Numerical analysis of a mathematical model of multiphase tissue growth

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Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

Signature:

Acknowledgments

First and foremost I want to thank Almighty God, for letting me through all the difficulties. I have experienced YOUR guidance day by day. YOU are the only one who let me finish my degree. I will keep on trusting YOU for my future.

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Abstract

In this thesis we study a mathematical system of equations that models multi phase tissue growth. The mathematical system comprises of three coupled equations: an advection diffusion equation for a scalar quantity that defines the volume fraction of one cell type and two constitutive relations for the pressure field and volume averaged velocity field.

A numerical discretisation of this mathematical model is derived using a coupled finite volume - finite difference scheme. Stability bounds on the approximate solution of a simplified version of the model are proved together with a convergence results relating the approximate solution to the weak solution of the simplified model.

In addition an efficient and reliable numerical scheme is implemented in the Matlab programming language to solve the numerical approximation of the full model and computational results are presented.

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Chapter 1

Introduction

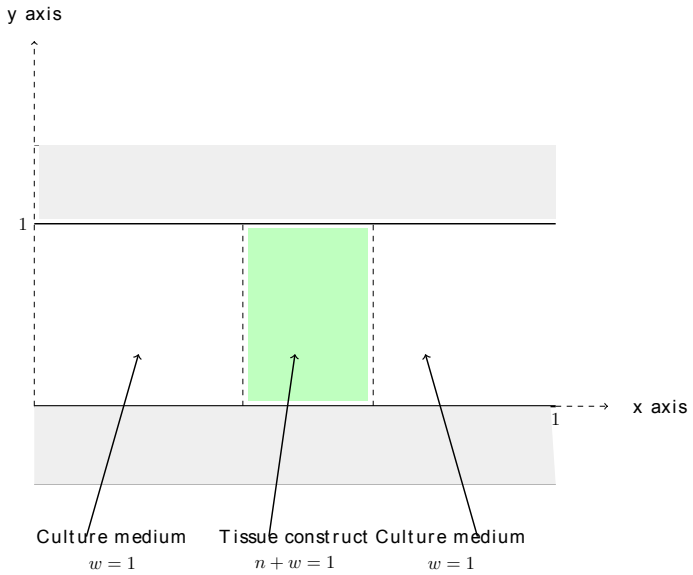
This thesis is concerned with the numerical analysis of the one dimensional formulation of the mathematical model of tissue growth presented and analysed in [12].

1.1 Mathematical models for tissue growth

In terms of scientific advancement, the study of tissue engineering has gained much popularity and attention due to its importance in many fields of science especially biology. It involves the replacement or repair, maintenance, enhancement of a tissue or a part of tissue like skin, blood vessel, muscle etc, see [17, 18]. Since the tissue needs certain mechanical and structural properties for proper functioning. In order to grow the tissue out of the body requires proper culture medium like bioreactor which is an advance tissue culture apparatus that attempts to stimulate a physiological environment in order to promote cell or tissue growth in culture medium. Tissue engineering is the combination of living cells, engineering, material methods and a suitable environment to improve or replace the tissue.

The mathematical model of tissue growth that we consider is one in which the tissue undergoes different phases/stages of growth. The tissue construct is a rich culture medium. The effects of nutrients rich culture on the growth of tissue are neglected so that the model can better study the effects of imposed flow on the response of cells. The presence of scaffold is also neglected where scaffold is supporting structure in the growth of tissue. The physical set up of the model

that is that of a two dimensional channel containing the tissue construct surrounded by culture medium as shown below.



The mathematical model that we consider in this thesis was studied in [12]. It is a two phase model that enables the effects of dynamic culture conditions on the growth of tissue to be analysed. The model describes the two phase fluids in a bioreactor which is a two dimensional channel with rigid walls that contains ”tissue construct”. The one phase consist of cells and extracellular matrix (ECM) and second phase represent culture medium. The model also describes the cell production and death as well. The imposed axial pressure drop generate the flow of culture medium which is represented by perfusion. It is important to note that “tissue construct” term distinguish the region occupied by the interacting cell and culture medium phases from the rest of the channel that contain only culture medium and we choose the interface between tissue construct and culture medium to be sharp/non-diffusive (i.e D=0).

A Cartesian coordinate system (x, y) is chosen in the corresponding coordinate direction (x, y) and the channel occupies space $0 \leq x \leq 1, 0 \leq y \leq 1$. The volume fraction of cell and culture medium phases are denoted by w, and n respectively and a volume averaged velocity u, pressure p. We now assume that the tissue undergoes one-dimensional growth parallel to $x - axis$

and associated pressure and velocity are functions of x and t only, where t represents time,

$$\frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} = (k_m - k_d)w + D\frac{\partial^2 w}{\partial x^2} \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{4k_m}{3}\frac{\partial^2 w}{\partial x^2} \quad \text{in } \Omega, \tag{1.2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial p}{\partial x} - \frac{k_m}{3}\frac{\partial w}{\partial x} \quad \text{in } \Omega, \tag{1.3}$$

where $\Omega = [0,1], t = [0,T]$, D is the diffusion coefficient and k_m and k_d are constants which are the rate of cell growth and cell death respectively. In general k_m and k_d will depend upon the cell's environment (e.g. nutrients availability, growth factors, stress). Here we take k_m and k_d to be constants for the uniform growth of tissue but they could be functions of x and t, which would imply that cell death results in a corresponding increase in culture medium.

The associated boundary conditions with the (1.1)-(1.3) are

$$\begin{aligned}
p(0,t) = 0 \quad & p(1,t) = 0 \quad \forall\, t > 0, \\
u(0,t) = 1 \quad & \frac{\partial u(1,t)}{\partial x} = 0 \quad \forall\, t > 0, \\
w(0,t) = 0 \quad & \text{if} \quad u(0,t) < 0 \quad \forall\, t > 0, \\
w(1,t) = 0 \quad & \text{if} \quad u(1,t) > 0 \quad \forall\, t > 0, \\
w(x,0) &= w_0(x) \quad \text{in } \Omega.
\end{aligned}$$

Other two phase models have also been studied e.g. Landsman & Please [24], Breward et al [25, 26], Franks et al. [23] and [19, 20, 21]; but in these models the solid characteristics of tissue neglected.

The diffusive term has been added to (1.1) whilst cells do exhibit the random motion, the tissue growth and perfusion induced flow fields are dominant mechanism for the cell movement, with diffusive effects assumed to be negligible [22, 23].

By choosing the interface between the tissue construct and culture medium sharp, corresponding

$D = 0$, the one dimensional model to

$$\frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} = (k_m - k_d)w \quad \text{in } \Omega_{\text{T}}, \tag{1.4}$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{4k_m}{3}\frac{\partial^2 w}{\partial x^2} \quad \text{in } \Omega_{\text{T}}, \tag{1.5}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial p}{\partial x} - \frac{k_m}{3}\frac{\partial w}{\partial x} \quad \text{in } \Omega_{\text{T}}, \tag{1.6}$$

$$p(0,t) = 0 \quad p(1,t) = 0 \quad \forall\, t > 0, \tag{1.7}$$

$$u(0,t) = 1 \quad \frac{\partial u(1,t)}{\partial x} = 0 \quad \forall\, t > 0, \tag{1.8}$$

$$w(0,t) = 0 \quad \text{if} \quad u(0,t) < 0 \quad \forall\, t > 0, \tag{1.9}$$

$$w(1,t) = 0 \quad \text{if} \quad u(1,t) > 0 \quad \forall\, t > 0, \tag{1.10}$$

$$w(x,0) = w_0(x) \quad \text{in } \Omega \tag{1.11}$$

where $\Omega = (0,1)$ and $\Omega_T = \Omega \times (0,T)$.

In Chapter 2 we derive a discrete coupled finite volume - finite difference approximation of (1.4)-(1.11). In Chapter 3 we assume that the velocity $u(x,t)$ in the above model is given and so by discarding (1.5) - (1.8) we consider the following simplified version of (1.4)-(1.11):

$$\frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} = (k_m - k_d)w \quad \text{in } \Omega_{\text{T}}, \tag{1.12}$$

$$w(0,t) = 0 \quad w(1,t) = 0 \quad \forall\, t > 0, \tag{1.13}$$

$$w(x,0) = w_0(x) \quad \text{in } \Omega \tag{1.14}$$

where $u(x,t)$ is a given smooth function that satisfies the boundary conditions

$$u(0,t) = 1, \quad \frac{\partial u(1,t)}{\partial x} = 0 \quad \forall\, t > 0. \tag{1.15}$$

In Chapter 3 we also analyse a finite volume approximation of the model (1.12)-(1.14) and we show that as the time step, Δt, and the mesh size, h, of the approximation, tend to zero, the approximate solution $\hat{w}_h$ of this discretization converges to the weak solution w of (1.12)-(1.14). In Chapter 4 we present some numerical results obtained from computationally solving the

numerical discretization presented in Chapter 2.

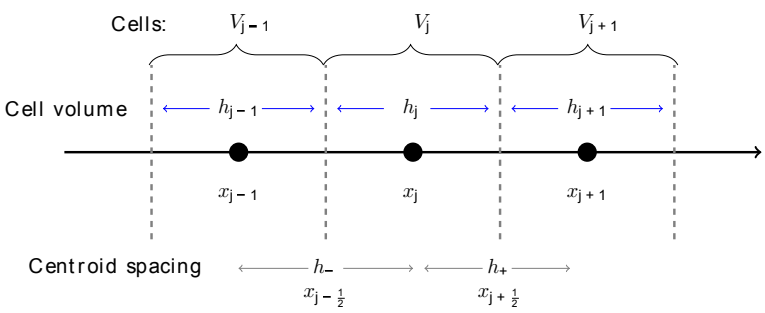
Chapter 2

Finite Volume - Finite Difference Approximation of the Model

In this chapter we derive a finite volume - finite difference approximation of (1.4)-(1.11). We first introduce some notation that will be useful in defining this approximation. We divide $\Omega := (0, L)$ into J subintervals of length h and we set $x_j = jh$, $j = 0, \ldots, J$ to be the nodes of the discretisation and we define $t^n = n\Delta t$. We define cell volumes as

$$V_j = \begin{cases} \left(0, \frac{h}{2}\right) & \text{for } j = 0 \\ \left(x_j - \frac{h}{2}, x_j + \frac{h}{2}\right) & \text{for } j \in [1, J-1] \\ \left(L - \frac{h}{2}, L\right) & \text{for } j = J \end{cases}$$

and shown below



and we define S_h, S_h^0, W_h and W_h^0 by

$$S_h = \{\chi \in C(\Omega) : \chi|_{(jh,(j+1)h} \text{ is linear } \forall\, j \in [0, J-1]\},$$

$$S_h^0 = \{\chi \in S_h : \chi(x_0) = \chi(x_J) = 0\},$$

$$W_h = \{\eta \in L^\infty(\Omega) : \eta|_{V_j} := \eta_j \;\forall\, j \in [0, J]\},$$

$$W_h^0 = \{\eta \in W_h : \eta|_{V_0} = \eta|_{V_J} = 0\}.$$

Before we derive the finite volume - finite difference approximation of our model we first describe the finite difference approximation method for linear problems for functions of one variable [1], and then we define the finite volume method for one space dimensional advection equations [6].

2.1 Finite difference method

The finite difference method for the linear second order boundary-value problem

$$y'' = p(x)y' + q(x)y + r(x), \quad 0 \le x \le L, \quad y(0) = \alpha, \quad y(L) = \beta, \tag{2.1}$$

involves the finite difference method for solving the boundary-value problems. The idea is to replace each of the derivatives in the differential equation with the appropriate difference-quotient approximation to approximate y' and y''. The particular difference quotient and step size h are chosen to maintain a specified order of truncation error. However, h cannot be chosen too small because of the instability of the derivative approximations.

We now follow the authors in [1] in describing the method:

First, we select the integer $N > 0$ and divide the interval $[0, L]$ into $(N + 1)$ equal subintervals whose endpoints are mesh points $x_i = ih$, for $i = 0, 1, ..., N + 1$, where $h = L/(N + 1)$. Choosing the step size h in this manner facilitates the application of a matrix algorithm, which solves a linear system involving an $N \times N$ matrix.

At the interior mesh points x_i, for $i = 1, 2, ..., N$, the differential equation to be approximated is

$$y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i). \tag{2.2}$$

Expanding y in a third Taylor polynomial about x_i evaluated at x_{i+1} and x_{i-1}, we have, assuming that $y \in C^4[x_{i-1}, x_{i+1}]$,

$$y(x_{i+1}) = y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\xi_i^+),$$

for some ξ^+ in (x_i, x_{i+1}), and

$$y(x_{i-1}) = y(x_i - h) = y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\xi_i^-),$$

for some ξ^- in (x_{i-1}, x_i). If these equations are added, we have

$$y(x_{i+1}) + y(x_{i-1}) = 2y(x_i) + h^2y''(x_i) + \frac{h^4}{24}[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)],$$

and solving for $y''(x_i)$ gives

$$y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{24}[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)].$$

The Intermediate Value Theorem can be used to simply this to

$$y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})] - \frac{h^2}{12}y^{(4)}(\xi_i), \tag{2.3}$$

for some ξ_i in (x_{i-1}, x_{i+1}). This is called the centered-difference formula for $y''(x_i)$.
A centered- difference formula for $y'(x_i)$ is obtained in a similar manner, resulting in

$$y'(x_i) = \frac{1}{2h}[y(x_{i+1}) - y(x_{i-1})] - \frac{h^2}{6}y'''(\eta_i), \tag{2.4}$$

for some η_i in (x_{i-1}, x_{i+1}).

The use of these centered-difference formulas in (2.2) results in the equation

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = p(x_i)[\frac{y(x_{i+1}) - y(x_{i-1})}{2h}] + q(x_i)y(x_i)$$
$$+r(x_i) - \frac{h^2}{12}[2p(x_i)y'''(\eta_i) - y^{(4)}(\xi_i)].$$

A finite-difference method with truncation error of order $0(h^2)$ results by using this equation together with the boundary conditions $y(0) = \alpha$ and $y(L) = \beta$ to define

$$w_0 = \alpha, \quad w_{N+1} = \beta$$

and

$$\left(\frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2}\right) + p(x_i)\left(\frac{-w_{i+1} - w_{i-1}}{2h}\right) + q(x_i)w_i = -r(x_i), \tag{2.5}$$

for each $i = 1, 2, ...N$.

In the form we will consider, (2.5) is rewritten as

$$-\left(1 + \frac{h}{2}p(x_i)\right)w_{i-1} + (2 + h^2q(x_i))w_i - \left(1 - \frac{h}{2}p(x_i)\right)w_{i+1} = -h^2r(x_i),$$

and the resulting system of equations is expressed in a tridiagonal $N \times N$ matrix.

2.2 Finite Volume method

Consider a simple one dimensional advection problem defined by the following partial differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial x} = 0, \quad 0 \leq x \leq L, \quad t \geq 0. \tag{2.6}$$

Here, $\rho = \rho(x, t)$ represents the state variable and $f = f(\rho(x, t))$ represents the flux or flow of ρ. Conventionally, positive f represents flow to the right while negative f represents flow to the left. We sub-divide the spatial domain $[0, L]$ into finite volumes or cells, V_i (defined above), with cell centres indexed as x_i. For a particular cell V_i, we can define the volume average value

of $\rho_i(t) = \rho(x,t)$ at time $t = t_1$ and $x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, as

$$\bar{\rho}_i(t_1) = \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x,t_1)dx, \tag{2.7}$$

and at time $t = t_2$ as

$$\bar{\rho}_i(t_2) = \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x,t_2)dx, \tag{2.8}$$

where $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$ represent the location of the upstream and downstream nodes respectively of the ith cell.

Integrating (2.6) in time, we have:

$$\rho(x,t_2) = \rho(x,t_1) - \int_{t_1}^{t_2} \frac{\partial f(x,t)}{\partial x} dt. \tag{2.9}$$

To obtain the volume average of $\rho(x,t)$ at time $t = t_2$, we integrate $\rho(x,t_2)$ over the cell volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and divide the result by $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ i.e.

$$\bar{\rho}_i(t_2) = \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left\{ \rho(x,t_1) - \int_{t_1}^{t_2} \frac{\partial f(x,t)}{\partial x} dt \right\} dx \tag{2.10}$$

Using integration by parts we obtain

$$\bar{\rho}_i(t_2) = \bar{\rho}_i(t_1) - \frac{1}{\Delta x_i} \Big(\int_{t_1}^{t_2} f_{i+\frac{1}{2}} dt - \int_{t_1}^{t_2} f_{i-\frac{1}{2}} dt \Big). \tag{2.11}$$

where $f_{i\pm\frac{1}{2}} = f(x_{i\pm\frac{1}{2}}, t)$.

We can therefore derive a semi - discrete numerical scheme for the above problem with cell centres indexed as i, and with cell edge fluxes indexed as $i \pm \frac{1}{2}$, by differentiating (2.11) with respect to time to obtain

$$\frac{d\bar{\rho}_i}{dt} + \frac{1}{\Delta x_i}[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}] = 0, \tag{2.12}$$

where values for the edge fluxes, $f_{i\pm\frac{1}{2}}$ can be reconstructed by interpolation or extrapolation of the cell averages. Equation (2.12) is exact for the volume averages; i.e., no approximations

have been made during its derivation.

If we assume that $\rho = \rho_i$ is constant on each cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, i.e. $\rho \in W_h$, and we take $f = \rho u$ then (2.6) takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

and the corresponding finite volume approximation is given by

$$\frac{d\rho_i}{dt} + \frac{1}{\Delta x_i}[(\rho u)_{i+\frac{1}{2}} - (\rho u)_{i-\frac{1}{2}}] = 0 \tag{2.13}$$

where $(\rho u)_{i+\frac{1}{2}} = \rho(x_i + \frac{h}{2}, t)u(x_i + \frac{h}{2}, t)$. Since

$$\rho(x,t) = \begin{cases} \rho_{i-1} & x \in (x_{i-\frac{3}{2}}, x_{i-\frac{1}{2}}) \\ \rho_i & x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \\ \rho_{i+1} & x \in (x_{i+\frac{1}{2}}, x_{i+\frac{3}{2}}) \end{cases}$$

we approximate the value of $\rho_{i+\frac{1}{2}}$ by either ρ_i or ρ_{i+1} and in the case of the upwind scheme we make our choice depending on the velocity u, in particular we set

$$\rho_{i+\frac{1}{2}} = \begin{cases} \rho_{i+1} & \text{if } u_{i+\frac{1}{2}} < 0 \\ \rho_i & \text{if } u_{i+\frac{1}{2}} > 0. \end{cases}$$

We obtain the difference method using the Taylor series in t to form the difference quotient

$$\frac{\partial \rho}{\partial t}(x_i, t_j) = \frac{\rho(x_i, t_{j+1}) - \rho(x_i, t_j)}{\Delta t} - \frac{k}{2}\frac{\partial^2 \rho}{\partial t^2}(x_j, \mu_{j+\frac{1}{2}}) \tag{2.14}$$

for some $\mu_{j+\frac{1}{2}} \in (t_j, t_{j+1})$.

Thus a fully discrete approximation of (2.13), known as an explicit method, is given by the following

$$\frac{h}{\Delta t}\left(\rho_j^{n+1} - \rho_j^n\right) = -\left(\rho_{j+1}^n[u_{j+\frac{1}{2}}^n]_- + \rho_j^n[u_{j+\frac{1}{2}}^n]_+ - \rho_j^n[u_{j-\frac{1}{2}}^n]_- - \rho_{j-1}^n[u_{j-\frac{1}{2}}^n]_+\right) \tag{2.15}$$

where $[a]_+ = \max(a, 0)$ and $[a]_- = \min(a, 0)$. The corresponding semi-implicit method (implicit in ρ and explicit in u) is given by

$$\frac{h}{\Delta t}\left(\rho_j^{n+1} - \rho_j^n\right) = -\left(\rho_{j+1}^{n+1}[u_{j+\frac{1}{2}}^n]_- + \rho_j^{n+1}[u_{j+\frac{1}{2}}^n]_+ - \rho_j^{n+1}[u_{j-\frac{1}{2}}^n]_- - \rho_{j-1}^{n+1}[u_{j-\frac{1}{2}}^n]_+\right). \quad (2.16)$$

2.3 A coupled finite volume - finite difference approximation of the model

We now present a coupled finite volume - finite difference approximation of (1.4)-(1.11), in particular we discretize (1.4) using the finite volume method and (1.5) and (1.6) using the finite difference method.

We set our approximation w_h^n to $w(x,t)$ to be such that $w_h^n \in W_h^0$ with

$$w_h^n(x) = w_j^n \text{ for all } x \in V_j,\ j \in [0, J], n > 0,$$

and we set our approximations u_h^n and p_h^n to $u(x,t)$ and $p(x,t)$ to be such that

$$u_h^n = \sum_{j=0}^{J} \eta_j u_j^n(x) \in S_h \quad \text{and} \quad p_h^n = \sum_{j=0}^{J} p_j^n \chi_j(x) \in S_h^0,$$

where χ_j is a piecewise linear basis function satisfying

$$\chi_j(x_k) = \delta_{jk} \ \ \forall\, j, k \in [0, J]. \quad (2.17)$$

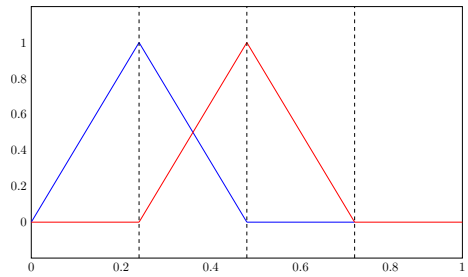


Figure 2.1: Piecewise linear function

We begin by approximating the initial data $w_0(x)$ and the boundary data in the following way

$$w_j^0 = \frac{1}{h}\int_{V_j} w_0(x)dx \quad \forall\, j \in [1, J-1], \tag{2.18}$$

$$w_0^0 = w_J^0 = 0. \tag{2.19}$$

In order to approximate (1.4) we use the semi-implicit upwinding finite volume scheme described in Section 2.2.

Similar to section (2.2) we have the following approximation of (1.4) which holds for all $j \in [1, J-1]$ and all $n \geq 0$:

$$\begin{aligned}\frac{h}{\Delta t}\left(w_j^{n+1}(1+(k_d-k_m)) - w_j^n\right) = \\ -\left(w_{j+1}^{n+1}[u_{j+\frac{1}{2}}^n]_- + w_j^{n+1}[u_{j+\frac{1}{2}}^n]_+ - w_j^{n+1}[u_{j-\frac{1}{2}}^n]_- - w_{j-1}^{n+1}[u_{j-\frac{1}{2}}^n]_+\right)\end{aligned} \tag{2.20}$$

where we recall that $u_{j+\frac{1}{2}}^n = \frac{1}{2}(u(x_{j+1}, t^n) + u(x_j, t^n))$.

We use a standard finite difference approximations of (1.5) and (1.6), see Section 2.1,

$$\frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{h^2} = \frac{4k_m}{3}\left(\frac{w_{j+1}^n - 2w_j^n + w_{j-1}^n}{h^2}\right) \quad \forall\, j \in [1, J-1] \tag{2.21}$$

$$\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} = \left(\frac{p_{j+1}^n - p_j^n}{h}\right) - \frac{k_m}{3}\left(\frac{w_j^{n+1} - w_j^n}{h}\right) \quad \forall\, j \in [1, J-1]. \tag{2.22}$$

From the boundary conditions (1.7)-(1.9) we set

$$w_0^n = w_J^n = 0,\ p_0^n = p_u,\ p_J^n = p_d,\ u_{\frac{1}{2}}^n = 1,\ \frac{u_J^n - u_{j-\frac{1}{2}}^n}{h} = 0 \tag{2.23}$$

with the approximation of the initial data $w_0(x)$ satisfying (2.18).

We note that, as a result of the upwinding scheme, in order to solve (2.20) for all interior nodes $j \in [1, J-1]$, we actually only need to define w_h^{n+1} at the inflow boundary nodes where $u_{\frac{1}{2}}^n > 0$ and $u_{J-\frac{1}{2}}^n < 0$. However in the later analysis we find it convenient to use the boundary conditions given in (2.23).

Chapter 3

Convergence of the numerical approximation

The main result of this chapter is a convergence result of the approximate solution w_h^n of a finite volume approximation of the simplified model (1.4)-(1.14) to the weak solution of this model. Before proving this convergence result we give include some useful analytical definitions and results from [7].

3.1 Useful definitions, notation and results

Definition. The norm of a function u is a scalar that satisfies:

1.$\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = 0$,

2.$\|\alpha u\| = |\alpha|\|u\|$ for any constant α, and

3.$\|u + \nu\| \leq \|u\| + \|\nu\|$.

Definition. Suppose $u, \nu \in L^1$, and α is a multiindex. We say that ν is the αth -weak derivative of u, written

$$D^\alpha u = \nu,$$

provided

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U \nu \phi dx$$

for all test functions ϕ.

In other words, if we are given u and if there happens to exist a function ν which verifies the above equation for all ϕ, we say that $D^\alpha u = \nu$ in the weak sense. If there does not exist such a function ν, then u does not possess a weak α^{th}-partial derivative.

Definition. If U is bounded open set in $\mathbb{R}^n$, we denote $u : \bar{U} \to \mathbb{R}^n$. Then the Holder space

$$C^{k,r}(\bar{U})$$

consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$\|u\|_{C^{k,r}(\bar{U})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,r}(\bar{U})}$$

is finite, where $C^{0,r}(\bar{U})$ the space of uniformly holder continuous functions with exponent r in U and $[D^\alpha u]$ is a semi-norm.
So the space $C^{k,r}(\bar{U})$ consist of those functions u that are k-times continuously differentiable and whose k^{th}- partial derivatives are Holder continuous with exponent r. Such functions are well behaved and furthermore the space $C^{k,r}(\bar{U})$ itself possesses a good mathematical structure.

Definition. The Sobolev space consist of all locally summable functions

$$u : U \to R$$

such that for each multiindex α with

$$|\alpha| \leq k$$

$D^\alpha u$ exists in the weak sense.

Remark. if $p = 2$, we usually write

$$H^k(U) = W^{k,2}(U), \qquad k = 0, 1, ...$$

The letter H is used, since as we will see $H^k(U)$ is Hilbert space. Note that $H^0(U) = L^2(U)$.

Definition. If $u \in W^{k,p}(U)$, we define its norm to be

$$\|u\|_{W^{k,p}(U)} = \begin{cases} (\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p dx)^{\frac{1}{p}}, & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq k} \text{ess sup}_U |D^\alpha u|, & (p = \infty). \end{cases}$$

Definition. (i) Let $\{u_m\}_{m=1}^\infty$, $u \in W^{k,p}(U)$. We say u_m converges to u in $W^{k,p}(U)$, written

$$u_m \to u \quad \text{in} \quad W^{k,p}(U),$$

provided

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0.$$

(ii) We write

$$u_m \to u \quad \text{in} \quad W^{k,p}_{loc}(U),$$

to mean

$$u_m \to u \quad \text{in} \quad W^{k,p}(V)$$

for each compactly embedded subset $V \subset\subset U$.

Definition. The total variation of a real-valued function f, defined on an interval $[a, b] \subset \mathbb{R}$ is the quantity

$$|f|_{TV[a,b]} = \sup_{P\in P} \sum_{i=0}^{nP-1} |f(x_{i+1}) - f(x_i)|$$

where the supremum is taken over the set $P = \{P = \{x_0, ..., x_{np}\}|P$ is a partition of $[a, b]\}$ of all partitions of the interval considered.

If f is differentiable and its derivative is Riemann-integrable, its total variation is the vertical component of the arc-length of its graph, that is to say

$$V_a^b(f) = \int_a^b |f^{'}(x)|dx.$$

A real-valued function f on the real line is said to be of bounded variation (a BV function) on

a chosen interval $[a, b] \subset \mathbb{R}$ if its total variation is finite, i.e.

$$f \in BV([a, b]) \Longleftrightarrow V_a^b(f) \leq +\infty.$$

Definition. The L^p space consist of all measurable functions $u : [0, t] \to X$ with

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|^p dt \right)^{\frac{1}{p}} < \infty$$

for $1 \leq p < \infty$, and

$$\|u\|_{L^\infty(0,T;X)} := \underset{0 \leq t \leq T}{ess\ sup} \|u(t)\| < \infty$$

Definition. The space

$$C([0, T]; X)$$

comprises all continuous functions $u : [0, T] \to X$ with

$$\|u\|_{C([0,T];X)} := \max_{0 \leq t < T} \|u(t)\| \leq \infty$$

Definition. Let X is a Banach space. We say a sequence $\{u_k\}_{k=1}^\infty \subset X$ converges weakly to $u \in X$, written

$$u_k \to u,$$

if

$$\langle u^*, u_k \rangle \to \langle u^*, u \rangle$$

for each bounded linear functional $u^* \in X^*$.

It is easy to check that if $u_k \to u$, then $u_k \to u$. It is also true that any weakly convergent sequence is bounded. In addition , if $u_k \to u$, then

$$\|u\| \leq \lim_{k \to \infty} \text{inf} \|u_k\|.$$

Definition. Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^\infty \subset X$ is

bounded. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and $u \in X$ such that

$$u_{k_j} \to u.$$

In other words, bounded sequences in a reflexive Banach space are weakly precompact. In particular, a bounded sequence in a Hilbert space contains a weakly convergent subsequence.

3.2 Weak formulation of the problem

Multiplying (1.4) by $\varphi \in W^{1,\infty}(\Omega_{\text{T}})$, integrating over Ω and from 0 to T, using integration by parts and noting (1.13) we obtain the following weak form of (1.4)-(1.14)

$$\int_0^T \int_0^1 w(\varphi_t - u\varphi_x + (k_d - k_m)\varphi)dxdt + \int_0^1 w_0\varphi(x,0)dx = 0 \quad \forall \varphi \in W^{1,\infty}(\Omega_{\text{T}}). \tag{3.1}$$

From Chapter 2 we have the following approximation of (1.4) - (1.14) which holds for all $j \in [1, J-1]$ and all $n \geq 0$:

$$\begin{aligned} \frac{h}{\Delta t}\left(w_j^{n+1}(1 + (k_d - k_m)) - w_j^n\right) = \\ -\left(w_{j+1}^{n+1}[u_{j+\frac{1}{2}}^n]_- + w_j^{n+1}[u_{j+\frac{1}{2}}^n]_+ - w_j^{n+1}[u_{j-\frac{1}{2}}^n]_- - w_{j-1}^{n+1}[u_{j-\frac{1}{2}}^n]_+\right) \end{aligned} \tag{3.2}$$

$$w_0^n = w_J^n = 0, \; n > 0, \tag{3.3}$$

$$w_j^0 = w(x_j, 0), \;\; \forall j \in [1, J-1] \tag{3.4}$$

where we have that $u_{j+\frac{1}{2}}^n = u(x_j + \frac{h}{2}, t^n)$, with $u(x,t)$ given.

3.3 Stability bounds

Before proving the convergence result we will obtain stability bounds on the approximate solution $\hat{w}_h^n$ of the finite volume approximation (3.2)-(3.4).
We assume that $u \in L^\infty(0,T;H^2(\Omega)) \cap W^{1,1}(0,T;L^\infty(\Omega))$ and that $u^n_{j+\frac{1}{2}} = u(x_j + \frac{h}{2}, t^n)$ satisfies

$$|u^n_{j+\frac{1}{2}}| \leq C_u, \ \frac{|u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}}|}{h} \leq C_{uu} \ \frac{|u^n_{j+\frac{3}{2}} - 2u^n_{j+\frac{1}{2}} + u^n_{j-\frac{1}{2}}|}{h^2} \leq C_{uuu}, \ |u^{n+1}_{j+\frac{1}{2}} - u^n_{j+\frac{1}{2}}| \leq C\Delta t. \tag{3.5}$$

In addition we assume that $w \in L^\infty(\Omega_T)$ and that

$$|w^n_j| \leq C_w. \tag{3.6}$$

Henceforth for the simplicity of notation we set $A = k_m - k_d$.

Lemma 3.3.1. *For all $h > 0$ and $0 < \Delta t < \frac{1-A}{C_{uu}}$, there exists a unique sequence $\{\hat{w}_h^n\}$ which solves (3.2) for all $n \geq 1$.*

Proof: We rewrite (3.2) as

$$\left(1 - \Delta t A + \frac{\Delta t}{h}([u^n_{j+\frac{1}{2}}]_+ - [u^n_{j-\frac{1}{2}}]_-)\right) w^{n+1}_j + \frac{\Delta t}{h}\left([u^n_{j+\frac{1}{2}}]_- w^{n+1}_{j+1} - [u^n_{j-\frac{1}{2}}]_+ w^+_{j-1} n + 1\right) = w^n_j$$

which is a system of equations of the form

$$L(u^n_h)\underline{w}^{n+1}_h = \underline{w}^n_h, \tag{3.7}$$

where $L(u^n_h)$ is a matrix with diagonal elements $L_{jj} \geq 1$, and off-diagonal elements $L_{jk} \leq 0$ for $j \neq k$. Since $\Delta t < \dfrac{1-A}{C_{uu}}$ and $x = [x]_+ + [x]_-$ where

$$[x]_+ = \max(x, 0)$$

$$[x]_- = \min(x, 0),$$

it follows that $L_{jj} - |L_{jj+1}| - |L_{jj+1}| = 1 - A + \frac{\Delta t}{h}(u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}}) \geq 1 - A - \Delta t C_{uu} > 0$, and hence L is strictly diagonally dominant and therefore invertible, giving a unique solution $\underline{w}_h^{n+1}$.
□

Remark. From Lemma 3.3.1 we see that for a unique solution of our scheme to exist we require

$$\Delta t < \frac{1 - A}{C_{uu}} \tag{3.8}$$

so from here onwards we assume Δt satisfies (3.8).

Lemma 3.3.2. *There exists a constant C independent of h and Δt, such that*

$$\sum_{n=0}^{N-1} \Delta t \sum_{j=0}^{J-1} |w_j^{n+1} - w_{j+1}^{n+1}|([u^n_{j+\frac{1}{2}}]_+ - [u^n_{j+\frac{1}{2}}]_-) \leq \frac{C}{\sqrt{h}}.$$

Proof: We rewrite (3.2) as

$$\frac{h}{\Delta t}(w_j^{n+1} - w_j^n) = -(w_{j+1}^{n+1} - w_j^{n+1})[u^n_{j+\frac{1}{2}}]_- - (w_j^{n+1} - w_{j-1}^{n+1})[u^n_{j-\frac{1}{2}}]_+ - w_j^{n+1}(u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}}) + hAw_j^{n+1} \tag{3.9}$$

and since $w_0^n = w_J^n = 0$ we can multiply (3.9) by w_j^{n+1} and sum from $j = 0$ to J to obtain

$$\begin{aligned}\sum_{j=0}^{J} \frac{h}{\Delta t}(w_j^{n+1} - w_j^n)w_j^{n+1} &= -\sum_{j=0}^{J}\Big(w_j^{n+1}(w_{j+1}^{n+1} - w_j^{n+1})[u^n_{j+\frac{1}{2}}]_- + w_j^{n+1}(w_j^{n+1} - w_{j-1}^{n+1})[u^n_{j-\frac{1}{2}}]_+\Big) \\ &+ \sum_{j=0}^{J}(w_j^{n+1})^2(hA - u^n_{j+\frac{1}{2}} + u^n_{j-\frac{1}{2}}).\end{aligned}$$

Noting that $[x]_+ = -[-x]_-$ and that $(a-b)a = \frac{1}{2}(a^2-b^2) + \frac{1}{2}(a-b)^2$ we have

$$\begin{aligned}
\sum_{j=0}^{J} \frac{h}{\Delta t}\left((w_j^{n+1})^2 - (w_j^n)^2\right) &\leq \sum_{j=0}^{J}(w_j^{n+1} - w_{j+1}^{n+1})^2[u_{j+\frac{1}{2}}^n]_- - (w_j^{n+1} - w_{j-1}^{n+1})^2[u_{j-\frac{1}{2}}^n]_+ \\
&+ 2\sum_{j=0}^{J}(w_j^{n+1})^2(hA - u_{j+\frac{1}{2}}^n + u_{j-\frac{1}{2}}^n) \\
&+ \sum_{j=0}^{J}\left((w_j^{n+1})^2 - (w_{j+1}^{n+1})^2\right)[u_{j+\frac{1}{2}}^n]_- - \sum_{j=0}^{J}\left((w_j^{n+1})^2 - (w_{j-1}^{n+1})^2\right)[u_{j-\frac{1}{2}}^n]_+ \\
&= \sum_{j=0}^{J}(w_j^{n+1} - w_{j+1}^{n+1})^2([u_{j+\frac{1}{2}}^n]_- - [u_{j+\frac{1}{2}}^n]_+) \\
&+ 2\sum_{j=0}^{J}\left((w_j^{n+1})^2(hA - u_{j+\frac{1}{2}}^n + u_{j-\frac{1}{2}}^n) + (w_j^{n+1})^2(u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n)\right) \\
&\leq \sum_{j=0}^{J}(w_j^{n+1} - w_{j+1}^{n+1})^2([u_{j+\frac{1}{2}}^n]_- - [u_{j+\frac{1}{2}}^n]_+) \\
&+ \sum_{j=0}^{J}(w_j^{n+1})^2\left(2hA - u_{j+\frac{1}{2}}^n + u_{j-\frac{1}{2}}^n\right). \qquad (3.10)
\end{aligned}$$

Summing (3.10) from $n=0$ to $N-1$, rearranging and using Lemma 3.3.1 gives

$$\begin{aligned}
\sum_{n=0}^{N-1}\Delta t\sum_{j=0}^{J}(w_{j+1}^{n+1} - w_j^{n+1})^2([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-) &\leq -\sum_{n=0}^{N-1}\sum_{j=0}^{J}h((w_j^{n+1})^2 - (w_j^n)^2) \\
&+ \sum_{n=0}^{N-1}\Delta t\sum_{j=0}^{J}h(w_j^{n+1})^2\left(\frac{|u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n|}{h} + 2A\right) \\
&\leq \sum_{j=0}^{J}h((w_j^0)^2 - (w_j^N)^2) + CT \leq C. \qquad (3.11)
\end{aligned}$$

From the Cauchy-Schwartz inequality we have

$$\begin{aligned}
&\sum_{n=0}^{N-1}\Delta t\sum_{j=0}^{J}|w_{j+1}^{n+1} - w_j^{n+1}|([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-) \\
&\leq \left(\sum_{n=0}^{N-1}\Delta t\sum_{j=0}^{J}(w_{j+1}^{n+1} - w_j^{n+1})^2([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-)\right)^{1/2}\left(\sum_{n=0}^{N-1}\Delta t\sum_{j=0}^{J}([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-)\right)^{1/2}.
\end{aligned}$$

From (3.5) we have

$$\sum_{j=0}^{J}([u^n_{j+\frac{1}{2}}]_+ - [u^n_{j+\frac{1}{2}}]_-) \leq \frac{C_u}{h}$$

and the result follows. □

We now establish further estimates on w^n_h.

Lemma 3.3.3. *Set Λ^* to be a constant. Then for all $n \geq 0$ and all $\Delta t \leq \min\left(\frac{1-A}{C_{uu}}, \frac{A+C_{uu}-(1-A)\Lambda^*}{\Lambda^* C_{uu}}\right)$ we have*

$$|\hat{w}^n_h|_{TV[0,L]} = \sum_{j=0}^{J-1}|w^n_{j+1} - w^n_j| \leq C(|\hat{w}^0_h|_{TV[0,L]} + \Lambda^*(n+1)\Delta t),$$

where

$$C = e^{\Lambda^*(n+1)\Delta t}.$$

Proof: From (3.2) we see that for $j = 1, \ldots, J-1$

$$\begin{aligned}(1-\Delta t A)w^{n+1}_j &= w^n_j - \frac{\Delta t}{h}\left(w^{n+1}_{j+1}[u^n_{j+\frac{1}{2}}]_- + w^{n+1}_j[u^n_{j+\frac{1}{2}}]_+ - w^{n+1}_j[u^n_{j-\frac{1}{2}}]_- - w^{n+1}_{j-1}[u^n_{j-\frac{1}{2}}]_+\right)\\ &= w^n_j - \frac{\Delta t}{h}\left((w^{n+1}_{j+1} - w^{n+1}_j)[u^n_{j+\frac{1}{2}}]_- + (w^{n+1}_j - w^{n+1}_{j-1})[u^n_{j-\frac{1}{2}}]_+\right)\\ &\quad -\frac{\Delta t}{h}w^{n+1}_j(u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}}). \end{aligned} \tag{3.12}$$

Hence for $j = 0, 1, \cdots, J-2$ we have

$$(1-\Delta t A)w^{n+1}_{j+1} = w^n_{j+1} - \frac{\Delta t}{h}\left((w^{n+1}_{j+2} - w^n_{j+1})[u^n_{j+\frac{3}{2}}]_- + w^{n+1}_{j+1}(u^n_{j+\frac{3}{2}} - u^n_{j+\frac{1}{2}}) + (w^{n+1}_{j+1} - w^{n+1}_j)[u^n_{j+\frac{1}{2}}]_+\right).$$

Setting $\xi^n_j = w^n_{j+1} - w^n_j$, gives the following for $j = 1, \ldots, J-2$

$$\begin{aligned}(1-\Delta t A)\xi^{n+1}_j &= \xi^n_j - \frac{\Delta t}{h}\left(\xi^{n+1}_{j+1}[u^n_{j+\frac{3}{2}}]_- + w^{n+1}_{j+1}(u^n_{j+\frac{3}{2}} - u^n_{j+\frac{1}{2}}) + \xi^{n+1}_j[u^n_{j+\frac{1}{2}}]_+ - \xi^{n+1}_j[u^n_{j+\frac{1}{2}}]_-\right)\\ &\quad +\frac{\Delta t}{h}\left(w^{n+1}_j(u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}}) + \xi^{n+1}_{j-1}[u^n_{j-\frac{1}{2}}]_+\right)\end{aligned}$$

$$\Rightarrow \left(1-\Delta tA-\frac{\Delta t}{h}[u^n_{j+\frac{1}{2}}]_- + \frac{\Delta t}{h}[u^n_{j+\frac{1}{2}}]_+\right)\xi^{n+1}_j = \xi^n_j - \frac{\Delta t}{h}[u^n_{j+\frac{3}{2}}]_-\xi^{n+1}_{j+1} + \frac{\Delta t}{h}[u^n_{j-\frac{1}{2}}]_+\xi^{n+1}_{j-1} - \Delta t\left(w^{n+1}_{j+1}(u^n_{j+\frac{3}{2}}-u^n_{j+\frac{1}{2}}) - w^{n+1}_j(u^n_{j+\frac{1}{2}}-u^n_{j-\frac{1}{2}})\right)$$

$$\Rightarrow \left(1-\Delta tA-\frac{\Delta t}{h}[u^n_{j+\frac{1}{2}}]_- + \frac{\Delta t}{h}[u^n_{j+\frac{1}{2}}]_+ + \Delta t(u^n_{j+\frac{3}{2}}-u^n_{j+\frac{1}{2}})\right)\xi^{n+1}_j$$
$$= \xi^n_j - \frac{\Delta t}{h}[u^n_{j+\frac{3}{2}}]_-\xi^{n+1}_{j+1} + \frac{\Delta t}{h}[u^n_{j-\frac{1}{2}}]_+\xi^{n+1}_{j-1} + \Delta t w^{n+1}_j(u^n_{j+\frac{3}{2}}-2u^n_{j+\frac{1}{2}}+u^n_{j-\frac{1}{2}}).$$

From (3.8) we have that $(1-A+\Delta t(u^n_{j+\frac{1}{2}}-u^n_{j-\frac{1}{2}}))>0$ and hence all the coefficients of ξ in the above equation are positive, hence for $j=1,\cdots,J-2$ we have

$$\left(1-\Delta tA-\frac{\Delta t}{h}[u^n_{j+\frac{1}{2}}]_- + \frac{\Delta t}{h}[u^n_{j+\frac{1}{2}}]_+ + \Delta t(u^n_{j+\frac{3}{2}}-u^n_{j+\frac{1}{2}})\right)|\xi^{n+1}_j|$$

$$\leq |\xi^n_j| - \frac{\Delta t}{h}[u^n_{j+\frac{3}{2}}]_-|\xi^{n+1}_{j+1}| + \frac{\Delta t}{h}[u^n_{j-\frac{1}{2}}]_+|\xi^{n+1}_{j-1}| + h\Delta t|w^{n+1}_j|\left|\frac{u^n_{j+\frac{3}{2}}-2u^n_{j+\frac{1}{2}}+u^n_{j-\frac{1}{2}}}{h^2}\right|. \tag{3.13}$$

Summing (3.13) from $j=1$ to $J-2$ and using (3.5) gives

$$\begin{aligned}(1-\Delta tA-C_{uu}\Delta t)\sum_{j=1}^{J-2}|\xi^{n+1}_j| &\leq \sum_{j=1}^{J-2}|\xi^n_j| + \frac{\Delta t}{h}\sum_{j=1}^{J-2}\left(-|\xi^{n+1}_{j+1}|[u^n_{j+\frac{3}{2}}]_- + |\xi^{n+1}_{j-1}|[u^n_{j-\frac{1}{2}}]_+\right)\\ &\quad + \frac{\Delta t}{h}\sum_{j=1}^{J-2}|\xi^{n+1}_j|\left([u^n_{j+\frac{1}{2}}]_- - [u^n_{j+\frac{1}{2}}]_+\right) - C_wC_{uuu}\Delta t\sum_{j=1}^{J-2}h\\ &= \sum_{j=1}^{J-2}|\xi^n_j| - C_wC_{uuu}\Delta t\sum_{j=1}^{J-2}h\\ &\quad + \frac{\Delta t}{h}\left(-|\xi^{n+1}_{J-1}|[u^n_{j-\frac{1}{2}}]_- + |\xi^{n+1}_1|[u^n_{\frac{3}{2}}]_-\right)\\ &\quad + \frac{\Delta t}{h}\left(-|\xi^{n+1}_{J-2}|[u^n_{j-2}]_+ + |\xi^{n+1}_0|[u^n_{\frac{1}{2}}]_+\right)\end{aligned} \tag{3.14}$$

Using (2.23) and (3.12) we have

$$\begin{aligned}(1-\Delta tA)\xi_0^{n+1} &= \xi_0^n + \frac{\Delta t}{h}\left(-\xi_1^{n+1}[u_{\frac{3}{2}}^n]_- - w_1^{n+1}(u_{\frac{3}{2}}^n - u_{\frac{1}{2}}^n) - \xi_0^{n+1}[u_{\frac{1}{2}}^n]_+\right)\\ &= \xi_0^n + \frac{\Delta t}{h}\left(-\xi_1^{n+1}[u_{\frac{3}{2}}^n]_- - hw_1^{n+1}(u_{\frac{5}{2}}^n - u_{\frac{3}{2}}^n) - \xi_0^{n+1}[u_{\frac{1}{2}}^n]_+\right)\\ &= \xi_0^n + \frac{\Delta t}{h}\left(-\xi_1^{n+1}[u_{\frac{3}{2}}^n]_- - h\xi_0^{n+1}(u_{\frac{5}{2}}^n - u_{\frac{3}{2}}^n) - \xi_0^{n+1}[u_{\frac{1}{2}}^n]_+\right)\end{aligned}$$

$$\Rightarrow (1-\Delta tA + \Delta t(u_{\frac{5}{2}}^n - u_{\frac{3}{2}}^n) + \frac{\Delta t}{h}[u_{\frac{1}{2}}^n]_+)\xi_0^{n+1} = \xi_0^n - \frac{\Delta t}{h}\xi_1^{n+1}[u_{\frac{3}{2}}^n]_-.$$

Noting from (3.8) that $1+\Delta t(u_{\frac{5}{2}}^n - u_{\frac{3}{2}}^n) > 0$ it follows that

$$((1-\Delta tA + \Delta t(u_{\frac{5}{2}}^n - u_{\frac{3}{2}}^n) + \frac{\Delta t}{h}[u_{\frac{1}{2}}^n]_+)|\xi_0^{n+1}| \le |\xi_0^n| - \frac{\Delta t}{h}\xi_1^{n+1}[u_{\frac{3}{2}}^n]_-$$

and similarly

$$(1-\Delta tA + \Delta t(u_{J+\frac{1}{2}}^n - u_{J-\frac{1}{2}}^n) - \frac{\Delta t}{h}[u_{J-\frac{1}{2}}^n]_-)|\xi_{J-1}^{n+1}| \le |\xi_{J-1}^n| + \frac{\Delta t}{h}\xi_{J-2}^{n+1}[u_{J-\frac{3}{2}}^n]_+.$$

Hence

$$(1-\Delta tA - C_{uu}\Delta t)|\xi_0^{n+1}| \le |\xi_0^n| - \frac{\Delta t}{h}|\xi_1^{n+1}|[u_{\frac{3}{2}}^n]_- - \frac{\Delta t}{h}|\xi_0^{n+1}|[u_{\frac{1}{2}}^n]_+$$

$$(1-\Delta tA - C_{uu}\Delta t)|\xi_{J-1}^{n+1}| \le |\xi_{J-1}^n| + \frac{\Delta t}{h}|\xi_{J-2}^{n+1}|[u_{J-\frac{3}{2}}^n]_+ + \frac{\Delta t}{h}|\xi_{J-1}^{n+1}|[u_{J-\frac{1}{2}}^n]_-.$$

Using the above definitions of $|\xi_0^{n+1}|$ and $|\xi_{J-1}^{n+1}|$ in (3.14) and noting that $\Delta t \le \frac{A+C_{uu}-(1-A)\Lambda^*}{\Lambda^* C_{uu}}$, we obtain

$$\sum_{j=0}^{J-1}|\xi_j^{n+1}| \le \frac{1}{1-\Delta tA - C_{uu}\Delta t}\sum_{j=0}^{J-1}|\xi_j^n| + C\Delta t \le (1+\Lambda^*\Delta t)\sum_{j=0}^{J-1}|\xi_j^n| + C\Delta t,$$

$$\begin{aligned}\Rightarrow \sum_{j=0}^{J-1}|\xi_j^{n+1}| &\le (1+\Lambda^*\Delta t)^{n+1}\sum_{j=0}^{J-1}|\xi_j^0| + \sum_{j=0}^{n}(1+\Lambda^*\Delta t)^j C\Delta t\\ &\le (1+\Lambda^*\Delta t)^{n+1}\sum_{j=0}^{J-1}|\xi_j^0| + (n+1)(1+\Lambda^*\Delta t)^{n+1}C\Delta t,\end{aligned}$$

$$\Rightarrow \sum_{j=0}^{J-1} |\xi_j^{n+1}| \leq e^{(n+1)\Lambda^*\Delta t} \left(\sum_{j=0}^{J-1} |\xi_j^0| + (n+1)\Lambda^*\Delta t \right),$$

and the result follows. □

Lemma 3.3.4. *There exists a constant C independent of h and Δt, such that*

$$||\hat{w}_h^{n+1} - \hat{w}_h^n||_{L^1(0,L)} = \sum_{j=1}^{J-1} h|w_j^{n+1} - w_j^n| \leq C\Delta t \quad \forall\, n \geq 0, \tag{3.15}$$

Proof: From (3.12) we have

$$h(w_j^{n+1} - w_j^n) = -\Delta t \left((w_{j+1}^{n+1} - w_j^{n+1})[u_{j+\frac{1}{2}}^n]_- + w_j^{n+1}(u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n + hA) + (w_j^{n+1} - w_{j-1}^{n+1})[u_{j-\frac{1}{2}}^n]_+ \right)$$

$$\begin{aligned} \Rightarrow h|w_j^{n+1} - w_j^n| &\leq \Delta t \left(-|\xi_j^{n+1}|[u_{j+\frac{1}{2}}^n]_- + |w_j^{n+1}|(|u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n| + Ah) + |\xi_{j-1}^{n+1}|[u_{j-\frac{1}{2}}^n]_+ \right) \\ &= \Delta t \left(-|\xi_j^{n+1}|[u_{j+\frac{1}{2}}^n]_- + h|w_j^{n+1}|(\left| \frac{u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n}{h} \right| + A) + |\xi_{j-1}^{n+1}|[u_{j-\frac{1}{2}}^n]_+ \right). \end{aligned}$$

Summing the above equation from $j = 1$ to $J - 1$, and applying the result of Lemma 3.3.3 and (3.5) gives (3.15). □

3.4 Convergence of the numerical approximation

Before we prove the main result of this section we introduce some notation and results.
We set $(\cdot,\cdot)_h$ to be the discrete replacement of the L^2 inner product over Ω, $(\cdot,\cdot)$, defined by the following 'lumped' integration rule

$$(\eta, \varphi)_h = \sum_{j=1}^{M} h\eta_j\varphi_j \quad \forall\, \eta, \varphi \in C(\overline{\Omega}), \tag{3.16}$$

where $\eta_j = \eta(x_j)$ are the nodal values of η_h. For every continuous function φ on $\overline{\Omega}$ we define the interpolation operator $\pi^h : C(\overline{\Omega}) \to S^h$, by $\pi^h\varphi \in S_h$ where $\pi^h\varphi = \varphi$ at every node x_j,

$j \in [0, J]$. From [2] we have the following results

$$|\varphi - \pi^h \varphi|_0 + h|\varphi - \pi^h \varphi|_1 \leq Ch^2 \quad \forall\, \varphi \in H^2(\Omega), \tag{3.17}$$

where C is a positive constant and

$$|\varphi|_0 = \left(\int_\Omega \varphi^2 dx\right)^{\frac{1}{2}}, \;\; |\varphi|_1 = \left(\int_\Omega |\varphi_x|^2 dx\right)^{\frac{1}{2}}.$$

It is well known that for all $\eta, \varphi \in S_h^0$ [16], we have

$$|(\eta, \varphi) - (\eta, \varphi)_h| \leq Ch^2 |\eta|_1 |\varphi|_1 \leq Ch|\eta|_0 |\varphi|_1, \tag{3.18}$$

and there exists constants C_1 and C_2 independent of h such that

$$C_1|\varphi|_0 \leq |\varphi|_h \leq C_2|\varphi|_0 \quad \forall\, \varphi \in S_h. \tag{3.19}$$

Let $f^n \in X$, for a function space X where f^n might be a finite element function. We define,

$$f_{\Delta t}(t) := f^n \qquad t \in [n\Delta t, (n+1)\Delta t), \tag{3.20}$$

$$f^+_{\Delta t}(t) := f^{n+1} \qquad t \in (n\Delta t, (n+1)\Delta t], \tag{3.21}$$

$$f^*_{\Delta t}(t) := f^n + (t - n\Delta t)\tfrac{(f^{n+1} - f^n)}{\Delta t} \qquad t \in [n\Delta t, (n+1)\Delta t]. \tag{3.22}$$

Lemma 3.4.1. *There exists a function $w \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; BV(\Omega))$ and a subsequences such that*

$$\hat{w}_{h,\Delta t} \to w \qquad \textit{strongly in } L^1(\Omega_T), \tag{3.23}$$

$$\hat{w}^+_{h,\Delta t} \to w \qquad \textit{strongly in } L^1(\Omega_T), \tag{3.24}$$

$$w_{h,\Delta t} \to w \qquad \textit{strongly in } L^1(\Omega_T), \tag{3.25}$$

$$w^+_{h,\Delta t} \to w \qquad \textit{strongly in } L^1(\Omega_T), \tag{3.26}$$

$$w_{h,\Delta t} \to w \qquad \textit{weakly in } L^2(\Omega_T), \tag{3.27}$$

as $h,\ \Delta t \to 0$.

Proof:

Using (3.15), (3.22) and Lemma 3.3.3 it follows that

$$||\hat{w}^*_{h,\Delta t}||_{L^\infty(0,T;BV(\Omega))} \leq C \quad \text{and} \quad ||\partial_t(\hat{w}^*_{h,\Delta t})||_{L^\infty(0,T;L^1(\Omega))} \leq C. \tag{3.28}$$

Since BV is compactly embedded in L^1, see [8], using [13] we conclude the existence of a function $w \in C([0,T];L^1(\Omega)) \cap L^\infty(0,T;BV(\Omega))$ such that

$$\hat{w}^*_{h,\Delta t} \to w \quad \text{strongly in } C([0,T];L^1(\Omega)). \tag{3.29}$$

To prove (3.23) we see that from (3.22), (3.29) and (3.15) it follows that

$$\begin{aligned}
\int_0^T ||\hat{w}_{h,\Delta t} - w||_{L^1(\Omega)} dt &\leq \int_0^T ||\hat{w}_{h,\Delta t} - \hat{w}^*_{h,\Delta t}||_{L^1(\Omega)} dt + \int_0^T ||\hat{w}^*_{h,\Delta t} - w||_{L^1(\Omega)} dt \\
&\leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} |t - n\Delta t| ||\frac{\hat{w}^{n+1}_h - \hat{w}^n_h}{\Delta t}||_{L^1(\Omega)} dt + \int_0^T ||\hat{w}^*_{h,\Delta t} - w||_{L^1(\Omega)} dt \\
&\longrightarrow 0 \quad \text{as } h, \Delta t \to 0.
\end{aligned} \tag{3.30}$$

Using (3.15) and (3.30) it follows that

$$\begin{aligned}
\int_0^T ||\hat{w}^+_{h,\Delta t} - w||_{L^1(\Omega)} dt &\leq \int_0^T ||\hat{w}^+_{h,\Delta t} - \hat{w}_{h,\Delta t}||_{L^1(\Omega)} dt + \int_0^T ||\hat{w}_{h,\Delta t} - w||_{L^1(\Omega)} dt \\
&\leq \sum_{n=0}^{N} \int_{t_n}^{t_{n+1}} ||\hat{w}^{n+1}_h - \hat{w}^n_h||_{L^1(\Omega)} dt + \int_0^T ||\hat{w}_{h,\Delta t} - w||_{L^1(\Omega)} dt \\
&\longrightarrow 0 \quad \text{as } h, \Delta t \to 0,
\end{aligned}$$

which proves (3.24). We now prove (3.25), to this end we note that since

$$\int_0^T \int_0^1 |w_{h,\Delta t} - w| dx dt \leq \int_0^T \int_0^1 |w_{h,\Delta t} - \hat{w}_{h,\Delta t}| dx dt + \int_0^T \int_0^1 |\hat{w}_{h,\Delta t} - w| dx dt,$$

using (3.23) it remains to show that

$$\int_0^T \int_0^1 |\hat{w}_{h,\Delta t} - w_{h,\Delta t}| dx dt \to 0 \text{ as } h \text{ and } \Delta t \to 0.$$

From the definitions of $w_{h,\Delta t}(x,t)$ and $\hat{w}_{h,\Delta t}(x,t)$ and Lemma 3.3.3 we have

$$\begin{aligned}
\int_0^T \int_0^1 |\hat{w}_{h,\Delta t}(x,t) - w_{h,\Delta t}| dx dt &= \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \sum_{j=0}^{J-1} \int_{jh}^{(j+1)h} |\hat{w}_{h,\Delta t} - w_{h,\Delta t}| dx dt \\
&\leq \frac{h\Delta t}{2} \sum_{n=0}^{N-1} \sum_{j=0}^{J-1} \left(|w_{j+1}^n - w_j^n| + |w_j^n - w_{j+1}^n| \right) \\
&= N\Delta t h \sum_{j=0}^{J-1} |w_{j+1}^n - w_j^n| \to 0 \text{ as } h \text{ and } \Delta t \to 0.
\end{aligned}$$

Thus (3.25) holds, and (3.26) can be proved in the same manner, furthermore using (3.5) and (3.25) we easily conclude (3.27). □

Lemma 3.4.2. *Let $w_{h,\Delta t}$, $u_{h,\Delta t}$ and $\phi_{h,\Delta t}$ be such that*

$$||u_{h,\Delta t}||_{L^\infty(\Omega_T)} \leq C, \quad ||\phi_{h,\Delta t}||_{L^\infty(\Omega_T)} \leq C, \quad ||w_{h,\Delta t}||_{L^\infty(\Omega_T)} \leq C, \tag{3.31}$$

$$\left.\begin{array}{ll}
u_{h,\Delta t} \longrightarrow u & \textit{strongly in } L^2(\Omega_T) \\
\phi_{h,\Delta t} \longrightarrow \phi & \textit{strongly in } L^2(\Omega_T) \\
w_{h,\Delta t} \longrightarrow w & \textit{strongly in } L^1(\Omega_T)
\end{array}\right\} \textit{ as } h, \Delta t \to 0. \tag{3.32}$$

Then

$$\int_{\Omega_T} u_{h,\Delta t} \phi_{h,\Delta t} w_{h,\Delta t} dx \longrightarrow \int_{\Omega_T} u \phi w dx \quad \textit{as } h, \Delta t \to 0. \tag{3.33}$$

Proof: Noting that

$$\begin{aligned}
\left|\int_{\Omega_{\mathrm{T}}}(u_{h,\Delta t}\phi_{h,\Delta t}w_{h,\Delta t}-u\phi w)dx\right| &\leq \left|\int_{\Omega_{\mathrm{T}}}(u_{h,\Delta t}-u)\phi_{h,\Delta t}w_{h,\Delta t}dx\right| \\
&\quad +\left|\int_{\Omega_{\mathrm{T}}}u(\phi_{h,\Delta t}-\phi)w_{h,\Delta t}dx\right| \\
&\quad +\left|\int_{\Omega_{\mathrm{T}}}u\phi(w_{h,\Delta t}-w)dx\right| \\
&\leq ||u_{h,\Delta t}-u||_{L^2(\Omega_{\mathrm{T}})}||\phi_{h,\Delta t}w_{h,\Delta t}||_{L^2(\Omega_{\mathrm{T}})} \\
&\quad +||\phi_{h,\Delta t}-\phi||_{L^2(\Omega_{\mathrm{T}})}||uw_{h,\Delta t}||_{L^2(\Omega_{\mathrm{T}})} \\
&\quad +\left|\int_{\Omega_{\mathrm{T}}}u\phi(w_{h,\Delta t}-w)dx\right| \\
&\leq ||u_{h,\Delta t}-u||_{L^2(\Omega_{\mathrm{T}})}||\phi_{h,\Delta t}w_{h,\Delta t}||_{L^2(\Omega_{\mathrm{T}})} \\
&\quad +||\phi_{h,\Delta t}-\phi||_{L^2(\Omega_{\mathrm{T}})}||uw_{h,\Delta t}||_{L^2(\Omega_{\mathrm{T}})} \\
&\quad +||w_{h,\Delta t}-w||_{L^1(\Omega_{\mathrm{T}})}||u\phi||_{L^\infty(\Omega_{\mathrm{T}})}.
\end{aligned}$$

Using (3.31) and (3.32) gives (3.33). □

We set

$$\hat{w}_h^0(x)|_{V_j} := w_j^0, \;\; \forall j \in [1, J-1] \tag{3.34}$$

then it is well known, see [13], that for all $w_0(x) \in BV(\Omega) \cap L^\infty(\Omega)$

$$w_h^0 \to w_0 \;\text{ strongly in } L^2(\Omega) \;\text{ as }\; h \to 0. \tag{3.35}$$

Theorem 3.4.3. *There exists a function $w \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; BV(\Omega))$ such that*

$$\hat{w}_{h,\Delta t} \to w \textit{ strongly in } L^1(\Omega_\tau), \tag{3.36}$$

where w is the unique solution of (3.1).

Proof: We note that for any continuous functions f, g we have

$$(f,g)_h = (\pi^h f, \pi^h g)_h = \int_\Omega \pi^h(fg)dx.$$

In the following we repeatedly use (3.18) for $f, g \in S_h^0$, where $||f||_{L^\infty(\Omega)} \leq C$ and g is the interpolant of a smooth function to deduce that

$$\lim_{h\to 0}((f,g)_h - (f,g)) = 0.$$

The existence of subsequences and a limit w that satisfies (3.36) follows directly from Lemma 3.4.1, so all that remains now is to prove that w is the unique solution of (3.1). To this end, from [5], we see that for any $\phi \in C^\infty(\Omega_{\mathrm{T}})$, the mesh functions $\phi_{h,\Delta t}(t)$ and $\delta_t(\phi_{h,\Delta t})(t)$ defined on $[0,T)$ by

$$\phi_{h,\Delta t}(t) = \pi^h\phi(n\Delta t) \equiv \phi_h^n, \;\; \forall\, t \in [n\Delta t, (n+1)\Delta t)$$

$$\delta_t(\phi_{h,\Delta t})(t) = (\phi_h^{n+1} - \phi_h^n)/\Delta t, \;\; \forall\, t \in [n\Delta t, (n+1)\Delta t)$$

have the approximation properties

$$\begin{aligned} t = n\Delta t, \qquad \pi^h\phi(t) &\to \phi(t) \qquad \text{strongly in } H_0^1(\Omega) \\ \phi_{h,\Delta t} &\to \phi \qquad \text{strongly in } L^2(0,T;H_0^1(\Omega)) \\ \delta_t(\phi_{h,\Delta t}) &\to \phi_t \qquad \text{strongly in } L^2(0,T;H_0^1(\Omega)). \end{aligned} \tag{3.37}$$

Next we note that (3.2) can be written as

$$\begin{aligned} h(w_j^{n+1} - w_j^n) \;\; = \;\; & -\frac{\Delta t}{2}\Big((w_{j+1}^{n+1} + w_j^{n+1})u_{j+\frac{1}{2}}^n - (w_j^{n+1} + w_{j-1}^{n+1})u_{j-\frac{1}{2}}^n\Big) + \Delta t h A w_j^{n+1} \\ & + \frac{\Delta t}{2}(w_{j+1}^{n+1} - w_j^{n+1})([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-) \\ & - \frac{\Delta t}{2}(w_j^{n+1} - w_{j-1}^{n+1})([u_{j-\frac{1}{2}}^n]_+ - [u_{j-\frac{1}{2}}^n]_-). \end{aligned} \tag{3.38}$$

Multiplying (3.38) by ϕ_j^n (where $\phi \equiv \pi^h\phi(n\Delta t)$, $\phi_0^n = \phi_J^n = 0$ and $\phi_j^N = 0$) and summing

from $j = 1$ to $J - 1$ and $n = 0$ to $N - 1$ gives

$$\begin{aligned}\sum_{n=0}^{N-1}\sum_{j=1}^{J-1} h(w_j^{n+1} - w_j^n)\phi_j^n &= -\frac{\Delta t}{2}\sum_{n=0}^{N-1}\sum_{j=0}^{J-1}(w_{j+1}^{n+1} + w_j^n)u_{j+\frac{1}{2}}^{n+1}(\phi_j^n - \phi_{j+1}^n) + \Delta t A\sum_{n=0}^{N-1}\sum_{j=0}^{J-1} h w_j^{n+1}\phi_j^n \\ &\quad + \frac{\Delta t}{2}\sum_{n=0}^{N-1}\sum_{j=0}^{J-1}(w_{j+1}^{n+1} - w_j^{n+1})([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-)(\phi_{j+1}^n - \phi_j^n) \\ &= -\Delta t\sum_{n=0}^{N-1}\int_0^1 w_h^{n+1}u_h^n(\phi_h^n)_x + \Delta t A\sum_{n=0}^{N-1}\sum_{j=0}^{J-1} h w_j^{n+1}\phi_j^n \\ &\quad + \Delta t\sum_{n=0}^{N-1}\sum_{j=0}^{J-1}(w_{j+1}^{n+1} - w_j^{n+1})([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-)(\phi_{j+1}^n - \phi_j^n).\end{aligned}$$

Noting that $\phi_j^N = 0$ for all $j \in [0, J]$ gives

$$\begin{aligned}-\sum_{j=1}^{J-1} h w_j^0\phi_j^0 - \sum_{n=0}^{N-1}\sum_{j=1}^{J-1} h w_j^{n+1}(\phi_j^{n+1} - \phi_j^n) &= -\Delta t\sum_{n=0}^{N-1}\int_0^1 w_h^{n+1}(u_h^n)(\phi_h^n)_x dx + \Delta t A\sum_{n=0}^{N-1}\sum_{j=0}^{J-1} h w_j^{n+1}\phi_j^n \\ &\quad + \Delta t\sum_{n=0}^{N-1}\sum_{j=0}^{J-1}(w_{j+1}^{n+1} - w_j^{n+1})([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-)(\phi_{j+1}^n - \phi_j^n).\end{aligned}$$

Using (3.16) we have

$$\begin{aligned}&-\left(w_h^0, \phi_{h,\Delta t}(0)\right)_h - \int_0^T \left(w_{h,\Delta t}^+(t), \delta_t(\phi_{h,\Delta t})(t)\right)_h dt \\ &= -\int_0^T\int_0^1 w_{h,\Delta t}^+(t)u_{h,\Delta t}(t)(\phi_{h,\Delta t})_x(t)dxdt + A\int_0^T \left(w_{h,\Delta t}^+, \phi_{h,\Delta t}\right)_h dt \\ &\quad + \Delta t\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}(w_{j+1}^{n+1} - w_j^{n+1})([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-)(\phi_{j+1}^n - \phi_j^n). \qquad (3.39)\end{aligned}$$

From Lemma 3.3.2 and the fact that $|\phi_{j+1}^n - \phi_j^n| \le Ch$, (since $\phi \in C^\infty(\Omega_{\mathrm{T}})$) we have,

$$\Rightarrow \Delta t\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}|(w_{j+1}^{n+1} - w_j^{n+1})([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-)(\phi_{j+1}^n - \phi_j^n)|$$

$$\begin{aligned} &\leq C\Delta t h \sum_{n=0}^{N-1}\sum_{j=1}^{J-1} |w_{j+1}^{n+1} - w_j^{n+1}|([u_{j+\frac{1}{2}}^n]_+ - [u_{j+\frac{1}{2}}^n]_-) \\ &\leq C\Delta t \sum_{n=0}^{N-1} h^{\frac{1}{2}}. \end{aligned} \tag{3.40}$$

Using (3.18), (3.27), (3.35) and (3.37), we can pass to the limit as h and Δt tend to zero in the left-hand side of (3.39) to obtain

$$\int_0^1 w_0(x)\phi(x,0)dx + \int_0^T \int_0^1 w\phi_t dxdt. \tag{3.41}$$

Also using (3.25), (3.35), (3.36), (3.37) and Lemma 3.4.2 with ϕ replaced by ϕ_x, we can pass to the limit as h and Δt tend to zero in the first two terms on the right-hand side of (3.39) to obtain

$$\int_0^T \int_0^1 wu\phi_x dxdt + A\int_0^T \int_0^1 w\phi dxdt. \tag{3.42}$$

From (3.40)-(3.42) it follows that letting h and Δt tend to zero in (3.39) gives

$$\int_0^1 w_0(x)\phi(x,0)dx + \int_0^T \int_0^1 w\phi_t + w(k_d - k_m)\phi dxdt = \int_0^T \int_0^1 wu\phi_x dxdt,$$

and hence w is a solution of (3.1).

Chapter 4

Numerical Results

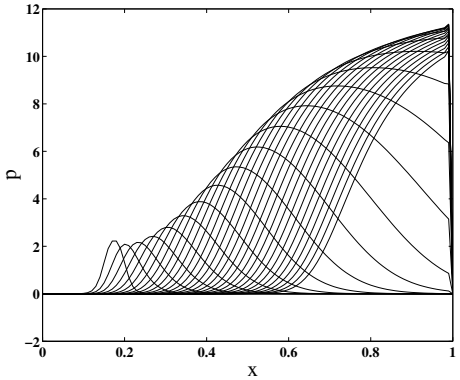
In this chapter we present some numerical results obtained from computationally solving the numerical discretization of Chapter 2 and Chapter 3. The solutions are obtained by using the coupled finite volume - finite difference method to the (2.20)-(2.23). The results of p, u and w are shown in Figures 4.1-4.8 for the different values of k_m, where we keep the value of k_d fixed, where k_m and k_d are the rates of cell growth and death respectively. Following figures shows that the cell growth is advected with the time and imposed flow u. The results for the pressure p are same as for w as it is directly proportional to w. The imposed flow advects the construct to the end of the domain; following the initial growth phase which is

$$w(x, 0) = 0.1[\text{tanh}(50(x - 0.15)) - \text{tanh}(50(x - 0.2))].$$

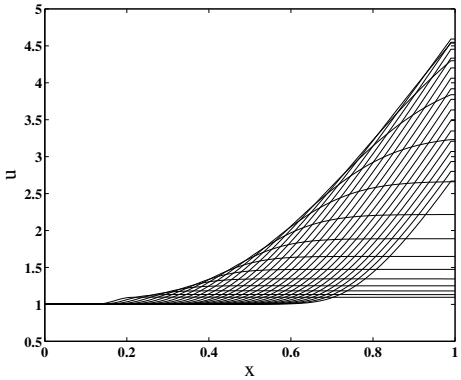
We see the comparison of cell growth phases in Figures 4.1-4.7 as t increases from 0 towards 0.9 by keeping the value of cell growth and death rate constant. Figures 4.1-4.4 shows the behavior of growth of tissue with that obtained in the case of $k_m = 10$ and $k_d = 0.1$, see [12], increases with the increase in N from 100 to 1000. In Figure 4.1(c), we see the tissue growth w towards the end domain boundary as time advance to final time. Identical behavior of pressure is seen in Figure 4.1(a) because pressure is directly proportional to tissue growth w. Velocity profile can be seen in Figure 4.1(b), which is constant prior to and after the tissue increasing approximately linearly within. We can not see the so much changes in the growth of cell w, pressure p and

velocity u as we increase the N from 100 to 200 and 500 as shown in 4.2, 4.3 respectively. But when we increase of the value of N to 1000 Figure 4.4, the cell growth increase rapidly which result in increase in pressure and velocity as well. We also computed the result by choosing the smaller and higher value of k_m, which is 1 and 25 respectively but by keeping the value of k_d same; 0.1 and $N = 1000$ as in Figure 4.5-4.7. While keeping the value of $k_m = 1.0$ we see the decrease in the pressure value; Figure 4.5(a) and in growth of tissue w; Figure 4.5(c) as time advances. The velocity also decrease with time step approach to final timestep; Figure 4.5(b). With the decrease in the value of k_m from 10 to 1.0, we notice cell growth decrease with decreasing the value of k_m; rate of cell growth. By taking value of k_m, 4 which is greater than previous value of 1.0 we see the obvious increasing growth change in the tissue growth w, pressure p where the velocity behavior is different, it start to decrease at the initial timestep but as time approaches to final timestep, we see increasing behavior, Figure 4.6(b). For choosing the highest value of $k_m = 25$, we see the big jumps in the tissue growth w, pressure p and velocity profile u with time step approach to final time. By keeping the value of k_m same but increasing the value of N from 1000 to 2000, we see solution converges more rapidly to final solution, see Figure 4.8.

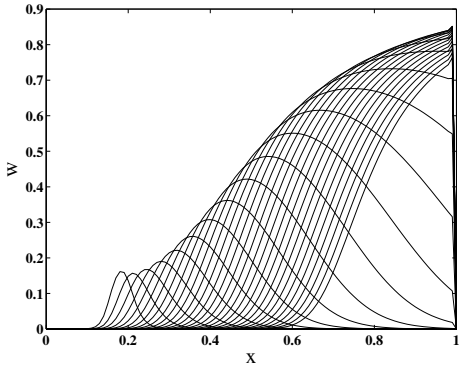
We conclude by increasing the value of N, we see the tissue undergoes the growth rapidly in term of pressure, velocity and volume, Figure 4.1-4.4 while increasing or decreasing the growth factor rate, k_m we see the big changes in the growth of tissue. Hence, the growth factor does have greater impact in the growth of tissue volume w, pressure p and velocity u.



(a)

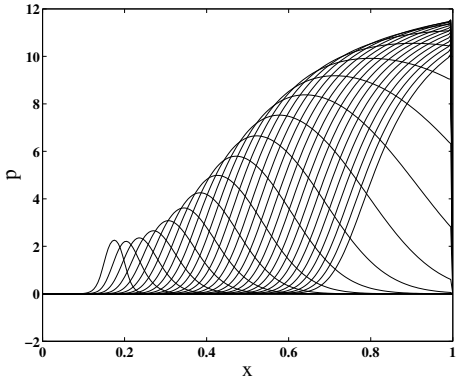


(b)

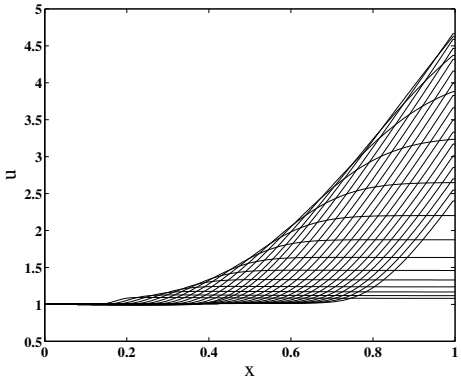


(c)

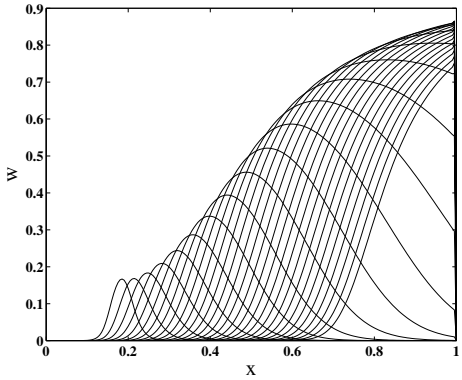
Figure 4.1: Plot of (a) pressure, p, and (b) the velocity, u, (c) the cell volume fraction, w, with $N = 100$ and for $t = 0 - 0.9$ with time steps 0.01: parameter values: $k_m = 10$, $k_d = 0.1$.



(a)

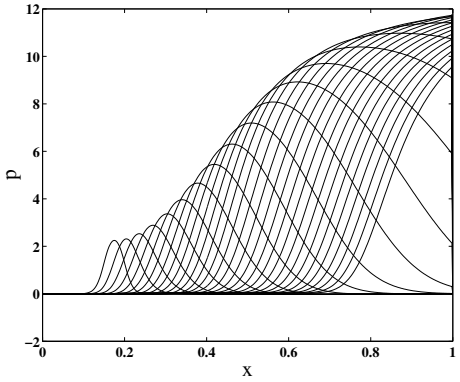


(b)

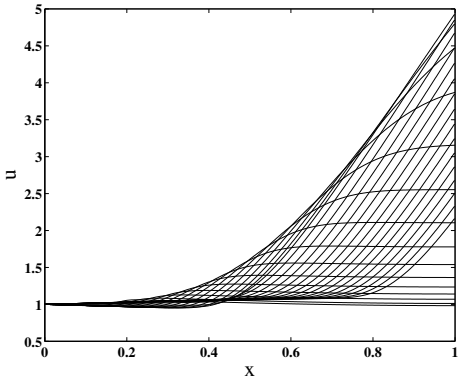


(c)

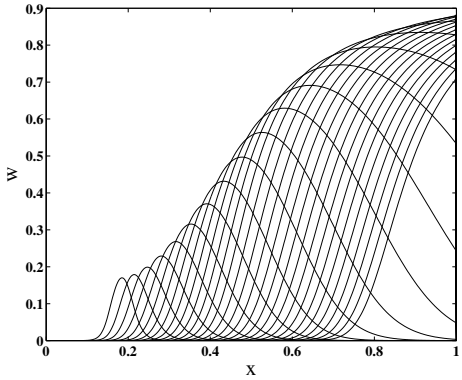
Figure 4.2: Plot of (a) pressure, p, and (b) the velocity, u, (c) the cell volume fraction, w, at t=0-0.9 with time steps 0.01: parameter values: $k_m = 10$, $k_d = 0.1$ and $N = 200$.



(a)

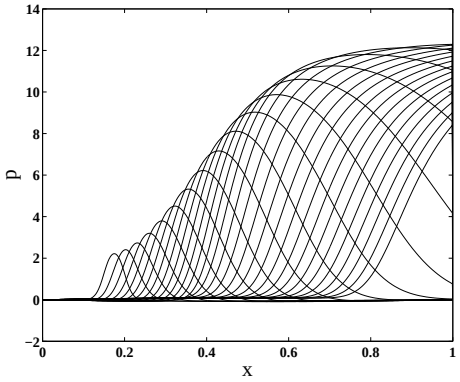


(b)

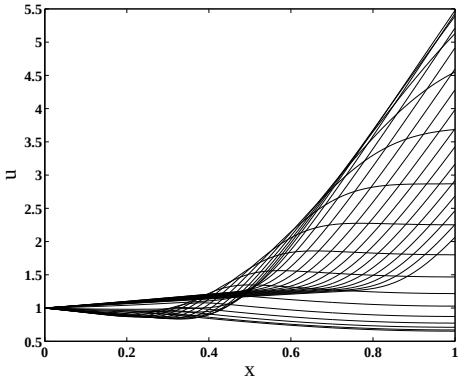


(c)

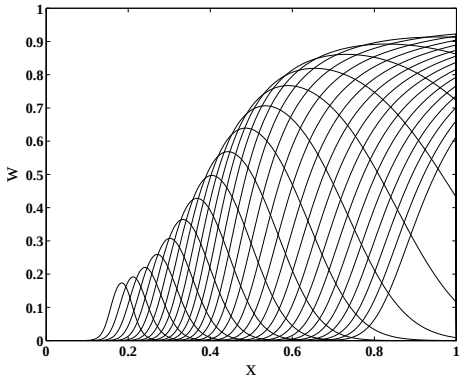
Figure 4.3: Plot of (a) pressure, p, and (b) the velocity, u, (c) the cell volume fraction, w,with $N = 500$ and at t=0-0.9 with time steps 0.01: parameter values: $k_m = 10$, $k_d = 0.1$.



(a)

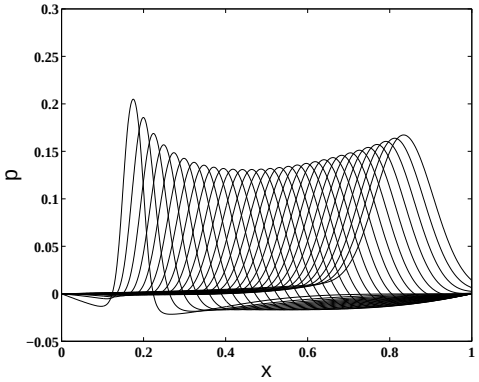


(b)

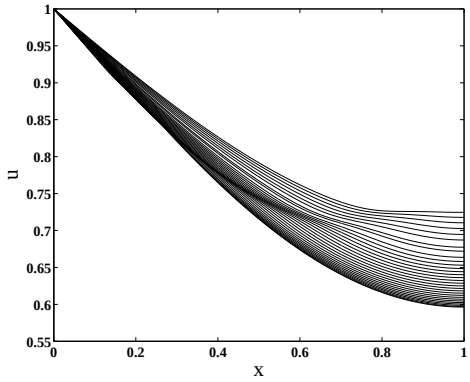


(c)

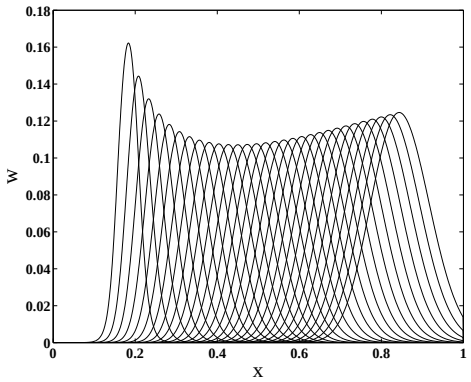
Figure 4.4: Plot of (a) pressure, p, and (b) the velocity, u, (c) the cell volume fraction, w, with $N = 1000$ and for t=0-0.9 with time steps 0.01: parameter values: $k_m = 10$, $k_d = 0.1$.



(a)

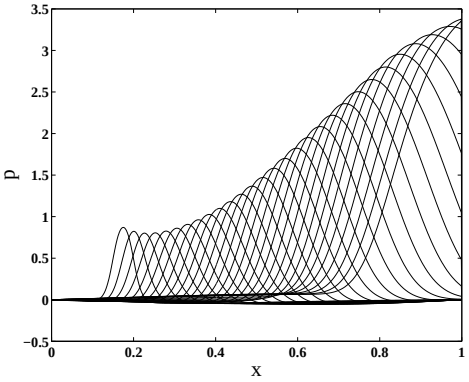


(b)

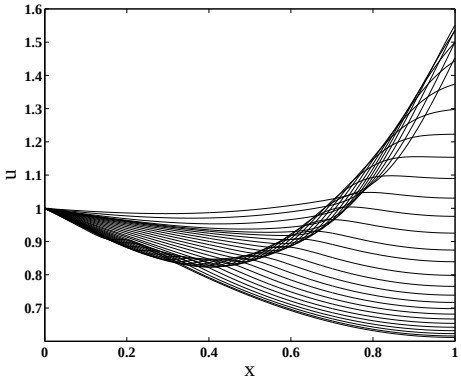


(c)

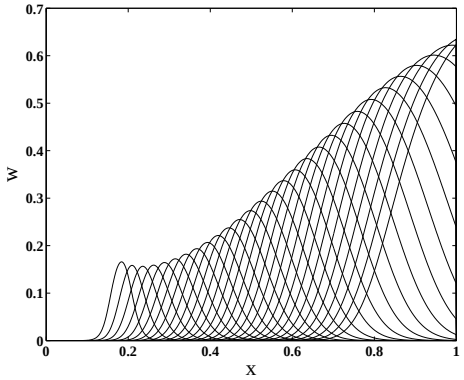
Figure 4.5: Plot of (a) pressure, p, and (b) the velocity, u, (c) the cell volume fraction, w, at t=0-0.9 with time steps 0.01: parameter values: $k_m = 1.0$, $k_d = 0.1$ where $N = 1000$.



(a)

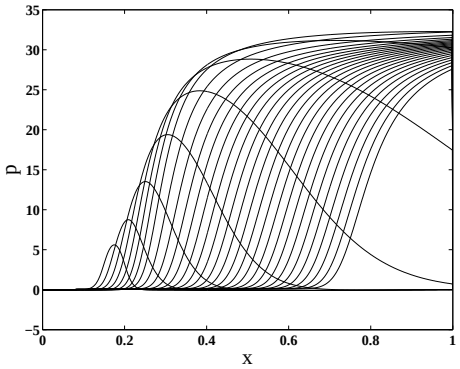


(b)

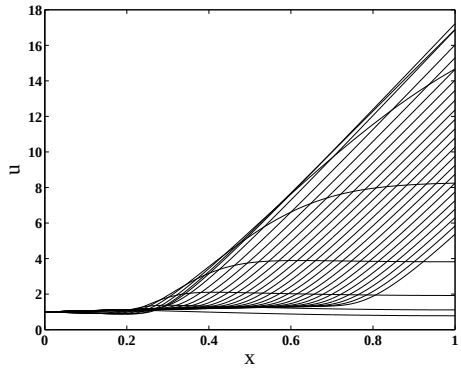


(c)

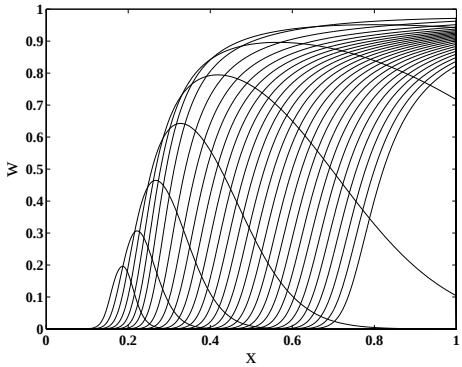
Figure 4.6: Plot of (a) pressure, p, and (b) the velocity, u, (c) the cell volume fraction, w, at t=0-0.9 with time steps 0.01: parameter values: $k_m = 4.0$, $k_d = 0.1$ and $N = 1000$.



(a)

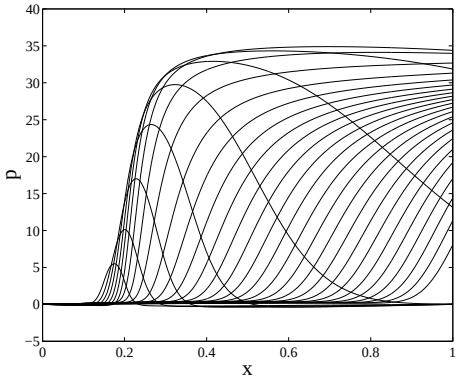


(b)

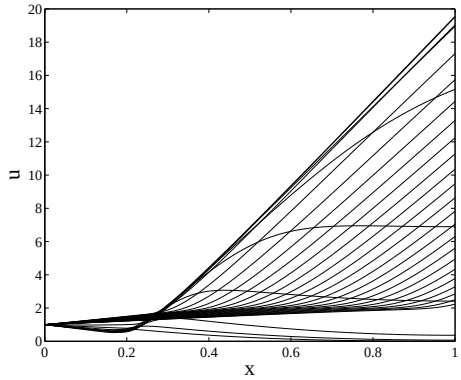


(c)

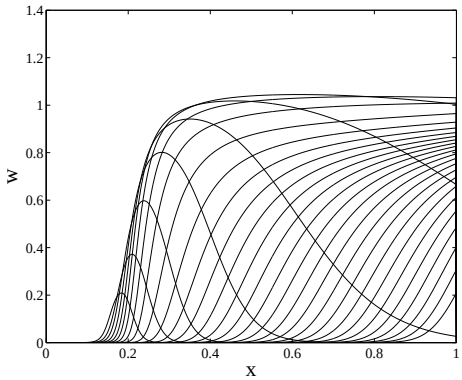
Figure 4.7: Plot of (a) pressure, p, and (b) the velocity, u, (c) the cell volume fraction, w, at t=0-0.9 with time steps 0.01: parameter values: $k_m = 25.0$, $k_d = 0.1$ where $N = 1000$.



(a)



(b)



(c)

Figure 4.8: Plot of (a) pressure, p, and (b) the velocity, u, (c) the cell volume fraction, w, at t=0-0.9 with time steps 0.01: parameter values: $k_m = 25.0$, $k_d = 0.1$ where $N = 2000$.

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