated over respectively – this is what is meant by the symbol $\sum\!\!\!\!\!\!\int$.[4] The measure associated to the continuous labels explicitly reads $d\lambda=\frac{1}{N}\prod_{n,i}d^2\lambda_n^i$ with N an appropriate normalization (to be determined) and as usual $d^2\lambda_n^i=id\lambda_n^i\wedge d\lambda_n^{*i}$ (no sum over i). The labels n and i will be related to the distribution of harmonics present and spacetime directions respectively. The requirements (a,b) are the minimal requirements for a state to be termed coherent [126] and to these we add the minimal string theory requirement (c).

One may construct[5] open string vertex operators using the A_n^A and A_n^I DDF operators for excitations in spatial directions tangent and transverse to the Dp-brane respectively with $A=\{1,\dots,p-1\}$ and $I=\{p+1,\dots,25\}$. (Note that $p\geq 1$, see Sec. 4.4.2). We shall here consider the construction of coherent state vertex operators with excitations in the directions tangent to the brane. Let us then consider the normalized open string DDF vertex operator,

$$V(\lambda)=\frac{g_{o,p}}{\sqrt{2p^+\mathcal{V}_\parallel}}C_\lambda\exp\Big(\sum_{n=1}^\infty\frac{1}{n}\lambda_n^A A_{-n}^A\Big)e^{ip_aX^a(z)}, \tag{5.2}$$

with $a=\{0,1,\dots,p\}$. We have found it convenient to define:[6]

$$g_{o,p}\equiv\frac{g_o}{\sqrt{V_\perp}},\qquad\text{with}\qquad \mathcal{V}_{d-1}\equiv V_\perp\mathcal{V}_\parallel,$$

with $V_\perp$ the volume of spacetime transverse to the Dp-brane, and $\mathcal{V}_\parallel$ the volume tangent to the brane (so that $V_\perp\mathcal{V}_\parallel$ is the total volume of spacetime transverse to x^+). [7] In parallel to (4.38) in particular, we thus define:

$$\begin{aligned}\mathcal{V}_\parallel&\equiv\lim_{p'\to p}(2\pi)\delta(p'^+-p^+)(2\pi)^{p-1}\delta^{p-1}(\mathbf{p}'-\mathbf{p}),\\ V_\perp&\equiv\lim_{p'\to p}(2\pi)^{d-1-p}\delta^{d-1-p}(\mathbf{p}'-\mathbf{p}).\end{aligned} \tag{5.3}$$

[4]We will normally not exhibit these additional labels explicitly, and hence write $V(\lambda)$ instead of $V(\lambda,\dots)$, or even $V(z)$ when there is no possibility for confusion with the mass eigenstates of the previous section.

[5]The necessary prerequisite for this subsection is Sec. 4.4.2, 4.3, and our open string conventions are given in Sec. 2.7 and Appendix K.

[6]The index p on $g_{o,p}$ denotes the dimensionality of the Dp-brane in which the string is propagating and should not be confused with the momentum of the vacuum p^a.

[7]The dimensionalities are such that $[g_{o,p}]=L^{\frac{d-2}{2}}L^{-\frac{d-1-p}{2}}=L^{\frac{p-1}{2}}$, so that $[g_{o,p}/\sqrt{2p^+\mathcal{V}_\parallel}]=1$ as required by unitarity.

The total volume of spacetime is $V_d = V_+\mathcal{V}_{d-1}$. The kinematic pre-factor and the normalization C_λ is chosen such that the vertex operator is normalized to 'one string in volume $\mathcal{V}_{d-1}$' as shown in (4.43) for the case of closed strings. p^a is the (tachyonic) vacuum momentum of the string, the DDF operators, A_n^A, defined in (4.73) and the normalization constant,

$$C_\lambda \equiv \exp\Big(-\sum_{n=1}^{\infty}\frac{1}{2n}|\lambda_n|^2\Big).$$

The vertex operators associated to ripples of the brane are related by T-duality [203, 4] to the vertices (5.2). The onshell constraints are given by (4.76), repeated here for convenience: $p\cdot q = 1/(2\alpha')$, $q^2 = 0$, and $p^2 = 1/\alpha'$. The polarization complex vectors $\{\lambda_n^A\}$ are defined such that $\lambda_n\cdot q = 0$, $\lambda_n^* = \lambda_{-n}$, and require [205] that $\sum_n|\lambda_n|^2 < \infty$ to ensure that the vertex is well behaved.

First of all we show that the vertex operator (5.2) is a coherent state. To prove this recall that a coherent state must by definition satisfy three properties: (a) it must be labelled by a set of continuous parameters, these here being $\{\lambda_n^A\}$, (b) there must exist a completeness relation of the form (5.1), and (c) it must transform correctly under the symmetries of string theory. (a) is trivially satisfied and the state remains correctly normalized for arbitrary values of the λ_n^A when $\sum_n|\lambda_n|^2 < \infty$. To prove that a completeness relation exists it is convenient to write (5.2) in operator form,

$$|V(\lambda,p)\rangle = \frac{1}{\sqrt{2p^+\mathcal{V}_\parallel}}\,C_\lambda \exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\lambda_n^A A_{-n}^A\Big)\,|0;p^a\rangle, \tag{5.4}$$

with the correspondence $|0;p^a\rangle \simeq g_{o,p}\,e^{ip_aX^a}$ and we use the relativistic normalization:

$$\langle 0;p^{a\prime}|0;p^a\rangle = 2p^+(2\pi)\delta(p^{\prime +}-p^+)(2\pi)^{p-1}\delta^{p-1}(\mathbf{p}'-\mathbf{p}).$$

Note primarily that from the DDF operator commutation relations, $V(\lambda)$ is an eigenstate of the annihilation operators, $A_{n>0}^A\cdot V(\lambda) \cong \lambda_{n>0}^A V(\lambda)$, from which on account of (5.2) it follows that states are not orthogonal, the inner product of two states being given by,

$$\big\langle V(\lambda,p')|V(\zeta,p)\big\rangle = \delta_{p',p}C_\lambda C_\zeta \exp\Big(\sum_{n>0}\frac{1}{n}\lambda_n^*\cdot\zeta_n\Big).$$

The factor $C_\lambda C_\zeta\exp\big(\sum_{n>0}\frac{1}{n}\lambda_n^*\cdot\zeta_n\big)$ reduces to unity when $\lambda_n^A = \zeta_n^A$, for all n, A,

$$\big\langle V(\lambda,p)|V(\lambda,p)\big\rangle = 1,$$

and $\delta_{p',p}$ is a *Kronecker* delta which reduces to unity when $p^{+\prime} = p^+$ and $\mathbf{p}' = \mathbf{p}$ and vanishes otherwise. Recall that coherent states are (when we choose $q^i = q^+ = 0$) eigenstates of momentum in the k^+ and $\mathbf{k}$ directions (but not in the k^- direction). So, as one would expect, these coherent states are over-complete, the overlap between any two being non-zero for a wide range of λ_n^i, ζ_n^i. From this expression we then deduce (by forming appropriate inner products and integrating) that there exists the completeness relation,

$$\mathbb{1} = \mathcal{V}_\parallel\int_0^\infty\frac{dp^+}{2\pi}\int_{\mathbb{R}^{p-1}}\frac{d^{p-1}\mathbf{p}}{(2\pi)^{p-1}}\int\Big(\prod_{n,A}\frac{d^2\lambda_n^A}{2\pi n}\Big)\,\big|V(\lambda,p)\big\rangle\big\langle V(\lambda,p)\big|,$$

with $d^2\lambda_n^A = id\lambda_n^A \wedge d\lambda_n^{*A}$. Finally, that the vertex operator (5.2) is physical follows from the fact that $L_{n\in\mathbb{Z}}$ commutes with all the DDF operators, $L_{n>0}$ annihilates the vacuum $e^{ip\cdot X(z)}$ and $L_0 \cdot e^{ip\cdot X(z)} \cong e^{ip\cdot X(z)}$. Therefore, $V(\lambda)$ satisfies the Virasoro constraints, $(L_0-1)\cdot V(\lambda)\cong 0$, $L_{n>0}\cdot V(\lambda)\cong 0$ and is hence physical. Recall from Sec. 4.4 that in addition all states formed from DDF operators are transverse to null states. We conclude that the string coherent state defining properties (a-c) are satisfied.

Let us now consider the corresponding local normal ordered representation of $V(\lambda)$, which in practice means subtracting all self contractions from the vertex (5.2). The vacuum $e^{ip\cdot X(z)}$ is already normal ordered and so the remaining self-contractions that need to be subtracted are those associated to contractions with one leg in the DDF operators and one leg in the vacuum. In Sec. 4.4 we computed the normal ordered representation of arbitrary covariant states. For the above coherent state this is obtained by using the integral representation of the DDF operators (4.64) in (5.2) and carrying out the operator products on account of the onshell constraints (given below (5.2)) and the property $\lambda_n\cdot q = 0$. The integrands of the DDF operators are to lie on the real axis as they are brought close to the vacuum which is also on the real axis, $z=\bar{z}$, and so the relevant propagator takes the form,

$$\langle X^a(z)X^b(w)\rangle = -(2\alpha')\eta^{ab}\ln(z-w). \tag{5.5}$$

From Fig. 4.2 where the open string DDF construction is exhibited it can be seen that this is the correct procedure – in the figure we have conformally mapped to the disc with boundary $z\bar{z}=1$ (instead of the upper half plane) where the propagator is again of the form (5.5) on the boundary (up to terms that drop out of correlation functions). We then compute all Wick contractions and subsequently analytically continue in the variable of integration and choose an integration contour that circles the vacuum. The same procedure can then be repeated, with additional DDF operators which may be brought close to the resulting state in the same manner as above and so on. The resulting normal ordered vertex assumes a particularly simple form when we assume in addition, $\lambda_{n>0}\cdot\lambda_{m>0}=0$, see (4.98). In this case the normal ordered open string coherent states are given by a linear combination of the traceless mass eigenstates (4.92),

$$V(\lambda) = \frac{g_{o,p}}{\sqrt{2p^+\mathcal{V}_\parallel}}\, C_\lambda \exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\lambda_n\cdot H_n(z)\, e^{-inq\cdot X(z)}\Big)\, e^{ip\cdot X(z)}, \tag{5.6}$$

the difference being that for open strings the dimensionless quantity $H_n(z)$ reads,

$$H_N^A(z) \equiv \sqrt{2\alpha'}p^A S_N(Nq;z) + P_N^A(z), \tag{5.7a}$$

$$P_N^A(z) = \sqrt{\frac{2}{\alpha'}}\sum_{m=1}^{N}\frac{i}{(m-1)!}\,\partial^m X^A(z) S_{N-m}(Nq;z). \tag{5.7b}$$

Had we not required, $\lambda_{n>0}\cdot\lambda_{m>0}=0$, we would have found instead for the case of a single mode, $\lambda^A = \{0,\dots,0,\lambda_n^A,0,\dots,0\}$, $n\neq 0$, for all $A=1,\dots,p-1$,

$$V(\lambda)=\frac{g_{o,p}}{\sqrt{2p^+\mathcal{V}_\|}}C_\lambda\sum_{g=0}^{\infty}\sum_{a=0}^{\lfloor g/2\rfloor}\frac{1}{a!(g-2a)!}\Big(\frac{1}{2n^2}\lambda_n\cdot\lambda_n\,\mathbb{S}_{n,n}\Big)^a\Big(\frac{1}{n}\lambda_n\cdot H_n\Big)^{g-2a}e^{i(p-gnq)\cdot X(z)},\tag{5.8}$$

which follows from (4.100). For later reference, define the quantity $U(\lambda)$ by the expression,

$$V(\lambda)\equiv\frac{g_{o,p}}{\sqrt{2p^+\mathcal{V}_\|}}C_\lambda U(\lambda)e^{ip\cdot X(z)}.$$

The more general open string coherent state for an arbitrary set $\{\lambda_n^A\}$ can be be written down directly on account of (4.98). Note that the vertex associated to (5.8) reduces to that derived from (5.6) when $\lambda_n\cdot\lambda_n=0$, $n\neq 0$, as it should.

Series expanding the exponential in (5.6) it is seen that the mass eigenstates in the underlying Hilbert space are polynomials in $\partial^\# X$, multiplied by $e^{i(p-\sum_n ns_nq)\cdot X(z)}$, for some sequence of positive integers, $\{s_1,s_2,\dots\}$, with $\sum_n ns_n$ equal to the level number. Also, $V(\lambda)$ is an eigenstate of momentum in the directions transverse to q^μ; given that $q^2=0$ one may take for example, $q^+=q^A=q^I=0$ and q^- non-vanishing (see also the discussion in Sec. 4.3), in which case one learns that $\hat p^A\cdot V(\lambda)=p^AV(\lambda)$ and $\hat p^+\cdot V(\lambda)=p^+V(\lambda)$, with[8] $\hat p^\mu=\frac{1}{\alpha'}\oint d\!\!\!{}^-z\,\partial X^\mu$. The full momentum expectation value is in turn given by,[9]

$$\langle\hat p^a\rangle=\big(p^a-N_eq^a\big),\qquad\text{and}\qquad\langle\hat p^2\rangle=-\frac{1}{\alpha'}\big(N_e-1\big),\tag{5.9}$$

where we have identified an *effective* level number,

$$N_e\equiv\sum_{n=1}^{\infty}|\lambda_n|^2,$$

in direct analogy to the generic DDF state momentum (4.67). These considerations imply that $V(\lambda)$ carries an effective mass associated to N_e, which is in agreement with the usual open string mass shell constraint, $m^2=(N-1)/\alpha'$, when N is identified with N_e. Notice that N_e is a continuous function of the $|\lambda_n|$ as required from the definition of a coherent state, not necessarily an integer. Therefore, coherent states can in particular have masses which are non-zero, but yet much smaller than the string scale (a common draw-back of mass eigenstates), or, in the opposite extreme, they may have large mass and represent macroscopic string states; although we have not yet proven that the states constructed are macroscopic.

From the well known properties of coherent states [126] we expect the limit $|\lambda_n|\gg 1$ to be associated to the macroscopic or long string limit. To show that this is indeed the case we next consider the open string coherent state (5.2) in lightcone gauge. Using the

[8] Here $\hat p^\mu=\hat p^\mu_{\rm open}=\frac{1}{\alpha'}\oint d\!\!\!{}^-z\,\partial X^\mu$ in this section only; in the rest of the paper, $\hat p^\mu=\hat p^\mu_{\rm closed}=\frac{2}{\alpha'}\oint d\!\!\!{}^-z\partial X^\mu$, see Sec. 2.6, and 2.7.

[9] Here as is standard in conformal field theory [207] we take $|V_{\rm in}\rangle=\lim_{z\to 0}V(z)|0\rangle$ with $|0\rangle$ the oscillator vacuum and $\langle V_{\rm out}|=(|V_{\rm in}\rangle)^\dagger$. We write $\langle A\rangle=\langle V_{\rm out}|A|V_{\rm in}\rangle$ for an operator A.

map discussed in Sec. 4.4 we immediately write down the lightcone gauge analogue of the covariant state (5.2),

$$|V(\lambda)\rangle_{\rm lc} = \frac{1}{\sqrt{2p^+\mathcal{V}_\parallel}}C_\lambda \exp\Big(\sum_{n=1}^\infty \frac{1}{n}\lambda_n\cdot\alpha_{-n}\Big)\big|0;p^+,p^A\big\rangle. \tag{5.10}$$

This is also an eigenstate of p^+, p^A, as was the covariant state above (when $q^+=q^A=0$). The contractions are associated to indices, A, and are transverse to the longitudinal, $\pm$, directions (with $v^\pm=\frac{1}{\sqrt{2}}(v^0\pm v^p)$ for some generic spacetime vector v^μ). This state is an eigenstate of the annihilation operators, $\alpha^A_{n>0}|V(\lambda)\rangle_{\rm lc}=\lambda^A_n|V(\lambda)\rangle_{\rm lc}$ and so the lightcone gauge position expectation value is given by (2.42),

$$\langle X^A(z,\bar z)-\hat x^A\rangle_{\rm lc}=(X^A(z,\bar z)-x^A)_{\rm cl},$$

with,

$$\big(X^A(z,\bar z)-x^A\big)_{\rm cl}=-i\alpha' p^A\ln|z|^2+i\Big(\frac{\alpha'}{2}\Big)^{1/2}\sum_{n\neq0}^{\infty}\frac{\lambda^A_n}{n}\big(z^n+\bar z^{-n}\big), \tag{5.11}$$

where we have identified $\langle\hat p^A\rangle$ with p^A (given that $q^A=0$). Equation (5.11) is the general solution to the equations of motion, $\partial\bar\partial X^A_{\rm cl}(z,\bar z)=0$, the constraints, $(\partial X_{\rm cl})^2=(\bar\partial X_{\rm cl})^2=0$ having been solved by the gauge choice:[10] $X^+_{\rm cl}(z,\bar z)=-i\alpha'p^+\ln|z|^2$, reached by the conformal map $z=e^{2iq\cdot X(z)}$, $\bar z=e^{2iq\cdot X(\bar z)}$ (recall that $q\cdot p=1/(2\alpha')$ for open strings). The corresponding longitudinal components of the position expectation value are likewise computed. On account of the operator equation, $\sqrt{2\alpha'}\alpha^-_n=\frac{1}{2p^+}\sum_{\ell\in\mathbb{Z}}:\alpha^i_{n-\ell}\alpha^i_\ell:$ (for $n\neq0$), and the fact that the coherent state is an eigenstate of $\alpha^A_{n>0}$ with eigenvalue λ^A_n one learns that,[11]

$$\big\langle X^-(z,\bar z)-\hat x^-\big\rangle_{\rm lc}=\big(X^-(z,\bar z)-x^-\big)_{\rm cl},$$

with

$$\begin{aligned}\big(X^-(z,\bar z)-x^-\big)_{\rm cl}=-i\frac{1}{p^+}\Big(\alpha'\mathbf{p}^2+\sum_{n=1}^\infty|\lambda_n|^2-1\Big)\ln|z|^2\\ +i\sum_{n\neq0}\frac{1}{n}\sum_{r\in\mathbb{Z}}\frac{1}{4p^+}\lambda_{n-r}\cdot\lambda_r\,\big(z^{-n}+\bar z^{-n}\big),\end{aligned} \tag{5.13}$$

with the definitions $\lambda^A_0\equiv\sqrt{2\alpha'}p^A$, $\mathbf{p}^2=p^Ap^A$. Finally, in the Dirichlet directions, on account of (2.42), it follows that,

$$\big\langle X^I(z,\bar z)-\hat x^I\big\rangle_{\rm lc}=\big(X^I(z,\bar z)-x^I\big)_{\rm cl}=0,$$

with

$$X^I(z,\bar z)=x^I-i\alpha'w^I\ln\frac{z}{\bar z}+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{\alpha^I_n}{n}\Big(\frac{1}{z^n}-\frac{1}{\bar z^n}\Big),$$

[10] Recall that $\tau=(\tau)_{\rm Euclidean}=i(\tau)_{\rm Minkowski}$, $z=e^{-i(\sigma+i\tau)}$, $\bar z=e^{i(\sigma-i\tau)}$.

[11] Recall that for open strings,

$$X^-(z,\bar z)-x^-=-i\alpha'\hat p^-\ln|z|^2+i\Big(\frac{\alpha'}{2}\Big)^{1/2}\sum_{n\neq0}\frac{\alpha^-_n}{n}\big(z^{-n}+\bar z^{-n}\big). \tag{5.12}$$

which shows that the open string coherent state vertex operators we have constructed are restricted to lie on a single Dp-brane, and that for vertices stretched between two parallel D-branes of the same dimensionality one can still work with these vertex operators provided the exponential factor given in (4.70) is inserted into the path integral.

The position operator is not a gauge invariant quantity and so the corresponding covariant gauge position expectation value, although of the form (5.11), would be a more complicated expression whose polarization tensors are not independent, being subject to the constraints $(\partial X)^2 = (\bar{\partial} X)^2 = 0$. Therefore, the covariant position expectation value is not a particularly useful quantity in practice because the classical solutions we want to match vertex operators to are not known in covariant gauge. The angular momentum on the other hand is a gauge invariant operator, $[L_n, J^{\mu\nu}] = 0$, and so a good consistency check is to show that both the covariant, $\langle J^{ab}\rangle_{\rm cov}$, and the lightcone, $\langle J^{ab}\rangle_{\rm lc}$, angular momentum expectation values are equal (in the unit norm representation) to the classical angular momentum, $J^{ab}_{\rm cl}$. Such an equivalence would support the conjecture that (5.2) and (5.10) are different manifestations of the same state and correspond classically to the lightcone gauge solution (5.11). The total angular momentum operator is the integral of the current associated to Lorentz invariance over a spacelike curve, say $|z|^2 = 1$ in the coordinates $z = e^{-i(\sigma+i\tau)}, \bar{z} = e^{i(\sigma-i\tau)}$, that cuts once across the string worldsheet [166]. For the open string,

$$J^{\mu\nu} = \frac{2}{\alpha'}\oint \bar{d}z X^{[\mu}\partial X^{\nu]}, \qquad \text{and} \qquad S^{\mu\nu} = -i\sum_{\ell=1}^{\infty}\frac{1}{\ell}\big(\alpha^{\mu}_{-\ell}\alpha^{\nu}_{\ell} - \alpha^{\nu}_{-\ell}\alpha^{\mu}_{\ell}\big), \tag{5.14}$$

with $a^{[\mu\nu]} = \frac{1}{2}(a^{\mu\nu} - a^{\nu\mu})$ and $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$. Due to the anti-symmetry there are no normal ordering ambiguities. $L^{\mu\nu}$ is the zero mode contribution[12] and we have used the doubling trick [114]. Notice furthermore that $S^{\mu\nu} = \sum_{\ell=1}^{\infty}\frac{2}{\ell}{\rm Im}\big(\alpha^{\mu}_{-\ell}\alpha^{\nu}_{\ell}\big)$. For simplicity focus on these non-zero mode components, $S^{\mu\nu}$, and consider first the components, S^{AB}. For the lightcone gauge classical computation we find, $S^{AB}_{\rm cl} = \sum_{n>0}\frac{2}{n}{\rm Im}\big(\lambda^{*A}_n\lambda^B_n\big)$, which follows from (5.11) and (5.14). In the lightcone gauge the quantity

$$\langle S^{AB}\rangle_{\rm lc} \equiv \langle V(\lambda)|S^{AB}|V(\lambda)\rangle_{\rm lc},$$

is computed using $S^{AB} = \sum_{\ell>0}\frac{2}{\ell}{\rm Im}\big(\alpha^{A}_{-\ell}\alpha^{B}_{\ell}\big)$, and (5.10). Given that $|V(\lambda)\rangle_{\rm lc}$ is an eigenstate of the annihilation operators it follows immediately that $\langle S^{AB}\rangle_{\rm lc} = \sum_{n>0}\frac{2}{n}{\rm Im}\big(\lambda^{*A}_n\lambda^B_n\big)$. Finally, the covariant gauge quantity

$$\langle S^{AB}\rangle_{\rm cov} = \langle V(\lambda)|S^{AB}|V(\lambda)\rangle_{\rm cov},$$

is also computed using $S^{AB} = \sum_{\ell>0}\frac{2}{\ell}{\rm Im}\big(\alpha^{A}_{-\ell}\alpha^{B}_{\ell}\big)$, and we are to identify $V(\lambda)$ with the covariant vertex operator (5.2), or, equivalently the operator state (5.4). For this computation one may readily derive the following commutators [208],

$$\big[\alpha^A_m, A^B_n\big] = m\delta^{A,B}B^n_m, \qquad \text{and} \qquad \big[A^A_n, B^m_\ell\big] = 0 = [B^n_m, B^\ell_r],$$

[12] In particular, in covariant gauge, $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and in lightcone gauge, $L^{ij} = x^i p^j - x^j p^i$, $L^{-i} = x^- p^i - \frac{1}{2}\big(x^i p^- - p^- x^i\big)$, $L^{-+} = \frac{1}{2}\big(x^- p^+ + p^+ x^-\big)$ and $L^{i+} = x^i p^+$ which may be interpreted either classically or quantum-mechanically.

with $B_m^n \equiv -i \oint đz\, z^{m-1}\, e^{inq\cdot X(z)}$, see Appendix K. Using these one can show primarily that

$$\alpha^A_{m>0} \cdot V(\lambda) \cong \sum_{n=1}^{\infty} \frac{m}{n} \lambda^A_n B^{-n}_m \cdot V(\lambda). \tag{5.15}$$

From the definition of B^{-n}_m and $[A^A_n, B^m_\ell] = 0$ follows the operator product,

$$B^{-n}_m \cdot V(\lambda) \cong\, : S_{n-m}(nq;z)\, e^{-inq\cdot X(z)} V(\lambda) :$$

From this latter expression and the properties (see Appendix J and K), $S_0 = 1$ and $S_{n<0} = 0$, we find that B^{-n}_m annihilates $V(\lambda)$ when $m > n$ and shifts the vacuum momentum, $p^a \to p^a - nq^a$, leaving the state otherwise unaltered, when $n = m$. From $(B^{-n}_m)^\dagger = B^n_{-m}$ we find that terms with $m > n$ similarly annihilate the out state, $V(\lambda)^\dagger$, in the expectation value $\langle S^{AB}\rangle_{\rm cov}$ where similar considerations apply. Therefore, only the term $n = m$ survives in the sum over n in (5.15). We thus find the covariant gauge expectation value, $\langle S^{AB}\rangle_{\rm cov} = \sum_{n>0} \frac{2}{n} {\rm Im}\big(\lambda^{*A}_n \lambda^B_n\big)$. Collecting the classical, lightcone gauge and covariant gauge computations, we have shown that,

$$\langle S^{AB}\rangle_{\rm cov} = \langle S^{AB}\rangle_{\rm lc} = \sum_{n>0} \frac{2}{n} {\rm Im}\big(\lambda^{*A}_n \lambda^B_n\big) = S^{AB}_{\rm cl}. \tag{5.16}$$

The angular momentum components in the longitudinal directions are similarly computed. For the lightcone gauge computation,

$$\langle S^{A-}\rangle_{\rm lc} = \langle V(\lambda)|S^{AB}|V(\lambda)\rangle_{\rm lc},$$

one can use the commutator $[\alpha^-_\ell, \alpha^A_{-n}] = n\alpha^A_{\ell-n}/(\sqrt{2\alpha'}p^+)$, but since $|V\rangle_{\rm lc}$ is an eigenstate of $\alpha^A_{n>0}$ with eigenvalue λ^A_n it is advantageous to use the expression, $\sqrt{2\alpha'}\alpha^-_\ell = \frac{1}{2p^+}\sum_{m\in\mathbb{Z}} : \alpha^A_m \alpha^A_{\ell-m} :$, in S^{A-}. This then leads to, $\langle S^{A-}\rangle_{\rm lc} = \frac{1}{\sqrt{2\alpha'}p^+}\sum_{\ell>0}\sum_{m\in\mathbb{Z}} \frac{1}{\ell}{\rm Im}\big(\lambda^{*A}_\ell \lambda_m \cdot \lambda_{\ell-m}\big)$, with $\lambda^A_0 \equiv \sqrt{2\alpha'}p^A$ as above. For the covariant gauge computation,

$$\langle S^{A-}\rangle_{\rm cov} = \langle V(\lambda)|S^{AB}|V(\lambda)\rangle_{\rm cov},$$

to match to the lightcone gauge we use lightcone *coordinates* where, $q^+ = q^A = 0$ and $q^- = -1/(2\alpha' p^+)$ (which solve the constraints $q^2 = 0$ and $p\cdot q = 1/(2\alpha')$). One can readily derive the commutators [208],

$$\big[\alpha^-_m, A^A_n\big] = n\sqrt{2\alpha'}q^- D^A_{m,n}, \qquad \big[A^A_\ell, D^B_{m,n}\big] = \ell\delta^{AB}E^{\ell+n}_m, \qquad \text{and} \qquad \big[A^A_\ell, E^n_m\big] = 0,$$

with $D^A_{m,n} = \sqrt{\frac{2}{\alpha'}}\oint đz z^m \partial X^A e^{inq\cdot X(z)}$ and $E^n_m = \oint đz z^m q\cdot\partial X e^{inq\cdot X(z)}$ (see Appendix K) from which follows the operator product,

$$\alpha^-_\ell \cdot V(\lambda) \cong \sqrt{2\alpha'}q^- \sum_{n=1}^{\infty}\Big(-\lambda_n\cdot D_{\ell,-n} + \sum_{m=1}^{\infty}\frac{1}{2}\lambda_n\cdot\lambda_m E^{-n-m}_\ell\Big)\cdot V(\lambda). \tag{5.17}$$

Consider the second term in this expression. Given that $[A_\ell, E^n_m] = 0$ one may commute the E^{-n-m}_ℓ through to hit the vacuum, $e^{ip\cdot X(z)}$, where the following operator product is required,

$$E^{-n-m}_\ell \cdot e^{ip\cdot X(z)} \cong\, : \sqrt{2\alpha'}q\cdot H_{n+m-\ell}\big((n+m)q;z\big)e^{i(p-n-m)\cdot X(z)} :, \tag{5.18}$$

with $\sqrt{2\alpha'}q \cdot H_0 = 1$ and $q \cdot H_{m<0} = 0$, the polynomial H_m having been defined in Appendix J.[13] In the expectation value, $\langle V^\dagger S^{A-} V\rangle$, this implies that we should only bring E_ℓ^{-n-m} to the right to hit $V(\lambda)$ if $n+m-\ell \le 0$. Of these, the $n+m-\ell = 0$ subset will shift the vacuum momentum, $p \to p-(n+m)q$, leaving the state otherwise unaltered, and the $n+m-\ell < 0$ subset will annihilate it. Therefore of the terms with $n+m-\ell \le 0$ in the sum over m only the $m = \ell - n$ term will contribute. The remaining terms with, $n+m-\ell > 0$, will not contribute either. These are to be commuted through to the out-state, $V^\dagger$, which is annihilated by them. In doing so these latter terms first encounter $\alpha_{-\ell}^A$ from S^{A-}. We here use the fact that $\langle V|\alpha_{-\ell}^A = \big(\alpha_\ell^A|V\rangle\big)^\dagger \cong \big(\sum_{n=1}^\infty \frac{\ell}{n}\lambda_n^A B^{-n}_{\ \ell}|V\rangle\big)^\dagger = \sum_{n=1}^\infty \frac{\ell}{n}\lambda_n^{*A}\langle V|B^n_{-\ell}$, and $[B^n_{-\ell}, E_r^{-m}] = 0$, so that the quantities, E_ℓ^{-n-m}, with $n+m-\ell > 0$ commute freely through to hit and annihilate the out state, $V^\dagger$, and so indeed only the term $m = \ell - n$ will survive in the second term in (5.17) in the computation of $\langle S^{A-}\rangle$.

Next consider the first term in (5.17). On account of the operator product,

$$D_{\ell,-n}^A \cdot e^{ip\cdot X(z)} \cong \,:H_{n-\ell}^A(nq;z)e^{i(p-nq)\cdot X(z)}:,$$

and the properties, $H_0^A = p^A$ and $H_{n<0}^A = 0$, we will commute the $D_{\ell,-n}^A$ through to hit the $e^{ip\cdot X(z)}$ vacuum when $n-\ell \le 0$. Of these the subset of $D_{\ell,-n}^A$ for which $n-\ell = 0$ shifts the vacuum momentum, $p^a \to p^a - nq^a$, leaving the state otherwise unaltered, whereas the subset satisfying $n-\ell < 0$ annihilates it. The $D_{\ell,-n}^a$ terms with $n-\ell > 0$ are to be commuted through to the out state, $V^\dagger$, in the expectation value $\langle V^\dagger S^{A-} V\rangle$, just like we did above for the E_ℓ^{-n-m} terms with $n+m-\ell > 0$. From the commutators, $\big[A_\ell^A, D_{m,n}^B\big] = \ell\delta^{AB}E_m^{\ell+n}$ and $\big[A_\ell^A, E_m^n\big] = 0$ we find that, $\Big[D_{\ell,-n}^A, \exp\big(\sum_{m>0}\frac{1}{m}\lambda_m \cdot A_{-m}\big)\Big] = \sum_{m>0}\lambda_m^A E_\ell^{-n-m}$. For the terms with $n-\ell \le 0$, for which $D_{\ell,-n}^A \cdot e^{ip\cdot X} \cong\, :\delta_{n,\ell}\sqrt{2\alpha'}p^A\, e^{i(p-n)\cdot X(z)}:$, we find,

$$\lambda_n\cdot D_{\ell,-n}\cdot V(\lambda) \cong \sum_{m>0}\lambda_n\cdot\lambda_m E_\ell^{-n-m}\cdot V(\lambda) + :\delta_{n,\ell}\sqrt{2\alpha'}\lambda_n\cdot p\, e^{-inq\cdot X(z)}V(\lambda): \qquad (n-\ell \le 0) \tag{5.19}$$

Now, the same argument that applied to the second term in (5.17) applies to the first term in (5.19) and so again only the $m = \ell - n$ term will contribute in the sum over m to the expectation value $\langle S^{A-}\rangle$. Finally, for the first term in (5.17), for which $n-\ell > 0$, we commute $\lambda_n \cdot D_{\ell,-n}$ through to the out state $V^\dagger$ using the fact that $[B_m^n, D_{\ell,-n}^A] = 0$ and $V^\dagger \cdot D_{\ell,-n}^A \cong \sum_{m>0}\lambda_{-m}^A V^\dagger \cdot E_\ell^{-n+m}$. The same argument as above applies and only the term $m = n-\ell$ contributes in the sum over m (which is consistent with $n-\ell > 0$ as m is positive).

Identifying $-q^-$ with $1/(2\alpha' p^+)$, the above considerations are summarized in the ex-

[13] See also comments below (K.11).

pression,

$$\begin{aligned}\left\langle V^\dagger S^{A-} V\right\rangle_{\text{cov}} &= \frac{1}{\sqrt{2\alpha' p^+}}\sum_{\ell>0}\frac{2}{\ell}\mathrm{Im}\Big\langle\Big(\lambda_\ell^{*A}B_{-\ell}^{\ell}\Big)\Big(\frac{1}{2}\sum_{n=1}^{\ell}\sum_{m>0}\lambda_n\cdot\lambda_m\delta_{n+m,\ell}E_\ell^{-\ell}\\&\qquad + \lambda_\ell\cdot\lambda_0 e^{-A\ell q\cdot X(0)} + \sum_{n=\ell+1}^{\infty}\sum_{m>0}\lambda_n\cdot\lambda_{-m}\delta_{n-m,\ell}E_\ell^{-\ell}\Big)\Big\rangle\\&= \frac{1}{\sqrt{2\alpha' p^+}}\sum_{\ell>0}\sum_{m\in\mathbb{Z}}\frac{1}{\ell}\mathrm{Im}\big(\lambda_\ell^{*A}\lambda_m\cdot\lambda_{\ell-m}\big),\end{aligned} \tag{5.20}$$

and this is in agreement with the lightcone gauge and classical computation. In going from the first to the second equality in (5.20) there are a number of steps. Let us write $f(n,m) = \lambda_n\cdot\lambda_m E_\ell^{-\ell}$. Focus on the second parenthesis and recall from (5.18) that one may replace $e^{-i\ell q\cdot X}$ in the second term with $E_\ell^{-\ell}$, which identifies the second term as $f(\ell,0)$. The delta function in the first term restricts the summations appearing, $\frac{1}{2}\sum_{n=1}^{\ell}\sum_{m>0}f(n,m)\delta_{n,\ell-m} = \frac{1}{2}\sum_{m=1}^{\ell-1}f(\ell-m,m)$, and when the resulting expression is combined with the second term, $\sum_{m=1}^{\ell-1}\to\sum_{m=0}^{\ell}$. Similarly, the delta function in the third term restricts the summations appearing according to $\sum_{n=\ell+1}^{\infty}\sum_{m>0}f(n,-m)\delta_{n,\ell+m} = \sum_{m<0}f(\ell-m,m)$. The second parenthesis in (5.20) is therefore equal to $\frac{1}{2}\sum_{m=0}^{\ell}f(\ell-m,m)+\sum_{m<0}f(\ell-m,m)$, half of the second term of which can be absorbed into the first term leading to $\frac{1}{2}\sum_{m=-\infty}^{\ell}f(\ell-m,m)+\frac{1}{2}\sum_{m<0}f(\ell-m,m)$. After a change of variables in the second term, $m'=m-\ell$ with $m'\in[\ell+1,\infty)$, these two terms can be combined into the expression $\frac{1}{2}\sum_{m\in\mathbb{Z}}f(\ell-m,m)$. On account of the fact that $\langle V^\dagger B_{-\ell}^{\ell}E_\ell^{-\ell}V\rangle = 1$ it follows that the first equality in (5.20) implies the second.

Collecting the classical, lightcone gauge and covariant gauge computations, we have shown that the longitudinal components of the angular momentum for the classical, lightcone gauge and covariant gauge computations are in agreement; in the wavepacket representation in particular,

$$\langle S^{-A}\rangle_{\text{cov}} = \langle S^{-A}\rangle_{\text{lc}} = \frac{1}{\sqrt{2\alpha' p^+}}\sum_{\ell>0}\sum_{m\in\mathbb{Z}}\frac{1}{\ell}\mathrm{Im}\big(\lambda_\ell^{*A}\lambda_m\cdot\lambda_{\ell-m}\big) = S_{\text{cl}}^{-A}. \tag{5.21}$$

The non-zero mode contributions to the angular momentum components involving S^{+-}, and S^{+A}, are all vanishing in the chosen coordinate system where $q^+=0$. Recall furthermore that $\lambda_0^i\equiv\sqrt{2\alpha'}p^i$.

We have shown that there is a one-to-one correspondence between the covariant vertex operators (5.2), lightcone gauge states (5.10) and classical macroscopic string evolution (5.11) and (5.12). The preceding angular momentum computations provide further support for the conjecture that the covariant and lightcone gauge descriptions are different manifestations of the same state, both of which have a classical interpretation.

5.2 Closed String

In close analogy to the open string case above, we define a closed string *coherent state*, $V(\lambda,\bar{\lambda},\dots)$, to be a state that:[14]

(a) is specified by a set of *continuous* labels $(\lambda,\bar{\lambda})=\{\lambda_n^i,\bar{\lambda}_n^i\}$ (with λ and $\bar{\lambda}$ associated to the left- and right-moving modes respectively of the string);

(b) there must exist a resolution of unity,

$$\mathbb{1}=\sum\!\!\!\!\!\!\!\int\int d\lambda d\bar{\lambda}\big|V(\lambda,\bar{\lambda},\dots)\big\rangle\big\langle V(\lambda,\bar{\lambda},\dots)\big|,$$

so that the $V(\lambda,\bar{\lambda},\dots)$ span the string Hilbert space, $\mathcal{H}$;

(c) it must transform correctly under all symmetries of the bosonic (or super-) string.

The dots "$\dots$" in $V(\lambda,\bar{\lambda},\dots)$ allow for the possibility that the vertex operator depends on additional continuous or discrete quantum numbers and these are all to be summed over in the completeness relation.[15] The unit operator on the left is defined with respect to $\mathcal{H}$,[16] $\mathbb{1}\cdot\big|V(\lambda,\bar{\lambda})\big\rangle\equiv\big|V(\lambda,\bar{\lambda})\big\rangle$. The measures for the case of interest explicitly read $d\lambda d\bar{\lambda}=\prod_{n,i}\frac{d^2\lambda_n^i d^2\bar{\lambda}_n^i}{N}$ with N a to-be determined normalization and as usual $d^2\lambda_n^i=id\lambda_n^i\wedge d\lambda_n^{*i}$ (no sum over i), and so on.

DLCQ Coherent States

In this subsection we construct closed string coherent states that satisfy the above definition. The construction will be naive and we will discover that internal consistency requires the underlying spacetime manifold be lightlike-compactified: $X^-\sim X^-+2\pi R^-$. Quantization on a lightlike compactified background is known as 'discrete lightcone quantization' (DLCQ) [209, 210, 211, 212]. In the following section we shall construct coherent states in a fully non-compact spacetime background.

The closed string coherent state candidate that we consider in this section is obtained by joining two copies of the open string state (5.2),

$$V(\lambda,\bar{\lambda},p)=\frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}C_{\lambda\bar{\lambda}}\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\lambda_n\cdot A_{-n}\Big)\exp\Big(\sum_{m=1}^{\infty}\frac{1}{m}\bar{\lambda}_m\cdot\bar{A}_{-m}\Big)e^{ip\cdot X(z,\bar{z})},\tag{5.22}$$

[14] I am adopting the rather general definition of a coherent state as given in [126] and minimally extend it to include the string theory requirements. For instance, under this definition, coherent states need not (in general) be eigenstates of the annihilation operators, $\alpha^\mu_{n>0}$, $\tilde{\alpha}^\mu_{n>0}$, in order for this definition to be satisfied.

[15] I will often not exhibit these latter labels explicitly, and hence write instead $V(\lambda,\bar{\lambda})$, or even $V_{\lambda\bar{\lambda}}$, all of which refer to the same object $V(\lambda,\bar{\lambda},\dots)$.

[16] I am being pedantic here for a subtle reason that will become clear later. Recall that the Hilbert space $\mathcal{H}$ is in general a background dependent quantity, and so the explicit realization of the unit operator, $\mathbb{1}$, is also background dependent.

with the normalization,

$$C_{\lambda\bar{\lambda}} = \exp\Big(\sum_{n=1}^{\infty} -\frac{1}{2n}|\lambda_n|^2 - \frac{1}{2n}|\bar{\lambda}_n|^2\Big),$$

chosen such that if we write $V(z,\bar{z}) = V(\lambda,\bar{\lambda},p)$, the most singular term in the operator product expansion is as in (4.43),

$$V(z,\bar{z})\cdot V(0,0) \cong \Big(\frac{g_c^2}{2p^+\mathcal{V}_{d-1}}\Big)\frac{1}{|z|^4} + \dots, \tag{5.23}$$

corresponding to 'one string in volume $\mathcal{V}_{d-1}$' as required by unitarity of the S-matrix, which was discussed in Sec. 4.3. In operator language, we have:

$$\langle V(\lambda,\bar{\lambda},p)|V(\lambda,\bar{\lambda},p)\rangle = 1, \qquad \text{with} \qquad |0,0;p\rangle \cong g_c\, e^{ip\cdot X(z,\bar{z})}.$$

This corresponds to a relativistic unit norm normalization with, see (4.54),

$$\langle 0,0;p'|0,0;p\rangle = 2p^+(2\pi)\delta(p^{+\prime} - p^+)(2\pi)^{d-2}\delta^{d-2}(\mathbf{p}' - \mathbf{p}).$$

Furthermore, $(\lambda,\bar{\lambda}) = \{\lambda_n^i, \bar{\lambda}_n^i\}$, are the polarization tensors, defined by, $\lambda_n\cdot q = 0$, $\lambda_n^* = \lambda_{-n}$, and $\sum_{n=1}^{\infty}|\lambda_n|^2 < \infty$, and similarly for the anti-holomorphic sector $\{\bar{\lambda}_n^i\}$. The real vectors p^μ and q^μ are as usual subject to the constraints (4.65), repeated here for convenience: $p\cdot q = 2/\alpha'$, $q^2 = 0$, and $p^2 = 4/\alpha'$.

First let us prove that the vertex operator (5.22) is a coherent state by showing that the defining properties (a-c) above are satisfied. (a) is trivially satisfied, the state is specified by the set of continuous labels $(\lambda,\bar{\lambda}) = \{\lambda_n^i, \bar{\lambda}_n^i\}$ and remains normalized for arbitrary values provided [205] $\sum_{n=1}^{\infty}|\lambda_n|^2 + |\bar{\lambda}_n|^2 < \infty$. To prove that (b) is satisfied note that primarily that $V(\lambda,\bar{\lambda})$ is an eigenstate of the annihilation operators, $A_{n>0}^i\cdot V \cong \lambda_n^i V$ and $\bar{A}_{n>0}^i\cdot V \cong \bar{\lambda}_n^i V$, which follows from the DDF operator commutation relations (4.69) and the corresponding anti-holomorphic expression with $\bar{A}_n$ replacing A_n. Therefore, we find the following inner product,

$$\big\langle V(\lambda,\bar{\lambda},p')|V(\zeta,\bar{\zeta},p)\big\rangle = \delta_{p',p}C_{\lambda\bar{\lambda}}C_{\zeta\bar{\zeta}}\exp\Big(\sum_{n>0}\frac{1}{n}\lambda_n^*\cdot\zeta_n + \frac{1}{n}\bar{\lambda}_n^*\cdot\bar{\zeta}_n\Big), \tag{5.24}$$

which reduces to unity when $(\lambda,\bar{\lambda}) = (\zeta,\bar{\zeta})$ and $p' = p$. Note that $\delta_{p',p}$ is a *Kronecker* delta which reduces to unity when $p^{+\prime} = p^+$ and $\mathbf{p}' = \mathbf{p}$ and vanishes otherwise. By then forming appropriate inner products and integrating we find that there exists the completeness relation,

$$\mathbb{1} = \mathcal{V}_{d-1}\int_0^\infty \frac{dp^+}{2\pi}\int_{\mathbb{R}^{24}}\frac{d^{24}\mathbf{p}}{(2\pi)^{24}}\int\Big(\prod_{n,A}\frac{d^2\lambda_n^A}{2\pi n}\Big)\Big(\prod_{n,A}\frac{d^2\bar{\lambda}_n^A}{2\pi n}\Big)\big|V(\lambda,\bar{\lambda},p)\big\rangle\big\langle V(\lambda,\bar{\lambda},p)\big|, \tag{5.25}$$

with $n = \{1,2,\dots,\infty\}$ or, more succinctly,

$$\mathbb{1} = \int d\mu(p)d\lambda d\bar{\lambda}\big|V(\lambda,\bar{\lambda};p)\big\rangle\big\langle V(\lambda,\bar{\lambda};p)\big|,$$

where we write $d\mu(p) = \mathcal{V}_{d-1}\frac{dp^+}{2\pi}\frac{d^{24}\mathbf{p}}{(2\pi)^{24}}$, $d\lambda = \prod_{n,i}\frac{d^2\lambda_n^i}{2\pi n}$, $d^2\lambda_n^i = id\lambda_n^i \wedge d\lambda_n^{*i}$, and similarly for the anti-holomorphic sector, $d\bar{\lambda}$, with $\bar{\lambda}_n^i$ replacing λ_n^i.[17] The phase space integrals are precisely as anticipated from Sec. 4.3 and in particular (4.40) for the sum over single string states. In the case of closed string coherent states therefore we see that the additional sums over quantum numbers in (4.40) correspond to integrals over the polarization tensors:

$$\not{\sum} = \int \bigg(\prod_{n,A}\frac{d^2\lambda_n^A}{2\pi n}\bigg)\bigg(\prod_{n,A}\frac{d^2\bar{\lambda}_n^A}{2\pi n}\bigg).$$

Finally, to show that (c) is satisfied we must prove that $V(\lambda,\bar{\lambda})$ satisfies the Virasoro constraints, $L_0\cdot V \cong V$, $L_{n>0}\cdot V \cong 0$. These are trivially satisfied given that: the DDF operators commute with the L_n, $\bar{L}_n$ for all n, and the vacuum $e^{ip\cdot X(z,\bar{z})}$ is physical, $L_0\cdot e^{ip\cdot X}\cong e^{ip\cdot X}$, $L_{n>0}\cdot e^{ip\cdot X}\cong 0$. Similar results hold for the antiholomorphic sector with $\bar{L}_n$ replacing L_n. Therefore, the vertex (5.22) is a coherent state and respects the string theory symmetries.

We postulated that closed string covariant coherent states are described by the vertex operator (5.22). These vertices however are not what we are looking for, and to see why let us normal order $V(\lambda,\bar{\lambda})$. To simplify the computation we need to assume as in the open string case that $\lambda_{n>0}\cdot\lambda_{m>0} = 0$ and similarly for the antiholomorphic sector (although the argument that follows applies for arbitrary polarization tensors as well). The normal ordering procedure has been explained in great detail in Sec. 4.4 for arbitrary mass eigenstates, the difference here being that the coherent state $V(\lambda,\bar{\lambda})$ is instead a linear superposition of mass eigenstates. As in the open string, the normal ordered version of (5.22) is obtained by using the integral representation of the DDF operators (4.64), the integration contour being taken around the vacuum $e^{ip\cdot X(z)}$ and $e^{ip\cdot X(\bar{z})}$ for the holomorphic and antiholomorphic sectors respectively. Holomorphy then allows us to shrink the contours and hence the computation only requires knowledge of the leading behaviour of the integrand close to the vacuum, which is determined by operator product expansions using the scalar propagator,

$$\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2. \tag{5.26}$$

This procedure leads to,

$$\begin{aligned} V(\lambda,\bar{\lambda}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,C_{\lambda\bar{\lambda}}\exp\bigg(\sum_{n=1}^\infty\frac{1}{n}\lambda_n\cdot H_n(z)e^{-inq\cdot X(z)}\bigg)& \\ \times\exp\bigg(\sum_{m=1}^\infty\frac{1}{m}\bar{\lambda}_m\cdot\bar{H}_m(\bar{z})e^{-inq\cdot X(\bar{z})}\bigg)\,e^{ip\cdot X(z,\bar{z})}&. \end{aligned}\tag{5.27}$$

Had we not assumed that $\lambda_{n>0}\cdot\lambda_{m>0}=0$, we would have found instead for the case of a single mode, $(\lambda,\bar{\lambda}) = \{0,\dots,0,\lambda_n^i,0,\dots,0;0,\dots,0,\bar{\lambda}_m^j,0,\dots,0\}$,

$$V(\lambda,\bar{\lambda}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\,C_{\lambda\bar{\lambda}}U(\lambda)\bar{U}(\bar{\lambda})e^{ip\cdot X(z,\bar{z})},$$

[17] We shall occasionally write $V_{\lambda\bar{\lambda}}(p)$, $V(\lambda,\bar{\lambda})$, $V(\lambda,\bar{\lambda};p)$, or even $V(z,\bar{z})$ (with $z,\bar{z}$ the worldsheet location where the vertex is inserted) to denote the same object $V(\lambda,\bar{\lambda},p)$.

with $U(\lambda)$ defined below (5.8),

$$U(\lambda)=\sum_{g=0}^{\infty}\sum_{a=0}^{\lfloor g/2\rfloor}\frac{1}{a!(g-2a)!}\Big(\frac{1}{2n^2}\lambda_n\cdot\lambda_n\,\mathbb{S}_{n,n}\Big)^a\Big(\frac{1}{n}\lambda_n\cdot H_n\Big)^{g-2a}\,e^{-ignq\cdot X(z)}$$

and $\bar{U}(\bar{\lambda})$ given by a similar expression with $\bar{\lambda}^i_m$, $\bar{z}$, $\bar{\mathbb{S}}_{m,m}(\bar{z})$ and $\bar{H}^i_m(\bar{z})$ replacing the corresponding holomorphic quantities. Note that the positive integers n,m need not be equal. The more general case of coherent states with more than a single harmonic and when $\lambda_{n>0}\cdot\lambda_{m>0}\neq 0$ can be deduced from (4.98).

The underlying Hilbert space consists of the states we are superimposing in order to construct the closed string coherent states. These can be obtained by series expanding the exponentials which leads to an expression of the form,

$$V(\lambda,\bar{\lambda})\propto\sum_{\{s_1,s_2,\dots\}=0}^{\infty}\mathrm{Pol}[\partial^{\#}X]e^{i(p-\sum_n ns_nq)\cdot X(z)}\times\sum_{\{\bar{s}_1,\bar{s}_2,\dots\}=0}^{\infty}\overline{\mathrm{Pol}}[\bar{\partial}^{\#}X]\,e^{i(p-\sum_m m\bar{s}_mq)\cdot X(\bar{z})},\tag{5.28}$$

with $\mathrm{Pol}[\partial^{\#}X]$ and $\overline{\mathrm{Pol}}[\bar{\partial}^{\#}X]$ being certain polynomials of the arguments which depend on the sets of uncorrelated positive integers $\{s_1,s_2,\dots\}$ and $\{\bar{s}_1,\bar{s}_2,\dots\}$ respectively. Let us write $N=\sum_{n=1}^{\infty}ns_n$ and $\bar{N}=\sum_{n=1}^{\infty}n\bar{s}_n$ for an arbitrary sequence of positive integers $\{s_1,s_2,\dots\}$ and $\{\bar{s}_1,\bar{s}_2,\dots\}$ respectively. We learn that the left- and right-moving momenta associated to a given mass eigenstate in (5.28) satisfy, $k^{\mu}_{\mathrm{L}}-k^{\mu}_{\mathrm{R}}=-\big(N-\bar{N}\big)q^{\mu}$, the associated total momentum being $k^{\mu}=\frac{1}{2}(k^{\mu}_{\mathrm{L}}+k^{\mu}_{\mathrm{R}})$. It is therefore clear that we are super-imposing mass eigenstates with asymmetric left-right momenta and so the manifold in which the coherent states live is in fact *compact*. This is an S^1 compactification in a direction specified by the *null* vector q^{μ}. We can read off the radius of compactification directly from $k_{\mathrm{L}}-k_{\mathrm{R}}$ or equivalently one may compute it by applying the operator, $\oint\big(đz\,\partial X^{\mu}+đ\bar{z}\,\bar{\partial}X^{\mu}\big)$, (that measures the total change in $X^{\mu}(z,\bar{z})$ in going once around the string [114]) to a mass eigenstate and identify the corresponding eigenvalue with $R^{\mu}w$, with w the winding number. This leads to $w=N-\bar{N}$ and $R^{\mu}=-\frac{\alpha'}{2}\,q^{\mu}$ and therefore: $R^2=0$. We learn that the underlying spacetime manifold is compactified in a *light-like* spacetime direction, that is we are considering the DLCQ [209] of string theory. Lightlike compactifications show up in the connection of M(atrix) models to string theories: DLCQ of M-theory has been conjectured [210] to be equivalent to U(N) super Yang-Mills (at *finite* N).[18] Although lightlike compactifications are in general rather non-trivial [212], various properties of a vertex operator in a lightlike compactified spacetime can be extracted rather straightforwardly as we show next.

To become more explicit go to a frame where $q^+=q^i=0$ and $q^-=-\frac{2}{\alpha'}R^-$ which implies the identification (with X^+ non-compact),

$$X^-\sim X^-+2\pi R^-.\tag{5.29}$$

This is shown schematically in Fig. 5.1. Let us go to the rest frame (in the lightcone gauge

[18] See for example, [209, 210, 211, 212] and also [213, 214, 215].

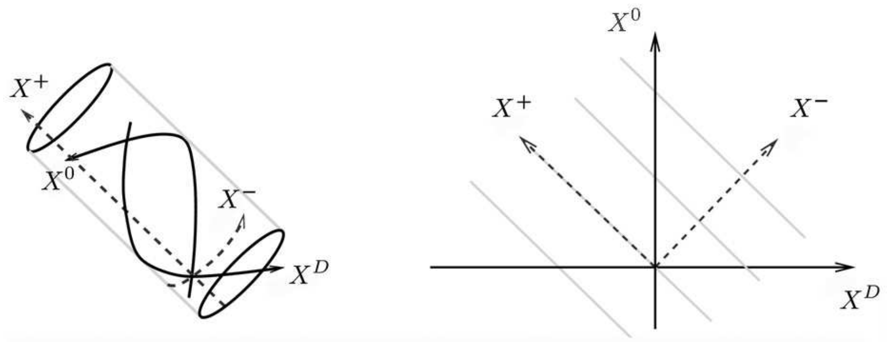

Figure 5.1: Lightlike spacetime compactification. The two-dimensional plane X^0-X^D is shown. In the figure on the right one is to identify the parallel grey lines such that $X^- \sim X^- + 2\pi R^-$. The future lightcone of a given spacetime event is specified by the dashed lines. The aforementioned identification leads to the equivalent $S^1 \times \mathbb{R}$ spacetime cylinder on the left. Signals slower than the speed of light and lightlike signals in the negative X^D direction always propagate up the cylinder in the positive X^+ direction. Lightlike signals in the positive X^D direction are stuck at $X^+ =$ const hypersurfaces. Causality is not violated (the spacetime is marginally causal).

sense) where in addition, $p^i = 0$. With this and the above ansatz for q^μ we can solve the constraints $p^2 = 4/\alpha'$, $p \cdot q = 2/\alpha'$ and $q^2 = 0$ which lead to the following expressions for the *total* momentum of a lightlike compactified mass eigenstate,

$$k^0 = \frac{1}{\sqrt{2}}\Big(\frac{1}{R^-} + m^2 R^-\Big), \qquad k^D = \frac{1}{\sqrt{2}}\Big(\frac{1}{R^-} - m^2 R^-\Big), \qquad \text{and} \qquad k^i = 0, \tag{5.30}$$

with $m^2 = \frac{2}{\alpha'}(N + \bar{N} - 2)$, the mass squared of the particular mass eigenstate in the superposition (5.28). That m^2 does not depend on R^- naively seems to imply that lightlike compactification does not change the mass spectrum of the uncompactified theory. However, the $L_0 - \bar{L}_0$ Virasoro constraint is already satisfied by the above state and so N need *not* equal $\bar{N}$: the Hilbert space, $\mathcal{H}$, contains all the usual states where $N = \bar{N}$ (and hence $w = 0$) but also includes additional states for which $N \neq \bar{N}$ (and $w \neq 0$) without breaking conformal invariance.

The Hilbert space $\mathcal{H}$ admits the orthogonal decomposition, $\mathcal{H} = \bigoplus_{w\in\mathbb{Z}} \mathcal{H}_w$, such that vertices $V_w \in \mathcal{H}_w$ wind around the lightlike direction with winding number w.[19] Given that winding number is conserved (i.e. commutes with the worldsheet Hamiltonian, $[L_0 + \bar{L}_0 - 2, \hat{W}] \cdot V_w \cong 0$), suggests that we may project out the winding states and thus obtain a vertex operator, $V_0 \in \mathcal{H}_0$, with (as we show below, see p. 120) coherent state properties which can be embedded in fully non-compact spacetime.[20]

Given that (5.30) is not of the standard form, $k = n/R$, for the total momentum in a compact dimension of radius R [114] one may wonder whether the corresponding

[19] The decomposition is orthogonal in the sense that $\langle V_m | V_n \rangle = \delta_{m,n}$. The interpretation of $\delta_{m,n}$ of course depends on the chosen normalization of the vertex operators, see e.g. (5.24).

[20] I would like to thank Joe Polchinski for suggesting that the projected states should also have coherent state properties.

wavefunctions are still single-valued[21] – single-valuedness of the wavefunction is the reason as to why one enforces $k = n/R$ in the first place. That they are single valued can be seen as follows. Translations along a compact dimension whose direction is specified by the vector R^μ are generated by, $\exp\big(2\pi i R\cdot\hat{p}\big) : X^\mu(z,\bar{z}) \to X^\mu(z,\bar{z}) + 2\pi R^\mu$, with $\hat{p}^\mu$ the total Noether momentum, $\hat{p}^\mu = \frac{1}{\alpha'}\oint\big(\bar{d}z\,\partial X^\mu(z) - \bar{d}\bar{z}\,\bar{\partial}X^\mu(\bar{z})\big)$. The excitations that appear in $V(\lambda,\bar{\lambda})$ (i.e. the polynomials of $\partial^\# X, \bar{\partial}^\# X$) commute with $\hat{p}$ and so single-valuedness of the vertex operator amounts to showing that

$$\exp\big(2\pi i R\cdot\hat{p}\big)\exp\Big(ik_{\rm L}\cdot X(z) + ik_{\rm R}\cdot X(\bar{z})\Big) = \exp\Big(ik_{\rm L}\cdot X(z) + ik_{\rm R}\cdot X(\bar{z})\Big),$$

for any mass eigenstate in the superposition. Carrying out the operator products on the left hand side (with the contour integrals encircling $z,\bar{z}$ and $k_{\rm L} = p - Nq$, $k_{\rm R} = p - \bar{N}q$) it follows that the above statement holds true for the individual mass eigenstates with lightlike winding and hence is also true for the closed string coherent states. We conclude that $V(\lambda,\bar{\lambda})$ is indeed single-valued under translations around the compact direction.[22]

Curiously, lightlike compactification seems to be invisible at the classical level when,

$$\sum_{n=1}^{\infty}|\lambda_n|^2 = \sum_{n=1}^{\infty}|\bar{\lambda}_n|^2,$$

which is none other than the statement of "classical level matching", $N_e = \bar{N}_e$, because $N_e = \sum_{n=1}^\infty|\lambda_n|^2$ and $\bar{N}_e = \sum_{n=1}^\infty|\bar{\lambda}_n|^2$ are what we would have found had we computed the expectation values of the number operators,

$$N = \sum_{n>0}\alpha_{-n}\cdot\alpha_n \qquad \text{and} \qquad \bar{N} = \sum_{n>0}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n,$$

and we have taken into account (5.24); that is, $\langle N\rangle = N_e$ and $\langle\bar{N}\rangle = \bar{N}_e$. Furthermore, classical level matching is required for consistency (see below). One way of seeing that lightlike compactification is invisible at the classical level is by directly computing the expectation value $\big\langle\hat{p}_{\rm L}^-\big\rangle - \big\langle\hat{p}_{\rm R}^-\big\rangle$ (with respect to the state (5.22)) and showing that it vanishes, as this would imply that $\langle X^-(z,\bar{z}) - x^-\rangle = -i\langle\hat{p}_{\rm L}^-\rangle\ln z - i\langle\hat{p}_{\rm R}^-\rangle\ln\bar{z} + \dots$ is single valued as one traverses a spacelike direction of the worldsheet which is classically only possible if X^- is non-compact, i.e. if $\langle X^-(z,\bar{z}) - x^-\rangle = -i\langle\hat{p}^-\rangle\ln|z|^2 + \dots$.

On account of (5.24), it follows that[23]

$$\big\langle X^-(z,\bar{z}) - x^-\big\rangle = -\,i\big(N_e - 1\big)R^-\ln z - i\big(\bar{N}_e - 1\big)R^-\ln\bar{z}. \tag{5.31}$$

Notice that only zero modes contribute to the position expectation value in the covariant gauge version of the state (5.22) for a reason that was first realized in [125], and which we expand on in the following paragraph. For the X^+ direction we find correspondingly,

$$\big\langle X^+(z,\bar{z}) - x^+\big\rangle = -\frac{\alpha'}{2}\frac{i}{R^-}\ln|z|^2. \tag{5.32}$$

[21] I would like to thank Diego Chialva for raising this question.

[22] This proves that the solution to the single-valuedness requirement that one normally considers, $k = n/R$, must be generalized in lightlike compactified spacetimes.

[23] For completeness I note also that $\frac{\alpha'}{2}\langle\hat{p}_{\rm L}^-\rangle = [N_e - 1]\,R^-$, $\frac{\alpha'}{2}\langle\hat{p}_{\rm R}^-\rangle = \big[\bar{N}_e - 1\big]R^-$ and $\langle\hat{p}_{\rm L}^+\rangle = \langle\hat{p}_{\rm R}^+\rangle = 1/R^-$ with $\hat{p}^\mu = \frac{1}{2}\big(\hat{p}_{\rm L}^\mu + \hat{p}_{\rm R}^\mu\big)$ and $\hat{p}_{\rm L,R}^\pm = \frac{1}{\sqrt{2}}\big(\hat{p}_{\rm L,R}^0 \pm \hat{p}_{\rm L,R}^D\big)$.

Recall that the operator $L_0 - \bar{L}_0$ generates spacelike worldsheet translations,

$$\big[L_0 - \bar{L}_0, X^\mu(z,\bar{z})\big] = \big(z\partial - \bar{z}\bar{\partial}\big)X^\mu(z,\bar{z}), \tag{5.33}$$

and that one of the physical state conditions is that states be invariant under such translations,

$$\exp[-i\epsilon(L_0 - \bar{L}_0)] \cdot V \cong V;$$

infinitesimally, $|\epsilon| \ll 1$, we have $(L_0 - \bar{L}_0) \cdot V \cong 0$. Computing the expectation value, $\big\langle\big[L_0 - \bar{L}_0, X^\mu(z,\bar{z})\big]\big\rangle = \big(z\partial - \bar{z}\bar{\partial}\big)\big\langle X^\mu(z,\bar{z})\big\rangle$, with respect to a physical state V it then follows that

$$\big(z\partial - \bar{z}\bar{\partial}\big)\big\langle X^\mu(z,\bar{z})\big\rangle = 0$$

must be satisfied by any such state. This in turn explains why there are only zero mode contributions in (5.31) and (5.32) (non-zero mode contributions would violate this condition), and secondly enforces classical level matching,

$$N_e = \bar{N}_e, \tag{5.34}$$

so as to ensure that the operator $\big(z\partial - \bar{z}\bar{\partial}\big)$ annihilates (5.31). Given that $V(\lambda,\bar{\lambda})$ has an effective mass given by $\langle m^2\rangle = \frac{2}{\alpha'}(N_e + \bar{N}_e - 2)$ it follows that the full momenta are given by, $\langle\hat{p}^-\rangle = \frac{1}{2}\langle m^2\rangle R^-$, $\langle\hat{p}^+\rangle = 1/R^-$, enabling one to write:

$$\big\langle X^\pm(z,\bar{z}) - x^\pm\big\rangle = -i\frac{\alpha'}{2}\langle p^\pm\rangle \ln|z|^2. \tag{5.35}$$

This implies that indeed as claimed above lightlike compactification seems to be invisible at the classical level. However, this result is not unique to lightlike compactifications. In particular, notice that the reasoning following (5.33) also applies in the case of spacelike compactifications, $x^i \sim x^i + 2\pi R$. In the case of spacelike compactifications, in particular, one finds the consistency requirement:

$$\langle p_L^i\rangle = \langle p_R^i\rangle.$$

Curiously, this seems to imply that toroidal compactification in general is invisible in such expectation values.

That only zero modes contribute to the expectation values (5.31) and (5.32) of course does not mean that the coherent state (5.22) does not have a classical interpretation, but rather implies that the condition for classicality, $\big\langle X^\mu(z,\bar{z})\big\rangle = X^\mu_{\rm cl}(z,\bar{z})$, with $\partial\bar{\partial}X^\mu_{\rm cl}(z,\bar{z}) = 0$ is not compatible with the symmetries of closed string theory when the gauge choice (covariant gauge in this example) does not fix the invariance under spacelike worldsheet translations [125]. Note that *any* covariant vertex operator must satisfy $\big(z\partial - \bar{z}\bar{\partial}\big)\big\langle X^\mu(z,\bar{z})\big\rangle = 0$, whether or not it has a classical interpretation. To get round this, one may fix the invariance of the state under such translations (as done in [125]) but this is somewhat messy and not practical for general states. Alternatively, one may pick a gauge that explicitly breaks the invariance under such translations from the outset, e.g. static gauge. To see this notice that[24] in static gauge, e.g. $X^0 = \alpha' p^0 \tau$, $X^D = R\sigma$ and $X^D \sim X^D + 2\pi R$, from the outset where it is manifest that spacelike worldsheet translation invariance, $\sigma \to \sigma + s$, is broken by the gauge choice. Here $\langle X^i(\sigma,\tau)\rangle = X^i_{\rm cl}(\sigma,\tau)$ can be satisfied non-trivially because in static gauge states of the form $e^{\lambda^i_n \alpha^i_{-n}} e^{\bar\lambda^i_n \tilde\alpha^i_{-n}}|0,0;p^i,p^D_{\rm L},p^D_{\rm R}\rangle$ are physical without requiring the existence of a lightlike compactification. Unfortunately, it is not known how to quantize the string in static gauge unless (starting from the Nambu-Goto action) one restricts to *small* fluctuations transverse to X^0, X^D with R large, in which case the leading term in the action becomes quadratic in the fields X^i and the path integral can be carried out perturbatively in $1/R$. We would like to discuss the construction of quantum states which correspond to arbitrary classical solutions (e.g. solutions with cusps where the above expansion would presumably not suffice) and so this is not the approach we shall take here. A better solution is possibly to instead replace the definition of classicality, $\big\langle X^\mu(z,\bar z)\big\rangle = X^\mu_{\rm cl}(z,\bar z)$, with,[25]

$$\big\langle :X^\mu(\sigma',\tau)X^\nu(\sigma,\tau):\big\rangle = \int_0^{2\pi} \bar{d}s\, X^\mu_{\rm cl}(\sigma'-s,\tau)X^\nu_{\rm cl}(\sigma-s,\tau), \tag{5.36}$$

modulo zero mode contributions (recall that $z = e^{-i(\sigma+i\tau)}$, $\bar z = e^{i(\sigma-i\tau)}$). Rather than fixing the invariance under σ-translations on the quantum side (as done in [125]) we average over σ-translations on the classical side.

The definition for classicality (5.36) is appropriate for states in any gauge (e.g. covariant or lightcone gauge) that does not fix the invariance under spacelike worldsheet translations and we will be making use of it when we present the construction of coherent states in non-compact spacetimes. For the states (5.22) however there is yet another solution which is even simpler – the solution is to go to lightcone gauge, because in lightcone gauge the presence of lightlike compactification breaks the invariance under such translations thus making the classical-quantum map, $\big\langle X^\mu(z,\bar z)\big\rangle = X^\mu_{\rm cl}(z,\bar z)$, possible.

Before we elaborate on the lightcone gauge construction, we would like to point out that one should be careful in drawing conclusions from statements of the form (5.35) when the expectation value is evaluated in covariant gauge. One can argue that it is not permissible to compute the expectation value of (5.33) given that $X^\mu(z,\bar z)$ is not a well defined conformal operator – although it is classically a worldsheet scalar, it can give rise to a conformal anomaly in the quantum theory.[26] In lightcone gauge there is no such subtlety because the constraints associated to quantum conformal symmetry are satisfied automatically by the gauge choice.

Above we mentioned that lightlike compactification breaks the invariance under worldsheet spacelike translations. To understand why this is the case recall that [166] in lightcone gauge the constraints $(\partial X)^2 = (\bar\partial X)^2 = 0$ reduce to the operator equations

[24] The following was suggested by Ashoke Sen and I would like to thank him for extensive very helpful discussions of these issues.

[25] This was suggested by Joe Polchinski and I am very grateful to him for this suggestion.

[26] For example, it is the presence of this anomaly that makes the object $e^{ip\cdot X(z,\bar z)}$ anomalous unless $p^2 = 4/\alpha'$.

$\alpha_0^- = \sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\big(L_0^\perp - 1\big)$, and $\tilde{\alpha}_0^- = \sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\big(\bar{L}_0^\perp - 1\big)$, with $L_0^\perp$, $\bar{L}_0^\perp$ the transverse Virasoro generators.[27] Therefore, level matching in lightcone gauge corresponds to the statement,

$$(\alpha_0^- - \tilde{\alpha}_0^-)|V\rangle_{\rm lc} = \sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}(L_0^\perp - \bar{L}_0^\perp)|V\rangle_{\rm lc}, \tag{5.37}$$

from which it follows that states compactified in a lightlike spacetime direction, for which $\alpha_0^- \neq \tilde{\alpha}_0^-$ (recall that α_0^- and $\tilde{\alpha}_0^-$ are the left- and right-moving momentum operators, $\sqrt{\frac{\alpha'}{2}}p_{\rm L}^-$ and $\sqrt{\frac{\alpha'}{2}}p_{\rm R}^-$ repsectively), are not invariant under spacelike worldsheet shifts, $(L_0^\perp - \bar{L}_0^\perp)|V\rangle_{\rm lc} \neq 0$. Therefore, the above argument which led to $\big(z\partial - \bar{z}\bar{\partial}\big)\big\langle X^i(z,\bar{z})\big\rangle = 0$ does not apply in lightlike compactified spacetimes, $X^- \sim X^- + 2\pi R^-$, thus implying that the classical-quantum map, $\big\langle X^\mu(z,\bar{z})\big\rangle = X^\mu_{\rm cl}(z,\bar{z})$, may be realized. We show next that indeed the lightcone gauge realization of the coherent states (5.22) can be mapped in this way to arbitrary general classical solutions.

According to the discussions in Sec. 4.4 the lightcone gauge version, $|V(\lambda,\bar{\lambda})\rangle_{\rm lc}$, of the vertex (5.22) is obtained by the mapping, $A^i_{-n} \to \alpha^i_{-n}$ and $g_c\, e^{ip\cdot X(z,\bar{z})} \to |0,0;p^+,p^i\rangle$, so that

$$\big|V(\lambda,\bar{\lambda})\big\rangle_{\rm lc} = \frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}C_{\lambda\bar{\lambda}}\exp\Big(\sum_{n=1}^\infty \frac{1}{n}\lambda_n\cdot\alpha_{-n}\Big)\exp\Big(\sum_{m=1}^\infty\frac{1}{m}\bar{\lambda}_m\cdot\tilde{\alpha}_{-m}\Big)\,|0,0;p^+,p^i\rangle. \tag{5.38}$$

This is similar to the open string case (5.10); it is an eigenstate of the annihilation operators, $\alpha^i_{n>0},\tilde{\alpha}^i_{n>0}$, with eigenvalues $\lambda^i_n,\bar{\lambda}^i_n$, and of the momenta $\hat{p}^+,\hat{p}^i$ with eigenvalues p^+,p^i, respectively. The vacuum is normalized as in (4.54),

$$\langle 0,0;p'|0,0;p\rangle = 2p^+(2\pi)\delta(p^{+\prime}-p^+)(2\pi)^{d-2}\delta^{d-2}(\mathbf{p}'-\mathbf{p}).$$

The position expectation value in the transverse directions is therefore given by,

$$\big\langle X^i(z,\bar{z}) - \hat{x}^i\big\rangle_{\rm lc} = \big(X^i(z,\bar{z}) - x^i\big)_{\rm cl},$$

with

$$\big(X^i(z,\bar{z}) - x^i\big)_{\rm cl} = -i\frac{\alpha'}{2}p^i\ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\big(\lambda^i_n\,z^{-n} + \bar{\lambda}^i_n\,\bar{z}^{-n}\big). \tag{5.39}$$

Furthermore, from the operator equations, $\alpha_n^- = \sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\big(L_n^\perp - \delta_{n,0}\big)$, we find that in the longitudinal directions,[28]

$$\big\langle X^-(z,\bar{z}) - \hat{x}^-\big\rangle_{\rm lc} = \big(X^-(z,\bar{z}) - x^-\big)_{\rm cl},$$

[27] Recall that the transverse Virasoro generators read, $L_0^\perp = \frac{\alpha'}{4}\hat{\mathbf{p}}_{\rm L}^2 + N^\perp$, $\bar{L}_0^\perp = \frac{\alpha'}{4}\hat{\mathbf{p}}_{\rm R}^2 + \bar{N}^\perp$, and $N^\perp = \sum_{n>0}\alpha^i_{-n}\alpha^i_n$, $\bar{N}^\perp = \sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n$.

[28] Recall that

$$X^-(z,\bar{z}) = x^- - i\frac{\alpha'}{2}\hat{p}_{\rm L}^-\ln z - i\frac{\alpha'}{2}p_{\rm R}^-\ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\big(\alpha_n^- z^{-n} + \tilde{\alpha}_n^-\bar{z}^{-n}\big)$$

and $L_n^\perp = \frac{1}{2}\sum_{r\in\mathbb{Z}} :\alpha^i_{n-r}\alpha^i_r:$.

with

$$
\begin{aligned}
\big(X^-(z,\bar z)-x^-\big)_{\rm cl} = -i\frac{1}{p^+}\Big(\frac{\alpha'}{4}\mathbf{p}^2+N_e-1\Big)\ln z - i\frac{1}{p^+}\Big(\frac{\alpha'}{4}\mathbf{p}^2+\bar N_e-1\Big)\ln \bar z \\
+\,i\sum_{n\neq 0}\frac{1}{n}\sum_{r\in\mathbb{Z}}\frac{1}{2p^+}\Big(\lambda_{n-r}\cdot\lambda_r z^{-n}+\bar\lambda_{n-r}\cdot\bar\lambda_r\bar z^{-n}\Big),
\end{aligned}
\tag{5.40}
$$

with the definitions $\lambda_0^i\equiv\sqrt{\frac{\alpha'}{2}}p^i$, $\bar\lambda_0^i\equiv\sqrt{\frac{\alpha'}{2}}p^i$, and $p^ip^i=\mathbf{p}^2$, and as discussed above we am to enforce classical level matching, $N_e=\bar N_e$. For completeness note also that

$$
X^+(z,\bar z) = -i\frac{\alpha'}{2}p^+\ln|z|^2.
$$

Notice that in the rest frame, $p^i=0$, the zero mode contribution in (5.40) is identical to that found in the covariant gauge (5.31) when $p^+=1/R^-$. The quantities (5.39) and (5.40) are none other than the general solutions to the equations of motion, $\partial\bar\partial X^\mu=0$, in lightcone gauge [216]. We therefore conclude that indeed the classical-quantum map, $\langle X^\mu(z,\bar z)\rangle_{\rm lc}=X^\mu_{\rm cl}(z,\bar z)$, can be realized in a spacetime with lightlike compactification when this map is carried out in lightcone gauge. This is in accordance with the above considerations. Note that this is specific to *lightlike*-compactified spacetimes and does not apply in spacelike compactifications, because this conclusion relied on the left-hand-side of (5.37) being non-vanishing.

Finally, before we construct closed string coherent states in fully non-compact spacetime let us show that the angular momentum of the covariant gauge, lightcone gauge and classical descriptions are all identical, as we did in the open string case (5.16) above. For the closed string,

$$
\begin{aligned}
J^{\mu\nu} &= \frac{2}{\alpha'}\Big(\oint d\!\!\!{}^-z X^{[\mu}\partial X^{\nu]}-\oint d\!\!\!{}^-\bar z X^{[\mu}\bar\partial X^{\nu]}\Big),\\
&= L^{\mu\nu}+S^{\mu\nu},
\end{aligned}
\tag{5.41}
$$

with the zero mode contribution denoted by $L_{\mu\nu}$ (given in a footnote on p. 105) and $S^{\mu\nu}=S^{\mu\nu}(\alpha)+S^{\mu\nu}(\tilde\alpha)$ with $S^{\mu\nu}(\alpha)=-i\sum_{\ell=1}^\infty\big(\alpha^\mu_{-\ell}\alpha^\nu_\ell-\alpha^\nu_{-\ell}\alpha^\mu_\ell\big)$ and a similar expression for the antiholomorphic sector, $S^{\mu\nu}(\tilde\alpha)$. We shall concentrate on the non-zero mode part: $S^{\mu\nu}$. The derivation is almost identical to the open string case and so we do not repeat it here, the only difference being that the open string normalization of the momentum is half that of the closed string: $\frac12 p_c=p_o$ (although we don't bother to keep the subscripts when the context is clear). We find that for the transverse directions,

$$
\langle S^{ij}\rangle_{\rm cov}=\langle S^{ij}\rangle_{\rm lc}=\sum_{n>0}\frac{2}{n}{\rm Im}\big(\lambda_n^{*i}\lambda_n^j+\bar\lambda_n^{*i}\bar\lambda_n^j\big)=S^{ij}_{\rm cl},
\tag{5.42}
$$

and for the longitudinal components,

$$
\langle S^{-i}\rangle_{\rm cov}=\langle S^{-i}\rangle_{\rm lc}=\sqrt{\frac{2}{\alpha'}}\sum_{m>0}\sum_{\ell\in\mathbb{Z}}\frac{1}{mp^+}{\rm Im}\big(\lambda^*_{m-\ell}\cdot\lambda^*_\ell\,\lambda^i_m+\bar\lambda^*_{m-\ell}\cdot\bar\lambda^*_\ell\,\bar\lambda^i_m\big)=S^{-i}_{\rm cl},
\tag{5.43}
$$

with in addition all components involving the + direction equal to zero. This correspondence provides further evidence for the conjecture that the covariant gauge vertex operator

(5.22) and the lightcone gauge state (5.38) describe the same physics (share identical correlation functions) and are different manifestations of the same state which classically have a lightcone gauge description given by (5.39) and (5.40).

Before delving into the coherent state construction in non-compact spacetimes it is worth noting that the requirement of a lightlike compactified background in the naive construction of the current section is the cost of working in a standard gauge, namely lightcone or covariant gauge where all the string technology for amplitude computations is well developed. It is also possible to construct closed string coherent states in a modified lightcone gauge [217], where the requirement of a lightlike compactified background, $X^- \sim X^- + 2\pi R^-$, gets shifted to the requirement of a spacelike compactified background, $X^D \sim X^D + 2\pi R^D$. Here, instead of making the lightcone gauge identification $X^+(z,\bar{z}) = -i\frac{\alpha'}{2}p^+ \ln|z|^2$, one chooses $X^+(z,\bar{z}) = -i\frac{\alpha'}{2}p^+_{\rm L}\ln z - i\frac{\alpha'}{2}p^+_{\rm R}\ln\bar{z}$, which in turn solves the constraints in a manner similar to the lightcone gauge case. Here however, with the additional freedom of choosing $p^+_{\rm L}$ and $p^+_{\rm R}$ independently it becomes possible to rotate the spacetime coordinate system in such a way that the resulting coherent states propagate in a spacelike rather than a lightlike compactified spacetime.[29]

Coherent States in Non-Compact Backgrounds.

We next construct coherent states in fully non-compact spacetimes. We showed above that the coherent state (5.22),

$$V(\lambda,\bar{\lambda},p) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}C_{\lambda\bar{\lambda}}\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\lambda_n\cdot A_{-n}\Big)\exp\Big(\sum_{m=1}^{\infty}\frac{1}{m}\bar{\lambda}_m\cdot\bar{A}_{-m}\Big)\,e^{ip\cdot X(z,\bar{z})}, \tag{5.44}$$

satisfies all the coherent state defining properties but only when the underlying spacetime manifold is compactified in a lightlike direction of spacetime. Below (5.30) we concluded that in addition to the usual states in the underlying Hilbert space which satisfy $N=\bar{N}$, there were additional states for which $N\neq\bar{N}$ and these correspond to states with lightlike winding. This suggests that starting from (5.44) we may truncate the underlying Hilbert space and project out all states with $N\neq\bar{N}$. The resulting states will be manifestly level-matched and will propagate consistently in fully non-compact (but also compact) spacetimes.

To project out all states with $N\neq\bar{N}$, thus leaving only $N=\bar{N}$ states in the underlying spectrum, we define a projection operator,

$$G_w = \int_0^{2\pi} \text{đ}s\, e^{is(\hat{W}-w)}, \qquad \text{with} \qquad \hat{W}\equiv\frac{\alpha'}{2}(\hat{p}^+_{\rm L}\hat{p}^-_{\rm L}-\hat{p}^+_{\rm R}\hat{p}^-_{\rm R}), \tag{5.45}$$

with $\hat{p}^\mu_{\rm L} = \frac{2}{\alpha'}\oint \text{đ}z\partial X^\mu$, $\hat{p}^\mu_{\rm R} = -\frac{2}{\alpha'}\oint \text{đ}\bar{z}\bar{\partial}X^\mu$, and $\hat{W}$ the lightlike winding number operator.

[29] I would like to thank Kostas Skenderis and Marika Taylor for bringing [217] to my attention.

The Virasoro constraints associated to level matching read,

$$
\begin{aligned}
L_0-\bar{L}_0&=\Big(\frac{\alpha'}{4}\hat{p}_{\rm L}^2+N\Big)-\Big(\frac{\alpha'}{4}\hat{p}_{\rm R}^2+\bar{N}\Big)\\
&=-\frac{\alpha'}{2}(\hat{p}_{\rm L}^+\hat{p}_{\rm L}^--\hat{p}_{\rm R}^+\hat{p}_{\rm R}^-)+\frac{\alpha'}{4}\big(\hat{\mathbf{p}}_{\rm L}^2-\hat{\mathbf{p}}_{\rm R}^2\big)+N-\bar{N}\\
&\equiv-\hat{W}+\frac{\alpha'}{4}\big(\hat{\mathbf{p}}_{\rm L}^2-\hat{\mathbf{p}}_{\rm R}^2\big)+N-\bar{N}\\
&=0,
\end{aligned}
\tag{5.46}
$$

from which the origin of the projector, G_w, becomes clear: when G_w is applied to arbitrary vertices it projects out all states in the underlying Hilbert space except for those with lightlike winding number w. In the case of interest when there are no transverse compact directions, $\mathbf{p}_{\rm L}^2=\mathbf{p}_{\rm R}^2=\mathbf{p}^2$, we may equivalently write the covariant expression

$$
\hat{W}=-\alpha' p\cdot\hat{w},
$$

where $p^\mu=\frac{1}{2}(p_{\rm L}^\mu+p_{\rm R}^\mu)$ is the momentum of the vacuum, $p^2=4/\alpha'$, $p\cdot q=2/\alpha'$, and $\hat{w}^\mu=\frac{1}{2}\big(\hat{p}_{\rm L}^\mu-\hat{p}_{\rm R}^\mu\big)$ is the winding vector (see Sec. 2.6).[30] Notice for example that for some generic vertex operator,

$$
\begin{aligned}
&\hat{W}\cdot P(\partial^\# X,\bar{\partial}^\# X)e^{i(p-Nq)\cdot X(z)}e^{i(p-\bar{N}q)\cdot X(\bar{z})}\\
&\qquad\qquad=(N-\bar{N})P(\partial^\# X,\bar{\partial}^\# X)e^{i(p-Nq)\cdot X(z)}e^{i(p-\bar{N}q)\cdot X(\bar{z})},
\end{aligned}
\tag{5.47}
$$

with $P(\partial^\# X,\bar{\partial}^\# X)$ the oscillator contribution that commutes with $\hat{W}$. Then, covariant vertex operators without lightlike winding are obtained by setting $w=0$ in (5.45) and are given by,

$$
V_0(\lambda,\bar{\lambda})\cong G_0\cdot V(\lambda,\bar{\lambda}),
\tag{5.48}
$$

the dot denoting operator product contractions, or normal ordering. Taking $V(\lambda,\bar{\lambda})$ to be the coherent state (5.44) we are to commute G_0 through the DDF operators, the relevant term giving $e^{is\hat{W}}e^{\sum_{n=1}^\infty\frac{1}{n}\lambda_n\cdot A_{-n}}=e^{\sum_{n=1}^\infty\frac{1}{n}e^{ins}\lambda_n\cdot A_{-n}}e^{is\hat{W}}$ with a similar relation for the anti-holomorpic sector, with e^{-ins} replacing e^{ins}. This follows from the Baker-Campbell-Hausdorff formula, the commutators $\big[\hat{W},A_{-n}^i\big]=nA_{-n}^i$, $\big[\hat{W},\bar{A}_{-n}^i\big]=-n\bar{A}_{-n}^i$, and the elementary Schur polynomial representation (J.1a) with

$$
a_s=\frac{1}{s!}\sum_{n=1}^\infty(ins)^s\frac{1}{n}\lambda_n\cdot A_{-n}.
$$

The resulting vertex operators are then the candidate quantum states to represent arbitrary classical loops in non-compact Minkowski spacetime:

$$
\begin{aligned}
V_0(\lambda,\bar{\lambda};p)=\frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\mathcal{C}_{\lambda\bar{\lambda}}\int_0^{2\pi}&\text{đ}s\exp\Big\{\sum_{n=1}^\infty\frac{1}{n}e^{ins}\lambda_n\cdot A_{-n}\Big\}\\
&\times\exp\Big\{\sum_{m=1}^\infty\frac{1}{m}e^{-ims}\bar{\lambda}_m\cdot\bar{A}_{-m}\Big\}e^{ip\cdot X(z,\bar{z})}
\end{aligned}
\tag{5.49}
$$

[30]The winding vector $\hat{w}$ is not to be confused with the *lightlike* winding operator $\hat{W}$.

with

$$\mathcal{C}_{\lambda\bar{\lambda}} = \Big[\int_0^{2\pi} \text{đ}s \exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}|\lambda_n|^2 e^{ins} + \frac{1}{n}|\bar{\lambda}_n|^2 e^{-ins}\Big)\Big]^{-1/2}$$

a normalization constant. The normalization as usual fixed by the 'one string in volume $\mathcal{V}_{d-1}$' condition, which leads to a unitary S-matrix. As discussed in Sec. 4.3, this is equivalent to fixing the most singular term in the operator product expansion as in (5.23), which in our conventions, as discussed there, is equivalent to requiring that the state have unit norm,

$$\langle V_0(\lambda,\bar{\lambda};p)|V_0(\lambda,\bar{\lambda};p)\rangle = 1, \qquad \text{with} \qquad |0,0;p\rangle \cong g_c\, e^{ip\cdot X}.$$

Note that the out state $V_0(\lambda,\bar{\lambda})^\dagger$ is given by $V_0(\lambda,\bar{\lambda})$ with $\{\lambda_n^*\}$, A_n and $-p$ replacing $\{\lambda_n\}$, A_{-n} and p respectively (corresponding to Hermitian conjugation in Minkowski signature worldsheet), and similarly for the anti-holomorphic sector.

We first check that (5.49) satisfies the defining properties (a-c) of a string coherent state as laid out in the beginning of this section. The properties (a,c) are trivially satisfied because the state is still specified by a set of continuous labels and the projection operator (5.45) does not alter the states in the underlying Hilbert space, $\mathcal{H}$. The Hilbert space is instead truncated[31] and so, given that any linear combination of physical states is also a physical state, the vertex (5.49) must be physical. To check that (b) is satisfied, i.e. that a completeness relation exists for the projected states, we start from the completeness relation associated to the unprojected states, the existence of which was established on p. 110,

$$\mathbb{1} = \int d\mu(p) d\lambda d\bar{\lambda} \big|V(\lambda,\bar{\lambda};p)\big\rangle\big\langle V(\lambda,\bar{\lambda};p)\big|.$$

Apply a projection operator, G_w, on either side of this expression to find that:

$$\mathbb{1}_w = \int d\mu(p) d\lambda d\bar{\lambda} \big|V_w(\lambda,\bar{\lambda};p)\big\rangle\big\langle V_w(\lambda,\bar{\lambda};p)\big|, \tag{5.50}$$

where we have defined, $G_w \equiv \mathbb{1}_w$, as G_w is none other than the unit operator, $\mathbb{1}_w$, with respect to the truncated Hilbert space, $\mathcal{H}_w$, which consists of all states with lightlike winding number w. To show this note that $|V_w\rangle = G_w|V\rangle$ and $G_w^2 = G_w$ (recall that G_w is Hermitian). From the latter two expressions it follows that

$$G_w|V_w\rangle = |V_w\rangle,$$

and so indeed $G_w = \mathbb{1}_w$. Thus, there exists a completeness relation for the projected states also, as required from the definition of a coherent state.

Note also that if we sum over w in (5.50) it follows that,

$$\mathbb{1} = \int d\mu(p) d\lambda d\bar{\lambda} \sum_{w=-\infty}^{\infty} \big|V_w(\lambda,\bar{\lambda};p)\big\rangle\big\langle V_w(\lambda,\bar{\lambda};p)\big|,$$

[31] The picture I have in mind here is, $\mathcal{H} = \bigoplus_{w\in\mathbb{Z}}\mathcal{H}_w$, with G_w such that, $G_w : \mathcal{H} \to \mathcal{H}_w$, and $G_w : \mathcal{H}_w \to \mathcal{H}_w$.

with $\mathbb{1}$ the unit operator with respect to the larger Hilbert space $\mathcal{H}$, so that $\mathbb{1}|V\rangle = |V\rangle$, and this serves as a consistency check.

The Hilbert space of interest here is $\mathcal{H}_0$ which is the *coherent state Hilbert space* associated to non-compact spacetimes. From the above considerations we conclude that (5.50) is indeed a resolution of unity with respect to $\mathcal{H}_w$, and have thus shown that the string coherent state defining properties are satisfied by the states $V_w(\lambda,\bar{\lambda};p)$. Next notice that because winding number is conserved, $\big[\hat{H},\hat{W}\big]\cdot V_w(\lambda,\bar{\lambda};p)\cong 0$, with $\hat{H}=L_0+\bar{L}_0-2$ the worldsheet Hamiltonian, the Hilbert space decomposition, $\mathcal{H}=\bigoplus_{w\in\mathbb{Z}}\mathcal{H}_w$, is indeed orthogonal; when all quantum numbers other than winding number are equal, $\langle V_m|V_n\rangle = \delta_{m,n}$ for vertices, $V_m\in\mathcal{H}_m$. We conclude that vertex operators,

$$V_0(\lambda,\bar{\lambda};p)\in\mathcal{H}_0,$$

can propagate in fully non-compact spacetimes, and have shown in particular that the vertex operator (5.49) is a closed string coherent state that can be consistently embedded in non-compact flat Minkowski spacetime.

In a scattering amplitude that involves say n coherent states V_0 and any number of non-coherent states, one can drop the G_0's in $n-1$ of these vertices. To see this let us look at an example, say the elastic massive string forward scattering amplitude from an arbitrary closed string coherent state, V_0,

$$\begin{aligned}\big\langle V_0^\dagger U^\dagger U V_0\big\rangle &= \big\langle (G_0V)^\dagger U^\dagger U(G_0V)\big\rangle\\ &= \big\langle V^\dagger U^\dagger U G_0^2 V\big\rangle\\ &= \big\langle V^\dagger U^\dagger U V_0\big\rangle,\end{aligned}\tag{5.51}$$

with, $U=P(\partial^{\#}X,\bar{\partial}^{\#}X)e^{ik\cdot X(z,\bar{z})}$, a vertex operator without lightlike winding, and we have used the fact that G_0 is Hermitian, commutes with U and squares to itself.

The inner product associated to the projected states can be derived from the properties,

$$A_n^i\cdot V_0\cong\lambda_n^iV_n,\qquad \bar{A}_n^i\cdot V_0\cong\bar{\lambda}_n^iV_n\qquad (n>0)\qquad\text{and}\qquad \langle V_n^\dagger V_m\rangle=\delta_{n,m},$$

which follow from the DDF operator commutation relations. From these it follows that the constructed coherent states are as usual over-complete,

$$\big\langle V_0(\lambda,\bar{\lambda};p')|V_0(\xi,\bar{\xi};p)\big\rangle = \delta_{p',p}\mathcal{C}_{\lambda\bar{\lambda}}\mathcal{C}_{\xi,\bar{\xi}}\int_0^{2\pi}\!\!d\!\!\!{}^-\! s\,\exp\Big(\sum_{n>0}\frac{1}{n}\lambda_n^*\cdot\xi_n\,e^{ins}+\frac{1}{n}\bar{\lambda}_n^*\cdot\bar{\xi}_n\,e^{-ins}\Big),$$

and this expression reduces to unity when $(\lambda,\bar{\lambda})=(\xi,\bar{\xi})$, and we have again made use of the fact that $G_0^2=G_0$. Note that $\delta_{p',p}$ is a Kronecker delta which reduces to unity when $p^{+\prime}=p^+$ and $\mathbf{p}'=\mathbf{p}$, with p and p' the momenta of the vacua associated to the in and out states, as above.

The normal ordered version of $V_0(\lambda,\bar{\lambda})$ analogous to (5.27) can be derived from (5.27) by computing the operator product, $V_0(\lambda,\bar{\lambda})\cong G_0\cdot V(\lambda,\bar{\lambda})$. In the particular case that $\lambda_{n>0}\cdot\lambda_{m>0}=0$, one finds an expression identical to (5.49) with $H_n^i(z)e^{-inq\cdot X(z)}$, $\bar{H}_n^i(\bar{z})e^{-inq\cdot X(\bar{z})}$ replacing A_{-n}^i, $\bar{A}_{-n}^i$ respectively, with an overall integral over s,

$$\begin{aligned}V_0(\lambda,\bar{\lambda};p)=\frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\mathcal{C}_{\lambda\bar{\lambda}}\int_0^{2\pi}\!\!d\!\!\bar{}\,s\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}e^{ins}\lambda_n\cdot H_n(z)e^{-inq\cdot X(z)}\Big)\\ \times\exp\Big(\sum_{m=1}^{\infty}\frac{1}{m}e^{-ims}\bar{\lambda}_m\cdot\bar{H}_m(\bar{z})e^{-imq\cdot X(\bar{z})}\Big)\,e^{ip\cdot X(z,\bar{z})}.\end{aligned}\tag{5.52}$$

This follows from the general result (4.98) and (5.49). Notice that this is still an eigenstate of $\hat{p}^+,\hat{p}^i$ if we make the choice $q^+,q^i=0$ with q^- non-zero, as was the unprojected state $V(\lambda,\bar{\lambda})$. Recall also that in the rest frame in addition to taking $p^i=0$ one is to take $H_n(z)\to P_n(z)$ as discussed in Sec. 4.4.

We now add a few comments on the extension of the result (5.52) to coherent states in spacetimes with spacelike compactifications. If we make the choice $q^+,q^i=0$ with q^- non-zero, and toroidally compactify d_c of the $d-2$ transverse (spacelike) dimensions with radii of compactification R^i,

$$x^i\sim x^i+2\pi R^i,$$

(with i as usual denoting directions transverse to $\pm$), then the vertex operator (5.52) generalizes trivially, and one is to simply replace $e^{ip\cdot X(z,\bar{z})}\to e^{ip_{\rm L}\cdot X(z)}e^{ip_{\rm R}\cdot X(\bar{z})}$, with $p_{\rm L}$ and $p_{\rm R}$ the left- and right-moving momenta, that are in turn related to the total momentum, p, by $p=\frac{1}{2}(p_{\rm L}+p_{\rm R})$, see Sec. 2.6.

It is possibly useful at this point to give an example. The simplest coherent state vertex operator where only $\lambda^i\equiv\lambda_1^i$ is non-vanishing and $\lambda\cdot\lambda=\bar{\lambda}\cdot\bar{\lambda}=\lambda^i_{n\neq\pm1}=\bar{\lambda}^i_{n\neq\pm1}=0$, follows from (5.52) and reads,

$$\begin{aligned}V_0(z,\bar{z})=\frac{g_c}{\sqrt{2p^+\mathcal{V}_{d-1}}}\mathcal{C}_{\lambda\bar{\lambda}}\int_0^{2\pi}\!\!d\!\!\bar{}\,s\,\exp\Big(ie^{is}\zeta\cdot\partial X\,e^{-iq\cdot X(z)}\Big)\\ \times\exp\Big(ie^{-is}\bar{\zeta}\cdot\bar{\partial}X\,e^{-iq\cdot X(\bar{z})}\Big)\,e^{ip\cdot X(z,\bar{z})},\end{aligned}\tag{5.53}$$

with

$$\zeta_\mu\equiv\lambda^i(\delta^i_\mu-p^iq_\mu),\qquad\bar{\zeta}^\mu\equiv\bar{\lambda}^i(\delta^i_\mu-p^iq_\mu),\qquad\text{and}\qquad|\zeta|^2=|\lambda|^2,\qquad|\bar{\zeta}|^2=|\bar{\lambda}|^2.$$

It is manifest that the s-integral serves to set the total number of holomorphic and anti-holomorphic worldsheet derivatives to be equal in every term of the series expansions of the exponentials.

Proceeding in a similar manner to the open string case, the lightcone gauge states corresponding to (5.49) are given by,

$$\begin{aligned}\big|V_0(\lambda,\bar{\lambda})\big\rangle_{\rm lc}=\frac{1}{\sqrt{2p^+\mathcal{V}_{d-1}}}\mathcal{C}_{\lambda\bar{\lambda}}\int_0^{2\pi}\!\!d\!\!\bar{}\,s\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}e^{ins}\lambda_n\cdot\alpha_{-n}\Big)\\ \times\exp\Big(\sum_{m=1}^{\infty}\frac{1}{m}e^{-ims}\bar{\lambda}_m\cdot\tilde{\alpha}_{-m}\Big)\,|0,0;p^+,p^i\rangle.\end{aligned}\tag{5.54}$$

We next consider the lightcone gauge classical solutions, $X^\mu_{\rm cl}(z,\bar z)$, corresponding to this state. Having projected out the lightlike winding states, worldsheet translation invariance is restored (in both lightcone and covariant gauges) and according to the above discussion the condition for classicality $\langle X^\mu(z,\bar z)\rangle = X^\mu_{\rm cl}(z,\bar z)$ is to be replaced by (5.36), rewritten here for convenience in the $(z,\bar z)=(e^{-i(\sigma+i\tau)},e^{i(\sigma-i\tau)})$ coordinate system with the zero mode contributions explicitly subtracted,

$$\big\langle :\big[X^\mu(z',\bar z')-\hat x^\mu\big]\big[X^\nu(z,\bar z)-\hat x^\nu\big]:\big\rangle = \int_0^{2\pi} \bar{d}s\,\big[X^\mu(z'e^{is},\bar z'e^{-is})-x^\mu\big]_{\rm cl}\big[X^\nu(ze^{is},\bar z e^{-is})-x^\nu\big]_{\rm cl}. \tag{5.55}$$

Given that we know the classical solution, i.e. the right-hand-side of (5.55), in lightcone gauge, see (5.39) and (5.40), we establish (5.55) for the projected states in lightcone gauge.

For the transverse directions, i,j, to evaluate the left hand side of (5.55) in the state (5.54), we make use of the closed string mode expansion,

$$X^i(z,\bar z)-\hat x^i = -i\frac{\alpha'}{2}\hat p^i\ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\big(\alpha^i_n\,z^{-n}+\tilde\alpha^i_n\,\bar z^{-n}\big), \tag{5.56}$$

and the fact that:

$$\alpha^i_{n>0}|V_w\rangle_{\rm lc} = \lambda^i_n|V_{w-n}\rangle_{\rm lc},\qquad \tilde\alpha^i_{n>0}|V_w\rangle_{\rm lc} = \bar\lambda^i_n|V_{w+n}\rangle_{\rm lc},\qquad \text{and}\qquad \langle V_n|V_m\rangle_{\rm lc}=\delta_{n,m}, \tag{5.57}$$

which follow from the oscillator commutation relations, $[\alpha^i_n,\alpha^j_m]=n\delta_{n+m,0}\delta^{ij}$, $[\tilde\alpha^i_n,\tilde\alpha^j_m]=n\delta_{n+m,0}\delta^{ij}$ and $G^\dagger_w=G_w$, $G^\dagger_wG_m=\delta_{w,m}G_m$. Furthermore, $\langle V_w|\alpha^i_{-m}=\lambda^{*i}_m\langle V_{w-m}|$ and $\hat p^i|V_0\rangle = p^i|V_0\rangle$. From these expressions we find that,

$$\begin{aligned}\big\langle :\big[X^i(z',\bar z')-x^i\big]\big[X^j(z,\bar z)-x^j\big]:\big\rangle &= -\Big(\frac{\alpha'}{2}\Big)^2 p^ip^j\ln|z'|^2\ln|z|^2\\ &+\frac{\alpha'}{2}\sum_{n\neq0}\frac{1}{n^2}\Big[\lambda^i_n\lambda^{*j}_n\Big(\frac{z}{z'}\Big)^n+\bar\lambda^i_n\bar\lambda^{*j}_n\Big(\frac{\bar z}{\bar z'}\Big)^n-\lambda^i_n\bar\lambda^j_n\Big(\frac{1}{z'\bar z}\Big)^n-\bar\lambda^i_n\lambda^j_n\Big(\frac{1}{\bar z'z}\Big)^n\Big]\end{aligned} \tag{5.58}$$

It is trivial to show that this expression is identical to the right-hand side of (5.55) when,

$$\big(X^i(z,\bar z)-x^i\big)_{\rm cl} = -i\frac{\alpha'}{2}p^i\ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\big(\lambda^i_n\,z^{-n}+\bar\lambda^i_n\,\bar z^{-n}\big), \tag{5.59}$$

thus proving that the definition of classicality (5.36) is satisfied by the projected coherent states in the transverse directions.

For the longitudinal directions, to evaluate the left-hand side of (5.55) in the state (5.54), we make use of mode expansions,

$$\begin{aligned}X^-(z,\bar z)-\hat x^- &= -i\frac{\alpha'}{2}\hat p^-\ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{1}{n}\big(\alpha^-_n z^{-n}+\tilde\alpha^-_n\bar z^{-n}\big)\\ X^+(z,\bar z) &= -i\frac{\alpha'}{2}\hat p^+\ln|z|^2\end{aligned} \tag{5.60}$$

We find that,

$$\begin{aligned}\big\langle :\big[X^-(z',\bar z')-x^-\big]\big[X^j(z,\bar z)-x^j\big]:\big\rangle &= -\Big(\frac{\alpha'}{2}\Big)^2\langle\hat p^-\rangle p^j\ln|z'|^2\ln|z|^2\\&+\frac{\alpha'}{2}\sum_{n\neq0}\frac{1}{n^2}\Big[\lambda_n^-\lambda_n^{*j}\Big(\frac{z}{z'}\Big)^n+\bar\lambda_n^-\bar\lambda_n^{*j}\Big(\frac{\bar z}{\bar z'}\Big)^n-\lambda_n^-\bar\lambda_n^j\Big(\frac{1}{z'\bar z}\Big)^n-\bar\lambda_n^-\lambda_n^j\Big(\frac{1}{\bar z' z}\Big)^n\Big],\end{aligned} \tag{5.61}$$

where we have found it convenient to write,

$$\lambda_n^- = \frac{1}{\sqrt{2\alpha'}}\sum_{r\in\mathbb{Z}}\frac{1}{p^+}\lambda_{n-r}\cdot\lambda_r,\qquad\text{and}\qquad \bar\lambda_n^- = \frac{1}{\sqrt{2\alpha'}}\sum_{r\in\mathbb{Z}}\frac{1}{p^+}\bar\lambda_{n-r}\cdot\bar\lambda_r.$$

This is computed using the fact that the α_n^- are determined entirely in terms of the α_n^i, according to (for $n\neq0$),

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\sum_{r\in\mathbb{Z}}:\alpha_{n-r}^i\alpha_r^i:,\qquad\text{and}\qquad \tilde\alpha_n^- = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\sum_{r\in\mathbb{Z}}:\tilde\alpha_{n-r}^i\tilde\alpha_r^i:,$$

from the relations (5.57), and from the commutation relations $[L_n^\perp,\alpha_m^i]=-n\alpha_{n+m}^i$ and $[\bar L_n^\perp,\tilde\alpha_m^i]=-n\tilde\alpha_{n+m}^i$ with $L_n^\perp=\sqrt{\frac{\alpha'}{2}}p^+\alpha_n^-$ and $\bar L_n^\perp=\sqrt{\frac{\alpha'}{2}}p^+\tilde\alpha_n^-$ (for $n\neq0$). The $n=0$ term yields the lightcone gauge Hamiltonian, $\hat p^-=\frac{1}{\sqrt{2\alpha'}}\big(\alpha_0^-+\tilde\alpha_0^-\big)$, with $\alpha_0^-=\sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\big(L_0^\perp-1\big)$, and $\tilde\alpha_0^-=\sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\big(\bar L_0^\perp-1\big)$, or:

$$\hat p^- = \frac{1}{\alpha'p^+}\big(L_0^\perp+\bar L_0^\perp-2\big).$$

The expectation value of the lightcone gauge Hamiltonian is in turn given by,

$$\langle\hat p^-\rangle = \frac{1}{\alpha'p^+}\Big(\frac{\alpha'}{2}\mathbf{p}^2+\sum_{n>0}|\lambda_n|^2+\sum_{n>0}|\bar\lambda_n|^2-2\Big),$$

exactly as for the DLCQ coherent states, and there is again thus an effective level number for the left- and right-movers $N_e=\sum_{n>0}|\lambda_n|^2$ and $\bar N_e=\sum_{n>0}|\bar\lambda_n|^2$ respectively. For the right-hand-side of (5.55), the computation is the same as for the transverse directions, given that the integrals do not see the polarization dependence, and so the result is as in (5.58) but with λ_n^- replacing λ_n^i in accordance with the above result.

Similarly, for the X^-X^- directions, the result is:

$$\begin{aligned}\big\langle :\big[X^-(z',\bar z')-x^-\big]\big[X^-(z,\bar z)-x^-\big]:\big\rangle &= -\Big(\frac{\alpha'}{2}\Big)^2\langle:(\hat p^-)^2:\rangle\ln|z'|^2\ln|z|^2\\&+\frac{\alpha'}{2}\sum_{n\neq0}\frac{1}{n^2}\Big[\lambda_n^-\lambda_n^{*-}\Big(\frac{z}{z'}\Big)^n+\bar\lambda_n^-\bar\lambda_n^{*-}\Big(\frac{\bar z}{\bar z'}\Big)^n-\lambda_n^-\bar\lambda_n^-\Big(\frac{1}{z'\bar z}\Big)^n-\bar\lambda_n^-\lambda_n^-\Big(\frac{1}{\bar z' z}\Big)^n\Big],\end{aligned} \tag{5.62}$$

whereas for the X^-X^+ and X^iX^+ directions only the zero modes contribute, because $\langle X^\mu-x^\mu\rangle=-i\frac{\alpha'}{2}\langle\hat p^\mu\rangle\ln|z|^2$ (with $\mu=\{\pm,i\}$),

$$\big\langle :\big[X^\mu(z',\bar z')-x^\mu\big]\big[X^+(z,\bar z)\big]:\big\rangle = -\Big(\frac{\alpha'}{2}\Big)^2\langle\hat p^\mu\rangle p^+\ln|z'|^2\ln|z|^2. \tag{5.63}$$

We have thus proven that (5.55) is indeed satisfied for the lightcone gauge coherent states (5.54), in all spacetime directions.

Furthermore, from (5.58) it follows that the *rms* transverse distance from the center of mass to an arbitrary point on the string, $r=\sqrt{\langle(\mathbf{X}(z,\bar{z})-\mathbf{x})^2\rangle}$, in the rest frame, $\mathbf{p}=0$, is given by,

$$r^2=\frac{\alpha'}{2}\sum_{n>0}\frac{1}{n^2}\Big(|\lambda_n|^2+|\bar{\lambda}_n|^2-2\mathrm{Re}\big(\lambda_n\cdot\bar{\lambda}_n e^{-2in\tau_{\mathrm{M}}}\big)\Big), \tag{5.64}$$

where we have Wick rotated back to a Minkowski signature worldsheet, $\tau=i\tau_{\mathrm{M}}$. The vertex operator (5.54) and by extension (5.52) clearly represents a macroscopic string when λ_n and $\bar{\lambda}_n$ satisfy,

$$\sum_{n>0}\frac{1}{n^2}\Big(|\lambda_n|^2+|\bar{\lambda}_n|^2-2\mathrm{Re}\big(\lambda_n\cdot\bar{\lambda}_n e^{-2in\tau_{\mathrm{M}}}\big)\Big)\gg 1.$$

Recall that one is to enforce $\sum_{n>0}|\lambda_n|^2<\infty$ and similarly for the antiholomorphic sector in order to ensure that the coherent state vertex operators are well behaved [205].

Let us compare the result (5.64) for the size of a string with the naive estimate for the length or size of a string, $\ell\sim\sqrt{\alpha' N_e}$, which follows from, $m_{\mathrm{eff}}^2\sim 4N_e/\alpha'$ and $m_{\mathrm{eff}}\sim\mu\ell$ (with $m_{\mathrm{eff}}=\langle m\rangle$, $\mu=1/(2\pi\alpha')$ the string tension and ℓ its length). Recall that $N_e=\sum_{n>0}|\lambda_n|^2$, and therefore,

$$\frac{r^2}{\alpha' N_e}\sim\frac{\sum_{n>0}\frac{1}{n^2}|\lambda_n|^2}{\sum_{n>0}|\lambda_n|^2}\leq 1.$$

For an arbitrarily excited cosmic string where arbitrarily large harmonics, n, contribute to N_e,

$$\ell\ll\sqrt{\alpha' N_e},$$

and so the naive estimate $\ell\sim\sqrt{\alpha' N_e}$ breaks down when the contribution of high harmonics is significant. This is of course to be expected, because the presence of high harmonics implies also that greater amounts of energy are concentrated in a smaller region of space.

We next show that the non-zero mode components of the angular momentum, S^{ij}, and S^{i-} associated to the covariant gauge coherent vertex operator (5.49), that associated to the corresponding lightcone gauge state (5.54) and that of the classical solutions (5.59) are all equal to the expressions found for lightlike compactified states (5.42) and (5.43), re-written here for convenience: for the transverse directions,

$$\langle S^{ij}\rangle_{\mathrm{cov}}=\langle S^{ij}\rangle_{\mathrm{lc}}=\sum_{n>0}\frac{2}{n}\mathrm{Im}\big(\lambda_n^{*i}\lambda_n^j+\bar{\lambda}_n^{*i}\bar{\lambda}_n^j\big)=S_{\mathrm{cl}}^{ij}, \tag{5.65}$$

and for the longitudinal components,

$$\langle S^{-i}\rangle_{\mathrm{cov}}=\langle S^{-i}\rangle_{\mathrm{lc}}=\sqrt{\frac{2}{\alpha'}}\sum_{m>0}\sum_{\ell\in\mathbb{Z}}\frac{1}{np^+}\mathrm{Im}\big(\lambda_{m-\ell}^*\cdot\lambda_\ell^*\,\lambda_m^i+\bar{\lambda}_{m-\ell}^*\cdot\bar{\lambda}_\ell^*\,\bar{\lambda}_m^i\big)=S_{\mathrm{cl}}^{-i}, \tag{5.66}$$

with in addition all components involving the + direction equal to zero. The derivation of these expressions is almost identical to that described in the open string coherent state section. The three modifications that are worth mentioning are: (a) the covariant and lightcone gauge projected vertex operators are not eigenstates of the annihilation operators, there being instead the relations (5.57) for the lightcone gauge and,

$$\alpha^i_{m>0} \cdot V_0(\lambda) \cong \sum_{n=1}^{\infty} \frac{m}{n} \lambda^i_n B^{-n}_m \cdot V_{-m}(\lambda). \tag{5.67}$$

for the covariant gauge; (b) there is a single s-integral due to the property mentioned with an example in (5.51) and so we do not need the relation analogous to (5.67) for the longitudinal direction; and (c) there exist the orthogonality relations, $\langle V^\dagger_n V_m\rangle_{\rm cov} = \delta_{n,m}$, $\langle V_n|V_m\rangle_{\rm lc} = \delta_{n,m}$ in covariant and lightcone gauge respectively.

Chapter 6

Graviton Emission Amplitude

In this chapter we consider an application of the cosmic string vertex operator construction of the previous section. The computation will nevertheless be somewhat incomplete, in the sense that we do not compare the results with the corresponding classical computation and we present the exact tree level result without making any approximations. Consequently, the result will also be harder to interpret – this section is work in progress.

We in particular compute the u-channel forward scattering graviton emission amplitude associated to the closed string covariant coherent states with first harmonics only excited (5.53),

$$V_0(z,\bar{z}) = \frac{g_c}{\sqrt{2p^+\mathcal{V}_{25}}}\,\mathcal{C}_{\lambda\bar{\lambda}}\int_0^{2\pi} \bar{d}s\, \exp\Big(ie^{is}\lambda\cdot\partial X\, e^{-iq\cdot X(z)}\Big)\exp\Big(ie^{-is}\bar{\lambda}\cdot\bar{\partial}X\, e^{-iq\cdot X(\bar{z})}\Big)\, e^{ip\cdot X(z,\bar{z})}, \tag{6.1}$$

the imaginary part of which yields the cross section for graviton emission.[1] We also check that in the appropriate limit the result reduces to the 4-graviton amplitude computation of Kawai, Lewellen and Tye [218].

The physical state conditions require that the polarization tensors satisfy, $\lambda\cdot\lambda = 0$; see the comments below (5.27). We will consider graviton emission from a coherent state at rest, $p^i = q^i = 0$, and so $p\cdot\lambda = q\cdot\lambda = 0$ (in fact $q\cdot\lambda = 0$ in all frames), and similarly for the antiholomorphic sector, $p\cdot\bar{\lambda} = q\cdot\bar{\lambda} = 0$. Furthermore, the normalization reads

$$\mathcal{C}_{\lambda\bar{\lambda}} = \Big[\int_0^{2\pi}\bar{d}s\,\exp\big(|\lambda|^2e^{is} + |\bar{\lambda}|^2e^{-is}\big)\Big]^{-1/2}.$$

We find it convenient in this section to take $\alpha' = 2$, so that the constraints on the momenta are, $p\cdot q = 1$, $q^2 = 0$ and $p^2 = 2$. Series-expanding the exponentials, it can be seen that these constraints are (as we showed in the previous chapter) the onshell conditions for the mass eigenstates that we are superimposing which gives rise to the above coherent states.

The relevant forward scattering process is depicted in Fig. 6.1, the imaginary part of

[1] Note the slight change of notation: we find it more useful here to make manifest the dependence of the vertex operator on the worldsheet insertion point, $V_0(z,\bar{z})$, rather than the continuous coherent state parameters, $V(\lambda,\bar{\lambda})$.

which yields the amplitude for a coherent state to emit a graviton,

$$V_g(z,\bar{z}) = \frac{g_c}{\sqrt{2k^+\mathcal{V}_{25}}}\, i\zeta\cdot\partial X(z)\, i\bar{\zeta}\cdot\bar{\partial}X(\bar{z})\, e^{ik\cdot X(z,\bar{z})}, \tag{6.2}$$

and subsequently go into anything. The graviton polarization tensor and momentum are such that $k^2 = 0$, $k^\mu\zeta_{\mu,\nu}$ and $\zeta_{\mu,\nu} = \zeta_{\nu,\mu}$. Without loss of generality we have formally written $\zeta_{\mu,\nu} = \zeta_\mu\bar{\zeta}_\nu$. That the coherent state is allowed to change after having emitted a graviton is a manifestation of the fact that we are automatically taking gravitational backreaction into account which is almost always neglected in the classical computations.

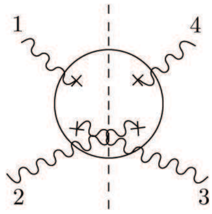

Figure 6.1: The u-channel forward scattering graviton emission amplitude, the imaginary part of which yields the cross-section for graviton emission. The coherent vertices are labelled by 1 and 4 and the graviton vertices by 2 and 3.

The non-trivial contribution to the dimensionless tree level S-matrix element for this process is given directly by the following path integral,

$$\begin{aligned}\langle f|(S-\mathbb{1})|i\rangle = C_{S_2}\int_{S_2} d^2z_4\, |z_{12}|^2|z_{13}|^2|z_{23}|^2 \\ \times\int \mathcal{D}X\, e^{-S[g,X]}V_0^\dagger(z_4,\bar{z}_4)V_g^\dagger(z_3,\bar{z}_3)V_g(z_2,\bar{z}_2)V_0(z_1,\bar{z}_1)\end{aligned} \tag{6.3}$$

with $z_{ij} = z_i - z_j$, the normalization [114], $C_{S_2} = 4\pi/g_c^2$, and the Polyakov action given in by, $S[g,X] = \frac{1}{4\pi}\int_\Sigma d^2z\,\partial_z X\cdot\partial_{\bar{z}}X + \dots$, where $d^2z = idz\wedge d\bar{z}$. Recall that there is no moduli integral at tree level given that the associated moduli space consists of a single point, see Appendix F.

Instead of working directly with the S-matrix, as discussed in (4.44) to (4.47), it is often more convenient to factor out the kinematic factors $1/\sqrt{2p^+\mathcal{V}_{d-1}}$ from the vertex operators, in which case the path integral yields:

$$\begin{aligned}&i(2\pi)^{26}\delta^{26}(P_f-P_i)\mathcal{M}(1,\dots,4) = \\ &\quad = C_{S_2}\int_{S_2} d^2z_4\,|z_{12}|^2|z_{13}|^2|z_{23}|^2\int\mathcal{D}X\,e^{-S[X]}V_0^\dagger(z_4,\bar{z}_4)V_g^\dagger(z_3,\bar{z}_3)V_g(z_2,\bar{z}_2)V_0(z_1,\bar{z}_1),\end{aligned} \tag{6.4}$$

where now it is understood that vertex operators are normalized as in (4.44). We thus work instead with the Lorentz scalar $\mathcal{M}(1,\dots,4)$ and to get an S-matrix element we divide by the appropriate kinematic factors as shown in (4.47),

$$S_{fi} = \delta_{fi} + i(2\pi)^{26}\delta^{26}(P_f-P_i)\frac{\mathcal{M}(1,\dots,4)}{\sqrt{2p_1^+\mathcal{V}_{d-1}}\dots\sqrt{2p_4^+\mathcal{V}_{d-1}}}.$$

Recall now that the s-integral in (6.1) arises from the operator product, $V_0 = G_0 \cdot V$, with G_0 a projector, $G_0^2 = G_0$ and $G_0^\dagger = G_0$, which only propagates states in the underlying Hilbert space without lightlike winding and V, in the normalization (4.44), the coherent vertex, $V = g_c C_{\lambda\bar\lambda} \exp\big(i\lambda\cdot\partial X\, e^{-iq\cdot X(z)}\big)\exp\big(i\bar\lambda\cdot\bar\partial X\, e^{-iq\cdot X(\bar z)}\big)\, e^{ip\cdot X(z,\bar z)}$, and $C_{\lambda\bar\lambda} = \exp(-\frac12|\lambda|^2 - \frac12|\bar\lambda|^2)$. Given that $[G_0, V_g] = 0$, it follows that $\langle V_0^\dagger V_g^\dagger V_g V_0\rangle = \langle (G_0V)^\dagger V_g^\dagger V_g (G_0 V)\rangle = \langle V^\dagger V_g^\dagger V_g V_0\rangle$, and therefore we need only retain a single s-integral in the scattering amplitude (6.3). This is essentially correct, the only subtlety being that although one of the two V_0 insertions can be replaced by V the normalization constant is still $\mathcal{C}_{\lambda\bar\lambda}$ for *both* insertions.

Due to the existence of, according to the Riemann-Roch theorem (F.12), three conformal Killing vectors (CKV) on S_2 there is [166, 114, 141] a residual PSL(2,$\mathbb{C}$) = SL(2,$\mathbb{C}$)/$\mathbb{Z}_2$ symmetry which has been used to fix three, z_1, z_2 and z_3, of the four vertices with a single remaining integral over z_4, see Appendix F. The standard factor $|z_{12}|^2|z_{13}|^2|z_{23}|^2$ arises from dividing by the volume of the gauge group [141]. We will eventually choose $z_1 = \infty$, $z_2 = 1$, $z_3 = 0$ and rename $z_4 = z$, and similarly for the antiholomorphic quantities, $\bar z_1 = \infty$, $\bar z_2 = 1$, $\bar z_3 = 0$, and $\bar z_4 = \bar z$, after having established the PSL(2,$\mathbb{C}$) invariance of the amplitude (which in turn ensures that it is independent of this choice). For this purpose it is convenient to factor out a PSL(2,$\mathbb{C}$) invariant measure,[2]

$$d\mu = d^2 z_4 \frac{|z_{13}|^2}{|z_{34}z_{14}|^2}, \tag{6.5}$$

because the integrand associated to this measure is then guaranteed to be a function of the cross-ratios, e.g. $z_{12}z_{34}/z_{13}z_{24}$ (see below), which are invariant under PSL(2,$\mathbb{C}$). Therefore, $d^2 z_4\, |z_{12}|^2|z_{13}|^2|z_{23}|^2 = d\mu |z_{12}z_{23}z_{34}z_{41}|^2$ and when we fix the coordinates as described above the measure reduces to, $d\mu = d^2 z/|z|^2$.

Carrying out the X-path integral leads to,

$$\mathcal{M}(1,\dots,4) = C\int d\mu\, |z_{12}z_{23}z_{34}z_{41}|^2 \int_0^{2\pi} \text{đ}s\Big(\sum_{a=0}^{\infty}\frac{1}{a!^2}M_a\prod_{i<j} z_{ij}^{k_{\mathrm{L}i}\cdot k_{\mathrm{L}j}}\Big)\Big(\sum_{b=0}^{\infty}\frac{1}{b!^2}\bar M_b\prod_{i<j}\bar z_{ij}^{k_{\mathrm{R}i}\cdot k_{\mathrm{R}j}}\Big), \tag{6.6}$$

with $k = \sum_i k_i$, the normalization, $C = 4\pi g_c^2 \mathcal{C}_{\lambda\bar\lambda}^2$, and the products in the last two factors

[2] Recall that under the global conformal group, PSL(2,$\mathbb{C}$), the coordinates, z_j, of S_2 transform according [141] to $z_j \to \frac{az_j+b}{cz_j+d}$, with a, b, c and d complex numbers such that $ad-bc=1$, under which:

$$dz_j \to \frac{dz_j}{(cz_j+d)^2}, \qquad \text{and} \qquad z_{ij} \to \frac{z_{ij}}{(cz_i+d)(cz_j+d)}.$$

ranging over $1 \le i, j \le 4$. The quantity M_a is defined by,

$$
\begin{aligned}
M_a =& \sum_{\ell=0}^{a-1}(a-\ell)(a-\ell)!\binom{a}{\ell}^2\Big[(1\cdot 2)(3\cdot 4)+(1\cdot 3)(2\cdot 4)\Big](1\cdot 4)^{a-\ell-1}\Big(\int(1)\Big)^{\ell}\Big(\int(4)\Big)^{\ell}\\
&+\sum_{\ell=0}^{a}(a-\ell)!\binom{a}{\ell}^2\Big[(2\cdot 3)+\int(2)\int(3)\Big](1\cdot 4)^{a-\ell}\Big(\int(1)\Big)^{\ell}\Big(\int(4)\Big)^{\ell}\\
&+\Bigg[\sum_{\ell=2}^{a}(a-\ell+2)!\binom{a}{\ell-2}\binom{a}{\ell}(1\cdot 2)(1\cdot 3)(1\cdot 4)^{a-\ell}\Big(\int(1)\Big)^{\ell-2}\Big(\int(4)\Big)^{\ell}\\
&\qquad+\sum_{\ell=1}^{a}(a-\ell+1)!\binom{a}{\ell-1}\binom{a}{\ell}(1\cdot 4)^{a-\ell}\Big[(1\cdot 3)\int(2)\\
&\qquad\qquad+(1\cdot 2)\int(3)\Big]\Big(\int(1)\Big)^{\ell-1}\Big(\int(4)\Big)^{\ell}+(1\leftrightarrow 4)\Bigg],
\end{aligned}
\tag{6.7}
$$

with a similar expression for $\bar{M}_b$ with $(i\cdot j)\to\overline{(i\cdot j)}$ and $\int(i)\to\int(\bar{i})$, and we have defined,

$$
(i\cdot j)=\frac{\zeta_i\cdot\zeta_j}{z_{ij}^2},\qquad \overline{(i\cdot j)}=\frac{\bar{\zeta}_i\cdot\bar{\zeta}_j}{\bar{z}_{ij}^2},\qquad \int(i)=\sum_{j\neq i}\frac{\zeta_i\cdot k_{\mathrm{Lj}}}{z_{ij}},\quad\text{and}\quad \int(\bar{i})=\sum_{j\neq i}\frac{\bar{\zeta}_i\cdot k_{\mathrm{Rj}}}{\bar{z}_{ij}}.
\tag{6.8}
$$

The combinatorial coefficients in M_a, $\bar{M}_b$ correspond to the number of permutations that leave the associated terms at fixed ℓ, a invariant.[3] Furthermore, we have written, $k_i^\mu = \frac{1}{2}(k^\mu_{\mathrm{Li}}+k^\mu_{\mathrm{Ri}})$ for the i^{th} vertex, and,

$$
\begin{aligned}
\{\zeta_1^\mu,\zeta_2^\mu,\zeta_3^\mu,\zeta_4^\mu\}&=\{e^{is}\delta^\mu_i\lambda^i,\zeta^\mu,\zeta^{*\mu},\delta^\mu_i\lambda^{*i}\},\\
\{\bar{\zeta}_1^\mu,\bar{\zeta}_2^\mu,\bar{\zeta}_3^\mu,\bar{\zeta}_4^\mu\}&=\{e^{-is}\delta^\mu_i\bar{\lambda}^i,\bar{\zeta}^\mu,\bar{\zeta}^{*\mu},\delta^\mu_i\bar{\lambda}^{*i}\},\\
\{k^\mu_{\mathrm{L1}},k^\mu_{\mathrm{L2}},k^\mu_{\mathrm{L3}},k^\mu_{\mathrm{L4}}\}&=\{p^\mu-aq^\mu,k^\mu,-k^\mu,-(p^\mu-aq^\mu)\},\\
\{k^\mu_{\mathrm{R1}},k^\mu_{\mathrm{R2}},k^\mu_{\mathrm{R3}},k^\mu_{\mathrm{R4}}\}&=\{p^\mu-bq^\mu,k^\mu,-k^\mu,-(p^\mu-bq^\mu)\}.
\end{aligned}
\tag{6.9}
$$

A fundamental consistency check is to show that the holomorphic and anti-holomorphic quantities in the integrand in (6.6) are invariant under PSL(2,$\mathbb{C}$), the global conformal group of S_2. Recall that the measure $d\mu$ is invariant. Closely examining the coordinate dependence of M_a (on account of momentum conservation and transversality of the polarization tensors) it can be seen that the following structure naturally arises,

$$
M_a(z_1,z_2,z_3,z_4)=(z_{14}^a z_{23})^{-2}F_a\Big(\frac{z_{12}z_{34}}{z_{13}z_{24}},\frac{z_{12}z_{34}}{z_{32}z_{14}}\Big),
\tag{6.10}
$$

with F_a a function of cross-ratios only. We define this function F_a by (6.10).[4] There is

[3] This counting does not include trivial permutations such as $(i\cdot j)\leftrightarrow(j\cdot i)$, $(i\cdot j)(k\cdot l)\leftrightarrow(k\cdot l)(i\cdot j)$, or $\int(i)\int(j)\leftrightarrow\int(j)\int(i)$.

[4] For 4-point functions there are six cross-ratios in total, three of which are,

$$
\frac{z_{12}z_{34}}{z_{13}z_{24}}\to\frac{-z}{1-z},\qquad \frac{z_{12}z_{43}}{z_{14}z_{23}}\to z,\qquad\text{and}\qquad \frac{z_{13}z_{42}}{z_{14}z_{32}}\to 1-z,
\tag{6.11}
$$

the other three being given by their inverses. Only two of these are independent [207]; more generally $N(\ge 4)$-point functions can always be written in terms of $N(N-3)/2$ cross-ratios. We have also noted the result of fixing the coordinates at $z_1=\infty$, $z_2=1$, $z_3=0$ and we have renamed $z_4=z$.

a similar expression for $\bar{M}_a$ with $\bar{z}_{ij}$, $\bar{\zeta}_i^\mu$, $k_{\rm Li}^\mu$ and b replacing z_{ij}, ζ_i^μ, $k_{\rm Ri}^\mu$ and a respectively; call the corresponding function of cross-ratios $\bar{F}_b$. Cross-ratios are invariant under PSL(2,$\mathbb{C}$) and therefore any such function F_a is also invariant under this symmetry group, see e.g. [207]. To establish then that the full amplitude, $\mathcal{A}$, is invariant, it suffices to show that the remaining explicit coordinate dependence depicted in (6.6) with the extra factor $(z_{23}z_{14}^a)^{-2}$ can be written in terms of cross-ratios. Using momentum conservation and the onshell conditions we find that,

$$
\begin{aligned}
z_{12}z_{23}z_{34}z_{41}(z_{23}z_{14}^a)^{-2}\prod_{i<j} z_{ij}^{k_{\rm Li}\cdot k_{\rm Lj}} &= \Big(\frac{z_{12}z_{34}}{z_{14}z_{23}}\Big)^{t(a)/2+1}\Big(\frac{z_{12}z_{34}}{z_{13}z_{24}}\Big)^{u(a)/2-(a-1)} \\
&= (-z)^{t(a)/2+u(a)/2-a+2}(1-z)^{-u(a)/2+a-1},
\end{aligned}
\tag{6.12}
$$

where in the first line it is seen that the left hand side is invariant under PSL(2,$\mathbb{C}$) which is what we set out to show, and in going from the first to the second line we have fixed the coordinates as described above. The Mandelstam variables in our conventions are given by,

$$
s(a) = -(k_{\rm L1}+k_{\rm L2})^2, \qquad t(a) = -(k_{\rm L1}+k_{\rm L4})^2, \qquad \text{and} \qquad u(a) = -(k_{\rm L1}+k_{\rm L3})^2, \tag{6.13}
$$

and we have written the result in terms of t and u because we are interested in the u-channel forward scattering, $t = 0$, limit, see Fig. 6.1. That the Mandelstam variables depend on the integer a which is summed over is a manifestation of the fact that coherent states are not mass eigenstates. Note also that $s+t+u = \sum_i m_i^2 = 4a-4$. We have therefore shown that the amplitude is indeed invariant under the global conformal group, PSL(2,$\mathbb{C}$).

The function F_a is a rather messy function of cross-ratios in general so we only exhibit its form explicitly for the case of interest, namely forward scattering which by unitarity will be related to the graviton emission cross-section. On account of the above considerations, we find that the PSL(2,$\mathbb{C}$) invariant, F_a, in the case of forward scattering but otherwise general polarization tensors and momenta, consistent with the symmetries, reduces (after a certain amount of algebra) to,

$$
\begin{aligned}
\frac{1}{a!^2}F_a = e^{ias}\sum_{\ell=0}^{a}\frac{\big(|\lambda|^a|\zeta|\big)^2}{(a-\ell)!\,\ell!^2}\Big(\frac{|\lambda\cdot k|}{|\lambda|}\Big)^{2\ell}\Bigg\{&(a-\ell)\Big(\frac{|\lambda\cdot\zeta|}{|\lambda||\zeta|}\Big)^2(-z)^{-2-\ell}(1-z)^{-\ell} \\
&+(a-\ell)\Big(\frac{|\lambda\cdot\zeta^*|}{|\lambda||\zeta|}\Big)^2(-z)^{-\ell}(1-z)^{-2-\ell}+\Big(\frac{|\zeta\cdot k_{\rm L1}|}{|\zeta|}\Big)^2(-z)^{-1-\ell}(1-z)^{-1-\ell} \\
&+\Big[1+2\ell(\ell-1)\frac{{\rm Re}[(\lambda\cdot\zeta)(\lambda\cdot\zeta^*)(\lambda^*\cdot k)^2]}{|\zeta|^2|\lambda\cdot k|^4}\Big](-z)^{-\ell}(1-z)^{-\ell} \\
&-2\ell\,\frac{{\rm Re}\big[(\lambda\cdot\zeta^*)(\zeta\cdot k_{\rm L1})(\lambda^*\cdot k)\big]}{|\zeta|^2|\lambda\cdot k|^2}(-z)^{-\ell}(1-z)^{-1-\ell} \\
&-2\ell\,\frac{{\rm Re}\big[(\lambda\cdot\zeta)(\zeta^*\cdot k_{\rm L1})(\lambda^*\cdot k)\big]}{|\zeta|^2|\lambda\cdot k|^2}(-z)^{-1-\ell}(1-z)^{-\ell}\Bigg\},
\end{aligned}
\tag{6.14}
$$

where we have fixed the three coordinates which can be done consistently after having established that F_a is a function of cross-ratios only; the $1/a!^2$ factor comes from the series expansion of the coherent vertices. Notice that for the soft graviton amplitude, where the $k \to 0$ limit becomes relevant and $\zeta \cdot k_1 = 0$, F_a would for instance be given by,

$$\begin{aligned}\frac{1}{a!^2}F_a^{(\text{soft})} \simeq e^{ias}\Big(\frac{1}{a!}\,|\lambda|^{2a}|\zeta|^2 + \frac{1}{(a-1)!}|\lambda\cdot\zeta|^2|\lambda|^{2(a-1)}(-z)^{-2}\\ + \frac{1}{(a-1)!}|\lambda\cdot\zeta^*|^2\,|\lambda|^{2(a-1)}(1-z)^{-2}\Big)\end{aligned} \tag{6.15}$$

Collecting the results (6.5), (6.6), (6.10) and (6.12) and substituting them into the expression for the amplitude (6.6) leads to,

$$\begin{aligned}\mathcal{M}(1,\dots,4) = 4\pi g_c^2\,\mathcal{C}^2_{\lambda\bar{\lambda}}\int d^2z\int_0^{2\pi} đs\Big(\sum_{a=0}^{\infty}\frac{1}{a!^2}F_a(-z)^{t(a)/2+u(a)/2-a+1}(1-z)^{-u(a)/2+a-1}\Big)\\ \times\Big(\sum_{b=0}^{\infty}\frac{1}{b!^2}\bar{F}_b(-\bar{z})^{t(b)/2+u(b)/2-b+1}(1-\bar{z})^{-u(b)/2+b-1}\Big).\end{aligned} \tag{6.16}$$

The s-integral sets $a=b$, and when we integrate out z, on account of (H.6), we find:

$$
\begin{aligned}
\mathcal{M}(1,\dots,4) &= \frac{(4\pi)^2}{\alpha'}\,g_c^2\mathcal{C}^2_{\lambda\bar\lambda}\sum_{a=0}^{\infty}\sum_{\ell,r=0}^{a}\frac{\big(|\lambda|^a|\zeta|\big)^2}{(a-\ell)!\,\ell!^2}\bigg(\sqrt{\frac{\alpha'}{2}}\frac{|\lambda\cdot k|}{|\lambda|}\bigg)^{2\ell}\frac{\big(|\bar\lambda|^a|\bar\zeta|\big)^2}{(a-r)!\,r!^2}\bigg(\sqrt{\frac{\alpha'}{2}}\frac{|\bar\lambda\cdot k|}{|\bar\lambda|}\bigg)^{2r}\\
&\times\bigg\{(a-\ell)\bigg(\frac{|\lambda\cdot\zeta|}{|\lambda||\zeta|}\bigg)^2H^{a-\ell-1,a-\ell+1}_{2\ell+2}+(a-\ell)\bigg(\frac{|\lambda\cdot\zeta^*|}{|\lambda||\zeta|}\bigg)^2H^{a-\ell+1,a-\ell-1}_{2\ell+2}\\
&+\frac{\alpha'}{2}\bigg(\frac{|\zeta\cdot k_1|}{|\zeta|}\bigg)^2H^{a-\ell,a-\ell}_{2\ell+2}+\bigg[1+2\ell(\ell-1)\frac{2}{\alpha'}\frac{\mathrm{Re}[(\lambda\cdot\zeta)(\lambda\cdot\zeta^*)(\lambda^*\cdot k)^2]}{|\zeta|^2|\lambda\cdot k|^4}\bigg]H^{a-\ell+1,a-\ell+1}_{2\ell}\\
&-2\ell\,\frac{\mathrm{Re}\big[(\lambda\cdot\zeta^*)(\zeta\cdot k_1)(\lambda^*\cdot k)\big]}{|\zeta|^2|\lambda\cdot k|^2}H^{a-\ell+1,a-\ell}_{2\ell+1}-2\ell\,\frac{\mathrm{Re}\big[(\lambda\cdot\zeta)(\zeta^*\cdot k_1)(\lambda^*\cdot k)\big]}{|\zeta|^2|\lambda\cdot k|^2}H^{a-\ell,a-\ell+1}_{2\ell+1}\bigg\}\\
&\times\bigg\{(a-r)\bigg(\frac{|\bar\lambda\cdot\bar\zeta|}{|\bar\lambda||\bar\zeta|}\bigg)^2\bar H^{2r+2}_{a-r-1,a-r+1}+(a-r)\bigg(\frac{|\bar\lambda\cdot\bar\zeta^*|}{|\bar\lambda||\bar\zeta|}\bigg)^2\bar H^{2r+2}_{a-r+1,a-r-1}\\
&+\frac{\alpha'}{2}\bigg(\frac{|\bar\zeta\cdot k_1|}{|\bar\zeta|}\bigg)^2\bar H^{2r+2}_{a-r,a-r}+\bigg[1+2r(r-1)\frac{2}{\alpha'}\frac{\mathrm{Re}[(\bar\lambda\cdot\bar\zeta)(\bar\lambda\cdot\bar\zeta^*)(\bar\lambda^*\cdot k)^2]}{|\bar\zeta|^2|\bar\lambda\cdot k|^4}\bigg]\bar H^{2r}_{a-r+1,a-r+1}\\
&-2r\,\frac{\mathrm{Re}\big[(\bar\lambda\cdot\bar\zeta^*)(\bar\zeta\cdot k_1)(\bar\lambda^*\cdot k)\big]}{|\bar\zeta|^2|\bar\lambda\cdot k|^2}\bar H^{2r+1}_{a-r+1,a-r}-2r\,\frac{\mathrm{Re}\big[(\bar\lambda\cdot\bar\zeta)(\bar\zeta^*\cdot k_1)(\bar\lambda^*\cdot k)\big]}{|\bar\zeta|^2|\bar\lambda\cdot k|^2}\bar H^{2r+1}_{a-r-1,a-r+2}\bigg\},
\end{aligned}
\tag{6.17}
$$

where we have restored α' by dimensional analysis and have defined,

$$
\begin{aligned}
H^{N,M}_K(u,t)&\equiv\frac{\Gamma(N-1-\frac{\alpha'}{4}s)\Gamma(M-1-\frac{\alpha'}{4}u)}{\Gamma(2+\frac{\alpha'}{4}t-K)},\\
\bar H^{\bar N}_{\bar M,\bar K}(u,t)&\equiv\frac{\Gamma(\bar N-1-\frac{\alpha'}{4}t)}{\Gamma(2+\frac{\alpha'}{4}s-\bar M)\Gamma(2+\frac{\alpha'}{4}u-\bar K)},
\end{aligned}
\tag{6.18}
$$

with the integers N,M,K and $\bar N,\bar M,\bar K$ ranging generically from 0 to ∞. For example, when $a=1$ the amplitude is proportional to $H^{0,0}_0\bar H^0_{0,0}$, which as we show below is precisely the 4-graviton amplitude[5] as one would expect.

In order to compute the cross section for graviton emission from the coherent state we need to extract the imaginary part from the above amplitude. Since the analytic continuation of $\Gamma(z)$ throughout the complex plane, $\mathbb{C}$, s known it follows that we know how to analytically continue the full amplitude throughout $(u,t)\in\mathbb{C}\times\mathbb{C}$ (recall that only two of the Mandelstam variables are independent). In particular, the Gamma function is analytic everywhere except for poles on the negative real axis, $z_{\rm poles}=-N$, for $N=0,1,2,\dots$, see Appendix H.4, and so the amplitude is analytic everywhere, except for poles located at,

$$
\begin{aligned}
s&=\frac{4}{\alpha'}\big(N_s-1\big), &\qquad N_s&=0,1,2,\dots\\
t&=\frac{4}{\alpha'}\big(N_t-1\big), &\qquad N_t&=0,1,2,\dots\\
u&=\frac{4}{\alpha'}\big(N_u-1\big), &\qquad N_u&=0,1,2,\dots
\end{aligned}
\tag{6.19}
$$

[5]That is, up to a normalization that is different due to the different normalization of the coherent state to that of the graviton.

These correspond precisely to the creation of intermediate string states with masses equal to the above values for s, t, u. We am interested in the forward scattering limit, $t \to 0$, and u-channel poles, see Fig. 6.1. Notice therefore that u essentially corresponds to the invariant mass of the intermediate (or final in the forward scattering sense) coherent state, after a graviton has been emitted. Of course, this is not quite correct because coherent states are not mass eigenstates, and in fact individual terms in the sum over mass eigenstates do *not* correspond to macroscopic string states. Nevertheless, a coherent state is a certain linear superposition of mass eigenstates, and the point is that mass eigenstates with arbitrarily large mass are expected to contribute; in fact on physical grounds we expect most of the contribution to come from an intermediate coherent state with $N_e \sim N_u$, where $N_e = \sum_{n\geq 0} |\lambda'_n|^2$ and λ'_n the polarization tensor associated to the final coherent state (see Chapter 5 for further details on N_e for general coherent states). Given furthermore that the final coherent state will certainly be macroscopic when the initial coherent state is macroscopic, we conclude that the contribution to the amplitude from the large N_u or large u region will be relevant. (There are subtleties here however as we show below.) This suggests that we use Stirling's approximation in the limit of large u, and $t \to 0$. Note however that we will keep t explicit until the imaginary part has been extracted. Stirling's approximation for the Gamma function, see Appendix H.4, is $\Gamma(z) \simeq z^{z-1/2}e^{-z}\sqrt{2\pi}$. This is a remarkably accurate expression for $|z| \gtrsim 1$. What is even more remarkable is that it averages over the infinite set of poles while automatically producing the correct [50] branch cut, the discontinuity in which is related to the imaginary part of the amplitude [219]. We hope to present this computation and the corresponding comparison with the classical computation (see the Sec. 7.3 for further details) in a forthcoming article. Below we add a few comments concerning the simplest case, namely the soft graviton emission amplitude.

Before discussing the soft graviton amplitude, a good consistency check is to show that the amplitude (6.17) in a certain limit reduces to that associated to forward scattering of 4 gravitons, computed by Kawai, Lewellen and Tye [218], which corresponds in particular to the $a = 1$ term in, $\mathcal{M}(1,\dots,4) = \sum_{a=0}^{\infty} \mathcal{M}(1,\dots,4)_a$. We find that the 4 graviton forward scattering amplitude reads (up to an overall normalization given that the coherent state normalization is different from the graviton normalization and with $\alpha' = 2$),

$$\mathcal{M}(1,\dots,4)_{a=1} = \frac{(4\pi)^2}{2}\, g_c^2 \mathcal{C}^2_{\lambda\bar{\lambda}} K\bar{K}\, \frac{\Gamma(-1-\frac{1}{2}s)\Gamma(-1-\frac{1}{2}t)\Gamma(-1-\frac{1}{2}u)}{\Gamma(2+\frac{1}{2}s)\Gamma(2+\frac{1}{2}t)\Gamma(2+\frac{1}{2}u)} \tag{6.20}$$

with the kinematic factors,

$$K = \Big(\frac{1}{2}s+1\Big)\Big(\frac{1}{2}t+1\Big)\Big(\frac{1}{2}u+1\Big)\bigg\{\Big(\frac{1}{2}s+1\Big)^{-1}\frac{tu}{4}|\lambda\cdot\zeta|^2+\Big(\frac{1}{2}u+1\Big)^{-1}\frac{st}{4}|\lambda\cdot\zeta^*|^2$$
$$+\frac{1}{2}t|\lambda|^2|\zeta\cdot k_1|^2+\Big(\frac{1}{2}s+1\Big)^{-1}\Big(\frac{1}{2}u+1\Big)^{-1}\frac{1}{2}t\Big(\frac{1}{2}t-1\Big)\Big(\frac{1}{2}t-2\Big)|\lambda\cdot k|^2|\zeta\cdot k_1|^2$$
$$+\Big(\frac{1}{2}t+1\Big)^{-1}\frac{su}{4}|\lambda|^2|\zeta|^2+\frac{1}{2}t|\zeta|^2|\lambda\cdot k|^2-\Big(\frac{1}{2}u+1\Big)^{-1}\frac{1}{2}t\Big(\frac{1}{2}t-1\Big)\,2\mathrm{Re}\big[(\lambda\cdot\zeta^*)(\zeta\cdot k_1)(\lambda^*\cdot k)\big]$$
$$+\Big(\frac{1}{2}s+1\Big)^{-1}\frac{1}{2}t\Big(\frac{1}{2}t-1\Big)2\mathrm{Re}\big[(\lambda\cdot\zeta)(\zeta^*\cdot k_1)(\lambda^*\cdot k)\big]\bigg\} \tag{6.21}$$

and

$$\bar{K} = \Big(\frac{1}{2}s+1\Big)\Big(\frac{1}{2}t+1\Big)\Big(\frac{1}{2}u+1\Big)\bigg\{\Big(\frac{1}{2}s+1\Big)^{-1}\frac{tu}{4}|\bar{\lambda}\cdot\bar{\zeta}|^2+\Big(\frac{1}{2}u+1\Big)^{-1}\frac{st}{4}|\bar{\lambda}\cdot\bar{\zeta}^*|^2$$
$$+\frac{1}{2}t|\bar{\lambda}|^2|\bar{\zeta}\cdot k_1|^2+\Big(\frac{1}{2}s+1\Big)^{-1}\Big(\frac{1}{2}u+1\Big)^{-1}\frac{1}{2}t\Big(\frac{1}{2}t-1\Big)\Big(\frac{1}{2}t-2\Big)|\bar{\lambda}\cdot k|^2|\bar{\zeta}\cdot k_1|^2$$
$$+\Big(\frac{1}{2}t+1\Big)^{-1}\frac{su}{4}|\bar{\lambda}|^2|\bar{\zeta}|^2+\frac{1}{2}t|\bar{\zeta}|^2|\bar{\lambda}\cdot k|^2-\Big(\frac{1}{2}u+1\Big)^{-1}\frac{1}{2}t\Big(\frac{1}{2}t-1\Big)\,2\mathrm{Re}\big[(\bar{\lambda}\cdot\bar{\zeta}^*)(\bar{\zeta}\cdot k_1)(\bar{\lambda}^*\cdot k)\big]$$
$$+\Big(\frac{1}{2}s+1\Big)^{-1}\frac{1}{2}t\Big(\frac{1}{2}t-1\Big)2\mathrm{Re}\big[(\bar{\lambda}\cdot\bar{\zeta})(\bar{\zeta}^*\cdot k_1)(\bar{\lambda}^*\cdot k)\big]\bigg\} \tag{6.22}$$

which is in agreement with the forward scattering limit of the expression obtained in [218], and this serves as a consistency check for the result (6.17).

The Soft-Graviton Limit

A simple amplitude that can be extracted from the above is the soft graviton amplitude. We have kept $|\zeta|^2, |\bar{\zeta}|^2$ in (6.17) explicit for clarity but the graviton normalization is such that it should be set to unity. Also, the quantities $\zeta\cdot k_1$, $\bar{\zeta}\cdot k_1$ can be set to zero if the graviton polarization tensor does not have any time-like components (recall that the initial coherent state is in the rest frame and so $k_1^i=0$). Given that $\frac{1}{2}u=-\frac{1}{2}(p-aq-k)^2$, it follows that in the soft graviton limit, where $k\to 0$, we have $\frac{1}{2}u=a-1$. However, in the Mandelstam variable-dependent coefficients, $H,\bar{H}$, we need to keep both u and t general for the time being. We find,

$$\begin{aligned}\mathcal{M}(1,\dots,4) &= \frac{(4\pi)^2}{\alpha'}\,g_c^2\mathcal{C}^2_{\lambda\bar{\lambda}}\\ &\sum_{a=0}^{\infty}\frac{1}{a!^2}(|\lambda|^2|\bar{\lambda}|^2)^a\bigg\{a\Big(\frac{|\lambda\cdot\zeta|}{|\lambda||\zeta|}\Big)^2H_2^{a-1,a+1}+a\Big(\frac{|\lambda\cdot\zeta^*|}{|\lambda||\zeta|}\Big)^2H_2^{a+1,a-1}+H_0^{a+1,a+1}\bigg\}\\ &\qquad\times\bigg\{a\Big(\frac{|\bar{\lambda}\cdot\bar{\zeta}|}{|\bar{\lambda}||\bar{\zeta}|}\Big)^2\bar{H}^2_{a-1,a+1}+a\Big(\frac{|\bar{\lambda}\cdot\bar{\zeta}^*|}{|\bar{\lambda}||\bar{\zeta}|}\Big)^2\bar{H}^2_{a+1,a-1}+\bar{H}^0_{a+1,a+1}\bigg\}.\end{aligned} \tag{6.23}$$

Notice that if the imaginary parts of the combinations $H\bar{H}$ appearing were independent of a, then we could use the fact that,

$$\mathcal{C}^2_{\lambda\bar{\lambda}}\sum_{a=0}^{\infty}\frac{1}{a!^2}\big(|\lambda|^2|\bar{\lambda}|^2\big)^a=1, \tag{6.24}$$

and so the only $\lambda,\bar{\lambda}$ dependence would be through the remaining terms, polarization tensors would not appear in the exponentials. A generic term $H\bar{H}$ is of the form,

$$H^{a+e_1,a+e_2}_{\delta'}\bar{H}^{\delta}_{a+e_3,a+3_4}=\frac{\Gamma(-x+\frac{1}{2}t+e_1)\Gamma(x+e_2)\Gamma(\delta-1-\frac{1}{2}t)}{\Gamma(-\delta'+2+\frac{1}{2}t)\Gamma(x+1-e_3-\frac{1}{2}t)\Gamma(-x+1-e_4)}, \tag{6.25}$$

with $\delta,\delta'=0$ or 2 and $e_i=\pm1$, and we have defined,

$$x\equiv a-1-\tfrac{1}{2}u=-(p-aq)\cdot k,$$

where in the second equality we have written u in terms of the original DDF momentum variables. For finite a, in the limit $k\to 0$ we see that $x\to 0$ and so the a-dependence of $H\bar{H}$ seems to drop out. However, a is not finite in general – the sum over a ranges from 0 to ∞, and so we first need to carry out the sum over a and then take the soft graviton limit $k\to 0$. This then suggests that x is the appropriate large variable that can be used in Stirling's approximation which in turn will lead to the imaginary part of $\mathcal{A}$. Consider an individual momentum eigenstate of momentum $k_1=p-aq$ and assume this emits a graviton of momentum k, resulting in a momentum k_1' state. Suppose all states are onshell, so that $m^2=-k_1^2$, $m'^2=-{k_1'}^2$ and $k^2=0$. Momentum conservation implies that $k_1=k_1'+k$, or $m^2-m'^2=-2(p-aq)\cdot k$, and so on physical grounds we expect $x\geq 0$. Therefore, in Stirling's approximation we will take x to be large and positive. We then find that the imaginary part of (6.25) is,

$$\mathrm{Im}\big(H^{a+e_1,a+e_2}_{\delta'}\bar{H}^{\delta}_{a+e_3,a+3_4}\big)=-\frac{\pi t}{2}\frac{(x-e_1-\frac{1}{2}t)^{e_1-\frac{1}{2}}(x+e_2)^{e_2-\frac{1}{2}}}{(x+1-e_3-\frac{1}{2}t)^{\frac{1}{2}-e_3}(x-1+e_4)^{\frac{1}{2}-e_4}}\Delta_{\frac{\delta}{2},\frac{\delta'}{2}},$$

where we have used the fact that $(-)^{e_1+e_4}=1$ and $\mathrm{Im}(-)^{\frac{1}{2}t}=\pi t/2$, and have defined

$$\frac{\Gamma(\delta-1-\frac{1}{2}t)}{\Gamma(-\delta'+2+\frac{1}{2}t)}\equiv\Delta_{\frac{\delta}{2},\frac{\delta'}{2}}=\begin{pmatrix}-2/t & -1\\ 1 & t/2\end{pmatrix}.$$

The various combinations $\delta,\delta'=0,2$ and $e_i=\pm1$ then yield the imaginary parts of each of the nine terms in (6.23).

Chapter 7

Discussion

We have presented a fairly complete discussion of massive vertex operators in bosonic string theory in a flat Minkowski background, a certain subclass of which (coherent states) may be identified with the macroscopic fundamental cosmic strings. We have presented in particular the construction of a complete set of mass eigenstate covariant normal ordered vertex operators and a complete set of (open and closed string) covariant coherent states with all constraints solved completely. The construction became possible by making use of DDF operators which enable one to translate between lightcone gauge states and covariant vertex operators. We then went on to discuss a simple amplitude computation involving these new vertex operators, in particular the graviton emission amplitude for a closed string coherent vertex operator with first harmonics only excited. In the next few paragraphs we briefly discuss and elaborate on the underlying structure that has been uncovered. We start with a discussion of the general covariant mass eigenstate vertex operators, and this is followed by a discussion of the more elaborate coherent state vertex operators.

7.1 The mass eigenstate vertex operators

One of the key features we have uncovered is that elementary Schur polynomials,[1] $S_m(nq;z)$, and the related polynomials, $H^i_n(z)$ and $\mathbb{S}_{m,n}(z)$, all of which are defined in Appendix J, play a fundamental role in the construction: arbitrary flat space vertex operators can be represented in terms of elementary Schur polynomials as we have shown explicitly in (4.99) and (4.98). The traceless subset of these is given by the vertex operators (4.92). These polynomials have useful integral representations which facilitate path integral computations.

Building on the observations of D'Hoker and Giddings [131], the use of DDF operators has enabled us to present an explicit one-to-one map between the lightcone gauge states and covariant normal ordered vertex operators. In the case of traceless polarization tensors there is a simple prescription: to construct the normal ordered vertex operator

[1] Or equivalently complete Bell polynomials.

corresponding to a given lightcone gauge state one is to make the replacements (4.93),

$$\begin{array}{ccc}\alpha^i_{-n} & \to & H^i_n(z) \\ \tilde{\alpha}^i_{-\bar{n}} & \to & \bar{H}^i_{\bar{n}}(\bar{z}) \\ |0,0;p^+,p^i\rangle & \to & g_c\, e^{i(p-Nq)^\mu X_\mu(z,\bar{z})}\end{array}$$

The spacetime vectors, p^μ, q^μ, are defined for the closed string in (4.65) and for the open string in (4.76), q^μ is transverse to all oscillator indices and the overall normalization and polarization tensors are then the same on both sides of the correspondence. States on both sides of this map have identical masses, angular momenta and we conjecture that they also share identical interactions. It would be useful to check this conjecture, possibly by performing amplitude computations on both sides of the correspondence and checking that there is agreement.

Due to the explicit presence of transverse indices on the resulting covariant vertex operators, one may wonder whether these are truly covariant (in the spacetime sense). The answer is that they are covariant but not manifestly so. This is made clear by the two examples (4.80) and (4.88) (the first of which has already been given in [131]), which have been re-written in such a way that the resulting polarization tensors and momenta can have all spacetime components non-vanishing, not just the transverse ones. These vertices can be inserted into covariant path integrals [114, 141] and one need not make the covariance manifest in order to do so.

Although the harmonic distribution of monomial vertex operators[2] is intimately connected with polarization tensors with specific Young tableaux symmetries [132], we have shown that general massive vertex operators can be constructed with polarization tensors which correspond to arbitrary irreducible representations of SO(25), and this is what one would expect from the lightcone gauge construction. Furthermore, we have shown that the monomial vertex operators are only useful for states with oscillators containing mode numbers (or harmonics) smaller than or equal to $D-1=25$, see (4.6).[3] Apart from the fact that one cannot write down a complete set of states by considering the monomial vertex operators [132], it is easy to see that for example there cannot exist a fully symmetric representation when higher harmonics are involved (i.e. worldsheet derivatives, $\partial^m X$ with $m>1$). This is because the harmonics correspond to the row of the Young tableau, and elements in any given column are anti-symmetrized, see Fig. 4.1. All these obstructions are resolved completely by the massive vertex operators presented in Sec. 4.4 which form a complete set with polarization tensors corresponding to arbitrary irreducible representations of SO(25), the little group of SO(25,1) for massive strings.

[2]By monomial vertex operators we mean expressions of the form (4.1), with polarization tensors as exhibited in Fig. 4.1).

[3]I would like to thank Steven Weinberg for correspondence concerning this point.

7.2 The coherent state vertex operators

The DDF construction has also enabled us to construct a complete set of closed and open string coherent state covariant vertex operators, i.e. states characterized by *continuous* labels (namely the polarization tensors λ_n^i, $\bar{\lambda}_n^i$), which transform correctly under all symmetries of bosonic string theory.[4] The exact definition of a coherent state vertex operator, that we suggest is appropriate in the context of superstring theory, can be found in the opening lines of Sec. 5.1 and Sec. 5.2, for the open and closed string respectively.[5] One of the most important features of these vertex operators is that they have a classical interpretation – what we mean by a state with a classical interpretation has been explained in the opening lines of Sec. 5. The size of these strings corresponding to these vertex operators is arbitrary and specified by the magnitude of the polarization tensors λ_n^i, $\bar{\lambda}_n^i$ – these states are (when $|\lambda_n|^2 \gg 1$) macroscopic with expectation values evolving according to the classical equations of motion, and may therefore be identified with a toy model version of the macroscopic fundamental cosmic strings.

The open string coherent states

The open string coherent states (5.2) are constructed from a linear superposition of the open string mass eigenstates of Sec. 4.4. The spacetime set-up we have in mind here corresponds to a vertex operator for an open string attached to a single Dp-brane or two parallel Dp-branes (of the same dimensionality), the so-called p-p string vertex operators NN and DD. It is likely that the more general p-p' vertex operators with possibly mixed boundary conditions ND and DN may be constructed from these along the lines of [202]. We have concentrated on strings with excitations within the D-brane worldvolume (i.e. polarization tensors with non-zero components in directions parallel to the brane), the corresponding transverse excitations which have the interpretation of ripples of the brane being related to these via T-duality [203, 4]. Apart from these, there are also open strings with excitations in both the transverse and tangent directions relative to the brane – we hope to present the details of these other possibilities in a separate article dedicated to the construction of open string coherent states on D-branes.

We have also provided a one-to-one correspondence between every open string covariant coherent state vertex operator, the corresponding lightcone gauge description and finally the classical solutions to which these vertex operators correspond to. We have computed the angular momentum and mass of these states and showed that there is agreement between these three descriptions.

The DLCQ closed string coherent states

[4] We have thus overcome the problems in the covariant coherent state construction encountered by Calucci [220], but see also [221, 125].

[5] Note that the naive definition that coherent states should be eigenstates of the annihilation operators is not in general compatible with the symmetries of string theory [125], as this would imply that $\langle X \rangle = X_{\rm classical}$, and this is not possible when states are invariant under spacelike worldsheet translations, see comments below (5.32).

The closed string coherent states we have considered are composed of two copies of the open string. The naive construction (5.22), of Sec. 5.2, with the corresponding lightcone gauge expression[6] (5.38), turns out to only be consistent in a spacetime with *lightlike compactification*, $X^- \sim X^- + 2\pi R^-$, see Fig. 5.1. The normal ordered expression has been given in (5.27) for the case of traceless polarization tensors. Although these states are presumably not phenomenologically relevant (at least if they are interpreted as cosmic strings because lightlike compactification breaks 4-dimensional Lorentz invariance), they serve as a good starting point for the more refined closed string coherent state construction of Sec. 5.2.

The lightlike compactified coherent states do nevertheless have interesting features and may have other applications: lightlike compactification also known as Discrete Lightcone Quantization (DLCQ) [209, 212] of M-theory has been conjectured [210] to be equivalent to *finite* N U(N) super Yang-Mills, see also [223, 224, 211] and [213, 214, 215][7]. A concise overview of these developments can be found in [225]. Although the present article is specific to the bosonic string, many of these results go through to the superstring as I hope to show in a forthcoming article. These coherent states have been shown to have certain perhaps surprising features: even though $X^- \sim X^- + 2\pi R^-$ the expectation value is single-valued: $\langle X^-(\sigma + 2\pi, \tau)\rangle = \langle X^-(\sigma, \tau)\rangle$ with all spacetime components being non-trivially consistent with the classical evolution, $\partial\bar{\partial}\langle X^\mu(z,\bar{z})\rangle = 0$, see (5.39), (5.40) and (5.34). This presumably implies that lightlike compactification is a quantum-mechanical effect which is invisible at the classical level.

There are certain subtleties here, related to whether the vertex operators are invariant under spacelike worldsheet shifts or not: when vertex operators are invariant under such shifts, the expectation value $\langle X^\mu(z,\bar{z})\rangle$ cannot satisfy the classical equations of motion non-trivially [125]. This is a gauge dependent issue and is not related to whether vertex operators have a classical interpretation or not. For example in lightcone gauge, lightlike compactification breaks the invariance under spacelike worldsheet shifts (while preserving conformal invariance) and this is why the expectation values are compatible with the equations of motion (5.39) and (5.40). Indeed, for every classical solution to the equations of motion there is a lightlike compactified coherent state with expectation values consistent with these equations of motion. These are subtle issues and have been explained in great detail in Sec. 5.2. For example, the covariant gauge version of the coherent state (5.22) is invariant under spacelike worldsheet shifts and so does not satisfy the equations of motion non-trivially: there is only the zero mode contribution (5.35) with a similar expression for the transverse indices.

We suggest that states with a classical interpretation that *are* invariant under spacelike worldsheet shifts should satisfy the equation (5.36), which may be interpreted as a definition of classicality for such states. In fact, this definition is relevant for most states with a

[6] A similar expression has appeared already in the literature, e.g. [222].

[7] I would like to thank Sanjaye Ramgoolam for a very interesting discussion on the Matrix Model – string theory correspondence.

classical interpretation: all states in lightcone or covariant gauge in a spacetime without lightlike compactification are invariant under such shifts, whether or not they have a classical interpretation. Static gauge on the other hand breaks the invariance under shifts and so instead the definition $\langle X\rangle = X_{\rm cl}$ is appropriate.

Another interesting feature is the mass shell constraint which is identical to the usual expression for non-compact spacetimes, $m^2 = 2(N+\bar{N}-2)/\alpha'$, but with N not necessarily equal to $\bar{N}$ (without breaking conformal invariance): the radius of compactification, R^-, does not appear in this expression. Furthermore, there is a rather curious dependence of the *total* zero mode momentum on R^-, see (5.30).

Finally, as a consistency check we have also shown that the covariant vertex operator (5.22) and the lightcone gauge state (5.38) have identical angular momenta in all spacetime directions which is in agreement with the corresponding classical computation, see (5.43) and (5.42). This, together with the fact that there is a one-to-one correspondence between the covariant and lightcone gauge states, supports our conjecture that the lightcone gauge states (5.38) and the covariant vertex operators (5.22) are different manifestations of the same states and therefore share identical interactions.

The non-compact closed string coherent states

Consistency in the naive closed string coherent state construction led to the requirement of a lightlike compactification of spacetime. Our main objective has been to construct covariant coherent state vertex operators that may be identified with the fundamental cosmic strings, and therefore the requirement of a lightlike compactification is possibly too constraining. In Sec. 5.2 we have shown that closed string coherent states can consistently be embedded in a spacetime without lightlike compactification: starting from the naive coherent states we project out the lightlike winding modes and end up with a vertex operator (5.49) that satisfies the definition of a coherent state and has a classical interpretation. The corresponding normal ordered vertex operator is given by (5.52) for the case of traceless polarization tensors. By projecting out the winding states, translation invariance is restored in both lightcone and covariant gauges and so the relevant definition of classicality is (5.36), which as we have shown (5.58) is satisfied by the projected states.

7.3 Graviton Emission Amplitude for Coherent States

We have computed the forward scattering graviton emission amplitude for a coherent state with first harmonics excited, including the leading order effects of gravitational backreaction in bosonic string theory. The result is expected to shed light on the long-standing question of how significant is gravitational backreaction in cosmic string evolution, and in particular close to cusps. Although the coherent state we have considered carries only first harmonics, it is possible with an appropriate choice of polarization tensors, see e.g. [58], for this state to exhibit a degenerate cusp, i.e. a cusp that persists throughout the loops motion. We hope to present a more complete discussion of these issues in a forthcoming article, where we compare the prediction of the amplitude computed in the last section of the current document with the analogous classical computation found in [82, 83] (where also backreaction was neglected), see also [55, 58]. It will also be interesting to determine how our results differ from the corresponding graviton emission from mass eigenstates [161, 54, 55, 56, 162, 58]. In [58] for example it was shown that the emission spectrum of a quantum mass eigenstate (representing classically a folded rotating closed string) does not exhibit the cusp-like behavior expected from the classical computation (where the spectrum is found to be proportional to $\sim \omega^{-4/3}$ for strings in 4 dimensions) – this clearly deserves further attention and it will be very interesting to find the corresponding results for the coherent states.

7.4 Outlook

An immediate application for the coherent state vertex operators is in fundamental cosmic string evolution: it is likely that these are then the correct vertex operators for the description of cosmic strings and it is now possible to search for discrepancies between the classical computations and the string theory predictions. Here the coherent states are useful not only because they correspond to an exact perturbative description of an arbitrarily excited macroscopic cosmic string, but because gravitational backreaction which is almost always neglected in the classical computations is automatically taken into account in string perturbation theory. In a forthcoming article I hope to present the first such computation of the gravitational radiation from cosmic string loops including the effects of gravitational backreaction.

A particularly interesting set-up is the gravitational radiation from strings with cusps which classically have been shown [75, 76] to lead to strong signals that may be detected in the gravitational wave experiments LIGO and LISA, although it is likely [117, 118] that the effect of extra dimensions can play a significant role in the damping of this signal. Cusps are likely to be a generic feature of string with junctions as well [64], although recent evidence [119] suggests that for such strings the kink signal plays a more significant role than does the gravitational wave signature form cusps. It might be that gravitational backreaction plays a significant role in all these computations [120], especially close to cusps and kinks on cosmic strings and therefore it is very important to carry out the corresponding string theory computations and check that there is agreement. In any case, given the quantum nature of fundamental cosmic strings, it is important to check that the evolution is predominantly classical and that quantum effects are small.

Another interesting avenue is the comparison of mass eigenstates and coherent states. A number of decay rate computations of mass eigenstate vertex operators have been carried out, see e.g. [51, 52, 54, 55, 56, 57, 162, 58, 59, 60], although explicit results have been limited to vertices on the leading trajectory (i.e. first harmonics only excited), where for example one does not expect to find non-degenerate cusps. At the qualitative level these are in line with one's geometrical classical expectation: mass eigenstate vertex operators corresponding classically to rotating circular loops are more stable than vertex operators corresponding to collapsed rotating loops for example [56], thus showing that these states do share at least certain characteristics of the classical evolution. However, as mentioned above the spectrum of gravitational radiation from mass eigenstates does not match the corresponding classical computation [58]. It will be very interesting to determine how the mass eigenstate amplitude computations compare with the corresponding coherent state vertex operator computations, the first computation of which has been given in the last section of the current document.

Finally, we mention also an analogy with standard point particle quantum mechanics. An important feature of harmonic oscillator coherent states is that in the presence of interactions an initial coherent state, $|\psi(0)\rangle$, remains a coherent state when the Hamiltonian is linear in the operators of the Heisenberg-Weyl group, H_4, e.g. a, $a^\dagger$, $\mathbb{1}$ and $a^\dagger a$ with $[a^\dagger, a] = 1$. That is to say, if $\hat{H}(t) = \hbar\omega a^\dagger a + j(t) a^\dagger + j^*(t) a$ and $j(t) \neq 0$, the solution to the Schrodinger equation, $i\partial_t|\psi\rangle = \hat{H}(t)|\psi\rangle$, reads [206], $|\psi(t)\rangle = \exp(\lambda(t)a^\dagger - \lambda^*(t)a)|0\rangle e^{-i\eta(t)}$, with $\lambda(t) = -ie^{-i\omega t}\int_0^t d\tau e^{i\omega\tau} j^*(\tau)$ and $\eta(t) = \frac{1}{2}\omega t + \int_0^t d\tau \mathrm{Re}[j(\tau)\lambda(\tau)]$. Therefore, in the presence of interactions the resulting state is a coherent state for all t, in accordance with the above statement. It is conceivable that this remains true in string theory, i.e. that coherent states evolve into coherent states at least at weak coupling, and it would be interesting to establish whether this is indeed the case. In the cosmic string context this is related to the question of what the final state of a radiating cosmic string is, or whether interactions preserve the classical nature of cosmic strings, questions that can be addressed using the coherent state vertex operators that we have constructed.

The developments presented here are expected to lead to greater insight into the observational prospects of cosmic strings, and in a wider sense of string theory.

Appendix A

String Theory Action

We here briefly describe the various contributions to the action of bosonic string theory. The relevant action is of the form,

$$S = S_G + S_\Phi + S_\mu + S_B + S_A + S_T.$$

The Polyakov term, S_G,

$$S_G[X, g] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-g}\, \nabla_\alpha X_\mu(\sigma) \nabla^\alpha X^\mu(\sigma). \tag{A.1}$$

is the essential ingredient and describes the embedding of the worldsheet into spacetime. Spacetime indices are contracted with $G_{\mu\nu}(X)$, the spacetime metric with $\mu = 0, \ldots, D-1$, and worldsheet indices are contracted with $g_{\alpha\beta}(\sigma)$, the worldsheet metric. We have the local coordinates σ^α with $\alpha = 0, 1$, and $X^\mu(\sigma)$ the embedding of the worldsheet into spacetime. The action is invariant under worldsheet diffeomorphisms; infinitesimally,

$$\begin{aligned} \delta\sigma^\alpha &= v^\alpha(\sigma), \\ \delta g_{\alpha\beta}(\sigma) &= \nabla_\alpha v_\beta + \nabla_\beta v_\alpha, \\ \delta X^\mu(\sigma) &= v^\alpha \partial_\alpha X^\mu, \end{aligned} \tag{A.2}$$

with $\nabla_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma^\sigma{}_{\alpha\beta} v_\sigma$ the Levi-Civita connection, and the corresponding finite diffeomorphisms are,

$$\begin{aligned} \sigma^\alpha &\to \sigma'^\alpha(\sigma) \\ g'_{\alpha\beta}(\sigma') &= \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha} \frac{\partial\sigma^\delta}{\partial\sigma'^\beta} g_{\gamma\delta}(\sigma), \\ X'^\mu(\sigma') &= X^\mu(\sigma) \end{aligned} \tag{A.3}$$

The action is also invariant under conformal worldsheet transformations,

$$\begin{aligned} \delta\sigma^\alpha &= 0, \\ \delta g_{\alpha\beta}(\sigma) &= 2\delta\phi(\sigma) g_{\alpha\beta}, \\ \delta X^\mu(\sigma) &= 0, \end{aligned} \tag{A.4}$$

and has a diffeomorphism invariance in spacetime, reflecting the fact that string theory is also a theory of gravity,

$$\begin{aligned} X^\mu &\to X'^\mu(X), \\ G'_{\mu\nu}(X') &= \frac{\partial X^\rho}{\partial X'^\mu}\frac{\partial X^\sigma}{\partial X'^\nu}G_{\rho\sigma}(X), \\ \delta\sigma^\alpha &= \delta g_{\alpha\beta}(\sigma) = 0. \end{aligned} \tag{A.5}$$

In the particular case of flat spacetime, $G_{\mu\nu}(X) = \eta_{\mu\nu}$, the last symmetry translates into Poincaré invariance,

$$\begin{aligned} X'^\mu(\sigma) &= \Lambda^\mu{}_\nu X^\nu(\sigma) + a^\mu, \\ \eta_{\mu\nu} &= \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}, \\ \delta\sigma^\alpha &= \delta g_{\alpha\beta}(\sigma) = 0, \end{aligned} \tag{A.6}$$

with a^μ a constant vector and Λ defined by the second equality (infinitesimally, $\Lambda \simeq 1+\omega$, the definition is $\omega_{\mu\nu} = -\omega_{\nu\mu}$). The action S_G and also S_N is invariant under all these symmetries. From Noether's theorem it follows that the Poincaré invariance (A.6) of the action leads to two conserved currents; associated to the translations, $\delta X^\mu = a^\mu$, is an energy-momentum current,

$$P^\alpha_\mu = -\frac{1}{\alpha'}\sqrt{-g}g^{\alpha\beta}\partial_\beta X_\mu, \tag{A.7}$$

and associated to the rotations, $\delta X^\mu = \Lambda^\mu{}_\nu X^\nu$, is the angular momentum current,

$$J^\alpha_{\mu\nu} = -\frac{1}{\alpha'}\sqrt{-g}g^{\alpha\beta}\big(X_\mu\partial_\beta X_\nu - X_\nu\partial_\beta X_\mu\big). \tag{A.8}$$

The associated conserved charges (momentum and angular momentum) follow from integrating the timelike components of these over a spacelike curve,

$$p_\mu = \int_0^{\sigma_{\max}} d\!\!{}^-\sigma P^{\tau_M}_\mu, \qquad \text{and} \qquad J_{\mu\nu} = \int_0^{\sigma_{\max}} d\!\!{}^-\sigma J^{\tau_M}_{\mu\nu}, \tag{A.9}$$

where we have defined $d\!\!{}^-\sigma = d\sigma/(2\pi)$, and (by convention) $\sigma_{\max} = \pi$ or 2π for open or closed strings respectively.

Other terms in the action that are required for consistency are the dilaton contribution, S_Φ, and antisymmetric tensor field contribution, S_B, with the set of background fields $\{G_{\mu\nu}, B_{\mu\nu}, \Phi\}$ corresponding to a massless multiplet,

$$S_B = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\epsilon^{\alpha\beta}\partial_\alpha X^\mu(\sigma)\partial_\beta X^\nu(\sigma)B_{\mu\nu}(X), \qquad S_\Phi = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{g}R_{(2)}\Phi(X).$$

Here $\epsilon^{\alpha\beta}/\sqrt{-g}$ transforms as a tensor, $\epsilon^{01} = -\epsilon^{10} = 1$, $\epsilon^{00} = \epsilon^{11} = 0$, and $R_{(2)}$ is the 2-dimensional scalar curvature. The quantities

$$G_{\mu\nu}(X)(dX^\mu \otimes dX^\nu + dX^\nu \otimes dX^\mu), \qquad \text{and} \qquad B_{\mu\nu}(X)\, dX^\mu \wedge dX^\nu,$$

are invariant under spacetime diffeomorphisms and $\Phi(X)$ is a scalar. Therefore, $S_G + S_B + S_\Phi$ is also invariant. When the dilaton has (by some unknown mechanism) acquired a constant vacuum expectation value, $\Phi(X) = \langle\Phi\rangle$, the term $S_\Phi = \langle\Phi\rangle\chi(\Sigma)$ is a topological invariant, with $\chi(\Sigma)$ the Euler characteristic of the worldsheet,

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{g}R_{(2)} = 2 - 2h - b.$$

h is the genus of the worldsheet and b is the number of boundaries. Therefore, the dilaton contribution to the path integral is (going to a Euclidean worldsheet and target space), $e^{iS_\Phi} \to e^{-S_\Phi} = g_s^{-2+2h+b}$, and $g_s = e^{\langle\Phi\rangle}$ is identified with a coupling constant: the sum over h, b generates the perturbation series expansion when $g_s \ll 1$. The region $g_s \gg 1$ is correspondingly therefore identified with non-perturbative string theory where the aforementioned series presumably breaks down.

There are also additional terms that can be added such as a tachyon term, S_T, and a term, S_μ, that is required for the renormalizability of the theory,

$$S_T = \frac{1}{\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{g}T(X), \qquad \text{and} \qquad S_\mu = \mu\int_\Sigma d^2\sigma\sqrt{g},$$

both of which break conformal invariance at the classical level, as does S_Φ, but this is restored at the quantum level. In the case of open strings it is also possible for the endpoints to carry charges with associated gauge field $A_\mu(X)$ with the following coupling,

$$S_A = iq\int_{\partial\Sigma} d\tau\partial_\tau X^\mu(\sigma)A_\mu(X),$$

with q the associated charge. Given that $dX^\mu A_\mu(X)$ is invariant under spacetime diffeomorphisms and $T(X)$ is a scalar, the terms $S_T + S_\mu + S_A$ are also invariant. There is also a U(1) gauge symmetry which acts as $\delta A_\mu = -\zeta_\mu/2\pi\alpha'$ and $\delta B_{\mu\nu} = \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu$, which leaves the combination $S_B + S_A$ invariant and leads to spacetime gauge invariance.

The choice of admissible backgrounds is heavily constrained by the requirement of conformal invariance, see e.g. [149]. Conformal invariance can be formulated as the requirement that the beta-functions associated to the couplings G, B, Φ, T in the action $S[X, g]$ vanish, which in turn ensures the resulting sigma model is conformally invariant:

$$\beta_G = \beta_B = \beta_\Phi = \beta_T = 0.$$

These equations can be identified with classical equations of motion, solutions to which lead to the admissible backgrounds. In this sense, every solution to these equations of motion gives rise to a different conformal field theory (CFT) with a different spectrum of states and so on. One such choice, and in fact the choice that is relevant in the current document, is:

$$\begin{aligned} G_{\mu\nu}(X) =&\eta_{\mu\nu}, \qquad B_{\mu\nu}(X) = 0, \qquad \Phi(X) = \langle\Phi\rangle, \\ &A_\mu(X) = 0, \qquad T(X) = 0. \end{aligned} \tag{A.10}$$

Appendix B

Complex Tensors

In this appendix we describe the local complex differential geometry on the worldsheet, Σ, to set the conventions that are used throughout the text.

For a given set of real coordinates (x, y) we define a complex set $(z, \bar{z})$ by $z = x + iy$, $\bar{z} = x - iy$ and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Then, we have $idz \wedge d\bar{z} = 2dx \wedge dy$ and we use the convention

$$d^2z \equiv idz \wedge d\bar{z} = 2dx \wedge dy,$$

throughout. Any two-dimensional Riemannian manifold is conformally flat, see e.g. [226], in the sense that a general metric $ds^2 = g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2$ can always by an appropriate coordinate transformation be written in terms of local conformally flat coordinates, so that $g = e^{2\phi(x,y)}(dx^2 + dy^2) = g_{z\bar{z}}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$. In this (later expression for the) metric, we have: ${\Gamma^z}_{zz} = \partial_z \ln g_{z\bar{z}}$ and $R_{(2)} = -g^{z\bar{z}}\partial_z\partial_{\bar{z}} \ln g_{z\bar{z}}$. We define a tensor V of conformal weight $(h, \bar{h})$ by

$$V = V_{\underbrace{z\ldots z}_{h}\underbrace{\bar{z}\ldots\bar{z}}_{\bar{h}}}(dz)^h(d\bar{z})^{\bar{h}} \in K^{(h,\bar{h})}, \tag{B.1}$$

with $K^{(h,\bar{h})}$ the space of tensors of weight $(h, \bar{h})$. We refer to the components of V as *conformal primary operators* and we occasionally write $\phi(z, \bar{z}) = V_{z\ldots z\bar{z}\ldots\bar{z}}$, when We do not want to specify the particular weights of the fields. *Vertex operators*, whose components are usually denoted by $V(z, \bar{z}) = V_{z\bar{z}}$, are defined as primary operators of weight (1,1),

$$V = \int_\Sigma d^2z\, V(z, \bar{z}) \in K^{(1,1)} \qquad \text{(vertex operators)}$$

(note that $\sqrt{g}g^{z\bar{z}} = 1$) and are therefore invariant under conformal transformations,

$$z \to f(z), \qquad \bar{z} \to \bar{f}(\bar{z}).$$

At the *quantum* level, we require correlation functions of primary operators $V \in K^{(h,\bar{h})}$ to remain in $K^{(h,\bar{h})}$ under conformal transformations, $z \to f(z)$, and $\bar{z} \to \bar{f}(\bar{z})$.

Define $K^{(n,0)} = K^n$. Using the metric $g_{z\bar{z}}$ to raise and lower indices there is an isomorphism $(n - m, 0) \sim (n, m)$ and one may therefore express all tensors in terms of holomorphic indices, e.g. we write, $g^{z\bar{z}}V_{\bar{z}} = V^z$, with $g^{z\bar{z}}g_{z\bar{z}} = 1$. Covariant derivatives satisfy $\nabla_z^{(n)} : K^n \to K^{n+1}$,

$$\begin{aligned}\nabla_z^{(n)} V &= (\partial_z - n\Gamma^z{}_{zz})V \otimes dz \\ &= (g_{z\bar{z}})^n \, \partial_z \, (g_{z\bar{z}})^{-n} \, V \otimes dz.\end{aligned} \tag{B.2}$$

We occasionally drop the index (n) from covariant derivatives when there is no ambiguity about the type of tensor it acts on. In addition, there is the Cauchy-Riemann operator $\partial_{\bar{z}}$; formally $\nabla_{\bar{z}}^n : K^n \to K^{n,1}$, so that

$$\nabla_{\bar{z}}^{(n)} V = \partial_{\bar{z}} V \otimes d\bar{z}. \tag{B.3}$$

According to the above identification we could also have written the Cauchy-Riemann operator as $\nabla^z_{(n)} : K^n \to K^{n-1}$, with

$$\nabla^z_{(n)} V = g^{z\bar{z}} \partial_{\bar{z}} V \otimes (dz)^{-1}. \tag{B.4}$$

We shall not in general display the differentials dz $(d\bar{z})$ in ∇_z $(\nabla_{\bar{z}})$ but include them in the definitions for concreteness.

The natural inner product between tensors $V_{1,2} \in K^n$ with respect to the metric g is

$$\big(V_1, V_2\big) = \int_\Sigma d^2z \sqrt{g} \, (g^{z\bar{z}})^n \, V_1^* \, V_2, \tag{B.5}$$

and we define the adjoint operators $\nabla_z^{(n)\dagger}$ and $\nabla^{z\dagger}_{(n)}$ with respect to this, $(V_1, \nabla_z^{(n)\dagger} V_2)_g \equiv (\nabla_z^{(n)} V_1, V_2)_g$. When $V_1 = V_2$ we also write $\|V\|^2 = \big(V, V\big)$. Using the definitions it follows that

$$\nabla_z^{(n)\dagger} = -\nabla^z_{(n+1)}, \qquad \nabla^{z\dagger}_{(n)} = -\nabla_z^{(n-1)}. \tag{B.6}$$

We can construct two, in general distinct, Laplacians using the differential operators (B.2) and (B.4)

$$\begin{aligned}\Delta^+_{(n)} &= -2\nabla^z_{(n+1)} \nabla_z^{(n)} \\ \Delta^-_{(n)} &= -2\nabla_z^{(n-1)} \nabla^z_{(n)},\end{aligned} \tag{B.7}$$

from which it follows that

$$\Delta^+_{(n)} - \Delta^-_{(n)} = nR_{(2)}. \tag{B.8}$$

Therefore, these two Laplacians are equal when acting on scalars, in which case $n = 0$, so we define $\Delta_{(0)} \equiv \Delta^+_{(0)} = \Delta^-_{(0)}$. The factor of -2 in the definitions (B.7) is conventional and is included so as to agree with the definition of the conventional Laplacian $\Delta_{(0)} = -\frac{1}{\sqrt{g}} \partial_\alpha \sqrt{g} g^{\alpha\beta} \partial_\beta$. In particular, for constant $g_{z\bar{z}}$, the Laplacian reads $\Delta_{(0)} = -2g^{z\bar{z}} \partial_z \partial_{\bar{z}}$, in agreement with both $\Delta^+_{(n)}$ and $\Delta^-_{(n)}$.

The Green's (or 2-d Stoke's) theorem,

$$\int_{\partial D} dx A_x + dy A_y = \int_D dx \wedge dy \, \big(\partial_x A_y - \partial_y A_x\big), \tag{B.9}$$

in complex coordinates, using the above conventions, takes the form

$$\int_{\partial D} d\bar{z} A_z + dz A_{\bar{z}} = \int_D dz \wedge d\bar{z} \left(\partial_z A_z - \partial_{\bar{z}} A_{\bar{z}}\right), \tag{B.10}$$

with the definitions $A_z = (A_x + iA_y)$ and $A_{\bar{z}} = (A_x - iA_y)$.

The string embedding $X : \Sigma \to \mathbb{R}^{26}$ is a scalar from the 2-dimensional point of view. Its derivatives are tensor fields in the sense of (B.1). In particular, $\partial X = \partial_z X dz$ is a tensor of weight $(1, 0)$, and using the derivatives (B.2) one can form tensors of weight $(\ell, 0)$ as follows $\nabla_z^{(\ell-1)} \ldots \nabla_z^{(1)}(\partial X)$. In practice we write this as $\nabla_z^{\ell-1}\partial_z X$ and may not in general (as mentioned above) display the differentials. When the $\Gamma^z{}_{zz}$ dependence drops out we shall write instead $\partial_z^l X$ or even $\partial^l X$ when there is no ambiguity and likewise for the anti-holomorphic counterpart.

Appendix C

Riemann Surfaces

In this section I provide a brief overview of the very basics of the theory of Riemann surfaces while emphasizing aspects that will better exhibit the connection with the Riemann theta function and prime form, which are the fundamental object that appear in string correlation functions. For further details and proofs that are omitted the reader is referred to the literature; [192, 227, 228] for a fairly formal but complete approach and [141, 186] for an approach closer to the underlying string physics.

Suppose Σ is a compact Riemann surface with complex structure defined on it. As a topological space it is completely determined, up to a diffeomorphism, by its genus h, i.e. the number of "handles" of the surface. Let $\Gamma(\Sigma,\Omega^1)$ denote the vector space of holomorphic 1-forms on Σ. From the Riemann-Roch-Atiyah-Singer index theorem,

$$\dim \mathrm{Ker}\nabla_z^{(n)} - \dim \mathrm{Ker}\nabla^z_{(n+1)} = \frac{1}{2}\,(2n+1)\,\chi(\Sigma) = (2n+1)\,(1-h), \tag{C.1}$$

it follows that the dimension of the vector space $\Gamma(\Sigma,\Omega^1)$ is equal to the genus h of Σ. To see this, notice that holomorphic one-forms live in $\mathrm{Ker}\nabla^z_{(1)}$. Then, from the index theorem we find that $\dim\Gamma(\Sigma,\Omega^1) = \dim\mathrm{Ker}\nabla^z_{(1)} = \dim\mathrm{Ker}\nabla_z^{(0)} - (1-h)$. But $\mathrm{Ker}\nabla_z^{(0)}$ is just a constant as $\nabla_z^{(0)}$ acts on scalars and so $\dim\mathrm{Ker}\nabla_z^{(0)} = 1$. We therefore see that $\dim\Gamma(\Sigma,\Omega^1) = h$, implying that there are h one-forms, call them ω_I, with $I = 1,\dots,h$, on a compact Riemann surface of genus h. Similarly, one can show that there are correspondingly h anti-holomorphic one-forms on Σ and we shall denote these by $\bar\omega_I$. The holomorphic and antiholomorphic one-forms ω_I and $\bar\omega_I$ generate the first cohomology group of the Riemann surface $H^1(\Sigma,\mathbb{C})$ and will be represented locally in analytic coordinates as $\omega_I = \omega_I(z)dz$ and $\bar\omega_I = \bar\omega_I(\bar z)d\bar z$ respectively.

Dual to these are the homology cycles of a Riemann surface. The first homology group of a compact Riemann surface is given by $H_1(\Sigma,\mathbb{Z}) = \mathbb{Z}^{2h}$. Let us then choose a canonical homology basis provided with by the cycles A_I, B_I, $I = 1,\dots,h$. Denote by $I(\sigma,\gamma)$ the intersection product of any two cycles $\sigma,\gamma\,(= n_I A_I + m_I B_I$, with n_I, m_I integers). Then,

$$I(A_I,A_J) = I(B_I,B_J) = 0, \qquad I(A_I,B_J) = -I(B_J,A_I) = \delta_{IJ} \tag{C.2}$$

The canonical basis A_I, B_I is not unique; any basis A'_I, B'_I with $A'_I = D_{IJ}A_J + C_{IJ}B_J$, $B'_I = B_{IJ}A_J + A_{IJ}B_J$ will satisfy (C.2) provided the $2h\times 2h$ matrix $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ is an element of the symplectic (or modular) group $\mathrm{Sp}(2h,\mathbb{Z})$: $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)^T \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, as can be explicitly verified. One may think of the modular group as being generated by 2π twists around the A_I and B_I cycles. Such twists are referred to as *Dehn twists*.

There exists a natural pairing between the first homology group $H_1(\Sigma,\mathbb{Z})$ and the first cohomology group $H^1(\Sigma,\mathbb{C})$, provided by the following line integral

$$\int : H_1(\Sigma,\mathbb{Z}) \times H^1(\Sigma,\mathbb{C}) \to \mathbb{C}. \tag{C.3}$$

We can represent this pairing explicitly by introducing a normalized basis of holomorphic 1-forms ω_I, such that:

$$\oint_{A_I} \omega_J = \delta_{IJ}, \qquad \oint_{B_I} \omega_J = \Omega_{IJ}, \tag{C.4a}$$

$$\Omega_{IJ} = \Omega_{JI}, \qquad \mathrm{Im}\,\Omega_{IJ} > 0. \tag{C.4b}$$

This pairing is independent of the choice of representatives of the equivalence classes. The first integral in (C.4a) defines the normalization of the ω_I and the second defines the *period matrix*, Ω_{IJ}, of the Riemann surface. The third and fourth relations (in (C.4b)) follow from the first and second Riemann bilinear identity respectively (see e.g. [192], p. 139, vol. 1 or [228], p. 231). In particular, for all holomorphic 1-forms ω, η,

$$\begin{aligned} &\sum_{I=1}^{h} \oint_{A_I} \omega \oint_{B_I} \eta - \oint_{B_I} \omega \oint_{A_I} \eta = 0, \qquad (1^{\mathrm{st}}\text{Riemann bilinear identity}) \\ &\mathrm{Im} \sum_{I=1}^{h} \oint_{A_I} \bar{\omega} \oint_{B_I} \omega > 0. \qquad (2^{\mathrm{nd}}\text{Riemann bilinear identity}) \end{aligned} \tag{C.5}$$

These two identities can be derived from the following equation. For all closed 1-forms ω and η, which may be holomorphic or antiholomorphic,

$$\int_\Sigma \omega \wedge \eta = \sum_{I=1}^{h} \oint_{A_I} \omega \oint_{B_I} \eta - \oint_{A_I} \eta \oint_{B_I} \omega, \tag{C.6}$$

which is also sometimes referred to as the Riemann bilinear identity. This later expression reduces to the first identity above when *both* ω and η are either holomorphic or antiholomorphic and implies the second when $\eta = \bar{\omega}$.

Given any base point p_0 we may associate to every point p on Σ a complex h-component vector $\mathbf{z}$ by the *Jacobi map* (referred to also as the *Abel map*):

$$\mathbb{I} : p \to \mathbf{z}(p) = \left(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_h \right). \tag{C.7}$$

This vector is unique up to periods (C.4a). We associate to Ω a lattice $L_\Omega \subset \mathbb{C}^h$, such that $L_\Omega \equiv \mathbb{Z}^h + \Omega\mathbb{Z}^h$. The vector $\mathbf{z}$ is an element of the complex torus $J(\Sigma)$ which is referred to as the *Jacobian variety* of Σ,

$$J(\Sigma) \equiv \mathbb{C}^h / L_\Omega = \mathbb{C}^h / (\mathbb{Z}^h + \Omega\mathbb{Z}^h). \tag{C.8}$$

At genus one, $h = 1$, the Jacobian variety reduces therefore to complex numbers z such that,

$$z \sim z + m + \tau n, \tag{C.9}$$

with $\tau = \tau_1 + i\tau_2$ the complex modulus of the torus, $\Omega = \tau$ and n, m integers. The modulus τ parametrizes the moduli deformations of the surface, and so for instance in a one-loop string amplitude the path integral would be over τ, which is to range over a single fundamental domain – a common choice being: $-1/2 \leq \tau_1 \leq 1/2$, $\tau_2 \geq 1$.

The space of matrices satisfying (C.4b), call it $\mathcal{H}_h$, is the *Siegel upper half space*, $\mathcal{H}_h = \{\Omega \in \mathbb{C}^h \,|\, \Omega_{IJ} = \Omega_{JI},\ \mathrm{Im}\Omega > 0\}$. The *Riemann theta function*, associated to Ω[1] is then defined for $\mathbf{z} \in J(\Sigma)$ by,

$$\vartheta(\mathbf{z}, \Omega) \equiv \sum_{n \in \mathbb{Z}^h} \exp\left\{2\pi i \left(\frac{1}{2} n^T \Omega n + n^T \mathbf{z}\right)\right\}. \qquad \text{(Riemann theta function)} \tag{C.10}$$

It is quasi-periodic (periodic up to a multiplicative factor) with respect to the lattice translations $\mathbf{z} \to \mathbf{z} + c$, with $c \in L_\Omega$, and is invariant under parity $\mathbf{z} \to -\mathbf{z}$:

$$\vartheta(\mathbf{z} + m + \Omega n, \Omega) = \exp\left\{2\pi i \left(-\frac{1}{2} n^T \Omega n - n^T \mathbf{z}\right)\right\} \vartheta(\mathbf{z}, \Omega) \qquad \text{(translations)} \tag{C.11a}$$

$$\vartheta(\mathbf{z}, \Omega) = \vartheta(-\mathbf{z}, \Omega) \qquad \text{(parity)} \tag{C.11b}$$

where $n, m \in \mathbb{Z}^h$. Notice that the RHS of (C.11a) is independent of m, thus implying that the Riemann theta function is invariant under the shift $\mathbf{z} \to \mathbf{z} + m$. In addition, from (C.11b) it follows that $\vartheta(0, \Omega) = 0$.

The theta function satisfies the heat equation,

$$\frac{\partial \vartheta(\mathbf{z}, \Omega)}{\partial \Omega_{IJ}} = \frac{1}{2\pi i} \frac{\partial^2 \vartheta(\mathbf{z}, \Omega)}{\partial \mathbf{z}_I \partial \mathbf{z}_J} \times \begin{cases} 1 & \text{for} \quad I \neq J \\ \frac{1}{2} & \text{"} \quad I = J \end{cases} \qquad \text{(heat equation)} \tag{C.12}$$

We can extend the definition of the theta function if we introduce rational characteristics $[{a \atop b}]$. The *Riemann theta function with (rational) characteristics* is defined by

$$\vartheta[{a \atop b}](\mathbf{z}, \Omega) \equiv \sum_{n \in \mathbb{Z}^h} \exp\left\{2\pi i \left(\frac{1}{2}(n+a)^T \Omega (n+a) + (n+a)^T (\mathbf{z} + b)\right)\right\}, \quad \forall\, a, b \in \mathbb{Q}^h. \tag{C.13}$$

It is also quasiperiodic with respect to lattice translations $\mathbf{z} \to \mathbf{z} + c$, with $c \in L_\Omega$,

$$\vartheta[{a \atop b}](\mathbf{z} + m + \Omega n, \Omega) = e^{2\pi i (a^T m - b^T n)} \exp\left\{2\pi i \left(-\frac{1}{2} n^T \Omega n - n^T \mathbf{z}\right)\right\} \vartheta[{a \atop b}](\mathbf{z}, \Omega). \tag{C.14}$$

In terms of the Riemann theta function,

$$\vartheta[{a \atop b}](\mathbf{z}, \Omega) = \exp\left\{2\pi i \left(\frac{1}{2} a^T \Omega a + a^T (\mathbf{z} + b)\right)\right\} \vartheta(\mathbf{z} + b + \Omega a, \Omega), \tag{C.15}$$

[1] Ω need not be identified with the Riemann surface period matrix in the definition of $\theta(\mathbf{z}, \Omega)$ but we shall do so.

and thus the original theta function is just $\vartheta\,(\mathbf{z},\Omega) = \vartheta[{}^{0}_{0}]\,(\mathbf{z},\Omega)$. The theta function with characteristics is invariant under parity, $\mathbf{z} \to -\mathbf{z}$, provided we also take $a, b \to -a, -b$, so that $\vartheta[{}^{a}_{b}]\,(\mathbf{z},\Omega) = \vartheta[{}^{-a}_{-b}]\,(-\mathbf{z},\Omega)$. This follows from (C.11b) and (C.15).

A very useful quantity that arises in the construction of the propagator (for both the bosonic and fermionic case) and therefore in all scattering amplitudes is the *prime form*, $E(z,w)$. The prime form generalizes the notion of distance between two points, $z - w$, on $\mathbb{C}$ to higher genus surfaces. In terms of the *Riemann theta function* it has the form [229, 192, 141]

$$E(z,w) = \frac{\vartheta[\delta]\left(\int_w^z \omega, \Omega\right)}{h_\delta(z) h_\delta(w)}, \tag{C.16}$$

and is quasi-periodic around the A_I and B_I cycles,

$$E(z + A_I, w) = E(z,w), \tag{C.17a}$$

$$E(z + B_I, w) = E(z,w) \exp\left(-\pi i \Omega_{II} + 2\pi i \int_w^z \omega_I\right). \tag{C.17b}$$

Appendix D

Conformal Symmetry

Let us consider first the d-dimensional case before specializing to the case of interest: $d = 2$. We work in a Minkowski spacetime $\mathbb{R}^{p,q}$ with flat metric $g_{\mu\nu} = \eta_{\mu\nu}$ of signature (p, q) so that $d = p + q$, although we will find it convenient to switch to a Euclidean signature when $d = 2$. In a local patch of the manifold, the corresponding line element is

$$ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta,$$

and the requirement that this remain invariant, $ds^2 = ds'^2$, under general diffeomorphisms, $x \to x' = f(x)$ determines $g'_{\alpha\beta}(x')$. The conformal group corresponds to the subset of these transformations under which

$$g'_{\mu\nu}(x') = \Omega(x)g_{\mu\nu}(x), \tag{D.1}$$

that is, the metric is invariant up to a scale transformation. Note that conformal transformations are nevertheless *implemented* by rescaling the metric without transforming the coordinates, i.e. $g'_{\mu\nu}(x) = \Omega(x)g_{\mu\nu}(x)$. Conformal transformations preserve angles in the sense that for two d-vectors v and w, $\Omega : \frac{v\cdot w}{\sqrt{v^2w^2}} \to \frac{v\cdot w}{\sqrt{v^2w^2}}$. Now consider infinitesimal transformations of the form,

$$x \to x' = x + \epsilon(x), \tag{D.2}$$

under which $g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \partial_\mu\epsilon_\nu(x) - \partial_\nu\epsilon_\mu(x) + \dots$. From (D.1),

$$\frac{2}{d}(\partial\cdot\epsilon)g_{\mu\nu} = \partial_\mu\epsilon_\nu(x) + \partial_\nu\epsilon_\mu(x). \tag{D.3}$$

This is the "conformal Killing equation", the solutions $\epsilon^\mu(x)$ correspond to infinitesimal conformal Killing vector fields (CKV) and these generate infinitesimal conformal transformations. The general solution is of the form,

$$\epsilon^\mu(x) = a^\mu + b^\mu{}_\nu x^\nu + c^\mu{}_{\nu\rho}x^\nu x^\rho, \quad (c^\mu{}_{\nu\rho} = c^\mu{}_{\rho\nu}) \tag{D.4}$$

where a, b and c are constants.[1] The first term corresponds to translations. In the second term $b_{\mu\nu}$ is a sum of a trace and an antisymmetric part,

$$b_{\mu\nu} = \lambda g_{\mu\nu} + \omega_{\mu\nu}, \quad (\omega_{\mu\nu} = -\omega_{\nu\mu}).$$

[1]Note also that there is no general solution to the conformal Killing equation unless we specify the background geometry, i.e. unless we specify $g_{\mu\nu}(x)$.

where the trace contribution corresponds to *dilatations* or *scale* transformations and the anti-symmetric piece corresponds to *rotations*, namely *Lorentz* transformations.[2] The tensor in the last term of the general solution is,

$$c_{\rho\mu\nu} = \frac{1}{d}(g_{\mu\rho}b_\nu + g_{\rho\nu}b_\mu - g_{\mu\nu}b_\rho).$$

where we have defined $b_\mu \equiv -\frac{1}{d}c^\alpha{}_{\alpha\mu}$. This term generates "special conformal transformations" (SCT); if we define the translation and inversion maps

$$\begin{aligned} T &: x^\mu \to x^\mu + b^\mu, \\ S &: x^\mu \to x^\mu / x^2, \end{aligned} \tag{D.5}$$

SCT correspond to the map

$$\begin{aligned} STS : x^\mu \to x'^\mu &= \frac{x^\mu + b^\mu x^2}{1 + 2(b\cdot x) + b^2x^2} \\ &\simeq x^\mu + b^\mu x^2 - 2(b\cdot x)x^\mu + \mathcal{O}(b^2), \end{aligned} \tag{D.6}$$

that is a translation followed and preceded by an inversion. The first line corresponds to an exact SCT which holds for *finite* b^μ (we shall not prove this), while the second line coincides with the infinitesimal form $c^\mu{}_{\nu\rho}x^\nu x^\rho$. The combined map STS as defined in (D.5) is defined globally even though S is only defined locally (it is singular at the origin). In total there are $d + d(d-1)/2 + 1 + d = (d+1)(d+2)/2$ independent parameters that generate conformal transformations and so we expect the same number of generators.

We would like to understand how conformal transformations act on the various fields, and in order to do so we first construct the conformal algebra, representations of which will correspond to the conformal fields we are looking for. Consider a set of fields, written collectively as $\Phi(x)$. A general infinitesimal transformation can be written as

$$x^\mu \to x'^\mu = x^\mu + \omega_a X^\mu_a, \tag{D.7}$$

$$\Phi(x) \to \Phi'(x') = \Phi(x) + \omega_a \mathcal{F}_a(\Phi(x)), \tag{D.8}$$

where $\{a\}$ is some arbitrary index structure, e.g. $\{\mu\}$, and ω_a are infinitesimal constant parameters corresponding for example to a^μ, $\omega^\mu{}_\nu$ or b^μ from the previous section. There is an implicit sum over independent a. In figure D.1 this transformation is shown schematically.

Let us now define a generator, G_a, of continuous symmetry transformations by

$$\delta_\omega\Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i\omega_a G_a \Phi(x), \tag{D.9}$$

where the sum is over independent generators. We can then determine G_a in the following manner.

[2]To see this, consider the defining property of the Lorentz group, $\Lambda^\alpha{}_\mu \eta_{\alpha\beta}\Lambda^\beta{}_\nu = \eta_{\mu\nu}$. Infinitesimally, we can take $\Lambda^\alpha{}_\mu \simeq \delta^\alpha{}_\mu + \omega^\alpha{}_\mu$; substitute this into the defining equation to obtain $\omega_{\mu\nu} = -\omega_{\nu\mu}$ to leading order.

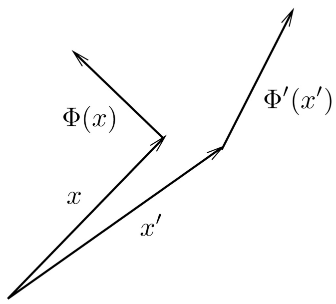

Figure D.1: Under an arbitrary continuous spacetime symmetry transformation $x \to x'$ the fields $\Phi(x) \to \Phi'(x')$.

With ω_a being small, let us Taylor expand $\Phi'(x')$ around x on account of (D.7), $\Phi'(x') = \Phi'(x) + \omega_a X_a \cdot \partial\Phi'(x) + \mathcal{O}(\omega^2)$. Dropping all $\mathcal{O}(\omega^2)$ terms we substitute this expression into (D.8) and hence find from (D.9) that $\delta_\omega \Phi(x) = -\omega_a X_a \cdot \partial\Phi'(x) + \omega_a \mathcal{F}_a(\Phi(x))$. Next, Taylor expand $\Phi'(x)$ around x' and make use of (D.8) once again to find

$$\delta_\omega \Phi(x) = -\omega_a X_a^\mu \partial_\mu \Phi(x) + \omega_a \mathcal{F}_a(\Phi(x)). \tag{D.10}$$

Then, from (D.10) and (D.9) we learn that

$$iG_a \Phi(x) = X_a^\mu \partial_\mu \Phi(x) - \mathcal{F}_a(\Phi(x)). \tag{D.11}$$

In general the transformation of the fields, paramterized by $\mathcal{F}_a(\Phi(x))$, will depend on the particular fields present but the *algebra* derived from the generators G_a should be independent of representation, independent of $\mathcal{F}_a(\Phi(x))$. We shall therefore choose a representation where $\Phi(x)$ transforms as a scalar, $\Phi'(x') = \Phi(x)$ and hence write (D.11) as

$$iG_a \Phi(x) = X_a^\mu \partial_\mu \Phi(x). \tag{D.12}$$

The commutation relations derived from the generators in (D.12) are also required to be satisfied by the generators in (D.11) and this in turn determines the possible forms of $\mathcal{F}_a(\Phi(x))$. In this sense we can go from the group structure to the field content of a given theory (rather than the other way round).

We can now apply these results to derive the generators of conformal transformations and the resulting algebra. From (D.12) it follows that we should determine each of the X_a^μ to derive the generators. For (D.7) and (D.2) to be consistent we require $\epsilon^\mu(x) = \omega_a X_a^\mu$. Let us then consider each of the possibilities for $\epsilon(x)$ separately:

- $\underline{\epsilon^\mu(x) = a^\mu \text{ (translations)}}$: For translations we need to set $\omega^a X_a^\mu = a^\mu$. Recalling that the ω_a correspond to infinitesimal parameters we find that this will hold true provided we take $\omega^a \to a^\nu$ and $X_a^\mu \to \delta^\mu{}_\nu$. Then, from (D.12) we find that the generator of translations, write $G_a \to P_\mu$, will be

$$P_\mu = -i\partial_\mu \qquad \text{(generator of translations)} \tag{D.13}$$

- $\underline{\epsilon^\mu(x) = \omega^\mu{}_\nu x^\nu \text{ (rotations)}}$: For rotations we need to set $\omega^a X_a^\mu = \omega^\mu{}_\nu x^\nu$. Let us then take $\omega^a X_a^\mu \to \omega^{\nu\rho} X^\mu{}_{\nu\rho}$ to find

$$\begin{aligned}
\omega^{\nu\rho} X^\mu{}_{\nu\rho} &= \omega^\mu{}_\rho x^\rho \\
&= \omega^{\nu\rho} \delta^\mu{}_\nu x_\rho \\
&= \omega^{\nu\rho} \frac{1}{2} \left(\delta^\mu{}_\nu x_\rho - \delta^\mu{}_\rho x_\nu \right),
\end{aligned} \tag{D.14}$$

where we have antisymmetrized the $\nu\rho$ indices to ensure that $X^\mu{}_{\nu\rho} = -X^\mu{}_{\rho\nu}$. Note that the symmetric contribution vanishes due to the antisymmetry of ω. We can now read off X from (D.14) and on account of (D.12), writing $G_a \to \frac{1}{2} L_{\mu\nu}$ (the factor of $\frac{1}{2}$ enforces no overcounting in (D.9)), we find that the generator of rotations is

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \qquad \text{(generator of rotations)} \tag{D.15}$$

- $\underline{\epsilon^\mu(x) = \lambda x^\mu \text{ (dilatations)}}$: For dilatations we set $\omega^a X_a^\mu = \lambda x^\mu$. This suggests we take $\omega^a X_a^\mu \to \lambda X^\mu$ to find $\lambda X^\mu = \lambda x^\mu$. Reading off X we find from (D.12), writing $G_a \to D$, that

$$D = -i(x \cdot \partial) \qquad \text{(generator of dilatations)} \tag{D.16}$$

- $\underline{\epsilon^\mu(x) = b^\mu x^2 - 2(b \cdot x) x^\mu \text{ (SCT)}}$: Proceeding as above for SCT we set $\omega^a X_a^\mu = b^\mu x^2 - 2(b \cdot x) x^\mu$. This suggests we take $\omega^a X_a^\mu \to b^\nu X^\mu{}_\nu$, leading to

$$\begin{aligned}
b^\nu X^\mu{}_\nu &= b^\mu x^2 - 2(b \cdot x) x^\mu, \\
&= b^\nu (\delta^\mu{}_\nu x^2 - 2 x_\nu x^\mu).
\end{aligned}$$

Reading off X and on account of (D.12), writing $G_a \to K_\nu$, we find

$$K_\nu = -i(x^2 \partial_\nu - 2 x_\nu (x \cdot \partial)) \qquad \text{(generator of SCT)} \tag{D.17}$$

From the above generators of conformal transformations one can verify after a certain amount of algebra that they satisfy the following algebra, which is isomorphic[3] to $\mathfrak{o}(p+1, q+1)$,

$$\begin{aligned}
&[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\
&[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}) \\
&[D, P_\mu] = i P_\mu \\
&[D, K_\mu] = -i K_\mu \\
&[K_\mu, P_\nu] = -2i(\eta_{\mu\nu} D - L_{\mu\nu}) \\
&[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu)
\end{aligned} \tag{D.18}$$

[3] Gothic letters are used here to label algebras; this is to distinguish algebras from groups which are denoted by capital letters in accordance with standard convention.

All other commutators vanish. The first two commutation relations correspond to the Poincaré subalgebra $\mathfrak{so}(p,q)$. Furthermore, one sees that there is also a slightly larger subalgebra that, in addition to the Poincaré symmetries, contains also dilatations. This has $d + d(d-1)/2 + 1$ generators P_μ, $L_{\mu\nu}$ and D obeying the first three commutation relations in (D.18). For $d = 2$ and $d = 4$ there are thus 4 and 11 generators respectively.

In total we have found that the *global* conformal algebra[4] $\mathfrak{o}(p,q)$ contains $d + d(d-1)/2 + 1 + d = (d+1)(d+2)/2$ generators, $d+1$ more generators than are present in the Poincaré algebra and is described by commutation relations (D.18). In $d = 2$ and $d = 4$ there are 6 and 15 generators respectively.

Let us now focus on $d = 2$ dimensions which is vastly richer than the general d-dimensional case considered above. The difference is due to an infinite local conformal symmetry that is generated in addition to the global conformal symmetry found there. This means that here the number of generators is in fact infinite. We will see below how this arises and derive the $d = 2$ classical conformal algebra, sometimes referred to as the Witt algebra in order to gain some insight that will be valuable in the corresponding quantum conformal algebra, referred to as the Virasoro algebra, which differs from the Witt algebra due to an anomaly.

It is convenient here to work in flat *Euclidean* space, $\eta_{\mu\nu} \to \delta_{\mu\nu}$, with $\mu = \{1,2\}$. A general infinitesimal coordinate transformation, $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, will generate a conformal transformation if it satisfies the conformal Killing equation (D.3), which in $d = 2$ reduces to[5] $(\partial \cdot \epsilon)\delta_{\mu\nu} = \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)$, equivalently,

$$\partial_1 \epsilon_2(x) = -\partial_2 \epsilon_1(x), \qquad \partial_1 \epsilon_1(x) = \partial_2 \epsilon_2(x). \tag{D.19}$$

However, these are just the *Cauchy-Riemann* equations and so there exists a holomorphic and an anti-holomorphic function $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ respectively, such that

$$\epsilon(z) = \epsilon_1(x) + i\epsilon_2(x),$$
$$\bar{\epsilon}(\bar{z}) = \epsilon_1(x) - i\epsilon_2(x).$$

We can interpret this result in the following way. If such holomorphic and anti-holomorphic functions, exist then the conformal Killing equation will be satisfied and hence so will the requirement of the transformation being conformal. We have introduced the complex coordinates[6] $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ with $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$, which under general infinitesimal coordinate transformations preserve their (anti-)holomorphy in the sense that

$$\begin{aligned} \Omega : z \to z' &= x_1' + ix_2' \\ &= x_1 + \epsilon_1 + i(x_2 + \epsilon_2) \\ &= z + \epsilon(z), \end{aligned}$$

[4] The resulting algebra is global because the general solution of (D.3) is defined globally, for all $x \in \mathbb{R}^d$.
[5] We shall not distinguish between upper or lower indices when working in Euclidean space.
[6] Note that $\partial \equiv \partial_z \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial_{\bar{z}} \equiv \partial/\partial\bar{z}$.

and likewise for $\bar{z}$. Therefore, (trivially generalizing to finite transformations) we have found that a 2-dimensional coordinate transformation is *conformal* provided it maps

$$\boxed{z \to f(z), \quad \bar{z} \to \bar{f}(\bar{z}),} \tag{D.20}$$

for *all* analytic functions $f(z), \bar{f}(\bar{z})$, *regardless* of the precise $z, \bar{z}$-dependance of $f(z)$ and $\bar{f}(\bar{z})$. Indeed, one can check that (D.1) is satisfied by first changing coordinates from $\{x\} = \{x_1, x_2\}$ to $\{x'\} = \{z, \bar{z}\}$, $ds^2 = \delta_{\mu\nu}dx^\mu dx^\nu = 2g_{z\bar{z}}dzd\bar{z}$, with $g_{z\bar{z}} = 1/2$ and then noticing that a conformal transformation takes the form (using the Jacobian $J = |\partial(f,\bar{f})/\partial(z,\bar{z})|$ and (D.20)) $\Omega : dzd\bar{z} \to |\partial f|^2 \, dzd\bar{z}$, from which we deduce that $\Omega(z,\bar{z}) = |\partial f|^2$. Therefore, all (anti-)holomorphic functions $f(z), \bar{f}(\bar{z})$ generate conformal transformations which is a remarkable result, unique to the case of 2 dimensions.

By inspection of (D.19) we see that $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ need not be defined globally on the Riemann sphere, $S^2 = \mathbb{C} \cup \infty$. This observation will in turn lead to both a local (the set of all not necessarily invertible holomorphic mappings) and a global conformal group, both of which will turn out to have physical implications in string theory. To see this let us write down the most general solution of (D.19) in complex coordinates which according to (D.20) will correspond to a Laurent series of $\epsilon(z)$,

$$\begin{aligned} \Omega : z \to z' &= z + \epsilon(z) \\ &= z + \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}. \end{aligned} \tag{D.21}$$

It immediately becomes manifest that not all choices of the infinitesimal c-number coefficients ϵ_n generate transformations that are globally defined on S^2; at $z = 0$ or ∞ the conformal transformation is ill-defined for arbitrary ϵ_n.

We next determine the generators of conformal symmetry. The generators of continuous symmetries, G_a, were defined in (D.9),

$$\delta_\omega \Phi(x) \equiv -i\omega_a G_a \Phi(x),$$

and for an arbitrary transformation $x^\mu \to x'^\mu = x^\mu + \omega_a X^\mu_a$ we saw that in the scalar representation of the fields, where $\Phi(x) \to \Phi'(x') = \Phi(x)$, the generators were found to be given by (D.12): $iG_a\Phi(x) = X^\mu_a \partial_\mu \Phi(x)$. Changing coordinates $(x_1, x_2) \to (z, \bar{z})$ and on account of (D.12), (D.7) and (D.21), we learn that[7]

$$\begin{aligned} i\sum_a \omega_a G_a &= \sum_a \omega_a X^\mu_a \partial_\mu \\ &= \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}\partial + \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n \bar{z}^{n+1}\bar{\partial} \\ &\equiv -\sum_{n=-\infty}^{\infty} \epsilon_n L_n - \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n \bar{L}_n, \end{aligned} \tag{D.22}$$

[7]It is to be understood that $\delta x = x' - x$ and likewise for $z, \bar{z}$.

where in the last line we defined the generators of $d = 2$ conformal transformations,

$$L_n = -z^{n+1}\partial, \quad \text{and} \quad \bar{L}_n = -\bar{z}^{n+1}\bar{\partial}. \tag{D.23}$$

The sum in (D.22) over a, μ is identified with the sum over independent generators, i.e. the sum over both anti-holomorphic and holomorphic sectors and the sum over n. The classical conformal algebra in 2 dimensions, the so-called *Witt algebra*, then follows directly from (D.23),

$$[L_n, L_m] = (n-m)L_{n+m}, \quad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m}. \tag{D.24}$$

All other commutators vanish. Therefore, the conformal algebra in 2-d is the direct sum $\mathcal{A} \oplus \bar{\mathcal{A}}$ of two isomorphic subalgebras (D.24) with $\dim(\mathcal{A}) = \dim(\bar{\mathcal{A}}) = \infty$ (there is an infinite number of generators). We may regard $z, \bar{z}$ as independent variables given that the algebras are independent and enforce the physical condition $\bar{z} = z^*$ at our convenience.

The algebra, as was expected from the above comments, is *local* as the generators are not defined globally on S^2. For general dimensionality, see comments below (D.18), the conformal algebra (defined globally) in $d = p+q$ dimensions for a flat Minkowski spacetime of signature (p, q) is $\mathfrak{o}(p+1, q+1)$, and so we expect to find the same subalgebra in (D.24) (with $p = 0$ and $q = 2$ in 2d Euclidean space) with the *same* number of generators, i.e. of the same dimensionality $\dim[\mathfrak{o}(1,3)] = (d+1)(d+2)/2|_{d=2} = 6$, as was found there. Indeed this is the case and, to see how this comes about, consider the conformal transformations generated by vector fields

$$v(z) = -\sum_n a_n L_n = \sum_n a_n z^{n+1}\partial.$$

Now we can make the following definition:

> *The global conformal group corresponds to the group of conformal transformations that are well defined and invertible on $S^2 = \mathbb{C} \cup \infty$.*

This condition implies that we should enforce the constraint $|v(z)| < \infty$ for all $z \in S^2$ in order to derive the global subalgebra. For this condition to be satisfied we see that $v(z)$ will be well behaved at $z = 0$ provided $a_n = 0$ for all $n < -1$. To probe the region $z \to \infty$, which corresponds to the only other region where a singularity may be encountered, make a conformal transformation $z \to -1/z$ which takes ∂ to $z^2\partial$. We then see that $v(z)$ will be well defined provided $a_n = 0$ for all $n > 1$. To summarize, $v(z)$ will be globally defined if $a_n = 0$ for all $|n| > 1$, i.e. $a_n = 0$ for all $n \neq \{-1, 0, +1\}$. The argument for the anti-holomorphic part is identical. This means that the relevant generators for the global conformal group should be $\{L_{-1}, L_0, L_1\} \cup \{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$. Indeed, from (D.24) it follows that the algebra associated with these generators closes and thus there is a subgroup of the local conformal group which corresponds to the global conformal group:

$$\begin{aligned} &[L_{\pm1}, L_0] = \pm L_{\pm1} \qquad [\bar{L}_{\pm1}, \bar{L}_0] = \pm\bar{L}_{\pm1} \\ &[L_{+1}, L_{-1}] = 2L_0 \qquad [\bar{L}_{+1}, \bar{L}_{-1}] = 2\bar{L}_0 \end{aligned} \tag{D.25}$$

This is the group $SL(2,\mathbb{C})/\mathbb{Z}_2 \simeq SO(1,3)$ which differs from $SU(2)$ by signs. It is perhaps curious that the 2-dimensional global conformal group is isomorphic to the 4-dimensional Lorentz group. The dimension of $SL(2,\mathbb{C})/\mathbb{Z}_2$ is indeed 6 as expected from the general d-dimensional case described above. The finite form of the global transformations (see equation (D.21))

$$\delta z = \epsilon_{-1} + \epsilon_0 z + \epsilon_{+1} z^2, \quad \delta \bar{z} = \bar{\epsilon}_{-1} + \bar{\epsilon}_0 \bar{z} + \bar{\epsilon}_{+1} z^2,$$

is

$$z \to z' = \frac{az+b}{cz+d}, \quad \bar{z} \to \bar{z}' = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}, \tag{D.26}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. The global conformal group with algebra (D.25) remains an exact symmetry group whereas the local conformal group with commutation relations (D.24) suffer from an anomaly. When the anomaly is taken into account, the Witt algebra is referred to as the *Virasoro algebra*, which is discussed in Sec. 3.1.

We can now also confirm that the generators of the global conformal group do indeed generate translations, rotations, dilatations and special conformal transformations on the complex plane, as was shown for the d-dimensional case in (D.17), (D.16), (D.15), and (D.13). In particular,

$$L_{-1} = -\partial \to e^{-\epsilon_{-1} L_{-1}} z = z + \epsilon_{-1} \qquad \text{(translations)}$$

$$i(L_0 - \bar{L}_0) = -i(z\partial - \bar{z}\bar{\partial}) \to e^{-i(\epsilon_0 L_0 - \bar{\epsilon}_0 \bar{L}_0)} z = (1 + i\epsilon_0) z \qquad \text{(rotations)}$$

$$L_0 + \bar{L}_0 = -z\partial - \bar{z}\bar{\partial} \to e^{-\epsilon_0 L_0 - \bar{\epsilon}_0 \bar{L}_0} z = (1 + \epsilon_0) z \qquad \text{(dilatations)}$$

$$L_1 = -z^2 \partial \to e^{-\epsilon_1 L_1} z = z + \epsilon_1 z^2 \qquad \text{(special conformal transformations)}$$

Appendix E

Path Integral over Embeddings

In this appendix we include a derivation of the path integral over embeddings at fixed worldsheet metric with source current, $J(z,\bar{z})$, in non-compact spacetimes [114]:

$$\Big\langle\!\Big\langle e^{i\int d^2z J(z,\bar{z})\cdot X(z,\bar{z})}\Big\rangle\!\Big\rangle = i(2\pi)^d\delta^d(J_0)e^{-\frac{1}{2}\int d^2z\int d^2z' J(z,\bar{z})\cdot J(z',\bar{z}')G'(z,z')}, \tag{E.1}$$

and a derivation of the corresponding functional derivative,

$$\prod_{l\in\mathcal{I}_A^n}\Big(-iD_l\frac{\delta}{\delta J_{\mu_l}(z_l,\bar{z}_l)}\Big)\exp\Big\{-\frac{1}{2}\int d^2z\int d^2z' J(z,\bar{z})\cdot J(z',\bar{z}')G(z,z')\Big\} = \sum_{k=0}^{\lfloor\mathcal{I}/2\rfloor}\sum_{\pi\in S_{\mathcal{I}}/\sim}$$
$$\prod_{l=1}^{k}\Big\{\eta^{\mu_{\pi(2l-1)}\mu_{\pi(2l)}}D_{\pi(2l-1)}D_{\pi(2l)}G(z_{\pi(2l-1)},z_{\pi(2l)})\Big\}\prod_{q=2k+1}^{\mathcal{I}}\Big\{i\int d^2z J^{\mu_{\pi(q)}}(z,\bar{z})D_{\pi(q)}G(z_{\pi(q)},z)\Big\}$$
$$\times\exp\Big\{-\frac{1}{2}\int d^2z\int d^2z' J(z,\bar{z})\cdot J(z',\bar{z}')G(z,z')\Big\},$$

with the expectation value $\langle\!\langle\dots\rangle\!\rangle$ defined by,

$$\langle\!\langle\dots\rangle\!\rangle \equiv \Big(\frac{4\pi^2\alpha'}{\int_\Sigma d^2z\sqrt{g}}\det{}'\Delta_{(0)}\Big)^{d/2}\int_{\mathcal{E}}\mathcal{D}X e^{-S_G[X,g]}\dots \tag{E.2}$$

The path integral (E.1) can be evaluated for arbitrary genus Riemann surfaces and topological information enters only via the Green's function $G(z,w)$ (and the Euler characteristic), the general (i.e. multi-loop) computation of which has been given in Appendix G. Let us first compute $\langle\!\langle 1\rangle\!\rangle$ for arbitrary worldsheet topology. The bosonic string theory action, $S_G[g,X]$, is given in (3.22),

$$S_G[g,X] = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\partial_z X\cdot\partial_{\bar{z}}X + \dots, \tag{E.3}$$

We will be working in flat (Euclidean) spacetime, $\delta_{\mu\nu}$, and assume the dilaton has acquired a vacuum expectation value $\langle\Phi\rangle$. The counter-term $\mu^2\int_\Sigma d^2z\sqrt{g}$ is not relevant for this computation.

Introducing the Laplacian (defined in Appendix B) $\Delta_{(0)} = -2g^{z\bar{z}}\partial_z\partial_{\bar{z}}$, we can rewrite $S_G[g, X]$ as

$$\begin{aligned} S_G[g, X] &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2z\sqrt{g}X\cdot\Delta_{(0)}X + \frac{1}{\alpha'}\oint_{\partial\Sigma}\frac{dz}{2\pi i}X\cdot\partial X, \\ &= \frac{1}{4\pi\alpha'}\left(X, \Delta_{(0)}X\right) \end{aligned} \tag{E.4}$$

The requirement that the surface term

$$S_{\partial\Sigma} = \frac{1}{\alpha'}\oint_{\partial\Sigma}\frac{dz}{2\pi i}X\cdot\partial X,$$

vanishes, leads to the various possible string configurations: for closed strings only there is no boundary and so $S_{\partial\Sigma} = 0$, for open strings we can impose Dirichlet boundary conditions, $X|_{\partial\Sigma} = 0$, on some of the coordinates (in the directions transverse the brane) or Neumann boundary conditions, $\partial X|_{\partial\Sigma} = 0$, (in the brane worldvolume directions). In (B.5) we identified the natural inner product between two rank-n tensors. Viewing the embedding X as a worldsheet scalar, i.e. $X \in K^0$ see (B.1), the unique Poincaré and diffeomorphism invariant metric $(\delta X, \delta X) = \|\delta X\|^2$ is,

$$\|\delta X\|^2 = \int_\Sigma d^2z\sqrt{g}\delta X\cdot\delta X. \tag{E.5}$$

To evaluate the path integral decompose X into a complete set of eigenstates, $\psi_n(z, \bar{z})$, of the Laplacian,

$$\begin{aligned} X^\mu(z, \bar{z}) &= \sum_n a_n^\mu\psi_n(z, \bar{z}), \\ &= X_0^\mu + X'^\mu(z, \bar{z}), \end{aligned} \tag{E.6}$$

such that

$$\Delta_{(0)}\psi_n = \lambda_n\psi_n. \tag{E.7}$$

Denote the zero mode $\psi_0 = \ker\Delta_{(0)}$ (i.e. $\lambda_0 = 0$) contribution by $X_0 = a_0\psi_0$ and other modes orthogonal to it by X', $(X_0, X') = 0$, leading to $\mathcal{D}X = (\prod_\mu dX_0^\mu)\mathcal{D}X'$. Again using the scalar inner product we have the orthogonal decomposition

$$(\psi_m, \psi_n) \equiv \int_\Sigma d^2z\sqrt{g}\psi_m\psi_n = \delta_{mn}. \tag{E.8}$$

Then it follows that the zero mode ψ_0 is given by

$$\psi_0 = \left(\int_\Sigma d^2z\sqrt{g}\right)^{-1/2}. \tag{E.9}$$

We next define the measure $\mathcal{D}X$ and do so by requiring

$$\begin{aligned} 1 &=: \int\mathcal{D}Xe^{-\|X\|^2/4\pi\alpha'}, \\ &= \int\prod_\mu dX_0^\mu e^{-\|X_0\|^2/4\pi\alpha'}\int\mathcal{D}X'e^{-\|X'\|^2/4\pi\alpha'}, \\ &= \left(\frac{4\pi^2\alpha'}{\int_\Sigma d^2z\sqrt{g}}\right)^{d/2}\int\mathcal{D}X'e^{-\|X'\|^2/4\pi\alpha'}, \end{aligned} \tag{E.10}$$

on account of (E.5) and (E.6)[1]. It then follows that the path integral over X is given by

$$
\begin{aligned}
\int \mathcal{D}X e^{-S_G[g,X]} &= \int \mathcal{D}X e^{-(X,\Delta_g X)/4\pi\alpha'}, \\
&= \left(\int \prod_\mu dX_0^\mu\right) \int \mathcal{D}X' e^{-(X',\Delta_g X')/4\pi\alpha'}, \\
&= \left(\int \prod_\mu dX_0^\mu\right) \left(\frac{4\pi^2\alpha'}{\int_\Sigma d^2z\sqrt{g}}\det{}'\Delta_g\right)^{-d/2}, \qquad \text{(E.11)}
\end{aligned}
$$

on account of the normalization (E.10). The prime on the determinant denotes the exclusion of zero mode contributions, these have been factored out and lead to the overall spacetime volume contribution $\left(\int \prod_\mu dX_0^\mu\right)$. Therefore, $\langle\langle 1\rangle\rangle = \left(\int \prod_\mu dX_0^\mu\right)$, as follows form the definition (E.2).

Let us next introduce a source $J(z,\bar{z})$ and insert this into the definition (E.2),

$$
\begin{aligned}
\langle\langle e^{i\int d^2z J\cdot X}\rangle\rangle &= \left(\frac{4\pi^2\alpha'}{\int_\Sigma d^2z\sqrt{g}}\det{}'\Delta_{(0)}\right)^{d/2} \int_{\mathcal{E}} \mathcal{D}X e^{-(X,\Delta_g X)/4\pi\alpha' + i\int d^2z J\cdot X}, \\
&= \left(\frac{4\pi^2\alpha'}{\int_\Sigma d^2z\sqrt{g}}\det{}'\Delta_{(0)}\right)^{d/2} \int \prod_\mu dX_0^\mu e^{iJ_0\cdot X_0} \int \mathcal{D}X' e^{-(X',\Delta_g X')/4\pi\alpha' + i\int d^2z J\cdot X'}, \\
&= (2\pi)^d \delta^d(J_0) \left(\frac{4\pi^2\alpha'}{\int_\Sigma d^2z\sqrt{g}}\det{}'\Delta_{(0)}\right)^{d/2} \int \mathcal{D}Y e^{-(Y,\Delta_g Y)/4\pi\alpha'} e^{-\frac{1}{2}\int d^2z\int d^2z' J\cdot J' G'(z,z')},
\end{aligned}
$$

where we have defined

$$
J_0 = \int_\Sigma d^2z J(z,\bar{z}). \qquad \text{(E.12)}
$$

In going from the second to the third equality we have completed the square and we have formally identified the Green's function $G'(z,z')$ with the inverse of the Laplace operator $\Delta_{(0)}^{-1}$. Notice that this excludes zero mode contributions (these have been factored out) and can hence be inverted. We have put a prime on the Green's function to denote this (but in most of the main body of the text we shall drop the prime, $G'(z,z') \to G(z,z')$). Now the remaining path integral excludes zero modes, is Gaussian and has been computed in (E.11) leading to

$$
\langle\langle e^{i\int d^2z J(z,\bar{z})\cdot X(z,\bar{z})}\rangle\rangle = i(2\pi)^d \delta^d(J_0) e^{-\frac{1}{2}\int d^2z\int d^2z' J(z,\bar{z})\cdot J(z',\bar{z}')G'(z,z')} \qquad \text{(E.13)}
$$

We have introduced an overall factor of i needed for the proper interpretation of (E.13) as an S-matrix (see e.g. [114]), or more specifically as the *non-trivial contribution* to the S-matrix. In essence, this factor arises from Wick rotating from Euclidean back to Minkowski space.[2]

[1] Recall that for Gaussian integration $\int dx\, e^{-x^2/2} = \sqrt{2\pi}$.

[2] The integration over embeddings is treated a bit more carefully in Moore and Nelson [182], p. 69, but the result is the same. In particular, it is not *a priori* clear that we are permitted to integrate over all embeddings without over-counting because the theory is in general invariant under spacetime diffeomorphisms which in flat spacetime in particular reduces to Poincaré transformations.

To compute scattering amplitudes we also need to the following result for functional derivatives of the above formula,

$$\prod_{l\in\mathcal{I}_A^n}\left(D_l\frac{\delta}{\delta J_{\mu_l}(z_l,\bar{z}_l)}\right)e^{-\frac{1}{2}\int d^2z\int d^2z' J(z,\bar{z})\cdot J(z',\bar{z}')G(z,z')}=$$
$$\sum_{k=0}^{\lfloor\mathcal{I}/2\rfloor}\sum_{\pi\in S_{\mathcal{I}}/\sim}\prod_{l=1}^{k}\Big\{-\eta^{\mu_{\pi(2l-1)}\mu_{\pi(2l)}}D_{\pi(2l-1)}D_{\pi(2l)}G(z_{\pi(2l-1)},z_{\pi(2l)})\Big\}$$
$$\times\prod_{q=2k+1}^{\mathcal{I}}\left\{-\int d^2z J^{\mu_{\pi(q)}}(z,\bar{z})D_{\pi(q)}G(z_{\pi(q)},z)\right\}e^{-\frac{1}{2}\int d^2z\int d^2z' J(z,\bar{z})\cdot J(z',\bar{z}')G(z,z')},\tag{E.14}$$

It is to be understood that the notation $\lfloor\mathcal{I}/2\rfloor$ in the sum over k indicates that the maximum value of k saturates the inequality $k\le\mathcal{I}/2$. $S_{\mathcal{I}}$ is the *symmetric group* of degree $\mathcal{I}$ [189], the group of all permutations of $\mathcal{I}$ elements, and the equivalence relation $\sim$ is such that $\pi_i\sim\pi_j$ with $\pi_i,\pi_j\in S_{\mathcal{I}}$ when they define the same element in (E.14). We have derived equation (E.14) by induction, and it can be thought of as the generalization of the functional version of,

$$\frac{\partial^n}{\partial x^n}e^{-\frac{1}{2}ax^2}=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{2^{-k}n!}{k!(n-2k)!}(-a)^k(-ax)^{n-2k}e^{-\frac{1}{2}ax^2}.$$

By induction, it is also possible to show that for a given k the number of terms that appear in the sum over permutations in (E.14) is indeed:

$$\frac{2^{-k}\mathcal{I}!}{k!(\mathcal{I}-2k)!},$$

as one would expect from the finite dimensional formula.

Appendix F

Moduli Space of Metrics

In this appendix we provide a rather detailed overview of the derivation of the path integral measure associated to metrics. For a more extensive treatment that is closest to our approach see e.g. [230, 150, 141, 226]. Note furthermore that compactification of string on compact spacetimes will not affect the measure which is purely local. In fact, the results of the current section also hold for arbitrary matter conformal field theories, arbitrary spacetime backgrounds that give rise to conformal field theories, not just flat Minkowski space.

We would like to determine the deformations of the metric that are orthogonal to diffeomorphisms,[1] $\mathrm{Diff}(\Sigma)$, and Weyl transformations, $\mathrm{Weyl}(\Sigma)$. We shall find that there are deformations of metric that cannot be reached by a combined $\mathrm{Weyl}(\Sigma) \ltimes \mathrm{Diff}(\Sigma)$ transformation. These are the *moduli deformations* and correspond to physically distinct (or gauge inequivalent) deformations. Denote by $\mathrm{Met}(\Sigma)$ the space of positive definite worldsheet metrics g. Then *moduli space* of genus h Riemann surfaces corresponds to the space of orbits of $\mathrm{Weyl}(\Sigma) \ltimes \mathrm{Diff}(\Sigma)$ in $\mathrm{Met}(\Sigma)$,

$$\mathcal{M}_h \equiv \mathrm{Met}(\Sigma)/\mathrm{Weyl}(\Sigma) \ltimes \mathrm{Diff}(\Sigma). \tag{F.1}$$

See also Fig. 3.3 for an example of a gauge slice. The semi-direct product symbol signifies that there is an overlap between $\mathrm{Diff}(\Sigma)$ and $\mathrm{Weyl}(\Sigma)$. This overlap is generated by conformal Killing vectors (CKV) and we shall neglect it for the time being; we will come back to it later, towards the end of this section. The metric deformations that are connected to the identity span the space of orbits of $\mathrm{Weyl}(\Sigma) \ltimes \mathrm{Diff}_0(\Sigma)$ in $\mathrm{Met}(\Sigma)$. This space is known as *Teichmüller space*, $\mathcal{T}_h$,

$$\mathcal{T}_h \equiv \mathrm{Met}(\Sigma)/\mathrm{Weyl}(\Sigma) \ltimes \mathrm{Diff}_0(\Sigma). \tag{F.2}$$

We shall construct the measure associated to metrics on $\mathcal{T}_h$. Then, to obtain the full measure we first define the mapping class group

$$\mathrm{MCG}_h \equiv \big(\mathrm{Diff}(\Sigma)/\mathrm{Diff}_0(\Sigma)\big)_h.$$

[1] $\mathrm{Diff}(\Sigma)$ contains both global diffeomorphisms, $\mathrm{Diff}_{\rm gl}(\Sigma)$, and diffeomorphisms connected to the identity, $\mathrm{Diff}_0(\Sigma)$.

With this definition, we have $\mathcal{M}_h = \mathcal{T}_h/\mathrm{MCG}_h$ and we obtain an integral over moduli space as follows,

$$\frac{1}{|\mathrm{MCG}_h|}\int_{\mathcal{T}_h}\mathcal{D}g = \int_{\mathcal{M}_h}\mathcal{D}g, \tag{F.3}$$

where $|\mathrm{MCG}_h|$ equals the number of elements (the cardinality) of the mapping class group. MCG is a discrete group which acts holomorphically with fixed points [141]. Given that

$$\mathcal{M}_h = \mathcal{T}_h/\mathrm{MCG}_h,$$

it is then seen that moduli space has the structure of an orbifold. For example, in the one-loop case $\mathcal{T}_1$ corresponds to the upper-half complex plane, defined in (C), $\mathcal{H}_1$, and the effect of dividing by $|\mathrm{MCG}|$ in (F.3) is equivalent to restricting the integration region to a fundamental domain, often denoted by $\mathcal{F}$, given by

$$\mathcal{M}_1 \equiv \mathcal{F} = \mathcal{H}_1/\mathrm{SL}(2,\mathbb{Z}).$$

After these introductory remarks let us now proceed with the decomposition of the measure. Focusing on metric deformations that are connected to the identity, the basic principle in the construction of the path integral measure associated to metrics is the following: we *orthogonally* decompose metric deformations,

$$\{\delta g\} = \{\delta g_{\mathrm{Weyl}}\} \oplus \{\delta g_{\mathrm{Diff}_0^\perp}\} \oplus \{\delta g_{\mathrm{mod}}\},$$

and include only moduli deformations, $\{\delta g_{\mathrm{mod}}\}$, in the path integral measure $\mathcal{D}g$, so as to ensure no over-counting. $\mathrm{Diff}_0^\perp(\Sigma)$ is the space of diffeomorphisms connected to the identity that cannot be reached by conformal transformations, i.e. $\delta g_{\alpha\beta} \propto g_{\alpha\beta}$. Note furthermore that deformations generated by conformal Killing vectors, $\{\delta g_{\mathrm{CKV}}\}$, are contained in $\{\delta g_{\mathrm{Weyl}}\}$. We shall elaborate on all these issues below. Having obtained the moduli deformations that are connected to the identity we may then use the prescription (F.3) in order to obtain the path integral measure over the full moduli space.

We work in conformal gauge:[2]

$$g = g_{z\bar{z}}\,(dz\otimes d\bar{z} + d\bar{z}\otimes dz) \in K^{(1,1)}. \tag{F.4}$$

An *arbitrary* deformation of metric close to the identity can always be written as $g+\delta g$ with

$$\delta g = 2\delta g_{z\bar{z}}|dz|^2 + \delta g_{\bar{z}\bar{z}}d\bar{z}^2 + \delta g_{zz}dz^2 \tag{F.5}$$

and this corresponds to the orthogonal decomposition,

$$\{\delta g\} = K^{(1,1)} \oplus K^{(2,0)} \oplus K^{(0,2)}. \tag{F.6}$$

We now go on to show that the space of tensors $K^{(1,1)}$ can be associated solely to Weyl variations of metric. In addition, we shall see that the space $K^{(2,0)} \oplus K^{(0,2)}$ contains the range of $\mathrm{Diff}_0(\Sigma)$ and *also* moduli deformations.

Arbitrary diffeomorphisms and Weyl transformations of g close to the identity take the following forms:[3]

[2]The space of tensors $K^{(n,m)}$ is defined in Appendix B.

[3]Covariant derivatives are defined in Appendix B.

- $\underline{\text{Diff}_0(\Sigma)}$:

$$\delta_{\text{Diff}_0} g_{\alpha\beta} = \nabla_\alpha \delta v_\beta + \nabla_\beta \delta v_\alpha \Rightarrow \begin{cases} \delta_{\text{Diff}_0} g_{z\bar{z}} = \left(\nabla_z^{(-1)} \delta v^z + \nabla^z_{(1)} \delta v_z\right) g_{z\bar{z}}, \\ \delta_{\text{Diff}_0} g_{zz} = 2\nabla_z^{(1)} \delta v_z, \\ \delta_{\text{Diff}_0} g_{\bar{z}\bar{z}} = 2\nabla_{\bar{z}}^{(-1)} \delta v_{\bar{z}}, \end{cases} \tag{F.7}$$

- $\underline{\text{Weyl}(\Sigma)}$:

$$\delta_{\text{Weyl}} g_{\alpha\beta} = 2\delta\sigma g_{\alpha\beta} \Rightarrow \begin{cases} \delta_{\text{Weyl}} g_{z\bar{z}} = 2\delta\sigma g_{z\bar{z}}, \\ \delta_{\text{Weyl}} g_{zz} = 0, \\ \delta_{\text{Weyl}} g_{\bar{z}\bar{z}} = 0,, \end{cases} \tag{F.8}$$

where the vector fields $\delta v_z \in K^{(1,0)}$ (and correspondingly $\delta v_{\bar{z}} \in K^{(0,1)}$)[4] generate diffeomorphisms and the scalar function $\delta\sigma : \Sigma \to \mathbb{R}$ generates Weyl transformations. Furthermore, use of $g_{z\bar{z}}$ has been made to raise and lower indices.

In the following two paragraphs we make two crucial observations upon which much of what follows will depend.

Notice from (F.7) and (F.8) that $\delta_{\text{Diff}_0} g_{z\bar{z}}$ can always be reached from a Weyl transformation and therefore its contribution is already taken into account in (F.8) by $\delta_{\text{Weyl}} g_{z\bar{z}} = 2\delta\sigma g_{z\bar{z}}$. We are then led to ask the following question: *Are there any arbitrary off-diagonal deformations of metric, $\delta g_{z\bar{z}}$, that cannot be written as $2\delta\sigma g_{z\bar{z}}$?* Well, the range of Weyl(Σ) is the full $K^{(1,1)}$ ($2\delta\sigma g_{z\bar{z}}$ can take arbitrary values) [150] and hence all deformations $\delta g_{z\bar{z}}$ can be written as Weyl deformations $\delta_{\text{Weyl}} g_{z\bar{z}}$ as given in (F.8)[5].

Likewise, for the diffeomorphisms $\delta_{\text{Diff}_0} g_{zz}$ we may ask the following question: *Are there any arbitrary diagonal deformations of metric, δg_{zz}, that cannot be written as $2\nabla_z^{(1)} \delta v_z$?* The answer to this question is yes and as such, deformations of this type correspond to *moduli deformations* (because they cannot be reached by the combined action of Weyl$(\Sigma) \ltimes$ Diff$_0(\Sigma)$). This is equivalent to saying that the range of Diff$_0(\Sigma)$ is not the full $K^{(2,0)} \oplus K^{(0,2)}$. We now proceed to prove this statement as follows.

Suppose that there are diagonal deformations in δg_{zz} that *are* orthogonal to $2\nabla_z^{(1)} \delta v_z$ and denote them by $\delta\phi_{zz}$:

$$\delta g_{zz} = 2\nabla_z^{(1)} \delta v_z + \delta\phi_{zz}. \tag{F.9}$$

[4] We shall not always refer explicitly to both holomorphic and anti-holomorphic components when no confusion should arise. We will mainly refer to the holomorphic sector and it is to be understood that identical arguments hold for the anti-holomorphic sector.

[5] Recall also the orthogonal decomposition (F.6) which prevents a tensor of one type from changing into a tensor of a different type under arbitrary variations.

Orthogonality is enforced by requiring that their inner product vanishes:

$$\begin{aligned} 0 &= \big(2\nabla^{(1)}_z\delta v_z, \delta\phi_{zz}\big) \\ &= \int d\mu_g\big(\nabla^z_{(1)}\delta v^z\big)\delta\phi_{zz} \\ &= \int d^2z g_{z\bar{z}}\delta v^z\big(-\nabla^z_{(2)}\delta\phi_{zz}\big). \end{aligned} \tag{F.10}$$

This should hold true for all vectors δv^z and hence it follows that $\delta\phi_{zz}$ will be orthogonal to $\delta_{\mathrm{Diff}_0}g_{zz}$ provided $\delta\phi_{zz} \in \mathrm{Ker}\nabla^z_{(2)}$. Now, $\nabla^z_{(2)}\delta\phi_{zz} = (g_{z\bar{z}})^{-1}\partial_{\bar{z}}\delta\phi_{zz}\otimes(dz)^{-1}$ and hence the requirement that $\delta\phi_{zz}$ be orthogonal to diffeomorphisms (and also Weyl transformations) is equivalent to requiring that it be holomorphic in z. The deformation $\delta\phi_{zz}$ is often referred to as a *holomorphic quadratic differential*. We thus have the following orthogonal decomposition

$$K^{(2,0)} = \underbrace{\mathrm{Range}\nabla^{(1)}_z}_{\delta_{\mathrm{Diff}_0}g_{zz}} \oplus \underbrace{\mathrm{Ker}\nabla^{(2)}_{\bar{z}}}_{\mathrm{moduli}}, \tag{F.11}$$

where $\nabla^{(2)}_{\bar{z}} = g_{z\bar{z}}\nabla^z_{(2)}$. We have therefore found that vectors $\phi_j = \mathrm{Ker}\nabla^{(2)}_{\bar{z}}$ live in the cotangent space of moduli $\mathcal{M}_h$. The number of moduli at a given genus h, determined by the range $j = 1, \dots, \mathrm{dim}\mathcal{M}_h$ with $\mathrm{dim}\mathcal{M}_h = \mathrm{dimKer}\nabla^{(2)}_{\bar{z}}$, follows from the *Riemann-Roch-Atiyah-Singer index theorem* (see e.g. [141]),

$$\dim_{\mathbb{C}}\mathrm{Ker}\nabla^{(n)}_z - \dim_{\mathbb{C}}\mathrm{Ker}\nabla^z_{(n+1)} = \frac{1}{2}\,(2n+1)\,\chi(\Sigma) = (2n+1)\,(1-h). \tag{F.12}$$

Taking $n=1$ we find that $\dim_{\mathbb{C}}\mathcal{M}_h = 3h - 3 + \dim_{\mathbb{C}}\mathrm{Ker}\nabla^{(1)}_z$. Tracing back the definitions of covariant derivatives we find that the second term on the right-hand side equals the number of independent solutions of $\partial_z v^{\bar{z}} = 0$. The solutions of this equation are called *conformal Killing vectors* and we shall elaborate on this connection below. The number of CKV's admitted by a genus h compact Riemann surface is [141, 114],

$$\dim_{\mathbb{C}}\ker\nabla^{(1)}_z = \dim_{\mathbb{C}}\ker\nabla^{(-1)}_{\bar{z}} = \begin{cases} 3 & \text{for} \quad h=0 \\ 1 & \text{"} \quad h=1 \\ 0 & \text{"} \quad h\geq 2 \end{cases} \tag{F.13}$$

To see this note that infinitesimal conformal transformations on a Riemann surface are locally of the form $z \to z + \epsilon(z)$. Laurent expanding gives $\epsilon(z) = \sum_{n=-\infty}^{\infty}\epsilon_n z^{n+1}$, every term in the sum being associated to a single generator L_n of the Virasoro algebra. CKV's are globally defined vectors, and so to determine the number of CKV's admitted by the Riemann surface we must require that the conformal transformation $z \to z + \epsilon(z)$ be globally well defined for each of the three cases:

- <u>$h=0$</u>: On S^2 $\epsilon(z)$ will be well defined at the origin provided $\epsilon_{n\leq -2} = 0$. From $S^2 = \mathbb{C}\cup\infty$ we see that it must also be well defined at infinity, and so we make a conformal transformation $z \to 1/z$, and notice that $\epsilon(z)$ will be well defined at the new origin provided $\epsilon_{n\geq 2} = 0$. We then find that infinitesimal conformal transformations of the form $z \to z+(a+bz+cz^2)$ are indeed globally defined and *three* linearly independent vectors a, bz, cz^2 may be identified with holomorphic CKV's. Similar remarks hold for the anti-holomorphic sector. A convenient basis for the CKV's is then,

$$\psi_1^z = 1, \qquad \psi_2^z = z, \qquad \text{and} \qquad \psi_3^z = z^2,$$

with similar expressions for $\psi_a^{\bar z}$, with $a = 1, 2, 3$. The above result for infinitesimal transformations generalizes as follows for finite transformations,

$$z \to \frac{az+b}{cz+d}, \qquad \text{with} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2,$$

with $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ the group of conformal automorphisms which act by Mobius transformations.

- $\underline{h=1}$: For the torus T^2 there is a single A- and a single B-cycle and the period matrix Ω is a single complex number, call it $\tau = \tau_1 + i\tau_2$, see (C.4). T^2 is then identified with the Jacobian variety (C.8), $J(\Sigma) = \mathbb{C}/\big(\mathbb{Z} + \tau\mathbb{Z}\big)$, so that we are to make the identification $z \sim z + m + \tau n$. Therefore, in order for the conformal transformation $z \to z + \epsilon(z)$ to be generated by CKV's we must require that $\epsilon(z)$ respect this periodicity: $\epsilon(z)$ must be doubly periodic. The only holomorphic doubly periodic functions are the (in general moduli-dependent) constants [231], and so there are two CKV's on T^2. These are the translations on the complex plane $\epsilon(z) = a$.

- $\underline{h \geq 2}$: In this case to show that there are no CKV's we assume the opposite: suppose that there is a solution to the defining equation $\nabla_z^{(1)}\bar v = 0$, with $\bar v = v_{\bar z} d\bar z$. Then, let us apply the differential operator $\nabla_z^{(-1)\dagger}$ to it giving $\Delta^+_{(-1)}\bar v = 0$, and on account of (B.8) conclude that $\Delta^-_{(-1)}\bar v = R_{(2)}\bar v$. Noting that a genus $h \geq 2$ Riemann surface is conformally related to a constant negative curvature surface, $R_{(2)} < 0$, we integrate $\Delta^-_{(-1)}\bar v$ versus $\bar v$ with respect to the inner product (B.5) and integrate by parts. This leads to $\|\nabla_z^{(-1)}\|^2 = R_{(2)}\|\bar v\|^2 < 0$, and we hence conclude that $\bar v = 0$. Therefore, there are no CKV's when $h \geq 2$: $\dim \mathrm{Ker}\nabla_z^{(-1)} = 0$. One can similarly show that, $\dim \mathrm{Ker}\nabla_{\bar z}^{(-1)} = 0$.

Given that $\dim_{\mathbb{C}}\mathcal{M}_h = 3h - 3 + \dim_{\mathbb{C}}\mathrm{Ker}\nabla_z^{(1)}$, we conclude that there are no moduli at genus $h = 0$, there is one complex modulus at genus $h = 1$ and $3h - 3$ complex moduli at genus $h \geq 2$. Note that for $h = 1$ the single modulus is to be identified with the period "matrix" $\Omega_{IJ} = \tau$, because the period matrix characterizes the complex structure of the surface, whereas for $h \geq 2$ the (complex) dimensionality of Ω_{IJ}, namely $h(h+1)/2$, is greater than the number of moduli, $3h - 3$, and so there are some redundant parameters in Ω_{IJ} – the moduli space at higher genus is in general unfortunately not well understood. (Notice however that also for $h = 2$ the entire period matrix is to be identified with a modulus.)

We will next introduce a set of coordinates on $\mathcal{M}_h$ and proceed as follows. An arbitrary metric close to (F.4) can be parametrized by

$$g+\delta g = 2\left(g_{z\bar z}+2\delta\sigma g_{z\bar z}\right)|dz+\delta\eta_{\bar z}{}^{z}d\bar z|^2, \tag{F.14}$$

where tensors of the type $\delta\eta_{\bar z}{}^{z}\in K^{(0,2)}$ and $\delta\bar\eta_{z}{}^{\bar z}\in K^{(2,0)}$ are referred to as *Beltrami differentials*. We have also used the fact that $\delta g_{z\bar z}$ can *always* be written as a Weyl transformation $2\delta\sigma g_{z\bar z}$ as we showed above. To leading order we deduce from (F.14), see also (F.5), that

$$\delta g = 4\delta\sigma g_{z\bar z}|dz|^2+\underbrace{2g_{z\bar z}\delta\eta_{\bar z}{}^{z}}_{\delta g_{\bar z\bar z}}d\bar z^2+\underbrace{2g_{z\bar z}\delta\eta_{z}{}^{\bar z}}_{\delta g_{zz}}dz^2. \tag{F.15}$$

Therefore,

$$\delta\eta_{\bar z}{}^{z}\in K^{(0,2)},\quad \delta\eta_{z}{}^{\bar z}\in K^{(2,0)}, \tag{F.16}$$

and so according to (F.11) we can project $\delta\eta$ onto moduli space and use the natural inner product to do so:

$$\big(\delta\eta,\phi_j\big)=\int_\Sigma dzd\bar z\,\delta\eta_{\bar z}{}^{z}\phi_{jzz},\quad \big(\delta\bar\eta,\bar\phi_j\big)=\int_\Sigma dzd\bar z\,\delta\eta_{z}{}^{\bar z}\phi_{j\bar z\bar z}. \tag{F.17}$$

This inner product is Weyl invariant (there is no dependence on $g_{z\bar z}$) and the kernel of $(\,\cdot\,,\phi_j)$ is the tangent space to the orbits of $\mathrm{Weyl}(\Sigma)\ltimes\mathrm{Diff}_0(\Sigma)$. For a given choice of ϕ_j and $\bar\phi_j$, the resulting *linear forms* define local complex coordinates $m_j,\bar m_j$ on moduli space $\mathcal{M}_h$:

$$\big(\delta\eta,\phi_j\big)=\delta m_j,\quad \big(\delta\bar\eta,\bar\phi_j\big)=\delta\bar m_j. \tag{F.18}$$

More generally, we may decompose $\delta\eta_{\bar z}{}^{z}$ according to

$$\delta\eta_{\bar z}{}^{z}=\nabla_{\bar z}^{(-1)}\delta v^z+\sum_{j=1}^{\dim\mathcal{M}_h}\delta m_j\mu_{j\bar z}^{z},\quad \delta\bar\eta_{z}{}^{\bar z}=\nabla_{z}^{(1)}\delta v^{\bar z}+\sum_{j=1}^{\dim\mathcal{M}_h}\delta\bar m_j\bar\mu_{jz}^{\bar z}, \tag{F.19}$$

with $\mu_{i\bar z}^{z}\in K^{(0,2)}$ (and correspondingly $\bar\mu_{iz}^{\bar z}\in K^{(2,0)}$) *Beltrami differentials* which can be chosen as follows

$$\mu_{j\bar z}^{z}=g^{z\bar z}\frac{\partial}{\partial m_j}g_{\bar z\bar z}(m_j,\bar m_j),\quad \bar\mu_{jz}^{\bar z}=g^{z\bar z}\frac{\partial}{\partial\bar m_j}g_{zz}(m_j,\bar m_j). \tag{F.20}$$

We thus derive from (F.19) and (F.11) that

$$\big(\delta\eta,\phi_j\big)=\sum_{i=1}^{\dim\mathcal{M}_h}\delta m_i\big(\mu_{i\bar z}^{z},\phi_{jzz}\big),\quad \big(\delta\bar\eta,\bar\phi_j\big)=\sum_{i=1}^{\dim\mathcal{M}_h}\delta\bar m_i\big(\bar\mu_{iz}^{\bar z},\bar\phi_{j\bar z\bar z}\big), \tag{F.21}$$

which is to be compared with (F.18). We see therefore that the *Beltrami differentials parametrize the gauge slice* and their corresponding projection onto ϕ (and $\bar\phi$) guarantees that we project our chosen slice onto moduli space. In particular, their projection onto ϕ_j guarantees that changes in moduli δm_j correspond to true deformations of metric. See also Fig. F.1.

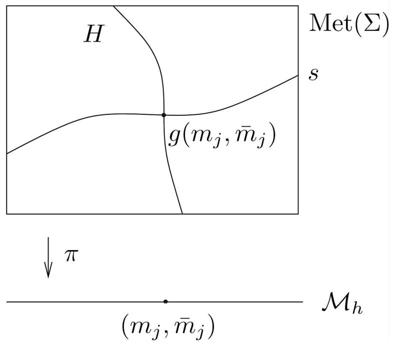

Figure F.1: A section s of moduli space $\mathcal{M}_h$ in Met(Σ), parametrized by $(m_j, \bar{m}_j)$. Note also that $H = \text{Weyl}(\Sigma) \ltimes \text{Diff}_0(\Sigma)$.

Now, from (F.15) we deduce that

$$\|\delta g\|^2 = 4\|\delta\sigma\|_g^2 + 2\|\delta\eta\|_g^2 + 2\|\delta\bar{\eta}\|_g^2, \tag{F.22}$$

leading to a measure

$$\mathcal{D}g = \mathcal{D}\sigma\mathcal{D}\eta\mathcal{D}\bar{\eta}. \tag{F.23}$$

Notice the relevant Jacobian for the change of coordinates $J = 1$. As we have derived above, $\delta\eta$ also admits an orthogonal decomposition (see (F.11) and (F.16)). Given the orthogonality of ϕ_j with Range$\nabla_{\bar{z}}^{(-1)}$ we can project $\delta\eta$ onto ϕ_j in the following manner:

$$\begin{aligned} 2\|\delta\eta\|^2 &= \|\nabla_{\bar{z}}^{(-1)}\delta v^z\|^2 + \sum_j \big(\delta\eta,\phi_j\big)\big(\phi_j,\phi_j\big)^{-1}\big(\phi_j,\delta\eta\big) \\ &= \big(\delta v^z, \Delta_{-1}^{-}\delta v^z\big) + \sum_{i,j,k} \delta m_i \delta m_k \big(\mu_i,\phi_j\big)\big(\phi_j,\phi_j\big)^{-1}\big(\phi_j,\mu_k\big), \end{aligned}$$

where use of (F.21) has been made and in the first term we have integrated by parts. Therefore, we may deduce that the measure associated to $\mathcal{D}\eta$ can be written as

$$\mathcal{D}\eta = \mathcal{D}'v^z \prod_{k=1}^{\dim\mathcal{M}_h} dm_k \det\big(\mu_i,\phi_j\big)\det{}^{-1/2}\big(\phi_j,\phi_j\big)\,\det{}'^{1/2}\Delta_{-1}^{-}, \tag{F.24}$$

where the prime indicates that we are not to integrate over CKV's or, equivalently, zero modes of $\nabla_z^{(1)}$ and $\nabla_{\bar{z}}^{(-1)}$.[6] It follows that the measure decomposes according to

$$\mathcal{D}g = \mathcal{D}\sigma\mathcal{D}'v \prod_{k=1}^{\dim\mathcal{M}_h} dm_k d\bar{m}_k \frac{|\det(\mu_i,\phi_j)|^2}{|\det(\phi_j,\phi_j)|}\det{}'\Delta_{-1}^{-}, \tag{F.25}$$

where $\mathcal{D}'v = \mathcal{D}'v^z\mathcal{D}'v^{\bar{z}}$.

[6]This restriction of the integration measure is required given that CKV's generate the overlap in $H = \text{Weyl}(\Sigma) \ltimes \text{Diff}_0(\Sigma)$, which in turn corresponds to (what is assumed throughout) to be a true symmetry of bosonic string (but also superstring) theory. We must integrate over configurations in the path integral that are orthogonal to the orbit of H and hence the integral over v must not include CKV contributions.

Now let us complete the description of the path integral measure associated to the space of metrics by making explicit the CKV contribution. Conformal Killing vectors live in the kernel of the operators $\nabla_z^{(1)}$ and $\nabla_{\bar{z}}^{(-1)}$ and if there are $2k$ such independent CKV's denote them respectively by $\psi_s^{\bar{z}}$ and ψ_s^z, $1 \leq s \leq k$. Now, the CKV vector space is finite dimensional, and we can read off k from (F.13), given that $k = \dim\ker\nabla_z^{(1)} = \dim\ker\nabla_{\bar{z}}^{(-1)}$. Let us then *orthogonally* decompose $\delta v^{\bar{z}}$ and δv^z

$$\delta v^{\bar{z}} = \delta\tilde{v}^{\bar{z}} + \delta\bar{a}^s\psi_s^{\bar{z}}, \quad \delta v^z = \delta\tilde{v}^z + \delta a^s\psi_s^z, \tag{F.26}$$

such that $\psi_s^{\bar{z}} \in \ker\nabla_z^{(1)}$ (and $\psi_s^z \in \ker\nabla_{\bar{z}}^{(-1)}$) while $\delta\tilde{v}$ generates orbits of $\mathrm{Diff}_0^{\perp}(\Sigma)$, with $\mathrm{Diff}_0^{\perp}(\Sigma)$ the subspace of $\mathrm{Diff}_0(\Sigma)$ orthogonal to CKV. It thus follows that

$$\begin{aligned} \|\delta v^{\bar{z}}\|^2 &= \|\delta\tilde{v}^{\bar{z}}\|^2 + \|\delta\bar{a}^s\psi_s^{\bar{z}}\|^2 \\ &= \|\delta\tilde{v}^{\bar{z}}\|^2 + \sum_{s,l=1}^{k}\delta\bar{a}^s\delta\bar{a}^l\big(\psi_s^{\bar{z}},\psi_l^{\bar{z}}\big), \end{aligned} \tag{F.27}$$

and likewise for the holomorphic part. Hence, we can read off the Jacobian for the change of coordinates $\delta v^{\bar{z}} \to \delta\tilde{v}^{\bar{z}}, \delta\bar{a}^s$ (note that $\mathcal{D}'\delta v \equiv \mathcal{D}\delta\tilde{v}$)

$$\mathcal{D}\delta v^{\bar{z}} = \det{}^{1/2}\big(\psi^{\bar{z}},\psi^{\bar{z}}\big)\mathcal{D}'\delta v^{\bar{z}}\, d^k\delta\bar{a}^s, \tag{F.28}$$

so that now the integral over $\delta v^{\bar{z}}$ is an integral over the full vector space $K^{(1,0)}$ associated to diffeomorphisms connected to the identity (see (F.7)) including zero modes of $\nabla_z^{(1)}$. Therefore, due to the fact that $K^{(1,0)} \otimes K^{(0,1)}$ is an orthogonal decomposition we deduce that the holomorphic contribution is entirely analogous with $\bar{z}$ replaced by z. We can then infer from (F.28) that[7]

$$\mathcal{D}v = \mathcal{D}'v\, d^k a\, d^k\bar{a}\, \big|\det\big(\psi^z,\psi^z\big)\big|, \tag{F.29}$$

where $\mathcal{D}v = \mathcal{D}v^z\mathcal{D}v^{\bar{z}}$. We therefore conclude that $\mathcal{D}'v$ in (F.25) is given by

$$\mathcal{D}'v = \mathcal{D}v\,\frac{\mathrm{vol}(\mathrm{CKV})^{-1}}{|\det(\psi^z,\psi^z)|}, \tag{F.30}$$

where we have identified the volume vol(CKV) with $\int d^k a d^k\bar{a}$; an integral that can be performed provided the integrand of the path integral is invariant under conformal Killing transformations, which is the case in critical string theory where there are no conformal anomalies provided the spacetime dimension $d = 26$. Note that for the specific case of genus $h \geq 2$ we have $\mathcal{D}'v = \mathcal{D}v$. The full path integral measure over metrics is thus given according to (F.25) and (F.30) by

$$\mathcal{D}g = \mathcal{D}\sigma\mathcal{D}v\prod_{k=1}^{\dim\mathcal{M}_h} dm_k d\bar{m}_k\,\frac{|\det(\mu_i,\phi_j)|^2}{|\det(\phi_j,\phi_j)|}\,\det{}'\Delta_{-1}^{-}\,\frac{\mathrm{vol}(\mathrm{CKV})^{-1}}{|\det(\psi^z,\psi^z)|}. \tag{F.31}$$

Thanks to the *uniformization theorem* [114, 141, 227] we may, by performing an appropriate conformal transformation, bring an arbitrary smooth metric to a metric of constant

[7] Recall that Jacobians of coordinate transformations on tangent spaces are equal to Jacobians of coordinate transformations on the corresponding base manifold.

curvature, the sign of the curvature depending on the genus of the surface. Given that in the path integral we must integrate along a slice orthogonal to conformal transformations, we may use this freedom to choose the gauge slice (which is in practice specified by a choice for the Beltrami differential) to be tangent to constant curvature metrics. This choice of gauge slice defines the *Weil-Petersson measure*:

$$d(\mathrm{WP}) = \prod_{k=1}^{\dim\mathcal{M}_h} dm_k d\bar{m}_k \frac{|\det(\mu_i,\phi_j)|^2}{|\det(\phi_j,\phi_j)|}, \tag{F.32}$$

and we shall evaluate this explicitly for the genus one surfaces. Therefore, writing $\mathcal{D}g = \mathcal{D}\sigma\mathcal{D}v\, d\mu_{\mathrm{WP}}$, we may now drop the measure associated to diffeomorphisms and conformal deformations given that the integrand[8] does not depend on these.[9] Finally therefore, using the notation of (E.2), and taking into account the effect of the mapping class group (F.3), we find that the full gauge fixed path integral takes the form

$$\langle\dots\rangle = \sum_{h=0}^{\infty} g_c^{-\chi(\Sigma)}\int_{\mathcal{M}_h} d(\mathrm{WP})\ \det{}'\Delta^{-}_{-1}\,\frac{\mathrm{vol(CKV)}^{-1}}{|\det(\psi^z,\psi^z)|}\left(\frac{4\pi^2\alpha'}{\int_\Sigma d^2z\sqrt{g}}\det{}'\Delta_{(0)}\right)^{-d/2}\langle\!\langle\dots\rangle\!\rangle \tag{F.33}$$

where "$\dots$" denotes vertex insertions of the form $V^{(1)}\dots V^{(n)}$.

One-Loop Amplitudes

Let us next compute the measure (F.33) at one-loop, that is the case when the worldsheet has the topology of a torus, T^2. Here there is a single A- and a single B-cycle and the period matrix, Ω_{IJ}, thus reduces to the single, see Appendix F, complex modulus of the torus, $\Omega_{IJ}\to\tau=\tau_1+i\tau_2$, $\tau_2>0$. Using the uniformization theorem we may map an arbitrary $h=1$ worldsheet via conformal transformations to a flat worldsheet $R_{(2)}=0$; that is, the complex z-plane with the identification $z\sim z+m+\tau n$. The torus is therefore identified with the Jacobian variety, $J(\Sigma)=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. The abelian differentials are globally defined and we may take $\omega(z)=\bar{\omega}(\bar{z})=1$, with the diffeomorphism invariant worldsheet distance given by $ds^2=2g_{z\bar{z}}dzd\bar{z}$, $g_{z\bar{z}}=1/2$. The isometry group is therefore generated by $z\to az+b$ with $|a|=1$ and b a complex constant. The only fixed-point free subgroup of the isometry group are the translations. These correspond to the two translations along the A- and B-cycles of the torus and define the conformal Killing vectors ψ, $\bar{\psi}$, see below.

Let us compute the measure associated to the moduli space of metrics, at genus one. There is a single modulus and so we need to determine the measure, $d\mu(\tau,\bar{\tau})$, in (F.33),

$$d\mu(\tau,\bar{\tau}) = d(\mathrm{WP})\ \det{}'\Delta^{-}_{-1}\,\frac{\mathrm{vol(CKV)}^{-1}}{|\det(\psi,\psi)|}\left(\frac{4\pi^2\alpha'}{\int_\Sigma d^2z\sqrt{g}}\det{}'\Delta_{(0)}\right)^{-13} \tag{F.34}$$

[8] The integrand is independent of the conformal factor only in 26 spacetime dimensions, when the counter-term μ^2 in the string action (3.22) is chosen appropriately [144] and provided the external states are onshell. We shall assume this is the case. This is related to the principle of ultralocality of Polchinski [144], see also p. 923, 931 in [141]

[9] There is a loose end here that we have not had time to cove, relating to the normalization $\mathcal{N} = \mathrm{Vol}(\mathrm{Diff}(\Sigma))\times\mathrm{Vol}(\mathrm{Conf}(\Sigma))$ and the cardinality $|\mathrm{MCG}_h|$, see D'Hoker and Phong [141], p. 931.

in terms of the modulus τ. Let us first consider the CKV contribution; let us choose a basis such that $\det(\psi,\psi)=(\psi,\psi)=1$, recall from above (or the Appendix for a more detailed explanation) that there is a single CKV at $h=1$. Then, given that these are holomorphic globally defined vectors we may write $\psi^z=a\partial_z$ and $\psi^{\bar z}=\bar a\partial_{\bar z}$, with $a,\bar a$ constants. From the definition of the inner product (B.5) we deduce that $(\psi,\psi)=\int_\Sigma d^2z\sqrt{g}|a|^2$. Therefore, in the coordinate system $z=\sigma^1+\tau\sigma^2$, $\bar z=\sigma^1+\bar\tau\sigma^2$ with $\sigma^1,\sigma^2\in[0,1]$, and using the conventions of Sec. B, it follows that $\int d^2z\sqrt{g}=\tau_2$, and so $a=\tau_2^{-1/2}$ (we have dropped the phase). Given that the CKV generates translations in the complex plane modulo lattice translations, $z\to z+\tau_2^{-1/2}(v_1+iv_2)\sim z+\tau_2^{-1/2}(v_1+iv_2)+m+\tau n$, we deduce that $\tau_2^{-1/2}(v_1+iv_2)$ lies in a parallelogram with endpoints at (say): 0, 1, τ and $\tau+1$. This parallelogram has volume $\int d^2z\sqrt{g}=\tau_2^{-1}\int d^2v$, leading to,

$$\mathrm{vol(CKV)}=\int d^2v=\tau_2^2,\qquad \text{when}\qquad \det(\psi,\psi)=1. \tag{F.35}$$

The gauge slice of constant curvature worldsheets, in this case $R_{(2)}=0$, defines the Weil-Peterson measure,

$$d(\mathrm{WP})=d\tau d\bar\tau\frac{|(\mu,\phi)|^2}{|(\phi,\phi)|}. \tag{F.36}$$

Recall that, see Appendix F, the (anti-)holomorphic quadratic differentials $\phi_{zz}=\ker\nabla_{\bar z}^{(2)}$ (and $\bar\phi_{\bar z\bar z}=\ker\nabla_z^{(-2)}$) are globally defined and orthogonal to gauge deformations of metric and hence are tangent to moduli space. On T^2, ϕ must therefore be of the form $\phi=\phi_{zz}dz\otimes dz=adz\otimes dz$ with $a\in\mathbb{C}$, leading to $(\phi,\phi)=\int d^2z\sqrt{g}(g^{z\bar z})^2(\phi_{zz})^*\phi_{zz}=4\tau_2|a|^2$, with respect to the inner product (B.5). The Beltrami differentials, $\mu^z_{\ \bar z}\in K^{(-1,1)}$, (and correspondingly $\bar\mu^{\bar z}_{\ z}\in K^{(1,-1)}$) are tangent to the gauge slice and the projections in (F.36) guarantee that the gauge slice is projected onto moduli space. We used the uniformization theorem to choose a gauge slice tangent to zero curvature metrics, $ds^2=|dz|^2=|d\sigma^1+\tau d\sigma^2|$, and this choice leads uniquely to explicit expressions for the Beltrami differentials.[10] From the definition

$$\mu^z_{\ \bar z}=g^{z\bar z}\frac{\partial}{\partial\tau}g_{\bar z\bar z}(\tau,\bar\tau),$$

it follows that the Beltrami differential is in turn determined by considering infinitesimal deformations of metric,

$$\delta g=|dz|^2\to|dz+\delta\tau g_{z\bar z}\mu^z_{\ \bar z}d\bar z|^2. \tag{F.37}$$

In the coordinates $z=\sigma^1+\tau\sigma^2$, $\bar z=\sigma^1+\bar\tau\sigma^2$, such deformations correspond to variations in τ, because under a small variation $\tau\to\tau+\delta\tau$,

$$\begin{aligned}|dz|^2=|d\sigma^1+\tau d\sigma^2|^2&\to|d\sigma^1+(\tau+\delta\tau)d\sigma^2|^2\\&\simeq\left|dz+\frac{i\delta\tau}{2\tau_2}d\bar z\right|^2.\end{aligned} \tag{F.38}$$

[10] Note that when we specify a metric $g_{z,\bar z}$ on the surface we can use this to raise and lower indices and so it becomes that there is very little difference between a quadratic and a Beltrami differential. If however we do not make such a choice of metric, it is the Beltrami differentials that should be thought of as being tangent to moduli space, see [141] p. 928.

Comparing with (F.37) leads to the following expression for the Beltrami differential,

$$\mu_{\bar{z}\bar{z}} = -\mu_{zz} = \frac{i}{2\tau_2}.$$

From above we have $\phi_{zz} = a$ and $(\phi, \phi) = 4|a|^2\tau_2$, and so given that $(\mu, \phi) = 2ia$, we deduce that the Weil-Peterson measure (F.36) reads,

$$d(\mathrm{WP}) = \frac{d\tau d\bar{\tau}}{\tau_2}.$$

Let us now consider the determinants of the differential operators appearing in the measure (F.34). From (B.7) and (B.8) it follows that on a flat worldsheet the two Laplacians that appear in the measure are equal,

$$\Delta_{(0)} = \Delta^{-}_{-1} = -2g^{z\bar{z}}\partial_z\partial_{\bar{z}}.$$

Therefore, we need to compute the spectrum of the Laplace operator $\Delta_{(0)}$ on T^2. In the same coordinate system as above, with metric $ds^2 = |dz|^2 = |d\sigma^1 + \tau d\sigma^2|^2$, $g_{z\bar{z}} = 1/2$ with $\sigma^1, \sigma^2 = [0, 1]$ and $\partial_z = \frac{i}{2\tau_2}(\bar{\tau}\partial_1 - \partial_2)$, $\partial_{\bar{z}} = -\frac{i}{2\tau_2}(\tau\partial_1 - \partial_2)$, we learn that, $\Delta_{(0)} = -\frac{1}{\tau_2^2}\big|\tau\partial_1 - \partial_2\big|^2$. In the σ^1, σ^2 coordinates, the worldsheet is a square of sides equal to 1, such that $\sigma^i \sim \sigma^i + 1$. Therefore, to compute the spectrum of the Laplace operator we need a complete set of eigenfunctions which satisfy this periodicity. The correct choice is $\psi_{m,n} = \exp\big(2\pi i n\sigma^1 + 2\pi i m\sigma^2\big)$, and so defining $\lambda_{m,n}$ according to $\Delta_{(0)}\psi_{m,n} = \lambda_{m,n}\psi_{m,n}$ we learn that the determinant (which excludes zero modes) is, $\prod_{(m,n)\neq(0,0)}\lambda_{m,n}$ or,

$$\det{}'\Delta_{(0)} = \prod_{m,n}{}' \frac{(2\pi)^2}{\tau_2^2}|m + \tau n|^2. \tag{F.39}$$

where we have written $\prod'_{m,n} = \prod_{(m,n)\neq(0,0)}$. Consider the first factor in the above product. Given that [232], $\prod'_{m,n} a = \big(\prod_{m\neq 0} a\big)\big(\prod_{n\neq 0} a\big)\big(\prod_{n\neq 0}\prod_{m\neq 0} a\big)$ and $\prod_{n=1}^{\infty} a = \exp[\ln\prod_{n>0} a] = \exp[(\ln a)\sum_{n>0} 1] = \exp[(\ln a)\zeta(0)]$, we find that $\prod_{n=1}^{\infty} a = a^{-1/2}$, given that $\zeta(0) = -1/2$. Therefore, $\prod'_{m\neq 0} a = 1/a$ and so $\prod'_{m,n}\frac{(2\pi)^2}{\tau_2} = \frac{\tau_2}{(2\pi)^2}$[11]. To compute the non-zero mode contribution to the determinant we can make use of the *Eisenstein series* and its properties. The Eisenstein series is defined by,

$$E(\tau, s) = \sum_{m,n}{}' \Big(\frac{\tau_2}{|m + \tau n|^2}\Big)^s,$$

and has a simple pole at $s = 1$. This enables us to write,

$$\det{}'\Delta_{(0)} = \frac{\tau_2}{(2\pi)^2} e^{-\partial_s E(\tau,s)}\Big|_{s=0}. \tag{F.40}$$

To evaluate the exponential consider the identity [226],

$$\pi^{-s}\Gamma(s)E(\tau, s) = \pi^{-(1-s)}\Gamma(1 - s)E(\tau, 1 - s),$$

[11] We have left out one of the two τ_2 factors because it is convenient to group it together with the non-zero mode pieces, as will soon become clear.

and in particular the $s \simeq 0$ region. Given that the pole is at $s = 1$, the strategy will be to Taylor expand this identity around $s = 0$ and solve for $\partial_s E(\tau, 0)$. The Taylor expansion of $E(\tau, s)$ on the left-hand side is, $E(\tau, s) \simeq E(\tau, 0) + s\partial_s E(\tau, 0) + \dots$. To consider the right-hand side in the neighborhood of $s = 0$, we need the limit,

$$\lim_{s\to 0} E(\tau, 1-s) = \lim_{s\to 0} \Big\{ -\frac{\pi}{s} + 2\pi\Big[\gamma - \ln 2 - \ln\big(\tau_2|\eta(\tau)|^4\big)\Big] + \mathcal{O}(s)\Big\},$$

with γ Euler's number and $\eta(\tau)$ the η-function. Plugging this and the aforementioned Taylor expansion into the above identity, and using the asymptotic form of the Gamma function $\Gamma(s) \simeq 1/s$ and $\Gamma(1) = 1$, it follows that $E(\tau, 0) = -1$ and $\partial_s E(\tau, 0) = 2\gamma - \ln\big(4\tau_2|\eta(\tau)|^4\big)$. Therefore, we learn that the determinant of the Laplacian is,

$$\det{}'\Delta_{(0)} = (e^{\gamma}\pi)^{-2}\, \tau_2^2|\eta(\tau)|^4. \tag{F.41}$$

The constant, $(e^{\gamma}\pi)^{-2}$, is often dropped, and the overall normalization is fixed by unitarity, or by comparing with the operator formalism result.

Collecting all the results, we can now write down the measure associated to moduli deformations for genus $h = 1$ amplitudes,

$$\begin{aligned} d\mu(\tau,\bar\tau) &= \frac{d\tau d\bar\tau}{2\tau_2}\, \tau_2^2|\eta(\tau)|^4\, \frac{1}{\tau_2^2}\left(\frac{4\pi^2\alpha'}{\tau_2}\, \tau_2^2|\eta(\tau)|^4\right)^{-13} \\ &= \frac{d\tau d\bar\tau}{2\tau_2}\, (4\pi^2\alpha'\tau_2)^{-13}|\eta(\tau)|^{-48}. \end{aligned} \tag{F.42}$$

The combinations $\tau_2|\eta(\tau)|^4$ and $d\tau d\bar\tau/\tau_2^2$ are invariant under $\mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$, under which

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \qquad \text{with} \qquad (ad - bc) = 1,$$

which is generated by $\tau \to \tau + 1$ and $\tau \to -1/\tau$. That is, the torus with modular parameter τ is equivalent to any other torus with modular parameter τ'. The group of large diffeomorphisms, the mapping class group, also contains transformations that flip both sides of the torus and so the full modular group is $\mathrm{SL}(2,\mathbb{Z})$. This is all as expected because the metric $|d\sigma^1 + \tau d\sigma^2|^2$ is invariant under $\sigma^1 \to a\sigma^1 + b\sigma^2$, and $\sigma^2 \to c\sigma^1 + d\sigma^2$ with a, b, c, d integers given that we are to identify $\sigma^1 \sim \sigma^1 + 1$ and $\sigma^2 \sim \sigma^2 + 1$. Under these large diffeomorphisms the parameter τ is mapped precisely to τ'. This means that when we integrate over the moduli τ_1, τ_2 we are to restrict the region of integration to within a *fundamental domain*, a convenient choice being [231],

$$\mathcal{M}_1 = \big\{\tau = \tau_1 + i\tau_2 \ \text{ with } \ -\tfrac{1}{2} \le \tau_1 \le \tfrac{1}{2}, \quad \tau_2 > 0, \quad |\tau| \ge 1\big\}.$$

The invariance under large diffeomorphisms is then fixed, although there remains a $\mathbb{Z}_2$ gauge invariance, $\sigma^1 \to -\sigma^1$, and $\sigma^2 \to -\sigma^2$, and so we need to include an overall factor of $1/2$ in the amplitude.

We have therefore shown that at one-loop, the amplitude (F.33) reduces to,

$$\langle \dots \rangle = \int_{\mathcal{M}_1} \frac{d\tau d\bar\tau}{4\tau_2}\, (4\pi^2\alpha'\tau_2)^{-13}|\eta(\tau)|^{-48}\langle\!\langle \dots \rangle\!\rangle, \tag{F.43}$$

where we have noted that $\chi(T^2) = 0$.

Appendix G

Scalar Green's Function

The scalar Green's function, $G(z,w)$ is the fundamental object which appears in all string amplitudes. We therefore consider it appropriate for completeness to present a rather complete overview of the definition of $G(z,w)$ broadly following the approach of [190, 141, 114]. For a general discussion of the Green's function on various two-dimensional surfaces see [233] and for particular emphasis on open strings see [177]. We shall concentrate on the form of the Green's function which is appropriate for arbitrary genus closed string calculations, namely in terms of the prime form, $E(z,w)$, which was first (possibly independently) described in [190] and [234].

In any local coordinate patch we may render the metric conformally flat, $g = g_{z\bar{z}}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$. We define the Green's function, $G(z,w)$, by the correlation function $\langle X(z,\bar{z}) \cdot X(w,\bar{w})\rangle$. This corresponds to the inverse of the Laplace operator (see Appendix B),

$$\Delta_{(0)} = -2g^{z\bar{z}}\partial_z\partial_{\bar{z}},$$

subject to certain boundary conditions. For closed strings the boundary conditions translate into periodicity requirements of $G(z,w)$ under translations around the A_I and B_I cycles of the Riemann surface – more about this later. Given that $\Delta_{(0)}$ is Hermitian, we may decompose $X(z,\bar{z})$ into a complete set of eigenstates, $\psi_n(z,\bar{z})$, of $\Delta_{(0)}$, $X^\mu(z,\bar{z}) = \sum_n a_n^\mu \psi_n(z,\bar{z})$, such that

$$\Delta_{(0)}\psi_n(z,\bar{z}) = \lambda_n \psi_n(z,\bar{z}). \tag{G.1}$$

Let us further require that this decomposition be orthogonal,

$$\begin{aligned} (\psi_m, \psi_n) &\equiv \int_\Sigma d^2z \sqrt{g}\psi_m\psi_n, \\ &= \delta_{mn}. \end{aligned} \tag{G.2}$$

However, $\Delta_{(0)}$ is not invertible in general due to the presence of zero modes and hence we need to be careful in interpreting $G(z,w)$ as the inverse of the Laplace operator. If we denote the zero modes $\ker\Delta_{(0)}$ by ψ_0, defined by $\lambda_0 = 0$ in (G.1), we deduce from (G.2) that

$$\psi_0 = \left(\int_\Sigma d^2z\sqrt{g}\right)^{-1/2}. \tag{G.3}$$

We may however invert $\Delta_{(0)}$ provided we restrict to the space of functions orthogonal to zero modes ψ_0 – we call the Green's function associated to this restricted space[1] $G'(z,w)$. This Green's function has then the explicit representation,

$$G'(z,w) = \sum_{n\neq 0} \frac{2\pi\alpha'}{\lambda_n} \psi_n(z,\bar{z})\psi_n(w,\bar{w}). \tag{G.4}$$

This is the inverse of $\Delta_{(0)}$ in the sense that, on account of (G.1) and (G.3),

$$\partial_z \partial_{\bar{z}} G'(z,w) = -\pi\alpha' \delta^2(z-w) + \frac{\pi\alpha' g_{z\bar{z}}}{\int_\Sigma d^2z\sqrt{g}}, \tag{G.5a}$$

$$\int_\Sigma d^2z\sqrt{g}\, G'(z,w) = 0. \tag{G.5b}$$

One often defines a (covariant) δ-function, $\delta^2(z,w)$, which in terms of the usual δ-function would read: $\delta^2(z,w) = \frac{1}{\sqrt{g}}\delta^2(z-w) = \sum_{n=0}^{\infty} \psi_n(z,\bar{z})\psi_n(w,\bar{w})$. This representation of $\delta^2(z-w)$ in terms of the basis $\{\psi_n(z,\bar{z})\}$ follows from the completeness of $X(z,\bar{z})$,

$$\begin{aligned} X(z,\bar{z}) &= \sum_n a_n \psi_n(z,\bar{z}) \\ &= \sum_n \left(\int_\Sigma d^2z' \sqrt{g'} \psi_n(z',\bar{z}') X(z',\bar{z}') \right) \psi_n(z,\bar{z}) \\ &= \int_\Sigma d^2z' \sqrt{g'} \left(\sum_n \psi_n(z,\bar{z}) \psi_n(z',\bar{z}') \right) X(z',\bar{z}'), \end{aligned} \tag{G.6}$$

whereas the second equation is a consequence of (G.4) given that the integrand, on account of (G.1), can be expressed as a total derivative. The Green's function may be defined by (G.5) up to an immaterial constant together with additional periodicity requirements determined (in the case of closed strings) by the genus of the Riemann surface.

The solution of (G.5a) should be that corresponding to a genus h compact Riemann surface. We are considering orientable surfaces and so the Green's function $G(z,w)$ should therefore be single-valued and periodic if we transport z around a cycle $n_I A_I + m_I B_I$.[2] Furthermore, as we consider only closed Riemann surfaces there are no distinguished regions in Σ at which $G(z,w)$ is required to vanish. When $z=w$ we see from (G.5a) that $G(z,w)$ should have a logarithmic singularity, $G(z,w) \sim -\frac{\alpha'}{2} \ln|z-w|^2 +$regular terms. To extend this solution to the entire surface we thus need the (hopefully unique) generalization of $z-w$ on $\mathbb{C}$ to higher genus Riemann surfaces. Such a generalization is provided by the prime form $E(z,w)$, defined in (C.16). Recall that $E(z,w)$ is a $(-1/2,-1/2)$ differential with a single zero at $z=w$, being analytic elsewhere. We therefore suspect that $G(z,w) \sim$

[1] We have placed a prime on the Green's function to remind the reader that zero modes are omitted. As we shall only be dealing with the Green's function which excludes zero mode contributions, we shall eventually drop the prime in what follows, $G'(z,w) \to G(z,w)$.

[2] We are being a little bit sloppy here. The coordinate z should really be thought of as the image $\mathbf{z}(p)$ of a point $p \in \Sigma$ under the Jacobi map, see (C.7). In particular, by transport z around a cycle $n_I A_I + m_I B_I$ we mean $\mathbf{z}(p) \to \mathbf{z}(p + n_I A_I + m_I B_I)$.

$-\frac{\alpha'}{2}\ln|E(z,w)|^2+$ regular terms is the required generalization. However, it is only *quasi*-periodic on Σ: if we transport z around the cycle $\gamma = n_I A_I + m_I B_I$, we have [192], $E(z,w) \to E(z,w)\exp 2\pi i\left(-\frac{1}{2}m_I\Omega_{IJ}m_J + m_I\int_z^w \omega_I\right)$ and therefore,

$$\ln|E(z,w)|^2 \to \ln|E(z,w)|^2 + 2\pi m_I(\mathrm{Im}\Omega)_{IJ}m_J - 4\pi m_I \mathrm{Im}\int_z^w \omega_I. \tag{G.7}$$

It then follows that if we add $\pi\alpha'\mathrm{Im}\int_z^w\omega_I(\mathrm{Im}\Omega)_{IJ}\mathrm{Im}\int_z^w\omega_J$ to $-\frac{\alpha'}{2}\ln|E(z,w)|^2$ the resulting expression,

$$G(z,w) = -\frac{\alpha'}{2}\ln|E(z,w)|^2 + \pi\alpha'\mathrm{Im}\int_z^w \omega_I\,(\mathrm{Im}\Omega)^{-1}_{IJ}\,\mathrm{Im}\int_z^w \omega_J + \text{regular terms} \tag{G.8}$$

will be single-valued and periodic around the A_I and B_I cycles of the Riemann surface. This is indeed (almost) the correct form for the Green's function on a closed Riemann surface of genus h. Note also that we have the freedom of adding holomorphic regular terms [233] in (G.8) which do not appear in the amplitudes, see e.g. [181]. However, this is not quite the end of the story given that this object is singular at $z = w$ and we shall therefore need to adopt a regularization prescription which removes this divergence and this is what we focus on next.

Close to $z = w$ the singular term in the Green's function is $-\frac{\alpha'}{2}\ln|z-w|^2$. We shall regularize this divergence by introducing a cut-off, a minimum distance $|\epsilon|$. We would like this cut-off to be present for arbitrary diffeomorphisms. The relevant diffeomorphism invariant quantity is $g_{z\bar z}|z-w|^2$ and so we should set $\lim_{w\to z} g_{z\bar z}|z-w|^2 = |\epsilon|^2$, or $\lim_{w\to z}|z-w|^2 = g_{z\bar z}^{-1}|\epsilon|^2$, leading to the following regularized expression for the Green's function at coincident points,

$$G_{\mathrm{R}}(z,z) = \frac{\alpha'}{2}\left(\ln g_{z\bar z} - \ln|\epsilon|^2\right) + \dots \tag{G.9}$$

The terms "$\dots$" are non-singular regular functions of $z, \bar z$ which are independent of $g_{z\bar z}$ and vanish in all amplitudes due to momentum conservation. In string amplitudes we can always absorb the coordinate and moduli independent cut-off $|\epsilon|$ into a renormalization of the vertex operator coupling constants g_c, see e.g. [235], given that $G(z,w)$ appears non-differentiated only in an over-all exponential. Therefore, we shall drop the cut-off term in (G.9) in what follows. From (G.8) on account of (G.9) we finally conclude that the correct regularized expression for the Green's function is

$$G(z,w) = \begin{cases} -\frac{\alpha'}{2}\ln|E(z,w)|^2 + \pi\alpha'\mathrm{Im}\int_z^w \omega_I\,(\mathrm{Im}\Omega)^{-1}_{IJ}\,\mathrm{Im}\int_z^w\omega_J + \dots, & \text{if } z \neq w \\ \frac{\alpha'}{2}\left(\ln g_{z\bar z} - \ln|\epsilon|^2\right) + \dots, & \text{if } z = w \end{cases} \tag{G.10}$$

in agreement with [190, 141]. The dots denote terms that do not appear in string amplitudes due to momentum conservation of the external states. The cutoff also vanishes when the external string states are on the mass shell after a wavefunction renormalization – this is elaborated on in the main text.

What remains to be shown is that the Green's function (G.10) satisfies (G.5). From (G.10) we have $\partial_z \partial_{\bar{z}} G(z,w) = -\pi\alpha' \delta^2(z-w) + \pi\alpha' \omega_I(z)(2\mathrm{Im}\Omega)^{-1}_{IJ}\bar{\omega}_J(\bar{z})$. Comparing with (G.5) we see that we are to make the identification,

$$\frac{g_{z\bar{z}}}{\int_\Sigma d^2z\sqrt{g}} \leftrightarrow \omega_I(z)(2\mathrm{Im}\Omega)^{-1}_{IJ}\bar{\omega}_J(\bar{z}), \tag{G.11}$$

which can be shown to be consistent with the Riemann bilinear identity (C.6) on account of the defining properties of the Abelian differentials (C.4). In particular, integrate both sides of the correspondence over Σ with measure $d^2z = idz \wedge d\bar{z}$ and use the fact that

$$i\int_\Sigma \omega_I \wedge \bar{\omega}_J = 2(\mathrm{Im}\Omega)_{IJ}.$$

We hence deduce that both sides of the correspondence when integrated reduce to unity.

Appendix H

Gamma Function Identities

We list a number of identities that are useful in manipulating gamma functions. The gamma and beta functions are defined respectively by

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x}, \qquad B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{H.1}$$

The following gamma function identities are used extensively throughout:

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \qquad \Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z) = \frac{\pi}{\cos \pi z}. \tag{H.2}$$

From these we find that:

$$\Gamma(n) = (n-1)! \qquad \Gamma(1/2) = \sqrt{\pi} \qquad \Gamma(-1/2) = -2\sqrt{\pi} \tag{H.3a}$$

$$\Gamma(\epsilon - n) \simeq \frac{1}{\epsilon n!} \quad (|\epsilon| \ll 1), \qquad \Gamma(n+\tfrac{1}{2}) = 2^{-n}\sqrt{\pi}(2n-1)!! \tag{H.3b}$$

A very useful (exact) representation of the gamma function is given by the Stirling formula [236],

$$\begin{aligned} \Gamma(z) &= z^{z-1/2} e^{-z} \sqrt{2\pi} \exp\Big(-\int_0^\infty dt\, \frac{P_1(t)}{z+t} \Big), \\ &\simeq z^{z-1/2} e^{-z} \sqrt{2\pi} \Big(1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots \Big), \qquad (|z| \to \infty, \quad |\arg z| < \pi) \end{aligned} \tag{H.4}$$

with $P_1(t) = t - [t] - 1/2$ the sawtooth function, [t] denoting the largest integer, n, with, $n \leq t$. The integral in the exponential tends to zero in every sector of complex numbers, $z = re^{i\theta}$, such that:

$$-\pi + \delta \leq \theta \leq \pi - \delta, \qquad \text{with} \qquad 0 < \delta < \pi,$$

Notice that this is potentially (depending on δ) the entire complex z-plane with the negative real axis (where the gamma function has poles) deleted. In the limit $|z| \to \infty$ dropping this integral becomes a better and better approximation as can be seen in the second line – this is Stirling's approximation. In terms of the beta function the leading order term in Stirling's approximation reads,

$$\begin{aligned} B(z,w) &\simeq \sqrt{2\pi} \frac{z^{z-1/2} w^{w-1/2}}{(z+w)^{z+w-1/2}}, && \text{for} \quad |z|, |w| \to \infty \\ B(z,w) &\simeq z^{-w}\Gamma(w), && \text{for} \quad |z| \to \infty,\ z/w = \text{fixed} \end{aligned} \tag{H.5}$$

which follows from (H.1) and (H.4). The following integral is required [141],

$$\int d^2z z^A(1-z)^B \bar{z}^{\bar{A}}(1-\bar{z})^{\bar{B}} = 2\pi \frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+2)} \frac{\Gamma(-1-\bar{A}-\bar{B})}{\Gamma(-\bar{A})\Gamma(-\bar{B})}, \tag{H.6}$$

which holds true provided $A-\bar{A}$ and $B-\bar{B}$ are integers, which is always the case in string theory.

Another useful identity is the Gauss multiplication formula,

$$\Gamma(nz) = (2\pi)^{-\frac{n-1}{2}} n^{nz-\frac{1}{2}} \Gamma(z)\Gamma(z+\tfrac{1}{n})\Gamma(z+\tfrac{2}{n})\ldots\Gamma(z+\tfrac{n-1}{n}) \tag{H.7}$$

Appendix I

Polarization Tensor for Monomial Vertex Operators

In this section we shall construct explicit representations for the polarization tensor which satisfy the physical state conditions associated to the massive monomial covariant gauge vertices of Sec. 4.1.

Let us primarily focus on the holomorphic part ζ of the polarization tensor $Z = \zeta \otimes \bar{\zeta}$. For massive states we may boost to the rest frame, $k = (k^0, 0, \dots, 0)$, and the condition $k \cdot \zeta = 0$ can then be satisfied if the zeroth component of every index vanishes: $\zeta_{\mu_1 \dots \mu_j \dots \mu_I} = 0$ when $\mu_j = 0$ for any j.

Traceless tensors whose indices are associated to a Young tableau correspond to irreducible representations of the orthogonal group. To construct such tensors we consider the partitioning $I = (|I_1|, |I_2|, \dots)$, which corresponds to a Young tableau with $|I_1|$ boxes in the 1^{st} row, $|I_2|$ in the 2^{nd}, and so on. Similarly, define the conjugate quantities, namely denote by $|J_j|$ the number of boxes in the j^{th} *column* for all columns j, which ranges from 1 to $q \equiv |I_1|$. We therefore have the following equivalent Young tableau partitioning in terms of the conjugate quantities $|J_j|$,

$$I = [|J_1|, |J_2|, \dots, |J_q|]. \tag{I.1}$$

According to the standard Young tableau prescription [189] for the construction of irreducible tensors with respect to $\mathrm{SO}(N)$, we write down a general tensor with as many indices as there are boxes in the Young tableau of interest. We then symmetrize the indices associated to rows and subsequently anti-symmetrize indices associated to columns of the tableau. The trace of such tensors is invariant under $\mathrm{SO}(N)$ rotations and so tensors constructed in this manner are not yet irreducible; the subspace of *traceless* tensors however does correspond to the space of irreducible representations of the orthogonal group. Therefore, we also require that the trace of ζ with respect to *any two* indices vanishes.

Let us introduce one complex vector e^μ_A for every row A of the tableau, namely $A = 1, \dots, m$. We then symmetrize the indices associated to the rows by writing down a tensor that is *trivially* symmetric with respect to any two spacetime indices when both label elements of the same row,[1]

$$\underbrace{e_1 \otimes e_1 \otimes \cdots \otimes e_1}_{|I_1|} \otimes \underbrace{e_2 \otimes e_2 \otimes \cdots \otimes e_2}_{|I_2|} \otimes \cdots \otimes \underbrace{e_m \otimes e_m \otimes \cdots \otimes e_m}_{|I_m|},$$

or alternatively,

$$\underbrace{e_1 \otimes e_2 \otimes \cdots \otimes e_{|J_1|}}_{|J_1|} \otimes \underbrace{e_1 \otimes e_2 \otimes \cdots \otimes e_{|J_2|}}_{|J_2|} \otimes \cdots \otimes \underbrace{e_1 \otimes e_2 \otimes \cdots \otimes e_{|J_q|}}_{|J_q|}, \tag{I.2}$$

where in the first and second line we have grouped the indices in the standard and conjugate form, i.e. in terms of rows and columns respectively. Note that the total number of boxes is the same in both cases, $|I_1| + \ldots |I_m| = |J_1| + \cdots + |J_q| = |I|$.[2] This is clearly not the most general form of a rank-$|I|$ tensor which is symmetric on the groups of indices associated to rows of a tableau.[3] In particular, this step largely reduces the space of available irreducible representations.[4] The remaining space however is sufficiently large in that there still is a very large number of non-trivial states that can be constructed from the above monomial. The physical implications of this restriction is obscured by the fact that these states also carry unphysical degrees of freedom.

We next anti-symmetrize the resulting object (I.2) on the indices associated to any given column by contracting the elements associated to the j^{th} column ($j = 1, \ldots, q$), which is composed of $|J_j|$ boxes, with the Levi-Civita symbol $\varepsilon_{AB\ldots}$ of dimensionality $|J_j|$. Therefore, the resulting polarization tensor will be of the form

$$\zeta = C_{|J_1|} \otimes C_{|J_2|} \otimes \cdots \otimes C_{|J_q|}, \tag{I.3}$$

with $|I| = |J_1| + |J_2| + \cdots + |J_q|$ and the completely anti-symmetric tensors C_p defined as $C_p = \frac{1}{p!}\varepsilon_{A_1\ldots A_p} e_{A_1} \otimes \cdots \otimes e_{A_p}$. In order for this polarization tensor to be physical it must satisfy the physical state conditions (4.4), which for the holomorphic sector read $\eta^{\mu_i\mu_j}\zeta_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_{|I|}} = 0$ and $k^{\mu_i}\zeta_{\mu_1\ldots\mu_i\ldots\mu_{|I|}} = 0$, and the normalization condition (4.3), namely $\zeta_{\mu_1\ldots\mu_{|I|}}\zeta^{*\mu_1\ldots\mu_{|I|}} = 1$. In terms of the bases vectors e^μ_A these three conditions read respectively:

$$e_A \cdot e_B = 0, \quad e^0_A = 0, \quad \text{and} \quad e_A \cdot e^*_B = \delta_{AB}, \tag{I.4}$$

[1] We suppress the spacetime indices of e^μ_A in the following.

[2] Recall that the total number of columns, q, equals the number of boxes in the first row, $q = |I_1|$, whereas the number of rows, m, equals the number of boxes in the first column, $m = |J_1|$.

[3] One may construct more general polarization tensors by introducing as many basis vectors as there are boxes, namely $|I|$ vectors e^μ_A, and subsequently sum over all permutations for every row independently. The resulting object is to replace (I.2). One may then proceed to the second step in the construction as described below, the computation is analogous but more tedious.

[4] Consider as an example the Young tableau (2) or equivalently [1, 1] in the notation (4.5) and (I.1) respectively. This would be equivalent to choosing from the space of symmetric rank-2 tensors which can in general be written as $a^ib^j + a^jb^i$ just the subset of tensors for which $a = b$, tensors of the form a^ia^j. This reduces the dimensionality from $d(d+1)/2$ to d.

for all [Young tableau] rows $A, B = 1, \ldots, m$. The second condition takes this form if we boost (for massive states) to the rest frame where $k = (k^0, 0, \ldots, 0)$. We may subsequently Lorentz-boost to more general frames if we so please and the above choice reflects the fact that under Lorentz transformations the polarization tensor transforms according to a real representation of the little group for k, namely SO(25).[5] These conditions are solved by the following basis expansion,

$$e^\mu_A = M^\mu{}_\nu N_{AB} \hat{e}^\nu_B, \tag{I.5}$$

with

$$k = \begin{pmatrix} k^0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \hat{e}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \hat{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \ldots, \qquad \hat{e}_{12} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \tag{I.6}$$

provided we take

$$M^\mu{}_\nu \eta_{\mu\rho} M^\rho{}_\sigma = \eta_{\nu\sigma}, \qquad M^0{}_j = 0, \qquad N_{AC} N_{BD} \delta_{CD} = \delta_{AB}, \tag{I.7}$$

with $j = 1, \ldots, 25$ and $A, B = 1, \ldots, 12$. That is, M and N must be elements of SO(25) and SO(12) respectively – we are free to fix $25(25-1)/2 = 300$ and $12(12-1)/2 = 66$ parameters in M and N respectively. It is the choice of the 66 parameters in N that lead to physically distinct polarization tensors for the left-movers – from (I.7) it is seen that the matrices M can only generate Lorenz transformations and this is a symmetry of the theory. Notice that we have restricted the number of basis vectors to a total of 12, namely we consider states with mode numbers $m \leq 12$. This is due to the fact that [189] the sum of the number of boxes of the first two columns must not exceed 25, in accordance with (4.6), and that we would like to in general consider irreducible representations with more than just a single column.

The anti-holomorphic part of the polarization tensor, namely $\bar{\zeta}^{\mu_1 \cdots \mu_{|\bar{I}|}}$, follows by direct analogy to the holomorphic case (I.3). We consider the expression,

$$\bar{\zeta} = \bar{C}_{|\bar{J}_1|} \otimes \bar{C}_{|\bar{J}_2|} \otimes \cdots \otimes \bar{C}_{|\bar{J}_{\bar{q}}|}, \tag{I.8}$$

with $|\bar{I}| = |\bar{J}_1| + |\bar{J}_2| + \cdots + |\bar{J}_{\bar{q}}|$ the total number of boxes of the tableau associated to the right-movers, $|\bar{J}_j|$ the number of boxes in column j and $\bar{q}$ the total number of columns, given by the size of the first row, $\bar{q} = |\bar{I}_1|$. The anti-symmetric tensor $\bar{C}_p$ in terms of its

[5] Recall that vertex operators transform like one-particle states under Lorentz transformations and that the irreducible representations of the full Poincaré group SO(25,1) are determined from an irreducible representation of the little group [189, 198].

components reads, $\bar{C}_p^{\mu_1 \ldots \mu_p} = \frac{1}{p!}\varepsilon_{A_1\ldots A_p}\bar{e}^{\mu_1}_{A_1}\ldots\bar{e}^{\mu_p}_{A_p}$. In direct analogy to the holomorphic case, the $\bar{e}_A$ must satisfy the conditions

$$\bar{e}_A \cdot \bar{e}_B = 0, \qquad \bar{e}^0_A = 0, \qquad \bar{e}_A \cdot \bar{e}^*_B = \delta_{AB}, \qquad \text{and} \qquad e_A \cdot \bar{e}_B = 0, \tag{I.9}$$

for all rows $A, B = 1, \ldots, \bar{m}$. This last condition is required in order to cancel potential anomalies of the form discussed by de Alwis [142]. To solve these constraints we make the following ansatz,

$$\bar{e}^\mu_A = \bar{M}^\mu{}_\nu \bar{N}_{AB}\hat{e}^\nu_B, \tag{I.10}$$

with $\bar{M}$ and $\bar{N}$ some constant real matrices, the constraints on which follow from the four conditions (I.9),

$$\bar{M}^\mu{}_\rho \eta_{\mu\nu}\bar{M}^\nu{}_\sigma = \eta_{\rho\sigma}, \quad \bar{M}^0{}_j = 0, \quad \bar{N}_{AC}\bar{N}_{BD}\delta_{CD} = \delta_{AB} \quad \text{and} \quad M = \bar{M}, \tag{I.11}$$

respectively. As for the holomorphic sector we see that $\bar{M}$ and $\bar{N}$ must be elements of SO(25) and SO(12) respectively. Any possible asymmetry between e_A and $\bar{e}_A$ is completely described by the real SO(12) matrices N_{AB} and $\bar{N}_{AB}$.[6] In particular, it follows from (I.5) and (I.10) that the left- and right-mover basis vectors of the polarization tensor are related:

$$\bar{e}_A = (\bar{N}N^{\mathrm{T}})_{AC}e_C. \tag{I.12}$$

By representing the left- and right-movers' polarization tensors by Young tableaux we allow explicitly for the possibility of having states with a large number of harmonics excited (provided $m, \bar{m} \leq 12$). By allowing in addition the basis vectors of the left-movers to be related by an SO(12) rotation to that of the right-movers' basis vectors, we are implicitly considering states with potentially (but not necessarily) asymmetric left-right excitations.

An example where left-right asymmetry can become important is in the context of cosmic strings. It is plausible that in massive string states left-right asymmetry is generic and such an asymmetry seems to be responsible for the presence of cusp-like features in the corresponding classical evolution of strings [83].

We have therefore constructed a rank-$|\mathcal{I}|$ physical polarization tensor $Z = \zeta \otimes \bar{\zeta}$ which under Lorentz transformations transforms according to a real representation of SO(25) (actually SO(24) as both $\hat{e}^0_A$ and $\hat{e}^{25}_A$ are empty and so there are no quantum fluctuations in the 0- and 25- directions in our construction unless one Lorentz transforms to a more general frame). The full polarization tensor Z takes the form

$$\begin{aligned} Z &= \zeta \otimes \bar{\zeta} \\ &= \big(C_{|J_1|} \otimes C_{|J_2|} \otimes \cdots \otimes C_{|J_q|}\big) \otimes \big(\bar{C}_{|\bar{J}_1|} \otimes \bar{C}_{|\bar{J}_2|} \otimes \cdots \otimes \bar{C}_{|\bar{J}_{\bar{q}}|}\big). \end{aligned} \tag{I.13}$$

Recall that it is the choice of the matrices N_{AB} and $\bar{N}_{AB}$ that gives rise to different polarization tensors and hence physically distinct states whereas the choice of M and $\bar{M}$ corresponds to a symmetry of the theory, in particular Lorentz rotations.

[6] The bar on $\bar{N}_{AB}$ is just a label and does not denote any kind of conjugation. These matrices are not to be confused with the worldsheet level operators.

Let us finally describe briefly the case when we have d non-compact and $26-d$ compact dimensions. We limit ourselves to the case when there are no quantum fluctuations in the compact dimensions. This is equivalent to setting $M^{\mu}{}_{\nu} = 0$ for $\mu = d+1, d+2, \ldots, d+(26-d)$, a case of interest being for example $d = 3$ for our universe. The polarization tensor is otherwise unaltered.

Appendix J

Schur Polynomials

In the current section we collect all the necessary polynomials that appear in the closed and open string covariant vertex operator construction of the main text.

Closed String Polynomials

Elementary Schur polynomials [145] are defined by the generating series,[1]

$$\sum_{m=0}^{\infty} S_m(a_1,\dots,a_m)z^m \equiv \exp\sum_{n=1}^{\infty} a_n\, z^n,$$

and read explicitly:

$$\begin{aligned} S_m(a_1,\dots,a_m) &= \sum_{k_1+2k_2+\dots+mk_m=m} \frac{a_1^{k_1}}{k_1!}\cdots\frac{a_m^{k_m}}{k_m!} && \text{(J.1a)}\\ &= -i\oint_0 \text{đ}w\, w^{-m-1}\exp\sum_{s=1}^{m} a_s w^s && \text{(J.1b)} \end{aligned}$$

with $\text{đ}w \equiv dw/(2\pi)$, $S_0=1$ and $S_{m<0}=0$. When $a_s=-\frac{1}{s!}inq\cdot\partial^s X(z)$, with q^μ defined in (4.65) we write $S_m(nq;z)\equiv S_m(a_1,\dots,a_m)$. For instance, when $a_s=-\frac{1}{s!}inq\cdot\partial^s X(z)$ or $a_s=-\frac{1}{s!}inq\cdot\bar\partial^s X(\bar z)$,

$$\begin{aligned} S_m(nq;z) &= \oint_0 \frac{dw}{2\pi i w}\, w^{-m}\exp\Big(-inq\cdot\sum_{s=1}^{m}\frac{w^s}{s!}\partial_z^s X(z)\Big), && \text{(J.2a)}\\ \bar S_m(nq;\bar z) &= -\oint_0 \frac{d\bar w}{2\pi i\bar w}\,\bar w^{-m}\exp\Big(-inq\cdot\sum_{s=1}^{m}\frac{\bar w^s}{s!}\partial_{\bar z}^s X(\bar z)\Big), && \text{(J.2b)} \end{aligned}$$

and when there is no ambiguity we shall write instead $S_m(nq)$ for the same object, and similarly for $\bar S_m(nq)$. The following Taylor series is useful,

$$e^{-inq\cdot X(z)} = \sum_{a=0}^{\infty} z^a S_a(nq;0)e^{-inq\cdot X(0)}.$$

[1] Elementary Schur polynomials, $S_m(\mathbf{x})$, are not to be confused with the Schur polynomials, $S_\lambda(\mathbf{x})$. Given a partition $\lambda=\{\lambda_1\geq\lambda_2\geq\dots\}$ these are related however, $S_\lambda(\mathbf{x})=\det(S_{\lambda_i-i+j}(\mathbf{x}))_{1\leq i,j,\leq|\lambda|}$.

Elementary Schur polynomials, S_m, are related to the complete Bell polynomials, B_m, according to, $S_m(a_1,a_2,\dots,a_m)=\frac{1}{m!}B_m(a_1,2!a_2,\dots,m!a_m)$. Properties of the latter have been studied in [194, 237, 238, 239, 240].

The following polynomials in $\partial^{\#}X$ and $\bar\partial^{\#}X$ are the fundamental building blocks in normal ordered covariant vertex operators and are recorded here for easy reference,

$$P_n^i(z)=\sqrt{\frac{2}{\alpha'}}\sum_{m=1}^n\frac{i}{(m-1)!}\,\partial^mX^i(z)S_{n-m}(nq;z), \tag{J.3a}$$

$$\bar P_n^i(\bar z)=\sqrt{\frac{2}{\alpha'}}\sum_{m=1}^n\frac{i}{(m-1)!}\,\bar\partial^mX^i(\bar z)\bar S_{n-m}(nq;\bar z). \tag{J.3b}$$

which when $\xi_{\dots i\dots}p^i$ is non-vanishing generalizes to

$$H_n^i(z)\equiv\sqrt{\frac{\alpha'}{2}}p^iS_n(nq;z)+P_n^i(z), \tag{J.4a}$$

$$\bar H_n^i(\bar z)\equiv\sqrt{\frac{\alpha'}{2}}p^i\bar S_n(nq;\bar z)+\bar P_n^i(\bar z). \tag{J.4b}$$

When necessary we shall also note the argument of the Schur polynomials by writing $P_n^i(mq;z)$ and $H_n^i(mq;z)$ although usually $n=m$ which is why we have written instead $P_n(z)$ and $H_n(z)$. For vertex operators whose lightcone gauge representation is not traceless, $\xi_{\dots i\dots j\dots}\delta^{ij}\neq0$, the following polynomials appear,

$$\mathbb{S}_{m,n}(z)\equiv\sum_{r=1}^n rS_{m+r}(mq;z)S_{n-r}(nq;z), \tag{J.5a}$$

$$\bar{\mathbb{S}}_{m,n}(\bar z)\equiv\sum_{r=1}^n r\bar S_{m+r}(mq;\bar z)\bar S_{n-r}(nq;\bar z), \tag{J.5b}$$

see (4.98). These polynomials have the properties, $S_0(nq;z)=\sqrt{\alpha'/2}\,q\cdot H_0(nq;z)=1$, and $H_0^i(nq;z)=\sqrt{\alpha'/2}\,p^i$ and vanish when the subscripts are negative. Explicitly, for the first few level numbers, $P_0^i(z)=0$, $P_1^i(z)=i\partial X^i(z)$, $P_2^i(z)=2\partial X^iq\cdot\partial X(z)+i\partial^2X^i(z)$, and so on, where we have taken $\alpha'=2$ for simplicity; also, $S_0(Nq)=1$, $S_1(Nq)=-iNq\cdot\partial X$, $S_2(Nq)=2(q\cdot i\partial X)^2-q\cdot i\partial^2X,\dots$

Open String Polynomials

In the open string sections of the main text we give explicit results for normal ordered vertex operators with excitations in the directions, $A=1,\dots,p-1$, tangent to the Dp-brane. The various polynomials that appear in the open string analogous to (J.3), (J.2), (J.4) and (J.5) of the closed string are in holomorphic language given respectively by,

$$S_N(nq;z) = \oint_0 \frac{dw}{2\pi i w}\, w^{-N} \exp\Big(- inq \cdot \sum_{s=1}^{m} \frac{w^s}{s!} \partial_z^s X(z) \Big), \tag{J.6a}$$

$$H_N^A(z) \equiv \sqrt{2\alpha'} p^A S_N(Nq;z) + P_N^A(z), \tag{J.6b}$$

$$P_N^A(z) \equiv \sqrt{\frac{2}{\alpha'}} \sum_{m=1}^{N} \frac{i}{(m-1)!} \partial^m X^A(z) S_{N-m}(Nq;z), \tag{J.6c}$$

$$\mathbb{S}_{m,n}(z) \equiv \sum_{r=1}^{n} r S_{m+r}(mq;z) S_{n-r}(nq;z), \tag{J.6d}$$

and further properties and examples for $N = 0, 1$ and 2 of these are given in Appendix K. The $\alpha' = 2$ results there correspond to $\alpha' = 1/2$ results here.

Appendix K

Commutators and Operator Products

As shown in Sec. 3.1, for two operators

$$A = \oint dz\, a(z), \qquad \text{and} \qquad B = \oint dw\, b(w),$$

there exists the interpretation,

$$[A, B] \cong A \cdot B = \oint_0 dw \oint_w dz\, a(z) \cdot b(w) \quad \text{and} \quad [A, b(w)] \cong A \cdot b(w) = \oint_w dz\, a(z) \cdot b(w), \tag{K.1}$$

the dot denoting an operator product expansion (OPE), where for a free scalar contractions are taken with respect to the propagator,

$$\langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln|z-w|^2.$$

Closed String DDF Operators and Covariant Commutators

The relevant components of the DDF operators are defined according to,

$$A_n^i = \sqrt{\frac{2}{\alpha'}} \oint đz\, \partial X^i e^{inq\cdot X(z)} \qquad \text{and} \qquad \bar{A}_n^i = -\sqrt{\frac{2}{\alpha'}} \oint đ\bar{z}\, \bar{\partial} X^i e^{inq\cdot X(\bar{z})}.$$

The spacetime vector q^μ is transverse to the spacelike indices i, and $q^2 = 0$. These satisfy an oscillator algebra,

$$\big[A_n^i, A_m^j\big] \cong n\delta^{ij}\delta_{n+m,0}, \qquad \text{and} \qquad \big[\bar{A}_n^i, \bar{A}_m^j\big] \cong n\delta^{ij}\delta_{n+m,0} \tag{K.2}$$

from which it follows that $(A_n^i)^\dagger = A_{-n}^i$. We define a vacuum according to, $\alpha_{n>0}^\mu \cdot e^{ip\cdot X(z,\bar{z})} \cong 0$ and $A_{n>0}^i \cdot e^{ip\cdot X(z,\bar{z})} \cong 0$ with,

$$p^2 = \frac{4}{\alpha'}, \qquad p\cdot q = \frac{2}{\alpha'}, \qquad \text{and} \qquad q^2 = 0.$$

From the above definition of the commutators we learn that,

$$\big[\alpha_m^\mu, A_n^i\big] = m\delta^{\mu,i} B_m^n + n\sqrt{\frac{\alpha'}{2}} q^\mu D_{m,n}^i, \qquad \big[\alpha_\ell^\mu, B_m^n\big] = n\sqrt{\frac{\alpha'}{2}} q^\mu B_{m+\ell}^n,$$

$$[\alpha^\mu_\ell, D^i_{m,n}] = \ell\delta^{\mu,i}B^n_{m+\ell}+n\sqrt{\frac{\alpha'}{2}}q^\mu D^i_{m+\ell,n}, \qquad [\alpha^\mu_m, E^n_\ell] = m\sqrt{\frac{\alpha'}{2}}q^\mu B^n_{m+\ell}-n\sqrt{\frac{\alpha'}{2}}q^\mu E^n_{m+\ell},$$

where following [208] we have defined,

$$B^n_m = \oint \frac{\bar{d}z}{iz}\, z^m\, e^{inq\cdot X(z)}, \qquad D^i_{m,n} = \sqrt{\frac{2}{\alpha'}}\oint \bar{d}z z^m \partial X^i e^{inq\cdot X(z)},$$

and

$$E^n_m = \oint \bar{d}z z^m q\cdot\partial X e^{inq\cdot X(z)}.$$

From these commutators and $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$, $(A^i_n)^\dagger = A^i_{-n}$, it follows that $(B^n_m)^\dagger = B^{-n}_{-m}$, $(D^i_{m,n})^\dagger = D^i_{-m,-n}$ and $(E^n_m)^\dagger = E^{-n}_{-m}$. In addition we learn that,

$$[A^i_\ell, D^j_{m,n}] = \ell\delta^{ij}E^{\ell+n}_m, \qquad [D^i_{-\ell,n}, D^j_{\ell,-m}] = \delta^{ij}\big(nE^{n-m}_0 - \ell B^{n-m}_0\big),$$

and, $[B^r_\ell, D^i_{m,n}] = [A^i_n, B^\ell_m] = [A^i_n, E^\ell_m] = 0 = [B^n_m, B^\ell_r] = [E^n_m, E^\ell_r] = [B^n_m, E^\ell_r]$.

On the chiral half of a (tachyonic) vacuum state, $e^{ip\cdot X(z)}$, one can readily compute the operator products,

$$B^{-n}_{\ m}\cdot e^{ip\cdot X(z)} \cong S_{n-m}(nq;z)\, e^{i(p-nq)\cdot X(z)}, \tag{K.3a}$$

$$D^i_{m,-n}\cdot e^{ip\cdot X(z)} \cong H^i_{n-m}(nq;z)e^{i(p-nq)\cdot X(z)}, \tag{K.3b}$$

$$E^{-n}_m\cdot e^{ip\cdot X(z)} \cong \sqrt{\frac{\alpha'}{2}}q\cdot H_{n-m}(nq;z)e^{i(p-nq)\cdot X(z)}, \tag{K.3c}$$

where the polynomials $S_{n-m}(nq;z)$ and $H^i_{n-m}(nq;z)$ have been defined below and we have made use of the Taylor expansion, $e^{-inq\cdot X(w)} = \sum_{a=0}^\infty (w-z)^a S_a(nq;z)e^{-inq\cdot X(z)}$. Note that in (K.3c) we have extended the definition of $H^i_{n-m}(nq;z)$, to include also longitudinal indices, $H^\mu_{n-m}(nq;z)$, without changing the form of the polynomial.

Open String DDF Operators and Vertex Operators

The relevant propagators on the upper half plane are,

$$\begin{aligned}
N:&\qquad \big\langle X^+(z,\bar z)X^-(w,\bar w)\big\rangle = \frac{\alpha'}{2}\Big(\ln|z-w|^2+\ln|z-\bar w|^2\Big),\\
N:&\qquad \big\langle X^A(z,\bar z)X^B(w,\bar w)\big\rangle = -\frac{\alpha'}{2}\delta^{AB}\Big(\ln|z-w|^2+\ln|z-\bar w|^2\Big),\\
D:&\qquad \big\langle X^I(z,\bar z)X^J(w,\bar w)\big\rangle = -\frac{\alpha'}{2}\delta^{IJ}\Big(\ln|z-w|^2-\ln|z-\bar w|^2\Big),
\end{aligned}\tag{K.4}$$

for the Neumann (N) or Dirichlet (D) directions respectively, with the normalization convention $\partial_z\partial_{\bar z}G(z,w) = -\pi\alpha'\delta^2(z-w)$, and $G(z,w) = \langle X(z,\bar z)X(w,\bar w)\rangle$.

To construct vertex operators we now distinguish between excitations tangent or transverse to the brane respectively,

$$A^A_n = \sqrt{\frac{2}{\alpha'}}\oint \bar{d}z\,\partial X^A(z)e^{inq\cdot X(z,\bar z)}, \quad \text{and} \quad A^I_n = \sqrt{\frac{2}{\alpha'}}\oint \bar{d}z\,\partial X^I(z)e^{inq\cdot X(z,\bar z)}, \tag{K.5}$$

and these act on the open string vacuum, $e^{ip\cdot X(z,\bar{z})}$, which is restricted to the real axis, $z=\bar{z}$. This procedure gives rise to vertex operators of the form,

$$V(z,\bar{z}) = C\xi_{ij\ldots}A^i_{-n_1}A^j_{-n_2}\ldots\, e^{ip\cdot X(z,\bar{z})}, \tag{K.6}$$

as explained in the main text. Self-contractions are subtracted using the correlation functions (K.4). The integrands of the DDF operators are to be restricted to the real axis, $z=\bar{z}$, and only after the normal ordering has been carried out are we to analytically continue the integrand in the complex plane so as to perform the contour integrations shown in (K.5). At this point the integrations should all be analytic in z.

Given that open string vertex operators live on the boundary of the worldsheet it is sometimes useful to represent them as holomorphic functions of a single variable, z. In the main text we concentrate on open string vertex operators with excitations in the directions tangent to the Dp-brane, and so it is possible to construct vertex operators using instead of (K.5) the DDF operator,

$$A^A_n = \sqrt{\frac{2}{\alpha'}}\oint \bar{d}z\,\partial X^A(z)e^{inq\cdot X(z)}, \tag{K.7}$$

with the corresponding vertex operators given by,

$$V(z,\bar{z}) = C\xi_{AB\ldots}A^A_{-n_1}A^B_{-n_2}\ldots\, e^{ip\cdot X(z)}, \tag{K.8}$$

in which case to obtain the normal ordered expression, the self-contractions are to be subtracted using the propagator,

$$N: \qquad \big\langle X^a(z)X^b(w)\big\rangle = -(2\alpha')\eta^{ab}\ln(z-w), \tag{K.9}$$

which follows from (K.4) by restricting the worldsheet arguments to the real axis. To carry out the contour integrations shown in (K.7) we analytically continue in z around the real axis and the contour is to contain the vacuum.

On a Minkowski signature worldsheet the DDF integrals are along the boundary of the worldsheet which is coincident with the Dp-brane.[1] The vacuum momenta p^μ and null vectors q^μ are restricted to lie within the D-brane worldvolume, see (2.41), and the q^μ are transverse to the DDF operators:

$$q^A = q^I = p^I = 0.$$

The onshell constraints for the open string are,

$$p^2 = \frac{1}{\alpha'}, \qquad p\cdot q = \frac{1}{2\alpha'}, \qquad \text{and} \qquad q^2 = 0, \tag{K.10}$$

[1] In Minkowski signature, where $z=e^{-i(\sigma-\tau)}$ and $\tau_{\text{Euclidean}} = i\tau_{\text{Minkowski}}$, we have instead,

$$\mathcal{A}^A_n = \frac{1}{\sqrt{2\alpha'}}\int_0^{2\pi}\bar{d}\tau\partial_\tau X^A e^{inq\cdot X}, \quad \text{and} \quad \mathcal{A}^I_n = \frac{1}{\sqrt{2\alpha'}}\int_0^{2\pi}\bar{d}\tau\,\partial_\sigma X^I e^{inq\cdot X},$$

with the integrals along the worldsheet boundary and the derivatives, ∂_τ and ∂_σ in the tangent and inward normal direction to the worldsheet boundary respectively. Note that: $\mathcal{A}^A_n|_{\text{Euclidean}} = \bar{A}^A_n + A^A_n$ and $\mathcal{A}^I_n|_{\text{Euclidean}} = \bar{A}^I_n - A^I_n$. The derivatives appearing in (K.5) are purely holomorphic because of the boundary conditions.

so as to ensure that the vertex operators (K.6) are onshell with mass spectrum $m^2 = -(p-Nq)^2 = (N-1)/\alpha'$ as appropriate for open strings. The contractions appearing in (K.10) are with respect to all spacetime indices μ.

Open String Covariant Commutators

In direct analogy to the closed string case above we learn that,

$$\big[\alpha_m^\mu, A_n^i\big] = m\delta^{\mu,i}B_m^n + n\sqrt{2\alpha'}q^\mu D_{m,n}^i, \qquad \big[\alpha_\ell^\mu, B_m^n\big] = n\sqrt{2\alpha'}q^\mu B_{m+\ell}^n,$$

$$\big[\alpha_\ell^\mu, D_{m,n}^i\big] = \ell\delta^{\mu,i}B_{m+\ell}^n + n\sqrt{2\alpha'}q^\mu D_{m+\ell,n}^i, \qquad \big[\alpha_m^\mu, E_\ell^n\big] = m\sqrt{2\alpha'}q^\mu B_{m+\ell}^n - n\sqrt{2\alpha'}q^\mu E_{m+\ell}^n,$$

where we have defined,

$$B_m^n = \oint \frac{đz}{iz}\, z^m\, e^{inq\cdot X(z)}, \qquad D_{m,n}^i = \sqrt{\frac{2}{\alpha'}}\oint đz z^m \partial X^i e^{inq\cdot X(z)},$$

and

$$E_m^n = \oint đz z^m q\cdot\partial X e^{inq\cdot X(z)}.$$

From these commutators and $(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$, $(A_n^i)^\dagger = A_{-n}^i$, it follows that $(B_m^n)^\dagger = B_{-m}^{-n}$, $(D_{m,n}^i)^\dagger = D_{-m,-n}^i$ and $(E_m^n)^\dagger = E_{-m}^{-n}$. In addition we learn that,

$$\big[A_\ell^i, D_{m,n}^j\big] = \ell\delta^{ij}E_m^{\ell+n}, \qquad \big[D_{-\ell,n}^i, D_{\ell,-m}^j\big] = \delta^{ij}\big(nE_0^{n-m} - \ell B_0^{n-m}\big),$$

and, $\big[B_\ell^r, D_{m,n}^i\big] = \big[A_n^i, B_m^\ell\big] = \big[A_n^i, E_m^\ell\big] = 0 = [B_m^n, B_r^\ell] = [E_m^n, E_r^\ell] = [B_m^n, E_r^\ell]$.

On the chiral half of a (tachyonic) vacuum state, $e^{ip\cdot X(z)}$, one can readily compute the operator products,

$$B_{\ m}^{-n}\cdot e^{ip\cdot X(z)} \cong S_{n-m}(nq;z)\, e^{i(p-nq)\cdot X(z)}, \tag{K.11a}$$

$$D_{m,-n}^i\cdot e^{ip\cdot X(z)} \cong H_{n-m}^i(nq;z)e^{i(p-nq)\cdot X(z)}, \tag{K.11b}$$

$$E_m^{-n}\cdot e^{ip\cdot X(z)} \cong \sqrt{2\alpha'}q\cdot H_{n-m}(nq;z)e^{i(p-nq)\cdot X(z)}, \tag{K.11c}$$

where the polynomials $S_{n-m}(nq;z)$ and $H_{n-m}^i(nq;z)$ have been defined below and we have made use of the Taylor expansion, $e^{-inq\cdot X(w)} = \sum_{a=0}^\infty (w-z)^a S_a(nq;z)e^{-inq\cdot X(z)}$. Note that in (K.11c) we have extended the definition of $H_{n-m}^i(nq;z)$, to include also longitudinal indices, $H_{n-m}^\mu(nq;z)$, without changing the form of the polynomial.

Appendix L

Gauge Invariant Position Operator

The position operator (2.37) is not a gauge invariant quantity, $[L_n, X^\mu(z,\bar z)] \neq 0$, and so cannot be inserted into covariant path integrals. It is sometimes useful to have an operator that is gauge invariant, and that does in many respects have the properties of a position operator, properties such as (2.38). We discuss this next.

Motivated by the isomorphism of the algebras satisfied by α_n^μ and A_n^i, see (2.38) and (4.69) respectively, and by the fact that the DDF operators are gauge invariant, $[L_n, A_m^i] = 0$ see Sec. 4.4, let us by direct analogy to (2.37) define the following position-like gauge invariant operator for the transverse indices [241, 242],

$$\mathsf{X}^i(z,\bar z) = \hat{\mathsf{x}}^i - i\frac{\alpha'}{2}\,\hat p^i \ln|z|^2 + i\Big(\frac{\alpha'}{2}\Big)^{1/2}\sum_{n\neq 0}\frac{1}{n}\big(A_n^i\, z^{-n} + \bar A_n^i\, \bar z^{-n}\big). \tag{L.1}$$

Here $\hat p^i = A_0^i = \alpha_0^i$ and $\mathsf{x}^i = \frac{\alpha'}{2} q_\mu J^{i\mu}$ with the null vector q^μ and the angular momentum operator $J^{i\mu}$ as defined above. On account of (K.1) one finds,

$$\big[A_n^i, A_m^j\big] = n\delta^{ij}\delta_{n+m,0}, \qquad \big[\mathsf{X}^i(z), \partial_\tau \mathsf{X}^j(z')\big] = \delta^{ij}\delta(\sigma-\sigma'), \qquad \text{and} \qquad [\mathsf{x}^i, p^j] = i\delta^{ij},$$

in direct analogy to (2.38). Unlike the standard position operator, $X(z,\bar z)$, however, the quantity (L.1) is gauge invariant given that the DDF operators and the zero modes $\hat{\mathsf{x}}^i$ and $\hat p^i$ commute with the Virasoro generators, $\big[L_n, \mathsf{X}^i(z,\bar z)\big] = 0$, and $[L_n, \hat{\mathsf{x}}^i] = [L_n, \hat p^i] = 0$, for all $n \in \mathbb{Z}$ (and similarly for $\bar L_n$), and therefore define sensible operators that may be inserted into covariant path integrals.

In fact, $\mathsf{X}^i(z,\bar z)$ can in some sense be thought of as the covariant version of the lightcone quantity $X^i(z,\bar z)$: the A_n^i reduce to the α_n^i when one restricts to lightcone gauge in which case (L.1) reduces to (2.37). We can use q^μ to define a lightcone time $q\cdot X(z,\bar z) = -i\ln|z|^2$ to find that (at least classically),

$$A_n^i\big|_{\rm l.c.} = \sqrt{\frac{2}{\alpha'}}\oint \bar{d}z\, \partial X^i(z) e^{inq\cdot X(z)}\Big|_{\rm l.c.} = \sqrt{\frac{2}{\alpha'}}\oint \bar{d}z\, \partial X^i(z)\, z^n = \alpha_n^i,$$

where we have formally factorized $q\cdot X(z,\bar z)$ into $q\cdot X(z) = -i\ln z$ and $q\cdot X(\bar z) = -i\ln \bar z$. We hence deduce that at least at the classical level,

$$\big(\mathsf{X}^i(z,\bar z) - \mathsf{x}^i\big)\big|_{\rm l.c.} = X^i(z,\bar z) - x^i.$$

We *conjecture* that this be elevated to a quantum-mechanical statement:

$$\big\langle V\big|F\big(\mathsf{X}^i(z,\bar{z})-\mathsf{x}^i\big)\big|V\big\rangle_{\rm cov}=\big\langle V\big|F\big(X^i(z,\bar{z})-x^i\big)\big|V\big\rangle_{\rm lc}, \tag{L.2}$$

for some well behaved functional $F(A)$ of the argument A. Here by $|V\rangle_{\rm cov}\cong V(z,\bar{z})$ we mean the covariant vertex operator (4.63),

$$V(z,\bar{z})=C\xi_{ij\dots,kl\dots}A^i_{-n_1}A^j_{-n_2}\dots\bar{A}^k_{-\bar{n}_1}\bar{A}^l_{-\bar{n}_2}\dots\, e^{ip\cdot X(z,\bar{z})}, \tag{L.3}$$

and $|V\rangle_{\rm lc}$ represents the corresponding lightcone gauge state (4.61),

$$|V\rangle_{\rm lc}=C\xi_{ij\dots,kl\dots}\,\alpha^i_{-n_1}\alpha^j_{-n_2}\dots\tilde{\alpha}^k_{-\bar{n}_1}\tilde{\alpha}^l_{-\bar{n}_2}\dots|0,0;p^+,p^i\rangle. \tag{L.4}$$

The expression (L.2) follows from the isomorphism of lightcone (in terms of the $\alpha^i_n,\tilde{\alpha}^i_n$) and covariant states (in terms of the $A^i_n,\bar{A}^i_n$), the isomorphism of the lightcone gauge and gauge invariant position operators, the fact that the states (L.3) and (L.4) have the same mass and angular momenta, the isomorphism of the corresponding oscillator algebras and finally from out main conjecture that the lightcone and covariant states, (L.4) and (L.3), share identical correlation functions (provided these are gauge invariant).

For example, (L.2) implies that the expectation value of the gauge invariant position operator in some covariant state tells us about the position expectation value of the *lightcone gauge description* of this covariant state.

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