Analysis of Eigenvalues and Conjugate Heat Kernel under the Ricci Flow

by

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in the

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Declaration

I hereby declare that this thesis has not been, and will not be, submitted in whole or in part to another University for the award of any other degree.

Signature:

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UNIVERSITY OF SUSSEX

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Abstract

Let M be an n-dimensional compact (or noncompact without boundary) manifold on which a one parameter family of Riemannian metrics $g(t), t \in [0,T)$ is defined. We say $(M, g(t))$ is a solution to the **Ricci flow** if it evolves by the following nonlinear system of weakly parabolic partial differential equation (PDEs)

$$\frac{\partial}{\partial t} g(x,t) = -2Rc(x,t), \qquad (x,t) \in M \times [0,T],$$

with $g(x,0) = g(0)$, where Rc stands for the Ricci curvature. We take the view-point of PDEs to study the above system in relation to some problems in Geometric Analysis, namely, monotonicity and estimates of eigenvalues and heat kernel of a Riemannian manifold.

This thesis is divided into three parts of five chapters. The first part as contained in Chapter one is purely introductory, where we give an expository overview of the theory and applications of the Ricci flow. Most of the results of this study are in part two. Precisely, in Chapter two, we derive monotonicity of the first nonzero eigenvalue of a Laplacian form operator under the action of Ricci flow ($n \geq 2$) using Perelman's idea of entropy formulas. The consequence of this allows us to rule out the existence of nontrivial breathers. We also give the conditions on which Einstein metrics shrink. Chapters three and four are devoted to the analysis of conjugate heat kernel, here, we couple the Ricci flow to the conjugate heat equation defined on M,

$$-\frac{\partial u}{\partial t} - \Delta_{g(t)} u + R_{g(t)} u = 0, \qquad (x,t) \in M \times [0,T],$$

where $\Delta_{g(t)}$ is the usual Laplace-Beltrami operator depending on the metric and the scalar curvature $R_{g(t)}$ is the metric trace of Rc. We obtain certain localized and global gradient estimates for all positive solutions to the conjugate heat equation. With the aid of Sobolev embedding, log-Sobolev inequality and a new entropy functional, we obtain some sharp upper bounds for the conjugate heat kernel, as a by-product, we derive log-Sobolev inequalities from Sobolev inequalities for the Ricci flow. The third part forms the last chapter, which is on the application of heat flow monotonicity approach to proving some functional-geometric inequalities, here, we deal with the family of Brascamp-Lieb inequalities as a model.

Dedication

To almighty GOD & my family

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Notation and Symbols

$B(x,r),\ B_r(x)$	Ball of radius r centred at x.
BLI	Brascamp-Lieb inequalities.
$const.$	Constant.
$cut\{x\}$	Cut locus of x.
$\nabla,\ \nabla_i$	Gradient / Covariant derivative.
$d(x,y)$	Geodesic distance between points x and y.
$\delta,\ div$	Divergence.
$\delta_y(\cdot)$	Dirac delta function centred at y
$\cdot$	Dot product or multiplication
$\circ$	Composition
n	Dimension of a space
(x^i)	A local coordinate chart
Γ_{ij}^k	Christoffel's symbols of a connection ∇ w.r.t. a local frame ∂_i.
$\partial_i = \frac{\partial}{\partial x^i}$	Partial derivative w.r.t. the coordinate x^i
$w.r.t.$	with respect to.
$\mathcal{F},\ \mathcal{W}$	Perelman's Energy / Entropy.
$\mathbb{R}^n$	n-dimensional Euclidean Space.
$\nabla\nabla f,\ Hess(f)$	Hessian of a function f.
$f_{ij}, \nabla_i\nabla_j f$	Tensor components of Hessian of f.
$\Delta,\ \Delta_g$	Laplace-Beltrami operator w.r.t metric g.
dx	Lebesgue measure
$\ln,\ \log$	Natural logarithm.
LSI	Logarithmic Sobolev inequalities.
LHS/RHS	Left hand side/Right hand side.
$g(X,Y), \langle X, Y\rangle$	metric or inner product for $X, Y \in T_pM$.
g_{ij}	Component of metric tensor.
$\mathcal{L}_X$	Lie derivative w.r.t vector field X.

$M, (M^n, g),\ (M, g)$	n-dimensional Riemannian manifold
ODE/PDE	Ordinary/Partial Differential Equation.
p'	Hölder's conjugate to a real number p such that $1/p + 1/p' = 1$.
p^*	Sobolev's conjugate to a real number p such that $1/p^* = 1/p - 1/n$.
P_t	Heat semigroup operator.
R	Scalar curvature tensor.
Rc, R_{ij}	Ricci curvature tensor / its components.
$Rm, R^k{}_{ijl}$	Riemannian curvature tensor / its components.
RF/NRF	Ricci flow / Normalized Ricci flow.
S^2T^*M	The set of $(0, 2)$-symmetric tensors over M.
$S^2_+T^*M$	The space of positive definite symmetric two tensors g_{ij}.
$diam(M)$	Diameter of M.
$\sigma(M)$	Spectrum of M.
$\lambda_1(M)$	The first nonzero eigenvalue of M.
$supp(f)$	The support of a function$f$.
$\otimes$	Tensor product.
$C_0^\infty(\Omega)$	Space of continuous functions with zero trace on the boundary (or with compact support).
$L^p = L^p(\Omega)$	The space of measurable functions on Ω such that $\|f\|^p$ is integrable for $p \in [1, \infty)$.
$\\|f\\|_p = \\|f\\|_{L^p}$	$\left(\int_\Omega \|f\|^p dx\right)^{\frac{1}{p}}$ for $f \in L^p$.
$L^\infty = L^\infty(\Omega)$	The space of measurable functions f on Ω for which there exists K and $\|f(x)\| \le K$ a.e. $x \in \Omega$.
$\\|f\\|_\infty = \\|f\\|_{L^\infty}$	$\min\{K > 0,\ \|f(x)\| \le K\ a.e.\} for f \in L^\infty$.
$W^{k,p}(\Omega)$	$:= \{f \in L^p(\Omega),\ \partial^\alpha f \in L^p(\Omega)\ \ \forall\ \ \|\alpha\| \le k\}$.
$W_0^{k,p}(\Omega)$	The closure of $C_0^\infty(\Omega)$.
$H^k(\Omega)$	$W^{k,2}(\Omega)$.
$H_0^k(\Omega)$	$W_0^{k,2}(\Omega)$.
$T_pM,\ T_p^*M$	Tangent space / cotangent spacet point p.
$T_s^r(M)$	(r, s)-tensor bundle over a manifold M.
$TM,\ T^*M$	Tangent bundle / cotangent bundle
$d\mu,\ d\mu_g$:	Volume form w.r.t a metric g.
dA :	Volume forn in dimension 2 (Area form).
$Vol(M)$:	Volume of manifold M.
$dV(x)$:	Volume form static manifold.

Introduction and Background

0.1 Introduction

The Ricci flow is a nonlinear system of geometric evolution partial differential equations on a Riemannian manifold. It has been a means of deforming a background metric to obtain an 'improved form' and has greatly helped to understand the geometry and topology of the underlying manifold. Since its introduction in 1982 by Richard Hamilton [87], it has gained stupendous interests among mathematicians and physicists, this is not unconnected with the breakthroughs associated with its applications, a very much celebrated of which, is the proof of Thurston's Geometrization Conjecture for 3-manifolds by Grisha Perelman [126, 127, 128] and the consequent solution to the longstanding Poincaré Conjecture. Indeed the Poincaré Conjecture had been listed as one of the seven Millennium Prize Problems by the Clay Mathematics Institute in 2000. The Ricci flow on a 2-Dimensional manifold also leads to a complete proof of the Poincaré-Koebe Uniformization Theorem by Richard Hamilton [88] and Bennett Chow [60]. Similarly, the recent proof of differentiable sphere theorem by Simon Brendle and Richard Schoen [35, 36] with the use of Ricci flow needs to be celebrated, see also [34]. The Ricci flow has raised hope of several applications in Mathematics, Physics and other Natural Sciences. It can in fact be said that Ricci flow is a connecting point for many fields of Mathematics be it Analysis, Geometry, Topology, Theoretical Physics and Applied Mathematics.

This thesis is divided into three parts of five chapters. The first part as contained in Chapter 1 is purely introductory, where we give an expository overview of the theory and applications of the Ricci flow. We take the view-point of PDEs to study this system of partial differential equations (Ricci Flow) in relation to some problems in Geometric Analysis, namely, monotonicity and estimates of eigenvalues and heat kernel of a Riemannian manifold. Although the Ricci flow is not strictly parabolic because of diffeomorphism invariance of the Ricci-tensor (presence of Bianchi identity), we notice that all the associated geometric quantities, especially curvatures evolve along the flow by heat-type equations. Combining this fact with some classical results relating to the spectrum and heat kernel of Riemannian manifolds, one is adequately equipped to investigate further the behaviour of these geometric objects under the Ricci flow. Motivated by this, we investigate the monotonicity property of eigenvalues of certain operator under the Ricci flow in Chapter 2. The basic idea here follows from the energy and entropy formulas of Perelman, the consequence of which we use to rule out the existence of nontrivial breathers. The results are extended to the normalized case and we are able to give conditions on which Einstein metrics shrink. It is well known that eigenvalues, eigenfunctions, Laplacian and Heat kernel are closely related, certain behaviours and applications of these objects become more obvious through one of the groundbreaking papers of Perelman [126]. We combine some classical idea of Peter Li and S-T Yau [112] with Perelman's to study the conjugate heat-type equation coupled to the Ricci flow and estimate its minimal positive solution, which we refer to as conjugate heat kernel in Chapters 3 and 4. While the estimates obtained in Chapter 3 are of Li-Yau-Hamilton Harnack type, Chapter 4 focuses on different approaches to obtain bounds for heat kernel. The main ingredients used in this chapter includes Sobolev embedding for Ricci flow and a new entropy functional, as a by-product, we derive log-Sobolev inequalities for the Ricci flow.

The last part of this thesis is contained in Chapter 5 which is on the elegant application of heat flow monotonicity to the proof of a family of functional-geometric inequalities, namely, Brascamp-Lieb inequalities. This chapter may be considered independent as the subject is interesting on its own right and we treat it as such by developing the theory from Euclidean-analytic point of view, but the connection is in the fact that the applications of such inequalities are becoming more obvious in PDEs and Geometric analysis, hence the need for their generalization to 'full' diffeomorphic setting.

The basic elements of Riemannian Geometry used in this thesis are included in the Appendix A while Appendix B gives detail background of eigenvalues of Laplacian and heat kernels of Riemannian manifolds. For completeness, the proofs of Perelman's energy and entropy formulas are presented in appendix C.

0.2 Riemannian Manifolds, Metrics and Connections

Here we give an overview of the concept of Riemannian manifold and its analysis, the aim of which is to fix notation that are adopted throughout this thesis. Appendices A and B illuminate further on the concept and provide other elements of Riemannian Geometry as transpired in the thesis. Informally, a manifold is a topological space whose each point has a neighbourhood which looks like Euclidean space. Lines and circles are one dimensional examples while the sphere and the torus are examples in 2-dimension, they are called surfaces. Riemannian manifolds are usually the object of study in Riemannian Geometry. It is a real smooth manifold equipped with inner product called Riemannian metric which varies from point to point on the tangent space. See the following [53, 54, 59, 79, 104] and other standard references on Riemannian Geometry for details.

Riemannian Metrics

For a manifold M and any point $p \in M$, the tangent space T_pM can be characterised as the set of derivations of algebra of germs at p of $\mathcal{C}^\infty$ functions on M i.e., the tangent vectors are directional derivatives [15]. Local coordinates (u^i) gives a basis for T_pM consisting of partial derivative operators $\frac{\partial}{\partial x^i}$.

A $(r,s)-tensor$ at $p \in M$ is an element of $T^r_s(M)$. We also define the bundle of $(r,s)-tensor$ on M by

$$T^r_s M = \bigcup_{p\in M}(T_pM) = \bigcup_{p\in M} T^*_p M \otimes T_p M. \tag{0.2.1}$$

Let TM and T^*M denote the tangent and cotangent bundles of a manifold M respectively. We also define a (r,s)-tensor $\mathbf{B}$ as a smooth section of the bundle $(T^*M)^{\otimes r} \otimes (TM)^{\otimes s}$. In local coordinate system (x^i), induced by a chart $\phi : U \to \mathrm{R}^n, U \subseteq M$, the tensor $\mathbf{B}$ has coordinate representation

$$\mathbf{B} = \mathbf{B}^{k_1,\ldots,k_s}_{l_1,\ldots,l_r} dx^{l_1} \otimes \ldots \otimes dx^{l_r} \otimes \frac{\partial}{\partial x^{k_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{k_s}}, \tag{0.2.2}$$

with

$$\mathbf{B}^{k_1,\ldots,k_s}_{l_1,\ldots,l_r} = \mathbf{B}\Big(\frac{\partial}{\partial x^{l_1}}, \ldots, \frac{\partial}{\partial x^{l_r}}, dx^{k_1}, \ldots, dx^{k_s}\Big).$$

Let $u, v \in T_pM$ with

$$u = u^i \frac{\partial}{\partial x^i}\Big|_p, \quad v = v^i \frac{\partial}{\partial x^i}\Big|_p,$$

we call $u^i \frac{\partial}{\partial x^i}$ vectors and $u_j dx^j$ covectors. We write ∂_i as an abbreviation for the $\frac{\partial}{\partial x^i}$ and ∇_i for the covariant derivative in the direction of $\frac{\partial}{\partial x^i}$.

Let (M^n, g) be an $n-$dimensional manifold M equipped with metric g. Our manifold M^n refers to a second countable Hausdorff topological space locally homeomorphic to some open subset of n-dimensional Euclidean space R^n.

Definition 0.2.1. *A Riemannian metric on a smooth manifold M is a tensor field g, section of the bundle S^2T^*M, symmetric and positive definite at each point $p \in M$. A Riemannian metric determines an inner product on each tangent space T_pM, written as*

$$\langle X, Y\rangle := g(X,Y), \quad (X, Y \in T_pM).$$

Let $\{x^i\}$ be a local coordinate system, a tensor field g can be locally expressed on U as

$$g = g_{ij} dx^i \otimes dx^j \quad \text{or} \quad g = g_{ij} dx^i dx^j,$$

where $\otimes$ denotes tensor product and $g_{ij} = g_{ji}$ is a smooth function on U. g therefore provides a bilinear function on T_pM at every point $p \in M$. The metric inverse of g_{ij} is denoted by g^{jk} so that $g_{ij}g^{jk} = \delta^k_i$.

Definition 0.2.2. *Any n-dimensional smooth manifold endowed with such a smooth metric described above is called a Riemannian manifold. Henceforth, the pair (M^n, g) is referred to as Riemannian manifold.*

We denote the induced volume element on manifold M by $d\mu = \sqrt{detg}dx^i$ and the components of the metric by $g_{ij} = g(\partial_i, \partial_j)$, where $\partial_i = \frac{\partial}{\partial x^i}$. The Levi-Civita connection is defined by

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k, \tag{0.2.3}$$

where ∂_k (which will be sometimes written as ∇_k) denotes the covariant derivative in the $\frac{\partial}{\partial x^k}$ direction, while its Christoffel's symbols are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{0.2.4}$$

The Riemannian curvature tensor of M is a $(1,3)$ tensor given by

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}^k \partial_k$$

and its component defined by

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^p \Gamma_{ip}^l - \Gamma_{ik}^p \Gamma_{jp}^l, \tag{0.2.5}$$

it is equivalent to a $(0,4)$ tensor

$$R_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma_{jk}^p \Gamma_{ipl} - \Gamma_{ik}^p \Gamma_{jpl}. \tag{0.2.6}$$

The curvature tensor satisfies the following identities

$$\begin{aligned} &\textbf{Symmetric identity}: \quad R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij} \\ &\textbf{First Bianchi identity}: \quad R_{ijkl} + R_{iklj} + R_{ilkj} = 0 \\ &\textbf{Second Bianchi identity}: \quad \nabla_r R_{ijkl} + \nabla_k R_{ijlr} + \nabla_l R_{ijrk} = 0. \end{aligned}$$

The Ricci curvature tensor of first kind is a symmetric $(0,2) - tensor$ defined by contracting the Riemannian tensor as follows

$$R_{ij} \equiv R_{ijk}^k = \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{ij}^p \Gamma_{pk}^k - \Gamma_{ik}^p \Gamma_{pj}^k, \tag{0.2.7}$$

while the scalar curvature is the trace of the Ricci tensor and defined by

$$R = g^{ij} R_{ij}. \tag{0.2.8}$$

When there is a chance of confusion, we use Rm, Rc and R to mean Riemannian tensor, Ricci tensor and Scalar curvature tensor respectively.

A Riemannian metric is said to be Einstein metric if its Ricci tensor is a scalar multiple of the metric at each point, that is, for some constant λ

$$R_{ij}(g) = \lambda g_{ij} \qquad everywhere.$$

Taking the trace of both sides gives a relation that involves Scalar curvature as

$$\lambda = \frac{1}{n} R,$$

where n is the dimension of the Manifold. Thus, the constant of proportionality λ for Einstein manifolds is related to the scalar curvature R and the Einstein condition is written as

$$R_{ij}(g) = \frac{1}{n} R g_{ij}.$$

We therefore define the Einstein tensor G as a $2-tensor$ field

$$G_{ij} = R_{ij}(g) - \frac{1}{2}R\, g_{ij}. \tag{0.2.9}$$

Its trace in n- dimensions, obtained by contracting with metric tensor g^{ij}, is

$$G = \frac{2-n}{2}R.$$

Einstein tensor is also called **trace-reversed Ricci tensor** because it is the negative of the Ricci tensor's trace. Taking the trace of the second Bianchi identity twice yields the Contracted Second Bianchi Identity

$$g^{ij}\nabla_i R_{jk} = \frac{1}{2}\nabla_k R,$$

which is equivalent to the Einstein tensor $R_{ij}(g) - \frac{1}{2}Rg_{ij}$ being divergence-free;

$$div\Big(R_{ij}(g) - \frac{1}{2}R\, g_{ij}\Big) = 0. \tag{0.2.10}$$

Notice that the first trace yields

$$g^{ir}\nabla_r R_{ijkl} = \nabla_j R_{kl} - \nabla_k R_{jl},$$

which implies that the divergence of Rm is the exterior covariant derivative of Rc considered as a 1-form with values in the tangent bundles. (This is done by multiplying the Second Bianchi Identity by $g^{ir}g^{kl}$).

Lemma 0.2.3. *([68, Lemma 6.57]) If g and h are two Riemannian metrics on an $n-$ dimensional Riemannian manifold and they are related by time scale factor ϕ (i.e $g = \phi h$), then, the various geometric quantities scale as follows*

$$g^{ij} = \frac{1}{\phi}h^{ij}, \qquad \Gamma^k_{ij\,(g)} = \Gamma^k_{ij\,(h)}, \qquad R^l{}_{ijk}(g) = R^l{}_{ijk}(h), \qquad R_{ijkl}(g) = \phi R_{ijkl}(h),$$

$$R_{ij}(g) = R_{ij}(h), \qquad R_{(g)} = \frac{1}{\phi}R_{(h)} \quad and \quad d\mu_{(g)} = \phi^{\frac{n}{2}}d\mu_{(h)}$$

Laplace-Beltrami Operator

Recall that the Laplacian of a function is defined as the divergence of the gradient of that function, that is,

$$\Delta f = \; div\; grad\; f = \nabla \cdot \nabla f.$$

The Laplacian operator Δ is a second-order differential operator in the n-Euclidean space. Now, extending the Laplacian to act on tensor bundles over a Riemannian Manifold (M^n, g), then it becomes the *connection Laplacian*, that is, the divergence of the covariant derivative.

Definition 0.2.4. *Let T be any tensor field defined on the tensor bundle $\mathcal{T}^k_l(M)$, the connection Laplacian is then, the trace of the second covariant derivative with metric g, denoted by*

$$\Delta T = tr_g \nabla^2 T, \tag{0.2.11}$$

where ∇ is the Levi-Civita connection associated with the metric g.

We can then write

$$\Delta T = g^{pq}(\nabla_p\nabla_q T)(\frac{\partial}{\partial x^{j_1}}, ... \frac{\partial}{\partial x^{j_l}}, dx^{i_1}, ...dx^{i_k}).$$

If the tensor bundle is simply $T^0 M$, we have

$$\begin{aligned}\Delta f &= div\ \nabla f = g^{ij}\nabla_i\nabla_j f\\ &= g^{ij}\Big(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\Big).\end{aligned}$$

This Laplacian is often called *Laplace-Beltrami Operator*.

Suppose (M, g) is an oriented smooth manifold with the volume defined as

$$Vol_g = \mu_g = \sqrt{detg}\, dx^1 \wedge ... \wedge dx^n,$$

where $'\wedge'$ is the wedge product. The divergence (div) of X on the manifold is defined as the scalar function

$$(divX)\mu_g := \mathcal{L}_X \mu_g, \tag{0.2.12}$$

where $\mathcal{L}_X$ is the Lie-derivative along vector field X. (Notice that Lie derivative of metric is expressed in local coordinate as $(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$). Then

$$div\ X\ = \frac{1}{\sqrt{detg}}\frac{\partial}{\partial x^i}(\sqrt{detg}X^i).$$

We also define the gradient of f as

$$(grad\ f)^i = \partial^i f = g^{ij}\frac{\partial}{\partial x^j}f.$$

Combining the definition of the gradient and divergence, we obtain the formula for the Laplace-Betrami operator on a scalar function f as

$$\Delta f = div\ grad\ f = \frac{1}{\sqrt{detg}}\frac{\partial}{\partial x^i}\sqrt{detg}\, g^{ij}\frac{\partial}{\partial x^j}f. \tag{0.2.13}$$

More generally, the Laplacian operator acting on tensor is given by

$$\Delta =\ div\ \nabla =\ trace_g \nabla^2 = g^{ij}\nabla_i\nabla_j = \nabla_i\nabla_i.$$

Consider the commutator of Δ and ∇ on any function $f \in C^\infty(M)$, we have the Ricci identity

$$\Delta\nabla_i f = \nabla_k\nabla_i\nabla_j f = \nabla_i\nabla_j\nabla_k f - R_{ijkl}\nabla_k f, \tag{0.2.14}$$

which implies that

$$\Delta\nabla_i f = \nabla_i\Delta f + R_{ij}\nabla_j f. \tag{0.2.15}$$

By Bochner formula, we define for a gradient vector field,

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle\nabla\Delta u, \nabla u\rangle + Rc(\nabla u, \nabla u). \tag{0.2.16}$$

A variant of which is given for any function $f \in C^\infty(M)$ as

$$\Delta|\nabla f|^2 = 2|\nabla\nabla f|^2 + 2R_{ij}\nabla_i f\nabla_j f + 2\nabla_i f\nabla_i(\Delta f). \tag{0.2.17}$$

0.3 Outline of the Thesis

This section mainly highlights the results of each chapter in this thesis. For each chapter, we give a detailed introduction, background and motivations of every studied topic. All the assumptions, statements of results and their proofs are stated explicitly as well. The Ricci flow will be a connecting point up to Chapter 4, while the last chapter will be treated independently.

Chapter 1

Chapter 1 is purely introductory, it provides detailed background on the theory and applications of the Ricci flow. We start with a brief history of the subject, we then present Ricci flow as a nonlinear evolution partial differential equation. We discuss some special solutions with examples, evolutions of geometric quantities under the flow and existence problem for the flow. We conclude this chapter with statements of maximum principles which are used extensively in the remaining chapters.

Chapter 2

In Chapter 2 we study evolution and monotonicity of the first eigenvalue of Laplacian-type operator under the Ricci flow via Perelman's energy functional. In Section 2.2, we introduce some classical energy functionals and lay emphasis on Perelman entropy and its geometric consequences. Let $(M^n, g_{ij}(t))$ be a closed manifold for a Riemannian metric $g_{ij}(t)$ and a smooth function f on M^n, Perelman's Energy functional $\mathcal{F}$ [126] on pairs (g_{ij}, f) is defined by

$$\mathcal{F}(g_{ij}(t), f) = \int_{M^n} (R + |\nabla f|^2) e^{-f} d\mu, \tag{0.3.1}$$

where g_{ij} and R are metric components and scalar curvature respectively. Perelman [126] proved

Lemma 0.3.1. *The coupled modified Ricci flow equation with a backward heat equation*

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2(R_{ij} + \nabla_i \nabla_j f) \\ \frac{\partial f}{\partial t} &= -\Delta f - R \end{aligned} \tag{0.3.2}$$

is a gradient flow for the energy functional $\mathcal{F}(g(t), f(t))$.

The above lemma is discussed in Lemma 2.2.2. Notice that the second equation in the coupling is a backward heat equation, which can be solved backward in time. Conjugating away the infinitesimal diffeomorphism converts (0.3.2) to

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2R_{ij} \\ \frac{\partial f}{\partial t} &= -\Delta f + |\nabla f|^2 - R. \end{aligned} \tag{0.3.3}$$

Precisely

$$\frac{d}{dt}\mathcal{F}(g_{ij}(t), f(t)) = 2\int_{M^n} |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu. \tag{0.3.4}$$

In particular $\mathcal{F}(g_{ij}(t), f(t))$ is monotonically nondecreasing in time and the monotonicity is strict unless $R_{ij} + \nabla_i\nabla_j f = 0$.

Perelman defines

$$\lambda(g_{ij}) = \inf\Big\{\mathcal{F}(g_{ij}, f) : f \in C_c^\infty(M), \int_M e^{-f} d\mu = 1\Big\}, \tag{0.3.5}$$

where the infimum is taken over all smooth functions f. Setting $e^{-f} =: u^2$, then the functional $\mathcal{F}$ is written as

$$\mathcal{F} = \int_{M^n} (Ru^2 + 4|\nabla u|^2) d\mu \quad \text{ with } \quad \int_M u^2 d\mu = 1. \tag{0.3.6}$$

Then $\lambda(g)$ is the first non zero (least) eigenvalue of the self adjoint modified operator $-4\Delta + R$ and the non-decreasing monotonicity of $\mathcal{F}$ implies that of λ. As an application, Perelman was able to rule out the existence of nontrivial steady or expanding Ricci breathers on closed manifolds.

In Section 2.3, we construct a new family of entropy functionals which proves to be monotonically nondecreasing.

Definition 0.3.2. *Let (M^n, g) be a closed n-dimensional Riemannian Manifold, $f : M^n \to \mathbb{R}$ be a smooth function on M^n, define a functional on pairs (g_{ij}, f) by*

$$\mathcal{F}_{\mathcal{B}} = \int_M \Big(\frac{1}{2}|\nabla f|^2 + R\Big) dm, \tag{0.3.7}$$

where $dm := e^{-f} d\mu$. (see Definition 2.3.2).

The functional $\mathcal{F}_{\mathcal{B}}$ is a variant of Perelman's energy functional $\mathcal{F}$, though expected to behave in a similar manner, it differs from the later by the introduction of constant $\frac{1}{2}$. We also define a family of functional $\mathcal{F}_{\mathcal{B}C}$ as

$$\mathcal{F}_{\mathcal{B}C} = \int_M (|\nabla f|^2 + 2CR) dm, \tag{0.3.8}$$

where $C \geq \frac{1}{2}, C \in \mathbb{R}$. When $C = \frac{1}{2}$, this is Perelman's $\mathcal{F}$ functional [126], $C = 1$ is a specific case we consider and $C = \frac{1}{2}k, k \geq 1$, we have Li-$\mathcal{F}_k$-family [109]. Precisely, we prove the following

Let $g_{ij}(t)$ and f solves the system (0.3.3) in the interval $[0, T)$, then,

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}}(g_{ij}, f) = \int_M |R_{ij} + \nabla_i\nabla_j f|^2 dm + \int_{M^n} |R_{ij}|^2 e^{-f} d\mu, \tag{0.3.9}$$

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}C} = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 dm + 2(2C-1)\int_M |R_{ij}|^2 dm \geq 0. \tag{0.3.10}$$

Particularly, define

$$\mu_C(g_{ij}) = \inf\Big\{\mathcal{F}_{\mathcal{B}C}(g_{ij}, f) : f \in C_c^\infty(M), \int_M e^{-f} d\mu = 1\Big\}, \tag{0.3.11}$$

where the infimum is taken over all smooth functions f. The normalisation $\int_M e^{-f} d\mu = 1$ makes $e^{-f} d\mu$ a probability measure and ensures a meaningful infimum. We prove (see Theorem 2.3.7)

Theorem 0.3.3. *Let $(M^n, g_{ij}(t)), t \in [0, T)$ be a solution of the Ricci flow, then, the least eigenvalue $\mu_C(g_{ij})$ of $(-2\Delta + CR)$ is diffeomorphism invariance and non-decreasing. The monotonicity is strict unless the metric is a steady gradient soliton.*

The above results are extended to the case of normalized flow in Section 2.4. The results here confirm that either expanding or steady breathers on compact manifold are necessarily Einstein. In the last section, we construct a new family of entropies over shrinkers (shrinking Ricci soliton), which allows us to obtain the conditions over which Einstein metric shrinks under both normalized and unnormalized Ricci flow. Although we assume that our manifold has no boundary, the results can be carried over directly to the case with empty boundary. Recall Perelman's shrinker entropy $\mathcal{W}$

$$\mathcal{W}(g, f, \tau) := \int_M \Big[\tau(R + |\nabla f|^2) + f - n\Big](4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu, \tag{0.3.12}$$

Here, Perelman [126] proved that if $(g(t), f(t), \tau(t))$ solves the following system

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i\nabla_j f), \\[2ex] \dfrac{\partial f}{\partial t} = -\Delta f - R + \dfrac{n}{2\tau}, \\[2ex] \dfrac{d\tau}{dt} = -1. \end{cases} \tag{0.3.13}$$

Then

$$\frac{d}{dt}\mathcal{W}(g, f, \tau) = \int_M 2\tau|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}|^2 u d\mu, \tag{0.3.14}$$

where $\int_M u d\mu$ is a constant. Similar to the above we have the following entropies over shrinkers

Definition 0.3.4. *Let (M, g) be a closed n-dimensional Riemannian Manifold, we define a family of entropy functional $\mathcal{W}_{\mathcal{B}C}$ as*

$$\mathcal{W}_{\mathcal{B}C} = \tau \int_M \Big[|\nabla f|^2 + 2C\Big(R + \frac{1}{\tau}(f - n)\Big)\Big] u d\mu, \tag{0.3.15}$$

where $C \geq \frac{1}{2}, C \in \mathbb{R}$. When $C = \frac{1}{2}$, this is Perelman's $\mathcal{W}$ entropy [126].

Theorem 0.3.5. *Let $(M, g_{ij}(t), f(t), \tau(t)), t \in [0, T)$, solve the system (0.3.13), where f evolves by a backward heat equation, then, $\mathcal{W}_{\mathcal{B}C}$ is monotonically non-decreasing. In particular, we have*

$$\frac{d}{dt}\mathcal{W}_{\mathcal{B}C} = 2\tau \int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}|^2 u d\mu + 2(2C - 1)\tau \int_M |R_{ij} - \frac{1}{2\tau} g_{ij}|^2 u d\mu \geq 0. \tag{0.3.16}$$

Moreover, the monotonicity is strict unless

$$R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij} = 0 \qquad \textit{and} \qquad R_{ij} - \frac{1}{2\tau} g_{ij} = 0. \tag{0.3.17}$$

(See Theorem 2.5.6).

Chapter 3

Chapter 3 of this thesis is organised as follows; Section 3.2 introduces the theory of conjugate heat equation and gives a quick review of how one can view Perelman's differential Harnack estimate as Li-Yau type and how it provides an alternative proof of a localised version of his entropy monotonicity formula. The main result of Section 3.3 is contained in Theorem 3.3.1, where we establish a point-wise differential Harnack inequality for all positive solutions of the conjugate heat equation on manifold evolving by the Ricci flow.

Theorem 0.3.6. *Let $u \in C^{2,1}(M \times [0,T])$ be a positive solution to the conjugate heat equation $\Gamma^* u = (-\partial_t - \Delta + R)u = 0$ and the metric $g(t)$ evolve by the Ricci flow in the interval $[0,T)$ on a closed manifold M with nonnegative scalar curvature. Suppose further that $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$, where $\tau = T - t$, then for all points $(x,t) \in (M \times [0,T])$, we have the Harnack quantity*

$$P = 2\Delta f - |\nabla f|^2 + R - \frac{2n}{\tau} \leq 0. \tag{0.3.18}$$

Then P evolves as

$$\frac{\partial}{\partial t} P = -\Delta P + 2\langle \nabla f, \nabla P\rangle + 2\Big|R_{ij} + \nabla_i \nabla_j f - \frac{1}{\tau} g_{ij}\Big|^2 + \frac{2}{\tau} P + \frac{2}{\tau}|\nabla f|^2 + \frac{4n}{\tau^2} + \frac{2}{\tau} R. \tag{0.3.19}$$

for all $t > 0$. Moreover $P \leq 0$ for all $t \in [0,T]$.

As an application of this, we derive the corresponding Harnack estimate under a mild assumption that the scalar curvature remains nonnegatively bounded. Here is the statement of the result (see Corollary 3.3.3 for the proof):

Corollary 0.3.7. *(Harnack Estimates). Let $u \in C^{2,1}(M \times [0,T))$ be a positive solution to the conjugate heat equation $\Gamma^* u = 0$ and $g(t), t \in [0,T)$ evolve by the Ricci flow on a closed manifold M with nonnegative scalar curvature R. Then for any points (x_1,t_1) and (x_2,t_2) in $M \times (0,T)$ such that $0 < t_1 \leq t_2 < T$, the following estimate holds*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \leq \Big(\frac{\tau_1}{\tau_2}\Big)^n \exp\Big[\int_0^1 \frac{|\gamma'(s)|^2}{2(\tau_1 - \tau_2)} ds + \frac{(\tau_1 - \tau_2)}{2} R\Big], \tag{0.3.20}$$

where $\tau_i = T - t_i, i = 1,2$ and $\gamma : [0,1]$ is a geodesic curve connecting points x_1 and x_2 in M.

We do hope the above results can be improved by removing the non negativity constraints on the curvatures. There is another important result in this section (Subsection 3.3.2), where we establish a localised form of the Harnack and gradient estimates obtained. The main idea is the application of the Maximum principle on some smooth cut-off function. It was the basic idea used by Li and Yau in [112], however our computation is more involved as the metric is also evolving. Let $d(x,y,t)$ be the geodesic distance between x and y with respect to the metric $g(t)$, define

$$\mathcal{Q}_{2\rho,T} := \{(x,t) \in M \times (0,T] : d(x,p,t) \leq 2\rho\}.$$

The following is the statement of the localised estimate (see Theorem 3.3.4):

Theorem 0.3.8. *Let $u \in C^{2,1}(M \times [0,T])$ be a positive solution to the conjugate heat equation $\Gamma^* u = (-\partial_t - \Delta + R)u = 0$ defined in geodesic cube $\mathcal{Q}_{2\rho,T} \subset M$ and the metric $g(t)$ evolves by the Ricci flow in the interval $[0,T]$ on a closed manifold M with bounded Ricci curvature, say $Rc \geq -Kg$, for some constant $K > 0$. Suppose further that $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$, where $\tau = T - t$, then for all points $\mathcal{Q}_{2\rho,T} \subset M$, we have the following estimate*

$$\frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} - R \leq \frac{4n}{1-4\delta n}\left\{\frac{1}{\tau} + C\left(\frac{1}{\rho^2} + \frac{\sqrt{K}}{\rho} + \frac{K}{\rho} + \frac{1}{T}\right)\right\}, \tag{0.3.21}$$

where C is an absolute constant depending only on the dimension of the manifold and δ is such that $\delta < \frac{1}{4n}$.

In Section 3.4, we introduce a dual entropy formula which surprisingly interpolates between Perelman's entropy [126] for conjugate heat equation on an evolving manifold and the Ni's modified entropy formula [122] for linear heat equation on static manifolds. Here, we will use $dV(x)$ instead of our usual notation $d\mu_{g(t)}$ of the volume form to indicate that volume is kept fixed throughout the time of evolution for the heat equation on a closed n-dimensional manifold $(M, g(t))$. The results described here are contained in our submitted paper [1].

Definition 0.3.9. *Let $u = u(x,t)$ be a positive solution to the heat equation*

$$\left(\frac{\partial}{\partial t} - \Delta\right)u(x,t) = 0. \tag{0.3.22}$$

Let $f : M \times (0,T] \to \mathbb{R}$ be smoothly defined as $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$ with $\int_M u(x,t)dV(x) = 1$. We introduce a generalized family of entropies by

$$\mathcal{W}_\epsilon(f,t) = \int_M \left[\frac{\epsilon^2 t}{4\pi}|\nabla f|^2 + f + \frac{n}{2}\ln\left(\frac{4\pi}{\epsilon^2}\right) - \frac{n\epsilon^2}{4\pi}\right]\frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}dV(x), \tag{0.3.23}$$

where $0 < \epsilon^2 \leq 4\pi$.

From this entropy formula, we also recover the corresponding differential Harnack inequality and gradient estimate for the fundamental solution (see Theorem 3.4.4), which in fact, holds for all positive solutions to the heat equation.

Theorem 0.3.10. *Let M be a closed Riemannian manifold. Assume that $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$ is a positive solution to the heat equation $\Gamma u = (\partial_t - \Delta)u = 0$, then, we have the following monotonicity formula for $\mathcal{W}_\epsilon(f,t)$ defined in (0.3.23)*

$$\frac{d}{dt}\mathcal{W}_\epsilon(f,t) = -\int_M \left[\frac{\epsilon^2 t}{2\pi}\left(\left|f_{ij} - \frac{\sqrt{\pi}}{\epsilon t}g_{ij}\right|^2 + R_{ij}f_i f_j\right) + \left(1 - \frac{\epsilon^2}{4\pi}\right)|\nabla f|^2\right]\frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}dV(x) \tag{0.3.24}$$

with (f,t) satisfying

$$\int_M \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}dV(x) = 1 \tag{0.3.25}$$

and $0 < \epsilon^2 \leq 4\pi$.

Theorem 0.3.11. *Let M be a closed manifold with nonnegative Ricci curvature and $H(x,y,t) = H = (4\pi t)^{-\frac{n}{2}}e^{-f}$ be the heat kernel, where H tends to a δ-function as $t \to 0$ and satisfies $\int_M HdV(x) = 1$. Then for all $t > 0$, we have*

$$P_\epsilon = \frac{\epsilon^2 t}{4\pi}\left(2\Delta f - |\nabla f|^2\right) + f + \frac{n}{2}\ln\left(\frac{4\pi}{\epsilon^2}\right) - \frac{n\epsilon^2}{4\pi} \leq 0. \tag{0.3.26}$$

Here, we have a family of entropy formulae for the conjugate heat equation on manifold evolving by the Ricci flow forward in time.

$$\mathcal{W}_\epsilon(g,f,\tau) = \int_M \left[\frac{\epsilon^2\tau}{4\pi}(R + |\nabla f|^2) + f - \frac{n\epsilon^2}{4\pi} + \frac{n}{2}\ln\left(\frac{4\pi}{\epsilon^2}\right)\right]\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}d\mu, \tag{0.3.27}$$

where $\tau = T - t > 0$ and $R = R(x,t)$ is the scalar curvature. Let $u = u(x,t)$ be a positive solution to the conjugate heat equation on a complete compact manifold with metric $g = g(x,t)$ evolving by the Ricci flow. Let

$$u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} \quad \text{satisfies} \quad \int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}d\mu = 1.$$

Then

$$(-\partial_t - \Delta + R)u = 0$$

and the following (see Theorem 3.4.13):

Theorem 0.3.12. *Let $(M, g(t)), t \in [0,T)$ be a solution of the Ricci flow $\partial_t g_{ij} = -2R_{ij}(g)$. Let $u : M \times [0,T) \to (0,\infty)$ solves the conjugate heat equation $(-\partial_t - \Delta + R)u = 0$. The entropy functional $\mathcal{W}_\epsilon(g,f,\tau)$ is nondecreasing by the formula*

$$\frac{d}{dt}\mathcal{W}_\epsilon(g,f,\tau) \geq \frac{\epsilon^2\tau}{2\pi}\int_M \left|R_{ij} + f_{ij} - \frac{1}{2\tau}g_{ij}\right|^2 ud\mu \geq 0 \tag{0.3.28}$$

for $0 < \epsilon^2 \leq 4\pi$.

As it is well known that entropy functional are intimately related to functional inequalities, we will apply the monotonicity proved in this section to derive a family of logarithmic Sobolev inequalities in the next chapter.

Chapter 4

The results of this chapter will appear in [2]. In the first part we prove an upper estimate on the conjugate heat kernel of the manifold evolving by the Ricci flow, it turns out that the estimate depends on the best constants in Sobolev inequality for the Ricci flow due to Q. Zhang in [157] and the bound on the scalar curvature. Here is the statement of the result (see Theorem 4.2.1):

Theorem 0.3.13. *Let $(M, g(x,t)), t \in [0,T]$ be a solution to the Ricci flow with $n \geq 3$ and $F(x,t;y,s)$ be the fundamental solution to the conjugate heat equation (conjugate heat kernel under Ricci flow). Then for a constant C_n depending on n only, the following estimate holds*

$$F(x,t;y,s) \leq \frac{C_n}{\left(\int_s^{\frac{t+s}{2}} \frac{e^{\frac{2}{n}P(\tau)}}{\alpha(\tau)A(\tau)}d\tau \cdot \int_{\frac{t+s}{2}}^{s} \frac{e^{-\frac{2}{n}P(\tau)}}{A(\tau)}d\tau\right)^{\frac{n}{4}}} \tag{0.3.29}$$

for $0 \le s < t \le T$, *where* $\alpha(\tau) = \frac{\rho^{-1} - \frac{2}{n}\tau}{\rho^{-1}}$, $R(g_0) \ge \rho$ *being the infimum of the scalar curvature taken at the initial time,* $P(\tau) = \int_s^t (B(\tau)A^{-1}(\tau) - \frac{1}{2}\phi(\tau))d\tau$, *with* $A(t)$ *and* $B(t)$ *being positive constants in the Zhang-Ricci-Sobolev inequality and* $\phi(t)$ *is the lower bound for the scalar curvature.*

In a special case where the scalar curvature is nonnegative at the starting time of the flow, one obtains a bound similar to the one in the fixed metric case.

Corollary 0.3.14. *Let the assumptions of the above theorem hold. Suppose further that the scalar curvature is nonnegative at time* $t = 0$ *(i.e.,* $R(x, 0) \ge 0$*). Then for a constant* $\tilde{C}_n$ *depending on* n *and the best constant in Euclidean Sobolev embedding, the following estimates hold*

$$F(x, t; y, s) \le \frac{\tilde{C}_n}{(t - s)^{\frac{n}{2}}} \tag{0.3.30}$$

for $0 \le s < t \le T$.

The exact value of $\tilde{C}_n$ is computed in the proof. Its value in the case $R(x, 0) = 0$ is different from that of the case $R(x, 0) > 0$.

In the second part of this chapter, upper estimates arising from the monotonicity of the $\mathcal{W}_\epsilon(f, t)$ entropy formula are obtained. The main ingredients used here are logarithmic Sobolev inequalities and ultracontractivity property of the conjugate heat semigroup. It is well known that Gross logarithmic Sobolev inequality [83] is equivalent to Nelson's hypercontractive inequality [121], both of which may imply ultracontractivity of the heat semigroup. (See [71, 72, 107, 152]). Our results establish this under the Ricci flow. Precisely, let there exists finite positive constants A_0 and B_0 depending only on n, g_0, lower bound for the Ricci curvature and injectivity radius of M. For any $u \in W^{1,2}(M, g_0)$, such that

$$\|u\|_{\frac{2n}{n-2}} \le A_0 \|\nabla u\|_2 + B_0 \|u\|_2, \tag{0.3.31}$$

where $m \ge 3$ and $\|\cdot\|_q = (\int_M |\cdot|^q d\mu_g)^{\frac{1}{q}}$, $1 \le p < \infty$. We can then write (0.3.31) as

$$\Big(\int_M u^{\frac{2n}{n-2}} d\mu_{g_0}\Big)^{\frac{n-2}{n}} \le A \int_M \Big(4|\nabla u|^2 + Ru^2\Big) d\mu_{g_0} + B \int_M u^2 d\mu_{g_0}, \tag{0.3.32}$$

where

$$A = \frac{1}{4}A_0, \ and \ B = \frac{1}{4}A_0 \sup R^-(\cdot, 0) + B_0$$

since $R(x, 0) + \sup R^-(\cdot, 0) = R^+(x, 0) - R^-(x, 0)$. R denotes the scalar curvature. We have the following (see Theorems 4.3.2 and 4.4.1):

Theorem 0.3.15. *Let* M *be a compact Riemannian manifold of dimension* $n \ge 3$. *Let the solution to the Ricci flow exist for all time* $t \in [0, T)$ *with bounded scalar curvature for all* $t \ge 0$. *Assume the Sobolev embedding (0.3.31) holds, then for finite positive constants* A *and* B *depending on* n, A_0, B_0, *lower bound for* R_{g_0} *and* T, *there hold for each* $t \in [0, T)$ *and* $u \in W^{1,2}(M)$

$$\Big(\int_M u^{\frac{2n}{n-2}} d\mu_g\Big)^{\frac{n-2}{2}} \le A \int_M \Big(|\nabla u|^2 + \frac{1}{4}Ru^2\Big) d\mu_g + B \int_A u^2 d\mu_g, \tag{0.3.33}$$

$$\int_M u^2 \ln u^2 d\mu_{g(t)} \leq \sigma^2 \int_M (4|\nabla u|^2 + Rv^2) d\mu_{g(t)} - \frac{n}{2} \ln \sigma^2 + (t + \sigma^2)\beta_1 + \frac{n}{2} \ln \frac{nA}{2e}, \tag{0.3.34}$$

if $\lambda_0 = \inf_{\|u\|_2=1} \int_M (4|\nabla u|^2 + Ru^2) d\mu_{g_0}$, *that is,* λ_0 *is the first eigenvalue of the operator* $-\Delta + \frac{1}{4}R$*. Here* $\sigma > 0, \quad \beta_1 = 4A_0^{-1}B_0 + \sup R^-(\cdot, 0)$.

Finally, for some constant C *depending on* n, t, T, A_0, B_0 *and* $\sup R(\cdot, 0)$*, there holds the following estimate*

$$F(x, T; y) \leq CT^{-\frac{n}{2}} \tag{0.3.35}$$

for the positive solution to the conjugate heat equation F *associated to the Ricci flow.*

The three results in the above theorem are essentially equivalent, their proofs occupy Sections 4.3 and 4.4.

Chapter 5

Chapter 5 is on the elegant application of heat flow monotonicity to the proof of a family of functional-geometric inequalities, namely, Brascamp-Lieb inequalities. This chapter may be considered independent from the remaining thesis as the subject is interesting in its own right and we treat it as such by developing the theory from Euclidean-analytic point of view.

The set up is as follows: For natural numbers $m, n, n_j \in \mathbb{N}, \ n \geq n_j, \ 1 \leq j \leq m$, define positive real numbers $p_j > 0$, such that

$$\sum_{j=1}^{m} p_j n_j = n. \tag{0.3.36}$$

Let $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be surjective linear maps from $\mathbb{R}^n$ onto $\mathbb{R}^{n_j}$ such that their common kernel $\cap_{j=1}^m B_j = \{0\}$. This condition forces $\sum_{j=1}^m p_j B_j^* A_j B_j$ to be isomorphism, where B_j^* is the adjoint of B_j and A_j is a positive-definite $n_j \times n_j$ matrix.

Brascamp-Lieb constant is defined as follows

$$D(p_j) = \frac{det\Big(\sum_{j=1}^m p_j B_j^* A_j B_j\Big)}{\prod_{j=1}^m \Big(\det A_j\Big)^{p_j}}. \tag{0.3.37}$$

In this case, each f_j is a centred gaussian function, i.e,

$$f_j = \exp(-\pi \langle A_j x, x \rangle). \tag{0.3.38}$$

Let C_1 and C_2 respectively be the smallest and largest constant such that for all $f_j, 1 \leq j \leq m$

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(B_j(x))\, dx \leq C_1 \prod_{j=1}^m \Big(\int_{\mathbb{R}^{n_j}} f_j\Big)^{p_j} \tag{0.3.39}$$

and

$$\int_{\mathbb{R}^n}^* \sup_{x = \sum_{j=1}^m p_j B_j^* x_j,\ x_j \in \mathbb{R}^{n_j}} \prod_{j=1}^m f_j^{p_j}(x_j)\, dx \geq C_2 \prod_{j=1}^m \Big(\int_{\mathbb{R}^{n_j}} f_j\Big)^{p_j}. \tag{0.3.40}$$

Here, the symbol $\int^*$ means outer integral. By a Theorem of Lieb [106], see also [13], both inequalities (0.3.39) (**Bracamp-Lieb inequalities**) and (0.3.40) (**Reverse Brascamp-Lieb inequalities**) are well known to be saturated by centred Gaussian functions. It was conjectured by Brascamp and Lieb in [32], that Gaussian functions give the best constants and proved by Lieb in [106] and simultaneously by Beckner in [18]. F. Barthe in [13] reproved Lieb's result using the method of optimal transport and simultaneously derived the dual result for the case of inequality (5.2.9) as conjectured by Lieb. In fact, the constants C_1 and C_2 can be computed explicitly as

$$C_1 = \sup_{f_j} \frac{\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(B_j(x))\, dx}{\prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j(x_j)\, dx_j \right)^{p_j}} = \left(\frac{\prod_{j=1}^m (\det A_j)^{p_j}}{\det \left(\sum_{j=1}^m p_j B_j^* A_j B_j \right)} \right)^{\frac{1}{2}} = D^{-\frac{1}{2}}$$

and

$$C_2 = \inf_{f_j} \frac{\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(B_j(x))\, dx}{\prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j(x_j)\, dx_j \right)^{p_j}} = \frac{\det \left(\sum_{j=1}^m p_j B_j^* A B_j \right)^{\frac{1}{2}}}{\prod_{j=1}^m (\det A_j)^{\frac{p_j}{2}}} = D^{\frac{1}{2}},$$

where both the infimum and the supremum are taken over the class of all Gaussian functions with minimum near the origin.

We discuss details of this using Barthe's arguments and show that many geometric inequalities actually have their generalisation in Brascamp-Lieb inequalities (see Lemma 5.2.2). In the above description of Brascamp-Lieb Inequality we have used linear transformation and Lebesgue measure, we like to submit that the strength of this family of inequalities is the flexibility to live in a more generalized setting. Hence, we collect some sort of variants due to various authors in section 5.2.

However, our major aim in this chapter is to prove the inequalities in (0.3.39) and (0.3.40) via heat flow monotonicity. This type of approach as noticed by Carlen-Lieb-Loss [52], Barthe-Cordero-Erasquin [16] and Bennett-Carbery-Christ-Tao [24] tends to generate sharp constants and identify extremisers. We prove this generalisation in Theorem 5.3.1 for linear setting and Theorem 5.4.6 for multilinear setting (the argument adopted here follows in spirit those of Bennett-Carbery-Christ-Tao [24, 25]).

At a first glance one may wonder if there is any connection at all between the subject of this chapter and those of the first part of this thesis. The last section of this chapter highlights where the connections lie. Firstly, we note that Brascamp-Lieb inequalities generalize Young's convolution inequality which is equivalent to Nelson's hypercontractive estimates and logarithmic Sobolev inequalities both of which are related to the entropy in 'Euclidean-Gaussian' setting. In fact, another connection is in the fact that the applications of such inequalities are becoming more obvious in PDEs and Geometric analysis, hence the need for their generalization to 'full' diffeomorphic setting. We do acknowledge that putting Brascamp-Lieb inequalities in a diffeomeorphic setting is not straightforward, and is the topic of current research.

Chapter 1

Basics of Ricci Flow

1.1 The Ricci Flow - An Overview and Brief History

Formally, the Ricci flow is a system of evolution for a one-parameter family of Riemannian metrics $g(x,t)$ on a smooth manifold M by the following nonlinear system of second order weakly parabolic partial differential equations

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \tag{1.1.1}$$

which becomes strictly parabolic modulo out the group of diffeomorphism of the underlying manifold. Here $R_{ij}(x,t)$ is the $(0,2)$-Ricci curvature tensor. The factor 2 in the system (1.1.1) is not very important but the negative sign ensures the flow remains forward in time, in fact, it shows that positive curvature is contracted while the negative curvature is dilated (we shall consider examples to illustrate this later). Introduced in a seminal paper by R. Hamilton in 1982 [87] as an initial value problem together with initial data $g(x,0) = g_0$, the Ricci flow was to attack geometrization conjecture (classification of three manifolds by William Thurston). In the same seminar paper Hamilton was able to show its short time existence though not by standard parabolic theory, he employed a "powerful analytic tool" called Nash-Moser implicit function theorem. Not quite later that Dennis DeTurck [73] found a simplified way of doing this, his approach popularly called DeTurck's trick follows from modification of the flow by a time-dependent change of variables, which breaks the diffeomorphism invariance of the flow equation, thus, making it parabolic.

Meanwhile, a classical problem in differential geometry is to find canonical metrics of Riemannian manifolds. By canonical metrics we mean metrics of constant curvature whose existence often yields useful geometric and topological implications. A well known example is the classification of Gauss curvature metrics of simply connected Riemannian surfaces, the uniformization theorem. The geometric flows play fundamental roles in achieving this objective, see R. Schoen [131] for Yamabe flow, Mullin [119] for curvature shortening flow, G. Huisken [98], mean curvature flow and Eells and Sampson [76] for harmonic maps heat flow. It can be said that Hamilton was motivated by the paper of Eells and Sampson [76] on harmonic maps heat flow to prove existence of harmonic maps into targets of non-positive sectional curvature. Considering his effort to extend the result of Eells and Sampson to the case of manifold with boundary [86]. His propositional idea was to study 3-manifolds with positive Ricci curvature and he obtained the following;

Theorem 1.1.1. *([87]). Let M be a closed 3-Riemannian manifold whose initial metric admits a strictly positive curvature, then, M admits a metric of constant sectional curvature. Moreover, M is diffeomorphic to the 3-sphere or its quotient by a finite group of isometries.*

The proof of the above consists in showing that volumetric version of (1.1.1) (Volume preserving Ricci flow (1.1.2)) is obtained by re-parametrizing in time scale and homothety for all time $t \in [0, \infty)$ and converges to a metric of constant sectional curvature. Invariably, the Poincaré conjecture follows immediately one is able to show that an homotopic spherical space form admits a positive Ricci curvature metrics. More generally, Elliptization conjecture would follow from showing that any closed 3-manifold with finite fundamental group admits a metric with positive Ricci curvature

Theorem 1.1.2. *(William Thurston's Elliptization Conjecture). A closed 3-manifold with finite fundamental group has a Riemannian metric of constant positive sectional curvature. Then, it is homeomorphic to the 3-sphere (via the covering map).*

Thus, proving the elliptization conjecture would prove the Poincaré conjecture as a corollary.

Theorem 1.1.3. *(The Poincaré Conjecture). Every simply connected closed 3-manifold is diffeomorphic to the 3-sphere $\mathbb{S}^3$.*

It was obvious that in applications we might need the volume of the manifold to be preserved throughout the evolution. To achieve this, Hamilton also introduced the normalized counterpart which differs from (1.1.1) by a cosmological constant

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \frac{2}{n} r g_{ij}(x,t). \tag{1.1.2}$$

Here, r denotes the average value of the scalar curvature of the metric $g(x,t)$. The evolution equations (1.1.1) and (1.1.2) are essentially equivalent, any solution to (1.1.1) can be transformed to that of (1.1.2) by a rescaling procedure.

Shortly after Hamilton's first result, Shing-Tung Yau suggested to him that the Ricci flow could be an appropriate tool to attack the structure theorem for three-manifolds in general. Hamilton then proved many important results ranging from the *uniformization of surfaces, maximum principle for tensors, Harnack estimates for curvatures, monotonicity of entropy formula*, all with Ricci flow. He actually laid the foundation for the programs towards a complete solution of the Poincaré conjecture and Thurston's geometrization conjecture via the Ricci flow. (Cf. [39] Collected Papers on Ricci Flow, Series in Geometry and Topology).

Basically, Hamilton's program states that starting with any given compact three-manifold endowed with an arbitrary initial metric without any curvature assumption, the Ricci flow may develop singularities in finite time. In this case, the unbounded regions (a small neighbourhoods of the points of singularities), need to be dealt with by performing topological surgeries on them and then continue to run the Ricci flow, this process should be repeated each time singularities are developed. If one can find only a finite number of surgeries during finite time interval and if the true behaviour of solution of the Ricci flow with surgery is well understood, then, one would see clearly the topological structure of the initial manifold. The major obstacles was the verification of what is now called Hamilton's Little Loop Lemma [92], which is a certain local injectivity radius estimate and the verification of discreteness of surgery times. In the late 2002 and 2003, G. Perelman [126, 128] came out with ingenuity that allowed him to remove the obstacles that remained in the program of Hamilton. His paper [127] gives details of how topological surgeries are performed on the 3-dimensional Ricci flow (see also [58, 94] for Ricci flow with surgeries on 4-manifolds). Perelman showed that all singularities are modelled by self-similar solutions (Ricci soliton). Perelman's breakthrough is unprecedented as it provides a complete proof of Poincaré conjecture which earned him a Fields medal, though he turned down the prize based on his opinion that Richard Hamilton deserved more credit.

1.2 Examples and Special Solutions of the Ricci Flow

1.2.1 Examples

The Ricci flow governs the evolution of a given metric which converges in some sense to an Einstein metric. A Riemannian metric is said to be Einstein if its Ricci curvature is a scalar multiple of the metric at each point, (see Section 0.2 and [27] for details on Einstein metrics).

Lemma 1.2.1. *Suppose (M, g) is an Einstein manifold with initial metric g_0, then the solution of the Ricci flow is governed by*

$$g(t) = (1 - 2\lambda_0 t)g_0.$$

Proof. Let $R_{ij}(g_0) = \lambda_0 g_0$ for some $\lambda_0 \in \mathbb{R}$. Let $g = \lambda(t)g_0$. Since Ricci tensor is scale-invariant, we have

$$R_{ij}(g) = R_{ij}(\lambda g_0) = R_{ij}(g_0) = \lambda_0 g_0.$$

Indeed

$$\frac{\partial}{\partial t} g_{ij} = \lambda' g_0 = -2R_{ij}(g) = -2\lambda_0 g_0.$$

The problem is reduced to solving the ODE

$$\lambda'(t) = -2\lambda_0, \;\; \lambda(0) = 1.$$

□

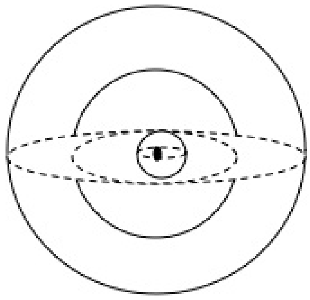

Figure 1.1: Shrinking Sphere

Examples 1.2.2. *(***Shrinking, Steady and Expanding Solutions***)*
The cases $\lambda > 0, \lambda = 0$ and $\lambda < 0$ corresponds to shrinking, steady and expanding solutions respectively.

Thus, running the Ricci flow with standard round metric on $\mathbb{S}^n$, Euclidean metric on $\mathbb{R}^n$ or Hyperbolic metric on $\mathbb{H}^n$ illustrates each case. In particular, for the round unit sphere $(\mathbb{S}^n, g_{can})$, we have

$$Rc(g_0) = (n-1)g_{can}$$

and the evolution is

$$g(t) = (1 - 2(n-1)t)g_{can}.$$

With this the sphere collapses to a point at a finite time $T = \frac{1}{2(n-1)}$ (called the singular point), that is, the Ricci flow on $\mathbb{S}^n$ has a finite time singularity where the diameter of the manifold goes to zero and the curvature explodes to $+\infty$. The Ricci flow is stationary on standard Euclidean metric $(\mathbb{R}^n, g_{Euc})$ and any other Ricci-flat manifold (Riemannian manifold with vanishing Ricci curvature), while hyperbolic metric (manifold with constant sectional curvature -1) expands homothetically for all time.

Example 1.2.3. *(***The Ricci flow on product manifolds***)*
Consider the geometry of product of spherical and euclidean metrics such as $\mathbb{S}^{n-k} \times \mathbb{R}^k$. The Ricci flow acts on the factor metrics separately, thus, as $\mathbb{S}^{n-k}$ is shrinking, $\mathbb{R}^k$ remains steady.

A perfect illustration for this is the 3-dimensional shrinking cylinder $\mathbb{S}^2 \times \mathbb{R}$, where $\mathbb{S}^2$ factor shrinks homothetically while $\mathbb{R}$ factor remains unchanged. As a consequence, the solution becomes singular in finite time and the manifold converges in the pointed Gromov-Hausdorff sense to $\mathbb{R}$.

The above examples have shown that an Einstein metric is a special solution of the Ricci flow. In particular, if it is of positive scalar curvature, it will shrink homothetically at finite time while that of negative scalar curvature expands homothetically for all times and the Ricci-flat Einstein metric is a stationary solution.

Figure 1.2: Shrinking Round Cylinder

1.2.2 Ricci Soliton

There is a larger class of self-similar solutions than the uniformly shrinking or expanding solutions given in the above examples. These special solutions are called the Ricci solitons. In this case, we modify the flow by a one-parameter group of diffeomorphisms φ_t and define a $time-$dependent vector field X from it.

Definition 1.2.4. *Let $\{\varphi_t\}, t \in I$ be a one-parameter family of diffeomorphisms, $\varphi_t : M \to M,$ and $\{g(t)\}_{t\in I}$ be a one-parameter family of Riemannian metrics defined on M. Given a smooth scalar function $\beta(t) > 0$, we call a solution $g(t)$ of (1.1.1) a Ricci sloliton, if it is a pull back of g_0, i.e.,*

$$g(t) = \beta(t)\varphi_t^* g_0. \tag{1.2.1}$$

This simply means that in a Ricci soliton all the Riemannian manifolds $(M^n, g(t))$ are isometric up to a scale factor that is allowed to vary with $time$. Therefore, the Ricci flow equation is equivalent to

$$Rc(g_0) + \frac{1}{2}\mathcal{L}_X g_0 = \lambda g_0 \tag{1.2.2}$$

for any $\lambda(t) = -\frac{1}{2}\beta'(t)$, where X is a vector field on M and $\mathcal{L}_X g_0$ is the Lie derivative of the evolving metric. If the vector field X is the gradient of a function, say f, then the solution is called a gradient Ricci soliton and (1.2.2) becomes

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}, \tag{1.2.3}$$

where λ is the homothety constant. The case $\beta'(t) < 0$, $\beta'(t) = 0$ or $\beta'(t) > 0$ corresponds to shrinking, steady or expanding gradient soliton. Clearly, a Ricci soliton is an Einstein metric if X vanishes identically.

Lemma 1.2.5. *Suppose the flow $(M, g(t)), 0 \le t < T$, where $g(t) = \beta(t)\varphi_t^* g_0$, is a solution of the Ricci flow, then, there exists a vector field X on M such that (M, g_0, X) satisfies (1.2.2). Conversely, given any solution (M, g_0, X) of (1.2.2), then, there exist a one-parameter family of diffeomorphism φ_t of M and scalar function $\beta(t)$ such that $g(t)$ of (1.2.1) solves the Ricci flow.*

Proof. Suppose $g(t) = \beta(t)\varphi_t^* g_0$ is a solution of the Ricci flow (1.1.1) and we assume that $\varphi_0 = Id_M$, then, it is seen at once that $\beta(0) = 1$, so that we have

$$\begin{aligned} -2Rc(g_0) = \frac{\partial g(t)}{\partial t}\Big|_{t=0} &= \beta'(0)\varphi_0 g_o + \beta(0)\varphi_0 \mathcal{L}_{Y(0)} g_0 \\ &= \beta'(0) g_0 + \mathcal{L}_{Y(0)} g_0, \end{aligned} \tag{1.2.4}$$

where $Y(t) = \frac{1}{\beta(t)} X$ is the family of vector fields generating the diffeomorphism φ_t. Comparing (1.2.2) and (1.2.4), we have $\lambda = -\frac{1}{2}\beta'(0)$ and $X = Y(0)$, hence g_0 satisfies (1.2.2).

Conversely, suppose that g_0 satisfies (1.2.2). Define

$$\beta(t) = 1 - 2\lambda t$$

and define a one-parameter family of vector fields $Y(t) = \frac{1}{\beta(t)} X(x)$. Let φ_t be the diffeomorphism generated by the family $Y(t)$, where $\varphi_0 = Id_M$ and define a smooth one-parameter family of a metrics on M by $g(t) = \beta(t)\varphi_t^* g_0$. Then,

$$\begin{aligned} \frac{\partial g(t)}{\partial t} &= \beta'(t)\varphi_t^* g_0 + \beta(t)\varphi_t^* g_0 \mathcal{L}_{Y(t)} g_0 \\ &= \varphi_t^*(\beta'(t) + \beta(t)\mathcal{L}_{Y(t)}) g_0 \\ &= \varphi_t^*(-2\lambda + \mathcal{L}_X) g_0 \\ &= \varphi_t^*(-2Ric(g_0)) = -2Rc(\varphi_t^* g_0) \end{aligned}$$

since $Ric(\alpha g) = Ric(g)$ for any $\alpha > 0$, it follows that

$$\frac{\partial g(t)}{\partial t} = -2Rc(g(t)).$$

Therefore $g(t)$ is a solution of the Ricci flow (1.1.1). □

Examples 1.2.6. *Rotationally symmetric Cigar and Bryant Solitons are stationary solutions to the Ricci flow.*

Consider a complete Riemann surface (Σ, g_Σ) with $g_\Sigma = u(x,t) g_{Euc}$, where g_{Euc} is the standard Euclidean metric on $\mathbb{R}^2$, $u(x,t) = \frac{1}{e^{4\epsilon t} + x^2 + y^2}$, and $\epsilon > 0$. Given the initial metric $g_\Sigma(0) = \frac{1}{1+x^2+y^2} g_{Euc}$, then (Σ, g_Σ) gives a stationary Ricci flow whose curvature decays exponentially. (Cigar soliton was introduced by R. Hamilton and called Witten's Black Hole in Physics). The analogy of Cigar soliton in higher dimension is referred to Bryant soliton. The Cigar soliton has positive curvature and is asymptotic to a cylinder of finite circumference at infinity. The Bryant soliton has positive sectional curvature, linear curvature decay and volume growth of geodesic balls of radius ρ on the order of $\rho^{(n+1)/2}$.

1.3 Evolution of Geometric Quantities

More interestingly, all the geometric quantities associated with the underlying manifold evolve as the Riemannian metric evolves along the Ricci flow, most importantly, the curvature tensors evolve by some nonlinear heat-type equations, this also serves as motivation to considering the behaviours of some other important geometric quantities such as eigenvalues and heat kernel of the manifold under the Ricci flow. The evolutions are summarised below, detailed proofs can be found in [4, 41, 69, 68, 87].

Lemma 1.3.1. *If a one-parameter family of metric $g(t)$ solves the Ricci flow (1.1.1), then, the inverse metric, the Christoffel's symbol, the volume element and the Laplacian evolve as follows*

$$\frac{\partial}{\partial t} g^{ij} = 2g^{ik}g^{jl}R_{kl} \tag{1.3.1}$$

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = -g^{kl}\Big(\partial_i R_{jl} + \partial_j R_{il} - \partial_l R_{ij}\Big) \tag{1.3.2}$$

$$\frac{\partial}{\partial t} d\mu = -R d\mu \tag{1.3.3}$$

$$\frac{\partial}{\partial t}\Delta_{g(t)} = 2R^{ij}\cdot\nabla_i\nabla_j. \tag{1.3.4}$$

Notice that Levi-Civita connection is not a tensor but it is determined by Christoffel's symbols whose time derivative is also a tensor.

Proof. Assuming that

$$\frac{\partial}{\partial t} g_{ij} = h_{ij},$$

where h_{ij} is a symmetric tensor. Recall also that

$$g^{ij}g_{jl} = \delta^i_l,$$

then

$$\Big(\frac{\partial}{\partial t} g^{ij}g_{jl}\Big) = 0 = \Big(\frac{\partial}{\partial t} g^{ij}\Big)g_{jl} + g^{ij}\Big(\frac{\partial}{\partial t} g_{jl}\Big).$$

Therefore

$$\Big(\frac{\partial}{\partial t} g^{ij}\Big)g_{jl} = -g^{ij}(h_{jl})$$

$$\frac{\partial}{\partial t}\Big(g^{ij}\Big) = -g^{ik}g^{jl}h_{kl}.$$

Taking $h_{ij} = -2R_{ij}$, part 1 of the lemma is proved.

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}\frac{\partial}{\partial t} g^{kl}\Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\Big) + \frac{1}{2}g^{kl}\Big(\partial_i\Big(\frac{\partial}{\partial t}g_{jl}\Big) + \partial_j\Big(\frac{\partial}{\partial t}g_{il}\Big) - \partial_l\Big(\frac{\partial}{\partial t}g_{ij}\Big)\Big)$$

working in a normal coordinate about a point p, we have $\partial_i g_{jk}(p) = 0$ for all i, j, k and from the fact that $\partial_i A_{jk} = \nabla_i A_{jk}$, we get

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{kl}\Big(\nabla_i\frac{\partial}{\partial t}g_{jl} + \nabla_j\frac{\partial}{\partial t}g_{il} - \nabla_l\frac{\partial}{\partial t}g_{ij}\Big).$$

Therefore

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{kl}\Big(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\Big). \tag{1.3.5}$$

Since both sides of (1.3.5) are components of tensors, it then holds as a tensor equation for any coordinate systems. Part 2 is proved.

In local coordinates, the volume form is written as $d\mu = \sqrt{detg}dx^1 \wedge ... \wedge dx^n$, then,

$$\frac{\partial}{\partial t}d\mu = \frac{\partial}{\partial t}\Big(\sqrt{detg}dx^i \wedge ... \wedge dx^n\Big).$$

By chain rule of differentiation

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\sqrt{detg}\Big) &= \frac{1}{2}\frac{1}{\sqrt{detg}}\frac{\partial}{\partial t}detg = \frac{1}{2}\frac{1}{\sqrt{detg}}\frac{\partial detg}{\partial g_{ij}}\frac{\partial g_{ij}}{\partial t}\\ &= \frac{1}{2}\sqrt{detg}g^{ij}h_{ij} = \frac{1}{2}trh\sqrt{detg},\end{aligned}$$

therefore

$$\frac{\partial}{\partial t}d\mu = \frac{1}{2}trh\, d\mu,$$

hence part 3 of the Lemma is proved.

Define

$$\Delta_g f = [g^{ij}(\partial_i\partial_j - \Gamma^k_{ij}\partial_k)f],$$

then,

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\Delta_g f\Big) &= \Big[g^{ij}(\partial_i\partial_j - \Gamma^k_{ij}\partial_k)f\Big]\\ &= \Big(\frac{\partial}{\partial t}g^{ij}\Big)\nabla_i\nabla_j f - g^{ij}\Big(\frac{\partial}{\partial t}\Gamma^k_{ij}\Big)\nabla_k f + g^{ij}(\partial_i\partial_j - \Gamma^k_{ij}\partial_k)\frac{\partial}{\partial t}f\\ &= \Big(\frac{\partial}{\partial t}g^{ij}\Big)\nabla_i\nabla_j f + \Delta\frac{\partial}{\partial t}f.\end{aligned}$$

Therefore

$$\frac{\partial}{\partial t}\Delta_g = \Big(\frac{\partial}{\partial t}g^{ij}\Big)\nabla_i\nabla_j, \tag{1.3.6}$$

where

$$\begin{aligned}g^{ij}\Big(\frac{\partial}{\partial t}\Gamma^k_{ij}\Big) &= g^{ij}\Big[\frac{1}{2}g^{kl}\Big(\nabla_i\frac{\partial}{\partial t}g_{jl} + \nabla_j\frac{\partial}{\partial t}g_{il} - \nabla_l\frac{\partial}{\partial t}g_{ij}\Big)\Big]\\ &= g^{kl}\Big[g^{ij}\nabla_i\Big(\frac{\partial}{\partial t}g_{jl}\Big) - \frac{1}{2}g^{ij}\nabla_l\Big(\frac{\partial}{\partial t}g_{ij}\Big)\Big]\\ &= 0\end{aligned}$$

by the contracted second Bianchi identity. Application of part 1 of the lemma to (1.3.6) proves part 4. □

Theorem 1.3.2. *(***Evolution of Curvature Tensors***). Let $g(t)$ be a solution to (1.1.1), the Riemannian curvature tensor (R_{ijkl}), Ricci curvature tensor (R_{ij}) and the scalar curvature tensor (R) respectively evolve as follows*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})\\ &\quad -(R^p_i R_{pjkl} + R^p_j R_{ipkl} + R^p_k R_{ijpl} + R^p_l R_{ijkp})\end{aligned} \tag{1.3.7}$$

$$\frac{\partial}{\partial t}R_{ij} = \Delta R_{ij} + 2g^{kp}g^{lq}R_{kijl}R_{pq} - 2g^{kl}R_{ik}R_{jl} \tag{1.3.8}$$

$$\frac{\partial}{\partial t}R = \Delta R + 2|R_{ij}|^2, \tag{1.3.9}$$

where $B_{ijkl} = -g^{pr}g^{qs}R_{piqj}R_{rksl} = -R^q_{pij}R^p_{qlk}$*, and* Δ *is the Laplace-Beltrami operator with respect to evolving metric* $g(t)$*.*

Proof. **Evolution of Riemann curvature tensor.** We write $(1,3)$-Riemann curvature tensor

$$R^p{}_{ijl} = \partial_i\Gamma^p_{jl} - \partial_j\Gamma^p_{il} + \Gamma^k_{ir}\Gamma^r_{jl} - \Gamma^p_{jr}\Gamma^r_{il}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}R^p{}_{ijl} &= \frac{\partial}{\partial t}\Big(\partial_i\Gamma^p_{jl} - \partial_j\Gamma^p_{il}\Big) + \Big(\frac{\partial}{\partial t}\Gamma^k_{ir}\Big)\Gamma^r_{jl} + \Gamma^k_{ir}\Big(\frac{\partial}{\partial t}\Gamma^r_{jl}\Big) \\ &\quad - \Big(\frac{\partial}{\partial t}\Gamma^p_{jr}\Big)\Gamma^r_{il} - \Gamma^p_{jr}\Big(\frac{\partial}{\partial t}\Gamma^r_{il}\Big) \\ &= \partial_i\Big(\frac{\partial}{\partial t}\Gamma^p_{jl}\Big) - \partial_j\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\Big).\end{aligned}$$

Now using the contraction $R_{ijkl} = g_{kp}R^p{}_{ijl}$, we have

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= g_{kp}\Big(\frac{\partial}{\partial t}R^p{}_{ijl}\Big) + \Big(\frac{\partial}{\partial t}g_{kp}\Big)R^p{}_{ijl} \\ &= g_{kp}\Big\{\nabla_i\Big[-g^{pq}(\nabla_j R_{lq} + \nabla_l R_{jq} - \nabla_q R_{jl})\Big] \\ &\quad - \nabla_j\Big[-g^{pq}(\nabla_i R_{lq} + \nabla_l R_{iq} - \nabla_q R_{il})\Big]\Big\} - 2R_{kp}R^p{}_{ijl} \\ &= \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} \\ &\quad + \nabla_j\nabla_i R_{lk} - \nabla_i\nabla_j R_{lk} - 2R_{kp}R^p{}_{ijl}.\end{aligned}$$

By interchanging covariant derivatives and using Bianchi identity property of Riemann curvature tensor, we have

$$\begin{aligned}\nabla_j\nabla_i R_{lk} - \nabla_i\nabla_j R_{lk} &= R_{ijkp}g^{pq}R_{qk} \\ &= -R_{ijlp}g^{pq}R_{qk} - R_{ijkp}g^{pq}R_{ql}\end{aligned}$$

and

$$R_{kp}R^p{}_{ijl} = R_{ijpl}g^{pq}R_{qk}$$

so that

$$\begin{aligned}\nabla_j\nabla_i R_{lk} - \nabla_i\nabla_j R_{lk} &= -2R_{ijpl}g^{pq}R_{qk} \\ &= -R_{ijlp}g^{pq}R_{qk} - R_{ijkp}g^{pq}R_{ql} - 2R_{ijpl}g^{pq}R_{qk} \\ &\quad - g^{pq}(R_{ijkp}R_{ql} + R_{ijpl}R_{qk}).\end{aligned}$$

Therefore

$$\frac{\partial}{\partial t}R_{ijkl} = \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} \\ - g^{pq}(R_{ijkp}R_{ql} + R_{ijpl}R_{qk}).$$

We then conclude the proof with the following claim: Let (M, g) be a smooth manifold, the Laplacian of the Riemannian tensor satisfies

$$\Delta R_{ijkl} \quad = \left\{ \begin{array}{r} \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} \\ -2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ +g^{pq}(R_{qjkl}R_{pi} + R_{iqkl}R_{pj}) \end{array} \right\}. \tag{1.3.10}$$

Consider the linearity of covariant derivative over the second Bianchi identity as follows

$$\nabla_p\nabla_q R_{ijkl} + \nabla_p\nabla_i R_{jqkl} + \nabla_p\nabla_j R_{qikl} = 0,$$

recall also that the connection Laplacian of any tensor field is the trace of the second covariant derivative with the metric g, therefore

$$\begin{aligned} \Delta R_{ijkl} = (tr_g\nabla^2 R)_{ijkl} = g^{pq}\nabla_p\nabla_q R_{ijkl} &= g^{pq}(-\nabla_p\nabla_i R_{jqkl} - \nabla_p\nabla_j R_{qikl}) \\ &= g^{pq}\nabla_p(-\nabla_i R_{jqkl} - \nabla_j R_{qikl}) \\ &= g^{pq}\nabla_p(\nabla_i R_{qjkl} - \nabla_j R_{qikl}). \end{aligned}$$

The first term on the RHS of the last equality is due to antisymmetric property. Now by commuting covariant derivative on this term we have

$$\nabla_p\nabla_i R_{qjkl} = \nabla_i\nabla_p R_{qjkl} + (R(\partial_i, \partial_p)R)(\partial_q, \partial_j, \partial_k, \partial_l), \tag{1.3.11}$$

using the second Bianchi identity and contracting with the metric on the first term on the RHS of (1.3.11), follow from this calculation $\nabla_i\nabla_p R_{qjkl} = \nabla_i\nabla_k R_{jqlp} - \nabla_i\nabla_l R_{jqkp}$, we arrive at

$$\begin{aligned} g^{pq}\nabla_i\nabla_p R_{qjkl} &= g^{pq}(\nabla_i\nabla_k R_{jqlp} - \nabla_i\nabla_l R_{jqkp}) \\ &= \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk}. \end{aligned}$$

The second term on the RHS of (1.3.11) can be written as

$$\begin{aligned} (R(\partial_i, \partial_p)R)(\partial_q, \partial_j, \partial_k, \partial_l) &= R^r_{ipq}R_{njkl} + R^r_{ipq}R_{qnkl} + R^r_{ipq}R_{qjnl} + R^r_{ipq}R_{qjkn} \\ &= g^{rs}(R_{piqs}R_{rjkl} + R_{pijs}R_{qrkl} + R_{piks}R_{qjrl} + R_{pils}R_{qjkr}) \end{aligned}$$

$$\begin{aligned} &g^{pq}(R(\partial_i, \partial_p)R)(\partial_q, \partial_j, \partial_k, \partial_l) \\ &\qquad = g^{pq}g^{rs}(R_{piqs}R_{rjkl} + R_{pijs}R_{qrkl} + R_{piks}R_{qjrl} + R_{pils}R_{qjkr}). \end{aligned}$$

Contracting each term on the right hand side of the last expression as follows

$$g^{pq}g^{rs}R_{piqs}R_{rjkl} = g^{rs}R_{is}R_{rjkl} = g^{pq}R_{iq}R_{pjkl} \tag{1.3.12}$$

$$\begin{aligned} g^{pq}g^{rs}R_{pijs}R_{qrkl} &= g^{pq}g^{rs}R_{pijs}(-R_{rkql} - R_{kqrl}) \\ &= g^{pq}g^{rs}(R_{pijs}R_{qlrk} - R_{pijs}R_{kqrl}) \\ &= R^{q}_{pij}R^{p}_{qlk} - R^{q}_{pij}R^{p}_{kql} = B_{ijlk} - B_{ijkl} \end{aligned} \tag{1.3.13}$$

$$g^{pq}g^{rs}R_{piks}R_{qjrl} = g^{pq}g^{rs}(-R_{ipsk}R_{qjrl}) = -B_{ikjl} \tag{1.3.14}$$

$$g^{pq}g^{rs}R_{pils}R_{qjkrl} = R^{q}_{ipl}R^{p}_{qjk}. \tag{1.3.15}$$

Combining the identities (1.3.11) -(1.3.15), we obtain

$$g^{pq}\nabla_i\nabla_p R_{qjkl} = \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_k R_{jk} \tag{1.3.16}$$

$$- (B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) + g^{pq}R_{iq}R_{pjkl}. \tag{1.3.17}$$

Therefore

$$\begin{aligned} \Delta R_{ijkl} &= g^{pq}\nabla_p\nabla_i R_{qjkl} - g^{pq}\nabla_p\nabla_j R_{qikl} \\ &= \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - (B_{jikl} - B_{ijlk} - B_{iljk} + B_{ikjl}) + g^{pq}R_{pjkl}R_{qi} \\ &\qquad - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} + (B_{jikl} - B_{jilk} - B_{jlik} + B_{jkil}) - g^{pq}R_{pjkl}R_{qj} \\ &= \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} + g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj}) \\ &\qquad - 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}), \end{aligned}$$

since B is a quadratic term in the Riemann tensor satisfying $B_{ijkl} = B_{jilk} = B_{klij}$. Equation (1.3.7) then follows immediately.

Proof of evolution of Ricci curvature tensor. Contracting Riemann curvature tensor we have

$$R_{ij} = g^{lk}R_{ijkl},$$

then

$$\frac{\partial}{\partial t}R_{ij} = \frac{\partial}{\partial t}\Big(g^{lk}R_{ijkl}\Big) = g^{lk}\frac{\partial}{\partial t}(R_{ijkl}) + \Big(\frac{\partial}{\partial t}g^{lk}\Big)R_{ijkl}.$$

Using formula (1.3.1), we obtain

$$\frac{\partial}{\partial t}R_{ij} = g^{lk}\frac{\partial}{\partial t}(R_{ijkl}) - 2g^{lp}g^{kq}R_{pq}R_{ijkl},$$

inserting the result of evolution of Riemann curvature tensor we obtain

$$\begin{aligned} \frac{\partial}{\partial t}R_{ij} = g^{lk}\Big\{&\Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &- (R^p_i R_{pjkl} + R^p_j R_{ipkl} + R^p_k R_{ijpl} + R^p_l R_{ijkp})\Big\} - 2g^{lp}g^{kq}R_{pq}R_{ijkl} \end{aligned}$$

and compute

$$
\begin{aligned}
g^{lk}\Delta R_{ijkl} &= \Delta R_{ij} \\
2g^{lk}(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) &= 2g^{lk}B_{ijkl} - 2g^{lk}(B_{iljk} + B_{ijlk}) - 2g^{lk}B_{iljk} \\
&= 2g^{lk}(B_{ijkl} - 2B_{iljk}) + 2g^{pr}g^{qs}R_{piqk}R_{jl}.
\end{aligned}
$$

It is obvious that $B_{ijkl} - 2B_{iljk} = 0$ from the Bianchi identity, similarly,

$$
\begin{aligned}
g^{lk}(R_i^p R_{pjkl} + R_j^p R_{ipkl} &+ R_k^p R_{ijpl} + R_l^p R_{ijkp}) \\
&= g^{pq}R_{qi}Rpj + g^{pq}R_{qj}R_{ip} + g^{pq}R_{qk}R_{ip} + g^{lk}g^{pq}R_{ijkp}R_{ql} \\
&= 2g^{pq}R_{qj}R_{jp} + 2g^{lk}g^{pq}R_{ijkp}R_{ql}.
\end{aligned}
$$

It therefore follows by putting these results together that

$$
\frac{\partial}{\partial t}R_{ij} = \Delta R_{ij} + 2g^{pr}g^{qs}R_{pjkq}R_{rs} - 2g^{pq}R_{qi}R_{jp},
$$

which ends the proof of evolution of Ricci curvature tensor.

Proof of evolution of Scalar curvature tensor. Similarly $R = g^{ij}R_{ij}$ and

$$
\begin{aligned}
\frac{\partial}{\partial t}R &= g^{ij}\frac{\partial}{\partial t}(R_{ij}) + \Big(\frac{\partial}{\partial t}g^{ij}\Big)R_{ij} \\
&= g^{ij}(\Delta R_{ij} + 2g^{kp}g^{lq}R_{kijl}R_{pq} - 2g^{kl}R_{ik}R_{jl}) - g^{ik}g^{jl}h_{kl}R_{ij} \\
&= \Delta R + 2g^{ik}g^{jl}R_{ij}R_{kl}.
\end{aligned}
$$

Equation (1.3.9) follows at once. This then completes the proof of Theorem 1.3.2. □

Similarly, we have the following evolution equations under the Normalized Ricci flow. Suppose $\tilde{g}(t)$ solves (1.1.2), we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\tilde{g}^{ij} = 2(\tilde{R}^{ij} - \frac{r}{n}\tilde{g}^{ij}) \\
&\frac{\partial}{\partial t}\tilde{R} = \tilde{\Delta}\tilde{R} + 2|\tilde{R}_{ij}|^2 - \frac{2r}{n}\tilde{R} \\
&\frac{\partial}{\partial t}d\tilde{\mu} = (r - \tilde{R})d\tilde{\mu} \\
&\frac{\partial}{\partial t}\tilde{\Delta}_{\tilde{g}} = 2\tilde{R}^{ij}\cdot\tilde{\nabla}_i\tilde{\nabla}_j - \frac{2r}{n}\tilde{\Delta}_{\tilde{g}}.
\end{aligned}
$$

1.4 Short-time Existence and Uniqueness

As we have remarked, the Ricci flow is a system of nonlinear weakly parabolic equation and the proof of its short-time existence by R. Hamilton does not follow from standard parabolic PDEs theory. In his proof [87], Hamilton used a ”powerful analytic tool” called Nash-Moser implicit function theorem. Not quite later that Dennis

DeTurck [73] found a simplified way of doing this, his approach is popularly called DeTurck's trick. In this section, we discuss the linearization of the Ricci tensor and show that the degeneracy of the equation is due to the diffeomorphism group of the manifold which acts as the gauge group of the flow. We then give a brief description of the DeTurck approach to establishing existence and uniqueness of the Ricci flow via a modified evolution equation, which turns out to be strictly parabolic. This enables us to apply the standard parabolic theory. See for instance Cao and Zhu [41], the books by Chow and Knopf [68] and Topping [147]. The papers [136, 137] contain details of the case when Ricci flow is defined on a complete noncompact manifold.

1.4.1 Linearization of the Ricci tensor

Our intention here is to get the linearised form of the Ricci tensor and understand what is meant for an evolution equation on a vector bundle to be parabolic. We shall, however, see that the Ricci flow is weakly parabolic for the metric g. Consider

$$\frac{\partial g_{ij}}{\partial t} = Q(g) = -2R_{ij}(g), \tag{1.4.1}$$

on the vector bundle S^2T^*M. We regard the Ricci tensor R_{ij} as a nonlinear partial differential operator with respect to the metric g, i.e.,

$$R_{ij} = Rc : \Gamma(S^2_+T^*M) \to \Gamma(S^2T^*M),$$

where $S^2_+T^*M$ is a space of positive definite symmetric tensor and S^2T^*M is a space of symmetric tensor.

Recall that the Riemannian curvature tensor given by

$$R^k_{ijl} := \partial_i\Gamma^k_{jl} - \partial_j\Gamma^k_{il} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il},$$

has its index lowered to $R_{ijkl} = g_{kp}R^p{}_{ijl}$ and its contraction

$$g^{kl}R_{ijkl} = R_{ij}$$

gives the Ricci tensor. Then we have

$$-2R_{ij} = -2\Big\{\partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^k_{kp}\Gamma^p_{ij} - \Gamma^k_{ip}\Gamma^p_{kj}\Big\}. \tag{1.4.2}$$

Since the Christoffel's symbols are defined by

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

$$\begin{aligned}
-2R_{ij} &= \partial_i\Big\{g^{kl}(\partial_k g_{jl} + \partial_j g_{kl} - \partial_l g_{kj})\Big\} - \partial_k\Big\{g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})\Big\}\\
&\quad + 2\Gamma^k_{kp}\Gamma^p_{ij} - 2\Gamma^k_{ip}\Gamma^p_{kj}\\
&= \partial_i\Big\{g^{kl}(\partial_j g_{kl})\Big\} - \partial_k\Big\{g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})\Big\} + 2\Gamma^k_{kp}\Gamma^p_{ij} - 2\Gamma^k_{ip}\Gamma^p_{kj}\\
&= g^{kl}\Big\{\partial_i\partial_j g_{kl} - \partial_i\partial_k g_{jl} - \partial_j\partial_k g_{il} + \partial_k\partial_l g_{ij}\Big\} + \textit{Lower Order Derivatives}.
\end{aligned}$$

Thus, the linearization of the Ricci tensor is (h is the variation of g)

$$-2\Big\{D(Rc)(h)\Big\}_{ij} = -2\frac{\partial}{\partial t}(Rc(g(t))\Big|_{t=0} = g^{kl}\Big\{\partial_i\partial_j h_{kl} - \partial_i\partial_k h_{jl} - \partial_j\partial_k h_{il} + \partial_k\partial_l h_{ij}\Big\}, \tag{1.4.3}$$

where $D(Rc_g) : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ and its principal symbol in the direction ζ is a bundle homomorphism

$$\hat{\sigma}[D(Rc_g)(\zeta)] : \Gamma(S^2_+T^*M) \to \Gamma(S^2T^*M),$$

which by replacing the covariant derivative ∂_i in (1.4.3) by the covector ζ_i, is defined as

$$\Big(\hat{\sigma}[D(Rc_g)](\zeta)(\tilde{g})\Big)_{ij} = -\frac{1}{2}g^{kl}\Big(\zeta_i\zeta_j h_{kl} - \zeta_i\zeta_k h_{jl} - \zeta_j\zeta_k h_{il} + \zeta_k\zeta_l h_{ij}\Big). \tag{1.4.4}$$

If the principal symbol is isomorphism for every 1-form ζ and some section h, then, we say that the nonlinear operator Q is elliptic at h and the corresponding evolution equation (1.4.1) is parabolic.

1.4.2 Ricci flow as weakly parabolic

Consider the principal symbol of the linearized equation obtained above, it is easy to see that this symbol will not certainly be elliptic, since for any ζ_i whatsoever, we can define $h_{jk} = \zeta_j\zeta_k$ and the symbol (1.4.4) evaluates to zero. The symbol possesses zero eigenvalues, which shows the equation is not strictly parabolic. To see this, consider (1.4.4) and assume ζ has length 1, since the function is homogeneous, we choose coordinates at a point such that

$$g_{ij} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

and $\zeta = (1, 0, ..., 0)$. A simple calculation yields

$$\Big(\hat{\sigma}[DQ(g)](\zeta)(h)\Big)_{ij} = h_{ij} + \delta_{i1}\delta_{j1}(h_{11} + ... + h_{nn}) - \delta_{i1}h_{1j} - \delta_{1j}h_{1i},$$

that is,

$$\begin{cases} \Big(\hat{\sigma}[DQ(g)](\zeta)(h)\Big)_{11} = h_{22} + h_{33} + \cdots + h_{nn} \\ \Big(\hat{\sigma}[DQ(g)](\zeta)(h)\Big)_{1j} = 0, \qquad j \neq 1 \\ \Big(\hat{\sigma}[DQ(g)](\zeta)(h)\Big)_{ij} = h_{ij}, \qquad i \neq 1, j \neq 1. \end{cases}$$

There is actually a good reason for the presence of these zero eigenvalues (see [87] for details). The fact that the principal symbol $\hat{\sigma}[D(Rc_g)](\zeta)$ of the nonlinear partial differential operator Rc_g has a nontrivial null space is intimately related to the fact that the Ricci curvature tensor has a property of diffeomorphism invariance, that is,

$$\phi^*(Rc(g)) = Rc(\phi^* g).$$

The failure of the ellipticity is due to the consequence of diffeomorphism invariance which also implies Bianchi identities.

1.4.3 DeTurck Approach

We are concerned with short-term existence and uniqueness of the Ricci flow, despite the fact that the linearized Ricci tensor is a non-strict elliptic second order differential operator.

Theorem 1.4.1. *(**Hamilton** [87]) Let (M, g) be a compact Riemannian manifold. Then there exists a constant $T > 0$ such that the initial value problem for the Ricci flow admits a unique smooth solution on M for all time $t \in [0, T)$.*

For noncompact case see W-X. Shi [136]. Next, we briefly describe DeTurck approach to proving the above theorem. The first step in this direction is to define a modified Ricci flow (Ricci-DeTurck flow) by adding an extra term from Lie derivative of the metric with respect to certain time-dependent vector field to the Ricci-Hamilton flow equation

$$\begin{cases} \dfrac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}(g) + \nabla_i W_j + \nabla_j W_i \\ \\ g(0) = g_0, \end{cases} \tag{1.4.5}$$

where

$$W_j = g_{jk} g^{pq} \Big((\Gamma_g)^k_{pq} - (\Gamma_{\tilde{g}})^k_{pq} \Big)$$

and $(\Gamma_{\tilde{g}})^k_{pq}$ are the Christoffel's symbols associated with Levi-Civita connection of the background metric $\tilde{g}$, and show that the system is strictly parabolic. Note that W_j is a time-dependent 1-form which is g-dual to the vector $W^k = g^{pq} \Big((\Gamma_g)^k_{pq} - (\Gamma_{\tilde{g}})^k_{pq} \Big)$. Now, it is easy to see that if $g_{ij}(x, t)$ solves the Ricci flow (1.1.1) and a one-parameter group of diffeomorphism φ_t is defined on M, then the pull-back metric

$$\tilde{g}_{ij}(x, t) = \varphi_t^* g_{ij}(x, t) \tag{1.4.6}$$

solves

$$\begin{cases} \dfrac{\partial}{\partial t} \tilde{g}_{ij}(x, t) = -2R_{ij}(x, t) + \nabla_i W_j + \nabla_j W_i = E(g_{ij}) \\ \\ \tilde{g}_{ij}(x, 0) = \tilde{g}_{ij}(x). \end{cases} \tag{1.4.7}$$

Notice that the RHS of (1.4.7) implies

$$\begin{aligned} E(g_{ij}) = g^{kl} \Big\{ \partial_i \partial_j g_{kl} - \partial_i \partial_k g_{jl} - \partial_j \partial_k g_{il} + \partial_k \partial_l g_{ij} \Big\} &+ \frac{1}{2} g^{pq} \Big\{ \partial_i \partial_q g_{pj} + \partial_i \partial_p g_{qj} - \partial_i \partial_j g_{pq} \Big\} \\ &+ \frac{1}{2} g^{pq} \Big\{ \partial_i \partial_q g_{pj} + \partial_i \partial_p g_{qj} - \partial_i \partial_j g_{pq} \Big\} + Lower\ Order\ Term \\ = g^{kl} \partial_k \partial_l g_{ij} + Lower\ Order\ Term. & \end{aligned}$$

Thus, its principal symbol is $(g^{kl} \zeta_k \zeta_l) \tilde{g}_{ij}$, i.e, $\Big(\hat{\sigma}[DE(g)](\zeta)(\tilde{g}) \Big)_{ij} = |\zeta|^2 \tilde{g}$, which gives an ellipticity condition. Hence the system (1.4.7) is strictly parabolic. The description above shows that the Ricci-DeTurck flow has a short-time solution on a compact manifold follow from the standard parabolic theory.

The next step is to modify such a solution in order to obtain a solution of the original Ricci-Hamilton flow (1.1.1). To do this, the following Lemma will be very useful.

Lemma 1.4.2. *Let W_t be a time-varying dependent vector field on a compact manifold M. Then, there exists a unique one parameter family of diffeomorphism $\phi_t : M \to M$ defined on the interval $0 \leq t \leq \infty$, such that*

$$\begin{cases} \dfrac{d\phi_t(x)}{dt} = -W(\phi_t(x), t), \quad x \in M, \quad t \in [0, T) \\ \phi_0 = Id_M. \end{cases} \tag{1.4.8}$$

As long as there exists a solution $g(t)$ of (1.4.7), the one parameter family of vector fields $W(t)$ (as defined by (1.4.5)) exists for $t \in [0, \epsilon)$. By solving the ODE in the Lemma above and by the compactness of M, all $\phi_t(p)$ exist and remain diffeomorphism for as long as diffeomorphism exists for $p \in M$ (see Lee [105, pp 451] for flows of time-dependent vector field) and solution $g(t)$ also exists for $t \in [0, T)$. Therefore, the family of the metric

$$g(t) := (\phi_t)_*(\bar{g}(t))$$

is a solution of (1.4.7) if $\bar{g}(t)$ also exists and indeed, $\bar{g}(0) = g(0) = g_0$ because $\phi_0 = Id_M$. Next, we show that $g(t)$ is a solution of the Ricci-DeTurck flow. Compute

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= \frac{\partial}{\partial t}\Big((\phi_t)_* \bar{g}(t)\Big) \\ &= (\phi_t)_* \frac{\partial}{\partial t}\Big(\bar{g}(t)\Big) + \mathcal{L}_{(\phi_t)_*(\frac{\partial}{\partial t}\phi_t)}\Big((\phi_t)_* \bar{g}(t)\Big) \\ &= (\phi_t)_* \Big[-2Rc(\bar{g}(t)) + \mathcal{L}_{(\phi_t)_*(\phi_t^*[W(t)])}\Big(g(t)\Big)\Big] \\ &= -2Rc(g(t)) + \mathcal{L}_{W(t)} g(t). \end{aligned}$$

Since the Ricci -DeTurck flow (1.4.7) is strictly parabolic, we are sure of a unique solution $g(t)$ and once we have $g(t)$, we can obtain the diffeomorphism ϕ_t by solving the non-autonomous ODE (1.4.8) in Lemma 1.4.2. We however observe that

$$\bar{g}(t) := \phi_t^*(g(t))$$

is a solution of Ricci-Hamilton flow. This is presented in the following Theorem.

Theorem 1.4.3. *The time- dependent metric $\bar{g}(t) := \phi_t^*(g(t))$ is a solution of the Ricci flow equation (1.1.1)*

Proof.

$$\begin{aligned}
\frac{\partial \bar{g(t)}}{\partial t} &= \frac{\partial \phi_t^* g(t)}{\partial t} = \frac{\partial}{\partial s}\Big|_{s=0}(\phi_{t+s}^* g(t+s)) \\
&= \phi_t^*\Big(\frac{\partial}{\partial t}g(t)\Big) + \frac{\partial}{\partial s}\Big|_{s=0}(\phi_{t+s}^* g(t)) \\
&= \phi_t^*(-2Rc(g(t)) + \mathcal{L}_{W(t)}g(t)) + \frac{\partial}{\partial s}\Big|_{s=0}(\phi_{t+s}^* g(t)) \\
&= -2Rc(\phi_t^* g(t)) + \phi_t^* \mathcal{L}_{W(t)}g(t)) + \frac{\partial}{\partial s}\Big|_{s=0}(\phi_{t+s}^* g(t)) \\
&= -2Rc(\phi_t^* g(t)) + \phi_t^* \mathcal{L}_{W(t)}g(t)) + \frac{\partial}{\partial s}\Big|_{s=0}\Big((\phi_t^{-1}\circ\phi_{t+s})^*\phi_t^* g(t))\Big) \\
&= -2Rc(\phi_t^* g(t)) + \phi_t^* \mathcal{L}_{W(t)}g(t)) - \mathcal{L}_{(\phi_t^{-1})_* W(t)}(\phi_t^* g(t)) \\
&= -2Rc(\phi_t^* g(t)).
\end{aligned}$$

The equality in the second to the last follow from the identity

$$\frac{\partial}{\partial s}\Big|_{s=0}\Big(\phi_t^{-1}\circ\phi_{t+s}\Big) = (\phi_t^{-1})_*\Big(\frac{\partial}{\partial s}\Big|_{s=0}\phi_{t+s}\Big) = (\phi_t^{-1})_* W(t).$$

The proof is complete. □

Next thing to show is that $\bar{g}(t)$ is indeed a unique solution for the Ricci flow.

On the uniqueness of the Ricci flow

The fact that the Ricci-DeTurck flow is strictly parabolic and satisfies the standard uniqueness conditions may not be enough to conclude that the solution to the Ricci flow is unique. This is simply due to the following argument; starting with two solutions of the Ricci flow which agree at time $t = 0$ on the same interval, modifying them by diffeomorphisms to get two solutions of the Ricci-DeTurck flow also with identical initial conditions so that the modified solutions may be the same. Notice that the diffeomorphisms used depend on the solutions to the Ricci flow themselves, so the modified solutions may not be the same if the solutions to the Ricci flow chosen differ. Then the uniqueness of the Ricci-DeTurck flow breaks down. For this, we need an alternative way of establishing the uniqueness of the Ricci flow and the way out is by reparametrization of diffeomorphism by the harmonic map heat flow which is equally strictly parabolic.

The argument is then reduced to a basic question: Do DeTurck's diffeomorphisms satisfy the harmonic map heat flow?. This has been answered in affirmative (Cf. Chow and Knopf [68, Lemmma 3.27] and Hamilton [92]). Therefore, we conclude with the following: Suppose $(M, \bar{g}(t))$ is a solution of the Ricci flow and that a family of diffeomorphism φ_t is a solution of the harmonic map heat flow with respect to $\bar{g}(t)$ and background metric $\widetilde{g}$, if $g(t) := (\varphi_t)_*\bar{g}(t)$ is a unique solution of the Ricci-DeTurck flow, we claim that a solution of $\bar{g}(t)$ of Ricci-Hamilton flow is unique. Suppose, we have two solutions of Ricci flow, $\bar{g}_1(t)$ and $\bar{g}_2(t)$, satisfying the initial condition $\bar{g}_1(0) = \bar{g}_2(0)$, choose $(\varphi_1)_t$, the harmonic map flow with respect to $\bar{g}_1(t)$ and $\widetilde{g}$, also, $(\varphi_2)_t$ as the harmonic map flow with respect to $\bar{g}_2(t)$ and $\widetilde{g}$. Then

$$g_1(t) = ((\varphi_1)_t)_* \bar{g}_1(t) \quad and \quad g_2(t) = ((\varphi_2)_t)_* \bar{g}_2(t)$$

are both solutions of the Ricci-DeTurck flow. Therefore, by uniqueness of solution of (1.4.7), $g_1(t) = g_2(t)$ for all $t \geq 0$ in their common interval of existence. Hence, both $(\varphi_1)_t$ and $(\varphi_2)_t$ satisfy the system of ODE in Lemma 1.4.2, generated by the same vector field W^k. Thus, $\varphi_1(x,t) = \varphi_2(x,t)$ as long as they exist, and

$$\bar{g}_1(t) = (\varphi_1)_t^* g_1(t) = (\varphi_2)_t^* g_2(t) = \bar{g}_2(t).$$

This concludes the uniqueness of the Ricci flow.

1.5 Ricci flow on Surfaces

Here, we briefly describe how closed surfaces can be deformed using Ricci flow. By a closed surface we mean a compact 2-manifold without boundary, if it is simply connected, then it is topologically equivalent to a 2-sphere. The Ricci flow was first understood in dimensions higher than 2, since it provides a complete classification of three manifolds (as mentioned earlier). Ricci flow is easily visualised on surfaces such as sphere, torus, cylinder, since they are more familiar, thus, the aforementioned ideas can be better explained in the Ricci flow on closed surfaces and geometric computations are carried out with less difficulties. In this case also, Ricci flow and Yamabe flow [1] are the same, which make the local existence of the flow easily obtainable. The Ricci flow in 2-dimension is conformal and if the total surface area should be preserved during the evolution, Ricci flow will definitely converge to a constant Gaussian curvature metric everywhere in the conformal class, that is, the limiting metric is conformal to the background metric and of course to metric $g(t)$ at any time t. This provides a proof of Uniformization Theorem of Poincaré and Koebe (See Hamilton [88], Chow [60] and Chen, Lu and Tian [57]. However, it is much more difficult to establish the convergence of the Ricci flow when the Euler characteristic of the surface is positive.

On surfaces M^2, the normalized Ricci flow becomes

$$\frac{\partial}{\partial t} g = (r - R)g, \tag{1.5.1}$$

where $r = A^{-1} \int_{M^2} R dA$, the average of the scalar curvature R of M^2, A is the total surface area, dA is the area element of metric g on M^2 and

$$\frac{\partial}{\partial t} A = \int_{M^2} (r - R) dA = 0. \tag{1.5.2}$$

Thus, the total surface area is preserved along the flow. The integral of R over the surface M^2 gives the Euler class $\chi(M^2)$. Recall the Gauss-Bonnet formula (1.5.3) on a closed surface (M^2)

$$\frac{1}{2\pi} \int_{M^2} K dA = \chi(M^2) = 2(1 - \gamma(M^2)), \tag{1.5.3}$$

[1] Yamabe flow is the negative L^2-gradient flow of the total scalar curvature of a Riemannian manifold in a given conformal class. It was also introduced by Richard Hamilton [88] to tackle Yamabe problem [131], see also [62].

where $\chi(M^2)$ is the Euler characteristic, $\gamma(M^2)$, the genus and K the Gaussian curvature of M^2. Here $2K = R$, then

$$\int_{M^2} R dA = 4\pi\chi(M^2). \tag{1.5.4}$$

In fact, Gauss Bonnet formula accounts for the relation between the topology and geometry of the underlying manifold as we can see that the sign of r can be determined explicitly even independent of g,

$$r = \chi(M^2) = \frac{1}{4\pi}\int_{M^2} R dA. \tag{1.5.5}$$

For example, if we consider a topological 2-sphere whose genus is 0, then, $\chi(M^2) = 2$ and $\int_{S^2} R d\mu = 8\pi$.

Uniformization of Surfaces

Uniformization theorem implies that every smooth surface admits a unique conformal metric, which classifies surfaces into three families using the sign of the curvature. This is a classical result in Riemannian geometry, that is, every simply connected surface is conformally equivalent to one among the Riemann spheres, the complex plane and the open disk. In this direction, Ricci flow greatly helps in the classification of closed two-dimensional manifolds into three families of constant positive, zero or negative curvature, as it is used in the classification of closed three manifold (Geometrization Conjecture). The procedure is to run Ricci flow on smooth surface and allow the metrics to be deformed over time such that the scalar curvature evolves as a reaction-diffusion equation and eventually becomes constant. The limiting metric is the uniformizing metric which classifies the universal covering space of the surface into one of the three canonical geometries. We now state an important result of Ricci flow on Surfaces due to R. Hamilton [88].

Theorem 1.5.1. *Let (M^2, g_0) be a closed surface, there exists a unique solution $g(t)$ of (1.5.1) for all time t. Moreover,*

1. *the metric g(t) converges to a metric g_∞ of constant curvature as $t \to \infty$, when $r \leq 0$.*
2. *If $R(g_0) > 0$, then the metric g(t) converges to a positive constant curvature metric at time $t \to \infty$.*

The above result together with the work of B. Chow [60] give a complete proof of the uniformization of surfaces using the Ricci flow. The main point of contention here lies in the class of positive Euler characteristics where the existence of gradient shrinking soliton uses Kazdan-Warner identity which itself assumes the uniformization theorem. Detail of this is contained in the book [68, Chapter 5], also see [57]) for a new proof of uniformization theorem without Kazdan-Warner identity.

1.6 Maximum Principles

The maximum principles are one of the fundamental properties possessed by second order parabolic equations. In general, the theories assert that any pointwise bounds that hold for a smooth solution of the heat equation at the initial time $t = 0$ also hold for all times $t > 0$. The theories are rich enough to admit geometric heat equation on compact manifolds, specifically, they are used to give the description of the Ricci flow and show that certain pointwise inequalities on the initial data of Ricci flow equation are preserved by the evolution. Consequently, we can obtain estimates on curvatures and show that Ricci flow preserves the nonnegativity of the curvature operator. We only give some statements of the maximum principles here. For general discussion, see Protter and Weinberger [129]. The theories have been developed for tensors and vector bundles by Hamilton in [87] and for general survey in the context of Ricci flow, see for instance the books [68, 69, 147].

Proposition 1.6.1. *Let $(M, g(t))$ be a family of Riemannian manifolds and $X(t)$ a family of smooth vector fields for $t \in [0, \infty)$. Suppose that $u \in C^{\infty}(M \times [0, \infty), \mathbb{R})$ satisfies the heat-type inequality*

$$\frac{\partial}{\partial t} u(x,t) \geq \Delta_g u(x,t) + \langle X, \nabla u \rangle (x,t). \tag{1.6.1}$$

Let there exists a constant $\alpha \in \mathbb{R}$ such that $u(x, 0) \geq \alpha$ for all $x \in M$, then $u(x,t) \geq \alpha$ for all $x \in M$ and $t \geq 0$.

The above proposition is the scalar maximum principle which compares the solution of the heat equation with that of the associated ordinary differential equation. It also applies to the heat equation when reaction term is introduced, the reaction term can either be linear or nonlinear. Consider the nonlinear heat equation with reaction term

$$\frac{\partial u}{\partial t} = \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u), \tag{1.6.2}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, $g(t)$ and $X(t)$ are as defined above. We call u a supersolution (or subsolution) if the equality in (1.6.2) is replaced with $" \geq "(or" \leq ")$.

Theorem 1.6.2. *Let $g(t)$ be a one-parameter family of Riemannian metrics on compact manifold M, let a C^2-function $u : M^n \times [0, T) \to \mathbb{R}$ be a supersolution of (1.6.2) i.e., it satisfies the follow inequality*

$$\begin{cases} \dfrac{\partial u}{\partial t} \geq \Delta_g u + \langle X, \nabla u \rangle + F(u) \\ \\ u(0) = \alpha \end{cases} \tag{1.6.3}$$

where $X(t)$ is a time-dependent family of vector field and F is locally Lipschitz continuos. Then

$$u(x,t) \geq \phi(t) \tag{1.6.4}$$

for all $(x, t) \in M \times [0, T]$, $(0 < T < \infty)$, where $\phi(t)$ satisfies the ordinary differential equation

$$\begin{cases} \dfrac{d\phi(t)}{dt} = F(\phi(t)) \\ \\ \phi(0) = \alpha. \end{cases} \tag{1.6.5}$$

Maximum principles are applied in Chapter 3.

Chapter 2

Eigenvalues and Entropy Monotonicity Formulas under The Ricci Flow

2.1 Introduction

Here, we consider an n-dimensional compact manifold $M^n, n \geq 2$, on which a one parameter family of Riemannian metrics $g_{ij}(t), t \in [0, \infty)$ is defined. We refer to $(M^n, g(t))$ as the solution of the Ricci flow, if it satisfies the following nonlinear evolution partial differential equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{2.1.1}$$

written in local coordinate, where R_{ij} is the Ricci curvature tensor of the manifold. The Ricci tensor can be linearised to obtain

$$R_{ij} = \frac{-1}{2}\Delta_g(g_{ij}) + Q_{ij}(g^{-1}, \partial g), \tag{2.1.2}$$

where Δ_g is the Laplace-Beltrami operator acting on manifold (M^n, g) and $Q_{ij}(g^{-1}, \partial g)$ is a lower order term, quadratic in inverse of g and its first order partial derivative. Hence, the Ricci flow equation is a heat-like (diffusion-reaction) equation. For example, in the usual Euclidean space, the Laplace-Beltrami operator is exactly the usual Laplace operator

$$\Delta = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x^i \partial x^j}, \tag{2.1.3}$$

where we can consider the eigenvalue problem for the Laplacian as follows

$$-\Delta u_i = \lambda_i u_i$$

and have the sequence

$$0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_i \longrightarrow \infty, \quad (i \to \infty)$$

as the eigenvalues of the Laplacian, repeated according to their geometric multiplicities, where any u_i corresponding to λ_i is the eigenfunction, the eigenspace being finite dimension. In this respect, various eigenvalue problems arise, such as

$$-\Delta u = \lambda u \quad in \;\; \Omega \;\subseteq \mathbb{R}^n, \quad \partial\Omega = \emptyset \tag{2.1.4}$$

so also Dirichlet ($u = 0 \; on \; \partial\Omega$) and Neumann ($\frac{\partial u}{\partial \nu} = 0 \; on \; \partial\Omega$), where ν is the unit normal vector exterior to the boundary of Ω, in case the boundary is nonempty. These can easily be generalised to the Riemannian Manifold (M^n, g) with or without boundary, where the Laplace-Beltrami operator is viewed as a self-adjoint operator on $L^2(M^n)$ and M^n has a pure point spectrum of a sequence of eigenvalues $\{\lambda_i\}_{i=1}^n$ and the eigenfunction u_i form orthonormal basis of $L^2(M^n)$ with $||u_i||_{L^2(M^n)} = 1$. Detail discussion on the spectrum of Riemannian manifold is included in Appendix B.

In this chapter, we consider boundaryless manifold (the results also holds when the boundary is empty). In this case, the first eigenvalue is equal to zero, because, here the constant functions are non trivial solutions of the eigenvalue problem, while the first eigenvalue is always positive if a boundary exists. Studying the behaviours of eigenvalues of Laplacian operator is not out of place as its properties such as monotonicity, multiplicity, asymptotic etc. provide us with rich information about the topology and geometry of the underlying manifold. In the first of his three groundbreaking papers [126], G. Perelman introduces the energy functional $\mathcal{F}$ and shows that it is non-decreasing along the modified Ricci flow coupled with certain conjugate heat equation. He establishes that monotonicity of $\mathcal{F}$ implies that of the first non-zero eigenvalue of the operator $-4\Delta + R$ and applies the monotonicity to rule out nontrivial steady and expanding breathers on compact manifold. Note that Ricci breathers correspond to periodic orbits which we do not usually expect since Ricci flow is a heat-type equation (see Definition 2.2.4 for detail). In [115], L. Ma shows that the eigenvalues of Laplace-Beltrami operator on compact domain of Riemannian manifold associated with the Ricci flow is non-decreasing but with nonnegativity assumption on the scalar curvature R. X. Cao has since extended this result to the eigenvalues of the operator $-\Delta + \frac{R}{2}$ [42]. In [43] the monotonicity of eigenvalue of $-\Delta + cR, c \geq \frac{1}{4}$ is established without sign assumption on the curvature operator and both compact steady and expanding Ricci breathers are trivial. In [108] a family of functional Li- $\mathcal{F}_k$, which happens to be nondecreasing under the Ricci flow is constructed and the result extended to rescaled Ricci flow in [109]. It turns out that the Ricci flow is a special case of the rescaled Ricci flow. More interestingly, these results can be extended to any other Laplacian-type operator under a closed Riemannian manifold. For instance, the first eigenvalue of p-Laplace operator ($p \geq 2$) with Einstein metric is monotonically non-decreasing [153], In this case, when $p = 2$, the main result coincides with that of [115]. See also [110] for results in harmonic maps flow coupled to the Ricci flow.

Throughout this chapter, we adopt Einstein summation convention, where the volume element on manifold $\sqrt{|g|}dx^1 \bigwedge ... \bigwedge dx^n = d\mu$, metric $g(\partial_i, \partial_j) = g_{ij}$, where $\partial_i = \frac{\partial}{\partial x^i}$ are the components of the metric. The Levi-Civita connection is defined by $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$, while its Christoffel's symbols are given by $\Gamma^k_{ij} = \frac{1}{2}g^{kl}\Big(\partial_i g_{jl} +$

$\partial_j g_{il} - \partial_l g_{ij}\Big)$, R_{ij} and R are the Ricci and scalar curvature tensors respectively, where $R = g^{ij}R_{ij}$, the trace of Ricci tensor. The contracted second Bianchi identity is given as $g^{ij}\nabla_i R_{jk} = \frac{1}{2}\nabla_k R$ and the inner product $\langle p, q\rangle := \int_{M^n} g^{ik}g^{jl}p_{ij}q_{kl}d\mu_g$ for any two tensors p and q. We sometimes write M instead of M^n to mean Manifold of dimension $= n$ without fear of confusion.

The rest of the chapter follows; in Section 2.2, we discuss some classical energy functionals and lay emphasis on Perelman entropy and its geometric consequences. In Section 2.3, we construct a new family of entropy functionals which proves to be monotonically nondecreasing, we also discuss the monotone properties of eigenvalues of the geometric operator "$-2\Delta + CR$", where $C \geq \frac{1}{2}$, and R, a scalar curvature, under the Ricci flow, while the results of Section 2.3 are extended to the case of normalized flow in Section 2.4. The results here confirm that expanding or steady breathers on compact manifold are necessarily Einstein. We also construct a new family of entropy over shrinkers which allows us to obtain conditions over which Einstein metrics shrink.

2.2 Classical Energy Functionals

2.2.1 Total Scalar Curvature

We obtain the derivative of the total scalar curvature on a closed manifold $(M^n, g(t))$ as

$$\frac{\partial}{\partial t}\int_M R d\mu = \int_M \Big(\frac{1}{2}(tr_g h)R - h^{ij}R_{ij}\Big)d\mu, \tag{2.2.1}$$

which coincides with the first variation of the classical Einstein-Hilbert functional $\mathcal{H} = \int_M R d\mu$ by using the following variation formulas

$$\frac{\partial g_{ij}}{\partial t} = h_{ij}, \qquad \frac{\partial R}{\partial t} = -\Delta(tr_g h) + \delta^2 h - \langle h, Rc\rangle,$$

where $\delta^2 h = g^{ij}g^{pq}\nabla_j\nabla_q h_{ip}$ and $\langle h, Rc\rangle = g^{ij}g^{kl}h_{ik}R_{jl}$. Specifically,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{H}(g_{ij}) &= \int_M \Big[-\Delta(tr_g h) + \delta^2 h - \langle h, Rc\rangle + \frac{R}{2}trh\Big]d\mu\\
&= \int_M \Big[\frac{R}{2}\langle g, h\rangle - \langle h, Rc\rangle\Big]d\mu\\
&= \int_M h^{ij}\Big(\frac{R}{2}g_{ij} - R_{ij}\Big)d\mu,
\end{aligned}$$

where $G_{ij} = R_{ij} - \frac{R}{2}g_{ij}$ is the Einstein tensor. Then, we have

$$\frac{\partial}{\partial t}\mathcal{H}(g_{ij}) = \int_M -h^{ij}G_{ij}d\mu = \int_M \langle h, \nabla\mathcal{H}(g)\rangle d\mu$$

and then obtain

$$\frac{\partial}{\partial t}g = \nabla\mathcal{H}(g) \tag{2.2.2}$$

as the gradient flow of $\mathcal{H}(g)$. For the gradient flow of the Einstein-Hilbert functional, we have (modified by multiplication factor 2)

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + Rg_{ij} = -2G_{ij}, \tag{2.2.3}$$

which is not parabolic even weakly, thus, we can not readily establish its solution even for a short time. We note that the weak part of (2.2.3) coincides with the Ricci flow, while the remaining term arises from the presence of the volume element $d\mu$, which itself is time-evolving.We shall however deal with this in Section 2.3.

Remark 2.2.1. *We call g stationary of $\mathcal{H}(g)$ if $\delta\mathcal{H}(g) = 0$ for all $h \in S^2T^*M$. Since $G_{ij} = G_{ji}$, then $G_{ij} = 0$ on M. Taking the trace, we have*

$$0 \equiv G = \frac{2-n}{2}R \tag{2.2.4}$$

So in dimension $n \neq 2$, this implies $R \equiv 0$ on M and therefore $Rc \equiv 0$ on M (Ricci flat manifold), then the functional becomes invariant under deformations. It is now clear that the Ricci flow is not a gradient flow of a functional over the space of smooth metric but can be formulated as a gradient-like flow. The key to achieving this is to look for functionals whose critical points are Ricci solitons, this is contained in the work of Perelman [126] as we briefly survey in the next section.

2.2.2 The Perelman's $\mathcal{F}$-Energy and its Consequences

In this section, we discuss Perelman's $\mathcal{F}$ -energy as introduced by Grisha Perelman in a truly groundbreaking paper [126] and give some of the geometric consequences of its monotonicity under the Ricci flow. Many authors have given exposition on this subject, some of which include [4, 64, 69, 101, 135].

Let $(M^n, g_{ij}(t))$ be a closed manifold for a Riemannian metric $g_{ij}(t)$ and a smooth function f on M^n, Perelman's Energy functional [126] on pairs (g_{ij}, f) is defined by

$$\mathcal{F}(g_{ij}(t), f) = \int_{M^n} (R + |\nabla f|^2) e^{-f} d\mu. \tag{2.2.5}$$

The introduction of function f has embedded the space of Riemannian metric in a larger space (see also [64, 101]). The energy functional (2.2.5) can be expressed in two other ways, namely

$$\mathcal{F}(g_{ij}(t), f) = \int_{M^n} (R + \Delta f) e^{-f} d\mu, \tag{2.2.6}$$

which clearly follows from the fact that

$$\int_M \Delta(e^{-f}) d\mu = 0 = \int_M (-\Delta f + |\nabla f|^2) e^{-f} d\mu$$

and

$$\mathcal{F}(g_{ij}(t), f) = \int_M (R + 2\Delta f - |\nabla f|^2) e^{-f} d\mu. \tag{2.2.7}$$

For any diffeomorphism $\varphi : M \to M$, we have

$$\mathcal{F}(\varphi^* g, f \circ \varphi) = \mathcal{F}(g, f)$$

and for any constants $\alpha > 0$ and β

$$\mathcal{F}(\alpha^2 g, f + \beta) = \alpha^{n-2}\, e^{-\beta} \mathcal{F}(g, f).$$

Thus, the $\mathcal{F}$-energy functional is diffeomorphism invariant but not scale invariant. Now taking the smooth variations of metric g and f as $\delta g_{ij} = h_{ij}$ and $\delta f = K$ for some function $K : M \to \mathbb{R}$ respectively with $H := tr_g h_{ij}$, we have the following variation formula

$$\begin{aligned}\delta\mathcal{F}(g_{ij}(t), f) = \int_M \Big[-\Delta H + \nabla_i\nabla_j h_{ij} - h_{ij}R_{ij} + 2\langle\nabla f, \nabla K\rangle - h_{ij}\nabla_i f\nabla_j f \\ +(R + |\nabla f|^2)(\frac{H}{2} - K)\Big] e^{-f} d\mu.\end{aligned} \tag{2.2.8}$$

Applying integration by parts to some terms in the variation formula (C.1.2), we obtain

$$\delta\mathcal{F}(g_{ij}(t), f) = \int_M \Big[-h_{ij}(R_{ij} + \nabla_i\nabla_j f) + (2\Delta f - |\nabla f|^2 + R)(\frac{H}{2} - K)\Big] e^{-f} d\mu. \tag{2.2.9}$$

Keeping the volume measure static, i.e., letting $dm := e^{-f}d\mu$, then $\delta(dm) = 0$ implies $H = 2K$ (see Appendix C for detail), and we can then consider the L^2-inner product on space of metric g with respect to the measure dm as $\langle p_{ij}, q_{ij}\rangle_M = \int_M \langle p_{ij}, q_{ij}\rangle dm$, then we have

$$\nabla\mathcal{F}^m(g) = -(R_{ij} + \nabla_i\nabla_j f). \tag{2.2.10}$$

This leads us to consideration of theL^2- gradient flow

$$h_{ij} = \frac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i\nabla_j f) \quad for\ \ 2\mathcal{F}^m(g),$$

where f is defined by the above. This is a Ricci flow modified by diffeomorphism generated by the gradient of f[1], indeed, it is equivalent to the Ricci flow. Perelman proved that the $\mathcal{F}$-energy functional is monotonically nondecreasing under the following coupled system of modified Ricci flow and backward heat equation.

Lemma 2.2.2. *The coupled modified Ricci flow equation with a backward heat equation*

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i\nabla_j f) \\ \\ \dfrac{\partial f}{\partial t} = -\Delta f - R \end{cases} \tag{2.2.11}$$

is a gradient flow.

Notice that the second equation in the coupling is a backward heat equation, which can be solved backward in time. Conjugating away the infinitesimal diffeomorphism converts the gradient flow (2.2.11) to

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij} \\ \\ \dfrac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R. \end{cases} \tag{2.2.12}$$

[1]The symmetric tensor $-(R_{ij} + \nabla_i\nabla_j f)$ is the L^2 gradient flow of the functional $\mathcal{F} = \int_M (R + |\nabla f|^2)dm$, where $f := \log\frac{d\mu}{dm}$. Thus, given a measure m, we may consider the gradient flow $(g_{ij})_t = -2(R_{ij} + \nabla_i\nabla_j f)$ for $\mathcal{F}^m(g)$. For general m, this flow may not exist even for a short time, however, when it exists, it is just the Ricci flow modified by diffeomorphism

This is done by invoking Lemma 1.2.5. Intuitively, if the diffeomorphism is generated by flowing along the time-dependent vector field $X(t) = \nabla f$, then, the equation for g and f become $(g_{ij})_t = -2(R_{ij} + \nabla_i\nabla_j f) + \mathcal{L}_{\nabla f} g$ and $f_t = -\Delta f - R + \mathcal{L}_{\nabla f} f$, where $\mathcal{L}_X$ is the Lie derivative along the vector field X. Using these together with the fact that $\mathcal{L}_{\nabla f} g = 2\nabla\nabla f$ and $\mathcal{L}_{\nabla f} f = |\nabla f|^2$, we will obtain the Ricci flow (2.2.12) (See (0.2.12) for the definition of Lie derivative).

Precisely

$$\frac{d}{dt}\mathcal{F}(g_{ij}(t), f(t)) = 2\int_{M^n} |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu. \tag{2.2.13}$$

In particular $\mathcal{F}(g_{ij}(t), f(t))$ is monotonically nondecreasing in time and the monotonicity is strict unless $R_{ij} + \nabla_i\nabla_j f = 0$.

Having established the nondecreasing monotone feature of the energy $\mathcal{F}$ for the Ricci flow, we now want to develop a control quantity for Ricci flow but we must eliminate f. Define

$$\lambda(g_{ij}) = \inf\left\{\mathcal{F}(g_{ij}, f) : f \in C_c^\infty(M), \int_M e^{-f} d\mu = 1\right\}, \tag{2.2.14}$$

where the infimum is taken over all smooth functions f. Setting $e^{-f} =: u^2$, then the functional $\mathcal{F}$ is written as

$$\mathcal{F} = \int_{M^n} (Ru^2 + 4|\nabla u|^2) d\mu \quad with \quad \int_M u^2 d\mu = 1. \tag{2.2.15}$$

Then $\lambda(g)$ is the first non zero (least) eigenvalue of the self adjoint modified operator $-4\Delta + R$ and the nondecreasing monotonicity of $\mathcal{F}$ implies that of λ. As an application, Perelman was able to rule out the existence of nontrivial steady or expanding Ricci breathers on closed manifolds. Let $u_0 > 0$ be the corresponding eigenfunction, then the following

$$-4\Delta u_0 + Ru_0 = \lambda(g_{ij})u_0$$

is satisfied with L^2-norm of u_0 equals 1 and $f_0 = -2\log u_0$ is a minimizer. $\lambda(g_{ij}) = \mathcal{F}(g_{ij}, f_0)$ and f_0 satisfy the equation

$$-2\Delta f_0 + |\nabla f_0|^2 - R = -\lambda(g_{ij}).$$

Below are the properties of the functional λ on a closed manifold (For detail see [68, pp. 206]).

1. **Lower bound for λ.**

   $$\mathcal{F}(g_{ij}(t), f) \geq \min_{x\in M} R(x) \int_M e^{-f} d\mu = \min_{x\in M} R(x) = R_{min}$$

   and in particular

   $$\lambda(g) \geq R_{min}.$$

2. **Diffeomorphism invariance.**

   If $\phi : M \to M$ is a diffeomorphism, then

   $$\lambda(\phi^* g) = \lambda(g).$$

3. **Existence of a smooth minimizer.**

   There exists $f \in C^\infty(M)$ with $\int_M e^{-f} d\mu = 1$ such that

$$\lambda(g) = \mathcal{F}(g, f) \quad i.e, \qquad \lambda(g) = \int_M (R + |\nabla f|^2) e^{-f} d\mu.$$

4. **Upper bound for λ**

$$\lambda(g) \leq \frac{1}{Vol(g)} \int_M R d\mu.$$

   This can be seen by choosing $f = \log Vol(g_{ij})$ such that

$$\int_M e^{-f} d\mu = 1 \quad and \quad \lambda(g) \leq \int_M (R + |\nabla f|^2) e^{-f} d\mu.$$

5. **Scaling.**

$$\lambda(cg) = c^{-1} \lambda(g).$$

On the Monotonicity of λ. The monotonicity of $\mathcal{F}$ implies the monotonicity of λ under the Ricci flow.

Lemma 2.2.3. *([64, Lemma 5.25] If $g_{ij}(t), t \in [0, T]$ is a solution to the Ricci flow, then,*

$$\frac{d}{dt} \lambda(g_{ij}(t)) \geq \frac{2}{n} \lambda^2(g_{ij}(t))$$

and $\lambda(g_{ij}(t))$ is nondecreasing in $t \in [0, T]$.

Proof. Let f_0 be a minimizer of $(g_{ij}(t_0), f_0)$ for any $t_0 \in [0, T]$, so that

$$\lambda(g_{ij}(t_0)) = \mathcal{F}(g_{ij}(t_0), f_0). \tag{2.2.16}$$

Solving the backward heat equation

$$\begin{cases} \dfrac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R. \\ \\ f(t_0) = f_0 \end{cases} \tag{2.2.17}$$

on $[0, T]$, then

$$\frac{d}{dt} \mathcal{F}(g_{ij}(t), f(t)) \geq 0, \qquad for \;\; all \;\; time \;\; t \leq t_0. \tag{2.2.18}$$

We then have that

$$\lambda(g_{ij}(t)) \leq \mathcal{F}(g_{ij}(t), f(t)) \qquad for \quad t \leq t_0 \tag{2.2.19}$$

since the constraint $\int_M e^{-f} d\mu$ is preserved under (2.2.17). We now recall the monotonicity formula for $\mathcal{F}$ under the Ricci flow

$$\frac{d}{dt} \mathcal{F}(g_{ij}(t), f(t)) = 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu. \tag{2.2.20}$$

By (2.2.16) - (2.2.19), we have

$$\lambda(g_{ij}(t)) \leq \mathcal{F}(g_{ij}(t), f(t)) \leq \mathcal{F}(g_{ij}(t_0), f_0) = \lambda(g_{ij}(t_0)) \tag{2.2.21}$$

and

$$\begin{aligned}
\frac{d}{dt}\lambda(g_{ij}(t))\Big|_{t=t_0} &\geq \frac{d}{dt}\mathcal{F}(g_{ij}(t), f(t))\Big|_{t=t_0} \\
&= 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu_{g(t_0)} \\
&\geq 2\int_M \frac{1}{n}(R + \Delta f)^2 e^{-f} d\mu_{g(t_0)} \\
&\geq \frac{2}{n}\Big(\int_M (R + \Delta f) e^{-f} d\mu_{g(t_0)}\Big)^2 \\
&= \frac{2}{n}\lambda^2(g_{ij}(t_0)),
\end{aligned} \tag{2.2.22}$$

where $f = f_0$ is the minimizer. Hence, it is clear from the above that $\lambda(g_{ij}(t))$ is nondecreasing under the Ricci flow. □

Definition 2.2.4. *(***Breathers***): A metric $g_{ij}(t)$ which solves the Ricci flow is called a breather if for some t_1, t_2, such that $t_1 < t_2$, the metric $g_{ij}(t_2) = \alpha\phi_t^* g_{ij}(t_1)$ for some constant $\alpha > 0$ and diffeomorphism $\phi_t : M \to M$. The cases $\alpha < 1, \alpha = 1$ and $\alpha > 1$ correspond to shrinking, steady and expanding breathers. Steady, shrinking or expanding Ricci solitons are trivial breathers for which metric $g_{ij}(t_1)$ and $g_{ij}(t_2)$ differ only by diffeomorphism and scaling for t_1 and t_2.*

Remark 2.2.5. *If we consider the Ricci flow as a dynamic system on the space of Riemannnian metrics modulo diffeomorphism and scaling, the Ricci breathers correspond to the periodic orbits while the Ricci soliton are fixed points. Since the Ricci flow is a heat-type equation, we expect that there are no periodic orbits except fixed points. For example, Ricci flat metric is a steady gradient soliton (i.e., a fixed point of a dynamic system). e. g. Hamilton's Cigar soliton on the 2-dimensional manifold $\Sigma = \mathbb{R}^2$ with conformal metric*

$$g_\Sigma = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

and the gradient function $f := \log\sqrt{1 + x^2 + y^2}$.

An important example of gradient shrinking soliton is the Gaussian soliton for which the metric g_{ij} is just the Euclidean metric on $\mathbb{R}^n$, $\alpha = 1$ and $f = -\frac{|x|^2}{2}$.

Proposition 2.2.6. *Let $g_{ij}(t)$ be a solution of the Ricci flow, we have*

1. *$\lambda(\phi_t^* g_{ij}) = \lambda(g_{ij})$ for any diffeomorphism $\phi_t : M \to M$.*

2. *$\lambda(g_{ij}(t))$ is nondecresing and the monotonicity is strict unless $R_{ij}+\nabla_i\nabla_j f=0$.*

3. *A steady breather is necessarily a steady gradient soliton.*

Proof. 1. $\lambda(\phi_t^* g_{ij}) = \inf \mathcal{F}(\phi_t^* g_{ij}, f)) = \inf \mathcal{F}(g_{ij}, f) = \lambda(g_{ij})$.

2. Let f_0 be a minimizer for any t_0 with $\int_M e^{-f} d\mu = 1$, we solve the backward heat equation

$$\begin{cases} \dfrac{\partial f}{\partial t} = -\Delta f - R + |\nabla f|^2 \\ \\ f(t_0) = f_0 \end{cases} \tag{2.2.23}$$

and obtain a solution $f(t)$ for $t \leq t_0$ which satisfies $\int_M e^{-f} d\mu = 1$ and the monotonicity formula (2.2.13). Since $\mathcal{F}(g_{ij}, f)$ is nondecreasing, then we have

$$\lambda(g_{ij}(t)) \leq \mathcal{F}(g_{ij}(t), f(t)) \leq \mathcal{F}(g_{ij}(t_0), f_0) = \lambda(g_{ij}(t_0)). \tag{2.2.24}$$

Suppose the monotonicity is not strict, i.e., for any solution $g_{ij}(t)$ for a Ricci flow, there exists $t_1 < t_2$ such that

$$\lambda(g_{ij}(t_2)) = \lambda(g_{ij}(t_1)).$$

Let $f(t_2)$ be the minimizer of $\mathcal{F}$ at time t_2 so that

$$\lambda(g_{ij}(t_2)) = \mathcal{F}(g_{ij}(t_2), f(t_2)).$$

Assuming that $f(t_2)$ solves the backward heat equation

$$\begin{cases} \dfrac{\partial f}{\partial t} = -\Delta f - R + |\nabla f|^2 \\ \\ f(t_2) = f_2 \end{cases} \tag{2.2.25}$$

for $t \in [t_1, t_2]$.

By monotonicity formula for $\mathcal{F}$ and the definition of λ, we have

$$\lambda(g_{ij}(t_1)) \leq \mathcal{F}(g_{ij}(t_1), f(t_1)) \leq \mathcal{F}(g_{ij}(t_2), f_2) = \lambda(g_{ij}(t_2)) \tag{2.2.26}$$

for all $t \in [t_1, t_2]$.

Since $\lambda(g_{ij}(t_1)) = \lambda(g_{ij}(t_2))$ and $\lambda(g_{ij}(t))$ is monotone, we have

$$\mathcal{F}(g_{ij}(t), f(t)) = \lambda(g_{ij}(t)) \equiv \ const. \quad for\ t \in [t_1, t_2].$$

Therefore $f(t)$ is the minimizer for $\mathcal{F}(g_{ij}(t))$ and $\frac{d}{dt}\mathcal{F}(g_{ij}(t), f(t)) \equiv 0$. Hence we have

$$\int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu \equiv 0$$

for all $t \in [t_1, t_2]$. Thus, $R_{ij} + \nabla_i \nabla_j f = 0$ for all $t \in [t_1, t_2]$.

3. We recall that gradient Ricci soliton satisfies

$$R_{ij}(g) + \nabla_i \nabla_j f = \alpha g_{ij},$$

then a steady breather (case $\alpha = 0$) is necessarily a steady gradient Ricci soliton.

□

2.2.3 Nonexistence of Nontrivial expanding Breathers.

This case is more subtle than the previous, to deal with it we need a scale invariant version

$$\tilde{\lambda}(g_{ij}) = \lambda(g_{ij}) V^{\frac{2}{n}}(g_{ij}),$$

here, $V = Vol(g_{ij})$ is the volume of the manifold M. The reason for this is that $\lambda(g)$ is not scale invariant, e.g., $\lambda(cg) = c^{-1}\lambda(g)$. Thus $\tilde{\lambda}(g_{ij})$ is normalized and we can see that $\tilde{\lambda}(cg) = \tilde{\lambda}(g)$ for any $c > 0$. So the invariant $\tilde{\lambda}$ is potentially useful for expanding and shrinking breathers. The scaling invariance is shown as follows; consider the scaling $\tilde{g} = c \cdot g$ and $\tilde{f} := f + \frac{n}{2}\ln c$ for any $c > 0$. We need to scale f since it must satisfy the normalization constraint $\int_M e^{-\tilde{f}} d\tilde{\mu} = \int_M e^{-f} d\mu = 1$. Then we scale

$$d\tilde{\mu} = \sqrt{det(\tilde{g})}dx = \sqrt{det(c \cdot g)}dx = \sqrt{c^n det(g)}dx = c^{\frac{n}{2}}\sqrt{det(g)}dx = c^{\frac{n}{2}} d\mu,$$

$$\int_M e^{-\tilde{f}} d\tilde{\mu} = \int_M e^{-f} e^{\ln c^{-n/2}} c^{n/2} d\mu = \int_M e^{-f} d\mu = 1, \quad \tilde{R} := R(\tilde{g}) = c^{-1}R(g)$$

$$\textit{and} \ \ |\nabla \tilde{f}|^2_{\tilde{g}} = \tilde{g}^{ij}\nabla_i \tilde{f} \nabla_j \tilde{f} = (cg^{ij})\nabla_i f \nabla_j f = c^{-1}|\nabla f|^2_g, \ \ \textit{since} \ \ \nabla_i \tilde{f} = \nabla_i f.$$

Using the above scalings we calculate

$$\begin{aligned}
\tilde{\lambda}(g(t)) = V^{\frac{2}{n}}(\tilde{g}) \cdot \lambda(\tilde{g}) &= \Big(\int_M d\tilde{\mu}\Big)^{\frac{2}{n}} \cdot \inf_f \left\{ \int_M \Big(R(\tilde{g}) + |\nabla \tilde{f}|^2_{\tilde{g}}\Big) e^{-\tilde{f}} d\tilde{\mu} \ \ with \int_M e^{-\tilde{f}} = 1 \right\} \\
&= \Big(\int_M c^{\frac{n}{2}} d\mu\Big)^{\frac{2}{n}} \cdot \inf_f \left\{ \int_M c^{-1}\Big(R + |\nabla f|^2_g\Big) c^{-n/2} e^{-f} c^{n/2} d\mu \ \ with \int_M e^{-f} e^{\ln c^{-n/2}} c^{n/2} d\mu = 1 \right\} \\
&= c^{\frac{n}{2}\cdot\frac{2}{n}} \Big(\int_M d\mu\Big)^{\frac{2}{n}} \cdot \inf_f \left\{ c^{-1}\int_M \Big(R + |\nabla f|^2_g\Big) e^{-f} d\mu \ \ with \ \ \int_M e^{-f} d\mu = 1 \right\} \\
&= V^{\frac{2}{n}}(g)\lambda(g).
\end{aligned}$$

We notice that the quantity $\tilde{\lambda}(g(t))$ is not monotone in general. Next we state and prove a result for the monotonicity of $\tilde{\lambda}(g(t))$ under Ricci flow when it is nonpositive.The proof is included for completeness, Cf [64] .

Proposition 2.2.7. *Let $\tilde{\lambda}(g_{ij}(t))$ be as described above;*

1. *$\tilde{\lambda}(g_{ij})$ is non decreasing along the Ricci flow whenever it is nonpositive and the monotonicity is strict unless $g_{ij}(t)$ is expanding gradient soliton.*

2. *An expanding breather is necessarily an expanding gradient soliton.*

Proof. Let $f(t)$ solve the backward heat equation

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R$$

at $t \leq t_0$ with $\int_M e^{-f} d\mu = 1$ and f_0 be a minimizer of

$$\lambda(g_{ij}(t)) = \inf \mathcal{F}(g_{ij}(t), f(t)), \qquad t = t_0.$$

From equation (2.2.22), we have that

$$\frac{d}{dt}\lambda(g_{ij}(t)) \geq \frac{d}{dt}\mathcal{F}(g_{ij}(t), f(t))\Big|_{t=t_0} = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu_{g(t_0)}.$$

Since $\tilde{\lambda}$ is Lipshchitz continuous, the time derivative exists in the sense of forward difference quotients. Hence we compute

$$\begin{aligned}
\frac{d}{dt}\tilde{\lambda}(g_{ij}(t)) &= \frac{d}{dt}\Big(\lambda(g_{ij}(t))V^{\frac{2}{n}}(g_{ij}(t))\Big) \\
&= V^{\frac{2}{n}}(g_{ij}(t))\frac{d}{dt}\lambda(g_{ij}(t)) + \lambda(g_{ij}(t))\frac{d}{dt}\Big(V^{\frac{2}{n}}(g_{ij}(t))\Big) \\
&\geq V^{\frac{2}{n}}\int_M 2|R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu + \frac{2}{n}V^{\frac{2-n}{n}}\lambda\frac{dV}{dt} \\
&= 2V^{\frac{2}{n}}\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu + \frac{2}{n}V^{\frac{2-n}{n}}\int_M (R+\Delta f)e^{-f} d\mu \int_M (-R) d\mu \\
&= 2V^{\frac{2}{n}}\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu - \frac{2}{n}V^{\frac{2-n}{n}}\int_M R d\mu \int_M (R+\Delta f)e^{-f} d\mu,
\end{aligned}$$

where we have used the identity $dV/dt = -\int_M R d\mu$ (Cf Lemma 1.3.1) and $\lambda = \inf \mathcal{F} = \int_M (R+\Delta f)e^{-f} d\mu$. Hence

$$\begin{aligned}
\frac{1}{2}V^{-\frac{2}{n}}\frac{d}{dt}\tilde{\lambda} &\geq \int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu - \frac{1}{n}\int_M (R+\Delta f)e^{-f} d\mu \cdot V^{-1}\int_M R d\mu \\
&= \int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{n}(R+\Delta f)g_{ij}|^2 e^{-f} d\mu + \frac{1}{n}\int_M (R+\Delta f)^2 e^{-f} d\mu \\
&\quad - \frac{1}{n}\int_M (R+\Delta f)e^{-f} d\mu \cdot \frac{\int_M R d\mu}{V} = RHS.
\end{aligned}$$

The last inequality is due to the following identity;

$$|R_{ij} + \nabla_i\nabla_j f|^2 = |R_{ij} + \nabla_i\nabla_j f - \frac{1}{n}(R+\Delta f)g_{ij}|^2 + \frac{1}{n}(R+\Delta f)^2.$$

Recall the upper bound of λ i.e, $\lambda \leq \frac{1}{V}\int_M R d\mu$ which implies

$$\lambda = \int_M (R+\Delta f)^2 e^{-f} d\mu \leq \frac{1}{V}\int_M R d\mu.$$

Suppose $\lambda(g_{ij}(t_0)) \leq 0$, then $\int_M (R+\Delta f)^2 e^{-f} d\mu \leq 0$ and the last term in the RHS becomes

$$\begin{aligned}
-\frac{1}{n}\int_M (R+\Delta f)e^{-f} d\mu \cdot \frac{\int_M R d\mu}{V} &\geq \frac{1}{n}\Big(-\int_M (R+\Delta f)e^{-f} d\mu\Big)\Big(\int_M (R+\Delta f)e^{-f} d\mu\Big) \\
&= -\frac{1}{n}\Big(\int_M (R+\Delta f)e^{-f} d\mu\Big)^2.
\end{aligned}$$

Therefore at $t = t_0$, we have

$$\begin{aligned}\frac{1}{2}V^{-\frac{2}{n}}\frac{d}{dt}\tilde{\lambda} &\geq \int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{n}(R+\Delta f)g_{ij}|^2 e^{-f}d\mu \\ &+ \frac{1}{n}\int_M (R+\Delta f)^2 e^{-f}d\mu - \frac{1}{n}\Big(\int_M (R+\Delta f)e^{-f}d\mu\Big)^2 \\ &\geq \int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{n}(R+\Delta f)g_{ij}|^2 e^{-f}d\mu\end{aligned}$$

since $\int_M e^{-f}d\mu = 1$, hence

$$\frac{d}{dt}\tilde{\lambda} \geq 2V^{\frac{2}{n}}\int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{n}(R+\Delta f)g_{ij}|^2 e^{-f}d\mu \geq 0.$$

This ends the proof of the first part of the theorem, noticing that equality holds if and only if

$$R_{ij} + \nabla_i\nabla_j f - \frac{1}{n}(R+\Delta f)g_{ij} = 0.$$

Thus, $g(t)$ is an expanding gradient soliton.

Next, we consider the evolution of the volume and we would necessarily have

$$\frac{dV}{dt} > 0, \qquad for\ some\ t \in [t_1, t_2].$$

Let $g(t)$ be an expanding breather on $[t_1, t_2]$ with $g(t_2) = \alpha\phi^* g(t_1)$, where $\alpha > 1$ and $\phi : M \to M$ is a diffeomorphism, We know that $V(g(t_2)) > V(g(t_1))$ and for some $t_1 < t_0 < t_2$

$$0 \leq \frac{d}{dt}\log V\Big|_{t=t_0} = \frac{-\int_M Rd\mu}{V(g(t_0))} \leq -\lambda(g(t_0))$$

by definition of $\lambda(g_{ij}(t))$. It follows that on an expanding breather on $[t_1, t_2]$,

$$\tilde{\lambda}(g(t)) = \lambda(g(t))V^{\frac{2}{n}}(g(t)) < 0, \quad for\ some\ t \in [t_1, t_2].$$

By statement 1, $\tilde{\lambda}(g(t))$ is increasing whenever it is negative, we then have

$$\tilde{\lambda}(g(t_1)) < \tilde{\lambda}(g(t_2)) < 0\ for\ all\ t \in [t_1, t_2]$$

unless we are on an expanding gradient soliton.

But the diffeomorphism and scaling invariance of $\tilde{\lambda}(g(t))$ imply

$$\tilde{\lambda}(g(t_1)) = \tilde{\lambda}(g(t_2)).$$

Therefore an expanding breather must be an expanding gradient soliton. □

We conclude this section with the following

Corollary 2.2.8. *([64, pp.213]). Expanding or steady breathers on a compact manifold are necessarily Einstein.*

Proof. From statement 1 of Proposition 2.2.7, we noticed that

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{n}(R + \Delta f) g_{ij} = 0$$

for an expanding breather $g(t)$. Since the monotonicity is strict, we have

$$R + \Delta f = C_1(t), \tag{2.2.27}$$

where C_1 is a constant depending on time t. This is enough to conclude that $g(t)$ could be steady gradient Ricci soliton (equivalently to steady breathers). We now suppose $g(t)$ is an expanding or steady gradient soliton. Combining (5.5.8) with the fact that

$$2\Delta f + R - |\nabla f|^2 = C_2(t),$$

(the last equation follows from the structure of gradient Ricci solitons [64, Proposition 1.15]), where C_2 is a constant depending on t. We have

$$\Delta f - |\nabla f|^2 = C_2.$$

Since $-\int \Delta(e^{-f}) d\mu = 0 = \int (\Delta f - |\nabla f|^2) e^{-f} d\mu$, then

$$\Delta f - |\nabla f|^2 \equiv 0.$$

Thus by strong maximum principle, we conclude that $f \equiv const.$ or since $0 = \int (\Delta f - |\nabla f|^2) e^{-f} d\mu = -2\int |\nabla f|^2 e^{-f} d\mu$, $f \equiv const.$ Hence $R_{ij} - \frac{1}{n} R g_{ij} = 0$ and g_{ij} is Einstein. (When $n = 2$, our conclusion is vacuous). □

Remark 2.2.9. *As a corollary of the above, we again see that expanding or steady solutions on closed manifolds are Einstein. In the case of shrinking solitons on closed manifolds, using the entropy functional, we shall see in a later section that they are necessarily gradient shrinking solitons.*

2.3 A New Family of Entropy Functionals

2.3.1 $\mathcal{B}$-Energy Functional

To circumvent the difficulty encounter under Einstein-Hilbert functional, we can replace the evolving measure $d\mu$ by some static measure dm and define a new functional

$$\mathcal{B} = \int_M R dm.$$

Now

$$\frac{d\mathcal{B}}{dt} = \int_M \left[(\Delta R + 2|R_{ij}|^2) dm + R \frac{\partial}{\partial t} dm \right] \tag{2.3.1}$$

since dm is static, we cannot apply divergence theorem which applies to evolving measure, we then set $dm := e^{-f}d\mu$ for scalar function $f : M \to \mathbb{R}$ and therefore obtain

$$\begin{aligned}\frac{d\mathcal{B}}{dt} &= \int_M (\Delta R + 2|R_{ij}|^2 - R\frac{\partial}{\partial t}f - R^2)e^{-f}d\mu \\ &= \int_M \Big[(\Delta R + 2|R_{ij}|^2 - R(-\Delta f + |\nabla f|^2 - R) - R^2\Big]e^{-f}d\mu \\ &= 2\int_M |R_{ij}|^2 e^{-f}d\mu + \int_M \Delta R e^{-f}d\mu + \int_M R(\Delta f - |\nabla f|^2)e^{-f}d\mu \\ &= 2\int_M |R_{ij}|^2 e^{-f}d\mu,\end{aligned}$$

where $\int_M \Delta R e^{-f}d\mu = \int_M R\Delta e^{-f}d\mu = \int_M R(-\Delta f + |\nabla f|^2)e^{-f}d\mu$ by using integration by parts.

Then, even by inspection, if the modified Ricci flow $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - 2\nabla_i\nabla_j f$ is considered as an L^2-gradient flow of Perelman's energy functional $\mathcal{F}$, we can easily conclude that the Ricci flow $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$ is also an L^2-gradient flow of our functional $\mathcal{B}$.

Theorem 2.3.1. *Let $(M^n, g_{ij}(t)), t \in [0, T)$ be a solution of the Ricci flow, then*

$$\frac{d}{dt}\mathcal{B}(g_{ij}, f) = 2\int_M |R_{ij}|^2 e^{-f}d\mu, \tag{2.3.2}$$

where $f = log\Big(\frac{d\mu}{dm}\Big)$ and satisfies

$$\frac{\partial}{\partial t}f = -\Delta f + |\nabla f|^2 - R. \tag{2.3.3}$$

In particular $\mathcal{B}(g_{ij}, f)$ is monotonically nondecreasing in time without sign assumption on the curvature operator and the monotonicity is strict unless $R_{ij} \equiv 0$. Moreover, there is no nontrivial Ricci breather except gradient steady Ricci soliton, which is necessarily flat.

Proof.

$$\frac{\partial}{\partial t}f = \frac{\partial}{\partial t}log\Big(\frac{d\mu}{dm}\Big) = \frac{1}{2}tr\Big(\frac{\partial}{\partial t}g_{ij}\Big) = \frac{1}{2}g^{ij}\Big[-2(R_{ij} + \nabla_i\nabla_j f)\Big] = -R - \Delta f.$$

Modulo the diffeomorphism out of $\frac{\partial}{\partial t}g_{ij} = -2(R_{ij} + \nabla_i\nabla_j f)$,

$$\frac{\partial}{\partial t}f = -\Delta f + |\nabla f|^2 - R$$

Then,

$$\frac{d}{dt}\mathcal{B}(g_{ij}, f) = 2\int_M |R_{ij}|^2 e^{-f}d\mu \geq 0,$$

where equality holds if and on if $R_{ij} \equiv 0$. This implies that $(M^n, g_{ij}(t))$ is Ricci flat (steady gradient Ricci soliton). □

2.3.2 The Entropy Formula and its Monotonicity

In this section, we construct a new entropy formula for the Ricci flow, the motivations for this are the behaviour of our functional $\mathcal{B}$ (Theorem 2.3.1) under the Ricci flow modulo diffeomorphism invariance and the classical results for Dirichlet energy functional for heat flow on Riemannian manifolds. It is well known that a typical heat equation for a function $f : M^n \times [0, \infty) \to \mathbb{R}$ on an n-compact manifold M (possibly without boundary) is a gradient flow for the classical Dirichlet energy functional

$$E(f) := \frac{1}{2}\int_{M^n} |\nabla f|^2 d\mu, \tag{2.3.4}$$

since there is natural L^2-inner product on S^2T^*M. An application of this is that any periodic (breather) solutions to the heat equation are harmonic function which in fact must be constant in M. The Li-Yau gradient estimate for the heat equation on complete Riemannian manifold suggests an entropy formula which was derived in [122] but proved to be monotone decreasing with non-negativity condition on Ricci curvature.

Definition 2.3.2. *Let (M^n, g) be a closed n-dimensional Riemannian Manifold, $f : M^n \to \mathbb{R}$ be a smooth function on M^n, define a functional on pairs (g_{ij}, f) by*

$$\mathcal{F}_{\mathcal{B}} = \int_M \Big(\frac{1}{2}|\nabla f|^2 + R\Big) dm, \tag{2.3.5}$$

where $dm := e^{-f} d\mu$.

The functional $\mathcal{F}_{\mathcal{B}}$ is a variant of Perelman's energy functional $\mathcal{F}$, though expected to behave in a similar manner, it differs from the later by the introduction of constant $\frac{1}{2}$.

Let $\delta g_{ij} = h_{ij}$ and $\delta f = K$, where $H = tr_g h_{ij}$, we have the first variation of $\mathcal{F}_{\mathcal{B}}$ as

$$\delta\mathcal{F}_{\mathcal{B}} = \int_M -h_{ij}(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2}\nabla_i f \nabla_j f) dm. \tag{2.3.6}$$

The coupled modified Ricci flow equation with a backward heat equation

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i\nabla_j f - \dfrac{1}{2}\nabla_i f\nabla_j f) \\[2ex] \dfrac{\partial f}{\partial t} = -R - \Delta f + \dfrac{1}{2}|\nabla f|^2 \end{cases} \tag{2.3.7}$$

is a gradient flow. Conjugating away the infinitesimal diffeomorphism converts (2.3.6) to (2.2.12).

Theorem 2.3.3. *Let $g_{ij}(t)$ and f solve the system (2.2.12) in the interval $[0, T)$, then,*

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}}(g_{ij}, f) = \int_M |R_{ij} + \nabla_i\nabla_j f|^2 dm + \int_{M^n} |R_{ij}|^2 dm. \tag{2.3.8}$$

Showing that $\mathcal{F}_{\mathcal{B}}(g_{ij}, f)$ is monotonically non-decreasing in time, however, the monotonicity is strict, unless $R_{ij} \equiv 0$ and f is a constant.

Proof.

$$\mathcal{F}_{\mathcal{B}} = \int_M \Big(\frac{1}{2}|\nabla f|^2 + R\Big) e^{-f} d\mu = \frac{1}{2}\int_M (|\nabla f|^2 + R) e^{-f} d\mu + \frac{1}{2}\int_M R e^{-f} d\mu,$$

therefore

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}}(g_{ij}, f) = \frac{1}{2}\frac{d}{dt}\mathcal{F} + \frac{1}{2}\frac{d}{dt}\mathcal{B}.$$

The result then follows. □

Definition 2.3.4. *Let (M^n, g) be a closed n-dimensional Riemannian Manifold, define a family of functional $\mathcal{F}_{\mathcal{B}C}$ as*

$$\mathcal{F}_{\mathcal{B}C} = \int_M (|\nabla f|^2 + 2CR)dm, \tag{2.3.9}$$

where $C \geq \frac{1}{2}, C \in \mathbb{R}$. When $C = \frac{1}{2}$, this is Perelman's $\mathcal{F}$ functional [126], $C = 1$ is a specific case we considered and $C = \frac{1}{2}k, k \geq 1$, we have Li-$\mathcal{F}_k$-family [108].

Remark 2.3.5. *Our functional $\mathcal{F}_{\mathcal{B}C}$ is a variant of Perelman functional which uses certain multiple of Dirichlet energy. Their monotonicities are consistent with each other.*

Theorem 2.3.6. *Let $(M^n, g_{ij}(t)), t \in [0, T)$ be a solution of the Ricci flow and f evolves by a conjugate heat equation or satisfies $e^{-f} = \frac{dm}{d\mu}$, then, under the coupled system (2.2.12), $\mathcal{F}_{\mathcal{B}C}$ is monotonically non-decreasing. In particular, we have*

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}C} = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 dm + 2(2C-1)\int_M |R_{ij}|^2 dm \geq 0. \tag{2.3.10}$$

Moreover, the monotonicity is strict unless $R_{ij} + \nabla_i\nabla_j f \equiv 0$, i.e., there is no nontrivial breathers except steady gradient Ricci soliton and the gradient function f is constant.

This shows that all steady breathers are gradient steady Ricci soliton with $f = 0$. An example of this is Hamilton cigar soliton (2- dimensional $\mathbb{R}^2$) with conformal metric $ds^2 = \frac{dx^2+dy^2}{1+x^2+y^2}$ and the gradient function $f = log\sqrt{1 + x^2 + y^2}$.

Proof. The proof follows from a direct computation based on the previous results.

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_{\mathcal{B}C} &= \frac{d}{dt}\int_M (|\nabla f|^2 + 2CR)dm \\
&= \frac{d}{dt}\Big(\int_M (|\nabla f|^2 + R)dm + (2C-1)\int_M Rdm\Big) \\
&= \frac{d}{dt}\mathcal{F} + (2C-1)\frac{d}{dt}\mathcal{B}.
\end{aligned}$$

The monotonicity formula (2.3.10) follows at once. Therefore

$$\frac{d}{dt}\mathcal{F}_{\mathcal{B}C}(g_{ij}, f) \equiv 0$$

if and only if $R_{ij} \equiv 0$ and f is a constant. □

2.3.3 Eigenvalues and their monotonicity

In this subsection, we discuss the monotonicity properties of the least eigenvalue of a self adjoint modified operator $-2\Delta + CR$ that occurs in our functional. This is important as it enables us gain controlled geometric quantity for the Ricci flow.

$$\mu_C(g_{ij}) = \inf\Big\{\mathcal{F}_{\mathcal{B}C}(g_{ij}, f) : f \in C_c^\infty(M), \int_M e^{-f}d\mu = 1\Big\}, \tag{2.3.11}$$

where the infimum is taken over all smooth functions f. The normalisation $\int_M e^{-f} d\mu = 1$ makes dm a probability measure and ensures a meaningful infimum.

Setting $e^{-f} =: u^2$, then, the functional $\mathcal{F}_{\mathcal{B}C}$ can be written in terms of u as

$$\mathcal{F}_{\mathcal{B}C} = \int_M (2|\nabla u|^2 + CRu^2) d\mu, \qquad with \int_M u^2 d\mu = 1. \tag{2.3.12}$$

Then $\mu_C(g_{ij}) = \lambda_1(-2\Delta + CR)$ is the least eigenvalue of the self-adjoint modified operator $(-2\Delta + CR)$. Let v be the corresponding eigenfunction, then, we have

$$-2\Delta v + CRv = \mu_C(g_{ij})v$$

and $f_C = -2 \, log \, v$ is a minimiser of

$$\mu_C(g_{ij}) = \mathcal{F}_{\mathcal{B}C}(g_{ij}, f_C).$$

By standard existence and regularity theories, the minimising sequence always exists.

Theorem 2.3.7. *Let $(M^n, g_{ij}(t)), t \in [0, T)$ be a solution of the Ricci flow, then, the least eigenvalue $\mu_C(g_{ij})$ of $(-2\Delta + CR)$ is diffeomorphism invariant and non-decreasing. The monotonicity is strict unless the metric is a steady gradient soliton.*

Proof. Let $\phi : M \to M$ be a one parameter family of diffeomorphism. For any diffeomorphism $\phi(t)$ we have

$$\mathcal{F}_{\mathcal{B}C}(\phi_t^* g_{ij}, f \circ \phi) = \mathcal{F}_{\mathcal{B}C}(g_{ij}, f),$$

then,

$$\mu_C(\phi_t^* g_{ij}(t)) = \mathcal{F}_{\mathcal{B}C}(\phi_t^* g_{ij}, f_C) = \mathcal{F}_{\mathcal{B}C}(g_{ij}(t), f_C) = \mu_C(g_{ij}(t)).$$

Solving the backward heat equation at any time $t \in [0, t_0)$ with initial condition $f(t_0) = f_0$, we know that f_0 is a minimizer with $\int_M e^{-f} d\mu = 1$. So our solution $f(t), t < t_0$ which satisfies $e^{-f} d\mu$ is also a minimizer. By Theorem 2.3.6, $\mathcal{F}_{\mathcal{B}C}(g_{ij}, f_c)$ is non-decreasing, then we have

$$\mu_C(g_{ij}(t)) = \inf \mathcal{F}_{\mathcal{B}C}(g_{ij}(t), f(t)) \leq \inf \mathcal{F}_{\mathcal{B}C}(g_{ij}(t_0), f(t_0)) = \mu_C(g_{ij}(t_0)).$$

Thus, μ_C is nondecreasing under the coupled Ricci flow. Suppose the monotonicity is not strict, then, for some times $t_1, t_2, t_1 < t_2$, the solution $g_{ij}(t)$ of the Ricci flow satisfies

$$\mu_C(g_{ij}(t_1)) = \mu_C(g_{ij}(t_2)).$$

If $f(t_1)$ is a minimizer of $\mathcal{F}_{\mathcal{B}C}(g_{ij}(t), f_c)$ at time t_1, so that

$$\mu_C(g_{ij}(t_1)) = \mathcal{F}_{\mathcal{B}C}(g_{ij}(t_1), f(t_1)).$$

But by the monotonicity of $\mathcal{F}_{\mathcal{B}C}$

$$\mathcal{F}_{\mathcal{B}C}(g_{ij}(t_1), f(t_1)) \leq \mathcal{F}_{\mathcal{B}C}(g_{ij}(t_2), f(t_2)), \quad t_1 < t_2,$$

$$= \mu_C(g_{ij}(t_2)).$$

The above inequality implies that

$$\frac{d}{dt}\mu_C(g_{ij}(t))\Big|_{t=t_2} \geq \frac{d}{dt}\mathcal{F}_{\mathcal{B}C}(g_{ij}(t_1), f(t_1)) \geq 0,$$

hence, the last part of the theorem follows clearly. □

We conclude this section with the fact that there is no compact steady Ricci breather other than Ricci flat metric, this is due to the diffeomorphism invariance of the eigenvalues. More details can be found in [42, Theorem 3], [87, 99], [108, Theorem 55], [126].

2.4 Monotonicity Formula under the Normalized Ricci Flow

The normalized Ricci flow (NRF) is given [68] as

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -2\tilde{R}_{ij} + \frac{2}{n}r\tilde{g}_{ij}, \tag{2.4.1}$$

where $r = (Vol_{\tilde{g}})^{-1}\int_M \tilde{R}d\tilde{\mu}$ is a constant, the average of the scalar curvature of M, and $Vol_{\tilde{g}} = \int_M d\tilde{\mu}$. The factor r appearing in (2.4.1) keeps the volume of the manifold constant. Here, we extend the results from previous sections (Theorems 2.3.1, 2.3.3, 2.3.6 and 2.3.7) to the case of the normalized Ricci flow. We recall that there is a bijection between the Ricci flow (2.1.1) and the NRF (2.4.1), if we choose a normalization factor $\phi := \phi(t)$ with $\phi(0) = 1$ such that $\tilde{g}(t) = \phi(t)g(t)$ and define a time scale $\tilde{t} = \int_0^t \phi(\tau)d\tau$, then $\tilde{g}(t)$ solves (2.4.1) whenever $g(t)$ solves (2.1.1).

Remark 2.4.1. *If $r = 0$, all the properties of the Ricci flow (2.1.1) including the monotonicity of the eigenvalues of Laplacian hold without further alteration.*

2.4.1 Monotonicity of the Entropy Formula

In this section, we extend some results in Section 2.3 to the case of NRF. Define a modified normalized Ricci flow by

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -2\tilde{R}_{ij} + \frac{2}{n}r\tilde{g}_{ij} - 2\tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}$$

and $\tilde{f} = log\left(\frac{dm}{d\tilde{\mu}}\right)$ i.e.,

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial t} &= \frac{1}{2}tr_g\frac{\partial}{\partial t}\tilde{g}_{ij} = \frac{1}{2}\tilde{g}^{ij}(-2\tilde{R}_{ij} + \frac{2}{n}rg_{ij} - 2\tilde{\nabla}_i\tilde{\nabla}_j\tilde{f})\\ &= -\tilde{R} + r - \tilde{\Delta}\tilde{f}.\end{aligned}$$

It is however clear that the coupled system

$$\begin{cases} \dfrac{\partial \tilde{g}_{ij}}{\partial t} = -2(\tilde{R}_{ij} - \dfrac{r}{n}\tilde{g}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}) \\ \\ \dfrac{\partial \tilde{f}}{\partial t} = -\tilde{\Delta}\tilde{f} - \tilde{R} + r \end{cases} \tag{2.4.2}$$

is equivalent to (using the same idea of conjugating away the infinitesimal diffeomorphism as was used to convert (2.2.11) to (2.2.12))

$$\begin{cases} \dfrac{\partial \tilde{g}_{ij}}{\partial t} = -2\tilde{R}_{ij} + \dfrac{2}{n}r\tilde{g}_{ij} \\ \\ \dfrac{\partial \tilde{f}}{\partial t} = -\tilde{\Delta}\tilde{f} + |\nabla f|^2 - \tilde{R} + r. \end{cases} \tag{2.4.3}$$

Now using Perelman's energy functional, $\tilde{\mathcal{F}} = \phi\mathcal{F}$ i.e., $\tilde{\mathcal{F}} = \int_M (|\tilde{\nabla}\tilde{f}|^2 + \tilde{R})e^{-\tilde{f}}d\tilde{\mu}$, we have

$$\begin{aligned} \frac{d\tilde{\mathcal{F}}}{dt} &= 2\int_M |\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r}{n}\int_M \tilde{g}^{ij}(\tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} + \tilde{R}_{ij})e^{-\tilde{f}}d\tilde{\mu} \\ &= 2\int_M |\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r}{n}\tilde{\mathcal{F}}. \end{aligned}$$

So, $\frac{d\tilde{\mathcal{F}}}{dt} \geq 0$ whenever $r \leq 0$. Thus we have proved the following;

Theorem 2.4.2. *Let $(\tilde{g}_{ij}, \tilde{f})$ solves (2.4.3) in the interval $[0, T)$, then*

$$\frac{d\tilde{\mathcal{F}}}{dt} = 2\int_M |\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r}{n}\tilde{\mathcal{F}} \geq 0, \tag{2.4.4}$$

when $r \leq 0$.

Theorem 2.4.3. *Suppose $\tilde{g}_{ij}(t)$ is a solution of (2.4.1) and we define energy functional*

$$\tilde{\mathcal{B}} = B(\tilde{g}_{ij}, \tilde{f}) = \int_M \tilde{R}e^{-\tilde{f}}d\tilde{\mu}, \tag{2.4.5}$$

then,

$$\frac{d\tilde{\mathcal{B}}}{dt} = 2\int_M |\tilde{R}_{ij}|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r}{n}\tilde{\mathcal{B}}. \tag{2.4.6}$$

Furthermore, $\tilde{\mathcal{B}}$ is non-decreasing whenever $r \leq 0$, where $\tilde{f} := log\left(\frac{dm}{d\tilde{\mu}}\right)$. The monotonicity is strict unless we are on Ricci flat metric.

Proof.

$$\begin{aligned} \frac{d\tilde{\mathcal{B}}}{dt} &= 2\int_M \frac{\partial \tilde{R}}{\partial t} - \tilde{R}\frac{\partial \tilde{f}}{\partial t} - \tilde{R}(r - \tilde{R})e^{-\tilde{f}}d\tilde{\mu} \\ &= 2\int_M \Big[\tilde{\Delta}\tilde{R} + 2|\tilde{R}_{ij}|^2 - \frac{2r}{n}\tilde{R} - \tilde{R}(-\tilde{\Delta}\tilde{f} + |\tilde{\nabla}\tilde{f}|^2 - \tilde{R} + r) - \tilde{R}(r - \tilde{R})\Big]e^{-\tilde{f}}d\tilde{\mu} \\ &= 2\int_M |\tilde{R}_{ij}|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r}{n}\int_M \tilde{R}e^{-\tilde{f}}d\tilde{\mu}, \end{aligned}$$

where we have used evolution of $\tilde{R}$ as obtained in Section (1.3) and evolution of $\tilde{f}$ as in (2.4.3). □

Therefore our new entropy functional (2.3.9) implies

$$\tilde{\mathcal{F}}_{BC} = \mathcal{F}_{\mathcal{B}C}(\tilde{g}_{ij}, \tilde{f}) = \int_M (|\tilde{\nabla}\tilde{f}|^2 + 2C\tilde{R})e^{-\tilde{f}}d\tilde{\mu} = \tilde{\mathcal{F}} + (2C-1)\tilde{\mathcal{B}}. \tag{2.4.7}$$

Hence

$$\begin{aligned}\frac{d}{dt}\tilde{\mathcal{F}}_{BC} &= 2\int_M |\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 e^{-\tilde{f}}d\tilde{\mu} + 2(2C-1)\int_M |\tilde{R_{ij}}|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r}{n}\tilde{\mathcal{F}} - 2(2C-1)\frac{r}{n}\tilde{\mathcal{B}} \\ &= 2\int_M |\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 e^{-\tilde{f}}d\tilde{\mu} + 2(2C-1)\int_M |\tilde{R_{ij}}|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r}{n}\tilde{\mathcal{F}}_{BC} \\ &\geq 0, \qquad (where\ \ r \leq 0).\end{aligned}$$

Theorem 2.4.4. *Let $\tilde{g}_{ij}(t), t \in [0, T)$ solves the normalized Ricci flow and $\tilde{f}$ the conjugate heat equation under the coupled system (2.4.3). Then, $\tilde{\mathcal{F}}_{BC}$ is monotonically non-decreasing when $r \leq 0$. More so, if $r = 0$, then the monotonicity is strict, unless the metric $\tilde{g}_{ij}(t)$ is Ricci flat and $\tilde{f}$ is a constant function.*

Our monotonicity formula does not classify the metric if r is negative, though this is not difficult to achieve, we need a little modification (This case is done by J. Li [109, Theorem 1.4]).

2.4.2 Monotonicity of the least eigenvalue under the NRF

Let $g(t)$ be an evolving solution of (2.4.1) on a compact Riemannian manifold, let $\tilde{\lambda}$ be the least nonzero eigenvalue of the modified operator $-2\tilde{\Delta} + C\tilde{R}, C \geq \frac{1}{2}$ at time t, i.e.,

$$\tilde{\lambda} = \inf \tilde{\mathcal{F}}_{BC} \quad with\ \ e^{-\tilde{f}}d\tilde{\mu}, \tag{2.4.8}$$

then, we have

$$\frac{d\tilde{\lambda}}{dt} = 2\int_M |\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 e^{-\tilde{f}}d\tilde{\mu} + 2(2C-1)\int_M |\tilde{R_{ij}}|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r}{n}\tilde{\lambda}, \tag{2.4.9}$$

when r is nonpositive. If r is strictly negative, we have the following version of Theorem 2.3.7.

Theorem 2.4.5. *The least eigenvalue of $-2\tilde{\Delta} + C\tilde{R}$ is diffeomorphism invariance and nondecreasing under the normalized Ricci flow. The monotonicity is strict unless we are on the Einstein metric.*

Proof. (a). The first part of the Theorem is modelled after the first part of the proof of Theorem 2.3.7.

(b). The second part can be seen using equation (2.4.9)

$$\frac{d}{dt}\tilde{\lambda} \geq 0, \quad where \quad r \leq 0.$$

(c). Examining (2.4.9), it is clear that it fails to classify the steady state of the least eigenvalue (as remarked in

[109]), so we need a modified form of (2.4.9) to tell the class of Einstein metric involved, however, we have

$$\begin{aligned}
\frac{d\tilde{\lambda}}{dt} &= 2\int_M \left|\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} + 2(2C-1)\int_M \left|\tilde{R_{ij}} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r\tilde{\lambda}}{n} \\
&\quad + \frac{4r}{n}\int_M \tilde{g}^{ij}(\tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} + \tilde{R_{ij}})e^{-\tilde{f}}d\tilde{\mu} - 2\int_M \left|\frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} + 4(2C-1)\frac{r}{n}\int_M \tilde{g}^{ij}\tilde{R_{ij}}\, e^{-\tilde{f}}d\tilde{\mu} \\
&\quad - 2(2C-1)\int_M \left|\frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} \\
&= 2\int_M \left|\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} + 2(2C-1)\int_M \left|\tilde{R_{ij}} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} - \frac{2r\tilde{\lambda}}{n} + \frac{4r}{n}\tilde{\mathcal{F}}_{\mathcal{B}C} - \frac{4Cr^2}{n} \\
&= 2\int_M \left|\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} + 2(2C-1)\int_M \left|\tilde{R_{ij}} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} + \frac{2r}{n}(\tilde{\lambda} - 2Cr) \\
&\geq 0
\end{aligned}$$

since by definition $\tilde{\lambda} \leq Cr$ ($\tilde{\lambda}$ being the least eigenvalue, see relation (3.3.11)). □

Corollary 2.4.6. *Under the normalized Ricci flow, the following monotonicity formula holds*

$$\frac{d\tilde{\lambda}}{dt} = 2\int_M \left|\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} + 2(2C-1)\int_M \left|\tilde{R_{ij}} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} \geq 0. \tag{2.4.10}$$

Equality is attained if and only if $\tilde{g}(t)$ is Einstein and $\tilde{f}$ is a constant gradient function.

Thus, we can rule out the existence of nontrivial expanding (or steady) gradient Ricci breathers when $r \leq 0$ except those that are gradient solitons. If $C = \frac{1}{2}$ and $r \leq 0$, we have the monotonicity formula

$$\frac{d\tilde{\lambda}}{dt} = 2\int_M \left|\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{r}{n}\tilde{g}\right|^2 e^{-\tilde{f}}d\tilde{\mu} + \frac{2r}{n}(\tilde{\lambda} - 2Cr) \geq 0 \tag{2.4.11}$$

which simply implies that expanding (or steady) breathers are necessarily expanding (or steady) soliton. Specifically when $C = \frac{1}{2}$, the fact that normalized Ricci flow preserve volume throughout the evolution and the fact that the eigenvalue $\tilde{\lambda}$ is invariant with respect to diffeomorphism and scaling will at once yield the Perelman's result for nonexistence of nontrivial expanding breather as discussed in the previous section. See [126, 64] for details and [109] for another version. In this case we just view $\tilde{\lambda}$ = Perelman's $\tilde{\lambda} = \lambda V^{\frac{2}{n}}$, we can therefore conclude that the monotonicity of our $\tilde{\lambda}$ under normalized Ricci flow is equivalent to the monotonicity of Perelman's $\tilde{\lambda}$ under unnormalized Ricci flow. Hence, we have as a corollary that expanding or steady breathers on compact manifold are necessarily Einstein.

2.5 Perelman's $\mathcal{W}$ -entropy Functional and Applications

Next, we introduce Perelman's $\mathcal{W}$-entropy as presented in [126]. This is a modification of $\mathcal{F}$-energy functional (discussed in the previous section) with inclusion of a positive scale parameter τ and combination of Nash entropy. These combine nicely and the resulting entropy yields useful applications. It was used in [126] to prove that shrinking breathers are necessarily shrinking gradient solitons, thus completing the proof of the existence of no nontrivial breathers other than gradient solitons and also to get a lower bound for the injectivity radius of the flow to complete no local collapsing theorem, (also known as Hamilton's Little Loop Conjecture).

2.5.1 The $\mathcal{W}$-entropy Functional and its Monotonicity

We define the $\mathcal{W}$-entropy functional (as in [126])

$$\mathcal{W}(g,f,\tau) := \int_M \Big[\tau(R+|\nabla f|^2)+f-n\Big](4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu, \tag{2.5.1}$$

where $g(t)$ is a Riemannian metric on n-compact manifold M, f is a smooth function on M and τ is a positive scale parameter. Recall from the previous sections, we have $\mathcal{F}$-energy functional and Nash entropy ($N(f) = \int_M f\, e^{-f}\, d\mu, u = e^{-f}$, under the gradient flow)

$$\mathcal{F}(g_{ij}(t),f) = \int_{M^n}(R+|\nabla f|^2)e^{-f}d\mu, \qquad N(u) = -\int_M u\, \log u\, du.$$

Denoting $\quad u := (4\pi\tau)^{-\frac{n}{2}}e^{-f} \quad with \quad \int_M u\, d\mu = 1$, equation (2.5.1) can be rewritten as

$$\mathcal{W}(g,f,\tau) = [\tau\mathcal{F}+N(f)]\,(4\pi\tau)^{-\frac{n}{2}} - n. \tag{2.5.2}$$

We note that the $\mathcal{W}$-entropy is invariant with respect to diffeomorphism and under simultaneous scaling of τ and g. That is, for any diffeomorphism $\phi : M \to M$, we have

$$\mathcal{W}(\phi_t^* g, \phi_t^* f, \tau) = \mathcal{W}(g,f,\tau)$$

and for any scaling factor $c(t)$, we have

$$\mathcal{W}(cg,f,c\tau) = \mathcal{W}(g,f,\tau).$$

The scaling is included in Appendix C (see Lemma C.2.4).

As in the previous section, let $\delta g_{ij} = h_{ij}$, $\delta\tau = \eta$ and $\delta f = K$ for some function $K : M \to \mathbb{R}$, where $H := g^{ij}h_{ij}$. We have the following

Lemma 2.5.1. *The first variation of $\mathcal{W}$-functional is*

$$\begin{aligned}\delta_{(h,K,\tau)}\mathcal{W}(g,f,\tau) = &\int_M -\tau h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\Big)u\, d\mu + \int_M \eta\Big(R+\Delta f - \frac{n}{2\tau}\Big)u\, d\mu \\ &+ \int_M \Big(\frac{H}{2}-K-\frac{n}{2\tau}\eta\Big)\Big[\tau\Big(R+2\Delta f - |\nabla f|^2\Big)+f-n-1\Big]u\, d\mu.\end{aligned}$$

Proof. See Lemma C.2.2 in Appendix C for the proof. □

2.5.2 The functional $\mathcal{W}$ and its gradient flow

Let us keep the volume measure fixed so that

$$\delta\Big(\int_M (4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu\Big) = 0 = \frac{H}{2} - K - \frac{n}{2\tau}\eta$$

and require that $\eta = -1$, thus τ is a quantity decreasing at a constant rate. We then obtain the gradient flow

$$\frac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i \nabla_j f),$$

with $\frac{d\tau}{dt} = \eta = -1$, where $f = -\ln u - \frac{n}{2}\ln(4\pi\tau)$ and $\frac{\partial f}{\partial t} = -\Delta f - R + \frac{n}{2\tau}$. Hence, we have the gradient flow in form of coupled modified Ricci flow

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i \nabla_j f), \\[2ex] \dfrac{\partial f}{\partial t} = -\Delta f - R + \dfrac{n}{2\tau}, \\[2ex] \dfrac{d\tau}{dt} = -1 \end{cases} \tag{2.5.3}$$

and the following

Proposition 2.5.2. *Let $(g(t), f(t), \tau(t))$ be a solution of the system (2.5.3), we have the identity*

$$\frac{d}{dt}\mathcal{W}(g, f, \tau) = \int_M 2\tau |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 u d\mu, \tag{2.5.4}$$

where $\int_M u d\mu$ is a constant. (See Proposition C.2.3 in Appendix C for the proof).

In a similar manner to the previous section, conjugating away the diffeomorphism generated by the vector field ∇f from the system (2.5.3), we have the coupled system of the Ricci flow-bakward nonlinear heat equation associated to the functional $\mathcal{W}$

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij}, \\[2ex] \dfrac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \dfrac{n}{2\tau}, \\[2ex] \dfrac{d\tau}{dt} = -1 \end{cases} \tag{2.5.5}$$

whose solutions are equivalent to those of the system (2.5.3) up to diffeomorphism and dilation.

As a corollary to the proposition (2.5.2), we have

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) \geq 0, \tag{2.5.6}$$

which is the $\mathcal{W}$-entropy monotonicity formula. Here, equality holds if and only if

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} = 0 \tag{2.5.7}$$

since we know that τ is a positive factor. The condition (2.5.7) implies that $g(t)$ is a shrinking gradient soliton which flows along ∇f. Before applying the monotonicity formula to complete the proof of the existence of no nontrivial breathers, we consider two functionals $\mu(g_{ij}, \tau)$ and $\nu(g_{ij}, \tau)$, using the $\mathcal{W}$-entropy functional.

Define the minimizing problem

$$\mu(g_{ij},\tau) := \inf\{\mathcal{W}(g_{ij},f,\tau) : f \in C^{\infty}(M), \int_M u d\mu = 1\}$$

and

$$\nu(g_{ij}) := \inf\{\mu(g_{ij},\tau) : \tau > 0\}.$$

Setting $v := e^{-\frac{f}{2}}$, to show existence of a minimizer , then the functional $\mathcal{W}$ can be expressed as

$$\mathcal{W}(g_{ij},f,\tau) = \int_M \Big[\tau\Big(Rv^2 + 4|\nabla v|^2\Big) - v^2\log v^2 - nv^2\Big](4\pi\tau)^{-\frac{n}{2}}d\mu$$

with $\int_M v^2(4\pi\tau)^{-\frac{n}{2}}d\mu = 1$.

We may assume that $4\pi\tau = 1$ without loss of generality and show that infimum of

$$I[v] = \int_M \Big[\tau\Big(Rv^2 + 4|\nabla v|^2\Big) - v^2\log v^2 - nv^2\Big](4\pi\tau)^{-\frac{n}{2}}d\mu$$

is achieved over a set

$$\mathcal{A} = \{v \in \mathbb{H}^1(M) : \int_M v^2 d\mu = 1\}.$$

By Sobolev compactness imbedding, $\mathcal{A}$ is weakly closed in $\mathbb{H}^1(M)$ and I is weakly lower semi-continuous. We can equally show that $I[\cdot]$ is coercive on $\mathcal{A}$. Let $v \in \mathcal{A}$, then using the inequality $\log x \leq x^{\frac{1}{n}}, \quad \forall\ x > 1$, interpolation and Hölder's inequalities, we have ($n > 2$)[2]

$$\begin{aligned}
\int_M v^2\log v^2 d\mu &\leq \int_{v>1} v^2\log v^2 \leq \int_{v>1} v^{2+\frac{2}{n}} \leq \int_M v^{2+\frac{2}{n}}d\mu, && (Interpolation\ \ inequalities)\\
&\leq \epsilon\int_M v^{2+\frac{4}{n}}d\mu \ +\ C(\epsilon)\int_M v^2 d\mu, && \Big(\epsilon > 0,\ \int_M v^2 d\mu = 1\Big)\\
&\leq \epsilon\Big(\int_M v^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\Big(\int_M v^2 d\mu\Big)^{\frac{2}{n}} +\ C(\epsilon), && (Holder's\ inequality)\\
&\leq \epsilon\Big(\int_M |\nabla v|^2 d\mu\Big) + C(\epsilon), && (Sobolev\ inequality).
\end{aligned}$$

Choosing $\epsilon = \tau$, we obtain[3]

$$\begin{aligned}
I[v] &= \int_M \Big[\tau\Big(Rv^2 + 4|\nabla v|^2\Big) - v^2\log v^2 - nv^2\Big](4\pi\tau)^{-\frac{n}{2}}d\mu\\
&\geq \int_M \Big[\tau\Big(Rv^2 + 4|\nabla v|^2\Big) - \tau|\nabla v|^2 - nv^2\Big]d\mu -\ C(\epsilon)\\
&\geq \int_M 3\tau|\nabla v|^2 d\mu + \inf_{x\in M}(\tau R - n) - C.
\end{aligned}$$

This proves the coercivity of $I[\cdot]$. By direct method in Calculus of variation, we can obtain positive minimizer of $I[u] \to \min$ in the set $\mathcal{A} = \{v \in \mathbb{H}^1(M) : \int_M v^2 d\mu = 1\}$. We can also minimize $I[\cdot]$ over the subset

$$\mathcal{A}^+ = \{v \in \mathbb{H}^1(M) : v > 0, \int_M v^2 d\mu = 1\}.$$

[2]Observe that for any $\epsilon > 0, \quad v^{1+2/n}v \leq \epsilon v^{2(1+2/n)} + \epsilon^{-1}v^2$.

[3]Notice that the monotonicity of Perelman's $\mathcal{W}$-entropy is equivalent to a version of logarithmic Sobolev inequality. Both are equivalent to the ultracontracitivity of the heat semigroup. We shall discuss this in the later chapter.

Then v satisfies the Euler-Langrange equation and it follws that $\mu(g,\tau)$ is achieved by a minimizer f_τ satisfying

$$\tau(2\Delta f_\tau - |\nabla f_\tau|^2 + R) + f_\tau - n = \mu(g).$$

Corollary 2.5.3. *([64, p.237], [126]) For any metric g on a closed manifold M and $\tau > 0$, $\mu(g,\tau) > -\infty$ and tends to zero as $\tau \to 0$.*

Proof. We follow Perelman's argument [126, pp.9]. Let $\tilde{\tau} > 0$ be so small such that the Ricci flow $g_{ij}(t)$ exists on the interval $0 \le t \le \tilde{\tau}$. Let $u := (4\pi\tau)^{-\frac{n}{2}} e^{-f}$ be the solution of the conjugate heat equation starting from δ-function at $t = \tilde{\tau}$, $\tau(t) = \tilde{\tau} - t$. Then $\mathcal{W}_{\tilde{\tau}} - \mathcal{W}_t \le 0$ and $\mathcal{W}(g_{ij}(t), f(t), \tau(t)) \to 0$ as $t \to \tilde{\tau}$. therefore by the monotonicity of $\mathcal{W}$, we have that

$$\mu(g_{ij}, \tilde{\tau}) \le \mathcal{W}(g_{ij}(0), f(0), \tau(0)) = \mathcal{W}(g_{ij}(0), f(0), \tilde{\tau}) < 0.$$

The proof of the inequality $\mathcal{W}(g_{ij}(0), f(0), \tilde{\tau}) < 0$ can be made more explicit, see the proof of Proposition 3.2 in [135]. □

Proposition 2.5.4. *$\mu(g_{ij}(t), \tau - t)$ is nondecreasing along the Ricci flow and the monotonicity is strict unless we are on a shrinking gradient soliton. A shrinking breather is necessarily a shrinking gradient soliton.*

Proof. Let $g(t)$ be a solution to the Ricci flow defined on some interval $[0,T]$ and $T < \tau$. Let f_0 be a minimizer of $\mu(g(t_0), \tau - t_0)$ for any time $t_0 \in [0,T]$. We then solve the backward heat equation

$$\begin{cases} \dfrac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \dfrac{n}{2\tau} \\ \\ f(t_0) = f_0 \end{cases} \tag{2.5.8}$$

and obtain a solution $f(t)$ for $t \le t_0$ which satisfies $\int_M (4\pi t)^{-\frac{n}{2}} e^{-f(t)} d\mu_{g(t)} = 1$.

To prove the first part of statement 1 of the proposition, it suffices to show that

$$\mu(g_{ij}(t), \tau - t) \le \mu(g_{ij}(t_0), \tau - t_0).$$

Let us choose another function φ with $\int_M (4\pi t_0)^{-\frac{n}{2}} e^{-\varphi} d\mu_{g(t_0)} = 1$, such that

$$\begin{aligned} \mu(g_{ij}(t_0), \tau - t_0) = (\tau - t_0) \int_M e^{-\varphi} \Big(R_{g(t_0)} + |\nabla\varphi|^2_{g(t_0)}\Big)(4\pi(\tau - t_0))^{-\frac{n}{2}} d\mu_{g(t_0)} \\ + \int_M e^{-\varphi}\Big(\varphi - \frac{n}{2}\log(4\pi(\tau - t_0)) - n\Big) d\mu_{g(t_0)}. \end{aligned}$$

Since

$$\int_M (4\pi t)^{-\frac{n}{2}} e^{-f(t)} d\mu_{g(t)} = \int_M (4\pi t_0)^{-\frac{n}{2}} e^{-\varphi} d\mu_{g(t_0)} = 1.$$

We have

$$\begin{aligned}\mu(g_{ij}(t),\tau-t) &\le (\tau-t)\int_M e^{-f(t)}\Big(R_{g(t)}+|\nabla\varphi|^2_{g(t)}\Big)(4\pi(\tau-t))^{-\frac{n}{2}}\,d\mu_{g(t)}\\&\quad+\int_M e^{-f(t)}\Big(f(t)-\frac{n}{2}\log(4\pi(\tau-t))-n\Big)d\mu_{g(t)}\\&\le(\tau-t_0)\int_M e^{-\varphi}\Big(R_{g(t_0)}+|\nabla\varphi|^2_{g(t_0)}\Big)(4\pi(\tau-t_0))^{-\frac{n}{2}}\,d\mu_{g(t_0)}\\&\quad+\int_M e^{-\varphi}\Big(\varphi-\frac{n}{2}\log(4\pi(\tau-t_0))-n\Big)d\mu_{g(t_0)}\\&=\mu(g_{ij}(t_0),\tau-t_0).\end{aligned}$$

Therefore, we have

$$\mu(g_{ij}(t),\tau-t)\le\mathcal{W}(g_{ij}(t),f(t),\tau-t)\le\mathcal{W}(g_{ij}(t_0),f(t_0),\tau-t_0)=\mu(g_{ij}(t_0),\tau-t_0)$$

for $t\le t_0$. Suppose we are on a shrinking gradient soliton, the second inequality is strict. The proves the first part of the theorem.

Let $g(t)$ be a shrinking breather on $[t_1,t_2]$ such that $g(t_2)=\alpha\phi^*g(t_1)$ for some $0<\alpha<1$. Since the $\mathcal{W}$-functional is scaling and diffeomorphism invariant, then, we have

$$\mu(g_{ij}(t_1),\tau-t_1)=\mu(\alpha g_{ij}(t_1),\alpha(\tau-t_1))=\mu(g_{ij}(t_2),\tau-t_2)$$

since τ is positive, say $\tau>0$,

$$\alpha(\tau-t_1)=\tau-t_2\Rightarrow\tau=\frac{t_2-\alpha t_1}{1-\alpha}.$$

Now define

$$\tau(t)=\frac{t_2-\alpha t_1}{1-\alpha}-t,$$

so that

$$\frac{d\tau}{dt}=1,\quad\tau(t_1)=\frac{t_2-t_1}{1-\alpha}\quad\tau(t_2)=\alpha\frac{t_2-t_1}{1-\alpha}\ \textit{ and }\ \tau(t_2)=\alpha\tau(t_1).$$

Let f_2 be a minimizer of

$$\{\mathcal{W}(g(t_2),f_2,\tau(t_2)):f\in C^\infty(M),\int_M(4\pi\tau)^{-\frac{n}{2}}d\mu=1\}$$

at $t=t_2$, so that $\mathcal{W}(g(t_2),f_2,\tau(t_2))=\mu(g(t_2),\tau(t_2))$.

Suppose $f(t)$ solves the backward heat equation above on $[t_1,t_2]$ with $f(t_2)=f_2$. By monotonicity formula and definition of μ, we have

$$\begin{aligned}\mu(g(t_1),\tau(t_1))&\le\mathcal{W}(g(t_1),f(t_1),\tau(t_1))\\&\le\mathcal{W}(g(t),f(t),\tau(t))\\&\le\mathcal{W}(g(t_2),f(t_2),\tau(t_2))=\mu(g(t_2),\tau(t_2))\end{aligned}$$

for all $t \in [t_1, t_2]$, Since $g(t_1) = \alpha\phi^* g(t_2)$ and $\tau(t_2) = \alpha\tau(t_1)$ by the diffeomorphism and scale invariance of μ, we have

$$\mu(g(t_1), \tau(t_1) = \mu(g(t_2), \tau(t_2)).$$

This and the fact that $\mathcal{W}(g(t), f(t), \tau(t))$ is monotone implies

$$\mathcal{W}(g(t), f(t), \tau(t)) = \mu(g(t), \tau(t)) \equiv const.$$

for $t \in [t_1, t_2]$. Thus $f(t)$ is a minimizer for $\mathcal{W}(g(t), f(t), \tau(t))$ and $\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) \equiv 0$, then, we have

$$\int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}|^2 e^{-f} d\mu = 0$$

for all $t \in [t_1, t_2]$ and hence $R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij} = 0$ for all $t \in [t_1, t_2]$. This completes the proposition. $\square$

2.5.3 A new family of entropy over shrinkers

Here, we define a new family of entropy and discuss its variation formula under the Ricci flow.

Definition 2.5.5. *Let (M, g) be a closed n-dimensional Riemannian Manifold, we define a family of entropy functional $\mathcal{W}_{\mathcal{B}C}$ as*

$$\mathcal{W}_{\mathcal{B}C} = \tau \int_M \Big[|\nabla f|^2 + 2C\Big(R + \frac{1}{\tau}(f - n)\Big)\Big] u d\mu, \tag{2.5.9}$$

where $C \geq \frac{1}{2}, C \in \mathbb{R}$. When $C = \frac{1}{2}$, this is Perelman's $\mathcal{W}$ entropy [126].

Theorem 2.5.6. *Let $(M, g_{ij}(t), f(t), \tau(t)), t \in [0, T)$, solve the system (2.5.5), where f evolves by a backward heat equation, then, $\mathcal{W}_{\mathcal{B}C}$ is monotonically non-decreasing. In particular, we have*

$$\frac{d}{dt}\mathcal{W}_{\mathcal{B}C} = 2\tau \int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}|^2 u d\mu + 2(2C-1)\tau \int_M |R_{ij} - \frac{1}{2\tau} g_{ij}|^2 u d\mu \geq 0. \tag{2.5.10}$$

Moreover, the monotonicity is strict unless

$$R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij} = 0 \qquad \text{and} \qquad R_{ij} - \frac{1}{2\tau} g_{ij} = 0. \tag{2.5.11}$$

Proof. As usual our proof is by direct computation, hence we write

$$\begin{aligned}
\mathcal{W}_{\mathcal{B}C} &= \tau\int_M \Big[|\nabla f|^2 + 2C\Big(R+\frac{1}{\tau}(f-n)\Big)\Big]ud\mu \\
&= \tau\int_M \Big[\Big(|\nabla f|^2 + R\Big) + \frac{1}{\tau}(f-n)\Big]ud\mu + (2C-1)\tau\int_M \Big[R+\frac{1}{\tau}(f-n)\Big]ud\mu \\
&= \mathcal{W} + (2C-1)\int_M (\tau R + f - n)ud\mu \\
\frac{d\mathcal{W}_{\mathcal{B}C}}{dt} &= \frac{d\mathcal{W}}{dt} + (2C-1)\int_M \Big[\tau\frac{\partial R}{\partial t} - R + \frac{\partial f}{\partial t}\Big]ud\mu \\
&= 2\tau\int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2 ud\mu \\
&\qquad + (2C-1)\int_M \Big[\tau(\Delta R + 2|R_{ij}|^2) - 2R - \Delta f + |\nabla f|^2 + \frac{n}{2\tau}\Big]ud\mu \\
&= 2\tau\int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2 ud\mu + (2C-1)\int_M \Big[2\tau|R_{ij}|^2 - 2R + \frac{n}{2\tau}\Big]ud\mu \\
&= 2\tau\int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2 ud\mu + 2(2C-1)\tau\int_M \Big[|R_{ij}|^2 - \frac{1}{\tau}R + \frac{n}{4\tau^2}\Big]ud\mu \\
&= 2\tau\int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2 ud\mu + 2(2C-1)\tau\int_M |R_{ij} - \frac{1}{2\tau}g_{ij}|^2 ud\mu \quad \geq 0.
\end{aligned}$$

□

Corollary 2.5.7. *Let $(M, g_{ij}(t), f(t), \tau(t)), t \in [0, T)$ solve the coupled system (2.5.5), then, there is no nontrivial shrinking Ricci breather other than shrinking gradient solitons.*

Corollary 2.5.8. *Let (M, g) be a closed Riemann manifold, then every shrinking Ricci breather must necessarily be Einstein.*

Theorem 2.5.9. *Let $(M, g_{ij}(t)), t \in [0, T)$ be a solution of the Ricci flow, then, the least eigenvalue $\mu_C(g_{ij})$ of $(-2\Delta + CR)$ is diffeomorphism invariance and non-decreasing. The monotonicity is strict unless the metric is a shrinking gradient soliton.*

2.5.4 Extension to the Normalized Ricci Flow

Here, we classify the class of metrics satisfying the normalized Ricci flow with respect to the entropy $\mathcal{W}$ and $\mathcal{W}_{BC}$ in the Theorems 2.5.10 and 2.5.11. Specifically, we show conditions upon which shrinking breathers are Einstein. For this purpose, we obtain the following coupled gradient flow-like for normalized Ricci flow associated to $\mathcal{W}$ -energy after a simple calculation:

$$\begin{cases}
\dfrac{\partial g_{ij}}{\partial t} = -2(R_{ij} - \dfrac{r}{n}g_{ij} + \nabla_i\nabla_j f) \\[2ex]
\dfrac{\partial f}{\partial t} = -\Delta f - R + r + \dfrac{n}{2\tau} \\[2ex]
\dfrac{d\tau}{dt} = -1.
\end{cases} \tag{2.5.12}$$

Conjugating away the diffeomorphism generated by the vector field ∇f from the above system gives an equivalent system

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij} + \dfrac{r}{n} g_{ij} \\ \\ \dfrac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + r + \dfrac{n}{2\tau} \\ \\ \dfrac{d\tau}{dt} = -1. \end{cases} \tag{2.5.13}$$

Using Perelman's $\mathcal{W}$ -entropy and diffeomorphism ϕ, we write $\tilde{\mathcal{W}} = \phi\mathcal{W}$, since $\mathcal{W}$ is invariant with respect to diffeomorphism. We therefore define the following

$$\tilde{\mathcal{W}} = \int_M \Big[\tau(\tilde{R} + |\tilde{\nabla}\tilde{f}|^2) + \tilde{f} - n\Big]\tilde{u}d\tilde{\mu} \tag{2.5.14}$$

and

$$\tilde{\mathcal{W}_{\mathcal{B}C}} = \tau \int_M \Big[|\tilde{\nabla}\tilde{f}|^2 + 2C\Big(\tilde{R} + \frac{1}{\tau}(\tilde{f} - n)\Big)\Big]\tilde{u}d\tilde{\mu}, \tag{2.5.15}$$

where $u := (4\pi\tau)^{-\frac{n}{2}} e^{-\tilde{f}} \quad with \quad \int_M \tilde{u}d\tilde{\mu}. = 1.$

Theorem 2.5.10. *Let $(M, \tilde{g}_{ij}(t), \tilde{f}(t), \tau(t), t \in [0,\infty)$ solve the system (2.5.12) or (2.5.13). Then $\tilde{\mathcal{W}}$ is monotonically nondecreasing. Furthermore, the monotonicity is strict unless the metric is a gradient shrinking soliton.*

Proof. By direct computation we have

$$\frac{d\tilde{\mathcal{W}}}{dt} = \tau\frac{d\tilde{\mathcal{F}}}{dt} - \tilde{\mathcal{F}} + \int_M \frac{\partial \tilde{f}}{\partial t}\tilde{u}d\tilde{\mu}. \tag{2.5.16}$$

Recall the evolutions of $\tilde{\mathcal{F}}$ in (2.4.4) and $\tilde{f}$ in (2.5.13) under the normalized Ricci flow and use them in (2.5.16). Then direct substitutions give

$$\begin{aligned}
\frac{d\tilde{\mathcal{W}}}{dt} &= 2\tau\int_M |\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2\tilde{u}d\tilde{\mu} - \frac{2r}{n}\tau\tilde{\mathcal{F}} - \tilde{\mathcal{F}} + \int_M \Big(-\tilde{\Delta}\tilde{f} - \tilde{R} + r + \frac{n}{2\tau}\Big)\tilde{u}d\tilde{\mu} \\
&= 2\tau\int_M |\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2\tilde{u}d\tilde{\mu} - 2\int_M \Big(\tilde{\Delta}\tilde{f} + \tilde{R}\Big)\tilde{u}d\tilde{\mu} + \frac{n}{2\tau}\int_M \tilde{u}d\tilde{\mu} - \frac{2r}{n}\tau\Big(\tilde{\mathcal{F}} - \frac{n}{2\tau}\Big) \\
&= 2\tau\int_M \Big[|\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f}|^2 - \frac{1}{\tau}\Big(\tilde{\Delta}\tilde{f} + \tilde{R}\Big) + \frac{n}{4\tau^2}\Big]\tilde{u}d\tilde{\mu} - \frac{2r}{n}\tau\Big(\tilde{\mathcal{F}} - \frac{n}{2\tau}\Big) \\
&= 2\tau\int_M |\tilde{R_{ij}} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{1}{2\tau}\tilde{g}_{ij}|^2\,\tilde{u}d\tilde{\mu} - \frac{2r}{n}\tau\Big(\tilde{\mathcal{F}} - \frac{n}{2\tau}\Big).
\end{aligned}$$

It suffices to uphold the claim that

$$\tilde{\mathcal{F}} - \frac{n}{2\tau} \leq 0.$$

The prove of this claim is reserved till Subsection 3.4.3 where it will become more evident. Using this we have

$$\frac{d\tilde{\mathcal{W}}}{dt} \geq 0.$$

Equality is attained here when we have the following

$$\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{1}{2\tau}\tilde{g}_{ij} \equiv 0 \quad \textit{and} \quad \tilde{\mathcal{F}} - \frac{n}{2\tau} \equiv 0,$$

which are also equivalent, taking the trace of the former gives the later. The implication of all these is that Ricci shrinking breathers must be gradient shrinking Ricci solitons. □

Theorem 2.5.11. *Let $(M, \tilde{g}_{ij}(t), \tilde{f}(t), \tau(t)), t \in (0,\infty)$ solve the system (2.5.12) or (2.5.13). Then $\tilde{\mathcal{W}}_{\mathcal{B}C}$ is monotonically nondecreasing. However, the monotonicity is strict unless the metric is a gradient shrinking soliton and Einstein.*

Proof. As in the proof of Theorem 2.5.6, we write

$$\frac{d\tilde{\mathcal{W}}_{\mathcal{B}C}}{dt} = \frac{d\tilde{\mathcal{W}}}{dt} + (2C-1)\int_M \Big[\tau\frac{\partial \tilde{R}}{\partial t} - \tilde{R} + \frac{\partial \tilde{f}}{\partial t}\Big]\tilde{u}d\tilde{\mu} \tag{2.5.17}$$

$$\mathbf{I} \quad + \quad \mathbf{II}.$$

From the last theorem, we have that

$$\mathbf{I} \quad = \quad 2\tau\int_M |\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{1}{2\tau}\tilde{g}_{ij}|^2\tilde{u}d\tilde{\mu} - \frac{2r}{n}\tau\Big(\tilde{\mathcal{F}} - \frac{n}{2\tau}\Big).$$

By direct substitution we have

$$\begin{aligned}
\mathbf{II} \quad &= (2C-1)\int_M \tau\Big[\Big(\tilde{\Delta}\tilde{R} + 2|\tilde{R}_{ij}|^2 - \frac{2r}{n}\tilde{R}\Big) - 2\tilde{R} - \tilde{\Delta}\tilde{f} + |\tilde{\nabla}\tilde{f}|^2 + \frac{n}{2\tau} + r\Big]\tilde{u}d\tilde{\mu} \\
&= (2C-1)\int_M \Big(2\tau|\tilde{R}_{ij}|^2 - 2\tilde{R} + \frac{n}{2\tau}\Big)\tilde{u}d\tilde{\mu} + (2C-1)\int_M \Big(-\frac{2r}{n}\tau\tilde{R} + r\Big)\tilde{u}d\tilde{\mu} \\
&= 2(2C-1)\tau\int_M |\tilde{R}_{ij} - \frac{1}{2\tau}\tilde{g}_{ij}|^2\tilde{u}d\tilde{\mu} - 2(2C-1)\frac{\tau}{n}r\int_M \Big(\tilde{R} - \frac{n}{2\tau}\Big)\tilde{u}d\tilde{\mu}.
\end{aligned}$$

Combining **I** and **II** with (2.5.17), we have the monotonicity formula

$$\frac{d\tilde{\mathcal{W}}_{\mathcal{B}C}}{dt} \geq 0$$

with $r \leq 0$ when $R \geq \frac{n}{2\tau}$ or $r \geq 0$ when $R \leq \frac{n}{2\tau}$. The monotonicity is strict unless the solution is a gradient shrinking soliton with

$$\tilde{R}_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j\tilde{f} - \frac{1}{2\tau}\tilde{g}_{ij} \equiv 0 \quad \textit{and} \quad \tilde{\mathcal{F}} - \frac{n}{2\tau} \equiv 0$$

and Einstein

$$\tilde{R}_{ij} - \frac{1}{2\tau}\tilde{g}_{ij} \equiv 0 \quad \textit{and} \quad \tilde{R} - \frac{n}{2\tau} \equiv 0.$$

This ends the proof. □

Chapter 3

Differential Harnack Estimates for Conjugate Heat Equation

3.1 Introduction

Gradient estimates and Harnack inequalities are indeed very powerful tools in geometric analysis as this is evident for examples in the work of P. Li and S-T. Yau (1986) and G. Perelman (2002). The paper of Li and Yau [112] paved way for the rigorous studies and many interesting applications of Harnack inequalities. In that paper, they derived gradient estimates for positive solutions to the heat operator defined on closed manifold with bounded Ricci curvature from which they obtained Harnack inequalities. These inequalities were in turn used to establish various lower and upper bounds on the heat kernel. They also studied manifolds satisfying Dirichlet and Neumann conditions. On the other hand, Perelman in [126] obtained a gradient estimate for the fundamental solution on compact manifold evolving by the Hamilton's Ricci flow. Perelman's results for Harnack estimates are unprecedented as they play a key factor in the proof of Poincaré conjecture. Meanwhile, shortly before Perelman's paper appeared online, C. Guenther [85] had found gradient estimates for positive solutions to the heat equation under the Ricci flow by adapting the methodology of Bakry and Qian [9] to time dependent metric case. As an application of her results, she got a Harnack-type inequality and obtained a lower bound for the fundamental solution. She also studied existence and basic properties of these solutions. S. Kuang and Q. Zhang [103] established a gradient estimate that holds for all positive solutions of the conjugate heat equation defined on a closed manifold whose metric is evolving by the Ricci flow. X. Cao [44] also used Perelman's approach to establish a differential inequality for all positive solutions to the conjugate heat equation under the Ricci flow with nonnegative condition on the scalar curvature. Immediate consequences of their results in [103] and [44] are Harnack-type inequalities. Q. Zhang [156] derived local gradient estimates for positive solutions to the heat equation coupled to the backward in time Ricci flow with lower bound assumption on the Ricci curvature. His gradient estimate was used to prove a Gaussian bound for the conjugate heat equation. In [8], they applied Zhang's method to prove both space-only and space-time gradient estimates for heat equation coupled to forward in time Ricci flow, they also study manifolds with nonempty convex boundary evolving under the Ricci flow. Ecker, Knopf, Ni and Topping [75] have a local result on gradient estimate in relation to mean value theorem and monotonicity for heat kernel.

The study of the Ricci flow coupled to heat-type equation arose from R. Hamilton's work [92], where he conceived the idea of investigating Ricci flow coupled to harmonic maps heat flow. He combined this with his previous results [89, 91] to study the formation of singularities in the Ricci flow. In [90], he provides a Harnack estimate on Riemannian manifolds with nonnegative positive curvature operator. Hamilton [88] also proved Harnack estimates for surfaces whose positive scalar curvature under the Ricci flow satisfies the heat equation with soliton potential. B. Chow [60] completed the proof of Harnack estimate for surfaces of positive scalar curvature in general. Thus, Harnack estimates of the Ricci flow on surfaces gives a control on curvature growth, while in higher dimension, one uses the Harnack estimates to classify the ancient solutions of nonnegative curvature operators. Perelman's Harnack-type estimate is used to prove noncollapsing of the metric under the Ricci flow. As useful as Harnack inequalities are, they have also been discovered in other geometric flows; See the following- H-D. Cao [38] H-D. Cao and L. Ni [40] and L. Ni [125] for heat equation on Kähler manifolds, B. Chow [61] for Gaussian curvature flow and [62] for Yamabe flow, also B. Chow and R. Hamilton [67], and R. Hamilton [93] on mean curvature flow. The following references among many others are found relevant [5, 45, 44, 63, 85, 113, 123, 138, 139, 140], see also the following monographs [71, 111, 132] for theory of Harnack inequalities and [64, 65, 68, 69, 71, 117, 118, 158]. for theory and applications of Ricci flow.

The rest of this chapter is organised as follows; the next section introduces the theory of conjugate heat equation and gives a quick review of how one can view Perelman's differential Harnack estimate as Li-Yau type and how it provides an alternative proof of a localised version of his entropy monotonicity formula. The main result of Section 3.3 is contained in Theorem 3.3.1, where we establish a point-wise differential Harnack inequality for all positive solutions of the conjugate heat equation on manifold evolving by the Ricci flow, as an application of this, we derive corresponding Harnack estimate under a mild assumption that the Ricci curvature remains nonnegatively bounded. There is another important result in this section (Subsection 3.3.2), where we establish a localised form of the Harnack and gradient estimates obtained. The main idea is the application of the Maximum principle and Bochner identity on some smooth cut-off function. It was the basic idea used by Li and Yau in [112], however our computation is more involved as the metric is also evolving. In Section 3.4, we introduce a dual entropy formula which surprisingly interpolates between Perelman's entropy [126] for conjugate heat equation on an evolving manifold and the Ni's modified entropy formula [122] for linear heat equation on static manifolds. From this entropy formula, we also recover the corresponding differential Harnack inequality and gradient estimate for the fundamental solution, which in fact, holds for all positive solutions to the heat equation. As it is well known that entropy functional are intimately related to functional inequalities, we will apply the monotonicity proved in this section to derive a family of logarithmic Sobolev inequalities in the next chapter. Larger part of the results presented in this chapter will appear in [1].

3.2 The Conjugate Heat Flow and Entropy Monotonicity Formula

3.2.1 The Conjugate Heat equation

All the results here are due Perelman [126].

Definition 3.2.1. *(**The Conjugate Heat Operator**). Let $\Gamma := \partial_t - \Delta$ be the heat operator acting on functions $u : M \times [0, T] \to \mathbb{R}$, where $M \times [0, T]$ is endowed with the volume form $d\mu(x)dt$. The conjugate (adjoint) to the heat operator Γ is defined by*

$$\Gamma^* := -\partial_t - \Delta_x + R, \tag{3.2.1}$$

where Δ_x is the Laplace-Beltrami operator with respect to space variable x and R is the scalar curvature.

We remark that for any solution $g(t), t \in [0, T]$ to the Ricci flow and smooth functions $u, v : M \times [0, T] \to \mathbb{R}$, the following identity holds

$$\int_0^T \int_M (\Gamma u) v d\mu(x)dt = \int_0^T \int_M u(\Gamma^* v) d\mu(x)dt. \tag{3.2.2}$$

By direct application of integration by parts and the fact that the functions u and v are C^2 with compact support (since M is compact) and using evolution of $d\mu$ under the Ricci flow, it follows that

$$\begin{aligned}
\int_0^T \int_M (\Gamma u) v d\mu(x)dt &= \int_0^T \int_M (\partial_t u - \Delta u) v d\mu(x)dt \\
&= \int_0^T \int_M (\partial_t u) v d\mu(x)dt - \int_0^T \int_M (\Delta u) v d\mu(x)dt \\
&= -\int_0^T \int_M u \partial_t v d\mu(x)dt - \int_0^T \int_M uv(\partial_t d\mu(x))dt - \int_0^T \int_M u\Delta v d\mu(x)dt \\
&= -\int_0^T \int_M u(\partial_t v - Rv + \Delta v) d\mu(x)dt \\
&= \int_0^T \int_M u(-\partial_t - \Delta + R) v d\mu(x)dt,
\end{aligned}$$

which proves the identity.

In a special case $u \equiv 1$, we have

$$\frac{d}{dt} \int_M v d\mu = -\int_M \Gamma^* v d\mu.$$

Proposition 3.2.2. *Let $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$ be a positive solution to the conjugate heat equation and $\partial_t \tau = -1$. The evolution equation*

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \tag{3.2.3}$$

is equivalent to the following evolution

$$\Gamma^* u = 0. \tag{3.2.4}$$

Proof.

$$\Gamma^* u = (-\partial_t - \Delta_x + R)(4\pi\tau)^{-\frac{n}{2}} e^{-f}.$$

By direct calculation, it follows that

$$\partial_t[(4\pi\tau)^{-\frac{n}{2}} e^{-f}] = (\frac{n}{2\tau} - \partial_t f)(4\pi\tau)^{-\frac{n}{2}} e^{-f}$$

$$\Delta[(4\pi\tau)^{-\frac{n}{2}} e^{-f}] = (-\Delta f + |\nabla f|^2)(4\pi\tau)^{-\frac{n}{2}} e^{-f}.$$

Then

$$\Gamma^* u = \Big(-\frac{n}{2\tau} + \partial_t f + \Delta f - |\nabla f|^2 + R\Big) u = 0.$$

Since $u > 0$, the claimed is then proved. □

In the next, we want to briefly look at how Perelman-Harnack estimates on conjugate heat kernel implies Li-Yau Harnack estimates. This section can be considered as a review of [126, section 9], where Perelman applied a localised version of his $\mathcal{W}$-entropy to prove his pseudolocality theorem. Let $g(t), t \in [0, T)$ be a solution of the Ricci flow and u be any C^∞ function on $M \times [0, T)$ such that $\Gamma^* u = 0$.

Definition 3.2.3. *We say that $H(x, \tau; y, \sigma)$ is a fundamental solution to the adjoint heat equation centred at (y, σ) for $x, y \in M,\ \sigma < t \in [0, T]$, if*

$$\Gamma^*_{x,\tau} H(x, \tau; y, \sigma) = 0$$

and

$$\lim_{\tau \to \sigma} H(x, \tau; y, \sigma) = \delta_y(x)$$

for any $x \in M$. The limit is in the sense of distributions.

Thus, $H(x, \tau; y, \sigma)$ is the unique minimal positive solution to the equation

$$\begin{cases} (-\partial_\tau - \Delta_{(x,\tau)} + R(x, \tau))H(x, \tau; y, \sigma) = 0 \\ \lim\limits_{\tau \to \sigma} H(x, \tau; y, \sigma) = \delta_y(x). \end{cases} \tag{3.2.5}$$

Lemma 3.2.4. *The conjugate heat kernel satisfies the following properties.*

1. $\int_M H(x, \tau; y, \sigma) d\mu_{(x,\tau)} = 1$

2. $H(x, \tau; y, \sigma) d\mu_{(x,\tau)}$ *is also the fundamental solution to* $\Gamma_{(y,\sigma)} = \partial_\sigma - \Delta_{(y,\sigma)}$.

We shall also need asymptotic behaviour of the fundamental solution to the conjugate heat equation for small τ. Let $d_\tau(x, y)$ be the distance function with respect to the metric $g(\tau)$.

Theorem 3.2.5. *([65, 69, 124]) (Ricci flow adjoint heat kernel parametrix.) Let $g(t), t \in [0,T)$ be a solution of the Ricci flow on closed n-dimensional manifold and $H(x,\tau;y,0)$ $(i.e, H(x,y,\tau))$ be the fundamental solution to the conjugate heat equation (3.2.5). Then, as $\tau \to 0$ we have*

$$H(x,\tau;y,0) \sim (4\pi\tau)^{-\frac{n}{2}} \exp\Big(-\frac{d_\tau^2(x,y)}{4\tau}\Big)\sum_{j=0}^{\infty} u_j(x,y,\tau)\tau^j. \tag{3.2.6}$$

By (3.2.6), it means that there exists $T>0$ and a sequence $u_j \in C^\infty(M\times M\times[0,T])$ such that

$$H(x,y,\tau) - (4\pi\tau)^{-\frac{n}{2}} \exp\Big(-\frac{d_\tau^2(x,y)}{4\tau}\Big)\sum_{j=0}^{k} u_j(x,y,\tau)\tau^j := w_k(x,y,\tau)$$

with

$$w_k(x,y,\tau) = O(\tau^{k+1-\frac{n}{2}}),$$

as $\tau \to 0$ uniformly for all $x,y \in M$. The function $u_0(x,y,\tau)$ can be chosen so that $u_0(x,x,0)=1$. The proof of this result is done by Garofalo and Lanconelli in [80] for the fundamental solution of the heat-type equation in divergence form when there is no zeroth order term $R(x,\tau)u(x,\tau)$. However, one can verify that the argument in [80] can be carried over to the case of the adjoint heat equation. Alternatively, one may write

$$H(x,\tau;y,\sigma) \sim E(x,\tau;y,\sigma)\sum_{j=0}^{\infty} u_j(x,y,\tau)\tau^j,$$

where $E(x,y,\tau)$ is the Euclidean Heat kernel.

3.2.2 Entropy Monotonicity Formula

Now, suppose that the Ricci flow $g(t)$ is defined for $t\in[0,T], T<\infty$ and let $u=(4\pi\tau)^{-\frac{n}{2}}e^{-f}$, where $\tau = T-t$, be a positive solution to the conjugate heat equation $\Gamma^* u = 0$. Define

$$v = [\tau(2\Delta f - |\nabla f|^2 + R) + f - n]u \tag{3.2.7}$$

Proposition 3.2.6. *[126] With the notation above, we have that*

$$v = [(T-t)(2\Delta f - |\nabla f|^2 + R) + f - n]u$$

satisfies

$$\Gamma^* v = -2(T-t)|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2(T-t)}g_{ij}|^2 u. \tag{3.2.8}$$

Moreover, if u tends to a $\delta-function$ as $t\to T$, then $v_H \le 0$ for all $t<T$, where $v_H = v$ with $u(x,\tau)$ replaced by $H(x,\tau;y,\sigma)$, the fundamental solution.

Proof.

$$v = [\tau(2\Delta f - |\nabla f|^2 + R) + f - n]u$$

Let $P = \tau(2\Delta f - |\nabla f|^2 + R) + f - n$ and $\partial_t \tau = -1$ since $\tau = T - t$.

$$\begin{aligned}
\Gamma^* P u &= (-\partial_t - \Delta + R)(Pu) \\
&= -\partial_t P \cdot u - P\partial_t u - \Delta P \cdot u - 2\langle \nabla P, \nabla u\rangle - P\Delta u + RPu \\
&= -\partial_t P \cdot u - \Delta P \cdot u - 2\langle \nabla P, \nabla u\rangle + P\,\Gamma^* u \\
&= -(\partial_t + \Delta)P \cdot u - 2\langle \nabla P, \nabla u\rangle.
\end{aligned}$$

We can write

$$\begin{aligned}
u^{-1}\Gamma^* P u &= -(\partial_t + \Delta)P - 2\langle \nabla P, u^{-1}\nabla u\rangle \\
&= -(\partial_t + \Delta)P + 2\langle \nabla P, \nabla f\rangle
\end{aligned}$$

since $f = -\ln u - \frac{n}{2}\ln(4\pi\tau)$ implies that $\nabla f = -\frac{\nabla u}{u} = -u^{-1}\nabla u$.

Let us compute $(\partial_t + \Delta)P$

$$\begin{aligned}
\frac{\partial P}{\partial t} &= \frac{\partial}{\partial t}\Big(\tau(2\Delta f - |\nabla f|^2 + R) + f - n\Big) \\
&= \frac{\partial}{\partial t}\tau\Big(2\Delta f - |\nabla f|^2 + R\Big) + \tau\frac{\partial}{\partial t}\Big(2\Delta f - |\nabla f|^2 + R\Big) + \frac{\partial}{\partial t}f \\
&= -\Big(2\Delta f - |\nabla f|^2 + R\Big) + \tau\frac{\partial}{\partial t}\Big(2\Delta f - |\nabla f|^2\Big) + \tau\frac{\partial}{\partial t}R + \frac{\partial}{\partial t}f.
\end{aligned}$$

Note that

$$\begin{aligned}
2\frac{\partial}{\partial t}\Delta f &= 2\frac{\partial}{\partial t}\Big[g^{ij}(\partial_i\partial_j f - \Gamma^k_{ij}\partial_k f)\Big] \\
&= 2\frac{\partial}{\partial t}(g^{ij})\partial_i\partial_j f + 2g^{ij}\partial_i\partial_j\frac{\partial}{\partial t}f - g^{ij}(\frac{\partial}{\partial t}\Gamma^k_{ij})\partial_k f - g^{ij}\Gamma^k_{ij}\partial_k\frac{\partial}{\partial t}f \\
&= 4R^{ij}\partial_i\partial_j f + 2g^{ij}(\partial_i\partial_j f - \Gamma^k_{ij}\partial_k)\frac{\partial}{\partial t}f \\
&= 4R^{ij}\partial_i\partial_j f + 2\Delta\frac{\partial}{\partial t}f,
\end{aligned}$$

where we have used $\frac{\partial}{\partial t}\Gamma^k_{ij} = 0$ and $\frac{\partial}{\partial t}(g^{ij}) = R_{ij}$ by Lemma 1.3.1, since it is possible to work in local normal coordinates.

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla f|^2 &= \frac{\partial}{\partial t}\Big(g^{ij}\partial_i f\partial_j f\Big) \\
&= 2R^{ij}\partial_i f\partial_j f + 2g^{ij}\partial_i f\partial_j\frac{\partial}{\partial t}f \\
&= 2R^{ij}\partial_i f\partial_j f + 2\langle \nabla f, \nabla\frac{\partial}{\partial t}f\rangle,
\end{aligned}$$

$$\frac{\partial}{\partial t}\Big(2\Delta f - |\nabla f|^2\Big) = 4R^{ij}\partial_i\partial_j f + 2\Delta\frac{\partial}{\partial t}f - 2R^{ij}\partial_i f\partial_j f - 2\langle \nabla f, \nabla\frac{\partial}{\partial t}f\rangle.$$

Combining these, we have

$$\frac{\partial}{\partial t}P = -(2\Delta f - |\nabla f|^2 + R) + \tau\Big(4R^{ij}\partial_i\partial_j f + 2\Delta\frac{\partial}{\partial t}f - 2R^{ij}\partial_i f\partial_j f - \langle \nabla f, \nabla\frac{\partial}{\partial t}f\rangle + \frac{\partial}{\partial t}R\Big) + \frac{\partial}{\partial t}f.$$

Next is to compute

$$\begin{aligned}\Delta P &= \Delta\Big[\tau(2\Delta f - |\nabla f|^2 + R) + f - n\Big]\\ &= \tau\Big[2\Delta(\Delta f) - \Delta|\nabla f|^2 + \Delta R\Big] + \Delta f\\ &= \tau\Big[2\Delta(\Delta f) - 2|\nabla\nabla f|^2 - 2\langle\nabla f, \Delta\nabla f\rangle - 2Ric(\nabla f, \nabla f) + \Delta R\Big] + \Delta f,\end{aligned}$$

we have used Bochner identity (0.2.17) in the last line. Therefore

$$\begin{aligned}\Big(\frac{\partial}{\partial t} + \Delta\Big)P &= -(2\Delta f - |\nabla f|^2 + R) + \tau\Big[4R^{ij}\partial_i\partial_j f + 2\Delta\frac{\partial}{\partial t}f - 2R^{ij}\partial_i f\partial_j f - 2\langle\nabla f, \nabla\frac{\partial}{\partial t}f\rangle\\ &+ 2\Delta(\Delta f) - 2|\nabla\nabla f|^2 - 2\langle\nabla f, \Delta\nabla f\rangle - 2Ric(\nabla f, \nabla f) + \frac{\partial}{\partial t}R + \Delta R\Big] + \frac{\partial}{\partial t}f + \Delta f\\ &= (-\Delta f + |\nabla f|^2 - R + \frac{\partial}{\partial t}f) + \tau\Big[4R^{ij}\partial_i\partial_j f - 2R^{ij}\partial_i f\partial_j f + 2\Delta(\frac{\partial}{\partial t}f + \Delta f + R)\\ &+ 2|Rc|^2 - 2\langle\nabla f, \nabla\frac{\partial}{\partial t}f\rangle - 2|\nabla\nabla f|^2 - 2\langle\nabla f, \Delta\nabla f\rangle - 2Ric(\nabla f, \nabla f)\Big]\\ &= (-\Delta f + |\nabla f|^2 - R + \frac{\partial}{\partial t}f) + \tau\Big[4R^{ij}\partial_i\partial_j f - 2R^{ij}\partial_i f\partial_j f + 2\Delta|\nabla f|^2 + 2|Rc|^2\\ &- \Delta|\nabla f|^2 - 2\langle\nabla f, \nabla|\nabla f|^2\rangle + 2\langle\nabla f, \nabla\Delta f\rangle + 2\langle\nabla R, \nabla f\rangle\Big].\end{aligned}$$

Now

$$\begin{aligned}2\langle\nabla P, \nabla f\rangle &= 2\langle\nabla(\tau(2\Delta f - |\nabla f|^2 + R) + f), \nabla f\rangle\\ &= 2\tau\Big[2\langle\nabla\Delta f, \nabla f\rangle - \langle\nabla|\nabla f|^2, \nabla f\rangle + \langle\nabla R, \nabla f\rangle\Big] + 2|\nabla f|^2.\end{aligned}$$

Therefore

$$\begin{aligned}&-\Big(\frac{\partial}{\partial t} + \Delta\Big)P + 2\langle\nabla P, \nabla f\rangle\\ &= (\Delta f - |\nabla f|^2 + R - \frac{\partial}{\partial t}f) - \tau\Big[4R^{ij}\partial_i\partial_j f - 2R^{ij}\partial_i f\partial_j f + \Delta|\nabla f|^2\\ &\quad + 2|Rc|^2 - 2\langle\nabla f, \nabla|\nabla f|^2\rangle + 2\langle\nabla f, \nabla\Delta f\rangle + 2\langle\nabla R, \nabla f\rangle\Big]\\ &\quad + 2\tau\Big[2\langle\nabla\Delta f, \nabla f\rangle - \langle\nabla|\nabla f|^2, \nabla f\rangle + \langle\nabla R, \nabla f\rangle\Big] + 2|\nabla f|^2\\ &= (\Delta f - |\nabla f|^2 + R - \frac{\partial}{\partial t}f) - \tau\Big[4R^{ij}\partial_i\partial_j f - 2R^{ij}\partial_i f\partial_j f + \Delta|\nabla f|^2\\ &\quad + 2|Rc|^2 + 2\langle\nabla\Delta f, \nabla f\rangle\Big]\\ &= (2\Delta f + 2R - \frac{n}{2\tau}) - \tau\Big[4R^{ij}\partial_i\partial_j f + 2|\nabla\nabla f|^2 + 2|Rc|^2\Big]\\ &= -2\tau\Big[2R^{ij}\partial_i\partial_j f + |\nabla\nabla f|^2 + |Rc|^2 - \frac{1}{\tau}(\Delta f + R - \frac{n}{4\tau})\Big]\\ &= -2\tau\Big[(R_{ij} + \nabla_i\nabla_j f)^2 - \frac{1}{\tau}(\Delta f + R - \frac{n}{4\tau})\Big]\\ &= -2\tau\Big[(R_{ij} + \nabla_i\nabla_j f)^2 - \frac{1}{\tau}(\Delta f + R) + \frac{n}{4\tau^2}\Big]\\ &= -2\tau\Big|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\Big|^2.\end{aligned}$$

Hence

$$u^{-1}\Gamma^*(Pu) = -2\tau|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2$$

and

$$\Gamma^* v = -2\tau\Big|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\Big|^2 u.$$

□

The proposition above implies a monotonicity formula for

$$v_H = [\tau(2\Delta f - |\nabla f|^2 + R) + f - n]H \leq 0$$

where $H = H(x, \tau; y, \sigma)$ is the conjugate heat kernel. This is Perelman's differential Harnack Inequality. It provides an alternative proof of the monotonicity formula for his $\mathcal{W}$-entropy. Indeed, from the above, we can develop an integral quantity, namely

$$\begin{aligned}\int_M v d\mu &= \int_M [\tau(2\Delta f - |\nabla f|^2 + R) + f - n]u d\mu \\ &= \int_M [\tau(2\Delta f - |\nabla f|^2 + R) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu \\ &= \mathcal{W}(g(t), f(t), \tau(t)).\end{aligned}$$

The consequence of which is a localised version of Perelman's $\mathcal{W}$-entropy monotonicity formula. Thus

$$\begin{aligned}\frac{d\mathcal{W}}{dt} = \frac{\partial}{\partial t}\int_M v d\mu &= \int_M (\partial_t v - Rv) d\mu \\ &= \int_M \Big(-\Gamma^* v - \Delta v\Big) d\mu \\ &= \int_M -\Gamma^* v d\mu \\ &= 2(T-t)\int_M \Big|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2(T-t)}g_{ij}\Big|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu\end{aligned}$$

Corollary 3.2.7. *[126] Under the same assumption as above on a closed manifold M, if u tends to a δ-function as $t \to T$ and $v \leq 0$ for all $t \in T$. Then it holds for any smooth curve $\gamma(t)$ in M as follows*

$$-\frac{d}{dt}f(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\gamma'(t)|^2) - \frac{1}{2(T-t)}f(\gamma(t), t). \tag{3.2.9}$$

Proof. We have from the monotonicity formula that

$$P = (T-t)(2\Delta f - |\nabla f|^2 + R) + f - n \leq 0,$$

which implies

$$\Delta f \leq \frac{1}{2}|\nabla f|^2 - \frac{1}{2}R - \frac{f}{2(T-t)} + \frac{n}{2(T-t)}$$

and the evolution of f that

$$\frac{\partial}{\partial t}f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2(T-t)}$$

which in turn implies

$$\frac{\partial}{\partial t}f + \frac{1}{2}R - \frac{1}{2}|\nabla f|^2 - \frac{f}{2(T-t)} \geq 0. \tag{3.2.10}$$

On the other hand

$$\begin{aligned} -\frac{d}{dt}f(\gamma(t),t) &= -\partial_t f - \langle \nabla f, \gamma'(t)\rangle \\ &\leq -\partial_t f + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|\gamma'(t)|^2, \end{aligned}$$

where we have used Young's inequality to obtain $-\langle \nabla f, \gamma'(t)\rangle \leq \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|\gamma'(t)|^2$. □

The inequalities of this type are referred to as Perelman's Harnack inequalities, they were originally proved by Li and Yau [112] for the solution of linear parabolic equations on Riemannian manifolds. Hamilton [89, 91] used his version called Li-Yau-Hamilton Harnack inequality for the solutions of backward heat equations on manifold to prove monotonicity formulas for certain parabolic flows. This result is very important in the study of the Ricci flow because the solutions which tend to dirac δ-function are essential to understand monotone functionals, which are part of the machinery developed by Perelman to tackle Poincaré conjecture.

3.3 Differential Harnack Estimates

Let $(M, g(t)), t \in [0,T]$ be a solution of the Ricci flow on a closed manifold. Let u be a positive solution to the conjugate heat equation, then we have the following coupled system.

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij} \\ \\ -\dfrac{\partial u}{\partial t} - \Delta_{g(t)}u + R_{g(t)}u = 0, \end{cases} \tag{3.3.1}$$

which we refer to as Perelman's conjugate heat equation coupled to the Ricci flow. We will prove Harnack estimates for all positive solution of the adjoint heat equation in the above system. A differential Harnack estimate of Li-Yau type yields a space-time gradient estimate for a positive solution to a heat-type equation, which when integrated compares the solution at different points in space and time. We will later apply the maximum principle to obtain a localized version of the estimates.

3.3.1 Harnack Inequality and Gradient Estimates.

The main result of this subsection is contained in Theorem 3.3.1 and as an application we arrived at Corollary 3.3.3, which gives the corresponding Li-Yau type gradient estimate for all positive solutions to the conjugate heat equation in the system (3.3.1).

Theorem 3.3.1. *Let $u \in C^{2,1}(M \times [0,T])$ be a positive solution to the conjugate heat equation $\Gamma^* u = (-\partial_t - \Delta + R)u = 0$ and the metric $g(t)$ evolve by the Ricci flow in the interval $[0,T)$ on a closed manifold M with* *nonnegative scalar curvature. Suppose further that* $u = (4\pi\tau)^{-\frac{n}{2}}e^{-f}$*, where* $\tau = T - t$*, then for all points* $(x,t) \in (M \times [0,T])$*, we have the Harnack quantity*

$$P = 2\Delta f - |\nabla f|^2 + R - \frac{2n}{\tau} \leq 0. \tag{3.3.2}$$

Then P *evolves as*

$$\frac{\partial}{\partial t}P = -\Delta P + 2\langle\nabla f, \nabla P\rangle + 2\Big|R_{ij} + \nabla_i\nabla_j f - \frac{1}{\tau}g_{ij}\Big|^2 + \frac{2}{\tau}P + \frac{2}{\tau}|\nabla f|^2 + \frac{4n}{\tau^2} + \frac{2}{\tau}R. \tag{3.3.3}$$

for all $t > 0$*. Moreover* $P \leq 0$ *for all* $t \in [0,T]$*.*

Note that $u = (4\pi\tau)^{-\frac{n}{2}}e^{-f}$ implies $\ln u = -f - \frac{n}{2}\ln(4\pi\tau)$ and we can write (3.3.2) as

$$\frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} - R - \frac{2n}{\tau} \leq 0, \tag{3.3.4}$$

which is similar to the celebrated Li-Yau [112] gradient estimate for the heat equation on manifold with nonnegative Ricci curvature.

We need the usual routine computations as in the following;

Lemma 3.3.2. *Let* (g,f) *solve the system (3.3.1) above. Suppose further that* $u = (4\pi\tau)^{-\frac{n}{2}}e^{-f}$ *with* $\tau = T - t$*. Then we have*

$$(\frac{\partial}{\partial t} + \Delta)\Delta f = 2R^{ij}\nabla_i\nabla_j f + \Delta|\nabla f|^2 - \Delta R$$

and

$$(\frac{\partial}{\partial t} + \Delta)|\nabla f|^2 = 4R_{ij}\nabla_i f\nabla_j f + 2\langle\nabla f, \nabla|\nabla f|^2\rangle + 2|\nabla\nabla f|^2 - 2\langle\nabla f, |\nabla R|^2\rangle.$$

Proof. By direct calculation

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta f) = \frac{\partial}{\partial t}(g^{ij}\partial_i\partial_j f) &= \frac{\partial}{\partial t}(g^{ij})\partial_i\partial_j f + g^{ij}\partial_i\partial_j\frac{\partial}{\partial t}f \\
&= 2R^{ij}\partial_i\partial_j f + \Delta(-\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}) \\
&= 2R^{ij}\nabla_i\nabla_j f - \Delta(\Delta f) + \Delta|\nabla f|^2 - \Delta R
\end{aligned}$$

then,

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} + \Delta\Big)\Delta f &= 2R^{ij}\nabla_i\nabla_j f - \Delta(\Delta f) + \Delta|\nabla f|^2 - \Delta R + \Delta(\Delta f) \\
&= 2R^{ij}\nabla_i\nabla_j f + \Delta|\nabla f|^2 - \Delta R
\end{aligned}$$

Part 1 is proved.

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla f|^2 &= 2R^{ij}\partial_i f\partial_j f + 2g^{ij}\partial_i f\partial_j\frac{\partial}{\partial t}f \\
&= 2R^{ij}\partial_i f\partial_j f + 2\langle\nabla f, \nabla(-\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau})\rangle \\
&= 2R^{ij}\nabla_i f\nabla_j f + 2\langle\nabla f, \nabla|\nabla f|^2\rangle - 2\langle\nabla f, \nabla\Delta f\rangle - 2\langle\nabla f, \nabla R\rangle
\end{aligned}$$

then,

$$\Big(\frac{\partial}{\partial t}+\Delta\Big)|\nabla f|^2 = 2R^{ij}\partial_i f\partial_j f + 2\langle\nabla f, \nabla|\nabla f|^2\rangle - 2\langle\nabla f, \nabla\Delta f\rangle - 2\langle\nabla f, \nabla R\rangle + \Delta|\nabla f|^2.$$

Using the Bochner identity

$$\Delta|\nabla f|^2 = 2|\nabla\nabla f|^2 + 2\langle\nabla f, \nabla\Delta f\rangle + 2Rc(\nabla f, \nabla f)$$

we obtain the identity in part (2). □

Proof. Proof of Theorem 3.3.1. Since $P = 2\Delta f - |\nabla f|^2 + R - \frac{2n}{\tau}$ and by direct computation and using Lemma 3.3.2, we have

$$\begin{aligned}
\Big(\frac{\partial}{\partial t}+\Delta\Big)P &= 2\Big(\frac{\partial}{\partial t}+\Delta\Big)\Delta f - \Big(\frac{\partial}{\partial t}+\Delta\Big)|\nabla f|^2 + \Big(\frac{\partial}{\partial t}+\Delta\Big)R - \frac{\partial}{\partial t}\Big(\frac{2n}{\tau}\Big)\\
&= 4R^{ij}\nabla_i\nabla_j f + 2\Delta|\nabla f|^2 - 2\Delta R - 4Rc(\nabla f, \nabla f) - 2\langle\nabla f, \nabla|\nabla f|^2\rangle\\
&\qquad - 2|\nabla\nabla f|^2 + 2\langle\nabla f, \nabla R\rangle + 2\Delta R + 2|Rc|^2 + \frac{2n}{\tau^2}\\
&= 4R^{ij}\nabla_i\nabla_j f + 2|Rc|^2 + \frac{2n}{\tau^2} - 2\langle\nabla f, \nabla|\nabla f|^2\rangle + 2\langle\nabla f, \nabla R\rangle\\
&\qquad + 2\Delta|\nabla f|^2 - 4Rc(\nabla f, \nabla f) - 2|\nabla\nabla f|^2\\
&= 4R^{ij}\nabla_i\nabla_j f + 2|Rc|^2 + \frac{2n}{\tau^2} - 2\langle\nabla f, \nabla|\nabla f|^2\rangle + 2\langle\nabla f, \nabla R\rangle\\
&\qquad + \Delta|\nabla f|^2 - 2Rc(\nabla f, \nabla f) + 2\langle\nabla f, \nabla\Delta f\rangle\\
&= 4R^{ij}\nabla_i\nabla_j f + 2|Rc|^2 + \frac{2n}{\tau^2} + 2|\nabla\nabla f|^2 - 2\langle\nabla f, \nabla|\nabla f|^2\rangle\\
&\qquad + 2\langle\nabla f, \nabla R\rangle + 4\langle\nabla f, \nabla\Delta f\rangle\\
&= 4R^{ij}\nabla_i\nabla_j f + 2|Rc|^2 + \frac{2n}{\tau^2} + 2|\nabla\nabla f|^2 + 2\langle\nabla f, \nabla P\rangle\\
&= 2|R_{ij} + \nabla_i\nabla_j f|^2 + \frac{2n}{\tau^2} + 2\langle\nabla f, \nabla P\rangle.
\end{aligned}$$

By direct computation we notice that

$$\Big|R_{ij} + \nabla_i\nabla_j f - \frac{1}{\tau}g_{ij}\Big|^2 = |R_{ij} + \nabla_i\nabla_j f|^2 - \frac{2}{\tau}(R + \Delta f) + \frac{n}{\tau^2},$$

which implies

$$2|R_{ij} + \nabla_i\nabla_j f|^2 + \frac{2n}{\tau^2} = 2\Big|R_{ij} + \nabla_i\nabla_j f - \frac{1}{\tau}g_{ij}\Big|^2 + \frac{4}{\tau}(R + \Delta f).$$

Also

$$\begin{aligned}
\frac{4}{\tau}(R + \Delta f) &= \frac{2}{\tau}(R + 2\Delta f) + \frac{2}{\tau}R\\
&= \frac{2}{\tau}P + \frac{2}{\tau}|\nabla f|^2 + \frac{4n}{\tau^2} + \frac{2}{\tau}R.
\end{aligned}$$

Therefore, by putting these together we have

$$\Big(\frac{\partial}{\partial t}+\Delta\Big)P = 2\langle\nabla f,\nabla P\rangle + 2\Big|R_{ij}+\nabla_i\nabla_j f-\frac{1}{\tau}g_{ij}\Big|^2+\frac{2}{\tau}P+\frac{2}{\tau}|\nabla f|^2+\frac{4n}{\tau^2}+\frac{2}{\tau}R,$$

which proves the evolution equation for P.

To prove that $P\leq 0$ for all time $t\in[0,T]$, we know that for small τ, $P(\tau)<0$. We can use the Maximum principle to conclude this. Notice that by the Perelman's $\mathcal{W}$-entropy monotonicity

$$R_{ij}+\nabla_i\nabla_j f-\frac{1}{\tau}g_{ij}\geq 0$$

and strictly positive except when $g(t)$ is a shrinking gradient soliton. So our conclusion will follow from a theorem in [46, Theorem 4].

For completeness we show this; by Cauchy-Schwarz inequality and the fact that $R=g^{ij}R_{ij}$ and $\sum_{i,j}g_{ij}=n$, we have

$$|R_{ij}+\nabla_i\nabla_j f-\frac{1}{\tau}g_{ij}|^2\geq\frac{1}{n}(R+\Delta f-\frac{n}{\tau})^2$$

and by definition of P

$$P+R+|\nabla f|^2 = 2(R+\Delta f-\frac{n}{\tau}).$$

Hence

$$2\Big|R_{ij}+\nabla_i\nabla_j f-\frac{1}{\tau}\Big|^2\geq\frac{1}{2n}(P+R+|\nabla f|^2)^2.$$

Putting the last identity into the evolution equation for P yields

$$\begin{aligned}\frac{\partial P}{\partial t} &\geq -\Delta P+2\langle\nabla P,\nabla f\rangle+\frac{1}{2n}(P+R+|\nabla f|^2)^2+\frac{2}{\tau}(P+R+|\nabla f|^2)+\frac{4n}{\tau^2}\\ &= -\Delta P+2\langle\nabla P,\nabla f\rangle+\frac{1}{2n}(P+R+|\nabla f|^2+\frac{2n}{\tau^2})^2+\frac{2n}{\tau^2}.\end{aligned}$$

This implies that

$$\frac{\partial P}{\partial\tau}\leq\Delta P-2\langle\nabla P,\nabla f\rangle-\frac{1}{2n}(P+R+|\nabla f|^2+\frac{2n}{\tau^2})^2-\frac{2n}{\tau^2}.$$

Then

$$\frac{\partial P}{\partial\tau}\leq\Delta P-2\langle\nabla P,\nabla f\rangle. \tag{3.3.5}$$

Applying the maximum principle to the evolution equation (3.3.5) yields clearly that $P\leq 0$ for all τ, hence, for all $t\in[0,T)$. □

The result here is an improvement on Kuang and Zhang's [103] since it holds with no assumption on the curvature. This result can also be compared with those of [45, 44] where they define a general Harnack quantity for conjugate heat equation and derive its evolution under the Ricci flow. We have the following as an immediate consequence of the above theorem.

Corollary 3.3.3. *(Harnack Estimates). Let $u \in C^{2,1}(M \times [0,T))$ be a positive solution to the conjugate heat equation $\Gamma^* u = 0$ and $g(t), t \in [0,T)$ evolve by the Ricci flow on a closed manifold M with nonnegative scalar curvature R. Then for any points (x_1, t_1) and (x_2, t_2) in $M \times (0,T)$ such that $0 < t_1 \leq t_2 < T$, the following estimate holds*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \leq \Big(\frac{\tau_1}{\tau_2}\Big)^n \exp\Big[\int_0^1 \frac{|\gamma'(s)|^2}{2(\tau_1-\tau_2)} ds + \frac{(\tau_1-\tau_2)}{2} R\Big], \tag{3.3.6}$$

where $\tau_i = T - t_i, i = 1,2$ and $\gamma : [0,1]$ is a geodesic curve connecting points x_1 and x_2 in M.

Proof. Let $\gamma : [0,1]$ be a minimizing geodesic connecting points x_1 and x_2 in M such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$ with $|\gamma'(s)|$ being the length of the vector $\gamma'(s)$ at time $\tau(s) = (1-s)\tau_1 + s\tau_2, \quad 0 \leq \tau_2 \leq \tau_1 \leq T$. Define $\eta(s) = \ln u(\gamma(s), (1-s)\tau_1 + s\tau_2)$. Clearly, $\eta(0) = \ln u(x_1, t_1)$ and $\eta(1) = \ln u(x_2, t_2)$.

Integrating along $\eta(s)$, we obtain

$$\ln u(x_2,t_2) - \ln u(x_1,t_1) = \int_0^1 \Big(\frac{\partial}{\partial s} \ln u(\gamma(s), (1-s)\tau_1 + s\tau_2)\Big) ds$$

i.e.,

$$\ln\Big(\frac{u(x_2,t_2)}{u(x_1,t_1)}\Big) = \ln u(\gamma(t), t)\Big|_0^1.$$

By direct computation, we have on the path $\gamma(s)$ that

$$\begin{aligned}\frac{\partial}{\partial s}\eta(s) = \frac{d}{ds}\ln u &= \nabla \ln u \cdot \gamma'(s) + \frac{\partial}{\partial t}\ln u \\ &= \frac{\nabla u}{u} \cdot \gamma'(s) - \frac{u_t(\tau_1-\tau_2)}{u} \\ &= (\tau_1-\tau_2)\Big(\frac{\nabla u}{u} \cdot \frac{\gamma'(s)}{\tau_1-\tau_2} - \frac{u_t}{u}\Big).\end{aligned}$$

From Theorem 3.3.1, we have

$$\frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} \leq R + \frac{2n}{\tau},$$

which implies

$$-\frac{u_t}{u} \leq \frac{1}{2}(R + \frac{2n}{\tau}) - \frac{|\nabla u|^2}{2u^2}.$$

By this, we have

$$\begin{aligned}\frac{d}{ds}\ln u &\leq (\tau_1-\tau_2)\Big(\frac{\nabla u}{u} \cdot \frac{\gamma'(s)}{(\tau_1-\tau_2)} - \frac{|\nabla u|^2}{2u^2} + \frac{1}{2}(R + \frac{2n}{\tau})\Big) \\ &= -\frac{(\tau_1-\tau_2)}{2}\Big(\frac{\nabla u}{u} - \frac{\gamma'(s)}{(\tau_1-\tau_2)}\Big)^2 + \frac{(\tau_1-\tau_2)}{2}\frac{|\gamma'(s)|^2}{(\tau_1-\tau_2)^2} + \frac{(\tau_1-\tau_2)}{2}\Big(R + \frac{2n}{\tau}\Big) \\ &\leq \frac{|\gamma'(s)|^2}{2(\tau_1-\tau_2)} + \frac{(\tau_1-\tau_2)}{2}\Big(R + \frac{2n}{\tau}\Big).\end{aligned}$$

Now integrating with respect to s, from 0 to 1, we have

$$\ln u\Big|_0^1 \leq \int_0^1 \frac{|\gamma'(s)|^2}{2(\tau_1-\tau_2)} + \frac{(\tau_1-\tau_2)}{2}\int_0^1 R ds + \ln\Big(\frac{\tau_1}{\tau_2}\Big)^n, \tag{3.3.7}$$

exponentiating both sides, we get

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \leq \Big(\frac{\tau_1}{\tau_2}\Big)^n \exp\Big[\int_0^1 \frac{|\gamma'(s)|^2}{2(\tau_1-\tau_2)}ds + \frac{(\tau_1-\tau_2)}{2}R\Big].$$

□

3.3.2 Localising the Harnack and Gradient Estimate

We establish a localised form of the Harnack and gradient estimates obtained in the last subsection. The main idea is the application of the Maximum principle on some smooth cut-off function. It was also the basic idea used by Li and Yau in [112], this type of approach has since become tradition. It has been systematically developed over the years since the paper of Cheng and Yau [56], see also [132, 154], however our computation is more involved as the metric is also evolving. This local estimate is desirable to extend our result to the case the manifold is noncompact [66], for example, in the monotonicity formula and mean value theorem considered in [75] a local version is needed.

A natural function that will be defined on M is the distance function from a given point, namely, let $p \in M$ and define $d(x,p)$ for all $x \in M$, where $dist(\cdot,\cdot)$ is the geodesic distance. Note that $d(x,p)$ is only Lipschitz continuous, i.e., everywhere continuously differentiable except on the cut locus of p and on the point where x and p coincide. It is then easy to see that

$$|\nabla d| = g^{ij}\partial_i d\, \partial_j d = 1 \;\; on \quad M \setminus \{\{p\} \cup cut(p)\}.$$

Let $d(x,y,t)$ be the geodesic distance between x and y with respect to the metric $g(t)$, we define a smooth cut-off function $\varphi(x,t)$ with support in the geodesic cube

$$\mathcal{Q}_{2\rho,T} := \{(x,t) \in M \times (0,T] : d(x,p,t) \leq 2\rho\},$$

for any C^2-function $\psi(s)$ on $[0,+\infty)$ with

$$\psi(s) = \begin{cases} 1, & s \in [0,1], \\ 0, & s \in [2,+\infty) \end{cases}$$

and

$$\psi'(s) \leq 0, \quad \frac{|\psi'|^2}{\psi} \leq C_1 \quad and \;\; |\psi''(s)| \leq C_2,$$

where C_1, C_2 are absolute constants depending only on the dimension of the manifold, such that

$$\varphi(x,t) = \psi\Big(\frac{d(x,p,t)}{\rho}\Big) \quad and \quad \varphi\Big|_{\mathcal{Q}_{2\rho,T}} = 1.$$

We will apply the maximum principle and invoke Calabi's trick [37] to assume everywhere smoothness of $\varphi(x,t)$ since $\psi(s)$ is in general Lipschitz. We need Laplacian comparison theorem to do some calculation on $\varphi(x,t)$. Here is the statement of the theorem; Let M be a complete n-dimensional Riemannian manifold whose Ricci curvature is bounded from below by $Rc \geq (n-1)k$ for some constant $k \in \mathbb{R}$. Then the Laplacian of the distance function satisfies

$$\Delta d(x,p) \leq \begin{cases} (n-1)\sqrt{k}\cot(\sqrt{k}\rho), & k>0 \\ (n-1)\rho^{-1}, & k=0 \\ (n-1)\sqrt{|k|}\coth(\sqrt{|k|}\rho), & k<0. \end{cases} \tag{3.3.8}$$

For detail of the Laplacian comparison theorem see [69, Theorem 1.128] or the books [111, 132]. We are now set to prove the localized version of the gradient estimate for the system (3.3.1).

Theorem 3.3.4. *Let $u \in C^{2,1}(M \times [0,T])$ be a positive solution to the conjugate heat equation $\Gamma^* u = (-\partial_t - \Delta + R)u = 0$ defined in geodesic cube $\mathcal{Q}_{2\rho,T}$ and the metric $g(t)$ evolves by the Ricci flow in the interval $[0,T]$ on a closed manifold M with bounded Ricci curvature, say $Rc \geq -Kg$, for some constant $K>0$. Suppose further that $u = (4\pi\tau)^{-\frac{n}{2}}e^{-f}$, where $\tau = T-t$, then for all points in $\mathcal{Q}_{2\rho,T}$ we have the following estimate*

$$\frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} - R \leq \frac{4n}{1-4\delta n}\left\{\frac{1}{\tau} + C\left(\frac{1}{\rho^2} + \frac{\sqrt{K}}{\rho} + \frac{K}{\rho} + \frac{1}{T}\right)\right\}, \tag{3.3.9}$$

where C is an absolute constant depending only on the dimension of the manifold and δ such that $\delta < \frac{1}{4n}$.

Proof. Recall the evolution equation for the differential Harnack quantity

$$P = 2\Delta f - |\nabla f|^2 + R - \frac{2n}{\tau},$$

$$\frac{\partial}{\partial t}P \geq -\Delta P + 2\langle \nabla f, \nabla P\rangle + 2|R_{ij} + \nabla_i\nabla_j f - \frac{1}{\tau}g_{ij}|^2 + \frac{2}{\tau}P + \frac{4n}{\tau^2} + \frac{2}{\tau}|\nabla f|^2,$$

using the non negativity of the scalar curvature Multiplying the quantity P by $t\varphi$, since φ is time-dependent we have at any point where $\varphi \neq 0$ that

$$\begin{aligned}
\frac{1}{\tau}\frac{\partial}{\partial t}(\tau\varphi P) &= \varphi\frac{\partial P}{\partial t} + \frac{\partial \varphi}{\partial t}P - \frac{\varphi P}{\tau} \\
&\geq \varphi\Big(-\Delta P + 2\langle\nabla f, \nabla P\rangle + \frac{2}{\tau}P + \frac{4n}{\tau^2} + \frac{2}{\tau}|\nabla f|^2\Big) \\
&\quad + 2\varphi|R_{ij} + \nabla_i\nabla_j f - \frac{1}{\tau}g_{ij}|^2 + \frac{\partial\varphi}{\partial t}P - \frac{\varphi P}{\tau} \\
&= -\Delta(\varphi P) + 2\nabla\varphi\nabla P + 2\langle\nabla f, \nabla P\rangle\varphi + P(\Delta + \partial_t)\varphi \\
&\quad + \frac{4n}{\tau^2}\varphi + \frac{\varphi P}{\tau} + \frac{2}{\tau}\varphi|\nabla f|^2 + 2\varphi|R_{ij} + \nabla_i\nabla_j f - \frac{1}{\tau}g_{ij}|^2.
\end{aligned}$$

The last equality is due to derivative test on (φP) at the minimum point as obtained in the condition (3.3.12) below. The approach is to estimate $\frac{\partial}{\partial t}(\tau\varphi P)$ at the point where minimum (or maximum) value for $(\tau\varphi P)$ is attained and do some analysis at the minimum (or maximum) point. We know that the support of $(\tau\varphi P)(x,t)$ is contained in $\mathcal{Q}_{2\rho} \times (0,T]$ since $Supp(\varphi) \subset \mathcal{Q}_{2\rho,T}$, where

$$\mathcal{Q}_{2\rho} := \{(x,t) \in M : d(x,p) \leq 2\rho\}, \;\; t = 0.$$

Now let (x_0, t_0) be a point in $\mathcal{Q}_{2\rho,T}$ at which $(\tau\varphi P)$ attains its minimum value. At this point, we have to assume that P is positive since if $P \leq 0$, we have the same estimate and $(\tau\varphi P)(x_0, t_0) \leq 0$ implies $(\tau\varphi P)(x, t) \leq 0$ for all $x \in M$ such that the distance $d(x, x_0, t) \leq 2\rho$ and the theorem will follow trivially.

Note that at the minimum point (x_0, t_0) we have by the derivative test that $(0 \leq \varphi \leq 1)$

$$\nabla(\tau\varphi P)(x_0, t_0) = 0, \quad \frac{\partial}{\partial t}(\tau\varphi P)(x_0, t_0) \leq 0 \quad and \quad \Delta(\tau\varphi P)(x_0, t_0) \geq 0. \tag{3.3.10}$$

We shall obtain a lower bound for $\tau\varphi P$ at this minimum point. Therefore

$$\begin{aligned} 0 \geq -\Delta(\varphi P) + 2\nabla\varphi\nabla P + 2\langle\nabla f, \nabla P\rangle\varphi + P(\Delta + \partial_t)\varphi + \frac{\varphi P}{\tau} \\ + \frac{4n}{\tau^2}\varphi + \frac{2}{\tau}\varphi|\nabla f|^2 + 2\varphi|R_{ij} + \nabla_i\nabla_j f - \frac{1}{\tau}g_{ij}|^2. \end{aligned} \tag{3.3.11}$$

By the argument in (3.3.10) and product rule we have

$$\nabla(\varphi P)(x_0, t_0) - P\nabla\varphi(x_0, t_0) = \varphi\nabla P(x_0, t_0)$$

which means $\varphi\nabla P$ can always be replaced by $-P\nabla\varphi$. Similarly,

$$-\varphi\Delta P = -\Delta(\varphi P) + P\Delta\varphi + 2\nabla\varphi\nabla P, \tag{3.3.12}$$

which we have already used before the last inequality. Notice that by direct calculation using product rule

$$\nabla\varphi\nabla P = \frac{\nabla\varphi}{\varphi}\cdot\nabla(\varphi P) - \frac{|\nabla\varphi|^2}{\varphi}P$$

and

$$2\langle\nabla f, \nabla P\rangle\varphi = \langle\nabla f, \nabla(\varphi P)\rangle - \langle\nabla f, \nabla\varphi\rangle P.$$

Putting the last two equations into (3.3.11) we have

$$\begin{aligned} 0 \geq -\Delta(\varphi P) + 2\frac{\nabla\varphi}{\varphi}\cdot\nabla(\varphi P) - 2\frac{|\nabla\varphi|^2}{\varphi}P + 2\langle\nabla f, \nabla(\varphi P)\rangle - 2\langle\nabla f, \nabla\varphi\rangle P \\ + P(\Delta + \partial_t)\varphi + \frac{\varphi P}{\tau} + \frac{4n}{\tau^2}\varphi + \frac{2}{\tau}\varphi|\nabla f|^2 + 2\varphi|R_{ij} + \nabla_i\nabla_j f - \frac{1}{\tau}g_{ij}|^2. \end{aligned}$$

By using the argument in (3.3.10)

$$\left.\begin{aligned} 0 \geq -2\frac{|\nabla\varphi|^2}{\varphi}P - 2\langle\nabla f, \nabla\varphi\rangle P + P(\Delta + \partial_t)\varphi + \frac{\varphi P}{\tau} \\ + \frac{4n}{\tau^2}\varphi + \frac{2}{\tau}\varphi|\nabla f|^2 + 2\varphi|R_{ij} + \nabla_i\nabla_j f - \frac{1}{\tau}g_{ij}|^2 \end{aligned}\right\}. \tag{3.3.13}$$

Observe that for any $\delta > 0$,

$$2|\nabla f||\nabla\varphi|P = 2\varphi|\nabla f|\frac{|\nabla\varphi|}{\varphi}P \leq \delta\varphi|\nabla f|^2 P + \delta^{-1}\frac{|\nabla\varphi|^2}{\varphi}P$$

$$2|\nabla f||\nabla\varphi|P \leq \delta\varphi|\nabla f|^4 P + \delta\varphi P^2 + \delta^{-1}\frac{|\nabla\varphi|^2}{\varphi}P \tag{3.3.14}$$

and also that

$$|R_{ij}+\nabla_i\nabla_j f-\frac{1}{\tau}g_{ij}|^2\geq\frac{1}{n}\Big(R+\Delta f-\frac{n}{\tau}\Big)^2.$$

It is equally clear that

$$P=2\Delta f-|\nabla f|^2+R-\frac{2n}{\tau}=2\Big(R+\Delta f-\frac{n}{\tau}\Big)^2-|\nabla f|^2-R,$$

which implies

$$(P+|\nabla f|^2+R)=2\Big(\Delta f+R-\frac{n}{\tau}\Big).$$

Therefore

$$2\varphi|R_{ij}+\nabla_i\nabla_j f-\frac{1}{\tau}g_{ij}|^2\geq\frac{\varphi}{2n}\Big(P+|\nabla f|^2+R\Big)^2.$$

Notice also that

$$\begin{aligned}(P+|\nabla f|^2+R)^2(y,s)&=(P+|\nabla f|^2+R^+-R^-)^2(y,s)\\&\geq\frac{1}{2}(P+|\nabla f|^2+R^+)^2(y,s)-(R^-)^2(y,s)\\&\geq\frac{1}{2}(P+|\nabla f|^2)^2(y,s)-(R^-)^2(y,s)\\&\geq\frac{1}{2}(P^2+|\nabla f|^4)(y,s)-(\sup_{\mathcal{Q}_{2\rho,T}}R^-)^2\\&\geq\frac{1}{2}(P^2+|\nabla f|^4)(y,s)-n^2K^2,\end{aligned}$$

where we have applied some inequalities, namely; $2(a-b)^2\geq a^2-2b^2$ and $(a+b)^2\geq a^2+b^2$ with $a,b\geq 0$ and a lower bound assumption on Ricci curvature, $R_{ij}\geq -K$, which implies $R=-nK \implies R^-\leq nK$ and $R=-R^-$. Hence

$$2\varphi|R_{ij}+\nabla_i\nabla_j f-\frac{1}{\tau}g_{ij}|^2\geq\frac{\varphi}{4n}P^2+\frac{\varphi}{4n}|\nabla f|^4. \tag{3.3.15}$$

Wherever $P<0$, we then obtain from (3.3.13) - (3.3.15) that

$$\begin{aligned}0\geq\Big(\frac{1}{4n}-\delta\Big)\varphi P^2+&\Big\{(\delta^{-1}-2)\frac{|\nabla\varphi|^2}{\varphi}+(\Delta+\partial_t)\varphi+\frac{\varphi}{\tau}\Big\}P\\&-\Big(\delta-\frac{1}{4n}\Big)\varphi|\nabla f|^4+\frac{2}{\tau}\varphi|\nabla f|^2+\frac{4n}{\tau^2}\varphi,\end{aligned}$$

using the inequality of the form $m|\nabla f|^4-n|\nabla f|^2\geq-\frac{n^2}{4m}$ and multiplying by φ again ($\varphi\neq 0$), we have a quadratic polynomial in (φP) which we use to bound (φP) in the following

$$\left.\begin{aligned}&\Big(\frac{1}{4n}-\delta\Big)(\varphi P)^2+\Big\{(\delta^{-1}-2)\frac{|\nabla\varphi|^2}{\varphi}+(\Delta+\partial_t)\varphi+\frac{\varphi}{\tau}\Big\}(\varphi P)\\&\qquad-\frac{4n}{\tau^2}\Big(\frac{1}{1-4n\delta}-1\Big)\varphi^2\leq 0.\end{aligned}\right\}. \tag{3.3.16}$$

Note that if there is a number $x \in \mathbb{R}$ satisfying inequality $px^2 + qx + r \le 0$, when $p > 0, q > 0$ and $r < 0$, then $q^2 - 4pr > 0$ and we then have the bounds

$$\frac{-q - \sqrt{q^2 - 4pr}}{2p} \le x \le \frac{-q + \sqrt{q^2 - 4pr}}{2p},$$

which clearly implies

$$\frac{-q - \sqrt{-4pr}}{p} \le x \le \frac{q + \sqrt{-4pr}}{p}.$$

Now, choosing δ such that $\delta < \frac{1}{4n}$ and denoting

$$Z = (\delta^{-1} - 2)\frac{|\nabla\varphi|^2}{\varphi} + (\Delta + \partial_t)\varphi,$$

we obtain

$$\tau_0 \varphi P \ge -\frac{4n}{1 - 4\delta n}\Big\{\tau_0 Z + \varphi + 4\varphi\sqrt{\delta n}\Big\}.$$

Moreover, since $\tau_0 \le \tau \le T$ and $0 \le \varphi \le 1$, we have

$$\tau P \ge -\frac{4n}{1 - 4\delta n}\Big\{\tau Z + 1 + C_3\Big\},$$

where C_3 depends on n and δ. It remains to estimate Z via appropriate choice of a cut function $\varphi : M \times [0, T] \to [0, 1]$ such that $\frac{\partial}{\partial t}\varphi, \Delta\varphi$ and $\frac{|\nabla\varphi|^2}{\varphi}$ have appropriate upper bounds. The main difficulty with this kind of approach lies in the fact that for any cut-off function, one gets different kind of estimates and therefore the cut-off function in use must be chosen so related to the result one is looking for.

Define a C^2-function $0 \le \psi \le 1$, on $[0, \infty)$ satisfying

$$\psi'(s) \le 0, \quad \frac{|\psi'|^2}{\psi} \le C_1 \quad and \quad |\psi''(s)| \le C_2$$

and define φ by

$$\varphi(x, t) = \psi\Big(\frac{d(x, x_0, t)}{\rho}\Big)$$

and we have the following after some computations

$$\frac{|\nabla\varphi|^2}{\varphi} = \frac{|\psi'|^2 \cdot |\nabla d|^2}{\rho^2 \varphi} \le \frac{C_2}{\rho^2},$$

and by the Laplacian comparison Theorem (3.3.8) we have

$$\Delta\varphi = \frac{\psi'\Delta d}{\rho} + \frac{\psi''|\nabla d|^2}{\rho^2} \le \frac{C_1}{\rho}\sqrt{K} + \frac{C_2}{\rho^2}$$

Next is to estimate time derivative of φ: consider a fixed smooth path $\gamma : [a, b] \to M$ whose length at time t is given by $d(\gamma) = \int_a^b |\gamma'(t)|_{g(t)} dr$, where r is the arc length. Differentiating we get

$$\frac{\partial}{\partial t}(d(\gamma)) = \frac{1}{2}\int_a^b \Big|\gamma'(t)\Big|^{-1}_{g(t)} \frac{\partial g}{\partial t}\Big(\gamma'(t), \gamma'(t)\Big) dr = \int_\gamma Rc(\xi, \xi) dr,$$

where ξ is the unit tangent vector to the path γ. For detail see [68, Lemma 3.11]. Now

$$\begin{aligned}\frac{\partial}{\partial t}\varphi = \psi'\Big(\frac{d}{\rho}\Big)\frac{1}{\rho}\frac{d}{dt}(d(x,p,t)) &= \psi'\Big(\frac{d}{\rho}\Big)\frac{1}{\rho}\int_\gamma Rc(\xi(s),\xi(s))ds \\ &\leq \frac{\sqrt{C_1}}{\rho}\psi^{\frac{1}{2}}K.\end{aligned}$$

Therefore

$$Z \leq \frac{C_2'}{\rho^2} + \frac{C_1}{\rho}\sqrt{K} + \frac{\sqrt{C_1}}{\rho}K + \frac{C_2}{\rho^2},$$

where C_2' depends on n and δ. Hence

$$\varphi P \geq -\frac{4n}{1-4\delta n}\left\{\frac{1}{\tau} + C\left(\frac{1}{\rho^2} + \frac{\sqrt{K}}{\rho} + \frac{K}{\rho} + \frac{1}{\tau}\right)\right\},$$

where $C = \max\{C_1, C_2, C_3\}$. The required estimate follows since both minimum and maximum points for (φP) are contained in the cube $\mathcal{Q}_{2\rho,T}$. □

3.4 Dual Entropy Formulae and the Gradient Estimates.

By now it is well known that entropy monotonicity formulas are closely related to the gradient estimate for the heat equation (forward or backward). Perelman's $\mathcal{W}$-entropy formula and Li-Yau gradient estimates are ones of several examples. Lei Ni [122] has considered this case for heat equation defined on a static manifold with nonnegative Ricci curvature. This section is fashioned after his paper but can be considered as a generalisation of some of his results in the paper. We introduce a family of dual entropy formula, dual in the sense that it generalises Ni's entropy formula for the forward heat equation on the one hand and also generalises Perelman's $\mathcal{W}$-entropy for the adjoint heat equation on the other hand.

3.4.1 Gradient Estimates for Heat Equation on Static Manifold.

In this subsection, we will use $dV(x)$ instead of our usual notation $d\mu_{g(t)}$ of the volume form to indicate that volume is kept fixed throughout the time of evolution for the heat equation on a closed n-dimensional manifold $(M, g(t))$.

Let $u = u(x,t)$ be a positive solution to the heat equation $\Gamma u(x,t) = 0$, i.e.,

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)u(x,t) = 0. \tag{3.4.1}$$

Let $f : M \times (0,T] \to \mathbb{R}$ be smoothly defined as $u = (4\pi t)^{-\frac{n}{2}}e^{-f}$ with $\int_M u(x,t)dV(x) = 1$. We introduce a generalized family of entropy by

$$\mathcal{W}_\epsilon(f,t) = \int_M \left[\frac{\epsilon^2 t}{4\pi}|\nabla f|^2 + f + \frac{n}{2}\ln\left(\frac{4\pi}{\epsilon^2}\right) - \frac{n\epsilon^2}{4\pi}\right]\frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}dV(x), \tag{3.4.2}$$

where $0 < \epsilon^2 \leq 4\pi$.

Throughout, we impose the condition of nonnegativity on the Ricci curvature of the underlying manifold $(M, g(t))$. We remark that if $\epsilon^2 = 4\pi$, we recover the Perelman's entropy as in the special case considered by Ni in [122]. From this entropy formula we later derive the corresponding differential inequality and gradient estimate for the fundamental solution, which in fact, holds for all positive solutions to the heat equation. The same entropy is used to examine the surprising relation that exists between the entropy formula for heat equation and the conjugate heat equation under the Ricci flow in the subsection after this.

Lemma 3.4.1. *Let $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$ be a positive solution to the heat equation $\Gamma u = 0$ on a closed Riemannian manifold M. Then*

$$(\partial_t - \Delta)|\nabla f|^2 = -2f_{ij}^2 - 2\langle \nabla f, \nabla |\nabla f|^2\rangle - 2R_{ij}f_i f_j \tag{3.4.3}$$

and

$$(\partial_t - \Delta)(\Delta f) = -2f_{ij}^2 - 2\langle \nabla f, \nabla |\nabla f|^2\rangle - 2\langle \nabla f, \nabla \partial_t f\rangle - 2R_{ij}f_i f_j. \tag{3.4.4}$$

Moreover, if $w = 2\Delta f - |\nabla f|^2$, then

$$(\partial_t - \Delta)w = -2f_{ij}^2 - 2R_{ij}f_i f_j - 2\langle \nabla w, \nabla f\rangle. \tag{3.4.5}$$

Proof. The proof follows from direct calculation as we did in Lemma 3.3.2. □

From this generalized entropy formula, we will derive the corresponding differential Harnack inequality for the fundamental solution to the heat equation on a static manifold. We remark that Kuang and Zhang [103] have a result in this direction, it is stated here below

Theorem 3.4.2. *([103].) Let M be a closed Riemannian manifold with nonnegative Ricci curvature. Let u be the fundamental solution to the heat equation with $f = -\ln u - \frac{n}{2}\ln(4\pi t)$, we have*

$$t(\alpha\Delta f - |\nabla f|^2) + f - \alpha\frac{n}{2} \leq 0 \tag{3.4.6}$$

for any constant $\alpha \geq 1$.

Indeed, if $\alpha = 2$, this is exactly the differential inequality

$$t(2\Delta f - |\nabla f|^2) + f - n \leq 0$$

proved in [122]. Dividing through by $\alpha \cdot t$, with $\alpha \geq 1$ and $t \geq 0$, we obtain

$$\Delta f - \frac{|\nabla f|^2}{\alpha} + \frac{f}{\alpha t} - \frac{n}{2t} \leq 0$$

as $t \to \infty$, which is precisely the Li-Yau gradient estimate. For $\alpha > 2$, the gradient estimate is an interpolation of Perelman's estimate and Li-Yau estimate. For $1 \leq \alpha \leq 2$, it is considered in [103], they remark that it can't be obtained directly from Perelman's gradient estimate and Li-Yau estimate. In Euclidean space $\mathbb{R}^n$, if u is the fundamental solution to the heat equation then (3.4.6) becomes an equality.

Proposition 3.4.3. *Let M be any closed manifold, $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$, any positive solution to the heat equation $\Gamma u = (\partial_t - \Delta)u = 0$ on $M \times (0.T]$. Denoting*

$$P_\epsilon = \frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi}, \tag{3.4.7}$$

where $0 < \epsilon^2 \le 4\pi$. Then

$$(\partial_t - \Delta)P_\epsilon \le -\frac{\epsilon^2 t}{2\pi}\Big(\Big|f_{ij} - \frac{\sqrt{\pi}}{\epsilon t}g_{ij}\Big|^2 + R_{ij}f_if_j\Big) - 2\langle\nabla P_\epsilon, \nabla f\rangle - \Big(1 - \frac{\epsilon^2}{4\pi}\Big)|\nabla f|^2. \tag{3.4.8}$$

Proof. Here we write

$$\widetilde{P_\epsilon} = \frac{\epsilon^2 t}{4\pi}w + \widetilde{f} + \frac{n}{2}\ln\Big(\frac{1}{\epsilon^2 t}\Big) - \frac{n\epsilon^2}{4\pi}.$$

Since $f = -\ln u - \frac{n}{2}\ln(4\pi t)$, taking $u = e^{-\widetilde{f}}$ implies $f = \widetilde{f} - \frac{n}{2}\ln(4\pi t)$. we notice also that $\nabla\widetilde{f} = \nabla f$, $\Delta\widetilde{f} = \Delta f$ and $\widetilde{f}_{ij} = f_{ij}$, then $(\partial_t - \Delta)\widetilde{f} = -|\nabla\widetilde{f}|^2 - \frac{n}{2t}$.

Now by direct differentiation and application of Lemma 3.4.1, we have the following computation

$$\begin{aligned}(\partial_t - \Delta)P_\epsilon &= \frac{\epsilon^2 t}{4\pi}(\partial_t - \Delta)w + \frac{\epsilon^2}{4\pi}w + (\partial_t - \Delta)\widetilde{f} + \frac{\partial}{\partial t}\Big(\frac{n}{2}\ln\Big(\frac{1}{\epsilon^2 t}\Big) - \frac{n\epsilon^2}{4\pi}\Big)\\ &= \frac{\epsilon^2 t}{4\pi}\Big(-2f_{ij}^2 - 2R_{ij}f_if_j - 2\langle\nabla w, \nabla f\rangle\Big) + \frac{\epsilon^2}{4\pi}(2\Delta f - |\nabla f|^2) - |\nabla f|^2 - \frac{n}{2t}\\ &= \frac{\epsilon^2 t}{4\pi}\Big(-2f_{ij}^2 - \frac{2\pi}{\epsilon^2}\frac{n}{t^2} - 2R_{ij}f_if_j\Big) + \frac{\epsilon^2}{4\pi}(2\Delta f - |\nabla f|^2) - 2\langle\frac{\epsilon^2 t}{4\pi}\nabla w, \nabla f\rangle - |\nabla f|^2.\end{aligned}$$

Notice that

$$2\langle\frac{\epsilon^2 t}{4\pi}\nabla w, \nabla f\rangle = 2\langle(\nabla P_\epsilon - \widetilde{f}), \nabla f\rangle = 2\langle\nabla P_\epsilon, \nabla f\rangle - 2|\nabla f|^2.$$

Then we have

$$\begin{aligned}(\partial_t - \Delta)P_\epsilon &\le -2\frac{\epsilon^2 t}{4\pi}\Big(f_{ij}^2 + \frac{\pi}{\epsilon^2}\frac{n}{t^2} - \frac{2\sqrt{\pi}}{\epsilon t}\Delta f + R_{ij}f_if_j\Big) - 2\langle\nabla P_\epsilon, \nabla f\rangle + \frac{\epsilon^2}{4\pi}|\nabla f|^2 - |\nabla f|^2\\ &= -\frac{2\epsilon^2 t}{4\pi}\Big(\Big|f_{ij} - \frac{\sqrt{\pi}}{\epsilon t}g_{ij}\Big|^2 + R_{ij}f_if_j\Big) - 2\langle\nabla P_\epsilon, \nabla f\rangle - \Big(1 - \frac{\epsilon^2}{4\pi}\Big)|\nabla f|^2.\end{aligned}$$

□

Theorem 3.4.4. *Let M be a closed Riemannian manifold. Assume that $u = (4\pi t)^{-\frac{n}{2}}e^{-f}$ is a positive solution to the heat equation $\Gamma u = (\partial_t - \Delta)u = 0$, then, we have the following monotonicity formula for $\mathcal{W}_\epsilon(f,t)$ defined in (3.4.2)*

$$\frac{d}{dt}\mathcal{W}_\epsilon(f,t) = -\int_M\Bigg[\frac{\epsilon^2 t}{2\pi}\Bigg(\Big|f_{ij} - \frac{\sqrt{\pi}}{\epsilon t}g_{ij}\Big|^2 + R_{ij}f_if_j\Bigg) + \Big(1 - \frac{\epsilon^2}{4\pi}\Big)|\nabla f|^2\Bigg]\frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}dV(x) \tag{3.4.9}$$

with (f,t) satisfying

$$\int_M \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}dV(x) = 1 \tag{3.4.10}$$

and $0 < \epsilon^2 \le 4\pi$.

Proof. Combining Proposition 3.4.3 with the fact that $\Gamma u = 0$ and $u\nabla f = -\nabla u$, we have

$$
\begin{aligned}
(\partial_t - \Delta)(P_\epsilon u) &= (\partial_t - \Delta)P_\epsilon \cdot u + P_\epsilon(\partial_t - \Delta)u - 2\langle \nabla P_\epsilon, \nabla u\rangle \\
&= -\frac{\epsilon^2 t}{2\pi}\Big(\Big|f_{ij} - \frac{\sqrt{\pi}}{\epsilon t} g_{ij}\Big|^2 + R_{ij} f_i f_j\Big)u - 2\langle \nabla P_\epsilon, \nabla f\rangle u \\
&- \Big(1 - \frac{\epsilon^2}{4\pi}\Big)|\nabla f|^2 u - 2\langle \nabla P_\epsilon, \nabla u\rangle.
\end{aligned}
$$

Integrating over M, we have

$$
\begin{aligned}
\int_M P_\epsilon u &= \int_M \Big[\frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi}\Big] u dV(x) \\
&= \int_M \Big[\frac{\epsilon^2 t}{4\pi}|\nabla f|^2 + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi}\Big] u dV(x) \\
&+ \frac{2\epsilon^2 t}{4\pi}\int_M (\Delta f - |\nabla f|^2) u dV(x) \\
&= \mathcal{W}_\epsilon(f,t),
\end{aligned}
$$

in the sense that the second integral in the RHS vanishes on a closed manifold since $(\Delta f - |\nabla f|^2)u = -\Delta u$. Therefore

$$
\begin{aligned}
\frac{d}{dt}\mathcal{W}_\epsilon(f,t) &= \frac{\partial}{\partial t}\int_M P_\epsilon u \, dV(x) \\
&= \int_M \Big(\frac{d}{dt}P_\epsilon \, u + P_\epsilon \frac{\partial}{\partial t} u\Big) dV(x) \\
&= \int_M \Big[(\partial_t - \Delta)P_\epsilon \, u + P_\epsilon(\partial_t - \Delta)u\Big] dV(x) \\
&= \int_M (\partial_t - \Delta)P_\epsilon \, u dV(x),
\end{aligned}
$$

where we have used integration by parts and $\Gamma u = 0$. Using the evolution $(\partial_t - \Delta)P_\epsilon$ from Proposition 3.4.3, we get the desired result. Moreover, if the manifold has nonnegative Ricci curvature, i.e, $R_{ij} \geq 0$, it becomes obvious from (3.4.9) that $d\mathcal{W}_\epsilon/dt \leq 0$. □

The monotonicity formula above may be viewed as a local version of the one discussed about in Chapter 2 of this thesis, (i.e., Perelman's $\mathcal{W}$-entropy formula). In what follows, we want to show that the local entropy satisfies a pointwise differential inequality for the heat kernel. We have the following fashioned after [122, Theorem 1.2] with the proof follows from the argument of [118, Proposition 3.6].

Theorem 3.4.5. *Let M be a closed manifold with nonnegative Ricci curvature and $H(x,y,t) = H = (4\pi t)^{-\frac{n}{2}} e^{-f}$ be the heat kernel, where H tends to a δ-function as $t \to 0$ and satisfies $\int_M H dV(x) = 1$. Then for all $t > 0$, we have*

$$
P_\epsilon = \frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi} \leq 0. \tag{3.4.11}
$$

Proof. Let h be any compactly supported smooth function for all $t_0 > 0$. Suppose $h(\cdot, t)$ is a positive solution to the backward heat equation $(\partial_t + \Delta)h = 0$, (This is Perelman's argument in [126, Corollary 9.3]), then, it is clear that $\frac{\partial}{\partial t}\int_M HhdV = 0$ and we have by direct calculation that

$$\begin{aligned}\frac{\partial}{\partial t}\int_M hP_\epsilon HdV(x) &= \int_M \Big[\partial_t h(P_\epsilon H) + h\partial_t(P_\epsilon H)\Big]dV(x)\\ &= \int_M \Big[(\partial_t + \Delta)h(P_\epsilon H) + h(\partial_t - \Delta)P_\epsilon H)\Big]dV(x)\\ &= \int_M h(\partial_t - \Delta)P_\epsilon HdV(x)\\ &\leq 0.\end{aligned}$$

The inequality is due to Theorem 3.4.4 since $R_{ij} \geq 0$. We are left to showing that for everywhere positive function $h(\cdot, t)$, the limit of $\int_M hP_\epsilon HdV(x)$ is nonpositive as $t \to 0$. We assume the claim apriori (i.e, $\lim_{t\to 0}\int_M hP_\epsilon HdV = 0$) and conclude the result.

For completeness, we devote the next effort to justifying the claim

$$\lim_{t\to 0}\int_M hP_\epsilon HdV \leq 0. \tag{3.4.12}$$

Our argument follows from [118], for detail see [122, 124, 126], the calculation in [103] is also similar. If H tends to a dirac δ-function, say at a point $p \in M$, for $t \to 0$, then f satisfies $f(x, t) \to \frac{d^2(p,\,x)}{4t}$. This is in relation to l-length of Perelman. [1] This yields

$$\lim_{t\to 0}\int_M fhHdV \leq \limsup_{t\to 0}\int_M \frac{d^2(p, x)}{4t}hHdV = \frac{n}{2}h(p, 0). \tag{3.4.13}$$

Meanwhile, by the strong Maximum principle we have $h(x, 0) > 0$ and $\lim_{t\to 0}\int_M hHdV = h(x, 0)$, hence by scaling argument, we assume that $h(x, 0) = 1$. All these will soon become clearer. Rewriting P_ϵ and using integrating by parts methods (namely, $\int_M(\Delta f - |\nabla f|^2)hHdV$), we have

$$\begin{aligned}\int_M P_\epsilon hHdV = \int_M \frac{\epsilon^2 t}{4\pi}(|\nabla f|^2 - \frac{n}{2t})hHdV - \frac{\epsilon^2 t}{2\pi}\int_M \langle\nabla f, \nabla h\rangle HdV\\ + \int_M fHhdV + \frac{n}{2}\Big[\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{\epsilon^2}{4\pi}\Big]\int_M Hh\, dV.\end{aligned}$$

Though, the H appearing in the last equation is actually the heat kernel on an evolving manifold in Ni's result [124] while h satisfies the forward heat equation, his argument still holds in our case, we only need the asymptotic behaviour of heat kernel on a fixed metric. We should also note that since $h(\cdot, t_0)$ is compactly supported and by strong maximum principle we have $h(\cdot, t_0)$, $|\nabla h(\cdot, t_0)|$ and $|\Delta h(\cdot, t_0)|$ bounded on M. This implies that there exists a bounded solution $h(\cdot, t_0)$. [2]

[1] see also remark after Corollary 4.3 of [118]. If $M \equiv \mathbb{R}^n$ and $p = 0$ is the origin, we have $l(x, t) = \frac{|x|^2}{4t}$ and so $u(x, t) = (4\pi t)^{-\frac{n}{2}}e^{-l(x,t)}$ is not only a lower bound to the heat kernel (which is true on every closed manifold M with $Rc \geq 0$) but it is in fact equal to the kernel. This is the only case where $l(x, t) = f(x, t)$.

[2] A solution given by the representation formula through the heat kernel to the backward heat equation is such a solution)

It turns out that we need to show that there exists a constant $B \geq 0$ which may depend on the geometry of the underlying manifold and independent of t as $t \to 0$, such that $\int_M P_\epsilon hHdV \leq B(n)$.

Now we claim that the first two terms on the right hand side of the last equation vanish as $t \to 0$, we can see this in the following argument. By integration by parts and the fact that $\nabla H = -H\nabla f$, we have

$$-t\int_M \langle \nabla f, \nabla h\rangle HdV = t\int_M \langle \nabla H, \nabla h\rangle dV = -t\int_M H\Delta h dV$$

is bounded since $|\Delta h|$ is bounded as stated earlier. Thus, the second term is bounded and goes to zero as $t \to 0$.

we need a bound of Li-Yau type to obtain a bound for the first term $|\nabla f|^2$, (Cf [55]), Here below is the statement of the result (see [65, Corollary 16.23] and Souplet and Zhang [133]).

Lemma 3.4.6. *[65, Corollary 16.23] On a complete Riemannian Manifold (M, g) with nonnegative Ricci curvature, the following estimate holds for the gradient of the heat kernel $H(x, y, t)$ and all $\delta > 0$,*

$$\frac{|\nabla H|^2}{H} \leq \frac{2H}{t}\Big(B(n) + \frac{d^2(x, y)}{(4-\delta)t}\Big) \tag{3.4.14}$$

for all x, y in M and $t > 0$.

By the above we have for the heat kernel in the present case that

$$t\int_M |\nabla f|^2 \leq 2\Big(B(n, \delta) + \frac{d^2(x, y)}{(4-\delta)t}\Big), \tag{3.4.15}$$

which is also clearly seen to be bounded from above as $t \to 0$ by the justification of asymptotic behaviour of the heat kernel. [3] We have now reduced the analysis to

$$\lim_{t\to 0}\int_M P_\epsilon hHdV \leq \limsup_{t\to 0}\int_M \Big(f + \frac{nq}{2}\Big)hHdV, \tag{3.4.16}$$

where $q = \ln(\frac{4\pi}{\epsilon^2}) - \frac{\epsilon^2}{4\pi}$. For simplicity, we can choose ϵ such that $\epsilon^2 \to 4\pi$ as $t \to 0$ so that the whole problem is reduced to finding

$$\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hHdV. \tag{3.4.17}$$

Using the asymptotic behaviour of the heat kernel, i.e, $f \approx \frac{d^2}{4t}$ as $t \to 0$. Recall (Cf. [80, 116, 124]) as $t \to 0$

$$H(x, y, t) \sim (4\pi t)^{-\frac{n}{2}} \exp\Big(\frac{d^2(x, y)}{4t}\Big)\sum_{j=o}^{\infty} u_j(x, y, t)t^j := w_k(x, y, t)$$

where $d^2(x, y)$ is the distance function and $w_k(x, y, t)$ satisfies uniformly for all $x, y \in M$

$$w_k(x, y, t) = O\Big(t^{k+1-\frac{n}{2}} \exp\Big(\frac{\delta d^2(x, y)}{4t}\Big)\Big)$$

[3] Better still the first term can be written as

$$\int_M \frac{\epsilon^2 t}{4\pi}(|\nabla f|^2 - \frac{n}{2t})hHdV = \frac{\epsilon^2 t}{4\pi}\int_M (\Delta f - \frac{n}{2t})hHdV = \frac{\epsilon^2 t}{4\pi}\int_M (\frac{|\nabla H|^2}{H^2} - \frac{\Delta H}{h} - \frac{n}{2t})hHdV,$$

where the last integrand is known to be nonpositive for all t by the Li-Yau gradient estimate.

and δ is just a number depending only on the geometry of (M, g). The function can be chosen such that $u_0(x, y, 0) = 1$. Though the above asymptotic result does not require any curvature assumption, a result due to Cheeger and Yau [55] states that on manifold with nonnegative Ricci curvature (which is our case), the heat kernel satisfies

$$H(x, y, t) \geq (4\pi t)^{-\frac{n}{2}} \exp\left(\frac{d^2(x, y)}{4t}\right)$$

which implies

$$f(x, t) \leq \frac{d^2(x, y)}{4t}.$$

Therefore

$$\begin{aligned}\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hH dV &\leq \lim_{t\to 0}\int_M \Big(\frac{d^2(x, y)}{4t} - \frac{n}{2}\Big)h(y, t)H(x, y, t)dV(y)\\ &= \lim_{t\to 0}\int_M \Big(\frac{d^2(x, y)}{4t} - \frac{n}{2}\Big)\frac{e^{-d^2(x,y)/4t}}{(4\pi t)^{\frac{n}{2}}}H(y, t)dV(y).\end{aligned}$$

It is easy to see that for any $\delta > 0$, the integration of the above integrand in the domain $d(x, y) \leq \delta$ converges to zero exponentially fast. Therefore

$$\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hH dV \leq \lim_{t\to 0}\int_{d(x,y)\leq\delta} \Big(\frac{d^2(x, y)}{4t} - \frac{n}{2}\Big)\frac{e^{\frac{-d^2(x,y)}{4t}}}{(4\pi t)^{\frac{n}{2}}}h(y, t)dV(y). \tag{3.4.18}$$

Whenever δ is chosen sufficiently small, $d(x, y)$ is asymptotically sufficiently close to the Euclidean distance. By standard approximation, we have

$$\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hH dV \leq \lim_{t\to 0}\int_{d(x,y)\leq\delta} \Big(\frac{|x-y|^2}{4t} - \frac{n}{2}\Big)\frac{e^{\frac{-|x-y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}h_p(y)dV(y), \tag{3.4.19}$$

where h_p is the pullback of $h(\cdot, 0)$ to the Euclidean space from the region $d(x, y) \leq \delta$.

Splitting the last integrand as in [103] we are left with

$$\begin{aligned}\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hH dV &\leq h_p(x)\lim_{t\to 0}\int_{\mathbb{R}^n} \Big(\frac{|x-y|^2}{4t} - \frac{n}{2}\Big)\frac{e^{\frac{-|x-y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}dV(y)\\ &= h_p(\cdot)\lim_{t\to 0}\int_{\mathbb{R}^n} \Big(\frac{|y|^2}{4t}\frac{e^{-\frac{|y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}\Big)dV(y) - \frac{n}{2}h_p(\cdot).\end{aligned}$$

The last equality is due to convolution properties of the heat kernel. Lastly we show that the RHS evaluates to 0. Recall, using standard Gauss integral, that

$$\int_{\mathbb{R}^n}|y|^2 e^{-\alpha|y|^2}dy = n\Big(\int_{-\infty}^{\infty} y^2 e^{-\alpha y^2}dy\Big)\Big(\int_{-\infty}^{\infty} e^{-\alpha y^2}dy\Big)^{n-1} = \frac{n}{2}\sqrt{\frac{\pi}{\alpha^3}}\cdot\Big(\sqrt{\frac{\pi}{\alpha}}\Big)^{n-1} = \frac{n}{2\alpha}\Big(\sqrt{\frac{\pi}{\alpha}}\Big)^{n},$$

so that we have

$$\int_{\mathbb{R}^n}\Big(\frac{|y|^2}{4t}\frac{e^{-\frac{|y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}\Big)dV(y) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\cdot\frac{n}{4t}\Big(\int_{-\infty}^{\infty} y^2 e^{-\frac{1}{4t}y^2}dy\Big)\Big(\int_{-\infty}^{\infty} e^{-\frac{1}{4t}y^2}dy\Big)^{n-1} = \frac{n}{2},$$

by taking $\alpha = 1/4t$ in the above. We can then conclude the claim. $\square$

3.4.2 Differential Harnack Estimates for the Heat Kernel

The following differential Harnack quantity for linear heat equation on static manifold follows immediately as an application of the results in the last subsection.

Corollary 3.4.7. *Let M be a closed manifold with curvature bounded from below by $Rc \geq 0$. Then we have*

$$\frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\Big(\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{\epsilon^2}{2\pi}\Big) \leq 0, \tag{3.4.20}$$

where $f = -\ln(4\pi t)^{\frac{n}{2}} H$ and H is the positive minimal solution to the heat equation

$$\Big(\frac{\partial}{\partial t} - \Delta_x\Big)H(x,y,t) = 0.$$

Remark 3.4.8. *Note that the quantity $2\Delta f - |\nabla f|^2$ can be expressed as $\frac{|\nabla u|^2}{u^2} - \frac{2u_t}{u}$ in terms of u, which is similar to Li-Yau gradient estimate [112] on a manifold with nonnegative Ricci curvature, $\frac{u_t}{u} - \frac{|\nabla u|^2}{u^2} + \frac{n}{2t} \geq 0$. This is equivalent to the differential Harnack inequality $2t\Delta f \leq n$, where $f = -\ln(4\pi t)^{\frac{n}{2}}u$, which can be regarded as a generalized Laplacian comparison theorem in space for Heat kernel on M.*

However, we have from (3.4.20) that

$$\begin{aligned} f &\leq \frac{n}{2}\Big[\frac{\epsilon^2}{2\pi} - \ln\frac{4\pi}{\epsilon^2}\Big] - \frac{\epsilon^2 t}{4\pi}(2\Delta f - |\nabla f|^2) \\ &\leq \frac{n}{2}\Big[\frac{\epsilon^2}{2\pi} - \ln\frac{4\pi}{\epsilon^2}\Big] - \frac{\epsilon^2 n}{8\pi} = \frac{n}{2}\Big[\frac{\epsilon^2}{4\pi} - \ln\frac{4\pi}{\epsilon^2}\Big]. \end{aligned}$$

Define

$$Q(x,t) = \frac{\epsilon^2}{\pi}tf(x,t) \tag{3.4.21}$$

$$(\partial_t - \Delta)Q(x,t) = \frac{\epsilon^2}{\pi}f(x,t) + \frac{\epsilon^2}{\pi}t(\partial_t - \Delta)f \leq \frac{n\epsilon^2}{2\pi}\Big[\frac{\epsilon^2}{4\pi} - \ln\frac{4\pi}{\epsilon^2}\Big]. \tag{3.4.22}$$

Still as $\epsilon = 2\sqrt{\pi}$ we recover Ni's generalized Laplacian.

From Corollary 3.4.7, we have the differential Harnack inequality as follows

$$\frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\Big(\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{\epsilon^2}{2\pi}\Big) \leq 0.$$

Multiplying through by $-\frac{2\pi}{\epsilon^2 t}$, we have

$$-\Delta f + \frac{1}{2}|\nabla f|^2 - \frac{2\pi}{\epsilon^2 t}f - \frac{n\pi}{\epsilon^2 t}\Big(\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{\epsilon^2}{2\pi}\Big) \geq 0$$

$$-\Delta f + \frac{1}{2}|\nabla f|^2 - \frac{2\pi}{\epsilon^2 t}f + \frac{n}{2t} - \frac{n\pi}{\epsilon^2 t}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) \geq 0.$$

Recall that $(\partial_t - \Delta)H = 0$ implies $\Delta f = \partial_t f + |\nabla f|^2 + \frac{n}{2t}$, then we have

$$-\partial_t f - \frac{1}{2}|\nabla f|^2 - \frac{2\pi}{\epsilon^2 t}f \geq \frac{n\pi}{\epsilon^2 t}\ln\Big(\frac{4\pi}{\epsilon^2}\Big)$$

$$\begin{aligned}\partial_t f + \frac{1}{2}|\nabla f|^2 &\leq -\frac{2\pi}{\epsilon^2 t} f - \frac{n\pi}{\epsilon^2 t}\ln\Big(\frac{4\pi}{\epsilon^2}\Big)\\ &= -\frac{2\pi}{\epsilon^2 t}\Big(f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big)\Big).\end{aligned}$$

From the Young's inequality we have on the path $\gamma(t)$ ($\gamma(t) : [t_1, t_2]$ is a minimizing geodesic connecting points x_1 and x_2 such that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$)

$$\begin{aligned}\frac{d}{dt} f(\gamma(t), t) &= \partial_t f + \langle \nabla f, \gamma'(t)\rangle\\ &\leq \partial_t f + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|\gamma'(t)|^2\\ &= \frac{1}{2}|\gamma'(t)|^2 - \frac{2\pi}{\epsilon^2 t}\Big(f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big)\Big)\end{aligned}$$

since we have from (3.4.20) that

$$f \leq \frac{n}{2}\Big(\frac{\epsilon^2}{4\pi} - \ln\frac{4\pi}{\epsilon^2}\Big),$$

inserting this quantity in the above inequality gives the following Harnack Estimates

$$\frac{d}{dt} f(\gamma(t), t) \leq \frac{1}{2}|\gamma'(t)|^2 - \frac{n}{4t}. \tag{3.4.23}$$

After the usual integration of (3.4.23) and exponentiation we have the following

Corollary 3.4.9. *With the notation and assumption of Corollary 3.4.7, we have the following differential Harnack estimates*

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \leq \Big(\frac{t_1}{t_2}\Big)^{\frac{n}{4}} \exp\Big[\frac{1}{2}\int_{t_1}^{t_2} |\gamma'(t)|^2 dt\Big]. \tag{3.4.24}$$

Remark 3.4.10. *If M is a closed manifold with nonnegative Ricci curvature and $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$ is the heat kernel on M. Then $\mathcal{W}_\epsilon(f, t_0) \geq 0$ for some $t_0 > 0$, if and only if M is isometric to Euclidean space $\mathbb{R}^n$. Recall that we have obtained that $\frac{d}{dt}\mathcal{W}_\epsilon(f, t) \leq 0$ and $\mathcal{W}_\epsilon(f, t) \leq 0$ which in turn imply that we must have $\mathcal{W}_\epsilon(f, t) \equiv 0$ for $0 \leq t \leq t_0$. For instance, in the case $\epsilon = 2\sqrt{\pi}$, we have*

$$|f_{ij} - \frac{1}{2t} g_{ij}|^2 = 0 \quad \textit{and} \quad f_{ij} - \frac{1}{2t} g_{ij} = 0.$$

Taking the trace of the above yields

$$t\Delta f - \frac{n}{2} = 0. \tag{3.4.25}$$

Because $f(x,t) \approx \widetilde{f}(x,t) = \frac{d^2(p,x)}{4t}$ for t small, we have $\lim_{t\to 0} 4tf = d^2(p, x)$. Hence (3.4.25) implies that

$$\Delta d^2(p, x) = 2n \tag{3.4.26}$$

so that we can get for the area $A_p(r)$ of $\partial B_p(r)$ and the volume $V_p(r)$ of the ball $B_p(r)$, the following quotient

$$\frac{A_p(r)}{V_p(r)} = \frac{n}{r}.$$

This shows that $V_p(r)$ is exactly the same as the volume function of Euclidean balls.

This argument shows that the Li-Yau Harnack inequality, which is equivalent to $2t\Delta f - n \leq 0$ for $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$ becomes an equality if and only if the manifold M with $Rc \geq 0$ is isometric to $\mathbb{R}^n$ and u is precisely the heat kernel. If $t = \frac{1}{2}$ and $M = \mathbb{R}^n$, the inequality $\mathcal{W}_\epsilon(f, t_0) \geq 0$ for $\epsilon^2 = 4\pi$, is equivalent to

$$\int_{\mathbb{R}^n} (\frac{1}{2}|\nabla f|^2 + f - n)(2\pi)^{-\frac{n}{2}} e^{-f} dV \geq 0 \tag{3.4.27}$$

for all f with the condition $\int_M (2\pi)^{-\frac{n}{2}} e^{-f} dV = 1$.

The above implies a sharp (Gross) logarithmic Sobolev inequality on $\mathbb{R}^n$. For details about logarithmic-Sobolev inequalities see for instance [83, 134, 152]. In the same vein our dual entropy also yields a version of logarithmic Sobolev inequality. (This will be discussed in Chapter 4).

Remark 3.4.11. *Note that $f_{ij} - \frac{\sqrt{\pi}}{\epsilon t} g_{ij} \geq 0 \implies \Delta f \geq \frac{n\sqrt{\pi}}{\epsilon t}$ which in turns $\implies \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \geq \frac{n\sqrt{\pi}}{\epsilon t}$.*

It turns out that $\mathcal{W}_\epsilon(f, t)$ being finite with u being the heat kernel, also has strong topological and geometric consequences. For instance, in the case M has nonnegative curvature, it implies that M has finite fundamental group. In fact one can show that M is of maximum volume growth if and only if the entropy $\mathcal{W}_\epsilon(f, t)$ is uniformly bounded for all $t \geq 0$, where u is the heat kernel. This analogy was originally discovered in [126] for ancient solution to the Ricci flow with bounded nonnegative curvature, where Perelman claims that ancient solution to the Ricci flow with nonnegative curvature operator is κ-noncollapsed if and only if the entropy is uniformly bounded for any fundamental solution to the conjugate heat equation.

Lastly, in this subsection we make some comment to show how sharp the dual entropy is for the heat equation. Recall

$$\mathcal{W}_\epsilon(f, t) = \int_M \Big[\frac{\epsilon^2 t}{4\pi}|\nabla f|^2 + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi}\Big] H dV \tag{3.4.28}$$

with $f = -\ln(4\pi t)^{\frac{n}{2}} H$ and $\int_M H dV = 1$ and $0 < \epsilon^2 \leq 4\pi$.

Rewrite $\mathcal{W}_\epsilon(f, t)$ as

$$\mathcal{W}_\epsilon(f, t) = \frac{\epsilon^2}{4\pi}\int_M (t|\nabla f|^2 + f - n) H dV + (1 - \frac{\epsilon^2}{4\pi})\int_M f H dV + \frac{n}{2}\ln\frac{4\pi}{\epsilon^2}\int_M H dV. \tag{3.4.29}$$

Hence, we have the following

Proposition 3.4.12. *For $0 < \epsilon^2 \leq 4\pi$, $f = -\ln(4\pi t)^{\frac{n}{2}} H$ with $\int_M H dV = 1$, we have the following monotonicity formula on a manifold with nonnegative Ricci curvature;*

$$\frac{d}{dt}\mathcal{W}_\epsilon(f, t) \leq -\frac{\epsilon^2}{2\pi} t \int_M \Big(|f_{ij} - \frac{1}{2t} g_{ij}|^2 + R_{ij} f_i f_j\Big) H dV. \tag{3.4.30}$$

Proof. The proof follows from a straight forward computation on $\mathcal{W}_\epsilon$ using the idea of [122, Theorem 1.1].

$$\frac{d}{dt}\mathcal{W}_\epsilon(f, t) = \frac{\epsilon^2}{4\pi}\frac{\partial}{\partial t}\Big(\int_M t|\nabla f|^2 + f - n\Big) H dV + (1 - \frac{\epsilon^2}{4\pi})\frac{\partial}{\partial t}\Big(\int_M f H dV\Big). \tag{3.4.31}$$

We are only left to justify the non-positivity of $\frac{\partial}{\partial t}\Big(\int_M fHdV\Big)$. Then we have by integration by parts

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int_M fHdV\Big) &= \int_M \Big(\frac{\partial}{\partial t}f\ H + f\frac{\partial}{\partial t}H\Big)dV \\ &= \int_M \Big(\frac{\partial}{\partial t}f\ H + f\Delta H + f(\frac{\partial}{\partial t} - \Delta)H\Big)dV \\ &= \int_M \Big(\frac{\partial}{\partial t} + \Delta\Big)fHdV \\ &= \int_M (2\Delta f - |\nabla f|^2 - \frac{n}{2t})HdV,\end{aligned}$$

where we have used the facts that $(\frac{\partial}{\partial t} - \Delta)H = 0$ and $\frac{\partial}{\partial t}f = \Delta f - |\nabla f|^2 - \frac{n}{2t}$. Taking $f = -\ln((4\pi t)^{\frac{n}{2}}H$, then the integrand in the RHS of the last equality becomes

$$2\Delta f - |\nabla f|^2 - \frac{n}{2t} = \frac{|\nabla H|^2}{H^2} - \frac{2\Delta H}{H} - \frac{n}{2t} \leq 0, \tag{3.4.32}$$

which is precisely the Li-Yau Harnack inequality since we are on nonnegative Ricci curvature manifold. Hence our claim. □

3.4.3 Generalization of $\mathcal{W}_\epsilon$-Entropy for Conjugate Heat Equation.

Here, we have a family of entropy formula for the conjugate heat equation on manifold evolving by the Ricci flow forward in time.

$$\mathcal{W}_\epsilon(g,f,\tau) = \int_M \Big[\frac{\epsilon^2\tau}{4\pi}(R + |\nabla f|^2) + f - \frac{n\epsilon^2}{4\pi} + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big)\Big]\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}d\mu, \tag{3.4.33}$$

where $\tau = T - t > 0$ and $R = R(x,t)$ is the scalar curvature. Let $u = u(x,t)$ be a positive solution to the conjugate heat equation on a complete compact manifold with metric $g = g(x,t)$ evolving by the Ricci flow. If

$$u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} \quad satisfies \quad \int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}d\mu = 1$$

then

$$\Gamma^* u = (-\partial_t - \Delta + R)u = 0 \tag{3.4.34}$$

and the following

Theorem 3.4.13. *Let* $(M, g(t)), t \in [0,T)$ *be a solution of the Ricci flow* $\partial_t g_{ij} = -2R_{ij}(g)$. *Let* $u : M \times [0,T) \to (0,\infty)$ *solves the conjugate heat equation* $(-\partial_t - \Delta + R)u = 0$. *The entropy functional* $\mathcal{W}_\epsilon(g,f,\tau)$ *is nondecreasing by the formula*

$$\frac{d}{dt}\mathcal{W}_\epsilon(g,f,\tau) \geq \frac{\epsilon^2\tau}{2\pi}\int_M \Big|R_{ij} + f_{ij} - \frac{1}{2\tau}g_{ij}\Big|^2 ud\mu \geq 0 \tag{3.4.35}$$

for $0 < \epsilon^2 \leq 4\pi$.

Proof. The entropy functional can be rewritten as

$$\mathcal{W}_\epsilon(g,f,\tau)=\frac{\epsilon^2}{4\pi}\int_M\Big(\tau(R+|\nabla f|^2)+f-n\Big)ud\mu+(1-\frac{\epsilon^2}{4\pi})\int_M fud\mu+\frac{n}{2}\ln\frac{4\pi}{\epsilon^2}. \tag{3.4.36}$$

By direct computation we obtain

$$\frac{d}{dt}\mathcal{W}_\epsilon(g,f,\tau)=\frac{\epsilon^2}{4\pi}\frac{\partial}{\partial t}\Big(\int_M\Big[\tau(R+|\nabla f|^2)+f-n\Big]ud\mu\Big)+(1-\frac{\epsilon^2}{4\pi})\frac{\partial}{\partial t}\Big(\int_M fud\mu\Big). \tag{3.4.37}$$

Recall the quantity v defined in (3.2.7) by $v=[\tau(2\Delta f-|\nabla f|^2+R)+f-n]u$ and that it satisfies the conjugate heat equation as in the following

$$\Gamma^*v=-2\tau|R_{ij}+f_{ij}-\frac{1}{2\tau}g_{ij}|^2u. \tag{3.4.38}$$

It is then clear that equation (3.4.37) is reduced to

$$\frac{d}{dt}\mathcal{W}_\epsilon(g,f,\tau)=\frac{\epsilon^2}{4\pi}\frac{\partial}{\partial t}\Big(\int_M v\,d\mu\Big)+(1-\frac{\epsilon^2}{4\pi})\frac{\partial}{\partial t}\Big(\int_M fud\mu\Big). \tag{3.4.39}$$

The problem is now reduced to evaluating the second integral on the RHS of the above equation (3.4.39), since we already know that

$$\frac{\partial}{\partial t}\Big(\int_M v\,d\mu\Big)=\int_M-\Gamma^*vd\mu. \tag{3.4.40}$$

Therefore we compute

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int_M fud\mu\Big)&=\int_M\Big(\frac{\partial}{\partial t}f\,u+f\frac{\partial}{\partial t}u-Rfu\Big)d\mu\\&=\int_M\Big(-\Delta f+|\nabla f|^2-R+\frac{n}{2\tau}\Big)ud\mu\\&\qquad+\int_M f(-\Delta u+Ru)d\mu-\int_M Rfud\mu\\&=\int_M(-2\Delta f+|\nabla f|^2)ud\mu+\int_M(\frac{n}{2\tau}-R)ud\mu,\end{aligned}$$

where we have used integration by parts on $-\int_M\Delta fu=-\int_M f\Delta u$. Rearranging the above we have

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int_M fud\mu\Big)&=\int_M(-R-2\Delta f+|\nabla f|^2)ud\mu+\frac{n}{2\tau}\int_M ud\mu\\&=-\int_M(R+|\nabla f|^2)ud\mu+\frac{n}{2\tau}\\&=-\mathcal{F}+\frac{n}{2\tau},\end{aligned}$$

where $\mathcal{F}=\int_M(R+|\nabla f|^2)ud\mu$ is the Perelman's energy functional introduced in [126], which we have discussed in Chapter 2 of this thesis. We however claim that

$$\frac{\partial}{\partial t}\Big(\int_M fud\mu\Big)=-\mathcal{F}+\frac{n}{2\tau}\geq 0. \tag{3.4.41}$$

By this claim and putting together equations (3.4.38), (3.4.39), (3.4.40) and (3.4.41), we have

$$\frac{d}{dt}\mathcal{W}_\epsilon(g,f,\tau)\geq\frac{\epsilon^2\tau}{2\pi}\int_M\Big|R_{ij}+f_{ij}-\frac{1}{2\tau}g_{ij}\Big|^2ud\mu\geq 0.$$

This has proved the theorem. Next is to justify the claim (3.4.41) above. Recall from Chapter 2 the evolution of $\mathcal{F}$

$$\frac{d}{dt}\mathcal{F}(g,f) = 2\int_M |R_{ij} + f_{ij}|^2 u d\mu. \tag{3.4.42}$$

Straightforward analysis, using elementary inequality and Cauchy-Schwarz inequality gives

$$|R_{ij} + f_{ij}|^2 \geq \frac{1}{n}(R + \Delta f)^2 \tag{3.4.43}$$

so that

$$\int (R + \Delta f) u d\mu \leq \Big(\int (R + \Delta f)^2 u d\mu\Big)^{\frac{1}{2}} \Big(\int u d\mu\Big)^{\frac{1}{2}},$$

therefore

$$\Big(\int (R + \Delta f) u d\mu\Big)^2 \leq \int (R + \Delta f)^2 u d\mu.$$

Hence by (3.4.42) and (3.4.43), we obtain

$$\frac{d}{dt}\mathcal{F} \geq \frac{2}{n}\int_M (R + \Delta f)^2 u d\mu. \tag{3.4.44}$$

We can then solve

$$\frac{d}{dt}\mathcal{F} \geq \frac{2}{n}\mathcal{F}^2, \qquad \mathcal{F} \geq 0.$$

This implies

$$\frac{d\mathcal{F}}{\mathcal{F}^2} \geq \frac{2}{n}dt$$

$$\Longrightarrow -\frac{1}{\mathcal{F}}\Big|_t^T \geq \frac{2}{n}(T - t) \Longrightarrow \frac{1}{\mathcal{F}(t)} - \frac{1}{\mathcal{F}(T)} \geq \frac{2}{n}\tau \Longrightarrow \frac{1}{\mathcal{F}(t)} \geq \frac{2}{n}\tau + \frac{1}{\mathcal{F}(T)}.$$

From here we can conclude as follows

(i). Suppose $\mathcal{F}(T) > 0$, then

$$\frac{1}{\mathcal{F}(t)} \geq \frac{2\tau}{n} \quad i.e, \mathcal{F}(t) \leq \frac{n}{2\tau}.$$

(ii). Suppose $\mathcal{F}(T) \leq 0$, then $\mathcal{F}(t) \leq 0 \leq \frac{n}{2\tau}$ for all $t \in [0, T)$, since we know that $\frac{d}{dt}\mathcal{F} \geq 0$.

Hence

$$\mathcal{F}(t) \leq \frac{n}{2\tau} \quad for\ t \in [0, T),$$

which proves the claim (3.4.41). □

Chapter 4

Bounds on the Conjugate Heat Kernel under Ricci Flow

4.1 Introduction

Let M be a compact Riemannian manifold endowed with metric $g(x,t)$ evolving by the Ricci flow in the interval $0 \leq t \leq T$, T is taken to be the maximum time of existence for the Ricci flow. Let $u(x,t)$ be a positive solution to the conjugate heat equation on $M \times [0,T]$, we consider the following coupled system

$$\begin{cases} \partial_t g_{ij}(x,t) = -2R_{ij}(x,t), & (x,t) \in M \times [0,T], \\ \\ (-\partial_t - \Delta + R(x,t))u(x,t) = 0, & (x,t) \in M \times [0,T], \end{cases} \tag{4.1.1}$$

where $R_{ij}(x,t)$ is the Ricci curvature of the the metric $g(t)$, its metric trace is called scalar curvature, denoted by $R(x,t)$ and Δ is the usual Laplace-Beltrami operator depending on the metric. We obtain several upper bounds for the fundamental solution (Heat kernel) to the conjugate heat equation defined on a compact manifold whose metric is evolving by the forward Ricci flow. The system above is associated to Perelman's monotonicity formula which has been proven to be of fundamental importance in the analysis of Ricci flow. Perelman [126] proved a lower bound for the heat kernel satisfying the above heat equation with application of the maximum principle and his reduced distance, an outstanding feature of the estimate is that it does not require explicit assumption on metric curvature, the information is being embedded in the reduced distance. In the present too, the bound obtained in the first part of this chapter needs no explicit curvature assumptions, it rather depends on the Zhang-Ricci-Sobolev constant [157], which in turn depends on the best constant in the usual Sobolev embedding controlled by the infimum of the Ricci curvature and the injectivity radius of the underlying manifold. The motivation for this was Q. Zhang's result in [156], where he obtained upper bounds for conjugate heat kernel under backward Ricci flow, such bounds depend on Yamabe constant or Euclidean Sobolev embedding constant. He further showed that this type of heat kernel upper bounds are proper extension of an on-diagonal upper bound in the case of a fixed manifold, where one obtains a bound of the form

$$F(x,t;y,s) \leq C(n) \max\{\frac{1}{(t-s)^{\frac{n}{2}}}, 1\}$$

with $C(n) > 0$ depending on n for all $t > s$ and $x, y \in M$. We also give a special case of weakly positive scalar curvature to support the above assertion. Recently, M. Băileşteanu [7] has adopted Zhang's approach to obtain similar estimate for the fundamental solution of the heat equation coupled to Ricci flow. Our calculation is based on the ideas of both papers cited above. We remark that the similarity in our results is a justification of the fact that heat diffusion on a bounded geometry with either static or evolving metric behaves like heat diffusion in Euclidean space, many a times, their estimates even coincide. A result of Cheeger and Yau [55] has revealed that the heat kernel of a complete manifold with bounded Ricci curvature can be compared with that of the space form whose curvature determines the lower bound for the manifold's Ricci curvature. In the second part of this chapter, sharp upper estimates arising from the monotonicity of an entropy formula are obtained. The main ingredients used here are logarithmic Sobolev inequalities and ultracontractivity property of the conjugate heat semigroup. It is well known that Gross logarithmic Sobolev inequality [83] is equivalent to Nelson's hypercontractive inequality [121], both of which may imply ultracontractivity of the heat semigroup. (See [71, 72, 107, 152]). These results will appear in [2].

In practice, the Ricci flow deforms and smoothens out irregularities in the metric to give a "nicer" form and thus, provides useful geometric and topological information on the manifold. The metric is bounded under the Ricci flow and nonnegativity of curvature operator is preserved during the flow. (Cf. [64, 68, 69] for more on this and detail of geometric and analytic aspect of the Ricci flow). For example, the evolution of scalar curvature is governed by the following differential inequality, since $|Rc|^2 \geq \frac{1}{n}R^2$,

$$\frac{\partial}{\partial t}R \geq \Delta R + \frac{2}{n}R^2.$$

Suppose $R(g_0) \geq \rho$, we can use maximum principle by comparing solution of the above inequality with that of the following ODE

$$\begin{cases} \dfrac{d\phi(t)}{dt} = \dfrac{2}{n}(\phi(t))^2 \\ \phi(0) = \rho, \end{cases} \tag{4.1.2}$$

solving

$$\phi(t) = \frac{1}{\frac{1}{\rho} - \frac{2}{n}t}.$$

Therefore

$$R(g(t)) \geq \phi(t) = \frac{1}{\frac{1}{\rho} - \frac{2}{n}t}. \tag{4.1.3}$$

Coupling Ricci flow to the heat equation can be associated with some physical interpretation in terms of heat conduction process. Precisely, the manifold M with initial metric $g(x,0)$ can be thought of as having the temperature distribution $u(x,0)$ at $t=0$. If we now allow the manifold to evolve under the Ricci flow and simultaneously allow the heat to diffuse on M, then, the solution $u(x,t)$ will represent the space-time temperature on M. Moreover, if $u(x,t)$ approaches δ-function at the initial time, we know that $u(x,t) > 0$, this gives another physical interpretation that temperature is always positive, whence we can consider the potential $f = \log u$ as an entropy or unit mass of heat supplied and the local production entropy is given by $|\nabla f|^2 = \frac{|\nabla u|^2}{u^2}$. Suffice to say that heat kernel governs the evolution of temperature on a manifold with certain amount of heat energy prescribed at the initial time.

We denote the fundamental solution to the conjugate heat equation by $F(x,\tau;y,\sigma) \in (M \times [0,T] \times M \times [0,T],\ \mathbb{R})$. We now give a formal definition and some important properties of conjugate heat kernel.

Definition 4.1.1. *We say that $F(x,\tau;y,\sigma)$ is a fundamental solution to the adjoint heat equation centred at (y,σ) for $x,y \in M, \sigma < \tau \in [0,T]$, if it satisfies the following system*

$$\begin{cases} (-\partial_t - \Delta_{(x,\tau)} + R(x,\tau))F(x,\tau;y,\sigma) = 0 \\ \\ \lim\limits_{\tau\to\sigma} F(x,\tau;y,\sigma) = \delta_y(x) \end{cases} \tag{4.1.4}$$

for any $x \in M$.

Thus, $F(x,\tau;y,\sigma)$ is the unique minimal positive solution to the equation which from henceforth we refer to as conjugate heat kernel.

Lemma 4.1.2. *The conjugate heat kernel satisfies the following properties.*

1. $\int_M F(x,t;y,s)d\mu(g(x,t)) = 1$

2. $F(x,t;y,0) = \int_M F(x,t;z,\frac{t}{2})F(z,\frac{t}{2};y,0)d\mu(g(z,\frac{t}{2}))$ *by the semigroup property (see Appendix B.4).*

3. $F(x,t;y,s)$ *is also the fundamental solution to the heat equation in (y,s)-variables i.e,*

$$\begin{cases} (\partial_s - \Delta_{(y,s)})F(x,t;y,s) = 0 \\ \\ \lim\limits_{s\to t} F(x,t;y,s) = \delta_x(y). \end{cases} \tag{4.1.5}$$

4. $\int_M F(x,t;y,s)d\mu(g(y,s)) \leq 1.$

Other important properties of heat kernel such as existence, uniqueness, smoothness, symmetry have been studied by many authors, C. Guenther in [85] and Garofalo and Lanconelli in [80] for examples.

4.2 Pointwise Upper Bound with Sobolev Inequality

In this section, we prove an upper estimate on the conjugate heat kernel of the manifold evolving by the Ricci flow, it turns out that the estimate depends on the best constants in Sobolev inequality for the Ricci flow due to Q. Zhang in [157] and the bound on the scalar curvature. The main result of this section is the following

Theorem 4.2.1. *Let $(M, g(x,t)), t \in [0,T]$ be a solution to the Ricci flow with $n \geq 3$ and $F(x,t;y,s)$ be the fundamental solution to the conjugate heat equation (conjugate heat kernel under Ricci flow). Then for a constant C_n depending on n only, the following estimate holds*

$$F(x,t;y,s) \leq \frac{C_n}{\left(\int_s^{\frac{t+s}{2}} \frac{e^{\frac{2}{n}P(\tau)}}{\alpha(\tau)A(\tau)}d\tau \cdot \int_{\frac{t+s}{2}}^{s} \frac{e^{-\frac{2}{n}P(\tau)}}{A(\tau)}d\tau\right)^{\frac{n}{4}}} \tag{4.2.1}$$

for $0 \leq s < t \leq T$, where $\alpha(\tau) = \frac{\rho^{-1} - \frac{2}{n}\tau}{\rho^{-1}}$, $R(g_0) \geq \rho$ being the infimum of the scalar curvature taken at the initial time, $P(\tau) = \int_s^t (B(\tau)A^{-1}(\tau) - \frac{1}{2}\phi(\tau))d\tau$, with $A(t)$ and $B(t)$ being positive constants in the Zhang-Ricci-Sobolev inequality and $\phi(t)$ is the lower bound for the scalar curvature.

As we mentioned earlier the approach requires no explicit condition on the curvature, and in a special case where the scalar curvature is nonnegative at the starting time of the flow, one obtains a bound similar to the one in the fixed metric case. Hence, the usefulness of this technique cannot be overemphasized, as it connects an analytic invariant (the best constant in the Euclidean Sobolev imbedding) to the geometry of the manifold M.

Corollary 4.2.2. *Let the assumptions of the above theorem hold. Suppose further that the scalar curvature is nonnegative at time $t = 0$ (i.e., $R(x,0) \geq 0$). Then for a constant $\tilde{C}_n$ depending on n and the best constant in Euclidean Sobolev embedding, the following estimates hold*

$$F(x,t;y,s) \leq \frac{\tilde{C}_n}{(t-s)^{\frac{n}{2}}} \tag{4.2.2}$$

for $0 \leq s < t \leq T$.

The exact value of $\tilde{C}_n$ is computed in the proof. Its value in the case $R(x,0) = 0$ is different from that of the case $R(x,0) > 0$.

In the next we give a brief discussion on the version of Sobolev embedding that will be used in the proof of the main results of this section (Theorem 4.2.1 and Corollary 4.2.2).

4.2.1 The Sobolev Embedding

Let (M,g) be an n-dimensional ($n \geq 3$) Riemannian manifold without boundary, it is well known that when M is compact the Sobolev space $H_1^q(M)$ is continuously embedded in $L^{q^*}(M)$ for any $1 \leq q < n$ and $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$. Here $H_1^q(M)$ is the completion of $C^\infty(M)$ with respect to the standard norm

$$\|u\|_q = \left(\int_M |\nabla u|^q d\mu(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q d\mu(g)\right)^{\frac{1}{q}} \tag{4.2.3}$$

and the embedding of $H_1^q(M)$ in $L^{q^*}(M)$ is critical. Similarly, the following Sobolev embedding inequality holds true; for any $\epsilon \geq 0$, there exists a positive constant $B_q(\epsilon)$ such that for any $u \in H_1^q(M)$

$$\left(\int_M |u|^{q^*} d\mu(g)\right)^{\frac{1}{q^*}} \leq (K(n,q)+\epsilon)\left(\int_M |\nabla u|^q d\mu(g)\right)^{\frac{1}{q}} + B_q(\epsilon)\left(\int_M |u|^q d\mu(g)\right)^{\frac{1}{q}}, \tag{4.2.4}$$

where $K(n,q)$, an explicit constant depending on n and q is the smallest constant having this property, ($K(n,q)$ is the best constant in the Sobolev embedding for $\mathbb{R}^n$). See Aubin [6], Hebey [95] and Talenti [145] also Taheri [141, 142, 144] for the impact of topology on Sobolev spaces and relation to twist theory). In other words, there exist positive constants A and B such that for all $u \in W^{1,2}(M,g)$, we have

$$\Big(\int_M u^{\frac{2n}{n-2}} d\mu(g)\Big)^{\frac{n-2}{n}} \leq A\int_M |\nabla u|^2 d\mu(g) + B\int_M u^2 d\mu(g). \tag{4.2.5}$$

On the compact manifold whose metric evolves along the Ricci flow, Q. Zhang, [157], S-Y Hsu [97] and R. Ye [155] have adopted Perelman $\mathcal{W}$-entropy monotonicity formula to derive various Sobolev embedding that holds for the case $n \geq 3$. In this section we shall make use of Zhang's version, here is the statement of his result;

Theorem 4.2.3. *([157] Let (M,g) be a compact Riemannian manifold with dimension $n \geq 3$ whose metric evolves by the Ricci flow in the interval $t \in [0,T]$. Let there exist positive constants A and B for the initial metric g_0 such that the following Sobolev inequality holds for any $u \in W^{1,2}(M,g_0)$*

$$\Big(\int_M u^{\frac{2n}{n-2}} d\mu(g_0)\Big)^{\frac{n-2}{n}} \leq A\int_M |\nabla u|^2 d\mu(g_0) + B\int_M u^2 d\mu(g_0). \tag{4.2.6}$$

Then, there exist positive functions of time $A(t)$ and $B(t)$ depending only on the initial metric g_0 and t such that for $u \in W^{1,2}(M,g(t)), t > 0$, it holds that

$$\Big(\int_M u^{\frac{2n}{n-2}} d\mu(g(t))\Big)^{\frac{n-2}{n}} \leq A(t)\int_M \Big(|\nabla u|^2 + \frac{1}{4}Ru^2\Big) d\mu(g(t)) + B(t)\int_M u^2 d\mu(g(t)), \tag{4.2.7}$$

where R is the scalar curvature of the metric $g(t)$. Moreover, if $R(x,0) > 0$, then $A(t)$ and $B(t)$ are in independent of t.

We have from the Sobolev embedding for $1 \leq q < n$ that $W^{1,q}(M)$ can be continuously embedded in $L^{q^*}(M)$, i.e, there exists a constant $C = C(n,q)$, such that

$$\|u\|_{L^{q^*}(M)} \leq C(n,q)\|u\|_{W^{1,q}(M)}$$

for all $u \in W^{1,q}(M)$. So by Holder's inequality we have ($p \geq q$)

$$\int_M |u|^p d\mu = \int_M |u|^q |u|^{p-q} \leq \Big(\int_M |u|^{\frac{qn}{n-q}} d\mu\Big)^{\frac{n-q}{n}} \Big(\int_M \|u\|^{\frac{n}{q}(p-q)} d\mu\Big)^{\frac{q}{n}}. \tag{4.2.8}$$

From the interpolation inequality

$$\int_M u^2 d\mu \leq \Big(\int_M u^{\frac{2n}{n-2}} d\mu\Big)^{\frac{n-2}{n+2}} \Big(\int_M u d\mu\Big)^{\frac{4}{n+2}} \tag{4.2.9}$$

for the case $q = 2$ and $n \geq 3$. Then by Sobolev inequality for manifold evolving by the Ricci flow we have

$$\int_M u^2 d\mu \leq \Big(A(t)\int_M \Big(|\nabla u|^2 + \frac{1}{4}Ru^2\Big) d\mu(g(t)) + B(t)\int_M u^2 d\mu(g(t))\Big)^{\frac{n}{n+2}}$$

$$\times\Big(\int_M u d\mu(g(t))\Big)^{\frac{4}{n+2}}.$$

Let $h(t) := \Big(\int_M u d\mu(g(t))\Big)^{\frac{4}{n+2}}$, the last inequality becomes

$$\begin{aligned}\int_M |\nabla u|^2 d\mu(g(t)) \geq \frac{1}{A(t)}\Big(h^{-1}(t)\int_M u^2 d\mu(g(t))\Big)^{\frac{n+2}{n}} \\ -\frac{B(t)}{A(t)}\int_M u^2\mu(g(t)) - \frac{1}{4}\int_M Ru^2\mu(g(t)).\end{aligned} \tag{4.2.10}$$

Thus, we have proved the following by using the bound on the scalar curvature (4.1.3) as discussed in the introduction.

Lemma 4.2.4. *With the hypothesis of Theorem 4.2.3 the following inequality holds*

$$\left.\begin{aligned}\int_M |\nabla u|^2 d\mu(g(t)) \geq \frac{1}{A(t)}\Big(h^{-1}(t)\int_M u^2 d\mu(g(t))\Big)^{\frac{n+2}{n}} \\ -\Big(\frac{B(t)}{A(t)} + \frac{\phi(t)}{4}\Big)\int_M u^2 d\mu(g(t))\Big)\end{aligned}\right\}. \tag{4.2.11}$$

4.2.2 Proof of Theorem 4.2.1

Proof. We suppose here and thereafter that $s = 0$ without loss of generality. Since $F(x,t;y,s)$ is the fundamental solution, it then follows from its semigroup property and Cauchy-Schwarz inequality that

$$\begin{aligned}F(x,t;y,0) &= \int_M F(x,t;z,\frac{t}{2})F(z,\frac{t}{2};y,0)d\mu(g(z,t)) \\ &\leq \Big(\int_M F^2(x,t;z,\frac{t}{2})d\mu(g(z,\frac{t}{2}))\Big)^{\frac{1}{2}}\Big(\int_M F^2(z,\frac{t}{2};y,0)d\mu(g(z,\frac{1}{2}))\Big)^{\frac{1}{2}}.\end{aligned}$$

Traditionally, deriving an upper bound for each of the terms in the right hand side of the last inequality suffices to settle the proof. In the present, the nature of the bounds to obtain depends on Sobolev embedding theorems on the manifold evolving by the Ricci flow. Now denote, say

$$\begin{aligned}V(t) &= \int_M F^2(x,t;y,s)d\mu(g(x,t)) \\ W(s) &= \int_M F^2(x,t;y,s)d\mu(g(y,s)).\end{aligned}$$

Thus, the pointwise estimates on the quantities $V(t)$ and $W(t)$ will determine an upper bound for the fundamental solution $F(x,t;y,s)$. Approaches to obtaining bound for each of the quantities $V(t)$ and $W(t)$ differ slightly due to the interpolation of the heat kernel between the conjugate heat equation in the variables (x,t) and the heat equation in the variables (y,s), i.e.,

$$\begin{aligned}(-\partial_t - \Delta_x + R(x,t))F(x,t;\cdot,\cdot) = 0 \\ (\partial_s - \Delta_y)F(\cdot,\cdot;y,s) = 0.\end{aligned}$$

We first treat the case when $F(x,t;y,s)$ solves the conjugate heat equation, that is, we want to estimate $V(t)$. The idea is to find an inequality involving $V(t)$. Hence

$$\begin{aligned} V'(t) &= \int_M (2F\partial_t F - RF^2)d\mu(x,t) \\ &= \int_M 2F(-\Delta F + RF)d\mu(x,t) - \int_M RF^2 d\mu(x,t) \\ &= 2\int_M |\nabla F|^2 d\mu(x,t) + \int_M RF^2 d\mu(x,t). \end{aligned}$$

Using Lemma 4.2.4, we arrive at

$$\begin{aligned} V'(t) &\geq 2A^{-1}(t)\Big(h^{-1}(t)\int_M F^2 d\mu(x,t)\Big)^{\frac{n+2}{n}} - 2\Big(B(t)A^{-1}(t) + \frac{1}{4}\phi(t)\Big)\int_M F^2 d\mu(x,t) \\ &\quad + \phi(t)\int_M F^2 d\mu(x,t) \\ &= 2A^{-1}(t)\Big(h^{-1}(t)\int_M F^2 d\mu(x,t)\Big)^{\frac{n+2}{n}} - \Big(2B(t)A^{-1}(t) - \frac{1}{2}\phi(t)\Big)\int_M F^2 d\mu(x,t). \end{aligned}$$

The problem is reduced to solving the following ODE

$$V'(t) + q(t)V(t) \geq 2A^{-1}(t)V(t)^{\frac{n+2}{n}}, \tag{4.2.12}$$

where $q(t) = 2B(t)A^{-1}(t) - \frac{1}{2}\phi(t)$. Equation (4.2.12) is due to the fact that under variables (x,t), the fundamental solution F satisfies

$$\int_M F(x,t;y,s)d\mu(x,t) = 1$$

and consequently then

$$h(t) = \Big(\int_M F d\mu\Big)^{\frac{4}{n+2}} = 1.$$

Notice that the resulting ODE (4.2.12) is true for any $\tau \in [s,t]$, we then solve it by using integrating factor method. Denote $Q(\tau) = \int q(\tau)d\tau$, the integrating factor is $e^{Q(\tau)}$, therefore we have

$$\Big(e^{Q(\tau)}V(\tau)\Big)' \geq 2A^{-1}(\tau)\Big(e^{Q(\tau)}V(\tau)\Big)^{\frac{n+2}{n}} e^{-\frac{2}{n}Q(\tau)}$$

integrating from s to t (by separation of variables) since it is true for all $\tau \in [s,t]$, with the facts that

$$\int_s^t \frac{\Big(e^{Q(\tau)}V(\tau)\Big)'}{\Big(e^{Q(\tau)}V(\tau)\Big)^{\frac{n+2}{n}}}d\tau = -\frac{n}{2}\Big(e^{Q(\tau)}V(\tau)\Big)^{-\frac{2}{n}}\Bigg|_s^t = \frac{n}{2}e^{Q(t)}V(t)$$

by taking the limit

$$\lim_{\tau\searrow s} V(\tau) = \int_M \lim_{\tau\searrow s} F^2(x,\tau;y,s)d\mu(x,t) = \int_M \delta_y^2(x)d\mu(x,s) = 0,\ [1]$$

[1] This follows from a property of dirac delta which says it is zero everywhere except at only one point where it is either undefined or has an infinite value.

we obtain the bound as follows

$$V(t) \leq \frac{\left(\frac{2}{n}\right)^{\frac{n}{2}} e^{-Q(t)}}{\left(2\int_s^t \frac{e^{-\frac{2}{n}Q(\tau)}}{A(\tau)} d\tau\right)^{\frac{n}{2}}} = \frac{\left(\frac{1}{n}\right)^{\frac{n}{2}} e^{-Q(t)}}{\left(\int_s^t \frac{e^{-\frac{2}{n}Q(\tau)}}{A(\tau)} d\tau\right)^{\frac{n}{2}}}.$$

Taking $C_n := \left(\frac{1}{n}\right)^{\frac{n}{2}}$, we arrive at

$$\int_M F^2(x,t;y,s)d\mu(x,t) = V(t) \leq \frac{C_n e^{-Q(t)}}{\left(\int_s^t A^{-1}(\tau)e^{-\frac{2}{n}Q(\tau)}d\tau\right)^{\frac{n}{2}}}. \tag{4.2.13}$$

The next is to estimate

$$W(s) = \int_M F^2(x,t;y,s)d\mu(y,s).$$

Due to the asymmetry of the equation, the computation is slightly different. We recall that $F(x,t;y,s)$ satisfies the heat equation in the variables (y,s), then we similarly have

$$\begin{aligned} W'(s) &= \int_M (2F\partial_s F - RF^2)d\mu(y,s) \\ &= \int_M 2F(\Delta F) - RF^2 d\mu(y,s) \\ &= -2\int_M |\nabla F|^2 d\mu(y,s) - \int_M RF^2 d\mu(y,s). \end{aligned}$$

Using Lemma 4.2.4 again we arrived at

$$\begin{aligned} W'(s) \leq -2A^{-1}(s)\Big(h^{-1}(s)\int_M F^2 d\mu(y,s)\Big)^{\frac{n+2}{n}} + 2\Big(B(s)A^{-1}(s) \\ +\frac{1}{4}\phi(s)\Big)\int_M F^2 d\mu(y,s) - \phi(t)\int_M F^2 d\mu(y,s) \\ = -2A^{-1}(s)\Big(h^{-1}(s)\int_M F^2 d\mu(y,s)\Big)^{\frac{n+2}{n}} + \Big(2B(s)A^{-1}(t) - \frac{1}{2}\phi(s)\Big)\int_M F^2 d\mu(y,s). \end{aligned} \tag{4.2.14}$$

We can further estimate the quantity $h(s) = \left(\int_M F d\mu\right)^{\frac{4}{n+2}}$. Notice that contrary to what was obtainable in the variable (x,t), $\int_M F(x,t;y,s)d\mu(y,s) \leq 1$, since the coordinate (x,t) are kept fixed here and we only integrate in (y,s). Therefore

$$\begin{aligned} \lambda'(s) &= \frac{d}{ds}\Big(\int_M F(x,t;y,s)d\mu(y,s)\Big) \\ &= \int_M \partial_s F(x,t;y,s)d\mu(y,s) - \int_M R(y,s)F d\mu(y,s) \\ &= \int_M \Delta_{y,s}F(x,t;y,s)d\mu(y,s) - \int_M R(y,s)F(x,t;y,s)d\mu(y,s) \\ &\leq -\phi(s)\int_M F(x,t;y,s)d\mu(y,s). \end{aligned}$$

The last inequality is due to the fact that we are on compact manifold, where $\int_M \Delta F d\mu = 0$ and by the lower bound on scalar curvature R due to the maximum principle.

Now for any $\tau \in [s,t]$

$$\lambda'(\tau) \le -\phi(\tau)\lambda(\tau)$$
$$\frac{\lambda'(\tau)}{\lambda(\tau)} \le -\phi(\tau) = -\frac{1}{\rho^{-1} - \frac{2}{n}\tau},$$

integrating this from s to t we get

$$\ln \lambda(t) - \ln \lambda(s) \le \frac{n}{2} \ln(\rho^{-1} - \frac{2}{n}\tau)\Big|_s^t$$
$$\frac{\lambda(t)}{\lambda(s)} \le \left(\frac{\rho^{-1} - \frac{2}{n}t}{\rho^{-1} - \frac{2}{n}s}\right)^{\frac{n}{2}} \implies \lambda(t) \le \left(\frac{\rho^{-1} - \frac{2}{n}t}{\rho^{-1} - \frac{2}{n}s}\right)^{\frac{n}{2}} \lambda(s),$$

we can show that $\lambda(s) \equiv 1$ as follows

$$\begin{aligned}\lambda(s) = \lim_{s \nearrow t} \int_M F(x,t;y,s)d\mu(y,s) &= \int_M \lim_{s\nearrow t} F(x,t;y,s)d\mu(y,t) \\ &= \int_M \delta_x(y) = 1,\end{aligned}$$

combining these we have

$$h(t) = \left(\frac{\rho^{-1} - \frac{2}{n}t}{\rho^{-1} - \frac{2}{n}s}\right)^{\frac{n}{2}\cdot\frac{4}{n+2}} = \left(\frac{\rho^{-1} - \frac{2}{n}t}{\rho^{-1} - \frac{2}{n}s}\right)^{\frac{2n}{n+2}} =: \alpha^{\frac{2n}{n+2}}.$$

By this (4.2.14) is now reduced to the following

$$\begin{aligned}W(s) \le -2A^{-1}(s)\alpha^{-2}(s)\Big(\int_M F^2 d\mu(y,s)\Big)^{\frac{n+2}{n}} \\ +(2B(s)A^{-1}(s) - \frac{1}{2}\phi(s)) \int_M F^2 d\mu(y,s),\end{aligned} \tag{4.2.15}$$

we are then to solve the following ODE

$$W'(s) \le -2A^{-1}\alpha^{-2} W(s)^{\frac{n+2}{n}} + r(s)W(s), \tag{4.2.16}$$

where $r(s) = 2B(s)A^{-1}(s) - \frac{1}{2}\phi(s)$. In the similar vein to the previous estimate, we also solve (4.2.16) using integrating factor method. Denote $R(\tau) = \int r(\tau)d\tau$, the integrating factor is $e^{-R(\tau)}$. Therefore we have

$$\Big(e^{-R(\tau)}W(\tau)\Big)' \le -2A^{-1}\alpha^{-2}\Big(e^{-R(\tau)}W(\tau)\Big)^{\frac{n+2}{n}} e^{\frac{2}{n}R(\tau)},$$

integrating from s to t since it is true for any $\tau \in [s,t]$ we have immediately

$$W'(s) \le \frac{\left(\frac{2}{n}\right)^{\frac{n}{2}} e^{R(s)}}{\left(2\int_s^t \frac{e^{\frac{2}{n}R(\tau)}}{\alpha^2(\tau)A(\tau)}d\tau\right)^{\frac{n}{2}}} = \frac{\left(\frac{1}{n}\right)^{\frac{n}{2}} e^{R(s)}}{\left(\int_s^t \alpha^{-2}(\tau)A^{-1}(\tau)e^{\frac{2}{n}R(\tau)}d\tau\right)^{\frac{n}{2}}},$$

hence

$$\int_M F^2(x,t;y,s)d\mu(y,s) = W(s) \le \frac{C_n e^{R(s)}}{\left(\int_s^t \alpha^{-2}(\tau)A^{-1}(\tau)e^{\frac{2}{n}R(\tau)}d\tau\right)^{\frac{n}{2}}}. \tag{4.2.17}$$

We can then see from the computation above that

$$V(\frac{t}{2}) = \int_M F^2\Big(x,t;z,\frac{t}{2}\Big)d\mu\Big(z,\frac{t}{2}\Big) = \frac{C_n e^{-Q(\frac{t}{2})}}{\Big(\int_s^t A^{-1}(\tau)e^{-\frac{2}{n}Q(\tau)}d\tau\Big)^{\frac{n}{2}}}$$

and

$$W(\frac{t}{2}) = \int_M F^2\Big(z,\frac{t}{2};y,0\Big)d\mu\Big(z,\frac{t}{2}\Big) = \frac{C_n e^{R(\frac{t}{2})}}{\Big(\int_s^t \Big(\frac{\rho^{-1}-\frac{2}{n}\tau}{\rho^{-1}}\Big)^{-2}A^{-1}(\tau)e^{\frac{2}{n}R(\tau)}d\tau\Big)^{\frac{n}{2}}}.$$

Here we choose

$$P(\frac{t}{2}) = \int_0^{\frac{t}{2}}\Big[B(\tau)A^{-1}(\tau) - \frac{1}{2}\phi(\tau)\Big]d\tau = Q(\frac{t}{2}) = R(\frac{t}{2}) \quad with \quad \phi(t) := \frac{1}{\rho^{-1}-\frac{2}{n}t}.$$

Finally we obtain the bound

$$F(x,t;y,s) \leq \frac{C_n}{\Bigg(\int_s^{\frac{t+s}{2}}\Big(\frac{\rho^{-1}-\frac{2}{n}\tau}{\rho^{-1}}\Big)^{-2}A^{-1}(\tau)e^{\frac{2}{n}P(\tau)}\cdot\int_{\frac{t+s}{2}}^{s}A^{-1}(\tau)e^{-\frac{2}{n}P(\tau)}d\tau\Bigg)^{\frac{n}{4}}}. \tag{4.2.18}$$

The required estimate follows immediately. □

4.2.3 Manifold with weakly positive scalar curvature

Note that if $R(x,0) \geq 0$, the maximum principle shows that it remains so as long as Ricci flow exists, for this case we obtain a Sobolev type embedding from Lemma 4.2.4

$$\int_M |\nabla u|^2 d\mu(g(t)) \geq \frac{1}{A}\Big(\int_M u^2 d\mu(g(t))\Big)^{\frac{n+2}{2}} - \frac{B}{A}\int_M u^2 d\mu(g(t)), \tag{4.2.19}$$

where A and B are absolute constant independent of time, in fact $A = K(n,2)$ is the best constant in Euclidean Sobolev embedding and B can be taken to be equivalent to zero when $R(x,0) = 0$.

In the case $R(x,0) > 0$, we have $\lambda'(\tau) \leq 0$ showing that $\lambda(\tau)$ is decreasing, that is $\lambda(\tau) \leq \lambda(s) = 1$, $\tau \in [s,t]$. This implies that $h(s) = h(t) = 1$, then (4.2.16) becomes

$$W'(s) \leq -2A^{-1}W(s)^{\frac{n+2}{n}} + r(s)W(s)$$

with $\tilde{r} = \frac{2B}{A}$ and we obtain the estimate

$$W(s) \leq \frac{C_n e^{\tilde{R}(s)}}{A^{-1}\Big(\int_s^t(\tau)e^{\frac{2}{n}R(\tau)}d\tau\Big)^{\frac{n}{2}}},$$

similarly

$$V(t) \leq \frac{C_n e^{-\tilde{R}(t)}}{A^{-1}\Big(\int_s^t(\tau)e^{-\frac{2}{n}R(\tau)}d\tau\Big)^{\frac{n}{2}}}.$$

Putting these together we have a counterpart estimate to (4.2.18) as follows

$$F(x,t;y,s) \leq \frac{C_n}{\left[A^{-2}\left(\int_s^{\frac{t+s}{2}} e^{\frac{2}{n}\tilde{R}(\tau)}d\tau \cdot \int_{\frac{t+s}{2}}^{s} e^{-\frac{2}{n}\tilde{R}(\tau)}d\tau\right)\right]^{\frac{n}{4}}}. \tag{4.2.20}$$

Here, the denominator of the last inequality is simplified to

$$\begin{aligned}\left[A^{-2}\left(\int_s^{\frac{t+s}{2}} e^{\frac{2}{n}\tilde{R}(\tau)}d\tau \cdot \int_{\frac{t+s}{2}}^{s} e^{-\frac{2}{n}\tilde{R}(\tau)}d\tau\right)\right]^{\frac{n}{4}} &= \left[\frac{n^2}{16B^2}\left(e^{\frac{4B}{nA}\cdot\frac{t+s}{2}} - e^{\frac{4B}{nA}\cdot s}\right)\left(e^{-\frac{4B}{nA}\cdot\frac{t+s}{2}} - e^{-\frac{4B}{nA}\cdot t}\right)\right]^{\frac{n}{4}} \\ &= \left[\frac{n^2}{16B^2}\left(1 - e^{-\frac{4B}{nA}\cdot\frac{t-s}{2}}\right)^2\right]^{\frac{n}{4}}.\end{aligned}$$

Therefore

$$F(x,t;y,s) \leq \frac{C_n}{\left[\frac{n}{4B}\left(1 - e^{-\frac{4B}{nA}\cdot\frac{t-s}{2}}\right)\right]^{\frac{n}{2}}} \leq \frac{\tilde{C}_n}{(t-s)^{\frac{n}{2}}}$$

by Taylor series expansion (i.e., $1 - e^{-z} \lesssim z$), where $\tilde{C}_n = C_n \cdot (2A)^{\frac{n}{2}}$.

In the case $R(x,0) = 0$, $B(t) \equiv 0$, $\tilde{R}(t) = \frac{B}{A}t \equiv 0$ and

$$F(x,t;y,s) \leq \frac{C_n}{\left[A_0^{-2}\left(\int_s^{\frac{t+s}{2}} d\tau \cdot \int_{\frac{t+s}{2}}^{s} d\tau\right)\right]^{\frac{n}{4}}} = \frac{C_n}{\left[A_0^{-1}\left(\frac{t-s}{2}\right)\right]^{\frac{n}{2}}} = \frac{\tilde{C}_n}{(t-s)^{\frac{n}{2}}}, \tag{4.2.21}$$

where $\tilde{C}_n = C_n \cdot (2A_0)^{\frac{n}{2}} = \left(\frac{2}{n}k(n,2)\right)^{\frac{n}{2}}$. This completes the proof of Corollary 4.2.2.

4.3 Logarithmic Sobolev Inequalities

Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$, the L^2-Sobolev constants of (M,g) is defined to be

$$C_S(M,g) := \sup\left\{\|u\|_{\frac{2n}{n-2}} - Vol^{-\frac{2}{n}}(M,g)\|u\|_2 : u \in C^1(M), \|\nabla u\|_2 = 1\right\}. \tag{4.3.1}$$

Thus, $C_S(M,g) < \infty$ is the smallest number for which the L^2-Sobolev inequality

$$\|u\|_{\frac{2n}{n-2}} \leq C_S(M,g))\|\nabla u\|_2 + \frac{1}{Vol^{\frac{2}{n}}}\|u\|_2 \tag{4.3.2}$$

holds true for all $u \in W^{1,2}(M,g)$. Here the L^p-norm of a measurable function f with respect to the metric g is defined by

$$\|f\|_{p,g} = \|f\|_p := \left(\int_M |f|^p d\mu\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \tag{4.3.3}$$

Entropy and monotonicity formulas are intimately related to Sobolev-type inequalities (or functional inequalities in general [143]). One of such entropies is Perelman's one described in Chapter 2, which is a log-entropy energy relating to logarithmic Sobolev inequalities. The Sobolev and logarithmic Sobolev inequalities are essentially equivalent, the latter however has an advantage of being dimensionless. As we can see, Sobolev inequality gives information about the size of a function in terms of those information about the derivatives' sizes of the function. This is usually measured in terms of L^p-norm with p as large as possible, where in (4.3.2) above we have $p = 2n/(n-2)$. The implication of this is the loss of effectiveness by Sobolev inequality as n becomes larger, hence, there was a need for an improvement. The problem of finding a replacement for Sobolev inequality which does not depend on dimension necessitated the invention of logarithmic Sobolev inequality (which remains correct even for $n = \infty$) in the seventies, beginning with the work of E. Nelson [121]. In 1975, L. Gross [83] proved the following Gaussian version of log-Sobolev inequalities

$$\int |f(x)|^2 \ln |f(x)| d\nu(x) \leq \int |\nabla f(x)|^2 d\nu(x) + \|f\|_2^2 \ln \|f\|_2 \tag{4.3.4}$$

for every weakly continuously differentiable function f with $\|\cdot\|$ being the L^2-norm, $\nabla f \in L^2$ and Gaussian measure $d\nu(x) = (2\pi)^{-\frac{n}{n}} \exp\{-\frac{|x|^2}{2}\}dx$ is defined in terms of Lebesgue measure, dx. Gross relates this to hypercontractivity inequality of Nelson [120], the log-Sobolev inequalities are also related to contractivity by Gross in [84] and ultracontractivity properties of heat kernel semigroup by Davies and Simon in [72]. In fact, they have become powerful fundamental tools in the analysis of functions which are uniform in the dimension of the underlying space (for more details, see Davies [71], Lieb and Loss [107] and Weissler [152]).

We will show here that a family of log Sobolev inequalities on Riemannian manifold evolving by the Ricci flow is a consequence of the L^2-Sobolev inequality and the monotonicity property of our generalized dual entropy $\mathcal{W}_\epsilon$ introduced in (3.4.34) of Chapter 3. It is well known that Sobolev and log Sobolev inequalities are both equivalent to a number of functional inequalities such as Nash, isoperimetric and heat kernel bound inequalities. In this section, we obtain a family of log Sobolev inequalities along the Ricci flow which is in turn used to derive a sharp upper bound of the form

$$\|u(\cdot, T)\|_\infty \leq e^{cT}(4\pi T)^{-\frac{n}{2}} \|u_0(\cdot, T)\|_1 \tag{4.3.5}$$

for the conjugate heat kernel. Evidently, such an estimate is sharp since a bound in the other direction can be obtained by simply taking $u_0 \in L^p$ to be a delta function in

$$u(\cdot, T) = H * u_0 = \int_M H(\cdot, T; y) u_0(y) d\mu(y). \tag{4.3.6}$$

Remarkably, one can adopt the approach of Davies [71] to establish this equivalence via the ultracontractive property of the conjugate heat semigroup. Notice that this approach has been adopted by some authors, we mention the following [97, 155, 157]. Another useful application of Logarithmic Sobolev inequality on manifold evolving by Ricci flow are found in [48].

4.3.1 Derivation of Log-Sobolev Inequalities along the Ricci Flow

From the results of Aubin [6] and Hebey [95] for complete manifolds whose Ricci curvature is bounded from below and injectivity radius is positive and bounded from above, we can assume the Sobolev embedding on the initial metric, since $(M, g(0))$ is a compact Riemannian manifold. Let there exist positive constants $A_0, B_0 < \infty$ such that for all $u \in W^{1,2}(M, g_0)$,

$$\|u\|_{\frac{2n}{n-2}} \leq A_0\|\nabla u\|_2 + B_0\|u\|_2, \tag{4.3.7}$$

where A_0 and B_0 depends only on n, g_0, lower bound for the Ricci curvature and injectivity radius. We can then write (4.3.7) as

$$\Big(\int_M u^{\frac{2n}{n-2}} d\mu_{g_0}\Big)^{\frac{n-2}{n}} \leq A\int_M \Big(4|\nabla u|^2 + Ru^2\Big) d\mu_{g_0} + B\int_M u^2 d\mu_{g_0}, \tag{4.3.8}$$

where

$$A = \frac{1}{4}A_0, \ and\ B = \frac{1}{4}A_0 \sup R^-(\cdot, 0) + B_0$$

since $R(x, 0) + \sup R^-(\cdot, 0) = R^+(x, 0) - R^-(x, 0)$. R denotes the scalar curvature. (See [157, Chapter 6]).

The usual way of deriving logarithmic Sobolev inequality follows from a careful application of Hölder's and Jensen's inequalities, since $\log u$ is a concave function in which case

$$\int \ln u dw \leq \ln \int u dw$$

with the assumption that $\int dw = 1$, we then obtain the following

Lemma 4.3.1. *For any $u \in W^{1,2}(M, g_0)$ with $\|u\|_2 = 1$*

$$\int_M u^2 \ln u^2 d\mu_{g_0} \leq \frac{n}{2}\ln\Big(A\int_M (4|\nabla u|^2 + Ru^2) d\mu_{g_0} + B\Big). \tag{4.3.9}$$

See [97, 155, 157] for the proof.

Inequalities in (4.3.9) are usually estimated further by the application of an elementary inequality of the form $\ln y \leq \alpha y - \ln\alpha - 1,\ \ \alpha, y, \geq 0$. Precisely, taking $y = A\int_M (4|\nabla u|^2 + Ru^2\Big) d\mu_{g_0} + B$ in (4.3.9) gives us

$$\begin{aligned}\int_M u^2 \ln u^2 d\mu_{g_0} &\leq \frac{n\alpha}{2}\Big\{A\int_M (4|\nabla u|^2 + Ru^2) d\mu_{g_0} + B\Big\} - \frac{n}{2}(1 + \ln\alpha) \\ &= \frac{n\alpha A}{2}\int_M (4|\nabla u|^2 + Ru^2) d\mu_{g_0} + \frac{n\alpha B}{2} - \frac{n}{2} - \frac{n}{2}\ln\alpha.\end{aligned} \tag{4.3.10}$$

We will now modify the arguments in both [155] and [157] to prove our results.

Theorem 4.3.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and the metric $g(t)$ evolve by the Ricci flow. Assume that L^2-Sobolev embedding (4.3.7) holds true with respect to the initial metric $g(0) = g_0$. Then, we have*

$$\int_M u^2 \ln u^2 d\mu_{g(t)} \leq \int_M \sigma^2(4|\nabla u|^2 + Ru^2) d\mu_{g(t)} - \frac{n}{2}\ln\sigma^2 + (t + \sigma^2)\beta_1 + \frac{n}{2}\ln\frac{nA}{2e} \tag{4.3.11}$$

$$if \quad \lambda_0 = \inf_{\|u\|_2 = 1}\int_M (4|\nabla u|^2 + Ru^2) d\mu_{g_0},$$

that is, λ_0 is the first eigenvalue of the operator $-\Delta + \frac{R}{4}$. Furthermore, if λ_0 is strictly positive for $R(\cdot, 0) > 0$, then

$$\int_M u^2 \ln u^2 d\mu_{g(t)} \leq \int_M \sigma^2(4|\nabla u|^2 + Ru^2) d\mu_{g(t)} - \frac{n}{2}\ln\sigma^2 + (t+\sigma^2)\beta_1 + \frac{n}{2}\ln\frac{nA}{2e} + \gamma\lambda_0(g_0), \tag{4.3.12}$$

where $\sigma > 0$, $\beta_1 = 4A_0^{-1}B_0 + \sup R^-(\cdot, 0)$ and $\gamma = \left(\frac{4\pi}{\epsilon^2} - \frac{1}{2}\right) > 0$.

Notice that the logarithmic Sobolev inequalities in the above theorem are uniform for all time but deteriorates as time becomes large. We however have as a corollary a uniform log Sobolev inequality along the Ricci flow without any time restriction provided only $\lambda_0(g_0) > 0$.

Corollary 4.3.3. *With the assumption of the above theorem, we have*

$$\int_M u^2 \ln u^2 d\mu_{g(t)} \leq \sigma^2 \int_M (|\nabla u|^2 + \frac{R}{4}u^2) d\mu_{g(t)} - \frac{n}{2}\ln\sigma^2 + c(M, g) \tag{4.3.13}$$

for all $u \in W^{1,2}(M, g)$ with $\|u\|_2 = 1$, where $c(M, g)$ is a constant depending on the dimension n, A_0, B_0 nonpositive lower bound for $R(g_0)$ and a positive lower bound for $\lambda_0(g_0)$.

Let us give the following brief remark before we state the proof of the theorem above.

Remark 4.3.4. *Our results confirm the assertion made by R. Ye in [155] that a uniform logarithmic Sobolev inequality such as the one in the corollary above without the assumption that $\lambda_0(g_0) \geq 0$ is false in general. We refer to Topping [147, Section 6.6] for detail on the Zeroth eigenvalue of $-\Delta + \frac{R}{4}$ and Chapter 2 of this thesis for the monotonicity of eigenvalues of Perelman's $\mathcal{F}$-energy. All the standard references on the subject of Ricci flow listed in the Bibliography give account of this as well.*

4.3.2 Proof of Theorem 4.3.2

Now, we consider a compact n-dimensional Riemannian manifold (M, g_0), $n \geq 3$. Let $g = g(t)$ be a smooth solution of the Ricci flow

$$\frac{\partial}{\partial t} g(x, t) = -2Rc(x, t) \tag{4.3.14}$$

on $M \times [0, T)$, for some finite time $T > 0$, with initial metric $g(x, 0) = g_0$.

Recall the family $\mathcal{W}_\epsilon$-entropy functional introduced in (3.4.34) of Chapter 3

$$\mathcal{W}_\epsilon(g, f, \tau) = \int_M \left[\frac{\epsilon^2\tau}{4\pi}(R + |\nabla f|^2) + f - \frac{n\epsilon^2}{4\pi} + \frac{n}{2}\ln\left(\frac{4\pi}{\epsilon^2}\right)\right] \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} d\mu, \tag{4.3.15}$$

where the number $\tau(t) = T - t > 0$ and the function $f \in C^\infty(M)$ smoothly satisfies

$$\int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} d\mu = 1.$$

Taking an L^2-solution $v = v(x,t)$ of the conjugate heat equation

$$\partial_t v = -\Delta v + Rv \tag{4.3.16}$$

to be $v = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$, we have proved that the $\mathcal{W}_\epsilon(g,f,\tau)$-entropy functional is nondecreasing for $0 < \epsilon^2 \leq 4\pi$ (see Theorem 3.4.13).

Relating the entropy $\mathcal{W}_\epsilon$ with the idea of logarithmic Sobolev inequalities we consider a function

$$u = \sqrt{v} = \frac{e^{-\frac{f}{2}}}{(4\pi\tau)^{\frac{n}{4}}} \tag{4.3.17}$$

such that $\int_M u^2 d\mu = 1$. We also notice that (4.3.17) implies $f = -\ln u^2 - \frac{n}{2}\ln\tau - \frac{n}{2}\ln(4\pi)$, hence the entropy (4.3.15) is rewritten as

$$\left.\begin{aligned} \mathcal{W}_\epsilon(g,f,\tau) = \frac{\epsilon^2}{4\pi}\int_M \Big[\tau(4|\nabla u|^2 + Ru^2) - u^2\ln u^2\Big]ud\mu - \frac{\epsilon^2}{4\pi}\frac{n}{2}\ln\tau \\ -\frac{\epsilon^2}{4\pi}\frac{n}{2}\ln(4\pi) + (1-\frac{\epsilon^2}{4\pi})\int_M fu^2 d\mu - \frac{n\epsilon^2}{4\pi} + \frac{n}{2}\ln\frac{4\pi}{\epsilon^2} \end{aligned}\right\}. \tag{4.3.18}$$

Define

$$\mathcal{W}^*_\epsilon(g,u,\tau) = \frac{\epsilon^2}{4\pi}\int_M \Big[\tau(4|\nabla u|^2 + Ru^2) - u^2\ln u^2\Big]ud\mu \tag{4.3.19}$$

and

$$\mu^*_\epsilon(g,u,\tau) = \inf\{\mathcal{W}^*_\epsilon(g,u,\tau) : \int_M u^2 d\mu = 1\}. \tag{4.3.20}$$

Set $T^* = t^* + \sigma^2$ and $\tau(t) = T^* - t$ for $0 \leq t \leq t^*$ for some fixed constant $\sigma > 0$. Then

$$\frac{d}{dt}\mathcal{W}_\epsilon(g,f,\tau) = \frac{d}{dt}\mathcal{W}^*_\epsilon(g,u,\tau) - \frac{n\epsilon^2}{8\pi}\frac{d}{dt}\ln\tau + (1-\frac{\epsilon^2}{4\pi})\frac{\partial}{\partial t}\int_M fu^2 d\mu + \frac{n}{2}\ln\frac{4\pi}{\epsilon^2} \geq 0,$$

where the last inequality is due to Theorem 3.4.13 (monotonicity of $\mathcal{W}_\epsilon(g,f,\tau)$), the proof of which also reveals that

$$\frac{\partial}{\partial t}\int_M fu^2 d\mu = -\mathcal{F} + \frac{n}{2\tau},$$

where $\mathcal{F} = \int_M (R+|\nabla f|^2)u^2 d\mu$ is the Perelman's energy functional. Let λ_0 be the first eigenvalue of the operator $-\Delta + \frac{R}{4}$, then, we know that $\lambda_0 = \inf_{\|u\|_2=1}\mathcal{F}$. Therefore we arrive at

$$\frac{d}{dt}\mathcal{W}^*_\epsilon \geq \frac{n\epsilon^2}{8\pi}\frac{d}{dt}\ln\tau + (1-\frac{\epsilon^2}{4\pi})\lambda_0.$$

To continue this argument, we should note that either (4.3.16) and (4.3.17) implies that the function $f = f(t)$ satisfies the following backward heat equation

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \tag{4.3.21}$$

with $u = u(x,t)$ satisfying

$$\frac{\partial u}{\partial t} = -\Delta u + \frac{|\nabla u|^2}{u} + \frac{R}{2}u \tag{4.3.22}$$

on $[0, t^*]$ with a given terminal value at $t + t^*$ and $g = g(t^*)$.

Let v_0 be a minimizer of the entropy $\mathcal{W}_\epsilon(g, f, \tau_0)$ for all v such that $\int_M v d\mu_{g(t_0)} = 1$. We can then solve heat equation (4.3.21) backward in time with initial data $f(t_0) = f_0$ and v_0 chosen at $t = t_0$. Let u_j be the solution of the conjugate heat equation (4.3.22) at $t = t_j$. We can define functions $f_j, j = 1, 2$ by $u_j = \frac{e^{-\frac{f_j}{2}}}{(4\pi\tau_j)^{\frac{n}{4}}}, j = 1, 2$. Then by the monotonicity of $\mathcal{W}_\epsilon(g, f, \tau)$-entropy (using Perelman's approach, see also Proposition 2.5.4 in Chapter 2), we have for $t_1, t_2 \in [0, t_0],\ t_1 \leq t_2$

$$\mu_\epsilon(g(t_1), \tau(t_1)) = \inf_{\|v_0\|_{g(t_1)}=1} \mathcal{W}_\epsilon(g(t_1), f(t_1), \tau(t_1)) \leq \mathcal{W}_\epsilon(g(t_1), f(t_1), \tau(t_1))$$

$$\leq \mathcal{W}_\epsilon(g(t_2), f(t_2), \tau(t_2)) = \inf_{\|v_0\|_{g(t_2)}=1} \mathcal{W}_\epsilon(g(t_2), f(t_2), \tau(t_2)) = \mu_\epsilon(g(t_2), \tau(t_2)).$$

It follows from the above that

$$\mu_\epsilon^*(g(t_1), \tau(t_1)) \leq \mu_\epsilon^*(g(t_2), \tau(t_2)) + \frac{n\epsilon^2}{8\pi} \ln \frac{\tau_1}{\tau_2}$$

for any $t_1 < t_2$, where $\tau_j = \tau(t_j), j = 1, 2$, where μ_ϵ^* is as defined in (4.3.20). Choosing $t_1 = 0$ and $t_2 = t^*$, we then obtain

$$\mu_\epsilon^*(g(0), t^* + \sigma^2) \leq \mu_\epsilon^*(g(t^*), \sigma^2) + \frac{n\epsilon^2}{8\pi} \ln \frac{t^* + \sigma^2}{\sigma^2} \tag{4.3.23}$$

since $0 < t^* < T$ is arbitrary, We can write (4.3.23) as

$$\mu_\epsilon^*(g(t), \sigma^2) \geq \mu_\epsilon^*(g(0), t + \sigma^2) + \frac{n\epsilon^2}{8\pi} \ln \frac{\sigma^2}{t + \sigma^2} \tag{4.3.24}$$

for all $t \in [0, T)$. [2]

We now complete the proof of Theorem 4.3.2.

Proof. The next is to apply (4.3.10) with $g = g_0$ to estimate $\mu_\epsilon^*(g(0), t + \sigma^2)$. For any function $u \in W^{1,2}(M, g)$ with $\|u\|_2 = 1$ and choosing

$$\frac{n\alpha A}{2} = t + \sigma^2 \implies \alpha = \frac{8(t + \sigma^2)}{nA_0},$$

then, the inequality in (4.3.10) becomes

$$\int_M u^2 \ln u^2 d\mu_{g_0} \leq (t + \sigma^2) \int_M (4|\nabla u|^2 + Ru^2) d\mu_{g_0} + \frac{n}{2} \frac{8(t + \sigma^2)}{nA_0} B - \frac{n}{2} \ln \frac{8(t + \sigma^2)}{nA_0} - \frac{n}{2}$$

$$= (t + \sigma^2) \int_M (4|\nabla u|^2 + Ru^2) d\mu_{g_0} + 4(t + \sigma^2) BA_0^{-1} - \frac{n}{2} \ln(t + \sigma^2)$$
$$+ \frac{n}{2}\Big(\ln A_0 + \ln n - 3\ln 2 - 1\Big).$$

Choosing $\epsilon^2 \leq 4\pi$ as before, it then follows that

$$\mu_\epsilon^*(g(0), t + \sigma^2) \geq \frac{n\epsilon^2}{4\pi} \Big\{ \frac{1}{2} \ln(t + \sigma^2) - \frac{4}{n}(t + \sigma) BA_0^{-1} - \frac{1}{2}(\ln A_0 + \ln n - 3\ln 2 - 1) \Big\}. \tag{4.3.25}$$

[2] The case $t = 0$ is optimal as equality is attained.

Combining (4.3.24) and (4.3.25), we obtain

$$\mu_\epsilon^*(g(t),\sigma^2) \geq \frac{n\epsilon^2}{8\pi}\ln\sigma^2 - \frac{n\epsilon^2}{\pi}(t+\sigma^2)BA_0^{-1} - \frac{n\epsilon^2}{8\pi}(\ln A_0 + \ln n - 3\ln 2 - 1), \tag{4.3.26}$$

which implies

$$\frac{\epsilon^2}{4\pi}\int_M \left[\sigma^2(4|\nabla u|^2 + Ru^2) - u^2\ln u^2\right]d\mu \geq \frac{n\epsilon^2}{8\pi}\ln\sigma^2 - \frac{n\epsilon^2}{\pi}(t+\sigma^2)BA_0^{-1} - \frac{n\epsilon^2}{8\pi}\ln\frac{nA_0}{8e}.$$

Therefore

$$\int_M u^2\ln u^2 d\mu \leq \int_M \sigma^2(4|\nabla u|^2 + Ru^2)d\mu - \frac{n}{2}\ln\sigma^2 + 4(t+\sigma^2)BA_0^{-1} - \frac{n}{2}\ln\frac{nA_0}{8e}. \tag{4.3.27}$$

Choosing $\beta_1 = 4BA_0^{-1} = 4A_0^{-1}(B_0 + A\sup R^-(x,0))$ and $A = \frac{A_0}{4}$, we obtain the result. □

With similar argument to the above, our entropy functional $\mathcal{W}_\epsilon(g,f,\tau)$ yields another version of inequality called Restricted Log Sobolev inequality since $\epsilon^2 \leq 4\pi$. We state this result in the next theorem and then give a sketch of the proof.

Theorem 4.3.5. *(Restricted Logarithmic Sobolev Inequality). With the same notation and assumptions in the last theorem, we have*

$$\begin{aligned}\int_M u^2\ln u^2 d\mu_{g(t)} \leq \frac{\epsilon^2}{4\pi}\int_M \sigma^2(4|\nabla u|^2 + Ru^2)d\mu_{g(t)} - \frac{n}{2}\ln\sigma^2 \\ +(t+\sigma^2)\beta_2 + \frac{n}{2}\ln\frac{n\pi A}{2\epsilon^2 e},\end{aligned} \tag{4.3.28}$$

where $\beta_2 = \frac{\epsilon^2}{\pi}(A_0^{-1}B_0 + \sup R^-(x,0))$.

Proof. Following the same line of argument as before, we rewrite (4.3.15) to get a counterpart of (4.3.18) as

$$\mathcal{W}_\epsilon(g,f,\tau) = \mathcal{W}_\epsilon^R(g,u,\tau) - \frac{n}{2}\ln\tau - \frac{n}{2}\ln(4\pi) - \frac{n\epsilon^2}{4\pi} + \frac{n}{2}\ln\frac{4\pi}{\epsilon^2} \tag{4.3.29}$$

with

$$\mathcal{W}_\epsilon^R(g,u,\tau) = \int_M \left[\frac{\epsilon^2}{4\pi}\tau(4|\nabla u|^2 + Ru^2) - u^2\ln u^2\right]ud\mu, \tag{4.3.30}$$

then we have

$$\frac{d}{dt}\mathcal{W}_\epsilon^R \geq \frac{n}{2}\frac{d}{dt}\ln\tau.$$

Let $\mu_\epsilon^R(g,\tau)$ be the infimum of $\mathcal{W}_\epsilon^R$ over all u satisfying $\|u\|_2^2 = 1(i.e, \int_M u^2 d\mu = 1)$. It then follows that

$$\mu_\epsilon^R(g(t_1),\tau(t_1)) \leq \mu_\epsilon^R(g(t_2),\tau(t_2)) + \frac{n}{2}\ln\frac{\tau_1}{\tau_2}$$

for any $t_1 < t_2$, where $\tau_j = \tau(t_j), j = 1,2$. Choosing $t_1 = 0$ and $t_2 = t^*$. Since $0 < t^* < T$, we then obtain a counterpart of (4.3.24) as follows

$$\mu_\epsilon^R(g(t),\sigma^2) \geq \mu_\epsilon^R(g(0),t+\sigma^2) + \frac{n}{2}\ln\frac{\sigma^2}{t+\sigma^2} \tag{4.3.31}$$

for all $t \in [0,T)$. We now apply (4.3.10) with $g = g_0$ to estimate $\mu_\epsilon^R(g(0), t+\sigma^2)$. For any function $u \in W^{1,2}(M,g)$ with $\|u\|_2 = 1$ and choosing

$$\frac{n\alpha A}{2} = \frac{\epsilon^2}{4\pi}(t+\sigma^2) \implies \alpha = \frac{2\epsilon^2(t+\sigma^2)}{4n\pi A} = \frac{2\epsilon^2(t+\sigma^2)}{n\pi A_0},$$

then, the inequality in (4.3.10) becomes

$$\begin{aligned}\int_M u^2 \ln u^2 d\mu_{g_0} &\le (t+\sigma^2)\frac{\epsilon^2}{4\pi}\int_M (4|\nabla u|^2 + Ru^2)d\mu_g + \frac{2\epsilon^2(t+\sigma^2)}{n\pi A_0}B - \frac{n}{2}\ln\frac{2\epsilon^2(t+\sigma^2)}{4n\pi A} - \frac{n}{2} \\ &= (t+\sigma^2)\int_M \frac{\epsilon^2}{4\pi}(4|\nabla u|^2 + Ru^2)d\mu_g + \frac{\epsilon^2}{\pi}(t+\sigma)BA_0^{-1} - \frac{n}{2}\ln(t+\sigma^2) \\ &\quad + \frac{n}{2}\Big\{\ln\frac{nA}{2} - \ln\frac{\epsilon^2}{4\pi} - 1\Big\}.\end{aligned}$$

It then follows that

$$\mu_\epsilon^R(g(0), t+\sigma^2) \ge \frac{n}{2}\ln(t+\sigma^2) - \frac{\epsilon^2}{\pi}(t+\sigma^2)BA_0^{-1} - \frac{n}{2}\Big\{\ln\frac{nA}{2e} - \ln\frac{\epsilon^2}{4\pi}\Big\}. \tag{4.3.32}$$

Combining with (4.3.31), we obtain

$$\mu_\epsilon^R(g(t), \sigma^2) \ge \frac{n}{2}\ln\sigma^2 - \frac{\epsilon^2}{\pi}(t+\sigma^2)BA_0^{-1} - \frac{n}{2}\Big\{\ln\frac{nA}{2e} - \ln\frac{\epsilon^2}{4\pi}\Big\}, \tag{4.3.33}$$

which implies

$$\int_M u^2\ln u^2 d\mu \le \frac{\epsilon^2}{4\pi}\int_M \sigma^2(4|\nabla u|^2 + Ru^2)d\mu \tag{4.3.34}$$

$$- \frac{n}{2}\ln\sigma^2 + \frac{\epsilon^2}{\pi}(t+\sigma^2)BA_0^{-1} + \frac{n}{2}\Big\{\ln\frac{nA}{2e} - \ln\frac{\epsilon^2}{4\pi}\Big\}. \tag{4.3.35}$$

The restricted log Sobolev inequality (4.3.28) for all $g(t)$ follows immediately by choosing β_2. □

4.4 Heat Kernel Bound via Log Sobolev Inequalities

In the next, we apply the logarithmic Sobolev inequality obtained in the last section to derive an upper bound for the conjugate heat kernel along the Ricci flow, demonstrating that there is a lot of geometric information embedded in such inequalities. The basic ideas, due to Davies and Simon [72], relate Nelson's hypercontractivity (see L. Gross [83]) to ultracontractivity (see also [71]). These ideas always yield estimates with sharp constants, we modify the argument in Q. Zhang [157] (see also [107, 158]) to prove our result.

Theorem 4.4.1. *Let $(M, g(x,t)), t \in [0,T]$ be a solution to the Ricci flow with $n \ge 2$ and $H(x,t;y)$ be the fundamental solution to the conjugate heat equation*

$$\Big(-\partial_t - \Delta + R(x,\tau)\Big)u(x,\tau) = 0. \tag{4.4.1}$$

Then, for some positive finite constant C depending on n, t, T, A_0, B_0 and $\sup R^-(\cdot, 0)$, there holds the following estimates

$$H(x, T; y) \leq CT^{-\frac{n}{2}}, \tag{4.4.2}$$

where $\partial_t \tau = -1$ and A_0, B_0 are as defined in the last section.

Without loss of generality, we may assume $u = u(x,t)$ to be a nonnegative solution of the conjugate heat equation (4.4.1) on the interval $[0, T]$, where $\partial_t \tau = -1$. Let $T > 0$ and $r(\tau) : [0, T] \to [1, \infty]$ be a continuously differentiable increasing function such that $r(0) = \infty$ and $r(T) = 1$. The function $r(\tau) = \frac{T}{\tau}$ gives a perfect example as we shall see below.

The idea here follows from the fact that if

$$u(x,t) = \int H(x,t;y) u_0(y) d\mu(y)$$

solves the heat equation, where $H(x,t;y)$ is the heat kernel, then

$$\sup_{u \neq 0} \frac{\|u(\cdot, t)\|_\infty}{\|u(\cdot, 0)\|_1} = \sup_{x,y} H(x,t;y),$$

we may obtain estimation of time derivative for the logarithms of the quantity

$$\|u\|_{r(t)} = \Big(\int_M |u|^{r(t)} d\mu_{g(t)} \Big)^{\frac{1}{r(t)}}$$

as follows

$$\int_0^T \frac{\partial}{\partial t} \ln \|u\|_{r(t)} dt = \ln \frac{\|u(\cdot, t)\|_\infty}{\|u(\cdot, 0)\|_1}.$$

Proof. By routine computation

$$\begin{aligned}
\partial_t \|u\|_{r(t)} &= \partial_t \Big(\int_M |u|^{r(t)} d\mu_{g(\tau)} \Big)^{\frac{1}{r(\tau)}} \\
&= -\frac{\dot{r}(\tau)}{r^2(\tau)} \|u\|_{r(\tau)} \ln \|u\|_{r(\tau)}^{r(\tau)} + \frac{\|u\|_{r(\tau)}^{1-r(\tau)}}{r(\tau)} \Bigg\{ \dot{r}(\tau) \int_M u^{r(\tau)} \ln u \, d\mu_{g(\tau)} \\
&+ r(\tau) \int_M \Big[u^{r(\tau)-1} \Big(-\Delta u + Ru \Big) + u^{r(\tau)}(-R) \Big] d\mu_{g(\tau)} \Bigg\}.
\end{aligned}$$

Multiply both sides by $r^2(\tau) \|u\|_{r(\tau)}^{r(\tau)}$,

$$\begin{aligned}
r^2(\tau) \|u\|_{r(\tau)}^{r(\tau)} \partial_t \|u\|_{r(t)} &= -\dot{r}(\tau) \|u\|_{r(\tau)}^{r(\tau)+1} \ln \|u\|_{r(\tau)}^{r(\tau)} + r(\tau) \dot{r}(\tau) \|u\|_{r(\tau)} \int_M u^{r(\tau)} \ln u d\mu_{g(\tau)} \\
&+ r^2(\tau) \|u\|_{r(\tau)} \int_M u^{r(\tau)-1} (-\Delta u) d\mu_{g(\tau)} + r^2(\tau) \|u\|_{r(\tau)} \int_M u^{r(\tau)-1} (Ru) d\mu_{g(\tau)} \\
&- r(\tau) \|u\|_{r(\tau)} \int_M u^{r(\tau)} R \, d\mu_{g(\tau)}.
\end{aligned}$$

By the application of integration by parts we have

$$\begin{aligned} r^2(\tau)\|u\|_{r(\tau)}\int_M u^{r(\tau)-1}(-\Delta u)d\mu_{g(\tau)} &= r^2(\tau)\|u\|_{r(\tau)}\int_M \nabla(u^{r(\tau)-1})\nabla u\, d\mu_{g(\tau)} \\ &= r^2(\tau)(r(\tau)-1)\|u\|_{r(\tau)}\int_M u^{r(\tau)-2}|\nabla u|^2 d\mu_{g(\tau)}, \end{aligned}$$

hence

$$\begin{aligned} r^2(\tau)\|u\|_{r(\tau)}^{r(\tau)}\,\partial_t\|u\|_{r(t)} = &-\dot r(\tau)\|u\|_{r(\tau)}^{r(\tau)+1}\ln\|u\|_{r(\tau)}^{r(\tau)} + r(\tau)\dot r(\tau)\|u\|_{r(\tau)}\int_M u^{r(\tau)}\ln u d\mu_{g(\tau)} \\ &+ r^2(\tau)(r(\tau)-1)\|u\|_{r(\tau)}\int_M u^{r(\tau)-2}|\nabla u|^2 d\mu_{g(\tau)} \\ &+ r(\tau)(r(\tau)-1)\|u\|_{r(\tau)}\int_M R\, u^{r(\tau)} d\mu_{g(\tau)}. \end{aligned}$$

Further dividing both sides by $\|u\|_{r(\tau)}$, we obtain

$$\begin{aligned} r^2(\tau)\|u\|_{r(\tau)}^{r(\tau)}\,\partial_t\Big(\ln\|u\|_{r(t)}\Big) = &-\dot r(\tau)\|u\|_{r(\tau)}^{r(\tau)}\ln\|u\|_{r(\tau)}^{r(\tau)} + r(\tau)\dot r(\tau)\int_M u^{r(\tau)}\ln u d\mu_{g(\tau)} \\ &+r^2(\tau)(r(\tau)-1)\int_M u^{r(\tau)-2}|\nabla u|^2 d\mu_{g(\tau)} \\ &+r(\tau)(r(\tau)-1)\int_M R\, u^{r(\tau)} d\mu_{g(\tau)}. \end{aligned} \tag{4.4.3}$$

Denoting

$$v = \frac{u^{\frac{r(\tau)}{2}}}{\|u^{\frac{r(\tau)}{2}}\|_2} \implies v^2 = \frac{u^{r(\tau)}}{\|u\|_{r(\tau)}^{r(\tau)}}\ \ \textit{then}\ \ |\nabla v|^2 = \frac{r^2(\tau)}{4\|u\|_{r(\tau)}^{r(\tau)}}\cdot u^{r(\tau)-2}|\nabla u|^2$$

and

$$\ln v^2 = \ln u^{r(\tau)} - \ln\|u\|_{r(\tau)}^{r(\tau)}.$$

Therefore

$$\begin{aligned} \dot r(\tau)\int_M v^2\ln v^2 d\mu_{g(\tau)} &= \dot r(\tau)\int_M \frac{u^{r(\tau)}}{\|u\|_{r(\tau)}^{r(\tau)}}\left\{\ln u^{r(\tau)} - \ln\|u\|_{r(\tau}^{r(\tau)}\right\}d\mu_{g(\tau)} \\ &= \frac{\dot r(\tau)r(\tau)}{\|u\|_{r(\tau)}^{r(\tau)}}\int_M u^{r(\tau)}\ln u^{r(\tau)} d\mu_{g(\tau)} - \dot r\ln\|u\|_{r(\tau}^{r(\tau)}. \end{aligned}$$

Plugging these into (4.4.3), we arrive at the following

$$\begin{aligned} r^2(\tau)\partial_t\Big(\ln\|u\|_{r(t)}\Big) &= \dot r(\tau)\int_M v^2\ln v^2 d\mu_{g(\tau)} + 4(r(\tau)-1)\int_M |\nabla v|^2 d\mu_{g(\tau)} \\ &\quad + r(\tau)(r(\tau)-1)\int_M Rv^2 d\mu_{g(\tau)} \\ &= \dot r(\tau)\int_M v^2\ln v^2 d\mu_{g(\tau)} + (r(\tau)-1)\int_M (4|\nabla v|^2 + Rv^2)d\mu_{g(\tau)} \\ &\quad + (r(\tau)-1)^2\int_M Rv^2 d\mu_{g(\tau)}. \end{aligned}$$

Using the choice $r(\tau)=\frac{T}{\tau}$, we have $\dot{r}(\tau)=-\frac{T}{\tau^2}$ and $r(\tau)-1=\frac{T-\tau}{\tau}$ so that we write the last equality as

$$\begin{aligned}
r^2(\tau)\partial_t\Big(\ln\|u\|_{r(t)}\Big) &= -\frac{T}{\tau^2}\int_M v^2\ln v^2 d\mu_{g(\tau)}+\frac{T-\tau}{\tau}\int_M(4|\nabla v|^2+Rv^2)d\mu_{g(\tau)}\\
&\quad+\Big(\frac{T-\tau}{\tau}\Big)^2\int_M Rv^2 d\mu_{g(\tau)}\\
&=\frac{T}{\tau^2}\Bigg\{\frac{\tau(T-\tau)}{T}\int_M(4|\nabla v|^2+Rv^2)d\mu_{g(\tau)}-\int_M v^2\ln v^2 d\mu_{g(\tau)}\Bigg\}\\
&\quad+\Big(\frac{T-\tau}{\tau}\Big)^2\int_M Rv^2 d\mu_{g(\tau)}.
\end{aligned}$$

From log-Sobolev inequality (4.3.11) point of view, we may choose

$$\sigma^2=\frac{4\tau(T-\tau)}{T}\le\frac{T}{4}$$

and we get

$$r^2(\tau)\partial_t\Big(\ln\|u\|_{r(t)}\Big)\ge\frac{T}{\tau^2}\Bigg\{\frac{n}{2}\ln\sigma^2-\frac{n}{2}\ln\frac{nA}{2e}-(t_0+\sigma^2)\beta_1\Bigg\}+\Big(\frac{T-\tau}{\tau}\Big)^2\int_M Rv^2 d\mu_{g(\tau)} \tag{4.4.4}$$

and [3]

$$\partial_t\Big(\ln\|u\|_{r(t)}\Big)\ge\frac{1}{T}\Bigg\{\frac{n}{2}\ln\frac{4\pi\tau(T-\tau)}{T}-\frac{n}{2}\ln\frac{n\pi A}{2e}-(t_0+\sigma^2)\beta_1-T\sup R^-(\cdot,0)\Bigg\}. \tag{4.4.5}$$

Notice that (since $\sigma^2\le\frac{T}{4}$)

$$\begin{aligned}
(t+\sigma^2)\beta_1+T\sup R^-(\cdot,0)&=4(t_0+\sigma^2)\Big(A_0^{-1}B_0+\frac{1}{4}\sup R^-(\cdot,0)\Big)+T\sup R^-(\cdot,0)\\
&\le(4t_0+T)A_0^{-1}B_0+\frac{1}{4}(4t_0+5T)\sup R^-(\cdot,0).
\end{aligned}$$

Denoting D by

$$D\equiv\frac{n}{2}\ln\frac{n\pi A}{2e}+(4t_0+T)A_0^{-1}B_0,$$

substituting into (4.4.5) and integrating the result from 0 *to* T, we have

$$\begin{aligned}
\ln\frac{\|u(\cdot,T)\|_{r(T)}}{\|u(\cdot,T)\|_{r(0)}}&\ge\frac{n}{2T}\int_0^T\ln\frac{4\pi\tau(T-\tau)}{T}dt-D-\frac{1}{4}(4t_0+5T)\sup R^-(\cdot,0)\\
&=\frac{n}{2}\ln(4\pi)-\frac{n}{2}\ln T-n+n\ln T-D-\frac{1}{4}(4t_0+5T)\sup R^-(\cdot,0)\\
&=\frac{n}{2}\ln(4\pi T)-n-D-(4t_0+5T)\sup R^-(\cdot,0).
\end{aligned}$$

This then yields

$$\ln\frac{\|u(\cdot,T)\|_1}{\|u(\cdot,T)\|_\infty}\ge\frac{n}{2}\ln(4\pi T)-n-D-\frac{1}{4}(4t_0+5T)\sup R^-(\cdot,0),$$

[3] $\left(\frac{T-\tau}{\tau}\right)^2\int Rv^2d\mu_{g(\tau)} = \frac{(T-\tau)^2}{\tau^2}\int(R^+ - R^-)v^2d\mu_{g(\tau)} \ge -\frac{(T-\tau)^2}{\tau^2}\sup_{(M,g(\tau))}R^-(\cdot,\tau)\int v^2d\mu_{g(\tau)} \ge -\frac{T^2}{\tau^2}\sup_{(M,g(\tau))}R^-(\cdot,\tau)$, where we have used the fact that most negative part of the scalar curvature is non increasing along the time.

which implies

$$\|u(\cdot,T)\|_\infty \leq \|u(\cdot,T)\|_1 \, \frac{\exp\{\frac{1}{4}(4t_0+5T)\sup R^-(\cdot,0)+D+n\}}{(4\pi T)^{\frac{n}{2}}}.$$

Because

$$u(x,T)=\int_M H(x,T;y)u(y,0)d\mu(y)_{g(\tau)},$$

where $H(x,T;y)$ is the conjugate heat kernel, then

$$H(x,T;y)\leq \frac{\exp(nD)}{(4\pi T)^{\frac{n}{2}}}\dot{\exp}\{\frac{1}{4}(4t_0+5T)\sup R^-(\cdot,0)\}.$$

This ends the proof of the estimate (4.4.2). $\square$

Chapter 5

Heat Flow Monotonicity and Functional-Geometric Inequalities

5.1 Introduction

This chapter discusses an elegant application of heat flow monotonicity to the field of functional inequalities with geometric inputs. We construct functionals involving the fundamental solution of heat equation in linear and multilinear settings. The basic properties of heat diffusion semigroup, most especially, smoothness, positivity, and Markovian properties play crucial roles in deriving monotonicity formulae which in turn produce the inequalities of the family of Brascamp-Lieb. Let us consider geometric inequality of the form

$$\lambda(\{f_j\} : 1 \leq j \leq m) \leq \Lambda(\{f_j\} : 1 \leq j \leq m),$$

where f_j are functionals defined on some functional spaces. Informally, the flow monotonicity approach to proving this type of inequality relies on making an appropriate choice of monotone quantity for the flow, say for instance nondecreasing, $Q(t)$ of time, such that $Q(t)$ goes to $\lambda(\cdot)$ and $\Lambda(\cdot)$ respectively as t approaches 0 from right and goes to ∞ from left, i.e.,

$$\lambda(\{f_j\} : 1 \leq j \leq m) = \lim_{t \searrow 0} Q(t) \leq \lim_{t \nearrow \infty} Q(t) = \Lambda(\{f_j\} : 1 \leq j \leq m).$$

This type of approach as noticed by Carlen-Lieb-Loss [52], Barthe-Cordero-Erasquin [16] and Bennett-Carbery-Christ-Tao [24] tends to generate sharp constants and identify extremisers.

Throughout this chapter, we work in Euclidean setting as all the geometric-inequalities discussed here were originally formed in Euclidean spaces, but we note that most of results presented lift favourably well into the more general setting and can as well fit into the setting of Riemannian manifolds, though, they may require some more technicalities. Firstly, we start the discussion with L^p-mixed norms and generalised Hölder's inequality, since the integral inputs are naturally L^p-functions and Hölder's inequalities are critical in the derivation of the geometric inequalities discussed. Secondly, we give detail background on this family of inequalities and then later present some results involving monotonicity formulas.

5.1.1 Generalized Hölder's Inequality

Theorem 5.1.1. *Let $f \in L^p(\Omega)$ be a nonnegative measurable function, $\Omega \subset \mathbb{R}^n$, a bounded domain and denote the product of m-L^p functions by $\prod_{j=1}^m f_j(x)$. For each $1 \le p_j \le \infty$ with $\sum_{j=1}^m p_j^{-1} = 1$, $j \in I_m = \{1, 2, ..., m\}$, then*

$$\prod_{j=1}^m f_j(x) \in L^1(\Omega)$$

and

$$\int_\Omega \Big| \prod_{j=1}^m f_j(x) \Big| dx \le \prod_{j=1}^m ||f_j(x)||_{L^{p_j}}. \tag{5.1.1}$$

More interestingly, there is a version of Hölder's inequality for L^p-mixed norms. By L^p - mixed norms [3], we have

$$\Big|\Big| f(x) \Big|\Big|_{L^{\mathbf{p}}} = \Big|\Big| \cdot \ \ \cdot \ \ \cdot \ \Big|\Big| \quad ||f_j(x)||_{L^{p_1}(dx_1)} \quad \Big|\Big|_{L^{p_2}(dx_2)} \cdot \ \cdot \ \cdot \quad \Big|\Big|_{L^{p_m}(dx_m)}. \tag{5.1.2}$$

Here, we have index vector $\mathbf{p} = (p_1, p_2, ..., p_m), 1 \le p_j \le \infty$ for $j \in I_m = \{1, 2, ..., m\}$. For a nonnegative measurable f, the number $||f_j(x)||_{L^{p_j}(dx_j)}$ means L^{p_j}-norm of a function $f = f(x_1, x_2, ..., x_m)$ with respect to the variable x_j, essentially,

$$||f_j(x)||_{L^{p_j}(dx_j)} = \Big(\int_{\Omega_j} |f(\cdot \ \cdot \ \cdot x_j \ \cdot \ \cdot \ \cdot)|^{p_j} dx_j \Big)^{\frac{1}{p_j}}, \quad 1 \le p_i \le \infty.$$

Generally speaking, a function $f(x_1, x_2, ...x_m)$ measurable in the product space (Ω, dx), $\Omega = \prod_{j=1}^m \Omega_j$, $dx = dx_1 dx_2 \cdots dx_m$, belongs to $L^{\mathbf{p}}(\Omega)$, if the number obtained after finding in succession the L^{p_1}-norm of f with respect to x_1, the L^{p_2}-norm of f with respect to x_2 until lastly L^{p_m}-norm of f with respect to x_m, is finite.

Remark 5.1.2. *The number obtained, finite or not, is denoted by $||f||_{\mathbf{p}}$ or $||f||_{L^{\mathbf{p}}}$. In all the above and in what follows $||\cdot||_p$ is not a norm for $0 \le p < 1$.*

Theorem 5.1.3. *(Mixed Norms Hölder's Inequality). Given $1 \le p_j, q_j, r_j \le \infty$, if $f_j \in L^{p_j}(\Omega)$ and $g_j \in L^{q_j}(\Omega)$ such that $p_j^{-1} + q_j^{-1} = r_j^{-1}, j \in I_m = \{1, 2, ..., m\}$. Then, it holds that*

$$||f(x)g(x)||_r \le ||f(x)||_p ||g(x)||_q,$$

where we have written the indices

$$p = (p_1, p_2, ..., p_m), \quad q = (q_1, q_2, ..., q_m) \quad and \quad r = (r_1, r_2, ..., r_m)$$

with $p^{-1} + q^{-1} = r^{-1}$.

Iterating the above inequality, we have mixed norms Hölder's inequality for a product of m-functions as follows

$$||\prod_{j=1}^{m} f_j(x)||_r \leq \prod_{j=1}^{m} ||f_j(x)||_{p_j}$$

with $\sum_{j=1}^{m} p_j^{-1} = r^{-1}$.

Corollary 5.1.4. *Given a nonnegative measurable functions $g_j, 1 \leq j \leq m$ and $p_j, r \geq 1$ such that $\sum_{j=1}^{m} p_j^{-1} = r^{-1}$, then*

$$\Big(\int_{\Omega} \Big|\prod_{j=1}^{m} g_j\Big|^r \, dx\Big)^{\frac{1}{r}} \leq \prod_{j=1}^{m} \Big|\Big|g_j\Big|\Big|_{p_j}, \tag{5.1.3}$$

Proof. We may use induction on m, starting with the case $m = 2$, then for any $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have

$$\int_{\Omega} \Big|\prod_{j=1}^{2} g_j\Big|^r \, dx = \int_{\Omega} g_1^r g_2^r \, dx \leq ||g_1||_p^r ||g_2||_{p'}^r \tag{5.1.4}$$

p' is the conjugate exponent. Set $p_1 = pr$ and $p_2 = p'r$ such that $p^{-1} + q^{-1} = r^{-1}$ as desired. Applying induction method

$$\Big|\Big|\prod_{j=1}^{m+1} g_j\Big|\Big|_r = \Big|\Big|\prod_{j=1}^{m} g_j \cdot g_{m+1}\Big|\Big|_r \leq \Big|\Big|\prod_{j=1}^{m} g_j\Big|\Big|_q \Big|\Big|g_{m+1}\Big|\Big|_{p_{m+1}}, \tag{5.1.5}$$

where $q^{-1} + p_{m+1}^{-1} = r^{-1}$. Since $\sum_{j=1}^{m} p_j^{-1} = q^{-1}$, we may use the induction hypothesis to conclude that

$$\Big|\Big|\prod_{j=1}^{m} g_j\Big|\Big|_q \leq \prod_{j=1}^{m} \Big|\Big|g_j\Big|\Big|_{p_j}. \tag{5.1.6}$$

Combining (5.1.5) and (5.1.6) proves (5.1.3). □

Generalization of Mixed Norms Hölder's Inequality

Let $I_m = \{1, 2, ..., m\} \subset \{1, 2, ..., n\} = I_n$ and $\mathbf{j} = (j_1, j_2, ..., j_m)$ be an m-tuple of integers in I_m. Given a point $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, let $x_j = (x_{j_1}, x_{j_2}, ..., x_{j_m})$ be a point in $\mathbb{R}^m$ and $dx_j = dx_{j_1} dx_{j_2} \cdot \cdot \cdot dx_{j_m}$ and $\Omega_{I_n} = \prod_{j=1}^{m} \Omega_j$.

Theorem 5.1.5. *Let f_j be a nonnegative measurable function depending on the $j - th$ component of x_j and $f_j \in L^{p_j}(\Omega_j)$, where $p_j \geq 1$ with $\sum_{j \in I_m} p_j^{-1} = 1$ and Ω_j is the orthogonal projection of Ω onto m-dimensional plane in $\mathbb{R}^n$ with the coordinates corresponding to the component of x_j. Then*

$$\prod_{j=1}^{m} f_j(x_j) \in L^1(\Omega_{I_n}) \tag{5.1.7}$$

and

$$\int_{\Omega_{I_n}} \Big|\prod_{j=1}^{m} f_j(x_j)\Big| dx_{I_n} \leq \prod_{j \in I_m} \Big(\int_{\Omega_{I_j}} \Big|f_j(x_j)\Big|^{p_j} \, dx_{I_j}\Big)^{\frac{1}{p_j}}. \tag{5.1.8}$$

A classical approach to proving the above statements is by combinatorial rearrangement and induction, see [3, Lemma 4.24] and [78, Theorem 2.1]. A special case of the generalized mixed norms Hölder's inequality is the Loomis-Whitney Inequality due to L. H. Loomis and H. Whitney (1949), [114] (see also [31]).

5.2 Brascamp-Lieb Inequalities

For a natural number m and $1 \leq j \leq m$, define m positive real numbers p_j, m vectors v_j in $\mathbb{R}^n$, spanning vectors and multilinear operator

$$\Lambda(f_1, f_2, \cdots, f_m) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(\langle x, v_j \rangle)^{p_j}\, dx,$$

where f_j, are nonnegative integrable functions. We define some constant C by the formula

$$C := \inf_{f_j} \frac{\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j\Big(\langle x, v_j \rangle\Big)^{p_j}\, dx}{\prod_{j=1}^{m} \Big(\int_{\mathbb{R}} f_j\, dx\Big)^{p_j}}. \tag{5.2.1}$$

By this, we can assert that C is the smallest constant for which an inequality of the form

$$\Lambda(f_1, f_2, \cdots, f_m) \leq C \prod_{j=1}^{m} \Big(\int_{\mathbb{R}} f_j\, dx\Big)^{p_j} \tag{5.2.2}$$

holds for all $f : \mathbb{R}^n \to [0, \infty)$. Equivalently, we can use the above to estimate the L^{p_j}-norm of f_j as follows

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(\langle x, v_j \rangle)^{p_j}\, dx \leq C \prod_{j=1}^{m} \Big(\int_{\mathbb{R}} f_j\, dx\Big)^{p_j}. \tag{5.2.3}$$

The best possible constant C in the above inequality is nondegenerate and can be computed explicitly, testing this on functions which are strictly positive near the origin, it also can be shown that C is strictly positive.

The above description is geometric in nature and in fact can be viewed as Brascamp-Lieb inequality which is originally due to H.J Brascamp and E. H. Lieb (1976) [32]. There, it was used to prove Young's convolution inequality and its converse. Many other geometric inequalities such as Loomis-Whitney inequality, multilinear Hölder's inequality, Brunn-Minkwoski inequality, Prékopa-Leindler inequality, Geometric Brascamp-Lieb inequalities e.t.c, find their natural unification and generalizatiion in Brascamp-Lieb inequality. This family of inequalities turns out to be a nice and powerful tool in geometric analysis, most especially in deriving sharp inequalities and estimating the best possible constant. Earlier on Beckner [18] in 1975 used it to re-derive a version of Nelson's hypercontractive inequality. Similarly, Weissler [152] relates this to heat semigroup estimates to obtain sharp logarithmic Sobolev inequalities and Sobolev-Nirenberg inequalities (see also [49, 51, 74]). In the 90's, K. Ball, [10], [11],[12], discovered the applications of Brascamp-Lieb inequality for volume estimate of convex bodies and reverse isoperimetric inequality.

We now state a standard definition of Brascamp-Lieb inequality and its converse, we then examine how it generalizes some geometric inequalities.

5.2.1 Brascamp-Lieb Constant and its converse

Definition 5.2.1. *For natural numbers $m, n, n_j \in \mathbb{N}$, $n \geq n_j$, $1 \leq j \leq m$, define positive real numbers $p_j > 0$, such that*

$$\sum_{j=1}^{m} p_j n_j = n. \tag{5.2.4}$$

Let $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be surjective linear maps from $\mathbb{R}^n$ onto $\mathbb{R}^{n_j}$ such that their common kernel $\cap_{j=1}^{m} ker\ B_j = \{0\}$. This condition forces $\sum_{j=1}^{m} p_j B_j^* A_j B_j$ to be isomorphism, where B_j^* is the adjoint of B_j and A_j is a positive-definite $n_j \times n_j$ matrix.

Brascamp-Lieb constant is defined as follows

$$D(p_j) = \frac{det\Big(\sum_{j=1}^{m} p_j B_j^* A_j B_j \Big)}{\prod_{j=1}^{m} \Big(\det A_j \Big)^{p_j}}. \tag{5.2.5}$$

In this case, each f_j is a centred Gaussian function, i.e,

$$f_j = \exp(-\pi \langle A_j x, x \rangle). \tag{5.2.6}$$

Equivalently, for nonnegative measurable functions $f_j \in L^{q_j}(\mathbb{R}^n)$, where $q_j = \frac{1}{p_j}$

$$D(p_j) = \inf_{f_j} \frac{\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(B_j(x))\, dx}{\prod_{j=1}^{m} \|f_j\|_{q_j}}. \tag{5.2.7}$$

The optimal constant defined by the formula (5.2.7) is achieved by using Gaussian function of the form (5.2.6) and computed explicitly by (5.2.5). Now, let C_1 be the smallest constant such that for all $f_j, 1 \leq j \leq m$

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{p_j}(B_j(x))\, dx \leq C_1 \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} f_j \Big)^{p_j} \tag{5.2.8}$$

and let C_2 be largest constant such that for all $f_j, 1 \leq j \leq m$

$$\int_{\mathbb{R}^n}^{*} \sup_{x = \sum_{j=1}^{m} p_j B_j^* x_j,\ x_j \in \mathbb{R}^{n_j}} \prod_{j=1}^{m} f_j^{p_j}(x_j)\, dx \geq C_2 \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} f_j \Big)^{p_j}. \tag{5.2.9}$$

Here, the symbol $\int^*$ means outer integral. By a Theorem of Lieb [106], see also [13], both inequalities (5.2.8) and (5.2.9) are well known to be saturated by centred Gaussian functions, that is, the optimal for the constants in both cases can be computed explicitly using centred Gaussian functions, thus

$$C_1 = \sup \left\{ \frac{\int_{\mathbb{R}^n} \prod_{j=1}^{m} g_j^{p_j}(B_j(x))\, dx}{\prod_{j=1}^{m} (\int_{\mathbb{R}^n} g_j)^{p_j}}, \quad g_j \ \ centred\ \ Gaussian\ \ on\ \ \mathbb{R}^{n_j} \right\} \tag{5.2.10}$$

$$C_2 = \inf \left\{ \frac{\int_{\mathbb{R}^n} \prod_{j=1}^{m} g_j^{p_j}(B_j(x))\, dx}{\prod_{j=1}^{m} (\int_{\mathbb{R}^n} g_j)^{p_j}}, \quad g_j \ \ centred\ \ Gaussian\ \ on\ \ \mathbb{R}^{n_j} \right\}. \tag{5.2.11}$$

This was initially conjectured by Brascamp and Lieb in [32], that Gaussian functions give the best constant and proved by Lieb in [106] and simultaneously by Beckner in [18]. F. Barthe in [13] reproved Lieb's result using the method of optimal transport and simultaneously derived the dual result for the case of inequality (5.2.9) as conjectured by Lieb. We next follow the same line of Barthe's argument to prove the following lemmas.

Lemma 5.2.2. *([13, Theorem 1]). With the notations of the definitions above*

$$C_1 = D^{-\frac{1}{2}} \qquad C_2 = D^{\frac{1}{2}} \quad \text{and} \quad C_1 \cdot C_2 = 1$$

using centred functions.

Proof. Let A_j be $n_j \times n_j$ be positive-definite matrices on $\mathbb{R}^{n_j}$ and let Q be the quadratic form on $\mathbb{R}^n$, defined by

$$Q(x) = \Big\langle \sum_{j=1}^{m} p_j B_j^* A_j B_j x, x \Big\rangle.$$

Define an associated Gaussian function

$$f_j(x) = \exp(-\pi \langle A_j x, x \rangle),$$

then, by standard calculation of Gauss integral

$$\int_{\mathbb{R}^{n_j}} f_j(x)\, dx = \int_{\mathbb{R}^{n_j}} \exp(-\pi \langle A_j x, x \rangle)\, dx = (\det A_j)^{-\frac{1}{2}}.$$

Similar computation yields

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{p_j}(B_j(x))\, dx = \int_{\mathbb{R}^{n_j}} \exp\Big(-\pi \Big\langle \sum_{j=1}^{m} p_j B_j^* A_j B_j x, x \Big\rangle \Big)\, dx = (\det Q)^{-\frac{1}{2}}.$$

Hence

$$C_1 = \sup \frac{\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{p_j}(B_j(x))\, dx}{\prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} f_j(x_j)\, dx_j \Big)^{p_j}} = \left(\frac{\prod_{j=1}^{m} (\det A_j)^{p_j}}{\det \Big(\sum_{j=1}^{m} p_j B_j^* A_j B_j \Big)} \right)^{\frac{1}{2}} = D^{-\frac{1}{2}}.$$

The supremum is taken over the class of all Gaussian functions with maximum near the origin.

Now define Q^* to be the dual quadratic form of Q, on $\mathbb{R}^n$ by

$$Q^*(x) = \sup \Big\{ \ |\langle x, y \rangle|^2 : \quad Q(y) \le 1 \Big\}$$

and

$$R(x) = \inf \Big\{ \ \sum_{j=1}^{m} p_j \langle A^{-1} x_j, x_j \rangle : \quad x = \sum_{j=1}^{m} p_j B_j^* x_j, \ \ x_j \in \mathbb{R}^{n_j} \Big\},$$

we know that $R(x) = Q^*(x)$ by a Theorem in [13]. Compute

$$|\langle x, y \rangle|^2 = \Big| \langle \sum_{j=1}^{m} p_j B_j^* x_j, \ \ y \rangle \Big|^2 = \Big| \sum_{j=1}^{m} \langle p_j^{\frac{1}{2}} A_j^{-\frac{1}{2}} x_j, \ \ p_j^{\frac{1}{2}} A_j^{\frac{1}{2}} B_j y \rangle \Big|^2.$$

By Cauchy-Schwarz inequality, we have

$$|\langle x,y\rangle|^2 \le \Big(\sum_{j=1}^m |p_j^{\frac{1}{2}} A_j^{-\frac{1}{2}} x_j|^2\Big)\Big(\sum_{j=1}^m |p_j^{\frac{1}{2}} A_j^{\frac{1}{2}} B_j y|^2\Big)$$

$$= \Big(\sum_{j=1}^m p_j\langle x_j,\ A_j^{-1}x_j\rangle\Big)\Big(\langle\sum_{j=1}^m p_j B_j^* A_j B_j y,\ y\rangle\Big).$$

We then compute

$$\int_{\mathbb{R}^{n_j}} \exp(-\pi\langle x,\ A_j^{-1}x\rangle)\,dx = \Big(\frac{1}{\det(A_j^{-1})}\Big)^{\frac{1}{2}} = (\det A_j)^{\frac{1}{2}}$$

and

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(B_j^* A^{-1} B_j x)\,dx = \int_{\mathbb{R}^n} \exp\Big(-\pi\Big\langle \sum_{j=1}^m p_j B_j^* A^{-1} B_j x,\ x\Big\rangle\Big)\,dx$$

$$= \left(\frac{1}{\det\Big(\sum_{j=1}^m p_j B_j^* A^{-1} B_j\Big)}\right)^{\frac{1}{2}} = \Big(\det\Big(\sum_{j=1}^m p_j B_j^* A B_j\Big)\Big)^{\frac{1}{2}}.$$

Therefore

$$C_2 = \inf \frac{\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(B_j(x))\,dx}{\prod_{j=1}^m \Big(\int_{\mathbb{R}^{n_j}} f_j(x_j)\,dx_j\Big)^{p_j}} = \frac{\det\Big(\sum_{j=1}^m p_j B_j^* A B_j\Big)^{\frac{1}{2}}}{\prod_{j=1}^m (\det A_j)^{\frac{p_j}{2}}} = D^{\frac{1}{2}},$$

The infimum is taken over the class of all Gaussian functions with maximum near the origin. □

We recall from [24] that the following three conditions are necessary for finiteness of constants $D, C_1,$ *or* C_2

i. Each B_j is surjective, ii. $\cap_{j=1}^m Ker B_j = \{0\}$, and iii. $\sum_{j=1}^m p_j n_j = n$.

Once the conditions above are satisfied, it can be proved for $n_j \times n_j$ positive-definite matrices, $\{A_j\},\ 1 \le j \le m$ and $\sum_{j=1}^m p_j B_j^* A_j B_j =: M : \mathbb{R}^n \to \mathbb{R}^n$, positive semi-definite transformation, that the following are equivalent

1. $\{A_j\}$ is globally extremisable to

$$\sup_{A_j>0}\left(\frac{\prod_{j=1}^m (\det A_j)^{p_j}}{\det\Big(\sum_{j=1}^m p_j B_j^* A_j B_j\Big)}\right)^{\frac{1}{2}}.$$

2. $\{g_j(x)\} = \exp(-\pi\langle A_j x,\ x\rangle$ gives an exremal for inequality (5.2.8).

3. M is invertible and we have $A_j^{-1} = B_j^* M^{-1} B_j$ for all $1 \le j \le m$ and $M \ge B_l^* A_l A B_l$ for all l.

See also Carbery [47] for detail proofs of the above.

In our description of Brascamp-Lieb Inequality so far, we have used linear transformation and Lebesgue measure, we like to submit that the strength of this family of inequalities is the flexibility to live in a more generalized setting. Its wide variety of applications results from the fact that we are not restricted to using transformation or even Lebesque measure only. Next, we collect some sort of variants due to various authors who have made Brascamp-Lieb active over the years. The following give perfect references ([10, 11, 12, 13, 14, 15, 16, 17, 24, 26, 19, 32, 50, 78, 106].

5.2.2 Brascamp-Lieb Inequality on Product Spaces

Consider arbitrary measure spaces (Ω, δ, μ) and $(\Omega_j, \delta_j, \mu_j)$. Let B_j be surjective linear maps from Ω onto Ω_j and p_j be exponents with $1 \leq p_j \leq \infty,\ 1 \leq j \leq m$. For the inputs $\{B_j, p_j\}, 1 \leq j \leq m$, there exists a finite constant $D = D(\{B_j\}_{j=1}^n, \{p_j\}_{j=1}^n)$, such that

$$\int_\Omega \prod_{j=1}^m f_j \circ B_j \, d\mu \leq D \prod_{j=1}^m \Big(\int_{\Omega_j} f_j^{p_j} \, d\mu_j \Big)^{\frac{1}{p_j}}, \tag{5.2.12}$$

whenever $f_j \in L^{p_j}(\Omega_j, \mu_j), 1 \leq j \leq m$, are nonnegative measurable.

An interesting illustration of this is when $\Omega = \prod_{j=1}^m \Omega_j,\ \delta = \prod_{j=1}^m \delta_j,\ \mu = \prod_{j=1}^m \mu_j$ and B_j is an orthogonal projection of Ω onto Ω_j, then B_j^* becomes an inclusion map for each j. Indeed, if $\Omega_j,\ 1 \leq j \leq m$, are nonzero subspaces of $\mathbb{R}^n,\ \Omega \subseteq \mathbb{R}^n$ and B_j are orthogonal projection of $\mathbb{R}^n$ onto Ω_j, then, for all nonnegative measurable functions f_j on Ω_j, we have

$$\int_\Omega \prod_{j=1}^m f_j \circ B_j \, \nu_n(x) \, dx \leq \prod_{j=1}^m \Big(\int_{\Omega_j} f_j^{p_j}(y) \, \nu_{n_j} dy \Big)^{\frac{1}{p_j}}, \tag{5.2.13}$$

where n_j is the dimension of Ω_j, $n_j \leq n$ for each j and $\nu(x)$ is the corresponding Gaussian measure with respect to Lebesgue measure dx. (see [78]).

5.2.3 Brascamp-Lieb Inequality on the Sphere

Let $\{e_j\},\ 1 \leq j \leq n$, be the standard orthonormal basis in $\mathbb{R}^n$, such that $B_j(x)$ on $\mathbb{S}^{n-1}$ is defined as $B_j(x) = e_j \cdot x$. Then, for all nonnegative measurable functions $f_j \in L^{p_j}([-1,\ 1]), 1 \leq j \leq n$

$$\int_{\mathbb{S}^{n-1}} \prod_{j=1}^n f_j(e_j \cdot x) d\mu \leq \prod_{j=1}^n \Big(\int_{\mathbb{S}^{n-1}} f_j^{p_j}(e_j \cdot x) d\mu \Big)^{\frac{1}{p_j}} \tag{5.2.14}$$

for all $p \geq 2$. In the case $p < 2$, Carlen, Lieb and Loss [52] have shown that it is possible for the quantity in the LHS to diverge while each integral on the RHS is finite. It has been shown as well that such a divergence is not possible if each f_j is square integrable. Indeed, for $p \geq 2$ and $n \geq 3$, there is equality in (5.2.14) if and only if f_j is constant for each j or at least one of $f_j's$ is identically zero. This is due to Carlen, Lieb and Loss [52].

5.2.4 Rank-One Brascamp-Lieb Inequality

For $m \geq n$, let $\{v_j\}, 1 \leq j \leq m$ be unit vectors in $\mathbb{R}^n$ and p_j be positive exponents such that

$$\sum_{j=1}^m p_j v_j \otimes v_j = \mathbb{I}_n$$

for each j, where $\mathbb{I}_n$ is the identity on $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(\langle v_j, x \rangle) \, dx \leq \prod_{j=1}^m \Big(\int_{\mathbb{R}} f_j \Big)^{p_j}. \tag{5.2.15}$$

Here, $\langle v_j, x\rangle$ is a linear functional on $\mathbb{R}^n$ and $f_j : \mathbb{R} \to [0, \infty)$ are measurable.

In this case, there will be equality if $f_j's$ are identically zero or $v_j's$ form an orthonormal basis of $\mathbb{R}^n$. F. Barthe in [13] proves strict inequality if none of the $f_j's$ is a Gaussian density. (The works of K. Ball [10, 11, 12] are in rank one).

5.2.5 Loomis-Whitney Inequality

Let $\{e_j\},\ 1 \le j \le n,\ n \ge 2$, be the standard basis of $\mathbb{R}^n$, let $\pi_j : \mathbb{R}^n \to e_j^{\perp}$ be the orthogonal projection onto e_j, where $e_j^{\perp}$ is the orthogonal complement of e_j. Then, the constant $D(p) = 1$ for each $p_j = \frac{1}{n-1}$, and $\sum_{j=1}^{n} p_j = \frac{n}{n-1}$, while the constant is infinite for any other value of p, then, we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(\pi_j(x))\, dx \le \prod_{j=1}^{n} \Big|\Big|f_j(\hat{x}_j)\Big|\Big|_{L^{n-1}(\mathbb{R}^{n-1})} \tag{5.2.16}$$

for $f_j \in L^{n-1}(\mathbb{R}^{n-1})$. Here $\hat{x}$ denotes the coordinate being omitted. Notice that the condition $\sum_{j=1}^{n} p_j = \frac{n}{n-1}$ defines a hyperplane containing the point $(\frac{1}{n-1}, \frac{1}{n-1}, \cdots, \frac{1}{n-1})$ which is the only point for which the constant is finite. See [25, 114].

5.2.6 Multilinear Hölder's Inequality

Consider the notations of definition (5.2.1), whece $B_j = \mathbb{I}_n$, euclidean identity on $\mathbb{R}^n$, i.e, $\mathbb{I}_n : \mathbb{R}^n \to \mathbb{R}^n$, we can compute the constant $D(p)$ to be 1, such that $\sum_{j=1}^{m} p_j = 1$, then, we have for all nonnegative measurable functions f_j

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(x)\, dx \le \prod_{j=1}^{m} \Big|\Big|f_j(x)\Big|\Big|_{p_j}. \tag{5.2.17}$$

5.2.7 Prékopa-Leindler

Let $0 < \lambda < 1$ and f, g, h, be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$h(\lambda x + (1-)y) \ge f(x)^{\lambda} g(y)^{1-\lambda}, \quad \forall x, y \in \mathbb{R}^n.$$

Then Prékopa-Leindler inequality says

$$\int_{\mathbb{R}^n} h(x)dx \ge \Big(\int_{\mathbb{R}^n} f(x)dx\Big)^{\lambda}\Big(\int_{\mathbb{R}^n} g(x)dx\Big)^{1-\lambda}. \tag{5.2.18}$$

This is equivalent to Brunn-Minkowski inequality when one sets $f = 1_A$ and $g = 1_B$, the indicator functions of set $A, B \subset \mathbb{R}^n$, see [33] (and also [29]). Now to show how Brascamp-Lieb inequality (BLI) generalizes Prékopa-Leindler inequality: consider the Reverse BLI (5.2.9), take $m = 2,\ n_1 = n_2 = n,\ B_1 = B_2 = \mathbb{I} : \mathbb{R}^n \to \mathbb{R}^n$, $p_1 = \lambda$ and $p_2 = 1 - \lambda$, we can also compute $D(p) = 1$. Then we have

$$\int_{\mathbb{R}^n}^{*} \sup \Big\{f_1(x_1)^{\lambda} f_2(x_2)^{1-\lambda} : x = \lambda x_1 + (1-\lambda)x_2\Big\}dx \ge \Big(\int_{\mathbb{R}^n} f_1(x)dx\Big)^{\lambda}\Big(\int_{\mathbb{R}^n} f_2(x)dx\Big)^{1-\lambda}. \tag{5.2.19}$$

5.3 Main Theorem Via Heat Flow (Linear Setting)

Let m, n be integers. For $1 \le j \le m$, let $p_j > 0$ be positive real numbers, define surjective linear maps $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ such that $B_j^* B_j = \mathbb{I}_n$, the Euclidean identity on $\mathbb{R}^n$ or equivalently $\sum_{j=1}^m p_j n_j = n$. Here the common kernel for $B_j's$ is trivial which forces B_j^* to be an isometric embedding in $\mathbb{R}^n$ for each j and $B_j^* B_j$ is an orthogonal projection from $\mathbb{R}^n$ to subspace $Im(B_j^*)$.

Theorem 5.3.1. *For $1 \le j \le m$ and nonnegative measurable functions f_j, then it holds that*

- **Brascamp-Lieb Inequality**

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(B_j(x))\,dx \le \prod_{j=1}^m \Big(\int_{\mathbb{R}^{n_j}} f_j\Big)^{p_j} \tag{5.3.1}$$

  and

- **Reverse Brascamp-Lieb Inequality**

$$\int_{\mathbb{R}^n}^* \sup_{x=\sum_{j=1}^m p_j B_j^* x_j,\ x_j\in\mathbb{R}^{n_j}} \prod_{j=1}^m f_j^{p_j}(x)\,dx \ge \prod_{j=1}^m \Big(\int_{\mathbb{R}^{n_j}} f_j\Big)^{p_j}. \tag{5.3.2}$$

5.3.1 Monotonicity Formula and the Proof of BLI

The aim of this section is to prove the inequality in (5.3.1) and (5.3.2). Here, we make use of the fundamental solution of the heat equation. Let $u(t,x) = P_t f(x)$ solve

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \ div\ (u(t,x)\nabla \log u(t,x)), \quad x\in\mathbb{R}^n, \quad t\in[0,\infty) \\ u(0,x) = f(x), \end{cases} \tag{5.3.3}$$

where P_t is the heat semigroup operator.

Setting $v := \log f(x)$ at $t = 0$, we have the equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + |\nabla v|^2 \\ v|_{t=0} = \log f \end{cases} \tag{5.3.4}$$

with the diffusion semigroup $v(t,x) = \log P_t f(x)$. Following the idea introduced in [52], we can define nonlinear heat semigroup

$$f(t,x) = \Big(P_t f(x))^2\Big)^{\frac{1}{2}}$$

to obtain a nonlinear heat flow

$$\frac{\partial f(t,x)}{\partial t}\Big|_{t=0} = \Delta f(x) + \frac{|\nabla f(x)|^2}{f(x)}.$$

Using the transformation $B_j(x)$ of $\mathbb{R}^n$ onto $\mathbb{R}^{n_j}$, $1 \le j \le m$, the nonlinear heat flow above is precisely written as

$$\frac{\partial f_j(t,B_j(x))}{\partial t} = \Delta f_j(t,B_j(x)) + \frac{|\nabla f_j(t,B_j(x))|^2}{f_j(t,B_j(x))}. \tag{5.3.5}$$

We now define the functional

$$\Phi(t) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(t, B_j(x))\, d\mu(x), \tag{5.3.6}$$

which is known to be differentiable and smoothly continuous by the smoothing properties of the heat kernel semigroup for all $t > 0$.

Lemma 5.3.2. *Let $v_j : \mathbb{R}^n \times [0, \infty) \to \mathbb{R},\ \ 1 \le j \le m$, be nonnegative solution of (5.3.4) such that v_j are rapidly decreasing at spatial infinity locally uniformly. Then $\Phi(t)$ is nondecreasing in time and specifically*

$$\Phi'(t) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{l \neq k} \Big[\nabla_l v_k - \nabla_k v_l\Big]^2 \prod_{j=1}^{m} f_j(t, B_j(x))\, d\mu(x). \tag{5.3.7}$$

Proof. Taking time derivative of $\Phi(t)$, and using (5.3.5), we have

$$\begin{aligned}
\Phi'(t) &= \frac{d}{dt}\Big(\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j\, d\mu(x)\Big) = \int_{\mathbb{R}^n} \Big(\sum_{k=1}^{m} \frac{\partial}{\partial t} f_k\Big) \prod_{j=1,\ j\neq k}^{m} f_j\, d\mu(x) \\
&= \int_{\mathbb{R}^n} \sum_{k=1}^{m} \left(\Delta f_k + \frac{|\nabla f_k|^2}{f_k}\right) \prod_{j=1,\ j\neq k}^{m} f_j\, d\mu(x) \\
&= \int_{\mathbb{R}^n} \sum_{k=1}^{m} \Big(\Delta f_k\Big) \prod_{j=1,\ j\neq k}^{m} f_j\, d\mu(x) + \int_{\mathbb{R}^n} \sum_{k=1}^{m} \Big(\frac{|\nabla f_k|^2}{f_k}\Big) \prod_{j=1,\ j\neq k}^{m} f_j\, d\mu(x).
\end{aligned}$$

Using integration by parts on the first term since the second integral is nonnegative, we have

$$\begin{aligned}
\Phi'(t) &= -\int_{\mathbb{R}^n} \sum_{k,\ l=1}^{m} \Big(\nabla f_k, \nabla f_l\Big) \prod_{j=1,\ j\neq k,l}^{m} f_j\, d\mu(x) + \int_{\mathbb{R}^n} \sum_{k=1}^{m} \Big(\frac{|\nabla f_k|^2}{f_k}\Big) \prod_{j=1,\ j\neq k}^{m} f_j\, d\mu(x) \\
&= -\int_{\mathbb{R}^n} \sum_{k,l=1}^{m} \left[\frac{\nabla f_k}{f_k} \cdot \frac{\nabla f_l}{f_l} - \frac{|\nabla f_k|^2}{f_k^2}\right] \prod_{j=1}^{m} f_j\, d\mu(x) \\
&= \frac{1}{2} \int_{\mathbb{R}^n} \sum_{k,l=1,\ k\neq l}^{m} \left[\frac{|\nabla f_k|^2}{f_k^2} + \frac{|\nabla f_l|^2}{f_l^2} - 2\frac{\nabla f_k}{f_k} \cdot \frac{\nabla f_l}{f_l}\right] \prod_{j=1}^{m} f_j\, d\mu(x) \\
&= \frac{1}{2} \int_{\mathbb{R}^n} \sum_{k\neq l}^{m} \left[\frac{\nabla f_k}{f_k} - \frac{\nabla f_l}{f_l}\right]^2 \prod_{j=1}^{m} f_j\, d\mu(x) \\
&= \frac{1}{2} \int_{\mathbb{R}^n} \sum_{k\neq l}^{m} \Big[\nabla v_k - \nabla v_l\Big]^2 \prod_{j=1}^{m} f_j\, d\mu(x).
\end{aligned}$$

There is equality in (5.3.7) if and only if

$$\frac{\nabla f_k}{f_k} - \frac{\nabla f_l}{f_l} = 0.$$

(i.e., if and only v_j is a constant for each j but $v_j \neq 0$.) Since we know that each f_j is strictly positive and each v_k is positive, smooth and bounded for all time $t > 0$, we therefore conclude that the quantity $\Phi(t)$ is nondecreasing for all $t > 0$. □

Proof. **of the inequality (5.3.1).** For any nonnegative measurable function $f_j, \quad 1 \leq j \leq m$, we have seen that the functional $\Phi(t)$ is nondecreasing for all $t > 0$. Then, we have by the monotonicity property of $\Phi(t)$ that the quantity

$$\tilde{\Phi}(t) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{p_j}(t, B_j(x)) d\mu(x)$$

is also nondecreasing for $0 < t < \infty$, therefore

$$\limsup_{t \to 0} \tilde{\Phi}(t) \leq \liminf_{t \to \infty} \tilde{\Phi}(t).$$

By Fatou's lemma, we have that

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{p_j}(B_j(x)) \leq \limsup_{t \to 0^+} \tilde{\Phi}(t). \tag{5.3.8}$$

Indeed, we have equality in (5.3.8), since $\lim_{t\to 0} P_t f = f$. It then suffices to prove that

$$\liminf_{t \to \infty} \tilde{\Phi}(t) \leq \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} f_j \Big)^{p_j}. \tag{5.3.9}$$

The proof of (5.3.9) can be made more rigorous but we give the outline here. Now, we observe that $f_j(t, x)$ depends on $B_j x$ not on x itself, then we have

$$f_j(t, B_j x) = (4\pi t)^{-\frac{n_j}{2}} \int_{\mathbb{R}^{n_j}} e^{-\|B_j x - z\|^2/4t} f_j(z)\, d\mu(z).$$

Notice that each f_j above solves the heat equation (5.3.3) with initial condition $f_j(0, x) = f_j(B_j(x))$ and we rewrite

$$\tilde{\Phi}(t) = (4\pi t)^{-\frac{\sum_{j=1}^m p_j n_j}{2}} \int_{\mathbb{R}^n} \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} e^{-||B_j x - z||^2/4t} f_j(z)\, d\mu(z) \Big)^{p_j} d\mu(x).$$

Noting also that $\sum_{j=1}^m p_j n_j = n$. By rescaling argument, using $u_\epsilon(x) \longrightarrow \epsilon^{-n} v(\frac{x}{\epsilon})$, $\epsilon > 0$, we then have the transformation (i.e., by making the change of variables $x = \epsilon y, \quad dx = \epsilon dy$),

$$\tilde{\Phi}(t) = \frac{\epsilon^n}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} e^{-\frac{\epsilon^2}{4t}||B_j y - z/\epsilon||^2} f_j(z)\, d\mu(z) \Big)^{p_j} d\mu(y).$$

Choosing a scaling factor $\epsilon^2 = 4\pi t$, by convolution property and Fubini's theorem, we have that

$$\liminf_{t \to \infty} \tilde{\Phi}(t) \leq \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} \Big| f_j(B_j y) \Big|\, d\mu(y) \right)^{p_j} \int_{\mathbb{R}^n} e^{-\pi||z||^2} d\mu(z).$$

The claim (5.3.9) then follows immediately, since by standard Gauss integral $\int_{\mathbb{R}^n} e^{-\pi||z||^2} d\mu(z) = 1$ or we just write

$$\liminf_{t \to \infty} \tilde{\Phi}(t) \leq \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} \Big| f_j(z) \Big|\, d\mu(z) \right)^{p_j} \int_{\mathbb{R}^n} \Big(e^{-\pi||B_j y||^2} \Big)^{p_j} d\mu(y)$$

where we calculate

$$\int_{\mathbb{R}^n} \Big(e^{-\pi||B_j y||^2} \Big)^{p_j} d\mu(z) = \int_{\mathbb{R}^n} \exp(-\pi \langle \sum_{j=1}^{m} p_j B_j^* B_j y, y \rangle) = \det(\mathbb{I}_n) = 1.$$

This completes the proof of (5.3.1). □

Remark 5.3.3. *Instead of heat semigroup approach used to prove the first inequality (5.3.1), we can apply the Ornstein-Uhlenbeck semigroup, with the generator $L := \Delta - \langle x,\ \nabla \rangle$ as introduced in [16] almost at the same time heat flow semigroup was being launched in [52]. Ornstein-Uhlenbeck semigroup describes a diffusion process with constant diffusion and linear drift. Their approach is an adaptation of the argument in [28] for the proof of Ehrhand's inequality vis a vis Prékopa-Leindler where a nonnegative Borel function f on $\mathbb{R}^n$ evolves by the Mehler formula*

$$P_t f(x) = \int f(x + \sqrt{t} y) d\mu_n(y).$$

In fact, the limiting flow

$$P_t f(x)_{t \to +\infty} \sim (2\pi t)^{-\frac{n}{2}} \Big(\int f_j d\mu(y) \Big)$$

yields Prékopa-Leindler. C. Borell's method was also adapted in [17] to derive Brunn-Minkwoski inequality. Reverse Brascamp-Lieb inequality generalises both Prékopa-Leindler and Brunn-Minkwoski inequalities. The description is as follows: The function $f(t,x) = P_t f(x)$ solves

$$\frac{\partial f}{\partial t} = Lf$$

and

$$\begin{cases} \dfrac{\partial V}{\partial t} = & LV + |\nabla V|^2 \\ V|_{t=0} = \log f. \end{cases} \tag{5.3.10}$$

Define

$$\alpha(t) = \int \prod_{i=1}^{n} (P_t f)^{p_j}\ d\mu_n,$$

which is known to be differentiable, using the Mehler's formula under appropriate assumption, then, the limiting flow results in $\alpha(0) \leq \alpha(+\infty)$ to conclude the inequality. In the next section, we adapt their argument to prove the reverse inequalities, theirs was used in the rank one setting.

Note that in our computation P_t is the usual heat semigroup operator.

Lemma 5.3.4. *For $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $x_j \in \mathbb{R}^{n_j}, 1 \leq j \leq m$, if $h : \mathbb{R}^n \to \mathbb{R}^+$ and $f_j : \mathbb{R}^{n_j} \to \mathbb{R}^+$ satisfy*

$$h(\sum_{j=1}^{m} p_j B_j^* x_j) \geq \prod_{j=1}^{m} f_j(x_j)^{p_j}, \qquad x_j \in \mathbb{R}^{n_j},$$

then

$$H(\sum_{j=1}^{m} p_j B_j^* x_j) \geq \sum_{j=1}^{m} p_j F_j(x_j), \qquad x_j \in \mathbb{R}^{n_j},$$

where $H = \log P_t h : \mathbb{R}^n \to \mathbb{R}$ and $F_j = \log P_t f_j : \mathbb{R}^{n_j} \to \mathbb{R}$.

Proof. Let $h : \mathbb{R}^n \to \mathbb{R}^+$and $f_j : \mathbb{R}^{n_j} \to \mathbb{R}^+$ be nonnegative measurable functions. Define

$$H(t,x) := \log P_t h : \mathbb{R}^n \to \mathbb{R}^+ = [0, \infty)$$

and

$$F_j(t,x) := \log P_t f_j : \mathbb{R}^{n_j} \to \mathbb{R}^+ = [0,\infty).$$

Consider the fundamental solution $P_t f(x)$ of heat dynamics

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta u(t,x) \\ u(0,x) = f(x). \end{cases} \tag{5.3.11}$$

Taking $V := \log P_t f(x)$ at $t = 0$, we have the positivity-preserving evolution equation

$$\begin{cases} \dfrac{\partial V}{\partial t} = \Delta V + |\nabla V|^2 \\ V|_{t=0} = \log f(x). \end{cases} \tag{5.3.12}$$

By the hypothesis of the Lemma

$$h(x) - \prod_{j=1}^{m} f_j(x_j)^{p_j} \geq 0,$$

where $\sum_{j=1}^m p_j B_j^* B_j x = x \in \mathbb{R}^n$ and $x_j \in \mathbb{R}^{n_j}$, then, the quantity $Q(t, x_1, x_2, ..., x_m) : \mathbb{R} \times \mathbb{R}^{n_1} \times ....\mathbb{R}^{n_m} \to \mathbb{R}^+$ defined by

$$Q(t, x_1, x_2, ..., x_m) = H(t, \sum_{j=1}^m p_j B_j^* B_j x) - \sum_{j=1}^m p_j F_j(t, x_j) \tag{5.3.13}$$

satisfies

$$Q(t, x_1, x_2, ..., x_m)\Big|_{t=0} \geq 0.$$

The idea here is to show that $Q(t,\cdot)$ remains nonnegative throughout the evolution. Our effort is to derive positivity-preserving evolution equation for $Q(t,x)$ and then invoke standard theory of Maximum Principle for parabolic equation. Noting that H and F_j are nonnegative and they satisfy the above equation (5.3.12) since h and f_j satisfy equation (5.3.11), then we obtain

$$\frac{\partial H}{\partial t}(t, \sum_{j=1}^m p_j B_j^* B_j x) = \Delta H(t, \sum_{j=1}^m p_j B_j^* B_j x) + |\nabla H(t, \sum_{j=1}^m p_j B_j^* B_j x)|^2 \tag{5.3.14}$$

and

$$\frac{\partial}{\partial t} \sum_{j=1}^m p_j F_j(t, x_j) = \sum_{j=1}^m p_j \Delta F_j(t, x_j) + \sum_{j=1}^m p_j |\nabla F_j(t, x_j)|^2. \tag{5.3.15}$$

The question now is; Does Q satisfy the same evolution equation satisfied by H and F_j ? Hence, it is expected that from (5.3.13) - (5.3.15)

$$\begin{aligned} \frac{\partial}{\partial t} Q(t, x_1, x_2, ..., x_m) = &\Big(\Delta H(t, \sum_{j=1}^m p_j B_j^* B_j x) - \sum_{j=1}^m p_j \Delta F_j(t, x_j)\Big) \\ &+ \Big(|\nabla H(t, \sum_{j=1}^m p_j B_j^* B_j x)|^2 - \sum_{j=1}^m p_j |\nabla F_j(t, x_j)|^2\Big). \end{aligned}$$

To answer the above question, we compute in a straightforward manner

$$\nabla_{x_j} Q = p_j B_j \nabla H - p_j \nabla F_j$$

$$\nabla_{x_j} \nabla_{x_k} Q = p_j p_k B_j (\nabla^2 H) B_k^* - \delta_{jk} p_j \nabla^2 F_j.$$

Now

$$\begin{aligned}\Delta Q &= \sum_{j=1}^{m} tr\Big[p_j B_j^* \nabla_{x_j} \nabla_{x_k} Q p_k B_k\Big] \\ &= tr\Big[\sum_{j=1}^{m} p_j B_j^* B_j (\nabla^2 H) \sum_{k=1}^{m} p_k B_k^* B_k\Big] - tr\Big[\sum_{j,k=1}^{m} \delta_{jk} p_j B_j^* (\nabla^2 F_j) B_k\Big],\end{aligned}$$

so we have

$$\begin{aligned}tr\Big[\sum_{j=1}^{m} p_j B_j^* B_j (\nabla^2 H) \sum_{k=1}^{m} p_k B_k^* B_k\Big] &= \sum_{j,k=1}^{m} tr\Big[B_j^* \Big(p_j p_k B_j (\nabla^2 H) B_k^*\Big) B_k\Big] \\ &= tr \nabla^2 H = \Delta H.\end{aligned}$$

Similarly

$$\begin{aligned}tr\Big[\sum_{j,k=1}^{m} \delta_{jk} p_j B_j^* (\nabla^2 F_j) B_k\Big] &= \sum_{j=1}^{m} tr\Big(B_j^* p_j (\nabla^2 F_j) B_j\Big) \\ &= \sum_{j=1}^{m} \nabla^2 F_j = \sum_{j=1}^{m} p_j \Delta F_j.\end{aligned}$$

By the identity $\sum_{j=1}^{m} p_j B_j^* B_j = \mathbb{I}_n$, we have that for all vector $v \in \mathbb{R}^n$

$$\begin{cases} v = \sum_{j=1}^{m} p_j B_j^* B_j v \\ and \\ |v|^2 = \langle v, v\rangle = \langle \sum_{j=1}^{m} p_j B_j^* B_j v, v\rangle = \sum_{j=1}^{m} p_j \langle B_j v, B_j v\rangle. \end{cases} \tag{5.3.16}$$

Let us now focus attention on the lower order term, we have calculated

$$\nabla_{x_j} Q = p_j B_j \nabla H - p_j \nabla F_j,$$

then

$$\begin{aligned}|\nabla_{x_j} Q|^2 = \langle \nabla_{x_j} Q, \nabla_{x_j} Q\rangle &= \sum_{j=1}^{m} p_j |B_j \nabla H - \nabla F_j|^2 \\ &= |\nabla H|^2 - \sum_{j=1}^{m} |\nabla F_j|^2.\end{aligned}$$

Therefore Q satisfies the evolution equation

$$\frac{\partial Q}{\partial t} = \Delta Q + |\nabla Q|^2,$$

while the principal term ΔQ is elliptic. Then, by standard Maximum Principle for Parabolic equation [77, 129], and since $Q(0,\cdot) \geq 0$, then $Q(t,\cdot) \geq 0$ for all $t > 0$.

Hence by (5.3.13)

$$H(t, \sum_{j=1}^{m} p_j B_j^* B_j x) - \sum_{j=1}^{m} p_j F_j(t, x_j) \geq 0. \tag{5.3.17}$$

This ends the proof. □

Corollary 5.3.5. *If*

$$h(x) \leq \prod_{j=1}^{m} f_j(B_j(x))^{p_j}, \qquad x \in \mathbb{R}^n,$$

then

$$P_t h(x) \leq \prod_{j=1}^{m} P_t f_j(B_j(x))^{p_j}, \qquad x \in \mathbb{R}^n.$$

Proof. **of the inequality (5.3.2)**. By the hypothesis of Lemma 5.3.4

$$h(x) = h(\sum_{j=1}^{m} p_j B_j^* x_j) \geq \prod_{j=1}^{m} f_j(x_j)^{p_j}, \qquad x_j \in \mathbb{R}^{n_j}, \ x \in \mathbb{R}^n,$$

we know that $Q(0,\cdot) \geq 0$ and we have shown that the inequalities are preserved by the heat flow, then the reversed BLI follows immediately from the limit $t \to +\infty$, when choosing $h(x) = \prod_{j=1}^{m} f_j(x)^{p_j}$. Clearly

$$P_t h(x) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{\|x-y\|^2}{4t}} h(y)dy$$

and

$$\begin{aligned} \prod_{j=1}^{m} P_t f_j^{p_j} &= \prod_{j=1}^{m} \Big((4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^{n_j}} e^{-\frac{\|x-z\|^2}{4t}} f_j(z)dz \Big)^{p_j} \\ &= (4\pi t)^{-\frac{\sum_{j=1}^{m} p_j n_j}{2}} \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} e^{-\frac{\|x-z\|^2}{4t}} f_j(z)dz \Big)^{p_j}. \end{aligned}$$

Since the Lemma implies $P_t h(x) \geq \prod_{j=1}^{m} P_t f_j(x_j)^{p_j}$, then, we have

$$\int_{\mathbb{R}^n} e^{-\frac{\|x-y\|^2}{4t}} h(y)dy \geq \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} e^{-\frac{\|x-z\|^2}{4t}} f_j(z)dz \Big)^{p_j}, \tag{5.3.18}$$

where we have used the decomposition identity $\sum_{j=1}^{m} p_j n_j = n$. Taking $t \to +\infty$, we arrived at

$$\int_{\mathbb{R}^n} h(y)dy \geq \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} f_j \Big)^{p_j}.$$

This ends the proof of (5.3.2). □

5.4 Multilinear Heat Flow and Generalised BLI

In this section, we study further into how heat flow can help provide extrimisable gaussian for Brascamp-Lieb constants, as a by-product, we give an explicit proof of the generalised Brascamp-Lieb inequality. Most of the materials here are from [24, 25].

Definition 5.4.1. *Consider the multilinear functional of the form*

$$F\Big(\{f_j\}, 1 \le j \le m\Big) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{p_j}(T_j(x))dx, \tag{5.4.1}$$

where $T_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ *is a surjective linear transformation,* $f_j : \mathbb{R}^{n_j} \to \mathbb{R}$*, is a nonnegative measurable functions and* $1 \le j \le m$ *and* p_j *are* m*-positive exponents.*

The question we set to revisit is this: For which m-tuples of p_j and f_j do we obtain

$$0 < \sup_{f_j} \frac{\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{p_j}(T_j(x))dx}{\prod_{j=1}^{m} ||f_j||_{L^{p_j}}} < \infty? \tag{5.4.2}$$

This question is not new and complete answers have been provided, though, through different approaches (see for instance [13, 24, 25, 17, 47, 149, 150]). In the previous section, we have as well discussed the cases when the Brascamp-Lieb constant is unity, here, we also follow the line of heat flow to give our answer in this respect. To start with, we give results of [24, 25] as lemmas. Here $\mathbb{H}, \mathbb{H}_j$ are Hilbert spaces of finite, positive dimensions, equipped with canonical Lebesque measure.

Lemma 5.4.2. *Let* $T_j : \mathbb{H} \to \mathbb{H}_j$ *be surjective linear transformation. Let* $f_j : \mathbb{H}_j \to \mathbb{R}$ *be nonnegative and (5.4.1) holds for all* $1 \le j \le m$*. Then*

$$D(p_j) = \sup_{f_j} \frac{\Big(F\{f_j\}, 1 \le j \le m\Big)}{\prod_{j=1}^{m} \Big(\int f_j\Big)^{p_j}} < \infty, \tag{5.4.3}$$

if and only if

$$\dim(\mathbb{H}) = \sum_{j=1}^{m} p_j \dim(\mathbb{H}_j) \tag{5.4.4}$$

and

$$\dim(V) \le \sum_{j=1}^{m} p_j \dim(T_j(V)) \tag{5.4.5}$$

for every vector subspace $V \subseteq \mathbb{H}$*.*

Here, $\dim(\mathbb{H})$ and $\dim(V)$ are the dimensions of Hilbert space $\mathbb{H}$ and a vector space V respectively. Without loss of generality, $\dim(\mathbb{H})$ can be associated with n of $\mathbb{R}^n$, while the dimension of $\mathbb{H}_j$ can be associated with n_j of $\mathbb{R}^{n_j}$. For necessity and sufficiency of this result see [24, 25]. A local variant of this result is also given in the following;

Lemma 5.4.3. *For all nonnegative measurable functions f_j and for every subspace $V \subset \mathbb{H}$. A necessary and sufficient condition for*

$$F_{loc}\Big(\{f_j\}\Big) \leq C\prod_{j=1}^{m}\Big(\int f_j\Big)^{p_j} \tag{5.4.6}$$

to hold such that constant $C < \infty$ exists is that

$$codim_H(V) \geq \sum_{j=1}^{m} p_j codim_{H_j}(T_j(V)), \tag{5.4.7}$$

where $codim_H(V)$ is the codimension of a vector subspace V of $\mathbb{H}$.

Comparing conditions (5.4.5) and (5.4.7), we see that (5.4.5) provides a necessary and sufficient condition governing a large scale geometry. The condition of Lemma (5.4.3) follows directly from that of Lemma (5.4.2) by Hölders inequality. [25, Remarks 7.1] also gives instances where Lemma (5.4.3) is not subsumed in Lemma (5.4.2).

In the next, we reprove the result of Barthe [13] to this setting of generalised Brascamp-Lieb inequality. Our discussion follows purely from heat monotonicity approach. We first state the following definitions.

Definition 5.4.4. *Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a positive definite symmetric linear transformation. A nonnegative $f : \mathbb{R}^n \to \mathbb{R}$ is said to be of class G, if it takes the form*

$$f(x) = (\det G)^{\frac{1}{2}} \int_{\mathbb{R}^n} \exp(-\pi\langle G(x-y),(x-y)\rangle)d\mu(y), \tag{5.4.8}$$

where μ is a finite positive measure on $\mathbb{R}^n$ with non zero total mass (if μ is a point mass, f is said to be of extreme class G).

This means that a class G function can be expressed as a convolution of the centred gaussian $\exp(-\pi\langle Gx, x\rangle)$ with a finite positive measure on $\mathbb{R}^n$. It can then be seen clearly (by using Fourier transform) that a class G functions are smooth and strictly positive. Positive measure themselves are also taken to be of class $+\infty$. A function of the class A, say $g(\cdot) = \exp(-\pi\langle A\cdot, \cdot\rangle)$ is of class G provided $A \leq G$, so the standard centred gaussian $\exp(-\pi\langle G\cdot, \cdot\rangle)$ will be called of class G.

We note that class A functions provide a class of solutions to the heat equation. Suppose a class A_j function u_j solves the heat equation

$$\partial_t u_j = div(A_j^{-1}\nabla u_j),$$

with initial data $u_j(0) = \mu$, where μ is finite positive measure. Then, by the fundamental solution to the heat equation

$$u_j(t,x) = (\det A_j/t)^{\frac{1}{2}} \int_{\mathbb{R}^{n_j}} \exp(-\pi\langle A_j(x-y),(x-y)\rangle/t)d\mu(y)$$

then

$$u_j(1,x) = (\det A_j)^{\frac{1}{2}} \int_{\mathbb{R}^{n_j}} \exp(-\pi\langle A_j(x-y),(x-y)\rangle)d\mu(y).$$

Definition 5.4.5. *(**Generalised Brascamp-Lieb Constants**). We now define the generalised Brascamp-Lieb constant*

$$D_G(p_j) = \sup_{f \ of \ class \ G} D(p_j)$$

$D(p_j)$ as defined in the section (5.2.1).

Here we have

$$D_G(p_j) = \sup_{A_j \le G_j} \frac{\det(\sum_{j=1}^m p_j T_j^* A_j T_j)}{\prod_{j=1}^m (\det A_j)^{p_j}},$$

if each f_j is of class G_j, where G_j are gaussians, then

$$D_G = \frac{det(\sum_{j=1}^m p_j T_j^* G_j T_j)}{\prod_{j=1}^m (\det G_j)^{p_j}} > 0.$$

In this case, each T_j is surjective and their common kernel $\cap_{j=1}^m KerT_j = \{0\}$ and $\sum_{j=1}^m p_j n_j = n$.

Theorem 5.4.6. *For $T_j, f_j, p_j, 1 \le j \le m$ as defined before, we have*

- **Generalised Brascamp-Lieb Inequality**

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(T_j(x))\, dx \le (D_G)^{-\frac{1}{2}} \prod_{j=1}^m \Big(\int_{\mathbb{R}^{n_j}} f_j \Big)^{p_j} \tag{5.4.9}$$

and

- **Generalised Reverse Brascamp-Lieb Inequality**

$$\int_{\mathbb{R}^n}^* \sup_{x=\sum_{j=1}^m p_j B_j^* x_j,\ x_j \in \mathbb{R}^{n_j}} \prod_{j=1}^m f_j^{p_j}(x)\, dx \ge (D_G)^{\frac{1}{2}} \prod_{j=1}^m \Big(\int_{\mathbb{R}^{n_j}} f_j \Big)^{p_j}. \tag{5.4.10}$$

Proof. Let $u(t,x) = P_t f(x)$ be the solution to the heat equation

$$\partial_t u = div \cdot \nabla u$$

with the initial data $u(x,0) = f(x)$. By the closure properties of heat equation (see [20]), the quantity $\tilde{u}(t,x) = \prod_{j=1}^m (P_t f_j)^{p_j}$ is a solution in the sense that

$$\begin{aligned} \frac{\partial}{\partial t} \prod_{j=1}^m u_j^{p_j} &= \sum_{k=1}^m p_k u_k^{p_k - 1} (\partial_t u_k) \prod_{j \ne k}^m u_j^{p_j} = \sum_{k=1}^m \frac{p_k}{u_k} (\partial_t u_k) \prod_{j=1}^m u_j^{p_j} \\ &=: \Delta \Big(\prod_{j=1}^m u_j^{p_j} \Big) \end{aligned}$$

and

$$\begin{aligned} div \cdot \nabla \Big(\prod_{j=1}^m u_j^{p_j} \Big) &= div \Big[\sum_{k=1}^m p_k u_k^{p_k-1} (\nabla u_k) \prod_{j \ne k}^m u_j^{p_j} \Big] = div \Big[\sum_{k=1}^m p_k \Big(\frac{\nabla u_k}{u_k} \Big) \prod_{j=1}^m u_j^{p_j} \Big] \\ &= \sum_{k=1}^m p_k \Big(\frac{\Delta u_k}{u_k} - \frac{|\nabla u_k|^2}{u_k^2} \Big) \prod_{j=1}^m u_j^{p_j} + \sum_{k=1}^m p_k \Big(\frac{\nabla u_k}{u_k} \Big) \Big(\frac{\nabla u_k}{u_k} \Big) \prod_{j=1}^m u_j^{p_j} \\ &= \sum_{k=1}^m p_k \Big(\frac{\Delta u_k}{u_k} \Big) \prod_{j=1}^m u_j^{p_j} = \sum_{k=1}^m \frac{p_k}{u_k} (\partial_t u_k) \prod_{j=1}^m u_j^{p_j}. \end{aligned}$$

Now, it is clear from previous section (5.3.2) that the quantity

$$\Phi(t)=\int_{\mathbb{R}^n}\prod_{j=1}^{m}u_j^{p_j}(t,T_j(x))d\mu(x)$$

is monotone nondecreasing for all time $t>0$. Indeed, we can look at it from this perspective; taking time derivative of $\Phi(t)$

$$\Phi'(t)=\int_{\mathbb{R}^n}\sum_{k=1}^{m}\frac{p_k}{u_k}(\partial_t u_k)\prod_{j=1}^{m}u_j^{p_j}=\int_{\mathbb{R}^n}\sum_{k=1}^{m}\frac{p_k}{u_k}(\Delta u_k)\prod_{j=1}^{m}u_j^{p_j}.$$

Let $V_j=\log u_j$ $(i.e, V_j(t,T_j(x))=\log u_j(t,T_j(x)))$, then $u_j=e^{V_j}$ and

$$\prod_{j=1}^{m}u_j^{p_j}=e^{\sum_{j=1}^{m}p_jV_j}\quad and\quad \Delta u_j=(\Delta V_j+|\nabla V_j|^2)e^{V_j}=(\Delta V_j+|\nabla V_j|^2)u_j.$$

Hence

$$\Phi'(t)=\int_{\mathbb{R}^n}\Big[\sum_{k=1}^{m}\frac{p_k}{u_k}\Big(\Delta V_k+|\nabla V_k|^2\Big)u_k\Big]e^{\sum_{j=1}^{m}p_jV_j}=\int_{\mathbb{R}^n}\Big[\sum_{k=1}^{m}p_k\Big(\Delta V_k+|\nabla V_k|^2\Big)\Big]e^{\sum_{j=1}^{m}p_jV_j}. \quad (5.4.11)$$

Using integration by parts on the first term of the (5.4.11), we have

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{k=1}^{m}p_k\Delta V_k e^{\sum_{j=1}^{m}p_jV_j}&=-\int_{\mathbb{R}^n}\sum_{k=1}^{m}p_k\nabla V_k\ \nabla(e^{\sum_{j=1}^{m}p_jV_j})\\&=-\int_{\mathbb{R}^n}\sum_{j\neq k}^{m}p_jp_k\nabla V_j\ \nabla V_k\ e^{\sum_{j=1}^{m}p_jV_j}\\&=-\int_{\mathbb{R}^n}|\sum_{k=1}^{m}p_k\nabla V_k|^2\ e^{\sum_{j=1}^{m}p_jV_j},\end{aligned}$$

putting this back into (5.4.11), we obtain

$$\Phi'(t)=\int_{\mathbb{R}^n}\Big[-|\sum_{k=1}^{m}p_k\nabla V_k|^2+\sum_{k=1}^{m}p_k|\nabla V_k|^2\Big]\ e^{\sum_{j=1}^{m}p_jV_j}.$$

Re-writing the first term of RHS of the last identity as follows (by uzing Jensen's inequality and $\sum_{k=1}^{m}p_k\leq 1$)

$$|\sum_{k=1}^{m}p_k\nabla V_k|^2\leq\sum_{k=1}^{m}p_k|\nabla V_k|^2,$$

reveals that $\Phi'(t)\geq 0$.

Next we have the following;

Lemma 5.4.7. *Let T_j be surjective linear transformation. Suppose there are gaussians $A_j\leq G_j$ for all j such that positive definite transformation*

$$M=\sum_{j=1}p_jT_j^*A_jT_j$$

is invertible and satisfies

$$A_j^{-1}=T_j^*M^{-1}T_j$$

for each $1 \leq j \leq m$. Let $\tilde{u}_j : \mathbb{R}^{n_j} \times \mathbb{R} \to \mathbb{R}$ be the solution to the heat equation

$$\partial_t \tilde{u}_j = div \cdot (A_j^{-1} \nabla \tilde{u}_j)$$

with initial data $\tilde{u}_j(1)$ at time $t = 1$. Suppose $\tilde{u}_j(1)$ is of class G_j, then

$$\tilde{u}_j(t,x) = (\det G_j/t)^{\frac{1}{2}} \int_{\mathbb{R}^{n_j}} \exp(-\pi \langle G_j(x-y),(x-y)\rangle/t)\, d\mu_j(y)$$

for some finite nonnegative measure μ_j on $\mathbb{R}^{n_j}, 1 \leq j \leq m$. Moreover, the quantity

$$\Psi(t) = \int_{\mathbb{R}^n} \prod_{j=1}^m \tilde{u}_j^{p_j}(t, T_j(x)) d\mu(x)$$

is monotone nondecreasing for all time $t \geq 1$. See Carbery [47] for detail.

The above lemma allows us to define a function

$$f_j(1,x) = \int_{\mathbb{R}^{n_j}} \exp(-\pi\langle A_j(x-y),(x-y)\rangle) d\mu_j(y)$$

of class A_j and define

$$F\Big(\{f_j\}, 1 \leq j \leq m\Big) = \int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(T_j(x)) d\mu(x)$$

and for $y \in \mathbb{R}^{n_j}$, we have

$$f_j(y) = (\det A_j)^{\frac{1}{2}} \int_{\mathbb{R}^{n_j}} \exp(-\pi\langle A_j(y-z),(y-z)\rangle) d\mu_j(z),$$

where $\partial_t f_j = div(A_j^{-1}\nabla f_j), f_j(0) = \mu_j$ for some nonnegative measure μ_j on $\mathbb{R}^{n_j}$. For each j we define measure $\tilde{\mu}_j$ on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} \varphi\, d\tilde{\mu}_j = \int_{\mathbb{R}^{n_j}} \varphi(T_j^*(T_jT_j^*)^{-1}y) d\tilde{\mu}_j(y),$$

(here we have used surjectivity of T_j) and we observe that for $x \in \mathbb{R}^n$,

$$\tilde{f}_j(T_jx) = (\det A_j)^{\frac{1}{2}} \int_{\mathbb{R}^{n_j}} \exp(-\pi\langle T_j^* A_j T_j(x-y),(x-y)\rangle)\, d\tilde{\mu}_j(x),$$

then

$$\begin{aligned}
\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{p_j}(T_j(x)) &= \int_{\mathbb{R}^n} \prod_{j=1}^m \left[(\det A_j)^{\frac{1}{2}} \int_{\mathbb{R}^{n_j}} \exp(-\pi\langle A_j(x-y),(x-y)\rangle) d\mu_j(y)\right]^{p_j} d\mu(x) \\
&= \int_{\mathbb{R}^n} \prod_{j=1}^m \Big(\det A_j\Big)^{\frac{P_j}{2}} \int_{\mathbb{R}^{n_j}} \exp(-\pi\langle \sum_{j=1}^m p_j T_j^* A_j T_j(x-y),(x-y)\rangle) d\mu_j(y).
\end{aligned}$$

Using Dominated convergence theorem, taking the limit as $t \to \infty$, we have that

$$\begin{aligned}
\lim_{t\to\infty} \Psi(t) &= \prod_{j=1}^m \Big(\det A_j\Big)^{\frac{P_j}{2}} \det\Big(\sum_{j=1}^m p_j T_j^* A_j T_j\Big)^{-\frac{1}{2}} \prod_{j=1}^m ||\mu_j||^{p_j} \\
&= \left(\frac{\prod_{j=1}^m \det A_j^{p_j}}{\det(\sum_{j=1}^m p_j T_j^* A_j T_j}\right)^{\frac{1}{2}} \prod_{j=1}^m \Big(\int_{\mathbb{R}^{n_j}} f_j\Big)^{p_j}.
\end{aligned}$$

By monotonicity property $\Psi(1) \leq \Psi(\infty)$ and we have, by Fatou's Lemma, that

$$\Psi(1) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{p_j}(T_j(x)).$$

Then, the result follows since by assumption $A_j \leq G_j$, we can say that f_j is of class G_j by limiting argument. □

Remark 5.4.8. *Putting BLI in a diffeomorphic setting seems not to be straightforward and of course it is a topic of current research. Bennett, Carbery and Wright [26] treat the simplest case of Loomis-Whitney on submanifolds, where no assumptions about the brackets of the underlying vector fields are made, they use combination of the method of refinement and a tensoring argument. Since then, there has been the work of Bejenaru, Herr and Tataru [19], also Bennett and Bez in [21] have attempted nonlinear BLI on submanifolds using an induction-on-scales which is in the spirit of Bourgain [30] and seems to build on the work of Bejenaru-Herr-Tataru[19]. See also Bennett, Bez, Carbery, Hundertmark [22] [23] for applications of heat flow monotonicity.*

5.5 Justification for Brascamp-Lieb inequalities

At a first glance one may wonder if there is any connection at all between the subject of this chapter and those of the first part of this thesis. This section highlights where the connections lie. Here we show that Brascamp-Lieb inequalities generalize Young's convolution inequality which is equivalent to Nelson's hypercontractive estimates and logarithmic Sobolev inequalities both of which are related to heat kernel bounds and the entropy in 'Euclidean-Gaussian' setting.

5.5.1 From BLI to Young's inequality to Log-Sobolev inequality

The study of monotonicity in time of $\|f\|_p$ is connected with the classical Young's inequality in sharp form for $p > 1$. The limiting case $p \to 1$ leads to the monotonicity in time of the entropy. Then all the functional inequalities can be put into a unified framework. For example, let $1 \leq p_j \leq \infty$ be as defined before such that $\sum_{j=1}^{m} p_j^{-1} = n - 1$, Toscani [148] has shown that the heat flow monotonicity implies, for $f_j \in L^{p_j}(\mathbb{R}^n)$,

$$\begin{aligned}\sup_x \Big| f_1 \star f_2 \star \cdots \star f_m \Big| &= \Big| \int_{(\mathbb{R}^n)^{(m-1)}} f_1(x_1) f_2(x_1 - x_2) \cdots f_m(x_{m-1})\, dx_1\, dx_2 \cdots dx_{m-1} \Big| \\ &\leq \prod_{j=1}^{m} C_{p_j}^n \|f\|_{p_j},\end{aligned}$$

where $C_{p_j}^2 = p_j^{1/p_j} / {p'_j}^{1/p'_j}, \quad 1/p_j + 1/p'_j = 1, \ \forall\, j.$

Notice that the original proof of the sharp form is due to Beckner [18] and Brascamp and Lieb [32]. Equality is attained if and only if $f_j's$ are Gaussians. The sharp form can be derived from BLI (5.2.9) by taking $m = 3,\ n_1 =$

$n_2 = n_3 = n$ and $B_j : \mathbb{R}^{2n} \to \mathbb{R}^n,\ i = 1, 2, 3$, such that $B_1(x, y) = x,\ B_2(x, y) = x - y$ and $B_3(x, y) = y$. In this case we can compute $D(p_j) = \widetilde{C}(p_1, p_2, p_3)^n = (C_{p_1} C_{p_2} C_{p_3})^n$. This yields

$$\int_{(\mathbb{R}^{2n})} f_1(x)^{p_1} f_2(x-y)^{p_2} f_3(y)^{p_3} dx\, dy \le \widetilde{C}(p_1, p_2, p_3)^n \Big(\int_{\mathbb{R}^n} f_1 \Big)^{p_1} \Big(\int_{\mathbb{R}^n} f_2 \Big)^{p_2} \Big(\int_{\mathbb{R}^n} f_3 \Big)^{p_3}. \tag{5.5.1}$$

Recall the heat semigroup $e^{-t\Delta} f = P_t \star f$, where P_t is the Gaussian

$$P_t(x, y) = e^{-t\Delta}(x, y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}.$$

For any function $f \in H^1)(\mathbb{R}^n)$ and a number $\epsilon > 0$, the Log-Sobolev inequality is given by

$$\int_{(\mathbb{R}^2)} |f|^2 \log \Big(|f|^2 / \|f\|_2^2 \Big) dx \le \frac{\epsilon^2}{\pi} \int_{(\mathbb{R}^2)} |\nabla f|^2 dx - n(1 + \log \epsilon) \|f\|_2^2 \tag{5.5.2}$$

with equality if and only if f is up to translation, a multiple of $\exp\Big(-\pi|x|^2/2\epsilon^2 \Big)$.

By Young's inequality we see that $e^{-t\Delta}$ maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ provided $p \le q$. Then the sharp log-Sobolev inequality above follows by differentiating a sharp Young's inequality at $p = q = 2$ for the heat semigroup. This follows by writing

$$\|e^{-t\Delta} f\|_q \le \Big(\frac{C_r C_p}{C_q} \Big)^n \|P_t\|_r \|f\|_p$$

with $1/p + 1/r = 1 + 1/q \iff 1/r' + 1/p + 1/q = 2$. One then evaluates Gaussian integer $\|P_t\|_r$ and obtains

$$\|e^{-t\Delta} f\|_q \le \Big(\frac{C_p}{C_q} \Big)^n \Bigg(\frac{4\pi t}{(1/p - 1/q)} \Bigg)^{-\frac{n}{2}(1/p - 1/q)} \|f\|_p.$$

Setting $q = 2$, this is essentially Nelson's hypercontractive inequality [121] and the log-Sobolev inequality follows after some elementary analysis (See Lieb and Loss' book [107]).

5.5.2 Brascamp-Lieb inequalities (BLI) and the entropy

The original motivation for putting BLI on the sphere $\mathbb{S}^{n-1}$ was to better understand a result on the subadditivity of the entropy on $\mathbb{S}^{n-1}$. Given a probability density μ on some measure space $(\mathbb{S}^{n-1}, \mu)$, define entropy

$$S(f) = \int_{\mathbb{S}^{n-1}} f \log f d\mu \tag{5.5.3}$$

provided $f \log f$ is integrable. Define $\phi_j(x)$ on $\mathbb{S}^{n-1}$ by $\phi_j(x) = e_j \cdot x$, where $\{e_1, ..., e_n\}$ denote the standard orthonormal basis in $\mathbb{R}^n$. Then one has (Carlen-Lieb-Loss [52])

$$\sum_{j=1}^{n} S(f(\phi_j)) \le 2S(f), \tag{5.5.4}$$

where $f(\phi_j)$ is the j^{th} marginal of f, $j = 1, 2, .., n$ and the constant 2 is the best possible.

Now given a probability measure μ on Riemannian manifold M, the entropy of a nonnegative function $f : M \to \mathbb{R}^n$ is defined by

$$S_0(f) = \int_M f \log f d\mu - \Big(\int_M f d\mu \Big) \log \Big(\int_M f d\mu \Big) = \int_M f \log f d\mu$$

with $\int_M f d\mu = 1$. Let $f > 0$ be a positive solution to the heat equation satisfying $\int_M f d\mu = 1$, then, a straightforward computation shows that

$$\frac{d}{dt}\mathcal{S}_0(f(t)) = -\int_M |\nabla \log f(x,t)|^2 f(x,t) d\mu =: -\mathcal{F}_0(f(t))$$

and

$$\frac{d^2}{dt^2}\mathcal{S}_0(f(t)) = -\frac{d}{dt}\mathcal{F}_0(f(t)) = -2\int_M \Big(|Hess \log f|^2 + Rc(\nabla \log f, \nabla \log f)\Big) f d\mu.$$

By the above calculation, the entropy $\mathcal{S}_0$ for a positive solution to the heat equation on manifold is seen to be monotone decreasing while its derivative is monotone nondecreasing on the condition that the Ricci curvature of M is nonnegative. This shows the entropy is convex. We then have Bakry-Emery log-Sobolev inequality

$$\mathcal{S}_0(f) \leq \frac{1}{2K}\mathcal{F}. \tag{5.5.5}$$

On the manifold whose Ricci curvature Rc satisfies $Rc \geq K$ for some constant $k > 0$.

Based on these we define the following

$$\mathcal{S}(f(t)) := \mathcal{S}_0(f(t)) + \frac{n}{2}\log(4\pi t) + \frac{n}{2} = \int_M \Big(\log f + \frac{n}{2}\log(4\pi t) + \frac{n}{2}\Big) f d\mu$$
$$\mathcal{F}(f(t)) := t\mathcal{F}_0(f(t)) - \frac{n}{2} = \int_M \Big(t|\nabla \log f|^2 - \frac{n}{2}\Big) f d\mu.$$

Here, we have normalized $\mathcal{S}$ so that it remains identically zero for all time when f is the heat kernel. It is easily shown that $\mathcal{S}$ is identically zero on $M = \mathbb{R}^n$, the Euclidean space when f is the Euclidean heat kernel.

Notice by a straightforward computation

$$-\frac{d}{dt}\Big(t\mathcal{S}(f(t))\Big) = \mathcal{F}(f(t)) - \mathcal{S}(f(t)) = \mathcal{W}((f)). \tag{5.5.6}$$

Obviously, the entropy $\mathcal{W}(f(t))$ reads

$$\mathcal{W}(f(t)) = \int_M \Big(t\frac{|\nabla f|^2}{f^2} - \log f - \frac{n}{2}\log(4\pi t) - n\Big) f d\mu.$$

and

$$\frac{d}{dt}\mathcal{W} = \frac{1}{t}\frac{d}{dt}\Big(t\mathcal{F}\Big) = -2t\int_M \Big(\Big|\nabla\nabla \log f - \frac{1}{2t}g\Big|^2 + Rc(\nabla \log f, \nabla \log f)\Big) f d\mu.$$

This is exactly Ni's result in [122] which states that $\mathcal{W}(f,t)$ is monotone nonincreasing on a closed manifold with nonnegative Ricci curvature. In the case the manifold is Ricci flat this is indeed Perelman's entropy monotonicity formula [126] on a metric evolving by the Ricci flow.

Notice also by the application of integration by parts $\mathcal{F}(f(t))$ can be written as

$$\mathcal{F}(f(t)) = \int_M -\Big(t\Delta \log f + \frac{n}{2}\Big) f d\mu. \tag{5.5.7}$$

This has a surprising connection to the Li-Yau gradient estimate. Clearly the quantity under the integral is equivalent to the Harnack quantity of Li-Yau

$$-\Big(t\Delta \log f + \frac{n}{2}\Big) = -t\Big(\frac{\Delta f}{f} - \frac{|\nabla f|^2}{f^2} + \frac{n}{2t}\Big).$$

Li-Yau gradient estimate [112] says $\mathcal{F}(f) \leq 0$ when $Rc \geq 0$, which implies

$$\frac{f_t}{f} - \frac{|\nabla f|^2}{f^2} + \frac{n}{2t} \geq 0.$$

This is in turn equivalent to

$$t\Delta \log f - \frac{n}{2} \leq 0, \tag{5.5.8}$$

which can be viewed as a generalized Laplacian comparison theorem. Indeed, the Laplacian comparison theorem on M is a consequence of (5.5.8) by applying inequality to the heat kernel and letting t tends to zero. One can also see that $\lim_{t\to 0} \mathcal{S}(f(t)) = 0$ for the heat kernel and hence $\mathcal{S}(f(t))$ is monotone increasing on nonnegative Ricci curvature manifold. Therefore, we have $\mathcal{W}(f,t) \geq 0$ for the heat kernel for some $t > 0$ if and only if M is isometric to $\mathbb{R}^n$.

Let M be a complete Riemannian manifold with nonnegative Ricci curvature, then at $t = 1/2$, $\mathcal{W}$ holds on M if and only if M is isometric to $\mathbb{R}^n$, (See also Weissler [152]). This is indeed equivalent to Gross logarithmic Sobolev inequalities [83] on $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla f|^2 + f - n\right)\frac{e^{-f}}{(4\pi t)^{-\frac{n}{2}}} \geq 0. \tag{5.5.9}$$

Thus, there is a strong relation between the log-Sobolev inequality and the geometry of the manifold. Chapter 3 of this thesis contains some applications of Ni's entropy, Perelman's entropy and Li-Yau Harnack inequalities while Chapter 4 has some Log Sobolev inequalities and their variants.

5.5.3 Final Remark

The problem of considering Brascamp-Lieb type inequalities on manifolds suggests looking at certain ”nonlinear” euclidean Brascamp-Lieb inequalities first, such as Bennett and Bez [21] and the simplest case of the so-called nonlinear Loomis-Whitney inequality, Bennett, Carbery, Wright [26]. Earlier, Tao and Wright [146] considered low dimensions where bracket assumptions are made. Since then there has been work of Bejenaru, Herr and Tataru [19]. These problems seem quite difficult, at least to go substantially beyond what is in the above papers.

The authors also attempted to use heat-flow, and the paper [21] above is the product of this attempt, see [21, Remark 2.1]. One of the difficulties with the heat-flow is that the equations need to depend on all of the mappings B_j; i.e. the j^{th} - heat equation wants to depend on B_k for all k. Quite whether heat flow can work is yet unclear, but Bennett and Bez in [21] were able to salvage an inductive argument that is morally a form of discrete-time heat flow . It's not clear that Riemannian geometry is the right setting either. Results in this direction are desirable.

Appendix A

Aspects of Geometric Analysis

This Appendix provides a quick overview of some of the main analytic tools used throughout this thesis. The detail can be found in any standard references on Geometric Analysis. For the purpose of this thesis we use [3, 54, 59, 69, 77, 79, 95, 96, 105, 111, 132, 158].

A.1 Integration and Divergence Theorem

Given an oriented Riemannian manifold (M, g) with or without boundary, with an oriented atlas of charts $(\mathcal{U}_\alpha, x_\alpha)$, $\alpha \in \mathbf{I}$, where $\mathbf{I}$ is some set. A function f on M is measurable if, for every chart $x : \mathcal{U} \to \mathbb{R}^n$, $f \circ x^{-1}$ is measurable on the image of $\mathcal{U}$ in $\mathbb{R}^n$. For every covering $\{x_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n : \alpha \in \mathbf{I}\}$ of M by charts with subordinate partition of unity $\{\phi_\alpha : \alpha \in \mathbf{I}\}$, the Riemannian measure is given by

$$d\mu_g = \sum_\alpha \phi_\alpha \sqrt{\det g_\alpha} dx^1_\alpha \cdots dx^n_\alpha,$$

where $dx^i_\alpha, i = 1, 2 \cdots n$ is the density of Lebesgue measure on $x_\alpha(\mathcal{U}_\alpha) \subseteq \mathbb{R}^n$. In particular, if X is a smooth vector field, its divergence, $divX$, measures the infinitesimal distortion of volume by the flow generated by X. In the case of manifold with boundary ∂M, the orientation on M defines an orientation on ∂M. Now let $\tilde{g}$ be an induced Riemannian metric on ∂M, then, we have the volume form of $\tilde{g}$ defined by

$$d\sigma_{\tilde{g}} = \iota_\nu d\mu_g|_{\partial M},$$

where ν denotes the outward unit normal vector field on ∂M and the interior product ι_X is a skew-derivation for vector field X (See Kobayashi and Nomizu [102, p. 35] and Jeffrey M. Lee [104, Chapter 9]). Thus, we have $\iota_X d\mu_g\Big|_{\partial M} = \langle X, \nu\rangle_g d\sigma_{\tilde{g}}$ and the divergence of X can be defined as a quantity satisfying $d(\iota_X d\mu) = \; divX d\mu$.

Theorem A.1.1. *([69, Theorem 1.47] Divergence Theorem). Let (M, g) be a compact oriented Riemannian man-*

ifold, X, a smooth vector field at least C^1 and ν, the outward unit normal vector field on ∂M. Then

$$\int_M divX d\mu = \int_{\partial M} \langle X, \nu\rangle_g d\sigma. \tag{A.1.1}$$

Furthermore, if M is closed, then

$$\int_M divX d\mu = 0. \tag{A.1.2}$$

From the divergence theorem we have the following consequences (for details see Chavel [54] and Jeffrey M. Lee [104].)

Theorem A.1.2. *(Integration by Parts). Let (M, g) be an oriented Riemannian manifold with functions $u, v \in C^\infty(M)$. Then*

$$\int_M [u\Delta v - v\Delta u] d\mu = \int_{\partial M} \Big(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\Big) d\sigma \tag{A.1.3}$$

on a compact manifold. Furthermore, if M is closed we have

$$\int_M [u\Delta v - v\Delta u] d\mu = 0. \tag{A.1.4}$$

If $v \in C^1$, we have

$$\int_M [v\Delta u - \langle \nabla u, \nabla v\rangle_g] d\mu = \int_{\partial M} u\frac{\partial u}{\partial \nu} d\sigma \tag{A.1.5}$$

on a compact manifold while

$$\int_M v\Delta u d\mu = -\int_M \langle \nabla u, \nabla v\rangle_g d\mu \tag{A.1.6}$$

on a closed manifold.

Lemma A.1.3. *Let X be a vector field, $X = X^i\partial_i$ and $f \in C_c^\infty(M)$ be smooth function with compact support on M. Then*

$$\langle -divX, f\rangle_g = \langle X, \nabla f\rangle_g = -\int_M \frac{1}{\sqrt{\det g}} f\partial_i(X^i\sqrt{\det g})\sqrt{\det g}\, dx^i.$$

Thus,

$$divX = \frac{1}{\sqrt{\det g}}\partial_i(X^i\sqrt{\det g}).$$

A.2 Sobolev Spaces and Inequalities

A.2.1 Weak Derivative and Euclidean Sobolev Spaces

We briefly recall some elementary facts about Sobolev spaces for open subsets of the Euclidean space. This is the setting upon which the theory of Sobolev spaces on Riemannian manifold is built. For our discussions here we refer to Hebey [95, 96]. Let Ω be a domain in $\mathbb{R}^n$. Let u be a locally integrable function on Ω. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a n-tuple of nonnegative integers, i.e., multi-index of length $|\alpha| = \sum_{j=1}^n \alpha_j$.

Definition A.2.1. *A function $v \in L^1_{loc}(\Omega)$ is called the α^{th}-weak (distributional) derivative of u if*

$$\int_\Omega v\varphi dx = (-1)^{|\alpha|} \int_\Omega u\partial^\alpha \varphi \, dx,$$

and we write $\partial^\alpha u = v$ for all $\varphi \in C_c^\infty(\Omega)$, where $\partial^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$.

Definition A.2.2. *For $p \geq 1$ and k, a nonnegative integer, we define the Sobolev spaces of order k as*

$$W^{k,p}(\Omega) = \Big\{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega), \ \forall \, |\alpha| \leq k\Big\},$$

with the norm

$$\|u\|_{k,p} := \begin{cases} \Big(\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u|^p dx \Big)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \\ \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)}, & p = \infty. \end{cases}$$

A.2.2 Sobolev Spaces and Embedding on Riemannian Manifold

Let (M, g) be a smooth Riemannian manifold, for k integer and $u : M \to \mathbb{R}$ smooth. Let $\nabla^k u$ denote the k^{th} covariant derivative of u and $|\nabla^k u|$ be its norm defined in a local chart by

$$|\nabla^k u| = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots j_k}.$$

For any $p \geq 1$ real and integer k, we set

$$\mathcal{C}^p_k(M) = \Big\{u \in C^\infty(M) : \int_M |\nabla^j u| d\mu < +\infty, \quad \forall j = 0, \cdots, k\Big\}.$$

When M is compact, one clearly has that $\mathcal{C}^p_k(M) = C^\infty(M)$ for any k and any $p \geq 1$. For $u \in \mathcal{C}^p_k(M)$, set also

$$\|u\|_{H^p_k(M)} = \sum_{j=0}^k \Big(\int_M |\nabla^j u|^p d\mu \Big)^{\frac{1}{p}}. \tag{A.2.1}$$

We define the Sobolev spaces $H^p_k(M)$ as follows:

Definition A.2.3. *Let (M, g) be a Riemannian manifold, k, an integer and $p \geq 1$ real, the Sobolev space $H^p_k(M)$ is the completion of $\mathcal{C}^p_k(M)$ with respect to $\| \cdot \|_{H^p_k}$.*

Theorem A.2.4. *Let (M, g) be a compact Riemannian manifold, Sobolev spaces $H^p_1(M)$ is continuously embedded in $L^p(M)$ for any $1 \leq p \leq n$ with L^p-norm defined by $\|u\|_p = \Big(\int_M |u|^p d\mu \Big)^{\frac{1}{p}}$.*

Note - $H^p_1(M)$ is the completion of $C^\infty(M)$ with respect to the standard norm

$$\|u\|_p = \Big(\int_M |\nabla u|^p d\mu \Big)^{\frac{1}{p}} + \Big(\int_M |u|^p d\mu \Big)^{\frac{1}{p}}.$$

Theorem A.2.5. *(Sobolev-Poincaré inequalities [96, Theorem 2.11]). Let (M, g) be a compact Riemannian manifold of dimension n, $q \in [1, n)$ be real and p real such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$. There exists a positive constant C such that for any $u \in H_1^q(M)$*

$$\Big(\int_M |u - \bar{u}|^p \, d\mu\Big)^{\frac{1}{p}} \le C(M)\Big(\int_M |\nabla u|^q \, d\mu\Big)^{\frac{1}{q}}, \tag{A.2.2}$$

where $\bar{u} = \frac{1}{Vol(M)} \int_M u d\mu$.

Theorem A.2.6. *Let (M, g) be a compact Riemannian manifold of dimension n. For any $p \in [1, n)$, $H_1^p(M) \subset L^{\frac{np}{n-p}}(M)$, i.e, there exists a positive constant $C(M, g)$ such that*

$$\Big(\int_M |u|^{\frac{np}{n-p}} \, d\mu\Big)^{\frac{n-p}{np}} \le C(M, g)\bigg(\Big(\int_M |\nabla u|^p d\mu\Big)^{\frac{1}{p}} + \Big(\int_M |u|^p d\mu\Big)^{\frac{1}{p}}\bigg) \tag{A.2.3}$$

for all $u \in H_1^p(M)$.

A.3 Laplacian Comparison Theorem

Two fundamental results in Riemannian Geometry are the Laplacian and Hessian comparison theorems for distance function. The idea of comparison theorems is to compare a geometric quantity on a Riemannian manifold with the corresponding quantity on a model space. Typically, in Riemannian Geometry, model spaces have constant sectional curvature. Now given $k \in \mathbb{R}$, define

$$H_k(r) := \begin{cases} (n-1)\sqrt{|k|}\cot(\sqrt{|k|}r), & if\ k > 0, \\ \dfrac{n-1}{r}, & if\ k = 0, \\ (n-1)\sqrt{|k|}\coth(\sqrt{|k|}r), & if\ k < 0. \end{cases} \tag{A.3.1}$$

The function $H_k(r)$ is the mean curvature of the $(n-1)$-sphere of radius r in the complete simply connected Riemannian manifold of constant sectional curvature k. The detail proof of the following theorem can be found in books [111] by P. Li and [132] by Schoen and Yau.

Theorem A.3.1. *Let (M, g) be a complete Riemannian manifold with $Rc \ge (n-1)k$, where $k \in \mathbb{R}$ and if $p \in M$, then for any $x \in M$, where $d(p, x)$ is smooth, we have*

$$\Delta d(p, x) \le H_k(d(p, x)) \tag{A.3.2}$$

on the whole manifold.

Remark A.3.2. *The Laplacian comparison theorem holds in the distribution sense, that is, for any nonnegative $\varphi \in C_c^\infty(M)$ with compact support, we have*

$$\int_M d(p, x)\Delta\varphi(x) d\mu(x) \le \int_M C_k H_k(d(p, x))\varphi(x) d\mu(x). \tag{A.3.3}$$

At a point x, where $d(p, x)$ is smooth the Laplacian of the distance function is the mean curvature of the distance sphere (i.e., $\Delta d(p, x) = H_k$). Thus, Theorem A.3.1 follows immediately.

Appendix B

Eigenvalues and Heat Kernels of Riemannian Manifolds

In this section, we restrict our discussions to compact manifold without boundary. While most of the results hold for manifold with empty boundary, an appropriate boundary condition must be prescribed in the case the boundary is nonempty. We refer to standard books on Geometric Analysis for details, see for examples Chavel [54], Davies [71], Grigor'yan [82], Li [111] and Schoen and Yau [132].

B.1 Eigenvalues

Let (M, g) be a compact Riemannian manifold without boundary. The eigenvalue problem on M consists in finding the pairs $(\lambda_i, \phi_i), i = 1, 2, ...$ which satisfies

$$\Delta\phi_i = -\lambda_i\phi_i, \tag{B.1.1}$$ [1]

where λ_i are real constants called the *eigenvalues*, ϕ_i are nonzero functions called the *eigenfunctions* and Δ is the usual Laplace-Beltrami operator, which is a self-adjoint elliptic operator on $H^1(M)$. By spectral theory, [54, 132] M has a pure point (discrete) spectrum of a sequence of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ and the eigenfunctions ϕ_i form an orthonormal basis of $H^1(M)$. The spectrum of M is Riemannian invariant, i.e, any two isometric Riemannian manifolds have the same spectrum. In each case, the eigenfunctions form a vector space of finite dimension (eigenspace) with the dimension referred to as the multiplicity of the eigenvalues. The implication of this is that certain topological information about the geometry of the manifold are extracted from the spectrum and vice versa. For this, many interesting questions arise in spectral geometry such as *"what information on (M, g) can be drawn*

[1]Two natural choices of boundary conditions usually prescribed on (B.1.1) whenever M has nonempty boundary are; $\phi\Big|_{\partial M} = 0$ for Dirichlet eigenfunction and $\partial_\nu\phi\Big|_{\partial M} = 0$ for Neumann eigenfunction, where ν is the outward unit normal vector field on M.

from the geometric information on $\sigma(M,g)$ (the spectrum of the manifold (M,g)) and vice versa?". Mark Kac [100] in 1966 asked "can one hear the shape of a drum?"

The **Weyl asymptotic formula (Hermann Weyl)** [54, 82] states that

$$\lambda_k \sim \frac{4\pi^2}{Vol(B_n)^2} \cdot \left(\frac{k}{Vol M}\right)^{\frac{2}{n}} \qquad as\ k \to \infty, \tag{B.1.2}$$

where $Vol(B_n)$ and $Vol(M)$ are the volume of the unit Euclidean Ball and M respectively.

For each λ_k, we may choose the corresponding eigenfunction ϕ_k and we obtain another important concept in the study of eigenvalues, the *Mini-Max Principle* as follows;

For any function $f \in L^2(M)$, we have

$$f = \sum_{k=0}^{\infty} \langle f, \phi_k \rangle_{L^2(M)} \phi_k$$

and

$$\|f\|^2_{L^2(M)} = \sum_{k=0}^{\infty} \langle f, \phi_k \rangle^2_{L^2(M)},$$

since $\{\phi_i\}_{i=1}^{\infty}$ is orthonormal. For the fact that Δ is a self-adjoint elliptic operator on $L^2(M)$, we have

$$\lambda_i = \inf_f \frac{\int_M |\nabla f|^2 d\mu}{\int_M |f|^2 d\mu}, \qquad \int_M f f_j d\mu = 0, \ \ i \geq 1. \tag{B.1.3}$$

This principle says in particular that $\lambda_1 > 0$ is the biggest constant for which the following inequality (**Poincaré inequality**) holds;

$$\int_M |\nabla f|^2 d\mu \geq C \int_M |f|^2 d\mu, \quad \forall f \in L^2(M) \ \ and\ C \leq \lambda_1 \ \ is\ a\ constant. \tag{B.1.4}$$

The study of eigenvalues of Laplacian further reveals the relationship between Sobolev embedding and Isoperimetric inequality. Faber and Krahn exploited this in the 1920s and obtained what is today known as Faber-Krahn inequality. Jeffrey Cheeger applied similar argument in the early 70s to estimating the first eigenvalue of the Laplacian [111]. A. Grigor'yan has also proved the equivalence of these inequalities. For more details see Chavel [54], Davies [71], Grigor'yan [82] and Schoen and Yau [132].

Theorem B.1.1. *(**Sobolev Inequality [111]**). Let M^n be a compact Riemannian manifold without boundary. Then there exists a constant $C_s > 0$ depending on $n, n \geq 2$, such that*

$$C_s \left(\int_M |f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq \int_M |\nabla f|, \quad \forall f \in L^2(M). \tag{B.1.5}$$

Theorem B.1.2. *(**Isoperimetric Inequality [111]**). Let M^n, $n \geq 2$ be a compact Riemannian manifold, Ω, a domain with a compact closure in M, then there exists a constant $C_I > 0$ independent of Ω, such that*

$$C_I \Big(Vol(\Omega)\Big)^{\frac{n-1}{n}} \leq Vol(\partial\Omega). \tag{B.1.6}$$

B.2 Bounds on Eigenvalues

P. Li and S-T. Yau developed methods of proving estimates on the least eigenvalues via the gradient estimates on the first eigenfunction.

Theorem B.2.1. *(**Lower Bounds** [111, 132]). Let (M, g) be a compact Riemannian manifold with nonnegative Ricci curvature. Then*

$$\lambda_1 \geq \frac{\pi^2}{4\, diam(g)^2}, \tag{B.2.1}$$

where $diam(g)$ is the diameter of (M, g).

Theorem B.2.2. *(**Eigenvalues Comparison Theorem** [132]). Let (M, g) be any n-dimensional Riemannian manifold with $Rc(M) \geq nk$. Then for any $x \in M$ and $r > 0$, we have*

$$\lambda_1(B(x, r)) \leq \lambda_1(B(k, r)), \tag{B.2.2}$$

where $B(k, r)$ denotes a ball with radius r in the simply connected n-dimensional model manifold. Equality in (B.2.2) holds if and only if $B(x, r)$ is isometric to $B(k, r)$.

Theorem B.2.3. *(**Monotonicity Formula for Eigenvalues**). Let (M, g) be a Riemannian manifold and $\Omega_1 \subset \Omega_2 \subset M$ be two relatively compact domains. Then*

$$\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2). \tag{B.2.3}$$

The inequality is strict if the interior of $\Omega_2 \backslash \Omega_1$ is not empty.

B.3 Heat Kernel

We define the *heat kernel* to be the fundamental solution of the heat equation on a compact Riemannian manifold (M, g) (dimension $n \geq 1$) with a δ- function as the initial data. Suppose $u \in C^\infty(M, [0, \infty))$ solves the heat equation

$$\begin{cases} \Big(\dfrac{\partial}{\partial t} - \Delta\Big) u(x, t) = 0, & x \in M, \quad t \in [0, \infty) \\ u(x, 0) = f(x), & f \in L^2(M). \end{cases} \tag{B.3.1}$$

It then follows that the solution $u(x, t)$ can be represented by

$$u(x, t) = \int_M H_M(x, y; t) f(y) d\mu(y). \tag{B.3.2}$$

Here, $H_M(x, y; t) \in C^\infty(M \times M \times \mathbb{R}_+)$ (or at least C^2 in the spatial variable and C^1 in the time variable) is the Heat kernel of M. The heat kernel can in turn be represented in terms of eigenvalues and eigenfunctions of Δ as follows

$$H_M(x, y; t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \phi_i(x) \phi_i(y), \tag{B.3.3}$$

the series which converges for $t > 0$ and $x, y \in M$. This is a unique positive solution and symmetric in x and y, however, the uniqueness is not true in general if M is non-compact. For detail see [54, 82, 111].

Example B.3.1. *The most familiar example is the* **heat kernel on** $\mathbb{R}^n$ *(although* $\mathbb{R}^n$ *is not a compact Riemannian manifold) which is given by*

$$H_{\mathbb{R}^n}(x,y;t) = (4\pi t)^{-\frac{n}{2}} \exp\Big\{-\frac{\|x-y\|^2}{4t}\Big\}. \tag{B.3.4}$$

It is symmetric in x and y and can be shown that

$$\lim_{t\to 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{\|x-y\|^2}{4t}} f(y)dy = f(x)$$

for all $f \in L^2(\mathbb{R}^n)$.

Heat Kernel of a Torus T_Γ *[54] is given by*

$$H_{T_\Gamma}(x,y;t) = (4\pi t)^{-\frac{n}{2}} \sum_{\gamma\in\Gamma} \exp\Big\{-\frac{\|x-y-\gamma\|^2}{4t}\Big\}. \tag{B.3.5}$$

Here, a Torus T_Γ is given by $\mathbb{R}^n/\Gamma$, where Γ is a lattice in $\mathbb{R}^n$. Let Γ^* be the dual lattice; $\Gamma^* = \{y^* \in \mathbb{R}^n : \langle x, y^*\rangle \in \mathbb{Z}, \forall\ x \in \Gamma\}$, the spectrum of T_Γ is given by $\sigma(T_\Gamma) = \{4\pi^2\|y^*\|^2 : y^* \in \Gamma^*\}$ and the associated eigenfunctions by $\phi_{y^*}(x) = e^{2i\pi\langle x,y^*\rangle}$.

The **heat kernel of** 3**-dimensional hyperbolic space** $\mathbb{H}^3_k$ *(of the constant sectional curvature $-k^2$) [81, 82] is given by*

$$H_{\mathbb{H}^3_k}(x,y;t) = (4\pi t)^{-\frac{3}{2}} \frac{\sqrt{k}r}{\sinh(\sqrt{k}r)} \exp\Big\{-\frac{\|x-y\|^2 - kt}{4t}\Big\}. \tag{B.3.6}$$

B.4 Properties of Heat Kernel

In addition to smoothness, existence and uniqueness, we briefly list other important properties of the heat kernel of a compact Riemannian manifold. For detail see [54, 81, 132, 85]. Let a Riemannian manifold M be compact, then, there exists the heat kernel $H_M(x,y;t) \in C^\infty(M \times M \times \mathbb{R}_+)$ such that

1. Heat equation

$$(\partial_t - \Delta_x)H(x,y;t) = 0, \qquad x,\ y\ \in M,\ t > 0$$

2. Initial condition

$$\lim_{t\to 0} H(x,y;t) = \delta_y(x),$$

   where $\delta_y(x)$ is a Dirac mass at y.

3. Symmetry

$$H(x,y;t) = H(y,x;t)$$

4. Semi-group property

$$H(x,y;t) = \int_M H(x,z;t-s)H(z,y;s)dz.$$

We note that property (ii) above implies that $\lim_{t\to 0}\int_M H(y,x;t)f(y)d\mu(y) = f(x)$ uniformly for every function f that is continuous on M and every $x \in M$.

Theorem B.4.1. *(***Comparison Theorem for Heat Kernel***). Let M be a complete Riemannian manifold with $Rc(M) \geq nk$. The heat kernel $H(x,y;t)$ of the ball $B(x,r)$ with centre x fixed in M satisfies*

$$H_B(x,y;t) \geq H_V(d(x,y),t), \tag{B.4.1}$$

where $H_V(d(x,y),t)$ is the heat kernel of the geodesic ball $V(k,r)$ in the model space of sectional curvature k.

In the above theorem which is due to Cheng and Yau [55], $H_B(d(x,y),t)$ can be thought of as a function on the geodesic ball $B(x,r)$ in an obvious way. It is smooth on $B(x,r)\backslash\{cut\{x\}\}$, where $\{cut\{x\}$ is the set of point on the cut-locus. The following facts about heat kernel are well known too.

Theorem B.4.2. *Let M be a compact Riemannian manifold, $\{u_i\}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions and λ_i be the corresponding eigenvalues, then*

$$\int_M H_M(x,x;t)d\mu(x) = \sum_{i=0}^{\infty} e^{-\lambda_i t}, \tag{B.4.2}$$

which is the trace of the heat kernel of M. Furthermore,

$$H_M(x,y;t) \sim (4\pi t)^{-\frac{n}{2}} \exp\Big(-\frac{d^2(x,y)}{4t}\Big)\sum_{i=0}^{\infty} u_i(x,y)t^i \tag{B.4.3}$$

as $t \to 0$.

In particular, the result of Cheng and Yau [55] implies that if (M,g) is a complete Riemannian manifold with nonnegative Ricci curvature, then

$$H_M(x,y;t) \geq (4\pi t)^{-\frac{n}{2}} \exp\Big(-\frac{d^2(x,y)}{4t}\Big). \tag{B.4.4}$$

Appendix C

$\mathcal{F}$- Energy and $\mathcal{W}$-Entropy Monotonicity Formulas

C.1 Monotonicity of Perelman's $\mathcal{F}$-Energy

The materials in this appendix are due to Perelman [126] and can be found in several books and papers such as [4, 41, 64, 69, 101, 135, 147].

Let $(M^n, g_{ij}(t))$ be a closed n-manifold for a Riemannian metric $g_{ij}(t)$ and a smooth function f on M^n, Perelman's Energy functional [126] on pairs (g_{ij}, f) is defined by

$$\mathcal{F}(g_{ij}(t), f) = \int_{M^n} (R + |\nabla f|^2) e^{-f} d\mu. \tag{C.1.1}$$

Now taking the smooth variations of metric g and f as $\delta g_{ij} = h_{ij}$ and $\delta f =: K$ respectively, where $H := tr_g h_{ij}$, we have the following variation formulas by routine calculations (see Chow et al [64, p. 191-193])

$$\begin{aligned}
\delta\Gamma_{ij}^k(g) &= \frac{1}{2} g^{kl}\Big(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\Big)\\
\delta\Gamma_{jk}^k(g) &= \frac{1}{2} g^{kl}\nabla_j h_{kl} = \frac{1}{2}\nabla_j H\\
\delta(e^{-f} d\mu) &= \Big(\frac{H}{2} - K\Big) e^{-f} d\mu\\
\delta|\nabla f|^2 &= h^{ij}\nabla_i f \nabla_j f + 2\langle \nabla f, \nabla K\rangle.
\end{aligned}$$

Lemma C.1.1. *In a normal coordinates system with variation formulas above, we have*

$$\begin{aligned}
\delta R_{ijk}^l &= \frac{\partial}{\partial x^i}\Big[\frac{1}{2} g^{lp}\Big(\nabla_j h_{kp} + \nabla_k h_{jp} - \nabla_p h_{jk}\Big)\Big] - \frac{\partial}{\partial x^j}\Big[\frac{1}{2} g^{lp}\Big(\nabla_i h_{kp} + \nabla_k h_{ip} - \nabla_p h_{ik}\Big)\Big]\\
\delta R_{jk} &= \frac{\partial}{\partial x^i}\Big[\frac{1}{2} g^{ip}\Big(\nabla_j h_{kp} + \nabla_k h_{jp} - \nabla_p h_{jk}\Big)\Big] - \frac{\partial}{\partial x^j}\Big[\frac{1}{2} g^{ip}\Big(\nabla_i h_{kp} + \nabla_k h_{ip} - \nabla_p h_{ik}\Big)\Big]\\
\delta R &= -\Delta H + \nabla_j \nabla_k h_{jk} - h_{jk} R_{jk}.
\end{aligned}$$

Proof. The proofs are similar to those of evolution of curvatures earlier obtained in Chapter 1. (See also [68]). $\square$

With the above variational formulas, the first variation of $\mathcal{F}$-functional is given by

$$\begin{aligned}\delta\mathcal{F}(g_{ij}(t),f) = \int_M \Big[&- \Delta H + \nabla_i\nabla_j h_{ij} - h_{ij}R_{ij} + 2\langle \nabla f, \nabla K\rangle - h_{ij}\nabla_i f\nabla_j f \\ &+(R+|\nabla f|^2)(\frac{H}{2}-K)\Big] e^{-f}d\mu.\end{aligned} \tag{C.1.2}$$

This variation formula above follows from a direct computation as follows

$$\begin{aligned}\delta\mathcal{F}(g_{ij}(t),f) &= \delta\Big[\int_M \Big(R+|\nabla f|^2\Big)e^{-f}d\mu\Big] \\ &= \int_M \delta\Big(R+|\nabla f|^2\Big)e^{-f}d\mu + \Big(R+|\nabla f|^2\Big)\delta\Big(e^{-f}d\mu\Big) \\ &= \int_M \Big[- \Delta H + \nabla_i\nabla_j h_{ij} - h_{ij}R_{ij} + 2\langle \nabla f, \nabla K\rangle - h_{ij}\nabla_i f\nabla_j f \\ &\qquad + (R+|\nabla f|^2)(\frac{H}{2}-K)\Big] e^{-f}d\mu.\end{aligned}$$

Hence we have

Lemma C.1.2. *[126]. The first variation of $\mathcal{F}$ is*

$$\delta\mathcal{F}(g_{ij}(t),f) = \int_M \Big[- h_{ij}(R_{ij}+\nabla_i\nabla_j f) + (2\Delta f - |\nabla f|^2 + R)(\frac{H}{2}-K)\Big] e^{-f}d\mu. \tag{C.1.3}$$

Proof. Applying integration by parts to some terms in the variation formula (C.1.2) to obtain

$$\begin{aligned}\int_M \langle\nabla f,\nabla K\rangle e^{-f}d\mu &= \int_M K\Delta(e^{-f})d\mu = \int_M K(-\Delta f + |\nabla f|^2)e^{-f}d\mu \\ \int_M \Delta H\ e^{-f}d\mu &= \int_M H\Delta(e^{-f})d\mu = \int_M H(-\Delta f + |\nabla f|^2)e^{-f}d\mu \\ \int_M \nabla_i\nabla_j h_{ij}e^{-f}d\mu &= \int_M -\langle\nabla e^{-f},\nabla h\rangle d\mu = \int_M h_{ij}\Big((\nabla f,\nabla f) - \nabla_i\nabla_j f\Big)e^{-f}d\mu.\end{aligned}$$

Therefore

$$\int_M (-\Delta H + 2\langle\nabla f,\nabla K\rangle)e^{-f}d\mu = 2\int_M (\Delta f - |\nabla f|^2)(\frac{H}{2}-K)e^{-f}d\mu,$$

putting all these into (C.1.2), we have

$$\begin{aligned}\delta\mathcal{F}(g,f) &= \int_M \Big[2(\Delta f - |\nabla f|^2)(\frac{H}{2}-K) + (R+|\nabla f|^2)(\frac{H}{2}-K)\Big]e^{-f}d\mu \\ &\quad + \int_M \Big[h_{ij}\Big((\nabla f,\nabla f) - \nabla_i f\nabla_j f\Big) - h_{ij}R_{ij} - h_{ij}\nabla_i f\nabla_j f\Big]e^{-f}d\mu \\ &= \int_M (\frac{H}{2}-K)(2\Delta f - |\nabla f|^2 + R)e^{-f}d\mu + \int_M -h_{ij}(R_{ij}+\nabla_i\nabla_j f)e^{-f}d\mu\end{aligned}$$

From where the lemma follows. $\square$

Theorem C.1.3. *Let $g_{ij}(t)$ and $f(t)$ evolve by*

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij} \\ \\ \dfrac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R. \end{cases} \tag{C.1.4}$$

Then

$$\frac{d}{dt}\mathcal{F}(g_{ij}(t), f(t)) = 2\int_{M^n} |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\mu.$$

In particular, $\mathcal{F}(g_{ij}(t), f(t))$ is monotonically nondecreasing in time and the monotonicity is strict unless $R_{ij} + \nabla_i\nabla_j f = 0$.

Proof. Recall the first variation of $\mathcal{F}(g, f)$ and use $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$. Notice also that $\frac{H}{2} = \frac{1}{2}trh = \frac{1}{2}g^{ij}(-2R_{ij}) = -R$ and $K = \frac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2$

$$\begin{aligned}
\frac{d}{dt}\mathcal{F} &= \int_M [-h_{ij}(R_{ij} + \nabla_i\nabla_j) + (\frac{H}{2} - K)(2\Delta f - |\nabla f|^2 + R)]e^{-f} d\mu \\
&= \int_M -(-2R_{ij})(R_{ij} + \nabla_i\nabla_j f) + \Big(\frac{1}{2}g^{ij}(-2R_{ij}) - \frac{\partial f}{\partial t}\Big)\Big(2\Delta f - |\nabla f|^2 + R\Big)e^{-f} d\mu \\
&= \int_M \Big[2R_{ij}(R_{ij} + \nabla_i\nabla_j f) + \Big(-R - \frac{\partial f}{\partial t}\Big)\Big(2\Delta f - |\nabla f|^2 + R\Big)\Big]e^{-f} d\mu \\
&= \int_M \Big[2R_{ij}(R_{ij} + \nabla_i\nabla_j f) + \Big(\Delta f - |\nabla f|^2\Big)\Big(2\Delta f - |\nabla f|^2 + R\Big)\Big]e^{-f} d\mu.
\end{aligned}$$

Computing the second term in the RHS of the last equality using the identity $-\Delta e^{-f} = (\Delta f - |\nabla f|^2)e^{-f}$, integration by parts and Ricci identity (see (0.2.14) - (0.2.15)), we have

$$\begin{aligned}
\int_M (\Delta f - |\nabla f|^2)(2\Delta f - |\nabla f|^2)e^{-f} d\mu &= \int_M -\Delta e^{-f}(2\Delta f - |\nabla f|^2) d\mu = \int_M -\nabla_i f\nabla_i(2\Delta f - |\nabla f|^2)e^{-f} d\mu \\
&= \int_M -\nabla_i f\Big[2\nabla_j(\nabla_i\nabla_j f) - 2R_{ij}\nabla_j f - 2\langle\nabla f, \nabla\nabla f\rangle\Big]e^{-f} d\mu \\
&= -2\int_M \Big[(\nabla_i f\nabla_j f - \nabla_i\nabla_j f)\nabla_i\nabla_j f - R_{ij}\nabla_i f\nabla_j f - \langle\nabla f, \nabla\nabla f\rangle\nabla f\Big]e^{-f} d\mu \\
&= 2\int_M \Big(|\nabla_i\nabla_j f|^2 + R_{ij}\nabla_i f\nabla_j f\Big)e^{-f} d\mu.
\end{aligned}$$

Similarly, using integration by parts and the contracted second Bianchi identity, we have

$$\begin{aligned}
\int_M (\Delta f - |\nabla f|^2)Re^{-f} d\mu &= \int_M (\Delta f)Re^{-f} d\mu - \int_M |\nabla f|^2 Re^{-f} d\mu \\
&= \int_M -\nabla_i f\nabla_j Re^{-f} d\mu \\
&= 2\int_M \nabla_i\nabla_j f R_{ij} e^{-f} d\mu - 2\int_M \nabla_i f\nabla_j f R_{ij} e^{-f} d\mu.
\end{aligned}$$

An alternative method is to fix the volume element $d\mu = \sqrt{\det g_{ij}}dx$ and then show that $\int_M e^{-f} d\mu$ is constant. In fact, a direct calculation gives

$$\frac{\partial}{\partial t}(e^{-f} d\mu) = (-\frac{\partial f}{\partial t} - R)e^{-f} d\mu = (\Delta f - |\nabla f|^2)e^{-f} d\mu = -\Delta(e^{-f} d\mu),$$

It then follows that

$$\frac{d}{dt}\int_M e^{-f}d\mu = -\int_M \Delta(e^{-f})d\mu = 0.$$

The result follows immediately. $\square$

C.2 Monotonicity of Perelman's $\mathcal{W}$-Entropy Functional

We define the entropy functional (as in [126])

$$\mathcal{W}(g,f,\tau) := \int_M \Big[\tau(R+|\nabla f|^2)+f-n\Big](4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu, \tag{C.2.1}$$

where $g(t)$ is a Riemannian metric on n-compact manifold M, f is a smooth function on M and τ is a positive scale parameter. Let $\delta g_{ij} = h_{ij}$, $\delta\tau = \eta$ and $\delta f = K$, where $H = g^{ij}h_{ij}$. We have the following

Lemma C.2.1. *Let* $u := (4\pi\tau)^{-\frac{n}{2}}e^{-f}$. *Then*

$$\delta(u\,d\mu) = \Big(\frac{H}{2} - K - \frac{n}{2\tau}\eta\Big)u\,d\mu.$$

Moreover, fixing the volume measure $(4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu$, *the relation* $\frac{H}{2} - K - \frac{n}{2\tau}\eta = 0$ *holds.*

Proof.

$$\begin{aligned}
\delta(u\,d\mu) &= \delta\Big[(4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu\Big]\\
&= \delta(e^{-f}d\mu)(4\pi\tau)^{-\frac{n}{2}} + \delta((4\pi\tau)^{-\frac{n}{2}})e^{-f}d\mu\\
&= \Big(\frac{H}{2} - K\Big)(4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu + \Big(-\frac{n}{2\tau}\Big)(4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu\\
&= \Big(\frac{H}{2} - K - \frac{n}{2\tau}\eta\Big)(4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu.
\end{aligned}$$

$\square$

Lemma C.2.2. *The first variation of* $\mathcal{W}$*-functional is*

$$\begin{aligned}
\delta_{(h,K,\tau)}\mathcal{W}(g,f,\tau) &= \int_M -\tau h_{ij}\Big(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\Big)u\,d\mu + \int_M \eta\Big(R + \Delta f - \frac{n}{2\tau}\Big)u\,d\mu\\
&+ \int_M \Big(\frac{H}{2} - K - \frac{n}{2\tau}\eta\Big)\Big[\tau\Big(R + 2\Delta f - |\nabla f|^2\Big) + f - n - 1\Big]u\,d\mu.
\end{aligned}$$

Proof. By straightforward calculation

$$\delta_{(h,K,\tau)}\mathcal{W}(g,f,\tau) = \delta\Big(\tau(4\pi\tau)\mathcal{F}(g,f) + \int_M (f-n)u\,d\mu\Big),$$

using the variation formula obtained for $\mathcal{F}$ in Lemma C.1.2, we have

$$\begin{aligned}
\delta\Big[\tau(4\pi\tau)^{-\frac{n}{2}}\mathcal{F}(g,f)\Big] &= \delta\Big[\tau(4\pi\tau)^{-\frac{n}{2}}\Big]\mathcal{F}(g,f) + \tau(4\pi\tau)^{-\frac{n}{2}}\delta\Big[\mathcal{F}(g,f)\Big] \\
&= (\eta - \frac{n}{2}\eta)(4\pi\tau)^{-\frac{n}{2}}\mathcal{F}(g,f) + \tau(4\pi\tau)^{-\frac{n}{2}}\int_M \Big[-h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f\Big) \\
&\qquad + \Big(2\Delta f - |\nabla f|^2 + R\Big)\Big(\frac{H}{2}-K\Big)\Big]e^{-f}d\mu \\
&= \Big(1-\frac{n}{2}\Big)\Big(\eta\int_M (R+|\nabla f|^2)e^{-f}d\mu\Big)(4\pi\tau)^{-\frac{n}{2}} \\
&\qquad + \int_M -\tau\, h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f\Big)(4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu \\
&\qquad + \int_M \tau\,\Big(\frac{H}{2}-K\Big)\Big(2\Delta f - |\nabla f|^2 + R\Big)(4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu \\
&= \int_M -\tau\, h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f\Big)u\,d\mu + \int_M \eta\Big(1-\frac{n}{2}\Big)\Big(R+|\nabla f|^2\Big)u\,d\mu \\
&\qquad + \int_M \tau\,\Big(\frac{H}{2}-K\Big)\Big(2\Delta f - |\nabla f|^2 + R\Big)u\,d\mu
\end{aligned}$$

and

$$\begin{aligned}
\delta\Big[\int_M (f-n)u\,d\mu\Big] &= \int_M \delta(f-n)\,u\,d\mu + \int_M (f-n)\delta(u\,d\mu) \\
&= \int_M K\,u\,d\mu + \int_M (f-n)\Big(\frac{H}{2}-K-\frac{n}{2\tau}\eta\Big)\,u\,d\mu \\
&= \int_M \Big[K + \Big(\frac{H}{2}-K-\frac{n}{2\tau}\eta\Big)(f-n)\Big]u\,d\mu.
\end{aligned}$$

Combining these we have

$$\begin{aligned}
\delta\mathcal{W}(g,f,\tau) &= \int_M -\tau\, h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f\Big)u\,d\mu + \int_M \eta\Big(1-\frac{n}{2}\Big)\Big(R+|\nabla f|^2\Big)u\,d\mu \\
&+ \int_M \tau\,\Big(\frac{H}{2}-K\Big)\Big(2\Delta f - |\nabla f|^2 + R\Big)u\,d\mu + \int_M \Big[K + \Big(\frac{H}{2}-K-\frac{n}{2\tau}\eta\Big)(f-n)\Big]u\,d\mu \\
&= \int_M -\tau\, h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f\Big)u\,d\mu + \int_M \eta\Big(1-\frac{n}{2}\Big)\Big(R+|\nabla f|^2\Big)u\,d\mu \\
&+ \int_M \Big(\frac{H}{2}-K\Big)\Big[\tau(2\Delta f - |\nabla f|^2 + R) + f - n\Big]u\,d\mu + \int_M \Big[K - \frac{n}{2\tau}\eta(f-n)\Big]\,u\,d\mu \\
&= \int_M -\tau\, h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f\Big)u\,d\mu + \int_M \eta\Big(R_{ij}+\nabla_i\nabla_j f\Big)u\,d\mu - \int_M -\eta(R+\Delta f)ud\mu \\
&+ \int_M \eta\Big(1-\frac{n}{2}\Big)\Big(R+|\nabla f|^2\Big)u\,d\mu + \int_M \Big(\frac{H}{2}-K\Big)\Big[\tau(2\Delta f - |\nabla f|^2 + R) + f - n\Big]u\,d\mu \\
&- \int_M \frac{n}{2}\eta\Big[(2\Delta f - |\nabla f|^2 + R) + f - n\Big]u\,d\mu + \int_M \frac{n}{2}\eta\Big[(2\Delta f - |\nabla f|^2 + R) + f - n\Big]u\,d\mu \\
&\qquad + \int_M \Big[K - \frac{n}{2\tau}\eta(f-n)\Big]ud\mu
\end{aligned}$$

$$
\begin{aligned}
&= \int_M -\tau\, h_{ij}\Big(R_{ij} + \nabla_i\nabla_j f\Big)u\, d\mu + \int_M \eta\Big(R_{ij} + \nabla_i\nabla_j f\Big)u\, d\mu + \int_M -\eta(R+\Delta f)ud\mu \\
&+ \int_M \eta\Big(1-\frac{n}{2}\Big)\Big(R + |\nabla f|^2\Big)u\, d\mu + \int_M \Big(\frac{H}{2} - K - \frac{n}{2\tau}\eta\Big)\Big[\tau(2\Delta f - |\nabla f|^2 + R) \\
&\qquad + f - n\Big]u\, d\mu + \int_M \frac{n}{2}\eta\Big(2\Delta f - |\nabla f|^2 + R\Big)u\, d\mu + \int_M K u d\mu
\end{aligned}
$$

$$
\begin{aligned}
&= \int_M \Big(-\tau h_{ij} + \eta g_{ij}\Big)\Big(R_{ij} + \nabla_i\nabla_j f\Big)ud\mu + \int_M \Big(\frac{H}{2} - K - \frac{n}{2\tau}\eta\Big)\Big[\tau(2\Delta f - |\nabla f|^2 + R) \\
&\qquad + f - n\Big]u\, d\mu + \int_M K u d\mu.
\end{aligned}
$$

Applying integration by parts method to the last term in the RHS of the last equality

$$
\begin{aligned}
\int_M K u\, d\mu = \int_M \delta f\, u d\mu &= -\int_M f\delta(ud\mu) \\
&= -\int_M f\Big(\frac{H}{2} - K - \frac{n}{2\tau}\Big)ud\mu,
\end{aligned}
$$

putting this back, we obtain the result. Hence we write

$$
\delta_{(h,K,\tau)}\mathcal{W}(g,f,\tau)\left\{
\begin{aligned}
&= \int_M \Big(-\tau h_{ij} + \eta g_{ij}\Big)\Big(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\Big)ud\mu \\
&+ \int_M \Big(\frac{H}{2} - K - \frac{n}{2\tau}\eta\Big)\Big[\tau(2\Delta f - |\nabla f|^2 + R) + f - n - 1\Big]ud\mu.
\end{aligned}\right. \tag{C.2.2}
$$

□

The functional $\mathcal{W}$ and its gradient flow

Let us keep the volume measure fixed so that

$$
\delta\Big(\int_M (4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu\Big) = 0 = \frac{H}{2} - K - \frac{n}{2\tau}\eta
$$

and require that $\eta = -1$, thus τ is a quantity decreasing at a constant rate. We then obtain the gradient flow

$$
\frac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i\nabla_j f)
$$

with $\frac{d\tau}{dt} = \eta = -1$, where $f = -\ln u - \frac{n}{2}\ln(4\pi\tau)$ and $\frac{\partial f}{\partial t} = -\Delta f - R + \frac{n}{2\tau}$. we have the following monotonicity formula

Proposition C.2.3. *Let $(g(t), f(t), \tau(t))$ be a soliton of the system (2.5.3), we have the identity*

$$
\frac{d}{dt}\mathcal{W}(g,f,\tau) = \int_M 2\tau|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2 ud\mu, \tag{C.2.3}
$$

where $\int_M ud\mu$ is a constant.

Proof. The result is obtained by a straightforward substitution of

$$\begin{aligned} h_{ij} &= \frac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i \nabla_j f), \\ K &= \frac{\partial f}{\partial t} = -\Delta f - R + \frac{n}{2\tau}, \\ \frac{\partial \tau}{\partial t} &= \eta = -1 \end{aligned}$$

into the first first variation of $\mathcal{W}$, i.e, equation (C.2.2), we then have

$$\begin{aligned} \frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = \int_M 2\tau\Big(R_{ij} + \nabla_i\nabla_j f\Big)\Big(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\Big)ud\mu \\ - \int_M \Big(R + \Delta f - \frac{n}{2\tau}\Big)ud\mu. \end{aligned}$$

The second term of RHS of the last equation can be written as

$$-\int_M \Big(R + \Delta f - \frac{n}{2\tau}\Big)ud\mu = 2\tau\int_M -\frac{1}{2\tau}g_{ij}\Big(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\Big)ud\mu.$$

The result follows by substituting these back into (C.2.2). □

Lemma C.2.4. *Let $\lambda > 0$ be any constant and $\phi : M \to M$ be any diffeomorphism. Then*

$$\mathcal{W}(\lambda \cdot g, f, \lambda \cdot \tau) = \mathcal{W}(g, f, \tau) \quad and \quad \mathcal{W}(\phi_t^* g, \phi_t^* f, \tau) = \mathcal{W}(g, f, \tau).$$

Proof. By straightforward computation

$$\begin{aligned} \mathcal{W}(\lambda \cdot g,\ f,\ \lambda \cdot \tau) &= \int_M \Big[\lambda \cdot \tau(R(\lambda \cdot g) + (\lambda \cdot g)^{ij}\nabla_i f \nabla_j f) + f - n\Big](4\pi\lambda\tau)^{-\frac{n}{2}} e^{-f}\sqrt{det(\lambda g)}dx \\ &= \int_M \Big[\lambda \cdot \tau(\lambda^{-1}R(g) + \lambda^{-1}g^{ij}\nabla_i f \nabla_j f) + f - n\Big]\lambda^{-\frac{n}{2}}(4\pi\tau)^{-\frac{n}{2}} e^{-f}\sqrt{\lambda^n det(g)}dx \\ &= \int_M \Big[\tau(R + |\nabla f|^2) + f - n\Big]\lambda^{-\frac{n}{2}}(4\pi\tau)^{-\frac{n}{2}} e^{-f}\lambda^{\frac{n}{2}}\sqrt{det(g)}dx \\ &= \mathcal{W}(g, f, \tau). \end{aligned}$$

The invariance under diffeomorphisms is clear since we are dealing with geometric quantities. One can use co-ordinates induced by ϕ for a proof. □

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