A COGNITIVE MODEL OF THE ROLES OF DIAGRAMMATIC REPRESENTATION IN SUPPORTING UNPRACTISED REASONING ABOUT PROBABILITY

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Declaration

I hereby declare that this thesis has not been and will not be, submitted in whole or in part to another University for the award of any other degree.

Signature:

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SUMMARY

Cognitive process accounts of the advantages conferred by diagrams in problem solving and reasoning have typically attempted to explain an idealised user or a reasoning system that has equivalent to practised knowledge of the task with the target representation. The thesis investigates the question of how diagrams support users in the process of solving unpractised problems in the domain of probability. The research question is addressed by the design and analysis of an empirical study and cognitive model.

The main experiment required participants (N=8) to solve a set of unpractised probability problems presented by combined text and diagram. Think-aloud and eye-movement protocols together with given solutions were used to infer the content and process of problem interpretation, solution interpretation and task execution strategies employed by participants. The data suggested that the diagram was used to facilitate problem solving in three different ways by: (a) supporting sub-problem identification, (b) supporting prior knowledge of diagrammatic sub-schemes used for interpreting a solution and (c) supporting the process of interpreting and testing the specific meaning of given problem instructions and self-generated solution instructions.

These empirical data were used to develop cognitive models of canonical strategies of the three identified phenomena:

- Sub-problem identification advantages are accounted for by proposing that the spatial semantics of diagrams coupled with competences of the visual-spatial processing system and opportunities for demonstrative interpretation strategies increase the probability of goal-relevant data being made available to central cognition for further processing.
- Framing advantages are accounted for by proposing that represented diagrammatic sub-schemes (e.g. part-whole portions, icon-arrays, 2D containers etc.) facilitate access to existing prior knowledge used to frame, derive, and reason about information analogically within that scheme.
- Advantages in instruction interpretation are related to the specificity of diagrams which support the opportunity to demonstratively test and evaluate the referential meaning of an instruction.

The cognitive model also investigates and evaluates assumptions about the prior knowledge for solving unpractised probability problems; a representational scheme for addressing the co-ordination of sub-goals; a deictic problem representation to support online processing of environmental information, a meta-cognitive processing scheme to address self-argumentation and intention tracking and visual and spatial competences to address the requirements of diagrammatic reasoning. The implications of the cognitive model are discussed with regard to existing accounts of diagrammatic reasoning, probability problem solving (PPS), and unpractised problem solving.

Table of Contents

List of Tables

List of Figures

Chapter 1: Introduction

Substantial research has been conducted on the advantages of using diagrams in reasoning and problem solving tasks. This research has typically focused on the information processing efficiency of diagrams for practised or equivalent to practised tasks. A number of factors have been uncovered identifying explanations of processing efficiency, including how diagrams allow inferences to be externally offloaded (e.g., Larkin & Simon, 1987; Shimojima, 1996; Stenning, Inder & Neilson, 1995; Lindsay, 1995); reduce the size of the search space or reasoning cases (e.g., Koedinger & Anderson; 1990; Stenning & Oberlander, 1995; Lane, Cheng & Gobet, 2000), facilitate recognition (Larkin & Simon, 1987; Koedinger & Anderson, 1990; Zhang, 1997); and limit visual search and the use of working memory (e.g., Larkin & Simon, 1995).

The cognitive benefits of diagrams arguably extend beyond their processing efficiency in executing a practised task. Diagrams, including visualisations of various kinds, are used as modelling tools in design domains such as architecture, engineering and computer programming; in domains of investigation and discovery in mathematics and science; and in teaching and demonstrating principles in science and mathematics. A general characteristic of all these problem contexts is that the target user (i.e. designer, investigator or learner) is typically employing the diagram to help support the process of interpreting the form of a solution by establishing how it follows from constraints of the problem.

Research on this issue from an information processing perspective appears to be less prevalent and well developed. Researchers in artificial intelligence and logic have long considered how diagrams are used as proofs or demonstration of the validity of abstractions because they possess constraints that allow what is possible to be directly observed (e.g., Sloman 1971; Lindsey 2002; Shimojima 2001b, Pylyshyn 2003). In a series of studies Cheng has provided empirical evidence that diagrams designed to model structural constraints in domains such as science and maths facilitate learning of the fundamental concepts and laws that underpin the target domain compared to representations that lack this property (Cheng 2002, 2011). Factors in the processing efficiency of diagrams (as described above) have been proposed to explain the cognitive support conferred by diagrams in related higher-order cognitive activities such as self-explanation (Ainsworth & Loizou, 2003), abduction (Thagard & Shelley, 1997) and discovery (Cheng & Simon, 1995), but this research is less developed.

In investigating this issue, the thesis focuses on the role of diagrams in supporting users in determining the form of a solution procedure to a set of probability word problems. To be clear, a solution procedure reflects the constraints or criteria used to determine the values of a solution for a particular problem instance that may be executed using different strategies. To make the research issues concrete, consider the following probability problem shown in Figure 1.1. Conditional probability problems are an interesting case because a large body of research has shown that the probability of non-expert participants deriving correct solutions is highly sensitive to manipulations of the presentation of the problem including the presence and the particular form of the representation of the problem situation (e.g., Gigerenzer & Hoffrage 1995; Brase 2009; Fox, & Levav 2004; Yamagishi 2003; Sloman, Over, Slovak, Stibel 2003).

The problem scenario, data structure and linguistic framing of the problem in Figure 1.1 is simpler and more intuitive than the typical conditional probability word problems tested and reported in the research literature; however, these problems still elicited incorrect solution procedure interpretations in a substantial proportion of participants tested. The correct answer to the problem is 1/3. The correct solution procedure can be expressed using the notation $|A \cap C|/|C|$ where A are members of the queried category (i.e. small), C are members of the conditional category (i.e. red) and the vertical symbols | | are notation for the cardinality of a set. The common incorrect solution procedure employed by participants for this problem can be described as $|A \cap C|/|U|$ where U is the universal set. The solution interpretation results from an omission in ruling out possibilities in the set U that are not in the set C. The role of this diagram in supporting interpretation of the correct solution procedure is potentially manifested in the presence and accessibility of the set structure and relative frequency between solution relevant sets A, C and U (i.e. subset & less than (A, C), subset & less than (C, U)). Such structural information may be less explicit and need to be inferred rather than observed in alternative problem presentation formats.

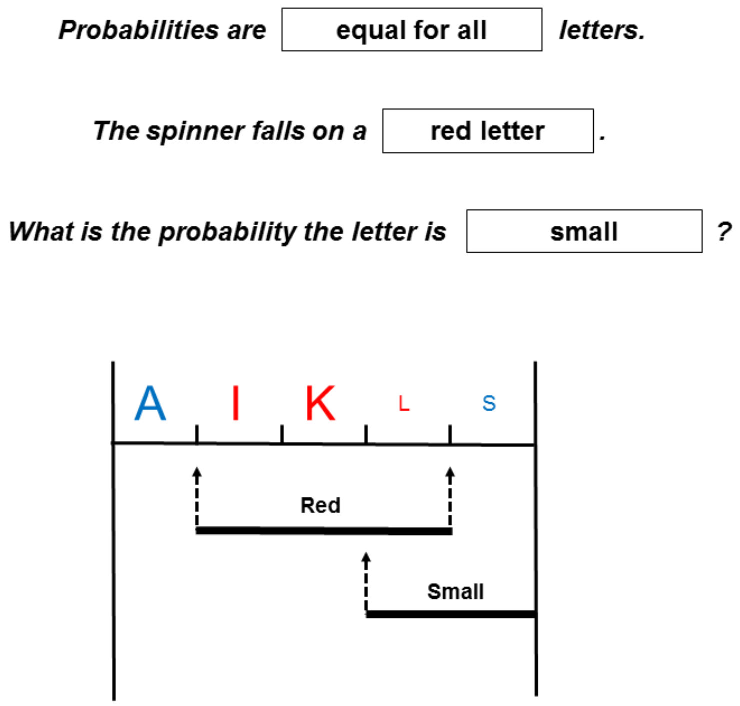


Figure 1.1. A conditional probability problem taken from the problems used in the main experiment.

1.1 Objectives

The main research question was to understand how diagrams support the process of determining the form of a solution procedure from the structural constraints of the problem in the domain of probability. The thesis reports empirical studies of participants solving a set of probability word problems with diagrams, who were relatively unfamiliar or unpractised at the task. Think-aloud protocols and eye-movement data were used to develop a cognitive model of the process of solving the problems and assess assumptions required to explain the nature of the problem solving observed.

The thesis considers existing evidence suggesting that the structural constraints of diagrams and the accessibility of information in diagrammatic formats support the interpretation of solutions. Diagrammatic accessibility can be viewed as a multifaceted and externally distributed phenomenon - depending on how properties of the external representation are coupled to exploit visual-spatial and high-level cognitive competences. The research requires accounting for how structural constraints of diagrams and different accessibility advantages are exploited by participants to support the interpretation of a solution procedure.

The research project addresses the research question in the domain of solving probability problems because a substantial research literature already exists which demonstrates the effects of a representation in determining the form of a solution procedure (e.g., Gigerenzer & Hoffrage, 1995; Yamagishi, 2003; Sloman, Over, Slovak & Stibel, 2003; Brase, 2009). Much of the research has focused on conditional probability problems and testing predictions of theories of probability problem solving (PPS) on representational effects. The thesis assumes that conditional probability problems involve the use of general knowledge and cognitive resources common to other probability problems and domains, and that the kinds of cognitive support of diagrams would therefore occur in other problems and task domains. The thesis therefore addresses the role of structural constraints and accessibility more broadly with different types of probability problems.

The project investigates a task context where participants have not learnt the required solution procedure for the class of problem instructions and are therefore required to adapt their existing knowledge to the problem. Addressing problem novelty in the particular context poses a greater requirement on higher-level cognitive activities such as meta-cognition, reasoning and sense-making (e.g., Klein, Phillips & Peluso, 2007; Tabachneck-Schijf, Koedinger & Nathan, 1994). The research requires identifying and accounting for these cognitive competencies through detailed empirical analysis of task performance and cognitive process modelling.

Addressing the research question requires the use and integration of the results of different methodologies including the analysis of external representations, task analysis of the problem, detailed protocol analysis (verbal and eye-movement) of participants solving the experimental problems and the development of explicit computational cognitive models. This kind of convergent methodology that is common in cognitive science research has not been fully adopted in the various strands of research on probability problem solving. Existing confusion and controversy in the research literature concerning what represented information is causally responsible for performance facilitation and what kinds of cognitive processes are involved; are considered to result from a lack of clarity and specificity about the nature of the representation, task and the process of problem solving. This assumption is an important motivation for the research undertaken for this thesis.

1.2 Outline

The thesis will be structured in the following way. There are two review chapters. Chapter 2 reviews research on diagrammatic representation and reasoning and Chapter 3 on PPS and the role of internal and external representation in cognitive accounts. In Chapter 4, the nature of the problems and task to be employed in the experiments and subsequently modelled will be outlined. Chapter 5 reports in abstract the preliminary findings of a pilot experiment. Chapter 6 reports in detail the findings of a main experiment which was later used for cognitive modelling. Chapter 7 provides an introduction to the ACT-R cognitive architecture used in the cognitive modelling research. Chapter 8 reports the design, analysis, simulation and evaluation of the cognitive model of participants solving the experimental problems detailed in Chapter 6. Chapter 9 reviews and discusses the research findings and implications of the research.

Chapter 2: Diagrammatic Cognition The aim of the chapter is to survey research related to the research goals. The chapter begins by discussing key informational properties of diagrams resulting from their structural constraints and how these properties can be seen to impact on interpretation and reasoning. The next section provides an overview of the cognitive support given by diagrams. The section is organised into three subsections: (1) accounts concerning the computational tractability of reasoning with diagrams, (2) factors associated with the potential accessibility of diagrams, and (3) research related to how people use diagrams in unfamiliar/unpractised contexts to understand a problem. The remaining section considers research on cognitive models and artificial intelligence frameworks for diagrammatic reasoning as a foundation to the cognitive modelling activities. The review suggests an absence of research providing an account of how diagrams are used in unpractised contexts; further, that computational accounts of diagrammatic reasoning have tended to be limited to highly abstract characterisations.

Chapter 3: External Representation in Probability Problem Solving The chapter details these different theoretical accounts or approaches to PPS including ecological rationality, the mental models theory, nested set and dual processing accounts. This is done as a prerequisite to scaffolding a review of the relevant set of research studies. This section is followed by a review of research studies that demonstrate experimental effects of the manipulation of the presentation of problem information. The review reveals a complex characterisation of effects of accessibility, which depend on the abstract features of the problem and the structure accessible in alternative presentations/representations of the problem. The thesis proposes that such effects are not specific to existing theoretical accounts of PPS, but instead consistent with a general information processing characterisation of problem solving and reasoning with external representations. The last section reviews empirical research that provide constraints on performance models of novice PPS participants. The aim of the review is to determine what empirical constraints exist on how novice participants actually go about solving PPS tasks in order to inform the development of cognitive models reported in this research.

Chapter 4: Problems and Tasks The chapter discusses the problems, representation and task used in the experiment and subsequently modelled. One purpose of the chapter is to comprehensively specify the rationale for choosing the problems and its particular attributes including the problem scenario, types of problem instructions being used, types of data structure employed and their mode of presentation/representation. The design of the problems and representations used in the experiment are motivated by a number of theoretical and methodological constraints. The chapter outlines constraints that are concerned with overcoming limitations on the theoretical interpretation of performance and constraints that are concerned with scaling down the complexity of the problem so that the behavioural data of the problem solving process is realistically tractable to analyse, interpret and be modelled in real time. A second purpose of the chapter is to outline a conceptual analysis of the problems, semantic analysis of the representation and analysis of the task in order to explicate how the research would be predicted to address the central research issues.

Chapter 5: Pilot Experiment The chapter discusses an initial pilot study designed to investigate the process of a solution procedure formulation (SPF). The chapter reports information on the types of solution procedure errors generated, time scale of problem solving and the nature of external visual attention strategies. The results of the pilot study, which are described in abstract, were used to assess the general feasibility of the experimental design and motivate subsequent changes for the main experiment.

Chapter 6: Main Experiment The chapter discusses the design of the main experiment. Verbal and eye-movement protocols are analysed to derive details about scheduling of subtasks, external attention and the internal process of interpretation, and reasoning. Several classes of errors in interpreting the required solution procedure using verbal protocols and the resulting solutions are analysed. Detailed examination of the process of participants' protocols reveal three important cases of the role of the diagram in determining the solution procedure employed: (1) in supporting recognition of unconsidered problem features that have implications on the form of the solution procedure, (2) facilitating existing prior knowledge of diagrammatic schemes used for framing the form of a solution procedure and (3) in supporting an interpretation of the correct meaning of an instruction.

Chapter 7: ACT-R Cognitive Architecture The chapter gives an overview of the ACT-R cognitive architecture and the rationale for its use in the research. This includes the types of knowledge considered and the structure and functions of different processing modules.

Chapter 8: Modelling Diagrammatic Reasoning about Probability The chapter begins by discussing the particular aims of the modelling research. The task model is then described hierarchically in terms of times scales of operation. The specification begins with the representational schemes and strategic knowledge involved in generic cognitive, meta-cognitive, visual and spatial processing routines. This is followed by a description of models of subtasks that supports activities such as comprehending the problem instructions, identifying sets, inferring possibilities, counting sets and formulating proportions. The subsequent sections describe how the component task knowledge comes together in solving the different problems and describes a model of canonical performance for each of the problems. The model is evaluated in terms of its competence, depth, generality, parsimony and functional coherence as well as its ability to reproduce consistent timings of the task performance and visual attention. The chapter then discusses implications and limitations of the model in its characterisation of diagrammatic accessibility, interpretation of solution procedures, the process of solving unpractised problems, visual spatial processing and diagrammatic reasoning, and its task knowledge for solving probability word problems.

Chapter 9: Conclusions The final chapter reviews the research and discusses broader implications and limitations. This chapter includes the scope and limitations of the main findings concerning the abstract roles of the diagrams in supporting the PPS task, and discusses the key finding on accessibility. The final section discusses the implications of additional findings concerning the nature of diagrammatic reasoning, unpractised problem solving and PPS.

Chapter 2: Diagrammatic Cognition

2.1 Introduction

The overarching aim of the chapter is to review existing research relevant to understanding the benefits of diagrams in formulating solution procedures in unfamiliar or unpractised problem solving tasks. The chapter has three component objectives. The first is to introduce the reader to key theoretical concepts in the study of representational systems. The purpose of the first objective is to lay the conceptual groundwork to critically and coherently evaluate existing research and to introduce concepts that will be referred to throughout the thesis. The second objective is to provide an overview of cognitive processing accounts for the support given by diagrammatic representations in problem solving tasks. These advantages are organised into three sections: the computational tractability of reasoning with diagrams, factors associated with the potential accessibility of diagrams, and existing research addressing how people use diagrams in unfamiliar/unpractised contexts to formulate solution procedures. The remaining section will outline general abstractions about cognitive models and artificial intelligence (AI) frameworks for diagrammatic reasoning. The aim of this section is to identify and assess key ideas critical to computational/information processing accounts of the cognitive benefits of diagrams and to identify their limitations, some of which will be addressed in this thesis.

2.1.1 Research on diagrammatic reasoning

The study of diagrammatic reasoning is an interdisciplinary field combining research from Psychology, Cognitive Science, Logic, Semiotics, Philosophy, AI, Information Visualization amongst others. Research on diagrammatic reasoning in Psychology and Cognitive Science has been directed towards a number of research goals. The list below includes some of the common research aims that can be derived from a broad literature review.

- General theories or accounts explaining why diagrams are more efficient for particular tasks than sentential representations (e.g., Larkin & Simon, 1987; Stenning & Oberlander, 1995).

- Cognitive accounts of using particular classes of diagrams in particular task domains such as data interpretation with graphs, theorem proving with geometry diagrams, mechanical reasoning with iconic diagrams (e.g., Pinker, 1990; Koedinger & Anderson 1990; Hegarty, 1992; Narayanan, Suwa & Motoda, 1995).
- Studies investigating which kinds of diagrams are better than others for certain kinds of tasks and why (e.g., Peebles & Cheng 2002).
- Individual differences in the preferences and efficacy of using diagrams compared to other representations (e.g., Cox, 1999**;** Hegarty & Sims, 1994).
- Principles and guidelines for the design and selection of representations (e.g., Cheng, 2002; Cheng, 2011; Kosslyn, 1989; Narayanan & Hegarty, 2002).

One important character of cognitive research in diagrammatic reasoning is that any cognitive processing account requires consideration of the interdependent relationship between representation, task and knowledge of the user. It is difficult to draw coherent conclusions without broader consideration of these factors. This is perhaps why methodologies employed in studies such as those cited above typically involve integrating experimental research findings with the outcomes of representational analysis, task analysis and/or computational cognitive modelling. It is also why interdisciplinary considerations are important to the cognitive study of diagrams. For these reasons the research reviewed in this chapter, as with the approach taken in this thesis, also adopts these converging methodologies.

2.2 Properties of diagrams related to processing advantages

Explaining how diagrams are processed depends on understanding the properties of diagrams. Classes of external representations (ERs) vary in complex multidimensional ways and are notoriously difficult to characterise. Many properties and perspectives of ERs have been discerned by researchers in the fields of Philosophy, Semantics, Artificial Intelligence and Cognitive Science. The thesis will address a subset of these issues relevant to the goals of the research. The section outlines three representational properties that have been considered central to explanations of the efficacy of diagrams. These properties are: (1) information and computational equivalence, (2) token and (3) constraint representation.

2.2.1 Information and computation

Larkin and Simon (1987) distinguished between informational and computational equivalence of representations. Whilst alternative representations may contain the same represented information, the computations required to process some expression of information may differ significantly between them. A representation contains information, if that information can be recovered from the representation through computations however elaborate. According to Larkin and Simon computational equivalence between representations can be seen to hold when the representations are informationally equivalent and the same information can be drawn with comparable ease and speed. Computational efficiency thus depends on the existence and speed by which operators processes the target information.

2.2.2 External representation of tokens

Probably the most general property of diagrams is that they use token symbols to model represented tokens. The term “model” is critical because the sense in which a diagram is intuitively viewed as a model is arguably based on the fact that representing tokens can be treated by a user as the represented tokens. There does not need to be a literal correspondence of modelled tokens. In iconic diagrams, it is typically the case that token symbols stand in for tokens in the represented state of affairs. However in abstract diagrams, token symbols do not necessarily have a literal token correspondence in the represented state of affairs, but may be transplanted in a system of ER to restructure relational information (e.g., tuples of data values as attributes of token points in a Cartesian graph, predicates as arcs in semantic networks, semantic classes as token regions in Euler diagrams, etc.).

The distinct way of representing tokens can be appreciated when contrasted with token representation using systems of a more sentential variety. For example, Figure 2.1a shows a PS-diagram (probability space diagram) of the same spinner situation. Each sub-unit stands in for a possible outcome on the spinner. Now consider Figure 2.1b in which the same state of affairs is described by a list of quasi-linguistic statements. The representation of tokens is different in Figure 2.1 because tokens are referred to by types of labels (e.g., let-1, let-2). In this case each token label does not actually stand in for a represented token. This distinguishing referential property of diagrams has been noted by a number of authors. Barwise and Etchemendy (1995) characterise the relation as part of a more general homomorphism between representing and represented. Stenning, Inder and Neilson (1995) call systems that represent tokens by tokens as *token reference systems,* and systems which represent tokens by labels (or descriptions) as *type reference systems*.

Some key properties of token reference systems (and diagrams) include the following. They involve using a single symbol for each represented token whereas type reference systems often require repeating the referring symbol in each expression of an attribution or relation. In token reference systems attributes of represented tokens are represented by attributes of tokens whereas this does not occur and is incompatible with type reference systems. Token symbols in token reference systems are the demonstrative subjects of represented semantic content. In type reference systems this is not normally the case. For example, in the PS-diagram, if you make the derivation that the K outcome has greater probability than the Q outcome, it is the token units in the diagram the relation is being made about.

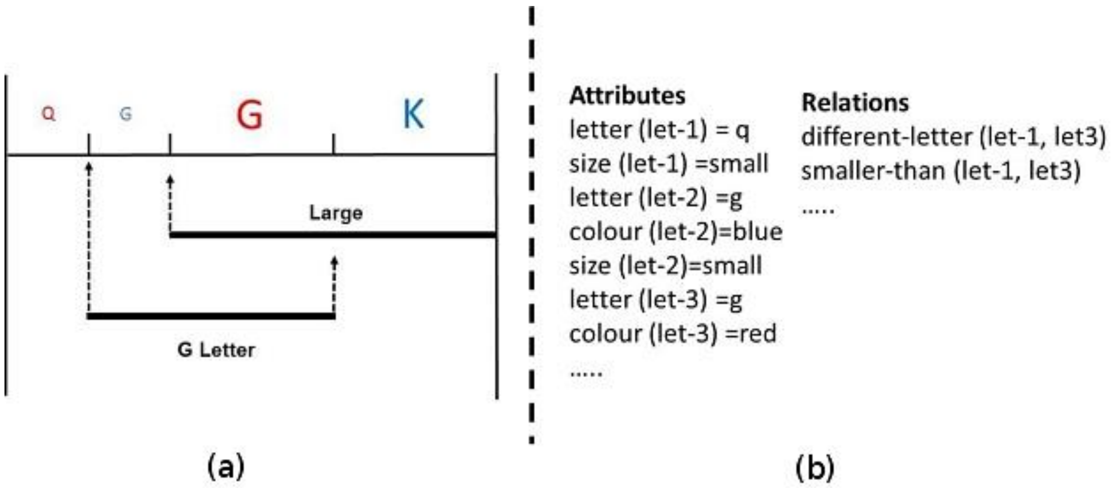


Figure 2.1. Representations of properties of outcomes of a letter spinner: (a) A PS-diagram in which each token outcome is represented by a sub-unit/letter token (e.g., token reference) and (b) a notation in which token outcomes are referred to by labels such as let-1 (i.e. type reference).

2.2.3 Representation of constraints

The representation of constraints in diagrams has been widely discussed by researchers and broadly interpreted as an important factor contributing to the efficiency of diagrams for certain kinds of tasks. There have been many different ways of characterising how constraints are represented in diagrams and other ERs. For example Palmer (1978) distinguished between intrinsic and extrinsic representation of relations. A representing relation is intrinsic if it has the same inherent logical constraints as the relation it represents. For example, transitivity would be intrinsic to the spatial containment relation when representing set membership. A relation is extrinsic if the constraints of the representing relation are not inherent to the representing relation and thus need to be imposed. Building on the analysis of Palmer, Shimojima (2001a) proposes the distinction of nomic and stipulative constraints that determine the states of affairs holding in a representation. A constraint between representing and represented state of affairs is a nomic constraint if it results from "natural laws" of the representation which include topological, geometric or physical laws. A constraint is stipulative if it results from a rule such as a condition on syntactic well-formedness. The differences between representations possessing nomic and stipulative constraints are proposed to closely mirror intuitive categorisations of diagrammatic and sentential systems of representations. Cheng (1996) considers constraints in terms of represented laws encoded within diagrams and appeals to how such laws are manifested in different cases or instantiations in which the relations of a law are represented. Stenning and Oberlander (1995) have framed the issue of constraints in terms of the general notion of representational specificity. According to these authors, the specificity of a representation corresponds to the extent to which it can express certain classes of information independently of expressing others. Linguistics systems are capable of expressing abstractions whereas constraints in diagrams typically limit this capacity. Researchers have also attempted to abstract different effects of constraints (e.g., Palmer, 1978; Barwise & Etchemendy, 1995; Stenning, Inder & Neilson, 1995; Shimojima, 1996; Shimojima, 2001a).

Constraints advantages resulting from modelling operations include updates to dependent attributes/relations that occur when modifications to representing attributes values of tokens are made (e.g., Palmer, 1978; Lindsay, 1995; Shimojima, 1996; Stenning, et al., 1995). As an example, consider changing the probability of outcome A (i.e. changing its width) in the PS-diagram of (1) of figure 2.2 such that its probability is set to twice its current probability as in (2) of figure 2.2. One can observe from (2) of Figure 2.2 the updated relations that indirectly occur. These effects occur for quantitative relations. For example, the probability of A is now larger than the outcomes C and D, G's probability becomes the same as A, the relative probability of each outcome C and D have changed from 1/5 to 1/6, and the probability of letters A and G are now 1/3. They also occur for qualitative relations. For example, if the letter C has its colour changed to blue as in (3) of Figure 2.2 then updated set relations between highlighted blue and consonant letter sets now partially overlap and all vowel letters are now a subset of blue letters. The updated attributes/relations that might otherwise need to be inferred are often described as being given for free in the representation. These updates can be observed in a wide variety of graphical representations and are often called free-rides (Shimojima, 1996).

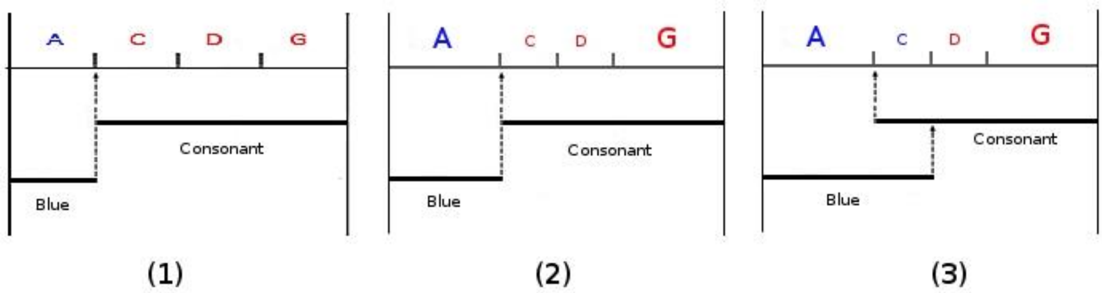


Figure 2.2. Updates to the values of attributes and relations as a result of modelling operations (i.e. free rides).

Free-rides reduce the inferential work that would otherwise be required to manually update relations using an alternative representation that lack such constraints (e.g., Lindsay, 1995; Shimojima, 1996). Free-rides in diagrams may also facilitate recognition of the consequences of edits to a representation and support look-ahead in problems that require consideration of complex dependencies (e.g., Barone & Cheng, 2005).

Another modelling advantage of constraints in diagrams has been called *auto-consistency* (or self-consistency) (e.g., Barwise & Etchemendy, 1995; Stenning, et al., 1995). This effect concerns enforcement of consistent expressions within a representation. For example, consider the axiom in probability theory that the probability of all outcomes must amount to one. The PS-diagram in Figure 2.3 (right) shows self-consistency with respect to this law because it is not possible in the correct use of the scheme to formulate probability values that are inconsistent with the law. This occurs because probability is modelled as the relative distance of part-whole relations between the sum and composite outcomes. The constraint is absent in the contingency table (left) and network diagram (middle) of figure 2.3, which employ numerical probability values, because there is nothing in the representing scheme that guarantees that the law holds in a specific instance. A representation that has auto-consistency with respect to certain classes of relations is synonymous with what Cheng (1996) calls Law Encoding Diagrams. The effects of auto-consistency in modelling restricts a user from formulating inconsistent expressions (e.g., Stenning, Inder & Neilson, 1995).

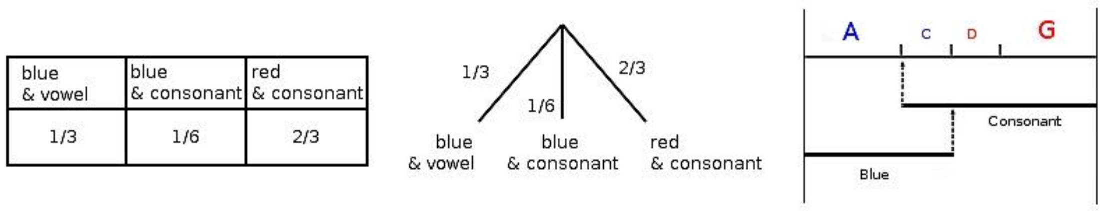


Figure 2.3. The law of that the probability of all outcomes must add to one is not encoded in and contingency table (left) or network diagram (middle) where numbers are used to designate probability values. The law is however encoded geometrically and is therfore auto-consistent in the PS-diagram (right)

Another advantage of constraints in diagrams concerns how their structure can be seen to demonstrate or explain why some abstraction must hold. For example, in the PS-diagram (Figure 2.3) the geometrical proportion that can be derived from the horizontal extent of the blue outcomes to the whole outcome space (i.e. 1/2) is equivalent to the proportion of the frequency of equal units that fall under the same categories (e.g., 3/6). The structure of the diagram arguably shows why this equivalence is the case and would hold for any instance. There are many contexts where diagrammatic structure supports the application of laws or universals, for example: in geometry theorem proving, where combinations of abstractions about parts of a configuration imply more general abstractions about the whole of a configuration (e.g., Koedinger & Anderson, 1990); and in representations of simple games, where geometrical properties of a configuration object imply winning opportunities in the game (e.g., Zhang 1997).

2.3 Computational tractability

An important component in understanding the effectiveness of diagrams in reasoning is their computational tractability, an issue which has been referred to by many researchers working in AI and cognitive science. Diagram configurations like visual scene configurations or pictures are rich in information, that is, they typically constrain many combinations of abstract relations such as those used in natural language and notations. If a system reasons with diagram configurations rather than abstract relations, it can exploit many more information constraints on each reasoning step to determine subsequent inferences. This inferential tractability requires that a cognitive or AI system has exemplar knowledge or instances of particular diagram configurations. The extent to which a cognitive system exploits these constraints is an empirical matter. The section discusses three accounts which indicate inferential tractability of diagrams in three different reasoning contexts: (1) inferring categorical abstractions in geometrical theorem proving, (2) inferring categories and values in corresponding representations, and (3) inferring diagram models of formal logic expressions.

2.3.1 Constraints on inferring abstractions in geometry

Koedinger and Anderson (1990) reported empirical data of students using diagrams to prove theorems in geometry. The verbal protocols they collected from participants formulating proofs suggested that skilled students were able to use geometry diagrams to focus on critical inferences and skip less important ones.

The authors developed a process model to explain the inferential efficiency of skilled students. The process model assumes that configurations in diagrams cue chunks in memory that function as operators for constructing the proof. The configuration chunks allow many steps to be executed in a single inference thereby explaining the inference skipping. They call these chunks diagram configuration schemas (DC-schema). Each DC-schema links a prototypical image of a diagram configuration, with a collection of facts about the configuration and a set of conditions for fact proving. The facts include a set of part-statements and a single whole-statement that hold about the particular configuration prototype. The part-statements are facts about local/part relations of a configuration, whereas the whole-statement is a global relation that holds about the whole of the configuration. The proof conditions in the schema specify the combinations of the partial set of part-statements that can be used to prove the remaining part-statements and whole-statements. The proof conditions therefore capture the constraints between part-whole relations of a geometrical configuration.

Establishing a proof is modelled as a process in which the system incrementally determines which set of DC-schemas hold starting from an initial set of premises. The initial premises are used to infer a lower order DC-schemas then, in a sequence of inferential steps, one DC-schema is used to prove another until a higher-order schema corresponding to the goal statement is established. Search though the space of DC-schemas is controlled by a set of heuristics. The authors claim that the model solves the problem in a diagrammatic search space that is both considerably smaller and more compact (it does not lead to dead ends) than other search spaces such as those implicated by the use of algebraic notations.

2.3.2 Inferring representation correspondences

Another example of computational tractability of diagrams comes from work on mapping between alternative diagrammatic representations. Lane, Cheng and Gobet (2000) report a model addressing how participants learn to re-represent circuit diagrams in an alternative system called AVOW diagrams. AVOW diagrams were designed by Cheng (2002) to support conceptual learning and allow exploitation of more effective problem solving strategies than other systems such as algebra. The modelled task requires that students re-represent a circuit diagram as an AVOW diagram in order to answer questions about the represented circuit which involves reading of information that occurs as a side effect in the AVOW diagram.

The authors report a model of student performance in the task using a cognitive architecture called CHREST+. In short, the architecture comprises of a discrimination network account of memory, which organises memorial instances into a collection of chunks. Each chunk is represented by a node. Nodes are linked together hierarchically based on their feature overlap, so that features of a particular instance are distributed across nodes. Chunk retrieval via recognition occurs by a similarity based search through the network of nodes/chunks until the most specific match is found. Perceptual chunks from instances of both diagrams are stored in the CHREST+ discrimination network. The model assumes that participants learn to associate perceptual chunks of the circuit and AVOW diagram configuration through equivalence links between nodes of corresponding networks, which allow retrieval of the corresponding AVOW diagram given successful identification of the circuit diagram. An important property of the mapping concerns the fact that distinct classes of circuit diagram configurations correspond in a one-to-one fashion to distinct AVOW diagram configurations. The property affords the cognitive system to learn relatively unambiguous mapping between the two forms of representations based on their visual structure. An important claim made by the authors is that the model is said to exhibit step skipping behaviour because configuration chunks can be drawn based on partial quantities of the diagrams configuration (e.g., electrical current or voltage).

2.3.3 Inferring possible models

Another account of the inferential tractability of diagrammatic representations comes from the work of Stenning and Oberlander (1995), who consider the issue in the context of model construction. The authors proposed a general theory of the differences between diagrams and sentential representations in reasoning. They claim that diagrams can be easier to reason with because they tend to be more specific systems of representation. Specific representations cannot represent some information abstraction without expressing other interdependent information due to representing constraints in the medium such as those discussed in Section 2.2.3.

The authors propose a computational explanation of why specific ERs, like diagrams, can be more efficient representations to reason with. Firstly, they draw on the analysis of Levesque (1988) to propose that a key property of making a system inferential tractability is limiting the number of cases (i.e. problem states) required to achieve some reasoning task. The authors suggest that because diagrams are specific, they can be represented as agglomerated diagram configurations (i.e. complete bindings of configuration attributes) rather than partial abstractions. This would limit the number of cases in an inferential task and the space of inferential alternatives at a given problem state. The argument is taken to provide a computational account of why reasoning should be more tractable with diagrams than sentential representations. Secondly, the authors propose a cognitive account of the computational arguments illustrated in the case of syllogistic reasoning with Euler circles. Their account assumes working memory systems also tends to use specific representations. In support of this claim they consider empirical evidence, which is often taken as support for imagistic representations or mental models, such as research on the n-term series problem. They suggest that the reason for the specificity of working memory representations is because they are implemented in neural networks. In support of this claim, the authors consider modelling research demonstrating that artificial neural networks (ANNs) are constraint satisfaction systems that enforce the specification of agglomerated representations they are trained on.

The authors also describe a particular kind of specificity that they consider critical to explaining the inferential efficiency of Euler circles using a reasoning strategy in which a conclusion is selected that involves minimal changes to topological relations between premise and conclusion diagrams. In doing so, they make the analogy between minimal transitions between problem states in this Euler circle problem space, and the so called continuity of content addressable representations coded in ANNs. Continuity in ANNs corresponds to the property that overlapping representations have closer hamming distances in the multi-dimensional coding space. ANNs trained on these neighbouring problems state configurations (i.e. ones that overlap or have minimal changes) would exhibit this semantic continuity. The researchers do not actually report the implementation of such models, but instead (see Stenning & Oberlander, 1994) abstractly describe a hybrid framework in which a rule based system would supervise strategic access to agglomerated patterns in a neural network memory module.

Two significant limitations of Stenning and Oberlander's (1994) account are worth noting. Firstly, given that most cognitive processing in the human mind is concerned with the construction and manipulation of abstract information used to control thought and action rather than just models of states of affairs in the world, the proposal that representational specificity results from a generic property of neural networks seems implausible. Stenning and Oberlander actually note this possible problem in relation to representations held by the articulatory loop. However, the main limitation of their theory is that it says nothing about accessibility of diagrammatic representations. Just because an ER enforces constraints on represented information does not guarantee that those constraints can be accessed and exploited by the cognitive system.

2.3.4 Summary

These three different accounts assume that efficiency in inferential tractability results from the system being able to represent problem states that correspond to diagram configurations and exploit inferential constraints on those configurations. The accounts (must) assume that inferential constraints on problem states are exploited in parallel (i.e. as combinations of satisfied part statements or through distributed constraints in ANNs). The accounts contrast in terms of the different cognitive processing context in which computational tractability is realised. These contexts are not exhaustive; there are other cognitive models and AI frameworks that report similar effects. One important difference between the accounts is that in Koedinger and Anderson's (1990) theory the computational tractability of problem states is exploited in process of deriving abstractions; whereas in Stenning and Oberlander's (1994) and Lane, Cheng and Gobet's (2000) model the effect occurs in the generation of models. The distinction highlights the fact that computational tractability of diagrammatic problem states can participate in explaining the effects of constraints in model construction and derivation contexts.

2.4 Accessibility

The accessibility of represented states and laws in diagrams are generally considered important factor in explaining their efficiency. Intuitive notions about accessibility of diagrams often appeal to vague visual processing explanations, but close analysis suggests that the issue is more complicated. Research within Psychology and Cognitive Science suggests different reasons, explanations and perspectives on the accessibility of diagrams. The section reviews research suggesting different possible accounts and components explaining diagrammatic accessibility including perceptual factors, recognition factors, search and memory, prior knowledge and skill, analogical correspondence, direct interpretation and visual abduction.

2.4.1 Perceptual salience of data

Several authors refer to mechanisms of perceptual grouping and salience effects of pre-attentive processing, in making relevant information accessible particular in diagrammatic representation of statistical data (e.g., Pinker, 1990). Perceptual grouping is automatically performed by the visual system, is bottom up or data driven and is normally impenetrable or unaffected by cognition and follows gestalt principles of perceptual organization such as similarity, continuity, common fate, etc. It has been taken as an important component in accounts of graph comprehension in which patterns of data points have meaningful interpretations (e.g., Pinker, 1990). Not all perceptual configurations convey meaning within a diagram. Perceptual grouping is exploited in diagrammatic semantics when the visual object processing system is used in the service of simplifying and substituting the representation of complex relation(s) with reified perceptual object configurations. Such configurations can be seen to support the accessibility of high-order expressions sometimes referred to as derivative meaning (Shimojima, 1999). Despite this, semantic bearing perceptual groupings may support, be irrelevant and even have negative effects on problem solving in a task (e.g., Ali & Peebles, 2013). These perceptual factors depend on the architecture of the visual processing system. The role of perceptual grouping in making meaning accessible in diagrams occurs only when perceptual groupings convey meaning about modelled objects and relations. The implication is that differences in accessibility based on perceptual salience only apply in this way to token reference systems such as diagrams that provide conditions for derivative meaning (Shimojima, 1999).

2.4.2 Recognition factors

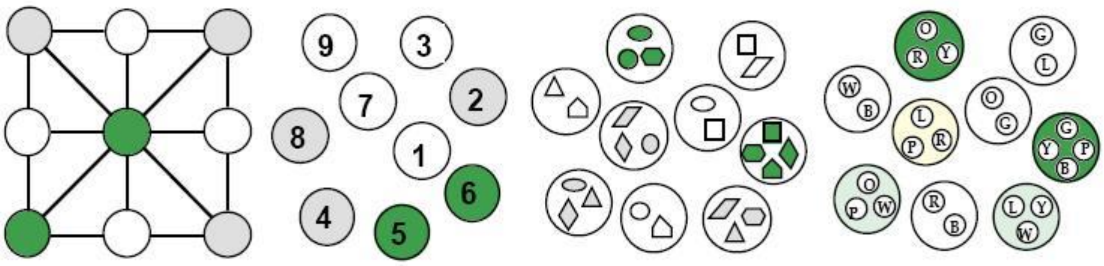
Larkin and Simon (1989) claimed that diagrams can often provide superior support in the recognition of information because such information may often be represented "explicitly" in diagrams compared to sentences. The recognition argument relates to the fact that abstract represented relations or attributes can be encoded as visual or spatial attributes of tokens in diagrams. This in turn allows them to be recognised as graphical attributes of visual objects. As an example, they consider how the emergent point of intersection between lines in a Cartesian graph of supply demand data conveys the equilibrium between the two variables. As a contrast, they consider the qualitative differences in accessing the implicitly represented information in an information equivalent table. In the graph case they suggest that the information would be automatically recognised by pattern specific productions, whereas the table would require numerous production steps to elaborate. They, however, point out that efficiencies in recognition depend on the possession of the relevant productions. Hence, quick and efficient recognition of equilibrium information in a graph is only likely to occur if a user has leant and practised this in the past and thus have the relevant productions.

A further related recognition advantages can be seen in cueing accounts of diagram configuration originally proposed by Koedinger and Anderson (1990). Recall that, in Koedinger and Anderson's (1990) model DC-schemas include reference to a prototype image, which can be matched against diagram configurations. In their model the image of a diagram configuration cues access to DC-schemas. Other accounts such as Lane, Cheng and Gobet's (2000) CHREST+ model also suggest that diagram configurations cue access to chunks. In their model, the chunk cueing occurs in the service of accessing a corresponding schema chunk for constructing a configuration in an alternative system of representation. It is worth noting that similar effects are also widely reported in chess playing in which expert chess players are able to select successful moves from game configurations without having to look-ahead through the many combinations of possibilities. Research suggests that they are able to do this by exploiting an extensive memory of different chess configurations (e.g., Chase & Simon, 1973). Perceptual chunking is a commonly considered basis for accessibility effects in diagrammatic reasoning, but is arguably limited to explanations of practised/expert problem solving behaviour.

An alternative perspective on recognition effects of alternative representations comes from the work of Zhang (1997). Zhang frames graphical cognition in a distributed cognition framework that proposes that: (1) structure and knowledge in ER tasks is distributed across internal and external ERs; (2) structure in ERs can be directly picked up by perceptual operators and acted on without deliberative processing or holding an internal model of represented information; and (3) ERs influence "*what information can be perceived, what processes can be activated, and what structures can be discovered*" (p. 179, Zhang, 1997).

In one study Zhang (1997) conducted a series of experiments using different isomorphisms of the tic tac toe (or noughts and crosses) problem. In the normal representation of the problem (Figure 2.4a), different classes of positions have different structural properties that can be exploited in game playing. Namely, certain sets of positions are visually symmetrical (i.e. corners, sides, centre); sets of visually symmetrical positions have the same number of winning opportunities (e.g., any corner has three possible ways of being in a win) and different symmetry groups have different numbers of winning opportunities (e.g., the centre position can win in four ways compared to three ways for corners and two ways for sides).

Zhang (1997) considers the symmetry of groups and number of winning opportunities associated with them to be perceptual invariants that provide action affordances; therefore, framing ER properties in terms of the ecological theory of vision (Gibson, 1986). One critical affordance proposed to be elicited by the standard representation is a choice strategy for selecting positions that have the most wins. Zhang created alternative isomorphic representations which carry the same structural information using combinations of different visual, spatial and semantic properties. As a result, the different isomorphisms vary in terms of their accessibility of different problem critical classes of information. The different isomorphisms are shown below in Figure 2.4.



(a) (b) (c) (d)

Figure 2.4. Tic Tac Toe isomorphism used in Zhang's (1997) study (p. 188). Each isomorphism differs in the accessibility of task relevant invariants.

Zhang (1997) tested participants playing against a computer program. In different experiments the strategy of the computer program was varied in terms of whether the bias afforded by the representation was consistent or not with preventing the opponent program from winning. One of the key findings was that the accessibility of invariant information in the different isomorphisms tended to be associated with eliciting the hypothesised biases. These biases occurred in experiments where they were either consistent or partially inconsistent with the opponent's game playing strategy. In short, Zhang argued that the findings support the view that invariant structure elicit strategic biases and that the claims were consistent with the distributed cognition framework of ER based problem solving outlined above.

What Zhang's (1997) research suggests is that the cognitive system can learn to anchor intentions and actions on perceptual configurations in an ER. When information in a representation has no direct perceptual correspondence or the correspondence has low salience then the formulation and/or subsequent selection of a strategy is less likely to occur. Zhang does not give an explicit model of the internal representations (i.e. perceptual) and processes that are supposed to occur in the recruitment of affordances. It has been suggested that affordances, in the sense used by Zhang, can be modelled by productions that have perceptual patterns as conditions as described by Vera and Simon (1993). If viewed in this way, Zhang's perceptual affordance account does not seem distinguishable from perceptual chunking accounts of recognition used to explain processing advantages of (learnt) diagram use (e.g., Larkin & Simon, 1987; Koedinger & Anderson, 1990).

Superior accessibility effects of perceptual chunking of diagram configurations is complicated by the fact that expressions of notations are also configurations which can be learnt perceptually and schematised like diagram configurations. Indeed, there is evidence and process models which also assume that users are able to recognise notation configurations and access relevant task knowledge accordingly (e.g., Anderson, 2005). There are, however, more important differences such as the fact that diagram configurations typically allow large amounts of information to be coded in a perceptually simple and distinct way. Mathematical expressions of equivalent information content involve complex arrangements of alphanumeric symbols that may often be too complicated to perceptually chunk. The combinatorial space of distinct expression in notational systems may also often be too vast for instance learning to be exploited to the same extent.

2.4.3 Search and working memory

The class of possible accessibility effects worthy of consideration include more indirect effects than those resulting from perceptual and recognising processes. Other effects also result from search and memory requirements. Larkin and Simon (1987) claimed that diagrams often limit the amount of search for information because different attributes of representing tokens that need to be used together are available at the same token location in diagrams. This is because diagrams are token reference systems. This can be contrasted with sentential representations in which symbols denoting different attributes of a represented token may be embedded in different sentences at different spatial locations. In comparable information integration tasks, sentential representation may often require matching extracted attributions about an individual across different sentences. Larkin and Simon's argument is concerned primarily with the search requirements. There are also factors relating to expectations about how relational information is manifested in token situations compared to sentential structures which may also be relevant to explaining search advantages.

Many researchers claim that diagrams and other ERs can function as memory aids (e.g., Larkin & Simon, 1987; Zhang, 1997). The availability of memory resources may also be considered an indirect factor in accessibility of a representation. Encoding of expressions typically involves some degree of serial or piecemeal processing of representing symbols that contribute to an expression. Accessibility is compromised when memory resources are not available to encode and interrelate expressions. Pylyshyn (2003) proposes that mechanisms of spatial indexes are exploited in visual reasoning and contribute to the efficiency of using diagrams because they allow the cognitive system to keep track of referents without the need to internally represent their contents in working memory. Other researchers have also developed similar accounts of spatial indexing to explain computational offloading in the environment (Ballard, Hayhoe, Pook & Rao, 1997). However, like other perceptual effects, the advantage is only meaningful in the space of token reference systems.

2.4.4 Prior knowledge and skill

Larkin and Simon (1987) make the general claim that diagrams often support perceptual inferences that are easy for the cognitive system to processes. In many cases they claim that these inferences depend on primitive visual spatial productions that, through high levels of practice, allow inferences to be made with very little effort. Their claim about visual inferences is independent of whether the inferred content is visual spatial or conceptual. Perceptual inferences are thus about the perceptual production conditions for inference and therefore related to the issue of recognition. As the efficacy of perceptual inference in diagrams is claimed to be based on having the necessary productions acquired through learning and practice, alternative non diagrammatic forms of representation may also have the potential to support highly efficient inferences. As an example, Larkin and Simon consider the case of a logician given extended practice in the use of logical notations, who could arguably make logical inferences from such expressions with comparable computational efficiency.

As noted earlier, Stenning and Oberlander's (1995) theory was basically silent about the arguably important role of accessibility in explaining advantages of diagrams in reasoning. In a subsequent article, Stenning, et al., (1995) seems to address this issue arguing that any account of the computational complexity of representations must be "supplemented" by an account of the availability of constraints that limit their expressiveness. The authors seem to propose that diagrams (presumably the representing ontology) allow users to tap into prior knowledge given only a few basic facts about the system such as meta-logical properties like transitivity of spatial containment. They claim, however, that systems which use an abstract syntax (i.e. maths notations) require extensive learning to exploit their expressiveness. They suggest that the availability of constraints in diagrams explains why they are often employed by student in early stages of learning about abstract mathematical domains, but that students often switch to more powerful abstract notations when they become knowledgeable of them and the domain. Larkin and Simon (1987) also seem to make a similar point. Recall that the authors argue that highly efficient perceptual inferences in diagrammatic reasoning are dependent on practice. They also suggest that such inferences are likely to occur with notations given sufficient practice and expertise. The kind of prior knowledge these authors appear to be appealing to is visual spatial knowledge that underpins peoples' ability to reason about relations such as containment and linear ordering (although they are admittedly less than explicit about what they mean).

Prior knowledge of how to reason in visual spatial domains is not only predicted from people's continual engagement in everyday physical activities, but by a considered recognition that a significant amount of abstract thought seems to involve visual spatial domains. According to cognitive linguistics and some psychologists, spatial cognitive domains for thinking about containment, paths, and so on, are used ubiquitously to conceptualise and reason about abstract domains**.** Lakoff and Johnson (1999) claim that people employ spatial schemas like containment to think about abstract relationships like class membership in order to project and exploit structural constraints available in the spatial domain, which are supposedly lacking in the abstract domains (for example, the authors consider the transitive structure of spatial containment to reason about sets). Lakoff and Nuñez (2000) propose a cognitive semantics account of mathematical concepts. Although the authors do not explicitly address diagrammatic reasoning, many of their assumptions about the structure and origin of mathematical concepts are based on integrated spatial domains in diagrammatic representations. The authors claim that mathematical concepts are constructed by conceptual blending of so called image schemas concerned with conceptual domains such as containment, paths, linear scales, orientations, etc. Independently of the reasons why spatial domains are apparently so ubiquitous in thought, the assertion that they are suggests some prior foundation for familiarity, particularly in the exploitation of structural constraints as suggested by Stenning, Inder and Neilson (1995).

Another kind of effect of prior knowledge in diagrammatic reasoning occurs in the use of iconic diagrams which depict physical objects configurations in recognizable forms. Research by Narayanan, et al. (1995) investigated these issues in the context of mechanical reasoning about labelled iconic diagrams. The authors reported research in a task that required predicting the mechanical operations of a device by incrementally hypothesising the interaction of casually connected components in the system. The task examines the use of prior knowledge not represented in the diagram such as the causal behaviour of components. The authors developed a task analysis and then collected data of a small number of participants solving the problem. The experimental task employed mixed protocols including verbal reports, pointing/tracing gestures, as well as drawing. The authors then developed a cognitive process model of their behaviour and evaluated it against the data. The task analysis and models explored assumptions about how the depicted information could be used to infer the sequence of processes in the device. The authors claimed that the task analysis and behavioural protocols supported the view that diagrams helped guide users' attention to causally related/connected components in reasoning about consequential behaviour of components; they also supported access to conceptual knowledge about represented mechanical components, inferred hypotheses about their behaviour, and helped users visualise/animate components in the diagram. Iconic diagrams are a class of ERs in which prior knowledge is purposefully exploited in making conceptual knowledge accessible about represented information.

2.4.5 Token reference & direct conceptualisation

Another reason why diagrams may have favourable accessibility in contrast to sentence based systems is because represented tokens in diagrams are directly conceptualised, whereas this is rarely the case in sentence based representations. Direct conceptualisation of represented tokens is typically afforded in diagrams because they are normally token reference systems in contrast to sentences, which are normally type reference systems. There are additional processing requirements implied in interpreting model information from type reference systems, including the necessary abstract syntactic processing and the requirement to internally represent individuals and selected relations between them (as opposed to perceiving them). These additional processing requirements in type reference systems arguably differentiate the accessibility of acquiring information compared to when the same information is represented in a token reference system.

2.4.6 Analogical correspondence

The view that diagrammatic ERs are analogical representations because they embody systematic structural correspondences to what they represent is widely discussed in the literature (e.g., Cheng, 2002; Myers & Konolige, 1995; Sloman, 1971; Gurr, 1998; Stenning, 2002). Some research suggests that the degree of structure correspondence between the diagram and its represented domain may contribute to the accessibility of diagrams. In research investigating the effectiveness of ERs for problem solving Cheng (2002) has proposed designing representations that closely model the conceptual structure of the represented domain because, according to the author, such representations will tend to be *semantically transparent*. Cheng uses the term semantic transparency as a synonym to the notion of semantic accessibility. Cheng's view of semantic transparency is not about visual properties of a representation, but about the interdependent system of correspondences that hold between the diagram and its represented domain – in other words, its analogical correspondence. Similar views come from the work of Stenning (2002). These authors also appeal to the systematicity property of analogical correspondence in diagrams. Systematicity is a term used in theories of analogy to refer to the depth of structural dependencies shared between two analogues. The correspondence property is interesting because research in analogical reasoning suggests that analogies are preferred when there is a systematic relation between them (e.g., Gentner, 1983). This would make sense because being able to reason validly in a representing domain about a represented domain depends on the systematicity that holds between the two analogues.

2.4.7 Visual abduction

There are perhaps more elusive aspects of the visual systems that are at work and are being exploited in diagrammatic reasoning. Pylyshyn (2003) alludes to the capacity of the visual system to make abductions from instances of diagrammatic representations. He claims that people are able to "see" universal properties that hold over a given set of instances (i.e. laws) from a single instance of a representation. For example, consider Figure 2.5 in which according to Pylyshyn one can see that if a line is drawn from the bottom vertices (D and C) to any point on the opposite side the line will intersect either at or below the mid-line of (m-m'). One can establish that this holds for any rectangle. Pylyshyn (2003) argues that the involvement of the visual systems in such activities "*goes beyond recognising that a certain pattern or property is present in a particular instance... visual perception appears to be the source of the generalization in the first instance*" (p. 447). Pylyshyn points out that going from particular instances to universals is a general property of the visual information processing system. He suggests that such computations are at work when people perceive and exploit visual spatial generalisation in diagrammatic reasoning.

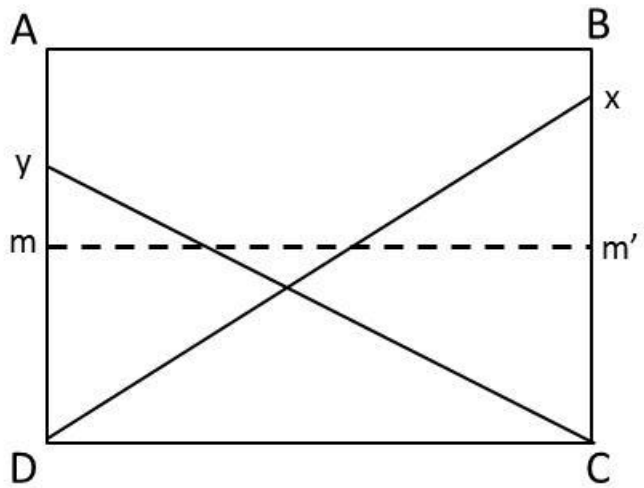


Figure 2.5. A simple demonstration of the capacity to apprehend universal constraints in diagrams. If a line is drawn from the bottom vertices (D and C) to any point on the opposite side, the line will intersect either at or below the mid-line of (m-m').

2.5 Abduction and Evaluation

An overarching objective of this research is to explore the hypothesis that diagrams help a user to figure out how to go about solving a problem (i.e. formulate a solution procedure). A central proposal is that diagrams confer processing advantages in facilitating the construction and evaluation of hypotheses, particularly of an explanatory nature. Explanation is not a monolithic phenomenon, but applies to different kinds of domain knowledge. Diagrams or representations that function as models more generally are used to support a kind of structural explanation in which structural constraints of the representation explain possibilities of a represented system (e.g., a Euler diagram may be used to explain why a conclusion logically follows from a set of premises).

There are a number of areas of diagrammatic reasoning research where abduction and evaluation of an explanatory hypothesis have been considered including learning, discovery, informal and formal contexts for making proofs. Whilst intuitions about the advantages of diagrams in supporting figuring out what to do are common, there have been no coherent cognitive models of such phenomena in the Cognitive Science literature. The following section will review research and accounts of related phenomena. These include research on the role of abduction in discovery with diagrams, adduction in the context of learning with diagrams, and the role of diagrams in evaluating the validity of hypotheses.

2.5.1 Abduction in discovery

Cheng and Simon (1995) reported a system called HUYGENS that simulates the process of law induction from diagrammatic representations. The idea is motivated by the conjecture that diagrammatic representations may have played an important role in helping early scientists like Huygens discover laws such as momentum conservation. The central argument is that constraints in the diagrams, which model laws of a domain, may have been abstracted in the course of experimentation and analysis with representations of different data sets. The authors developed the HUYGENS system in order to investigate and evaluate the hypothesis. HUYGENS discovers the law of momentum conservation from a class of 1-D diagrams. The diagrammatic scheme used models the variables (i.e. mass and velocity) and relations of a particular collision between two objects. The length of parallel line segments are used to represent the proportional mass of objects and their velocity before and after the collision.

HUYGENS models the process of constructing diagrams from data and comparing high-order relations of the represented data to find underlying laws, which it can formulate in algebraic terms. The HUYGENS system comprises diagrammatic data structures, operators for constructing and modifying diagrams, regularity spotters and heuristics for controlling search. The system has operators that generate and modify diagrammatic representations to reveal relations between variables. Operators plot variables such as line segments, determine arithmetic relations by arranging lines in geometrical configurations (e.g., the add operator redraws lines end to end), and re-represent diagram configurations as normalised variables. Regularity spotters identify relations that are common to different data sets represented in diagrams, such as whether differences between types of lines are constant, equal or involve the same relative quantity to other lines, etc. HUYGENS discovers the law of momentum conservation in a series of cycles of operator application and regularity spotting. When a pattern is detected, the system infers a law about the pattern. The system uses several domain specific heuristics to select data and operators.

The authors claim that the efficiency of the induction process is facilitated by accessibility advantages in search and recognition afforded by the diagram. The system is able to exploit powerful operators and regularity spotters, such as considering triplets of lines in one go. The authors claim that the power of these rules reduce the size of the search space relative to the search space implicated by the use of algebraic notations. The HUYGENS system does not explicitly commit to any empirical constraints on cognitive processing or attempt to model how in real time a scientist could make such a discovery. Whilst the authors explicate the operators and patterns-spotters used in the model, no qualification is given about how cognitively realistic they are intended to be.

2.5.2 Abduction in learning

Another context where abductive reasoning is exploited with diagrammatic ERs is in learning. One line of research is the so called self-explaining effect. Self-explaining was a term employed by Chi, Bassok, Lewis, Reimann and Glaser (1989) to describe the meta-cognitive strategies of students who, in the course of learning, are more likely to engage in explaining ideas, justifying solution procedures and monitoring their own problem solving behaviour. The research of Chi et al. (1989) suggest that students who engage in self-explaining behaviour typically perform better in learning tasks than students who do not. Self-explaining is not a very well defined psychological construct. Attempts at explicating exactly what the process of self-explaining entails was addressed in computational modelling research reported by VanLehn, Jones and Chi (1992), who modelled self-explaining as involving abductive reasoning.

Although initial research on self-explaining effects were not about diagrammatic representations, the research prompted others to connect the findings of Chi, et al. (1989) with observations about diagrammatic reasoning, resulting in conjectures that diagrams may facilitate self-explanation in the process of learning (Cox, 1999**;** Brna, Cox & Good, 2001). Empirical evidence in support of this hypothesis comes from a study reported by Ainsworth and Loizou (2003). In their experiment, two groups of participants studied the human circulatory system. One group received exclusively text based learning materials, whereas the other group received materials that included a labelled iconic diagram. The authors found that the students who received the materials with a diagrammatic illustration performed better in the post test and developed more self-explanations than those students who received text only materials. The authors claim that their findings support the view that diagrams promote self-explaining and suggest several possible reasons including freeing up cognitive resources (e.g., working memory), the specificity of the representations and making critical information (e.g., causal) available that would otherwise need to be inferred from text.

Another line of research relevant to this issue is work on learning with Law Encoding Diagrams (LEDs). Cheng (1996) proposed that represented laws encoded by constraints in diagrams support learning in abstract domains such as Mathematics and Physics. Cheng hypothesised that advantages of law encoding diagrams exploited by scientists in discovery may also generalise to conceptual learning. He claims that law encoding diagrams support learning of coherent networks of knowledge because they allow users to apprehend the laws of the domain from different instances of represented situations and from the different perspectives normally available in complex diagrams. Cheng is therefore referring to is a capacity of LEDs to support modelling and abstraction activities that are explained by laws modelled by the diagrammatic system. Several empirical studies with novel LEDs give some support to these arguments. These studies involved comparisons of LEDs with traditional ERs often involving mathematical notations based on a combination of semantic analyses and longitudinal learning experiments. Cheng appeals to notions of conceptual integration and explanatory coherence in attempting to frame why such benefits may occur, but provides no process explanation of the abductive component of learning with LEDs.

2.5.3 Evaluation: Testing and verification

Another important function of diagrams concerns the evaluation or verification of inferences about abstractions that hold in some represented model. According to Peirce's framework, inferred explanatory hypotheses are subsequently verified by deductive methods. A canonical context of verifying abstractions using diagrams occurs in cases of demonstrating a visual proof in domains such as Geometry. A simple example illustrated by Lindsay (2002) is the visual proof of Pythagoras such that the area of a parallelogram equals the product of its height and base length as shown below.

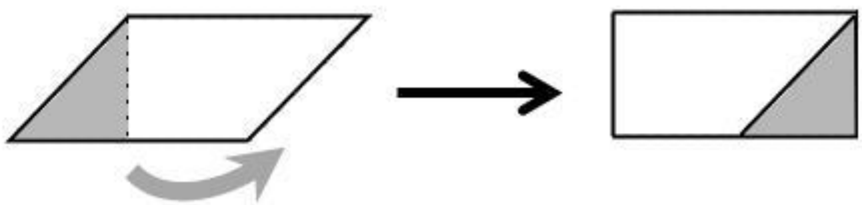


Figure 2.6. A simple visual proof demonstrating that the area of a parallelogram is the product of the length of its height and base.

The above example shows a classical diagrammatic proof which involves manipulating the structure of the diagrams, which can be done by redrawing or mental animation. According to traditional views on mathematical reasoning, only proofs expressed in precise formal languages are valid. Under this view, diagrams are apparently unsound for making proofs despite the fact that diagrams are widely used in mathematical demonstrations. According to Barwise and Etchemendy (1995), this view occurred because of errors made by earlier mathematicians that were incorrectly attributed to the accompanying diagrams used in the proof rather than human error. This assumption has been challenged by a number of researchers working on diagrammatic reasoning in fields of logic and AI (e.g., Sloman, 1971; Barwise & Etchemendy, 1995; Stenning & Oberlander 1995).

Researchers contest to the view that diagrams are ubiquitously employed as proofs of the consistency of represented information (Lindsay, 2002; Shimojima, 2001b) more often than formal proofs (e.g. Barwise & Etchemendy, 1995). Using diagrams to access and verify explanatory hypotheses is not only confined to the mathematicians' goals of proof formulation, but is more generally employed in contexts of diagrammatic cognition in which solution procedures are not known to a user (at least with some level of certainty). These include the contexts of learning and discovery, as well as unfamiliar or unpractised problem solving contexts. In each case, explanatory hypotheses need to be verified.

Some researchers have considered the use of diagrams in evaluating the validity of inferences as qualitatively different to establishing validity using sentential ERs. For example, Sloman (1971) identifies different senses of truth, a logical sense which can be established by the form of sentential expressions and a sense which is established by observing the way things are in the world. The observational sense of truth comes in to play when using diagrams to verify the validity of some inference. Lindsay (2002) also distinguishes between the formal sense of demonstrating a proof from demonstrating a proof with diagrams. Lindsay characterises the diagrammatic sense of proof as a way of allowing a user to understand the validity of a proof in an "experiential" way.

What is it about diagrams that support this sense of establishing the validity of some inference? Shimojima (2001b) provides one of the most elaborate discussions on this issue. He uses the term *consistency proof* to describe the use of diagrams in demonstrating valid state of affairs and discusses several examples. A canonical case described is the context of planning the layout of furniture in a room using a scaled iconic diagram. Shimojima's main aim is to propose a logical account of the semantic mechanisms that underpin why diagrams satisfy this activity. He attributes this capacity in diagrams as being dependent on the auto-consistency property as discussed previously. Recall that auto-consistency concerns a capacity exclusive to certain classes of diagrams to maintain that its expressions are logically consistent. This occurs for different kinds of relations including set and arithmetic. According to Shimojima, the auto-consistency property depends on particular types of matching constraints between the representation and the modelled state of affairs. Shimojima also suggests that the operations involved in using diagrams as consistency proofs involve what Magnani's (2001) calls manipulative abductions, which is a term used to refer to the manipulation of scientific devices including diagrams to formulate explanatory hypotheses.

As apprehending the consistency of expressions of an ER is a basis for judging the validity of an interpretation, then the property is a plausible determinant of the evaluative utility of diagrams in reasoning. The representational account proposed by Shimojima (2001b) is also consistent with other accounts of the efficiency of reasoning with diagrams (e.g., Sloman, 1971; Stenning, Inder & Neilson, 1995). Shimojima questions why diagrammatic models are readily taken as valid given that they are only representations after all. As auto-consistency is an information property of diagrams, a cognitive explanation of its exploitation needs to account for its accessibility as suggested by Stenning, et al., (1995). How and in what way users understand the consistency property of diagrams appears to be an elusive and interesting question.

Understanding properties such as auto-consistency is also viewed as depending on having something like a theory of the system of the representation. The theory is what allows the diagram to be used as an explanation support tool. The theory would allow one to understand how different expressive possibilities are constrained by the system. This kind of assumption is explicit in a program reported by Lindsay (2002), which uses diagrams to demonstrate theorems in Geometry. Lindsay construes the system as a competence model claiming that its understanding of a proof can be characterised as "*the process of confirming that transformations of representations are correct with respect to the system's underlying repertoire of permitted transformations, and thus that the situation, fact, or event that is understood is consistent with the system's "theory" of the subject"* (p. 267).

Some researchers have also appealed to the nature of the visual system in attempting to shed light on consistency proof properties of diagrams. As discussed earlier, Pylyshyn (2003) claimed that the visual system plays a role in determining generalisation that holds from particular diagrammatic instances. Lindsay (2002) also suggests that diagrams exploit evolutionary adapted abilities to process physical laws or constraints.

These kinds of claims are not uncommon and there is clearly intuitive appeal to these ideas, but they are difficult to evaluate without explicit analytical demonstrations and modelling which are invariably lacking.

2.5.4 Meta-cognitive activities and cognitive resources

The capacity of diagrams to support abduction, hypothesise evaluation and other meta-cognitive activities may also be related to possible reduced cognitive demands of using diagrams. For example, Thagard and Shelley (1997) suggest that performing abduction through visual reasoning, including the use of diagrams, may have stronger advantages over sentential reasoning. They claim that pictorial or diagrammatic representations may limit the amount of search for relevant inferences about an explanatory hypothesis compared to sentential rule based representations. The authors envisage processing advantages in terms production rules that infer the transformation of one graphical state to another as visualisations/mental animations. This is compatible with the claims of Cheng & Simon (1995), who also report search and recognition advantages in the HUYGENS model. Formulating and evaluating alternative explanatory hypotheses are likely to depend heavily on executive resources and working memory. If many basic operations on diagrams free-up cognitive resources then these resources may be more likely to be made available to engage in meta-cognitive activities.

2.6 Cognitive architecture of diagrammatic reasoning

The purpose of this section is to summarize some basic developments in AI research and cognitive modelling and identify abstractions about the nature of representation and processing in diagrammatic reasoning.

2.6.1 AI systems

A number of researchers in AI have proposed computational frameworks for diagrammatic reasoning. Ideas in AI research have been used to inform and constrain cognitive theorising and modelling. Early research by Funt (1980) reported a system called WHISPER which was capable of inferring the trajectory of a collapsing physical structure by a cyclical process of modifying and encoding changes to a diagrammatic representation. The system uses a retina consisting of a collection of spatially arranged processors that compute parts of a diagram in parallel, and perceptual primitives, which perform tests on image input. Glasgow and Papadias (1995) proposed an array based formalism for representing the spatial structure of visual images. The formalism has been used to justify and inspire the use of array representations in the cognitive modelling of diagrammatic reasoning (e.g., Narayanan, et al., 1995). Lindsay (1995) proposed a knowledge representation ontology for diagrams comprising of a broad set of construction and retrieval processes, which have been used as a foundation for subsequent proposals. Chandrasekaran et al. (2004) have proposed a general framework they call the *diagrammatic representational system,* which was initially developed in the context of applied AI, but has been adopted by some researchers in models developed in cognitive architectures such as SOAR and ACT-R. The framework includes a visual object ontology including points, curves and regions, and routines which extract qualitative and quantitative spatial relations between tokens, project non-veridical objects on represented scenes and manipulate the state of the representation. The framework is proposed to be uncommitted to what kinds of data structure/ format are used to represent the diagrammatic information.

2.6.2 Cognitive models with weak architectural commitments

As described, a number of cognitive models of diagrammatic reasoning have also been developed that have made weak commitments about the underlying cognitive architecture. For example, Larkin & Simon (1987) employed an abstract production system modelling framework using predicate data structures to model both sentential and diagrammatic information. Cheng and Simon's (1995) HUYGENS model uses attribute value triples to represent 1-d line segments, which are organised recursively in accordance with the structure of the modelled diagram. The system includes production based operators for constructing and modifying diagrams and domain specific pattern matchers for detecting relations between diagrams. The model by Narayanan, et al. (1995) used a spatial array representation to model low level "depictive" features of the diagram and "descriptive" frames to model higher-level diagrammatic relations. The array representation in their model is specified by filling array locations with symbolic labels thus, according to the authors, giving rise to its "*shape, geometry and configuration*". Diagrams frames encode attributes of object such as point, lines and areas and conceptual frames represent conceptual attributes of objects. The system also has a number of operators for reading, writing, indexing, scanning and visualisation operations. Koedinger and Anderson’s (1990) diagram configuration schemas models relations and constraints configuration prototypes. The schemas can be viewed as mapping between perceptual representations (i.e. image slots), semantic knowledge (i.e. part-whole relations) and operators (i.e. condition to prove) associated with their use. Lane, Cheng and Gobet’s (2000) CHREST+ model has a simulated visual attention mechanism, which can encode limited information from a diagram and has limited capacity short term memory. Chunks akin to diagram configuration schemas are modelled in a discrimination network, although details about the information content of chunks are not reported.

2.6.3 Cognitive models with stronger architectural commitments

There are also recent models of diagrammatic reasoning reported in literature that integrate explicit assumptions about the architectures, particularly of the visual processing system. For example, the CAMERA architecture includes a visual iconic buffer and distinct what and where buffers. The visual spatial system has its own specific operators modelled by production rules. The visual iconic buffer is modelled using a bit map representation and associated operators are used to extract visual and spatial predicate representations (or node link structures), which are subsequently placed in the what and where buffers. The what and where buffers and associated operators function as a visual short term memory system. The buffers do not have any capacity limits so the system may maintain an unrealistic description of a diagram in problem solving.

Several models of diagrammatic reasoning have been implemented in ACT-R, such as models of graph based reasoning (Peebles & Cheng, 2002), and Geometry problem solving (Stocco & Anderson, 2008). The standard ACT-R architecture shares visual spatial processing components with CAMERA and other general cognitive architectures. The architecture consists of a visual module, which contains what and where processing systems, including buffers and associated processing functions. It also has an imaginal module, which is typically used to model the maintenance and manipulation of visual spatial representations. Unlike the visual module, the imaginal module does not have a committed chunk ontology or fixed processing functions. ACT-R also has an iconic visual memory in which pre-attentive features of object bindings are represented in a retinotopic frame of reference. ACT-R can only attend/process a single visual or spatial object chunk at a time. Higher order chunks representing spatial and visual relations between attended or imagined objects need to be processed in the imaginal buffer. ACT-R's visual module thus has resource limitation on the attention and maintenances of visual spatial object representations. All operations are initiated by a central production system. These models, which inherit the constraints of a developed cognitive architecture, have been relatively unspecific about general knowledge ontologies for diagrammatic reasoning – with perhaps the exception of Matessa, Archer and Mui (2007), who take ideas from the DRS framework.

Mimicking spatial constraints

Many of the models or systems discussed commit to a base level representation. Such representations are either implicit models of spatial coordinate systems or explicit data structures that subsume a spatial co-ordinate system (e.g., bitmap, array etc.). In some models, the level of representation is also taken to model a more specific cognitive structure hypothesised from empirical research such as CAMERA's bitmap model of the visual icon buffer. The purpose of these representations, arguably, is to replicate the effects of constraints resulting from the spatial properties of diagrams, so that certain computations are offloaded on the internal representation of the diagram (e.g., free rides). The spatial coordinate systems are of course modelled by numerical co-ordinate systems. There are a couple of points worth noting here. Possible constraints in an internal representation of space need not exploit anything spatial in the representing medium. Indeed, whether assumptions about constraints in an internal representation result from properties of the representing medium or are constructed by inferential processes cannot be established in any straight forward way. Inferential efficiency of internal diagrammatic representations (e.g., free rides in mental animations) may have more to do with automaticity or minimal control requirements than the information structure of internal representation.

Exploiting inheritance

Any picture/diagram contains many combinations of relations between representing objects. For any given task, a cognitive system will encode, represent or memorise only a small subset of those available. Many cognitive models, cognitive architectures and AI frameworks assume that visual and spatial relations are locally derived from a base level representation on a need-be basis. The base level representation, as stated, may have a coordinate system, which provides the necessary constraints on implicated objects' relations. This avoids a combinatorial explosion in relations that a cognitive system would have to deal with and maintain. It is also broadly consistent with research suggesting that what information is processed and retained in visual spatial tasks is more limited and goal directed than suggested by phenomenological impressions.

2.7 Summary and conclusion

The review of research suggests that understanding advantages of reasoning with diagrams requires consideration of different interdependent components of a user, task and representation. ERs differ in complex multidimensional ways, but there are key properties of diagrams that appear to be critical in general explanations of their efficiency.

- Diagrammatic systems are typically more inferentially tractable than sentential systems from a computation perspective. Inferential tractability results from the amount of information in a representation. This tractability is realised in modelling and derivation activities. The inferential efficacy of diagrams has been demonstrated in research on AI, via task and representational analysis observed experimentally with users and has been modelled computationally in various contexts. The inferential tractability of a representation does not by itself guarantee its cognitive exploitation.
- The accessibility of information expressed by diagrammatic systems is also a critical factor in explaining their processing efficiency. The kinds of information accessible in diagrams includes represented states of affairs, transformational possibilities and laws that underpin the represented model. Accessibility in diagrams and ERs more generally is a multifaceted phenomena in which different combinations of representational properties, perceptual and cognitive processing may combine in explaining the processes of accessibility. Accessibility mechanisms play a role in explaining how the inferential tractability of diagrams can be exploited.
- The hypothesised cognitive benefits of diagrams in the formulation and evaluation of solution procedures (i.e. figuring out what to do) appears to depend

on the meta-cognitive opportunities they confer. These meta-cognitive opportunities derive from representational properties such as auto-consistency and relate to the capacity to access and evaluate explanatory hypothesis. The suggested benefits of abductive reasoning is a common theme in accounts of diagrammatic reasoning, where solution procedures are unknown to the user such as in learning and discovery. Opportunities to evaluate explanatory hypotheses arise through the capacity to "demonstratively" test and verify them on the representation. Such activities exploit a user theory of its consistency and (perhaps) assumptions about physical laws that are built in to the visual spatial processing system. Other indirect effects may arise through the computational efficiency in carrying out solution procedures with diagrams, which free up processing resources for meta-cognitive activities.

The research review highlights some general limitations and corresponding requirements for the future study of diagrammatic reasoning. These include (a) principled modelling of diagrammatic reasoning in empirically constrained cognitive architectures, (b) empirical study and computational model of abductive and verification based cognition with diagrams, and (c) the study of the process of accessibility and inferential tractability in unpractised/unfamiliar diagrammatic problem solving.

- Many of the computational accounts of diagrammatic reasoning reported in the literature are abstract or implemented in general cognitive architectures that enforce empirically informed constraints on processing. In particular, realistic commitments about working memory limitations, visual spatial processing and cognitive control are typically limited. There have been models in empirically grounded architectures such as ACT-R, but these models are largely uncommitted about the underlying ontology and architecture for reasoning with diagrams. A timely contribution will be to explicitly evaluate commitments to modelling diagrammatic cognition in principled and more empirically constrained architectures and models.

- Although abduction and verification is an important theme in work on diagrammatic reasoning, little cognitive research has been done examining in detail how such processing is interleaved in unpractised/unfamiliar problem solving contexts. Existing empirical cognitive research on these issues appears to be confined to learning (e.g., Ainsworth & Loizou, 2003), whereas contexts such as discovery (e.g., Cheng & Simon, 1995) and mathematical reasoning (e.g., Lindsay, 2002) have been approached from analytical, AI and modelling perspectives. Research on the use abduction and verification in unpractised diagrammatic problem solving tasks will complement research and address more generic assumptions about advantages of diagrams that have been assumed by researchers.

- Conceptual or computational modelling in diagrammatic reasoning has tended to focus on models of practised problem solving behaviour for example, in the case of modelling domain experts (e.g., Koedinger & Anderson, 1990), students that have undergone training with a diagrammatic system (Lane, Cheng & Gobet, 2000), participants that have undergone repeated experimental trials (Peebles & Cheng, 2002; Zhang, 1997) or competence models of effective strategies that would, if performed, take a user practice to master (e.g., Stenning & Oberlander, 1995; Lindsay, 2002). Other models simulate the computational requirements of performing a task in a manner that abstracts over factors that would differentiate levels of learning or expertise (e.g., Larkin & Simon, 1987; Cheng & Simon, 1995). The upshot is that research that comprehensively examines accessibility mechanisms and inferential efficiencies in unfamiliar and unpractised problem solving contexts where solution procedures are not known to participants would be a timely activity.

Chapter 3: External Representation in Probability Problem Solving

3.1 Introduction

3.1.1 Aims & motivation

The general research question of how diagrams confer advantages on the formulation of solution procedures through semantic accessibility applies to a range of different domains. The research project chose to focus on the domain of probability problem solving (PPS) because it satisfies several important or necessary conditions relevant to addressing the research question. These include:

Problem properties

- *Problems afford analytical formulation of solution procedures.* PPS problems are a characterisable class of convergent mathematical problems in which correct solutions depend on the logical structure of the problem data. Probability problems therefore afford logical composition of solution procedures with a derivable explanatory structure (i.e. a proof either visual or formal).
- *Problems are sufficiently challenging for investigating solution procedure formulation.* The domain of PPS appears to be abstract and notoriously difficult for students to learn, reflect on and correctly put to use. Research has also appealed to the counter intuitive nature of many classes of problems in PPS (e.g., Shimojo & Ichikawa, 1989; Fox & Levav, 2004). People's general knowledge of PPS and domains used in PPS (e.g., intuitive set theory) are typically considered partially coherent, even following tutoring (e.g., Corter & Zahner, 2007; O'Connell, 1999; Cheng 2011).

User requirement

- *Novice participants have sufficient knowledge and skills to formulate solution procedures***.** Individuals, at least in developed civilizations, have a rudimentary competence for PPS which they acquire as children (e.g., Piaget & Inhelder, 1975; Girotto & Gonzales 2008; Falk & Wilkening, 1998). The knowledge required to formulate solution procedures depend on general abstract domains for which novice participants possess naive theories or schemas (i.e. chance, sets theory, arithmetic and belief modelling).

External representation

- *PPS tasks are supported by diagrams.* PPS tasks typically require the specification of a model of the problem situation in terms of classes of information (i.e. set structure, proportional magnitudes) that diagrams are well suited and conventionally used to represent. Indeed, the dependence of PPS on external representations is evident by the variety of diagrams and graphics that are conditionally used in teaching probability such as Venn diagrams, network diagrams, outcome lists and contingency tables (e.g., Cheng, 2011; Corter & Zahner, 2007).
- *PPS tasks elicit accessibility effects.* There exists a significant body of experimental research demonstrating that PPS is notoriously sensitive to the way the problems are presented, including linguistic framing and numerical representation (e.g., Fox & Levav, 2004; Sloman, Over, Slovak & Stibel, 2003; Gigerenzer & Hoffrage, 1995; Girotto & Gonzalez, 2001). Amongst this research are a number of studies which appear to show effects of diagrams or graphics on the correct formulation of solution procedures (e.g., Brase 2009; Cheng, 2011; Yamagishi, 2003; Sloman, et al., 2003).

Taken collectively these facts suggest that PPS is an ideal task domain for investigating the research question.

3.1.2 Cognitive research on probability

Written characterisations about probability can be traced as far back to philosophical writings of Aristotle. However, it apparently was not until the 17th century that formal conceptions of probability began to appear (e.g., Good, 1959). Hence, probability at least in the mathematical sense is a recent cultural development. Indeed, informal assessment of chance and possibility is likely to implicate knowledge and skills that are fundamental to everyday thinking and independent of recent cultural developments in probability.

What probability is and how it should be conceptualised has and continues to be a subject of debate by philosophers and mathematicians. According to Good (1959), several different ways of thinking about probability have been developed over the last few hundred years. One common distinction that is made is between subjective sense of probability (in which probability is understood in terms of degrees of belief assigned to propositions) and the so called frequentist sense of probability (in which probability is understood objectively in terms of outcome frequencies that result in experimental setups).

Mixed interpretations and debates about the proper characterisation of probability have also arguably been reflected in cognitive psychological characterisations of probability, perhaps, in part, because of its conceptually elusive nature. There are, however, other factors about probability and cognition that have also introduced confusion and varied opinions, including the distinction between people's cognitive theories/schemas of probability, and systems of the cognitive architectural that learn and respond according to probabilities of sampled distributions (e.g., declarative memory).

Cognitive research on reasoning about probability has focused on a number of areas typically implicating researchers from different academic backgrounds.

- Research investigating strategies and errors in heuristic approaches to making probability judgements are often called non-extensional reasoning about probabilities (Tversky & Kanehman, 1974). This research has aimed at deriving implications for understanding errors in probability judgements in natural and critical decision making contexts such as medical, legal and financial domains.
- Research examining the development of children’s understanding and competence of probability at different stages of intellectual development (e.g., Piaget & Inhelder, 1975; Falk & Wilkening, 1998; Girotto & Gonzalez, 2008). The research has been used to constrain theoretical accounts of the source and nature of knowledge and processes that underpin reasoning and judgements about probability.
- Educationally oriented research investigating the relationship problem representation and performance in PPS tasks (e.g., Cheng, 2011; Corter & Zahner, 2007; O’Connell 1999). Such research tends to be directed at deriving

pedagogical implications about the nature of task and representation in teaching and learning.

- Research oriented an understanding why certain classes of probability problems are difficult or counter intuitive for individuals (e.g., Shimojo & Ichikawa, 1989). Many classes of problems that tend to elicit certain errors have been identified often involving conditional probabilities (e.g., Prisoner problem, Monty Hall Dilemma). This research is mainly directed at determining theoretical accounts of reasoning about probability, although pedagogical implications may sometimes be appealed to (e.g., Johnson-Laird, Legrenzi, Girotto, Legrenzi & Caverni, 1999; Gigerenzer & Hoffrage, 1995).

This excludes for consideration research that is not strictly about PPS in the mathematical sense. The relevant distinction is often called extensional or non-extensional reasoning about probability.

3.1.3 Extensional *vs.* non-extensional reasoning

Extensional reasoning[1] about probability may be 'roughly' considered a mathematical or analytical approach to determining probability of some possibility. Johnson-Laird, et al., (1999) eloquently define extensional reasoning about probability as "*inferring the probability of an event from the different ways it could occur*" (p. 63). Extensional reasoning therefore assumes that the problem solver needs to exhaustively represent the set of relevant outcomes (i.e. extension) to determine a calculated solution. Extensional reasoning is notably considered to depend in part on deductive rather than inductive reasoning (e.g., Johnson-Laird, et al., 1999).

Extensional reasoning about probabilities can be distinguished from *non-extensional* reasoning about probability, which does not require modelling sets of relevant possibilities. Examples of such research come from influential studies of Tversky and Kanheman (e.g., Tversky & Kanheman , 1974). Their research aimed at determining to what extent human reasoning about probabilities accords with the results of Bayesian

[1] In this thesis we also use the term *probability problem solving* (e.g., Corter & Zahner, 2007) to involve extensional rather than non-extensional reasoning about probability. The term 'problem solving' highlights the broad cognitive requirements of the probability tasks, which are a more realistic characterisation.

calculus. Tversky and Kanheman (1983) observed that participants often made errors in their probability judgements that appeared to be consistent with the use of heuristics strategies. Their studies identified a number of different types of errors or biases based on different heuristics such as the representativeness and availability heuristics. As an example, consider the canonical problem below, which is known to elicit the so called *conjunction fallacy*.

Linda is 31 years old, single, outspoken, and very bright. She majored in Philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Which is more probable?

Linda is a bank teller.

Linda is a bank teller and is active in the feminist movement.

Figure 3.1. The Linda problem known to elicit the conjunction fallacy.

Tversky and Kanheman (1983) found that when participants were given this problem information 85% of them considered the second option (i.e. Linda is a bank teller and active in the feminist movement) as being more probable. The popular answer is incorrect because of set constraints. A conjunction of outcomes cannot have a greater probability than any of the single outcomes alone because the conjunction set must be in either of the sets of its conjuncts. In this particular example, the researchers suggested that participants employ a representativeness heuristic to make intuitive judgements. That is, participants judge the incorrect hypothesis as being more probable because it is more representative of the assumed data (i.e. the description of Linda). Such reasoning strategies are often characterised as inductive in contrast to the proposed deductive approach of extensional reasoning (e.g., Johnson-Laird, et al., 1999).

An important question, which has been raised by a number of authors, concerns the conditions in which participants choose to employ extensional strategies rather than non-extensional reasoning strategies (e.g., Johnson-Laid, et al., 1999). Problems used in empirical studies invoking extensional reasoning often involve a numerical specification of problem data, whereas problems invoking non-extensional reasoning do not and may also request judgements rather than calculated values. In addition, assumption about participant knowledge of the data in the problem description also appears to be a critical factor. Problems that evoke non-extensional reasoning typically depend on learnt expectations (e.g., people involved in political activism will be concerned about social justice), whereas problems that evoke extensional reasoning involve arbitrary data (i.e. probabilities or frequencies of outcomes) that are novel to the problem solver.

3.1.4 Chapter plan

The purpose of this chapter is to review relevant research on representational effects on solution procedure formulation in PPS tasks (aka. extensional reasoning about probability). In section 3.2, we will consider theoretical frameworks within which PPS has been conceptualised. The main aim of this section is to provide the reader with an initial conceptual outline to interpret subsequent issues on PPS addressed in the chapter. In section 3.3, research will be reviewed on accessibility effects of PPS tasks mainly identifying effects of solution procedure formulation. The main aim of this section is to understand the variety and scope of accessibility effects in PPS as well as the validity of specific theoretical interpretations. In section 3.4, we will consider empirical research and assumptions about prior knowledge, cognitive strategies and information characterisations of novice PPS. The information will be required to constrain interpretations of performance effects and the development of task analysis and process models. In section 3.5, we will summarise the research considering limitation on existing research, constraints on information processing account of PPS and theoretical implications regarding the fundamental research question being addressed in this thesis.

3.2 Theoretical approaches to PPS

The following section outlines some of the main theories and accounts of PPS and subtypes (i.e. conditional reasoning) in the research literature. Doing this at the outset will afford appreciation of the theoretical context behind different research themes and issues that will be separately addressed in subsequent sections of this chapter. The section will consider the following theories: *ecological rationality approach*, *mental models theory*, *nested set theories*, *dual processing theories* and accounts that focus on particular aspects of PPS such as strategies. The section will end by comparing and contrasting these different approaches.

3.2.1 Ecological rationality & natural sampling

The Ecological Rationality theory argues that PPS, particularly problems involving conditional probabilities, depend on the recruitment of adaptively evolved cognitive algorithms that compute probabilities from natural sampling. The term natural sampling is used to refer to sequential acquisition of outcome experiences overtime. In support of natural sampling, the authors consider research consistent with the proposal that human and animal minds have evolved mechanisms to learn and behave according to the statistical structure of outcomes in their environment. To contextualise this idea Gigerenzer and Hoffrage (1995) provide an example of a fictitious scenario of a doctor who, over time, would witness patients with different symptoms, some of whom turn out to have some disease rather than another. According to the authors, such observations alone would allow the doctor to make subsequent estimations about the conditional probability of a patient having some previously observed disease given some symptom. The critical claim they focus on is that such estimations are based on exposure to frequencies of observed cases. The cognitive algorithms that have purportedly evolved for estimating probabilities based on natural sampled cases are proposed to have a specific input format, namely frequencies of observed cases.

Explanations about input format are used to make predictions about people's ability to make Bayesian probability estimations as a function of the way the problem is presented. In other words, the authors appeal to an accessibility hypothesis to support their theory. According to their account, performance should be facilitated in conditional probability problems to the extent that the data of the problem matches the format of the natural sampling algorithm. What they call frequency formats, which include whole number specification of possible cases, should facilitate performance relative to probability, which are normalised fractions. Part of their argument is also motivated by the independent explanatory claim that probability formats are more computationally complex than frequency formats because, according to their analyses, the latter requires less operations, attention to fewer units of information (e.g., base rates can be ignored), and permits the posterior distribution to be computed from frequencies per se.

Proponents of this view have acquired empirical evidence that frequency formats better facilitate reasoning than probability formats in conditional probability tasks. There are varieties of particular interpretations of natural sampling accounts. For example, Barbey and Sloman (2007) distinguish between three related accounts. Some researchers have proposed the existence of specialised evolutionary adapted modules for computing probabilities based on natural sampling (Cosmides & Tooby 1996). In response to criticisms, proponents more recently adapted the interpretation of frequency formats of natural sampling arguing that natural sampling algorithms are tuned to the parsing of whole objects and events (e.g., Brase, 2009).

There are a number of problems with this account. Those criticisms dealing problem presentation predictions of the theory will be addressed in the next section. The other main problems with this account are conceptual. As noted by Johnson-laird et al. (1999), it is difficult, if not impossible, to test evolutionary theories. However, the biggest problem is that the idea that individuals apply an evolutionary adapted Bayesian cognitive algorithm to solve word problems has little plausibility. Such cognitive algorithms are viewed as if they were part of the cognitive architecture like those implicated in lower level visual processes. This is a counter intuitive characterisation for a task that clearly requires high-level deliberative planning, reasoning and evaluation to arrive at a solution. Indeed, such tasks require participants to figure out what the problem is and how to solve it, and it is these processes that play a significant role in performance differences (others may be errors in carrying out a plan). At least some of the reasoning behind carrying out steps of the task appear to be available to conscious awareness as has been reported in studies involving verbal protocol analysis (e.g., Shimojo & Ichkawa, 1989; Fox & Levav, 2004; etc.) implying that they are unlikely to be the results of evolved cognitive algorithms. Note that participants in their experiment spend on average approximately 5 minutes to solve each problem (reportedly 15 problems in first session taking on average 73 minutes), which is consistent with the idea that they were figuring out how to solve the problem, that is, formulating a solution procedure.

3.2.2 Mental models theory

The Mental Models theory (e.g., Johnson-Laird, et al., 1999) is aimed specifically at naïve extensional reasoning about probability. Mental model theory was originally developed to explain research in deductive reasoning and language comprehension. In short, the theory proposes that individuals, who are naive to formal probability, solve problems by constructing mental models. The authors define mental models as a "*representation of a possibility that has a structure and a content that captures what is common to the different ways in which the possibility might occur*" (p. 66, Johnson-Laird, et al., 1999). A central property of mental models that distinguish them from other kinds of representations is that they represent only true possibilities. An example of the mental models that should be constructed for the disjunctive statement such as '*There is a circle or there is a triangle, but not both'* would be |circle| |triangle| where each closed bracket represents a model (Johnson-laird et al. actually use an array notation in which rows are models). The construction of mental models is a typical rather than an absolute prediction. According to the theory, an individual may also, under certain circumstances, construct fully explicit models which represent what is false through negation. Fully explicit models for the statement would be |circle ¬triangle||¬circle triangle|.

The theory attempts to explain performance on a range of different types of probability problems varying in terms of the logical connective used to describe the query and premises. Emphasis in the theory is given to the deductive nature of the task and the requirements to construct models from the premises. A key characterisation of the theory is that predictions of constructed models correspond to different partitions of possibilities. The theory is described by a set of principles which are, with one exception (i.e. the truth principle), specific to PPS rather than general mental models theory. A central and intrinsic principle of the theory is the *truth principle,* which states that participants will tend to construct mental models of true possibilities rather than fully explicit models. The other principles stated by the theory are basically procedural specifications for dealing with features of probability problems namely assumptions of equiprobability (*equiprobability principle*), how probabilities are quantified (*proportionality principle*), how and when to deal with unequal possibilities (*numerical representation*), and how conditional probabilities are determined (*subset principle*).

The central component of mental models theory is its account of the extensional representation that people use. However, this account makes few commitments about the nature of internal representations employed in PPS. Such proposals abstract over the different ways that participants may internally model problem situations. These include the particular dimensions of a problem they represent, how representing referents are conceptualised, and if they use abstract ERs what kind of structure is being represented and what kinds of constraints are being exploited from it.

Another point worth noting is that many of the principles of the theory have no relation to mental models theory per se, and there is no new explanatory information gained from their integration. For example, the equiprobability principle, in short, states that individuals will assume equiprobability unless given information to the contrary. Probability is a case in which the authors attempt to glean some coherence between mental models theory and the principle when they claim "*equiprobability applies to mental models and mental models represents only what is true within true possibilities*" (p. 69, Johnson-Laird, et al., 1999). But there is nothing in mental models theory which constrains or shows any explanatory connection for the assumption of equiprobability. It is wise to note that the term mental models has multiple and sometimes unspecific meanings in cognitive science literature, which should not be confused with the particular theory of PPS reported by Johnson-Laird, et al. (1999). It is uncontroversial that PPS depends on models (internal or external) of relevant outcomes irrespective of any commitment to this particular theory.

3.2.3 Nested-set accounts

The nested-set account was proposed by a number of authors in direct response to the natural sampling accounts of frequency formats (e.g., Mellers & McGraw, 1999; Tversky & Kahneman, 1983; Yamagishi, 2003; Sloman, et al., 2003; Girotto & Gonzales, 2001). The nested-set account proposes that normative performance on extensional reasoning about probability depends on the representing the set structure of the problem situation. As such, the account instead explains the effect of frequency format in terms of providing more effective cues to access the set structure of prior and posterior outcomes. In other words, performance is facilitated to the extent that subset relations are accessible or can be formulated from the representation of the problems. In addition to the numerical representation used in the word problem description, the account has also been used to make predictions about the facilitative effects of diagrams in PPS (e.g., Yamagishi, 2003; Sloman, et al., 2003; Brase, 2009). Although the account has been predominately used to explain performance on conditional probability problems, it has also been proposed to predict facilitation for a range of different problems and not just conditional probability problems (Barbey & Sloman, 2007).

The nested set account has a close affiliation with the mental models theory of PPS because both assume that such tasks require extensional representations of relevant outcomes and operations to determine subset relations. Indeed, mental models theory has also been classified as an exemplar of nested accounts (e.g., Sloman, et al., 2003; Barbey & Sloman, 2007). A weakness of nested accounts is that no clear information processing explanations of how subset structure facilitates performance has been provided. Indeed, reports of such accounts are typically devoid of any kinds of process description of how individuals go about solving probability problems.

Another criticism, concerns the vagueness about what constitutes an extensional representation of sets. For example, Sloman, et al. (2003) take the Venn diagram representation as a canonical representation of sets, but such representations are essentially spatial containment analogies rather than literal models of sets as spatially discernible collections. This has arguably lead to confused predictions about what graphical representations and combinations of semantic information best represent sets and best facilitate performance (e.g., Brase, 2009).

Proponents of nested set accounts may be focusing too narrowly by considering only the accessibility of set structure in explaining performance in extensional reasoning tasks. This is because set structure is not the only kind of extensional information critical to PPS (e.g., proportional relations). One may argue that such accounts of facilitation would be more appropriately cast in terms of the accessibility of relevant relational structure of the problem situation more generally than set relations per se.

3.2.4 Dual processing accounts

Barbey and Sloman (2007) proposed a dual processing account of reasoning about probabilities with a focus on explaining the so-called based rate neglect phenomena. Their account is a specific version of a dual processing theory of reasoning for which there are others. The account distinguished between an associative and rule-based reasoning system (sometimes called type 1 and 2 systems). According to Barbey and

Sloman (2007) a "*primitive associative judgement system*" (type 1) makes responses based on principles of similarity/memory retrieval. The rule based system (type 2), on the other hand, is involved in representing referents and computing necessary set operations from the representation (i.e. extensional reasoning). The rule-based system depends on working memory resources and operations that are deliberative and effortful.

According to the theory, base rate neglect results from the use of the associative system, whereas performance facilitation occurs when the rule based system correctly employs rules. The dual processing account subsumes other accounts that distinguish between a form of extensional reasoning about probability that requires the representation of referents/instances of a category such as mental models theory and the nested-set accounts of conditional PPS. Like the natural sampling account, Barbey and Sloman (2007) allude to the issue of specificity of a representation to processing operations. The authors assume that relevant rules apply to representations in which the set structure is "*transparent for problem solving"* (p. 244, Barbey & Sloman, 2007) (i.e. the nested set hypothesis/theory).

3.2.5 Other accounts of PPS

There are other accounts that are not committed to general abstract theories, but specific models of aspects of PPS. For example, researchers have attempted to specify abstract process stage models of PPS taking inspiration from earlier information processing accounts of mathematical word problems (e.g., Zahner & Corter, 2010). Other accounts have focused more specifically on models of strategies/solution procedures in conditional probability problems (e.g., Fox & Levav, 2004).

3.2.6 Summary

The review identifies similarities and differences between theoretical accounts. Nested set, mental models and dual processing theories also assume that users need to form extensional representations of the problem situation. However, representation has a central role in all account of PPS. Specifically, the manipulation of the accessibility of external represented information has been the common leverage for making theoretical claims about PPS. Mental models commit to minimal predictions about solution procedures, whereas natural sampling simply assume the use of what they construe as

Bayesian algorithms, given appropriate conditions. Aside from solution procedures few commitments are made about the nature of information processing in PPS tasks. It is an interesting point that, with the exception of nested-set accounts, these theories are grand theories of reasoning more generally and are not tied to the domain of PPS per se. Motivations to determine one general theoretical account rather than another arguably obscures the value of more specific information about the nature PPS and explanations of PPS performance. Theoretical commitments about PPS are perhaps weak because of this.

3.3 Accessibility effects of external representations

The following section discusses research on accessibility effects of ERs on performance, particularly the formulation of solution procedures, in PPS tasks. Much of the research on ER accessibility effects has focused on errors in conditional probability problems, with a particular emphasis on the determination of competing nested set and natural sampling accounts. This chapter will discuss accessibility effects from different sources of ERs, including the text presentation/representation of the problem, diagrammatic representation of the problem situation, and concrete physical manipulations of problem scenario artefacts. The section will outline a number of reported ER accessibility effects in PPS including, in the following order: hypothesised effects of the quantification format/ERs in text presentations, the representation of set structure in diagrams, the representation of token/instance structure in diagrams, textual presentation effects on partitioning of possibilities, and concrete physical manipulations of partition edits. In the final part of the section, these research findings will be synthesised and evaluated allowing alternative considerations to be surmised.

3.3.1 Probability calculus

Much of the research on accessibility effects in PPS task has been limited to conditional reasoning about probability. In these tasks, the Bayesian calculus for computing conditional probabilities has been taken as the normative way of determining a solution. The simplest form of Bayes theorem is shown by the formula below (Figure 3.2). The variables in the formula are values of propositions. In the equation, the conditional probability that a hypothesis H holds given evidence E is conventionally called the posterior probability written as P(H|E). The posterior probability can be calculated from the probability of both hypotheses P(H) and evidence P(E) occurring alone normally called prior or marginal probabilities and the probability of the evidence given the hypothesis P(E|H), normally called the likelihood. Bayesian calculus specifies the quantification of possibilities at a level of abstraction higher than specific counts. To get values to plug into a Bayesian formula, one needs to do determine a normalised probability for each term. These normalised values contain less information than counts of outcomes/possibilities; for example, they do not carry information about base rates (e.g., Gigerenzer & Hoffrage, 1995). It is not an intuitive formula to understand because of this abstraction. Determining conditional probabilities does not require the use of Bayesian calculus.

$$p(H|E) = \frac{p(E|H) \times p(H)}{p(E)}$$

Figure 3.2. Bayesian calculus in its simplest form.

3.3.2 Accessibility effects of frequency presentation from text

Perhaps one of the most widely addressed and most controversial issues in the PPS research literature is the contentious issue of why so called frequency format presentations facilitate normative Bayesian performance solutions in determining conditional probabilities. The research programme proposed by Gigerenzer and colleagues was, in part, motivated by disputed arguments concerning the generality of Tversky and Kahneman's (1983) claim that people untrained on the probability calculus were prone to make unconservative probability judgements and that such findings apparently depart from the view of humans as a normative Bayesian reasoners. Gigerenzer and colleagues had instead proposed that performance of such participants could be significantly improved if the format of the problem data was presented in a way that the mind had naturally evolved to deal with, that is, if presented in a frequency rather than probability format. Gigerenzer and Hoffrage (1995) reported a series of experiments that aimed to test and investigate this hypothesis.

According to Gigerenzer and Hoffrage (1995), frequency formats are presentations of data about the problem situation that specify counts of sets of outcomes. In contrast, probability formats are problem presentations where the relevant data about sets of outcomes are given as normalised probabilities using decimals, percentages or fractions. Table 3.1 shows an information equivalent conditional probability problem, taken from Gigerenzer and Hoffrage's (1995) study, presented using frequency (b and d) or probability formats (a and c).

Menu/ Format	**Problem description**
Standard/ Probability (a)	The probability of breast cancer is 1% for women at age forty who participate in routine screening. If a woman has breast cancer, the probability is 80% that she will get a positive mammography. If a woman does not have breast cancer, the probability is 9.6% that she will also get a positive mammography. A woman in this age group had a positive mammography in a routine screening. What is the probability that she actually has breast cancer? ___%
Standard/ Frequency (b)	10 out of every 1,000 women at age forty who participate in routine screening have breast cancer. 8 of every 10 women with breast cancer will get a positive mammography. 95 out of every 990 women without breast cancer will also get a positive mammography. Here is a new representative sample of women at age forty who got a positive mammography in routine screening. How many of these women do you expect to actually have breast cancer? ___ out of ___
Short/ Probability (c)	The probability that a woman at age forty will get a positive mammography in routine screening is 10.3%. The probability of breast cancer and a positive mammography is 0.8% for a woman at age forty who participates in routine screening. A woman in this age group had a positive mammography in a routine screening. What is the probability that she actually has breast cancer? ___%
Short/ Frequency (d)	103 out of every 1,000 women at age forty get a positive mammography in routine screening.8 out of every 1,000 women at age forty who participate in routine screening have breast cancer and a positive mammography. Here is a new representative sample of women at age forty who got a positive mammography in routine screening. How many of these women do you expect to actually have breast cancer? ___ out of ___

Table 3.1. An example of crossed versions of format and menu for the mammography problem in Gigerenzer & Hoffrage's (1995) study (p. 688).

The researchers tested predictions made by the proposals in a set of experiments involving word problems such as that shown in Table 3.1. The experiments involved a design in which frequency (Table 3.1b and Table 3.1d) and probability formats of word problems (Table 3.1a and Table 3.1c) were crossed with presentations that either specified the base rate information (Table 3.1a and Table 3.1b) or omitted it (Table 3.1c and Table 3.1d). Gigerenzer and Hoffrage (1995) call the latter presentation differences short and long menus, respectively. According to their rationale, the information presentations involving frequency format and short menus should facilitate producing a normative solution because they more closely match the input format of the mind's natural sampling algorithms. In a series of experiments involving multiple problems, the authors observed that that the same problem information presented in frequency formats was substantially more likely to facilitate correct normative solutions than when presented in a probability format, presentations which short menus provided greater facilitation than presentations with long menus in the probability format. The menu manipulation was also found to have little effect on facilitation in both short and standard versions of the frequency format condition (frequency/short = 50%; frequency/standard = 46%; probability/short = 28%; probability/standard = 16%). According to the authors, incorrect responses were arrived at by several different algorithms that were all classified as non-Bayesian.

This study and others since have been taken to support natural sampling account of human probability estimations (e.g., Brase, 2009). A number of researchers have been critical of the natural sampling explanation. For example, Johnson-Laird, et al. (1999) claim that "*the mere use of frequencies does not constitute what they call a natural sample*" (p. 81), in which they seem to be pointing out what are arguably incoherent differences between making probability calculations from word problems and making probabilistic judgements based on natural sampling. Other researchers have demonstrated that frequency presentations can also be normalised and be as difficult for people to solve as percentages/decimals – disputing that frequency format, at least in the general sense, is not the critical factor in performance facilitation (e.g., Johnson Laird, et al., 1999; Girotto & Gonzalez, 2001).

Several researchers have proposed an alternative explanation of the observed advantages of frequency formats, they claim instead that frequency formats facilitate performance in the relevant task because they help participants visualise the set structure between prior and posterior outcomes (Girotto & Gonzalez, 2001; Johnson-Laird, et al., 1999;

Mellers & McGraw, 1999; Sloman, et al., 2003; Yamagishi, 2003). The reasoning behind the explanation is summarised by Sloman, et al. (2003), who state that (a) descriptions of frequencies elicit an internal representation of instances of a category (i.e. tokens or individuals) rather than properties of a category; (b) set structure can be revealed by the representation of instances; and (c) nested set relations are "*cognitively transparent for problem solving*" when the set structure is revealed (p. 298).

Hoffrage, Gigerenzer, Krauss and Martignon (2002) had claimed that critics of the natural sampling account had misinterpreted the meaning of frequency format and its relation to natural sampling claiming that their initial proposal meant "natural" frequency formats that have the structure of naturally sampled frequencies and are therefore said to carry base rate information (p. 348). Hoffrage, et al. (2002) argue that nested sets are just a property of natural frequency formats that alone are not sufficient to account for the critical facilitation effects (p. 349), whereas others have argued the contrary (e.g., Johnson-Laird, et al., 1999; Girotto & Gonzalez, 2001; Sloman, et al., 2003; Yamagishi, 2003). It is interesting at this point to note that, of all the commentary regarding and the controversy between these issues, there is no explicit task/representational analysis or model of any of the proposed accounts.

There have been a numbers of studies which aimed at challenging and testing the alternative nested set hypothesis. Many of these studies have used diagrammatic representations in presentations of the problem. Whilst being informative about the relevant theoretical accounts of naïve probability from which they are motivated, the research also has important explanatory implication in research on diagrammatic reasoning, in particular, on hypothesised advantages of diagrams conferred on interpreting solution procedures.

3.3.3 Accessibility effects of set structure in diagrams

Other researchers have used diagrammatic representations conveying the set structure to investigate and evaluate the nested set hypothesis. Yamagishi (2003) compared performance on conditional probability problems using the manufacturing problem scenarios described in Figure 3.3. In two of the experiments (1 & 2), a 2 X 2 design was used in which word problem conditions that had either frequency or probability formats were crossed with problems in which a roulette diagram (shown in Figure 3.3b) was either present or absent. Groups of participants were assigned to one of each of the crossed combinations. The condition with the diagram present was hypothesised to make the critical set relations more accessible. In both experiments, Yamagishi found that, when the diagram was absent, the frequency condition elicited more correct responses than the probability condition. However, in conditions where the diagram was present there was no statistically significant differences in performance between frequency and probability word problem conditions. The correct response rates in either diagram condition was significantly greater than the frequency word problem without a diagram.

A factory manufactures 1200 artificial gemstones daily. Among the 1200, 300 gemstones are blurred, 300 are cracked, and 600 contain neither. An inspection machine removes all cracked gemstones and retains all clear gemstones. However the machine removes half of the blurred gemstones. How many gemstones pass the inspection and how many are blurred?

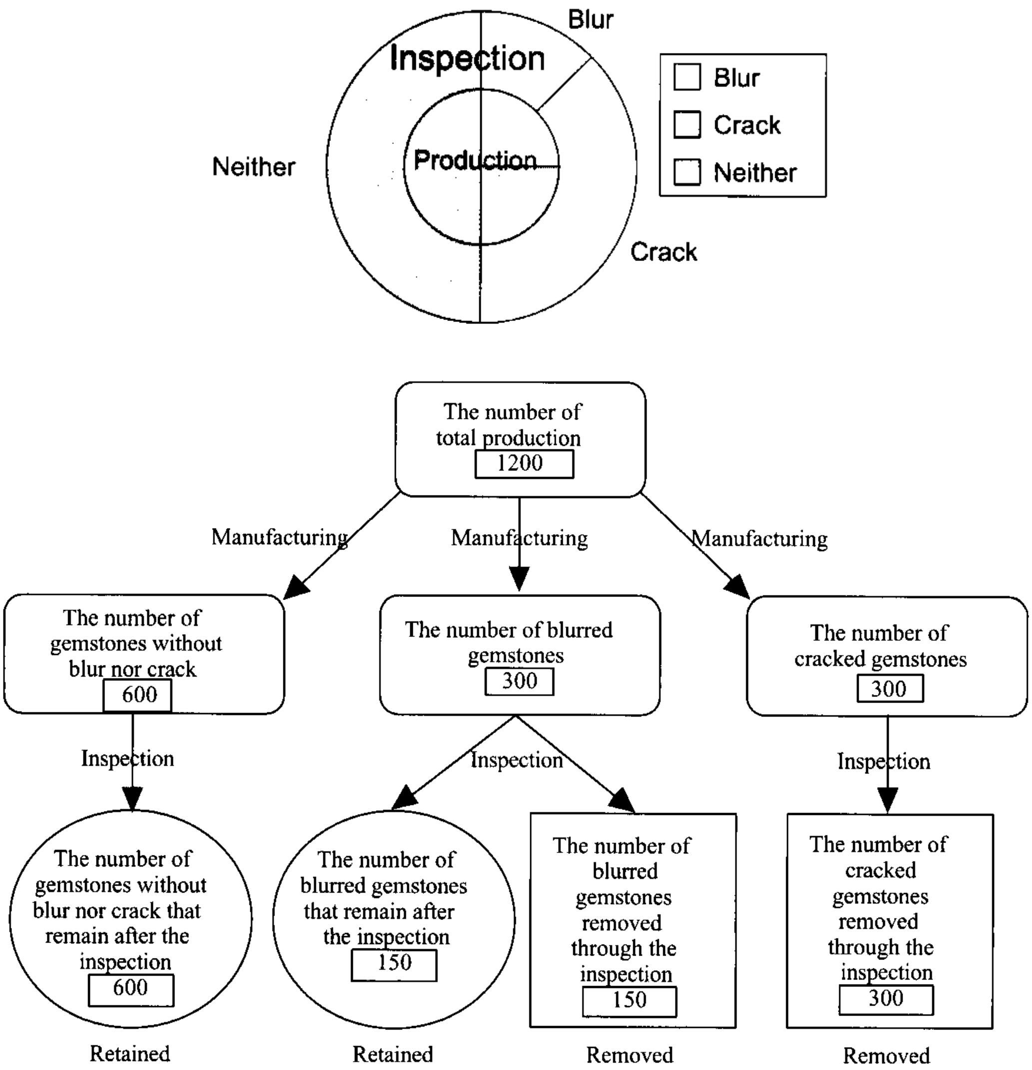


Figure 3.3. The frequency format instruction (top), roulette wheel diagram (middle), and the network diagrams (bottom) used in the study of Yamagishi (2003).

In a third experiment, Yamagishi (2003) replicated the crossed design of the previous experiments, but substituted the no diagram conditions with conditions involving a tree-diagram that was judged to make the critical subset relations less accessible (Figure 3.3c). The alternative diagram conditions allowed graded comparisons of the effects of the accessibility of nested set information on facilitating correct responses. Yamagishi found that correct performance rates were consistently better over both instruction format conditions for participants who received the roulette diagram than those who received the tree diagram. Moreover, the presence of either diagram substantially improved performance compared to conditions without a diagram as observed experiments 1 and 2. A significant difference was found between participants who received alternative instruction format conditions with the tree diagram, but not for with the roulette diagram. The former finding replicates the results of experiments 1 and 2.

In all of the experiments, frequency formats of the word problems had substantially less effect on performance than the presence of a diagram that expressed the critical set relations. The results were argued to support the claim that the accessibility of critical nested sets were a more significant factor in determining correct performance than frequency presentation; and they were consistent with the proposal that the effect of frequency presentations facilitates performance because they make the relevant nested sets easier to visualise. According to Yamagishi (2003), as the roulette diagram used in the study did not present information in terms of frequencies coupled with the observation that the effects of the diagrams on performance was substantially greater than the effect of frequency instruction; the possibility that the set structure of the diagram facilitated frequency interpretation of the problem was ruled out. Yamagishi (2003) explained the performance facilitation resulting from the presence of the diagrams arguing that graphical representations such as those used in the study "*take advantage of peoples' automatic visual computations in grasping the relationship between prior and posterior probabilities*" (p. 105).

Similar accessibility effects of set structure in diagrams have been observed elsewhere. For example, Sloman, et al. (2003) reported experiments which manipulated the text presentation of the problem and the existence of an Euler circle diagram depicting the critical subset relations. Their results lead them to propose that the presence of a diagram facilitated performance only if the nested sets were not already accessible in the word problem. The authors took the findings as support of the nested set hypothesis.

3.3.4 Accessibility of whole object representation in diagrams

One version of the natural sampling account of frequency format proposes that natural sampling mechanisms are adapted to parse whole object representations (e.g., Brase, 2009). According to this view, the more that the problem data matches this natural sampling format of whole objects, the greater performance rates should be facilitated. In a series of experiments, Brase (2009) tested a version of a conditional probability problem in which participants were assigned to one of four ER conditions. In a control condition, the problem was presented without the aid of a graphical representation. Otherwise, the problem was presented with either an unfilled Venn diagram (Figure 3.4a); a Venn diagram filled with dots (Figure 3.4b) and a diagram involving a rectangle matrix of spatial grouped icons (Figure 3.4c) in which the number of icons matched the frequency given in the problem statement. The rationale and predictions given by Brase for the experimental designs are as follows. If the nested account was correct then all representations should facilitate performance. However, if the natural sampling account holds then the presence of "individuated entities" (e.g., Icon condition) should facilitate correct solutions because the entities are supposed to elicit a frequency interpretation. According to Brase, in the case of the filled Venn diagram condition, if the natural sampling account is correct then there should be some facilitation compared to the empty Venn diagram, but not as much as the Icon condition. If the nested-set account holds, Brase predicted that there should be no difference in performance between Venn diagram filled and empty conditions.

In line with Brase's (2009) predictions of the natural sampling account, it was found that (a) the icon representation group performed better than the Venn group; (b) the filled Venn group performed slightly better than the empty Venn group; and (c) there was no significant difference between empty Venn and no diagram group. Brase also reported another two experiments designed to rule out alternative interpretations. Of particular interest is the third experiment, which replicated the initial experiment, except the icon diagram was modified so that icons from different sets were randomly spaced rather than spatial grouped, and dots in the filled Venn condition were modified such that their frequency corresponded to the values given in the word problem. In this study Brase found the ungrouped icon diagram facilitated performance equally as well as the grouped icon diagram, whereas there was no difference between the Venn diagrams. The facilitation of the Icon diagram condition on correct performance was consistently observed in all three experiments. Brase (2009) interpreted the results as supporting the ecological rationality interpretation claiming that "*representation that better approximate natural sampling frequencies tend to elicit better Bayesian reasoning*" (p. 380).

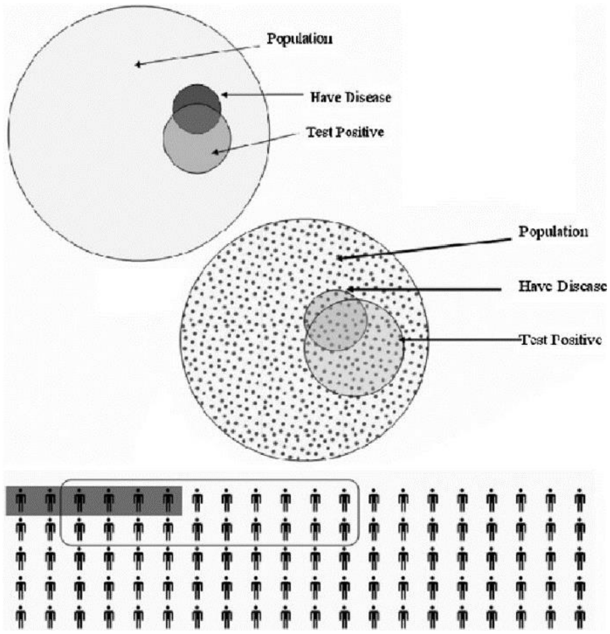


Figure 3.4. Different diagram conditions used in experiment 1 of Brase's (2009) study (a) unfilled Venn diagram (top), (b) filled Venn diagram (middle), and (c) Icon diagram (bottom).

Brase's (2009) findings are interesting and are difficult to reconcile with the findings of Yamagishi (2003) and Sloman, et al. (2003). There are a couple of criticisms worth noting. Firstly, the icon diagram not only represents a token referential model of possible outcome tokens (i.e. icons in Brase's terminology), but also uses a scheme that clearly expresses both nested set and magnitude relations between sets of outcomes. The magnitude relations are a side effect of using a 2-d grid of icons in the icon diagram condition. Indeed, the choice of a matrix organisation is an appropriate one because it highlights these relations, although, according to Brase, the choice was taken to de-emphasise subset grouping (p. 379). The implication is that the additional magnitude is a possible reason for the facilitation of the representation. Note that the roulette diagram in Yamagishi's study also clearly expressed magnitude relations; hence, it may be the expression of this information that plays a facilitator role.

There may be more global properties of the icon diagram that could play an explanatory role in facilitating performance, namely, how semantically accessible the diagram is as whole. The integrated semantics of the icon diagram combine in making the representation more intuitive to understand, perhaps because of its greater specificity or concreteness. The Venn diagrams are clearly more abstract in this respect.

3.3.5 Accessibility effects of event partition from text

Another issue relating to the accessibility of problem presentation concerns the explicitness of the partitions of alternative possibilities. Various presentation factors have been shown to influence the partition of the problem. Experiments demonstrating such effect have been approached from different theoretical perspectives. For example, Johnson-Laird, et al. (1999) reported an experiment which aimed to test the truth principle of the mental models theory of PPS. Recall that the truth principle states that mental models represent true possibilities. The authors hypothesised that the principle should predict biased solutions in the interpretation of certain problem presentations. They tested participants on a set of problems designed to elicit the bias and also a set of control problems. Biased problems predicted probability estimations that were different for mental model and fully explicit model interpretations, whereas control problems predicted the same solution for both mental model and fully explicit model interpretation.

There is a box in which there is at least a red marble or else there is a green marble and there is a blue marble.
Given the preceding assertion what is the probability of the following situation?
In the box there is a red marble and a blue marble.

Figure 3.5. An example of a biased problem in which proposed solutions that follows from a mental model interpretation differs to a fully explicit model interpretation.

An example of a biased problem provided by the authors (p. 74, Johnson-Laird, et al., 1999) is shown in Figure 3.5. According to the authors, the mental model theory predicts that participants should construct two models for the problem: |red| and |green blue|

which would predict a probability of 0% because no model is consistent with the queried state of affairs. However, when correctly interpreted as fully explicit models there are four models for the problem namely |red green ~blue|, |red ~green blue|, |red ~green ~blue|,|~red green blue|. Assuming equiprobability, the explicit models would predict an unbiased estimate of 25%. The authors tested participants on 18 problems; half involving biased and half involving control problems. The problems involved different connectives in premises and questions, which were balanced across experimental conditions. Approximately two thirds of responses in both conditions were consistent with the predicted responses above chance level (p. 75, Johnson-Laird, et al., 1999). According to the authors, participants' solutions fit the predictions of mental model theory, namely, that naïve participants will tend to reason from partitions that are consistent with mental models rather than fully explicit models. A number of subsequent experiments, which investigated the accessibility of the partition of problem data, have also been reported (Girotto & Gonzalez, 2001; Fox and Levav, 2004).

3.3.6 Accessibility effects of the edit partition from physical manipulations

Fox and Levav (2004) reported an experiment providing evidence that the accessibility of edit information can influence the partitioning of the possibilities in conditional probability problems. The authors investigated reasoning with an adapted version of the infamous *Monte Hall Problem*. In one experiment, participants were told they would be dealt a set of five cards face down, two to the participant and three to the dealer. The participants were also told that that they would be awarded a dollar if at the end of the game they had a hand with the ace. The experimenter then told the participants that "*I'm going to look at my hand, then indicate two cards that are not an ace. After that I will ask you if you want to trade your cards for my cards*" (p. 629, Fox and Levav, 2004). In one condition, the target cards are indicated by pointing to them, in the other condition the two cards are identified by physically turning them over. After identifying the cards, the experimenter asks the participant what the probability is that the ace was in the experimenter's hand rather than the participant's hand.

For either condition, the probability does not change because knowing that two out of the three cards is an ace does not provide any further information. Hence the probability of the dealers hand remains 3/5 rather than 1/3 and the participant should switch hands.

The authors predicted that turning over the cards will make the incorrect edit partition more concrete and accessible than the pointing to condition, which would require mentally transforming the partition with five outcomes. Fox and Levav (2004) found that significantly more participants in the card turning over condition incorrectly reported a third than in the card pointing condition (66% vs. 48%). The card pointing condition also resulted in three times as many correct responses of 3/5 than the card turning over condition (26% vs. 8%). If the effect is merely to do with concreteness, these findings add to the suggestion that issues of accessibility in conditional reasoning problems may be more complicated than the either nested-set or frequency accessibility.

3.3.7 Summary

The section has reviewed a number of studies that have shown accessibility effects of ERs, linguistic descriptions and other communicated information in the domain of PPS. These accessibility effects appear to influence what information people deem as relevant to solving the problems and what solution procedures they formulate and end up executing. In this review, we have taken ERs broadly to include text descriptions of the word problem, numerical ERs of the problem data embedded within words problems, as well as accompanying diagrams or graphics of the problem situation. Accessibility of information in ERs have been found to influence the determination of different subtasks in formulated solution procedures, including conceptualisation of what constitute possible outcomes in the partition of the problem and whether the partition should be edited.

3.4 Constraints on models of PPS

The aim of the following section is to outline empirical research and analysis that provide constraints on models of novice PPS. The aim of this activity is to understand and establish constraints on how novice participants actually go about solving PPS tasks, which will be used to inform the development of cognitive models reported in this research. It is an interesting fact that there has been so much debate and confusion concerning accounts of PPS, which has driven significant amounts of empirical research. Despite this, there have been no detailed process models of PPS put forward. Indeed, characterisation of novice PPS are arguably vague. Needless to say, many confusions and overly narrow considerations that have prevailed in the research field may be more readily dispelled with specific modelling methodologies. The other purpose of this section is to identify limitations on existing models of PPS. The section will discuss evidence for the prior knowledge and solution procedures or strategies used by participants novice to PPS tasks, the dependence of representation in PPS, information processing accounts of errors, and what factors determine processing in PPS tasks.

3.4.1 Prior knowledge and assumptions in novice PPS

A crucial intuition in PPS is the interpretation of probability as a ratio, which has been taken as central to some accounts of solution procedures in naive extensional reasoning about probability. For example, the mental models theory proposes the *proportionality principle,* which states that individuals take the probability of an event to depend on the proportion of models in which the target event occurs (Johnson-Laird, et al., 1999). The partition-edit-count model of conditional probability assumes that individuals take the probability to be a ratio of interchangeable events partitioned from the sample space (Fox & Levav, 2004). Both of these accounts report empirical research that is consistent with this intuitive assumption. The intuition appears to be present in childhood. For example, Piaget and Inhelder (1975) reported that children develop an understanding of probability as a proportion between favourable and total cases by the age of 10 or 11. Falk & Wilkening (1998) found evidence for this in children as young as nine. Girotto and Gonzalez (2008) observed that, from the age of five years old, children appear to be able to use posterior information to make decisions in uncertain conditions and judgements about random outcomes. In a latter assessment of research on children intuitions about probability, the authors claim that "*children, like adults, base their decisions and judgements under uncertainty on an extensional evaluation of possibilities, considering and comparing the various ways in which an outcome may or may not occur*" (p. 338-339, Girotto & Gonzales, 2008). Whilst understanding how to numerically quantify a probability involving equiprobable alternatives as ratio or fraction is likely to be learnt through instruction, the extensional assessment of probability as proportion of possibilities (perhaps in analogical sense) may not need an explicit instructional basis. Children may develop an intuitive theory through everyday interaction in the same way that they develop intuitive theories of quantities, collections, physical causality, etc.

Another intuitive assumption about probability is equiprobability. Empirical evidence that participants assume equiprobability comes from several sources. For example, Shimojo & Ichikawa (1989) reported that some participants who solved *the three prisoner problem* exhibited what the authors termed as the *number-of-cases* intuition. The intuition corresponds to a method for determining the probability of an outcome from a set of possibilities by dividing one by the number of alternative outcomes. The three prisoners problem does not provide definite prior probabilities of alternative outcomes. The intuition, which was derived through interviewing participants, is proposed to play a role in misleading participants' interpretation on how to solve the three-prisoner problem. Further evidence for assumed equiprobability comes from an experiment reported by Johnson-Laird, et al. (1999). Recall that the assumption of equiprobability is a principle of mental models theory of naive extensional reasoning. The researchers tested the hypothesis that participants would assume equiprobability using variations of a marble-box problem, in which a model of alternatives was given as an inclusive disjunction (as show below), but the form of the question differed between problems. The different question forms used in their experiment can be formally stated as P(A); P(A and B); P(A and not-B); P (not-A and not-B). The problems allow one to predict equiprobability from the solutions given. For example, the equiprobable answer for the problem in Figure 3.6 is 67%.

There is a box in which there is a black marble, or a red marble, or both.

Given the preceding assertion, according to you, what is the probability of the following situation?

In the box there is a black marble with or without another marble.

Probability: %

Figure 3.6. Example of a problem used by Johnson-Laird, et al. (1999) to investigate equiprobability assumptions.

The authors found that participants did not question the omission of outcome probabilities and solutions given for problems involving different question forms tended to match those predicted from assuming equiprobability. The observation that participants did not question equiprobability does not mean they did not entertain this issue. Conventions in assumptions of equiprobability with random probability devices such as dice and roulette wheels in education and culture may allow participants to justify the assumptions of equiprobability with little concern.

3.4.2 Task and external representation in PPS

External representations have been considered critical to PPS learning and problem solving. Conventions in teaching probability typically involve guidelines for the conditional use of representations. For example, Cheng (2011) derived representation procedures for different PPS task based on a survey of the educational literature. According to his analysis, different representations are normally instructed with different kinds of problem conditions.

Although, there have been a number of studies testing the controlled effects of different ERs and information presentation in PPS, there have been few studies which examined the independent selection and use of ERs in PPS tasks. In two of experiments, Corter and Zhaner (2007) and Zhaner and Corter (2010) examined these factors using a range of different probability problems. In one study Corter and Zahner tested graduate students on a set of eight probability problems. The students had received initial teaching in probability on an introductory statistics course, which had included the use of various ERs. The problems that were tested comprised of four types: combinations, sequential, permutations and conditional probability problems. The authors found that students used a variety of representations for different problems. These forms of ERs were classified and the prevalence of their use in problem was computed. The classification and prevalence reported were spatial reorganisation of given information (96%), pictures (85%), novel schematic representations (65%), trees (84%), outcomes listings (39%), contingency tables (8%) and Venn diagrams (4%).

Corter and Zahner (2007) claimed to find regularities in the use of ERs, for example: pictures with sequential, combination and permutation problems, trees with conditional problems, re-organisations with conditional and combination problems, and novel schematics with permutation problems. Although these specific findings are likely to be dependent on many task factors (e.g., problem type, user knowledge, etc.), they suggest participants do depend on different ERs to solve the task and that ER selection is task dependent. Participants in the study sometimes used multiple ERs for a problem, apparently showing signs of changing ERs and this varied for different problems. The authors found that the use of ERs was differentially related to performance in complex ways. In some cases, the use of ERs was found to be negatively related to correct performance, whereas in others, they were found to be positively related.

3.4.3 Novice strategies and solution procedures in PPS

In the following subsection we consider research and accounts of solution methods and strategies employed by novice participants in PPS tasks. Characterisations of solution procedures have been proposed at different levels of specificity. At the most abstract level, some researchers have focused on the general phases of PPS over broad classes of problems. Zhaner and Corter (2010) propose a stage model of PPS involving (a) text comprehension, (b) problem representation, (c) strategy formulation and selection, (d) strategy execution, and (e) solution checking, which occurs only sometimes. The authors conducted a study involving 34 participants solving 18 problems of different types (i.e. Joint Events, Conditional Probability, Independent Events, Combinations, Fundamental Principle of Combinatorics, and Permutations) in which verbal protocols of participants were used to estimate the order and interleaving of processing phases. The authors report time estimations based on the number of utterances. According to their findings, participants spent over half of the PPS time on problem representation (56%), and only 5% on text comprehension, whereas the remaining time was approximately equally divided on strategy formulation (19%) and execution (20%). The authors also report that participants tended to follow the order just specified with some back and forth transitions, mainly between problem representation and strategy formulation in iterated solution attempts.

Gigerenzer and Hoffrage (1995) suggested three ‘cognitive algorithms’ by which people may arrive at normative Bayesian solutions to conditional probability problems. These include: (a) algorithms that accord with specification of Bayesian formula, (b) algorithms that accord with a simplification of the Bayesian formula (shortcut algorithms), and (c) algorithms that involve diagrammatic or pictorial representations, but that accord with a Bayesian specification of the formula (pictorial algorithms).

The authors describe a shortcut algorithm which involves incorrectly omitting and relating the quantification of sets of outcomes. Such algorithms may apparently be selected under conditions in which the problem data would provide a similar result if it were done using Bayesian calculus. For example, the so called rare event shortcut (p. 690, Gigerenzer and Hoffrage, 1995), which was derived from participants' protocols, involves taking the complement of P(E & ~H)/P(E & H) as the solution if P(E & ~H) is less than P(E & H). An example of a pictorial algorithm observed from an experimental participant is also described. The algorithm involves the use of a beam diagram in which portions of the beam correspond to the scaled quantities of sets specified in the sample space. The cognitive algorithm they describe involves segmenting (cutting out in the example) base rate then hit rate and false alarm portions of the beam then conjoining them in such a way that they correspond to the formula *P(E&H)/P(E&H) + P(E&~H)*. In addition to Bayesian algorithms, the authors also report the use of non-Bayesian algorithms which they sub-classify as joint occurrence, Fisherian, and likelihood subtraction. Joint occurrence was reported to be the most common non-Bayesian algorithm, which involves taking the solution as the *P(H & E)* or *P(H)P(E|H)* and, thus, neglecting the false alarm rate.

Although Gigerenzer and Hoffrage (1995) classify scratchings and resulting solution as belonging to several categories of solution procedures, which are specified as mathematical statements, there is little detail or explanation in their accounts. The authors choose to classify solutions in terms of Bayesian or non-Bayesian methods, but this classification is questionable because one can arrive at a normative solution as given by the Bayesian calculus through methods that are not Bayesian, but are correct in the normative solution sense as stated by others (e.g., Johnson-Laird, et al., 1999). The classification may be appropriately described as solution procedures that generate Bayesian equivalent solutions or not.

Another theoretical approach that makes specific claims about the nature of solution procedures is the mental models theory. The theory's commitments about solution procedures are mainly specified in terms of its principles. Based on these principles and other assumptions solution procedure commitments can be expressed by the following rules:

- If a premise is given then construct the set of mental models of the premises in which each model represents what is true in a true possibility (*truth principle*).
- If certain rare circumstances hold, construct fully explicit models which represent what is false in a true possibility.
- If no information is given about the probability of alternatives in the premise then assume by default that each model is an equiprobable alternative (*equiprobability principle*).
- If numeral probabilities are given in the premise then tag models with the relevant values (*numerical principle*).
- If equiprobability is assumed then the probability of an event is calculated by the ratio of the proportion of models in which they occur.
- If the problem is conditional and equiprobability is assumed then the conditional probability P(H|E) is taken to be the proportion of H that is the subset of E relative to E (subset principle).
- If the problem is conditional and frequencies are given for alternatives in the premise then calculate the conditional probability P(H|E) as the proportion of H that is the subset of E relative to E (subset principle).

A similar characterisation of solution procedures has been proposed by Fox and Levav's (2004). These authors explicitly present a strategic account of extensional reasoning about conditional probabilities, which they call *partition-edit-count*. Simply put, the solution procedure model proposes that (see p. 637, Fox and Levav, 2004):

- The sample space is subjectively partitioned into a set of elementary possibilities. The term subjective is used to highlight that the partitioning is sensitive to various factors such as the presentation of the problem.
- Any possibilities that can be eliminated based on conditional information are edited out.
- The possibilities that remain are counted.
- The probability is then determined by taking *"the ratio of the number of focal events to the total number of events*."

The procedural account is argued to be supported by a series of experiments conducted by the authors that manipulated the accessibility of partition and edit information in the problem description, which systematically influences predicted partition and editing procedures derived from solutions and informal protocol analysis.

Accounts of solution procedures, such as mental models and partition-edit and count, are abstract about the specificity of people's action knowledge for PPS tasks. For example, is the subset principle of mental models as readily applied to all cases of set structure in conditional problems? If not, behavioural predictions are not only over general, but the knowledge possessed by individuals does not correspond to the level of abstraction implied by the theory.

3.4.4 Types of errors

Whilst abstract features of solution procedures have been proposed in particular cognitive theories of PPS, these accounts are largely uncommitted to information processing properties of solution procedure formulation and implementation. As the different phases of problem solving are implemented in a resource bounded cognitive system, how different requirements of task (e.g., problem comprehension, formulation, planning and execution of solution procedures) are co-ordinated and implemented in terms of particular cognitive and perceptual motor events is a critical part of explanations of task performance. As discussed, performance differences as a function of task conditions (i.e. ER, descriptive framing) may be determined by a number or combination of information processing issues.

Research relevant to these kinds of questions was reported by O’Connell (1999), who described the results of a study investigating the nature of errors made by novice participants in PPS tasks. The author used an initial classification of types of errors using a larger sample (N = 180) of students solving 93 problems of different types. At the most general level errors were classified as belonging to one of four categories: (a) text comprehension/misunderstanding errors; (b) procedural errors that result from the “faulty applications of formulas or rules”; (c) conceptual errors that result from “difficulties with probability concepts”, and (d) arithmetic errors that result from mistakes in calculations. The authors identified 110 errors and classified the errors into types of 8 text comprehension, 10 procedural, 11 conceptual and 1 arithmetic. Of these, the most common errors made by participants were procedural (44.7%), followed by text comprehension (22.7%), conceptual (19.1%) and arithmetic (9.0), and the remaining were unclassified. These finding reveal that there are different reasons for generating an incorrect solution other than solution procedure formulation (conceptual errors in their terminology) and that one needs to be mindful about distinguishing between different causes of solutions. The high frequency of procedural and arithmetic errors (and possibly text comprehension) suggest that the coordination and monitoring of the results of cognitive processes in a resource bound cognitive architecture is also a significant factor in explaining PPS performance.

3.4.5 Summary

Few clear details have been discerned about what knowledge is employed in novice PPS, how the knowledge is realised in the cognitive system of the problem solver and how people use the knowledge to compose and implement solution procedures. Abstract theoretical commitments typically correspond to common sense knowledge or known normative instructions about how probability word problems should be solved (although this is partly due to their intuitive status). In order to understand PPS and specify cognitive models, many more details about it need to be determined. A critical point worth noting is that theories do not address how novice participants formulate solution procedures in PPS tasks, but instead only address what solution procedures participants are predicted to execute. Solution procedure formulation is a particularly salient feature of the tasks given that the experimental problems are typically unfamiliar, conceptually challenging and time consuming for individuals to solve.

3.5 ER accessibility and solution procedure formulation in PPS

The chapter has outlined cognitive research on accessibility effects of external problem information in PPS. Accessibility effects in PPS tasks arise from different information sources including text descriptions, numerical ERs, diagrams and concrete physical manipulations of problem outcomes. Alternative theoretical accounts have been proposed to explain PPS or subcategories of PPS, such as conditional probability. All accounts make predictions about how the accessibility of certain classes of problem information facilitate correct performance, but differ in the scope and specific classes of information considered relevant.

3.5.1 Project relevant research limitations

A number of limitations of current theories and accounts can be identified. These limitations are opportunities to be addressed by the thesis.

- Theoretical accounts of PPS make few commitments about the nature of people's knowledge, how the knowledge is represented and organised, how they actually solve tasks, and what information processing constraints govern their cognitive behaviour.
- Conceptual problems in theoretical accounts of PPS have concerned central empirical issues of what classes of information influence PPS performance and how to classify solution procedures. The confusions are arguably a result of a lack of systematic analysis of the cognitive task and representation combined with impoverished empirical information about the actual process of PPS.
- Whilst accessibility effects are central prediction of PPS accounts, proposals about how external information interfaces with the internal processes responsible for observed performance facilitation are absent. As discussed in Chapter 2, accessibility of information in ERs may occur for a number and combination of reasons.
- Research on accessibility effects does not distinguish between incorrect solutions that result from the formulation of solution procedures from those that result from the execution of solution procedures. Results in such studies typically suggest an effect of solution procedure formulation, but those that result from execution per se are not distinguished. This prevalence of different kinds of formulation and execution errors is supported by O'Connell's (1999) study.
- Research conducted to address accessibility effects of representation in PPS have generally proceeded by examining the solutions generated by participants rather than the 'process' by which participants generate and apply the solution procedures.
- Theoretical accounts, whilst making some abstract commitments about predicted solution procedures in PPS, do not specify why people chose to use them. Solution procedures in PPS have a logical explanatory structure. For example, part of the solution procedure for determining conditional probabilities can be

justified by determining the possibilities that could occur within sets relating to prior posterior outcomes.

- Theoretical accounts do not address what is arguably the most central characterisation of PPS performance in accessibility experiments, namely, how people figure out and evaluate how to go about solving a problem. Instead, theories merely identify abstraction of solution procedures that are purported to be used.
- Empirical research on accessibility effects of PPS has been largely focussed on a narrow subclass of probability word problems (i.e. conditional probabilities) and the extent to which performance matches irrelevant Bayesian solution procedures.
- Empirical research in PPS, particularly concerning accessibility effects on performance, has been largely driven by attempts to test and confirm abstract domain independent theories or theoretical approaches rather than develop specific models.

3.5.2 Project relevant constraints

The research review has also provided the following information that can be used to inform cognitive models of PPS and the study of accessibility effects.

- Accessibility of problem information can influence performance through several different mediums, including the framing of content of text description, numerical ERs, diagrams, and concrete physical manipulation of problem elements.
- Accessibility of problem information has been demonstrated to influence different components of solution procedure formulation, including how to partition the problem and determining whether to rule out possibilities.
- Accessibility effects through diagrams have been observed for different classes of problem information including set structure, frequency structure, and possibly relations of proportionality.
- Novice participants possess an intuitive schema/s for understanding probability as a proportion of equiprobable alternatives (Johnson-Laird, et al., 1999). The schema may have a non-instructional basis (e.g., Girotto & Gonzalez, 2008).

- PPS involves several problem solving phases including comprehension, representation, solution procedure formulation and execution. Meta-cognitively demanding activities such as solution procedure formulation and problem representation constitute significant portions of problem solving episodes and may involve iterated attempts (e.g., Zhaner & Corter 2010).
- Evidence suggests that solution errors in PPS result from a combination of factors, including the correct comprehension, formulation, and execution of solution procedures (e.g., O'Connell, 1999).

3.5.3 Proposals

Analysis of the PPS research coupled with research on diagrammatic reasoning and other considerations support the following tentative proposals of relevance to the empirical and analytical enquiry developed in this thesis.

- PPS is not (normally) a monolithic cognitive task, but involves integrating naive theories and action schemas about sets, proportions, chance/possibility, belief and perspective taking together with more specific elementary knowledge of procedures used to calculate probability and solve mathematical word problems that are learnt in school. Solving the kinds of PPS tasks used in experiments is not simply a case of initiating or selecting a stored cognitive algorithm. The conceptual integration and co-ordination of these cognitive theories/schemas are a major meta-cognitive burden to the problem solver. Realistic accounts of non-routine PPS performance should be sensitive to these factors.
- Research demonstrating that participants have default assumptions about probability and solution procedures suggests that certain errors may result from failing to recognise their misapplication. Such proposals implicate an import role for meta-cognitive skills used to monitor and interrupt cognitive activities in the light of inconsistencies in PPS tasks.
- At the most abstract level, information accessibility is capable of influencing performance in three kinds of problem solving phases: formulation, evaluation and implementation of a solution procedure. Distinguishing effect in different phases is a critical step for understanding information accessibility in PPS.

- Errors in the execution of a plan may result in it being side-tracked, delayed and ultimately resulting in generation of an incorrect solution. Errors may include a failure to carry out necessary problem solving steps, mistakenly executing an unplanned step or retrieving incorrect information about the problem situation. Such errors will include what O'Connell (1999) classified as procedural and arithmetic errors.
- Solution procedure formulation and evaluation is facilitated by opportunities conferred by diagrams to hypothesise and evaluate explanations of solution procedures. As described in Chapter 2, such advantages depend on the presence and accessibility of properties of auto-consistency.
- The accessibility of information may affect solution procedure execution by cueing meta-cognitive processes that interrupt actions and direct attention to inconsistent problem information and actions plans. Such accessibility may arise through combinations of cognitive mechanism, task requirements and representation properties.
- Accessibility effects in PPS should be understood in the resource bounded cognitive architecture in which they occur. It is proposed that limited processing resources interact in the generation of accessibility effects. For example, freeing up cognitive resources may allow the scheduling of meta-cognitive processing or attention (not just visual) to problem critical information in an ER.
- Accessibility effects should not only determine the correctness of the solution, but also the particular problem solving trajectory and determined processing timings in a problem solving episode. Detailed measures of the 'process' of PPS are central to specific accounts.

Chapter 4: Problems and Tasks

The first aim of the chapter is to discuss the content and presentation of the problems used in the research, including the methodological constraints and rationale for choosing the problems. The second aim is to outline empirical and analytically based predictions of how the problems are solved, including: the nature of the task and underlying constraints of the task environment, the information and knowledge required for solving the task, and in what ways cognitive strategies may differ. The problems are used, with minor variations, in both experiments reported in subsequent chapters (Chapter 5 and Chapter 6), and the specific computational models are discussed in the remaining chapters. This section, therefore, describes core assumptions that underpin the experimental and modelling methodology discussed in the remainder of the thesis.

This chapter will be divided into four sections. In section 4.1 the abstract structure and content of the probability problems will be outlined, including the chosen probability scenario, the nature of the problem instruction and the informational structure of alternative probability problem situations. This will be followed by section 4.2, outlining a specification, analysis and empirical justifications on the way the problems were presented to address the research questions, including the chosen format of the problem instructions and representation of the problem situation. In section 4.3, the meaning of problems, implied interpretation and the assumed knowledge possessed by naïve probability problem solving (PPS) participants shall be outlined. In section 4.4, the nature of task shall be outlined, including a consideration of solution procedure (SP) errors.

4.1 Probability problems

4.1.1 Problem scenario

The problem scenario of the probability problem employed in the PPS experiments refers to their particular story context. Commonly employed classes of PPS problem scenarios include medical diagnosis, games and randomisation artefacts. Performance on PPS tasks are likely to be influenced by the complexity and familiarity of the problem scenario. As discussed in Chapter 3, the problem scenario employed in many PPS studies are often complicated and may be unfamiliar to naïve PPS participants. In such cases, participants need to make all kinds of unpractised inferences specific to the problem scenario in addition to the abstract structure of the probability problem. The potential implication is that observed performance, particularly negative performance that is taken as a measure of participants' ability to reason normatively about probabilities, may depend erroneously on the familiarity of the problem scenario rather the abstract probability problem structure.

A single problem scenario was chosen for all problem instances employed in the research. The chosen problem scenario required estimating the probability of spinning a multi-sided letter spinner. The main reason for choosing a randomisation artefact was that it was assumed to be familiar to all the naive PPS participants in the study. Randomisation artefacts are commonly employed in games and education based probability problems and people are likely to have developed schemas for thinking about them. Randomisation artefacts also appear to be intuitive to understand. The prevalence and history of randomisation artefacts in different cultures is presumably partly due to the 'relatively' uncomplicated way that they afford understanding and extensional reasoning about chance and probability. This perhaps has something to do with them being both concrete instruments for implementing chance effects and at the same time (3D diagrammatic) external representations of chance in a manner that systematically integrates both perspectives of user functions. In summary, the choice of a single, familiar and intuitive problem scenario will reduce the potential confounding cognitive burden of using unfamiliar problem scenarios. It will also help to meet constraints on determining a tractable methodology for empirical analysis and cognitive modelling outlined in the introduction by, for example, reducing the time taken, complexity and heterogeneity of strategies in solving the PPS problems.

An important constraint in the design of the experiments is that participants should solve the problem using the data structure of problem situation embedded in the representation. Hence, the choice of the letter spinner subclass of the randomization artefact, rather than a more familiar randomization artefact such as die, eliminates the possibility of using prior knowledge of the data structure of the problem situation (e.g., the number and identity of the sides of a die) and possible solution instances (e.g., the probability of an unbiased throw is one in six) to solve a problem. As the letter spinner is essentially analogical to common randomisation artefacts like die, it is assumed that participants will have no problem in generalising their declarative knowledge and skills of more familiar instances. In keeping with this constraint concerning the use of prior knowledge, the model/data of the spinner also needs to be different for each problem instance so that knowledge of the problem situation and specific solution instances memorised in previous trials of the experiment cannot be applied in subsequent trials. Different models should therefore vary in terms of the number and identity of letter sides of the spinner.

4.1.2 Problems instructions

The term *problem instruction* as used in the PPS literature refers to a part of the word problem that can be distinguished from the graphical representation of the problem situation. Problem instructions involve an abstract specification of the actions and goals to be achieved and are normally expressed using natural language. In the studies reviewed, they also normally involve some specification of the data of the problem situation, which can also be represented diagrammatically. The distinction between instruction and representation of the problem situation depends on a combination of types of format (linguistic vs. diagrammatic) and content (intentional actions/goals vs. extensional problem situation/data structure), although this is not clear cut.

Heterogeneous problem instructions. Unlike the vast majority of existing PPS research that focussed on conditional probability problems (see Chapter 3), this research aimed to address more heterogeneous problem instructions, which required derivation of the structure of set and probability relations from the data of the represented problem situation for different problem solving goals. The motivation for this aim is based on the hypothesis that accessibility effects of representations in PPS are general to a broad range of inferences about the structure of the problem situation, as they also appear to be in non PPS domains. This is in contrast to the almost exclusive concern with conditional probabilities problems reported in the research literature. This hypothesis is also consistent with a model of the use of generic problem solving skills and cognitive resources in PPS as proposed in Chapter 3 rather than specialised processing system accounts of PPS tasks as proposed by authors such as Gigerenzer and colleagues (e.g., Gigerenzer et al. 1995).

A further motivation for employing heterogeneous problem instruction types concerns the potential for determining a more specific understanding of the information processing requirements of PPS tasks and accessibility effects. A cognitive model is arguably more prone to gloss over internal information conditions used by the cognitive system to select actions when the possible courses of actions to be modelled are limited. Different models of a task can generate the same processing steps with different specification of internal representations/information conditions. Internal information conditions specified in cognitive models (i.e. operator specificity) can be either over or under specific and it is difficult for a modeller to specify with some accuracy what information is being used without considering different contexts for which related knowledge may be recruited. Indeed, extending the tasks/problems of a process model may often require making more specific distinctions about the internal information conditions in order for the model to reproduce appropriate behaviour over the alternative problem conditions. Generally speaking, it is assumed that the more problems that need to be modelled the more constraints are likely to become available about the functional nature of modelled knowledge (i.e. its processing role in different problems). These constraints therefore can arguably help the modeller gain a more specific and systematic understanding the relationship between knowledge and information processing and improve the potential validity of a process model.

Partitioning goals and data. A further aim of the design of the problem instructions was to minimise the amount of information about the problem situation in the problem instruction so that the problem instruction is more exclusively a specification of the goals, whereas that the representation/diagram is more exclusively a specification of problem situation needed to solve the problem. Hence, unlike previous PPS studies, the problem instruction should not contain sufficient information to solve the problem alone and participants should need to use the representation of the problem situation for this class of information. An important advantage of this scheme is that the task is more controlled with respect to limiting variation between participants in their chosen degree of reliance on information presented in either the verbal instruction or the graphically represented problem situation. It also ensures that the main locus of accessibility effects of information about the problem situation reside in the graphical representation of the problem situation, which is of course the central focus of investigation.

Scaled-down problems. A last general issue for the problem instructions is to satisfy constraints on methodological tractability. Solving probability problems similar to those reported in the research literature is conceptually demanding and may engage participants for long periods. For example, Gigerenzer and Hoffrage (1995) report the total time spent by participants in solving their set of problems, which works out as equivalent to spending approximately 5 minutes on average per problem. Researchers often reported that even relatively simple problem solving tasks initiate a wide range of cognitive and perceptual-motor strategies in participants. The problem instructions thus need to be sufficiently demanding to evoke some degree of reflection and the need for a solution procedure formulation (SPF), but not too demanding so as to evoke overly long solving times and highly heterogeneous problem solving strategies. Minimizing the amount of strategic variation is a necessary requirement to allow some degree of structure to be obtained from detailed protocol analysis and permit the development of pragmatically tractable process models. The problems also need to be made feasible for people to solve on a display without having to make external notes because this controls the representational strategies and makes the use of eye-movement protocol analysis more feasible.

Problem instruction design. The design of the problem instructions were formulated to satisfy the stated research goals and methodological constraints. All of the problem instructions used in the research involved estimating the probability of an outcome of a single trial. There are five types of problem instructions: simple, queried disjunction, queried conjunction, conditional and unequal probabilities. These types can be analysed in terms of three instruction dimensions:

- Whether the prior probabilities of possible outcomes in the trial are equal or not.
- Whether conditional information about the category of the trial outcome is to be assumed or not.
- The number of attributes and types of operators used to specify the category of the queried outcome.

Table 4.1 shows the three dimensions of variation of problem instruction and the set of possible classes of values for the problems used in the research. A key feature of the scheme is that alternative problems are differentiated by the value of a single instruction dimension. The scheme has the methodological advantage that the interpretation of participants resulting solutions (i.e. a probability ratio) can be more clearly attributed to particular types of solution procedure (SP) because the space of incorrect SPs for each problem is minimised.

SPF framing of instruction design. The ‘simple’ problem instruction can be viewed as a kind of baseline problem because its values arguably constitute the most culturally prevalent type of instruction (i.e. equiprobable, non-conditional problems involving a single category query) whose solution procedure is predicted to be generally familiar to naive PPS participants. One may view the simple instruction dimension values (cell values of Table 4.1 without shading) as corresponding to practised problem attributes of the PPS schema of naïve PPS participants. For the non-simple instruction dimension values (shaded cell values of Table 4.1), we would expect that corresponding problem attributes are not practised.

<table>
<tr><th></th><th colspan="3">Instruction dimension</th></tr>
<tr><th>Problem type</th><th>Priors?</th><th>Stated outcome?</th><th>Queried outcome?</th></tr>
<tr><td>Simple</td><td>Equal chances</td><td rowspan="2">No unique category</td><td rowspan="3">Single category</td></tr>
<tr><td>Unequal probabilities</td><td>Division of unequal chances</td></tr>
<tr><td>Conditional</td><td rowspan="3">Equal chances</td><td>Single unique category</td></tr>
<tr><td>Queried conjunction</td><td rowspan="2">No unique category</td><td>Conjunction category</td></tr>
<tr><td>Queried disjunction</td><td>Disjunction of categories</td></tr>
</table>

Table 4.1. Types of problem instructions used in the research as defined by the values of their instruction dimensions.

4.1.3 Problem data

The data of the probability problems may vary in terms of the specific variables of the problem situation namely the number of outcome possibilities, categories of outcomes and their probabilities. Values of the problem data determine higher-order set and probability relations. These higher-order relations are potential experimental manipulations which influence the difficulty of problem solving because they vary in terms of the requirements of information processing (e.g., processing steps, required working memory resources and executive/process scheduling control) and because of their differential familiarity, which determine whether SPF is required.

Design specifics of the problem data related to an instruction type are different in experiments 1 and 2; hence, these will be discussed in those respective chapters. There are, however, some methodologically motivated generalities about the data structure of the problem situation with regard to both experiments that are worth mentioning at this point. The required arithmetic complexity of a probability problem which depends on routine procedures and required cognitive resources of SP execution can be distinguished from the interpretive demands of SPF. Whilst the former may influence the latter, experimental designs aimed at minimising effects of arithmetic complexity should more clearly reveal the details of representational influences of SPF in task performance. Recall in Chapter 2 that this is one of a number of confounds that could arguably contribute to the proposed difficulty of problems reported in a number of influential PPS studies that were attributed to information format rather than the data.

In addition (as stated) problems should be solvable online without external workings in order to make performance on the task more tractable to analyse and model. As a result, three properties of the problem data should be constrained. Firstly, a sample (or frequency) frame is desirable for the problem data such that outcomes are given in terms of sample tokens rather than relative probability values of category occurrences. Secondly, the number of sample outcomes to be dealt with should be kept relatively minimal. Thirdly, for problems involving unequal probabilities the simplest case should be preferable i.e. a simple division of two unequal sets with the relative proportion being 2:1.

4.2 Problem format

All problem presentations involved a text based specification of the problem instructions and a corresponding PS-diagram (probability space diagram) representing the data structure of the problem situation.

4.2.1 Problem instruction format

In order to simplify the instruction encoding component of the task, allow participants to easily access text frame values on repeated occasions without having to search though elaborate text, and to increase the accuracy of identification and interpretation of participant's eye-movements, the problem instructions in the experiments were formatted using a generic text frame. The text frame scheme also has the advantage of not requiring a complex model of text processing to model participants' performance. The text frame in both experiments comprise of three fixed frame slots with different values for each problem. The specific frame used in experiment 2 is shown below. The first (top) line specifies whether the outcomes should be considered equal or not using the frame slot "Probabilities are | | letters:" with values could be equal (i.e. "equal for all") or unequal (e.g. "double for red") depending on the trials instruction type. The second line specifies if and what conditional information is known using the frame slot "The spinner falls on a | |" with values that could be unconditional (i.e. "letter") or conditional (e.g., "consonant letter"). The third line states the question using the frame slot "What is the probability the letter is:" and values, which could be a single letter category (e.g., "G"), a single letter property (e.g., blue), a conjunction of letter properties (e.g., "red and small") or a disjunction of letter properties (e.g., "large or blue or both"). Table 4.2 shows the text frame with values of a simple problem instruction used in the second experiment. The first experiment involves an almost identical text frame with small differences to the wording.

Instruction statement type	Frame entry	Instruction value type	Frame value
Priors statement	*Probabilities are | ? | letters*	Equal probabilities	*equal for all*
		Unequal probabilities	*e.g. double for red*
Outcome statement	*The spinner falls on a | ? |*	Non conditional	*letter*
		Conditional	*e.g. red letter*
Query statement	*What is the probability the letter is | ? | ?*	Simple	*e.g. large*
		Conjunction	*e.g. blue & small*
		Disjunction	*e.g. red or large or both*

Table 4.2. Text frame entries, values used to specify problem instructions.

4.2.2 Problem situation format

Experimentally manipulated variations in the format of PS-diagrams were chosen to represent the problem situation. PS-diagrams were chosen as a diagrammatic format for the research for three main reasons. PS-diagrams are representational systems specifically designed to support learning probability and in certain contexts are considered to improve on the use of traditional heterogeneous systems for such purposes (Cheng, 2011). Firstly, compared to traditional forms of representation used in probability, PS-diagrams provide a more comprehensive scheme that represents set relations using both referential identity and containment relations, and represents probability relations using geometrical configurations. For single trial problems, the information represented spatially includes the same information as in icon diagram representations of single trial probability situations such as those used in the study of Brase (2009) and the roulette diagram of Yamagishi's (2003) study. Recall that in both studies the corresponding diagrams facilitated the best PPS performance. With regard to comparisons of the icon diagram, the PS-diagram improves by representing the probability of individual outcomes by the relative size of outcome units. Note that a single trial PS-diagram can be viewed as an extended icon diagram when used to represent an outcome sample (frequency frame) rather than the probability of outcome categories. With regards to the roulette diagram, the PS-diagram improves on information richness by representing both referential identity of outcomes (i.e. frequency frame) and their probability.

The information in PS-diagrams is argued to be accessible (or semantically transparent) for novice users. This claim is supported by a combination of representational analysis and empirical research findings on the use of the diagram. Specifically, in a longitudinal learning study Cheng (2011) found improved performance with participants trained on PS-diagrams compared to participants trained on a conventionally assigned assortment of traditional ERs (e.g., Venn, network diagrams and contingency tables) or algebraic notations. The results of Brase (2009) and Yamigishi (2003) also support the accessibility because the strongest diagrams in their study involve representing spatial (set) and geometrical (probability) schemes that are subsumed in the PS-diagram.

A second reason for choosing PS-diagrams, is that the semantic richness of the format allows the scheme to be degraded in a more systematic way to assess the presence and accessibility of represented information than conducted in previous studies using conventional or ad hoc ERs. Thirdly the choice of using PS-diagrams rather than ad hoc ERs with similar levels of required information integration, is because PS-diagrams are capable of modelling much more complex problems than investigated in this research, which would allow extensions of research to complex problems to exploit existing research developments with PS-diagrams (e.g., a process model of solving simpler problems using PS-diagrams could be extended to solve more complex problems rather than starting again from scratch using a completely different ER). In addition, it is notable that PS-diagrams are a purposely designed ER for learning probability in education such that studies involving them could potentially have greater applied significance.

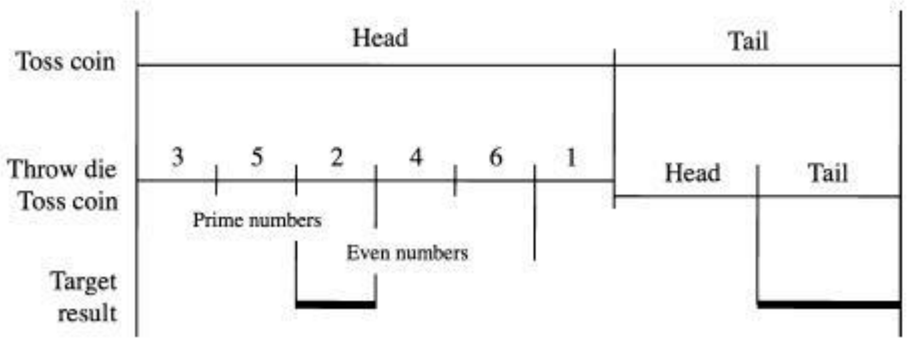


Figure 4.1. Cheng's (2011) sample of a PS-diagram, (p. 482).

Format and semantics. A canonical PS-diagram is shown in Figure 4.1. An important characteristic of PS-diagrams is that they represent within a unitary system many of the critical relations and constraints (or axioms) of the domain. It simultaneously allows meaning to be derived from alternative perspectives and levels of abstraction that are normally only partially available through traditional representational systems. As this research only employed single trial probability problems, the following description of the PS-diagrams will be limited (for a more complete description see Cheng (2011)).

In the complete system, probability problems are modelled in a two dimensional space in which trials are represented across the vertical dimension and outcome instances or classes within a particular trial are represented across the horizontal dimension. For our purposes, only a single trial diagram will need to be described. Each segment in a trial represents one of the collectively exhaustive set of outcomes. The data structures of the problems used in the experiment involve a sample framing of outcomes although the system is capable of representing a non-sample framing. The total width of the conjoined outcome segments represents the total probability of exhaustive possibilities in the model and is called the probability space. The probability of any outcome is represented by the relative width of the representing outcome segment relative to the probability space. In modelling a problem situation, it is conventional to selectively arrange outcomes segments into neighbouring groups to spatially model abstract set relations between outcomes such as intersection, union, disjointedness. The scheme also includes marking lines below the probability space which mark out sets of interest to the problem. The containment relations between the marking lines also provide an additional abstract and perhaps more salient expression of set relations in a similar way to Venn and Euler diagrams.

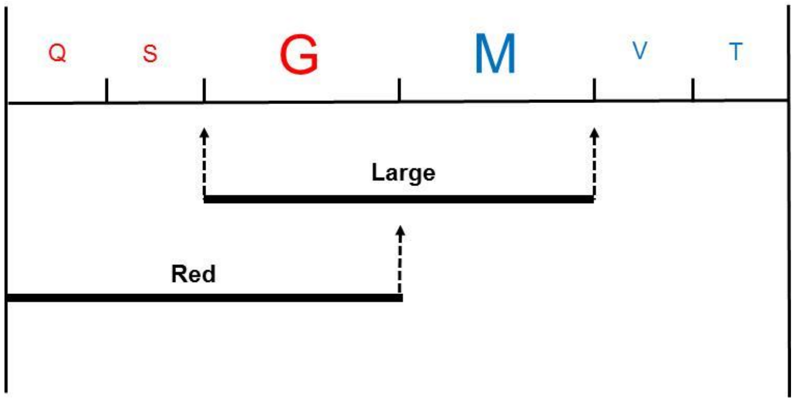


Figure 4.2. Format of PS-diagram used in experiment 2 with weighted outcomes and an overlapping data structure.

Structural constraints The diagram provides a relatively accessible consistency proof of the problem situation. The consistency proof results from laws of probability and set membership that are represented by spatial and geometrical relations in the diagram. A complete specification of the laws modelled by the system can be found in Cheng (2011). Constraints on probability are revealed by a number of representational properties such as that *it is not possible to construct a complete representation where outcome partitions do not sum to unity,* which is an example of self-consistency. Another example of self-consistency is that the relative probability proportion is consistent irrespective of the derived proportion (e.g., proportion of *large* to the whole, *large and red* to *large,* etc.). Specifying the probability of an outcome determines the probability of remaining outcomes, which is an example of a free-ride. Constraints on set relations are revealed by self-consistency of expressions; for example, if A is a subset of B and B is a subset of C it is not possible to represent situations where A is not a subset of C. Similarly, in specifying the first two abstractions, the relation between A and C follows, which is an example of a free ride. As modelling in PS-diagrams proceeds by specifying the probability and category of outcomes, all set relations that result are free-rides.

Accessibility of relations and constraints. As argued, the presence of constraints does not determine their accessibility. This is influenced by whether the user is in possession of a theory/schema of the spatial scheme embedding the constraints. Quantitative probability is represented in traditional ERs such as tree diagrams and contingency tables using numerical symbols, often expressed in terms of relative probabilities rather than sample frequencies. There is good reason to hypothesise that the accessibility of the quantitative structural constraints of probability may also be critical to SPF in probability problems. As discussed earlier, evidence for this claim derives from the studies of Brase (2009) and Yamagishi (2003), who both used representations that encoded the relative proportion of probability, although they attributed performance facilitation exclusively to the accessibility of either set structure of the icon format (i.e. a token reference representation of a sample frame). In the PS-diagram, probability relations are embedded within part-whole configurations between any nested set of outcomes derived in the diagram. The particular part-whole configuration which underpin probability constraints is arguably intuitive because it taps in to geometrical knowledge of a class of related schemes for interpreting proportional structure that apply to everyday interactions with a range of physical objects and artefacts.

Set relations are represented in traditional PPS ERs such as Venn and tree diagrams. As discussed, the apprehension of nested sets is argued to be a critical factor in determining performance in extensional reasoning about conditional probabilities. In the PS-diagram, a range of set relations can be derived between outcomes by either relations of referential identity or containment relations between groups or marker lines. Set relations represented in the diagram (e.g., proper subset, disjoint, overlap, intersection, complement, union) are required to compute the possible reference of individuals in sets in accord with instruction criteria such as conjunction, disjunction and conditional. The way set relations are modelled via relations of referential identity and relations of containment are assumed to be intuitive in the sense that participants would be expected to have well practised schemas for interpreting and reasoning about sets within these spatial domains.

4.3 Concepts and knowledge

4.3.1 Meaning of probability problems

Explicating key assumptions about the meaning of the classes of probability word problems used in the experiments, including intentional (conceptual) and extensional (models) aspects, is viewed as a preliminary methodological requirement for the characterisation of observed measures of interpretation in the experimental tasks and in implementing knowledge for cognitive models of participants' performance. This is particularly critical because of the conceptually difficult nature of probability.

Probability word problems typically require the quantification of possible states of affairs. Whilst solution procedures may not always be based on reasoning from initial principles (assumed constraints) of the problems, the proposed SPF requirements of the experimental problems are assumed to depend on the recruitment of such principles. Reasoning from first principles in probability word problems implicates abstract reasoning about category identity, the differentiation of alternative hypothetical models, the co-coordination of meta-cognitive or self-relating intentions towards models, and assumptions about past/future and definite/indefinite state of affairs. Even in the relatively simple experimental problems the required abstractions and subtle distinctions in conceptualisations of problem information is complex when analysed.

Category abstraction and set structure. In the research problems, querying the probability is a request to determine the quantification of the possibility of a category of outcome occurring in a hypothetical model of a trial. Only an outcome category is being queried rather than the referential identity of a particular outcome individual. The particular category of the outcome is a minimal specification rather than unique specification of the outcome category such that any alternative with the minimal description is considered a possibility irrespective of other categories it may have. Categorical queries of probability are a convention in probability word problems. In the experimental problems these conditions enforce that solutions require determining the referential identity of the queried category, which in turn implicates the domain of sets because set relations are functions required to determine the referential identity from a category description. Note that it is only because probability word problems conventionally query categories rather than referents that the derivation of set relations are critical to solving such problems.

Interpretation of models. According to Johnson-Laird et al. (1999), each alternative/possibility in extensional reasoning about probability is represented as a distinct mental model. Whilst there is reason to be sceptical about, at least, the generality of this claim (see Chapter 3), it is assumed that the experimental problems require differentiating models/states of affairs of outcome categories for implementing particular subtasks. The use of the term *model* here is understood in terms of what it represents (i.e. a distinct state of affairs) rather than how it is represented (e.g., a specific representing format in the mind).

4.3.2 Prior PPS knowledge

Theoretical accounts of PPS typically do not factor in what specific knowledge naïve PPS participants do and do not posses about the PPS domain. Johnson-Laird, et al., (1999) specify a set of principles of extensional reasoning of probability, but does not commit to an interpretation of how these principles are manifest in cognitive processes. Fox and Levav (2004) propose what appears to be an account of reasoning about conditional probabilities, which roughly corresponds to a hypothesised SP for conditional PPS. Whilst the solution procedure provides constraints on the interpretation of procedural knowledge, little can be gleaned about the underlying declarative knowledge and prior assumption possessed by naive PPS participants.

The experimental problems have been designed based on the assumption that participants have learnt through previous experience how to solve a stereotypical class of probability problems which involve:

- a sample (or frequency) framing of outcomes
- outcomes are equiprobable
- the probability of single trial is queried
- no conditional information is given
- the queried outcome category is identified by a single attribute

These problem features are common of probability problems involving random artefact scenarios such as dice, roulette. The type of problem described is equivalent to the experimental baseline problem instruction specified in section 4.1.2. This stereotypical PPS schema is proposed to implicate the following interdependent default assumptions that may not be explicitly considered by participants unless the problem instruction specifies conditions perceived to be unusual. These assumptions are important in interpretation of PPS performance and in their initiation in cognitive models.

- *Sample referents as outcome surrogates*. Outcomes are a particular kind of an event construal, namely the resulting state of a trial event. However, under this assumption, at least with certain kinds of problems, "distinct" possible outcomes are taken to correspond to the distinct referential identity of sample objects in the problem situation (e.g., side of a die, cards of a pack, balls in a bag). This default assumption may play a part in so called frequency format effects because naïve PPS participants are more able to apprehend a solution procedure when it is presented in a frame consistent with this assumption (e.g. Brase 2009).

- *Outcomes have equal probability.* This assumption corresponds with Johnson-Laird, et al.'s (1999) equiprobability principle - namely that participants assume that possible outcomes in certain stereotypical word problems are to be taken as equiprobable as a matter of course. Recall that in Johnson-Laird, et al.'s study participants were observed to assume equiprobability without question in problems despite the absence of any information about the likelihood of possible outcomes. Whilst probabilities are explicitly stated in the experimental problems, the assumption is important because it is proposed to be a central condition for the initiation of equiprobable quantification procedures.

- *Probability is a proportion of sample referents.* This is the assumption that the numerical values of a probability ratio are the count of sample outcome referents. In other words, quantification of probability is derived from the number of outcome referents rather their likelihood. This assumption which concerns the quantification subtask is dependent on the last two assumptions, but is additionally required to solve the problem.

4.4 Task and strategy

The aim of the section is to characterise the nature of the experimental task, identify key task requirements, constraints, critical diagrammatic information that may be relevant to solving the problem and possible sources of SP errors for each problem type discussed below.

4.4.1 General task structure

Cheng (2011) distinguished between modelling, interpretation and calculation phases of PPS with PS-diagrams. Zhaner and Corter (2010) propose a stage model of PPS involving (a) text comprehension, (b) problem representation, (c) strategy formulation and selection, (d) strategy execution, and (e) solution checking, which occurs only sometimes. Analysis of the experimental problems conform roughly to both Cheng and Zhaner and Corter proposals with the exclusion of the modelling phase because the diagrammatic models are pre-specified.

At the most general level of the experimental problems, two phases of problem solving can be roughly distinguished: possibility determination and probability quantification, with the former generally preceding the latter. Both include interpretive and procedural/SP execution components. Possibility interpretation depends on the representation of set relations because set relations need to be computed to determine the referential identity of possible individuals in models of the categorical specified problem instruction. The relevant SPF components of the disjunction, conjunction and conditional problem types are therefore primarily concerned with formulating an interpretation of possibility. Quantification of possibility, on the other hand, depends on the representation of probability. The biased probability problem type is intended to address SPF in the phase of possibility quantification.

At a more specific level, the task of solving each of the different experimental problems can be broken down into several general subtasks including comprehending the instruction frame, identifying corresponding sets of referred categories, using the diagram to determine logically possible models of the problem instruction, quantifying the probabilities, and reporting the answer. The analysis excludes checking behaviour which may occur for different subtasks. With the exception of the reporting subtask, each subtask is performed in the service of a subgoal to acquire a particular class of information. Interdependent constraints exist between information requirements of subtasks; for example, possible outcomes of interest must be identified before being quantified. Due to the information compositional nature of the task, alternative SPs can be differentiated at the subtask level so that each subtask can be viewed as a SP component.

4.4.2 Task requirements

Table 4.3 shows the instruction type and corresponding SPs for the different kinds of problems. The SP for each instruction type is general in the sense that it applies to all types of data structures, which have different relations between referent sets of interest. Each SP corresponds to a solution to a sub-problem that can be read-off directly from the PS-diagram. Table 4.3 shows visualisations of the solution read-offs applied to three data structures involving the different types of set relations, overlap, proper subset and disjoint. Note that information in the diagram can be used to determine correct solutions in less direct ways that rely more heavily on internal cognitive algorithms.

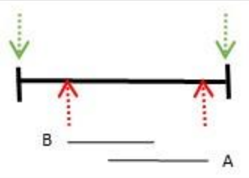
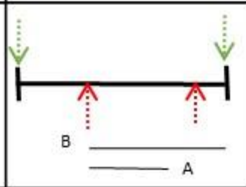
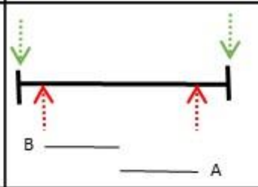
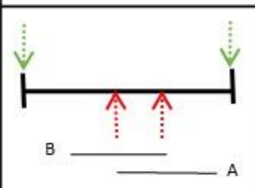
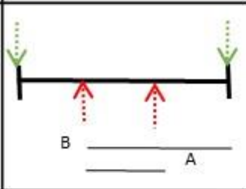
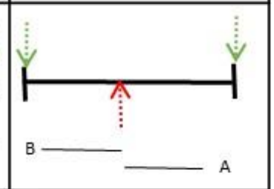
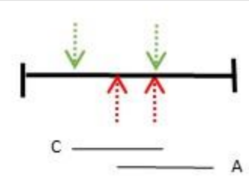
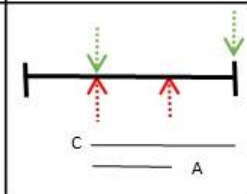
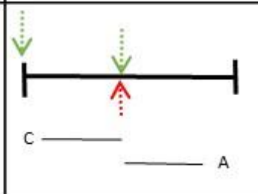
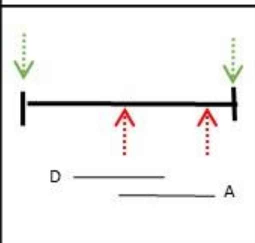
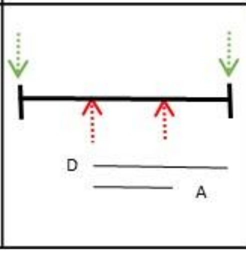
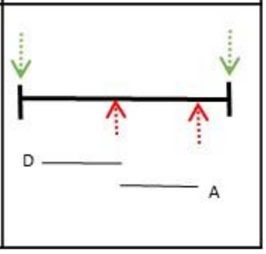
Instruction type	Data structure	Read off/ embedded solution form		
	SP component	Overlap	Subset	Disjunction
Queried disjunction P(A or B)	A ∪ B			
Queried conjunction P(A and B)	A ∩ B			
Conditional P(A|C)	A ∩ C | C			
Biased P(A) assuming P(D} = 2* P{~D}	||A|/|U||			

Table 4.3. SP component for instruction type and data structure for the problems. Images depict solution read-offs in a PS-Diagram for each data structure. The simplest expression of the SP using spatial/geometrical operators are shown on the left.

Conditional problem instruction. The solution to the conditional sub-problems are referential and require limiting targets set to C. The general referential solution that can be read-off for the sub-problem is the intersection A and C relative to C (i.e. A∩C|C). On conditional/disjoint and conditional/subset problems the procedure of limiting A to C is required. On conditional/overlap and conditional/subset problems the procedure of U to C is required.

Queried disjunction instruction The solution to the disjunction sub-problems are referential and require determining the referents in alternative models of the queried category. The general solution that can be read-off is the union of the target sets. In disjunction/subset and disjunction/overlap, participants are required to take the union in each model. However, in the disjunction/disjoint problems either the union or the conjoined sets (which can be summed) will provide the correct solution because there is no intersection between A and B.

Queried conjunction instruction. The solution to the conjunction sub-problems is referential and requires determining the set of referents that have both categories A&B specified in the conjunction statement for numerator quantification. Hence, the referential solution that can be read off is AUB. In conjunction/subset and conjunction/overlap the set AUB need to be quantified. In the disjunction/disjoint there are no members of AUB so the probability is zero.

Biased outcomes. The solution to the corresponding sub-problem is quantitative and involves deriving the proportion of members in the queried category A relative to U. There are two procedures available: (1) taking a normalised frequency of A to U (i.e. |A|/|U|), or (2) reading off the geometrical fraction of A to U (i.e. ||A|/U||). For all biased problems determining the denominator value requires normalising two different sets in U. In the biased/subset and biased/disjoint only one set needs to be normalised in A. In the biased/overlap problem two sets need to be normalised for A.

4.4.3 Solution procedure formulation

The different problems should require SPF with respect to sub-problems associated with the problem specific instructions. In Chapter 2, we hypothesised that the constraint related properties and the accessible representation of structural constraints possible in diagrams may facilitate SPF by supporting abduction and evaluations of candidate SPs. SPs can be viewed as functions that are partially coded into structural constraints of diagrams as illustrated in Chapter 2 (e.g., a lined up array of object representing quantity by size can encode an addition function). Under this view, the representing scheme modelling the structurally coded function may be advantageously exploited by users in explaining an SP correspondence of the represented function. Mapping between the arguments (input) and solution (output) information simultaneously represented by a diagram are related by acts of derivative meaning relating these states (i.e. a visual demonstration/proof of consistency). The representing structure is judged as explanatory by virtue of its consistency. Consistency is a property of diagrams that analogically model representing structural constraints. A function (i.e. SP) that can be derived in a diagram can explained through its structural constraints if those constraints are judged to be consistent – that is, if the users treats the representations as a consistency proof of the relevant function.

It is hypothesised that explanations will be actively sought by users to justify SPs in unfamiliar problem contexts. The representing scheme carrying the structural constraints can therefore influence the formulation of SP hypotheses if that scheme is sufficiently accessible and its consistency can be judged. If users are able to derive an SP consistent function analogy in the diagram this could facilitate the construction of a candidate SP and its evaluation by allowing the diagram to be a test bed for an SP hypothesis.

The results or solutions of SPF for the problem instructions are the intersection of A & B for the conjunction problem, the union of A and B for the disjunction problem, the subset of A in B relative to B for the conditional problem, and the weighted sum of outcomes in the unequal probabilities problem. The input data used to compute these functions are the sets in the diagram referred to in the experimental instructions. Possible derived sets are partitions of the represented state of affairs, which may corresponds to |A & ~ B| ,|B & ~A|, |A & B| depending on the data structure. Meaning derivations that relate the input and solution in the scheme used to explain why a general or a data specific sub-case of an SP holds. For the conditional probability problem, it is the derivation of possibility constraints from set structure of derived referents in the queried and assumed outcome categories. For the disjunction problem, it is the derivation of shared referential identity between derived models of referents in the categories of the disjunction statement. For the unequal outcomes, it is the derivation of a uniform scheme for adding outcome probabilities and deriving their relative proportion. Functions corresponding to SPs of different problem instructions have different degrees of structural encoding in the PS-diagram. For example, SPF for the unequal probabilities problem appears to be most strongly supported in the PS-diagram – partly because a number of constraints of the arithmetic functions for probability quantification are structurally coded in the PS-diagram scheme.

4.4.4 Solution procedure errors

In Chapter 3 we reviewed research which indicated that SP errors can result for a number of different reasons including SPF. For example, O'Connell (1999) observed errors that were classified as belonging to one of four categories: (a) text comprehension/ misunderstanding errors ; (b) procedural errors result from the "*faulty applications of formulas or rules*"; (c) conceptual errors that result from "*difficulties with probability concepts*"; and (d) arithmetic errors result from mistakes in calculations. Given our particular framing and terminology SP errors are considered to result in instruction comprehension, SPF and SP execution.

- SPF errors occur in the mapping of the instruction interpretation to a solution procedure plan. SPF errors result from a failure to critical consider logical constraints between instruction interpretations and hypothesised solution procedures. As discussed, the diagram may be used to guide SPF by revealing constraints on the mapping between the instruction and the SP. The formulation errors may be likened to what O'Connell (1999) calls conceptual errors.

- Misinterpretation of problem instruction errors may occur when the user incorrectly interprets the intended meaning of the problem instruction. Misinterpretation errors may come from a failure to integrate the meaning of statements (e.g., conditional and query), confuse values of referred categories as well as misunderstanding the intended meaning of individual expressions (e.g., what models are being queried by a disjunction). As full instruction interpretation does not need to be immediately formed upon reading the corresponding text (participants may encode the lexical information required for determining a postponed interpretation), the distinction between problem instruction interpretation errors and SPF errors is somewhat vague. Such cases, whilst exploiting constraint on interpretation, may result in errors in the absence of checking procedures because this class of strategies implicate initial superficial encoding of instructions.

- SP execution errors occur in carrying out of a solution procedure. Even if a solution procedure plan is correctly formulated errors can also be made in executing that solution procedure. There are a range of errors that could result. For example, procedures implicated by a previously formulated plan may not be carried out, inappropriate procedures may be incorrectly implemented that were inconsistent with an initial plan, and participants may also incorrectly retrieve or select categories (set identities) and instances (e.g. mathematical facts), which result in errors. SP execution errors therefore include what O'Connell (1999) call procedural and arithmetic (which may also be procedural).

As the main goals of the research concern SPF, we will focus on outlining the main predicted SPF errors associated the different problem instructions and leave execution and text interpretation errors, which implicated more numerous and varied possibilities. For conditional problems, predicted errors in SPF include (a) failing to limit A to $A \cap C$ and (b) failing to limit C to U because of an absence in interpreting the conditional nature of the problem instruction and/or implications of the set structure between A, C and U. For conjunction and disjunction problems, predicted SPF errors include problems in translating the referential criteria for possibilities of the queried categories from the queried statement. For biased problems, predicted incorrect SPF errors include failing to normalise or correctly normalise the frequency possibilities from sets A and/or U to determine the proportion.

4.5 Summary & conclusion

The chapter has outlined the design of the experiment problems and the main motivation for choices of the problem types and choice of the format. The problems designed were less complicated than those used in previous studies, but helped to overcome a number of methodological problems associated with interpretation problems identified in experimental designs used in previous studies (e.g., unfamiliar problem scenario, linguistic and arithmetic complexity, presentation and representation confounds, etc.) and in making the particular research methods to be used in this research more tractable for the use of fine grained protocol analysis and computational process modelling. The chapter has also specified assumptions about the nature of the task including a characterisation of the conceptual distinctions central to the problems, an overview of the structure of the task, the task requirements for each SPF critical part of the problems, how the diagram is predicted to support SPF for different problems, the predicted errors in SPF, and assumption about prior knowledge of participants. The details unpack a number of assumptions to be addressed and, where appropriate, built on in the research. The analysis of the problems and tasks also reveal the methodological feasibility of these components of the experimental framework for addressing the main research aims, that is, understanding the process of diagrammatic accessibility effects on SPF formulation.

Chapter 5: Pilot Experiment

5.1 Introduction

The main aim of the pilot study was to test whether the types of problems and presentation format were feasible for the research goals. That is, whether the problems were sufficiently challenging to require solution procedure formulation (SPF), but at the same time simple enough to permit methodologically tractable protocol analysis and cognitive modelling. A second aim of the experiment was to test a manipulation of the accessibility of represented information in determining the form of a solution procedure for the different types of problems. A third aim was to test an experimental design required to separate these effects of SPF from effects on the execution of a solution procedure. The last main issue was to test out and assess the interpretability and informativeness of using eye-movement and verbal protocol analysis for the particular experimental task. The detailed rationale for the design of this pilot and main experiment are described in Chapter 4.

5.2 Method

5.2.1 Participants

The study involved eleven participants who were either postgraduate (n = 9), undergraduate (n = 1) or a research fellow (n = 1) at the Department of Informatics at the University of Sussex. Six participants were male and five were female with a median age of 27 years.

5.2.2 Problems

The experiment involved a class of three simple practice problems and three of each of the four types of experimental problems: disjunction, conjunction, conditional and biased problems, as described in Chapter 4. The three instances of each problem type always involved a different data set. All problems involved a spinner scenario and referred categories in the problem instruction were either letter types, colour or consonant/vowel status of letters. The problem instructions were presented within a frame scheme as described in Chapter 4, which includes invariant text entries for the priors statement (i.e. Probabilities are:____), outcome statement (i.e. Spinner falls on: _______), and query statement (i.e. What is the probability of:_______). The instruction frame values and spinner data models used for each problem and trial type are shown Table 5.1 and an example of a trial display is shown in Figure 5.1.

Problem type	**Trial type**	**Priors**	**Outcome**	**Query**	**Model**
Disjunction	formulation	equal	unknown	blue or vowel	|F|Q|O|E|K|
	transfer -	equal	unknown	consonant or red	|U|E|G|N|K|U|
	transfer +	equal	unknown	consonant or blue	|U|W|X|E|O|
Conjunction	formulation	equal	unknown	consonant and blue	|U|P|G|T|O|E|
	transfer -	equal	unknown	vowel and blue	|Z|C|A|U|E|O|J|N|
	transfer +	equal	unknown	consonant and red	|E|A|K|L|M|N|O|
Conditional	formulation	equal	blue	E	|A|B|C|D|E|F|G|
	transfer -	equal	consonant	G	|O|K|G|H|E|
	transfer +	equal	red	A	|F|U|V|A|X|Z|
Biased	formulation	double red	unknown	Z	|E|Z| K | Q |F|O|
	transfer -	double consonant	unknown	D	|A| D | Q |J |E|
	transfer +	double blue	unknown	T	|U|T| M | E |S|

Table 5.1. Instruction frame values and spinner model for each problem and trial type in the pilot experiment. The columns contain values of the prior statement, outcome statement and the models in the diagram.

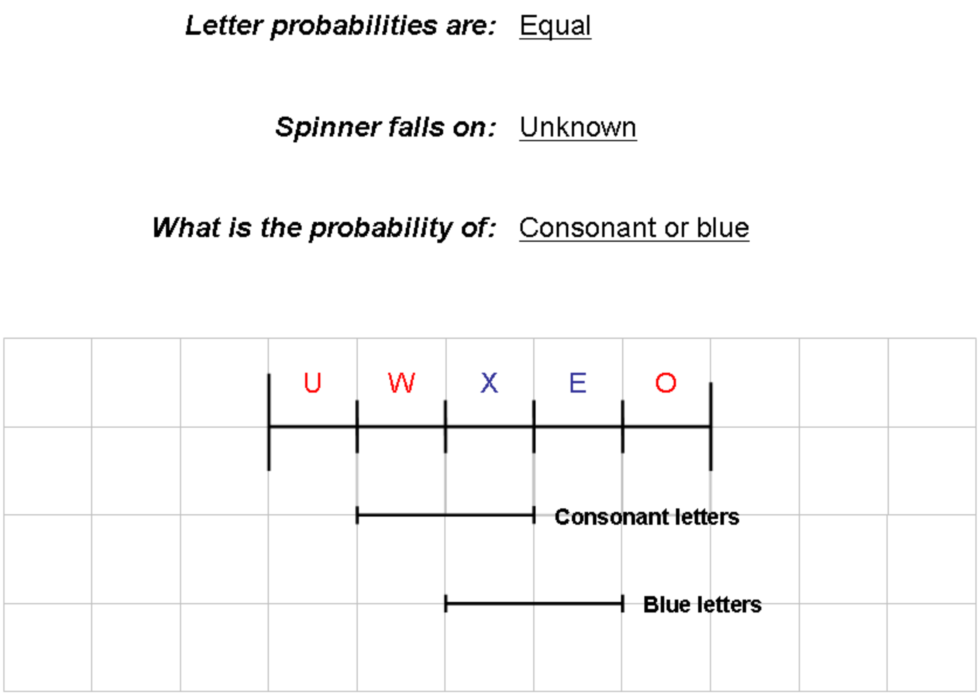


Figure 5.1. Example of a trial display used in the pilot experiment.

5.2.3 Design

The experiment involved a manipulation of representational accessibility. The manipulation involved testing the presence or absence of representational features on solution formulation. The features manipulated depended on the problem type. For conjunction, disjunction and conditional problems, where the sub-problem involves determining possibilities from the set structure of referred categories, the features that were either present or omitted were the marker-lines that expressed the set structure of referred categories. For the biased problems, which involve a quantitative sub-problem, the feature that was either present or omitted was the representation of probability, which is coded geometrically by the relative width of PS-units (probability space units). The marker-lines should facilitate interpretation because they highlight the set structure of the target sets of three problem types through a 1-D form of spatial containment. The geometrical representation of probability should facilitate solution procedure interpretation in the biased problem because it expresses relative probability of outcomes in a part-whole structure. It was hypothesised that the feature present condition would facilitate interpretation of the correct solution procedure for each of the problem types such that performance time would be less and a correct solution would be greater for the feature present group members.

The experimental manipulations also have effects on strategic possibilities used for the execution of some subtasks. For example, the marker lines allow sets to be identified and related without considering the categories of individual letters of PS-units. The geometrical representation of probability allows the geometrical proportion to be read-off without identifying the probability of individual PS-units and computing the denominator/ numerator from these details. In an attempt to separate the effect of the experimental manipulation on interpreting a solution procedure from the effects of just executing one, additional trials involving each of the same type of problems were also included in the experiment.

Specifically, the problems were presented in blocks containing three trials of each problem type. The first trial of each block is called a formulation trial because it is the trial where participants are predicted to initially derive a solution procedure. The second and third trials are called transfer trials because they are trials where participants are expected to transfer the solution procedure formulated in the formulation trial. The transfer trials can be viewed as a type of control or baseline trial. The performance difference between formulation and transfer trials is predicted to represent a difference between the requirement to formulate and execute a solution procedure (formulation trial) or just execute a recalled procedure (transfer trial). Performance differences include trial time, solution correctness and potential interpretation processes indicated via eye-movement and verbal protocols. All participants were required to complete two transfer trials on each problem type: one with the target feature present (transfer +) and one with the target feature absent (transfer -). The rationale for having the participants complete one of each transfer trial is that it allows between group differences in just executing a solution procedure to be compared under both representational conditions. Note that all trial instances involve different data, so no problem instance is the same. The structure of the experimental design is shown in Table 5.2.

Groups	Problem Type Block		
	Formulation	Transfer -	Transfer +
Feature Present	+	-	+
Feature Absent	-	-	+

Table 5.2. Illustration of the design of a problem type block used in pilot experiment. Cell values indicated whether the trial had the target feature present (i.e. +) or absent (i.e. -).

To explore the effects of different performance measures, two groups of participants conducted the task under different instruction/measurement conditions. Six participants performed the task on a head-mounted Eyelink eye-tracker without making verbal protocols. This group were instructed to solve the problems as quickly as possible and provide a solution as soon as they were sure of it. A second group of five participants performed the experiment on a Tobii head-free eye-tracker and were instructed to provide ongoing verbal protocols, but unlike the first group no time constraints were instructed for task performance as this constraint could interfere with the production of think-aloud protocols. The former condition would potentially provide a more sensitive measure of problem solving time, whereas the latter should potentially provide a more sensitive measure of interpretation strategy. Note that measure conditions were crossed with the experimental representation condition.

5.2.4 Procedure

Participants were given elaborate written experimental instructions, which indicated how to interpret the instruction frame for each problem type. The experimenter then checked that participants understood the task by asking them to provide a solution to a printed example in the experiment instruction. Participants first completed a block of practice trials then completed four blocks of experimental trials for each problem type. Blocks of problem types were performed in the following order: conjunction, disjunction, conditional and biased. Problem type trials were performed in blocks of consecutive trials to maximise transfer. A fixed order of problem type blocks was used, so possible transfer/practice effects between problem types would be the same for the small group of participants used in the experiment as an exhaustive counter balanced design was not compatible with the sample size.

5.3 Results

Details of the pilot experiment will only be outlined broadly due to space and relevance constraints. The section will focus on performance in the formulation trials, specifically the nature of solution procedure errors, and the allocation of visual attention and effects of the measurement and representation conditions. As protocols were generally sparse for the five participants instructed to give them; analysis of them will be omitted.

A generic template was used to segment diagram elements into interest areas. Specifically, each PS-unit, set-marker/label pairing, statement entry and value corresponded to a unique interest area as shown in Figure 5.2. A template for processing the eye-fixation data was developed in Excel that took a model of the content of the presentation and assigned codes identifying attributes of fixated objects.

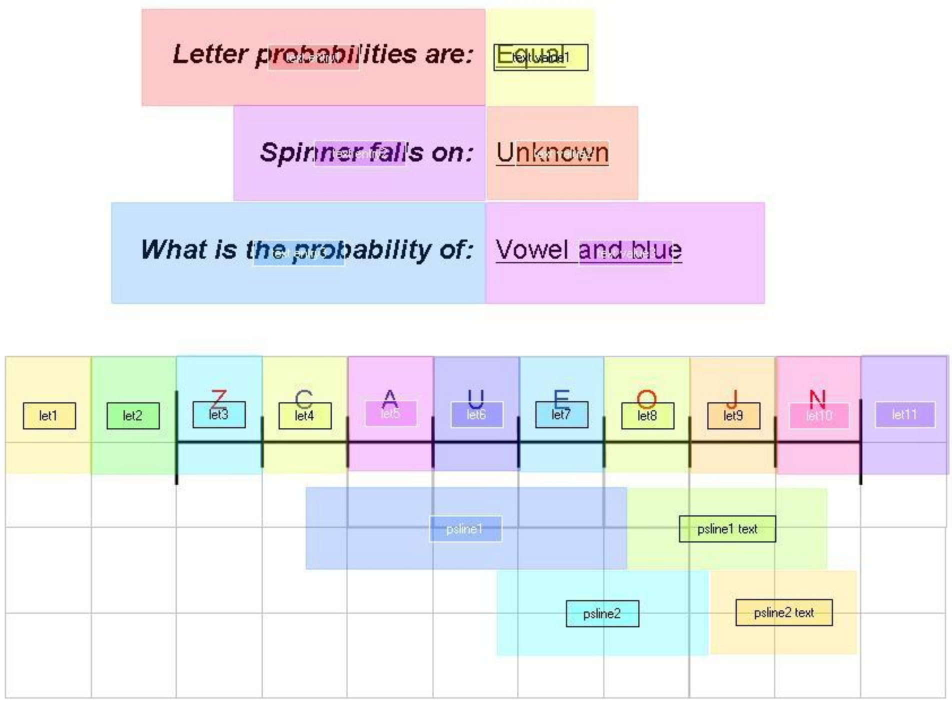


Figure 5.2 Interest area template used for eye-movement analysis in the pilot experiment

Initial inspection of scan paths revealed a difficulty in their interpretation because of the number of eye-movements in a trial. To support interpretation, an alternative visualisation of eye-movements was developed in a purpose specific software application. The alternative visualisation represents the position of eye-movements only on the horizontal axes, but represents the type of objects by the colour of the eye-movement icon. Hence, white icons are fixations to PS-units, green icons marker lines and the different shades of burgundy correspond to different statements in the problem instruction. The vertical axes encodes the time, order and duration of eye-fixations. A snapshot of the visualisation software is illustrated in Figure 5.3B and the scan path for a subject on the conditional formulation trial is shown in Figure 5.3A. The interest area segments of the corresponding problem display has colour coded blocks at the top of the scan path visualisation. Following the scan path from the start at the top one can observe that the first eye-movement is to the PS-unit D, then the two brief fixations left at PS-unit B, then fixations to the priors text entry and then the priors text values, and so on. At a higher-level, one can make observations like: (a) there is a first pass encoding of the problem instruction at the beginning of the trial lasting about 3 seconds; (b) the participants re-encoded text value on three occasions after first pass text encoding; (c) most of the fixations in the diagram are on the PS-unit with the letter E and, to a less extent, PS-units involving blue letters, both of which are the referred categories; and (d) there is a scan of the diagram starting at about 3.5 seconds and, again, at about 9 seconds that probably represents counting and so on.

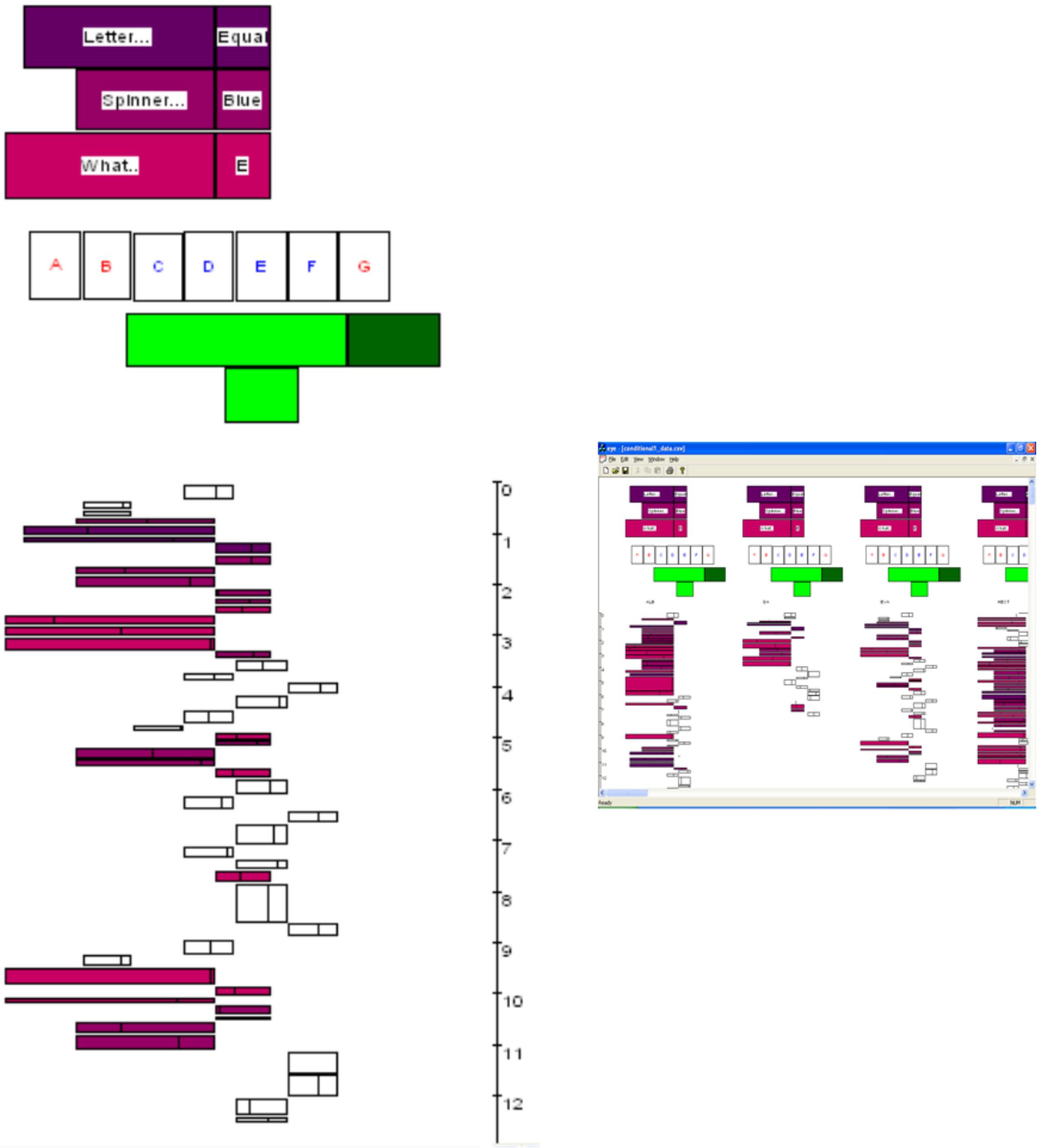


Figure 5.3. (A) A close up example of a scan path of on the conditional problem (left), and (B) a screen shot of the software developed to visualise the scan paths (right).

5.3.1 Solution Procedures

On average, participants gave correct solutions on 66 % of formulation trials and 70% of transfer trials. The graph in Figure 5.4 shows relative differences in correct solution probability between the two representation groups on formulation and transfer trials. The percentage of correct solutions was greater for the feature present than the feature absent group on formulation trials (80% vs. 50%) and transfer trials (75% vs. 65%), but this difference was pronounced on the formulation trials, where the representation conditions are different. The difference between feature absent and feature present groups in terms of the frequency of correct response for each participant on formulation trials was statistically significant ($t(9) = 2.6$, $P < 0.05$).

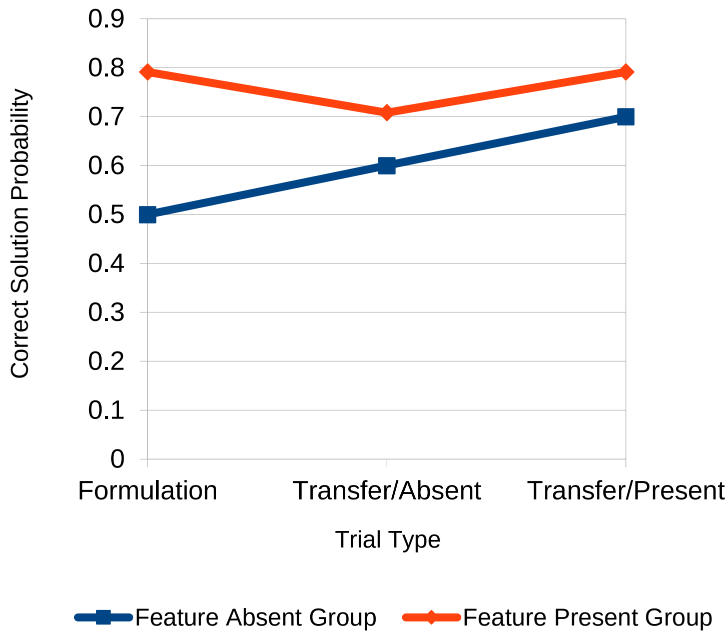


Figure 5.4 Solution probability as a function of representation group and trial type.

Table 5.3 shows the number of participants who gave a correct solution in each group for each problem and trial type. The participants who gave incorrect solutions are shown in parentheses. This data suggests, in general, that participants tended to use the same solution procedure across transfer trials. One can observe from the table that some participants repeatedly produce an incorrect solution. Indeed, 73% of the errors made by participants on the formulation trial were repeated on one or both of the transfer trials. Note that although there is repetition of errors for a problem by particular participants, different participants produce errors on different problems. Only one participant (S3) produced correct solutions to all problems. Details of solution errors are described below.

Correct and incorrect performance on each problem type is summarised below. Table 5.3 shows the frequency of participants who provided correct solutions.

- On the conditional formulation problem one participant in the feature absent group gave a solution consistent with incorrectly deriving the denominator from |U| rather than |C|, which implies failing to determine the conditional trial possibilities. Note that two additional participants (S8, S11) made this error on subsequent transfer trials.

Problem	Representation	Formulation	Transfer -	Transfer +
Disjunction	**Present (N=6)**	5 (S6)	5 (S6)	5 (S6)
	Absent (N=5)	2 (S9, S10, S11)	3 (S10, S11)	2 (S10, S11, S7)
Conjunction	**Present (N=6)**	2 (S1, S2, S5, S6)	3 (S1, S2, S6)	3 (S1, S2, S6)
	Absent (N=5)	2 (S8, S9, S11)	4 (S9)	4 (S9)
Conditional	**Present (N=6)**	6	5 (S4)	5 (S4)
	Absent (N=5)	4 (S11)	3 (S11, S8)	4 (S11)
Biased	**Present (N=6)**	6	4 (S1, S6)	6
	Absent (N=5)	2 (S7, S8, S10)	2 (S7, S8, S10)	4 (S7)

Table 5.3: Frequency of correct solutions for each problem, trial and representation conditions. The list in parenthesis identifies the participants whose solution was incorrect.

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- On the disjunction problem, one participant in the feature present group gave an incorrect answer compared to three participants in the feature absent group. S6 of the feature present group reported a probability consistent with |A|/|B|. Of the feature absent group, incorrect solutions were consistent with taking the numerator from |A| + |B| (S9), whatever set out of |A| or |B| is the largest (S10) and |A∩B| (S11). Note that these solution procedure errors were made consistently across transfer trials for S6, S10 and S11.
- For the conjunction problem, four (S1, S2, S5, S6) of the feature present and three of the feature absent group (S8, S9, S11) gave incorrect solutions. Incorrect answers were all consistent with participants taking the numerator from |AUB| rather than |A∩B|, except for S6, who reported a probability consistent with |A|/|B|. Four participants (S1, S2, S6, S9) continued to make the same error on transfer trials.

- For the biased problem, three of the feature absent group answered incorrectly. S7's solution was consistent with taking the denominator from $|\underline{D}|$[2]. S8 and S10 gave denominators that were close, but incorrect values to $|\underline{U}|$ suggesting they may have resulted from arithmetic or counting errors. Note, however, that on transfer trials the solutions of S8 and S10 are consistent with failing to normalise either |U| or |A|. In addition, some participants (S1, S6) also made this class of error in a transfer trial after providing a correct normalised solution in the formulation trial.

5.3.2 Problem solving time

As would be expected, the verbalisation group (M = 26) tended to take longer in seconds to solve the formulation problems than the non-verbalisation group (M = 16) and the difference was statistically significant ($t(42) = 3.655$, $p < 0.001$) and the trend was consistent across transfer trials. The problem solving time in seconds for formulation trials tended to be less for the feature present group (M = 18) compared to the feature absent (M = 23) group, but the effect was not statistically significant ($t(28.1) = 1.6$, $p > 0.05$), indeed this trend also reversed in transfer trials. When pooled together, the average problem solving times for the whole sample are highly similar for the formulation (M = 21), transfer-absent (M = 21) and transfer-present trials (M = 20). The average differences in problem solving time in seconds between problem types was shortest for the conditional (M = 16), similar for the conjunction (M = 20), disjunction (M = 20), and longest for the biased problem (M = 26).

5.3.3 External attention

For the formulation trials, overall participants spent on average 10 seconds attending to diagram elements compared to 7 seconds attending to instruction text and only about 1 second attending marker lines and/or the accompanying labels in cases where present (i.e. feature present trials). On average, only 1 second of time was spent attending to the blank areas surrounding the text and diagram suggesting that presentation information was in near constant use throughout the trial. Figure 5.5 shows the average relative

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[2]The notation $|\underline{X}|$ means the normalised cardinality of set X, see Chapter 6.

allocation of attention to classes of interest areas as measured by accumulated fixation durations for the different combinations of representation and measure conditions.

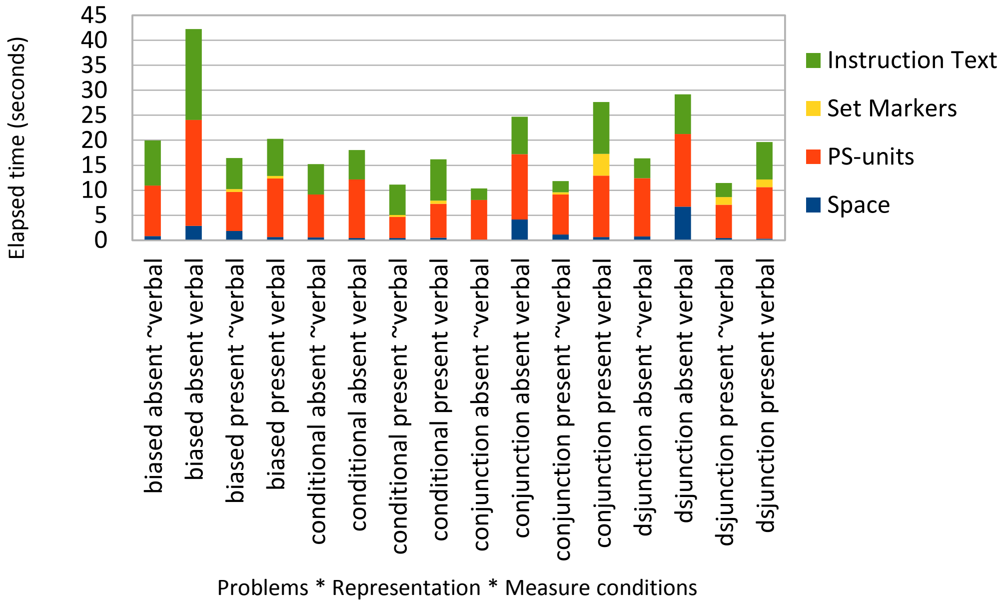


Figure 5.5. Average attention time spent on different classes of information element for the different problems, representation and measure groups.

Whilst participants tended to read the instruction before interpreting the diagram, participants appeared to refer back and forth between diagram and text in the course of solving the problems. On average, on a formulation trial, the number of switches of attention between PS-units and instruction text was 9, between PS-units and set markers 2, and between set markers and instruction text less than 1.

5.4 Discussion

The frequency of solution procedure errors made by participants suggest that the problems were sufficiently challenging, but the time taken to solve the problems suggest they are sufficiently tractable for the methodological goals of detailed process analysis and modelling. The analysis of solutions indicate a range of problem specific errors, some of which occurred for multiple participants suggesting potential regularities for the classes of problem. These results therefore suggest potentially interesting problems and errors worthy of further empirical investigation and analysis. Visual attention on presentation elements appears to be constant during the problem solving episode and frequent co-referencing of information between the diagram and text instruction indicate continual dependence of externally presented information in the service of limiting working memory requirements. Participants were able to use the diagrams and problem instructions and solve these problems online without making further notes. Collectively these results suggest that the generic problems types and the type of presentation format are adequate for addressing the main research goals.

The presence of marker lines were employed as an independent variable to distinguish conditions of the accessibility of set structure. However, for the formulation group, attention time to marker lines and co-referencing of marker lines were proportionally infrequent in comparison to PS-units. If marker lines were employed by participants to determine the reference of sets and set relations then one would arguably expect more attention and co-referencing than was observed. Attention to marker-lines may be influenced by their salience and connectedness to the array of PS-units, therefore changes to their connectedness may better support their consideration.

One of the main limitations with the design of the problems concern differences in accessibility of referred categories used in the experiment, i.e. colour, consonant/vowel status and letter form, which are not systematically controlled across formulation and transfer trials. The marked differences in accessibility were realised by modelling the process of set identification in the task using ACT-R (subsequent to the data collection), revealing that consonant/vowel status of letters needs to be retrieved from declarative memory, whereas colour and letter form comparisons require no such retrieval (at least in ACT-R) because colour is a visual property available in the visual buffer of attended visual objects that can be matched against the buffered search criteria held in a goal or imaginal buffer. In addition, grouping of objects into sets requires qualitatively different strategies as colour grouping can be performed in parallel by pre-attentive processes. Consonant/vowel status grouping, on the other hand, requires serial testing and aggregating of referred letters. The analysis suggest that modifications to the design of subsequent experiments must include either systematic control or balancing of the accessibility of referred categories.

In terms of solution procedure errors one of the most unexpected results was the frequency of errors on conjunction problems that were consistent with applying the union of A and B, which is the correct procedure for the disjunction problem. The results suggest that participants may have incorrectly transferred the solution procedure from the disjunction problems to the conjunction problems, as the later trial block immediately followed the former. A key similarity between the two problem types (other than both involving queries of two categories related by an operator) is that interpreting the disjunction query correctly (instruction interpretation) when followed logically implies interpreting all members of both A and B as the queried possibilities i.e. AUB (solution interpretation). Therefore, this incorrect generalisation may have arisen from a confusion between instruction interpretation and the corresponding referential solution interpretation between the two problems. The incorrect generalisation may also be influenced by the verbally economical format of the problem instruction frame, which could be improved to make the instruction more clear.

The data suggest that measurement conditions that include both verbal protocols and eye-movements are the best alternative for understanding facilitation effects of the representational format. This is because inspection of eye-movement using the graphical representation developed indicates that strategies and visual-spatial procedures are difficult to interpret from the data. The eye-movements provide useful constraints on interpreting the problem solving process, but this interpretation can be made more specific with verbal protocols.

Conclusion The initial pilot experiment provides an indication of the time scales of problem solving, the type and frequency of errors and the order and distribution of attention to presented information. The experiment has also provided information about the empirical feasibility of the problems, the problem presentation and alternative measurement conditions. The study has also indicated some potential limitations on the design of the initial pilot study including the lack of verbal protocols that could be due to the verbalisation instructions; possible errors that could be due to the economical nature of the instruction frame; and differing cognitive properties of the outcome sets (i.e. letters) that could differentially influence the accessibility of a set as well as the cognitive work involved in deriving a solution across problem instances.

Chapter 6: Main Experiment

6.1 Introduction

The main experiment was designed to gather eye-movement and verbal protocol data of the step by step process by which participants solved experimental problems with the problem and presentational dimensions described in Chapter 4. The study is an extension of the initial pilot and was motivated by the pilot study to adopt an experimental design compatible with a more in-depth analysis of the strategies and solution interpretation process. The results of the pilot study suggested that the class of problems were feasible for participants to solve in reasonable time without the need to make external notes or sketches. However, the time spent on reflecting about some of the problems by some participants coupled with the frequency and nature of solution procedure errors suggest that the problems chosen provided an appropriate balance between methodological requirements and research aims.

Representational role. The central, and first aim of the experiment was to investigate the role of the diagram in determining how to solve the problems under different problem instructions. Unlike the pilot study, the experimental design does not manipulate an independent variable of representation format. The absence of a representation manipulation was motivated by the assumption that it is possible to identify aspects of the use of the diagrammatic representation on the process of facilitating problem identification and solution conceptualisation from details of the protocols. However, such an approach does not provide comparative information about the degree of facilitation for solving the experimental problems for alternative representations. Research reviewed in Chapter 2 identified a number of studies in which enhanced facilitation on correct performance was observed with diagrams in comparison to text and alternative diagrammatic formats differing in the accessibility of goal relevant information. As such comparative effects have been robustly demonstrated, the aim of experiment 2 was to gather data exclusively about the process by which diagrammatic facilitation in probability problem solving (PPS) occurs.

It was predicted that the structural constraints and accessibility of the diagrammatic format, namely the iconic (or token referential) representation of possibilities, the containment representation of set structure, and the geometrical representation of probability, will be exploited by participants in two ways. Firstly, it is hypothesised that participants will employ read-off solution procedures to sub-problems afforded by the representation schemes in the PS-diagram (probability space diagram) as described in Chapter 4. The solution read-off procedures should be preferred over more abstract cognitively demanding solution strategies. The read-off solutions will be evidenced by verbal protocols indicating the identity of quantified and related sets in conjunction with the values of the solutions given. Secondly, it is hypothesised that the same structural constraints of the PS-diagram format will be exploited by participants in determining the form of the solution procedure to particular sub-problems. This will be evidenced by protocols that identify that a solution is directly read-off from the diagram in conjunction with protocols indicating that the solution interpretation was not known or planned in advance.

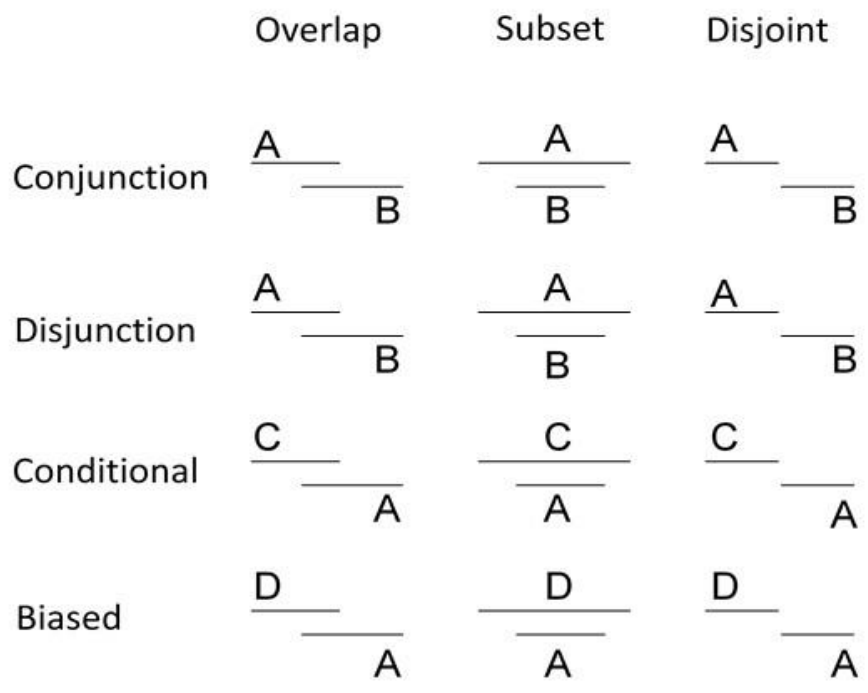


Figure 6.1 The set relation between members of the two referred categories of a problem were manipulated so that it was either disjoint, a proper subset or overlapping relation. Letters are referred category members i.e. A = first category of the query, B = second category of the query, C = category of the conditional outcome, D = category with double probability).

Set structure. A second aim of the experiment was to investigate the effects of the nature of the set structure of the problem data on performance. The instruction of each experimental problem type refers to two categories. To examine variation in set structure, the set relation between members of the two referred categories of a problem were manipulated so that it was either disjoint, a proper subset or overlapping relation. Hence, for the conjunction and disjunction problems the manipulated set structures were between A and B, for the conditional problems between A and C, and the biased problems between A and D as shown in Figure 6.1.

Recall that different types of set structures for the same type of problem instruction not only pose different cognitive requirements on solution procedure execution (e.g., cognitive steps, use of cognitive resources), but also impose different solution procedure requirements as specified in Table 4.3 (Chapter 4). As well as a general solution procedure for each problem type, there are also correct solution procedures for different data structures of a problem type. Take, for example, the conditional problem instances where the set structure is varied between the sets A and C. Note the general solution form for all problems is $|A \cap C|/|C|$. However, applying the form $|A|/|C|$ will provide the correct solution for the conditional/subset problem, but not for the conditional/overlap problem because the derivation of $|A \cap C|$ is not a requirement. For the disjoint problem, applying the form $|A \cap C|/|U|$ will result in the correct solution because the value of $|A \cap C|$ alone is sufficient to determine the correct value of zero probability.

It is hypothesised that the data structure of the problem will differentially influence the interpretation of the form of the solution procedure. It is also hypothesised that the diagram will support the derivation of a general solution procedure form over different data structure cases. Evidence for particular solution procedures applied by participants will be derived from participants' protocols, namely the verbalised identity of set relations and quantified sets.

6.2 Method

6.2.1 Participants

Eight participants were all educated to graduate level or above and either in postgraduate study (n = 6), working in post-doctoral research (n = 1) or in non-academic employment (n = 1). Five participants were male and three were female with a median age of 31. Participants completed questions about their mathematics education. Based on their reports, all had studied GSCE maths or equivalent, six had studied A-level maths or equivalent, five had studied intermediate statistics, and two had studied Bayesian theory. All participants were offered payment for their participation.

6.2.2 Problems

The experiments involved the same types of problem instructions and problems scenario (i.e. spinner trial) described in Chapter 4. There were three practice trials and twelve experimental trials. The twelve experimental trials comprised of the four types of problem instructions (conjunction, disjunction, conditional and biased) crossed with the three types of set structure (overlap, subset, disjoint). Each experimental trial therefore differed in terms of an instruction type or set structure of problem critical set. The spinner problem scenario always involved a spinner with letter sides, and referred categories in the instruction were always the visual properties colour (red or blue) and size (large or small) of the letters that are sufficiently similar in accessibility.

6.2.3 Presentation

The problem instructions were presented in a generic frame format using the same class of three statement types: probability, conditionality and query. The instruction frame used in experiment 2 and the types of frame values are shown in Table 6.1 (see appendix 2).

Statement type	Problem dimension	Frame statement
Priors	Equiprobable	Probabilities are (equal for all) letters
	¬Equiprobable	Probabilities are (double for D) letters
Outcome	¬Conditional	The spinner falls on a (letter)
	Conditional	The spinner falls on a (C letter)
Query	Single category query	What is the probability the letter is (A)?
	Conjunction query	What is the probability the letter is (A and B)?
	Disjunction query	What is the probability the letter is (A or B or both)?

Table 6.1. Types of frame statements slots and values (in brackets) used in experiment 2.

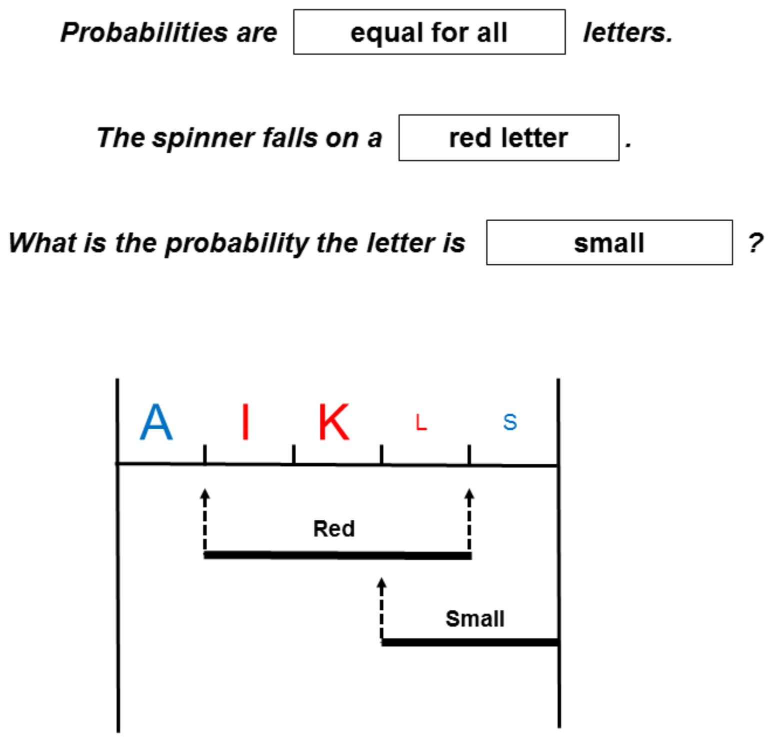


Figure 6.2. An example presentation for the conditional/overlap problem.

6.2.4 Procedure

All participants were given experimental instructions before completing the experiment (see appendix 1). The instructions contained an example of a simple problem and elaborate explanations of how to interpret the frame meanings. Participants performed the experiment on a Tobii T120 eye-tracker that recorded their eye-movements as well as audio-video recording of their speech and facial expressions. Participants completed the trials in three blocks consisting of three practice trials in block 1, six test trials in block 2 and six test trials in block 3. Eye-movements were calibrated before each block commenced. Each block started with an instruction screen and was followed by a set of problem displays each separated by break display. Participants clicked the mouse to move between displays. Participants were instructed to report the answer as soon as they were sure of it and then click on the button to commence the next trial. All subjects were instructed to provide on-going verbal protocols. The trials were presented in randomised order. Two randomisation trial orders were created so that half the subjects did one order and half of the subjects did the other.

6.3 Results

6.3.1 Data classification

The problem displays were segmented into interest areas as in the pilot experiment. A scheme was also developed for representing verbal protocols to provide a uniform, concise and explicit vocabulary for the purpose of supporting quantitative analysis of protocols and the economical documentation of sequences of problem solving steps. An initial review of the content of participants verbal protocols indicate that they could be divided into:

(1) Information encoded from the diagram and text such as frequencies, set relations, proportions and instruction statement.

(2) Higher-order projected inferences based on information in (1) involving propositions about possibility, proportion and normalisation.

(3) Meta-relations indicating how propositions of (1) and/or (2) are used in the service of reasoning and problem solving via relations of implication, exception, interruption and explanation.

The main categories of encoded and projected propositions are documented in Table 6.3. Statements in (1) and (2) are encoded in a kind of predicate argument form and using set notation symbols. Arguments are either types of text statements or referred sets in the diagram. Text statements and text values are written as types such as prior, outcome or query. Sets are identified by variable letters (e.g., A) that indicate their problem role (e.g., member of the first category of the query) rather than problem specific category values such as the colour blue. Arguments of sets can be combinations of categories and related with operators such as conjunction (&) and negation (~). Quantitative operators are also used including the frequency of a set (e.g., $|A|$), the geometrical quantity (e.g., $||A||$) and normalised frequency (e.g., $|\underline{A}|$). Hence, if a participant says "*there are two red and large*" and red is the queried category and large is the double category and there are actually two then the statement would be translated as $\boldsymbol{|A\&D| = 2}$. The arguments used are listed in Table 6.2.

Variable Type	Meaning
query	the query statement
outcome	the outcome statement
priors	the priors statement
A	first category of the query
B	second category of the query
C	category of the conditional outcome
D	category with double probability

Table 6.2: Argument vocabulary translating protocols.

Category	Code rule	Examples	Coding examples
Encode	Reading aloud a statement or statement value on first occasion.	"probabilities are equal for all letters"	encode (priors = equal)
Re-encode	Verbalising a statement or statement value on a subsequent occasion accessed via reading and/or memory retrieval.	*"its a blue letter"* *"it falls on a letter"*	re-encode (outcome = C) re-encode (outcome = U)
Counting/ Counting normalised	Counting aloud using a regular or normalisation strategy e.g., count-remainder or count-double.	*"there's 1, 2, 3, 4, ,5 ,6 letters"* *"that's 2, 3, 4 letters"*	counting: |U|= 6 count-remainder: |A|= 4
Frequency/ Frequency normalised	Verbalising the frequency/normalised frequency of a set.	*"There are three blue letters"* *"I have double for large letters so its like having 8 letters"*	|A| = 3 |U| = 8
Set	Verbalising a set relations *or set reference.*	*"All the reds are large"* *"there aren't any small blue letters"* *"some of the blue are big"* *"blue"*	subset (A,~B) empty (A, C) overlap (A, B) set(A)
Possibility	Verbalising projected possibility values of a set.	"*its only falls on four*" "*we have seven possible letters*" "*it can't be small and red*"	limited-trial-possibilities (C) trial-possibilities (U) impossible-query (A&B)
Denominator/ Numerator	Verbalising a numerator or denominator role assignment.	"its out of five"	denominator = |U| = 5
Arithmetic operations	Verbalising arithmetic operations such as addition, subtraction and multiplication.	"one of six plus two of six " "one half times two of six"	|A|/|U| + |B|/|U| ||AUB/|U|| * |B|/|U|
Frequency probability/ Geometrical probability	Verbalising a probability proportion in which the result and intermediate steps indicate the proportion was derived from component frequencies or the geometrical structure of the PS-diagram.	"*Its 2 out of 6 for blue*" "ah its a half anyway"	P(A) = |A|/|U| = 2/6 P(A) = ||A|/|U|| = 1/2

Table 6.3. Coding used for classifying verbalised derivation operations.

Meta-relations were identified via conjunctions and other intervening terms of phrases that relate to concurrent propositions such as *so*, *but*, *if*, *because*, *hold on,* etc. The translation of the terms identifies the argument structure expressed by verbal protocols. Note that meta-relations are composed and coded as constituent roles. The types of meta-relations are documented below in Table 6.4.

Category	**Code rule**	**Example**	**Code**
Conditional	Where a given/conclusion is indicated in a statement by given terms such as "if", "well", "as" "since" and/or a conclusion terms such as "so."	"one of them is small so the probability is one third"	GIVEN $\|A\cap C\| = 1$ CONCLUSION $P(A) = \|A\cap C\|/\|C\|$ $= 1/3$
Exception	Where an exception to a default interpretation is indicated by terms such as "but."	"I have six letters but I have double for large letters"	NORMALLY denominator = $\|U\|$ $= 6$ BUT re-encode (priors = D or ~D)
Interruption	Where a result of some interpretation or calculation is abruptly interrupted indicated by phrases such as "no", "wait", "hold on."	"One in..... no no, the spinner falls on a large letter"	INTERRUPTED $P(A) = \|\|A/\|U\|\| =$ $1/2$ CONSIDERED re-encode (outcome = C)
Explain	Where some choice or interpretation is explained or justified by one or more facts using terms such as "because." .	"its out of five *cos* |because| it falls on a letter"	EXPLAIN: denominator = $\|U\|$ $= 5$ BECAUSE re-encode (outcome =U)

Table 6.4. Meta-relation classification used in protocol analysis.

The translation to the notation allows the verbal protocols to be presented in a way that reveals what acquired and projected information was verbalised, its order, and how the information was used, therefore providing some indication of the argument structure that participants constructed in the course of solving the problem.

6.3.2 Problem solving time

The solving time on a problem was measured by the duration from the first to the last fixation of the trial. The median time to solve a problem was approximately 28 seconds.

The longest time spent solving a problem was 150 seconds and the shortest time took 10 seconds. The median solving time was approximately double or more for trials involving disjunction (M = 49) and biased (M = 37) problem instructions than on trials involving conjunction (M = 20) and conditional (M = 22) problem instructions. Table 6.5 presents descriptive statistics of the problem solving times for each problem. One can observe that the greatest dispersion in problems solving times, as measured by the inter-quartile range, is for the biased/subset and disjunction/overlap problem.

Instruction	Data structure	Median	Inter-quartile Range	Minimum	Maximum
Biased	Overlap	34	10	25	101
	Disjoint	38	14	20	150
	Subset	58	49	27	144
Conditional	Overlap	22	5	15	28
	Disjoint	17	5	12	29
	Subset	24	5	13	70
Conjunction	Overlap	22	11	16	38
	Disjoint	18	9	12	58
	Subset	19	14	10	45
Disjunction	Overlap	49	37	20	125
	Disjoint	31	19	18	51
	Subset	61	12	25	97

Table 6.5. Descriptive statistics of the problem solving time for each problem.

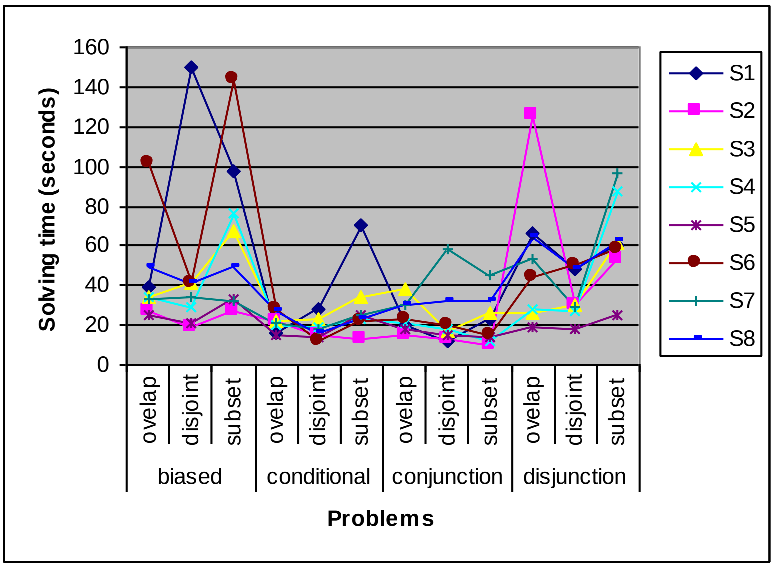


Figure 6.3. Problem solving time for each participant, for each problem. Each line represents a participant. Problems are not presented in the order in which they were solved.

Figure 6.3 shows the problem solving time for each participant on each problem. One can observe that there are several outlier solving times in the data for disjunction and biased problems for the participants S1, S2 and S6. These deviant solving times result from the particular participants reaching an impasse and taking some time to determine a solution procedure.

6.3.3 Task scheduling and cognitive strategies

General scheduling order. At an abstract level, the scheduling of subtasks tended to follow a specific order as evidenced by protocols. Participants tended to:

1. Initially read the problem statements from top to bottom aloud.
2. Identify members of referred categories and their set structure (i.e., A, B, C, D).
3. Enumerate the trial possibilities (U if the outcome was unconditional, or C if conditional).
4. Enumerate the possibilities of the queried outcome.
5. Report an answer.
6. Check the answer.

Depending on the problem, the protocols also indicated intermediate steps including statement re-encoding, re-identifying sets, determining intermediate frequencies and probabilities, as well as engaging in acts reflection and reasoning. Details of subtask scheduling are described below.

Initial instruction comprehension. The instructions were consistently read aloud in full from top to bottom by six of the eight participants before interpreting the data in the diagram (i.e. first pass instruction comprehension). The two exceptions to this first pass instruction comprehension behaviour were S5 and S2. S5 tended to interleave acts of reading statements of the instruction and identifying sets of referred categories in the diagram. S2 showed a tendency to just read out the statement values and skip reading the frame text.

Enumeration. Participants appeared to use a combination of serial counting and subitizing to determine the cardinality of sets. Serial counting was indicated by the consecutive verbalisation of count states (i.e. one, two, three, four, etc.) and occurred most commonly when enumerating U. Subitizing was indicated by reporting the frequency of a set in the absence of verbalising count states. This tended to occur for small subsets (i.e. 2 or 3) of PS-units (problem space units) when indicated by a referring category in the text (i.e. A, B, C, D). For biased probability problems, participants appeared to use either one of two strategies to quantify outcomes according to a normalisation scheme. The *remainder-strategy* involved taking the count of double PS-units, multiplying the sets by 2 then counting the remainder from the resulting value. The *double-count* strategy involved counting a PS-unit once or twice depending on whether it had double probability (i.e. twice) or not (i.e. once). The *double-count* strategy was the most common strategy for biased probability problems.

Search, set structure derivation and co-reference. Participants' performance on conditional and conjunction disjoint problems indicate that they are likely to initially identify the set structure of referred categories before proceeding with any form of quantification. Visual re-identification of sets and set structure occurs at different stages depending on the problem. Over different problems there are different patterns of diagram scanning and cross referencing between referred categories in text statements, corresponding members in the diagram and marker-lines (see the next section for quantitative details) reflecting task requirement and difficulties associated with a problem.

Proportions. Protocols indicate that participants commonly derived frequency based proportions. Geometrical proportions were only employed on a small number of trials and tended to occur in problems where the solution proportion involved a half. This is arguably because of the increased visual-spatial fidelity of a ½ proportion can be more readily recognised and its accuracy assessed with greater confidence. Verbalising a geometrical proportion requires translating a visual spatial representation to a numerical representation. Verbalising a geometrical proportion only when fidelity constraints are satisfied suggests that the cognitive system must encode the proportion to check if these constraints are satisfied first. This suggests geometrical proportions may be encoded at some level as a matter of course, but only selected for numerical translation and verbalisation when an accuracy trade-off is satisfied.

Checking and monitoring. Participant's protocols typically revealed periods between verbalising an initial solution and then clicking the mouse to move on to the next trial. Within this period, many trials involve verbal protocols that indicate checking/ confirming the initial solution by, for example, repeating the answer after a pause or using confirmation indicating terms such as “yes”. Inspection of protocols suggest what is checked varies and may involve check of frequencies, categories as well as solution interpretations.

6.3.4 External attention

The fixations made to classes of interest areas were tabulated in order to derive a measure of the relative amount of attention spent by each participant on each problem. For simplification, the analysis considers fixation of five classes of interest areas: the PS-unit area of the diagram, the set markers, the probability statement, the conditional statement and the query statement. As shown in Figure 6.4, the relative time spent on a type of statement depends on whether a referring category was present. So, for example, the average fixation time on the probability statement was 8.5 seconds in biased problems, but only 2.7 seconds in others, the average time on the conditional statement was 4.9 seconds in conditional problems, but 3.2 in others. The average total fixation time for the query statement is greatest for the disjunction problems (M = 17.4 seconds), which is substantially greater than the conjunction problems (M = 6.6 seconds) even though they express the same number of category values. These results confirm the earlier proposal that some of the attention to the query statement in the disjunction problem is driven by instruction interpretation requirements for solution procedure formulation rather than just rehearsing, re-accessing or checking category values. The amount of attention to PS-units for the difficult problems appeared to far exceed the requirements to identify and enumerate the target set to determine the probability proportion. The amount of attention is consistent with participants reflecting on how to solve the problem rather than employing a practised solution procedure. Note that the mean amount of fixation time for a problem on the blank-space surrounding the diagram and text is approximately 2 seconds, suggesting that the information in the problem presentation was almost constantly in use.

Analysis of attention shifts between different features of the problem presentation where derived from consecutive fixations. The purpose of the analysis was to provide some quantification of attentional strategies for the problems and differences between different problems instructions and types of data structure. Note that such changes in fixations are a conservative estimation of attention shift because it is possible to attend to a different element E without making an eye-movement if E is in close enough proximity. This point is relevant to interpretation of eye-movement on PS-units because eye-movement replays have shown that participants count the total set of PS-units without necessarily making eye-movement to all the units, particularly the end ones.

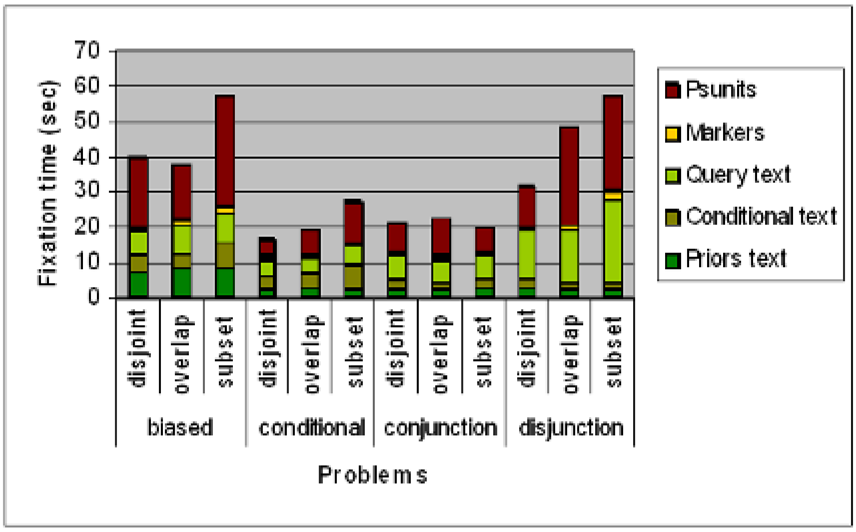


Figure 6.4. The relative fixation time on different interest areas for each experimental problem average across participants.

Diagram scanning. In order to assess the amount of scanning or eye-movement visits made to PS-units of the diagram, the frequency of fixations to PS-units from any of the different token interest areas (including different token PS-units) were tabulated for each participant. Participants on average made 45 eye-movement visits to PS-units in solving an experimental problem, but there appeared to be substantial differences in this frequency between types of problems. The mean number of fixation visits to PS-units ranged from 18 for the conditional/disjoint to 97 for the biased/subset problem. As there are between seven and nine PS-units in a diagram and only some PS-units are critical to the problem, the frequencies indicate again the high frequency of attention shifts to different PS-units in the course of problem solving. Figure 6.5 shows the average frequency for each problem. As can be seen, the largest number are for biased and disjunction problems, which is consistent with the problem solving times of these problems.

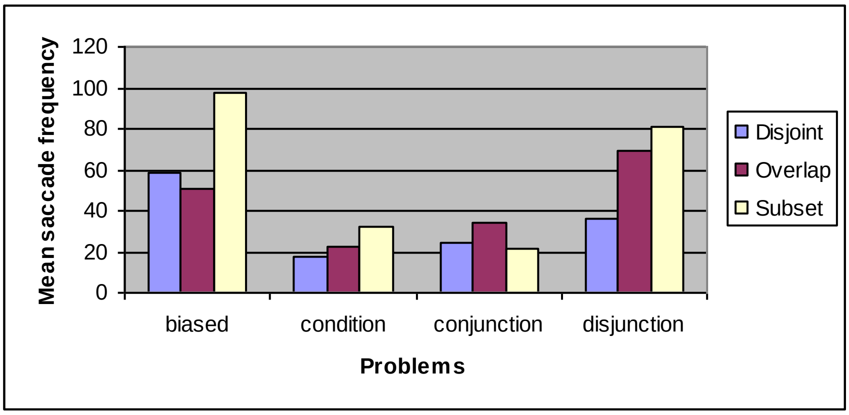


Figure 6.5: Frequency of saccades made to PS-units of the diagram for each of the twelve experimental problems averaged over participants.

The average number of times either or both markers were fixated by participants whilst solving a problem was 5.3. On average, markers on the biased/disjoint problem received the lowest number of fixations (M = 3.8), whereas markers on the disjunction/subset problem received the highest number of marker fixations (M = 9.3). Attention shifts to the markers were not that frequent, probably because determining this configuration information requires a couple of attention steps.

Diagram and text integration. To get a measure of the frequency of interrelating text and diagram elements, the saccades between text and diagram interest areas (i.e. text -> diagram or diagram -> text) were derived. The mean number of fixation switches made by participant between text and diagram elements over all problems was 15.3. This indicates again that, in general, attention shifting and interrelating of text and diagrams elements were frequent in the task because there are only three statements and only two statements contain category information. Although participants tended to exhibit a relatively contained first pass of comprehending the full set of text instructions, the act of referring back to the text was common in the process of problem interpretation and solution procedure execution using the diagram. The graph below (Figure 6.6) shows that the highest frequencies of this tended to occur with the biased and disjunction problems. Note that each type of problem instruction has two category values to encode, so differences between problems in terms of switching between text and diagram are likely to be the result of more than the need to rehearse or re-access a category value. Rather, these differences between problems may also be influenced by the requirement to re-address the interpretation of the problems instructions.

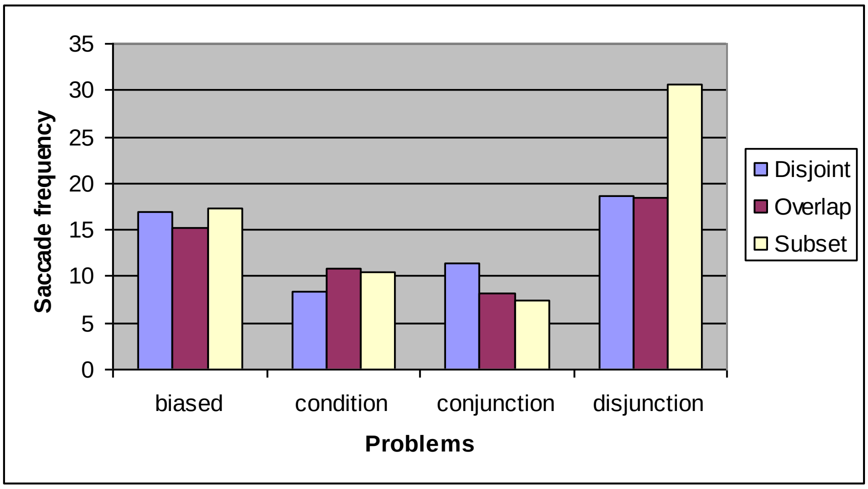


Figure 6.6: Frequency of saccades between text and diagram elements for each of the twelve experimental problems averaged over participants.

6.3.5 Problem interpretations

The section focuses on the process of problem interpretation. As protocols revealed that the conditional, biased and disjunction problems with subset and overlapping set structure were the trials that elicited the most reflection and appeared to be the most challenging to participants, the section will focus on performance on these problems.

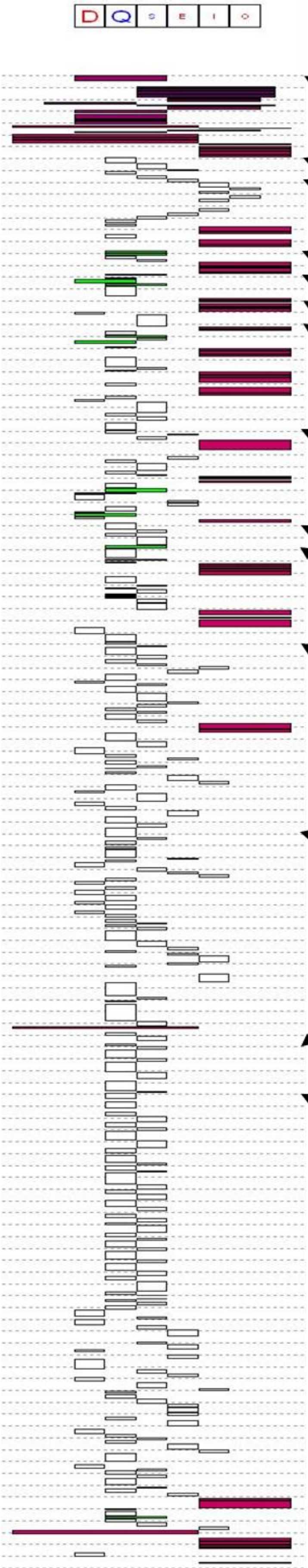
Disjunction. For the disjunction/disjoint problem calculating |A| + |B| will provide the correct numerator and calculating |A|/|U| + |B|/|U| will provide the correct answer. The difference between the two is superficial because |A|/|U| + |B|/|U| will simply require adding the numerators as the denominator is a constant. Indications of either suggest the interpretation or framing of an addition procedure. As can be seen in Table 6.6, on the disjoint problem, one group of participants (S2, S3, S6, S8) provided protocols consistent with adding probabilities of each set because they either verbalised the frequencies for each model (S3), the probabilities of each model (S2, S6) or explicitly verbalised adding them together (S2, S8). Only S4 and S5 provided no verbalisations of component models, which is consistent with taking the union of the possibilities of the alternative models. An exception is S7, who incorrectly answered with component probabilities for each model – presumably due to an instruction misinterpretation.

Subject	Disjoint		Overlap		Subset	
S1	N/A	|AUB|/|U|	N/A	|AUB|/|U|	N/A	|AUB|/|U|
S2	|U|, |A|/|U|, |B|/|U|, |A∩B|/|U|	|A|/|U|+|B|/|U||+ |A∩B|/|U	|U|, |A|, |A|/|U|+ |B|/|U|+|A∩B|/|U|.....	A|/|U| * |B|/|U ……..…	|U|, |A|/|U|+|B|/|U|, |U|	|AUB|/|U|
S3	|U|, |A|, |B|	|A|+|B|/|U|	|U|, |A|, |B|, |A∩B|	|AUB|/|U|		|AUB|/|U|
S4		|AUB|/|U|	|U|	|AUB|/|U|	|A|/|U|, |B|/|U|, |A∩B|/|U|	|A∩B|/|U|
S5		|AUB|/|U|	|A|, |B|, |A∩B|	||AUB|/|U||		|AUB|/|U|
S6	|A|/|U|, |B|/|U|, |A∩B|/|U|	||A|/|U|+|B|/|U||+ |A∩B|/|U	|A|/|U|, |B|/|U|, |A∩B|/|U|, |A|, |B∩~A|, |B∩A|, |AUB|, |A∩B|/|U|,	|A∩B|/|U|	|A|/|U| , |B|/|U|	|A∩B|/|U|
S7		|A|/|U|, |B|/|U|, |A∩B|/|U|	|A|/|U|, |B|/|U|, |A∩B|/|U|	|A|/|U|, |B|/|U|, |A∩B|/|U|	|A|/|U|, |B|/|U|, |A∩B|/|U|	|A|/|U|, |B|/|U|, |A∩B|/|U|
S8	|U|, |A|/|U| + |B|/|U|+ |A∩B|/|U|	|A|/|U|+|B|/|U||+ |A∩B|/|U	|A|/|U| + |B|/|U|, |A∩B|/|U|	||AUB|/|U||	|A|/|U|, |B|/|U|, |A∩B|/|U|	A|/|U|, |B|/|U|, |A∩B|/|U|

Table 6.6. Quantitative derivations on the disjunction problems. S1's derivations are not included due to problems with the audibility of his/her verbalisations. The left column represents verbalised quantitative derivations in order excluding the final solution. The right column represents derivation consistent with final solution given. Shaded cells are incorrect solutions.

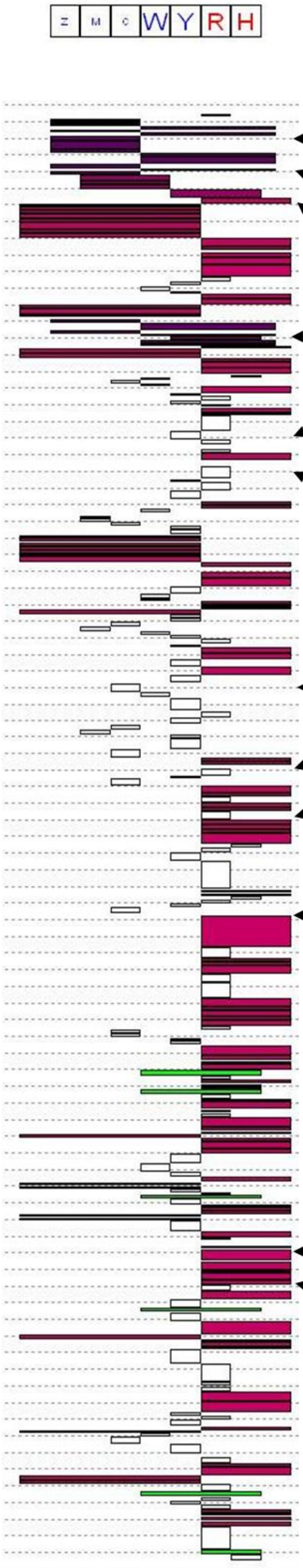
For the disjunction/overlap and disjunction/subset problems, calculating |A| + |B| will not provide the correct numerator and calculating |A|/|U| + |B|/|U| will not provide the correct answer. Instead only |AUB| will provide the correct numerator on these problems For the disjunction/overlap problem (S2, S3, S5, S6, S8) and/or for the disjunction/subset problem (S2, S4, S6) six participants provided protocols consistent with employing an initial adding procedure, but none of them gave the result as a final solution. Participants S4 (disjunction/subset) and S6 (disjunction/subset and disjunction/overlap) proceeded to give incorrect solutions consistent with taking the numerator from |A∩B|. S2's verbal protocols indicated that he/she generated an incorrect solution by multiplying the probabilities from each model (see figures 6.7 and 6.8). The remaining trials of this group of six participants are consistent with them taking |AUB| as the numerator or the probability from the geometrical proportion of ||AUB||/||U|| as their solution. Participants whose protocols were not consistent with the use of an initial addition procedure include subjects S3, S5 (disjunction/subset) and S4 (disjunction/overlap), who only provided correct union solutions as indicated in the solution paths of figures 6.9 and 6.10. Other participants, S7 (subset and overlap) and S8 (subset) gave component probabilities as solutions on problems.

Whilst applying the incorrect addition procedure on these problems, verbal protocols of some participants indicate that they noticed the procedure yielded a high (i.e. disjunction/overlap) or impossible probability (i.e. disjunction/subset) resulting in participants applying an alternative solution procedure. For some subjects, this was indicated by explicit verbalisation e.g., S2, S3, S6, S8. For many other participants, this was indicated by a pause either following or in the middle of executing the addition procedure – suggesting reflection on the sub-problem. Examples of the coded solution paths derived from verbal protocols of participants, who abandoned the addition procedure, are shown in figures 6.7 and 6.8.



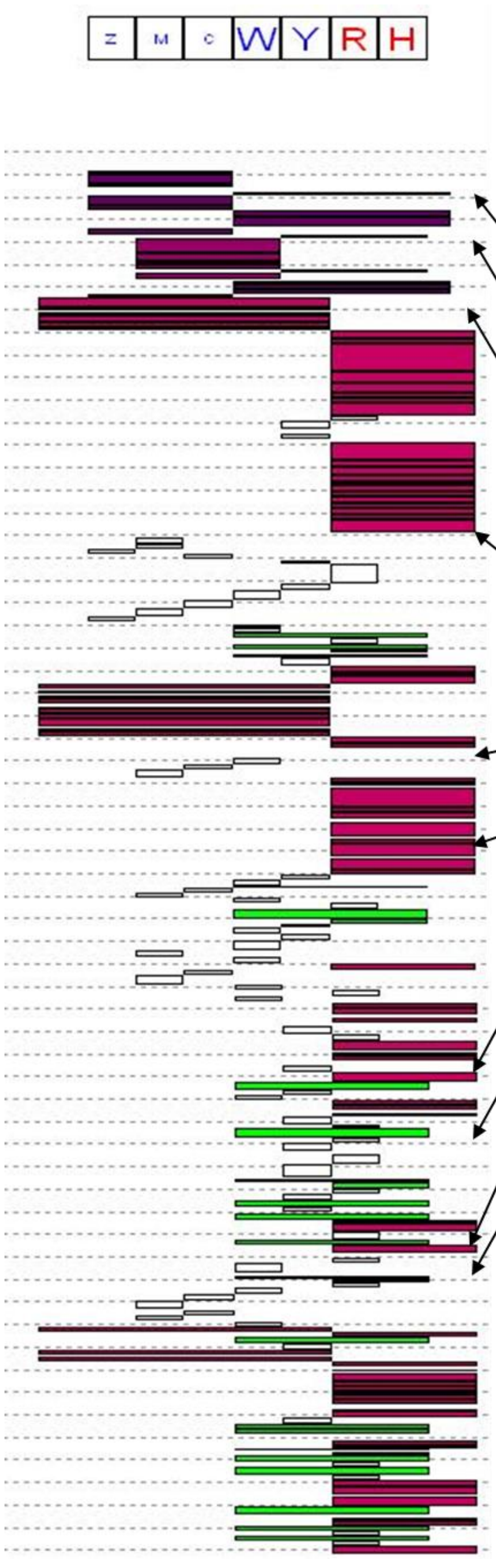
<table>
<tr><th colspan="3">S2 Disjunction/ Overlap</th></tr>
<tr><td>0.52</td><td></td><td>press button (start)</td></tr>
<tr><td>0.52</td><td>and mm equal for all letters</td><td>encode(priors) = equal</td></tr>
<tr><td>0.58</td><td>and blue</td><td>encode(query) = A or B</td></tr>
<tr><td>1</td><td>1 2 3 4 5 6</td><td>counting: |U| = 6</td></tr>
<tr><td>1.06</td><td>and then 2 of them</td><td>|A| = 2</td></tr>
<tr><td>1.08</td><td>2 of 6 are blue</td><td>|A|/|U| = 2:6</td></tr>
<tr><td>1.1</td><td>and 2 of them are large</td><td>|B|= 2</td></tr>
<tr><td>1.12</td><td>for both there is only one of six that is both</td><td>|A&B|/U| = 1:6</td></tr>
<tr><td>1.21</td><td>2 of 6 Is</td><td>|?|/|U| = 2:6</td></tr>
<tr><td>1.29</td><td>1, 2 is large</td><td>counting: |B| = 2</td></tr>
<tr><td>1.31</td><td>mmm</td><td>REFLECTING</td></tr>
<tr><td>1.39</td><td>1 of 6 + 2 of 6 + 2 of 6 that makes it a lot, 5</td><td>|A|/|U| + |B|/|U| + |A&B|/U| = 5/|
INTERRUPTED</td></tr>
<tr><td>1.55</td><td>blue is 2 is? is 3 and then…????</td><td>|A| + |?|
REPEAT/CHECK</td></tr>
<tr><td>2.13</td><td>mmm</td><td>REFLECTING</td></tr>
<tr><td>2.17</td><td>one half times 2 of 6 makes it 1 of 6 plus a quarter times a quarter times 4 of 6, that's 4 of 6</td><td>? * ? * ? = ? + ? * ?</td></tr>
<tr><td>2.52</td><td>so 2 of 6</td><td>CONCLUSION
P(A or B) = 2:6</td></tr>
<tr><td>2.54</td><td>yeah 2 of 6 I think</td><td>CONFIRMATON
P(A or B) = 2:6</td></tr>
<tr><td>2.57</td><td></td><td>press button (end)</td></tr>
</table>

Figure 6.7. Coded protocols of S2 on the dijunction/overlap problem, who began adding probabilities for each model, but abandoned the procedure and gave an incorrect answer consistent with multiplying sets of possibilities.



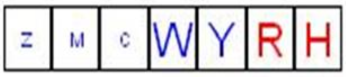
S4 Disjunction/ Subset		
2.50		press button (start)
2.52	probabilities are equal for all letters	encode(priors) = equal
2.54	the spinner falls on a letter	encode(outcome) = U
2.56	what is the probability the letter is large or	encode(query) = A or B or A&B
3.04	mmm large or red,	re-encode(query) = A or B A&B
3.10	large or red or both	re-encode(query) = A or B A&B
3.12	mmm	REFLECTING
3.25	large which is 4 out of 7	|A|/|U| = 4/7
3.30	red is 2 out of 7	|B|/|U| = 2/7
3.33	and large and red is 2 out of 7	|A&B|/|U| = 2/7
3.39		REFLECTING
3.59	large or red or both	re-encode(query) = A or B or A&B
4.01	mmm	REFLECTING
4.16	2 out of 7	|A∩B|/|U| = 2/7
4.17		press button (end)

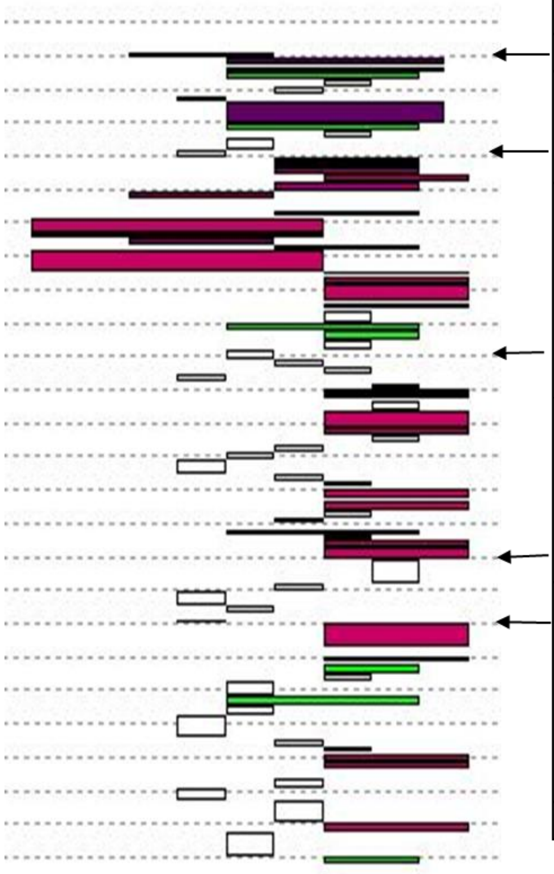
Figure 6.8. Coded protocols of S4 on the disjunction/subset problem, who began adding probabilities for each model, but abandoned the procedure and gave an incorrect answer consistent with taking the numerator from |A∩B|.



S3 Disjunction/ subset		
3.46		press button (start)
3.48	probabilities are equal for all letters	encode(priors) = equal
3.5	the spinner falls on a letter	encode(outcome) = U
3.53	what is the probability the letter is large or red or both	encode(query) = A or B or A&B
4.03	so i have 1,2,3,4,5,6, 7 letters	GIVEN counting: |U| = 7
4.13	and the probability that its large or red or both	re-encode(query) = A or B or A&B
4.17		REFLECTING
4.27	large	set(A)
4.3	or red	set(B)
4.35	or both	set(A&B)
4.36	i would say 4 out of 7	CONCLUSION |AUB|/|U| = 4:7
4.47		press button (end)

Figure 6.9. Coded protocols of S3 on the disjunction/subset problem, who gave a correct answer consistent with interpreting the numerator from |AUB| without initially trying an alternative procedure. This answer is preceded by a reflective period, verbalising the sets and scanning the relevant set structure.





S5 Disjunction/ subset		
0.45		press button (start)
0.46	probabilities are equal	encode(priors) = equal
0.49	whats the probability the letter is large of red or both	encode(query) = A or B
0.55	well all the red ones are large anyway so	GIVEN subset (B, A)
1.01	so, its 4 over 7	CONCLUSION |AUB|/|U| = 4:7
1.03	is it? think it is(mumble), Yeah 4 over 7	|AUB|/|U| = 4:7 CONFIRMED
1.09		press button (end)

Figure 6.10. Coded protocol of S5 on the disjunction/subset problem, who gave a correct answer consistent with interpreting the numerator from |AUB| without initially trying an alternative procedure. This solution appears to be hypothesised by recognising that the queried alternatives are all members of A which is equivelent to the union.

On the disjunction/overlap and disjunction/subset problems we can assume that participants who answered correctly on these problems took the numerator as |AUB| because there does not appear to be an alternative interpretation consistent with the answer. The observations that some participants initially attempted an incorrect solution procedure together with the refection times of all participants before reporting a solution suggest that these participants did not plan the union solution interpretation on instruction comprehension, but instead appeared to derive the form of the solution with the use of the diagram.

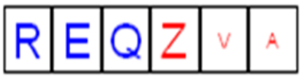
Eye-movement protocols indicate participants tended to make frequent switches of attention back and forth to the disjunction statement and the queried possibilities of those models in the diagram before reporting an answer (see figures 6.7 to 6.10). The average number of switches from the query statement to the diagram or back again was 29 for the disjunction/subset, 18 for the disjunction/overlap and 17 for the disjunction/disjoint. The pattern of eye-movements coupled with verbal repetitions of the disjunction statement suggests that participants were not clear about the meaning expressed by the disjunction or its referential interpretation. It is suggested that the repeated encoding and consideration of the disjunction statement and its corresponding sets reflect a bi-direction interpretation strategy in which participants use both sources (text and diagram) of information that they are uncertain about to determine a coherent interpretation of the solution.

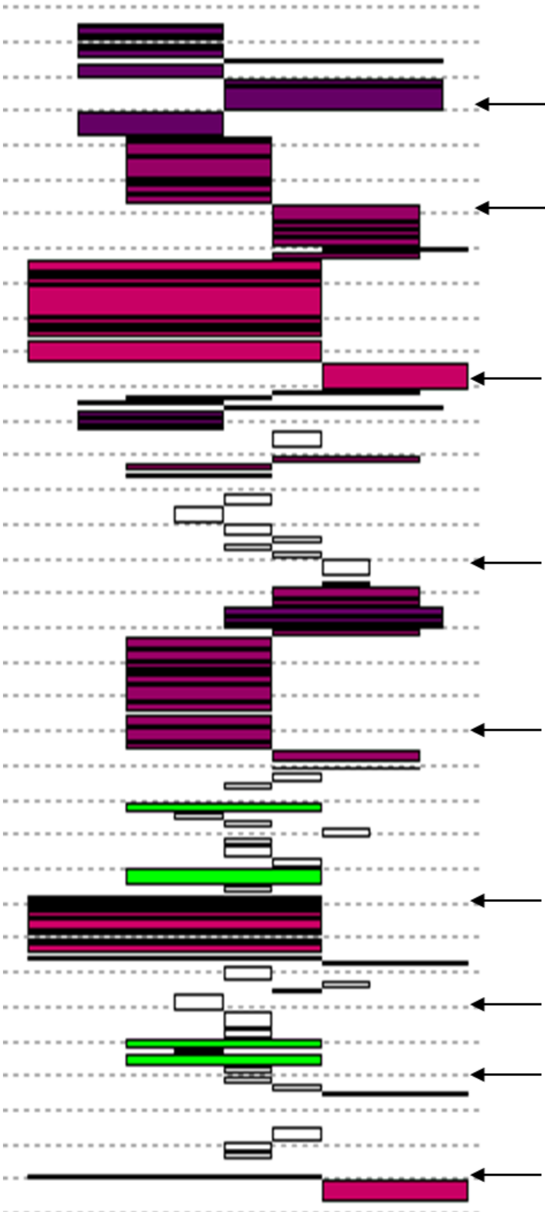
Conditional problem. Both the conditional/subset and conditional/overlap problems require participants to interpret the set C, rather than U, as the trial possibilities. The conditional/disjoint and conditional/overlap problems require participants to interpret the set A&C, rather than A, as the queried possibilities. Table 6.7 shows the quantitative derivations and solutions given by participants on each problem. As can be seen, the most common error made by four participants (S1, S2, S4, S7) involved incorrectly interpreting members of U rather than C as the possibilities on the disjunction/subset and/or disjunction/overlap problems.

Subject	Overlap		Subset		Disjoint	
S1	N/A	|A&C|/|U|	N/A	|A|/|C|	N/A	|A&C|
S2	|U|	|A&C|/|U|	|U|	|A|/|U|		|A&C|
S3	|C|, |A&C|	|A&C|/|C|	|U|, |C|, |A&C|	|A&C|/|C|		|A&C|
S4	|A&C|	|A&C|/|U|	|U|	|A|/|U|		|A&C|
S5	|U|, |C|	|A&C|/|C|	||A|/|U||	|A|/|C|		|A&C|
S6	|C|, |A&C|	|A&C|/|C|	||A|/|U||, |C|	|A|/|C|		|A&C|
S7		|A|/|U|	|A|, |-A|	|A|/|U|	|U|, |A|	|A|/|U|
S8	|C|, |A&C|	|A&C|/|C|	|C|	|A|/|C|	|C|	|A&C|

Table 6.7. Verbalised quantitative derivations for the conditional problems. The left column represents verbalised quantitative derivations in order excluding the final solution. The right column represents derivation consistent with final solution given. Shaded cells are incorrect solutions.

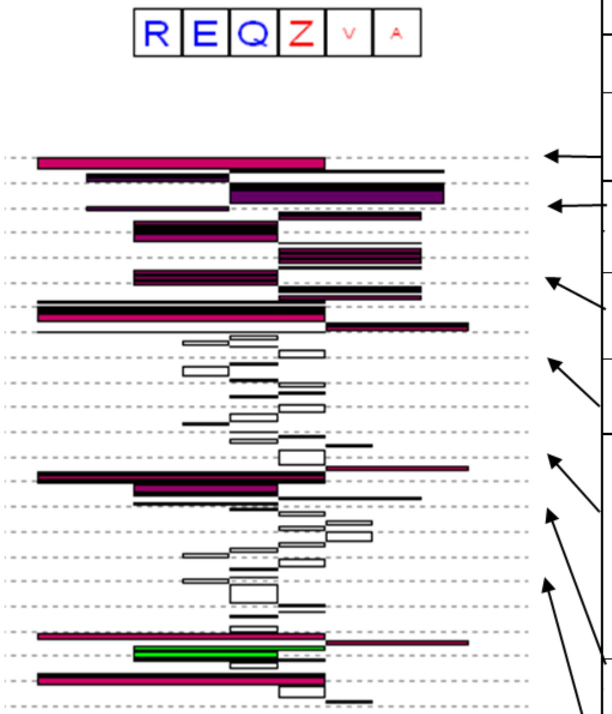
On the conditional/subset problem three subjects (S2, S4, S7) gave incorrect answers consistent with taking U as the trial possibilities. The protocols of three (S3, S5, S6) of the remaining five (S1, S3, S5, S6, S8), who correctly solved the conditional/subset problem indicate that they initially derived |U| (S3) or derived the geometrical proportion ||A|||/|U|| (S5, S6) after the first pass reading of the instruction. This suggests that this subgroup (S3, S5, S6) had not initially considered the need to limit the trial possibilities of U to C. Subsequent eye-movement revisit to the conditional statement was preceded by verbalisations such as "but" (S3, S1) indicating recognition of an exception to derive the denominator from |U| or "*no no*" (S6), "*ah no*" (S5) indicating abrupt interruption of an initial solution derived from ||A|/|U||. After re-encoding the outcome statement and returning to the diagram these participants formed an interpretation of C as the trial possibilities. Examples of coded verbal protocols of two participants S3 (6.11) and S6, (6.12) who appear to show initial consideration of U as the possibilities for denominator quantification, are shown below.





S3 Conditional/subset		
0		press button (start)
0.03	probabilities are equal for all letters	encode(priors) = equal
0.06	the spinner falls on a large letter	encode(outcome) = C
0.11	what is the probability the letter is blue	encode(query) = A
0.16	so there are 2, 3 ,4 there are six letters	denominator = |U| = 6 [NORMALLY]
0.21	but the spinner only falls on a large letter,	BUT re-encode(outcome) = C, limited (C)
0.26	which four are large	|C| = 4
0.29	the probability the letter is blue	re-encode(query) = A
0.31	since 3 of the large letters are blue	GIVEN |A∩C| = 3
0.34	3 out of 4	CONCLUSION |A∩C|/|C| = 3:4
0.36		press button (end)

Figure 6.11. Coded protocol of S3 on the conditional/subset problem. This participant answered correctly but after recognising an exception to taking the denominator from |U|.

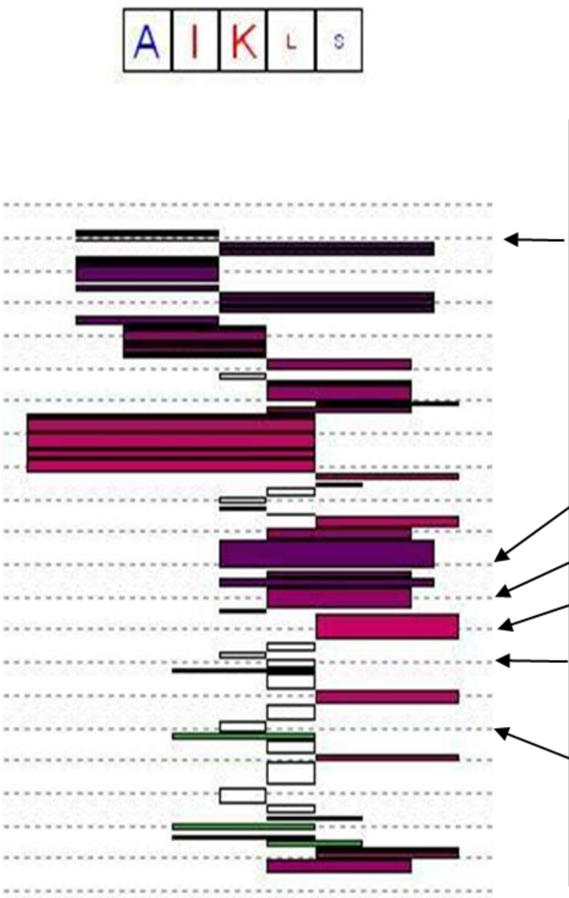


S6 Conditional/subset		
4.19		press button (start)
4.20	probabilities are equal for all letters	encode (priors) = equal
4.22	the spinner falls on a large large letter	encode (outcome) = C
4.25	what is the probability the letter is blue	encode (query) = A
4.28	one in….	||A|/|U|| = 1/2 INTERRUPTED
4.32	no no, the spinner falls on a large letter	EXCEPTION re-encode (outcome) = C
4.34	so it only falls on 4	CONCLUSION limited(C), |C| = 4
4.37	so its 3 in 4 is the answer	CONCLUSION |A|/|C| = 3:4
4.41		press button (end)

Figure 6.12. Coded protocol of S6 on the conditional/subset problem. This participant answered correctly after appearing to interrupt an initial solution in which the denominator is taken from |U|.

In the conditional/disjoint problem only one participant (S7) responded incorrectly. Of the remaining group, five participants (S2, S3, S4, S5, S6) did not verbalise any quantitative derivations. This is consistent with participants immediately deriving that the probability is zero from recognition of the disjoint relation between sets A and C in the diagram before deciding on any form of quantification. Some participants showed surprise to initial recognition that A&C is an empty set as indicated by the tone of their voice. The fact that participants tended to derive the probability zero immediately after reading the instruction by observing A and C are disjoint suggests that they interpreted the form of the queried possibilities as being the set A&C from reading the instruction.

On the conditional/overlap problem, only three participants (S2, S4, S7) gave incorrect answers because they gave answers consistent with interpreting U as the trial possibilities. Note that these are a subset of participants who also failed to interpret C as the trial possibilities on the conditional/subset problem. However, although they derived incorrect denominators, S2 and S4 provided numerators consistent with interpreting the queried possibilities as A&C on the conditional/overlap problem indicating that they understood the outcome information as conditional, but failed to make both of the required conditional inferences (see figure 6.13). Note also that on this problem only one participant (S5), who answered correctly, initially derived |U| compared to four on the conditional/subset problem. For some participants this was not the first instance of the conditional problem allowing them to recognise the solution implication of the conditional information on first pass instruction interpretation as indicated in figure 6.14.



S4 Conditional/overlap		
2.16		press button (start)
2.17	probabilities are equal for all letters	encode (priors) = equal
	the spinner falls on a red letter	encode (outcome) = C
	what is the probability the letter is small	encode (query) = A
2.27	equal for all	re-encode (prior) = equal
2.28	red	re-encode (outcome) = C
2.29	small	re-encode (query) = A
2.3	theres only 1 small letter	[GIVEN]
		limited (A,C),
		|A&C| = 1
2.32	so 1 out of 5	CONCLUSION
		|A&C|/|U|= 1/5
2.36		press button (end)

Figure 6.13. Coded protocols of S4 on the conditional/overlap problem. The participant provides an incorrect answer consistent with correctly taking the numerator from |AnC| but incorrectly taking the denominator from |U| rather than |C| .

S6 Conditional/overlap		
0.48		press button (start)
0.49	the probabilities are equal for all letters	encode (priors) = equal
0.53	the spinner falls on a red letter	encode (outcome) = C
0.57	the spinner only falls on a red letter	limited possible category (C)
0.59	what is the probability the letter is small	encode (query) = A
1.04	so the probabilities are equal for all letters, it falls (mumble)	re-encode (priors) = equal re-encode (outcome) = C [GIVEN]
1.08	so there are only three things it could fall on	CONCLUSION limited possibilities (C), |C| = 3
1.1	one of which is small	|A∩C| = 1
1.12	1 in 3	|A∩C|/|C| 1:3
1.15		press button (end)

Figure 6.14. Coded protocols of S6 on the conditional/overlap problem. Following a previose conditional instrucution trial the participant recognises the requirement to limit trial possibilities upon first pass interpretation of the instruction.

Biased problems. The solution to the biased problem can be derived in two ways. The frequency procedure requires interpreting a scale to normalise possibilities that have double and non-double probability then using one of several possible strategies for counting them. An alternative procedure is deriving the probability from the geometrical proportion of queried and trial possibilities. Table 6.8 shows the quantitative derivations verbalised by participants and resulting solutions. For all biased problems, three participants repeatedly gave incorrect solutions (biased/overlap: S1, S4, S6; biased/subset: S1, S6; biased/disjoint: S6).

- **Excluded possibilities.** On the biased/overlap problem, S4 correctly derived the denominator from |U|, but incorrectly took the numerator from only |A&D|, which omitted |A&~D|. S1 also made the same mistake on this problem taking the numerator from only |A&D|, but also omitted normalisation of the denominator.

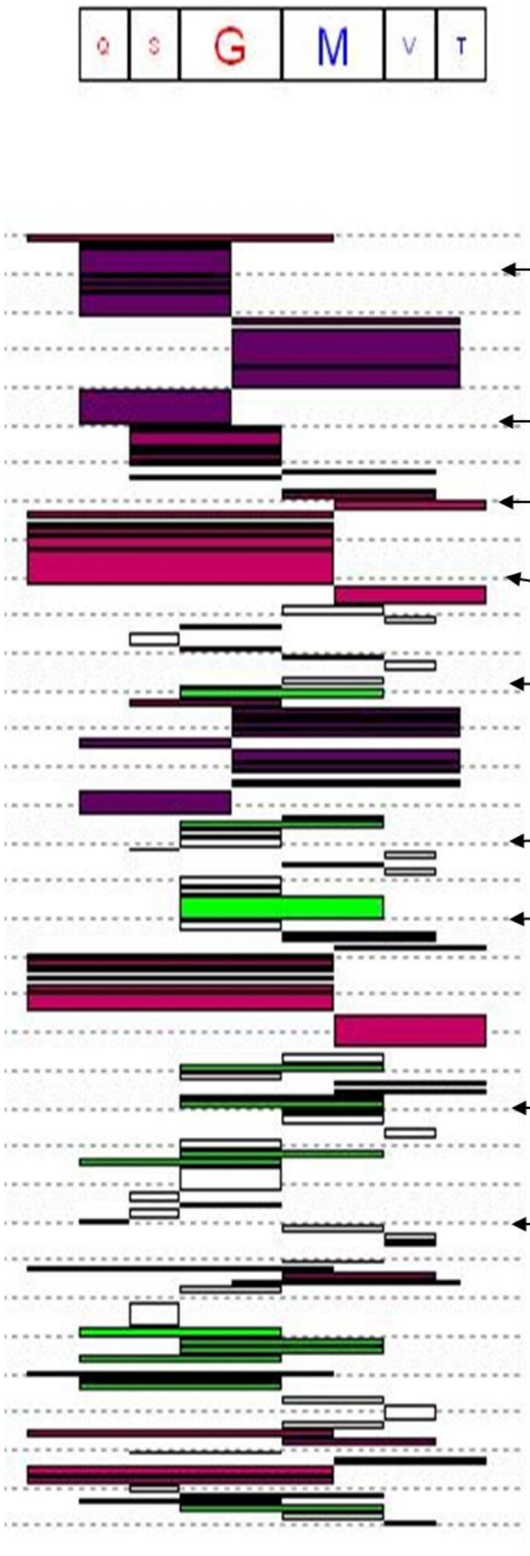
- **Normalisation omissions.** For the biased/overlap problem, S6 provided the correct numerator value derived from |A|, but like S1 did not apply normalisation to the denominator. On the biased/subset, S1 also gave a solution consistent with making the same error. A similar mistake was also made, but corrected by S5 on the biased/subset problem in which S5 appeared to have initially omitted normalising the numerator whilst reporting a fraction involving a normalised denominator value as indicated by the verbalisation ”*oh not two over eight, four over eight*”. These normalisation omissions replicate errors observed in the pilot experiment.

- **Normalisation misinterpretations.** On the biased/disjoint problem, S6 chose to derive the denominator from |U|-|D| which appears to be some way of normalising |D|. On the biased/subset trial, S6 derives the denominator first from |U|-|D| then rejects it and chooses the value of 10 for the denominator apparently because it is consistent with |D|*2|/|U|*2 + |~D|*2|/|U|*2. S6 thus appears to derives the resulting solution from |A|/|U|*2**.** Each of these procedures appears to be incorrect attempts at providing a normalised solution of |U|.

<table>
<tr><th>Subject</th><th colspan="2">Overlap</th><th colspan="2">Subset</th><th colspan="2">Disjoint</th></tr>
<tr><td>S1</td><td>N/A</td><td>|A&D|/|U|</td><td>N/A</td><td>|A|/|U|</td><td>N/A</td><td>|A|/|U|</td></tr>
<tr><td>S2</td><td>|U|, |A|</td><td>|A|/|U|</td><td>|D|, |U|</td><td>|A|/|U|</td><td>|U|</td><td>|A|/|U|</td></tr>
<tr><td>S3</td><td>|U|, |U|, |A|</td><td>|A|/|U|</td><td>|U|, |D|, |U|, |D&A|</td><td>|A|/|U|</td><td>|U|,|D|, |U|, |A|, |A&D|</td><td>|A|/|U|</td></tr>
<tr><td>S4</td><td>|D|, |D|, |U|, |A&D|, |A&D|</td><td>|A&D|/|U|</td><td>|A|, |D|, |~D|</td><td>|A|/|U|</td><td>|D|, |D|, |U|</td><td>|A|/|U|</td></tr>
<tr><td>S5</td><td>|D|, |U|</td><td>||A|/|U||</td><td>|U|, |A|/|U|</td><td>|A|/|U|</td><td>|U|, |U|,</td><td>|A|/|U|</td></tr>
<tr><td>S6</td><td>||A|/|U||, |?|/|?|, ||A|/|U||, |A&~D|/|U|, |A&D|/|U|</td><td>|A|/|U|</td><td>|U|- |D|, |?|/|?|, |U|, |?|/|U|, |U|, |D|*2|/|U|*2| , |~D|*2|/|U|*2|,|D&A|, |~D&A|</td><td>|A|/|U|*2</td><td>|U|, |U|-|D|</td><td>|A|/|U|-|D|</td></tr>
<tr><td>S7</td><td></td><td>|A|/|U|</td><td></td><td>|A|/|U|</td><td>|D|</td><td>|A|/|U|</td></tr>
<tr><td>S8</td><td>|U|</td><td>||A|/|U||</td><td>|U|, |D|</td><td>||A|/|U||</td><td>|U|, |D|, |U|</td><td>|A|/|U|</td></tr>
</table>

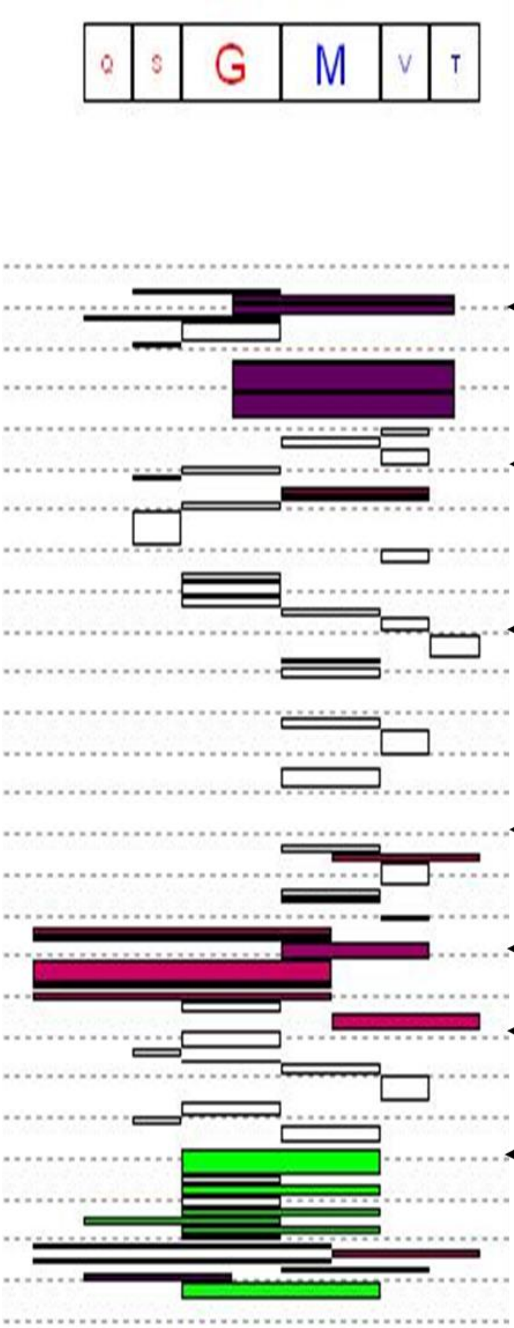
Table 6.8. Verbalised quantitative derivations for the biased problems. The left column represents verbalised quantitative derivations in order excluding the final solution. The right column represents a derivation consistent with final solution given. Shaded cells are incorrect solutions.

On correct solutions, as can be observed in Table 6.8, participants tended to verbalise normalised frequencies for |U|, |A| and sometimes |D| before providing a solution. The frequencies |D| and |D| were verbalised as either part of a *count-remainder* strategy or in the context of describing the condition for deriving normalised frequencies |U| or |A|. In deriving a normalisation scale some participants (S3, S2, S4) on multiple problems made verbal protocols that suggested they interpreted large PS-units that represent double probability as being equivalent to two letter outcomes that have non-double probability as exampled in figure 6.15. This is indicated by verbalisations such as "*it's like if I had eight letters in total*" (S3), "*double for blue means there's six for blue* " (S2), *"means we have six blue letters*" (S4).



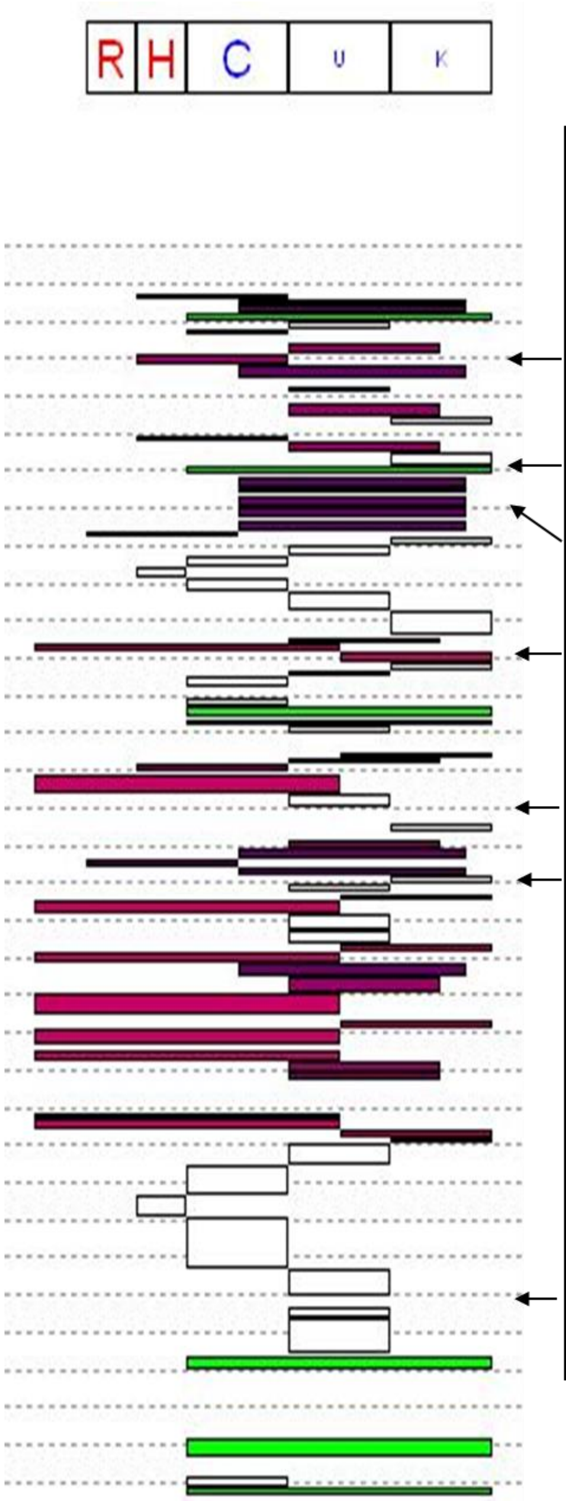
S3 Biased/overlap		
1.27		press button (start)
1.28	probabilities are double for larrge letters	encode(priors) = D, ~D
1.32	the spinner falls on a letter	encode(outcome) = U
1.34	what is the probability the letter red	encode(query) = A
1.36	so i have six letters	NORMALLY denominator = |U| = 6
1.39	but i have double for large letters	BUT re-encode (priors) = D, ~D
1.43	so its like having 8 letters	CONCLUSION |U| = 8
1.45	the probability that it's red is	re-encode(query) = A
1.5	1 2 3	counting double: |A| = 4
1.53	4 out of 8	|A|/|U| = 4:8
2		press button (end)

Figure 6.15. Coded protocols of S3 on the biased/overlap problem. The participant arrives at the correct solution by appearing to frame the double probability letters as equivelent to two letter outcomes.



<table>
<tr><th colspan="3">S5 Biased/overlap</th></tr>
<tr><td>0.18</td><td></td><td>press button (start)</td></tr>
<tr><td>0.19</td><td>so probabilities are double for large letters</td><td>encode (priors) = D, ~D</td></tr>
<tr><td>0.23</td><td>so there’s one, two , three four five, six letters</td><td>counting: |U| = 6</td></tr>
<tr><td>0.27</td><td>two of them are double</td><td>|D| = 2</td></tr>
<tr><td>0.32</td><td>so thats gonna be, so its gonna be over 8</td><td>CONCLUSION
|U| = 8</td></tr>
<tr><td>0.35</td><td>whats the probability the letters are red</td><td>re-encode(query) = A</td></tr>
<tr><td>0.37</td><td>so its half anyway</td><td>CONCLUSION
||A|/|U|| = 1:2</td></tr>
<tr><td>0.4</td><td>so it’s a half</td><td>CONCLUSION
||A|/|U|| = 1:2
CONFIRMED</td></tr>
<tr><td>0.43</td><td></td><td>press button (end)</td></tr>
</table>

Figure 6.16. Coded protocols of S5 solution procedure on the biased/overlap problem. The participant appears to provide the correct solution by noticing the geometrical proportion between target sets.



S5 Biased/subset		
0.42		press button (start)
0.45	so its falling on a letter	encode(outcome) = U
0.48	probabilities double for blue so	encode(priors) = D, ~D
0.49	so 1,2,3,4,5,6,7, 8	CONCLUSION counting double: |U| = 8
0.53	probability its small is errr	encode(query) = A
0.57	oh not 2 over 8	|A|/|U| = 2:8 INTERRUPTED
0.59	four over 8	|A|/|U| = 4:8
1.1	i'll go 4 over 8, im not sure about that, i think, yeah a half	||A|/|U| = 4:8, |A|/|U|| = ½ CONFIRMED
1.14		press button (end)

Figure 6.17. Coded protocols of S5 solution procedure on the biased/subset problem. The participant provides a correct solution by using a weighted count stratgy.

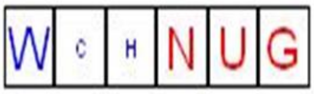
Three (S8, S5, S6) participants produced verbal protocols consistent with deriving geometrical proportions (see figure 6.16 for an example of this strategy). For S5, recognising the geometrical proportion appears to have overridden using a frequency procedure: "*It's gonna be over eight, What's the probability the letters are? so its half anyway*" (S5, biased/overlap). For S8, he/she immediately recognised when the queried possibilities are identified: "*the probability the letters is small? is a half, lets look at the diagram, naught point five"* (S8, biased/subset). Another of participant's initial solution (S6, biased/overlap) also seemed to be derived from the geometrical proportion, but then gave an incorrect solution based on a frequency derivation. Other participants appear to use a weighted count strategy, an example of which is shown in figure 6.17.

Conjunction problems. Conjunction problems were all answered correctly by seven of the eight participants in contrast to experiment 1. Only participant S7 incorrectly answered on the conjunction/subset problem. S7 incorrectly determined the probability from the frequencies of type values rather than tokens of possibilities, but gave correct solutions on the other two problems.

Subject	Overlap		Subset		Disjoint	
S1	N/A	|A&B|/|U|	N/A	|A&B|/|U|	N/A	|A&B|/|U|
S2	|U|	|A&B|/|U|	|U|	|A&B|/|U|		|A&B|/|U|
S3	|U|, |A|, |B&A|	|A&B|/|U|	|U|, |A|, |B&A|	|A&B|/|U|		|A&B|/|U|
S4	|U|	|A&B|/|U|		|A&B|/|U|		|A&B|/|U|
S5	|U|	|A&B|/|U|	|U|	|A&B|/|U|		|A&B|/|U|
S6	|U|	|A&B|/|U|		|A&B|/|U|		|A&B|/|U|
S7	|A|/|U|, |B|/|U|	|A&B|/|U|	|V(A)|, 1/|V(A)|, |V(B)|, 1/|V(B)|	1/|V(A)| * 1/|V(B)|	|A|/|U|, |B|/|U|	|A&B|/|U|
S8	|U|	|A&B|/|U|	|A|/|U|	|A&B|/|U|		|A&B|/|U|

Table 6.9. Verbalised quantitative derivations for the conjunction problems. The left column represents verbalised quantitative derivations in order excluding the final solution. The right column represents derivation consistent with final solution given. Shaded cells are incorrect solutions. |V(X)| is the number of values of the dimension of X e,g. |V(blue)| is 2 because there are 2 values blue and red.

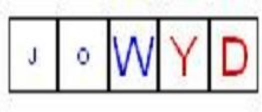
Note that on the disjoint problem in which the probability is zero only one participant verbalised making a quantitative derivation before giving a solution. Like the conditional/disjoint problem, the fact that participants (with the exception of S7) do not take |U| suggests that they encode the set structure of the referred categories before choosing to do any form of quantification and this act is not normally verbalised. As they cannot know before the act whether a set is disjoint and quantification is not needed, the observation suggests identifying the set structure of referred categories before performing any quantification (i.e. |U|) may be a matter of course for participants (see figure 6.18 as an example).

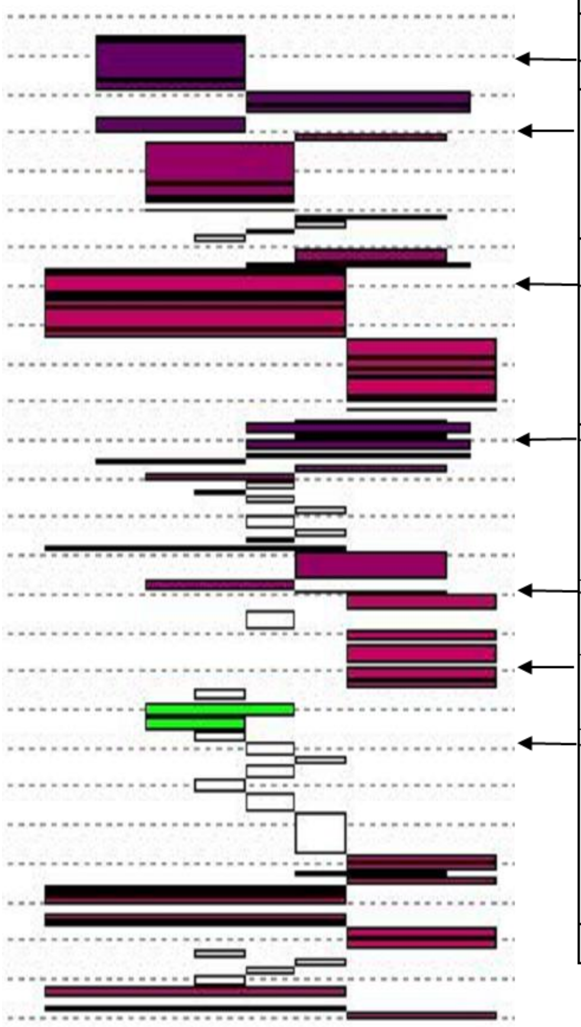


S2 Conjunction/disjoint		
3.26		press button (start)
3.27	equal for all	encode(priors) = equal
3.29	falls on a letter	encode(outcome) = U
3.31	small and red	encode(query) = A & B
3.34	all reds are big	GIVEN subset (B&-A)
3.36	so I would say none	CONCLUSION $\lvert A \cap B \rvert = 0$
3.39	END	press button (end)

Figure 6.18. Coded protocols of S2 on the conjunction/disjoint problem who provides the correct answer.

On the conjunction/overlap and conjunction/subset problems participants tend to derive |U|. In addition, only two participants appear to make quantitative derivations of either A or B alone before giving a solution. Note that for S3 this seems to be part of a strategy for determining the frequency $|A \cap B|$ because |A| is derived then $|B \cap A|$ is derived from the set A as indicated in the protocols of figure 6.19. These observations together with the relatively shorter problem solving times and correct response rates are consistent with the view that participants tended to have little difficulty in interpreting the queried possibilities for the conjunction problem. Coded examples of performance on the conjunction problem are shown below.





S3 conjunction/subset		
3.18		press button (start)
3.19	probabilities are equal for all letters	encode(priors) = equal
3.21	the spinner falls on a letter	encode(outcome) = U
3.25	what is the probability the letter is blue and small	encode(query) = A & B
3.29	so we have five letters	CONCLUSION |U| =5
3.33	three of them are blue	|A| =3
3.35	only 2 of them are small	|B∩A| =2
3.37	so it would be 2 out of 5	CONCLUSION |A∩B|/|U|
3.43	END	press button (end)

Figure 6.19. Coded protocols of S3 on the conjunction/subset problem who provided the correct answer.

Summary. The results of the experiment reveal details about the strategies typically employed in solving the problems, how participants interpreted the represented information, and the solution procedure errors made. To recap: observed errors replicate some of those observed in experiment 1; the protocols suggest that some participants sometimes used solution read-off procedures afforded by the diagram as described in Chapter 4 in contrast to known or adapted arithmetic algorithms that operate on probabilities; the diagrammatic scheme was sometimes used in the process of problem interpretation and solution procedure formulation and the diagram facilitated recognition of unplanned consequences of problem data. Analysis of eye-movement patterns replicate the continuous dependence of the problem presentation in the process of problems solving and the apparent in situ use of externalised information.

6.4 Discussion

The section discusses the implications of the experimental data. The discussion is divided into 5 sections. Section 1 discusses the effects of the changes to the methodology, section 2 discusses the patterns of solution procedures errors and what information they provide about representation and reasoning processes in the task. Section 3 discusses the interpretation processes on correct solution procedures, and section 4 describes the problem solving strategies that were implemented to support information acquisition for problem interpretation and the executions of planned solutions procedures.

6.4.1 Methodology

The changes to the design of the problem presentation and experimental instructions appeared to have the predicted effects on performance. The presence of examples in the verbal protocol instruction appeared to elicit greater verbalisation by participants during the task, but this, in turn, increased the time to perform the task. The requirement to read the text aloud substantially extended the time of first pass reading of the problem instruction. The changes to the conjunction statements resulted in a major reduction in solution procedure errors for conjunction problems compared to experiment 1. The changes to the disjunction statement resulted in some participants assuming that a component answer was required for each model expressed by the disjunction, which did not occur in the first experiment. These instruction presentation changes show how sensitive participants' performance is to the wording of problem instructions – even though such problems are clearly less conceptually challenging than those typically reported in experimental research.

6.4.2 Problem interpretations

Conditional problems

A common interpretation error in conditional problems was failing to infer the limitation of trial possibilities in problems where this was required (i.e. conditional/subset and conditional/overlap). Participants who gave a correct solution on these problems showed signs of initially making this omission or considering a default interpretation that implied this omission in the course of the trial. It cannot be the case that participants who failed to infer the limitation of trial possibilities did not observe that C was a subset of U and was a smaller set than U because these same participants used C in deriving limitations on queried possibilities from $A \cap C$ in conditional/disjoint and conditional/overlap problems. This also implies that all participants (except perhaps S7) did interpret the outcome statement as conditional information and that the problem for these participants was specifically inferring the implications of the limited trial possibilities, even though they showed evidence of accessing all the required information necessary to make the inference.

The importance of the instruction in making conditional inferences is suggested by the pattern of errors on different conditional problems. Namely, the observation that participants who omitted inferring limited trial possibilities did correctly infer limited queried possibilities whenever it was required. A simple explanation of this pattern can be attributed to differences in how informative the problem instruction was on the former, but not the latter inference. It is proposed that when participants interpret the query statement, they infer that the outcome of the queried category must also have the category of the conditional because the meaning of the instruction designates models of the same outcome token. The information allows participants to infer the criteria for possibilities in the queried set as being those that are members of both the queried and conditional category, directly from the instruction. This is consistent with protocols indicating participants immediately responded in surprise after recognising that the sets in the queried and conditional category are disjoint on first inspection of the diagram; presumably because these participants had already formulated this expectation before searching for possibilities in the queried category.

Participants solved the conditional problems in a relatively short time and thus engaged in less reflection about the problem (compared to biased and disjunction problems) suggesting that the problems posed less of a challenge independently of whether participants gave correct solutions or not. Participants who gave an incorrect answer did not spend time deliberating about the conditional information implying that they did not foresee the need to question or assess the correctness of their interpretation. Making conditional inferences about limited possibilities is arguably an everyday cognitive activity. However, it is likely that the requirement for such inferences need appropriate conditions to be framed. Given this claim is correct, the omission errors observed in this experimental context may result from a problem in setting the required framing conditions for initiating the inference of limited possibilities. Whilst the structural relation between U and C may facilitate this framing the information alone is not sufficient in the problem context.

For those participants who answered correctly, the observation that these participants began counting U and provided verbal reports suggesting interruption of the goal to derive the denominator from U suggests that they did not anticipate the requirement and solution implication of this information upon reading the instruction. Rather, these participants initially adopted a default interpretation of what counts as the trial possibilities (i.e. U), despite inferring the conditional dependences of C on the queried possibilities. The interruption occurred after verbalising $|A|/|U|$ or $|U|$. This is normally at the juncture where participants focus on the queried possibilities that would have included consideration of the conditional set relation and implied possibilities between A and C. One possibility is that the conceptual context in conjunction with recognition of the subset relation between C and U may have provided the necessary cues for initiating consideration of the limited possibilities of U in participants who recovered from the omission. Another possibility is that these participants set a prospective goal to consider the stated outcome information when addressing the queried possibilities because, on reading the instruction, they had inferred its implication on queried possibilities (but not the trial possibilities).

Disjunction problems

The main interpretation error for the disjunction probability problems was with the disjunction/overlap and disjunction/subset. For these problems a number of participants appeared to derive the numerator by adding the frequency of each model or by adding the probability of each model in the disjunction query. The procedure may result from an interpretation of the disjunctions as involving sets of mutually exclusive categories of possibilities and/or an omission in considering the implication of the set structure between alternative sets of possibilities on the value of the resulting numerator. Given the latter is true this suggests that participants' interpretation underpinning the use of the adding procedure is partially correct, but under-specific because it fails to consider the set structure between alternative sets.

None of the participants produced a resulting solution consistent with this error. Instead, participants interrupted and abandoned the procedure. It is clear from the verbalisations of some participants the interruption is based on the observation that the resulting probability is too high. This invites the question of what information participants use as a basis for making this judgement? One possibility is that they make this judgment by comparing the result of summing the probabilities of each model (i.e. |A|/|U| + ||B|/U| + |A&B|/|U|) to the relative geometrical proportion of ||AUB||/||U|| or the numerator of the result against some impression of the quantity of |AUB|. Another possibility is that some quantitative apprehension of the complement of AUB is used and judged against the remainder of the resulting probability. Whatever the basis of the judgement may be, the observations of the judgement suggest that participants must employ checking strategies to evaluate the results of calculations as a matter of course.

Verbalisations seem to suggest that the solution was based on an initial guess. As most participants answered correctly, the guess is clearly not an arbitrary choice of a set, but based on one or more facts that constrain and/or explain the result. The information that the union is the correct solution follows from several observations that could have been made: (1) the value of AUB is less than the value of the add-probabilities procedure, which was the reason for rejecting it; (2) the reason why adding probabilities is higher that AUB is because the former does not involve repeating the count of A∩B; (3) AUB is also consistent with an interpretation of the query as involving no mutually exclusive categories. The diagram provides conditions for demonstrating the meaning and implications of an instruction whether given or self-generated.

Biased problems

The most common error on the biased problem involved omitting the normalisation of the frequency of a sets used in the probability proportion. This error was also performed by participants in the pilot experiment. One possible reason for the omission is that participants just made a slip. That is, they planned or understood the requirement, but accidentally failed to execute it. However, the verbal protocols of S6 suggest that this was not the case, at least for this participant. The participant seemed to believe that the double probability information does not change how the denominator must be quantified, but only the queried possibilities. This implies that S6 did not apprehend the constraint that assuming double probability of some PS-Units changes the relative probability of non-double PS-units.

The protocols provide evidence that participants appeared to use the geometrical representation of probability to interpret a solution to the quantification sub-problem. These participants interpreted double probability PS-units as being equivalent to two units that have half the probability. This conceptualisation has a diagrammatic correspondence in which double probability units can be viewed as made of two non-double probability units because they have double the width. Participants appeared to be interpreting a geometrical normalisation scale that they used to adapt a frequency based sub-procedure for determining the probability proportion as though the problem were equiprobable (e.g., $|A|/|U|$) by taking the frequency of normalised units in A and U rather than actual token possibilities. The diagram also has a potential role in expressing why the denominator can be taken from this normalised frequency. A denominator in a fraction is an expression of a number of uniform unit of relative quantity to a whole. Counting the trial possibilities to get the denominator is equivalent to deriving a denominator from a relative proportion/probability of an outcome because there is an assumption of uniform quantity of the number of denominator units. The diagram thus shows this equivalence through its structural constraints. The assumption that participants interpreted the fraction scheme and understood the equivalence between the geometrical and frequency based derivation of the proportion on the biased problem is supported by participants who spontaneously read-off the geometrical proportion on these problems to provide a solution.

The diagram expresses frequency and probability information simultaneously, but does so in a format that shows constraints on the equivalence of expression from either perspective which can be demonstrated. Participants appear to exploit knowledge of this equivalence in interpreting a frequency based solution from the geometrical structure of the diagram using a normalisation scale, taking opportunities to derive the geometrical proportion to determine a probability.

6.4.3 Errors

Solution procedure errors. Experiment 2 replicates many of the types of errors reported in the pilot experiment, such as omitting the limitation on trial possibilities implied by the conditional statement and problem data, failing to normalise the frequency of either trial or queried possibilities in the biased problem, and interpreting the intersection rather than the union of models of queried possibilities in the disjunction problem. Collectively, the data suggests that the errors may be regularities for the types of problems and presentation format tested in the experiment. The incorrect solutions given are, in general, consistent with error in incorrectly determining the form of a solution procedure rather than errors in task execution.

Task execution errors. These were not common but, when observed, appeared to involve the incorrect derivation of information from the diagram that was subsequently corrected. The protocols revealed several cases of incorrectly deriving set relations and incorrectly enumerating sets. These errors occurred even though the data was simple and the presentation format was arguably highly accessible.

Implications of errors. The analysis reveals a mixed bag of problem solving errors that fall under a similar classification to those reported by O’Connell (1999). We have distinguished between errors that result from failing to consider implications of problem data, errors that result from a difficulty in interpreting the logical constraints of the solution, and task execution errors involving the incorrect derivation of information. The latter two can be classified as conceptual errors and procedural errors in O’Connell's framework.

6.4.4 Format advantages

The study has aimed to identify the process of problem interpretation at a methodologically tractable time scale. Performance on these different problems highlights different roles of the diagram in determining a correct solution. Three different roles are identified: (1) sub-problem identification, (2) framing a solution procedure and (3) establishing and distinguishing the specific referential meaning or alternative meanings of a given or self-generated instruction.

Sub-problem identification. The protocols of some participants suggest that they employed the diagram in the identification of sub-problem requirements. Recognition of the implications of structural information is observed in three problems, notably, the conditional, but also the disjunction and biased problems. This type of support is most explicit in the conditional problem in which the relevant information was the subset structure between C and U and its implication in interpreting a limitation of trial possibilities. In the disjunction/subset and disjunction/overlap, it is the implication of the overlapping structure between alternative sets that determines what procedure legally constitutes the reference and cardinality of the set of queried possibilities. On the biased problem, some participants made verbal protocols that expressed consideration of a default equiprobable assumption and engaged in redundantly counting U before deriving a normalised frequency.

These cases described are ones where critical sub-problem features (e.g. the set structure of C and U) of are not initially considered before a solution or parts of a solution are planned. The specificity of the diagram ensures that the relevant information conditions (e.g., set structure) are present. In addition, the context is one where default assumptions about what solution procedure to apply are made by problem solvers and will be applied only if information condition are recognised to override such assumption. This suggests that structure of probability problems is such that problem solvers can act in accordance with an assumption of default problem features (e.g., equiprobability, unconditional) unless additional information is recognised and framed as relevant. This could explain why diagrams can have a particular utility in probability problem solving because they facilitate recognition of potentially omitted sub-problem features for determining a solution.

Framing a solution procedure. Another form of cognitive support suggested by the experiment is the use of the form of the scheme in framing the form of the solution procedure. This is most clearly suggested by protocols of participants solving biased problem. The scheme's geometrical representation of probability was exploited by participants in interpreting how the denominator could be derived from the diagram by seeing how non-double probability units could be used as a uniform scale unit and how this interpretation was a consistent or equivalent way of determining the denominator. The framing in question concerns the apprehension of a scheme or system. It is not only the feature of the representation that is encoded (e.g., the width of PS-units), but how the feature is part of a scheme (e.g., part-whole/geometrical fraction scheme). This is because one needs to apprehend why interpreting double probability units as two outcomes and using this normalisation rule to enumerate them provides an equivalent proportion of the denominator and numerator values. The scheme is an instance of familiar part-whole geometrical schemes, such as pie charts. The framing is tapping into existing prior knowledge of diagrammatic systems. As described in Chapter 2, diagrammatic systems can be seen as more general than conventional external representations (like pie charts) and can include ubiquitous types of spatial analogies or metaphors, such as the trajectory metaphor of time that is used in everyday thinking as well as conventionalised external representations. Geometrical proportions of spatial extent may also constitute a common informal spatial analogy/representational scheme for modelling proportions or fractions of non-geometrical data.

Solution read-offs. Analysis of participants' protocols suggest that participants typically employed the solution read-offs on problems that involved referential sub-problems as required by the disjunction, conjunction and conditional problems. On the biased problems, the solution read-off that involved deriving the probability from a geometrical proportion was used by a small number of participants. In disjunction problems, some participants employed an abstract add-probabilities procedure implying an interpretation of derived component possibilities. S1 also showed cases of applying abstract procedures on other problems and, interestingly, unlike the other participants, showed a tendency to make eye-movements to blank regions of the problem display consistent with his verbalised abstract procedures.

6.4.5 Strategies

The experimental data provides information about the nature of subtask scheduling, the encoding of problem information and the argument structure employed in the problem solving episode.

Subtask scheduling. A number of strategic regularities in the scheduling of subtasks were observed that were general to all and specific to some of the problems. These regularities suggest an underlying knowledge structure of subtask execution order that had been initially constructed and reused over the course of performing the experimental trials. Due to the small number of practice episodes in the experiment, this knowledge must be represented declaratively rather than being part of procedural knowledge. Initial task ordering choices will be based on different factors including experimental instructions, inferences of logical subtask dependencies, generic knowledge about word problems and PPS tasks as well as generic task ordering heuristics. These subtask scheduling regularities suggest the requirement for the cognitive model to incorporate a declarative representation of the task schedule.

Argument structure. Verbal protocols of some participants expressed an underlying argument structure**.** Participants were not instructed to justify or explain their solutions, suggesting that argument verbalisations were motivated by the problem demands. The argument verbalisations were also commonly made in situ rather than retrospectively after a full plan and solution had been worked out; this suggests that the content of argument verbalisations tended to be a reflection of participants' ongoing thinking and was a reflection of cognitive representations and processes that were functionally involved in task performance. Expressions of exception, implication and explanation indicate that participants represent the meta-cognitive role of acquired and projected information in higher-order meta-cognitive relations. The roles identify patterns of specific cognitive activities; for example, the verbalisation of an exceptions (indicated by terms such as "but") identifies the consideration and rejection of a default interpretation that is overridden by some piece of information. This invites questions about how to characterise and model such meta-cognitive states algorithmically and what functional role these representational states play in the reasoning process.

Attention allocation. Eye-movements tended to be continuously made to text and diagram interest areas. Eye-movements protocols reveal many repeated saccades to problem relevant information in the diagram and text. The frequent eye-movement revisits could be explained in different ways including (a) a choice strategy in which visual access is used as a substitute for memory retrieval and maintenance, (b) as a meta-cognitive strategy to visually rehearse previously encoded items, (c) as a side effect of reconstructive episodic memory in which retrieved information activates eye-movements associated with the encoding source, or (d) as a general search heuristic in problem and solution interpretation in which processing is directed at repeatedly re-encoding target information until something comes to mind. Neither of these explanatory abstractions are necessarily mutually exclusive. This data provides a challenge to account for the eye-movement patterns and the frequency of eye-movement repetitions in a cognitive model that is consistent with one or more of these abstractions.

6.4.6 Conclusion

The second experiment provided richer verbal protocols than the first pilot experiment. These protocols aided identification of subtask scheduling and solution interpretation strategies of participants. The frequency of solution procedure errors, time taken to perform the task and the period of apparent reflection suggest that participants typically did not have practised solution procedures that could applied to all features of the problems. Particular solution read-offs afforded by the representational format were spontaneously employed by participants for a number of problems together with abstract arithmetic procedures that operate on proportions and frequencies. Analysis of the protocols provide evidence that in some cases the diagram appeared to support the process of problem solving by (1) facilitating unplanned consideration of data implications on a solution procedure, and (2) facilitating solution procedure formulation by providing a frame or scheme for deriving a solution interpretation and (3) allowing participants to demonstrate the specific meaning and implications of a solution interpretation.

Chapter 7: ACT-R Cognitive Architecture

The chapter briefly outlines the ACT-R cognitive architecture used in the project. The models were developed in the most recent version of ACT-R, called ACT 6.0. The next section discusses the rationale for choosing the cognitive architecture, followed by an overview of ACT-R, the symbolic and sub-symbolic constraint on knowledge in ACT-R, and the modular structure of ACT-R.

7.1 Rationale for modelling framework

The ACT-R cognitive architecture was chosen because it was judged to best satisfy the most requirements for the research goals. A major pragmatic requirement given the complexity of the to-be-modelled task was that the architecture needed to be suitable to support analysis and experimentation with different strategies and knowledge representation possibilities. An abstract production system modelling approach was judged to be best suited to these requirements for which ACT-R is an instance. In addition, a cognitive architecture was also required to provide a framework for modelling interactions with external diagrams. ACT-R has perceptual motor modules and a well-developed modelling framework for supporting the development of such models.

In addition to the pragmatic requirements satisfied by ACT-R, this cognitive architecture also has a number of empirical and theoretical credentials that made its choice favourable over other cognitive architectures. ACT-R has long history of empirically informed research development and has been used to model a wide a variety of empirical phenomena, from reaction time tasks to complex problem solving. Many of these models are either publicly available or described in sufficient detail to inform the development of new models. Many aspects of ACT-R, in particular its theory of memory, have been empirically assessed on the ability to make fine grained predictions about the time course of processing, not just modelling accounting for qualitative aspects of behaviour data. The production of real time metrics is another important criteria for choosing ACT-R to model the target tasks because such data can be matched against the verbal and eye-movement protocols collected in the experiments allowing temporal constraints to be exploited in the development and evaluation of cognitive models. This is particular important given the goals to formulate explanations of behaviour that are grounded in specific details of processing.

7.2 Overview of architecture

ACT-R is a theory of a hybrid cognitive architecture that is implemented in a computational modelling framework. ACT-R partitions knowledge into declarative and procedural types, which are modelled by chunks of attribute value pairings and production rules respectively. The selection of both types of knowledge are governed by mathematical theories of sub-symbolic processing, which determine the continuous activation levels of chunks in declarative memory and utility values assigned to production rules in procedural memory. The architecture consists of a central procedural module and a set of specialised processing modules that deal with cognitive, perceptual and motor operations. Each of these modules interacts through the contents of one or more of its buffers. The specialised modules perform local operations and their buffers hold declarative chunks that result. A central production system selects productions in a serial manner based on the states of modules and the contents of their buffers. Selected productions route information between modules in parallel which initiate modular operations changing the contents of their buffers. Changes to the state of buffers cause new productions to fire, which in turn, move the cognitive system in to a different state. Cognition thus unfolds through cycles of parallel interactions between modules mediated by the central procedural module.

7.3 Symbolic and sub-symbolic knowledge

ACT-R theory assumes two kinds of knowledge: declarative and procedural, each of which is taken to have a distinct functional role within the cognitive architecture.

Declarative knowledge. According to ACT-R theory, a distinguishing property of declarative knowledge is that can be used for different purposes and is normally considered accessible to introspective processing. In ACT-R, types of knowledge encoded declaratively include semantic facts, episodic memories, goals or intentional states, knowledge of how to do things, spatial relations, perceptual states and motor plans. Declarative knowledge is organised into discrete knowledge structures called chunks. A chunk in ACT-R holds a set of references to other chunks in its slots.

Two kinds of values of slots may be distinguished; ones that refers to chunks corresponding to semantic types (e.g., the identity of a number) and one that refer to chunks that are episodic representations (e.g., an episodic memory trace of encoding a number). The difference between the two cases concerns their access. Semantic chunks are part of long term memory for which similarity associations (which are not chunks) exist (e.g., the semantic chunks for the numbers one and two will be semantically related in a fully specified model). This allows access to be based on similarity based retrieval or partial matching as it is often termed. Episodic chunks have no similarity relations between other chunks in declarative memory hence their content is opaque. In these cases, the slot reference is often construed as holding a pointer to the chunk.

Slots are type defined with the type normally specifying the attribute or role of the referred chunk. Consider the canonical example shown in Figure 7.1, which represents the fact that 2 + 3 = 5. The chunk is made up of three slots; *arg1*, *arg2* and *sum*. Each slot is bound (or said to hold) to the reference of another chunk, which corresponds to the value of the chunk's slot; for example, the chunks for numbers *two*, *three* and *five* in the example.

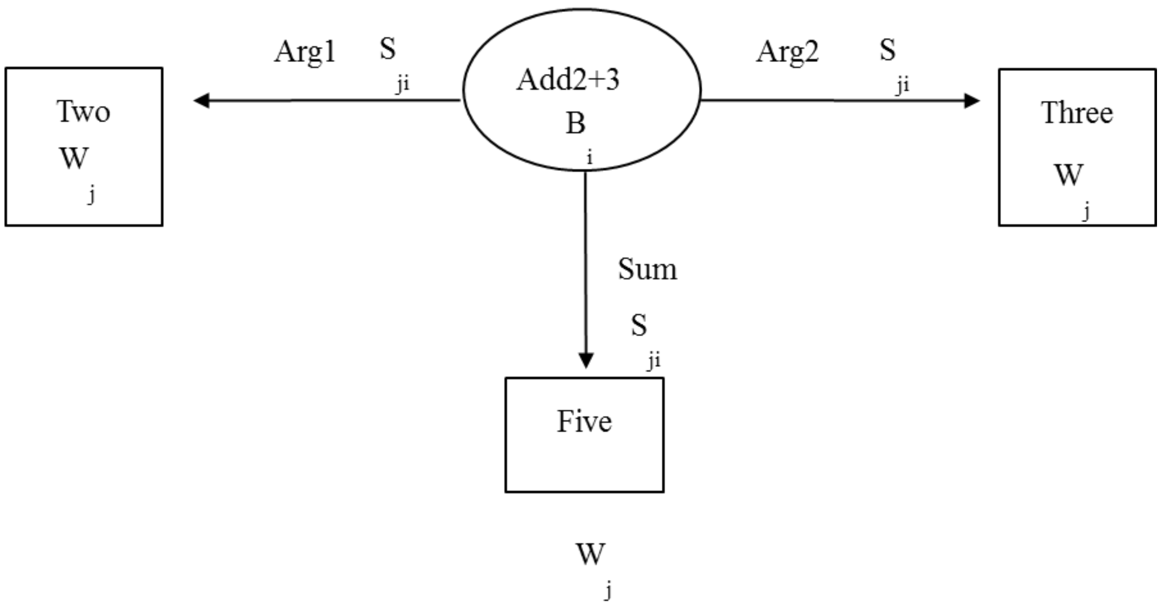


Figure 7.1. Sub-symbolic properties for a declarative chunk adapted from Anderson (2007).

The sub-symbolic level of chunks are characterised in term of their activation (see Anderson, 2007; Anderson et al., 2004). The activation level of a chunk determines, at a given point in time, whether and how quickly it can be retrieved. Chunks have a base level activation and can receive an associative activation from other chunks in buffers. The activation A_i of a chunk $_i$ is calculated by the following equation:

$$A_i = B_i + \sum_j W_j S_{ji}$$

In the above equation, *Bi* corresponds to the base-level activation of the chunk *i*, *Wj* is described as the attentional weighting of those source chunks in slots of the chunk in the current buffer, and *Sji* is the strength of association between source chunks *j* to *i*. The associative part of the equation consists of the attentional weights *Wj* that are set to the result of 1/*n* where *n* is the number of sources of activation. The *Sij* are computed as:

$S_{ji}=S - \ln(fan_{ji})$

where the so called *fan i* is the number of chunks associated with source *j*. The value of *S* is a parameter that is typically set at 2. Base level learning determines the change in the base level activation of a chunk as a function of its use. This is defined by:

$$B_i = \ln\left(\sum_{j=1}^{n} t_j^{-d}\right).$$

where *n* is the number of presentations of $chunk_i$ to declarative memory, t_j is the time elapsed since the *j*th presentation of the $chunk_i$ and *d* is a decay parameter. The equation reflects the log odds that a chunk will reoccur as a function is past appearance (Anderson, 2007). The probability that a chunk *i* will be retrieved, depends on whether its activation exceeds a threshold. The equation that governs the probability of a chunk being retrieved is:

$$P_i = \frac{1}{1 + e^{-(A_i - T)/S}}$$

The parameter *T* is the retrieval threshold and *S* is value that determines noise in the activation level. If the chunk is retrieved, the retrieval time is defined as:

$$T_i = Fe^{-A_i}$$

where *F* is a latency factor.

Procedural knowledge. Procedural knowledge is knowledge of how to perform a particular skill such as operate a keyboard, drive a car, make a cup of tea or solve a quadratic equation. Unlike declarative knowledge, units of procedural knowledge are purpose specific and unavailable to introspective processes. Procedural knowledge is organised into primitive units called productions. Productions in ACT-R relate a set of conditions about the states of the cognitive system with a set of actions that change that state. The information stored in the condition side of a production is a specification of the content of chunks held in buffers and the processing state of modules required for the production to fire (i.e. to be executed). The information stored in the action side specifies the transfer of information between buffers and processing requests made to modules. In ACT-R models, productions have a specific functional meaning, that is, they are specifications or action-selection contingencies established through a form of reinforcement learning. They are not to be confused with intuitive notions of rules. There are several different ways in which rules can be said to be represented or emerge within ACT-R and productions only constitute one specific functional sense.

Productions in ACT-R models are idealised knowledge structures modelled at a high level of abstraction. There are different levels of abstraction that productions can be modelled at including connectionist and computational neuroscience levels. For example, current neural level accounts propose that productions are stored in neural circuits in substructures of the basal ganglia and emerge through complex dynamic interactions within these circuits (Stocco & Anderson, 2008). The critical point is that the choice of modelling abstraction is a matter of pragmatics rather than an exclusive theoretical commitment. ACT-R is typically used to model high-level and complex cognitive phenomena for which the role of strategy and contribution of all components of the cognitive system is an important explanatory feature of the model.

Productions in ACT-R have constraints on what pattern matching capabilities and actions are possible. The conditions specified in productions need only be partial specifications of chunks in buffers and the processing states of modules. The contents of chunks specified in productions are slots that may have constants, variables or may be empty. If a variable is specified in a production then a condition is signalled that a chunk must be bound to that slot irrespective of its content, if a constant is specified then the chunk with a specific semantic value must be bound. An empty slot is treated as not present in the chunk. Productions in ACT-R can be seen as implementing three classes of operators between slot values of conditions: an AND operator (i.e. each slot value specified), an equivalence operator (i.e. slot values that have the same variable or constant value), and negation (i.e. where a variable of constant value does NOT hold). A single production cannot respond to a disjunction of slot values; hence the system would need to have different productions for each of the disjunctive possibilities.

Productions' conditions can apply to queries about the processing state of modules. Information made available by modules include whether a chunk is present in the buffer, whether a buffer chunk was created by the corresponding module in response to a request, whether a module is busy processing information (where multiple processes are possible what class of process is being performed). The main reason for the processing state queries is to determine the availability of processing resources so that resources are not requested when they are used and can be used as soon as they become available. Actions that can be taken by productions include modifying multiple slot values of chunks within buffers with either constants or variables, copying complete chunks to buffers, clearing chunks from buffers and sending complex retrieval or perceptual commands to modules via partial chunk specifications.

Productions can be static or dynamic. In static productions the slot type is a specified constant whereas in dynamic productions the slot type can also be bound to a variable. Consider the two productions in Figure 7.2, where the retrieved value of the *sum* slot of an addition fact in the retrieval buffer is copied to the imaginal buffer. In the static production the identity of the *sum* slot is specified in the retrieval and imaginal chunks of the production. This means that the production will only respond when the specific semantic conditions hold (i.e. the *sum* slot pattern across buffers).

Now consider the dynamic production, which can do the same job. In the dynamic production the *sum* slot is not specified, but instead, a variable slot is specified labelled =result that could take on the value of the *sum* slot (one could also imagine the same production responding to the result of a multiplication or subtraction fact). For a slot to be a variable in a dynamic production, it must also be the value of at least one static slot as shown in the example (i.e. =goal> result [=result]). This production could fire for any case where the value of the result slot is a variable slot retrieved from declarative memory. On the action side, in the example, the variables for the slot *=result* and its value *=value* are copied to the chunk in the imaginal buffer. Note that it does not matter if the slot bound to the variable =result is part of the imaginal chunk as the dynamic production mechanism allows chunks in buffers to be extended with new slots.

Static production

Condition
Goal>
  result [sum]
Retrieval>
  sum [=value]
Imaginal>
  sum [?]

Action
Imaginal>
  sum [=value]

Dynamic production

Condition
Goal>
  result [=result]
Retrieval>
  =result [=value]
Imaginal>
  =result [?]

Action
Imaginal>
  =result [=value]

Figure 7.2. Static and dynamic productions that can carry out the same operation. The dynamic production provides a more general operation in which the retrieved value of some result is updated in the imaginal buffer.

Productions in ACT-R also have a sub-symbolic level, which determine their selection probability in addition to symbolic pattern matching. The selection probability of a production is based on a measure of its utility to the current goal. As production utility is not used in the model, the mathematical details of production utility computations will not be discussed further.

7.4 Modular structure

ACT-R 6.0 has several specialised processing modules that deal with specific functions of central, perceptual and motor cognition (see Figure 7.2). Central cognitive modules include a goal module that keeps track of processing intentions, a declarative memory module which processes the storage and retrieval of memories, an imaginal module that holds and manipulates problem states, and a procedural module which selects productions based on the state of the cognitive system. Visual and auditory modules handle requests to search and process modality specific information, a manual motor module controls hand movements, and a speech module processes requests to generate vocalisations. Modules have one or more buffers, which act as an interface to the module. Chunks may be created in buffers, modified and maintained over the course of processing. Modules can only hold and execute processes on a single chunk at time. In addition to the content of their buffers, modules can signal abstract information about their availability for processing. The core modules of the architecture have been empirically linked to particular brain regions, where the primary neural circuitry behind the modules is believed to take place. These associations have been based on a combination of general findings in cognitive neuroscience as well as specific studies which explored bold response predictions of ACT-R models in fMRI tasks.

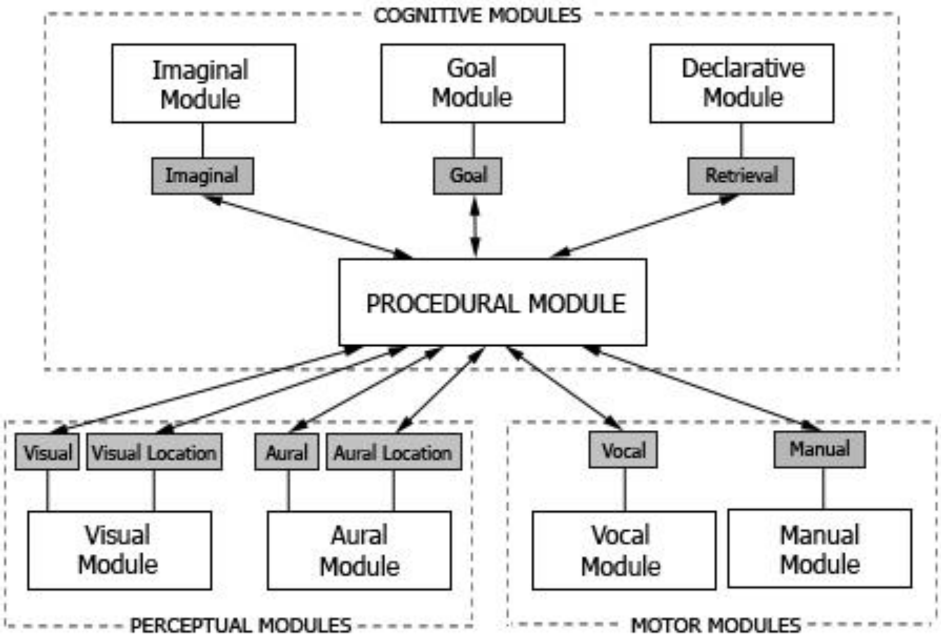


Figure 7.3. Modular architecture of ACT-R 6.0.

Procedural module. The procedural module is involved in the learning, storage, selection and execution of productions. The module is proposed to be implemented in the basal ganglia and associated neural structures. According to the current theory, the striatum is involved in recognizing patterns in buffers, the palladium performs conflict resolution and thalamus participates in controlling the execution of productions. According to Anderson et al. (2004), a production rule can be considered a "*specification of a cycle from the cortex, to the basal ganglia, and back again*" (p. 1038). The procedural module can only execute one production at time and each production takes approximately 50ms to execute.

Goal module. The goal (or control) module contains a buffer which holds chunks that contains abstract control information. This information is specific to particular intentions carried out by the system. The content of the control buffer is determined by productions that make requests to construct and modify control chunks. The goal module is said to track the state of the cognitive system as the execution of a cognitive task unfolds. For example, if a request is made to retrieve information from declarative memory then the procedural system may update the goal buffer with a control state specifying that a retrieval is being made. This allows productions to respond to the result of the retrieval request conditionally on the goal state. Control states can thus be seen to provide the necessary conditions to prune the selection of concurrent productions in a goal directed way.

Declarative module. The declarative module includes mechanisms for accessing declarative memories and a buffer for holding chunks that result. The module is associated with the functioning of the lateral prefrontal cortex. Retrieval requests are made to the module by productions that provide a partial specification of chunk. The information specified can be viewed as retrieval cues that constrain search for the appropriate memory. The retrieval module can execute a partial matching search so that chunks that are similar, but do not exactly match the retrieval constraints, have a chance of being retrieved. If the retrieval is successful a chunk matching the request is generated in the retrieval buffer. If the retrieval fails, a module state is generated which signals an error. The retrieval takes some interval of time in accordance with its activation equations.

Type	**Modules**	**Module function**	**Buffers**	**Buffer Properties**	**Knowledge**
Cognitive	Procedural	Action selection, reinforcement, learning	Procedural	Input, output	Productions
	Goal	Construct & modify intentional states	Control State	Input, Output, Maintenance, Modifiable	Unspecified chunks
	Declarative	Memory retrieval, declarative learning	Retrieval	Input, Output, Maintenance	Unspecified chunks
	Imaginal	Construct and modify problem states/images	Imaginal	Input, Output, Maintenance, Modifiable	Unspecified chunks
Perceptual	Visual	Visual search & attention processing	Visual location	Input, Output, Maintenance	Visual-location chunks
			Visual Object	Input, Output, Maintenance	Visual-object chunks
	Auditory	Auditory search & attention processing	Auditory location	Input, Output, Maintenance	Auditory-location chunks
			Auditory Object	Input, Output, Maintenance	Auditory-object chunks
Motor	Manual	Hand movements	Manual	Input	Manual command chunks
	Speech	Speech production	Vocal	Input	Speech command chunk

Table 7.1. A tabulation of properties of modules in ACT 6.0.

Imaginal module. The imaginal module is part of central cognition and has a single buffer. Its main functions are the maintenance and manipulation of internal representations of states of a problem. Maintenance of chunks in the imaginal buffer are normally required when the cognitive system needs to interrelate or integrate serially acquired information from perception and memory. One may think of the imaginal module as a system for processing representations of an external environment (images) because chunks represented in the imaginal module typically carry information about real or imaginary external state of affairs. It can be contrasted with the goal module that represents internal states about the cognitive system. Information changes to the imaginal buffer consume processing time. The module is proposed to be associated with functions computed by the posterior parietal cortex.

Perceptual modules. The visual module models abstract aspects of visual attention processing sufficient to make rough predictions about the time to encode visual information in complex tasks. The module comprises of visual location and visual object buffers corresponding to the distinction between so called where and what processing streams. Productions make requests to search for objects by specifying search constraints. If the search proves successful a location chunk is placed in a corresponding location buffer. The visual location chunk holds information about the position of an attended location. In the case of vision, the position information is specified in a retinotopic frame of reference. To process the perceptual features of the located object, the procedural system must make a request to the module to shift attention to the location of the object. If the request is successful, a corresponding visual object chunk that binds the different perceptual features of the object together (e.g., colour, form, size, etc.) is placed in a visual buffer. These processes take time to execute.

In ACT-R, tags called FINST (Finger Instantiations) are assigned to visual object chunks by default whenever a visual object is attended. The FINST mechanisms implemented in ACT-R is a conservative model of the original theory of spatial indexes/FINST developed by Pylyshyn (1994). In ACT-R the main (and limited) function of FINST is to allow the system to use the attended status of visual objects as a search criterion. For example, if the system is trying to find objects that have/have not recently been attended, productions can specify this constraint in the search requests sent to the visual module. As this mechanism resides in the visual module, central cognition cannot directly access the identity of FINST bindings. Central cognition can only establish that a FINST is assigned to an object by serially requesting a search for objects that have been recently attended and then attending to the result. The assignment of a FINST to a visual object has a default decay parameter of a few seconds so what counts as recently attended is with respect to a short time window.

Motor modules. The architecture also includes a module for processing speech and a module for processing manual motor actions. In both cases, processing requests are made by productions that transfer commands via chunks that specify symbolic parameters of a motor action. Once transferred to the module, these chunks are not maintained and no new chunks are created as a result of the motor process. Indeed, unlike other modules in ACT-R, the motor modules do not hold chunks in buffers that the procedural system can directly access.

7.5 Conclusion

The ACT-R cognitive architecture satisfies a number of requirements for modelling performance in the experiment. It has a modular organisation that embodies constraints on available resources and the time course of processing and maintenance, a sub-symbolic theory of memory that constrains and predicts the time course of knowledge access and a framework for modelling interactions with an external display. These architectural constraints are supported by empirical research.

Chapter 8: Modelling Diagrammatic Reasoning about Probability

8.1. Introduction

The aim of the modelling research is to develop a cognitive model that simulates the process by which participants solved the experimental problems. This is to be achieved by exploring and explicating the set of possible assumptions that are required to provide an explanation of participants' observed solution procedures and strategies. The purpose of this exploratory modelling investigation is to acquire information about the task model possibilities, information processing roles of the diagram in the unpractised PPS (probability problem solving) experimental task and its potential implications in understanding other PPS and problem solving tasks using diagrammatic representations.

8.1.1. Chapter outline

Section 8.2 describes generic modular mechanisms, representations and processes concerned with visual and spatial processes, with meta-cognitive processing associated with the goal module and with problem representation, and processing associated with the imaginal module and declarative inference. Section 8.3 describes the main probability concepts and how PPS knowledge is represented and organised. Section 8.4 describes how basic subtasks are implemented in the model such as reading, identifying sets, counting and determining proportions. Section 8.5 describes models of unpractised problem solving which aim to address the process of how problem identification, solution interpretation and solution procedure formulation occurs. Section 8.6 discusses the findings and implications of the modelling research.

8.1.2. Modelling aims

The main focus of the cognitive modelling is to investigate the role of the diagram in the formulation of a solution procedure. The experiment identified roles of the diagrammatic format on solution procedure interpretation. It was argued that these types of cognitive support are dependent on the information properties of the diagram (e.g. specificity, modelled constraints, token referential, visual spatial ontology) and their accessibility.

The cognitive modelling research aims to address more specifically how these different solution procedure interpretation advantages are realised at an information processing level. The general modelling aims also require addressing a set of modelling sub-problems posed by the modelling task: (1) the nature and use of PPS relevant knowledge possessed by participants, (2) the nature of generic representations and processes employed in unpractised problems solving activities, and (3) the nature of visual spatial processing implicated in the diagrammatic reasoning activities.

Evolution of cognitive model. The model has been developed incrementally as both a pragmatic strategy of model development, but also in response to the increased understanding of requirements and limitations of modelling possibilities that naturally come about when engaging in such activities. One assumption applied in the development of the cognitive model was to develop subtask strategies and other routines as independent collection of knowledge with an independent control structure. There are several arguments supporting this theoretical assumption and modelling strategy. (1) Whilst this kind of approach is common in software development and engineering (e.g., object oriented programming) its motivation in this context also has a theoretical and empirical basis. Many theories of cognitive phenomena assume that cognition implicates a hierarchy of processing that involve combining primitives to make up complex processes. (2) The task being modelled implicates a number of routine subtasks and lower-level cognitive routines that are executed across all problems (e.g., instruction comprehension, object/set identification, enumeration, proportion construction). (3) As the task is unpractised it is required that knowledge from different domains or schemas be brought together in solving the task rather that assuming proceduralisation of a unitary strategy derived from the aggregated optimisation of different subtask domains. (4) Treating the routine subtasks as relatively fixed parameters was employed as prerequisite to developing the sequence of interpretation processes that occur between the execution of subtasks. Initial models of the task were practised models that included knowledge of goals specifying parameters of a subtask. The practised model was used to initially test and evaluate the subtask strategic knowledge.

8.2 Generic modular processes and mechanisms

The section outlines modular processing mechanisms and knowledge that are specific to the cognitive model and the modelling task. Four main classes of cognitive functions and corresponding modules are discussed: (1) visual-object processing using the visual module, (2) spatial processing using the spatial module, (3) meta-cognitive processing using the goal module, (4) the processing of integrated problem information using the problem state/imaginal module.

8.2.1. Visual object processing

Visual representations. ACT-R's visual module provides an abstract model of visual attention. The chunk ontology of visual object representations is limited to text elements, lines and dialog buttons, which are not sufficient for modelling visual interactions with complex diagrams. The visual chunk ontology was therefore extended for modelling visual attention with the PS-diagram (probability space diagram) and problem display features used in the experiments. Table 8.1 lists the ontology.

Display components	**Visual object category**
Text	Text-frame
	Text-statement
	Text-entry
	Text-value
	Text-unit
Diagram	PS-space
	PS-unit-group
	PS-unit
	PS-letter
	PS-unit-boundary
	Marker-line

Table 8.1. Visual object chunks in the modelling framework.

The extended ontology is built around the assumption that visual objects in the problem display are individuated at different levels of granularity. This is normally the case in graphical cognition because represented information is typically embedded in hierarchically organised parts of diagram configuration. For example, the model assumes that the problem solver is able to distribute attention to a whole PS-diagram, subgroups of PS-units (problem space units), individual PS-units, letters within PS-units, and vertical boundary lines that separate neighbouring PS-units. The scheme supports multi-scale visual attention as have been previously reported in ACT-R models elsewhere (e.g., Anderson, 1997). The ontology of visual object types is summarised in Table 8.1. Each type of location chunk corresponds to a possibility for visual attention. Visual objects formed by groupings are based on distinct and salient properties. In the modelling framework, the visicon, which is a model of the external display, contains the space of potential individuations at different levels of granularity.

To model minimal functions of peripheral vision, additional slots were added to visual objects chunks. The chunks hold peripheral information only about the existence per se (not attribute bindings) of surrounding objects. Peripheral surround information is modelled minimally with a single slot that holds a ground part of a figure-ground pattern. The pattern indicates that there are peripheral objects to the left "?#", right "#?", both left and right "?#?" or neither "#" modelled by the corresponding notation. The peripheral information is used to make scanning decisions in low level visual attention routines implemented in the model. Evidence that the cognitive system has access to knowledge of the presence/absence of visual objects in peripheral vision is supported by studies in reading (e.g., McConkie & Rayner 1975; Rayner 1975). In addition, the observation that eye-movements are rarely made to the last letters in the PS-unit array is consistent with the claim that participants processing systems must have initially detected the end of the array via peripheral vision whilst fixating on a preceding rather than end PS-unit (one can also verify this by inspecting any of the presented scan paths of participants in Chapter 6).

In the model, perceptual groupings are represented by single visual object chunks. Note, however, that by default visual object chunks in ACT-R only represent a single value of an attribute at a time. In the model, when a group has multiple values of a visual attribute (e.g., a group of large letters containing subgroups of red and blue letters) then the corresponding visual-object chunk does not contain a value for that visual attribute (e.g., colour) indicating the value is indeterminate. In order to specify this information, the system must serially attend to each subgroup having the single target visual attribute. The constraint, however, has an upside. The information that an attribute of a chunk of a visual group is unspecified is used to indicate that there are subgroups with different values of that attribute. This is exploited by visual attention productions that implement decisions about whether to scan component groupings. This modelling constraint (not ACT-R) is motivated by and consistent with the Boolean map theory of visual attention (Huang & Pashler, 2007) that claims the visual system can only process/attend to one single value of a visual attribute of an object at a time, but can process values of different attributes in parallel.

8.2.2. Spatial representation and processing

In addition to representation and processing requirements of the visual object processing module, ACT-R also requires a spatial processing capacity to adequately model the diagrammatic reasoning in the PPS tasks. ACT-R 6.0 has limited architectural commitments for modelling spatial processing. Critical spatial processing activities that need to be addressed include the derivation of topological relations of spatial containment, relations of magnitude and orientation, and capacity to identify or individuate spatially defined groupings such as the intersection and union of two or more groups. The section discusses solutions to these problems.

Logan and Sadler's (1996) computational theory of spatial apprehension proposes that abstract conceptual spatial relations are selected with intermediate representations from a perceptual spatial representation that is analogous to a spatial array representation, but contains spatial relation information only implicitly. Gunzelmann and Lyon (2007) propose certain architectural features for a general model for spatial processing competence in ACT-R based on a survey of empirical research. Their proposals include: replacing the existing imaginal module with a spatial module and several additional buffers including an egocentric buffer to represent visual locations in egocentric frames of reference; an environmental buffer to represent spatial and magnitude relations between visual objects in an exocentric frame of reference; and an episodic buffer to hold episodic representations that index chunks from the different visual spatial buffers. Matessa and Brockett (2007) proposed a particular implementation of a framework developed by Chandrasekaran et al. (2004) for diagrammatic reasoning. In summary, their proposals include: the existence of routines that determine spatial relations through search requests to the visual location system; make metric comparisons to visual objects in the visual buffer; and project non-veridical objects (e.g., paths) on perceived scenes using a combination of visual and imaginal processing. These authors however do not report the empirical basis or an evaluation of the feasibility of the proposals.

A spatial buffer with modular functions was added to the existing architecture to support the spatial processing required for reasoning with the PS-diagram in the experimental task. To be clear, the additions are not intended as a general model of human spatial processing abilities. The spatial processing additions to ACT-R are based on recent empirical results and theoretical claims as well as analytical consideration of the feasibility of the proposed scheme. The constraints implemented in the model are summarised below.

Parallel indexing hypothesis. Computing spatially relational information between selected objects requires indexing the target objects (e.g., Pylyshyn, 1994, 2000). FINST theory proposes that the cognitive system has a capacity of about four spatial indexes that are used to compute spatial relations. Relations of relative position between objects may be computed from a representation of a set of two to five objects files (Hayworth, Lescroart, Biederman, 2011) as described by the neural object files theory (Xu and Chun, 2009). Studies of visual short term memory supports an estimate of three to four slots, where each slot holds the bindings of a unique visual object (Luck and Vogel, 1997), groups of objects (Kong, Schunn and Wallstrom, 2010), or pairs of attended visual objects (Clevenger and Hummel, 2014).

Spatial specificity hypothesis. There is a functional requirement to derive spatial relations from a perceived base representation of space, which is informationally specific. The base representation could be the absolute position of objects indexed in an abstract spatial array or retinotopic co-ordinate system. Such representations are assumed in computational theories of diagrammatic reasoning. A base perceptual spatial representation is proposed in Logan and Sadler's (1996) computational theory of spatial apprehension, which in turn provides information conditions to select abstract conceptual spatial relations (e.g., left-of) by an intermediate process of template matching. Note that such a base representation is also implicit in the ACT-R framework as the absolute location of visual-objects that are encoded as visual-location chunks in a retinotopic co-ordinate system. Hence, a spatial buffer in ACT-R holding N visual-location chunks can be viewed to represent a local retinotopic base representation of selected visual objects.

Research on the organisation of visual short term memory (VSTM) suggests that objects in visual short term memory are coded within a global spatial configuration (Jiang, Ohlson and Jung, 2000). This is also consistent with evidence for a recent structural description account of spatial encoding in which the cognitive system explicitly represents a configuration of relative spatial relations between represented objects files (Hayworth, et al., 2011). It is also consistent with evidence and the model of visual spatial short term memory capacity and representation proposed by Clevenger and Hummel (2014) in which (about two) pairs of objects can be bound to a representation of all spatial relations. According to their research, the number of relations between a pair have no additional working memory resource cost. The authors explain this resource independence by proposing that the role bindings take the forms of what they call *stacked relations,* which involve combined roles between pairs of objects (e.g., left-of-near-to-larger-than) in VSTM. All of these accounts are supported by evidence suggesting immediate spatial representations of perceived objects are specific/configurational and these spatial configurations representations involve relations of relative position.

Global and local spatial hypothesis. Scene and graphical interpretation often require tracking objects at different levels of granularity. A number of studies provide evidence that VSTM simultaneously represent global (i.e. a group of visual objects) and local information (individuals or subgroups of visual objects) (see Kong, Schunn and Wallstrom, 2010). The model of VSTM proposed by Kong, et al. (2010) proposes that the cognitive system represents objects in VSTM at different levels of granularity using the same VSTM capacity resources of a few item/object slots. Their model also proposes that the cognitive system encodes global information first and uses this information to guide the encoding of local information whilst maintaining the global information in VSTM and thereby keeping track of the bigger picture.

Model. The extended architecture includes a spatial buffer, which holds a chunk containing four slots that reference attended visual-location chunks. Note that each visual-location chunk encodes position in a retinotopic co-ordinate system, so this information can be viewed as a perceptual base representation. The visual-location slots have roles that identify the currently attended object (target) and one or two previously attended objects (referents). The location chunks of visual objects can only be indexed in the spatial buffer when the corresponding visual object is attended to. In the model, this indexing is done automatically by a modular function (that is not part of standard ACT-R) in response to an update request by an attention production.

In addition to slots holding the locations of objects, the spatial chunk also holds a transient representation of the spatial configuration between corresponding target object and one or two referent objects. This information is a representation of the relative position of objects. This is intended to model a specific representation of the spatial configuration between selected objects that functions as an information interface for productions of central cognition. In the model this is represented by a set of spatial values that are automatically computed by modular functions on attention shifts from the retinotopic position of location chunks indexed in the spatial buffer. Note that this spatial representation exists at a short timescale in the order of 100s of milliseconds because its content changes on each attention shift (the imaginal buffer is used to hold a more permanent representation of selected spatial information). The modular functions implemented in the model can be viewed to constitute visual routines (e.g. Ullman 1984) that are part of a visual spatial processing system.

This spatial configuration representation provides conditions for abstract (e.g., left-of) conceptual spatial relations to be selected for purposes such as reasoning and communication consistent with Logan and Sadler's theory of spatial apprehension. In the model, the selection and mapping is carried out by productions. These productions respond to/recognise patterns of the spatial configuration and particular goal conditions and select a conceptual spatial relation appropriate to the goal context and generate the conceptual information in the imaginal buffer (which buffers conceptual information only in this model). In principle, the conceptual spatial relations could also be retrieved from declarative memory using configuration features as retrieval cues and memory instances that map configurations to stereotypical conceptual spatial relations, which would in turn be consistent with the template matching account of spatial apprehension theory of Logan & Sadler. However, given the theoretical assumption of proceduralisation in ACT-R, this would occur only if only one assumes that production learning has not been achieved between spatial and goal context information, which one would assume to exist for common visual spatial contexts.

In the model, the spatial processing is initiated by generic attention shifting productions. This is similar to a model of a spatial encoding task reported by Johnson, Wang and Zhang (2003) in which relations between concurrently attended objects are automatically generated on attention shifts. The request by an attention shift production results in the spatial buffer being updated with a chunk containing the spatial structure information while attention is being requested to an object. As the attention shift production/s are generic and depend on limited control information, the spatial update may be viewed as automatic and context independent. However, the content of spatial configuration information is controlled by central cognition in two ways. Firstly, the configuration information is local and between objects concurrently selected for visual attention by central cognition. Secondly, as will be shown, complex n-array spatial relations requiring multiple attention shifts that result from derivations of sets such as the union and intersection of subsets depend on specific search requests using spatially buffered object locations as operation arguments. Note that the spatial representation is transient – as soon as attention is shifted to a new location the representation will change. Attention shift invariant spatial information needs to be encoded from the spatial buffer and updated in the imaginal buffer which is more like a working memory.

The spatial chunks share a similar slot ontology of visual search requests that are part of the existing ACT-R framework. A distributed feature based representation is assumed to be consistent with most models of visual processing, but these features make up a specific representation. The spatial chunk uses the direction slots to indicate the results of the direction of the attention shift from the referent to the target represented by qualitative values (e.g. left-of). The module also computes relative differences in size between the target and referent represented by a numerical value that represents an analogical code (modelled as the target size divided by the referent size). These difference would be assumed to have a rough approximate value. To represent conditions for the derivation of relations of spatial containment, a single slot is used to represent boundary configuration values. When attention is shifted to an object whose region is fully inside the boundaries of the previously attended object, this will result in binding *boundary: within.* When attention is shifted to an object whose region overlaps with the previous object, this will result in binding *boundary: overlap*. When the boundary is shared the binding *boundary: connected* will result. When there is no shared boundary the binding *boundary: separate* will result. These slots always represent a binary relation between the current (targ-loc) and previously attended object (ref-loc1). One can view the contents of the spatial buffer as representing relations of attention changes between consecutive objects i.e. changes in reference, direction, boundary and size.

Note that the use of linguistic notation for spatial attribute values in the spatial buffer is only for communication purposes – they are intended to model non-linguistic spatial codes computed from attention changes. These codes could be notationally represented in other ways that provide equivalent functionality to the model, including numerical codes or so called stacked relations. The key hypothesis implemented in the model is that production rules can interpret the values of a specific representation of a spatial structure between two or a few selected objects.

Spatial index slots		
	target-loc	(e.g. visual-location22)
	referent-loc1	
	referent-loc2	
Spatial value slots		
	direction	(e.g. left-of)
	relative-size	(e.g. .5)
	boundary	(e.g. within, overlap, separate)

Figure 8.1. Limited ontology of features employed to represent the spatial structure.

Whenever a production attends to an object location in response to a search request, the spatial module updates the buffer with information designating the locations and spatial structure between the current and previously attended object/s. Hence, the relative size differences, boundary and direction values between external objects are generated in the spatial buffer automatically on attention shifts. Updating of locations operates by a first-in-last-out policy in agreement with empirical justification of other models of working memory. Hence, on each attention shift the current location is bound to the target location slot and the previous target binding is bound to the next referent slot.

Spatially derived groupings. The derivation of object groups (union, intersection, complement) are determined by search processes of the visual module, which take the contents of the spatial buffer as search parameters. For example, in determining the union of two sets, the locations of the two spatial indexed sets maintained in the spatial buffer are used as parameters to search for an object that satisfies the grouping of the indexed objects in the spatial buffer. Hence, for a union of two sets, this involves a minimum of three attention shifts: to object A, then B, then the grouping corresponding to AUB. Note that the resulting location of the search request is always the approximate centroid of the visual group. When attention is shifted to the location of the derived visual group (e.g., intersection, union, etc.), the visual object chunk representing that group is generated in the visual object buffer and an imaginal buffer slot (e.g. members-union) with the reference to the visual group. Spatial buffer conditions are used to determine whether a derived grouping is possible. For example, in determining the intersecting group, the binding *boundary: overlap* between spatial buffered referents must hold.

8.2.3. Meta-cognitive and control operations

A second important issue in the modelling work is concerned with the requirements for meta-cognitive processing. Meta-cognitive processing requirements in this model include the selection of goal requirements and task strategies, the co-ordination and keeping track of online processing, the interruption and diversion of cognitive activities, and the interrelation of information in reasoning/argumentation.

Strategy tracking The chunk in the goal buffer has slot values that reflect strategic variables that designate what the system is doing at different time-scales. The top level *subtask* slot holds information about the subtask strategy being performed (e.g., counting, reading, identification, etc.), the *step* slot holds abstract information about the step in the subtask (e.g., starting, scanning, checking, complete, etc.) and the more fine grained *operation* slot holds the module operations being performed (e.g., retrieving, searching, updating, attending, etc.). Operation and step slots are modified directly by productions of routines that generate the slot values. The content of the subtask slot may be copied from a declarative memory chunk and interpreted by generic interpretation productions or may be directly produced by a production.

Role (slot)	**Meaning**	**Example**
Subtask	The subtask strategy being executed	e.g., Subtask: counting
Step	The step in the subtask strategy	e.g., Step: start
Operation	The operation requested for processing by a module	e.g., Operation: retrieving

Table 8.2. Levels of strategic control information represented by goal chunks used in the model.

Meta-role bindings

The goal chunks also represent the meta-cognitive role of the problem roles bound in the imaginal buffer. Recall the imaginal buffer holds a representation of a perceived or imagined environmental problem state. The control buffer chunk, in contrast, holds meta-role slots. These slots include goal information, for example, whether a type of information is a requirement or result of some subtask. They also include self-argumentation information, for example, whether some information is tagged as a rejected, given case, conclusion, exception, etc. The value of meta-role slots are values that are slots in the imaginal buffer and are typically matched in dynamic binding productions. The scheme implemented in the goal buffer uses a slot-to-variable-slot binding and matching as illustrated in Figure 8.2.

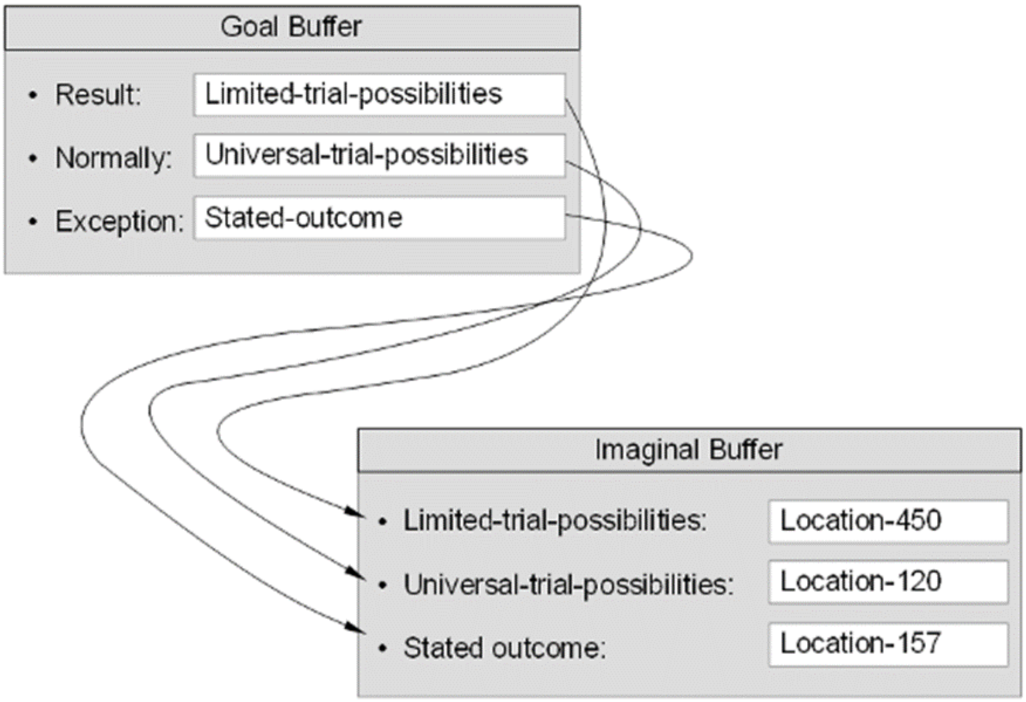


Figure 8.2. Illustration of *slot-to-variable-slot* matching in the goal buffer in which meta-roles are matched to problem-role slots and values in the imaginal buffer

This scheme is a candidate model of the requirement to represent recursive relations of meta-cognitive information in working memory. It is proposed that such meta-relations must be represented in a buffer in order to be used to trigger productions by providing conditions for decision making and the selection of actions.

Intention/goal tracking.

The goal buffer also holds goal information which represents an abstract functional interpretation of the goal that is independent of the strategy for acquiring the information. The need to represent and dissociate goal and strategic meta-cognitive information is a functional requirement. For example, if the goal is to get the number of queried possibilities one could implement different strategies such as counting or subitizing elements in a diagram, retrieving the value from memory if present or reading the number from a text based instruction if given. Whatever the case the same abstract goal concepts would apply which constitutes context independent schematic knowledge. Goal information is structured according to the following meta-roles by binding the meta-role to the identity of problem role in the imaginal buffer. These include: (1) the role of the required data (e.g., *Require [part-frequency]*), the role of the data that the requirement is derived from (e.g., *Object [queried-possibilities]*) and the function interpretation of the procedure (not the strategy) for getting the requirement (e.g., *SP [frequency]*). When the requirement is achieved its status is set as a result by binding the role of the requirement to the result (e.g. *Result [part-frequency*]). The goal scheme is used by the system to designate and track over time the goal focus and meta-role of role bound problem information in the imaginal buffer via slot-to-variable-slot binding. The goal meta-roles and their definition are listed below in Table 8.3.

Functional correspondence	**Meta-Role**	**Meaning**
Output	Require	The role of the data whose value is required.
	Result	The role of the data whose value has been achieved.
Function	SP	The relational definition of the procedure for determining the result/requirement.
Input	Object	The role of the data that the result/requirement is derived from using the SP

Table 8.3. Descriptions of goal meta-role slots in the imaginal buffer.

Argument tracking The unpractised nature of the task also requires participants to track the meta-role of problem information represented in the imaginal buffer in the process of reasoning. These requirements are consistent with verbal protocols as described in chapter 6 which suggest that participants engage in a form of self-argumentation. Recall that the meta-roles implied by participants' verbal protocols include relations of exception, rejection/revision, explanation and case conclusions. Note that the function of the goal module is in establishing and tracking the interrelation between problem roles that are proposed to underpin reported self-argumentation. Rather than the goal module actually carrying out the reasoning, inferred or acquired information is generated and selected for imaginal representation by perceptual and the declarative modules but the units of role specific information are tagged with meta-roles and maintained in the goal buffer via a slot binding of the form *meta-role [problem-role]*. Implementing the hypothesis that such meta-roles are managed by a meta-cognitive control subsystem is consistent with the general functional meta-cognitive role ascribed to the goal module in ACT-R models. These meta-roles are used to constrain the appropriate selection of productions in reflective/meta-cognitive processing in a way that allows the system to respond in non-reactive way to perceptual and problem states.

Role	**Meaning**
Unexpected	The role of the data whose value is unexpected.
Rejected	The role of the data whose value is rejected.
Case	The role of the data used to infer or derive a conclusion.
Conclusion	The role of the inferred or derived information
Normally	The role of the requirement normally employed given no data exception.
Exception	The role of the data that is a condition for making an exception.
Explanadum	The role of the data or case that needs explaining
Explanan	The relation of the case or law that explains some data

Table 8.4. Example of how the meta-role slots in the goal buffer express a meta-role of an attribute and its values in the imaginal buffer.

Self-instructions. As described in Chapter 6, certain regularities in the execution of subtasks were observed. Some of these regularities partly derived from information and instruction constraints, others are clearly choice based. In addition, participants show recognition and learning of sub-problem experiences, which appear to be used in decision making and reasoning. In order to account for these observations, the model implements a form of instance based memory of goals. In the model, these previously constructed instances can be retrieved and used to make a decision about what to do next, judge whether problem features are unusual or familiar and, if so, how they were previously used. We will call these instance based representations self-instructions that are hypothesised to be initially constructed in practice trials, but are continually modified over the course of the experimental trials as participants are confronted with more complex instances of sub-problems that require changes to SPs and requirements. The main slots used in the self-instructions are a subset of those of the goal chunks.

Slot	Meaning/function
Result	The type/role of result of the last goal.
Require	The role of the requirement to be achieved.
SP	The relational definition of the procedure for determining the requirement.
Object	The role of the data the requirement is derived from.
Subtask	The subtask strategy used to determine the requirement.

Table 8.5. Descriptions of main self-instruction slots in a goal chunk.

These chunks are retrieved from declarative memory by a generic production *Retrieve-next-step* when a result has been achieved and there are no pending subgoals or processing requests (*Goal> step [complete], operation [nil], pending [nil]*). *Retrieve-next-step* simply uses the current problem role bound to the result in the goal buffer and requests retrieval of a goal using the current result value as a retrieval constraint for next goal (*+Retrieval> result [=result]*). This is basically a generic production firing to recall what is normally required after the current result has been achieved. When the self-instruction chunk is retrieved and condition constraints match, a subtask specific production fires to initiate the goal by mapping the slot values of the self-instruction in the retrieval buffer to the goal buffer (e.g., *=Goal> require [=require], SP [=SP] subtask [=subtask], object [=object], etc.*). Note that the productions for implementing retrieved self-instructions are specific to the subtask type because they test matching requirements of the self-instruction in their conditions.

The analysis described in Chapter 4 indicates that the problem can be decomposed into a set of interdependent information requirement goals. So, for example, one information requirement is to interpret the queried outcome, another is to determine the queried possibilities from the queried outcome, and another is to determine the part frequency from the queried possibilities. Transitions between subtasks are mediated by the retrieval and interpretation of self-instructions, when available, that identify the required class of information given the acquisition of a known class of information. This network of self-instructions can be viewed to model the prior episodic knowledge of the self-instruction dependencies formulated by participants that is initially constructed in the practice trials and modified over the course of the experimental trials. This network, which is implemented in the ACT-R model, is shown in Figure 8.3. Note that the order of scheduling given by the conditional goals states of the self-instructions were designed to match the canonical sub-task scheduling observed by participants.

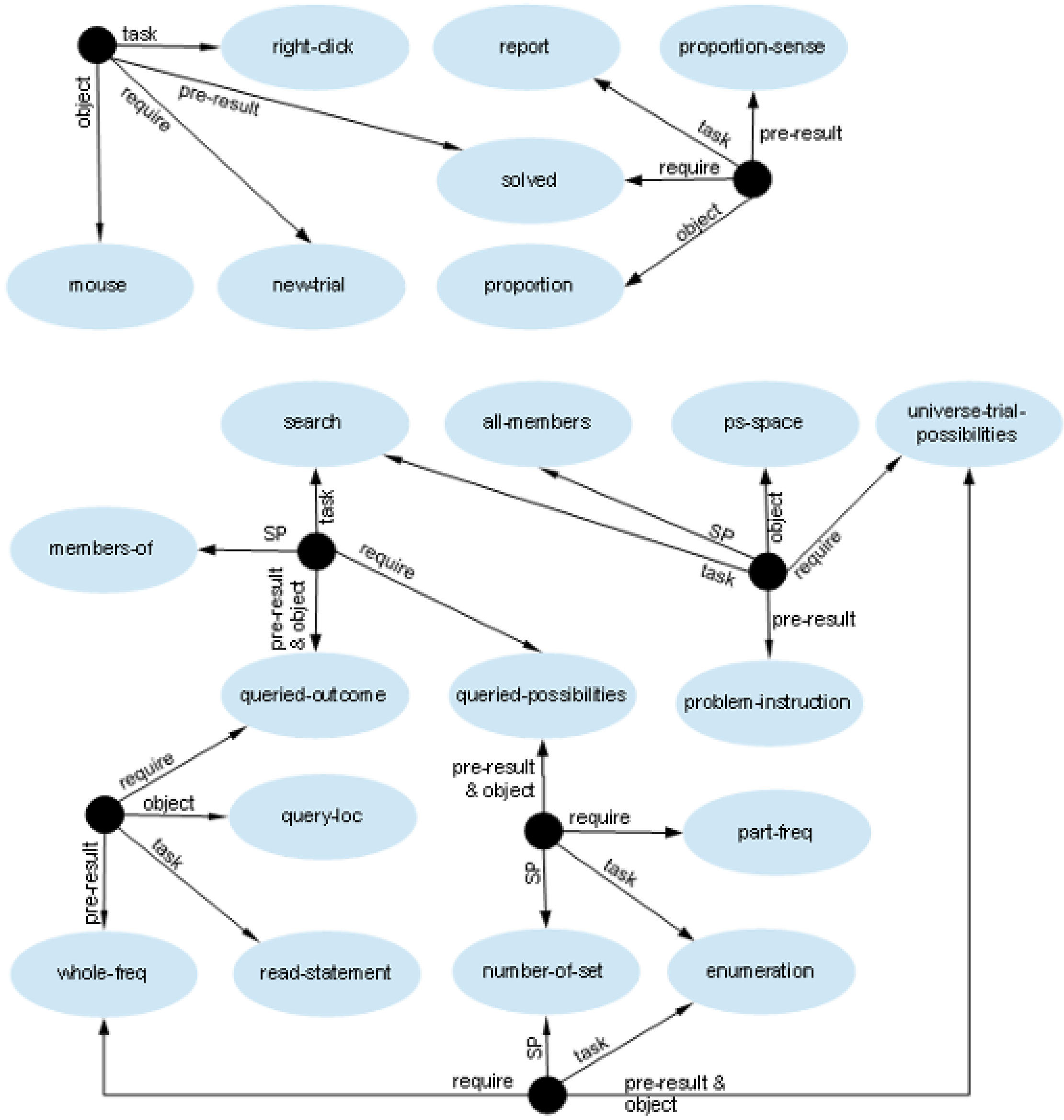


Figure 8.3. PPS self-instruction network.

8.2.4. Problem representation and knowledge

The cognitive model integrates accumulated information about the problem that it gathers over the instruction and diagram comprehension episodes using chunks which are incremented in the imaginal buffer. Each imaginal chunk contains two levels of information. At the top level, all imaginal chunks contain an abstract problem role and case problem role, which represent a classification of the information content of the chunk in relation to the problem or task. The lower order slots designate specific problem information about environmental referents using role-to-referent binding slots. Imaginal chunks are hierarchically organised so that subordinate chunks are related to super-ordinate chunks semantically by their matching problem roles.

This representational scheme for accumulated problem information can be viewed to result in a kind of meta-memory of problem state information in which the system is able to know the kind of information that it has acquired without knowing the particular data. To access the data from a sub-chunk, the system must retrieve the sub-chunk containing the target information. Retrieval occurs by domain specific productions using the problem role as a retrieval cue in conjunction with other information depending on the type of subtask.

The model implements the assumption that participants solve the probability problems by classifying and assigning known problem roles to data derived from the instruction and diagrams. These problems roles are intended to be the declarative part of the models schematic knowledge. As the problem roles are used to frame specific problem data in terms of their categorised role in the probability problem, they are assumed to apply to different problems varying in terms of data, structure and problem scenario. The cognitive model uses and modifies such problem roles for different experimental problems. The existence of such roles in the model is considered a functional requirement in order to connect problem data to prior declarative and procedural knowledge employed to solve a problem.

As an example, the problem role labelled *queried-outcome* represents the generic role of a model of an outcome being queried in a probability problem. The problem roles are intended to represent a higher-order classification of a combination of abstractions (e.g., outcome (x), category (y), probability (z), has (x, y), has (x, z), situation (x, y, z), supposed (x, y), unknown (z), queried (z), etc.) that are compressed through prior learning. The use of problem roles such as *queried-outcome* in the model are not a pragmatic modelling convenience, but are based on the hypothesis that conceptual knowledge is available and often used as units of working memory states that carry compressed content.

Recall that all imaginal chunks are assigned a problem role and case values, which identify the overarching classification of chunks of information constructed in the imaginal buffer resulting from some subtasks. Table 8.7 reviews the main PPS problem roles and Figure 8.4 gives an example of their potential reference.

Problem Role	**Meaning**
Priors	An interpretation of the prior probabilities.
Stated-outcome	An interpretation of the stated outcome.
Queried-outcome	An interpretation of the queried outcome.
Universe-trial-possibilities	The initial universal set of trial possibilities.
Limited-trial-possibilities	The reduced/limited set of trial possibilities given conditional information
Queried-possibilities	The set of all possibilities of the queried outcome.
Limited-queried-possibilities	The reduced/limited set of queried possibilities given information about a unique outcome category.
Part-frequency	The frequency of the part set of the proportion.
Whole-frequency	The frequency of the whole set of the proportion.
Part-set	The part set of the proportion.
Whole-set	The whole set of the proportion.
Proportion-sense	The relative value of the proportion.

Table 8.6. Main problem roles and their meaning.

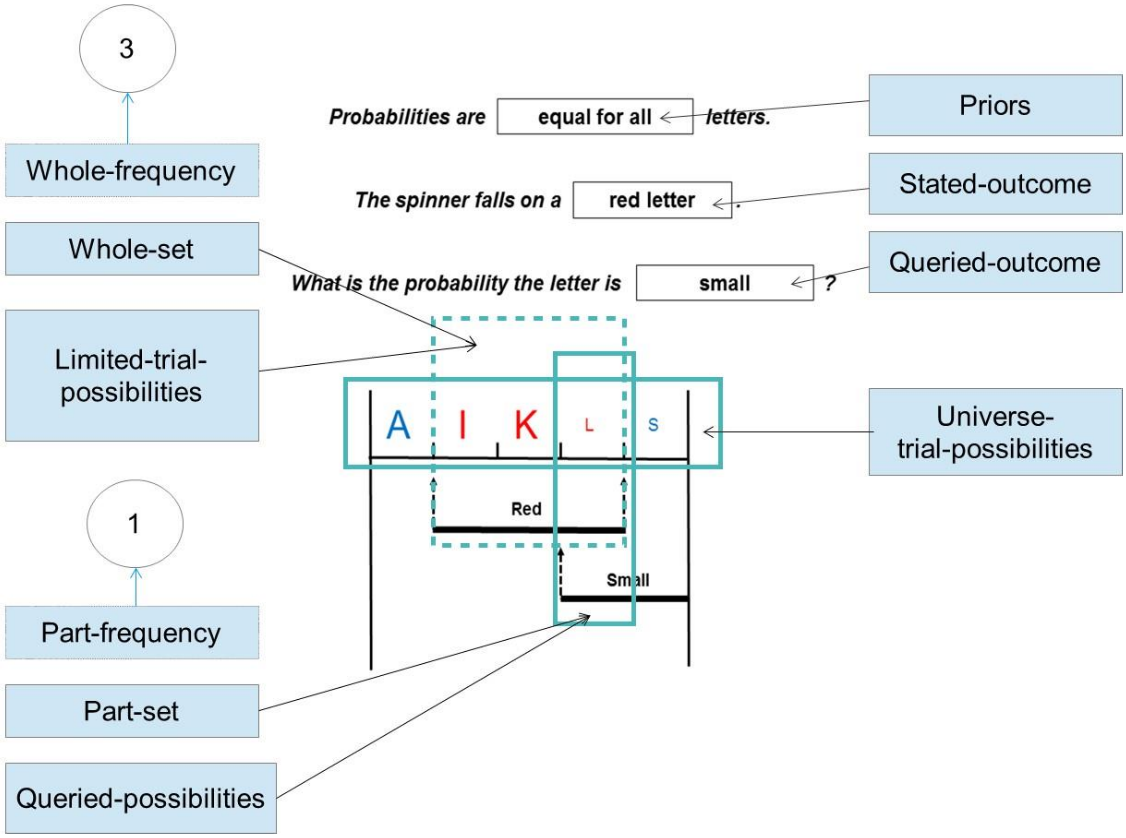


Figure 8.4. Example of the potential reference of the problem roles on the conditional/overlap problem.

8.2.5. Declarative Inference

A canonical ACT-R model performs a task with a goal and/or imaginal chunk that contains slots that are incrementally filled in the course of the task or subtask. Such chunks are often used to specify binding requirements and indicate the status of bindings. These functional assumptions are consistent with models of practiced performance specific to a task.

As the model is concerned with un-practiced performance it requires a more flexible and incremental way of determining the contents of goal and imaginal representations (i.e. slot and values). In the model when declarative semantic knowledge is required to frame referents via role-to-referent slot bindings that are not available in a self-instruction such roles needs to be retrieved from declarative memory. The process can be viewed as declarative inference making. This process is modelled by productions that respond to specific problem state conditions in the imaginal or perceptual buffers and retrieve a chunk containing the sematic role/s applicable to framing the context. These roles are value/s of the retrieved chunks that are converted to slots and bound to designated referents in the imaginal buffer by dynamic binding productions. At least two production steps are needed to infer new roles because the semantic knowledge needs to be retrieved and evaluated before being bound to some value in the imaginal buffer. The semantic knowledge employed in inferences are chunks representing conceptual transformations which specify the new role to bind and the currently represented role of target referent that is the recipient of the new binding.

8.4 Basic subtasks

In this section, we discuss some of the basic subtask strategies that are implemented by the PPS model. These strategies constitute general skills or schemas recruited and scheduled during the task such as instruction comprehension, identifying set members and relations, determining possibilities, enumerating sets and deriving proportions. The control structure of each subtask gives rise to a finite set of paths of strategic possibilities that are realised in various problem simulations.

8.4.1. Instruction interpretation

The model does not attempt to capture the intricacies of reading and comprehending the problem statements because text processing is not central to the research. Several constraints exist that follow from an analysis of available processing strategies with the text frame and strategies suggested by participants' protocols.

Each statement of the text frame expresses a particular type of problem information allowing participants to familiarise and anticipate the expected type of information after completing practice trials. Evidence that participants quickly learn the type and case of the problem information associated with a text frame is discussed in Chapter 6. The types of text frames are available at fixed spatial locations of the problem presentation allowing participants to learn where to look for particular types of problem information. The permanence of the location of the text frames can function as a mnemonic aid for accessing information. Evidence that participants learnt where to look for types of problem information is suggested by the experimental data.

Text frame structures are a ubiquitous way of economically presenting information (e.g., forms, specification lists, programming and mathematical notations, etc.); hence, participants would be expected to have existing knowledge of them and cognitive strategies for processing them. Protocols indicate that participants recognise immediately when the value of a statement is familiar/usual or not. This is described in Chapter 6.

In interpreting text statements, the model encodes the frame entry and then the frame values. When the lexical meanings of the frame entry are encoded, the model retrieves a previously constructed interpretation of the frame entry meaning and its associated problem role and uses the memory to classify the problem role of the frame values. When encoding the lexical meaning of the frame values, the system incrementally builds the chunk representing an interpretation of the statements meaning. When a statement has been interpreted, the system evaluates the familiarity of the statement with respect to the problem. When a statement has been encoded, the system moves on to the next statement.

Lexical encoding routine. In reading the statement at a lower level, the model runs through basic lexical encoding operations. From the left, the model iteratively finds a neighbouring word (*find-first-word-in-text-line*, *find-next-word-and-update*), attends to the word (*attend-unattended-word*), retrieves and verbalises the lexical meaning of words (*retrieve-lexical-meaning*), updates the lexical meaning in a lexical buffer (*find-next-word-and-update*), and processes the sound of the verbalised word (*searching-vocalisation*). The lexical chunks include a slot for the text form, semantic attribute and value of the word (e.g. text: "blue", attribute: colour, value: blue). The strategy is consistent with the observation that participants tended to read aloud the problem instructions word-for-word in experiment 2 as required by the experimental instructions. The control structure for these lower level routine productions is shown in Figure 8.5.

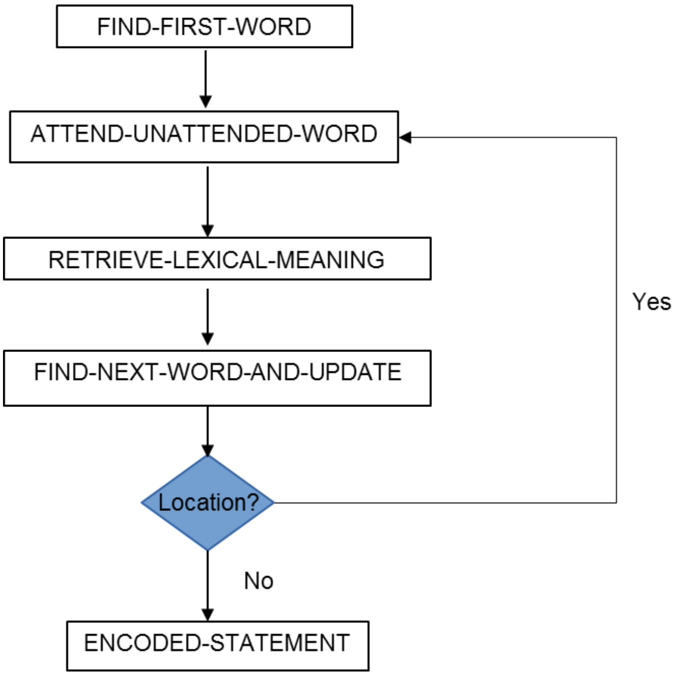


Figure 8.5. Flow for lexical encoding routine.

Statement categorisation. When the lexical meanings of the words in the text frame (i.e. Probabilities are [ ] , The spinner falls on [ ] , What is the probability of a [ ]) have been encoded, the generic production *Retrieve-frame-entry-expectation* fires and retrieves a memory of a previous interpretation experience of the corresponding text entry using the buffered lexical elements (*+Retrieval> lexical1 [=lexical1], lexical2 [=lexical2], etc.*). When retrieved, the interpretation chunk includes previously inferred information about the statement's problem role in the task. (*=Retrieval> role [=role]*. Hence, the problem role in the interpretation chunk of priors statement is represented by the binding *role: priors,* of the outcome statement *role: stated-outcome* and of the query statement *role: queried-outcome*. If the chunk is retrieved, *Set-requirement-for-expected-frame-role* fires and sets an intention to classify the interpretation of the subsequent text value/s in terms of the currently retrieved problem role by updating the requirement state in the goal buffer (*Goal> require[=role]).* This process corresponds to the goal directed and problem specific framing of the values of the corresponding statements.

Statement interpretation. After encoding the lexical meaning of text entry value/s, the system constructs a single interpretation chunk in the imaginal buffer holding the meaning of the statement which includes the referred categories and the problem role of the interpreted objects. The interpretation is carried out by a class of meaning specific interpretation productions that bind the updated meaning in a lexical buffer of the retrieved lexical chunks to the interpretation chunk. In addition, this class of interpretation productions also bind the encoding location of the text interpretation from the visual module. Statement interpretation is an ongoing/incremental process that includes syntactic interpretation hence the one shot interpretation strategy reported in the model is a simplification of the process that abstracts over the syntactic processing requirements.

Examples of the chunks containing interpretations of the different kinds of problem statements are shown below. Note that slots containing a pointer to the encoding location of the interpretation are also present, but omitted from the figures below. In interpreting the prior statement, the model will construct a chunk containing information about the relative probability of two categories of letters (as in the biased problem instruction) or about the equiprobability of letters (as in the remaining equiprobable problems).

role	priors
case	unequal-priors
-----	
universe-kind	PS-letter
for	all
if-attribute	colour
colour	blue
then-attribute	relative-probability
relative-probability	2-to-1
"the probabilities are |double for blue| letters"	

role	priors
case	equal-priors
----	
universe-kind	PS-letters
for	all
attribute	relative-probability
relative-probability	1-to-1
"the probabilities are |equal for all| letters"	

Figure 8.6. Chunks holding information about a model interpreted from the prior probability statement for statements designating (A) unequal and (B) equal probabilities, left and right, respectively.

In interpreting the category of the stated-outcome, the chunks created are of the following form: the chunk will contain information only about the type of object in non-conditional problem instructions (see Figure 8.7A) or hold additional information about an attribute of the object in conditional problem instructions (see Figure 8.7B).

role	stated-outcome
case	universal-outcome
---------	
outcome	spinner-fall-on
for	individual
kind	PS-letter
“the spinner falls on a | letter|”	

role	stated-outcome
case	categorised-outcome
---------	
outcome	spinner-fall-on
for	individual
kind	PS-letter
attribute	colour
colour	blue
“the spinner falls on a |blue letter|”	

Figure 8.7. Chunks holding information about a model interpreted from the conditional statement for unconditional (A) and conditional (B) examples – left and right, respectively.

In interpreting the category of the queried-outcome, the chunks created are of the following form: the chunk will contain either a single attribute, or a conjunction of attributes (for conditional and conjunction problem instructions).

role	queried-outcome
case	single-queried-outcome
------	
outcome	spinner-fall-on
universe-kind	PS-letter
for	individual
queried-attribute	colour
colour	blue
query-attribute	probability

role	queried-outcome
case	conjunction-queried-outcome
------	
outcome	spinner-fall-on
universe-kind	PS-letter
for	individual
queried-attribute	colour
queried-attribute2	height
colour	blue
height	large
query-attribute	probability

role	queried-outcome
case	disjunction-queried-outcome
------	
outcome	spinner-fall-on
universe-kind	PS-letter
for	individual
queried-attribute	colour
colour	red
query-attribute	probability

role	queried-outcome
case	conditional-queried-outcome
outcome	spinner-fall-on
universe-kind	PS-letter
for	individual
queried-attribute	colour
stated-attribute	height
colour	blue
height	large
query-attribute	probability

Figure 8.8. Chunks holding information about a model interpreted from the outcome statement for unconditional (A) and conditional (B) examples.

The chunks holding an interpretation of the priors and the stated outcome are interpreted from their corresponding frame statement in isolation. However, the queried-outcome chunk is also constructed from the outcome and query statements because both refer to the same outcome token. Namely, when participants comprehend "*the spinner falls on a [e.g., letter/red letter]*" followed by "*what is the probability the letter is "[e.g., red]*" they are proposed to infer that the outcome referred to in the query statement is the same individual as referred to in the last statement, and so, combine the models. Specifically, in interpreting the queried-outcome, the model retrieves the stated-outcome interpretation chunk and increments it with the category information of the queried statement so that it also inherits the category of the stated-outcome chunk. This encoding process is used to explain why participants showed little or no deliberation in limiting the queried possibilities to A∩C when required as described in Chapter 6.

Statement evaluation. When an interpretation has been constructed the model evaluates the familiarity of the statement interpretation. This is carried out by the production *recall-if-im-familiar-with-case*, which attempts to retrieve a previous self-instruction, which includes the categorised case and role of the statement (Retrieval> case =case role =role). The evaluative production *familiar-case* fires when a retrieved goal/self-instruction matches the current case and role. The production *unfamiliar-case* fires when the retrieval fails and notes that the problem case is unfamiliar by binding the particular case categorisation to the meta-role *exception* in the goal buffer (e.g., *Goal>*

exception[categorised-outcome]). When the interpretation is complete, the production *recall-where-i-was-at* retrieves the superordinate problem chunk and *represent-and-update-where-i-was-at* dynamically binds the problem role of the interpreted chunk as a slot of the problem chunk with the encoding location of the interpretation as its value. Binding the encoding location serves as a cue to retrieve the interpreted chunk and to locate its corresponding text externally in the superordinate problem chunk.

8.4.2. Identifying member sets

Participants derive sets of possibilities in the diagram based on category membership. The interpretation of eye-movement data and cognitive modelling analysis suggest the following modelling constraints:

- The greater relative frequency of attention on PS-units compared to set-markers suggest participants use the iconic representation of PS-units as the main source to identify corresponding sets, but may use set markers in complementary way to check and possibly support the apprehension set relations.
- Participants' eye-movements suggest they tend to iteratively scan groups of PS-units making sequences of fixations on consecutive neighbours, in both, left to right and right to left directions. These fixations are typically short in length.
- The diagram is composed of visual grouping of PS-units and it is assumed that these groups support visual pop-out/pre-attentive processes that can be used to guide attention to perceptual groups.
- Participants appear to notice set relations between referred groups immediately after first pass instruction encoding/initial inspection of the diagram.
- Research in visual working memory suggests visual-spatial encoding of groupings proceeds by initially attending to target groups globally before encoding local elements of that group.
- Participants should have generic knowledge for understanding the role of the marker lines and label in the diagram. This is because information presentations commonly employ highlighting of text and graphics by underlining or circling and/or labelling to support comprehension.
- The set identification subtask is carried out in four phases: (1) setting the goal, (2) globally identifying the exhaustive set and ruling any other members, (3)

scanning the group to encode its detail, and (4) checking the consistency of the group with its corresponding marker lines.

Whist it is possible to create a model in ACT-R that solves the problem by simply selecting and encoding perceptual groups in one shot attention shifts, the amount of eye-movements exhibited by participants and short length of the fixations are inconsistent with this kind of visual encoding model. Two additional assumptions about group identification strategies can be made to reconcile these issues. Firstly, after participants initially identify a required perceptual group, they serially scan the group to encode the detail of subgroups or sub-elements. This is done in a left-to-right or right-to-left fashion and is consistent with eye-movement protocols. This kind of visual interrogation may result from highly practised routines that support a requirement of increasing the detail of a visual representation that is difficult to model in ACT-R's visual representations. Secondly, the model assumes participants attend to the group of PS-units outside the identified group in order to attentively check that there are no remaining members of a target category. In ACT-R, productions can detect whether individuals in a specified region do not have a value of a visual property from the module state (i.e. state = error) that results on a failed search. It seems reasonable that attention shifts should also be used to confirm the absence of individuals having category but by attending globally to groups of observed candidates.

Setting the goal The goal to initiate identifying sets of referred categories is made by productions in response to a retrieved self-instruction to determine the possibilities of targets sets, i.e. universe-trial-possibilities and queried-possibilities. The production fires and binds the requirement and subtask to the goal buffer (e.g., *Goal> subtask [search], require [queried-possibilities], object[queried-outcome]*). The category information used to search for the set is contained in the chunks interpreted from the text statement, which are retrieved from declarative memory in response to the self-instruction.

Global identification When a chunk containing the identity of the model is successfully retrieved, one of a set of search productions fires (e.g., *find-kind-with*-single-property*, find-kind-with-conjunction*-property) to initiate the search for a group. These search productions bind attribute types to argument slots in the goal buffer and send a request for a visual location containing the visual category or categories of the particular kind.

The search productions use dynamic binding. When the search starts, the goal buffer sets the state of the search to none (i.e. *Goal> step[none]*). If the visual module returns a location, a generic attention production fires to attend to the group and updates the spatial buffer (i.e. *attend-to-unattended-location-and-update-spatial*). If the features of the attended group match the selected search features in the imaginal buffer, a production (e.g., *update-and-find-next-group*) updates the location of the group to a members slot in the imaginal buffer and the state of the search to some (e.g., *Goal> step[some]; Imaginal> members[=screen-pos]*) and makes a requests to the visual module for another group matching the set.

If after finding and initial group the search request for an additional group fails (indicating that no other objects matching the attribute can be detected pre-attentively), one of a class of production fires to attentively check the remaining group/s (e.g. *check-left-side, check-right-side*). These productions use peripheral information in the currently attended visual chunk to decide where to look and request a location of the largest remaining group to either side of the attended group (e.g., *+Visual-location> segments [largest], kind[=kind], > screen-x [current], screen-y [current]*). If the attended remaining group has a uniform value of the searched property that does not match the search criteria, a production (e.g. *return-back-to-single-category-group*) update the step to indicate that the search has been exhausted (*=Goal> step [exhausted]*) and requests re-indexing of the previously indexed location of the visual group bound to the imaginal buffer (*+Visual-location> =members*).

Scanning. When the exhaustive group has been determined *(=Imaginal> members [=members], Goal> step[exhausted]*) at a global level, the scanning subroutine is initiated to interrogate and encode details of the perceptual group. The production *find-first-left-subgroup* fires in response to the exhausted state and a property discontinuity as indicated by an absent perceptual binding of the attended visual object group, which signals the existence of differing subgroups and requests the location of the first subgroup on the left *(+Visual-location> kind [=kind], objects [largest], > screen-x [current] < screen-x [current], screen-y [current]*). When a subgroup is attended the production *find-next-subgroup* keeps firing until a visual location is not returned at which point *completed-subgroup-scanning* responds to the failed module state of the search request and set the subtask step to scanned.

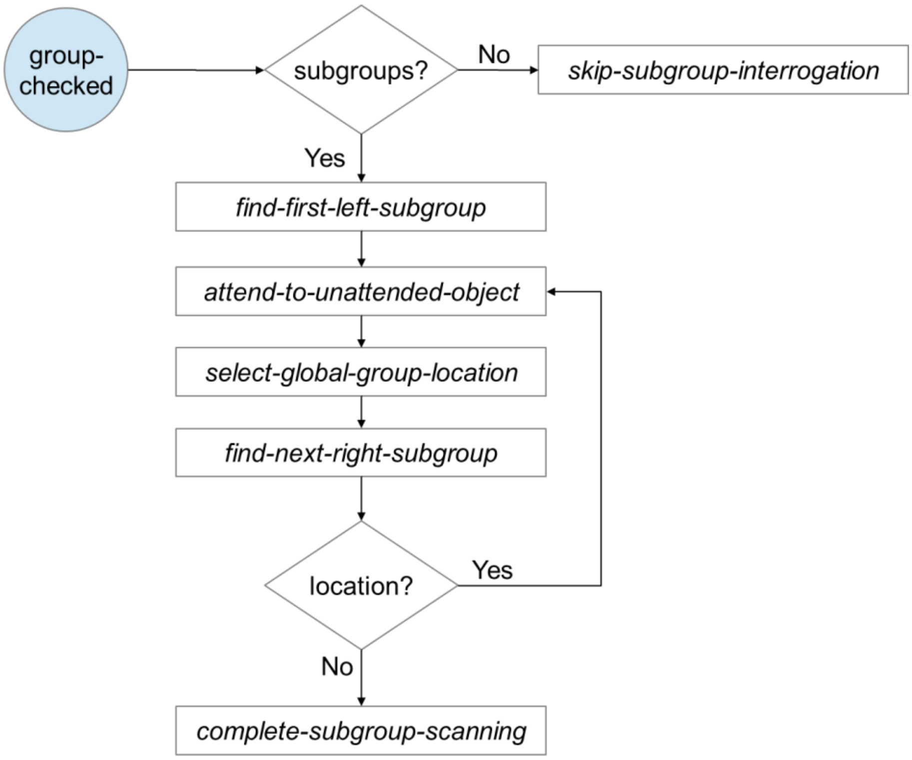


Figure 8.9. Control flow of scanning routine used to identify subgroups.

Marker and label checking. As part of the subtask, the model may also check that the marker lines and label are consistent with the group. In response to the step binding and the absence of a binding of the label and marker lines in the imaginal buffer the production *find-marker-label* fires and request the visual-location of the vertically corresponding label then *retrieve-label-meaning fires* retrieves the lexical meaning of the text. *Update-matching-label* fires in response to the matching category of the search set and retrieved text attribute/value meaning and then binds the location of the text to a *label* role in the imaginal buffer thus indexing the role of the text. The production *find-marker-line* then requests the location of the neighbouring marker-line. After attending to the marker-line the production *Update-and-return-to-members* re-indexes the members and *matches-marker-set-dimensions* fires in response to the direction and relative size in the spatial buffer indicating the marker matches the horizontal position and extension of the target set and sets the step to check.

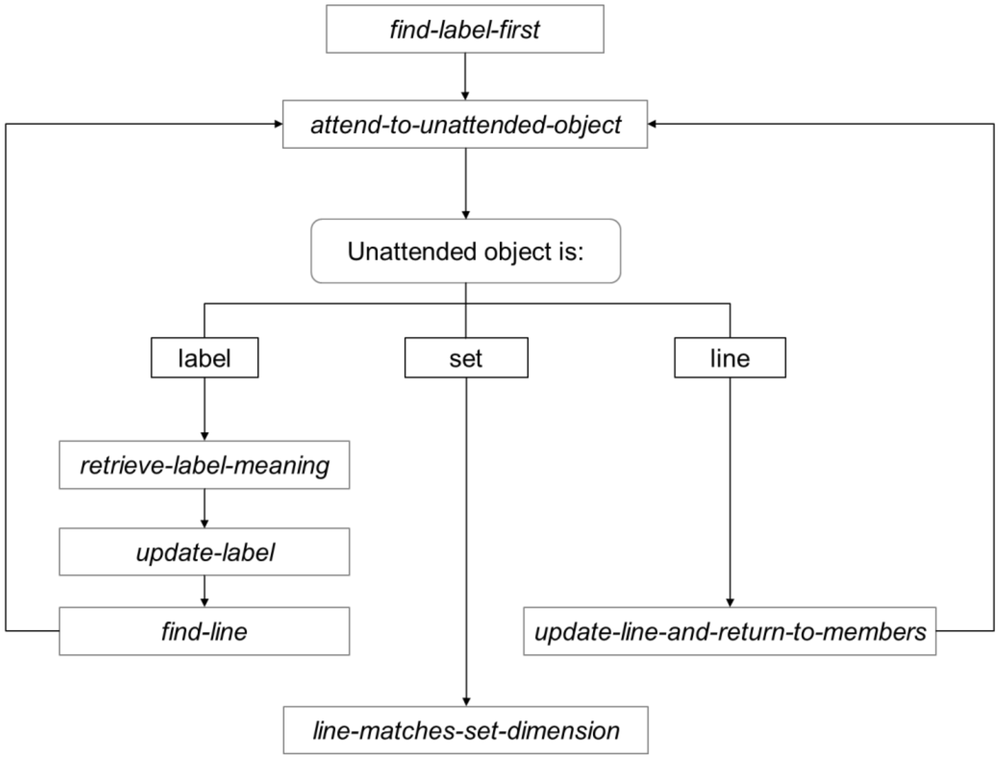


Figure 8.10. Control flow for checking marker line and label.

8.4.3. Possibility derivation

The model implements productions for determining possibility. Constraints on possibility are listed below:

- Analysis of the model requirements indicate that unless participants have learnt instances that map particular solutions to set structure and instruction conditions then they must determine what individuals are possible in order to assign the frequency roles of the probability proportion to those derived sets.
- Observed verbalisations indicate that participants (at least sometimes) ascribe possibility to relevant sets. Verbalisations indicating construals of possibility occur most often in conditional problems requiring the limitation of trial possibilities from U to C.
- There is analytical and empirical support for the proposal that the operators, rules or mechanisms involved in the derivation of possibility operate on input models of individuals or sets rather than internal category descriptions (unless data specific instances have already been learnt).

- Determining the possibility of a set is a process that appears to require deliberate initiation, but whose probability is influenced by external information cues.

Model.

The model implements inference productions for testing and ascribing roles of possibility. This means that inferences of possibility are with respect to models of individuals assigned particular problems roles and the inferences of possibility are particular possibility problems roles. The productions have goal and problem state dependent conditions and make case inferences using specific retrieval constraints. The model of inferences of possibility is based on the assumption that participants will have learnt particular cases through prior educational and cultural experiences and that reasoning about possibility based on more primitive and generic knowledge and cognitive strategies would on reflection imply more complexity and time consumed than was observed in derivation of possibility by participants in the experiment because it would imply the requirement to conceptualise and determine how to connect problems roles of sets of individuals that would be novel to the system

The possibility inference productions in the model make inferences based on the roles of co-referred models of sets of individuals. In the modelling framework the mental models are deictic/indexical representations of individuals in the imaginal buffer that bind problem roles to the reference of individuals in the diagram. The functional character of the representation is a requirement for the possibility inference productions that compute co-reference constraints between roles of referred sets in their conditions. For example, the production *limit-universe-trial-possibilities* respond to the condition that there exists a universe of trial possibilities a set of members of a stated outcome category and that the former is a subset if the latter (i.e. =Imaginal> universe-trial-possibilities: [->U], stated-outcome: [->O], members: [->C], members-of: [->Q] subset [->C], subset-of [U].

8.4.4. Proportion

Another important subtask of the modelled PPS task is framing, computing and evaluating proportions of the sets of possibilities. The model is based on the following constraints as listed below:

- Participants typically frame the probability proportion spatially. This is suggested by the percentage of verbal protocols where the reported solution was described using spatial relation terms such ("three out of six" or "three in six" rather than fraction terms such as "three sixes" or "point five".
- Participants frame the problem role of derived frequency as soon as they are determined. This is suggested by participants who made verbalisations indicating whole frequency (or denominator) framing of enumerated possibilities (e.g., "its out of six").
- Participants compute and evaluate the sense of a proportion and do so as soon as the part and whole frequency values are made available as suggested by participants that abandon a proportion based on it being too high or not summing to unity.
- Participants evaluate proportions as too high suggesting some geometrical sense of a proportion must be used a bases for the judgment.
- Participants selectively report proportion senses from computed from different inputs. As well as from frequencies, some participants appear to derive proportions from the geometrical size of proportion sets. The latter strategy was only used for high visual fidelity fractions (e.g., ½)**.**
- Participants appear to compute the geometrical structure of a proportion as a matter of course as suggested by the selective ability to report it numerically and the ability to evaluate frequency derived proportions.

Model.

The model's representation of the probability proportion includes the roles that hold a representation of the reference of the corresponding spatial part and whole roles of the sets (i.e. *whole-set* and *part-set)* of the proportion from which the frequencies are derived from. The roles of the derived values of the proportion frame are the *part-frequency*, *whole-frequency* and *proportion-sense.*

When both, the *part-frequency* and *whole-frequency* values are available in the imaginal buffer, the production *retrieve-proportion-sense* fires and attempts to retrieve a proportion instance from declarative memory using the frequency values as retrieval constraints (+*Retrieval*> *part-frequency [=part-freq], whole-frequency [=whole-freq]*). If successful, *update-relative-quantity* production fires and updates the retrieved *proportion-sense* value in the imaginal buffer (*=Imaginal> proportion-sense [=proportion-sense]*) allowing it to be evaluated. In addition, the *proportion-sense,* which is assumed to be a spatial code, may also be represented with some fidelity in the spatial buffer if the groups indexed in the *part-set* and *whole-set* roles are simultaneously also indexed in the spatial buffer (and therefore have their relative size computed). If the fidelity of the computed *relative-size* is familiar, the production *retrieve-proportion-fraction* fires to get the component frequencies by retrieving a proportion instance with the *relative-size* value as a retrieval cue (*+Retrieval> proportion-sense [=relative-size]*) in order to bypass deriving the frequencies of component sets.

part-frequency [two]
part-set [=part-location]
whole-frequency [four]
whole-set [=whole-location]
proportion-sense [0.5]

Figure 8.11. An example of roles and bindings used in determining a proportion.

Despite the proportion-sense being analogical, when the inputs are numerical symbols the model attempts to retrieve a proportion instance from declarative memory using the semantic codes for the numerical values as retrieval cues. The retrieval assumption is motivated by the reasoning that if a *proportion-sense* value is repeatedly computed from scratch by mapping counts to modelled analogical input then memory instances of these mappings are likely to be generated for common cases such as small number numerator/denominator proportions. The availability of such chunks allows instance based retrieval strategies as commonly assumed for arithmetic facts in ACT-R models.

8.4.5. Enumeration

To acquire the *part-frequency* and *whole-frequency* values, the system needs to implement an enumeration procedure on sets of linearly arranged letter units. The protocol's data identifies three main kinds of common enumeration strategies that are selected depending on the task requirements and problem context: (a) subitizing for small sets, (b) counting of individual objects in a linear direction for large sets, and (c) weighted counting using a count twice strategy for double probability PS-units in biased probability problems (see below).

- Participants show evidence of using subitizing strategies for small sets such as A, B, C or D.
- Participants show tendency to serially count in a spatially linear manner for large sets such as U using left to right or right to left strategies.
- Participants employ one of two strategies for counting units on biased problems: (a) double-count strategy and the (b) add-count strategy.
- Participants employ add numerator strategies in disjunction problems.

Subitizing. The model of subitizing implements a model similar to the recognition account of object groupings as reported by Peterson & Simon (2000). As linear configurations of objects are commonly observed structures, it is assumed that participants have generic enumeration instances for linear configurations in declarative memory that can be generalised to the arrangements of PS-units in the diagram. These chunks relate an object segmentation pattern with a semantic count value and the vocal code of the count (e.g., *segments: 2, count: two, vocal "two"*). For simplicity, numbers are used in the model to stand in for the cardinality of the object segment pattern. If the system is attending to a visual group and has set the subtask to enumerating and the visual-object chunk indicates that a number of segments is less than 5 (*=Visual> segments [=segments], < segments [5]*), then the *retrieve-subitized-count* production fires and requests retrieval of the numerical count of the object grouping matching the segments value (*+Retrieval> segments [=segments]*) and modifies the subtask (*Goal> subtask [subitize]*). If the retrieval is successful, the production *update-done-subitize-count* fires and dynamically updates the retrieved count (*=Imaginal> =requirement [=count])*, vocalizes the count (*+Vocal> string [=vocal])* and updates the subtask step and result state in the goal buffer (*=Goal> step [complete], result [=requirement]*).

Serial counting. The production *find-first-left-to-count* fires and requests the visual location of the first left object in the indexed array. The count strategy uses chunks in declarative memory that hold the semantic codes of the current count, next count and a vocalisation code of the next count number (e.g., *current-count[two], next-count [three], vocal ["three"]*). Initially, the *retrieve-first-count* production fires to retrieve the first count chunk (+*Retrieval*> *current-count [start]*), then on each subsequent count iteration the *retrieve-next-count* production fires when a newly attended visual-object appears in the visual buffer and requests retrieval of a chunk containing a count that is equal to the current count (+*Retrieval*> *current-count [=current-count]).* The *hold-current-count-and-find-next-right* fires to update the imaginal buffer with the next count value *(=Imaginal*> *count [=next-count]*), makes a request to vocalise the count number (+*Vocal*> *string [=vocal-count]*) and requests the visual location of the next visual object to count (+*Visual-location*> *kind [=letter-unit], > screen-x [current], nearest [current]*). When a new visual location cannot be found in the array, the production *done-count* fires to update the final count value, terminate the routine and dynamically updates the result binding in the imaginal buffer (*=Goal*> *step [complete], result[=required]; =Imaginal*> *=required [=count]*). The flow diagram in Figure 8.12 summarises the strategy.

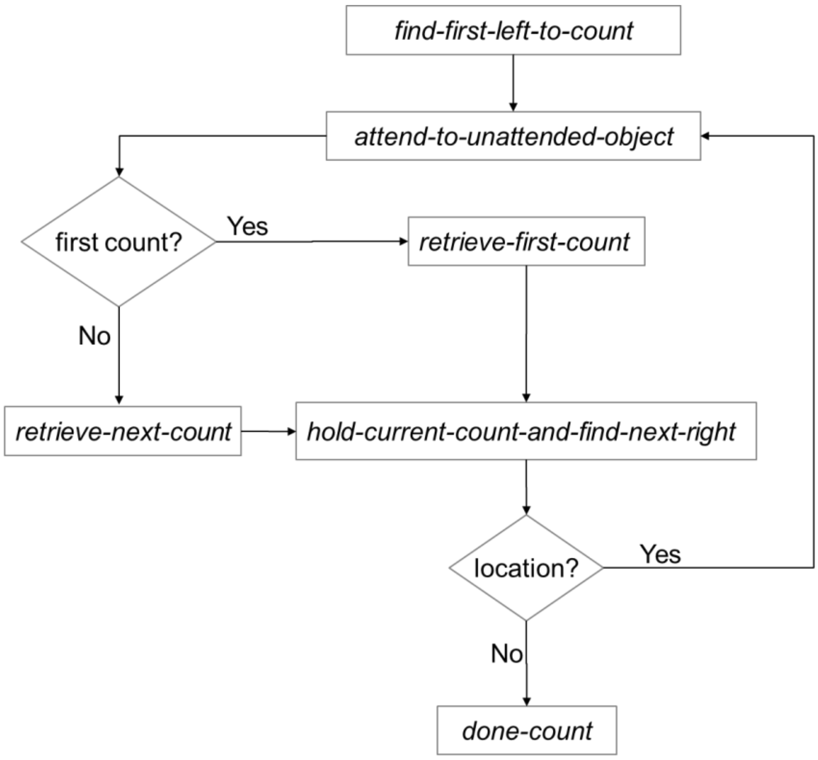


Figure 8.12. Flow of serial count strategy.

Weighted count strategy. The *double-count* strategy involved counting each item the weighted number of times (i.e. twice for letters with double probabilities) before moving on to the next item. Although this was not the only strategy observed for the biased probability problems, only this strategy was implemented because of the more peripheral importance of the count strategy to the research aims. The implemented strategy uses the letter category of double probability letters as a condition for determining whether to double weight a count for that object or not. In interpreting the self-instruction, the imaginal slots containing the identity of the count condition and the number of times to count are set to the goal buffer and these variable slots and their values are bound to the imaginal buffer (i.e. *=Goal> arg2 [=times], arg4[=category]; =Imaginal> =times [=times-value], =category [=category-value]*). If the attended visual object chunk does not match the condition in the imaginal buffer, the *retrieve-next-count-for-unweighted-object* fires to retrieve a count. When the count is retrieved, the production *update-count-and-find-next-object* fires and requests the location of the next count object. If the visual object does match, the condition *retrieve-next-count-for-weighted-object* fires and sets an iteration goal state so, when the count is retrieved, *update-and-retrieve-next* will keep firing until the value of the iterate-state (i.e. *=Imaginal> iterate-state [=times-value], =times [=times-value]*) slot matches the value of times slot. When this occurs, *update-last-count-and-find-next-object* fires, which updates the last count and requests the location of the next count object. Figure 8.13 shows the control flow of the strategy.

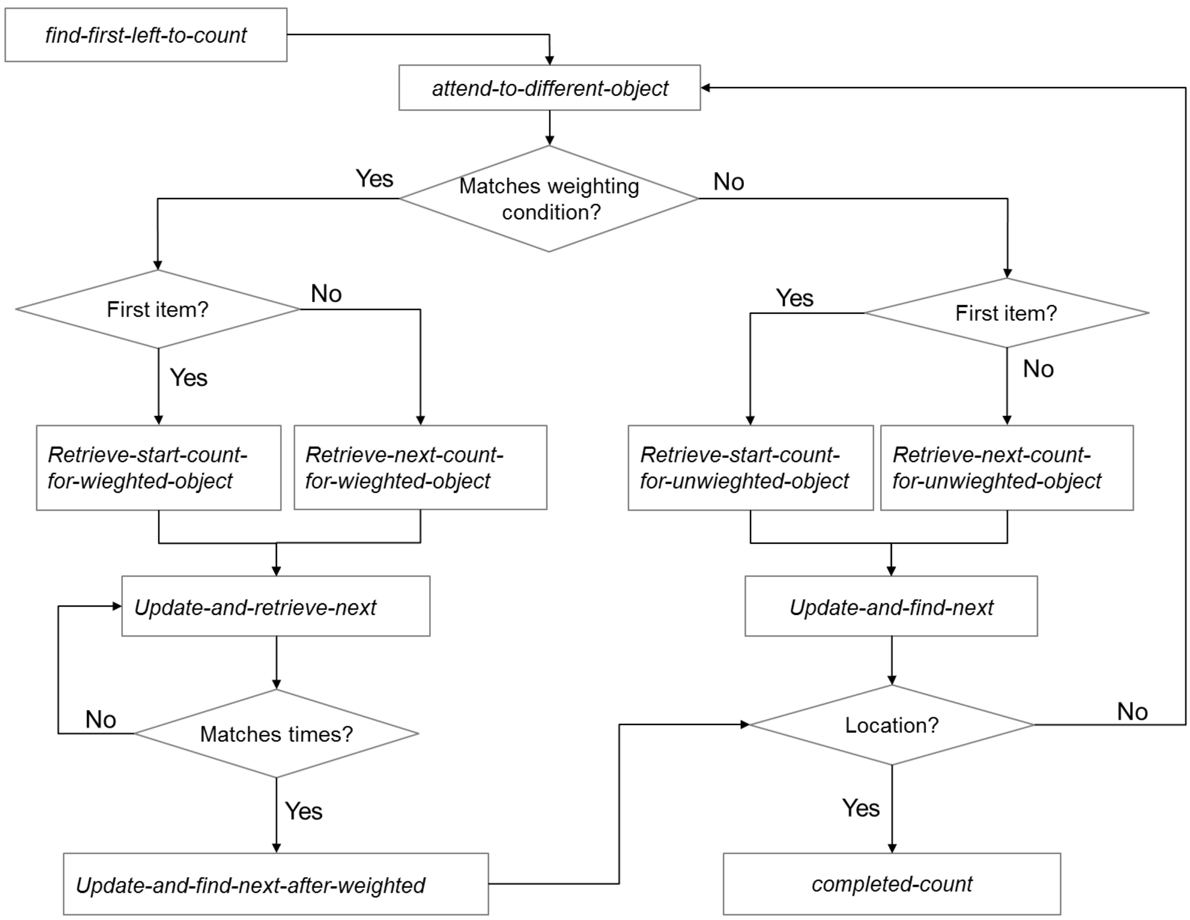


Figure 8.13. Flow of weighted count strategy.

8.5. Unpractised models of correct performance

The aim of the section is to discuss the models of unpractised problem solving developed to understand the process of identifying sub-problem exceptions, instruction interpretation and solution framing in the process of solution procedure formulation as suggested by the protocols observed in the experiment. The issues to be addressed concern the nature of the model of the task and its implications. The research presented attempts to create a model of canonical problem solving strategies observed by a subset of participants.

8.5.1. Practice

As well as the experimental instructions, participants had three practice trials to construct reusable self-instructions in the goal buffer. The simulation of completing a practice trial that lasts approximately 21 seconds. The details of subtasks can be interpreted from the preceding sections on subtasks. The task model of the simple problem provides a template and point of reference to understand the task models of the experimental problems because it results from a subtask schedule encoded in its self-instruction network and because a subset of subtasks of the simple problem are reused in the all experimental problems. The task model goes through the following steps where each step corresponds to a particular requirement/sub-goal in the model.

Step	Main role bindings
1. Encode prior statement and interpret a priors chunk	role [priors] priors [->P]
2. Encode the outcome statement and interpret a stated-outcome chunk	role [stated-outcome] stated-outcome [->O]
3. Encode the queried statement and interpret a queried-outcome chunk	role [queried-outcome] queried-outcome [->Q]
4. Indexes the array of PS-units and interpret it as the universe-trial-possibilities	universe-trial-possibilities[->U]
5. Interpret the whole-set of the proportion from the universe-trial-possibilities, then enumerate them and interpret the result as the whole-frequency	role [probability-proportion] whole-set [->U] whole-frequency [=whole-freq]
6. Re-encode the stated-outcome	role [queried-outcome] queried-outcome [->Q]
7. Index members of the queried-outcome in the diagram and interpret them as queried-possibilities	queried-possibilities [->A] members [->A] members-of [->Q]
8. Interpret the part-set of the proportion from the queried-possibilities then enumerate them and interpret the result as the part-frequency	part-set [->A] part-frequency [=part-freq]
9. Retrieve the proportion-sense of the part-frequency relative to the whole-frequency and evaluate the frequency proportion against it geometrical structure.	sense [=proportion-sense]

Table 8.7. Main steps and role bindings for practice problems. Key: [=?] = semantic binding, [->A] =spatial index binding.

8.5.2. Conjunction

Participants appear to solve the conjunction problem with little deliberation.

Model

The model runs though the subtasks in the same way as the simple problem with minor differences (see appendix 3.a. for a model trace). The model interprets the *queried-outcome* as involving a conjunction of unique categories and proceeds as normal. When the self-instruction to determine the queried-possibilities is retrieved, the model implements a goal to determine members of the category of *queried-outcome* as normal.

The only deviation of the default solution path occurs when the model is confronted with the conjunction/disjoint problem.

- When the system has established that there are no members matching the queried-outcome categories but that there are members of each category of the queried-outcome the production *note-disjoint-pattern-implies-empty-set* fires and binds the members slot in the imaginal buffer to empty. In interpreting the result of the requirement the production *update-no-queried-possibilities* fires and interprets that there are no queried-possibilities by binding the *empty* value to the *queried-possibilities* role.

- When the self-instruction is retrieved to quantify the *queried-possibilities* the enumeration production *retrieve-empty-set-count* fires and retrieves the zero count semantic mapping of an empty set and the enumeration production *update-zero-count* dynamically updates the zero count to the variable of the requirement in the imaginal buffer which is the *part-frequency* in the particular case.

- In response to the zero *part-frequency* binding in the imaginal buffer the production *retrieve-zero-proportion* fires and retrieves the proportion chunk containing the zero proportion sense *(+Retrieval> part-frequency [zero)]*. The production *update-zero-proportion* then updates the analogical proportion sense in the imaginal buffer *(=Imaginal> sense [0]).* After retrieving the self-instruction to report the result a series of routine productions proceed to report a zero proportion result.

8.5.3. Conditional

The strategic model of the conditional problems was developed in accord with the following constraints.

- On conditional/subset and conditional/overlap problems some participants incorrectly omitted limiting the trial possibilities despite evidence of processing the conditional information. These participants also tended not to deliberate about the conditional information.
- On conditional/subset and conditional/overlap problems some participants appeared to initially omit limiting the trial possibilities consistent with the default procedure but later corrected the omission. Some protocols are consistent with a

co-ordinated delay in framing the limitation on trial possibilities. Other protocols are consistent with abrupt recognition of a framing requirement.

- On the conditional/disjoint problem some participants immediately detect disjoint relations between members of the stated and queried outcome categories upon initial inspection of the diagram.
- On condition/overlap and conditional/disjoint problems all participants provide solutions consistent with correctly interpreting the constraints of the stated outcome category on queried possibilities when required - including those participants who omitted limiting the trial possibilities. This suggested all participants interpreted the conditional information from the stated outcome but some had problems judging whether its full implications were considered.
- Some participants who recognised the limitation in trial possibilities spent additional time reflecting on its implications in determining the quantification of the target probability.
- Participant's verbalisations are consistent with considering taking the whole-frequency from |U| as default and treating the conditional information as an exception condition to this procedure.
- Participant's verbalisations indicate a series of steps in self-argumentation including verbalising an exception relation.
- After limiting the trial-possibilities on the first occasion, some participants verbalised recognition of this requirement upon reading the stated outcome on the next conditional problem.
- After limiting the trial possibilities on the first occasion participants did not apply the incorrect default procedure of taking the whole-frequency from |U| on the next conditional problem.

The main features of the implemented strategy are shown in Table 8.8. (see appendix 3.c. for a model trace)

Step	**Imaginal role bindings**
Encode prior statement and interpret a priors chunk	role [priors] priors [->P]
Encode the outcome statement and interpret a stated-outcome chunk	**role [stated-outcome]** **stated-outcome [->O]**
Encode the queried statement and interpret a queried-outcome chunk	**role [queried-outcome]** **queried-outcome [->Q]**
Indexes the array of PS-units and interpret it as the universe-trial-possibilities	universe-trial-possibilities[->U]
Interpret the whole-set of the proportion from the limited-trial-possibilities, then enumerate them and interpret the result as the whole-frequency	whole-set [->U] whole-frequency [=whole-freq]
Re-encode the stated-outcome	role [queried-outcome] queried-outcome [->Q]
Index members of the stated-outcome in the diagram and interpret them as limited-trial-possibilities	**members [->C]** **members-of [->O]** **limited-trial-possibilities [->C]**
Reinterpret the whole-set of the proportion from the limited-trial-possibilities, then enumerate them and interpret the result as the whole-frequency	**whole-set [->C]** **whole-frequency [=whole-freq]**
Re-encode the queried-outcome	role [queried-outcome]
Index members of the conditional queried-outcome in the diagram and interpret them as queried-possibilities	**queried-possibilities [->A&C]**
Interpret the part-set of the proportion from the queried-possibilities then enumerate them and interpret the result as the part-frequency	part-set [->A&C] part-frequency [=part-freq]
Retrieve the proportion-sense of the part-frequency relative to the whole-frequency and evaluate the frequency proportion against it geometrical structure.	sense [=proportion-sense]

Table 8.8. Steps and main role bindings on conditional/subset and conditional/overlap problems assuming un-practiced model.

Instruction encoding. The model implements the assumptions that the probability query is interpreted as conditional on first-pass instruction comprehension. Recall that participants tended to provide answers and verbalisations consistent with limiting queried possibilities when required irrespective of whether they omitted limiting the trial possibilities. When the task model interprets the chunk of the probability query it retrieves the stated-outcome chunk and creates a chunk with the category of both the stated and queried-outcome. The resulting chunk represents an interpretation of a queried outcome that has both categories. The justification for this modelling assumption is that both statements express reference to the same individual.

Incorrect framing and execution of universe-trial-possibilities. The task model initially determines the *universe-trial-possibilities* then proceeds to determine the *whole-frequency* from it by using the default self-instructions in steps 4 and 5. Obtaining the *whole-frequency* from |U| thus occurs as normal because the system has not yet interpreted the implication of the stated outcome category on the trial possibilities.

Inferring limited trial possibilities. The model implements the assumption that the target exception is recognised after recalling the stated outcome information and referring to the statement and the corresponding members in the diagram. It is not clear from the protocols whether recall of the information is cued by observing the corresponding diagram set or is planned by the agent because it is implicated in determining the queried possibilities. The model implements the latter assumption. Productions create a subgoal to recall details of a frame statement when addressing the probability query (i.e. when the self-instruction to re-encode the probability query is retrieved) if its role is tagged as unexpected. A second subgoal is initiated to index members of the category in the diagram.

The model implements the hypothesis that the conditional inference of *limited-trial-possibilities* is triggered conditionally after binding the *universe-trial possibilities* role on first exposure to the problem because it is an inference that overrides or revises this initial belief. This is consistent with participants only making this inference after appearing to initially consider the *universe-trial-possibilities*. After identifying the members of the *stated-outcome* the production *detect-limited-trial-possibilities* fires and retrieves the case from declarative memory. An update production fires in response to the retrieved case and dynamically binds the *limited-trial-possibilities* role to the reference of members of the stated-outcome in the imaginal buffer. The *detect-limited – trial-possibilities* production is shown in Figure 8.16. Note that the production condition fires in response to abstract roles bound to the reference of individuals and sets (i.e. a model) rather than particular category information and outcome instances and is therefore capable of generalising to different contexts using this frame.

```
(p detect-limited-trial-possibilities
   isa meta-cognitive
   step demonstrated
  =imaginal>
   isa pps-image
   universe-trial-poss =universe
   given-outcome =given-outcome
   members =given-members
   members-subset-of =universe
   members-of =given-outcome
  ?retrieval>
   state free
==>
  =goal>
   step interrupting
   operation retrieving
   normally universe-trial-poss
   exception given-outcome
  +retrieval>
   isa case-relation
   relation trial-poss-of-stated-outcome)
```

Figure 8.14 The production detect-limited-trial-possibilities

The inference provides conditions for meta-roles bindings associated with exception processing (normally, exception) for the revised interpretation which are implemented by *detect-limited-trial-possibilities* and a subsequent update production *frame-limited-trial-possibilities*. This is because the inference overrides a default goal requirement. These information states represent the rejection of the initially intended *universe-trial-possibilities* (i.e. *Normally [universe-trial-possibilities]*) but exclusively to the information context (i.e. *Exception [given-outcome]*) and relates it to a revised conclusion (i.e. *Conclusion [limited-trial-possibilities]*). This meta-cognitive relation is used to represent the change in the subtask requirement over the course of the subtask.

Recall that on subsequent conditional problems some participants show immediate recognition of the requirement to limit the trial possibilities on first-pass instruction comprehension and do not verbalise an exception argument when determining the trial possibilities, if done on the first occasion. In the model this can be replicated by simply retrieving the self-instruction (meta-role features of the goal) created on the initial problem that will have the *limited-trial-possibilities* requirement. This means that the model bypasses the need to make the limited trial possibilities inference and engage in exception processing instead relying on a form of instance based goal memory.

Limiting queried-possibilities. The retrieved default self-instruction chunk is used to identify the exhaustive set of member of the queried outcome. As the *queried-outcome* is interpreted as a conditional model of an individual containing the category of both the queried and *stated-outcome* and the default self-instruction specifies a requirement to determine members of the *queried-outcome* (i.e. *require [queried-possibilities], SP [members-of], Object [queried-outcome]*) then normal interpretation productions are able to apply the default self-instruction. This leads to a conjunction search in which only referents that match the queried category that are also in the set of the conditional category get selected. The behaviour of the model is consistent with the observation above that all participants made this inference because the text frame more explicitly specifies this information and that participants tend to make this inference without signs of deliberation.

8.5.4. Biased probability

The experimental data suggested the role of the diagram in framing a solution on biased problems.

- The mapping of the geometrical probability of PS-units was not specified in the experimental instructions and needed to be interpreted by participants.
- The verbal protocols of some participants suggest that they employed the geometrical representation of probability in the process of solution interpretation as indicated by verbal protocols suggesting the conceptualisation of double probability icons (representing token outcomes) as equivalent to two non-double probability icons.
- Some participants also determine the probability proportion from the geometrical fraction scheme when its visual fidelity is high. The verbal protocols indicate that geometrically derived proportion appeared to be suddenly recognised rather than being the result of a deliberately planned strategy.

- Some participants use a particular strategy which we called weighted-count in which double probability units are serially counted twice.

Model

An overview of the main steps on the biased problems for an unpractised model of the problem is shown in table 8.9 (see appendix 3.b. for a model trace).

Step	**Main imaginal role bindings**
Encode prior statement and interpret a priors chunk	**role [priors]** **priors [->P]**
Encode the outcome statement and interpret a stated-outcome chunk	role [stated-outcome] stated-outcome [->O]
Encode the queried statement and interpret a queried-outcome chunk	role [queried-outcome] queried-outcome [->Q]
Indexes the array of PS-units and interpret it as the universe-trial-possibilities	universe-trial-possibilities[->U]
Interpret the whole-set of the proportion from the universe-trial-possibilities, then enumerate them	whole-set [->U] whole-frequency [=whole-freq]
Re-encode priors	**role [priors]** **priors [->P]**
Index members of the double and non-double probability icons and detect their size difference	**double-size-parts[->D]** **single-size-parts[->S]**
Hypothesise a representational correspondence and explain the correspondence by assuming a fraction scheme	**fraction-scheme [->U]**
Interpret normalized frequency units from the diagram and use weighted-count strategy to determine the whole-frequency	**frequency-units[->S]** **whole-frequency [=whole-freq]**
Re-encode the stated-outcome	role [queried-outcome] queried-outcome [->Q]
Index members of the queried-outcome in the diagram and interpret them as queried-possibilities	queried-possibilities [->A] members [->A] members-of [->Q]
Interpret the part-set of the proportion from the queried-possibilities then apply a weighted count and interpret the result as the part-frequency	**part-set [->A]** **part-frequency [=part=freq]**
Retrieve the proportion-sense of the part-frequency relative to the whole-frequency and evaluate the frequency proportion against it geometrical structure.	sense [=proportion-sense]

Table 8.9. An overview of the main steps on the biased problems for an unpractised model.

As the interpretation of the priors statement is a comparative relation between categories of individuals; the system initiates a goal to reference members of the compared categories. *Set-goal-to-find-compared-objects* selects the subtask whilst productions *Find-comparison-set*, *find-compared-set* and *update-compared-set* with visual attention productions implement the demonstrative routine and binds the reference of the sets to *compared-set* and *comparison-set* roles in the imaginal buffer.

Detecting icon size differences. The icons size difference could have been initially detected at different points in the problem solving process and using different strategies or cognitive resources such as pre-attentive pop out, memory expectation mismatch and spatial processing. The model implements the latter case because it is assumed that comparisons based on visual-spatial processing strategies would still be required to get relative size information and therefore involves the least commitment. In the process of relating the sets of icons the spatial structure between them is computed and the relative size differences between them is made available in the spatial buffer. This is implemented by low level productions that compare the size of instances in both sets (*find-size-object-in-compared-set, find-size-object-in-comparison-set*). This result provides conditions for the production *detect-double-size-difference* to fire in response to the difference in the relative size between PS-unit icons in the diagram and comparative roles of focal target sets. The retrieval request produces a *double-size-difference* case relation. The roles of the chunk (i.e. *single-size-part, double-size-part*) are then incrementally converted to slots in the imaginal buffer and bound to indexes of attended visual instances by a series of productions (*index-single-size-parts, index-double-size-parts*). This process demonstratively implements the interpretation of the diagram being composed of double and single size unit parts which is consistent with the verbal protocols of some participants.

```
(p detect-double-size-difference
  =goal>
   isa meta-cognitive
   step compare-size
  =imaginal>
   isa imaginal
   compare-members =targ-set
   comparison-members =ref-set
  =spatial>
   isa spatial
   relative-size 2
   loc =targ
   ref-loc =ref
  ?retrieval>
   state free
==>
  =goal>
   operation retrieving
   step interrupting
  +retrieval>
   isa case-relation
   relation double-size-difference)
```

Figure 8.15. The production detect-parts-size-difference.

ER-scheme interpretation. Indexing the different size parts provide conditions for the production *detect-representational correspondence* to fire in response to the analogical correspondence between the represented relative size differences of the PS-units parts and the relative probability of the referred categories. When fired, this production retrieves the *fraction-scheme-mapping* case to explain the double size difference case and changes the meta-role of the case relation to the explanandum slot. When this categorisation is retrieved indexing and update productions (*index-whole-scheme, update-whole-scheme*) attribute the scheme to the diagram by dynamically converting the scheme role value to a slot and binding the attended reference of the diagram in the imaginal buffer (i.e. *=Imaginal> =er-scheme [=diagram-location]*) and updating the *fraction-scheme-of-mapping* explanation by binding the relation to the explanan slot.

```
(p detect-representational-correspondence
  =goal>
   isa meta-cognitive
   step interrupting
   case-relation double-size-difference
  =imaginal>
   isa magnitude-comparison
   then-attribute =relative-probability
   =relative-probability 2-to-1
   compare-members =targ-set
   comparison-members =ref-set
   double-size-parts =targ-set
   single-size-parts =ref-set
  ?retrieval>
   state free
 ==>
  =goal>
   step reflecting
   operation retrieving
   case-relation nil
   explanandum double-size-difference
  +retrieval>
   isa case-relation
   relation fraction-scheme-mapping
)
```

Figure 8.16. The production detect-representational-correspondence.

Interpreting normalised frequency units. The model implements the hypothesis that interpreting the diagram as a part-whole fraction scheme in which the size (in this case width) of parts represent a fraction of a whole allows schematic knowledge of the use of the scheme to be employed to solve the sub-problem of quantifying the probability proportion. The model makes use of the knowledge that the *whole-frequency* for determining the proportion is a number of equal size parts of the whole that are divisible/normalised to the part and whole of the target proportion.

The critical issue for this sub-problem is defining a solution procedure for determining the proportion as the requirement for the sub-problem (i.e. *whole-frequency*). Access to this knowledge is modelled using inference and update productions. The model has a production *hypothesise-normalised-frequency-units* which retrieves the case of *normalised-frequency-units* if the goal is to determine a proportion and the representation is a fraction scheme. Subsequent productions *(update-normalised-frequency-units*) bind the relation *normalised-frequency-units* to the SP slot in the goal buffer which specifies a solution procedure interpretation. A subsequent production *select-single-whole-freq-units* dynamically binds a *whole-frequency-units* role slot to the spatial index of the *single-size-parts* set in the imaginal buffer and requests visual attention to the set. These productions demonstratively implement the interpretation of single size PS-units as common units for determining a whole-frequency/denominator. The declarative knowledge and associated productions are considered part of a schema for the part-whole fraction scheme. The production implementing the initial inference *hypothesize-normalized-frequency-units* is shown Figure 8.17. The inference occurs after interpreting the fraction scheme.

```
(p hypothesize-normalized-frequency-units
  =goal>
   isa meta-cognitive
   step reflecting
   require whole-freq
   SP ?
  =imaginal>
   isa imaginal
   fraction-scheme =diagram
   whole-freq-units ?
  ?retrieval>
   state free
==>
  =goal>
   operation retrieving
  +retrieval>
   isa case-relation
   relation normalized-frequency-units)
```

Figure 8.17. The production hypothesize-normalized-frequency-units

Implementing the count strategy. Given that the system has now interpreted a solution it needs to select and plan a task strategy for implementing the solution procedure. Only one strategy is implemented in the model termed the weighted-count strategy. Whilst the self-instruction part of the goal has the requirement, SP and object values reflecting a complete solution procedure interpretation, it still does not have a subtask strategy. The subtask strategy is initially selected by the production *select-weighted-count-strategy* and this selection involves binding the *count-weighted* value to the subtask slot in the goal buffer. The additional parameters of the subtask strategy include the perceptual condition for the repeated count and the number of repetitions. These parameters are implemented by a series of productions *set-weighted-object-condition*, *retrieve-multiplier* and *update-multiplier*. The resulting settings corresponds to a plan of counting a unit twice if it has the double probability category.

8.5.5. Disjunction problem

The model of performance on the disjunction problem was developed to satisfy the following constraints.

- Participants determine the reference of each referred set of the disjunction query one set at a time as exhibited by eye-movements
- On subset and overlap problems some participants appear to initially employ an incorrect add-probabilities procedure (i.e. P(A) + P(B) +P(A&B)) which suggests they interpret the possibilities of the disjunction query as a conjunction of members of each model (instead of the union of members)
- All participants who appear to incorrectly employ the add-probabilities procedure latter abandon it because they judge the resulting value as greater than unity and/or too high.
- Participants tend to make a high number of repeated eye-movements back and forth between the disjunction query and its referred sets suggesting bidirectional interpretation of the correspondence between the interpretation of the probability query and the structure of the data in the diagram in order to determine the form of the correct SP for the queried possibilities requirement (i.e. the union)

Model

An overview of the main steps on the disjunction/subset and disjunction/overlap problems for an unpractised model of the problem is shown below in Table 8. 10 (see appendix 3.d. for a model trace)

Step	**Main imaginal role bindings**
Encode prior statement and interpret a priors chunk	role [priors] priors [->P]
Encode the outcome statement and interpret a stated-outcome chunk	role [stated-outcome] stated-outcome [->O]
Encode the queried statement and interpret a queried-outcome chunk	**role [queried-outcome] queried-outcome [->Q]**
Indexes the array of PS-units and interpret it as the universe-trial-possibilities	universe-trial-possibilities[->U]
Interpret the whole-set of the proportion from the universe-trial-possibilities, then enumerate them and interpret the result as the whole-frequency	role [probability-proportion] whole-set [->U] whole-frequency [=whole-freq]
Re-encode the stated-outcome	role [queried-outcome] queried-outcome [->Q]
Infers members-of-each SP and indexes each model of the queried outcome	**queried-possibilities [->A] members [->A] queried-possibilities [->B] members2 [->B] queried-possibilities [->A&B] members3 [->A&B] members-of [->Q]**
Infer sum-of-each SP and enumerate each set then and interpret the sum as the part-frequency and the part set as the union of the sets	**part-set [->AUB] part-frequency [=part=freq]**
Retrieve the proportion-sense of the part-frequency relative to the whole-frequency and evaluate the frequency proportion against it geometrical structure and interrupt because of a mismatch	**sense [=proportion-sense]**
Elaborate and explain the mismatch and hypothesise a union SP and backtrack to the queried-possibilities requirement	**too-high [=part-freq] repeated-count [->AUB]**
Demonstrate the union interpretation of the SP	**union-members [->AUB]**
Use the model of the union correspondence to infer and test an alternative meaning of the instruction	**queried-possibilities[->AUB]**
Interpret the part-set of the proportion from the revised queried-possibilities then enumerate them and interpret the result as the part-frequency	part-set [->AUB] part-frequency [=part-freq]
Retrieve the proportion-sense of the part-frequency relative to the whole-frequency and evaluate the frequency proportion against it geometrical structure.	sense [=proportion-sense]

Table 8.10. An overview of the steps on the disjunction/subset and disjunction/overlap problems for an unpractised model of the problem.

Addition procedure. The model implements the hypothesis that the initial choice to use the add-probabilities SP is based on an incorrect interpretation of what counts as possibilities of the disjunction query. It is proposed that participants who use this procedure interpret the disjunction query as implying possibilities that are referentially equivalent to the conjunction of members of each model in the disjunction query. This incorrect interpretation is insensitive to the particular set structure between members of the disjunctive models and it is assumed that the implications of different set structure possibilities are not considered by participants when forming this interpretation. Quantifying the probability of these queried possibilities appeared to be carried out using either one of two equivalent procedures (a) by adding together members of each set to get the part-frequency (i.e. |A| + |B| + |A&B| / |U|) or by forming a probability proportion for each set and adding them together (i.e. |A|/|U| + |B|/|U| + |A&B|/|U|)

Interrupting and rejecting the add-probabilities. The model implements the hypothesis that abandonment of the add-probabilities procedure is made as a result of comparing the summed probabilities to the geometrical proportion of the union of disjunction sets. There are two cases (1) when the resulting proportion is judged as too high when less than unity and (2) when the resulting proportion is judged as too high and is greater than unity. The model contains a production *detect-part-freq-too-high* which fires if the proportion sense derived from a *part-frequency* and *whole-frequency* is “roughly” higher than the geometrical proportion irrespective of the context and retrieves the *part-freq-too-high* case. When retrieved an update production evaluates the match and binds the *too-high* role to the *part-frequency* value in the imaginal buffer.

```
(P* detect-part-frequency-too-high
    =goal>
     isa meta-cognitive
     require sense
     step demonstrated
    =imaginal>
     isa imaginal
     part-set  =part-set
     whole-set  =whole-set
     sense =sense
    =spatial>
     isa spatial
     relative-size =relative-size
     loc =part-set
     ref-loc =whole-set
     !eval!( >  (-  =sense =relative) 0.1)
    ?retrieval>
     state free
    ==>
    =goal>
     operation retrieving
     step interrupting
    +retrieval>
     isa case-relation
     relation part-freq-too-high
     )
```

Figure 8.18. The production detect-part-frequency-too-high.

Explaining the outcome. The modelled strategy assumes that participants explain that the *part-frequency* is too high because the count is derived from repeated counting of overlapping sets and then use the explanation to infer an alternative SP (i.e. *Sum-of-union*) to replace the current SP (i.e. *sum-of-each*). Seeking the explanation is triggered by an inference production *retrieve-counting-overlap-case* that retrieves a case to explain (i.e. *part-frequency-too-high*) and sets the case relation value to the explanandum slot. A subsequent production (i.e. *update-overlap-explanation*) evaluates and updates and binds the retrieved relation *(repeated-count-of-overlapping-sets)* to the explanan slot.

Given an explanation for the *part-frequency* being too high is hypothesised to provide conditions to infer the *sum-of-union* relation as the new SP. The production *Retrieve-non-repeated-count-of-overlapping-members-case* fires and retrieves a case containing the *sum-of-union* action role given a retrieval constraint of *non-repeated-count* outcome role. A subsequent update production (*update-action-hypothesis-for-case*) evaluates the matching case (i.e. *overlapping-members*) and binds the action role *sum-of-union* to the SP slot. A series of production backtracking to the previous dependent requirement. (i.e. *queried-possibilities*) and infer the dependent sub-procedure (i.e. *union-of-each*).

Demonstrating the union. The production *select-union-correspondence* initiates a routine to demonstrate the new solution procedure and a series of productions then demonstrate the union correspondence between referred categories in the diagram. This involves making a correspondence between each referred category and set in the union by re-indexing their locations in the imaginal buffer. This is consistent with observed repeated eye-movements between referring terms and sets of the disjunction query.

Reinterpreting the instruction meaning. It is hypothesised that participants use the hypothesised union solution interpretation of queried possibilities to reinterpret the instruction meaning consistent with a referential interpretation of the solution. The model includes a production *detect-each-or-meaning-mismatch* that detects the mismatch between the *each-or* interpretation and its referential union interpretation. This production fires and infers an alternative case of the "or" relation (i.e. any-or) that is consistent with the union correspondence. The retrieved case then triggers the productions *update-any-or-meaning-match* which evaluates the match to the union correspondence and sets the value to the conclusion meta-role slot and modifies its semantic interpretation. A subsequent production *map-instruction-interpretation-to-result* fires in response to the case match and updates the union interpretation of the queried-possibilities as the new result. With the revised interpretation of the queried possibilities the model proceeds with the remaining steps as normal.

8.5.6. Evaluation

The evaluation of the model is targeted at the components of the modelling framework as presented as well as the implementation of the problem specific strategic knowledge as described. The evaluation is considered with respect to the competence and generality of the model, its simplicity and parsimony, its ability to explain and make predictions about experimental performance, its consistency or match to the experimental data and existing empirical research. Sample protocol traces generated by the model are shown in appendix 3 and the detailed ACT-R traces of the model output on all problems are available on the accompanying disk.

Problem solving time. The model generates problem solving times comparable to the times taken by participants. Figure 8.19 shows a line graph expressing median problem solving times of participants on each problem and the problem solving times generated by the model assuming practice on only the practice problems. The correlation coefficient between the two data sets is .9. Similarities between the model and data can be accounted for by similar times in subtask processing and subtask requirements of a problem. Note the use of median comparison times are intended to indicate the extent that the model is in the range of comparable processing times rather than to test predictions of performance of an average participant. As presented, the problem solving times were highly variable between subjects.

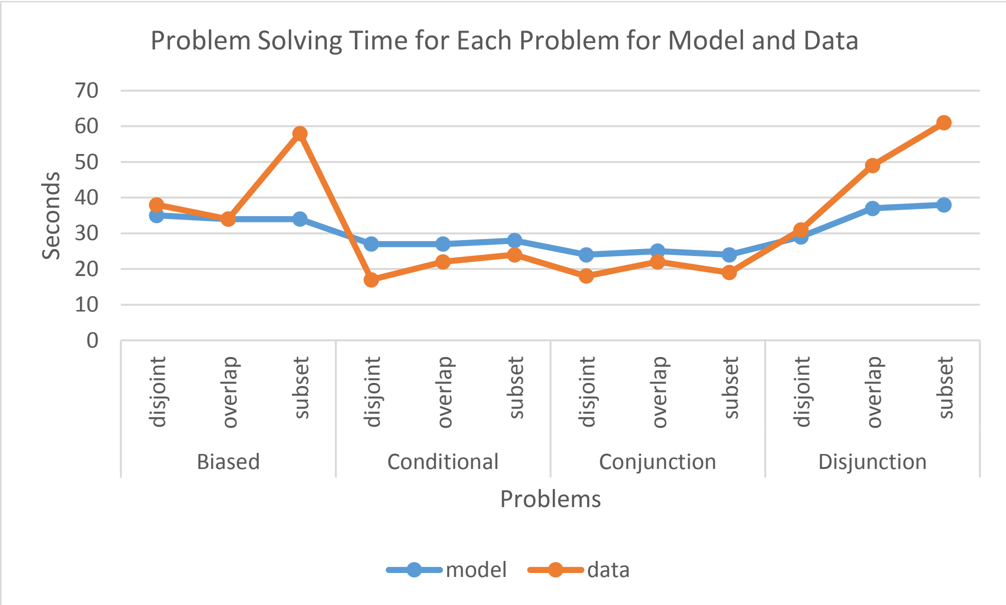


Figure 8.19. Median problem solving time for each participant on each experimental problem in comparison to problem solving times generated by model.

Visual attention. The model generates traces of attention shifts to presentation elements (see appendix 3). Whilst the mapping between attention and fixations is not one-to-one there is a close enough correspondence to consider comparisons. Whilst the model generates a large number of attention shifts simulating the observed scanning behaviour, the number is still not comparable to the fixations produced by participants on the same problems. One reason for this is that some attention shifts generated by the model whilst carrying out other operations are unrealistically long (up to approximately 2 seconds in some cases). Fixations may result from wondering as well as goal directed behaviour and no attempts were made to implement such mechanisms because of a lack of explanatory information. The average number of fixations to PS-units on a problem in the experiment was 45 which compares to an average of 28 (ranging from 15 to 40) generated by the model.

The average numbers of attention shifts between text and diagram elements generated by the model on a problem was 5 (ranging from 3 to 11) compared to an average of 15 saccades observed between the same elements on a problem in the experiment. Figure 8.20 shows the relative time attended to interest areas for the different problems (see figure 6.5 for a comparison). Consistent with the experimental data, the amount of attention time on a particular instruction element increased when that element contained solution critical referred categories. The amount of time on an instruction element is also determined by solution difficulty. So for example simulated attention time to the query statement was highest for the disjunction problem, a pattern reflected in the experimental data. However, the absolute average simulated value of 13 seconds of attention time was still less compared to an observed mean of 17 seconds of fixation time for a disjunction query in the experiment.

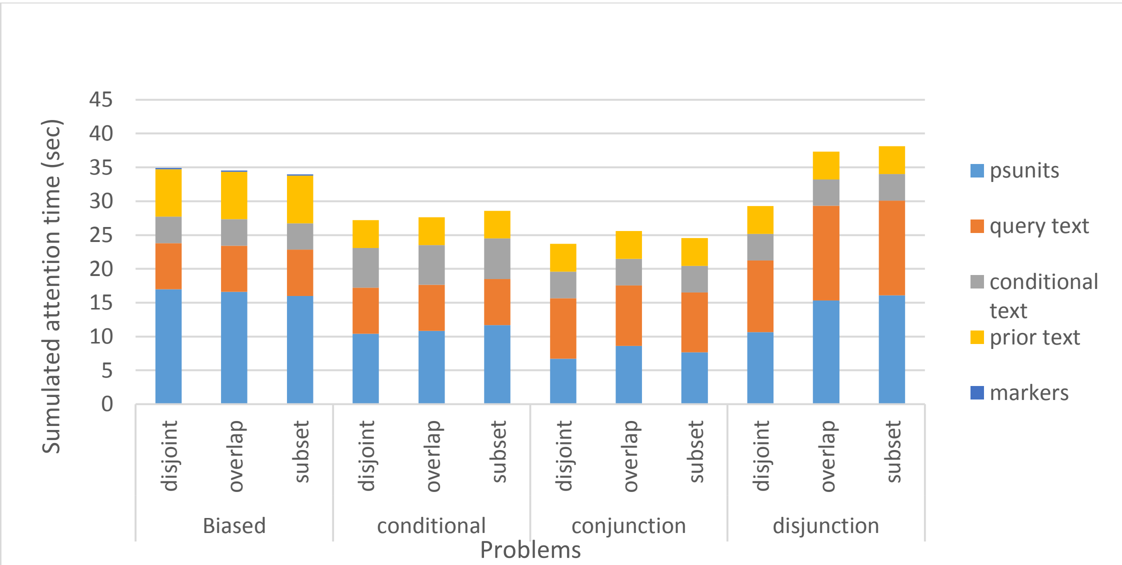


Figure 8.20. Simulated attention time for each problem generated by the model.

Spatial processing The addition of the spatial buffer, representation scheme, architectural processing functions as well as strategic knowledge, although limited, is able to provide spatial processing competences sufficient for carrying out the task and processing spatial information that the empirical data suggests participants engage in. The model of spatial processing is parsimonious because its shares or is compatible with existing assumptions in ACT-R. The feature based representation has similar attributes to the existing ACT-R ontology for visual search hence the features that are represented after an attention shift from one object to another are equivalent to the spatial feature possibilities for specifying a search request. Moreover, higher-level spatial strategies such as relative magnitude comparisons, finding a grouping (union) or spatial overlap (intersection) rely on the use of existing visual module functions in conjunction with the spatial buffer.

The assumption that the representation is specific but local to objects is consistent with recent empirical evidence on VSTM. Note however that the assumption that multiple spatial relations or features are computed in parallel is consistent with but undetermined by evidence that spatial memory representations are specific. However, this additional assumption makes the critical prediction that the parallel generation of a specific but local spatial representation in response to selectively attended objects provides unrequested spatial information as a side effect that can aid recognition of goal relevant but not necessary pre-considered information. This can explain why a diagram can facilitate inference and reframing of the problem attributes thus providing an accessibility advantage conferred by diagrams. Parallel generation of alternative visual features of objects is proposed in theories of visual attention and ACT-R. Parallel processing appears to be a general feature of sub-systems within the cognitive architecture (ACT-R for example assumes this for all within-module processing).

Self-instruction. The self-instruction scheme is used by the system in all problems. The scheme is motivated by and predicts or is in principle capable of making predictions about participants' protocols and solutions. The existence of the scheme in the model predicts and explains regularities in subtask scheduling observed from experimental protocols, the incorrect transfer and inappropriate interpretation and execution of procedures to sub-problems. It also provides one operational conception of a default solution procedure that is overridden only when conditions are recognized to interrupt. Such a scheme is required to explain how participants can process a particular solution interpretation as an exception (as observed in participants' protocols) because exception recognition implies consideration of what is default or normal (i.e., a retrieved self-instruction in the model).

The attributes of self-instruction were used for all subtasks in the model problems suggesting some scope for generality. Whilst this tentative processing scheme has not been tested on other types of problems there is good reason to suppose that the constant attributes would generalize to other problem domains (with possible extension) because they constitute the fundamental types of goal information. The processing scheme can be viewed as simple or minimal because it involves a chunk with a small number of constant attributes and associated productions. The processing scheme is parsimonious because the same attributes are a subset of those used in the goal buffer. Reusing the self-instruction would increase its base level activation and adapting or creating a new self-instruction will result in a new self-instruction chunk in declarative memory. Therefore, goal tracking, goal formation and goal memory are part of the same parsimonious system using an invariant representation.

Argument structure The system of representing and processing meta-relations is used across different problems as the roles of meta-relations reflect generic information states used in tracking reasoning and self-argumentation. The existence of meta-relations in the model is motivated by self-argumentation evident from participants' verbal protocols. Hence, the generation of the states roughly correlate with the main problem states verbalized in protocols. The states are given a functional role in constraining production selection in the model and are therefore explanatory in modelling unpractised cognition. Although not modelled the existence of this information can in principle be used to select future actions by serving as learnt declarative rule instances in memory. This is consistent with protocols of some participants who after verbalizing a state of exception on detection of a new trial problem feature, when presented with the same problem feature on a subsequent trial, express recognition of what to do but without again verbalising a state of exception.

Deictic problem representation The system of representing and processing framed environmental information is used across different problems. The system is a functionally required part of the model that provides conditions for inference generation and action selection that depend on the co-reference of role slots bound to pointer representations of individuals. This model is consistent with empirical findings and theories of deictic representations that propose higher-level concepts and programs can be bound by pointer representations to environmental objects. It is also consistent with accounts of cross modal episodic representations that propose that visual and spatial features of an encoding site are bound together with interpreted semantic or conceptual features.

8.6. Discussion

The discussion reviews the main challenges addressed by the model and findings derived from the modelling activity as well as implications and limitations of these findings. The discussion will consider in the following order findings and implications concerning the roles of diagrams in PPS, the nature of visual spatial processing in the diagrammatic PPS task, unpractised problem solving, the modelling approach employed and general conclusions.

8.6.1. Diagrams and PPS

Co-reference constraints. The ACT-R model derives probability by testing what is possible based on set membership and assigning respective possibility roles. Hence, like other analytical reasoning domains, the cognitive model needs to use a token model of the data in order to analytically derive a solution consistent with the structural constraints of the problem. The PS-diagram used is token referential, in which diagram tokens stand in for possible outcomes. The requirement for the use of token referential representations of possibilities have an empirical and theoretical basis. This proposal is supported by and consistent with the mental model theory and nested set accounts of extensional reasoning about probability. It is also consistent with broader evidence and theoretical proposals that the human mind tends to employ specific representations/models in reasoning about states of affairs (e.g., Johnson-Laird, 1983; Stenning and Oberlander, 1995).

The possible requirement for the representation of token models in reasoning about probability (like many other analytical problem domains) may depend in part on the need to process co-reference constraints on individuals in reasoning. The cognitive model uses a deictic problem representation in which role ascriptions are bound to a representation of the reference of sets (i.e. location chunks) of possibilities in the diagram, which function as indexes. This deictic character of the problem representation is connected to how inferences are computed by the model. Productions that initiate inferences do so by recognising patterns. The pattern matching is not just based on semantic matches to roles or attributes, but also on the co-referential bindings to individuals indexed by the problem representation. Hence, the production triggered to infer limited trial possibilities involves recognising from the problem representation state that X are members of the stated outcome, that Y is the universal set, and that X is a subset of Y thus using the co-reference of X and Y in different roles.

The computationally efficient exploitation of co-reference constraints in reasoning as implemented in the model may explain performance advantages and the proposed efficacy of reasoning about probability with diagrams or model like representations. Using indexes to external individuals, as implemented in the model, appears to be the most algorithmically simple and a direct way of exploiting co-reference constraints on individuals in reasoning. This is also a plausible hypothesis because it is coherent on a semantic level. Note the cognitive capacity and ubiquity of demonstrative thought is evident by the existence and use of demonstrative terms in language (e.g., “this”) and is consistent with developments of mathematical formalisations of thought employed in logic and knowledge engineering that use deictic or referential variables (e.g. ∃x, letter (x), blue (x)).

This proposal is similar and related to proposals made by Larkin and Simon (1987) in their analysis of the advantages of location indexing in diagrams. Recall that Larkin and Simon's analysis of processing advantages were concerned with the constraint that the representing properties of diagrammatic objects are co-located. The model has shown how location indexing supports co-reference constraints involving propositions about projected meaning (e.g. possibility) that are modelled by problem roles about representing diagrammatic objects in the cognitive model. In simple terms, this view holds that deictic reference is a generic information processing competence conferred by diagrams. Diagrams thus support reasoning in PPS because they exploit a deictic competence in the representation of co-reference constraints that are used in pattern matching for inference generation.

This co-reference hypothesis has implications in understanding performance in conditional reasoning tasks. The debate about whether nested sets, frequency formats, or whole object properties of the representations of problem data are responsible for performance facilitation observed in conditional probability reasoning tasks was argued to be problematic because of the potential role of each of these different kinds of information in the reasoning task (i.e. set structure vs frequency). If the co-reference hypothesis property is correct then understanding the facilitation effects of diagrams is not just a matter of what particular content of the domain expressed by a representation is responsible for performance facilitation, as implied by nested set accounts. It also provides an alternative re-framing of the hypothesis that whole object representation is responsible for performance facilitation as reported by Brase (2009). Rather than having anything to do with the theoretical proposals proposed by ecological rationalists/natural sampling, the effect would be an implicated result of the token referential property of the representation of outcomes and the appropriate conceptual framing (by the diagram semantics) of the opportunities for direct deictic representation and required inferences exploited in determining co-reference constraints. Under this view the difference in performance observed between icon and Venn diagrams in Brase's study would therefore correspond to differences in the framing of these deictic processing opportunities as the icon diagram represents the sample individuals by tokens, whereas the Venn diagram more abstractly frames sets by enclosed regions.

Prior diagrammatic knowledge. The model implements the hypothesis that (at least some) participants exploited prior knowledge of diagrammatic schemes in the process of solution procedure formulation. This was demonstrated with a model of solving the biased probability problems, in which recognition of the unequal size PS-units provides conditions for hypothesising a potential mapping of their size difference and categorisation of the implicated diagram scheme (i.e. fraction-scheme). Interpreting the new mapping and framing the scheme interpretation, in turn, provided conditions to interpret an equivalent whole-frequency set of equiprobable possibilities and a procedure for deriving a normalised frequency from this set.

This process can be viewed as involving an accessibility effect of the particular diagrammatic scheme on prior knowledge or schemas for common diagrammatic schemes. Note that this kind of account can be distinguished from diagram configuration chunk accounts (e.g., Koedinger and Anderson, 1990; Lane, Cheng and Gobet, 2000) and other accounts that assume exemplar/instance based knowledge of diagram configurations (e.g., Stenning and Oberlander, 1995) because it is not the recognition of a learnt configuration instance that is the critical source of knowledge, but instead, the classification and framing of a more abstract system or scheme of representation.

This kind of activity modelled in the biased problem would be expected in a number of PPS studies described in Chapter 3 because the diagrams employed are often novel, ad hoc representations that incorporate familiar representation schemes and there is no reported instruction administered to participants on the semantics of the representations. Hence, participants would need to make sense of the mappings and infer implications of the framing of the diagrammatic scheme as implemented in the cognitive model.

The strategic characterisation of performance can be viewed as a framing effect of the conceptual structure expressed by the diagram. Recall that the conceptual structure encoded in a representation is hypothesised to aid problem solving and reasoning and is proposed to be connected to the apprehension of the underlying represented laws that underpin the modelled domain (e.g., Cheng 2002). In the cognitive model, representing laws that underpin the domain are not explicitly derived or declaratively represented. In the cognitive model, such laws are tacit in the inferential competence associated with a diagrammatic scheme as implemented in the pattern matching conditions of productions that initiate inferences. In our view, explicating such laws into declarative representations would require explicit educational instruction or engaging in the kind of analytical activities and goals attributed to a semantic analyst. However, behaving in accordance with an understanding of such laws, only requires the know-how knowledge. As stated, this know-how knowledge that is tapped into constitutes a more generic and perhaps more primitive prior knowledge of diagrammatic schemes.

Specificity, demonstration and semantic interpretation. An important role of the diagram modelled in the disjunction problem was the use of the diagram in facilitating the correct interpretation of the specific meaning of a given problem instruction or self-generated solution procedure instruction. The modelled strategy depended on interrelating several sources of information, including ruling out and explaining the fault of the add-probabilities procedure, hypothesising an alternative union interpretation and an alternative instruction interpretation. One of the important abstractions of this process account is that the specific nature of the diagram was exploited in demonstratively interpreting the implicated meaning of the add-probabilities procedure (i.e.

understanding that it implies the repeated inclusion of possibilities for overlapping sets) and demonstratively interpreting how an alternative interpretation of the disjunction problem instruction was consistent with the union solution procedure. It was hypothesised that this was achieved by actively interpreting and testing the correspondence between referred categories and corresponding sets in the diagram.

The processing requirements of this strategy were consistent with the extended reflection time and larger number of observed eye-movement switches between instructions and referred sets in the diagram, which results from attention being devoted to interpreting the target correspondences. The potential of the diagram to be used to investigate, elaborate and evaluate the implied meaning of a self-instructed solution procedure or a given problem instruction through action and feedback interactions with the diagram, is an important problem solving role that can be exploited in diagrams. It assumes that the conceptual interpretation of a solution procedure that constitutes part of a self-instruction may be vague in the sense that it could imply more than one different solution procedure with different consequences.

8.6.2. Deictic representation and processing

The cognitive model can be viewed to provide a distributed account of the cognitive task. Rather than represent a complete model of the diagram, the cognitive model holds a deictic representation in the imaginal buffer comprising of indexes to the locations of a limited set of visual objects. Productions automatically initiate attention to the referred objects when the chunks of these roles are being processed. Visual properties of an object are only available for one object/group at a time in the visual buffer and are accessed as needed. The deictic use of external information is consistent with theories of spatial indexing that have been proposed to explain how the cognitive system is capable of processing environmental information in situated cognitive tasks in a way that is more computationally efficient, representationally leaner and can be viewed to connect or ground the relation of internal cognitive information (e.g., symbols, programs, etc.) to external objects (e.g., Pylyshyn, 2000; Ballard et al., 1997).

The ACT-R model solves the problems by demonstratively assigning conceptual roles to referent groups in the diagram. These role bindings, in turn, provide the conditions for connecting knowledge, inferring new knowledge and deciding on actions. The modelled strategies incorporate a set of demonstrative routines associated with interpreting the requirements and results of a diagrammatic subtask. These demonstrative routines instantiate the referential meaning of a computed relation. For example, in the proportion subtask a demonstrative routine binds external referents of the problem roles *whole-set* and *part-set* as arguments to determine values of their corresponding *whole-frequency* and *part-frequency*. This is a functional requirement and consistent with the phenomenology of thinking "*its this set relative to that set*". When the proportion sense is computed from the *part-frequency* and *whole-frequency* values, a strategy is initiated to automatically re-index the referents of these roles to relate the objects of the proportion result thus providing referential meaning to the abstracted quantities. This, in turn, provides conditions for interpreting the geometrical proportion of the referent in conjunction with the proportion derived from the frequency. Such actions are implied by participants' data. For example, in the proportion case, some participants appear to selectively choose to report a geometrical fraction when its fidelity allows it to be mapped to reportable numerical representation of a fraction; thus suggesting the information conditions are sampled as a matter of course thus implying the existence of such demonstrative strategies.

8.6.3. Visual and spatial processing in diagrammatic PPS

Attention allocation. One of the challenges of the model was attempting to explain the frequency and nature of eye-movement patterns observed in the diagrammatic PPS task. The comparison of the model did not provide a close fit either in frequency or patterns observed. Whilst the correspondence between actual eye-movements as observed and attention shifts as modelled will not be one-to-one, there are other good reasons to explain a critical limitation of the model. Notably the patterns of eye-movements observed was substantially more stochastic than the attention shifts predicted by the model.

In order to predict eye-movement patterns, the model incorporated: (a) lower level goal/requirement independent attention routines that scanned sub-groups to identify discontinuities when attention to a group had been made; (b) a search strategy that involves attending to sets to rule out that they had properties as well as attending to sets that have target properties; (c) productions that automatically initiate attention to the location source of comprehended information when that information is retrieved; and (d) attention made as a result of strategies that make online demonstration of relations and attributions about representing objects in the diagram. All of these factors are empirically and analytically motivated and add to making attention shifts appear more cognitively plausible. Other possibilities include the existence of repeated eye-movements to support the rehearsal of visual and spatial representations in the diagram (e.g., Tremblay, Saint-Aubin and Jalbert, 2006), to re-encode forgotten information (e.g., Peebles and Cheng), or to check retrieved information the system meta-cognitively deems unreliable.

The hypothesis that when participants think about a piece of information they automatically index and attend to the encoding location of the information source is an established empirical phenomena reported in the research literature and proposed to result in part from the intra-modal nature of episodic representations such that the encoding experience results in an episodic representation that links visual-spatial and comprehended information that is later retrieved (e.g., Ferreira, Apel & Henderson., 2008). This hypothesis is also consistent with general findings of episodic encoding reactivation in the brain (e.g. Danker and Anderson, 2010).

The model implements these attention shifts by productions that respond to the location information encoded in the retrieved chunks and sends attention requests to the retrieved location. An alternative model of this phenomena would involve an architecture in which retrieval requests produce the modality specific chunks in the buffers where they were created. This is unlike ACT-R, which holds retrieved chunks in an abstract retrieval buffer. If the alternative model was correct, associated actions would then be a side-effect of cognitive architecture constraints rather than learnt procedural knowledge. The hypothesis was implemented to partly explain the large number of corresponding eye-movements between task related information such as corresponding referred categories, their sets and marker-lines in the diagram. This hypothesis may also help explain eye-movement changes in other diagrammatic reasoning tasks. For example, Stocco and Anderson (2008) reported an inconsistency in the eye-movements predicted by their model and the larger quantity of eye-movements observed by participants in a geometry theorem proving task.

Perceptual grouping. The model incorporates the visual capacity for perceptual grouping which, like individual visual objects, is generated by the visual module in response to search and attention requests. The development of the model explicated three main processing advantages of perceptual grouping in the PPS task. Perceptual grouping provided simpler parallel rather than serial operations for identifying task relevant sets, is used to determine relations of set membership by their spatial structure, and simplifies the internal problem representation and the tracking of sets into a minimal number of location indexes. The model explicitly incorporates all three processing advantages that depend on the use of representations that exploit perceptual grouping and the visual system's capacity for processing such information. These advantages are also arguably exploited in the reported PPS studies that use diagrams, such as the roulette wheel diagram (Yamagishi, 2003), the Venn diagram and grouped icons diagram (Brase, 2009).

Visual salience. The model does not incorporate pre-attentive processing or explicitly model the effects of perceptual salience of diagram features because this was not a developed property of the ACT-R architecture. A recent visual attention module for ACT-R incorporating pre-attentive vision has been proposed by Nyamsuren and Taatgen (2013). Their module incorporates a visual icon memory, which holds peripheral/pre-attentive visual objects and a stochastic model of search that uses top-down (production specified) and bottom-up (salience based activation) information to determine what object locations of the limited icon memory are selected. It should be noted that, in our model, a restricted advantage of pre-attentive processing competence is assumed in the capacity of the visual module to accept requests for search and attention to perceptual groups because the existence of perceptual groups in the model are created based on their assumed salience (only perceptual groups formed by an invariant value of a visual dimension across its elements are specified in the model's visicon).

An arguably potential limitation of the lack of pre-attentive mechanisms is the possible attentional biasing effect of visual salience that may influence cognition to positive or negative ends in the task. For example, additional features such as discontinuities in letter properties and marker line patterns in peripheral vision may bias attention and, therefore, influence higher-level processing of those visual features. Such effects can however be matched by productions that choose to scan letters and diagram elements as a general heuristic strategy, which is incorporated in the existing model. The assumption that information in the diagram is encoded as a result of active visual-spatial engagement is consistent with the requirements of developing a coherent cognitive model of the task and explaining some of the experimental observations as discussed. It is unclear to what extent pre-attentive vision could play a bottom-up role in influencing the content of high-level thought. However, it is proposed that pre-attentive vision may provide a more realistic prediction about eye-movement patterns, but may, in turn, require more sophisticated strategic knowledge for error correction to cope with the stochastic nature of location selection on attention shifts. The prediction of incorporating a pre-attentive model would be expected to contrast with the attention shifts generated by our model, which follow somewhat rigid strategically predictable patterns.

Spatial processing. A further modelling challenge was to develop a spatial processing system that would satisfy modelling requirements of the diagrammatic PPS task, explain assumed accessibility advantages of diagrams whilst being consistent with empirical research on visual spatial processing. The model implements the hypothesis that a specific non-conceptual representation of the spatial structure between selectively indexed objects is processed and generated automatically on attention shifts in a spatial buffer. Whilst the referents of the spatial structure are selected in a top-down way via productions; the actual processing of the structure is assumed to be local to a spatial subsystem or module in response to location indexing/attention changes.

The model of spatial processing system provides information advantages in the diagrammatic reasoning task. The representation of spatial structure is specific and generated automatically on attention shifts. The main functionally important abstraction is that it provides conditions for the selection of goal relevant conceptual spatial relations. This is because the system does not need to serially test for relations that may not exist, as the information of any abstraction is implicit, it provides the opportunity for spatial pattern recognition. This property would be true of any perceived base representation just so long as there are the appropriate pattern matching operators available. This property means that the system can notice abstract conceptual spatial relations as a result of selecting concurrent objects for attention in the same way that the system can notice when searching and attending to a large letter that it happens to be blue as a side-effect, because the results of the search request is a specific representation of the letter. It is proposed that this advantage in spatial processing is general in diagrammatic reasoning and can be considered an important accessibility advantage exploited in diagrammatic PPS.

8.6.4. Unpractised problem solving

Another challenge was the attempt to model the unpractised character of the problem solving process. There are three main co-dependent features of the model that are central to the modelling of unpractised problem solving: (1) the meta-cognitive representations of the goal buffer, (2) the self-instruction scheme, and (3) the dynamic binding mechanisms involved in knowledge integration.

Reasoning and argumentations. ACT-R's theoretical commitments are generally focused towards modelling routine action based behaviour, which is perhaps why it has minimal commitments for modelling high-level reflective processing compared to cognitive architectures like SOAR (Newell, 1990). The model implements the process of generating declarative inferences using hypothesis specific retrieval productions. These productions are intended to model hypothesis generation in which the retrieved content is intended to represent a conceptual case applicable to the internal problem state. The conditions of these productions are specific to the case. One possible limitation of the model is the absence of the use of bottom-up activation based processing in reasoning. Whilst spreading activation and instance based memory are features that can be used in hypothesis generation, it appears difficult to implement with complex heterogeneous knowledge. In diagrammatic reasoning tasks it is possible to imagine how bottom-up activation could influence/bias the retrieval contents from activation emanating from perceptual, spatial and imaginal buffers containing processed environmental representations. This could also be viewed to constitute an accessibility effect not addressed by the research. The potential role of bottom-up activation of diagrammatic information on the accessibility of perceptual inferences is an interesting line of future research on the role of diagrams in supporting solution interpretation.

Argumentation tracking The model included the use of meta-roles in the goal buffer to track the argumentation relations of acquired information in the course of solution interpretation. These were motivated by the observed protocols, their functional role in cognitive control and the requirement for a more detailed purposive representation of the use of task knowledge. Rather than just accumulating slot bindings, the use of meta-roles structure the use of problem information in the task model. It also provides abstract conditions for productions that operate at the level of meta-relations (meta-cognitive), which is a functional requirement and logical implication in unpractised reasoning and problem solving tasks. The model provides a novel way of characterising the representation of reasoning/argumentation in the ACT-R architecture as it frames argumentation, in part, as a meta-cognitive tracking process. Such roles can be seen to correspond to meta-cognitive relations of knowledge when their meaning is analysed from an information processing perspective (e.g. doing something normally or something being an exception requires a meta-cognitive judgement about remembered experiences). Brain imaging research on reasoning could be an important source of evidence in testing this kind of model, namely that such representations are meta-cognitive and processed in a different subsystem to other kinds of conceptual knowledge.

Self-instructions. The model uses generic productions to retrieve declarative goal instances containing self-instruction information to determine what to do next based on what was done before. Some form of instance based memory would be predicted for such a task before proceduralization could take place and is supported by suggested cases of negative transfer evident from solutions in the pilot and main experiment.

The scheme in the model is similar to the declarative operator scheme reported by Anderson (2007), where operator chunks are taken to be decoding's of task instructions that are interpreted by generic dynamic binding productions to control task execution. One key difference with the self-instruction scheme is that it is part of the goal chunk containing the invariant parameters required to carry out its corresponding subtask. This arguably makes the self-instruction scheme a more parsimonious explanation of the kinds of activities supported because both creation, encoding, use and reuse are part of an interrelated system. Although not tested, it would also imply predictions about negative transfer, confusion errors and concurrent changes in the subtask requirements as more self-instructions are adapted and goal chunks are dropped into declarative memory over problems.

A second key difference in the self-instruction scheme is that the information contained in the chunks are subtask specifications rather than specifications of individual operators. This is considered more cognitively plausible because it is proposed that instructions are not likely to be interpreted by people at the level of individual operators, but instead at the level of subtask goals. In addition, unlike the operator scheme, self-instructions can be abstract and potentially vague with respect to their detailed strategic implementation such that strategic details of a self-instruction may be filled in on the go. This reflects the assumption that the cognitive system uses its existing knowledge resources to implement self-generated and given instructions in a flexible and context sensitive way.

Knowledge integration. Another important information processing competence required to explain unpractised problem solving is the arbitrary (or near arbitrary) capacity to integrate information about the problem in the problem representation. This capacity appears to be uncontroversial given the imaginal buffer is viewed as working memory resource, but ACT-R chunks are typed and productions respond to chunk types (however, in ACT-R 6.1, released in December 2014, chunk types are explicitly no longer part of the model, but only exist as part of the modelling notation). The model makes use of dynamic binding productions to add slots to an imaginal chunk, but this involves suspending the type restriction for problem chunks so that productions can respond to the contents of the chunk rather than its type. This is done by predefining the space of all possible required slots in an imaginal chunk even though most will be dynamically bound and only a subset will have bindings at any given time.

Dynamic binding productions are a recent feature of the ACT-R architecture that have been used to model the translation of retrieved declarative operator representations into actions within the system's own cognitive repertoire (Anderson, 2007). The use of dynamic binding in the PPS model is used for a range of purposes, including inference generation strategies, strategies involved in setting up, and responding to meta-cognitive binding (i.e. meta-roles to problem role relations) in argumentation and intention tracking.

8.6.5. Modelling methodology

The reported cognitive modelling research has focused on modelling the strategies inferred from participants' performance data. Creating a model of inferred strategies enforces the need to specify particular knowledge and information requirements and the processes involved in transitions between concurrent cognitive states. Whilst developing a coherent runnable model is an overarching goal, the approach taken has also attempted to develop strategic models that satisfy a number of principled modelling constraints. These include:

The model assumes that subtask action strategies (e.g., searching, counting, reading) have an autonomous control structure and that the systems must integrate these different knowledge domains in unfamiliar/non-optimised problem tasks in order to reach a higher-order goal. This assumption about the nature of knowledge is implied by the requirement for any system to recruit knowledge in novel ways in an unfamiliar task – a type of flexibility which constitutes a fundamental human cognitive competence.

The model explicitly proposes modular specific representational formats, which provide a structured representational system for the cognitive architecture and constrains modelling possibilities. These formats are empirically and analytically motivated and can be seen as compatible with a number tacit assumptions that already exist about the nature of knowledge and processing in the cognitive architecture. Tentative generic strategic knowledge has been implemented for various types of purposes including visual-spatial and meta-cognitive processing. Although preliminary, these representations have been developed to generalise across tasks rather than be ad hoc.

The focus of generic representation and strategies or routines were prevalent in earlier cognitive modelling research (e.g., Newell, 1990; Anderson, 1978), but became de-emphasized; perhaps influenced by the development of the rational analysis theory, which views the structure of knowledge as highly adaptive and context dependent (Anderson, 1990). An abstract assumption made in the development of the model is that the existence and use of context independent knowledge is a functional requirement of unpractised cognition so must also co-exist with highly context specific knowledge acquired through practice. More recent research has returned to considerations of generic forms of knowledge, for example, as discussed, Anderson (2007) reported generic operator representations for implementing a task from instructions. Taatgen (2014) has appealed to the modelling of generic primitive operators that can be used as constraints in a model, like features of the cognitive architecture. The modelling approach developed here is therefore timely and compatible with very recent theoretical aims and developments in the cognitive modelling community.

Modelling evaluation. The evaluation of the model considers the matching of temporal data, behavioural profiles, as well as explanatory properties of the cognitive model. The evaluation choices were motivated by several factors. The unfamiliar nature of the task requires a greater focus in providing a coherent underlying task model and making the coherence of the task model an important part of the evaluation. A principled modelling approach has been attempted proposing a number of separate processing schemes that come together in simulating task performances. The complex, extended and heterogeneous data profiles of the small sample of participants require a focus on detailed cognitive properties of the task performance that are common to participant subgroups such as the conceptual knowledge, strategies and argumentation. The general aims of research are focused on understanding the process of interpretation in the task as well as attempting to develop a model using structured representations and processes.

The emphasis of the evaluation is arguably different to current trends, especially with ACT-R that focus on data fitting of gross temporal behavioural patterns using statistical measures. Temporal data fitting may be viewed as appropriate to modelling short optimized task performance where the strategic possibilities are limited and where (normally) sub-symbolic and other constraints of the cognitive architecture are the focus of the modelling explanation. Not all contemporary cognitive modelling research is concerned with temporal data fitting, especially when the research is aimed at understanding high-order cognition rather than routine action oriented cognition. For example, research with alternative cognitive architectures such as CLARION (Sun, 2002) and Polyscheme (Cassimatis, 2006) focus on qualitative results and broader competence implications. Whilst temporal data fitting can be an important evaluation constraint, a number of researchers have pointed to other factors that are important in evaluation. Cassimatis, Bello, Langley (2008) for example propose additional criteria such as breadth, ability and parsimony of cognitive models, which in their view, can be used in a complementary way or as an alternative to data fitting evaluation.

Chapter 9: Conclusions

The remaining chapter will consider some of the main findings and their broader implications. Three findings will be discussed in the following order: (1) the cognitive roles of the diagram in the task, (2) the multi-component nature of diagrammatic accessibility (3) the nature of PPS competence

9.1. Diagrammatic roles

At an abstract level, the experimental protocols of participants and the cognitive modelling used to investigate explanations of participants' performance suggest the diagrams in the study supported three main roles in the problem solving and reasoning process: (a) the demonstrative routines and visual-spatial competences of the cognitive architecture support sub-problem identification and solution procedure adaptation; (b) prior instrumental knowledge of representing sub-schemes expressed by the diagrams support the framing of a solution interpretation; and (c) the specific representational property of the diagram supports the opportunity to demonstratively elaborate and test out the specific meaning and implications of a comprehended or self-generated instruction interpretation.

These main findings are sufficiently abstract that they are not likely to be specific to the domain of PPS, but have requirement contexts that are likely to apply to other problem solving and reasoning domains, particularly involving mathematical word problems. The requirement contexts of diagrammatic problem solving tasks are summarised as follows.

- Sub-problem identification would be required in any context where a model of the data needs to be inferred from an abstract description or used where a description is incomplete and where the particular structure of the model determines whether default assumptions need to be revised.
- The exploitation of prior knowledge of representational schemes will be required where a particular way of representing a model of the problem situation provides one or more familiar sub-schemes that support framing of the problem data and also of the interrelation between equivalent ways of framing the problem data in a solution interpretation.

- Demonstratively testing the meaning of a comprehended or self-generated instruction will be required whenever the instruction may tend to be interpreted in a vague or abstract way by a user and where the specific nature of the diagram is capable of unambiguously expressing the determinate implicated meaning corresponding to a single or alternative interpretations of the vague or abstract instruction.

9.2. Diagrammatic accessibility and advantages

The empirical and modelling research findings have provided a context for understanding the multi-component accessibility of diagrams. The research has shown that performance on the PPS task exploits properties of diagrams and their accessibility in task execution and directly or indirectly influences the interpretation of a problem and solution. In Chapter 2 several kinds of accessibility were reported based on a review of the empirical literature. We compare the findings of this research with issues considered in the literature.

Perceptual grouping. The role of parallel processing and visual grouping is well documented in the diagrammatic reasoning literature. These considerations appear to be concerned with the issues of high-order meanings expressed by the perceptual form of grouping, such as the shape of a scatter plot, or form of a configuration and the ease of recognition of such information (e.g., Larkin and Simon, 1987; Shimojima, 1999; Koedinger and Anderson, 1990). The research goes further in considering additional functions, including simplifying the referential requirements of an internal problem representation, and thus, supporting the process of keeping track of goal relevant information. The model also implements the hypothesis that, for the modelled problems, perceptual groups are an additional information condition required to generate inferences that depend on co-reference constraints of demonstratively referred sets.

Proof, explanation and evaluation. Theoretical discussion regarding the use of diagrams in proof and explanation was considered. The central claim that is derivable from the theoretical literature is the role of structural constraints of diagrams in being able to observe and understand in an explanatory way why some proposition must hold (e.g., Sloman, 1971; Lindsay, 2002). The hypothesis that participants exploited the determinate and token referential character of the diagrams in elaborating and evaluating problem and self-generated solution instructions is consistent with this line of reasoning. Eye-movements of participants are also consistent with the constraints of online interpretation of problem and solution information. The cognitive model offered an explicit characterisation of this observation by proposing a demonstrative form of problem representation in the imaginal buffer that connects conceptual knowledge in the form of abstract problem roles to the spatial reference of attended objects in the environment, productions that elicit attention to referenced objects when meanings about those objects are being processed and demonstrative strategies and related roles that frame the referential context by referring to objects in deriving abstract information. These acts of deictic reference coupled with the specific nature of the diagram provide a range of information conditions that are critical to solving the tasks that can be directly observed and are likely to be motivated by the need to demonstrate that an interpretation is valid and consistent with the structural constraints on data.

Spatial processing. The role of spatial processing competence in diagrammatic accessibility has been considered by researchers investigating the facilitations of diagrams and PPS. The exploitation of the spatial processing system in reasoning with a diagram is typically attributed to the processing efficiency of the visual-spatial processing system. This rather unspecific characterisation is due to a lack of explicit models or theories of spatial processing of the kind required in diagrammatic reasoning. We have already discussed the hypothesised role of deictic spatial processes and co-reference constraints in reasoning. The research also suggests two additional proposals about this. Firstly, the model includes the proposal that the implemented spatial processing system generates specific representations of the local spatial structure that provides information conditions for the recognition of more abstract conceptual and spatial relations used in higher cognition. Secondly, implementing the task model demonstrates that even with this spatial processing advantage the system need to actively employ demonstrative routines to relate target set and individuals in the process of framing and reasoning about the data.

Although there exist diagrammatic reasoning systems incorporating spatial processing competencies, the empirical bases for the systems as well as the capacity to make any kind of experimental predications are lacking. Although the main assumptions in the implemented model of spatial processing reported here are abstract and tentative, they are analytically motivated and have theoretical and empirical support and could be potentially tested though further experimentation.

Knowledge accessibility. In Chapter 2 we discussed the role of prior knowledge in diagrammatic accessibility. Spatial schemes are ubiquitous in diagrammatic representation and everyday thought more generally. The interpretation of some participants protocols suggested they use prior knowledge of the diagrammatic fraction scheme in interpreting the mapping of the scheme and framing the represented data in a solution interpretation. A cognitive model of the strategic process was developed. Our analysis suggests that the particular phenomena addressed can be differentiated from instance/exemplar based diagram configuration accounts which involved the more concrete knowledge structures derived from highly practised contexts. The research reports findings that a more abstract form of diagrammatic knowledge exists that applies to categories of schemes rather than configuration instances per se. Such categories are hypothesised to provide conditions for recruiting instrumental kinds of knowledge associated with the scheme. The nature of such knowledge is not well understood and has only been addressed in very abstract terms in the model. This is no doubt an important avenue of future research.

9.3. The Nature of PPS competence

The data and interpretation of modelled strategies provided information about the goals, declarative knowledge and strategies employed in the PPS task.

Problem frames. The protocols of participants indicate that (at least sometimes) they derive the proportion from the sets they interpret possible given the problem instruction and data. Verbal reports identifying possibility interpretations were most prevalent on conditional problems where the ruling out of possibilities was required. The framing of the problem in terms of a proportion of possibilities is consistent with mental models theory (Johnson-Laird, et al., 1999) and the partition-edit-count model (Fox and Levav, 2004) and with empirical research suggesting that such framing is present in early childhood (Girotto and Gonzales, 2008). The verbal protocols also indicate a tendency to determine the probability proportion from the frequency of the target sets, which requires the assumption of equiprobability. This is also consistent with mental models theory and the partition-edit-count model and research on probability intuitions (e.g., Shimojo and Ichikawa, 1989).

Errors. The small group of participants made an assortments of errors on different problems. These errors result from a failure to consider or correctly interpret either the problem instruction, problem data or solution procedure. The incorrect solutions of some participants suggest that they may also make errors by incorrectly generalising previous solution procedure instances. The assortment of these different classes of errors are similar to the findings of O'Connell (1999), although our categorisation is different. The different kinds of observed errors that are related to different features of the problem presentation supports a broader approach to understanding the role of representation in PPS in comparison to the majority of research on this issue that is arguably overly pre-occupied with conditional probability problems.

Generic knowledge. Participants did not initially recognise the solution procedure requirement upon reading the instructions but instead tended to work it out with the aid of the diagram; suggesting participants were initially unfamiliar with the solution procedure to particular sub-problems. Some participants' protocols suggested they engaged in self-argumentation in the process of solving problems implying the meta-cognitive requirement to track the interdependent roles of acquired information rather than simply execute a learnt procedure. Participants seemed to adapt solution interpretations and procedures to fit with their existing conception of the problem. To model the task depended on implementing a number of inferences and cognitive strategies that are independent or unrestricted to the domain of probability. In addition, many of the adapted solution interpretations can be viewed as the results of establishing its equivalence to a known requirement. These observations suggest that participants solve the PPS problems using generic types of limited task knowledge and that their performance is consistent with a generic view of reasoning and problem solving competencies. This appears to rule out the need to allude to specialised modules or mechanisms for PPS and extensional reasoning about probability as proposed by ecological rationalist proponents.

Summary

In summary the research has employed a converging approach to investigate the roles of diagrams in supporting solution procedure formulation in a PPS task. The experimental data has suggested abstract roles of the diagram in facilitating a correct solution procedure. It has also led to methodological developments of a novel visualisation for more clearly representing scan paths and a tentative classification system of tabulating verbalisations about the problem and self-argumentation. The cognitive modelling research has led to the development of a number of new proposals for modelling the task domain that could be applied to other domains. These include the meta-cognitive processing scheme, self-instruction scheme, deictic processing scheme and spatial processing schemes. These specific proposals have been aggregated to produce cognitive simulations of the modelled PPS task that has been used to provide an account for observed performance and has generated insights about the nature of this distributed cognitive task.

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Appendix 1. Main experimental instructions

Data recording

In this experiment the eye-tracking equipment will record your eye-movements, speech and video images of your facial area. The reasons for recording verbalisation and eye-movement protocols is to allow us to acquire information on what people think about and look at as they are solving the problems.

Concurrent verbalisation
Whilst you are doing the experiment you are requested to try and verbalise your immediate thoughts as soon as you become aware of them. You do not need to explain what you are thinking about or concern yourself with how your verbalisations may appear. Just try to verbalise whatever you are thinking about as soon as the thoughts come to your mind. Examples may include but are not restricted to

- The things you are looking at or searching for
- The calculations you are making
- The reasoning you are doing
- The plans you are making
- The way you are thinking about things

You will be given three practice trials to help you get familiar with doing this.

Confidentiality
Data concerning the identity of participants will be kept private and confidential. If you have any further questions or concerns please address them to the experimenter before you begin.

Instructions

In the following experiment you will be presented with a series of problems displayed on a computer monitor. Each problem will require you to calculate the probability of the spin of a letter spinner. A letter spinner is an object like that pictured in figure 1 which has several sides each containing a coloured letter of a particular size. When a spinner is spun it can fall on only one of the several sides. The number of sides and letters on the letter spinner will change for every problem you will be given.

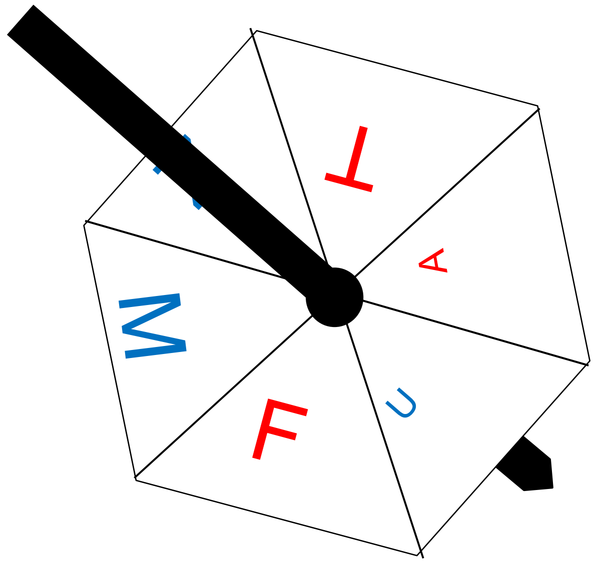


Figure 1. Depiction of a six sided letter spinner

Each problem display will involve a text section and a diagram as shown in figure 2. The text section states information about the spinner event and the probability that you are required to estimate.
The content of the text boxes in each sentence of the text section changes over different problems. All other text will remain the same for each of the different problems.

The first sentence states what is known about the probability of each letter coming up. In the example below it states that all the letters have an equal probability. In other problems it may say that some letters have double the chance of others.

The second sentence says something about the outcome. In the example in figure 2 it simply states the spinner falls on a letter. In other problems it may say the spinner fall on a letter with a particular colour or size.

The third sentence states the probability that you need to calculate. In the figure 2 it simply asks for the probability of getting an F. In other problems the questions may be about getting letters with different colours and/or sizes.

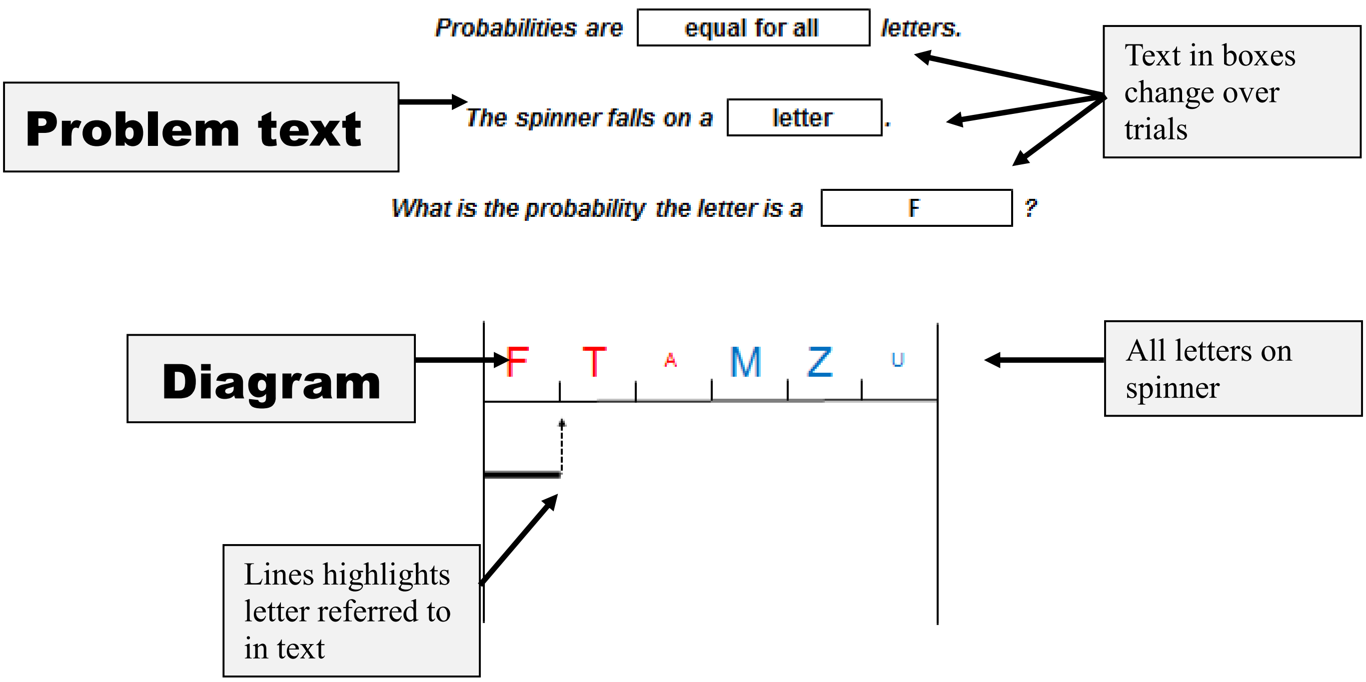


Figure 2. Example of a problem display

Below the text is a diagram that shows the letters on the spinner. Letters stated in the text are highlighted by lines drawn below the letters (see under letter F in Figure 2). Look at figure 2 and familiarize yourself with the display.

You should use the information presented in the text and diagram to help determine the correct answer. When you are confident that you know the correct answer click the mouse and then say the answer.

You will be given three practice problems before you complete the 12 experiment problems. The experiment problems will be more difficult than the practice problems.

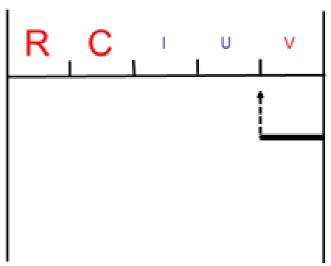
If you are unsure about anything or have any questions please ask the experimenter.

Appendix 2. Problems used in experiment 2

Probabilities are [equal for all] *letters.*

The spinner falls on a [letter] .

What is the probability the letter is a [V] ?

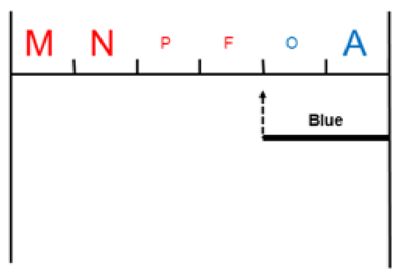


Practice 1

Probabilities are [equal for all] *letters.*

The spinner falls on a [letter] .

What is the probability the letter is [blue] ?

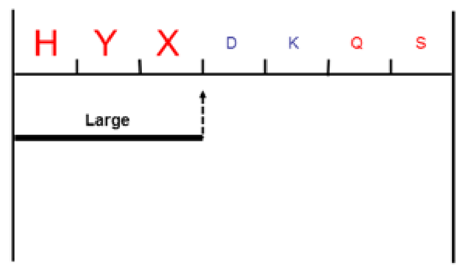


Practice 2

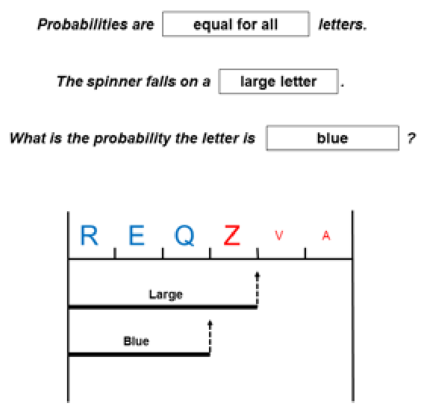
Probabilities are [equal for all] *letters.*

The spinner falls on a [letter] .

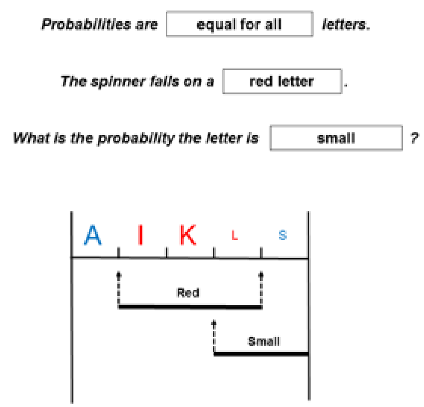
What is the probability the letter is [large] ?



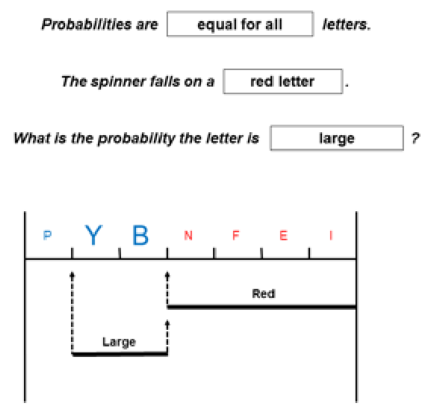
Practice 3



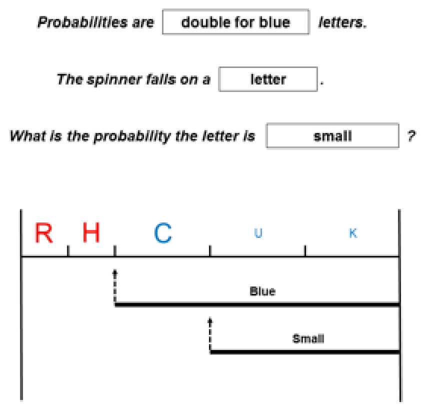
Conditional/Subset



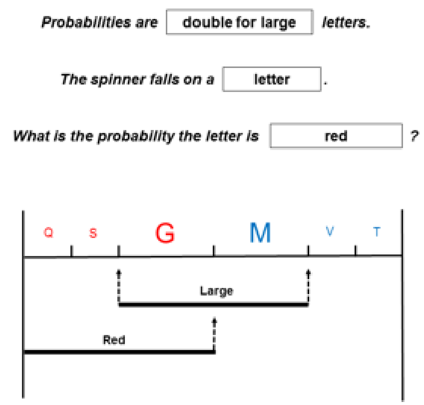
Conditional/Overlap



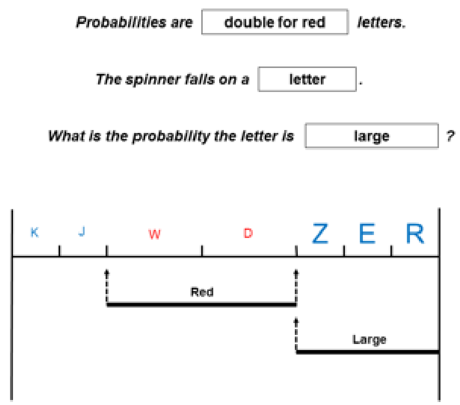
Conditional/Disjoint



Biased/Subset



Biased/Overlap

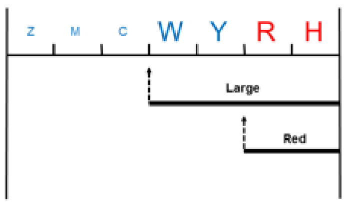


Biased/Disjoint

Probabilities are equal for all *letters.*

The spinner falls on a letter .

What is the probability the letter is large or red or both ?

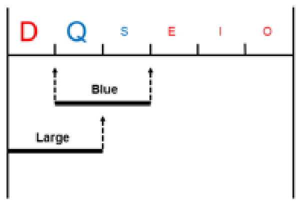


Disjunction/Subset

Probabilities are equal for all *letters.*

The spinner falls on a letter .

What is the probability the letter is blue or large or both ?

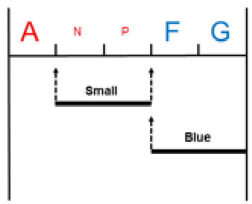


Disjunction/Overlap

Probabilities are equal for all *letters.*

The spinner falls on a letter .

What is the probability the letter is small or blue or both ?

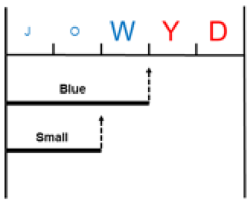


Disjunction/Disjoint

Probabilities are equal for all *letters.*

The spinner falls on a letter .

What is the probability the letter is blue and small ?

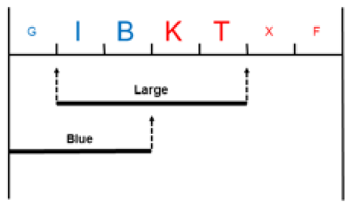


Conjunction/Subset

Probabilities are equal for all *letters.*

The spinner falls on a letter .

What is the probability the letter is large and blue ?

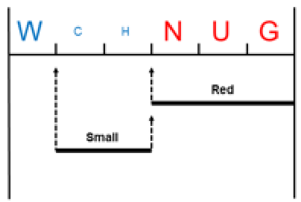


Conjunction/Overlap

Probabilities are equal for all *letters.*

The spinner falls on a letter .

What is the probability the letter is small and red ?



Conjunction/Disjoint

Appendix 3. Sample of model protocol traces

Appendix 3.a. Model protocol trace for the conjunction/subset problem

```
type time xpos ypos fixation object

(eye 150 320 100 185 Probabilities are)
(eye 335 320 100 335 probabilities)
(vocal 470  probabilities)
(eye 670 382 100 1100 are)
(vocal 1370  are)
(eye 1770 500 100 200 equal for all)
(eye 1970 500 100 335 equal)
(vocal 2105  equal)
(eye 2305 542 100 450 for)
(vocal 2555  for)
(eye 2755 570 100 1500 all)
(vocal 2905  all)
(eye 4255 320 150 185 The spinner falls on a)
(eye 4440 320 150 335 the)
(vocal 4575  the)
(eye 4775 362 150 400 spinner)
(vocal 4925  spinner)
(eye 5175 413 150 500 falls)
(vocal 5475  falls)
(eye 5675 455 150 450 on)
(vocal 5925  on)
(eye 6125 476 150 535 a)
(vocal 6260  a)
(eye 6660 500 150 1500 letter)
(vocal 6810  letter)
(eye 8160 200 200 185 What is the probability the letter is)
(eye 8345 220 200 335 What)
(vocal 8480  What)
(eye 8680 248 200 400 is)
(vocal 8880  is)
(eye 9080 275 200 385 the)
(vocal 9215  the)
(eye 9465 330 200 350 probability)
(vocal 9600  probability)
(eye 9815 387 200 735 the)
(vocal 10350  the)
(eye 10550 425 200 400 letter)
(vocal 10700  letter)
(eye 10950 460 200 700 is)
(vocal 11250  is)
(eye 11650 500 200 200 blue & small)
(eye 11850 500 200 335 blue)
(vocal 11985  blue
```

```
type time xpos ypos fixation object

(eye 12185 535 200 400 &)
(vocal 12385  &)
(eye 12585 549 200 685 small)
(vocal 12720  small)
(eye 13270 500 200 1200 blue & small)
(eye 14470 360 380 535 |J|O|W|Y|D|)
(vocal 14855  theres)
(eye 15005 160 360 450 J)
(vocal 15355  one)
(eye 15455 260 360 335 O)
(vocal 15740  two)
(eye 15790 360 360 350 W)
(vocal 16090  three)
(eye 16140 460 360 500 Y)
(vocal 16540  four)
(eye 16640 560 360 900 D)
(vocal 16940  five)
(eye 17540 500 200 185 blue & small)
(eye 17725 500 200 335 blue)
(vocal 17860  blue)
(eye 18060 535 200 400 &)
(vocal 18260  &)
(eye 18460 549 200 685 small)
(vocal 18595  small)
(eye 19145 500 200 950 blue & small)
(eye 20095 260 380 235 |J|O|W|)
(eye 20330 460 380 185 |W|Y|D|)
(eye 20515 260 380 185 |J|O|W|)
(eye 20700 160 360 185 J)
(eye 20885 260 360 185 O)
(eye 21070 360 360 235 W)
(eye 21305 260 380 185 |J|O|W|)
(eye 21490 210 380 1085 |J|O|)
(vocal 21625  both)
(vocal 22025  theres)
(vocal 22225  two)
(eye 22575 360 380 235 |J|O|W|Y|D|)
(eye 22810 210 380 1885 |J|O|)
(vocal 23195  so)
(vocal 23495  two)
(vocal 23845  in)
(vocal 24145  five)

Reported answer:  two in five
```

Appendix 3.b. Model protocol trace for the biased/overlap problem

type time xpos ypos fixation object

(eye 150 320 100 185 Probabilities are)
(eye 335 320 100 335 probabilities)
(vocal 470 probabilities)
(eye 670 382 100 1100 are)
(vocal 1370 are)
(eye 1770 500 100 200 double for large)
(eye 1970 500 100 335 double)
(vocal 2105 double)
(eye 2305 549 100 550 for)
(vocal 2655 for)
(eye 2855 577 100 500 large)
(vocal 3005 large)
(eye 3355 500 100 1050 double for large)
(eye 4405 320 150 185 The spinner falls on a)
(eye 4590 320 150 335 the)
(vocal 4725 the)
(eye 4925 362 150 400 spinner)
(vocal 5075 spinner)
(eye 5325 413 150 500 falls)
(vocal 5625 falls)
(eye 5825 455 150 450 on)
(vocal 6075 on)
(eye 6275 476 150 535 a)
(vocal 6410 a)
(eye 6810 500 150 1500 letter)
(vocal 6960 letter)
(eye 8310 200 200 185 What is the probability the letter is
(eye 8495 220 200 335 What)
(vocal 8630 What)
(eye 8830 248 200 400 is)
(vocal 9030 is)
(eye 9230 275 200 385 the)
(vocal 9365 the)
(eye 9615 330 200 350 probability)
(vocal 9750 probability)
(eye 9965 387 200 735 the)
(vocal 10500 the)
(eye 10700 425 200 400 letter)
(vocal 10850 letter)
(eye 11100 460 200 700 is)
(vocal 11400 is)
(eye 11800 500 200 1800 red)
(vocal 11950 red)
(eye 13600 510 380 535 |Q|S|G|M|V|T|)
(vocal 13985 theres)
(eye 14135 160 360 450 Q)
(vocal 14485 one)
(eye 14585 260 360 335 S)
(vocal 14870 two)
(eye 14920 410 360 350 G)
(vocal 15220 three)
(eye 15270 610 360 500 M)
(vocal 15670 four)
(eye 15770 760 360 350 V)
(vocal 16070 five)
(eye 16120 860 360 950 T)
(vocal 16470 six) (eye 17070 500 100 185 double for large
(eye 17255 500 100 335 double)
(vocal 17390 double)

type time xpos ypos fixation object

(eye 17590 549 100 550 for)
(vocal 17940 for)
(eye 18140 577 100 500 large)
(vocal 18290 large)
(eye 18640 500 100 1200 double for large)
(eye 19840 510 380 235 |G|M|)
(eye 20075 710 380 235 |M|V|T|)
(eye 20310 860 360 185 T)
(eye 20495 610 360 250 M)
(eye 20745 710 380 100 |M|V|T|)
(eye 20845 510 380 250 |G|M|)
(eye 21095 510 380 485 |Q|S|G|M|V|T|)
(vocal 21480 so)
(eye 21580 710 380 800 |M|V|T|)
(eye 22380 510 380 235 |Q|S|G|M|V|T|)
(eye 22615 160 360 335 Q)
(vocal 22900 one)
(eye 22950 260 360 350 S)
(vocal 23250 two)
(eye 23300 410 360 850 G)
(vocal 23600 three)
(vocal 24050 four)
(eye 24150 610 360 750 M)
(vocal 24450 five)
(vocal 24850 six)
(eye 24900 760 360 350 V)
(vocal 25200 seven)
(eye 25250 860 360 1000 T)
(vocal 25650 eight)
(eye 26250 500 200 1535 red)
(vocal 26385 red)
(eye 27785 310 380 235 |Q|S|G|)
(eye 28020 710 380 185 |M|V|T|)
(eye 28205 310 380 185 |Q|S|G|)
(eye 28390 160 360 185 Q)
(eye 28575 260 360 185 S)
(eye 28760 410 360 235 G)
(eye 28995 310 380 185 |Q|S|G|)
(eye 29180 510 380 185 |Q|S|G|M|V|T|)
(eye 29365 310 380 185 |Q|S|G|)
(eye 29550 310 580 185 ---)
(eye 29735 310 380 700 |Q|S|G|)
(vocal 30185 theres)
(eye 30435 310 380 235 |Q|S|G|)
(eye 30670 160 360 335 Q)
(vocal 30955 one)
(eye 31005 260 360 350 S)
(vocal 31305 two)
(eye 31355 410 360 1150 G)
(vocal 31655 three)
(vocal 32105 four)
(eye 32505 510 380 250 |Q|S|G|M|V|T|)
(eye 32755 310 380 1935 |Q|S|G|)
(vocal 33140 so)
(vocal 33440 four)
(vocal 33840 in)
(vocal 34140 eight)

Reported answer: four in eight

Appendix 3.c. Model protocol trace for the conditional/overlap problem

type time xpos ypos fixation object

(eye 150 320 100 185 Probabilities are)
(eye 335 320 100 335 probabilities)
(vocal 470 probabilities)
(eye 670 382 100 1100 are)
(vocal 1370 are)
(eye 1770 500 100 200 equal for all)
(eye 1970 500 100 335 equal)
(vocal 2105 equal)
(eye 2305 542 100 450 for)
(vocal 2555 for)
(eye 2755 570 100 1500 all)
(vocal 2905 all)
(eye 4255 320 150 185 The spinner falls on a)
(eye 4440 320 150 335 the)
(vocal 4575 the)
(eye 4775 362 150 400 spinner)
(vocal 4925 spinner)
(eye 5175 413 150 500 falls)
(vocal 5475 falls)
(eye 5675 455 150 450 on)
(vocal 5925 on)
(eye 6125 476 150 535 a)
(vocal 6260 a)
(eye 6660 500 150 200 red letter)
(eye 6860 500 150 335 red)
(vocal 6995 red)
(eye 7195 528 150 1500 letter)
(vocal 7345 letter)
(eye 8695 200 200 185 What is the probability the letter is)
(eye 8880 220 200 335 What)
(vocal 9015 What)
(eye 9215 248 200 400 is)
(vocal 9415 is)
(eye 9615 275 200 385 the)
(vocal 9750 the)
(eye 10000 330 200 350 probability)
(vocal 10135 probability)
(eye 10350 387 200 735 the)
(vocal 10885 the)
(eye 11085 425 200 400 letter)
(vocal 11235 letter)
(eye 11485 460 200 700 is)
(vocal 11785 is)
(eye 12185 500 200 1800 small)
(vocal 12335 small)
(eye 13985 360 380 535 |A|I|K|L|S|)

type time xpos ypos fixation object

(vocal 14370 theres)
(eye 14520 160 360 450 A)
(vocal 14870 one)
(eye 14970 260 360 335 I)
(vocal 15255 two)
(eye 15305 360 360 350 K)
(vocal 15605 three)
(eye 15655 460 360 500 L)
(vocal 16055 four)
(eye 16155 560 360 900 S)
(vocal 16455 five)
(eye 17055 528 150 1435 letter)
(vocal 17190 letter)
(eye 18490 360 380 235 |I|K|L|)
(eye 18725 560 360 185 S)
(eye 18910 360 380 185 |I|K|L|)
(eye 19095 260 360 185 I)
(eye 19280 360 360 185 K)
(eye 19465 460 360 235 L)
(eye 19700 360 380 185 |I|K|L|)
(eye 19885 360 380 185 |A|I|K|L|S|)
(eye 20070 360 380 1750 |I|K|L|)
(vocal 20120 but)
(vocal 20470 only)
(vocal 21270 three)
(eye 21820 500 200 1535 small)
(vocal 21955 small)
(eye 23355 510 380 235 |L|S|)
(eye 23590 260 380 185 |A|I|K|)
(eye 23775 510 380 185 |L|S|)
(eye 23960 460 360 185 L)
(eye 24145 560 360 235 S)
(eye 24380 510 380 185 |L|S|)
(eye 24565 460 360 1085 L)
(vocal 24700 both)
(vocal 25100 theres)
(vocal 25300 one)
(eye 25650 360 380 235 |I|K|L|)
(eye 25885 460 360 1885 L)
(vocal 26270 so)
(vocal 26570 one)
(vocal 26920 in)
(vocal 27220 three)

Reported answer: one in three

Appendix 3.d. Model protocol trace for the disjunction/subset problem

```
type time xpos ypos fixation object

(eye 150 320 100 185 Probabilities are)
(eye 335 320 100 335 probabilities)
(vocal 470  probabilities)
(eye 670 382 100 1100 are)
(vocal 1370  are)
(eye 1770 500 100 200 equal for all)
(eye 1970 500 100 335 equal)
(vocal 2105  equal)
(eye 2305 542 100 450 for)
(vocal 2555  for)
(eye 2755 570 100 1500 all)
(vocal 2905  all)
(eye 4255 320 150 185 The spinner falls on a)
(eye 4440 320 150 335 the)
(vocal 4575  the)
(eye 4775 362 150 400 spinner)
(vocal 4925  spinner)
(eye 5175 413 150 500 falls)
(vocal 5475  falls)
(eye 5675 455 150 450 on)
(vocal 5925  on)
(eye 6125 476 150 535 a)
(vocal 6260  a)
(eye 6660 500 150 1500 letter)
(vocal 6810  letter)
(eye 8160 200 200 185 What is the probability the letter is)
(eye 8345 220 200 335 What)
(vocal 8480  What)
(eye 8680 248 200 400 is)
(vocal 8880  is)
(eye 9080 275 200 385 the)
(vocal 9215  the)
(eye 9465 330 200 350 probability)
(vocal 9600  probability)
(eye 9815 387 200 735 the)
(vocal 10350  the)
(eye 10550 425 200 400 letter)
(vocal 10700  letter)
(eye 10950 460 200 700 is)
(vocal 11250  is)
(eye 11650 500 200 200 large or red or both)
(eye 11850 500 200 335 large)
(vocal 11985  large)
(eye 12185 542 200 450 or)
(vocal 12435  or)
(eye 12635 563 200 385 red)
(vocal 12770  red)
(eye 13020 591 200 350 or)
(vocal 13155  or)
(eye 13370 612 200 685 both)
(vocal 13505  both)
(eye 14055 500 200 1200 large or red or both)
```

```
type time xpos ypos fixation object

(eye 15255 460 380 535 |Z|M|C|W|Y|R|H|)
(vocal 15640  theres)
(eye 15790 160 360 450 Z)
(vocal 16140  one)
(eye 16240 260 360 335 M)
(vocal 16525  two)
(eye 16575 360 360 350 C)
(vocal 16875  three)
(eye 16925 460 360 500 W)
(vocal 17325  four)
(eye 17425 560 360 350 Y)
(vocal 17725  five)
(eye 17775 660 360 400 R)
(vocal 18125  six)
(eye 18175 760 360 900 H)
(vocal 18475  seven)
(eye 19075 500 200 185 large or red or both)
(eye 19260 500 200 335 large)
(vocal 19395  large)
(eye 19595 542 200 450 or)
(vocal 19845  or)
(eye 20045 563 200 385 red)
(vocal 20180  red)
(eye 20430 591 200 350 or)
(vocal 20565  or)
(eye 20780 612 200 685 both)
(vocal 20915  both)
(eye 21465 500 200 1100 large or red or both)
(eye 22565 610 380 235 |W|Y|R|H|)
(eye 22800 360 380 185 |Z|M|C|W|Y|)
(eye 22985 610 380 185 |W|Y|R|H|)
(eye 23170 460 360 185 W)
(eye 23355 560 360 185 Y)
(eye 23540 660 360 185 R)
(eye 23725 760 360 235 H)
(eye 23960 610 380 250 |W|Y|R|H|)
(eye 24210 710 380 235 |R|H|)
(eye 24445 360 380 185 |Z|M|C|W|Y|)
(eye 24630 710 380 185 |R|H|)
(eye 24815 660 360 185 R)
(eye 25000 760 360 285 H)
(eye 25285 710 380 185 |R|H|)
(eye 25470 610 380 235 |W|Y|R|H|)
(eye 25705 710 380 250 |R|H|)
(eye 25955 710 380 485 |R|H|)
(vocal 26090  both)
(eye 26440 610 380 385 |W|Y|R|H|)
(vocal 26725  four)
(eye 26825 710 380 1050 |R|H|)
(vocal 27125  two)
(vocal 27475  is)
(vocal 27775  six)
```

Appendix 4.d. Model protocol trace for the disjunction/subset problem continued

type time xpos ypos fixation object
(eye 27875 710 380 1300 |R|H|)
(vocal 28175 two)
(vocal 28525 is)
(vocal 28825 eight)
(eye 29175 610 380 100 |W|Y|R|H|)
(eye 29275 710 380 150 |R|H|)
(eye 29425 610 370 235 HRYW)
(eye 29660 460 380 235 |Z|M|C|W|Y|R|H|)
(eye 29895 610 370 1435 HRYW)
(vocal 30480 no)
(vocal 30730 so)
(eye 31330 500 200 185 large or red or both)
(eye 31515 500 200 335 large)
(vocal 31800 large)
(eye 31850 610 380 185 |W|Y|R|H|)
(eye 32035 500 200 365 large)
(eye 32400 542 200 335 or)
(eye 32735 563 200 335 red)
(vocal 33020 red)
(eye 33070 710 380 185 |R|H|)
(eye 33255 563 200 365 red)
(eye 33620 591 200 335 or)
(eye 33955 612 200 335 both)
(vocal 34240 both)
(eye 34290 710 380 185 |R|H|)
(eye 34475 612 200 815 both)
(vocal 34890 no)
(vocal 35190 so)
(eye 35290 610 370 750 HRYW)
(vocal 35690 four)
(eye 36040 460 380 285 |Z|M|C|W|Y|R|H|)
(eye 36325 610 370 1935 HRYW)
(vocal 36710 so)
(vocal 37010 four)
(vocal 37410 in)
(vocal 37710 seven)

Reported answer: four in seven