Stability Analysis of Non-Smooth Dynamical Systems with an Application to Biomechanics

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0.1 Statement

I hereby declare that this thesis has not been and will not be, submitted in whole or in part to another University for the award of any other degree.

P. Stiefenhofer

0.2 Summary

This thesis discusses a two dimensional non-smooth dynamical system described by an autonomous ordinary differential equation. The right hand side of the differential equation is assumed to be discontinuous. We provide a local theory of existence, uniqueness and exponential asymptotic stability and state a formula for the basin of attraction. Our conditions are sufficient. The theory generalizes smooth dynamical systems theory by providing contraction conditions for two nearby trajectories at a jump. Such conditions have only previously been studied for a two dimensional nonautonomous differential equation. We provide an example of the theory developed in this thesis and show that we can determine stability of a periodic orbit without explicitly calculating it. This is the main advantage of our theory. Our conditions require to define a metric. This however, can turn out to be a difficult task, and at present, we do not have a method for finding such a metric systematically. The final part of this thesis considers an application of a nonsmooth dynamical system to biomechanics. We model an elderly person stepping over an obstacle. Our model assumes stiff legs, and suggests a gait strategy to overcome an obstacle. This work is in collaboration with Professor Wagner's research group at Institute for Sport Science at the University of Münster. However, we only present work developed independently in this thesis.

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Chapter 1

Introduction

Dynamical systems theory has a long history and has been proven a powerful tool of mathematical analysis in understanding behavioral phenomena in a wide range of problems in many scientific fields including physics, engineering, biology, and economics to mention a few only. The theory itself widely relies on the system evolution being described by a smooth function. However, assuming smoothness excludes a wide range of problems that arise in practice. Dynamical systems with non-smooth evolution paths are models with a switching where a system changes between two states of nature. Such systems are called piece-wise smooth systems if each state of nature is defined by a smooth system evolution function.

This thesis considers a dynamical system described by a two dimensional autonomous ordinary differential equation with discontinuous right hand side given by

$$\dot{x} = f(x)$$

with $f \in C^1\left(\mathbb{R}^2 \backslash (x_1, 0), \mathbb{R}^2\right)$. We consider the case where f is discontinuous for $x_2 = 0$. Other conditions such as i.e. $x_2 = c$ where $c \in \mathbb{R}$ or $x_1 = 0$ could be considered without altering the theory. Such systems are introduced in Filippov [16] who also provides the conditions for existence and uniqueness of its non-smooth solution.

Such systems arise in many applied problems in mechanical and electrical engineering. A large number of applications of autonomous ordinary differential equations with discontinuous right hand side can be found in [13], [6], [29]. This list of references is indicative but not exhaustive. Giesl [19] considers an application of a nonsmooth dynamical system described by a nonautonomous differential equation to motors with dry friction. In this thesis, we will also consider an application of an ordinary differential equation with discontinuous right hand side. We consider an application in the field of biomechanics where we model an elderly person stepping over an obstacle.

The aim of this thesis is to provide sufficient conditions for the existence, uniqueness and exponential asymptitic stability of a nonsmooth periodic orbit, and for a set to belong to its basin of attraction. Such conditions have been provided by Borg [5], and generalized by Hartman and Olech [30], Leonov, Noack and Reitman [33], Giesl [18], and Giesl [17] in the case of smooth periodic orbits. Giesl and Rasmussen consider a generalization of Borg's criterion to almost period differential equations [32]. In two dimensions, such conditions have been derived by Giesl [19] who considers a periodic nonautonomous differential equation with discontinuous right hand side. However, in this case the nature of the problem is different to our since one of the dimensions is time, and time moves in forward direction. Moreover, the nature of the nonautonomous case requires no time synchronization.

We now briefly discuss Borg's criterion, which is the main condition on which our result relies. Borg [5] provides a local contraction condition. Using this contraction property, he gives sufficient conditions showing the existence and uniqueness of a limit cycle. In order to discuss this condition, we consider a differential equation

$$\dot{x} = f(x)$$

with $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. We denote the flow which maps the initial point x_0 at time $t = 0$ to the solution at time t by $S_t x_0$. For each point $x \in \mathbb{R}^n$ we define

$$\begin{aligned} L(x) &:= \max_{\|v\|=1, v \perp f(x)} L(x, v) \\ \text{where } L(x, v) &:= \langle Df(x)v, v\rangle \end{aligned}$$

and where $\langle . \rangle$ denotes the Euclidean scalar product and Df the Jacobian matrix

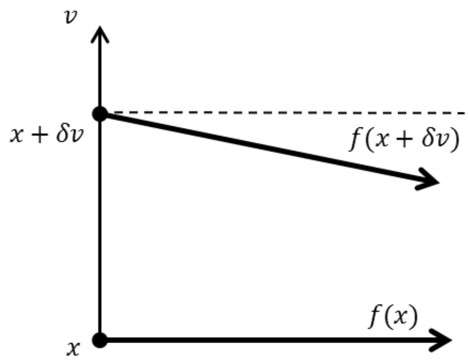

Figure 1.1: Borg's criterion

of f. Borg's criterion

$$L(x) < 0$$

states that within a neighborhood of a given point $x \in \mathbb{R}^n$ two adjacent trajectories move towards each other in forward time. Hence, the criterion is a local contraction property.

We want to show that $L(x) < 0$ is a sufficient condition for two adjacent trajectories to move towards each other. For example, consider the points $x \in \mathbb{R}^n$ and $x + \delta v \in \mathbb{R}^n$ in the phase space. Let $\delta > 0$, $v \perp f(x)$, and $\|v\| = 1$. Then in order for two adjacent trajectories through the points x and $x + \delta v$ to move towards each other it must hold that

$$\begin{aligned} 0 &> \langle f(x + \delta v), v \rangle \\ &\approx \langle f(x) + \delta Df(x)v, v \rangle \\ &= \delta \langle Df(x)v, v \rangle \text{ since } v \perp f(x). \end{aligned}$$

where

$$L(x, v) := \langle Df(x)v, v \rangle.$$

Hence, if $L(x) < 0$ then locally, two adjacent trajectories move towards each other. Borg provides the following theorem under slightly different assumptions:

Theorem 1 (Version of Borg [5]) *Let $\emptyset \neq K \subset \mathbb{R}^n$ be a compact, connected and positively invariant set which contains no equilibrium. Let $L(x) < 0$ hold for all*

$x \in K$ with

$$\begin{aligned} L(x) &:= \max_{\|v\|=1, v\perp f(x)} L(x,v) \\ \text{where } L(x,v) &:= \langle Df(x)v, v\rangle. \end{aligned}$$

Then there exists one and only one periodic orbit $\Omega \subset K$. Ω is exponentially asymptotically stable and its basin of attraction $A(\Omega)$ contains K.

This theorem is proven in Giesl [17], corollary (1.4). Borg provides a sufficient condition for existence and uniqueness of a limit cycle using above contraction property. He shows that if this condition holds within a compact set then this set belongs to the basin of attraction of a unique periodic orbit. A first generalization of Borg's criterion to unbounded sets is provided by Hartman and Olech [30]. They show existence and uniqueness of a periodic orbit for unbounded sets. They provide sufficient conditions only. A generalization of this result is given in Giesl [18] who also provides necessary conditions for a limit cycle and its basin of attraction. The theorem uses a Riemannian metric instead of the Euclidean scalar product as a metric.

The matrix valued function $M \in C^1(\mathbb{R}^n, \mathbb{R}^{n\times n})$ is called a Riemannian metric if for each $x \in K$ the matrix $M(x)$ is symmetric and positive definite, where $K \subset \mathbb{R}^n$ is a nonempty, compact, connected and positively invariant set containing no equilibria.

Theorem 2 (Giesl [18]) *Let $\emptyset \neq K \subset \mathbb{R}^n$ be a compact, connected and positively invariant set which contains no equilibrium. Moreover, let $M(x)$ be a symmetric and positive definite Riemannian metric and let $\max\limits_{x\in K} L_M(x) := -\nu < 0$, with*

$$\begin{aligned} L_M(x) &:= \max_{v^T M(x) v = 1, v^T M(x) f(x) = 0} L_M(x,v) \\ \text{where } L_M(x,v) &:= v^T \left[M(x)Df(x) + \frac{1}{2}M'(x) \right] v \end{aligned}$$

and $M'(x)$ is the orbital derivative of $M(x)$, i.e., $M'(x) = (\frac{d}{dt})M(S_t x)\mid_{t=0}$.

Then there exists one and only one periodic orbit $\Omega \subset K$. Ω is exponentially asymptotically stable and the real parts of all non trivial Floquet exponents are less than or equal to $-\nu < 0$. Moreover, the basin of attraction $A(\Omega)$ contains K.

This result generalizes Borg's criterion by considering a point-dependent scalar product based on a Riemannian metric given by a matrix valued function, $v^T M(x) w = \langle v, w \rangle$. A similar result is provided by Stenström [39] who also considers a Riemannian metric but on a Riemannian manifold. Moreover he provides a definition of L_M which holds also for equilibria and shows that the set K is isomorphic to the torus. The $\mathbb{R}^n$ space is considered as a special case with a Riemannian metric given by a constant matrix. Moreover Stenström [39] shows the contraction property by proof by contradiction and cannot prove the exponential stability of a periodic orbit. Leonov et al provides sufficient conditions of exponential stability of a periodic orbit in a series of papers summarized in the book [38]. They introduce the idea of time synchronization between two adjacent solutions in [31]. In that paper they synchronize the time of two adjacent trajectories $S_t x$ and $S_t q$ perpendicular to $f(S_t x)$ with respect to the Euclidean metric [31]. Besides Borg's criterion, this is another relevant tool of analysis of our theory to be developed in the main chapter of this thesis.

Smooth dynamical systems theory provides a number of approaches dealing with the existence of a periodic orbit. Such are perturbation theory and averaging methods (cf. [25], [22], [41]) or the Poincaré-Bendixon theory for two dimensional systems. The Bendixon criterion of nonexistence of a periodic orbit can be used as a useful tool to show uniqueness [15].

Smooth dynamical systems theory also provides an established theory of stability of periodic orbits. Classical stability theories include linearization, Floquet theory, and Lyapunov theory. Results regarding linearization around the periodic orbit are discussed in (cf. [2], [25], [42]). Floquet theory requires to show that all nontrivial real parts of the Floquet exponents are strictly negative, except for the trivial one. This can be shown via Poincaré map (cf. [23]). A disadvantage of this theory, however, is that we need to determine the periodic orbit explicitly in order to apply it directly, which might be difficult in practice. Further criteria regarding linearizations around a periodic orbit are given in (cf. [14] and [15]) which are results for two dimensional systems. Stability criteria using Lyapunov functions are discussed in ([10], [26], [43]) for example. Moreover, we can also use Lyapunov functions in order to determine the basin of attraction of a periodic orbit [40].

1.1 Motivation

The aim of this thesis is to provide a theory of existence, uniqueness and exponentially asymptotically stability of a non-smooth dynamical system defined by an autonomous ordinary differential equation

$$\dot{x} = f(x)$$

where f is a discontinuous function at $x_2 = 0$ and $x \in \mathbb{R}^2$. With $f := f^{\pm}$ we have

$$\dot{x} = f^{\pm}(x) = \begin{cases} f^{+}(x) & \text{if } x_2 > 0 \\ f^{-}(x) & \text{if } x_2 < 0. \end{cases} \tag{1.1}$$

Our theory relies on results of smooth dynamical systems theory such as Borg's criterion [5] and a time synchronization of adjacent solutions [31] and concepts developed in Giesl [18], and [17]. Moreover, a stability theory for nonsmooth periodic differential equations of the form

$$\dot{x} = f(t, x)$$

where f is a discontinuous function at $x = 0$ and $x \in \mathbb{R}$, and periodic in time, i.e. $f(t, x) = f(t + T, x)$ with constant minimal periodic time $T > 0$ already exists [19]. With $f := f^{\pm}$ the system of equations is given by

$$\dot{x} = f^{\pm}(t, x) = \begin{cases} f^{+}(t, x) & \text{if } x > 0 \\ f^{-}(t, x) & \text{if } x < 0. \end{cases} \tag{1.2}$$

Conditions: Consider $\dot{x} = f(t, x)$, where $f \in C^1(\mathbb{R} \times (\mathbb{R} \backslash 0), \mathbb{R})$ and $f(t, x) = f(t + T, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. Each of the function $f^{\pm}$ can be extended continuously up to $x = 0$. Let the same hold for their derivatives $f_x^{\pm}(t, x)$. Moreover let $f^{+}(t, 0) - f^{-}(t, 0)$ be a C^1- function with respect to time t, and assume that for all $t \in [0, T]$ at least one of the inequalities holds: $f^{+}(t, 0) < 0$ or $f^{-}(t, 0) > 0$.

Giesl [19] proves the following theorem:

Theorem 3 (Giesl [19]) *Let the conditions hold. Assume that $W^{\pm} : \mathbb{R} \to \mathbb{R}_0^{\pm}$ are differentiable functions with $W^{\pm}(t + T, x) = W^{\pm}(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}_0^{\pm}$. Let*

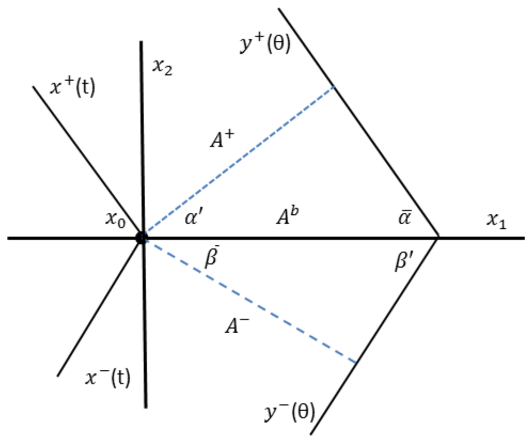

Figure 1.2: Main idea

its orbital derivative W' exist, and be continuously extendable up to $x = 0$. Let K be a nonempty, connected, positively invariant and compact set, such that the following conditions hold with constants $\nu, \epsilon > 0$

- $f_x(t, x) + W'(t, x) \leq -\nu < 0$ *for all* $(t, x) \in K$ *with* $x \neq 0$.
- $\frac{f^-(t,0)}{f^+(t,0)} e^{W^-(t,0) - W^+(t,0)} \leq e^{-\epsilon} < 1$ *for all* $(t, x) \in K$ *with* $f^-(t, 0) < 0$
- $\frac{f^+(t,0)}{f^-(t,0)} e^{W^+(t,0) - W^-(t,0)} \leq e^{-\epsilon} < 1$ *for all* $(t, x) \in K$ *with* $f^+(t, 0) > 0$

Then there is one and only one periodic orbit Ω with period T in K. Ω is exponentially asymptotically stable with exponent $-\nu$ and for its basin of attraction we have the inclusion $K \subset A(\Omega)$.

We prove a similar result. However, our context is a dynamical system described by equation (1.1) instead of the dynamical system described by equation (1.2). The result, theorem 5 is stated in chapter two where a proof is also provided. However, we already provide some intuition of the main result here.

Here we wish to provide some intuition for the jumping conditions only. Jumps occur at a point x_0 where one of the solutions $x(t)$ or $y(t)$ hit the $x_2 = 0$ axis. See figure 1.2. We consider an equation of the form (1.1) with constant coefficients. Consider the situation depicted in figure 1.2, where $\alpha', \bar{\alpha}, \beta', \bar{\beta}$ are given angles. We want to show contraction of the distance A^+ and A^-, i.e.

$A^- < A^+$, where A defines the distance between two adjacent solutions $x^{\pm}(t)$ and $y^{\pm}(\theta)$ over the jump where t and θ is the time of the corresponding solution. A^b is the distance between the two solutions when $x_2, y_2 = 0$. Geometrically, we have

$$\begin{aligned} \frac{A^+}{A^b} &= \sin \bar{\alpha} \\ \frac{A^-}{A^b} &= \sin \beta'. \end{aligned}$$

Hence we need to show that

$$\frac{A^-}{A^+} = \frac{\sin \beta'}{\sin \bar{\alpha}} < 1,$$

which in terms of components of $f^{\pm}$ (see figure 1.3) and a weight function $e^{W^{\pm}}$ with

$$\begin{aligned} A^+ &:= \parallel y^+(\theta_0) - x_0 \parallel \cdot e^{W^+(x_0)} \\ A^- &:= \parallel y^-(\theta_0) - x_0 \parallel \cdot e^{W^-(x_0)} \end{aligned}$$

and $(y^{\pm}(\theta_0) - x_0) \perp f^{\pm}(x_0)$ yields

$$\frac{A^-}{A^+} = \frac{\frac{f_2^-(x_0)}{\sqrt{(f_1^-(x_0))^2 + (f_2^-(x_0))^2}}}{\frac{f_2^+(x_0)}{\sqrt{(f_1^+(x_0))^2 + (f_2^+(x_0))^2}}} e^{W^-(x_0) - W^+(x_0)} < 1, \tag{1.3}$$

where W is a function to be found which is context specific. The condition (1.3) implies that two adjacent solutions contract over the jump interval. From figure 1.3 we observe that both, $f_2^+ < 0$ and $f_2^- > 0$ must be satisfied for a jump to occur in $+/-$ direction. These conditions are weaker in theorem 3 which allows for sliding motion. Intuition for the contraction of two nearby trajectories in the smooth case has already been provided in the section on a brief discussion of Borg's criterion.

The next chapter formalizes the above idea and presents a complete proof. Chapter three discusses an example of a dynamical system described by equation (1.1). In part one of chapter three, we calculate its non-smooth periodic

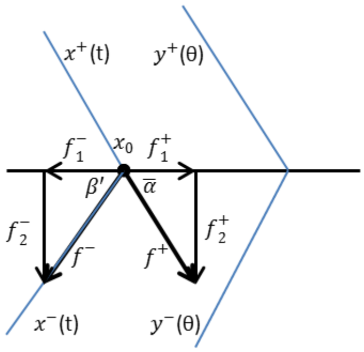

Figure 1.3: Vector components

orbit explicitly and show its stability by application of classic dynamical systems theory. In part two of the same chapter we consider an application of our theory developed in chapter two. We show the stability of a periodic orbit without actually calculating its explicit solution. This is the main advantage of our theorem. Chapter four is an application of a dynamical system described by equation (1.1) to biomechanics. We model an elderly person stepping over an obstacle and provide a strategy of how to do so in a robust way. This work is in collaboration with the research group at the Institute of Sport and Exercise Sciences of the University of Münster lead by Professor Wagner. The final chapter is a conclusion and an outlook for future work.

Chapter 2

Existence, Uniqueness, and Stability of a Non-Smooth Periodic Orbit

2.1 Definitions and notation

We consider a nonsmooth dynamical system defined by an autonomous ordinary differential equation

$$\dot{x} = f(x)$$

where f is a discontinuous function at $x_2 = 0$ and $x \in \mathbb{R}^2$. The discontinuity of $f \in C^1(\mathbb{R}^2 \backslash (\mathbb{R} \times \{0\}), \mathbb{R}^2)$ implies that the phase space $\mathbb{X} = \mathbb{R}^2$ is divided into subspaces $\mathbb{X} = \mathbb{X}^+ \cup \mathbb{X}^0 \cup \mathbb{X}^-$, where $\mathbb{X}^+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$, $\mathbb{X}^- = \{x \in \mathbb{R}^2 : x_2 < 0\}$, and $\mathbb{X}^0 = \{x \in \mathbb{R}^2 : x_2 = 0\}$. By defining $f := f^{\pm}$ where $f(x) = f^+(x)$ if $x \in \mathbb{X}^+$, and $f(x) = f^-(x)$ if $x \in \mathbb{X}^-$, we have

$$\dot{x} = f(x) = \begin{cases} f^+(x) & \text{if } x \in \mathbb{X}^+ \\ f^-(x) & \text{if } x \in \mathbb{X}^-. \end{cases} \tag{2.1}$$

We restrict ourselves to a set of assumptions which according to a sequence of results by Filippov [16] guarantees global existence, uniqueness, and continuous dependence on the initial condition of solutions of the differential equation (2.1). Let the flow of the system given by (2.1) be defined by $S_t(x_0) := (x_1(t), x_2(t)) \in \mathbb{X}$, where $(x_1(t), x_2(t)) \in \mathbb{X}$ is the solution of (2.1) with initial value $((x_1(0), x_2(0)) = x_0$. Hence the flow $S_t x_0$ maps the initial point x_0 at time $t = 0$ to a point $x(t)$ at time $t \geq 0$. And adjacent trajectory is defined by $S_t(x_0 + \eta) := (y_1(t), y_2(t)) \in \mathbb{X}$, where $(y_1(t), y_2(t)) \in \mathbb{X}$, where $\|\eta\| > 0$.

Assumption 1 *Consider equation (2.1). We assume*

- $f \in C^1(\mathbb{R}^2 \backslash (\mathbb{R} \times \{0\}), \mathbb{R}^2)$.
- *Each function* $f^{\pm}(x)$ *with* $x \in \mathbb{X}^+$ *or* $x \in \mathbb{X}^-$ *can be extended to a continuous function up to* $x \in \mathbb{X}^0$.
- *Each function* $Df^{\pm}(x)$ *with* $x \in \mathbb{X}^+$ *or* $x \in \mathbb{X}^-$ *can be extended to a continuous function up to* $x \in \mathbb{X}^0$.
- $f^+(x_1, 0) - f^-(x_1, 0)$ *is a* C^1*-function with respect to* x_1.
- *For all* $(x_1, 0) \in \mathbb{X}^0$ *it holds that* $f_2^+(x_1, 0) \cdot f_2^-(x_1, 0) > 0$.

The assumption $f_2^+(x_1, 0) \cdot f_2^-(x_1, 0) > 0$ for all $(x_1, 0) \in \mathbb{X}_0$ states that

$$\begin{aligned} f_2^+(x_1, 0), f_2^-(x_1, 0) &< 0 \text{ or} \\ f_2^+(x_1, 0), f_2^-(x_1, 0) &> 0. \end{aligned}$$

This assumption excludes all sliding phenomena and shall be relaxed in future work. It implies a jump in $+/-$ direction if both $f_2^+, f_2^- < 0$ or a jump in $-/+$ direction if both $f_2^+, f_2^- > 0$.

Let the assumption 1 hold. Let $\mathcal{P}$ denote the power set. Then we can define a set valued function

$$F : \mathbb{R}^2 \to \mathcal{P}(\mathbb{R}^2)$$

by

$$\begin{aligned} F(x_1, x_2) &:= \{f(x_1, x_2)\} \text{ for } x_2 \neq 0 \\ F(x_1, 0) &:= \{\alpha f^+(x_1, 0) + (1 - \alpha) f^-(x_1, 0) : \alpha \in [0, 1]\} \text{ for } x_2 = 0. \end{aligned}$$

According to Filippov [16], a solution of equation (2.1) is defined to be an absolutely continuous function $x : [a, b] \to \mathbb{R}^2$ which satisfies the differential inclusion

$$\dot{x}(t) \in F(x(t))$$

for almost all $t \in [a, b]$. $x^{\pm}(t)$ is a solution of the smooth differential inclusion (2.1) if $x_2 > 0$ or $x_2 < 0$. The next theorem is provided by Filippov with proofs given in section 7 Theorem 1 and Theorem 2, section 8 Theorem 1, and section 10 Theorem 2 [16]. Existence of a solution follows from the fact that F is non-empty, closed, convex, and bounded for each $x \in \mathbb{X}$ and uniqueness requires that for all $(x_1, 0) \in \mathbb{X}_0$ either the strict inequalities $f_2^+(x_1, 0) > 0$ and $f_2^-(x_1, 0) > 0$ or the strict inequalities $f_2^+(x_1, 0) < 0$ and $f_2^-(x_1, 0) < 0$ hold.

Theorem 4 (Filippov [16]) *Let the conditions 1 hold. Then for each initial condition $x_0 \in \mathbb{R}^2$ there exists a solution to the initial value problem*

$$\begin{aligned} \dot{x}(t) &\in F(x(t)) \textit{ with initial condition} \\ x(t_0) &= x_0 \end{aligned}$$

for almost all t on an interval $t \in [t_0, t_0 + b)$ with $b > 0$. The solution can be extended up to a maximal interval $[t_0, t_0 + b)$ and if $b < \infty$ then $\| x(t) \| \to \infty$ for $t \to t_0 + b$. Moreover, the solution is unique and depends continuously on the initial value (t_0, x_0).

The aim of this chapter is to provide sufficient conditions for existence and uniqueness of an exponentially asymptotically stable periodic orbit. Moreover, these conditions imply that a subset of the phase space belongs to the basin of attraction of the periodic orbit. To show this requires introducing the conditions of orbital and exponentially asymptotic stability of a solution of equation (2.1). We first provide some definitions before stating the main result.

Definition 1 (Positively invariant set) *K is positively invariant if $S_t x_0 \in K$ for all $t \geq 0$ with $x_0 \in K$.*

Definition 2 *Let $K \subset \mathbb{R}^2$ and $K \neq \varnothing$ be a compact, connected and positively invariant set which contains no equilibria. Moreover, set*

1. *$K^+ := K \cap \{x \in \mathbb{R}^2 : x_2 > 0\}$*
2. *$K^- := K \cap \{x \in \mathbb{R}^2 : x_2 < 0\}$.*

Definition 3 (Flows)

- $S_t x_0 =: S_t^+ x_0$ *with* $x_0 \in K$ *if* $S_t x_0 \subset K^+$ *for all* $t \in (t_{j-1}^+, t_j^-)$*, where* $j \in \{2n : n \in \mathbb{N}_0\}$ *is an index.* $S_\theta(x_0 + \eta) =: S_\theta^+(x_0 + \eta)$ *with* $(x_0 + \eta) \in K$ *if* $S_\theta(x_0 + \eta) \subset K^+$ *for all* $t \in (t_{j-1}^+, t_j^-)$ *and* $\theta \in (\theta_{j-1}^+, \theta_j^-)$*.*
- $S_t x_0 =: S_t^- x_0$ *with* $x_0 \in K$ *if* $S_t x_0 \subset K^-$ *for all* $t \in (t_{j-1}^+, t_j^-)$*.* $S_\theta(x_0 + \eta) =: S_\theta^-(x_0 + \eta)$ *with* $(x_0 + \eta) \in K$ *if* $S_\theta(x_0 + \eta) \subset K^+$ *for all* $t \in (t_{j-1}^+, t_j^-)$ *and* $\theta \in (\theta_{j-1}^+, \theta_j^-)$ *and index* $j \in \{2n - 1 : n \in \mathbb{N}_0\}$*.*
- $S_t^+ x_0$ *switches to* $S_t^- x_0$ *with* $x_0 \in K$ *when either* $S_t^+ x_0 \in K^0$ *or* $S_t^+(x_0 + \eta) \in K^0$ *with* $t \in [t_{j-1}^-, t_{j-1}^+]$ *and* $j \in \{2n : n \in \mathbb{N}_0\}$*.*
- $S_t^- x_0$ *switches to* $S_t^+ x_0$ *with* $x_0 \in K$ *when either* $S_t^- x_0 \in K^0$ *or* $S_t^-(x_0 + \eta) \in K^0$ *with* $t \in [t_{j-1}^-, t_{j-1}^+]$ *and index* $j \in \{2n - 1 : n \in \mathbb{N}_0\}$*.*

Definition 4 (Periodic orbit) *A periodic orbit* Ω *of the system (2.1) is a set defined by*

$$\Omega := \{S_t(x_0) : t \in [0, T],\ \textit{such that}\ S_T(x_0) = x_0\} \subset \mathbb{X},\ \textit{with minimal period}\ T > 0.$$

We now discuss the concept of stability of a periodic orbit. A periodic orbit Ω is called orbitally stable if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all points $z \in \mathbb{X}$ we have that

$$dist(z, \Omega) \leq \delta \Rightarrow dist(S_t(z), \Omega) \leq \epsilon$$

for all $t \geq 0$, where the distance $dist(.,.)$ measures the distance between a point z and a compact set $\Omega \subset \mathbb{R}^2$,

$$dist(z, \Omega) := \min_{w \in \Omega} \|z - w\|.$$

We say that a periodic orbit Ω is exponentially asymptotic stable if there are constants $\nu, \iota > 0$ such that for all points $z \in \mathbb{X}$ we have that

$$dist(z, \Omega) \leq \iota \Rightarrow dist(S_t z, \Omega) e^{-\nu t} \xrightarrow{t \to \infty} 0.$$

While orbital stability says that orbits in a neighborhood of Ω remain close under under the flow, the concept of exponential asymptotic stability provides a condition such that orbits in a neighborhood of $A(\Omega)$ converge to Ω.

We now define a neighborhood $A(\Omega)$ of Ω consisting of a set of points x_0 in $\mathbb{X}$ such that the distance between $S_t(x_0)$ and Ω vanishes as $t \to \infty$. The set $A(\Omega)$ is called the basin of attraction of a periodic orbit Ω.

Definition 5 (Basin of attraction) *The basin of attraction $A(\Omega)$ of an exponentially asymptotically stable orbit Ω of (2.1) is the set defined by*

$$A(\Omega) := \{x_0 \in \mathbb{X} : dist(S_t x_0, \Omega) \xrightarrow{t\to\infty} 0\}.$$

2.2 Main result

We prove the following main result. Let $\max\limits_{x \in K^\pm} L_{W^\pm}(x) := -\nu < 0$.

Theorem 5 (Main Result) *Let assumption 1 hold, and let $\emptyset \neq K \subset \mathbb{R}^2$ be a compact, connected and positively invariant set with $f^\pm(x) \neq 0$ for all $x \in K^\pm$. Moreover, assume that $W^\pm : \mathbb{X}^\pm \to \mathbb{R}$ are continuous functions and let the orbital derivatives $(W^\pm)'$ exist and be continuous functions in $\mathbb{X}^\pm$ and continuously extendable up to $\mathbb{X}_0$. We set $K^0 := \{x \in\ K : x_2 = 0\}$. Let the following conditions hold:*

1. $L_{W^+(x)} := \max\limits_{\|v^+\| = e^{-W^+(x)}, v^+ \perp f^+(x)} L_{W^+}(x, v^+) \le -\nu < 0$
   $L_{W^+}(x, v^+) := e^{2W^+(x)} \left\{ (v^+)^T \left[Df^+(x) \right] v^+ + \langle \nabla W^+(x), f^+(x) \rangle \|v^+\|^2 \right\}$
   for all $x \in K^+$.

2. $L_{W^-(x)} := \max\limits_{\|v^-\| = e^{-W^-(x)}, v^- \perp f^-(x)} L_{W^-}(x, v^-) \le -\nu < 0$
   $L_{W^-}(x, v^-) := e^{2W^-(x)} \left\{ (v^-)^T \left[Df^-(x) \right] v^- + \langle \nabla W^-(x), f^-(x) \rangle \|v^-\|^2 \right\}$
   for all $x \in K^-$.

3. $\frac{f_2^-(x)}{f_2^+(x)} \cdot \frac{\sqrt{\left(f_1^+(x)\right)^2 + \left(f_2^+(x)\right)^2}}{\sqrt{\left(f_1^-(x)\right)^2 + \left(f_2^-(x)\right)^2}} e^{W^-(x) - W^+(x)} < 1$
   for all $x \in K^0$ with $f_2^+(x) < 0$, $f_2^-(x) < 0$.

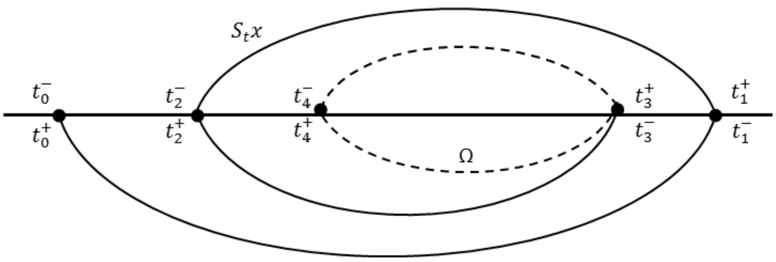

Figure 2.1: Time index orbit

4. $\frac{f_2^+(x)}{f_2^-(x)} \cdot \frac{\sqrt{\left(f_1^-(x)\right)^2+\left(f_2^-(x)\right)^2}}{\sqrt{\left(f_1^-(x)\right)^2+\left(f_2^-(x)\right)^2}} e^{W^+(x)-W^-(x)} < 1$
for all $x \in K^0$ with $f_2^+(x) > 0$, $f_2^-(x) > 0$.

Then there is one and only one periodic orbit $\Omega \subset K$. Moreover, Ω is exponentially asymptotically stable with exponent $-\nu < 0$ and for its basin of attraction the inclusion $K \subset A(\Omega)$ holds.

The aim of this chapter is to prove theorem 5. The main idea of the proof is to consider long run (LR) and short run (SR) cases separatively. The long run is a case which is associated with time structures where neither of the solutions changes phase space $\mathbb{X}^+$ or $\mathbb{X}^-$. During this time segment, solutions are smooth and we show that the distance between two adjacent smooth solutions decreases. This part relies on results from smooth dynamical systems theory and theorems provided by Giesl [18]. Short run cases are cases associated with short time intervals where the right hand side of the differential equation (2.1) is discontinuous. Hence short run time intervals are associated with jumps. We show that in the short run, the distance between to adjacent solutions decreases. We show this by comparing the distance between solutions when the first of the solutions enters $\mathbb{X}^0$ and the distance between solutions when the last solution leaves $\mathbb{X}^0$. Moreover, we show that if one solution changes sign the other also changes sign within a short time interval. The time between the two adjacent solutions is synchronized. Once contracting conditions are shown in each case, i.e., jumps in $+/-$ and $-/+$ direction, we patch the solutions together and show global contraction of adjacent solutions.

Next we formalize LR and SR time structures.

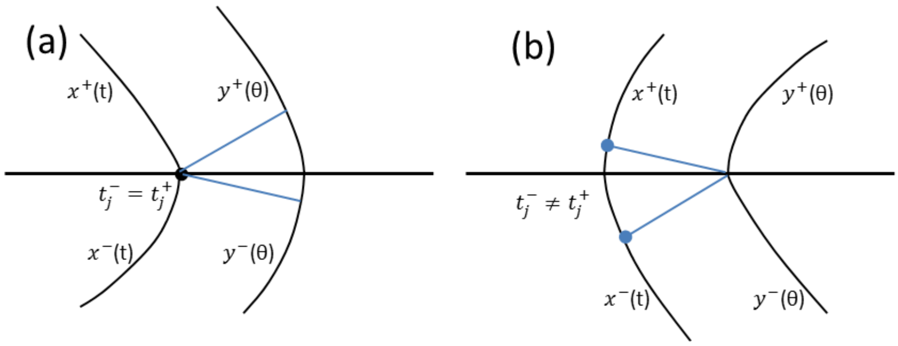

Figure 2.2: Time index at jump

Definition 6 (Long run time index of $S_t x_0, x_0 \in K$)

$$\mathcal{G}^- := \left\{ \cup_j (t_{j-1}^+, t_j^-) : j \in \{2n-1 : n \in \mathbb{N}_0\} \right\}$$

$$\mathcal{G}^+ := \left\{ \cup_j (t_{j-1}^+, t_j^-) : j \in \{2n : n \in \mathbb{N}_0\} \right\}$$

For all $t \in \mathcal{G}^\pm$ with $x_0 \in K$ solutions $S_t^\pm x_0 \in \mathbb{X}^\pm$ are smooth.

Definition 7 (Short run time index of $S_t x_0, x_0 \in K$)

$$\mathcal{J}^\mp := \left\{ \cup_j [t_j^-, t_j^+] : t_j^- \neq t_j^+, j \in \{2n-1 : n \in \mathbb{N}_0\} \textit{ and } n \in \mathbb{N}_0 \right\}$$

$$\mathcal{J}^\pm := \left\{ \cup_j [t_j^-, t_j^+] : t_j^- \neq t_j^+, j \in \{2n : n \in \mathbb{N}_0\} \textit{ and } n \in \mathbb{N}_0 \right\}$$

For all $t \in \mathcal{J}^\pm$ with $x_0 \in K$ a solution $S_t^+ x_0 \in \mathbb{X}^+$ jumps to $S_t^- x_0 \in \mathbb{X}^-$ when $S_t x_0 \in \mathbb{X}^0$. For all $t \in \mathcal{J}^\mp$ with $x_0 \in K$ a solution $S_t^- x_0 \in \mathbb{X}^-$ jumps to $S_t^+ x_0 \in \mathbb{X}^+$ when $S_t x_0 \in \mathbb{X}^0$.

$$\mathcal{I}^\mp := \left\{ \cup_j [t_j^-, t_j^+] : t_j^- = t_j^+, j \in \{2n-1\} \textit{ and } n \in \mathbb{N}_0 \right\}$$

$$\mathcal{I}^\pm := \left\{ \cup_j [t_j^-, t_j^+] : t_j^- = t_j^+, j \in \{2n\} \textit{ and } n \in \mathbb{N}_0 \right\}$$

For all $t \in \mathcal{I}^\pm$ solutions S_t^+ jump to S_t^-. For all $t \in \mathcal{I}^\mp$ solutions S_t^- jump to S_t^+.

Short run intervals are time intervals $t \in \mathcal{J}^\pm$ or $t \in \mathcal{I}^\pm$ where equation (2.1) is associated with a righthand side discontinuity. We observe that $\mathcal{I}$ structures are time structures at which time is frozen at $t_j^- = t_j^+$. $\mathcal{J}$structures are time structure where an adjacent solution is frozen. Hence, we next consider the time structure of an adjacent solution.

It remains to formalize the time structure of an adjacent solution of $S_t^+ x_0$ denoted by $S_\theta^+(x_0 + \eta)$ with initial condition $\|\eta\| \leq \frac{\delta}{2}$. We want to synchronize the time of two nearby solutions such that $\left(S^+_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S_t^+ x\right)^T f^+(S_t^+ x) = 0$ holds. This requires to define a multi valued mapping $\mathcal{T}$ where $\mathcal{T} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\theta = \mathcal{T}(t)$. We show that such a mapping exists and state its properties. We now define the complete time structure.

Definition 8 (Synchronized (LR,SR) time)

$$\theta_j = \mathcal{T}(t) := \begin{cases} \mathcal{T}(t) & \textit{if } t \in \mathcal{G}^+ \\ \mathcal{T}(t_j^+) & \textit{if } t \in \mathcal{J}^\pm \\ \left(\mathcal{T}(t), \lim\limits_{t \to t_j^+} \mathcal{T}(t)\right) & \textit{if } t \in \mathcal{I}^\pm \end{cases} \tag{2.2}$$

for all $j \in \{2n : n \in \mathbb{N}_0\}$ and $n \in \mathbb{N}_0$. For the opposite jump direction we define θ_j similarly.

$$\theta_j = \mathcal{T}(t) := \begin{cases} \mathcal{T}(t) & \textit{if } t \in \mathcal{G}^- \\ \mathcal{T}(t_j^+) & \textit{if } t \in \mathcal{J}^\mp \\ \left(\mathcal{T}(t), \lim\limits_{t \to t_j^+} \mathcal{T}(t)\right) & \textit{if } t \in \mathcal{I}^\mp \end{cases} \tag{2.3}$$

for all $j \in \{2n-1 : n \in \mathbb{N}_0\}$ and $n \in \mathbb{N}_0$.

The pairs $(t, \theta) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ determine the complete time structure of two adjacent trajectories. Figure 2.3 (a) illustrates the time structure in the smooth case. In this scenario both solutions, $x^+(t)$ and $y^+(\theta)$ enter the axis $x_2 = 0$ simultaneously. Moreover, $x^-(t)$ and $y^-(\theta)$ leave it simultaneously. Figure 2.3 depicts a situation where $x^+(t)$ enters the axis x_2 first and $y^-(\theta)$ leaves it last or vice versa.

In figure 2.4 (a) we show a situation where $y^+(\theta)$ enters the $x_2 = 0$ axis first and $y^-(\theta)$ leaves it last. The remaining picture in figure 2.4 (b) shows the opposite, where $x^+(t)$ enters the $x_2 = 0$ axis first and $x^-(t)$ leaves it last.

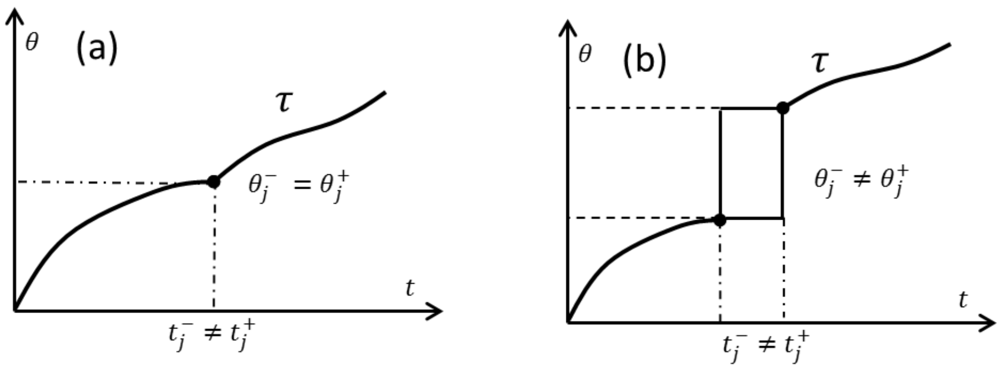

Figure 2.3: Time structure 1

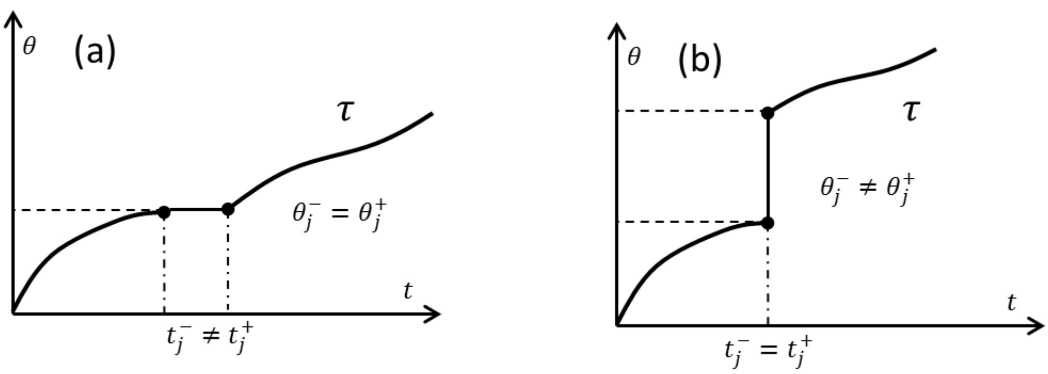

Figure 2.4: Time structure 2

The proof of theorem 5 proceeds according to the main steps outlined below. The idea of the proof is to first consider each long run and short run case independently before patching all cases together. Once the contraction property is shown in each individual case ω-limit set properties are established, before stability and the basin of attraction are studied.

1. For each long run case (LR) and short run case (SR) we define a time-dependent distance function between two solutions. We then show that for two nearby points x and $x+\eta$ such that $(x+\eta) \perp f^{\pm}(x)$ with $x \in K$ the distance between solutions decreases exponentially, i.e. $\|S_t x - S_{\mathcal{T}(t)}(x+\eta)\| e^{W(S_t x)} \to 0$ as $(t) \to \infty$. (Section 2.3 and 2.4)

2. We show for all cases that for two nearby points x and $x+\eta$ in K that their ω-limit set is the same. (Section 2.5)

3. We show that in all cases the ω-limit is the same for all points in a neighborhood. (Section 2.5)

4. We show that the ω-limit set is the same for all points in K. (Section 2.6)

5. We show that this ω-limit set is an exponentially asymptotically stable orbit Ω. (Section 2.7)

6. We show that the rate of the exponent ν decreases. (Section 2.6)

7. We show that the basin of attraction $A(\Omega)$ contains the set K. (Section 2.7)

Definition 9 (Riemannian metric) *The matrix valued function $M = (M^{\pm}) \in C^1(\mathbb{X}^{\pm}, \mathbb{R}^{2\times 2})$ is called a Riemannian metric, if for each $x \in K$ the matrix $M(x)$ is:*

- *symmetric*
- *positive definite.*

We consider a special case of a metric

$$M^{\pm}(x) := Ie^{2W^{\pm}(x)}, \text{ where } I \text{ is an identity matrix.}$$

with $W^{\pm} \in C^1(\mathbb{X}^{\pm}, \mathbb{R})$.

In the two dimensional case we can obtain necessary conditions by considering a Riemannian metric given by a weight function. The reason for this is that for a two-dimensional system as considered here there is only a one-dimensional family of vectors $v \perp f(x)$. Let $W + \pm := W$.

Definition 10 *Let the time-dependent distance function* $A : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *by*

$$A(t) := \sqrt{\left(\left(S_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S_t x\right)^T e^{2W(S_t x)} \left(S_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S_t x\right)\right)} \tag{2.4}$$

2.3 Long run case

2.3.1 Smooth case +

This part of the proof is similar to proofs in smooth dynamical systems theory. We prove the following result, which is a special case of the version proved by Giesl [18] who considers a phase space in $\mathbb{R}^n$ and a distance function depending on a Riemannian metric defined by a matrix valued function. In our case the distance function is weighted by a function e^W.

Proposition 1 (Step 1, LR case +) *Let the assumptions of theorem 5 hold. Then for each* $k \in (0,1)$ *there are constants* $\delta > 0$ *and* $C \geq 1$ *such that for all* $x \in K$ *and all* $\eta \in \mathbb{R}^2$ *with* $\eta^T f(x) = 0$ *and* $\| \eta \| \leq \frac{\delta}{2}$ *there exists a diffeomorphism* $\mathcal{T}_x^{x+\eta} : \mathcal{G}^+ \to \mathbb{R}_0^+$ *which satisfies* $\frac{2}{3} \leq \dot{\mathcal{T}}_x^{x+\eta}(t) \leq \frac{4}{3}$ *and*

$$\left(S^+_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S^+_t x\right)^T f^+(S^+_t x) = 0$$

for all $t \in \mathcal{G}^+$. $\mathcal{T}_x^{x+\eta}$ depends continuously on η. Moreover we have

$$\parallel S^+_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S^+_t x \parallel \leq Ce^{-\nu(1-k)t} \parallel \eta \parallel . \tag{2.5}$$

for all $t \in \mathcal{G}^+$.

Proof. This proposition is proven in some steps. We will first introduce a time-dependent distance function between two solutions with adjacent starting points denoted by x and $x+\eta$ such that $\eta^T f^+(x) = 0$ holds.

Step (i): Pick any $x \in K^+$ and $\eta \in \mathbb{R}^2$ so that $\eta^T f^+(x) = 0$ and $\parallel \eta \parallel \leq \frac{\delta}{2}$. The synchronization of time between two trajectories through the points x and $x+\eta$ is achieved by defining $\mathcal{T}_x^{x+\eta}(t)$ implicitly by

$$Q^+(\mathcal{T}, t, \eta) := \left(S^+_{\mathcal{T}}(x+\eta) - S^+_t x\right)^T f^+(S^+_t x) = 0, \tag{2.6}$$

where $\mathcal{T} = \mathcal{T}_x^{x+\eta}(t)$. We will show that $\mathcal{T}_x^{x+\eta}(t)$ is well defined locally near $t = 0$ by the implicit function theorem and by equation (2.6) since $\frac{\partial Q_{\mathcal{T}}(0,0,\eta)}{\partial \mathcal{T}} \neq 0$ and $Q^+(\mathcal{T}, t, \eta)$ continuously depends on η. Moreover, by Sandberg [34] Theorem 1 and Theorem 2, $\mathcal{T}_x^{x+\eta}(t)$ is globally defined for all $t \in \mathcal{G}^+$.

Step (ii): By definition 10 we have a time-dependent distance function $A^+ : \mathcal{G}^+ \to \mathbb{R}^+_0$ given by

$$A^+(t) := \sqrt{\left(\left(S^+_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S^+_t x\right)^T e^{2W^+(S^+_t x)} \left(S^+_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S^+_t x\right)\right)} \tag{2.7}$$

using definition 9. This can be simplified to

$$A^+(t) := e^{W^+(S^+_t x)} \| S^+_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S^+_t x \|$$

We set $v(t) := \frac{\left(S^+_{\mathcal{T}}(x+\eta) - S^+_t x\right)}{A^+(t)}$ which is defined since $A^+(t^+_{j-1}) \neq 0$ for $j \in \{2n\}$ with $n \in \mathbb{N}$ implies that $A^+(t) \neq 0$ for all $t \geq t^+_{j-1}$. Hence we have

$$\left(S^+_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S^+_t x\right) = A^+(t)v(t)$$

where $v(t)$ is a vector in $\mathbb{R}^2$ with $\sqrt{e^{2W^+(S_t^+x)}\|v(t)\|^2} = 1$. We now show that $\| v(t) \|$ is bounded above and below. We set

$$\begin{aligned}
\lambda_m &:= \min_{x\in K^+\cup K^0} e^{2W^+(S_t^+x)} \\
\lambda_M &:= \max_{x\in K^+\cup K^0} e^{2W^+(S_t^+x)}.
\end{aligned}$$

and obtain for all $\xi \in \mathbb{R}^2$ and all $x \in K$ with

$$\lambda_m \| \xi \|^2 \leq e^{2W^+(S_t^+x)}\|\xi\|^2 \leq \lambda_M \| \xi \|^2 \tag{2.8}$$

$$\| Ie^{2W^+(S_t^+x)}\xi \| \leq \lambda_M \| \xi \| \tag{2.9}$$

By (2.8) we have

$$\begin{aligned}
\lambda_m \| v^+(t) \|^2 &\leq e^{2W^+(S_t^+x)}\|v^+(t)\|^2 \leq \lambda_M \| v^+(t) \|^2 \\
\lambda_m &\leq \frac{e^{2W^+(S_t^+x)}\|v^+(t)\|^2}{\| v^+(t) \|^2} \leq \lambda_M \\
\sqrt{\lambda_m} &\leq \sqrt{\frac{e^{2W^+(S_t^+x)}\|v^+(t)\|^2}{\| v^+(t) \|^2}} \leq \sqrt{\lambda_M}
\end{aligned}$$

which by little algebraic manipulation and by $\sqrt{e^{2W^\pm(S_t^+x)}\|v^+(t)\|^2} = 1$ yields

$$\sqrt{\frac{1}{\lambda_M}} \leq \| v^+(t) \| \leq \sqrt{\frac{1}{\lambda_m}}$$

as requested.

We show that the distance function between solutions decreases exponentially. (i) We parameterize the time of the trajectory $S_{\mathcal{T}}$ and (ii) define a time-dependent distance function based on a special case of a Riemannian metric [38]. (iii) We show that the distance function decreases exponentially. By equation (2.6) we have for all $t \in \mathcal{G}^+$ until a jump that $v^T(t)f^+(S_t^+x) = 0$. By the implicit function theorem we calculate

$$\dot{\mathcal{T}}_x^{x+\eta}(t) = -\frac{\partial Q_t^+(\mathcal{T}, t, \eta)}{\partial Q_{\mathcal{T}}^+(\mathcal{T}, t, \eta)}.$$

We have

$$\begin{aligned}\partial Q_t^+(\mathcal{T},t,\eta) &= -\|f^+(S_t^+x)\|^2 + \left(S_{\mathcal{T}}^+(x+\eta) - S_t^+x\right)^T D_x f^+(S_t^+x) f^+(S_t^+x) \\ &= -\|f^+(S_t^+x)\|^2 + A^+(t)v^T(t) f^+(S_t^+x) \\ &\quad + A^+(t)v^T(t) D_x f^+(S_t^+x) f^+(S_t^+x). \end{aligned} \quad (2.10)$$

By the mean value theorem we have

$$\begin{aligned}\partial Q_{\mathcal{T}}^+(\mathcal{T},t,\eta) &= f^+(S_{\mathcal{T}}^+(x+\eta))^T f^+(S_t^+x) \\ &= \| f^+(S_t^+x) \|^2 \\ &\quad + A^+(t)\left(\int_0^1 D_x f(S_t^+x + \lambda A^+(t)v(t)) d\lambda v(t)\right)^T \times f^+(S_t^+x) \end{aligned} \quad (2.11)$$

$\mathcal{T}_x^{x+\eta}(t)$ is locally well defined since as we next show $\partial Q_{\mathcal{T}}(0,0,\eta) \neq 0$. We also show that $\mathcal{T}_x^{x+\eta}(t)$ is a strictly monotone increasing. By (2.8) we have

$$\begin{aligned}\lambda_m \| \eta \|^2 &\leq e^{2W^+(S_t^+x)}\|\eta\|^2 \leq \lambda_M \| \eta \|^2 \\ \sqrt{\lambda_m} \| \eta \| &\leq A^+(t_{j-1}^+) \leq \sqrt{\lambda_M} \| \eta \| \end{aligned}$$

Hence, $A^+(t_{j-1}^+) \leq \sqrt{\lambda_M} \| \eta \|$ and by $\|\eta\| \leq \frac{\delta}{2}$ and choosing δ' with

$$\delta := \frac{\sqrt{\lambda_m}}{\sqrt{\lambda_M}}\delta'$$

such that $A^+(t_{j-1}^+) \leq \sqrt{\lambda_M} \| \eta \| \leq \sqrt{\lambda_m}\frac{\delta'}{2}$. Hence for small enough $t \in \mathcal{G}^+$ the continuous function A^+ satisfies $A^+(t) \leq \sqrt{\lambda_m}\delta'$. We now define δ'. Since both, f^+ and $e^{2W^\pm(S_t^+x)}$ are continuous functions for all $x \in K^+$ we have the following bounds

$$0 < f_m^+ \leq \| f^+(x) \| \leq f_M^+ \quad (2.12)$$

and

$$e^{2W^+(S_t^+x)} \leq M_D. \quad (2.13)$$

We define

$$\epsilon := min\left(\frac{k}{3}, \frac{2}{3}k\nu\frac{\lambda_m}{M_D}\right) \leq \frac{1}{3}, \quad (2.14)$$

with $k \in (0,1)$ and $\nu := -max_{x \in K^+ \cup \mathbb{X}_0} L_M(x) > 0$. Also Df^+ is uniformly continuous on $K^+ \cup K^0$. Thus there exists $\delta_1 > 0$ with $\|\xi\| \leq \delta_1$ such that for all $x \in K^+$ and $\xi \in \mathbb{R}^2$

$$\|Df^+(x) - Df^+(x+\xi)\| \leq \frac{\nu k \lambda_m}{3(1+\epsilon)\lambda_M} \tag{2.15}$$

holds. For all points $q \in \overline{K_{\delta_1}}$, where $\overline{K_{\delta_1}}$ is the closure of the set K with diameter δ_1 and center x defined by $\overline{K_{\delta_1}} := \overline{\{x : dist(x,K) \leq \delta_1\}}$ there is a positive constant such that

$$\|Df^+(q)\| \leq f_D^+. \tag{2.16}$$

We now define

$$\delta' := min\left(\delta_1, \frac{\lambda_m (f_m^+)^2 \epsilon}{f_M^+\left(M_D + (2+\epsilon) f_D^+ \lambda_M\right)}\right) \tag{2.17}$$

Since K is positively invariant and supposing that $A^+(t) \leq \sqrt{\lambda_m}\delta'$ it holds that $S_t x + \lambda A^+(t) v(t) \in \overline{K_{\delta_1}}$. Using (2.16) and (2.17) it holds that $\| \int_0^1 D_x f(S_t^+ x + \lambda A^+(t) v(t)) d\lambda \| \leq f_D^+$.

Definitions (2.8), (2.9),(2.12),(2.13), and (2.16) together with equations (2.10), and (2.11) imply

$$\begin{aligned}
\dot{\mathcal{T}}_x^{x+\eta}(t) &\leq \frac{\|f^+(S_t^+ x)\|^2 + \delta' f_M^+ (M_D + f_D^+ \lambda_M)}{\|f^+(S_t^+ x)\|^2 - \delta' f_M^+ f_D^+ \lambda_M} \\
&\leq 1 + \frac{\delta' f_M^+ (M_D + 2 f_D^+ \lambda_M)}{\lambda_m (f_m^+)^2 - \delta' f_M^+ f_D^+ \lambda_M} \\
&\leq 1 + \epsilon \leq \frac{4}{3} \text{ by (2.17)}
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\dot{\mathcal{T}}_x^{x+\eta}(t) &\geq \frac{\|f^+(S_t^+ x)\|^2 + \delta' f_M^+ (M_D + f_D^+ \lambda_M)}{\|f^+(S_t^+ x)\|^2 - \delta' f_M^+ f_D^+ \lambda_M} \\
&\geq 1 - \frac{\delta' f_M^+ (M_D + 2 f_D^+ \lambda_M)}{\lambda_m (f_m^+)^2 - \delta' f_M^+ f_D^+ \lambda_M} \\
&\geq 1 - \epsilon \geq \frac{2}{3} \text{ by (2.17)}
\end{aligned}$$

We conclude that $\mathcal{T}_x^{x+\eta}(t)$ is strictly increasing for all $t \in \mathcal{G}^+$ until a jump, and in particular $\partial Q_{\mathcal{T}}(0,0,\eta) \neq 0$. By similar calculations as above, the inverse map satisfies $\frac{3}{4} \leq \dot{t}(\mathcal{T}_x^{x+\eta}) \leq \frac{3}{2}$. We will later show that $\mathcal{T}_x^{x+\eta}(t)$ is globally defined, including at jumps.

Part (iii): Next, we show that the distance between two orbits decreases over time. Hence, we show that the evolution path A^+ as defined in (2.7) decreases exponentially. To show this, we calculate the temporal derivative of $(A^+)^2$ and apply by definition 9 that $M^+(x) = [M^+(x)]^T = Ie^{2W^+(S_t^+x)}$. We also use that $v^T(t)[Ie^{2W^+(S_t^+x)}](S_t^+x)f^+(S_t^+x) = 0$. Now, by equation (2.7) we have

$$(A^+)^2(t) = \left(S_{\mathcal{T}}^+(x+\eta) - S_t^+x\right)^T M^+(S_t^+x)\left(S_{\mathcal{T}}^+(x+\eta) - S_t^+x\right)$$

which by using definition 9 yields

$$(A^+)^2(t) = e^{2W^+(S_t^+x)}\|S_{\mathcal{T}}^+(x+\eta) - S_t^+x\|^2$$

We calculate $\frac{d(A^+)^2(t)}{dt} = 2(A^+)(t)(\dot{A^+})(t)$. We have

$$\begin{aligned}
\frac{d(A^+)^2(t)}{dt} &= 2e^{2W^+(S_t^+x)}\left(S_{\mathcal{T}}^+(x+\eta) - S_t^+x\right)^T\left(f^+(S_{\mathcal{T}}^+(x+\eta))\frac{d\mathcal{T}_x^{x+\eta}(t)}{dt} - f^+(S_t^+x)\right) \\
&+ \left(S_{\mathcal{T}}^+(x+\eta) - S_t^+x\right)^T(e^{2W^+(S_t^+x)})'\left(S_{\mathcal{T}}^+(x+\eta) - S_t^+x\right) \\
&= 2A^+(t)\dot{\mathcal{T}}_x^{x+\eta}(t)v^T(t)(e^{2W^+(S_t^+x)})f^+(S_t^+x + A^+(t)v(t)) \\
&\quad +(A^+)^2(t)v^T(t)(e^{2W^+(S_t^+x)})'v(t)
\end{aligned}$$

and by the mean value theorem we obtain

$$\begin{aligned}
&= 2(A^+)^2(t)\dot{\mathcal{T}}_x^{x+\eta}(t)v^T(t)(e^{2W^+(S_t^+x)})\left(\int_0^1 D_xf^+(S_t^+x + \lambda A^+(t)v(t))d\lambda v(t)\right) \\
&\quad +(A^+)^2(t)\dot{\mathcal{T}}_x^{x+\eta}(t)v^T(t)(e^{2W^+(S_t^+x)})' \\
&\quad +(A^+)^2(t)(1 - \dot{\mathcal{T}}_x^{x+\eta}(t))v^T(t)(e^{2W^+(S_t^+x)})'v(t).
\end{aligned}$$

Since for small $t \in \mathcal{G}^+$ and all $\lambda \in [0,1]$ it holds that $\|\lambda A^+(t)v(t)\| \leq \delta'$ we can apply the bound given by equation (2.15). We obtain

$$\begin{aligned}
\leq\ & 2(A^+)^2(t)\dot{\mathcal{T}}_x^{x+\eta}v^T(t)\left[(e^{2W^+(S_t^+x)})D_xf^+(S_t^+x)+\frac{1}{2}(e^{2W^+(S_t^+x)})'\right]v(t) \\
& +v^T(t)Ie^{2W^+(S_t^+x)}\left(\int_0^1\left[D_xf^+(S_t^+x+\lambda A^+(t)v(t))d\lambda-D_xf^+(S_t^+x)\right]d\lambda v(t)\right) \\
& (A^+)^2(t)|1-\dot{\mathcal{T}}_x^{x+\eta}(t)|\cdot|v^T(t)(e^{2W^+(S_t^+x)})'v(t)|.
\end{aligned}$$

With $v^T(t)f^+(S_t^+x)=0$ we have

$$\begin{aligned}
L_M(S_t^+x) &= v^T(t)\left[(e^{2W^+(S_t^+x)})D_xf^+(S_t^+x)+\frac{1}{2}(e^{2W^+(S_t^+x)})'\right]v(t) \\
&= \left\{(v^+)^TDf^+(x)v^++\langle\nabla W^+(x),f^+(x)\rangle\|v^+\|^2\right\}e^{2W^+(S_t^+x)}.
\end{aligned}$$

With $L_M(S_t^+x):=-\nu$ and bounds (2.16) we obtain

$$\begin{aligned}
2(A^+)^2(t)\dot{A}^+(t) &\leq -2\nu(A^+)^2(t)\dot{\mathcal{T}}_x^{x+\eta}(t)+2\frac{(A^+)^2(t)\dot{\mathcal{T}}_x^{x+\eta}(t)\lambda_M}{\lambda_m}+\frac{\nu k\lambda_m}{3(1+\epsilon))\lambda_M} \\
&\quad +(A^+)^2(t)|1-\dot{\mathcal{T}}_x^{x+\eta}(t)|\frac{M_D}{\lambda_m} \\
&\leq \left(-2\nu+2\epsilon\nu+\frac{2}{3}k\nu+\epsilon\frac{M_D}{\lambda_m}\right)(A^+)^2(t)
\end{aligned}$$

which by little algebra and definition 2.14 yields

$$\begin{aligned}
2(A^+)^2(t)\dot{A}^+(t) &\leq -2(1-k)\nu(A^+)^2(t) \\
\dot{A}^+(t) &\leq -(1-k)\nu.
\end{aligned}$$

Solving this equation yields

$$A^+(t)\leq A^+(0)e^{-\nu(1-k)t}\leq\sqrt{\lambda_m}\frac{\delta'}{2}e^{-\nu(1-k)t}. \tag{2.18}$$

Equation (2.18) in particular shows that $A^+(t)\leq A^+(t_{j-1}^+)\leq\sqrt{\lambda_m}\frac{\delta'}{2}$ for all $t\in\mathcal{G}^+$ and $j\in\{2n\}$ and $n\in\mathbb{N}_0$ here we have assumed that $\min t\in\mathcal{G}^+$ is 0. Hence equation (2.18) is a prolongation argument which states that both $\mathcal{T}_x^{x+\eta}(t)$ and $A^+(t)$ are defined for all $t\in\mathcal{G}^+$ by [35]. Moreover, equation (2.18) also shows

$$\begin{aligned}\sqrt{\lambda_m} \parallel S^+_{\mathcal{T}}(x+\eta) - S^+_t x \parallel &= A^+(t) \\ &\leq A^+(t^+_{j-1})e^{-\nu(1-k)t} \\ &\leq \sqrt{\lambda_M} \parallel \eta \parallel e^{-\nu(1-k)t}.\end{aligned}$$

Hence the contraction property (2.5) follows with

$$C := \frac{\sqrt{\lambda_M}}{\sqrt{\lambda_m}} \geq 1.$$

The next proposition considers the (LR) smooth case when $t \in \mathcal{G}^-$. This is the case when solutions live in the negative phase space $\mathbb{X}^-$ for the entire time interval considered. Together, proposition 1 and proposition 2 prove the smooth cases of part one of the outline of the strategy of the proof of the main result, theorem 5. Since the proof follows closely what we have already shown, we provide some steps only.

2.3.2 Smooth case -

This case is similar to the smooth case "+"). We hence only provide some steps of the proof.

Proposition 2 (Step 1, LR case -) *Let the assumptions of theorem 5 hold. Then for each $k \in (0,1)$ there are constants $\delta > 0$ and $C \geq 1$ such that for all $x \in K$ and all $\eta \in \mathbb{R}^2$ with $\eta^T f(x) = 0$ and $\parallel \eta \parallel \leq \frac{\delta}{2}$ there exists a diffeomorphism $\mathcal{T}^{x+\eta}_x : \mathcal{G}^- \to \mathbb{R}^+_0$ which satisfies $\frac{2}{3} \leq \dot{\mathcal{T}}^{x+\eta}_x(t) \leq \frac{4}{3}$ and*

$$\left(S^-_{\mathcal{T}^{x+\eta}_x(t)}(x+\eta) - S^-_t x\right)^T f^-(S^-_t x) = 0$$

for all $t \in \mathcal{G}^-$. $\mathcal{T}^{x+\eta}_x$ depends continuously on η. Moreover we have

$$\parallel S^-_{\mathcal{T}^{x+\eta}_x(t)}(x+\eta) - S^-_t x \parallel \leq Ce^{-\nu(1-k)t} \parallel \eta \parallel . \tag{2.19}$$

for all $t \in \mathcal{G}^-$.

Proof. (i) Pick any $x \in K^-$ with so that $\eta^T f^+(x) = 0$ and $\| \eta \| \leq \frac{\delta}{2}$. The synchronization of time between two trajectories through the points x and $x+\eta$ is achieved by defining $\mathcal{T}_x^{x+\eta}(t)$ implicitly by

$$Q^-(\mathcal{T}, t, \eta) := \left(S^-_{\mathcal{T}}(x+\eta) - S^-_t x\right)^T f^-(S^-_t x) = 0, \tag{2.20}$$

where $\mathcal{T} = \mathcal{T}_x^{x+\eta}(t)$.

By equation (2.20) we have for all $t \in \mathcal{G}^-$ until a jump that $v^T(t) f^-(S^-_t x) = 0$. By the implicit function theorem we calculate

$$\dot{\mathcal{T}}_x^{x+\eta}(t) = -\frac{\partial Q^-_t(\mathcal{T}, t, \eta)}{\partial Q^-_{\mathcal{T}}(\mathcal{T}, t, \eta)}.$$

$$\begin{aligned} \partial Q^-_t(\mathcal{T}, t, \eta) &= -\|f^-(S^-_t x)\|^2 + \left(S^-_{\mathcal{T}}(x+\eta) - S^-_t x\right)^T D_x f^-(S^-_t x) f^-(S^-_t x) \\ &= -\|f^-(S^-_t x)\|^2 + A^-(t) v^T(t) f^-(S^+_t x) \\ &\quad + A^-(t) v^T(t) D_x f^-(S^+_t x) f^-(S^-_t x). \end{aligned} \tag{2.21}$$

By the mean value theorem we have

$$\begin{aligned} \partial Q^-_{\mathcal{T}}(\mathcal{T}, t, \eta) &= f^-(S^-_{\mathcal{T}}(x+\eta))^T f^-(S^-_t x) \\ &= \| f^-(S^-_t x) \|^2 \\ &\quad + A^-(t) \left(\int_0^1 D_x f(S^-_t x + \lambda A^-(t) v(t)) d\lambda v(t) \right)^T \times f^-(S^-_t x) \end{aligned} \tag{2.22}$$

(ii) We now define the time-dependent distance function $A^- : \mathcal{G}^- \to \mathbb{R}^+_0$ by

$$A^-(t) := e^{W^-(S^-_t x)} \| S^-_{\mathcal{T}_x^{x+\eta}}(x+\eta) - S^-_t x \|$$

We set $v(t) := \frac{\left(S^-_{\mathcal{T}_x^{x+\eta}}(x+\eta) - S^-_t x\right)}{A^-(t)}$ which is defined since $A^-(t^+_{j-1}) \neq 0$ implies that $A^-(t) \neq 0$ for all $t \geq t^+_{j-1}$ and $j \in \{2n-1\}$ with $n \in \mathbb{N}_0$. Hence we have

$$\left(S^+_{\mathcal{T}_x^{x+\eta}}(x+\eta) - S^+_t x\right) = A^+(t) v(t)$$

where $v(t)$ is a vector in $\mathbb{R}^2$ with $\sqrt{e^{2W^-(S_t^+x)}\|S^-_{\mathcal{T}_x^{x+\eta}}(x+\eta) - S_t^- x\|^2} = 1$. For all $x \in K^-$ there are values with $0 < \lambda_m \leq \lambda_M < \infty$ such that

$$\begin{aligned} \lambda_m \parallel \xi \parallel^2 &\leq e^{2W^-(S_t^+x)}\|\xi\|^2 \leq \lambda_M \parallel \xi \parallel^2 &(2.23)\\ \parallel Ie^{2W^-(S_t^-x)}\xi \parallel &\leq \lambda_M \parallel \xi \parallel &(2.24)\end{aligned}$$

hold for all $\xi \in \mathbb{R}^2$ and all $x \in K$ with

$$\begin{aligned} \lambda_m &:= \min_{x\in K^-\cup K^0} e^{2W^-(S_t^-x)}\\ \lambda_M &:= \max_{x\in K^-\cup K^0} e^{2W^-(S_t^-x)}. \end{aligned}$$

By (2.23) we have

$$\begin{aligned} \lambda_m \parallel v^-(t) \parallel^2 &\leq e^{2W^-(S_t^-x)}\|v^-(t)\|^2 \leq \lambda_M \parallel v^-(t) \parallel^2\\ \lambda_m &\leq \frac{e^{2W^-(S_t^-x)}\|v^-(t)\|^2}{\parallel v^-(t) \parallel^2} \leq \lambda_M\\ \sqrt{\lambda_m} &\leq \sqrt{\frac{e^{2W^-(S_t^-x)}\|v^-(t)\|^2}{\parallel v^-(t) \parallel^2}} \leq \sqrt{\lambda_M} \end{aligned}$$

which by little algebraic manipulation and by $\sqrt{e^{2W^-(S_t^-x)}\|v^-(t)\|^2} = 1$ yields

$$\sqrt{\frac{1}{\lambda_M}} \leq \parallel v^-(t) \parallel \leq \sqrt{\frac{1}{\lambda_m}}$$

as requested.

We now show that $\mathcal{T}_x^{x+\eta}(t)$ is strictly increasing, hence $\partial Q(0,0,\eta) \neq 0$. By (2.23) we have

$$\begin{aligned} \lambda_m \parallel \eta \parallel^2 &\leq e^{2W^-(S_t^-x)}\|\eta\|^2 \leq \lambda_M \parallel \eta \parallel^2\\ \sqrt{\lambda_m} \parallel \eta \parallel &\leq A^-(t_{j-1}^+) \leq \sqrt{\lambda_M} \parallel \eta \parallel \end{aligned}$$

Hence, $A^-(t^+_{j-1}) \leq \sqrt{\lambda_M} \parallel \eta \parallel$ and by $\|\eta\| \leq \frac{\delta}{2}$ and choosing δ' with

$$\delta := \frac{\sqrt{\lambda_m}}{\sqrt{\lambda_M}}\delta'$$

such that $A^-(t^+_{j-1}) \leq \sqrt{\lambda_M} \parallel \eta \parallel \leq \sqrt{\lambda_m}\frac{\delta'}{2}$. Hence for small enough $t \geq \mathcal{G}^-$ the continuous function A^+ satisfies $A^+(t) \leq \sqrt{\lambda_m}\delta'$. We now define δ'. Since both, f^- and $Ie^{2W^-(S_t^- x)}$ are continuous functions for all $x \in K^-$ we have the following bounds

$$0 < f_m^- \leq \parallel f^-(x) \parallel \leq f_M^- \tag{2.25}$$

and

$$e^{2W^-(S_t^- x)} \leq M_D. \tag{2.26}$$

We define

$$\epsilon := min\left(\frac{k}{3}, \frac{2}{3}k\nu\frac{\lambda_m}{M_D}\right) \leq \frac{1}{3}, \tag{2.27}$$

with $k \in (0,1)$ and $\nu := -max_{x\in K^-\cup\mathbb{X}_0} L_M(x) > 0$. Also Df^- is uniformly continuous. Thus there exists $\delta_1 > 0$ with $\|\xi\| \leq \delta_1$ such that for all $x \in K^-$ and $\xi \in \mathbb{R}^2$

$$\|Df^-(x) - Df^-(x+\xi)\| \leq \frac{\nu k\lambda_m}{3(1+\epsilon)\lambda_M} \tag{2.28}$$

holds. For all points $q \in \overline{K_{\delta_1}}$, there is a positive constant such that

$$\|Df^+(q)\| \leq f_D^+. \tag{2.29}$$

We now define

$$\delta' := min\left(\delta_1, \frac{\lambda_m (f_m^-)^2\epsilon}{f_M^-\left(M_D + (2+\epsilon)f_D^-\lambda_M\right)}\right) \tag{2.30}$$

Since K is positively invariant and supposing that $A^-(t) \leq \sqrt{\lambda_m}\delta'$ it holds that $S_t x + \lambda A^-(t)v(t) \in \overline{K_{\delta_1}}$. Using (2.29) and (2.30) it holds that $\| \int_0^1 D_x f(S_t^- x + \lambda A^-(t)v(t))d\lambda\| \leq f_D^-$.

Definitions (2.23), (2.24),(2.25),(2.26), and (2.29)together with equations (2.21), and (2.22) imply

$$
\begin{aligned}
\dot{\mathcal{T}}_x^{x+\eta}(t) &\leq \frac{\|f^-(S_t^- x)\|^2 + \delta' f_M^-(M_D + f_D^- \lambda_M)}{\|f^-(S_t^- x)\|^2 - \delta' f_M^- f_D^- \lambda_M} \\
&\leq 1 + \frac{\delta' f_M^-(M_D + 2 f_D^- \lambda_M)}{\lambda_m (f_m^-)^2 - \delta' f_M^- f_D^- \lambda_M} \\
&\leq 1 + \epsilon \leq \frac{4}{3} \text{ by (2.30)}
\end{aligned}
$$

Similarly we have

$$
\dot{\mathcal{T}}_x^{x+\eta}(t) \geq 1 - \epsilon \geq \frac{2}{3} \text{ by (2.30)}
$$

We conclude that $\mathcal{T}_x^{x+\eta}(t)$ is strictly increasing for all $t \geq \mathcal{G}^-$ until a jump, and in particular $\partial Q_{\mathcal{T}}(0,0,\eta) \neq 0$. By similar calculations as above, the inverse map satisfies $\frac{3}{4} \leq \dot{t}(\mathcal{T}_x^{x+\eta}) \leq \frac{3}{2}$.

We calculate $\frac{d(A^-)^2(t)}{dt} = 2(A^-)(t)(\dot{A^-})(t)$. We have

$$
\begin{aligned}
\frac{d(A^-)^2(t)}{dt} &= 2e^{2W^-(S_t^- x)} \left(S_{\mathcal{T}}^-(x+\eta) - S_t^- x\right)^T \left(f^-(S_{\mathcal{T}}^-(x+\eta)) \frac{d\mathcal{T}_x^{x+\eta}(t)}{dt} - f^-(S_t^- x) \right) \\
&+ \left(S_{\mathcal{T}}^-(x+\eta) - S_t^- x\right)^T (e^{2W^-(S_t^- x)})' \left(S_{\mathcal{T}}^-(x+\eta) - S_t^- x\right) \\
&= 2A^-(t) \dot{\mathcal{T}}_x^{x+\eta}(t) v^T(t) (e^{2W^-(S_t^- x)}) f^-(S_t^+ x + A^+(t) v(t)) \\
&\quad + (A^-)^2(t) v^T(t) (e^{2W^+(S_t^- x)})' v(t).
\end{aligned}
$$

By the mean value theorem we obtain

$$
\begin{aligned}
&= 2(A^-)^2(t) \dot{\mathcal{T}}_x^{x+\eta}(t) v^T(t) (e^{2W^-(S_t^- x)}) \left(\int_0^1 D_x f^-(S_t^- x + \lambda A^-(t) v(t)) d\lambda v(t) \right) \\
&\quad + (A^-)^2(t) \dot{\mathcal{T}}_x^{x+\eta}(t) v^T(t) (e^{2W^-(S_t^- x)})' \\
&\quad + (A^-)^2(t) (1 - \dot{\mathcal{T}}_x^{x+\eta}(t)) v^T(t) (e^{2W^-(S_t^+ x)})' v(t)
\end{aligned}
$$

Since for small $t \in \mathcal{G}^-$ and all $\lambda \in [0,1]$ it holds that $\|\lambda A^-(t) v(t)\| \leq \delta'$ we can apply the bound given by equation (2.28). We obtain

$$
\begin{aligned}
\leq\ & 2(A^-)^2(t)\dot{\mathcal{T}}_x^{x+\eta}v^T(t)\left[(e^{2W^-(S_t^+x)})D_xf^+(S_t^-x)+\frac{1}{2}(e^{2W^-(S_t^-x)})'\right]v(t)\\
& +v^T(t)Ie^{2W^-(S_t^+x)}\left(\int_0^1\left[D_xf^-(S_t^+x+\lambda A^-(t)v(t))d\lambda-D_xf^-(S_t^-x)\right]d\lambda v(t)\right)\\
& (A^-)^2(t)|1-\dot{\mathcal{T}}_x^{x+\eta}(t)|\cdot|v^T(t)(e^{2W^-(S_t^-x)})'v(t)|.
\end{aligned}
$$

With $v^T(t)f^+(S_t^-x)=0$ we have

$$
\begin{aligned}
L_M(S_t^-x) &= v^T(t)\left[(e^{2W^-(S_t^-x)})D_xf^-(S_t^-x)+\frac{1}{2}(e^{2W^-(S_t^+x)})'\right]v(t)\\
&= \left\{(v^-)^TDf(x)v^-+\langle\nabla W^-(x),f^-(x)\rangle\|v^-\|^2\right\}e^{2W^-(S_t^+x)}.
\end{aligned}
$$

With $L_M(S_t^-x):=-\nu$ and bounds (2.29) we obtain

$$
\begin{aligned}
2(A^-)^2(t)\dot{A}^-(t) &\leq -2\nu(A^-)^2(t)\dot{\mathcal{T}}_x^{x+\eta}(t)+2\frac{(A^-)^2(t)\dot{\mathcal{T}}_x^{x+\eta}(t)\lambda_M}{\lambda_m}+\frac{\nu k\lambda_m}{3(1+\epsilon))\lambda_M}\\
&\quad +(A^-)^2(t)|1-\dot{\mathcal{T}}_x^{x+\eta}(t)|\frac{M_D}{\lambda_m}\\
&\leq \left(-2\nu+2\epsilon\nu+\frac{2}{3}k\nu+\epsilon\frac{M_D}{\lambda_m}\right)(A^-)^2(t)
\end{aligned}
$$

which by little algebra and definition 2.27 yields

$$
\begin{aligned}
2(A^-)^2(t)\dot{A}^-(t) &\leq -2(1-k)\nu(A^-)^2(t)\\
\dot{A}^-(t) &\leq -(1-k)\nu.
\end{aligned}
$$

Solving this equation yields

$$
A^-(t)\leq A^-(t_{j-1}^+)e^{-\nu(1-k)t}\leq\sqrt{\lambda_m}\frac{\delta'}{2}e^{-\nu(1-k)t}. \tag{2.31}
$$

Equation 2.31 in particular shows that $A^-(t)\leq A^-(t_{j-1}^+)\leq\sqrt{\lambda_m}\frac{\delta'}{2}$ for all $t\in\mathcal{G}^-$. Hence equation 2.31 is a prolongation argument which states that both $\mathcal{T}_x^{x+\eta}(t)$ and $A^-(t)$ are defined for all $t\in\mathcal{G}^-$ and $j\in\{2n-1\}$ with $n\in\mathbb{N}_0$.

Moreover, equation 2.31 also shows

$$\begin{aligned}\sqrt{\lambda_m} \parallel S^-_{\mathcal{T}}(x+\eta) - S^-_t x \parallel &= A^-(t)\\ &\leq A^-(0)e^{-\nu(1-k)t}\\ &\leq \sqrt{\lambda_M} \parallel \eta \parallel e^{-\nu(1-k)t}.\end{aligned}$$

Hence the contraction property 2.5 follows with

$$C := \frac{\sqrt{\lambda_M}}{\sqrt{\lambda_m}} \geq 1.$$

■

This concludes the first part of point one of the strategy of the proof outlined earlier. For (LR) solutions living entirely in the positive or negative phase space we have shown that the distance between two adjacent solutions decreases exponentially. To show this, we considered a time synchronization between solutions, and applied Borg's contraction property. We now consider the short run case.

2.4 Short run case

The aim of this section is to complete the proof of theorem 5 as outlined in point one of the strategy of the proof. This requires to show that the distance between two adjacent solutions decreases in the short run (SR). We consider the time structures $t \in \mathcal{I}$ or $t \in \mathcal{J}$. These time intervals are associated with discontinuities of f and jumps occurring in the +/- and -/+ direction.

We now prove that the distance between two adjacent trajectories at a frozen time $t^-_j = t^+_j$ with $\theta^-_j \neq \theta^+_j$ at jumps in +/- with $j \in \{2n\}$ for $n \in \mathbb{N}_0$ contracts. This corresponds to the case where the solution $x^+(t)$ hits the axis $x_2 = 0$ first and $x^-(t)$ leaves it last. We also prove that the distance between two adjacent trajectories at a frozen time $\theta^-_j = \theta^+_j$ with $t_j \neq t_j$ at jumps in +/- with $j \in \{2n\}$ for $n \in \mathbb{N}_0$ contracts. This corresponds to the case where the solution $y^+(t)$ hits the axis $x_2 = 0$ first and $y^-(t)$ leaves it first. Moreover we show that the mapping $\mathcal{T}$ is well defined. This requires to derive bounds on its magnitude.

The proof of proposition 3 below is divided into two four main parts. We consider the jump direction from positive phase space to negative phase space indicated by (+/-). The jumps in the other direction follow the same proof with only minor changes and is therefore omitted.

Part one:

- Case I (+/-): Solution $x^+(t)$ hits the axis $x_2 = 0$ first, then $y^+(\theta)$ hits this axis with a delay. We have the following time structure:

$$t \in \mathcal{I}^{\pm} \text{ with } t_j^- = t_j^0 \text{ and } \theta_j^- \neq \theta_j^0.$$

- Case II (+/-): Solution $x^-(t)$ leaves the axis $x_2 = 0$ last, then $y^-(\theta)$ leaves this axis with a delay. We have the following time structure:

$$t \in \mathcal{I}^{\pm} \text{ with } t_j^0 = t_j^+ \text{ and } \theta_j^0 \neq \theta_j^+.$$

The four cases are illustrated in figure 2.5, 2.6, 2.7, and 2.8.

Part two:

- Case III (+/-): Solution $y^+(t)$ hits the axis $x_2 = 0$ first, then $x^+(\theta)$ hits this axis with a delay. We have the following time structure:

$$t \in \mathcal{J}^{\pm} \text{ with } t_j^- \neq t_j^0 \text{ and } \theta_j^- = \theta_j^0.$$

- Case IV (+/-): Solution $y^+(t)$ leaves the axis $x_2 = 0$ first, then $x^+(\theta)$ leaves this axis with a delay. We have the following time structure:

$$t \in \mathcal{J}^{\pm} \text{ with } t_j^0 \neq t_j^+ \text{ and } \theta_j^0 = \theta_j^+.$$

Part three:

Part three shows that above four cases can be patched together in the following ways:

- Case I and case II
- Case I and case IV
- Case III and case IV
- Case III and case II.

For each combination of cases we show that when one of the solutions changes phase space the other will follow. Moreover, we conclude that the distance between solutions at the beginning of a jump compared to the distance between the same solutions at the end of the jump decreases. We also provide upper time bounds for each case. Hence, we refer to these cases as short run cases (SR).

Part four:

Part four is a trivial case. It covers all situations where two adjacent solutions have the following time structure:

$$t_j^- = t_j^0 = t_j^+$$

and

$$\theta_j^- = \theta_j^0 = \theta_j^+.$$

This part covers the extreme cases where both trajectories are perpendicular to the axis $x_2 = 0$. This scenario, as there is no time delay in either trajectory, reduces to the smooth case. Contraction of the distance function between solutions in the smooth case is already shown in proposition 1 and proposition 2.

Part one of the proof discusses the scenarios where the solution x^+ enters the switching manifold at $x_2 = 0$ first and x^- leaves the switching manifold first. An illustration of these scenarios is given in figure 2.9. Part two of the

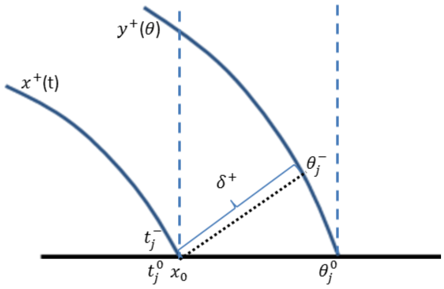

Figure 2.5: Case I.

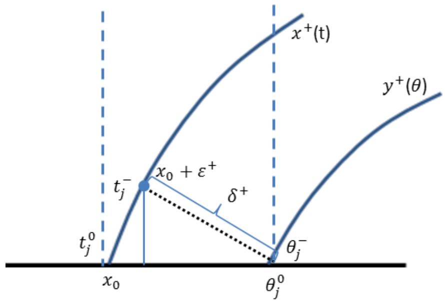

Figure 2.6: Case III.

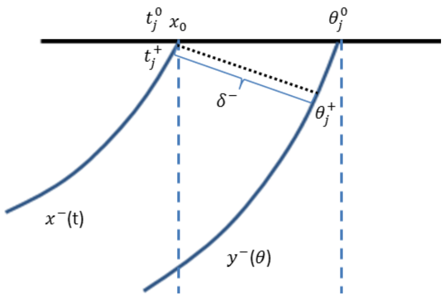

Figure 2.7: Case II.

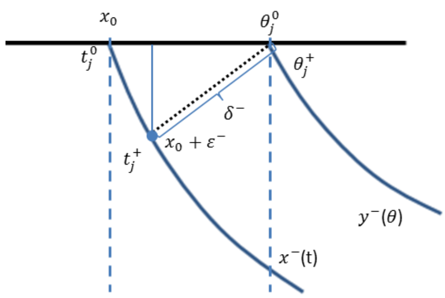

Figure 2.8: Case IV.

proof of proposition 3 is associated with the cases where the y^+ solution enters and y^- leaves the switching manifold last. Note that following conditions hold for $\varepsilon^{\pm}, \delta^{\pm} > 0$:

- $f^{\pm}(x_0) \perp \delta^{\pm}$
- $f^{\pm}(x_0 + \varepsilon^{\pm}) \perp \delta^{\pm}$.

2.4.1 Part one: Cases I and II

Proposition 3 (Jump case (+/-)) *Let the assumptions of theorem 5 hold. Moreover assume that there are constants $\delta, \nu, m, M, N_1, N_2 > 0$ such that for all $\eta \in \mathbb{R}^2$ with $\| \eta \| \leq \delta$ there is a piecewise multi valued mapping $\mathcal{T}_x^{x+\eta}(t)$ defined by equation (2.3) which satisfies for all $t \in \mathcal{I}^{\pm}$ or for all $t \in \mathcal{J}^{\pm}$*

$$m \leq \triangle \mathcal{T}_x^{x+\eta}(t) \leq M. \tag{2.32}$$

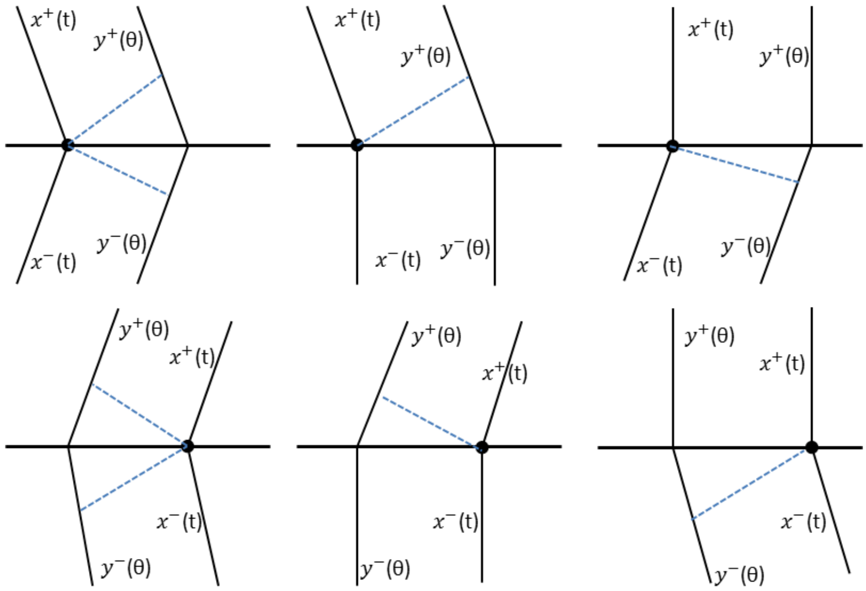

Figure 2.9: Scenarios covered by case I and II.

and

$$(S_{\mathcal{T}^+(t)}(x+\eta) - S_t^+ x) \cdot f^+(S_t^+ x) = 0.$$

Also for all $t \in [t_j^-, t_j^+]$.

$$A^-(t_j^+) < A^+(t_j^-)e^{-\nu t}. \tag{2.33}$$

For all $t \in \mathcal{I}^\pm$ *and* $\theta^* \in [\theta_j^-, \theta_j^0]$ *with*

$$|\theta_j^0 - \theta_j^-| \leq N_1. \tag{2.34}$$

and $y_2^+(\theta^*) \geq 0$.

For all $t \in \mathcal{I}^\pm$ *and* $\theta^{**} \in [\theta_j^0, \theta_j^+]$ *with*

$$|\theta_j^+ - \theta_j^0| \leq N_2. \tag{2.35}$$

and $y_2^-(\theta^{**}) \leq 0$.

Proof. In part one of the proof, we consider the cases depicted in figure 2.5 and figure 2.6. Together, these two cases cover the scenarios graphically presented in figure 2.9.

We begin the proof by showing some properties of $\mathcal{T}$. Continuous dependency of $\mathcal{T}$ on η was shown in proposition 1 and proposition 2.

We can define constants $c_1 > 0$ and $c_2 > 0$ such that

$$|t_j^- - t_{j-1}^+| \geq c_2 \tag{2.36}$$
$$|t_j^+ - t_j^-| \leq c_1. \tag{2.37}$$

By proposition 1 we have

$$\frac{2}{3} \leq \dot{\mathcal{T}} \leq \frac{4}{3},$$

We consider the time interval $\tau \in (t_{j-1}^+, t_j^-)$ with $t_j^- > t_{j-1}^+$. Hence, by integration we obtain

$$\frac{2}{3}\tau \leq \mathcal{T} \leq \frac{4}{3}\tau, \tag{2.38}$$

To show (2.32) we consider the time interval $\tau \in (t_{j-1}^+, t_j^-) \cup [t_j^-, t_j^- + c_2]$ with $t_j^- > t_{j-1}^+$ and $c_2 > 0$. By (2.38) and the prolongation of the time interval we have

$$\begin{aligned}
\theta(\tau) &\geq \frac{2}{3}\left(\frac{t_j^-}{t_j^- + c_2}\right) \cdot \tau \\
&= \frac{2}{3}\left(\frac{1}{1 + \frac{c_2}{t_j^-}}\right) \cdot \tau \\
&\geq \frac{2}{3}\left(\frac{1}{1 + \frac{c_2}{c_2}}\right) \cdot \tau =: m_1
\end{aligned}$$

$$\begin{aligned}
\theta(\tau) &\leq \frac{4}{3}\left(\frac{t_j^-}{t_j^- + c_2}\right) \cdot \tau \\
&= \frac{4}{3}\left(\frac{1}{1 + \frac{c_2}{t_j^-}}\right) \cdot \tau \\
&\leq \frac{4}{3}\left(\frac{1}{1 + \frac{c_2}{c_2}}\right) \cdot \tau =: M_1
\end{aligned}$$

Next we consider the time interval $\tau \in \left\{(t_{j-1}^+, t_j^-) \cup [t_j^-, t_j^- + c_1]\right\}$ with $t_j^- >$

t_{j-1}^{+} and $c_1 > 0$. By (2.38) and the prolongation of the time interval we have

$$\begin{aligned}\theta(\tau) &\geq \frac{2}{3}\left(\frac{t_j^-}{t_j^- + c_1}\right)\cdot\tau \\ &= \frac{2}{3}\left(\frac{1}{1+\frac{c_1}{t_j^-}}\right)\cdot\tau \\ &\geq \frac{2}{3}\left(\frac{1}{1+\frac{c_1}{c_1}}\right)\cdot\tau =: m_2\end{aligned}$$

$$\begin{aligned}\theta(\tau) &\leq \frac{4}{3}\left(\frac{t_j^-}{t_j^- + c_1}\right)\cdot\tau \\ &= \frac{4}{3}\left(\frac{1}{1+\frac{c_1}{t_j^-}}\right)\cdot\tau \\ &\leq \frac{4}{3}\left(\frac{1}{1+\frac{c_1}{c_1}}\right)\cdot\tau =: M_2\end{aligned}$$

$$\begin{aligned} m &:= min\,\{m_1, m_2\} \\ M &:= max\,\{M_1, M_2\}\end{aligned}$$

It remains to define constants c_1, c_2. We define $c_1 := \delta_2 > 0$ which holds uniformly. For the constant c_2 we consider $d := K \cap \{x_2 = 0\}$. From

$$max_{x\in K} \mid f_1(x) \mid= s$$

and

$$t \cdot s = d$$

we obtain by $d \leq \int_0^t f_1(x(\tau))d\tau$

$$c_2 := \frac{d}{\max_{x\in K} \mid f_1 \mid} \leq t.$$

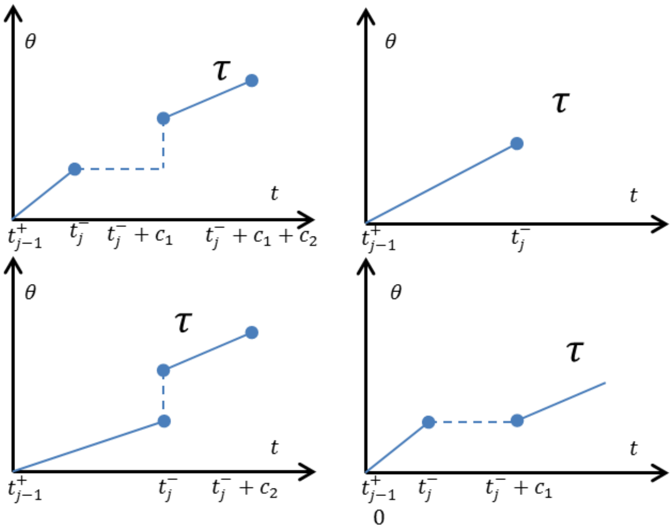

Figure 2.10: Monotonicity

This shows (2.32).

We consider the case where x is frozen at the point $x_2 = 0$. We want to show that at time t_0 and $x_0 \in K^0$

$$\frac{A^-(t_0)}{A^+(t_0)} < 1,$$

where $A^+(t_0)$ is the distance between two adjacent solutions leaving in the positive phase space, and $A^-(t_0)$ is the distance between two adjacent solutions leaving in the negative phase space. Above condition requires to calculate $\frac{A^+(t_0)}{r_b}$ and $\frac{A^-(t_0)}{r_b}$, where r_b is the distance between two adjacent solutions when both hit the axis $x_2 = 0$.

1. We calculate $\frac{A^+(t_0)}{r_b}$, case I.
2. We calculate $\frac{A^-(t_0)}{r_b}$, case II.
3. We calculate $\frac{A^-(t_0)}{A^+(t_0)}$, combining the outcome of case I and II.

We have defined a time dependent distance function $A^+ : \mathcal{G}^{\pm} \to \mathbb{R}_0^+$ by

$$A^+(t) := \sqrt{\left(\left(S^+_{\mathcal{T}}(x+\eta) - S_t^+ x\right)^T e^{2W^+(S_t^+ x)} \left(S^+_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S_t^+ x\right)\right)}$$

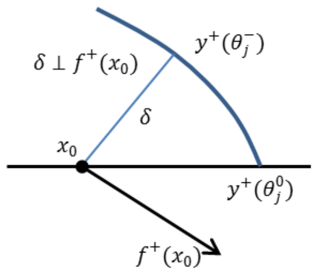

Figure 2.11: Case I

which by definition 3 for all $t \in \mathcal{G}^+$ yields

$$A^+(t) := \sqrt{\left(y^+(\mathcal{T}(t)) - x^+(t)\right)^T e^{2W^+(x^+(t))} \left(y^+(\mathcal{T}(t)) - x^+(t)\right)}$$

which can be rewritten as

$$A^+(t) = e^{W^+(x^+(t))} \parallel y^+(\mathcal{T}(t)) - x^+(t) \parallel . \tag{2.39}$$

Similarly by definition 3 for all $t \in \mathcal{G}^-$ we have

$$A^-(t) := \sqrt{\left(y^-(\mathcal{T}(t)) - x^-(t)\right)^T e^{2W^-(x^-(t))} \left(y^-(\mathcal{T}(t)) - x^-(t)\right)}$$

which can be rewritten as

$$A^-(t) = e^{W^-(x^-(t))} \parallel y^-(\mathcal{T}(t)) - x^-(t) \parallel . \tag{2.40}$$

We consider (2.39) and (2.40) for time $t \in \mathcal{I}^\pm$.

$$K^0 := \{x \in \mathbb{X} : \mathbb{X} \cap (\mathbb{R} \times \{0\})\} .$$

Case I

By conditions of theorem 5, K^0 is a non-empty and compact set which contains no equilibrium. Let $G > 0$ be a constant such that for all $x_0 \in K^0$ satisfying

$f_2^+(x_0) < 0$, $f_2^+(x_0) = -c \leq -G$,

$$-G := \max_{x_0 \in K^0, f_2^+(x_0)<0} f_2^+(x_0)$$

Let $\varepsilon_2 \leq \frac{G}{2}$. Also let there be a constant $D > 0$ such that for all $x_0 \in K^0$ satisfying $f_2^+(x_0) < 0$, $f_1^+(S_t^+ x_0) = d \leq D$,

$$D := \max_{x_0 \in K^0, f_2^+(x_0)<0} |f_1^+(x_0)|$$

Let $\varepsilon_1 \leq \frac{D}{2}$.

Given $(\varepsilon_1, \varepsilon_2) > 0$ there is a $(b_1, b_2) > 0$ so that we can construct a box in the positive phase space with center x_0 by

$$B_{(x_0)}^+ := \left\{(y_1, y_2) \in K \cap [x_0^1 - b_1, x_0^1 + b_1] \times [0, b_2]\right\}$$

such that for all $y \in B_{(x_0)}^+$

$$\begin{aligned} -\frac{D}{2} \leq \mid d \mid -\varepsilon_1 \leq -d - \varepsilon_1 \quad \leq & \\ f_1^+(y) \quad \leq & \quad d + \varepsilon_1 \leq \mid d \mid +\varepsilon_1 \leq \frac{3D}{2} \qquad (2.41) \\ -\frac{3G}{2} \leq -c - \varepsilon_2 \quad \leq & \quad f_2^+(y) \leq -c + \varepsilon_2 \leq -\frac{G}{2}. \qquad (2.42) \end{aligned}$$

Consider a solution $y_2^+(\theta)$ with $y_2^+(\theta_j^-) \in B_{(x_0)}^+$ and $y_2^+(\theta_j^-) > 0$. Now, we want to show that there is a time θ^* such that the solution $y_2^+(\theta^*) = 0$. A solution $y_2^+(\tau) > 0$ decreases to $y_2^+(\theta^*) = 0$ since $f_2^+(y_2^+(\tau)) \leq \frac{-G}{2} < 0$ for all $y_2^+(\tau) \in B_{(x_0)}^+$ with $f_2^+(y_2^+(\tau)) \in [-c - \varepsilon_2, -c + \varepsilon_2]$ and $\tau \in [\theta_j^-, \theta^*]$. We have

$$\begin{aligned} y_2^+(\theta) &= y_2^+(\theta_j^-) + \int_{\theta_j^-}^{\theta_j^0} f_2^+(y_2^+(\tau))d\tau \\ y_2^+(\theta) - y_2^+(\theta_j^-) &= \int_{\theta_j^-}^{\theta_j^0} f_2^+(y_2^+(\tau))d\tau. \end{aligned}$$

Define $\delta_2 := y_2^+(\theta_j^-)$ and $e_2 := \frac{1}{\tau^*} \int_{\theta_j^-}^{\theta^*} [f_2^+(y_2^+(\tau)) + c]d\tau$. Then since $y_2^+(\theta^*) = 0$ and $|e_2| \leq \varepsilon_2$ we have by equation (2.42) that

$$-\delta_2 = -c \cdot \theta^* + e_2 \cdot \theta^*$$

$$\frac{\delta_2}{c - e_2} \quad = \quad \theta^*. \tag{2.43}$$

Hence

$$\frac{\delta_2}{c + |e_2|} \quad \leq \quad \theta^*.$$

Thus there is a time $\theta^* \in [\frac{\delta_2}{c+\varepsilon_2}, \frac{\delta_2}{c-\varepsilon_2}]$ such that $y_2^+(\theta^*) = 0$.

We have shown that for frozen time $t_j^- = t_j^+$ with $j \in \{2n\}$ and $n \in \mathbb{N}_0$ of solution $x^+(t)$ there is a time $\theta^* :=| \theta_j^0 - \theta_j^- |$ of solution $y^+(\theta)$ such that $N_1 := \frac{\delta_2}{c-\varepsilon_2} \geq \theta^*$. We have shown (2.34).

Now, we use θ^* in (2.43) in order to determine $y_1^+(\theta^*)$. We have

$$y_1^+(\theta^*) \quad = \quad y_1^+(\theta_j^-) + \int_{\theta_j^-}^{\theta^*} f_1^+(y^+(\tau))d\tau$$

Define $\delta_1 := y_1^+(\theta_j^-) - x_1^+(t_j^0)$ and $e_1 := \frac{1}{\tau^*}\int_{\theta_j^-}^{\theta^*}[f_1^+(y^+(\tau)) - d]d\tau$, and since $|e_1| \leq \varepsilon_1$ we have

$$y_1^+(\theta^*) - x_1^+(t_j^0) \quad = \quad \delta_1 + \theta^* \cdot [d + e_1]$$

and by equation (2.43) we obtain

$$y_1^+(\theta^*) - x_1^+(t_j^0) \quad = \quad \delta_1 + \frac{\delta_2}{c - e_2} \cdot [d + e_1].$$

Using $\delta_1 f_1^+(x_0) + \delta_2 f_2^+(x_0) = 0$ yields

$$\begin{aligned}
y_1^+(\theta^*) - x_1^+(t_j^0) \quad &= \quad \delta_1 - \left(\frac{\delta_1 f_1^+(x_0)}{(-f_2^+(x_0) - e_2) f_2^+(x_0)}\right) \cdot [f_1^+(x_0) + e_1] \\
y_1^+(\theta^*) - x_1^+(t_j^0) \quad &= \quad \delta_1 \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + e_1 \cdot f_1^+(x_0) + e_2 \cdot f_2^+(x_0)}{(f_2^+(x_0))^2 + e_2 \cdot f_2^+(x_0)}\right].
\end{aligned}$$

We have the following bounds

$$\begin{aligned}
\mid \delta_1 \mid \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 - \varepsilon_1 \mid f_1^+(x_0) \mid -\varepsilon_2 \mid f_2^+(x_0) \mid}{(f_2^+(x_0))^2 + \varepsilon_2 \mid f_2^+(x_0) \mid}\right] &\leq \\
|y_1^+(\theta^*) - x_1^+(t_j^0)| &\leq \\
\mid \delta_1 \mid \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + \varepsilon_1 \mid f_1^+(x_0) \mid +\varepsilon_2 \mid f_2^+(x_0) \mid}{(f_2^+(x_0))^2 - \varepsilon_2 \mid f_2^+(x_0) \mid}\right] &
\end{aligned}$$

We need to verify the bounds. This only requires to check that

$$(f_2^+(x_0))^2 - \varepsilon_2 |f_2^+(x_0)| \geq E > 0.$$

We have $|f_2^+(x_0)| = c$. Hence from $(f_2^+(x_0))^2 - \varepsilon_2|f_2^+(x_0)|$ we obtain

$$\begin{aligned}
c(c - \varepsilon_2) &\geq c(c - \frac{G}{2}) \\
&\geq G(G - \frac{G}{2}) = \frac{G^2}{2} > 0.
\end{aligned}$$

as required.

We consider $|y_1^+(\theta^*) - x_1^+(t_j^0)|$ and define

$$r_b :=\mid \delta_1 \mid \cdot \left|\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + e_1 \cdot f_1^+(x_0) + e_2 \cdot f_2^+(x_0)}{(f_2^+(x_0))^2 + e_2 \cdot f_2^+(x_0)}\right| = |y_1^+(\theta^*) - x_1^+(t_j^0)|.$$

The calculation for $A^+(t_0)$ follows from

$$\begin{aligned}
A^+(t_0) &= \sqrt{\delta_1^2 + \delta_2^2} \\
&= \sqrt{\delta_1^2 + \delta_1^2 \frac{\left(f_1^+(x_0)\right)^2}{\left(f_2^+(x_0)\right)^2}} \\
&= \mid \delta_1 \mid \frac{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}{\mid f_2^+(x_0) \mid}
\end{aligned}$$

We now calculate $\frac{A^+(t_0)}{r_b}$. We have

$$
\begin{aligned}
\frac{A^+(t_0)}{r_b} &= \frac{\mid \delta_1 \mid \frac{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}{|f_2^+(x_0)|}}{\mid \delta_1 \mid \left|\frac{(f_1^+(x_0))^2+(f_2^+(x_0))^2+f_1^+(x_0)\cdot e_1+e_2\cdot f_2^+(x_0)}{(f_2^+(x_0))^2+e_2\cdot f_2^+(x_0)}\right|} \\
&= \frac{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}{|f_2^+(x_0)|}\cdot\frac{|(f_2^+(x_0))^2+e_2\cdot f_2^+(x_0)|}{|(f_1^+(x_0))^2+(f_2^+(x_0))^2+f_1^+(x_0)\cdot e_1+e_2\cdot f_2^+(x_0)|} \\
&= \frac{|f_2^+(x_0)+e_2|}{\left|\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}+\frac{f_1^+(x_0)\cdot e_1+e_2\cdot f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}\right|}.
\end{aligned}
$$

Let

$$
F^+(e_1,e_2) := \frac{|f_2^+(x_0)+e_2|}{\left|\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}+\frac{f_1^+(x_0)\cdot e_1+e_2\cdot f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}\right|} \tag{2.44}
$$

with bounds

$$
\begin{aligned}
\frac{\mid f_2^+(x_0)\mid -\varepsilon_2}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}+\frac{\varepsilon_1\cdot|f_1^+(x_0)|+\varepsilon_2\cdot|f_2^+(x_0)|}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}} &\leq \\
F^+(e) &\leq \\
\frac{\mid f_2^+(x_0)\mid +\varepsilon_2}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}+\frac{-\varepsilon_1\cdot|f_1^+(x_0)|-\varepsilon_2\cdot|f_2^+(x_0)|}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}}
\end{aligned}
$$

We need to show that the denominator E of $F^+(e_1,e_2)$ is strictly positive. This only requires to check that

$$
(f_2^+(x_0))^2-\varepsilon_2|f_2^+(x_0)|+(f_1^+(x_0))^2-\varepsilon_1|f_1^+(x_0)|\geq E>0.
$$

We have $|f_2^+(x_0)|=c$ and $[|f_2^+(x_0)|-\varepsilon_2]=[c-\varepsilon_2]$. Hence from $(f_2^+(x_0))^2 - \varepsilon_2|f_2^+(x_0)|$ we obtain

$$
\begin{aligned}
c(c-\varepsilon_2) &\geq c(c-\frac{G}{2}) \\
&\geq G(G-\frac{G}{2})=\frac{G^2}{2}.
\end{aligned}
$$

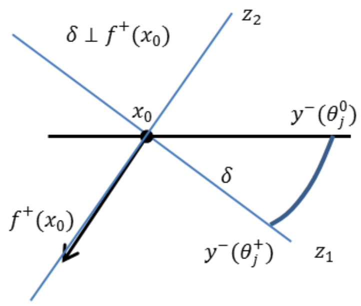

Figure 2.12: Case II

From $(f_1^+(x_0))^2 - \varepsilon_1|f_1^+(x_0)|$ we obtain

$$\begin{aligned}(f_1^+(x_0))^2 - \varepsilon_2|f_1^+(x_0)| &\geq -\varepsilon_1|f_1^+(x_0)| \\ &\geq -\varepsilon_1\frac{3D}{2} \\ &\geq -\frac{G^2}{4},\end{aligned}$$

with $\varepsilon_1 \leq min\left(\frac{1}{2}, \frac{G^2}{6D}\right)$. Hence it follows that

$$\frac{G^2}{2} - \frac{G^2}{4} \geq E > 0$$

as required.

Also, we have

$$\big(|\ f_2^+(x_0)\ | - \varepsilon_2\big)^2 > 0$$

since $\varepsilon_2 \leq \frac{G}{2}$, hence

$$\begin{aligned}\big(|\ f_2^+(x_0)\ | - \varepsilon_2\big) &= (|-c\ | - \varepsilon_2) \\ &\geq \left(c - \frac{G}{2}\right) \\ &\geq \left(G - \frac{G}{2}\right) > 0.\end{aligned}$$

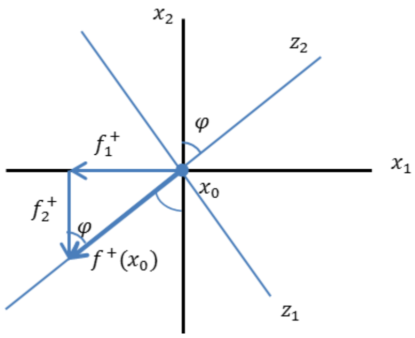

Figure 2.13: Rotation

Case II

We now want to calculate $\frac{A^-(t_0)}{r_b}$. This requires to consider a coordinate rotation through x_0. Hence, let

$$P = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$$

be an orthogonal rotation matrix changing a new orthonormal basis B' to its old orthonormal basis B and

$$P^{-1} = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix}$$

performing the reverse transformation. Now we want to determine the angle φ generating the new coordinate system. We have at $x_0 \in K$ that

$$\frac{f_1^+(x_0)}{f_2^+(x_0)} = \tan(\varphi) \Rightarrow \varphi.$$

We now introduce the notation under a coordinate rotation $P^{-1}(\varphi)$.

$$\begin{aligned} \xi &:= P^{-1}x \\ \dot{\xi} &:= g^+(\xi) \\ \dot{x} &:= f^-(x) \end{aligned}$$

then we have the following relationships:

$$\dot{\xi} = P^{-1}\dot{x} = P^{-1}f^{-}(x) = P^{-1}f^{-}(P\xi) = g^{+}(\xi).$$

Define

$$\begin{aligned}\zeta &:= P^{-1}y \\ \dot{\zeta} &:= g^{+}(\zeta) \\ \dot{y} &:= f^{-}(y),\end{aligned}$$

then we have the following relationships:

$$\dot{\zeta} = P^{-1}\dot{y} = P^{-1}f^{-}(y) = P^{-1}f^{-}(P\zeta) = g^{+}(\zeta).$$

By conditions of theorem 5 K^0 is a non-empty and compact set which contains no equilibrium. Let $G_z > 0$ be a constant such that for all $\xi_0 \in K^0$ satisfying $g_2^+(\xi_0) < 0$, $g_2^+(\xi_0) = -c_z \leq -G_z$,

$$-G_z := \sup_{\xi_0 \in K^0, g_2^+(\xi_0)<0} g_2^+(\xi_0)$$

Let $\varepsilon_2 \leq \frac{G_z}{2}$. Also let there be a constant $D_z > 0$ such that for all $\xi_0 \in K^0$ satisfying $g_2^+(\xi_0) < 0$ and $g_1^+(\xi_0) = d_z \leq D_z$,

$$-D_z := \sup_{\xi_0 \in K^0, g_2^+(\xi_0)<0} \mid g_1^+(\xi_0) \mid$$

Let $\varepsilon_1 \leq \frac{D_z}{2}$. We choose $\varepsilon^z > 0$ then there is a $b^z = (b_1^z, b_2^z) > 0$ so that we can construct a box in the positive phase space with center ξ_0 by

$$B^+_{(\xi_0)} := \left\{(\zeta_1, \zeta_2) \in K \cap [\xi_0^1 - b_1^z, \xi_0^1 + b_1^z] \times [0, b_2^z]\right\}$$

such that

$$
\begin{aligned}
-\frac{D_z}{2} \leq \mid d_z \mid -\varepsilon_1 \leq -d_z - \varepsilon_1 &\leq \\
g_1^+(\zeta) &\leq d_z + \varepsilon_1 \leq \mid d_z \mid +\varepsilon_1 \leq \frac{3D_z}{2} \qquad (2.45)\\
-\frac{3G_z}{2} \leq -c_z - \varepsilon_2 &\leq \\
g_2^+(\zeta) &\leq -c_z + \varepsilon_2 \leq -\frac{G_z}{2} \qquad (2.46)
\end{aligned}
$$

Remark 1 *In the new coordinate system, we now move forward in time along a trajectory in order to find time θ^{**}. Hence, in the new coordinates we let θ^{**} be the time when $\zeta_2^+(\theta^{**}) = 0$. This is the point when $\zeta^+(\theta^{**}) = y^-(\theta^{**})$ where $y^-(\theta^{**})$ is the notation of this point without coordinate transformation.*

Now, we want to show that there is a time θ^{**} such that the solution $\zeta_2^+(\theta)$ starting at $\zeta_2^+(\theta_j^0) > 0$ reaches $\zeta_2^+(\theta^{**}) = 0$. A solution $\zeta_2^+(\tau) < 0$ decreases to $\zeta_2^+(\theta^{**}) = 0$ since $g_2^+(\zeta) < 0$ for all $\zeta^+(\tau) \in B^+_{(\xi_0)}$ with $g_2^+(\zeta) \in [-c_z - \varepsilon_2, -c_z + \varepsilon_2]$. We have

$$
\begin{aligned}
\zeta_2^+(\theta) &= \zeta_2^+(\theta_j^0) + \int_{\theta_j^0}^{\theta^{**}} g_2^+(\zeta) d\tau \\
\zeta_2^+(\theta) - \zeta_2^+(\theta_j^0) &= \int_{\theta_j^0}^{\theta^{**}} g_2^+(\zeta) d\tau.
\end{aligned}
$$

Define $\sigma_2 := \zeta_2^+(\theta_j^0)$ and $e_2 := \frac{1}{\tau^*}\int_{\theta_j^0}^{\theta^{**}} [g_2^+(\zeta) + c_z] d\tau$. Then since $\zeta_2^+(\theta^{**}) = 0$ and $\mid e_2 \mid \leq \varepsilon_2$ we have

$$
-\sigma_2 = -c_z \cdot \theta^{**} + e_2 \cdot \theta^{**}
$$

$$
\frac{\sigma_2}{c_z - e_2} = \theta^{**} \qquad (2.47)
$$

Hence

$$
\frac{\sigma_2}{c_z + |e_2|} \leq \theta^{**}. \qquad (2.48)
$$

Hence there is a time interval with $\theta^{**} \in [\frac{\sigma_2}{c_z - \varepsilon_2}, \frac{\sigma_2}{c_z + \varepsilon_2}]$ such that $\zeta_2^+(\theta^{**}) = 0$. We have found a time θ^{**} in the new coordinate system. Time θ^{**} is invariant under a coordinate transformation.

We have shown that for frozen time $t_{j-1}^- = t_{j-1}^+$ of solution $x^-(t)$ there is a time $\theta^{**} :=\mid \theta_j^+ - \theta_j^0 \mid$ of solution $y^-(\theta)$ such that $N_2 := \frac{\delta_2}{c-\varepsilon_2} \geq \theta^{**}$. We have shown (2.35)

Next, we now consider trajectories in the negative space without a coordinate transformation. We want to determine $A^-(t_0)$.

By conditions of theorem 5, K^0 is a non-empty and compact set which contains no equilibrium. Let $G^- > 0$ be a constant such that for all $x_0 \in K^0$ satisfying $f_2^+(x_0) < 0$, $f_2^+(x_0) = -c^- \leq -G^-$,

$$-G^- := \max_{x_0 \in K^0, f_2^+(x_0)<0} f_2^+(x_0)$$

Let $\varepsilon_2^- \leq \frac{G^-}{2}$. Also let there be a constant $D^- > 0$ such that for all $x_0 \in K^0$ satisfying $f_2^+(x_0) < 0$, $f_1^+(S_t^+ x_0) = d^- \leq D^-$,

$$D^- := \max_{x_0 \in K^0, f_2^+(x_0)<0} |f_1^+(x_0)|$$

Let $\varepsilon_1^- \leq \frac{D^-}{2}$.

Given $\varepsilon = (\varepsilon_1^-, \varepsilon_2^-) > 0$ there is a $b = (b_1^-, b_2^-) > 0$ so that we can construct a box in the negative phase space with center x_0 by

$$B_{(x_0)}^- := \left\{ (y_1^-, y_2^-) \in K \cap [x_0^1 - b_1^-, x_0^1 + b_1^-] \times [0, -b_2^-] \right\}$$

such that

$$\begin{aligned} -\frac{D^-}{2} \leq - \mid d^- \mid -\varepsilon_1^- \leq d^- - \varepsilon_1^- \quad &\leq \\ f_1^-(y^-) \quad &\leq \quad d^- + \varepsilon_1^- \leq \mid d^- \mid + \varepsilon_1^- \leq \frac{3D^-}{2} \qquad (2.49) \\ -\frac{3G^-}{2} \leq -c^- - \varepsilon_2^- \quad &\leq \\ f_2^-(y^-) \quad &\leq \quad -c^- + \varepsilon_2^- \leq -\frac{G^-}{2}. \qquad (2.50) \end{aligned}$$

We are given an ε^- and choose $b^- > 0$ small enough so that $y^-(\tau) \in B_{(x_0)}^-$.

Given $b^- > 0$ and $0 < \phi < \frac{1}{2}\pi$ we show that there is a $b_z > 0$ with $\zeta^+(\tau) \in \widehat{B}^+_{(\xi_0)}$ for all $\tau \in [0, \theta^{**}]$. We define

$$\widehat{B}^+_{(\xi_0,\phi)} := \left\{(\zeta^1, \zeta^2) \in K \cap [b^z_1(\phi), 0] \times [0, b^z_2(\phi)]\right\}.$$

with

$$\begin{aligned} b^z_1 &:= \max_{0<\phi<\frac{1}{2}\pi} b^z_1(\phi) \\ b^z_2 &:= \max_{0<\phi<\frac{1}{2}\pi} b^z_2(\phi) \end{aligned}$$

where

$$\begin{aligned} b^z_1(\phi) &= b^-_1 \cos(\phi) \\ b^z_2(\phi) &= b^-_2 \sin(\phi). \end{aligned}$$

such that $\zeta^+(\tau) \in \widehat{B}^+_{(\xi_0)}$ for all $\tau \in [0, \theta^{**}]$ and $\widehat{B}^+_{(\xi_0)} \subset B^-_{(x_0)}$. Hence, there is $\varepsilon^- > 0$ with b^- small enough such that $(y^-(\tau), \zeta^+(\tau)) \in \left\{B^-_{(x_0)} \cap \widehat{B}^+_{(\xi_0,\theta)}\right\}$. Finally, for given $\varepsilon > 0$ set $\|b\| = \|\delta\|$ where

$$\begin{aligned} r_b \frac{\mid f^+_2(x_0) \mid -\varepsilon_2}{\sqrt{\left(f^+_1(x_0)\right)^2 + \left(f^+_2(x_0)\right)^2 + \frac{\varepsilon_1 \cdot |f^+_1(x_0)| + \varepsilon_2 \cdot |f^+_2(x_0)|}{\sqrt{\left(f^+_1(x_0)\right)^2 + \left(f^+_2(x_0)\right)^2}}}} &\leq \\ A^+(t_0) &\leq \\ r_b \frac{\mid f^+_2(x_0) \mid +\varepsilon_2}{\sqrt{\left(f^+_1(x_0)\right)^2 + \left(f^+_2(x_0)\right)^2 + \frac{-\varepsilon_1 \cdot |f^+_1(x_0)| - \varepsilon_2 \cdot |f^+_2(x_0)|}{\sqrt{\left(f^+_1(x_0)\right)^2 + \left(f^+_2(x_0)\right)^2}}}} & \end{aligned}$$

with $A^+(t_0) = \|\delta\|$ such that $y^+(\tau) \in B^+_{x_0}$ for all $\tau \in [0, \theta^*]$. It remains to show that for given ε^- we have $y^-(\tau) \in B^-_{(x_0)}$ for all $\tau \in [0, \theta^*]$.

Now, we have

$$y^-_1(\theta^{**}) = y^-_1(\theta^0_j) + \int_{\theta^0_j}^{\theta^{**}} f^-_1(y^-(\tau))d\tau$$

Define $\delta_1^- := y_1^-(\theta_j^0) - x_1^-(\theta_j^0)$ and $e_1^- := \frac{1}{\theta^{**}} \int_{\theta_j^0}^{\theta^{**}} [f_1^-(y_1^-(\tau)) - d^-] d\tau$, and since $|e_1^-| \leq \varepsilon_1^-$ we have

$$y_1^-(\theta^{**}) - x_1^-(\theta_j^0) \quad = \quad \delta_1^- + t^{**} \cdot [d^- + e_1^-]$$

and by equation (2.47) $\frac{\sigma_2}{c_z - e_2} = \frac{\delta_2^-}{c^- - e_2}$ in the non transformed coordinates we obtain

$$y_1^-(t^{**}) - x_1^-(t_j^0) \quad = \quad \delta_1^- + \frac{\delta_2^-}{c^- - e_2^-} \cdot [d^- + e_1^-].$$

Using $\delta_1^- f_1^-(x_0) + \delta_2^- f_2^-(x_0) = 0$ yields

$$\begin{aligned} y_1^-(t^{**}) - x_1^-(t_j^0) &= \delta_1^- - \left(\frac{\delta_1^- f_1^-(x_0)}{(-f_2^-(x_0) - e_2^-) f_2(x_0)} \right) \cdot [f_1^-(x_0) + e_1^-] \\ y_1^-(t^{**}) - x_1^-(t_j^0) &= \delta_1^- \left[\frac{(f_1^-(x_0))^2 + (f_2^-(x_0))^2 + e_1^- \cdot f_1^-(x_0) + e_2^- \cdot f_2^-(x_0)}{(f_2^-(x_0))^2 + e_2^- \cdot f_2^-(x_0)} \right]. \end{aligned}$$

We have the following bounds

$$\begin{aligned} \mid \delta_1^- \mid \left[\frac{(f_1^-(x_0))^2 + (f_2^-(x_0))^2 - \varepsilon_1^- \mid f_1^-(x_0) \mid - \varepsilon_2^- \mid f_2^-(x_0) \mid}{(f_2^-(x_0))^2 + \varepsilon_2^- \mid f_2^-(x_0) \mid} \right] &\leq \\ |y_1^-(t^{**}) - x_1^-(t_j^0)| &\leq \\ \mid \delta_1^- \mid \left[\frac{(f_1^-(x_0))^2 + (f_2^-(x_0))^2 + \varepsilon_1^- \mid f_1^-(x_0) \mid + \varepsilon_2^- \mid f_2^-(x_0) \mid}{(f_2^-(x_0))^2 - \varepsilon_2^- \mid f_2^-(x_0) \mid} \right] & \end{aligned}$$

We consider $\mid y_1^-(t^{**}) - x_1^-(t_j^0) \mid$ and define

$$r_b := \mid \delta_1^- \mid \cdot \left| \frac{(f_1^-(x_0))^2 + (f_2^-(x_0))^2 + e_1^- \cdot f_1^-(x_0) + e_2^- \cdot f_2^-(x_0)}{(f_2^-(x_0))^2 + e_2^- \cdot f_2^-(x_0)} \right| = |y_1^-(t^{**})|.$$

The calculation for $A^-(t_0)$ follows from

$$\begin{aligned} A^-(t_0) &= \sqrt{(\delta_1^-)^2 + (\delta_2^-)^2} \\ &= \sqrt{(\delta_1^-)^2 + (\delta_1^-)^2 \frac{\left(f_1^-(x_0)\right)^2}{\left(f_2^-(x_0)\right)^2}} \\ &= \mid \delta_1^- \mid \frac{\sqrt{\left(f_1^-(x_0)\right)^2 + \left(f_2^-(x_0)\right)^2}}{\mid f_2^-(x_0) \mid}. \end{aligned}$$

We now calculate $\frac{A^-(t_0)}{r_b}$. We have

$$\begin{aligned}
\frac{A^-(t_0)}{r_b} &= \frac{|\ \delta_1^-\ |\ \frac{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}{|f_2^-(x_0)|}}{|\ \delta_1^-\ |\ \left|\frac{(f_1^-(x_0))^2+(f_2^-(x_0))^2+f_1^-(x_0)\cdot e_1^-+e_2^-\cdot f_2^-(x_0)}{(f_2^+(x_0))^2+e_2^-\cdot f_2^+(x_0)}\right|} \\
&= \frac{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}{|f_2^-(x_0)|}\cdot\frac{|(f_2^-(x_0))^2+e_2^-\cdot f_2^-(x_0)|}{|(f_1^-(x_0))^2+(f_2^-(x_0))^2+f_1^-(x_0)\cdot e_1^-+e_2^-\cdot f_2^-(x_0)|} \\
&= \frac{|f_2^-(x_0)+e_2^-|}{\left|\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}+\frac{f_1^-(x_0)\cdot e_1^-+e_2^-\cdot f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}\right|}.
\end{aligned}$$

Let

$$F^-(e) := \frac{|f_2^-(x_0)+e_2^-|}{\left|\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}+\frac{f_1^-(x_0)\cdot e_1^-+e_2^-\cdot f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}\right|} \tag{2.51}$$

with bounds

$$\begin{aligned}
\frac{|\ f_2^-(x_0)\ | -\varepsilon_2^-}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}+\frac{+\varepsilon_1^-\cdot|f_1^-(x_0)|+\varepsilon_2^-\cdot|f_2^-(x_0)|}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}} &\leq \\
F^+(e) &\leq \\
\frac{|\ f_2^-(x_0)\ | +\varepsilon_2^-}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}+\frac{-\varepsilon_1^-\cdot|f_1^-(x_0)|-\varepsilon_2^-\cdot|f_2^-(x_0)|}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}
\end{aligned}$$

We need to show that the denominator of $F^-(e^-) \geq E^- > 0$. This only requires to check that

$$(f_2^-(x_0))^2 - \varepsilon_2|f_2^-(x_0)| + (f_1^-(x_0))^2 - \varepsilon_1|f_1^-(x_0)| \geq E > 0.$$

We have $|f_2^-(x_0)| = c$ and $[|f_2^-(x_0)| - \varepsilon_2^-] = [c - \varepsilon_2^-]$. Hence from $(f_2^-(x_0))^2 - \varepsilon_2^-|f_2^-(x_0)|$ we obtain

$$\begin{aligned}
c^-(c^- - \varepsilon_2^-) &\geq c^-(c^- - \frac{G^-}{2}) \\
&\geq G^-(G^- - \frac{G^-}{2}) = \frac{(G^-)^2}{2}.
\end{aligned}$$

From $(f_1^-(x_0))^2 - \varepsilon_1^-|f_1^-(x_0)|$ we obtain

$$\begin{aligned}(f_1^-(x_0))^2 - \varepsilon_2^-|f_1^-(x_0)| &\geq -\varepsilon_1^-|f_1^-(x_0)| \\ &\geq -\varepsilon_1^-\frac{3D^-}{2} \\ &\geq -\frac{(G^-)^2}{4},\end{aligned}$$

with $\varepsilon_1^- \leq min\left(\frac{1}{2}, \frac{(G^-)^2}{6D^-}\right)$. Hence it follows that

$$\frac{(G^-)^2}{2} - \frac{(G^-)^2}{4} = E^- > 0$$

as required.

Also, we have

$$\left(\mid f_2^-(x_0) \mid -\varepsilon_2^-\right)^2 > 0$$

since $\varepsilon_2^- \leq \frac{G^-}{2}$, hence

$$\begin{aligned}\left(\mid f_2^-(x_0) \mid -\varepsilon_2^-\right) &= \left(\mid -c^- \mid -\varepsilon_2^-\right) \\ &\geq \left(c^- - \frac{G^-}{2}\right) \\ &\geq \left(G^- - \frac{G^-}{2}\right) > 0.\end{aligned}$$

Combining case I and case II

Step (iii) requires to show that $\frac{A^+(t_0)}{A^-(t_0)}e^{w^+ - w^-} < 1$.

By case I and case II we have

$$
\begin{aligned}
\frac{A^-(t_0)}{A^+(t_0)} &= \frac{\frac{A^-(t_0)}{r_b}}{\frac{A^+(t_0)}{r_b}} \\
&= F^-(\varepsilon^-)e^{w^-}\frac{1}{f^+(\varepsilon)e^{w^+}} \\
&= \left(\frac{\frac{f_2^-(x_0)+|\varepsilon_2^-|}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2+\frac{f_1^-(x_0)|\varepsilon_1^-|+|\varepsilon_2^-|f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}}}{\frac{f_2^+(x_0)+|\varepsilon_2|}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2+\frac{f_1^+(x_0)|\varepsilon_1|+|\varepsilon_2|f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}}}}\right)e^{w^- - w^+} \\
&\leq 1-\rho < 1
\end{aligned}
$$

where $\rho > 0$, since for all $x_0 \in K^0$ we can chose $(\varepsilon_1, \varepsilon_2)$ at the beginning small enough.

Remark 2 *We have that*

$$
\lim_{e\to 0} F(e) = \frac{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}} e^{w^-(x^1,0)-w^+(x^1,0)} < 1. \tag{2.52}
$$

which shows (2.33).

■

2.4.2 Part two: Cases III and IV

We consider the case where y is frozen at the point $x_2 = 0$.

We want to show that at time t_0 and $x_0 \in K^0$

$$
\frac{A^-(t_0)}{A^+(t_0)} < 1,
$$

where $A^+(t_0)$ is the distance between two adjacent solutions leaving in the positive phase space, and $A^-(t_0)$ is the distance between two adjacent solution leaving in the negative phase space. Above condition requires to calculate $\frac{A^+(t_0)}{r_b}$ and $\frac{A^-(t_0)}{r_b}$, where r_b is the distance between two adjacent solutions when both hit the axis $x_2 = 0$.

1. We calculate $\frac{A^+(t_0)}{r_b}$, case III
2. We calculate $\frac{A^-(t_0)}{r_b}$, case IV
3. We calculate $\frac{A^-(t_0)}{A^+(t_0)}$, combining case III and case IV.

Proposition 4 (Jump case (+/-)) *Let the assumptions of theorem 5 hold. Moreover assume that there are constants* $\delta, \nu, m, M, N_1, N_2 > 0$ *such that for all* $\eta \in \mathbb{R}^2$ *with* $\| \eta \| \leq \delta$ *there is a piecewise multi valued mapping* $\mathcal{T}_x^{x+\eta}(t)$ *defined by equation (2.3) which satisfies for all* $t \in \mathcal{I}^{\pm}$ *or for all* $t \in \mathcal{J}^{\pm}$

$$m \leq \triangle \mathcal{T}_x^{x+\eta}(t) \leq M. \tag{2.53}$$

and

$$(S_{\mathcal{T}^+(t)}(x+\eta) - S_t^+ x) \cdot f^+(S_t^+ x) = 0.$$

Also for all $t \in [t_j^-, t_j^+]$.

$$A^-(t_j^+) < A^+(t_j^-) e^{-\nu t}. \tag{2.54}$$

For all $t \in \mathcal{I}^{\pm}$ *and* $t^* \in [\theta_j^-, t_j^0]$ *with*

$$|t_j^0 - t_j^-| \leq N_1. \tag{2.55}$$

and $y_2^+(t^*) \geq 0$.

For all $t \in \mathcal{I}^{\pm}$ *and* $t^{**} \in [t_j^0, t_j^+]$ *with*

$$|t_j^+ - t_j^0| \leq N_2. \tag{2.56}$$

and $y_2^-(t^{**}) \leq 0$.

Proof. (2.53) is already shown in part one of the proof and is hence omitted. It remains to show (2.54), (2.55), and (2.56).

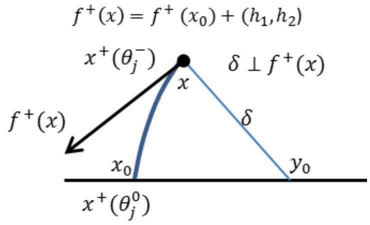

Figure 2.14: Case III

Case III

By conditions of theorem 5, K^0 is a non-empty and compact set which contains no equilibrium. Let $G > 0$ be a constant such that for all $x_0 \in K^0$ satisfying $f_2^+(x_0) < 0$, $f_2^+(x_0) = -c \leq -G$,

$$-G := \max_{x_0 \in K^0, f_2^+(x_0) < 0} f_2^+(x_0)$$

Let $\varepsilon_2 \leq \frac{G}{2}$. Also let there be a constant $D > 0$ such that for all $x_0 \in K^0$ satisfying $f_2^+(x_0) < 0$, $f_1^+(S_t^+ x_0) = d \leq D$,

$$D := \max_{x_0 \in K^0, f_2^+(x_0) < 0} |f_1^+(x_0)|$$

Let $\varepsilon_1 \leq \frac{D}{2}$.

Given $(\varepsilon_1, \varepsilon_2) > 0$ there is a $(b_1, b_2) > 0$ so that we can construct a box in the positive phase space with center x_0 by

$$B^+_{(x_0)} := \{(x_1, x_2) \in K \cap [x_0^1 - b_1, x_0^1 + b_1] \times [0, b_2]\}$$

such that

$$\begin{aligned} -\frac{D}{2} \leq - \mid d \mid - \varepsilon_1 \leq d - \varepsilon_1 \quad &\leq \\ f_1^+(x_1, x_2) \quad &\leq \quad d + \varepsilon_1 \leq \mid d \mid + \varepsilon_1 \leq \frac{3D}{2} \qquad (2.57) \\ -\frac{3G}{2} \leq -c - \varepsilon_2 \quad &\leq \quad f_2^+(x_1, x_2) \leq -c + \varepsilon_2 \leq -\frac{G}{2}. \qquad (2.58) \end{aligned}$$

Consider a solution $x_2^+(t)$ with $x_2^+(t_j^-) \in B^+_{(x_0)}$ and $x_2^+(t_j^-) > 0$. Now, we want to show that there is a time t^* such that the solution $x_2^+(t^*) = 0$. Define the point $x_0 := x(t^*)$, where $x_2^+(t^*) = 0$. Hence, $x_0 \in K^0$. We also define $t_j^0 =: t^*$.

Hence there is $(h_1, h_2) > 0$ such that $x^+(t_j^-) = x_0^+ + (h_1, h_2)$.

A solution $x_2^+(\tau) > 0$ decreases to $x_2^+(t^*) = 0$ since $f_2^+(x) \leq \frac{-G}{2} < 0$ for all $x^+(\tau) \in B^+_{(x_0)}$ with $f_2^+(x^+(\tau)) \in [-c - \varepsilon_2, -c + \varepsilon_2]$ and $\tau \in [0, t^*]$. We have

$$\begin{aligned} x_2^+(t) &= x_2^+(t_j^-) + \int_{t_j^-}^{t^*} f_2^+(x^+(\tau))d\tau \\ x_2^+(t) - x_2^+(t_j^-) &= \int_{t_j^-}^{t^*} f_2^+(x^+(\tau))d\tau. \end{aligned}$$

Define $\delta_2 := x_2^+(t_j^-)$ and $e_2 := \frac{1}{\tau^*}\int_{t_j^-}^{t^*}[f_2^+(x^+(\tau)) + c]d\tau$. Then since $x_2^+(t^*) = 0$ and $|e_2| \leq \varepsilon_2$ we have by equation (2.58) that

$$-\delta_2 = -c \cdot t^* + e_2 \cdot t^*$$

$$\frac{\delta_2}{c - e_2} = t^*. \qquad (2.59)$$

Hence

$$\frac{\delta_2}{c + |e_2|} \leq t^*.$$

Hence there is a time $t^* \in [\frac{\delta_2}{c+\varepsilon_2}, \frac{\delta_2}{c-\varepsilon_2}]$ such that $x_2^+(t^*) = 0$.

We have shown that for frozen time $\theta_j^0 = \theta_j^-$ of solution $y^+(t)$ there is a time $t^* :=\mid t_{j-1}^- - t_{j-1}^0 \mid$ of solution $x^+(t)$ such that $N_1 := \frac{\delta_2}{c-\varepsilon_2} \geq t^*$. We have shown (2.55)

Now, we use t^* in order to determine $x_1^+(t^*)$. We have

$$x_1^+(t^*) = x_1^+(t_j^-) + \int_{t_j^-}^{t^*} f_1^+(x^+(\tau))d\tau$$

Define $\delta_1 := x_1^+(t_j^0) - y_1^+(\theta_j^-)$ and $e_1 := \frac{1}{\tau^*}\int_{t_j^-}^{t^*}[f_1^+(x^+(\tau)) - d]d\tau$, and since $|e_1| \leq$

ε_1 we have

$$= \quad \delta_1 + t^* \cdot [d + e_1]$$

and by equation (2.43) we obtain

$$= \quad \delta_1 + \frac{\delta_2}{c - e_2} \cdot [d + e_1].$$

Using $\delta_1 \left[f_1^+(x_0) + h_1\right] + \delta_2 \left[f_2^+(x_0) + h_2\right] = 0$ yields

$$\begin{aligned}
&= \quad \delta_1 - \left(\frac{\delta_1 \left[f_1^+(x_0) + h_1\right]}{(-f_2^+(x_0) - e_2) \left[f_2^+(x_0) + h_2\right]} \right) \cdot [f_1^+(x_0) + e_1] \\
&= \quad \delta_1 \left[1 - \left(\frac{\left[f_1^+(x_0) + h_1\right] \cdot \left[f_1^+(x_0) + e_1\right]}{(-f_2^+(x_0) - e_2) \left[f_2^+(x_0) + h_2\right]} \right) \right] \\
&= \quad \delta_1 \left[\frac{(f_2^+(x_0) + e_2) \left[f_2^+(x_0) + h_2\right]}{(f_2^+(x_0) + e_2) \left[f_2^+(x_0) + h_2\right]} - \left(\frac{\left[f_1^+(x_0) + h_1\right] \cdot \left[f_1^+(x_0) + e_1\right]}{(-f_2^+(x_0) - e_2) \left[f_2^+(x_0) + h_2\right]} \right) \right] \\
&= \quad \delta_1 \left[\frac{(f_2^+(x_0) + e_2) \left[f_2^+(x_0) + h_2\right] + \left[f_1^+(x_0) + h_1\right] \cdot \left[f_1^+(x_0) + e_1\right]}{(f_2^+(x_0) + e_2) \left[f_2^+(x_0) + h_2\right]} \right] \\
&= \quad \delta_1 \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + (e_1 + h_1) \cdot f_1^+(x_0) + (e_2 + h_2) \cdot f_2^+(x_0) + h_1 e_1 + h_2 e_2}{(f_2^+(x_0))^2 + (e_2 + h_2) \cdot f_2^+(x_0) + e_2 h_2} \right].
\end{aligned}$$

We have the following bounds

$$\begin{aligned}
|\delta_1| \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 - (2\varepsilon_1) \cdot |f_1^+(x_0)| - \varepsilon_1^2 - (2\varepsilon_2) \cdot |f_2^+(x_0)| - \varepsilon_2^2}{(f_2^+(x_0))^2 + 2\varepsilon_2 \cdot |f_2^+(x_0)| + \varepsilon_2^2} \right] &\leq \\
|x_1^+(t^*) - y_1^+(\theta_j^0)| &\leq \\
|\delta_1| \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + (2\varepsilon_1) \cdot |f_1^+(x_0)| + \varepsilon_1^2 + (2\varepsilon_2) \cdot |f_2^+(x_0)| + \varepsilon_2^2}{(f_2^+(x_0))^2 - 2\varepsilon_2 \cdot |f_2^+(x_0)| - \varepsilon_2^2} \right] &\leq
\end{aligned}$$

We need to verify the bounds. This only requires to check that

$$(f_1^+(x_0))^2 + (f_2^+(x_0))^2 - (2\varepsilon_1) \cdot |f_1^+(x_0)| - \varepsilon_1^2 - (2\varepsilon_2) \cdot |f_2^+(x_0)| - \varepsilon_2^2 \geq E > 0.$$

a well as

$$
\begin{aligned}
(f_2^+(x_0))^2 - (2\varepsilon_2)\cdot|f_2^+(x_0)| - \varepsilon_2^2 &\geq c^2 - 2\varepsilon_2 c - \varepsilon_2^2 = c(c-2\varepsilon_2) - \varepsilon_2^2 \\
&\geq G(G-2\varepsilon_2) - \varepsilon_2^2 \\
&\geq G(G-2\frac{G}{9}) - \frac{G}{81}^2 = \frac{62}{81}G^2 > 0
\end{aligned}
$$

for $\varepsilon_2 := \frac{G}{9}$.

We also have

$$
\begin{aligned}
(f_2^+(x_0))^2 - (2\varepsilon_1)\cdot|f_2^+(x_0)| - \varepsilon_1^2 &\geq c^2 - 2\varepsilon_1 c - \varepsilon_2^2 = c(c-2\varepsilon_2) - \varepsilon_1^2 \\
&\geq G(G-2\varepsilon_1) - \varepsilon_1^2 \\
&\geq G(G-2\frac{G}{3}) - \frac{G}{9}^2 = \frac{2}{9}G^2 > 0
\end{aligned}
$$

for $\varepsilon_2 := \frac{G}{3}$. Then it follows that

$$
\frac{62}{81}G^2 - \frac{2}{9}G^2 = \frac{44}{81}G^2 > 0
$$

as requested.

We consider $|x_1^+(t^*) - y_1^+(\theta_j^0)|$ and define

$$
\begin{aligned}
r_b &:= \\
&\quad |\ \delta_1\ | \times \\
&\quad \left| \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + (e_1+h_1)\cdot f_1^+(x_0) + (e_2+h_2)\cdot f_2^+(x_0) + h_1e_1 + h_2e_2}{(f_2^+(x_0))^2 + (e_2+h_2)\cdot f_2^+(x_0) + e_2h_2} \right] \right| \\
&= |x_1^+(t^*) - y_1^+(\theta_j^0)|.
\end{aligned}
$$

The calculation for $A^+(t_0)$ follows from

$$
A^+(t_0) = \sqrt{\delta_1^2 + \delta_2^2}
$$

and using $\delta_1\left[f_1^+(x_0)+h_1\right]+\delta_2\left[f_2^+(x_0)+h_2\right]=0$ yields

$$
\begin{aligned}
&= \sqrt{\delta_1^2+\delta_1^2\frac{\left(f_1^+(x_0)+h_1\right)^2}{\left(f_2^+(x_0)+h_2\right)^2}} \\
&= \mid \delta_1 \mid \frac{\sqrt{\left(f_1^+(x_0)+h_1\right)^2+\left(f_2^+(x_0)+h_2\right)^2}}{\mid f_2^+(x_0)+h_2 \mid}.
\end{aligned}
$$

We now calculate $\frac{A^+(t_0)}{r_b}$. We have

$$
\begin{aligned}
\frac{A^+(t_0)}{r_b} &= \frac{\mid \delta_1 \mid \frac{\sqrt{\left(f_1^+(x_0)+h_1\right)^2+\left(f_2^+(x_0)+h_2\right)^2}}{|f_2^+(x_0)+h_2|}}{\mid \delta_1 \mid \left[\frac{(f_1^+(x_0))^2+(f_2^+(x_0))^2+(e_1+h_1)\cdot f_1^+(x_0)+(e_2+h_2)\cdot f_2^+(x_0)+h_1e_1+h_2e_2}{(f_2^+(x_0))^2+(e_2+h_2)\cdot f_2^+(x_0)+e_2h_2}\right]} \\
&= \frac{\sqrt{\left(f_1^+(x_0)+h_1\right)^2+\left(f_2^+(x_0)+h_2\right)^2}}{|f_2^+(x_0)+h_2|}\times \\
&\quad \frac{|(f_2^+(x_0))^2+(e_2+h_2)\cdot f_2^+(x_0)+e_2h_2|}{|(f_1^+(x_0))^2+(f_2^+(x_0))^2+(e_1+h_1)\cdot f_1^+(x_0)+(e_2+h_2)\cdot f_2^+(x_0)+h_1e_1+h_2e_2|} \\
&= \frac{1}{\left(\sqrt{\left(f_1^+(x_0)+h_1\right)^2+\left(f_2^+(x_0)+h_2\right)^2}|f_2^+(x_0)+h_2|\right)}\times \\
&\quad \frac{|(f_2^+(x_0))^2+(e_2+h_2)\cdot f_2^+(x_0)+e_2h_2|}{|(f_1^+(x_0))^2+(f_2^+(x_0))^2+(e_1+h_1)\cdot f_1^+(x_0)+(e_2+h_2)\cdot f_2^+(x_0)+h_1e_1+h_2e_2|} \\
&= \frac{|(f_2^+(x_0))^2+(e_2+h_2)\cdot f_2^+(x_0)+e_2h_2|}{B\left((f_1^+(x_0))^2+(f_2^+(x_0))^2\right)+B\cdot A} \\
&= \frac{|(f_2^+(x_0))^2+(e_2+h_2)\cdot f_2^+(x_0)+e_2h_2|}{B+B\frac{A}{\left((f_1^+(x_0))^2+(f_2^+(x_0))^2\right)}} \\
&= \frac{|(f_2^+(x_0))^2+(e_2+h_2)\cdot f_2^+(x_0)+e_2h_2|}{B\left(1+\frac{A}{\left((f_1^+(x_0))^2+(f_2^+(x_0))^2\right)}\right)}
\end{aligned}
$$

where $A=|(e_1+h_1)\cdot f_1^+(x_0)+(e_2+h_2)\cdot f_2^+(x_0)+h_1e_1+h_2e_2|$ and $B=\sqrt{\left(f_1^+(x_0)+h_1\right)^2+\left(f_2^+(x_0)+h_2\right)^2}|f_2^+(x_0)+h_2|$.

We define

$$F^+(e_1, e_2, h_1, h_2) \quad := \quad \frac{|(f_2^+(x_0))^2 + (e_2 + h_2) \cdot f_2^+(x_0) + e_2 h_2|}{B\left(1 + \frac{A}{\left((f_1^+(x_0))^2 + (f_2^+(x_0))^2\right)}\right)}.$$

Remark 3 *We observe that*

$$\lim_{e_1, e_2, h_1, h_2 \to 0} F^+(e_1, e_2, h_1, h_2) = \frac{f_2^+(x_0)}{\sqrt{f_1^+(x_0) + f_2^+(x_0)}},$$

This can be seen from

$F^+(e_1, e_2, h_1, h_2) =$

$$\frac{|(f_2^+(x_0))^2 + (e_2 + h_2) \cdot f_2^+(x_0) + e_2 h_2|}{\left(\sqrt{\left(f_1^+(x_0) + h_1\right)^2 + \left(f_2^+(x_0) + h_2\right)^2} |f_2^+(x_0) + h_2|\right)\left(1 + \frac{|(e_1 + h_1) \cdot f_1^+(x_0) + (e_2 + h_2) \cdot f_2^+(x_0) + h_1 e_1 + h_2 e_2|}{\left((f_1^+(x_0))^2 + (f_2^+(x_0))^2\right)}\right)}.$$

We have the bounds

$$\frac{|(f_2^+(x_0))^2| - (2\varepsilon_2) \cdot |f_2^+(x_0)| - \varepsilon_2^2}{\left(\sqrt{\left(f_1^+(x_0) - \varepsilon_1\right)^2 + \left(f_2^+(x_0) - \varepsilon_2\right)^2} \left(|f_2^+(x_0)| - \varepsilon_2\right)\right)\left(1 - \frac{(e_1 + \varepsilon_1) \cdot |f_1^+(x_0)| + (\varepsilon_2^2) \cdot |f_2^+(x_0)| + \varepsilon_1^2 + \varepsilon_2^2|}{\left((f_1^+(x_0))^2 + (f_2^+(x_0))^2\right)}\right)}$$
$$\leq F^+(e, h) \leq$$
$$\frac{|(f_2^+(x_0))^2| + (2\varepsilon_2) \cdot |f_2^+(x_0)| + \varepsilon_2^2}{\left(\sqrt{\left(f_1^+(x_0) + \varepsilon_1\right)^2 + \left(f_2^+(x_0) + \varepsilon_2\right)^2} \left(|f_2^+(x_0)| + \varepsilon_2\right)\right)\left(1 + \frac{(e_1 + \varepsilon_1) \cdot |f_1^+(x_0)| + (\varepsilon_2^2) \cdot |f_2^+(x_0)| + \varepsilon_1^2 + \varepsilon_2^2|}{\left((f_1^+(x_0))^2 + (f_2^+(x_0))^2\right)}\right)}$$

where $| e_1 \cdot h_1 | \leq \varepsilon_1$ and $| e_2 \cdot h_2 | \leq \varepsilon_2$ and box small enough.
We need to show that

$$F^+(e_1, e_2, h_1, h_2) \neq 0$$

Now similarly as before, with $| e_1 \cdot h_1 | \leq \varepsilon_1$ and $| e_2 \cdot h_2 | \leq \varepsilon_2$ we need to verify the bounds. This only requires to check that

$$(f_1^+(x_0))^2 + (f_2^+(x_0))^2 - (2\varepsilon_1) \cdot |f_1^+(x_0)| - \varepsilon_1^2 - (2\varepsilon_2) \cdot |f_2^+(x_0)| - \varepsilon_2^2 \geq E > 0.$$

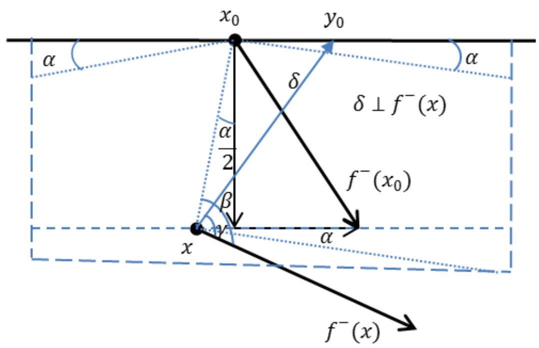

Figure 2.15: Case IVa

a well as

$$\begin{aligned}
(f_2^+(x_0))^2 - (2\varepsilon_2) \cdot |f_2^+(x_0)| - \varepsilon_2^2 &\geq c^2 - 2\varepsilon_2 c - \varepsilon_2^2 = c(c - 2\varepsilon_2) - \varepsilon_2^2 \\
&\geq G(G - 2\varepsilon_2) - \varepsilon_2^2 \\
&\geq G(G - 2\frac{G}{9}) - \frac{G}{81}^2 = \frac{62}{81}G^2 > 0
\end{aligned}$$

for $\varepsilon_2 := \frac{G}{9}$.

We also have

$$\begin{aligned}
(f_2^+(x_0))^2 - (2\varepsilon_1) \cdot |f_2^+(x_0)| - \varepsilon_1^2 &\geq c^2 - 2\varepsilon_1 c - \varepsilon_2^2 = c(c - 2\varepsilon_2) - \varepsilon_1^2 \\
&\geq G(G - 2\varepsilon_1) - \varepsilon_1^2 \\
&\geq G(G - 2\frac{G}{3}) - \frac{G}{9}^2 = \frac{2}{9}G^2 > 0
\end{aligned}$$

for $\varepsilon_2 := \frac{G}{3}$. Then it follows that

$$\frac{62}{81}G^2 - \frac{2}{9}G^2 = \frac{44}{81}G^2 > 0$$

as requested.

Case IV

This case is depicted in figure 2.16. Some ideas are presented in figure 2.15. Given x_0 there is $f^-(x_0)$ with $0 < \alpha < \frac{1}{2}\pi$ by conditions of theorem 5. By

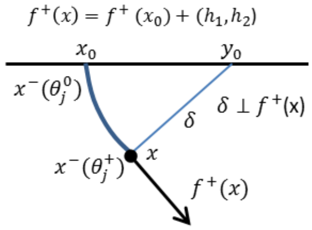

Figure 2.16: Case IVb

the implicit function theorem we find x such that $f^-(x) \perp \delta$. Observe that $0 < \alpha < \frac{1}{2}\pi$ and $\beta > \frac{1}{2}\pi$ with S_τ^- continuous. Then using $f^-(x) \perp \delta \approx f^-(x_0) + (h_1, h_2) \perp \delta$ we derive the results in a similar way as previously.

We now want to calculate $\frac{A^-(t_0)}{r_b}$. This requires to consider a coordinate rotation through (y_0). Hence, let

$$P = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

be an orthogonal rotation matrix changing a new orthonormal basis B' to its old orthonormal basis B and

$$P^{-1} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

performing the reverse transformation. Now we want to determine the angle ϕ generating the new coordinate system. We have at $y_0 \in K$ that

$$\frac{f_1^-(y_0)}{f_2^-(y_0)} = \tan(\phi) \Rightarrow \phi.$$

We now introduce the notation under a coordinate rotation $P^{-1}(\phi)$.

$$\begin{aligned} \xi &:= P^{-1}x \\ \dot{\xi} &:= g^+(\xi) \\ \dot{x} &:= f^-(x) \end{aligned}$$

then we have the following relationships

$$\dot{\xi} = P^{-1}\dot{x} = P^{-1}f^{-}(x) = P^{-1}f^{-}(P\xi) = g^{+}(\xi).$$

With

$$\begin{aligned} \zeta &:= P^{-1}y \\ \dot{\zeta} &:= g^{+}(\zeta) \\ \dot{y} &:= f^{-}(y), \end{aligned}$$

we have

$$\dot{\zeta} = P^{-1}\dot{y} = P^{-1}f^{-}(y) = P^{-1}f^{-}(P\zeta) = g^{+}(\zeta).$$

By conditions of theorem 5 K^0 is a non-empty and compact set which contains no equilibrium. Let $G_z > 0$ be a constant such that for all $\xi_0 \in K^0$ satisfying $g_2^+(\xi_0) < 0$, $g_2^+(\xi_0) = -c_z \leq -G_z$,

$$-G_z := \sup_{\xi_0 \in K^0, g_2^+(\xi_0)<0} g_2^+(\xi_0)$$

Let $\varepsilon_2 \leq \frac{G_z}{2}$. Also let there be a constant $D_z > 0$ such that for all $\xi_0 \in K^0$ satisfying $g_2^+(\xi_0) < 0$ and $g_1^+(\xi_0) = d_z \leq D_z$,

$$-D_z := \sup_{\xi_0 \in K^0, g_2^+(\xi_0)<0} \mid g_1^+(\xi_0) \mid$$

Let $\varepsilon_1 \leq \frac{D_z}{2}$. We choose $\varepsilon^z = (\varepsilon_1^z, \varepsilon_2^z) > 0$ then there is a $b^z = (b_1^z, b_2^z) > 0$ so that we can construct a box in the positive phase space with center ξ_0 by

$$B^{+}_{(\xi_0)} := \left\{(\zeta_1, \zeta_2) \in K \cap [\xi_0^1 - b_1^z, \xi_0^1 + b_1^z] \times [0, b_2^z]\right\}$$

such that

$$
\begin{aligned}
-\frac{D_z}{2} \leq \mid d_z \mid -\varepsilon_1 \leq -d_z - \varepsilon_1 &\leq \\
g_1^+(\zeta) &\leq d_z + \varepsilon_1 \leq \mid d_z \mid + \varepsilon_1 \leq \frac{3D_z}{2} \qquad (2.60) \\
-\frac{3G_z}{2} \leq -c_z - \varepsilon_2 &\leq \\
g_2^+(\zeta) &\leq -c_z + \varepsilon_2 \leq -\frac{G_z}{2} \qquad (2.61)
\end{aligned}
$$

Now, we want to show that there is a time t^{**} such that the solution $\zeta_2^+(t)$ starting at $\zeta_2^+(t_j^0) > 0$ reaches $\zeta_2^+(t^{**}) = 0$. A solution $\zeta_2^+(\tau) < 0$ decreases to $\zeta_2^+(t^{**}) = 0$ since $g_2^+(\zeta) < 0$ for all $\zeta^+(\tau) \in B^+_{(\xi_0)}$ with $g_2^+(\zeta) \in [-c_z - \varepsilon_2, -c_z + \varepsilon_2]$. We have

$$
\begin{aligned}
\zeta_2^+(t) &= \zeta_2^+(t_j^0) + \int_{t_j^0}^{t^{**}} g_2^+(\zeta) d\tau \\
\zeta_2^+(t) - \zeta_2^+(t_j^0) &= \int_{t_j^0}^{t^{**}} g_2^+(\zeta) d\tau.
\end{aligned}
$$

Define $\sigma_2 := \zeta_2^+(t_j^0)$ and $e_2 := \frac{1}{\tau^*} \int_{t_j^0}^{t^{**}} [g_2^+(\zeta) + c_z] d\tau$. Then since $\zeta_2^+(t^{**}) = 0$ and $\mid e_2 \mid \leq \varepsilon_2$ we have

$$-\sigma_2 = -c_z \cdot t^{**} + e_2 \cdot t^{**}$$

$$\frac{\sigma_2}{c_z - e_2} = t^{**} \qquad (2.62)$$

Hence

$$\frac{\sigma_2}{c_z + |e_2|} \leq t^{**}. \qquad (2.63)$$

Hence there is a time interval with $t^{**} \in [\frac{\sigma_2}{c_z - \varepsilon_2}, \frac{\sigma_2}{c_z + \varepsilon_2}]$ such that $\zeta_2^+(t^{**}) = 0$. We have found a time t^{**} in the new coordinate system. Time t^{**} is invariant under a coordinate transformation.

We have shown that for frozen time $\theta_j^0 = \theta_j^-$ of solution $y^-(\theta)$ there is a time $t^{**} :=\mid t_j^+ - t_j^0 \mid$ of solution $x^-(t)$ such that $N_2 := \frac{\delta_2}{c - \varepsilon_2} \geq t^{**}$. We have shown

(2.56).

Next, we now consider trajectories in the negative space without a coordinate transformation. We want to determine $A^-(t_0)$.

By conditions of theorem 5, K^0 is a non-empty and compact set which contains no equilibrium. Let $G^- > 0$ be a constant such that for all $x_0 \in K^0$ satisfying $f_2^+(x_0) < 0$, $f_2^+(x_0) = -c^- \leq -G^-$,

$$-G^- := \max_{x_0 \in K^0, f_2^+(x_0) < 0} f_2^+(x_0)$$

Let $\varepsilon_2^- \leq \frac{G^-}{2}$. Also let there be a constant $D^- > 0$ such that for all $x_0 \in K^0$ satisfying $f_2^+(x_0) < 0$, $f_1^+(S_t^+ x_0) = d^- \leq D^-$,

$$D^- := \max_{x_0 \in K^0, f_2^+(x_0) < 0} |f_1^+(x_0)|$$

Let $\varepsilon_1^- \leq \frac{D^-}{2}$.

Given $\varepsilon = (\varepsilon_1^-, \varepsilon_2^-) > 0$ there is a $b = (b_1^-, b_2^-) > 0$ so that we can construct a box in the negative phase space with center x_0 by

$$B_{(x_0)}^- := \left\{(x_1, x_2) \in K \cap [x_0^1 - b_1^-, x_0^1 + b_1^-] \times [0, -b_2^-]\right\}$$

such that

$$\begin{aligned} -\frac{D^-}{2} \leq - \mid d^- \mid -\varepsilon_1^- \leq d^- - \varepsilon_1^- &\leq \\ f_1^-(x) &\leq d^- + \varepsilon_1^- \leq \mid d^- \mid + \varepsilon_1^- \leq \frac{3D^-}{2} \quad (2.64) \\ -\frac{3G^-}{2} \leq -c^- - \varepsilon_2^- &\leq \\ f_2^-(x) &\leq -c^- + \varepsilon_2^- \leq -\frac{G^-}{2}. \quad (2.65) \end{aligned}$$

We are given an ε^- and choose $b^- > 0$ small enough so that $x^-(\tau) \in B_{(x_0)}^-$. Given $b^- > 0$ and $0 < \phi < \frac{1}{2}\pi$ we show that there is a $b_z > 0$ with $\zeta^+(\tau) \in \widehat{B}_{(\xi_0)}^+$

for all $\tau \in [0, t^{**}]$. We define

$$\widehat{B}^{+}_{(\xi_0,\phi)} := \left\{ (\zeta^1, \zeta^2) \in K \cap [b_1^z(\phi), 0] \times [0, b_2^z(\phi)] \right\}.$$

with

$$\begin{aligned} b_1^z &:= \max_{0<\phi<\frac{1}{2}\pi} b_1^z(\phi) \\ b_2^z &:= \max_{0<\phi<\frac{1}{2}\pi} b_2^z(\phi) \end{aligned}$$

where

$$\begin{aligned} b_1^z(\phi) &= b_1^- \cos(\phi) \\ b_2^z(\phi) &= b_1^- \sin(\phi). \end{aligned}$$

such that $\zeta^+(\tau) \in \widehat{B}^+_{(\xi_0)}$ for all $\tau \in [0, t^{**}]$ and $\widehat{B}^+_{(\xi_0)} \subset B^-_{(x_0)}$. Hence, there is $\varepsilon^- > 0$ with b^- small enough such that $(y^-(\tau), \zeta^+(\tau)) \in \left\{ B^-_{(x_0)} \cap \widehat{B}^+_{(\xi_0,\phi)} \right\}$. Finally, for given $\varepsilon > 0$ set $\|b\| = \|\delta\|$ where

$$\begin{aligned} r_b \frac{\mid f_2^+(x_0) \mid -\varepsilon_2}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2 + \frac{\varepsilon_1 \cdot |f_1^+(x_0)| + \varepsilon_2 \cdot |f_2^+(x_0)|}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}}} &\leq \\ A^+(t_0) &\leq \\ r_b \frac{\mid f_2^+(x_0) \mid +\varepsilon_2}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2 + \frac{-\varepsilon_1 \cdot |f_1^+(x_0)| - \varepsilon_2 \cdot |f_2^+(x_0)|}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}}} & \end{aligned}$$

with $A^+(t_0) = \|\delta\|$ such that $y^+(\tau) \in B^+_{x_0}$ for all $\tau \in [0, t*]$. It remains to show that for given ε^- we have $x^-(\tau) \in B^-_{(x_0)}$ for all $\tau \in [0, t^*]$.

Now, we have

$$x_1^-(t^{**}) = x_1^-(t_j^0) + \int_0^{t^{**}} f_1^-(x^-(\tau)) d\tau$$

Define $\delta_1^- = x_1^-(t_j^0) - y_1^-(t_j^0)$ and $e_1^- = \frac{1}{t^{**}} \int_{t_j^0}^{t^{**}} [f_1^-(x^-(\tau)) - d^-] d\tau$, and since

$|e_1^-| \leq \varepsilon_1^-$ we have

$$= \quad \delta_1^- + t^{**} \cdot [d^- + e_1^-]$$

and by equation (2.47) $\frac{\sigma_2}{c_z - e_2} = \frac{\delta_2^-}{c^- - e_2}$ in the non transformed coordinates we obtain

$$= \quad \delta_1^- + \frac{\delta_2^-}{c^- - e_2^-} \cdot [d^- + e_1^-].$$

Using $\delta_1^- \left[f_1^-(x_0) + h_1^-\right] + \delta_2^- \left[f_2^-(x_0) + h_2^-\right] = 0$ yields

$$\begin{aligned}
&= \delta_1^- - \left(\frac{\delta_1^- \left[f_1^-(x_0) + h_1^-\right]}{(-f_2^-(x_0) - e_2^-)\left[f_2^-(x_0) + h_2^-\right]} \right) \cdot [f_1^-(x_0) + e_1^-] \\
&= \delta_1^- \left[1 - \left(\frac{\left[f_1^-(x_0) + h_1^-\right] \cdot \left[f_1^-(x_0) + e_1^-\right]}{(-f_2^-(x_0) - e_2^-)\left[f_2^-(x_0) + h_2^-\right]} \right) \right] \\
&= \delta_1^- \left[\frac{(f_2^-(x_0) + e_2^-)\left[f_2^-(x_0) + h_2^-\right]}{(f_2^-(x_0) + e_2^-)\left[f_2^-(x_0) + h_2^-\right]} - \left(\frac{\left[f_1^-(x_0) + h_1^-\right] \cdot \left[f_1^-(x_0) + e_1^-\right]}{(-f_2^-(x_0) - e_2^-)\left[f_2^-(x_0) + h_2^-\right]} \right) \right] \\
&= \delta_1^- \left[\frac{(f_2^-(x_0) + e_2^-)\left[f_2^-(x_0) + h_2^-\right] + \left[f_1^-(x_0) + h_1^-\right] \cdot \left[f_1^-(x_0) + e_1^-\right]}{(f_2^-(x_0) + e_2^-)\left[f_2^-(x_0) + h_2^-\right]} \right] \\
&= \delta_1^- \left[\frac{(f_1^-(x_0))^2 + (f_2^-(x_0))^2 + (e_1^- + h_1^-) \cdot f_1^-(x_0) + (e_2^- + h_2^-) \cdot f_2^-(x_0) + h_1^- e_1^- + h_2^- e_2^-}{(f_2^-(x_0))^2 + (e_2^- + h_2^-) \cdot f_2^-(x_0) + e_2^- h_2^-} \right].
\end{aligned}$$

We have the following bounds

$$\begin{aligned}
|\delta_1^-| \left[\frac{(f_1^-(x_0))^2 + (f_2^-(x_0))^2 - (2\varepsilon_1^-) \cdot |f_1^-(x_0)| - (\varepsilon_1^-)^2 - (2\varepsilon_2^-) \cdot |f_2^-(x_0)| - (\varepsilon_2^-)^2}{(f_2^-(x_0))^2 + 2\varepsilon_2^- \cdot |f_2^-(x_0)| + (\varepsilon_2^-)^2} \right] &\leq \\
|x_1^+(t^*) - y_1^-(\theta_j^0)| &\leq \\
|\delta_1^-| \left[\frac{(f_1^-(x_0))^2 + (f_2^-(x_0))^2 + (2\varepsilon_1^-) \cdot |f_1^-(x_0)| + (\varepsilon_1^-)^2 + (2\varepsilon_2^-) \cdot |f_2^-(x_0)| + (\varepsilon_2^-)^2}{(f_2^-(x_0))^2 - 2\varepsilon_2^- \cdot |f_2^-(x_0)| - (\varepsilon_2^-)^2} \right] &\leq
\end{aligned}$$

We need to verify the bounds. This only requires to check that

$$(f_1^+(x_0))^2 + (f_2^+(x_0))^2 - (2\varepsilon_1) \cdot |f_1^+(x_0)| - \varepsilon_1^2 - (2\varepsilon_2) \cdot |f_2^+(x_0)| - \varepsilon_2^2 \geq E > 0.$$

a well as

$$
\begin{aligned}
(f_2^+(x_0))^2 - (2\varepsilon_2)\cdot|f_2^+(x_0)| - \varepsilon_2^2 &\geq c^2 - 2\varepsilon_2 c - \varepsilon_2^2 = c(c-2\varepsilon_2) - \varepsilon_2^2 \\
&\geq G(G-2\varepsilon_2) - \varepsilon_2^2 \\
&\geq G(G-2\frac{G}{9}) - \frac{G}{81}^2 = \frac{62}{81}G^2 > 0
\end{aligned}
$$

for $\varepsilon_2 := \frac{G}{9}$.

We also have

$$
\begin{aligned}
(f_2^+(x_0))^2 - (2\varepsilon_1)\cdot|f_2^+(x_0)| - \varepsilon_1^2 &\geq c^2 - 2\varepsilon_1 c - \varepsilon_2^2 = c(c-2\varepsilon_2) - \varepsilon_1^2 \\
&\geq G(G-2\varepsilon_1) - \varepsilon_1^2 \\
&\geq G(G-2\frac{G}{3}) - \frac{G}{9}^2 = \frac{2}{9}G^2 > 0
\end{aligned}
$$

for $\varepsilon_2 := \frac{G}{3}$. Then it follows that

$$\frac{62}{81}G^2 - \frac{2}{9}G^2 = \frac{44}{81}G^2 > 0$$

as requested.

We consider $|x_1^+(t^*) - y_1^-(\theta_j^0)|$ and define

$$
\begin{aligned}
r_b^- &:= \\
&= |\ \delta_1\ | \times \\
&= \left|\left[\frac{(f_1^-(x_0))^2 + (f_2^-(x_0))^2 + (e_1^- + h_1^-)\cdot f_1^-(x_0) + (e_2^- + h_2^-)\cdot f_2^-(x_0) + h_1^- e_1^- + h_2^- e_2^-}{(f_2^-(x_0))^2 + (e_2^- + h_2^-)\cdot f_2^-(x_0) + e_2^- h_2^-}\right]\right| \\
&= |x_1^+(t^*) - y_1^-(\theta_j^0)|.
\end{aligned}
$$

The calculation for $A^-(t_0)$ follows from

$$A^-(t_0) = \sqrt{(\delta_1^-)^2 + (\delta_2^-)^2}$$

and using $\delta_1^- \left[f_1^0(x_0) + h_1^-\right] + \delta_2^- \left[f_2^-(x_0) + h_2^-\right] = 0$ yields

$$
\begin{aligned}
&= \sqrt{(\delta_1^-)^2 + (\delta_1^-)^2 \frac{\left(f_1^-(x_0) + h_1^-\right)^2}{\left(f_2^-(x_0) + h_2^-\right)^2}} \\
&= \mid \delta_1^- \mid \frac{\sqrt{\left(f_1^-(x_0) + h_1^-\right)^2 + \left(f_2^-(x_0) + h_2^-\right)^2}}{\mid f_2^-(x_0) + h_2^- \mid}.
\end{aligned}
$$

We now calculate $\frac{A^+(t_0)}{r_b}$. We have

$$
\begin{aligned}
\frac{A^+(t_0)}{r_b^-} &= \frac{\mid \delta_1^- \mid \frac{\sqrt{\left(f_1^-(x_0)+h_1^-\right)^2 + \left(f_2^-(x_0)+h_2^-\right)^2}}{|f_2^-(x_0)+h_2^-|}}{\mid \delta_1^- \mid \left[\frac{(f_1^-(x_0))^2 + (f_2^-(x_0))^2 + (e_1^- + h_1^-)\cdot f_1^-(x_0) + (e_2^- + h_2^-)\cdot f_2^-(x_0) + h_1^- e_1^- + h_2^- e_2^-}{(f_2^-(x_0))^2 + (e_2^- + h_2^-)\cdot f_2^-(x_0) + e_2^- h_2^-}\right]} \\
&= \frac{\sqrt{\left(f_1^-(x_0) + h_1^-\right)^2 + \left(f_2^-(x_0) + h_2^-\right)^2}}{|f_2^-(x_0) + h_2^-|} \times \\
&\quad \frac{|(f_2^-(x_0))^2 + (e_2^- + h_2^-)\cdot f_2^-(x_0) + e_2^- h_2^-|}{|(f_1^-(x_0))^2 + (f_2^-(x_0))^2 + (e_1^- + h_1^-)\cdot f_1^-(x_0) + (e_2^- + h_2^-)\cdot f_2^-(x_0) + h_1^- e_1^- + h_2^- e_2^-|} \\
&= \frac{1}{\left(\sqrt{\left(f_1^-(x_0) + h_1^-\right)^2 + \left(f_2^-(x_0) + h_2^-\right)^2}|f_2^-(x_0) + h_2^-|\right)} \times \\
&\quad \frac{|(f_2^-(x_0))^2 + (e_2^- + h_2)\cdot f_2^-(x_0) + e_2^- h_2^-|}{|(f_1^-(x_0))^2 + (f_2^-(x_0))^2 + (e_1^- + h_1^-)\cdot f_1^-(x_0) + (e_2^- + h_2^-)\cdot f_2^-(x_0) + h_1^- e_1^- + h_2^- e_2^-|} \\
&= \frac{|(f_2^-(x_0))^2 + (e_2^- + h_2^-)\cdot f_2^-(x_0) + e_2^- h_2^-|}{B^-\left((f_1^-(x_0))^2 + (f_2^-(x_0))^2\right) + B^- \cdot A^-} \\
&= \frac{|(f_2^-(x_0))^2 + (e_2^- + h_2^-)\cdot f_2^-(x_0) + e_2^- h_2^-|}{B^- + B^- \frac{A^-}{\left((f_1^-(x_0))^2 + (f_2^-(x_0))^2\right)}} \\
&= \frac{|(f_2^-(x_0))^2 + (e_2^- + h_2^-)\cdot f_2^-(x_0) + e_2^- h_2^-|}{B^-\left(1 + \frac{A^-}{\left((f_1^-(x_0))^2 + (f_2^-(x_0))^2\right)}\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
A^- &= |(e_1^- + h_1^-)\cdot f_1^-(x_0) + (e_2^- + h_2^-)\cdot f_2^-(x_0) + h_1^- e_1^- + h_2^- e_2^-| \\
B^- &= \sqrt{\left(f_1^-(x_0) + h_1^-\right)^2 + \left(f_2^-(x_0) + h_2^-\right)^2}|f_2^-(x_0) + h_2^-|.
\end{aligned}
$$

We define

$$F^-(e^-,h^-) \quad := \quad \frac{|(f_2^-(x_0))^2 + (e_2^- + h_2^-)\cdot f_2^-(x_0) + e_2^- h_2^-|}{B^-\left(1 + \frac{A^-}{\left((f_1^-(x_0))^2+(f_2^-(x_0))^2\right)}\right)}.$$

Remark 4 *We observe that*

$$\lim_{e^-,h^-\to 0} F^-(e^-,h^-) = \frac{f_2^-(x_0)}{\sqrt{f_1^-(x_0)+f_2^-(x_0)}},$$

This can be seen from

$$F^-(e^-,h^-) = \frac{|(f_2^-(x_0))^2+(e_2^-+h_2^-)\cdot f_2^-(x_0)+e_2^-h_2^-|}{\left(\sqrt{\left(f_1^-(x_0)+h_1^-\right)^2+\left(f_2^-(x_0)+h_2^-\right)^2}|f_2^-(x_0)+h_2^-|\right)\left(1+\frac{|(e_1^-+h_1^-)\cdot f_1^-(x_0)+(e_2^-+h_2^-)\cdot f_2^-(x_0)+h_1^-e_1^-+h_2^-e_2^-|}{\left((f_1^-(x_0))^2+(f_2^-(x_0))^2\right)}\right)}$$

We have the bounds

$$\frac{|(f_2^-(x_0))^2|-(2\varepsilon_2^-)\cdot|f_2^-(x_0)|-(\varepsilon_2^-)^2}{\left(\sqrt{\left(f_1^-(x_0)-\varepsilon_1^-\right)^2+\left(f_2^-(x_0)-\varepsilon_2^-\right)^2}\left(|f_2^-(x_0)|-\varepsilon_2^-\right)\right)\left(1-\frac{(e_1^-+\varepsilon_1^-)\cdot|f_1^-(x_0)|+((\varepsilon_2^-)^2)\cdot|f_2^-(x_0)|+(\varepsilon_1^-)^2+(\varepsilon_2^-)^2|}{\left((f_1^-(x_0))^2+(f_2^-(x_0))^2\right)}\right)}$$
$$\le F^-(e^-,h^-) \le$$
$$\frac{|(f_2^-(x_0))^2|+(2\varepsilon_2^-)\cdot|f_2^-(x_0)|+(\varepsilon_2^-)^2}{\left(\sqrt{\left(f_1^-(x_0)+\varepsilon_1^-\right)^2+\left(f_2^-(x_0)+\varepsilon_2^-\right)^2}\left(|f_2^-(x_0)|+\varepsilon_2^-\right)\right)\left(1+\frac{(e_1^-+\varepsilon_1^-)\cdot|f_1^-(x_0)|+((\varepsilon_2^-)^2)\cdot|f_2^-(x_0)|+(\varepsilon_1^-)^2+(\varepsilon_2^-)^2|}{\left((f_1^-(x_0))^2+(f_2^-(x_0))^2\right)}\right)}$$

We need to show that

$$F^-(e^-,h^-) \neq 0$$

This can be shown as a similar way as previously and is therefore omitted.

Combining case III and IV

In step (iii) we show that $\frac{A^+(t_0)}{A^-(t_0)}e^{w^+-w^-} < 1$.

$$
\begin{aligned}
\frac{A^-(t_0)}{A^+(t_0)} &= \frac{\frac{A^-(t_0)}{r_b^-}}{\frac{A^+(t_0)}{r_b}} \\
&= f^-(e^-)e^{w^-}\frac{1}{f^+(e)e^{w^+}} \\
&= \left(\frac{\frac{|(f_2^-(x_0))^2+(e_2^-+h_2^-)\cdot f_2^-(x_0)+e_2^-h_2^-|}{B^-\left(1+\frac{A^-}{\left((f_1^-(x_0))^2+(f_2^-(x_0))^2\right)}\right)}}{\frac{|(f_2^+(x_0))^2+(e_2+h_2)\cdot f_2^+(x_0)+e_2h_2|}{B\left(1+\frac{A}{\left((f_1^+(x_0))^2+(f_2^+(x_0))^2\right)}\right)}}\right)e^{w^- - w^+} \\
&\leq 1-\rho<1
\end{aligned}
$$

with $\rho > 0$.

We conclude that

$$
\lim_{e,h,e^-,h^-\to 0} F(e,h,e^-,h^-) = \frac{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}}e^{w^-(x_1,0)-w^+(x_1,0)}. \tag{2.66}
$$

This follows from remark 3 and remark 4. We have shown (2.33)

Remark 5 *(2.33) is shown by equation (2.68) and equation (2.69). We have derived the jumping condition (+/-) of theorem 5. The way we have shown this is by considering four cases, where case I and case II are considered jointly, and case III and case IV are also considered jointly. The conditions hold for any combination of the four cases. However, it remains to provide bounds on $\mathcal{T}(t)$ for combination of case $I-IV$ and case $II-III$. This is shown in lemma1*

Lemma 1 (Combined cases bounds) *Let the assumptions of theorem 5 hold. Moreover let $\mathcal{T}_x^{x+\eta}:\mathbb{R}_0^+\to\mathbb{R}_0^+$ be given in proposition 5. Then for all $t\in\mathcal{J}^\pm\cup\mathcal{I}^\pm$*

$$
m \leq \triangle\mathcal{T}(t) \leq M. \tag{2.67}
$$

Proof. We consider the time interval $\tau\in(t_{j-1}^+,t_j^-)\cup[t_j^-,t_j^-+c_1+c_2]$ with $t_j^->t_{j-1}^+$ and $c_1,c_2>0$. By (2.38) and the prolongation of the time interval we have

$$
\begin{aligned}
\theta(\tau) &\geq \frac{2}{3}\left(\frac{t_j^-}{t_j^- + c_1 + c_2}\right)\cdot\tau \\
&= \frac{2}{3}\left(\frac{1}{1 + \frac{c_1}{t_j^-} + \frac{c_2}{t_j^-}}\right)\cdot\tau \\
&\geq \frac{2}{3}\left(\frac{1}{2 + \frac{c_1}{c_2}}\right)\cdot\tau =: m
\end{aligned}
$$

$$
\begin{aligned}
\theta(\tau) &\leq \frac{4}{3}\left(\frac{t_j^-}{t_j^- + c_1 + c_2}\right)\cdot\tau \\
&= \frac{4}{3}\left(\frac{1}{1 + \frac{c_1}{t_j^-} + \frac{c_2}{t_j^-}}\right)\cdot\tau \\
&\leq \frac{4}{3}\left(\frac{1}{2 + \frac{c_1}{c_2}}\right)\cdot\tau =: M.
\end{aligned}
$$

■

2.4.3 Part three: All cases I-IV and their combinations

We state the jump conditions for all cases in +/- direction.

1. Case I and II

$$
\lim_{e\to 0} F(e) = \frac{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}} e^{w^-(x^1,0)-w^+(x^1,0)} < 1. \tag{2.68}
$$

This is obtained by combining equations (2.44) and (2.51).

2. Case III and IV

$$\lim_{e,h,e^-,h^-\to 0} F(e,h,e^-,h^-) = \frac{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}} e^{w^-(x_1,0)-w^+(x_1,0)} < 1. \tag{2.69}$$

This is obtained by combining equations (2.60) and (2.66)

3. Case I and IV

$$\lim_{e,e^-,h^-\to 0} F(e,e^-,h^-) = \frac{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}} e^{w^-(x_1,0)-w^+(x_1,0)} < 1. \tag{2.70}$$

This is obtained by combining equations (2.44) and (2.66)

4. Case I and II

$$\lim_{e,h\to 0} F(e,h,) = \frac{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}} e^{w^-(x_1,0)-w^+(x_1,0)} < 1. \tag{2.71}$$

This is obtained by combining equations (2.51) and (2.60)

2.4.4 Part four: Trivial cases

Trivial jump cases occur when

$$t_j^- = t_j^0 = t_j^+ \quad and \quad \theta_j^- = \theta_j^0 = \theta_j^+.$$

In +/- direction the smooth condition

$$L_{W^+(x)} := \max_{\|v^+\|=e^{-W^+(x)}, v^+\perp f^+(x)} L_{W^+}(x,v^+) \leq -\nu < 0$$

holds.

2.4.5 Jumps in opposite direction -/+

We state proposition 5 without proof. The proof is similar to the proof of propositions 3 and 4.

Proposition 5 (Jump case (-/+)) *Let the assumptions of theorem 5 hold. Moreover assume that there are constants* $\delta, \nu, m, M, N_1, N_2 > 0$ *such that for all* $\eta \in \mathbb{R}^2$ *with* $\| \eta \| \leq \delta$ *there is a piecewise multi valued mapping* $\mathcal{T}_x^{x+\eta}(t)$ *defined by equation (2.3) that continuously depends on* η *and satisfies for all* $t \in \mathcal{I}^{\pm}$ *or for all* $t \in \mathcal{J}^{\pm}$

$$m \leq \triangle \mathcal{T}_x^{x+\eta}(t) \leq M. \tag{2.72}$$

and

$$(S_{\mathcal{T}(t)}(x+\eta) - S_t x) \cdot f(S_t^+ x) = 0.$$

Also for all $t \in [t_j^-, t_j^+]$.

$$A^-(t_j^+) < A^+(t_j^-)e^{-\nu t}. \tag{2.73}$$

For all $t \in \mathcal{I}^{\pm}$ *and* $\theta^* \in [\theta_j^-, \theta_j^0]$ *with*

$$|\theta_j^+ - \theta_j^-| \leq N_1. \tag{2.74}$$

and $y_2^+(\theta^*) \geq 0$.

For all $t \in \mathcal{I}^{\pm}$ *and* $\theta^{**} \in [\theta_j^0, \theta_j^+]$ *with*

$$|t_j^+ - t_j^-| \leq N_2. \tag{2.75}$$

and $y_2^-(\theta^{**}) \leq 0$.

The proof of this proposition follows the same structure as the proof of proposition 3 and proposition 4. We therefore only state that

$$\lim_{e \to 0} F(e) = \frac{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}}{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2 + \left(f_2^-(x_0)\right)^2}}} e^{w^+(x_0) - w^-(x_0)} < 1$$

$$\lim_{e,h,e^-,h^- \to 0} F(e,h,e^-,h^-) = \frac{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}}{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2 + \left(f_2^-(x_0)\right)^2}}} e^{w^+(x_0) - w^-(x_0)} < 1.$$

Similarly as in the +/- case combinations of cases do not alter the main condition. We do not state them. ■

2.5 Joint case and ω-limit set

We now consider the time interval $t \in (t_{j-1}^{+}, t_j^{+}) \cup [t_{j-1}^{+}, t_j^{+}]$ and show that the distance between two adjacent solutions decreases. We also show show for all cases that for two nearby points x and $x + \eta$ in K that the ω-limit set is the same.

Proposition 6 (Joint case) *Let the assumptions of theorem 5 hold. Then there are constants $\delta > 0$ and $C \geq 1$ such that for all $x \in K$ and for all $\eta \in \mathbb{R}^2$ with $\| \eta \| \leq \delta/2$*

$$A(t_{j-1}^{-}) \leq A(t_{j-1}^{+})e^{-\mu t} \text{ for all } t \geq 0. \tag{2.76}$$

Moreover, we have

$$\omega(x) = \omega(x + \eta). \tag{2.77}$$

Proof.

We now show the contraction property of the distance function.

- We show that ν defined over a smooth time interval is strictly larger than μ defined over the same time interval including the subsequent time interval.
- We show that the distance function is decreasing for all positive time.

(i) By equations (2.18) and (2.31) we have

$$\begin{aligned} A(t) &\leq e^{-\mu t} A(t_{j-1}^{-}) \text{ for all } t \in [t_{j-1}^{-}, t_{j-1}^{+}] & (2.78) \\ A(t) &\leq e^{-\mu t} A(t_j^{-}) \text{ for all } t \in [t_j^{-}, t_j^{+}] & (2.79) \end{aligned}$$

Equations (2.78) and (2.79) show the contraction rate μ over each jumping interval in +/- and in -/+ direction. We now state similar equations for the smooth intervals with contraction rate ν. We have

$$
\begin{aligned}
A(t) &\leq e^{-\nu t} A(t^+_{j-1}) \text{ for all } t \in (t^+_{j-1}, t^-_j) &(2.80)\\
A(t) &\leq e^{-\nu t} A(t^+_j) \text{ for all } t \in (t^+_j, t^-_{j+1}). &(2.81)
\end{aligned}
$$

We consider the time interval $[t_1^-, t_1^+] \cup (t_1^+, t_2^-)$. Hence by equation (2.78) and equation (2.80) we obtain

$$
\begin{aligned}
e^{-\nu(t_2^- - t_1^+)} A^+(t_1^+) &\leq e^{-\mu(t_2^+ - t_1^+)} A^+(t_1^+)\\
\nu(t_2^- - t_1^+) &\leq \mu(t_2^+ - t_1^+)\\
\mu &\leq \nu \left(\frac{t_2^- - t_1^+}{t_2^+ - t_1^+} \right). &(2.82)
\end{aligned}
$$

We define

$$
\begin{aligned}
S &:= (t_j^- - t_{j-1}^+) \geq c_2 \text{ for } j = 1, 2, 3, ... &(2.83)\\
J &:= (t_j^+ - t_j^-) \leq c_1 \text{ for } j = 1, 2, 3, ... &(2.84)
\end{aligned}
$$

where constants $c_1, c_2 > 0$ are defined by

$$c_1 := \delta > 0$$

For the constant c_2 we consider $d := K \cap \{x_2 = 0\}$. From

$$max_{x \in K} \mid f_1(x) \mid = s$$

and

$$t \cdot s = d$$

we obtain by $d \leq \int_0^t f_1(x(\tau)) d\tau$

$$c_2 := \frac{d}{\max_{x \in K} \mid f_1 \mid} \leq t.$$

Equation (2.82) with bounds (2.83) and (2.84) and extension of time interval $t_2^+ - t_1^+ = (t_2^- - t_1^+) + (t_2^+ - t_2^-)$ yields

$$\mu = \nu \left(\frac{t_2^- - t_1^+}{(t_2^+ - t_1^+) + (t_2^+ - t_2^-)} \right) \leq \nu \left(\frac{c_2}{c_2 + c_1} \right) = \nu \left(\frac{1}{1 + \frac{c_1}{c_2}} \right)$$

Since $c_1 = \delta$ we can choose δ small enough so that μ gets as close to ν as we wish. From

$$\begin{aligned} A(t_1^+ + t) &\leq e^{-\mu t} A(t_1^+) \text{ for all } t \in \left\{ (t_1^+, t_2^- -) \cup [t_2^- - t_2^+] \right\} & (2.85) \\ A(t_1^+ + t) &\leq e^{-\mu t} A(t_1^+) \text{ for all } t \in \left\{ (t_2^+, t_3^- -) \cup [t_3^- - t_3^+] \right\} & (2.86) \end{aligned}$$

we have

$$A(t_1^+ + \tau) \leq e^{-\mu\tau} A(t_1^+) \text{ for all } \tau \in \left\{ (t_1^+, t_2^- -) \cup [t_2^- - t_2^+] \cup (t_2^+, t_3^- -) \cup [t_3^- - t_3^+] \right\}$$

which generalizes to $\tau \geq 0$, by

$$A(t_1^+ + \tau) \leq e^{-\mu\tau} A(t_1^+) \text{ for all } \tau \in \left\{ (t_{j-1}^+, t_j^- -) \cup [t_j^- - t_j^+] \cup (t_j^+, t_{j+1}^- -) \cup [t_{j+1}^- - t_3^+] \right\} \tag{2.87}$$

This shows (2.76). It remains to show (2.77).

Now, we show that all points $x + \eta$ with $\eta \in \mathbb{R}^2$, $\eta \perp f(x)$, and $\| \eta \| \leq \delta/2$ have the same ω-limit set as the point x. We first show the inclusion $\omega(x) \subset \omega(x + \eta)$. Assume there is a $w \in \omega(x)$. Then we have a strictly increasing sequence $t_i \to \infty$ satisfying $\| w - S_{t_i} x \| \to 0$ as $i \to \infty$. Because of condition (2.76) of proposition 6 and properties of $\mathcal{T}$ given in propositions 3, 5 there is a sequence $\mathcal{T}(t_i)$ that satisfies

$$\mathcal{T}(t_i) \to \infty \text{ as } i \to \infty,$$

and

$$A^-(t_i) \leq A^+(\mathcal{T}(t_i)) e^{-\mu t_i} \text{ as } i \to \infty.$$

This proves that $S_{\mathcal{T}(t_i)}(x + \eta) \to w$ and $w \in \omega(x + \eta)$.

We now show that the inclusion $\omega(x+\eta) \subset \omega(x)$. Assume there is a $w \in \omega(x+\eta)$. Then we have a strictly increasing sequence $\theta_i \to \infty$ satisfying $\| w - S_{t_i}x \| \to 0$ as $i \to \infty$. Because of condition (2.76) of proposition 6 and properties of $\mathcal{T}$ given in propositions 3, 5 there is a sequence $\mathcal{T}^{-1}(\theta_i)$ that satisfies $\mathcal{T}^{-1}(\theta_i) \to \infty$ as $i \to \infty$. This proves that $S_{\mathcal{T}(t_i)}(x+\eta) \to w$ and $w \in \omega(x+\eta)$.

This concludes the proof of proposition 6. ∎

Proposition 6 shows that all points of a neighborhood of x living in a hyperplane consisting of the points $(x+\eta) \perp f(x)$ with $\|\eta\| \leq \frac{\delta}{2}$ have the same $\omega-$limit set.

Proposition 7 *Let assumption of theorem5 be satisfied. Then for all* $x, y \in K$

$$\emptyset \neq \omega(x) = \omega(y) =: \Omega$$

Proof. Let $x_0 \in \Omega \setminus K^0$. Since for all $t \geq 0$ we have $S_t x_0 \subset K$, which is a compact set, hence

$$\emptyset \neq \omega(x_0) =: \Omega \subset K.$$

Now, pick an arbitrary point $x \in \Omega \setminus K^0$. By proposition 6 we have $\omega(x) = \omega(y)$ for all y in a neighborhood of x. Hence

$$\begin{aligned} K_1 &:= \{x \in K : \omega(x) = \omega(x_0)\} \\ K_2 &:= \{x \in K : \omega(x) \neq \omega(x_0)\} \end{aligned}$$

are open sets. Since $K = K_1 \dot{\cup} K_2$ and $p_0 \in K_1$ with K connected, it must be that K_2 is empty and $K_1 = K$ ∎

2.6 Omega limit set for all points in a neighborhood

2.6.1 Main proposition

The next result generalizes proposition 6 to a full neighborhood of x. It provides the conditions for a point x to belong to an exponentially asymptotically stable periodic orbit. A good theory of $\omega-$limit sets for smooth dynamical systems is provided by Giesl [17], [18], and [19] for a non-smooth dynamical system. This part of the proof relies on these results.

We define $F(t)$ as a smooth function in a direction which is not perpendicular to $f(S_t x)$.

$$F(t) := \frac{f(S_t x)}{\|f(S_t x)\|}.$$

Proposition 8 *Let assumptions of theorem 5 hold. Let $x \in K^{\pm}$ satisfy $x \in \omega(x)$. Assume there is a continuous map $F : \mathbb{R}_0^+ \to \mathbb{R}^2$ with $\|F(t)\| = 1$ and $\langle F(t), f(S_t x)\rangle > 0$ for all $t \geq 0$. Furthermore, assume that there are constants $\delta, \nu > 0$, $C \geq 1$, and $m_1, m_2, m_3, M_1, M_2, M_3 > 0$ and such that for all $\eta \in \mathbb{R}^2$ with $\eta \perp F(0)$ and $\| \eta \| \leq \delta$ there is a piecewise multi valued mapping $\mathcal{T}_x^{x+\eta} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\mathcal{T}_x^{x+\eta}(t_i)$ depends continuously on η and satisfies*

1. *for all $t_i \in \mathcal{G}^{\pm}$*

$$m_1 \leq \dot{\mathcal{T}}(t_i) \leq M_1. \tag{2.88}$$

2. *for all $t_i \in \mathcal{I}^{\pm}$*

$$m_2 \leq \triangle\mathcal{T}(t_i) \leq M_2. \tag{2.89}$$

3. *for all $t_i \in \mathcal{J}^{\pm}$*

$$m_3 \leq \triangle\mathcal{T}(t_i) \leq M_3. \tag{2.90}$$

$$\langle S_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S_t x, F(t)\rangle = 0 \tag{2.91}$$

and

$$\|S_{\mathcal{T}_x^{x+\eta}(t)}(x+\eta) - S_t x\| \leq Ce^{-\mu t} \| \eta \| \tag{2.92}$$

for all $t \geq 0$. Then x is a point of an exponentially asymptotically stable period orbit Ω.

Proof. The proof of proposition (8) consists of four main steps.

1. Characterization of solutions near a point $x \in K$.
2. Definition of a correction mapping π.

3. Definition of a Poincaré-like map $\mathcal{P}$.

4. Existence of period orbit Ω and its stability.

2.6.2 Characterization of solutions near a point $x \in K$

We first introduce a new coordinate system and then characterize the behavior of $x \in K$ in terms of $f(x)$. The next lemma shows that trajectories starting within a neighborhood can only move within a cone.

We center the new coordinate system at the point $x \in K$.

- Define by $y(q)$ the scalar amount in $F(0)$-direction.
- Define by $x(q)$ the vectorial amount in $F(0)^{\perp}$-direction.

We now define a Hyperplane which consists of the points $q \in \mathbb{R}^2$ where $y(q) = 0$ holds.

$$H := \left\{ q \in \mathbb{R}^2 : x + F(0)^{\perp}, \text{ and } y(q) = 0 \right\}.$$

For arbitrary $q \in \mathbb{R}^2$, we define

$$y(q) \quad := \quad \langle q - x, F(0) \rangle \in \mathbb{R} \tag{2.93}$$

$$x(q) \quad := \quad q - x - y(q)F(0) \in F(0)^{\perp} \tag{2.94}$$

By equation (2.94) we can express

$$q = x + y(q)F(0) + x(q) \tag{2.95}$$

and $\|q - x\|^2 = |y(q)|^2 + \|x(q)\|^2$. We also define $f(q)$ for all $q \in \mathbb{R}^2$. Hence we define

$$\lambda(q) \quad := \quad \langle f(q) - f(x), F(0) \rangle \in \mathbb{R} \tag{2.96}$$

$$u(q) \quad := \quad f(q) - f(x) - \lambda(q)F(0) \in F(0)^{\perp} \tag{2.97}$$

Thus by equation (2.97) be obtain

$$f(q) = f(x) + \lambda(q)F(0) + u(q). \tag{2.98}$$

We have the following upper bounds on $\lambda(q)$ and $u(q)$. By assumptions of theorem 5 we know that f is continuous in x. Hence there is a δ_1 with $0 < \delta_1 \leq \delta$ such that for all $q \in B_{\delta_1}$ the following bounds hold:

$$\begin{aligned} \mid \lambda(q) \mid \ &\leq \ \frac{1}{2}\alpha_0 &(2.99)\\ \| \ u(q) \ \| \ &\leq \ \| \ f(x) \ \|, &(2.100) \end{aligned}$$

with

$$\alpha_0 \ := \ \langle F(0), f(x) \rangle > 0 \tag{2.101}$$

We can finally characterize solutions for all $q \in B_{\delta_1}$. We show that adjacent trajectories can only move within a cone.

Lemma 2 *Let $S_t q \in B_{\delta_1}$ hold for all $t \in [0, \tilde{\tau}]$ with $\tilde{\tau} > 0$. Then for all $t \in [0, \tilde{\tau}]$ and all $\tau_1 \leq \tau_2 \leq \tilde{\tau}$ the following bounds hold:*

$$\begin{aligned} \frac{1}{2}\alpha_0 \leq &\quad \tfrac{d}{dt} y(S_t q) &\leq \frac{3}{2}\alpha_0 &\qquad (2.102)\\ \frac{1}{2}\alpha_0(\tau_2 - \tau_1) \leq &\quad y(S_{\tau_2} q) - y(S_{\tau_1} q) &\leq \frac{3}{2}\alpha_0(\tau_2 - \tau_1) &\qquad (2.103)\\ &\textit{and} & & \\ \| \ x(S_{\tau_2} q) - x(S_{\tau_1} q) \ \| &\quad \leq & k_0(y(S_{\tau_2} q) - y(S_{\tau_1} q)), &\qquad (2.104) \end{aligned}$$

where $k_0 := 4\frac{\|f(p)\|}{\alpha_0}$.

Proof. We first show inequality (2.102). By equation (2.95) we have

$$S_t q = x + y(S_t q)F(0) + x(S_t q).$$

Hence by differentiation we conclude that

$$\begin{aligned} f(S_t q) &= \frac{d}{dt} S_t q \\ &= \frac{d}{dt} y(S_t q) + \frac{d}{dt} x(S_t q). \end{aligned} \tag{2.105}$$

By equation (2.94) we have $x(S_t q) \perp F(0)$ for all $t \in [0, \tilde{\tau}]$, we conclude that $\frac{d}{dt} x(S_t q) \perp F(0)$ hods too. By (2.95) we have

$$f(S_t q) = f(x) + \lambda(S_t q) F(0) + u(S_t q)$$

Using (2.96) and (2.97) yields

$$f(S_t q) = f(x) + \langle f(S_t q) - f(x), F(0) \rangle + f(S_t q) - f(x) - \lambda(S_t q) F(0)$$

which by little algebraic manipulation and using $\frac{d}{dt} y(S_t q) = f(S_t q)$ yields

$$0 = \frac{d}{dt} y(S_t q) F(0) - f(x) + f(x) F(0) - \lambda(S_t q).$$

Using (2.101) and rearranging yields

$$\frac{d}{dt} y(S_t q) F(0) = \alpha_0 + \lambda(S_t q). \tag{2.106}$$

Equation (2.106) with bound (2.99) yields condition (2.102) as required. Since we consider the time interval $t \in [0, \tilde{\tau}]$ with $0 \leq \tau_1 \leq \tau_2 \leq \tilde{\tau}$ conditio (2.103) follows from

$$\int_{\tau_1}^{\tau_2} \frac{d}{dt} y(S_t q) dt = y(S_{\tau_2} q) - y(S_{\tau_1} q)$$

and bounds of condition (2.102). Now, we multiply (2.106) by $\frac{d}{dt} x(S_t q)$ and with $\frac{d}{dt} x(S_t q) \perp F(0)$ we obtain

$$\| \frac{d}{dt} x(S_t q) \|^2 = \langle f(S_t q), \frac{d}{dt} x(S_t q) \rangle$$

which by (2.95) becomes

$$= \langle f(x) + u(S_t q), \frac{d}{dt} x(S_t q) \rangle$$

$$\| \frac{d}{dt} x(S_t q) \| \leq \langle \| f(p) \| + \| u(S_t q) \| . \tag{2.107}$$

Hence

$$\begin{aligned}
\| x(S_{\tau_2}) - x(S_{\tau_1}) \| &= \| \int_{\tau_1}^{\tau_2} \frac{d}{dt} x(S_t q) dt \| \\
&\leq \int_{\tau_1}^{\tau_2} \| \frac{d}{dt} x(S_t q) dt \| \\
&\leq \int_{\tau_1}^{\tau_2} (\|f(x)\| + \|u(S_t q)\| dt \text{ by } (2.107) \\
&\leq 2(\tau_2 - \tau_1) \|f(x) \text{ by } (2.100) \\
&\leq k_0 (y(S_{\tau_2} q) - y(S_{\tau_1} q)) \text{ by } (2.103).
\end{aligned}$$

which proves condition (2.104). This concludes the prove of lemma 2. ■

2.6.3 Definition of a correction mapping π

We define the operator π which maps nearby points to the hyperplane H along trajectories. We show that there is some short time interval such that trajectories through points in a small ball around $x \in K$ intersect the hyperplane $H := x + F(0)^{\perp}$.

Lemma 3 *Let*

$$\pi : \begin{cases} B_{\delta_2(p)} & \to & H := x + F(0)^{\perp} \\ q & \mapsto & \pi(q) \end{cases} \tag{2.108}$$

be a continuous map defined by

$$\pi(q) = S_{t^* q} q,$$

where $t^*(q)$ is a continuous function satisfying $|t^*(q)| \leq \frac{2\delta_2}{\alpha_0} =: t_0$ for all $q \in B_{\delta_2}(p)$.

Then for $x^* \in H \cap B_{\delta_2}(p)$ we have

$$\parallel \pi(q) - x^* \parallel \leq (k_0 + 1) \parallel q - x^* \parallel . \tag{2.109}$$

Proof. We only consider the case $y(q) \leq 0$. We consider the continuous function $y(S_\tau q)$ and show that for $q \in B_{\delta_2}(x)$ $y(S_\tau q)$ vanishes for some time $\tau = t^*$. We show this by condition (2.103) of lemma 2. Then we show by contradiction that t^* is close enough to zero so that a trajectory $S_\tau q$ remains in $B_{\delta_1}(x)$ for all time $\tau \in [0, t^*]$. Finally, we show condition (2.109) using condition (2.104) of lemma 2. Since we consider the case $y(q) \leq 0$ we have that as long as $S_\tau q \in B_{\delta_1}(x)$ with $\tau \geq 0$ we have that by condition (2.103) of lemma 2 $y(S_\tau q) \geq y(q) + \frac{\tau}{2}\alpha_0$. For $\tilde{\tau} = -\frac{2}{\alpha_0} y(q) \geq 0$ we have $y(S_{\tilde{\tau}} q) \geq 0$. Observe that $\mid \tilde{\tau} \mid \leq \frac{2}{\alpha_0}\delta_2 = t_0$. The existence of a time $t^* \in [0, \tilde{\tau}]$ such that $y(S_{\tilde{\tau}} q) = 0$ is satisfied is implied by the intermediate vale theorem. Uniqueness of t^* follows from lemma 2 as $y(S_\tau q)$ is monotonously increasing inτ. Now, by the implicit function theorem we can define the continuous function $t^*(q)$ by $y(S_{\tilde{\tau}} q) = 0$. Since y and S_t are continuous functions, it follows that t^* is continuous. That proves that the projection mapping π is also continuous as required.

Next, we show by contradiction that $\tilde{\tau}$ is close enough to zero so that a trajectory $S_\tau q$ remains in $B_{\delta_1}(x)$ for all time $\tau \in [0, \tilde{\tau}]$. Assume the contrary. Let there be a $\tau_0 \in [0, \tilde{\tau}]$ with $\parallel S_{\tau_0} q - x \parallel = \delta_1$ and $\parallel S_\tau q - x \parallel < \delta_1$ for all $\tau \in [0, \tau_0]$. Then by (2.98), (2.99), and (2.100) we have

$$\parallel f(q) \parallel \leq 2 \parallel f(x) \parallel + \frac{1}{2}\alpha_0$$

for all $q \in B_{\delta_1}(x)$. This yields

$$\begin{aligned} \delta_1 &= \parallel S_{\tau_0} q - x \parallel \\ &\leq \parallel \int_0^{\tau_0} f(S_\tau q) d\tau \parallel + \parallel q - x \parallel \\ &\leq \mid \tilde{\tau} \mid \epsilon_0 + \delta_2 \\ &\leq \delta_2 \left(\frac{2\epsilon_0}{\alpha_0} + 1 \right) = \frac{\delta_1}{2}. \end{aligned}$$

Hence a contradiction.

In the final step of the proof we need to show property 2.109. By (2.93) we have

$$
\begin{aligned}
y(q) &= \langle q - x, F(0) \rangle \\
&= \langle q - x^*, F(0) \rangle + \langle x^* - x, F(0) \rangle
\end{aligned}
$$

with $\langle x^* - x, F(0) \rangle = 0$. Hence $|y(q) \leq \| q - x^* \|$. Condition (2.104) of lemma 2 implies that

$$
\| x(\pi(q)) - x(q) \| \leq k_0 \mid t(q) \mid \leq k_0 \| q - x^* \| .
$$

We conclude the following

$$
\begin{aligned}
\| \pi(q) - x^* \| &= \| x(\pi(q)) - x(x^*) \| \\
&\leq \| x(\pi(q)) - x(q)) \| + \| x(x(q)) - x(x^*) \| \\
&\leq (k_0 + 1) \| q - x^* \| .
\end{aligned}
$$

This concludes the proof of lemma 3. ■

2.6.4 Definition of a Poincaré-like map $\mathcal{P}$

In the last two steps of the proof of proposition 8 we need to show that a period orbit Ω exists and that it is exponentially asymptotic stable. Existence of a periodic orbit Ω requires to define a Poincaré-like map $\mathcal{P}$ which maps open sets into themselves, $\mathcal{P} : U_0 \to U_0$, where the compact sets U_0 are subsets of the hyperplane H. Once $\mathcal{P}$ is defined, we show that it is contracting, which shows existence of Ω.

Now we define a Poincaré-like map $\mathcal{P}$ and show that the diameter of $\mathcal{P}^k(U_0)$ decreases. Let's consider a point $x \in K$ such that $x \in \omega(x)$. Set $\delta^* := \frac{\delta_2}{2(k_0+1)}$. Thus there is a minimal time period $T^* \geq 3t_0 + 2\frac{ln[2C(k_0+1)]}{\nu}$ so that $S_{T^*}x \in B_{\delta^*}(x)$. By lemma 3 there is a $T_1 \in [T^* - t_0, T^* + t_0]$ such that the point $x_1 := S_{T_1}x = \pi(S_{T^*}x)$ that by condition (2.109) of lemma 3 it holds that $x_1 \in H$ and $\| x_1 - x \| \leq \frac{\delta_2}{2}$. Observe that $t_1 \geq 2\left(t_0 + 2\frac{ln[2C(k_0+1)]}{\nu}\right)$.

Now, set $\delta_3 := 2\|x_1 - x\| \leq \delta_2 \leq \delta_1 \leq \delta$. By condition (2.92) of proposition 8 a point q in the compact set U_0 with $U_0 := H \cap \overline{B_{\delta_3}(x)}$ will move to a point q_1 by $S_{T_x^q(T_1)}q$ which satisfies $\| q_1 - x_1 \| \leq Ce^{-\nu T_1} \| q - x \| \leq \frac{\delta_3}{2(k_0+1))} < \delta_2$. The point $\pi(q_1)$ satisfies $\pi(q_1) \in H$ by lemma 3, Moreover, by condition (2.109) we have that $\| \pi(q_1) - x_1 \| \leq (k_0+1) \| q_1 - x_1 \| \leq \frac{\delta_3}{2}$.

Hence, we are now in a position to define the Poincaré-like map by

$$\mathcal{P}: \begin{cases} U_0 & \to & U_0 \\ q & \mapsto & \pi(S_{T_x^q(T_1)}q). \end{cases} \tag{2.110}$$

Remark 6 *The Poincaré-like map $\mathcal{P}$ is a return map. However, it is not necessarily the first return map to the hyperplane H.*

It remains to prove that $\mathcal{P}(U_0) \subset U_0$. By calculation

$$\| \mathcal{P}(x) - q \| \leq \| \mathcal{P}(x) - x_1 \| + \| x_1 - x \| \leq \frac{\delta_3}{2} + \frac{\delta_3}{2}.$$

Note that $\mathcal{P}$ is continuous. Continuity of $\mathcal{P}$ directly follows from continuity of π, S_T and T_x^q. By definition of the projection map π we have $\mathcal{P}(q) = S_{\tau(q)}q$ for a continuous map τ with $\tau(q) \geq \frac{T_1}{2} - t_0 \geq \frac{ln[2C(k_0+1)]}{\nu} > 0$ for all $q \in U_0$.

We now show that the diameter of the sets defined by the return map decrease. We first define the compact sets U_k by $\mathcal{P}$.

$$U_k := \mathcal{P}^k(U_0) \text{ for all } k \in \mathbb{N}. \tag{2.111}$$

Lemma 4 *Let the compact sets $U_k \subset H$ be defined for all $k \in \mathbb{N}$ by the return map $\mathcal{P}^k$ in definition (2.111). Moreover, define for all $k \in \mathbb{N}$ the points x_k by*

$$x_k := \mathcal{P}^k(U_0) \in U_k.$$

Then the following properties hold for all $k \in \mathbb{N}$

$$\begin{aligned} U_k &\subset U_{k-1} & (2.112) \\ diam U_k &\leq \frac{\delta_3}{2^{k-1}}. & (2.113) \end{aligned}$$

Proof. We first show statement (2.112). For $k = 1$ we have

$$U_1 = \mathcal{P}(U_0) \subset U_0.$$

For $k \geq 2$ we have

$$\mathcal{P}^k(U_0) = \mathcal{P}^{k-1}\mathcal{P}(U_0) \subset \mathcal{P}^{k-1}(U_0).$$

Since $\mathcal{P}$ is continuous, it follows that since U_k are images of U_{k-1} under $\mathcal{P}$ that U_k are compact sets by induction.

We now show statement (2.113). We remarked that $\mathcal{P}$ does not necessarily have to be a first return map. We now take this into consideration and show that we reach the same points no matter whether we apply π after each return or only once at the end. We provide a characterization of U_k.

The points $p_k$belong to the forward trajectory through the point x. Hence we define T_k such that $x_k = \mathcal{P}(x_{k-1}) = S_{T_k x_{k-1}}$ for all $k \in \mathbb{N}$. Also $x_0 := p$. Moreover, we know that $T_k \geq \frac{ln[2C(k_0+1)]}{\nu} > 0$. We now set for all $k \in \mathbb{N}$

$$V_k := \left\{ S_{T_x^q(\sum_{i=1}^k T_i)} q : q \in U_0 \right\}$$

and

$$q_k := S_{T_x^q(\sum_{i=1}^k T_i)} q$$

and claim that $\mathcal{P}^k(q) = \pi(q_k)$ holds for all $q \in U_0$ and all $k \in \mathbb{N}$. In particular we have the characterization $U_k = \pi(V_k)$.

Now, pick any $k \in \mathbb{N}$. Then we already know that $\pi(x_k) = x_k = \mathcal{P}^k(x)$ and $x_k \in U_k \cap \pi(V_k)$. This is the claim for $q = x$. Moreover , we have that $U_k, \pi(V_k) \subset H$, and all points of both U_k and $\pi(V_k)$ can be written as $S_{\tau_i(q)} q$ with $q \in U_0$, where τ_i are continuous functions. This is used to prove that $\mathcal{P}^k(q) = \pi(q_k)$ holds for all $q \in U_0$ and all $k \in \mathbb{N}$.

Let's consider

$$Q(\tau, q) = \langle x_k - p, F(0) \rangle \text{ for all } q \in U_0.$$

By (2.95) and (2.99) we have that $Q(\sum_{i=1}^k T_{i,x}) = \langle x_k - x, F(0) \rangle = 0$ and $\partial_\tau Q(\tau, q) = \langle f(S_\tau q), F(0) \rangle \geq \frac{\alpha_0}{2} > 0$ for all $S_\tau q \in B_{\delta_1}$. In particular $\partial_\tau Q(\sum_{i=1}^k T_{i,x}) \neq 0$. Hence by the implicit function theorem there is a unique continuous function

$\tau(q)$ near x such that $Q(\tau(q), q) = 0$. This is equivalent to $S_{\tau(q)}q \in H$. Since both, τ_1 and τ_2 are such functions, they have to coincide near the point x. Hence, by a prolongation we obtain $\tau_1 = \tau_2$ on $U_0 \subset B_{\delta_1(x)}$. Thus for $q_k = S_{T_x^q(\sum_{i=1}^k T_i)}q$ we have that $\mathcal{P}^k(q) = \pi(q_k)$ as we wanted to show.

We now want to prove statement(2.113). Hence we consider a point $q \in U_0$. Thus for $q_k = S_{T_x^q(\sum_{i=1}^k T_i)}q$ we have that $\mathcal{P}^k(q) = \pi(q_k)$ as we have shown above. From condition (2.92) of proposition 8 we obtain that

$$\| q_k - x_k \| \leq Ce^{-\nu \sum_{T_i^k}} \| q - x \| \leq (k_0 + 1)C \frac{\delta_3}{[2C(k_0 + 1)]^k}.$$

Note that both C and $(k_0 + 1) \geq 1$. By condition (2.109) of lemma 3 this yields

$$\| \mathcal{P}^k(q) - x_k \| = \| \pi(q_k) - x_k \| \leq \frac{\delta_3}{[2^k(k_0 + 1)]^k} \text{ for all } k \in \mathbb{N}.$$

As

$$\begin{aligned} diam U_k &= \max_{q',q'' \in U_0} \| \mathcal{P}^k(q') - \mathcal{P}^k(q'') \| \\ &\leq \max_{q' \in U_0} \| \mathcal{P}^k(q') - x_k \| + \max_{q'' \in U_0} \| x_k - \mathcal{P}^k(q'') \| \\ &\leq 2\frac{\delta_3}{2^k}. \end{aligned}$$

This completes the proof. ■

2.6.5 Existence of period orbit Ω and its stability

By construction of a sequence of compact sets U_k with decreasing diameter, we have shown in lemma 4 that for all $k \in \mathbb{N}_0$ there is one and only one point $\tilde{x}$ which lies in all sets. We know that $\tilde{x}$ is a fixed point of $\mathcal{P}$ since $\mathcal{P}(\tilde{x})$ lies in all compact sets U_k as well. Given a fixed point, there is a shortest time $T > 0$ so that $S_T\tilde{x} = \tilde{x}$ and thus $\tilde{x}$ is a point of the periodic orbit Ω. Since $\tilde{x} = x_\eta$ with $\eta \perp F(0)$ and $\| \eta \| \leq \delta$ we know by condition (2.92) of proposition 8 that both points x and $\tilde{x}$ have the same ω-limit sets. Thus $x \in \omega(x) = \omega(\tilde{x}) = \Omega$ and x is a point of the periodic orbit Ω.

Main proposition generalized

Now we prove the following result.

Proposition 9 *Define* $\delta_m := \min_{q\in H, \|q\|=\delta, t\in[0,T]} \| S_{T_x^q(t)}q - S_t p \| > 0$. *Then there are constants* $\delta_2', t_0' > 0$ *such that for each* $q \in \mathbb{R}^2$ *with* $dist(q, \Omega)) \leq \delta_2'$ *there is a* t *with* $| t' | \leq t_0'$ *such that* $S_t q = S_\theta x + \eta$ *with* $\theta \in [0,T]$, $\| \eta \| \leq \delta_m$ *and* $\langle \eta, F(0) \rangle = 0$.

Proof. Similar to the early part of the proof of proposition 8 we define new coordinates. The main difference here is that we consider coordinates for all points $S_\theta p$ with $\theta \in [0,T]$. Hence by (2.94), (2.93), (2.96) and (2.97) we obtain

$$x_\theta(q) := S_\theta q - S_\theta x - y_\theta(q)F(\theta) \in F(\theta)^\perp \tag{2.114}$$

$$y_\theta(q) := \langle S_\theta q - S_\theta x, F(\theta)\rangle \in \mathbb{R} \tag{2.115}$$

$$\lambda_\theta(q) := \langle f(S_\theta q) - f(S_\theta x), F(\theta)\rangle \in \mathbb{R} \tag{2.116}$$

$$u_\theta(q) := f(S_\theta q) - f(S_\theta x) - \lambda_\theta(q)F(\theta) \in F(\theta)^\perp \tag{2.117}$$

Since $[0,T]$ is a compact set the constants

$$\begin{aligned} f_M &:= \max_{\theta\in[0,T]} \| f(S_\theta x) \| \\ \alpha_m &:= \min_{\theta\in[0,T]} \langle F(\theta), f(S_\theta x)\rangle > 0 \\ \alpha_M &:= \max_{\theta\in[0,T]} \langle F(\theta), f(S_\theta x)\rangle > 0 \end{aligned}$$

exist.

We can choose $\delta_1' > 0$ such that for all q

$$|y_\theta(q)| \leq \frac{1}{2}\alpha_m \tag{2.118}$$

$$\| u_\theta(q) \| \leq f_M \tag{2.119}$$

there is a $\theta \in [0,T]$ with $\| S_\theta q - S_\theta x \| \leq \delta_1'$.

We define

$$
\begin{aligned}
k_0' &:= \frac{4}{\alpha_m} f_M \\
\epsilon_0' &:= 2f_M + \frac{1}{2}\alpha_m \\
\delta_2' &:= min\left(\frac{1}{2}\frac{\delta_1'}{\frac{2\epsilon_0'}{\alpha_m}+1}, \frac{\delta_m}{k_0'+1}\right) \\
t_0' &:= \frac{2\delta_2'}{\alpha_m}.
\end{aligned}
$$

Proposition 10 (Generalized lemma 2) *Let $S_t q \in B_{\delta_1'}(S_\theta x)$ hold for all $t \in [0, \tilde{\tau}]$ with $\tilde{\tau} > 0$. Then for all $t \in [0, \tilde{\tau}]$ and all $\tau_1 \leq \tau_2 \leq \tilde{\tau}$ the following bounds hold:*

$$
\frac{1}{2}\alpha_m \leq \quad \tfrac{d}{dt} y_\theta(S_t q) \quad \leq \frac{1}{2}\alpha_m + \alpha_M \tag{2.120}
$$

$$
\frac{1}{2}\alpha_m(\tau_2 - \tau_1) \leq \quad y_\theta(S_{\tau_2} q) - y_\theta(S_{\tau_1} q) \quad \leq \left(\frac{1}{2}\alpha_m + \alpha_M\right)(\tau_2 - \tau_1) \tag{2.121}
$$

and

$$
\| \, x_\theta(S_{\tau_2} q) - x_\theta(S_{\tau_1} q) \, \| \quad \leq \quad k_0' \left(y_\theta(S_{\tau_2} q) - y_\theta(S_{\tau_1} q) \right) \tag{2.122}
$$

Proof. We first show inequality (2.120). By equation (2.114) we have

$$
S_t q = S_\theta x + y_\theta(S_t q) F(\theta) + x_\theta(S_t q).
$$

Hence by differentiation we conclude that

$$
\begin{aligned}
f(S_t q) &= \frac{d}{dt} S_t q \\
&= \frac{d}{dt} y_\theta(S_t q) + \frac{d}{dt} x_\theta(S_t q).
\end{aligned} \tag{2.123}
$$

By equation (2.114) we have $x(S_t q) \perp F(\theta)$ for all $t \in [0, \tilde{\tau}]$, we conclude that $\frac{d}{dt} x_\theta(S_t q) \perp F(\theta)$ holds too. Moreover from equation (2.114) we obtain

$$
S_\theta q = S_\theta x + y_\theta(q) F(\theta) + x_\theta(q). \tag{2.124}
$$

By (2.124) we have

$$f(S_t q) \quad = \quad f(S_\theta x) + \lambda_\theta(S_t q)F(\theta) + u_\theta(S_t q)$$

Using (2.116) and (2.117) yields

$$f(S_t q) \quad = \quad f(S_\theta x) + \langle f(S_t q) - f(S_\theta p), F(\theta)\rangle + f(S_t q) - f(S_\theta x) - \lambda_\theta(S_t q)F(\theta)$$

which by little algebraic manipulation and using $\frac{d}{dt}y_\theta(S_t q) = f(S_t q)$ yields

$$0 \quad = \quad \frac{d}{dt}y_\theta(S_t q)F(\theta) - f(S_\theta x) + f(x)F(\theta) - \lambda_\theta(S_t q).$$

Using (2.101) and rearranging yields

$$\frac{d}{dt}y_\theta(S_t q)F(\theta) = \alpha_0 + \lambda_\theta(S_t q). \tag{2.125}$$

Equation (2.125) with bound (2.118) yields condition (2.120) as required.

Since we consider the time interval $t \in [0, \tilde{\tau}]$ with $0 \leq \tau_1 \leq \tau_2 \leq \tilde{\tau}$ condition (2.121) follows from

$$\int_{\tau_1}^{\tau_2} \frac{d}{dt}y_\theta(S_t q)dt = y_\theta(S_{\tau_2} q) - y_\theta(S_{\tau_1} q)$$

and bounds of condition (2.120). Now, we multiply (2.125) by $\frac{d}{dt}x_\theta(S_t q)$ and with $\frac{d}{dt}x_\theta(S_t q) \perp F(\theta)$ we obtain

$$\parallel \frac{d}{dt}x_\theta(S_t q) \parallel^2 \quad = \quad \langle f(S_t q), \frac{d}{dt}x_\theta(S_t q)\rangle$$

which by (2.124) becomes

$$= \quad \langle f(S_\theta x) + u_\theta(S_t q), \frac{d}{dt}x_\theta(S_t q)\rangle$$

$$\| \frac{d}{dt} x_\theta(S_t q) \| \quad \leq \quad \langle \| f(S_\theta x) \| + \| u_\theta(S_t q) \| \, . \tag{2.126}$$

Hence

$$\begin{aligned} \| x_\theta(S_{\tau_2}) - x_\theta(S_{\tau_1}) \| &= \| \int_{\tau_1}^{\tau_2} \frac{d}{dt} x(S_t q) dt \| \\ &\leq \int_{\tau_1}^{\tau_2} \| \frac{d}{dt} x_\theta(S_t q) dt \| \\ &\leq \int_{\tau_1}^{\tau_2} (\|f(x)\| + \|u_\theta(S_t q)\| dt \text{ by } (2.126) \\ &\leq 2(\tau_2 - \tau_1) \|f(x) \text{ by } (2.118) \\ &\leq k_0(y(S_{\tau_2} q) - y(S_{\tau_1} q)) \text{ by } (2.121). \end{aligned}$$

which proves condition (2.122). This concludes the prove of lemma 10. ■

Lemma 5 (generalized lemma 3) *Let*

$$\pi'_\theta : \begin{cases} B_{\delta'_2(S_\theta p)} & \rightarrow \quad H' := S_\theta x + F(\theta)^\perp \\ q & \mapsto \quad \pi_\theta(q) \end{cases} \tag{2.127}$$

be a continuous map defined by

$$\pi'_\theta(q) = S_{t'(q)} q,$$

where $t'(q)$ is a continuous function satisfying $|t'(q)| \leq \frac{2\delta_2}{\alpha_0} =: t'_0$ for all $q \in B_{\delta'_2(S_\theta p)}$. Then for $x' H' \cap B_{\delta'_2(S_\theta x)}$ we have

$$\| \pi'_\theta(q) - S_\theta x^* \| \leq (k'_0 + 1) \| S_\theta q - S_\theta x' \| \, . \tag{2.128}$$

Proof. We only consider the case $y_\theta(q) \leq 0$ we have that as long as $S_\tau q \in B_{\delta'_1}(S_\theta x)$ with $\tau \geq 0$ we have that by condition (2.121) of lemma 10 $y_\theta(S_\tau q) \geq y_\theta(q) + \frac{\tau}{2}\alpha_0$. For $\tilde{\tau} = -\frac{2}{\alpha_0} y_\theta(q) \geq 0$ we have $y_\theta(S_{\tilde{\tau}} q) \geq 0$. Observe that $| \tilde{\tau} | \leq \frac{2}{\alpha_0}\delta'_2 = t'_0$. The existence of a time $t' \in [0, \tilde{\tau}]$ such that $y_\theta(S_{\tilde{\tau}} q) = 0$ is satisfied is implied by the intermediate vale theorem. Uniqueness of t' follows from lemma 10 as $y_\theta(S_\tau q)$ is monotonously increasing inτ. Now, by the implicit function theorem we can define the continuous function $t'(q)$ by $y_\theta(S_{\tilde{\tau}}q) = 0$. Since y_θ and S_t are continuous functions, it follows that t' is continuous. That proves that the projection mapping π_θ is also continuous as required.

Next, we show by contradiction that $\tilde{\tau}$ is close enough to zero so that a trajectory $S_\tau q$ remains in $B_{\delta_1'}(S_\theta x)$ for all time $\tau \in [0, \tilde{\tau}]$. Assume the contrary. Let there be a $\tau_0 \in [0, \tilde{\tau}]$ with $\| S_{\tau_0} q - S_\theta p \| = \delta_1'$ and $\| S_\tau q - S_\theta p \| < \delta_1'$ for all $\tau \in [0, \tau_0]$. Then by

$$f(S_t q) \quad = \quad f(S_t x) + \lambda_\theta(S_t q) F(\theta) + u_\theta(S_t q)$$

and bounds (2.118) we have

$$\| f(S_\tau q) \| \leq 2 \| f(S_\theta x) \| + \frac{1}{2}\alpha_0$$

for all $q \in B_{\delta_1'}(S_\theta x)$. This yields

$$\begin{aligned}
\delta_1' &= \| S_{\tau_0} q - S_\theta x \| \\
&\leq \| \int_0^{\tau_0} f(S_\tau q) d\tau \| + \| S_\tau q - S_\theta x \| \\
&\leq | \tilde{\tau} | \epsilon_0 + \delta_2 \\
&\leq \delta_2 \left(\frac{2\epsilon_0}{\alpha_0} + 1 \right) = \frac{\delta_1'}{2}.
\end{aligned}$$

Hence a contradiction.

In the final step of the proof we need to show property 2.128. By (2.115) we have

$$\begin{aligned}
y_\theta(q) &= \langle S_\tau q - S_\theta x, F(\theta) \rangle \\
&= \langle S_\tau q - S_\theta x', F(\theta) \rangle + \langle S_\theta x' - S_\theta x, F(\theta) \rangle
\end{aligned}$$

with $\langle S_\theta x' - S_\theta x, F(\theta) \rangle = 0$. Hence $|y_\theta(q) \leq \| S_\tau q - S_\theta x' \|$. Condition (2.122) of lemma 10 implies that

$$\| x_\theta(\pi_\theta(q)) - x_\theta(q) \| \leq k_0' \mid t(q) \mid \leq k_0' \| S_\tau q - S_\theta x' \| .$$

We conclude the following

$$
\begin{aligned}
\| \pi_\theta(q) - S_\theta x' \| &= \| x_\theta(\pi_\theta(q)) - x_\theta(S_\theta x') \| \\
&\leq \| x_\theta(\pi_\theta(q)) - x_\theta(q)) \| + \| x_\theta(x_\theta(q)) - x_\theta(S_\theta x') \| \\
&\leq (k'_0 + 1) \| S_\tau q - S_\theta x' \| .
\end{aligned}
$$

This concludes the proof of lemma 5. ■

Thus by lemma (B.3 general) we have

$$\pi'_\theta : B_{\delta'_2}(S_\theta x) \to S_\theta x + F(\theta)^\perp.$$

Hence, we can write $\pi'_\theta(q) = S_{t'(q)} q = S_\theta x + \eta$ with $\eta \perp F(\theta)$ and $|t'(q)| \leq t'_0$. Now. by condition (equivalent of 41) we have that

$$\| \eta \| \leq (k'_0 +) \delta'_2 \leq \delta_m.$$

■

This concludes the proof of lemma 10 . ■

2.7 Stability

Now still need to complete the proof by showing that a periodic orbit is stable.

Finally we prove that the periodic orbit $\Omega := \{S_\theta x : \theta \in [0, T]\}$ is exponentially asymptotically stable. We define

$$
\begin{aligned}
H_0 &:= H \cap \overline{B_\delta(x)} \\
\mathcal{P}_\theta H_\theta &:= \left\{ S_{T^q_x(\theta)} : q \in H_0 \right\} \\
E &:= \bigcup_{\theta \in [0,T]} \mathcal{P}_\theta H_\theta .
\end{aligned}
$$

By condition (2.92) of proposition 8 the points of E are attracted exponentially fast by the periodic orbit Ω. We show that the trajectory of each point of a neighborhood of Ω meets a point of E in finite time.

We have that for all $\theta \in [0, T]$

$$\mathcal{P}_\theta H_\theta \supset \{S_\theta(x+\eta) : \|\eta\| \leq \delta_m, \text{ and } \eta \perp F(\theta)\}.$$

Thus by lemma 10 we know that for all points q of the neighborhood $\Omega_{\delta'_2}$ of a periodic orbit Ω

$$S_t q \in \bigcup_{\theta \in [0,T]} \mathcal{P}_\theta H_\theta \subset E$$

holds for all time $| t | \leq t'_0$. This contraction property in finite time shows that the periodic orbit Ω is exponentially asymptotically stable.

This concludes the proof of proposition 8 as required. Moreover, theorem 5 is shown by propositions 1, 2, 3, 5, 6, and 8.

■

2.8 Conclusion

This chapter discussed the main result of this thesis. We provide conditions for the existence, uniqueness, and exponentially asymptotically stability of a periodic orbit, and for a set to belong to its basin of attraction. the conditions are not necessary and sufficient. A converse theorem needs to be proved. In future work, we will show that given an exponentially asymptotically stable orbit a function W always exist. A converse theorem would provide a good tool to prove that a certain set belongs to the basin of attraction of a unique exponentially asymptotically stable orbit.

Chapter 3

An Example

We consider an example. In the first part of this chapter we show stability of a a solution by application of conventional theory. This requires to calculate the explicit solution of a periodic orbit and then by application of Poincaré theory show that its is asymptotically stable. Then in the second part of this chapter, we study the same example from the perspective of our theory developed in chapter two. The main advantage of our theory is that we do not have to calculate the explicit solution. However, a disadvantage is that it requires to find a function W. The theory does not provide a method how to find such a function.

3.1 Conventional method

We consider an explicit calculation of a periodic orbit of a non-smooth dynamical system described by equation (2.1). The calculation of a periodic orbit is performed by transforming the original problem into a problem in polar coordinates. The explicit solution is then found by separation of variables. We show stability of this orbit by application of a Poincaré map.

3.1.1 The model

We consider a differential equation

$$\dot{x} = f(x)$$

given by

$$\left.\begin{array}{rl} f_1^+(x) = & x_1\left((R^+)^2 - ((x_1)^2 + (x_2)^2)\right) - x_2 \\ f_2^+(x) = & x_2\left((R^+)^2 - ((x_1)^2 + (x_2)^2)\right) + x_1 \end{array}\right\} \text{if } x_2 > 0. \tag{3.1}$$

and

$$\left.\begin{array}{rl} f_1^-(x) = & x_1\left((R^-)^2 - ((x_1)^2 + (x_2)^2)\right) - x_2 \\ f_2^-(x) = & x_2\left((R^-)^2 - ((x_1)^2 + (x_2)^2)\right) + x_1 \end{array}\right\} \text{if } x_2 < 0, \tag{3.2}$$

where $R^{\pm} > 0$ are some constants.

This is a dynamical system of the form of equation (2.1) discussed in the main chapter of this thesis. Such equations may appear in mechanical engineering. An example of a similar equation is shown in the next chapter where an application of such an equation is considered in the context of a biomechanics problem. The aim here is to show existence and uniqueness by calculating its solution via conventional theory. We also show that the periodic orbit is stable.

3.1.2 An explicit calculation of a periodic orbit

We want to find a solution of this model. Hence we transform the model in polar coordinates. We use the formulas

$$\dot{r}^{\pm} = \frac{1}{r^{\pm}}(x_1^{\pm}\dot{x}_1^{\pm} + x_2^{\pm}\dot{x}_2^{\pm}) \tag{3.3}$$

$$\dot{\theta}^{\pm} = \frac{x_1^{\pm}\dot{x}_2^{\pm} + x_2\dot{x}_1^{\pm}}{(r^{\pm})^2}. \tag{3.4}$$

Using equations (3.3) and (3.4) we obtain

$$\dot{r}^{\pm} = r^{\pm}\left((R^{\pm})^2 - (r^{\pm})^2\right) \tag{3.5}$$

$$\dot{\theta}^{\pm} = 1. \tag{3.6}$$

The differential equation of the phase path is given by

$$\frac{\frac{dr^{\pm}}{dt}}{\frac{d\theta^{\pm}}{dt}} = \frac{dr^{\pm}}{d\theta^{\pm}} = \frac{r^{\pm}\left((R^{\pm})^2 - (r^{\pm})^2\right)}{1}.$$

This differential equation can be solved by the method of separating variables. Thus separating variables and integrating both sides yields

$$\int \frac{dr^{\pm}}{r^{\pm}\left((R^{\pm})^2-(r^{\pm})^2\right)} = \int \frac{d\theta^{\pm}}{1}. \tag{3.7}$$

In order to find a solution of the l.h.s of equation (3.7) we apply partial sums. Hence, we have

$$\int \frac{1}{r^{\pm}(R^{\pm}-r^{\pm})(R^{\pm}+r^{\pm})}dr^{\pm} = \int \frac{A}{r^{\pm}}dr^{\pm} + \int \frac{B}{(R^{\pm}-r^{\pm})}dr^{\pm} + \int \frac{C}{(R^{\pm}+r^{\pm})}dr^{\pm}. \tag{3.8}$$

We solve

$$1 = A^{\pm}(R^{\pm}-r^{\pm})(R^{\pm}+r^{\pm}) + B^{\pm}r^{\pm}(R^{\pm}+r^{\pm}) + C^{\pm}r^{\pm}(R^{\pm}-r^{\pm}). \tag{3.9}$$

in order to obtain values for the constants A,B, and C. Hence equation (3.9) with $r^{\pm}=0$ yields $A^{\pm}=\frac{1}{(R^{\pm})^2}$. Equation (3.9) with $r^{\pm}=R^{\pm}$ yields $B^{\pm}=\frac{1}{2(R^{\pm})^2}$, and with $r^{\pm}=-R^{\pm}$ yields $C^{\pm}=-\frac{1}{2(R^{\pm})^2}$. Substituting these constants back into equation (3.8) yields

$$\begin{aligned}
&= \int \frac{\frac{1}{(R^{\pm})^2}}{r^{\pm}}dr^{\pm} + \int \frac{\frac{1}{2(R^{\pm})^2}}{(R^{\pm}-r^{\pm})}dr^{\pm} + \int \frac{-\frac{1}{2(R^{\pm})^2}}{(R^{\pm}+r^{\pm})}dr^{\pm} \\
&= \int \frac{1}{(R^{\pm})^2 r^{\pm}}dr^{\pm} + \int \frac{1}{2(R^{\pm})^2(R^{\pm}-r^{\pm})}dr^{\pm} - \int \frac{1}{2(R^{\pm})^2(R^{\pm}+r^{\pm})}dr^{\pm} \\
&= \frac{1}{(R^{\pm})^2}\int \frac{1}{r^{\pm}}dr^{\pm} + \frac{1}{2(R^{\pm})^2}\int \frac{1}{(R^{\pm}-r^{\pm})}dr^{\pm} - \frac{1}{2(R^{\pm})^2}\int \frac{1}{(R^{\pm}+r^{\pm})}dr^{\pm}.
\end{aligned}$$

Now solving equation (3.7) yields

$$\begin{aligned}
\frac{1}{(R^{\pm})^2}\ln r^{\pm} - \frac{1}{2(R^{\pm})^2}\ln(R^{\pm}-r^{\pm}) - \frac{1}{2(R^{\pm})^2}\ln(R^{\pm}+r^{\pm}) &= \theta^{\pm}+c_1^{\pm} \\
\frac{1}{2(R^{\pm})^2}\ln r^{\pm} + \frac{1}{2(R^{\pm})^2}\ln r^{\pm} - \frac{1}{2(R^{\pm})^2}\ln(R^{\pm}-r^{\pm}) - \frac{1}{2(R^{\pm})^2}\ln(R^{\pm}+r^{\pm}) &= \theta^{\pm}+c_1^{\pm} \\
\frac{1}{2(R^{\pm})^2}\ln\left(\frac{(r^{\pm})^2}{(R^{\pm}-r^{\pm})(R^{\pm}+r^{\pm})}\right) &= \theta^{\pm}+c_1^{\pm}. \\
\left(\frac{(r^{\pm})^2}{(R^{\pm}-r^{\pm})(R^{\pm}+r^{\pm})}\right) &= e^{2(R^{\pm})^2(\theta^{\pm}+c_1^{\pm})}.
\end{aligned}$$

Hence we have

$$\left(\frac{(r^{\pm})^2}{(R^{\pm}-r^{\pm})(R^{\pm}+r^{\pm})}\right) = e^{2(R^{\pm})^2(\theta^{\pm}+c_1^{\pm})} \tag{3.10}$$

with $D^{\pm} = e^{c^{\pm}}$, $r_0^{\pm} = r_0^{\pm}(\theta_0^{\pm})$, and $\theta_0^{\pm}$ we obtain

$$D^{\pm} = \frac{(r_0^{\pm})^2}{((R^{\pm})^2-(r_0^{\pm})^2)} e^{-2\theta_0^{\pm}(R^{\pm})^2}. \tag{3.11}$$

Substituting equation (3.11) into equation (3.10) yields

$$\begin{aligned}
\left(\frac{(r^{\pm})^2}{(R^{\pm}-r^{\pm})(R^{\pm}+r^{\pm})}\right) &= e^{2\theta^{\pm}R^{\pm})^2} D^{\pm} \\
\left(\frac{(r^{\pm})^2}{((R^{\pm})^2-(r^{\pm})^2)}\right) &= \frac{(r_0^{\pm})^2}{((R^{\pm})^2-(r_0^{\pm})^2)} e^{-2(\theta_0^{\pm}-\theta^{\pm})(R^{\pm})^2}
\end{aligned}$$

which after some algebra yields

$$(r^{\pm})^2 = \frac{\left(\frac{(r_0^{\pm})^2}{((R^{\pm})^2-(r^{\pm})^2)} e^{-2\theta_0^{\pm}(R^{\pm})^2}(R^{\pm})^2\right) e^{2\theta^{\pm}(R^{\pm})^2}}{1+\left(\frac{(r_0^{\pm})^2}{((R^{\pm})^2-(r^{\pm})^2)} e^{-2\theta_0^{\pm}(R^{\pm})^2}\right) e^{2\theta^{\pm}(R^{\pm})^2}}$$

and simplifies to

$$(r^{\pm})^2 = \frac{(R^{\pm})^2}{e^{-2(\theta^{\pm}-\theta_0^{\pm})(R^{\pm})^2}\left(\frac{(R^{\pm})^2-(r_0^{\pm})^2}{(r_0^{\pm})^2}\right)+1}. \tag{3.12}$$

Now we consider the positive orbit. Hence let $\theta_0^+ = 0$, and $\theta_1^+ = \pi$, then

$$(r_1^+)^2 = \frac{(R^+)^2}{e^{-2\pi(R^+)^2}\left(\frac{(R^+)^2-(r_0^+)^2}{(r_0^+)^2}\right)+1}. \tag{3.13}$$

Let's consider the negative orbit. Hence let $\theta_1^- = \pi$, and $\theta_2^- = 2\pi$, then

$$(r_2^-)^2 = \frac{(R^-)^2}{e^{-2\pi(R^-)^2}\left(\frac{(R^-)^2-(r_1^-)^2}{(r_1^-)^2}\right)+1}. \tag{3.14}$$

We now define the Poincaré map

$$P^2 : (r_0^+)^2 \to (r_1^+)^2 = (r_1^-)^2 \to (r_2^-)^2$$

using equations (3.13) and (3.14) equation by

$$P^2(r_0^+) = \frac{(R^-)^2}{e^{-2\pi(R^-)^2}\left(\frac{(R^-)^2 - \frac{(R^+)^2}{e^{-2\pi(R^+)^2}\left(\frac{(R^+)^2-(r_0^+)^2}{(r_0^+)^2}\right)+1}}{\frac{(R^+)^2}{e^{-2\pi(R^+)^2}\left(\frac{(R^+)^2-(r_0^+)^2}{(r_0^+)^2}\right)+1}}\right)+1}. \tag{3.15}$$

We want to find a fixed point

$$P^2(r_0^+) = (r_0^+)^2$$

and show that

$$|P'(\rho_0^i)| < 1.$$

with $\rho_i := (r_i)^2$ and $i = 0, 1$. Now, we calculate

$$\bar{P}^2(r_0^+) = (r_0^+)^2 \tag{3.16}$$

using equation (3.13). We have by equation (3.13)

$$\begin{aligned} (r_1^+)^2 &= \frac{(R^+)^2}{e^{-2\pi(R^+)^2}\left(\frac{(R^+)^2-(r_0^+)^2}{(r_0^+)^2}\right)+1} \\ &= \frac{A^2}{\frac{B^2-(r_0^+)^2}{(r_0^+)^2} + \frac{A^2}{B^2}} \end{aligned}$$

with

$$\begin{aligned} A^2 &= e^{2\pi(R^+)^2}(R^+)^2 \\ B^2 &= (R^+)^2. \end{aligned}$$

Then using (3.16) we obtain

$$\begin{aligned}
\frac{A^2(r_0^+)^2}{B^2-(r_0^+)^2+\frac{A^2}{B^2}(r_0^+)^2} &= (r_0^+)^2 \\
A^2(r_0^+)^2 &= (r_0^+)^2\left[B^2-(r_0^+)^2+\frac{A^2}{B^2}(r_0^+)^2\right],
\end{aligned}$$

hence

$$r_0^+ = 0$$

is a solution. Other solutions are given by

$$\begin{aligned}
A^2 &= \left[B^2-(r_0^+)^2+\frac{A^2}{B^2}(r_0^+)^2\right] \\
A^2-B^2 &= (r_0^+)^2\left[\frac{A^2}{B^2}-1\right] \\
(r_0^+)^2 &= \frac{A^2-B^2}{\frac{A^2}{B^2}-\frac{B^2}{B^2}} \\
&= \frac{A^2-B^2}{A^2-B^2}B^2
\end{aligned}$$

which by substitution of $B^2=(R^+)^2$ and simplification yields the solution

$$r_0^+ = \sqrt{(R^+)^2} = R^+.$$

Now, we calculate

$$P^2(r_0^+) = (r_0^+)^2 \tag{3.17}$$

using equation (3.15). We have by equation (3.15)

$$\begin{aligned}
(r_2^-)^2 &= \frac{(R^-)^2}{e^{-2\pi(R^-)^2}\left(\frac{(R^-)^2-(r_1^-)^2}{(r_1^-)^2}\right)+1} \\
&= \frac{a^2}{\frac{b^2-(r_1^-)^2}{(r_1^-)^2}+\frac{a^2}{b^2}}
\end{aligned}$$

with

$$\begin{aligned} a^2 &= e^{2\pi (R^-)^2}(R^-)^2 \\ b^2 &= (R^-)^2. \end{aligned}$$

Then using (3.17) we obtain

$$\begin{aligned}
\frac{a^2 (r_1^-)^2}{b^2 - (r_1^-)^2 + \frac{a^2}{b^2}(r_1^-)^2} &= (r_0^+)^2 \\
a^2 (r_1^-)^2 &= (r_0^+)^2 \left[b^2 - (r_1^-)^2 + \frac{a^2}{b^2}(r_1^-)^2 \right] \\
\frac{a^2}{(r_0^+)^2} \left[\frac{B^2 - (r_0^+)^2}{(r_0^+)^2} + \frac{A^2}{B^2} \right] &= \left[b^2 + (\frac{a^2}{b^2} - 1) \right] \left[\frac{B^2 - (r_0^+)^2}{(r_0^+)^2} + \frac{A^2}{B^2} \right] \\
\frac{a^2}{(r_0^+)^2} &= \frac{b^2 (r_0^+)^2}{\frac{A^2 (r_0^+)^2}{B^2 - (r_0^+)^2 + \frac{A^2}{B^2}(r_0^+)^2}} + \frac{a^2 - b^2}{b^2} \\
&= \frac{b^2 \left[B^2 - (r_0^+)^2 + \frac{A^2}{B^2}(r_0^+)^2 \right]}{A^2 (r_0^+)^2} + \frac{a^2 - b^2}{b^2} \\
\frac{a^2 - b^2}{b^2} &= \frac{1}{(r_0^+)^2} \left[a^2 - \frac{b^2 B^2 - b^2 (r_0^+)^2 + \frac{A^2}{B^2} b^2 (r_0^+)^2}{A^2} \right] \\
\left(\frac{a^2 - b^2}{b^2} \right) (r_0^+)^2 &= (r_0^+)^2 \left(b^2 - \frac{A^2}{B^2} b^2 \right) + A^2 a^2 - b^2 B^2 \\
A^2 a^2 - b^2 B^2 &= (r_0^+)^2 \left(A^2 \left[\frac{a^2 - b^2}{b^2} \right] + b^2 \left[\frac{A^2 - B^2}{B^2} \right] \right) \\
(r_0^+)^2 &= \frac{A^2 a^2 - b^2 B^2}{\left(A^2 \left[\frac{a^2 - b^2}{b^2} \right] + b^2 \left[\frac{A^2 - B^2}{B^2} \right] \right)}
\end{aligned}$$

which after substitution of A, B, a, b yields

$$\begin{aligned}
(r_0^+)^2 &= \frac{e^{2\pi (R^+)^2}(R^+)^2 e^{2\pi (R^-)^2}(R^-)^2 - (R^-)^2 (R^+)^2}{\left(e^{2\pi (R^+)^2}(R^+)^2 \left[\frac{e^{2\pi (R^-)^2}(R^-)^2 - (R^-)^2}{(R^-)^2} \right] + (R^-)^2 \left[\frac{e^{2\pi (R^+)^2}(R^+)^2 - (R^+)^2}{(R^+)^2} \right] \right)} \\
&= \frac{(R^+)^2 (R^-)^2 \left[e^{2\pi ((R^+)^2 (R^-)^2)} - 1 \right]}{(R^+)^2 (R^-)^2 \left[\frac{e^{2\pi ((R^+)^2 (R^-)^2)}}{(R^-)^2} - \frac{e^{2\pi (R^-)^2}}{(R^-)^2} + \frac{e^{2\pi (R^+)^2}}{(R^+)^2} - \frac{1}{(R^+)^2} \right]} \\
&= \frac{(R^+)^2 (R^-)^2 \left[e^{2\pi ((R^+)^2 (R^-)^2)} - 1 \right]}{(e^{2\pi (R^+)^2} - 1)\,(e^{2\pi (R^-)^2}(R^+)^2 + (R^-)^2)}
\end{aligned}$$

Hence we have the solution

$$r_0^+ = \sqrt{\frac{(R^+)^2(R^-)^2\left[e^{2\pi((R^+)^2(R^-)^2)}-1\right]}{\left(e^{2\pi(R^+)^2}-1\right)\left(e^{2\pi(R^-)^2}(R^+)^2+(R^-)^2\right)}}.$$

3.1.3 Stability of the periodic orbit

Next we want to show stability of the periodic orbit. This requires to check that the absolute value of the derivative of the Poincaré map is less than one at the calculated fixed point.

Let the radius r go from initial value r_0 to r_1 and then to r_2, and let $\rho_i = r_i^2$. It is enough to show that $\Pi(\rho_0) = P^2(r_0^2)$ satisfies $|\Pi'(\rho_0)| < 1$ (so everything for the squares): Indeed, we have

$$\begin{aligned}
\frac{dr_2}{dr_0} &= \frac{d\sqrt{\rho_2}}{d\rho_0}\frac{d\rho_0}{dr_0} \\
&= \frac{1}{2}\rho_2^{-1/2}\frac{d\rho_2}{d\rho_0}2r_0 \\
&= \frac{d\rho_2}{d\rho_0}\frac{r_0}{r_2}.
\end{aligned}$$

Note that at the fixed point (periodic orbit) we have $r_0 = r_2$, so that $\frac{dr_2}{dr_0} = \frac{d\rho_2}{d\rho_0}$.

Now let's calculate $\frac{d\rho_2}{d\rho_0}$. We have

$$\rho_1 = \frac{(R^+)^2 e^{2\pi(R^+)^2}}{\frac{(R^+)^2}{\rho_0} - 1 + e^{2\pi(R^+)^2}}, \tag{3.18}$$

$$\rho_2 = \frac{(R^-)^2 e^{2\pi(R^-)^2}}{\frac{(R^-)^2}{\rho_1} - 1 + e^{2\pi(R^-)^2}}. \tag{3.19}$$

In particular we have from the first equation

$$\rho_1\left(\frac{(R^+)^2}{\rho_0} - 1 + e^{2\pi(R^+)^2}\right) = (R^+)^2 e^{2\pi(R^+)^2}. \tag{3.20}$$

Then

$$
\begin{aligned}
\Pi'(\rho_0) &= \frac{d\rho_2}{d\rho_1}\frac{d\rho_1}{d\rho_0} \\
&= \frac{(R^-)^2 e^{2\pi(R^-)^2}}{\left(\frac{(R^-)^2}{\rho_1} - 1 + e^{2\pi(R^-)^2}\right)^2}\frac{(R^-)^2}{\rho_1^2} \\
&\quad \times \frac{(R^+)^2 e^{2\pi(R^+)^2}}{\left(\frac{(R^+)^2}{\rho_0} - 1 + e^{2\pi(R^+)^2}\right)^2}\frac{(R^+)^2}{\rho_0^2} \\
&= \frac{(R^-)^4 e^{2\pi(R^-)^2}(R^+)^4 e^{2\pi(R^+)^2}}{\left(\frac{(R^-)^2}{\rho_1} - 1 + e^{2\pi(R^-)^2}\right)^2 (R^+)^4 e^{4\pi(R^+)^2}\rho_0^2} \text{ using (3.20)} \\
&= \frac{(R^-)^4 e^{2\pi(R^-)^2} e^{2\pi(R^+)^2}}{\left(\frac{(R^-)^2}{\rho_1} - 1 + e^{2\pi(R^-)^2}\right)^2 \rho_0^2 e^{4\pi(R^+)^2}}.
\end{aligned}
$$

The fixed point (periodic orbit) condition is $\rho_2 = \rho_0$, i.e. see second equation (3.18)

$$
\begin{aligned}
\rho_0 &= \frac{(R^-)^2 e^{2\pi(R^-)^2}}{\frac{(R^-)^2}{\rho_1} - 1 + e^{2\pi(R^-)^2}} \\
\left(\frac{(R^-)^2}{\rho_1} - 1 + e^{2\pi(R^-)^2}\right)^2 \rho_0^2 &= (R^-)^4 e^{4\pi(R^-)^2}
\end{aligned}
$$

This gives

$$
\begin{aligned}
\Pi'(\rho_0) &= \frac{e^{2\pi(R^-)^2} e^{2\pi(R^+)^2}}{e^{4\pi(R^-)^2} e^{4\pi(R^+)^2}} \\
&= e^{-2\pi((R^-)^2 + (R^+)^2)} < 1
\end{aligned}
$$

By inspection of this inequality we conclude that the periodic orbit is stable.

3.2 Application of the new method

In many applications of non-smooth dynamical systems calculation of explicit solutions may be difficult. We want to establish the existence and stability of a periodic orbit without explicit calculation of the periodic orbit. Our theory developed in chapter two allows us to do so.

We consider the differential equation of above example

$$\dot{x} = f(x)$$

given by equations (3.1) and (3.2).

The aim is to calculate the following conditions of theorem 5:

1. $L_{W^+(x)} := \max_{\|v^+\|=e^{-W^+(x)}, v^+ \perp f^+(x)} L_{W^+}(x, v^+) < 0$
   $L_{W^+}(x, v^+) := e^{2W^+(x)} \left\{ (v^+)^T \left[Df^+(x) \right] v^+ + \langle \nabla W^+(x), f^+(x) \rangle \|v^+\|^2 \right\}$
   for all $x \in K^+$ with $x_2 > 0$.

2. $L_{W^-(x)} := \max_{\|v^-\|=e^{-W^-(x)}, v^- \perp f^-(x)} L_{W^-}(x, v^-) < 0$
   $L_{W^-}(x, v^-) := e^{2W^-(x)} \left\{ (v^-)^T \left[Df^-(x) \right] v^- + \langle \nabla W^-(x), f^-(x) \rangle \|v^-\|^2 \right\}$
   for all $x \in K^-$ with $x_2 < 0$.

3. $\frac{f_2^-(x_1,0)}{f_2^+(x_1,0)} \cdot \frac{\sqrt{\left(f_1^+(x_1,0)\right)^2 + \left(f_2^+(x_1,0)\right)^2}}{\sqrt{\left(f_1^-(x_1,0)\right)^2 + \left(f_2^-(x_1,0)\right)^2}} e^{W^-(x_1,0) - W^+(x_1,0)} < 1$
   for all $x \in K^0$ with $f_2^+(x_1, 0) < 0$, $f_2^-(x_1, 0) < 0$.

4. $\frac{f_2^+(x_1,0)}{f_2^-(x_1,0)} \cdot \frac{\sqrt{\left(f_1^-(x_1,0)\right)^2 + \left(f_2^-(x_1,0)\right)^2}}{\sqrt{\left(f_1^-(x_1,0)\right)^2 + \left(f_2^-(x_1,0)\right)^2}} e^{W^+(x_1,0) - W^-(x_1,0)} < 1$
   for all $x \in K^0$ with $f_2^+(x_1, 0) > 0$, $f_2^-(x_1, 0) > 0$.

3.2.1 The smooth conditions

From condition one above we have

$$\begin{aligned} L_{W^+}(x, v^+) &= e^{2W^+(x)} \left\{ (v^+)^T \left[Df^+(x) \right] v^+ + \langle \nabla W^+(x), f^+(x) \rangle \|v^+\|^2 \right\} \\ &= e^{2W^+(x)} \left[(v^+)^T Df^+(x) v^+ + (W^+(x))' \|v^+\|^2 \right] \end{aligned}$$

Hence our aim is to calculate $L_{W^+}(x, v^+) < 0$ which requires to check that

$$e^{2W^+(x)}\left[(v^+)^T Df^+(x)v^+ + (W^+(x))'\|v^+\|^2\right] < 0.$$

Let $v = \begin{pmatrix} -f_2^+ \\ f_1^+ \end{pmatrix}$. We first calculate

$$(v^+)^T\left[Df^+\right]v^+ = \begin{pmatrix} -f_2^+ \\ f_1^+ \end{pmatrix}^T \begin{pmatrix} (R^+)^2 - r^2 - 2x_1^2 & -2x_1x_2 - 1 \\ -2x_1x_2 + 1 & (R^+)^2 - r^2 - 2x_2^2 \end{pmatrix} \begin{pmatrix} -f_2^+ \\ f_1^+ \end{pmatrix}$$

where

$$\begin{aligned}
&= \left((R^+)^2 - r^2 - 2x_1^2\right)(f_2^+)^2 - f_1^+ f_2^+(-4x_1x_2) + \left((R^+)^2 - r^2 - 2x_2^2\right)(f_1^+)^2 \\
&= \left((R^+)^2 - r^2 - 2x_1^2\right)\left(x_2^2((R^+)^2 - r^2)^2 + x_1^2 + 2x_1x_2((R^+)^2 - r^2)\right) \\
&\quad + \left((R^+)^2 - r^2 - 2x_2^2\right)\left(x_1^2((R^+)^2 - r^2)^2 + x_2^2 - 2x_1x_2((R^+)^2 - r^2)\right) \\
&\quad +4x_1x_2\left[x_1x_2((R^+)^2 - r^2)^2 - x_1x_2 + (x_1^2 - x_2^2)((R^+)^2 - r^2)\right] \\
&= ((R^+)^2 - r^2)^3 r^2 - 4x_1^2x_2^2((R^+)^2 - r^2)^2 + ((R^+)^2 - r^2)r^2 - 2(x_1^4 + x_2^4) \\
&\quad -4x_1x_2[x_1^2 - x_2^2]((R^+)^2 - r^2) \\
&\quad 4x_1^2x_2^2\left[((R^+)^2 - r^2)^2 - 1\right] + 4x_1x_2(x_1^2 - x_2^2)((R^+)^2 - r^2)^2 \\
&= ((R^+)^2 - r^2)^3 r^2 + ((R^+)^2 - r^2)r^2 - 4x_1^2x_2^2 - 2x_1^4 - 2x_2^4 \\
&= ((R^+)^2 - r^2)^3 r^2 + ((R^+)^2 - r^2)r^2 - 2(x_1^2 + x_2^2)^2 \\
&= ((R^+)^2 - r^2)^3 r^2 + ((R^+)^2 - r^2)r^2 - 2r^4,
\end{aligned}$$

which can be rewritten as

$$v^T\left[Df\right]v = (R^+)^6 r^2 - 3(R^+)^4 r^4 + 3(R^+)^2 r^6 - r^8 + (R^+)^2 r^2 - 3r^4.$$

The second part of the condition requires to calculate $(W^+(x))'\|v^+\|^2$. This in turn requires to define W. We start by calculating

$$\begin{aligned}
\|v^+\|^2 &= \left(\sqrt{(f_1^+)^2 + (-f_2^+)^2}\right)^2 \\
&= r^2\left[((R^+)^2 - r^2)^2 + 1\right]
\end{aligned}$$

Next, we define W^+. Let's assume that $W^+ = W^+(r, \theta)$. Also, our model, given by equations (3.1) and (3.2) can be written in polar coordinates as

$$\begin{aligned} \dot{r}^{\pm} &= r^{\pm}\left((R^{\pm})^2 - (r^{\pm})^2\right) && (3.21) \\ \dot{\theta}^{\pm} &= 1. && (3.22) \end{aligned}$$

Then the orbital derivative of W^+ is given by

$$(W^+)' = W_r^+ \dot{r} + W_\theta^+ \dot{\theta} = W_r^+ r\left((R^+)^2 - r^2\right) + W_\theta^+.$$

We now define W.

$$\begin{aligned} W^+(r,\theta) &= 2\left(-(R^+)^2 \ln r + \frac{r^2}{2}\right) + 1.9(\theta + 2\pi) \\ W^-(r,\theta) &= 1.9 \cdot \theta \end{aligned}$$

with

$$\begin{aligned} W_r^+ &= \frac{2}{r}((-R^+)^2 + r^2) \\ W_r^- &= 0 \end{aligned}$$

and

$$\begin{aligned} W_\theta^+ &= 1.9 \\ W_\theta^- &= 1.9 \end{aligned}$$

thus

$$\begin{aligned} \left(W^+(r,\theta)\right)' &= -2((r^2 - R^+)^2)^2 + 1.9 \\ \left(W^-(r,\theta)\right)' &= 1.9 \end{aligned}$$

We now collect the partial results and state the smooth conditions $L_{W^+}(x, v^+) <$

0

$$\Big\{(R^+)^6r^2 - 3(R^+)^4r^4 + 3(R^+)^2r^6 - r^8 + (R^+)^2r^2 - 3r^4 + r^2\left[((R^+)^2 - r^2)^2 + 1\right]\left(-2((r^2 - R^+)^2)^2 + 1.9\right)\Big\} < 0 \tag{3.23}$$

for $r \in [R^-, R^+]$.

and $L_{W^-}(x, v^-) < 0$

$$\Big\{(R^-)^6r^2 - 3(R)^4r^4 + 3(R^-)^2r^6 - r^8 + (R^-)^2r^2 - 3r^4 + r^2\left[((R^-)^2 - r^2)^2 + 1\right]\cdot 1.9\Big\} < 0 \tag{3.24}$$

for $r \in [R^-, R^+]$. $K = \{r \in \mathbb{R} : R^- \leq r \leq R^+\}$ is a positively invariant set.

3.2.2 The jump conditions

Now, we wan to calculate the jumping conditions a the point $x_0 = (x_1, 0)$, with $x_1 < 0$ (condition 3). Hence, we start by calculating

$$\frac{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2 + \left(f_2^-(x_0)\right)^2}}}{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}} e^{W^-(x_0) - W^+(x_0)}. \tag{3.25}$$

From equation (3.1) we have

$$f_2^+ = x_2^+\left((R^+)^2 - ((x_1^+)^2 + (x_2^+)^2)\right) + x_1^+ \tag{3.26}$$

and from equation (3.2) we have

$$f_2^- = x_2^-\left((R^-)^2 - ((x_1^-)^2 + (x_2^-)^2)\right) + x_1^-. \tag{3.27}$$

We first calculate $\sqrt{\left(f_1^\pm(x_0)\right)^2 + \left(f_2^\pm(x_0)\right)^2}$. Hence

$$\begin{aligned}\left(f_1^\pm(x_0)\right)^2 + \left(f_2^\pm(x_0)\right)^2 &= \left\{x_1^\pm\left[(R^\pm)^2 - ((x_1^\pm)^2 + (x_2^\pm)^2)\right] - x_2^\pm\right\}^2 \\ &\quad \times \left\{x_2^\pm\left[(R^\pm)^2 - ((x_1^\pm)^2 + (x_2^\pm)^2)\right] + x_1^\pm\right\}^2.\end{aligned}$$

Let $a = ((x_1^\pm)^2 + (x_1^\pm)^2) - (R^\pm)^2)$. Then we have

$$\begin{aligned}
&= ((-a)x_1^\pm - x_2^\pm)^2 + ((-a)x_2^\pm + x_1^\pm)^2 \\
&= (ax_1^\pm)^2 + (ax_2^\pm)^2 + (x_1^\pm)^2 + (x_2^\pm)^2 \\
&= \left\{\left[(x_1^\pm)^2 + (x_1^\pm)^2\right] - (R^\pm)^2\right\}^2 (x_1^\pm)^2 + \\
&\quad \left\{\left[(x_1^\pm)^2 + (x_1^\pm)^2\right] - (R^\pm)^2\right\}^2 (x_2^\pm)^2 + (x_1^\pm)^2 + (x_2^\pm)^2
\end{aligned}$$

which after using $(r^\pm)^2 = (x_1^\pm)^2 + (x_2^\pm)^2$ yields

$$\begin{aligned}
&= ((r^\pm)^2 - (R^\pm)^2)^2(x_1)^2 + ((r^\pm)^2 - (R^\pm)^2)^2(x_2)^2 + (r^\pm)^2 \\
&= ((r^\pm)^2 - (R^\pm)^2)^2((x_1)^2 + (x_2)^2) + (r^\pm)^2 \\
&= ((r^\pm)^2 - (R^\pm)^2)^2(r^\pm)^2 + (r^\pm)^2.
\end{aligned}$$

Hence, we obtain

$$\sqrt{\left(f_1^\pm(x_0)\right)^2 + \left(f_2^\pm(x_0)\right)^2} = \sqrt{((r^\pm)^2 - (R^\pm)^2)^2(r^\pm)^2 + (r^\pm)^2}. \tag{3.28}$$

We can now calculate the main condition (3.25) using equations (3.26), (3.27), and (3.28). We obtain

$$\frac{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}} e^{W^-(x_0)-W^+(x_0)} = \frac{\frac{x_2^-\left((R^-)^2-((x_1^-)^2+(x_2^-)^2)\right)+x_1^-}{\sqrt{((r^-)^2-(R^-)^2)^2(r^-)^2+(r^-)^2}}}{\frac{x_2^+\left((R^+)^2-((x_1^+)^2+(x_2^+)^2)\right)+x_1^+}{\sqrt{((r^+)^2-(R^+)^2)^2(r^+)^2+(r^+)^2}}} e^{W^-(x_0)-W^+(x_0)}$$

and since $x_2^\pm = 0$, we obtain

$$= \frac{\frac{x_1^-}{\sqrt{((r^-)^2-(R^-)^2)^2(r^-)^2+1}}}{\frac{x_1^+}{\sqrt{((r^+)^2-(R^+)^2)^2(r^+)^2+1}}} e^{W^-(x_0)-W^+(x_0)}$$

which since $x_1^+ = x_1^-$ and $r^+ = r^-$ yields

$$= \sqrt{\frac{(r^2 - (R^+)^2)^2 + 1}{(r^2 - (R^-)^2)^2 + 1}} e^{W^-(x_0)-W^+(x_0)}.$$

We need to check that

$$[(r^2-(R^+)^2)^2+1]e^{2(W^-(x_0)-W^+(x_0))} < (r^2-(R^-)^2)^2+1. \tag{3.29}$$

Hence we need to show that

$$W^-(x_0)-W^+(x_0) \quad < \quad \frac{1}{2}\ln\left(\frac{(r^2-(R^-)^2)^2+1}{(r^2-(R^+)^2)^2+1}\right). \tag{3.30}$$

For the jump in the opposite direction we do a similar derivation.

$$\frac{\frac{f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2+\left(f_2^+(x_0)\right)^2}}}{\frac{f_2^-(x_0)}{\sqrt{\left(f_1^-(x_0)\right)^2+\left(f_2^-(x_0)\right)^2}}}e^{W^+(x_0)-W^-(x_0)} \quad = \quad \frac{\frac{x_2^-\left((R^+)^2-((x_1^-)^2+(x_2^-)^2)\right)+x_1^-}{\sqrt{((r^+)^2-(R^+)^2)^2(r^+)^2+(r^+)^2}}}{\frac{x_2^-\left((R^-)^2-((x_1^-)^2+(x_2^-)^2)\right)+x_1^-}{\sqrt{((r^-)^2-(R^+)^2)^2(r^-)^2+(r^-)^2}}}e^{W^-(x_0)-W^+(x_0)}$$

and since $x_2^\pm = 0$, $x_1^+ = x_1^-$ and $r^+ = r^-$ we obtain

$$= \quad \sqrt{\frac{(r^2-(R^-)^2)^2+1}{(r^2-(R^+)^2)^2+1}}e^{W^+(x_0)-W^-(x_0)}.$$

We need to check that

$$[(r^2-(R^-)^2)^2+1]e^{2(W^+(x_0)-W^-(x_0))} < (r^2-(R^+)^2)^2+1. \tag{3.31}$$

This is equivalent to checking that

$$W^+(x_0)-W^-(x_0) \quad < \quad \frac{1}{2}ln\left(\frac{(r^2-(R^+)^2)^2+1}{(r^2-(R^-)^2)^2+1}\right) \tag{3.32}$$

We conclude by collection of conditions (3.23), (3.24), (3.30), and (3.32) that

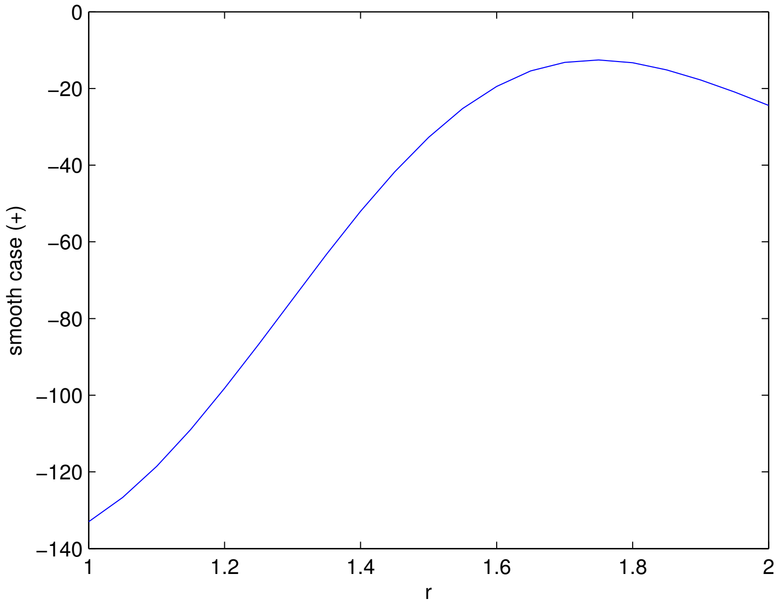

Figure 3.1: Condition 1: smooth(+).

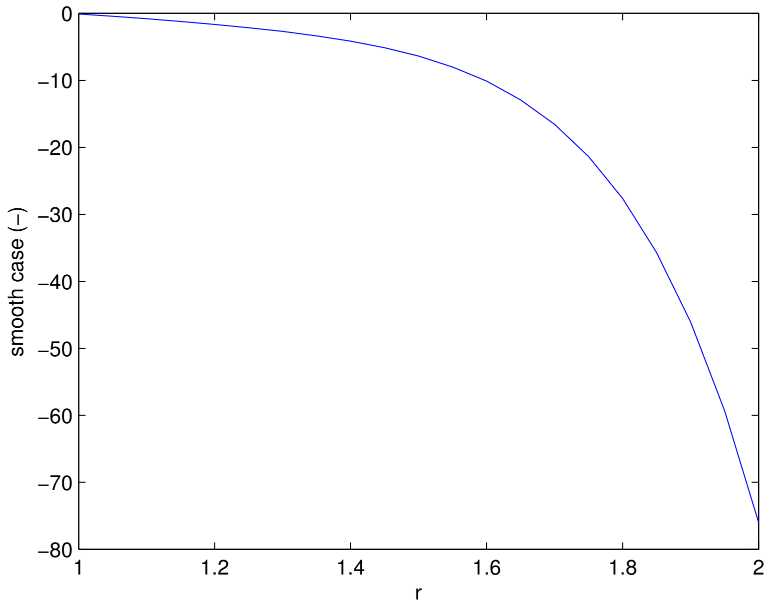

Figure 3.2: Condition 2: smooth(-).

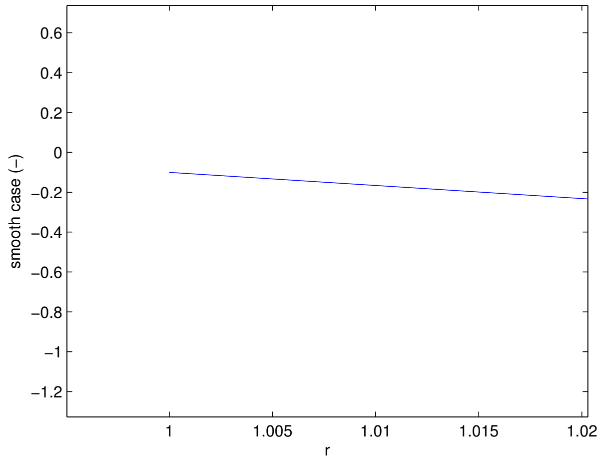

Figure 3.3: Condition: smooth(-) at r=1.

we need to show that (Condition 1)

$$\Big\{(R^+)^6r^2 - 3(R^+)^4r^4 + 3(R^+)^2r^6 - r^8 + (R^+)^2r^2 - 3r^4 + \\ r^2\left[((R^+)^2 - r^2)^2 + 1\right]\left(-2((r^2 - R^+)^2)^2 + 1.9\right)\Big\} \quad < \quad 0,$$

(Condition 2)

$$\Big\{(R^-)^6r^2 - 3(R)^4r^4 + 3(R^-)^2r^6 - r^8 + (R^-)^2r^2 - 3r^4 + \\ r^2\left[((R^-)^2 - r^2)^2 + 1\right]\cdot 1.9\Big\} \quad < \quad 0$$

(Condition 3), where $x_1 < 0$, $\theta = \pi$ and

$$\begin{aligned} W^-(x_0) &= 1.9\cdot\pi \\ W^-(x_0) &= 2(-(R^+)^2\ln x_1 + \frac{(x_1)^2}{2}) + 1.9\cdot 3r \end{aligned}$$

$$W^-(x_0) - W^+(x_0) - \left(\frac{1}{2}\ln\left(\frac{(r^2-(R^-)^2)^2+1}{(r^2-(R^+)^2)^2+1}\right)\right) \quad < \quad 0$$

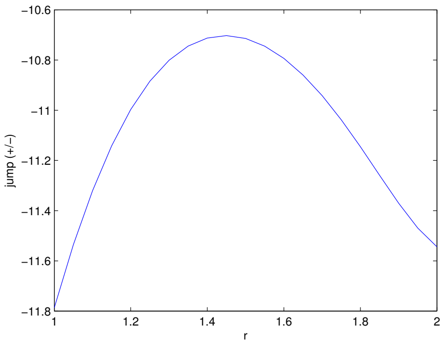

Figure 3.4: Condition 3: jump(+/-).

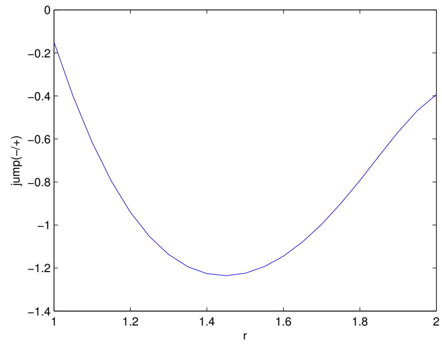

Figure 3.5: Condition 4: jump(-/+).

(Condition 4), where $x_1 > 0$, $\theta = 2\pi$ and

$$\begin{aligned} W^-(x_0) &= 1.9 \cdot 2\pi \\ W^-(x_0) &= 2(-(R^+)^2 \ln x_1 + \frac{(x_1)^2}{2}) + 1.9 \cdot 2r \end{aligned}$$

$$W^+(x_0) - W^-(x_0) - \left(\frac{1}{2} \ln \left(\frac{(r^2 - (R^+)^2)^2 + 1}{(r^2 - (R^-)^2)^2 + 1}\right)\right) \quad < \quad 0$$

which holds for all $r \in [R^-, R^+]$ with

$$\begin{aligned} R^- &:= 1 \\ R^+ &:= 2 \end{aligned}$$

by inspection of the graphs, in figure 3.1, figure 3.2, figure 3.3, figure 3.4, and figure 3.5.

3.3 Conclusion

This chapter discusses an example of a dynamical system of the form (2.1) as discussed in chapter two given by equations (3.1) and (3.2). In the first part, we apply classical smooth dynamical systems theory. Thus we first calculate an explicit solution by separation of variables and then show that the periodic orbit is stable by application of a Poincaré map. We then consider an alternative approach using our theory developed in chapter 2. To show that a periodic orbit is exponentially asymptotically stable requires to calculate the four conditions of theorem 5 given by (3.23), (3.24), (3.30), and (3.32). However, this requires finding the functions $W^{\pm}$ which might not be easy in more difficult applications. We managed to find such functions and by inspection of the graphs in figure 3.1, figure 3.2, figure 3.3, figure 3.4, and figure 3.5 we conclude that the conditions are all satisfied.

Chapter 4

Biomechanics Application

4.1 A system of inverted nonsmooth pendula: Modeling an elderly person stepping over an obstacle

We derive a mechanical model of human motion where an elderly person decides to step over an obstacle rather than avoiding it. Such a decision may be deliberate or forced due to a sudden appearing obstacle in his/her way. The model is represented by a system of ordinary differential equations with discontinuous right hand side. The discontinuity of the system of differential equations describes the physical property that the motion considered requires a periodic switching from one leg to the other. Assuming stiff legs it follows that the physical characteristics of the model are those of a system of two inverted pendula with a switching between them, where motion on each leg is represented by an inverted pendulum. We consider properties of the periodic orbits of such a physical model for large angles and small angles represented by a linearized differential equation. We also provide a notion of stability and show robustness of the linearized model in terms of an external force acting on the person's center of gravity at a particular point on the periodic orbit in lateral direction. The model shows that increasing the angle between legs increases robustness linearly. This implies that an individual reduces the risk of falling due to stepping over an obstacle by increasing the angle between legs.

4.2 Motivation

In an ageing population the number of accidents related to falls of elderly people is typically rising. Current NHS financial costs associated with falls and fall-related accidents in the UK are estimated at more than £2.3 billion a year according to NICE, clinical guideline 161 published in June 2013. It is shown that people aged 65 and older have the highest risk of falling. It is reported that at least 30 percent of the people older than 65 and 50 percent of the people older than 80 fall at least once a year. Accidents due to falling do not only create a financial cost, they also incur a human cost such as distress, pain, loss of confidence, loss of independence, injury, and mortality. Injuries due to falls are the most common cause of mortality in the UK for people aged over 75. Hence, falls are a common and serious problem for older people.

Stepping over an obstacle is a typical daily life scenario associated with the risk of falling. The process of stepping over an obstacle presents a particularly challenging task for an elderly person. This chapter considers the problem where an elderly person decides to step over an obstacle rather avoiding it by walking around it. This decision may be forced due to a suddenly appearing obstacle in his/her way or deliberate. In any scenario, such a decision introduces some risk of falling because the person needs to change the way s/he controls the balance. Accidents related to gait and balance disorders and weaknesses account for 17 percent of all causes of fall in older people and are the second main cause of falling after "accidents and environmental hazards"[1].

There are many parameters affecting a person's balance. Such are: changes in vision, sensory information, cognitive function, and physical constraints for example. In addition, motion patterns are also likely to affect a person's balance. There are different ways a person can step over an obstacle. For example, the person could jump over the obstacle. This however, requires strong physical and mental abilities. Alternatively the person could step quickly or slowly over the obstacle and therefore, motion patterns depend on time.

We consider an individual with stiff legs where the individual's balance measured by the location of the person's center of gravity is described by a system of two inverted pendula. Associated with each leg is an inverted pendulum describing the person's motion of center of gravity. While the person's weight is on the left leg and the right leg off ground, an associated pendu-

[1] According to a report called "Recurrent Falls" in Patient.co.uk.

lum models the person's balance. Similar is true when the person's weight is shifted on the right leg with the left leg in the air. Each case describes a situation in which the person stands on one leg and tries to maintain balance. This is a static scenario in the sense that the person does not move over an obstacle. Motion over an obstacle occurs when the person switches between the two inverted pendula. Switching occurs when both legs are on the ground. Hence, stepping over an obstacle is possible by switching from one pendulum to the other.

The aim of this chapter is to consider a simple mechanical model with which some of the conditions associated with the risk of falling of an elderly person stepping over an obstacle can be studied. We are interested in robustness of the model in terms of variations of initial conditions, such as i.e. the angle between legs. For that purpose, we first define a notion of stability and then study robustness using a time elapse equation, where robustness is defined in terms of an external force acting laterally on the model at a particular point on the periodic orbit. The aim is to derive a relationship between robustness and initial angles between legs. Two models are considered, a nonlinear model and a linearized model. The linearized model is sufficiently rich in structure in order to establish the robustness conditions. We show for the linearized model that the relationship between the initial angle between legs and an external force acting on the model is linear. An increase in the initial angle between legs reduces the risk of falling of an elderly person when stepping over an obstacle.

4.3 Literature

Judging by our everyday experience legged locomotion appears a rather simple task. We hope, walk and run without thinking about it, and yet the interaction between the skeletal system, muscles, tendons and nerves necessary to generate locomotion is quite complex.

According to Alexander however, global leg behavior seems surprisingly simple, suggesting a spring-like behavior [1]. This spring-like behavior motivates an elastic model of legged locomotion initially introduced by Blickhan [4]. This model is referred to in the literature as the spring-loaded inverted pendulum or shortly SLIP model. Others have studied this energy-conservative model (cf. [9], and [21]).

A main problem of interest in the context of legged locomotion is stability. For example we want to know whether stable locomotion can be maintained under small perturbations. Seyfarth et al show that the SLIP model for running exhibits a mechanical self-stabilizing property for an appropriate choice of initial conditions, such as velocity, leg stiffness and angle of attack [3]. Blum et al [37] show that the basin of attraction can be enlarged by introducing a control mechanism such as a swing leg control. In their model, variation of leg parameters prior to touchdown compensates for perturbations of ground level and thus, allows to access previously unstable periodic solutions and even further stabilize already stable solutions. Since parameters are held constant during ground contact, the SLIP model with swing leg control remains energy conservative.

The model discussed so far is a purely mechanical model. In order to move towards a biomechanics model requires introducing muscles, tendons, and nerves. For example, muscles mainly have visco-elastic properties which may explain the landing-takeoff asymmetry observed in running (cf. [8], and [27]) and hopping [28], a property that is not inherent in the conservative SLIP model with fixed parameters. In addition, these studies show that leg length, i.e. distance between center of mass and center of pressure, is larger at takeoff than at touchdown. The force length relationships for human running presented in Lipfert [27] also indicate that stiffness decreases during ground contact. This landing is supported by measurements on joint level. There are already a number of studies considering spring-mass models with either variable rest-length (cf.[12],[11]) or variable stiffness (cf.[7],[20]) during contact.

A common approach to improve explanatory and predictive power of the SLIP model is to increase its structural complexity, following the template-anchor concept introduced in the paper by Full and Koditschek [24], e.g. by adding a trunk [36]). Additional structures, however, complicate analysis and therefore, fundamental insights might be overlooked.

In this chapter, we consider a fundamentally different approach to modeling legged locomotion compared to the SLIP literature which implicitly assumes that legged individuals are sufficiently flexible and for whom walking is second nature. Our model is motivated by observing gait patterns of elderly people who seem less agile and strong compared to a young person. A spring-mass model, hence, seems less likely to explain the gait pattern of interest to us.

4.4 The basic theory

Stepping over an obstacle requires motion of mass m representing the center of gravity of a person. This motion is described by a second order differential equation. This equation is derived via energy conservation method, where

$$E_T = E_P + E_K$$

states that the total energy E_T of the physical system is determined by the sum of potential energy $E_P = mgh$ and kinetic energy $E_K = \frac{1}{2}mv^2$. h is the height measured between two angular positions of m along the vertical axis and v is the speed of motion of the mass m. Conservation of energy implies

$$\Delta E_T = 0.$$

Hence,

$$mgh = \frac{1}{2}mv^2,$$

from which we obtain $v = \sqrt{2gh}$. From the formula of the arc length $s = l\gamma$, where l is the length of a leg (cord of pendulum) and γ is the angular displacement it follows that $\frac{d\gamma}{dt} = \frac{1}{l}\sqrt{2gh}$. From the geometry of the pendulum and assuming an initial condition $y_0 = l\cos\gamma_0$ and assuming that after some swing m is at position $y_1 = l\cos\gamma$, it follows that $h = l(\cos\gamma - \cos\gamma_0)$. Substituting h in $\frac{d\gamma}{dt} = \frac{1}{l}\sqrt{2gh}$ we obtain the first integral equation given by

$$\frac{d\gamma}{dt} = \sqrt{\frac{2g}{l}(\cos\gamma - \cos\gamma_0)}.$$

By differentiation of the first integral equation, we obtain the second order differential equation of the pendulum.

$$\frac{d^2\gamma}{dt^2} + \frac{g}{l}\sin\gamma = 0. \tag{4.1}$$

4.5 The model

4.5.1 Definitions, assumptions, and notation

We consider a scenario where there is an obstacle in a person's way. The person decides to step over this obstacle rather avoiding it. Assuming stiff legs, the person periodically shifts his/her balance between the left L and the right R leg. The angle between the legs is denoted by α. It is measured from leg R to L in counterclockwise direction and is held constant during the transition phase. A minimum angle α_{min} is required in order to successfully overcome the obstacle. This information is known from the context of the situation. Let m denote the mass representing the person's center of gravity. It is connected with the leg R or L depending on which part of the periodic orbit m presently travels. A supporting straight line $\bar{m}$ goes through the center of mass m at angle $\frac{\alpha}{2}$.

Let $\bar{g}$ be the gravity line perpendicular to the ground going through the center of gravity represented by mass m. The angle between the gravity line $\bar{g}$ and the leg R, measured from $\bar{g}$ to R in clockwise direction, is represented by $\gamma_R \geq 0$. The angle between the gravity line $\bar{g}$ and the leg L, measured from $\bar{g}$ to L in clockwise direction, is represented by $\gamma_L \leq 0$. When both legs are on the ground we observe a discontinuity between γ_R and γ_L where γ_R jumps to γ_L or vice versa. Hence, when γ_R changes sign its value jumps from $\frac{\alpha}{2}$ to $-\frac{\alpha}{2}$ or vice versa when a change in sign of γ_L is considered. This discontinuity occurs at the point where a switching between the two inverted pendula R and L occurs.

Let the angle between the middle line $\bar{m}$ and the gravity line $\bar{g}$ be denoted by β where β is measured from $\bar{m}$ to $\bar{g}$ in clockwise direction. We observe that $\beta > 0$ when m is described by R and $\beta < 0$ when m is described by L. In the case when both legs are on the ground, we observe that $\beta = 0$, and $\gamma_R = \frac{\alpha}{2}$ switches to $\gamma_L = -\frac{\alpha}{2}$ and vice versa depending on the direction of β. At variance to γ_R or γ_L, we observe that β shows no discontinuous behavior when R switches to L and vice versa.

We now consider the change in direction of β when the person is shifting his/her weight from leg R and L off ground to leg L and R off ground. While shifting the weight on R we observe that $\beta \geq 0$ increases firstly from 0 to $\frac{\alpha}{2}$ or les. On the other hand $\gamma_R \geq 0$ decreases from $\frac{\alpha}{2}$ until it eventually becomes

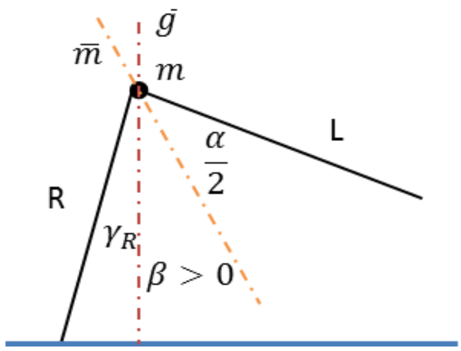

Figure 4.1: The model

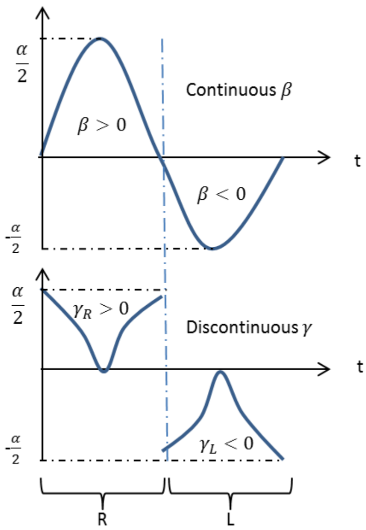

Figure 4.2: Angles

zero. At the right extreme, $\beta \geq 0$ decreases from $\frac{\alpha}{2}$ until it eventually becomes zero while γ_R increases from zero to $\frac{\alpha}{2}$ again at which point it jumps to γ_L with value $-\frac{\alpha}{2}$. Motion of m on L follows a similar pattern with opposite signs. Hence $\beta \leq 0$ initially decreases to $-\frac{\alpha}{2}$ and then increases to zero again, while γ_L increases from $-\frac{\alpha}{2}$ to zero and then decreases to $-\frac{\alpha}{2}$ again at which point it switches to γ_R. We have described a full oscillation of mass m from R to L to R in terms of α, β, γ_R, and γ_L.

In the next subsections, we derive a mechanical model of an elderly person overcoming an obstacle by oscillating his/her center of gravity from leg R to L and back from L to R and so on. The equations of motion describing the trajectories of mass m are those of a system of two inverted pendula. Motion of m on each leg is expressed by a nonautonomous second order ordinary differential equation (ODE). There is also a switching between the inverted pendula R and L. At the switching point, mass m traveling on R continues its journey on L or vice versa depending on its direction. This model requires considering three cases: (I) leg R is on the ground and L in the air, (II) leg L is on the ground and R in the air, and finally (III) both legs are on the ground and switching occurs. The study of a trajectory of mass m requires a model involving all three cases. We derive such a model in the next three subsections.

4.5.2 Case I: R is on the ground

We consider the movement of mass m on leg R and L off ground. We have

$$\begin{aligned}\gamma_R &< \frac{\alpha}{2} \\ \beta &> 0.\end{aligned}$$

In terms of γ_R, and $0 < \gamma_R < \frac{\alpha}{2}$, we have by equation (4.1) for $\gamma = \gamma_R$

$$-\ddot{\gamma}_R = -\frac{g}{l}\sin(\gamma_R).$$

In terms of β, since $\frac{\alpha}{2} = \gamma_R + \beta$, and $\frac{\alpha}{2} > \beta > 0$, we obtain

$$\ddot{\beta} = -\frac{g}{l}\sin(\frac{\alpha}{2} - \beta). \tag{4.2}$$

The two models are equivalent since

$$\ddot{\gamma}_R = -\frac{g}{l}\sin(\gamma_R) = -\frac{g}{l}\sin(\frac{\alpha}{2} - \beta) = -\ddot{\beta},$$

where, $0 < \beta < \frac{\alpha}{2}$, $\frac{\alpha}{2} > \gamma_R > 0$, and $\frac{\alpha}{2} = \gamma_R + \beta$.

4.5.3 Case II: L is on the ground

We consider the movement of mass m on leg L and R off ground. We have

$$\begin{aligned}\gamma_L &> -\frac{\alpha}{2} \\ \beta &< 0.\end{aligned}$$

In terms of γ_L, and $0 > \gamma_L > -\frac{\alpha}{2}$, we have by equation (4.1) for $\gamma = -\gamma_L$

$$\ddot{\gamma}_L = \frac{g}{l}\sin(\gamma_L).$$

In terms of β, since $\frac{\alpha}{2} = \gamma_L - \beta$, and $-\frac{\alpha}{2} < \beta < 0$, we obtain

$$\ddot{\beta} = \frac{g}{l}\sin(\frac{\alpha}{2} + \beta). \tag{4.3}$$

The two models are equivalent since

$$-\ddot{\gamma}_L = \frac{g}{l}\sin(\gamma_L) = \frac{g}{l}\sin(\frac{\alpha}{2} + \beta) = -\ddot{\beta},$$

where $0 > \beta > -\frac{\alpha}{2}$, and $-\frac{\alpha}{2} < \gamma_L < 0$.

4.5.4 Case III: Switching between the two cases

We now consider the case where the motion of mass m switches from the inverted pendulum R to the inverted pendulum L and vice versa. The system switches at $\beta = 0$. In this position, both legs are on the ground and

$$\ddot{\beta} = \frac{g}{l}\begin{cases} \sin(\beta - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ \sin(\beta + \frac{\alpha}{2}) & \text{, if } \beta < 0 \end{cases}. \tag{4.4}$$

Note that this model describes a periodic orbit as the sum of two trajectories, one for each leg. In the first part of the next section we will provide a Lyapunov function and an equation describing the periodic orbits of the model. We then derive a time elapse equation for a simplified model, where $\sin(\beta)$ is approximated by β. This linearized model is sufficiently simple but rich in structure in order to derive a simple relationship between ε, β_0 and α, where ε is an exogenous force acting on the model. Asymptotic stability in the usual sense fails to hold. However, we determine stability in terms of an external force acting on the model. We say that the system is robust under such a perturbation if a perturbation does not shift a current trajectory to a different energy level which is on a trajectory outside a defined separatrix. We then study robustness of the linearized model and derive conclusions.

4.6 The time elapse equation

We have derived a model (4.4) depending on the conditions of β, where β is continuous. This is at variance to the model initially depending on γ which is discontinuous. In this section we find a time elapse equation for the linear case, where $\sin(\beta) \approx \beta$. Therefore, we first study model (4.4) by transforming a autonomous system of second order ODE's into a system of first order autonomous ODE's. We then define a Lyapunov function and derive the equations describing a full periodic orbit. Finally, we provide an equation for the time elapse of a periodic orbit of the linearized model.

From model (4.4) we obtain a system of first order ODE's

$$\begin{aligned} \dot{\beta} &= \omega \\ \dot{\omega} &= \frac{g}{l}\begin{cases} \sin(\beta - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ \sin(\beta + \frac{\alpha}{2}) & \text{, if } \beta < 0. \end{cases} \end{aligned}$$

We define a Lyapunov function V by

$$V(\beta,\omega) := \frac{1}{2}\omega^2 + \frac{g}{l}\begin{cases} \cos(\beta - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ \cos(\beta + \frac{\alpha}{2}) & \text{, if } \beta < 0 \end{cases} = const = \frac{g}{l}\begin{cases} \cos(\beta_0 - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ \cos(\beta_0 + \frac{\alpha}{2}) & \text{, if } \beta < 0 \end{cases}$$

and using $\frac{d}{dt}V(\beta(t),\omega(t)) = \nabla V \cdot f(\beta(t),\omega(t))$ obtain

$$V'(\beta,\omega) = \begin{cases} \omega \cdot \frac{g}{l} sin(\beta - \frac{\alpha}{2}) - \frac{g}{l} \cdot sin(\beta - \frac{\alpha}{2}) \cdot \omega & \text{, if } \beta \geq 0 \\ \omega \cdot \frac{g}{l} sin(\beta + \frac{\alpha}{2}) - \frac{g}{l} \cdot sin(\beta + \frac{\alpha}{2}) \cdot \omega & \text{, if } \beta < 0 \end{cases} = 0.$$

The contour of the Lyapunov function V shows stable and unstable orbits of the system of inverted pendula. These orbits depend on the initial conditions of α, and β_0. In terms of $\omega = \dot{\beta}$, we obtain an equation for the phase paths for fixed values of $C = const = V(\beta,\omega)$

$$\omega = \omega_R + \omega_L, \tag{4.5}$$

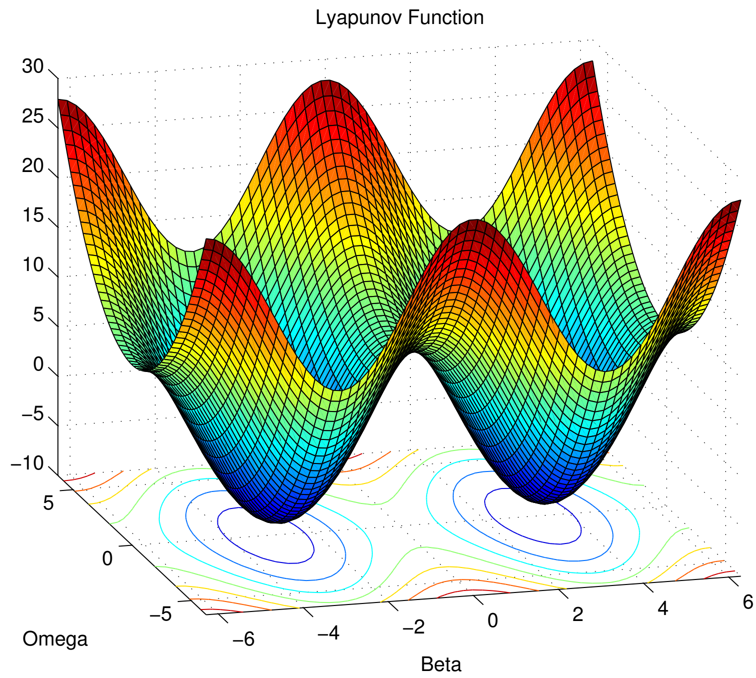

Figure 4.3: Lyapunov function

where

$$\begin{aligned} \omega_R &= \pm\sqrt{2C - 2\frac{g}{l}\cos(\beta - \frac{\alpha}{2})} \text{ if } \beta \geq 0 \\ \omega_L &= \pm\sqrt{2C - 2\frac{g}{l}\cos(\beta + \frac{\alpha}{2})} \text{ if } \beta < 0. \end{aligned} \tag{4.6}$$

The positive and negative values of ω_R together describe the part of the orbit of mass m when leg R is fixed and L off ground. The left leg L contributes to the description of mass m via positive and negative values of ω_L. The picture shows some orbits for different initial conditions represented by the constant C. The picture shows that ω produces unstable orbits for $C \geq \frac{g}{l}$. Such orbits are separatrices and oscillations with no physical relevance to our model. We will show later that we are interested in orbits which lie inside the separatrix.

4.6.1 Solution for small angles

When angles are small, then we can consider a linearized version of the model above. Hence, let $sin(\beta) \approx \beta$. We want to calculate the time T of a periodic orbit. In the form of a second order differential equation, we have

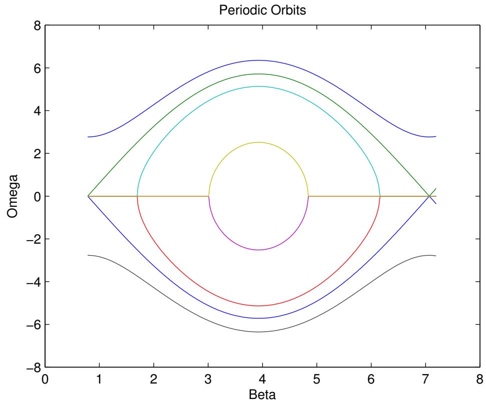

Figure 4.4: Orbits

$$\ddot{\beta} = \frac{g}{l} \begin{cases} (\beta - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ (\beta + \frac{\alpha}{2}) & \text{, if } \beta < 0 \end{cases}. \tag{4.7}$$

The homogenous equation is given by

$$\ddot{\beta} - \frac{g}{l}\beta = 0. \tag{4.8}$$

We can find a solution of this differential equation via characteristic equation. The characteristic equation is given by

$$\lambda^2 = \frac{g}{l}.$$

Hence, $\lambda = \pm\sqrt{\frac{g}{l}}$. The general solution of the homogenous equation (4.8) is given by

$$\beta(t) = c_1 e^{\sqrt{\frac{g}{l}}t} + c_2 e^{-\sqrt{\frac{g}{l}}t},$$

which, with $\beta = \frac{\alpha}{2}$ as a constant yields the solution of the inhomogeneous equation

$$\beta(t) = c_1 e^{\sqrt{\frac{g}{l}}t} + c_2 e^{-\sqrt{\frac{g}{l}}t} \pm \frac{\alpha}{2},$$

depending on $\beta \geq 0$ or $\beta < 0$. We now solve the initial value problem of a second order differential equation, and use the observation that the solution is a special case with initial conditions $\beta(0) = \beta_0$, $\dot{\beta}(0) = 0$ since the roots of the characteristic equation satisfy $\lambda_1 = -\lambda_2$. Hence,

$$
\begin{aligned}
\beta(0) &= \beta_0 < \frac{\alpha}{2} \\
&\Rightarrow \; c_1 + c_2 \pm \frac{\alpha}{2} = \beta_0,
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\beta}(0) &= \sqrt{\frac{g}{l}}(c_1 - c_2) = 0 \\
&\Rightarrow \; c_1 = c_2 = \frac{\beta_0 \mp \frac{\alpha}{2}}{2}.
\end{aligned}
$$

Hence, in follows that

$$
\beta(t) = \left(\beta_0 \mp \frac{\alpha}{2}\right) \cosh\left(\sqrt{\frac{g}{l}}t\right) \pm \frac{\alpha}{2}.
$$

The next step requires to use the formula for the time interval of an orbit. From $\beta(t) = \pi$

$$
0 = \left(\beta_0 - \frac{\alpha}{2}\right) \cosh\left(\sqrt{\frac{g}{l}}t\right) + \frac{\alpha}{2}
$$

we obtain for $\beta(t) = 0$

$$
\begin{aligned}
cosh\left(\sqrt{\frac{g}{l}}t\right) &= \frac{\frac{-\alpha}{2}}{\beta_0 - \frac{\alpha}{2}} \\
&= \frac{1}{-\frac{2}{\alpha}\beta_0 + 1} \\
t &= \frac{arcosh\left(\frac{1}{1-2\frac{\beta_0}{\alpha}}\right)}{\sqrt{\frac{g}{l}}}.
\end{aligned}
$$

The formula for time T of a full period orbit of the linearized model is obtain by considering a full oscillation, hence 4 times t, which then becomes

$$
T = \frac{4}{\sqrt{\frac{g}{l}}} arcosh\left(\frac{1}{1 - 2\frac{\beta_0}{\alpha}}\right). \tag{4.9}
$$

We apply equation (4.9) in the characterization of robustness of our model. Intuitively, we expect that for a fixed value of T, a small increase in α increases

β_0 proportionally. Consequently we expect the region of stable orbits to increase for a proportional increase in both parameters. In the next section we will define a notion of stability and show robustness of the linearized model. Essentially there are three key ideas involved in demonstrating robustness. First, a periodic orbit is robust if it lies inside a defined separatrix. This is a property of the Lyapunov function. We provide the conditions on β_0 and α producing this separatrix. Second, we define an external force $\varepsilon(\beta_0, \alpha)$ acting on the model. Associated with this force, we define an unique stable periodic orbit, ω_ε. We then characterize all stable β_0. These β_0 produce unique orbits inside ω_ε satisfying the perturbation conditions. Finally, we demonstrate robustness of our model by showing the effects of changes in α on β_0 and ε. The effects of a change in β_0 on α and ε are also evident from the proof.

4.6.2 Stability and robustness of the linearized model

We want to show robustness of or model in terms of changes in α. To show this we progress along three steps.

(1) We define for a fixed value of α its associated separatrix ω_C. The choice of α satisfies $\alpha \geq \alpha_{min}$, where α_{min} is the minimum angle require in order to successfully overcome an obstacle.

(2) We then pick the unique stable periodic orbit ω_ε through $\overline{\beta_0}$ which lies inside the separatrix. This orbit is laterally stable at $\overline{\beta_0}$ because $\overline{\beta_0} + \varepsilon(\alpha, \overline{\beta_0})$ is another orbit inside the separatrix, where $\varepsilon(\alpha, \overline{\beta_0})$ is an exogenously determined perturbation through the choice of $\alpha, \overline{\beta_0}$. We characterize all stable β_0 associated with α and $\overline{\beta_0}$.

(3) We apply the time elapse equation (4.9) of the linearized model to show the effect of a change in α from α to α_{new} on $\overline{\beta_0}$. This relation is then used to show robustness in terms of $\varepsilon(T, \alpha_{new})$ for fixed T.

We can determine using equation (4.4) periodic orbits for different values of C. We are interested in values of C which satisfy $\frac{g}{l} > C > -\frac{g}{l}$. For such values of C, we know that periodic orbits are stable as they are inside the orbit defining a separatrix. Now, let's consider the separatrix, where $C = \frac{g}{l}$ must be

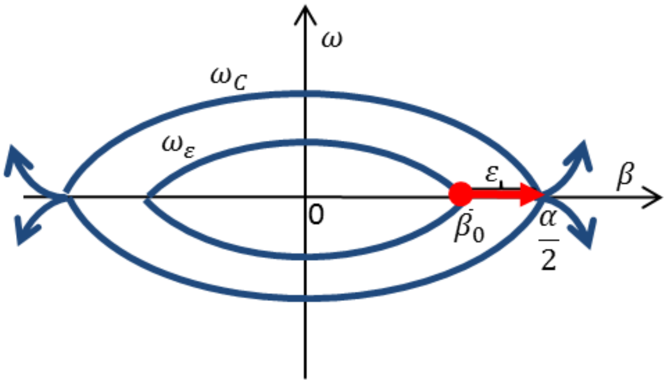

Figure 4.5: Robustness

satisfied. We assume that

$$C = \frac{g}{l}\cos\left(\beta_0 \mp \frac{\alpha}{2}\right).$$

Then it must be that since for

$$C = \frac{g}{l}\cos\left(\beta_0 \mp \frac{\alpha}{2}\right) = \frac{g}{l}$$

$$\begin{aligned} cos\left(\beta_0 \mp \frac{\alpha}{2}\right) &= 1 \\ \beta_0 \mp \frac{\alpha}{2} &= 0 \\ \beta_0 &= \pm\frac{\alpha}{2}. \end{aligned}$$

Step 1: For a fixed value of α, we determine the separatrix ω_C via equation (4.9) . For all $\beta_0 < \frac{\alpha}{2}$ we know from the properties of the Lyapunov function that C is such that (4.9) produces orbits inside the separatrix, which hence are stable by properties of the pendulum. Note that the choice of α is such that $\alpha \geq \alpha_{min}$ where α_{min} is the minimum angle between R and L required in order for a person to successfully step over a given obstacle.It is assumed known from the context of the situation.

Step 2: We now also fix $\overline{\beta_0}$ and define stability in terms of an external force

ε acting on the model. We formulate ε in terms of parameters $\overline{\beta_0}$, and $\frac{\alpha}{2}$ by

$$\varepsilon\left(\overline{\beta_0}, \frac{\alpha}{2}\right),$$

where

$$\varepsilon = \frac{\alpha}{2} - \overline{\beta_0}.$$

This notion of stability considers the case where an external force ε acts on mass m at point $\overline{\beta_0}$ in direction $\beta > \overline{\beta_0}$ when motion of mass m is at the right extreme (or $\beta < \overline{\beta_0}$ for left extreme) of the periodic orbit. Then the system is stable subject to a perturbation ε for all β_0 satisfying $\beta_0 + \varepsilon < \frac{\alpha}{2}$. For a perturbation ε we observe that the system is stable for all $\beta_0 \in (0, \overline{\beta_0})$[2]. The periodic orbit ω_ε associated with α and $\overline{\beta_0}$ is given by equation (4.6).

Step 3: We now want to show that a change in α from α to α_{new} affects the stability interval $(0, \overline{\beta_0})$ and associated robustness interval ε given by $(\overline{\beta_0}, \frac{\alpha}{2})$. Hence, in addition to α and $\overline{\beta_0}$ we also fix T in equation (4.9). Then for any $K \in \mathbb{R}$ we obtain

$$\begin{aligned} T &= \frac{4}{\sqrt{\frac{g}{l}}} arcosh\left(\frac{1}{1 - 2\frac{K \cdot \overline{\beta_0}}{K \cdot \alpha}}\right) \\ &= \frac{4}{\sqrt{\frac{g}{l}}} arcosh\left(\frac{1}{1 - 2\frac{\overline{\beta_0}}{\alpha}}\right) \\ &= \frac{4}{\sqrt{\frac{g}{l}}}\left(arcosh\frac{1}{(1-k)}\right). \end{aligned}$$

From this we directly observe that $\frac{\overline{\beta_0}}{\frac{\alpha}{2}} = k$, where $k = constant$. This yields

$$\overline{\beta_0} = k \cdot \frac{\alpha}{2}, k \in (0,1).$$

We can now reformulate $\varepsilon(\alpha, \overline{\beta_0})$ in terms of $\varepsilon(\alpha, k)$ which becomes

$$\varepsilon(\alpha, k) = \frac{\alpha}{2} - k\frac{\alpha}{2} = \frac{\alpha}{2}(1-k).$$

[2]Note that the system is also stable when a force ε acts on m at β_0 and $\beta < \beta_0$, when motion on leg L is considered.

We have shown that T in (4.9) is invariant for any constant K. Hence, for fixed T let

$$\begin{aligned}\frac{\alpha_{new}}{2} &:= K \cdot \frac{\alpha}{2}\\ \overline{\beta_{0,new}} &:= K \cdot \overline{\beta_0}.\end{aligned}$$

Then

$$k = \frac{2 \cdot K \cdot \overline{\beta_0}}{K \cdot \alpha} = \frac{2 \cdot \overline{\beta_{0,new}}}{\alpha_{new}} = k(T).$$

Robustness then follows from

$$\varepsilon(\alpha, T) = \frac{\alpha}{2}(1 - k(T)), \text{ for } k(T) \in (0,1).$$

We have shown that robustness of our model is a linear relationship between ε and α. Increasing α increases robustness ε.

4.7 Conclusion

This chapter considers the situation where an elderly person decides to step over an obstacle rather than avoiding it. This is a daily life situation potentially leading to accidents due to the risk of falling. Associated with such accidents are personal suffering, private and financial costs. The paper develops a mechanical model of human motion and addresses stability and robustness conditions leading to a reduction of risk of falling of elderly people.

This paper characterizes all stable β_0 for fixed α depending on a notion of stability which considers an external force ε acting on the model. It then shows for fixed time T a positive linear relationship between ε and α suggesting that increasing α (and hence β_0) increases robustness ε of the model.

Assuming well informed economic agents, the insights of this paper are expected to contribute to private an public cost reductions of accidents related to falls of elderly people through appropriate prophylaxis.

Chapter 5

Conclusion

This thesis discusses a theory of existence, uniqueness, and exponentially asymptotically stability of non-smooth periodic orbits, and provides a condition for a set $A(\Omega)$ to belong to the basin of attraction. Chapter two provides the main theorem and its proof. The theorem states sufficient conditions. We expect to the generalize this theorem to a converse theorem. We know from smooth dynamical systems theory that converse theorems exist, where Borg's criterion is used to prove the existence of an exponentially asymptotically stable periodic orbit of an autonomous differential equation and to determine its basis of attraction. It is a main advantage that for the formulation of Borg's criterion, no special information concerning the periodic orbit is needed. The criterion only makes use of the fact that, within the basis of attraction, adjacent solutions approach each other in forward time. Establishing necessary conditions would be very useful to prove that a certain set belongs to the basin of attraction of a unique exponentially asymptotically stable orbit. Moreover, if there is such an orbit, then there exists an associated metric such that Borg's criterion is satisfied. However, it is still a difficult problem to find such a metric. Future work is concerned with establishing necessary and sufficient conditions.

The advantage of theorem 5 is that it enables us to determine the stability of a periodic orbit without its explicit calculation. This is a desirable result in applied mathematics. We have seen in chapter 3 that even for simple equations that calculations of explicit solutions can be difficult. However, finding a metric, remains an art.

Chapter 4 discusses a model in terms of an equation as introduced in chapter two. The model describes a scenario where an elderly person steps over an obstacle. This joint work is in collaboration with the research group lead by Professor Wagner at the Sports Science Institute of the University of Münster. We have developed the model within the dynamical systems research group at the University of Sussex and then have validated it empirically and experimentally in Wagner's labor. In a series of experiments, subjects, represented by sports students were asked to perform various tasks involving stepping over an obstacle with stiff legs. Data was collected and validated. Early results suggest to support our model predictive power. However, this is a project in progress and for that reason, no empirical results are discussed in this thesis. Moreover, the project seems sufficiently interesting and has recently attracted further specialists from the health sector. The model developed in this thesis is very simple at this stage. We are considering extensions establishing the links between skeletal system, muscles, tendons and nerves. Moreover, we are also considering the option of conducting field experiments with the aim of bringing theory and practise closer.

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