Harmonic Analysis in Non-Euclidean Geometry: Trace Formulae and Integral Representations

by

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Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

Signature:

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Abstract

This thesis is concerned with the spectral theory of the Laplacian on non-Euclidean spaces and its intimate links with harmonic analysis and the theory of special functions. More specifically, it studies the spectral theory of the Laplacian on the quotients $\mathscr{M} = \Gamma\backslash G/K$ and $\mathscr{X} = G/K$, where G is a connected semisimple Lie group, K is a maximal compact subgroup of G and Γ is a discrete subgroup of G. It builds upon the special cases of compact and noncompact hyperbolic surfaces $\mathscr{M} = \Gamma\backslash\mathbf{H}$ where $\Gamma \subset PSL(2,\mathbf{R})$ is a Fuchsian group and $\widetilde{G} := PSL(2,\mathbf{R})$ is the projective special linear group of all 2×2 real matrices with determinant 1 (the latter is the group of orientation-preserving isometries of the Poincaré upper half-plane $\mathbf{H} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ and by a Fuchsian group we mean a discrete cofinite subgroup of $PSL(2,\mathbf{R})$ that acts properly discontinuously on $\mathbf{H}$) and it generalises these techniques to explicit constructions of various spectral functions on n-dimensional symmetric spaces $\mathscr{X}$, most notably, when $\mathscr{X}$ = the unit sphere $\mathbf{S}^n$; the Euclidean space $\mathbf{R}^n$; the real projective space $\mathbf{RP}^n$; the complex projective space $\mathbf{CP}^n$; the hyperbolic upper half-space $\mathbf{H}^n$; and the hyperbolic unit ball $\mathbf{D}^n$. The main tools in contemplating this throughout are the use of various spectral estimates and identities, integral representations and the indispensable trace formulae.

The thesis consists of six chapters including an introduction whose contents are summarised and briefly outlined in the following paragraphs. The reader is referred to the main text for further coverage and details.

We describe the action of the group $\widetilde{G}$ on $\mathbf{H}$ and the construction of a hyperbolic Riemann surface $\mathscr{M}$ as the quotient $\Gamma\backslash\mathbf{H}$. Also discussed are Eisenstein series as automorphic functions on a hyperbolic surface $\mathscr{M}$; this is important because the continuous spectrum of a noncompact hyperbolic surface $\mathscr{M}$ is well understood in terms of Eisenstein series.

Explicit trace formulae for compact and noncompact hyperbolic surfaces $\mathscr{M}$ are derived by decomposing Γ into conjugacy classes of $\gamma \in \Gamma$. An important application of the trace formula is the computation of the trace of the heat operator on $\mathscr{M}$. Having successfully established the trace formula for a noncompact $\mathscr{M}$; in particular for the modular surface $\mathscr{M} = SL(2,\mathbf{Z})\backslash\mathbf{H}$, we apply the trace formula to determine the determinant of the Laplacian $\widetilde{\Delta} - s(1-s)$ ($s \in \mathbf{C}$) on $SL(2,\mathbf{Z})\backslash\mathbf{H}$. Our results lead to new determinant expressions for the det $\widetilde{\Delta} - s(1-s)$ for some special values of $s \in \mathbf{R}$, $s > 0$, in line with the works of P. Sarnak, A. Chang, B. Osgood, K. Okikiolu. The general version of the Selberg spectral expansion formula for automorphic functions $f \in L^2(\mathscr{M})$, namely the Parseval inner product formula is computed for nonholomorphic Eisenstein series for the modular group $SL(2,\mathbf{Z})$, and this inner product formula is new in the literature.

We compute explicitly the Poisson kernels on $\mathbf{S}^n$, $\mathbf{B}^n$, $\mathbf{H}^n$ and $\mathbf{D}^n$, in terms of special functions; we give explicit integral representations for the Euclidean Poisson kernel. These apart from being interesting in their own right lead to various identities that are novel in the context of special functions. We present fractional and integral representations of the heat kernels on compact symmetric spaces $\mathscr{X} = \mathbf{S}^n$ and $\mathscr{X} = \mathbf{CP}^n$; the heat kernels on $\mathbf{S}^n$ which are obtained in terms of series involving the Gegenbauer polynomial using the device of the Gegenbauer transform, are transformed by applying the Riemann-Liouville fractional derivative formula. New integral representations for the heat kernels on complex projective spaces $\mathbf{CP}^n$ are obtained. We express the traces of the heat kernels on these compact symmetric spaces in terms of the Euclidean Poisson kernel.

Indeed let $K_{\mathscr{X}}$ be the heat kernel associated to the Laplacian $\Delta_{\mathscr{X}}$ on a compact symmetric space $\mathscr{X}$. We show that the Minakshisundaram-Pleijel asymptotic expansion

$$\begin{aligned}\operatorname{tr} e^{-t\Delta_{\mathscr{X}}} &= \int_{\mathscr{X}} K_{\mathscr{X}}(t,x,x)\,\mathrm{dVol}(x)\\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}}\left(a_0^n + a_1^n t + a_2^n t^2 + \cdots + a_k^n t^k + O\left(t^{k+1}\right)\right)\end{aligned}$$

holds for the particular case of the sphere $\mathscr{X} = \mathbf{S}^n$. Using Jacobi's theta functions we give a precise and relatively simple description of the Minakshisundaram-Pleijel coefficients a_k^n for the Euclidean spheres $\mathbf{S}^n$. Apart from giving a description of the heat trace coefficients in this case it is nice to see that the expansion of the heat operator on symmetric spaces can be expressed purely in terms of Jacobi's theta functions and their higher order derivatives.

Finally, we discuss integral representations of spectral functions on a noncompact symmetric space, namely we consider the generalised spherical functions, heat kernel, Green function and Generalised Mehler-Fock integral formula for the real hyperbolic space $\mathscr{X} = \mathbf{H}^n$. Using the Green function, the Poisson kernel and spectral resolutions of the Laplacian in $\mathbf{H}^n$ we derive the generalised Mehler-Fock inversion formula, and in particular we extend the formula to the heat kernel in $\mathbf{H}^n$ by appropriately choosing a spectral function. New integral representations for general eigenfunctions of the Laplacian in $\mathbf{H}^n$ are obtained and these integral representations turn out to be integral transforms of harmonic functions in the Euclidean unit ball.

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Richard Olu Awonusika
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Contents

List of Symbols

$\mathbf{B}^n$	n-dimensional Euclidean unit ball
C	$\approx 0.5772156649015...$ the Euler-Mascheroni constant
$\mathfrak{C}$	vector space of automorphic forms
$\mathbf{C}$	set of complex numbers
$C_0^\infty(\Omega)$	space of infinitely differentiable functions with compact support in Ω
$d(x, x')$	distance between two points x, x' on a Riemannian manifold
$d\nu_n$	volume measure on $\mathbf{S}^n$
$d\mu_{\mathscr{X}}$	volume measure on $\mathscr{X} = \mathbf{H}^n, \mathbf{D}^n$
$\mathbf{D}$	Poincaré unit disc
$\mathbf{D}^n$	n-dimensional Poincaré unit ball
$\mathbb{D}_n$	Laplacian on the Euclidean space $\mathbf{R}^n$
$\mathcal{D}_\Gamma$	fundamental domain of Γ
$\Delta_{\mathbf{D}^n}$	Laplacian on $\mathbf{D}^n$
$\widetilde{\Delta}$	Laplacian on $\mathbf{H}$, $\mathscr{M}$
$\Delta_{\mathbf{H}^n}$	Laplacian on $\mathbf{H}^n$, $n \geq 3$
$\Delta_{\mathbf{S}^n}$	Laplacian on the unit sphere $\mathbf{S}^n$
$\Delta_{(M,g)} = \Delta$	Laplace-Beltrami operator on a Riemannian manifold (M, g) associated with the Riemannian metric g
$E(z, s)$	nonholomorphic Eisenstein series
$\{\gamma\}_p$	primitive conjugacy class of $\gamma \in \Gamma$
$\widetilde{\mathcal{G}}_s$	resolvent kernel (Green's function) on $\mathbf{H}$
$\mathcal{G}_s$	Green's function on $\mathscr{M}$
$G_{\mathbf{H}^n}$	Green's function on $\mathbf{H}^n$
Γ	Fuchsian group of the first kind
h_γ	axis of γ
$\mathbf{H}$	hyperbolic upper half-plane
$\mathbf{H}^n$	n-dimensional hyperbolic upper half-space
$\mathfrak{H}_k^{n-1}, \mathfrak{h}_k^{n-1}$	space of homogeneous harmonic polynomials/ spherical harmonics of degree k in n dimensions
κ	Gaussian/sectional curvature
$\widetilde{K}$	heat kernel on $\mathbf{H}$
$K_{\mathbf{S}^n}$	heat kernel on the unit sphere $\mathbf{S}^n$
$K_{\mathbf{H}^n}$	heat kernel on $\mathbf{H}^n$
$\ell(\gamma)$	length of γ
$\mathscr{L}_{\mathscr{M}}$	hyperbolic length spectrum of $\mathscr{M}$
$L^2(\Omega)$	space of square integrable functions on Ω
M_k^{n-1}	dimension of $\mathfrak{H}_k^{n-1}$/multiplicity of eigenvalues on $\mathbf{S}^{n-1}$
$\mathscr{M} = \Gamma \backslash \mathbf{H}$	hyperbolic surface
$\mathscr{N}_{\mathscr{M}}$	hyperbolic length spectrum counting function
$\mathbf{N}$	set of natural numbers
$\mathbf{N}_0$	$\mathbf{N} \cup \{0\}$
$\mathscr{N}(\lambda)$	hyperbolic eigenvalue λ counting function
$\mathscr{N}_\rho$	hyperbolic lattice point counting function, $\rho = d(z, z')$, $z, z' \in \mathbf{H}$
ν_n	volume of $\mathbf{S}^n$
$P^{\mathscr{X}}, P_{\mathscr{X}}$	Poisson kernel on the space $\mathscr{X}$

π_k	orthogonal projection from $L^2\left(\mathbf{S}^{n-1}\right)$ onto $\mathfrak{H}_k^{n-1}$
Π_Γ	limit set of Γ
$\mathfrak{P}_k^{n-1}$	space of homogeneous polynomials of degree k in n dimensions
$\Phi_k^{\mathscr{X}}$	spherical functions on the space $\mathscr{X}$ of degree k
$\mathbf{R}$	set of real numbers
$\mathcal{R}_s$	resolvent operator
$\mathbf{R}^n$	n-dimensional Euclidean space
$\mathbf{S}^n$	n-dimensional unit sphere
$\mathbf{T}^n$	n-dimensional torus
$\operatorname{tr} A$	trace of operator A
$\Theta(t)$	trace of the wave operator on $\mathscr{M}$
$\Theta(t), \theta(t)$	trace of heat operator on $\mathscr{M}$
$W(t, \cdot, \cdot)$	wave kernel on $\mathscr{M}$
$W(t)$	wave operator on $\mathscr{M}$
$W_{\mathbf{H}^n}$	wave kernel on $\mathbf{H}^n$
$\widetilde{W}(t, \cdot, \cdot)$	wave kernel on $\mathbf{H}$
Y_k	spherical harmonic of degree k
$\mathcal{Z}_k$	zonal harmonic of degree k
$Z(s)$	Selberg zeta function of $s \in \mathbf{C}$
$Z_{\mathcal{I}}(s), Z_{\mathcal{P}}(s)$	zeta functions of $s \in \mathbf{C}$ associated with identity and parabolic elements of Γ
$\mathbf{Z}$	set of integers

Dedication

To The Glory Of The Almighty God & My Family

Chapter 1

Introduction and Overview

1.1 Historical Background of Non-Euclidean Geometries

Non-Euclidean geometry was first discovered by N. Lobachevski, C. F. Gauss and J. Bolyai in the 1820s. Later the mathematical works on non-Euclidean geometry was first published by N. Lobachevski in 1829 and J. Bolyai in 1832. Perhaps because of the controversy surrounding the idea of non-Euclidean geometry at that time, the results of Gauss were not published; this is the reason some authors call the hyperbolic upper half-space the Lobatchevsky upper half-space. Models for hyperbolic geometry, namely the upper half-space model, the unit disc model and the projective disc model (i.e., the Kleinian model) of n-dimensional hyperbolic geometry were first introduced by Liouville, then by E. Beltrami in 1868, and then by Klein in 1870; see e.g., Milnor et al. [110] for the exposition of the history of hyperbolic geometry, and Ratcliffe [139] for a highly detailed introduction of the subject. The Liouville-Beltrami upper half-plane model was later rediscovered by Poincaré in 1882, the reason some author refers to hyperbolic upper half-plane the Poincaré upper half-plane (as we call it in this thesis). The upper half-space also arises quite naturally in connection with the theory of binary Hermitian forms, see Elstrodt et al. [57, Ch. 9] for an exposition of this topic.

The mystery of why Euclid's parallel postulate could not be proved remained unsolved for over two thousand years, until the discovery of non-Euclidean geometry and its Euclidean models revealed the impossibility of any such proof (Greenberg [67]); see Greenberg [67] for a rigorous treatment of the foundations of Euclidean geometry and an introduction to hyperbolic geometry (with emphasis on its Euclidean models). According to Greenberg [67], Albert Eistein mentioned that he would not have developed his theory of relativity without the discovery of hyperbolic geometry. The discovered models of hyperbolic geometry were later used in the investigations of discontinuous groups; see e.g. Poincaré [129], Riemann [140, pp. 272-287], Selberg [147], Maass [105], Terras [167], Beardon [22] and the extensive list of references therein. With the investigation of Lie groups and symmetric spaces by Lie, Cartan and others, further progress was made from the 1880s to the 1930s.

The name "hyperbolic" which is due to Klein and comes from the Greek word *hyperballein* (meaning to throw beyond) can be explained in two ways. The first explanation for the name "hyperbolic" is that given two geodesic rays which originate from the ends of a geodesic segment perpendicular to both of them, then the non-Euclidean distance between these two rays will increase. The second justification for the name "hyperbolic" emanates from the fact that the upper half-plane $\mathbf{H}$ has a constant Gaussian curvature -1 and thus $\mathbf{H}$ looks like a hyperboloid of one sheet or a hyperbolic paraboloid (e.g. $z = x^2 - y^2$ at the origin). See Terras [167, Ch. 3] for this description and the references therein.

1.2 Riemannian Symmetric Spaces

For analysis on Riemannian manifolds (see Appendix A for a discussion on Riemannian manifolds) to be explicit we need to consider those Riemannian manifolds with symmetries; these symmetric properties allow computations of spectral functions and spectral invariants associated to the Laplace-Beltrami operator to be explicit. For instance, given a Riemannian manifold with nontrivial topology the spectrum and eigenfunctions of the Laplacian are difficult to calculate, we can improve this situation by putting symmetries on the manifold.

We proceed our discussion with the following definitions: (i) a *Lie group* is a group that is also a smooth manifold, in which the group operations of multiplication and inversion are smooth maps, (ii) a connected space is a topological space that cannot be represented as the union of two or more disjoint nonempty open subsets, (iii) the Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space to G at the identity, provided with an operation called *the Lie bracket*, (iv) a Lie group is called semisimple if its Lie algebra is semisimple; a Lie algebra is called semisimple if its only commutative ideal is $\{0\}$. For example, the special linear group $SL(n)$ and special orthogonal group $\mathbf{SO}(n)$ (over $\mathbf{R}$ or $\mathbf{C}$) are semisimple, (iv) the center of a group G consists of all those elements x in G such that $xy = yx$ for all y in G; this is a normal subgroup of G.

Let G be a connected semisimple Lie group with finite centre and K a maximal compact subgroup of G. We call $\mathscr{X}$ a homogeneous space of a Lie group G, a group of transformation, or a group of motion, if every point $x \in \mathscr{X}$ can be carried by motion into every other point. We also say then that G *acts transitively on* $\mathscr{X}$. The quotient $\mathscr{X} = G/K$ is then called a *homogeneous space.* Let x_0 be a fixed point in $\mathscr{X}$. The set of transformations that carry x_0 into x form a closed subgroup K of G. This subgroup is called the *stability group of* x_0. What is interesting in this situation is that because the space $\mathscr{X}$ is homogeneous, the machinery for differential calculus becomes readily available.

A symmetric space is a Riemannian manifold M such that for any $p \in M$, there is a symmetry (geodesic-reversing isometry) $\tau_p : M \to M$, preserving the Riemannian metric. That is for any $x \in \mathscr{X} = M$, there is some $\tau_x \in G =$ isometry group of $\mathscr{X}$ with the property that $\tau_x(x) = x$; we call the isometry τ_x the symmery at x. So if $\mathscr{X}$ is any homogeneous space, (i.e., the group of isometry of $\mathscr{X}$ acting transitively on $\mathscr{X}$), then $\mathscr{X}$ is symmetric if and only if there exists a symmetry τ_x (i.e., an isometry satisfying $\tau_x(x) = x$ for some $x \in \mathscr{X}$). We can also say that a homogeneous space is *symmetric* if the covariant derivative of the Riemann tensor vanishes, i.e., the curvature tensor is parallel.

There are 4 main types of symmetric spaces – the Euclidean spaces, compact Lie groups, quotients of compact Lie groups, and quotients of noncompact semisimple Lie groups (see Terras [167] for elaboration on this classification and Helgason [77, 78, 79, 80, 81, 82, 83, 84] for a thorough discussion of harmonic analysis on these symmetric spaces). The differential operators on these symmetric spaces are G-invariant.

Examples of compact symmetric spaces include the sphere $\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbf{RP}^n = \mathbf{SO}(n+1)/\{\pm I\} = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbf{CP}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$ (of real dimension $2n$), the quaternionic projective space $\mathbf{QP}^n = \mathbf{Sp}(n+1)/\mathbf{Sp}(n)\times\mathbf{Sp}(1)$ (of real dimension $4n$) and the Cayley projective plane $\mathbf{CayP}^2 = \mathbf{F}_4/\mathbf{Spin}(9)$ (of real dimension 16); all these spaces are simply connected except the real projective space $\mathbf{RP}^n$ which is doubly connected. In the case of compact symmetric spaces $\mathscr{X} = G/K$, G (real or complex) and K (real or complex) are connected compact Lie groups. Later in Chapter 5 of this thesis we shall discuss spectral functions on the sphere $\mathbf{S}^n$, the real projective space $\mathbf{RP}^n$ and the complex projective space $\mathbf{CP}^n$.

Examples of noncompact symmetric spaces include the real hyperbolic space $\mathbf{H}^n = \mathbf{SO}_0(n,1)/\mathbf{SO}(n)$, the complex hyperbolic space $\mathbf{CH}^n = \mathbf{SU}(n,1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$, the quaternionic hyperbolic space $\mathbf{QH}^n = \mathbf{Sp}(n,1)/\mathbf{Sp}(n)\times\mathbf{Sp}(1)$ and the Cayley hyperbolic plane $\mathbf{CayH}^2 = \mathbf{F}_4^*/\mathbf{Spin}(9)$; in this case, G is a connected (real or complex) semisimple Lie group of noncompact type, while K (real or complex) is compact. Chapter 6 of this thesis is devoted to integral representations of spectral functions on the real hyperbolic space $\mathbf{H}^n$.

Some of the Lie groups of Real and Complex Matrices are described below.

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

$$
\begin{aligned}
GL(n,\mathbf{K}) -&\ \text{the group of nonsingular } n\times n \text{ matrices with entries in } \mathbf{K};\\
SL(n,\mathbf{R}) -&\ \text{the group of real } n\times n \text{ matrices of determinant } 1;\\
SL(n,\mathbf{C}) -&\ \text{the group of complex } n\times n \text{ matrices of determinant } 1;\\
\mathbf{U}(n,1;\mathbf{K}) -&\ \text{the group of matrices in } GL(n+1,\mathbf{K}) \text{ which leave invariant}\\
&\ \text{the Hermitian form } a_1\bar{b}_1+\cdots+a_n\bar{b}_n - a_{n+1}\bar{b}_{n+1},\quad a_j,b_j\in\mathbf{K};\\
\mathbf{O}(n,1) =&\ \mathbf{U}(n,1;\mathbf{R}),\ \mathbf{U}(n,1)=\mathbf{U}(n,1;\mathbf{C});\\
\mathbf{SO}(n,1) =&\ \mathbf{O}(n,1)\cap SL(n+1,\mathbf{R}),\ \mathbf{SU}(n,1)=\mathbf{U}(n,1)\cap SL(n+1,\mathbf{C});\\
\mathbf{SO}_0(n,1) -&\ \text{the identity component of } \mathbf{SO}(n,1);\\
\mathbf{U}(n,\mathbf{K}) -&\ \text{the group of matrices in } GL(n,\mathbf{K}) \text{ which stabilise the form } (a,b)_{\mathbf{K}};\\
\mathbf{O}(n) =&\ \mathbf{U}(n,\mathbf{R}) - \text{ the orthogonal group};\\
\mathbf{U}(n) =&\ \mathbf{U}(n,\mathbf{C}) - \text{ the unitary group};\\
\mathbf{SO}(n) =&\ \mathbf{O}(n)\cap SL(n,\mathbf{R}) - \text{ the rotation group};\\
\mathbf{SU}(n) =&\ \mathbf{U}(n)\cap SL(n,\mathbf{C});\\
\mathbf{U}(n)\times\mathbf{U}(1) =&\ \left\{\begin{pmatrix} A & 0\\ 0 & e^{i\theta}\end{pmatrix} : A\in\mathbf{U}(n),\ \theta\in\mathbf{R}\right\};\\
\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1)) =&\ (\mathbf{U}(n)\times\mathbf{U}(1))\cap SL(n+1,\mathbf{C}).
\end{aligned}
$$

References for these Lie groups include Baker [17], Curtis [46], Arvanitogeōrgos [4], Fegan [59], Helgason [82, 83, 84], Flensted-Jensen [60], Terras [168], Volchkov and Volchkov [174], Faraut [58].

Two closely related models of hyperbolic spaces, namely the upper half-space which we also denote by $\mathbf{H}^n$:

$$\mathbf{H}^n = \left\{w=(x,y) : x\in\mathbf{R}^{n-1}, y>0\right\},\quad n\geq 2, \tag{1.1}$$

and the Poincaré unit ball

$$\mathbf{D}^n = \{x \in \mathbf{R}^n \,:\, |x| < 1\}\,, \quad n \geq 2,$$

will be studied. The two-dimensional hyperbolic space (i.e., the case $n = 2$) is called the *hyperbolic plane* of which there are two common models - the first is the *Poincaré upper half-plane* denoted $\mathbf{H}^2 = \mathbf{H}$ and defined by

$$\mathbf{H} = \{z = x + iy \in \mathbf{C} \,:\, \operatorname{Im} z = y > 0\}\,;$$

and the second is the *Poincaré unit disc* denoted $\mathbf{D}^2 = \mathbf{D}$ and defined by

$$\mathbf{D} = \{z \in \mathbf{C} \,:\, |z| < 1\}\,.$$

For details of these models of the hyperbolic space, see Sections 1.5 and 1.6. Other models of the hyperbolic space include the Klein, hyperboloid, and hemisphere models (see Thurston [169], Strichartz [157]).

In summary we shall discuss the spectral theory of the Laplacians on the following Riemannian manifolds:

- the hyperbolic Riemann surfaces $\Gamma\backslash\mathbf{H}$ (here Γ is the Fuchsian group of the first kind);
- the unit spheres $\mathbf{S}^n$ and the unit balls $\mathbf{B}^n$ in $\mathbf{R}^n$;
- the real projective spaces $\mathbf{RP}^n$;
- the complex projective spaces $\mathbf{CP}^n$;
- the Poincaré unit ball $\mathbf{D}^n$; and
- the hyperbolic upper half-space $\mathbf{H}^n$.

Further discussions of spectral theory of a general rank one quotient $\Gamma\backslash G/K$ and the higher rank case $\mathbf{SL}(n,\mathbf{Z})\backslash\mathbf{SL}(n,\mathbf{R})/\mathbf{SO}(n)$ are contained in Awonusika and Taheri [8].

1.3 The Heat Kernels on Compact Symmetric Spaces

Let $\mathscr{X} = G/K$ be a compact symmetric space. Let $\Delta_{\mathscr{X}}$ be the Laplace-Beltrami operator (Laplacian) on $\mathscr{X}$, $C(k)$ the k-th eigenvalue of $\Delta_{\mathscr{X}}$ and d_k the multiplicity of the eigenvalue $C(k)$. Given two points $x, y \in \mathscr{X}$, one of them can be translated to the origin by an action of the group G, say $x \in \mathscr{X}$; the origin is held fixed while the other endpoint $y \in \mathscr{X}$ is rotated about it on an orbit of K. The geodesic distance from any point $y \in \mathscr{X}$ to the point x is denoted by $\theta = d(x, y)$; a subspace orthogonal to the orbits of K is called a *maximal torus*. That is, a maximal torus is a maximal, totally geodesic, flat submanifold passing through the origin. The dimension of the maximal torus is called the *rank* of a compact symmetric space $\mathscr{X}$.

Below we give a list of rank one compact symmetric spaces that are of interest to us, with their corresponding spectral and geometric data.

- $\Delta_{\mathscr{X}} = -\frac{\partial^2}{\partial\theta^2} - \left(m_{2\alpha}\cot\theta + \frac{1}{2}m_\alpha\cot(\theta/2)\right)\frac{\partial}{\partial\theta}$,

- $C(k) = k(k+2\rho)$, $\rho = \frac{1}{2}\left(m_{2\alpha} + m_\alpha\right)$,
- $d_k = \frac{2(k+\rho)\Gamma(k+2\rho)\Gamma\left(\frac{m_{2\beta}+1}{2}\right)\Gamma\left(k+\frac{N}{2}\right)}{k!\Gamma(2\rho+1)\Gamma\left(\frac{N}{2}\right)\Gamma\left(k+\frac{m_{2\alpha}+1}{2}\right)}$,
- $\mathrm{Vol}(\mathscr{X}) = 2^N \pi^{\frac{N}{2}} \frac{\Gamma\left(\frac{m_{2\alpha}+1}{2}\right)}{\Gamma\left(\frac{N+m_{2\alpha}+1}{2}\right)}$,
- $\mathbf{S}^n : m_{2\alpha} = n-1,\ m_\alpha = 0,\ N = n$,
- $\mathbf{RP}^n : m_{2\alpha} = n-1,\ m_\alpha = 0,\ N = n,\ C(k) = 2k(2k+2\rho)$,
- $\mathbf{CP}^n : m_{2\alpha} = 1,\ m_\alpha = 2(n-1),\ N = 2n$.

If a group G acts on a symmetric space $\mathscr{X}$ a number of invariance properties of the heat kernel can be derived which give further information and suggest new geometrical approaches to the solution of the associated heat equation. The orthonormalised eigenfunctions (called spherical functions) of the Laplacian can be identified with suitable matrix elements of the finite-dimensional irreducible unitary representations π_k of the symmetric group G. We use spherical irreducible unitary representation of G to transform the heat kernel (A.3) on an arbitrary compact Riemannian manifold M to the heat kernel on a compact symmetric space $\mathscr{X} = G/K$. Henceforth we shall write $C(k) = \lambda_k$, bearing in mind that for the real projective space $\mathbf{RP}^n$, $C(k) = \lambda_{2k}$, $k \geq 0$. In order to move from the heat kernel on an arbitrary compact Riemannian manifold to the heat kernel on a compact symmetric space it is necessary to discuss the finite-dimensional irreducible representation of a topological group G.

A representation of a topological group G is key in understanding the group G and its invariants. We start with the definition of a representation of a topological group. Let $\mathscr{V}$ be a (finite-dimensional) vector space and let $\mathscr{G}(\mathscr{V})$ be the group of invertible continuous maps from $\mathscr{V}$ to $\mathscr{V}$.

Definition 1.1. *A representation of a topological group G is a pair $(\pi, \mathscr{V})$ where $\mathscr{V}$ is a finite-dimensional vector space and $\pi : G \to \mathscr{G}(\mathscr{V})$ is a continuous group homomorphism of G to the group $\mathscr{G}(\mathscr{V})$, such that the resulting map $G \times \mathscr{V} \to \mathscr{V}$, given by*

$$(x, v) \mapsto \pi(x)v,$$

is continuous.

For $v \in \mathscr{V}$, the action of G on $\mathscr{V}$ is by right translation:

$$(\pi(x)v)(y) = v(yx), \qquad x, y \in G. \tag{1.2}$$

Definition 1.2. *The representation $(\pi, \mathscr{V})$ is unitary if*

$$(\pi(x)u, \pi(x)v) = (u, v)$$

for all $u, v \in \mathscr{V}$ and every $x \in G$. The representation $(\pi, \mathscr{V})$ is said to be finite-dimensional if $\mathscr{V}$ is finite-dimensional and the dimension of $\mathscr{V}$, dim($\mathscr{V}$) is the dimension of the representation. A subspace $\mathscr{V}_0$ of $\mathscr{V}$ is called invariant or G-invariant if $x \cdot \mathscr{V}_0 \subset \mathscr{V}_0$ for all $x \in G$. A representation π is called irreducible if $\mathscr{V}$ has no closed π-invariant subspaces.

Remark 1.1. *Any representation $(\pi, \mathscr{V})$ of G always has at least two invariant subspaces; these are $\mathscr{V}_0 = \{0\}$ and $\mathscr{V}_0 = \mathscr{V}$, and they are called the trivial invariant subspaces of the representation $(\pi, \mathscr{V})$ of G.*

Definition 1.3. *A finite-dimensional representation $(\pi, \mathscr{V})$ of G is completely reducible if*

$$\mathscr{V} = \mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_k,$$

where $\pi(x)\mathscr{V}_i \subset \mathscr{V}_i$, for all $x \in G$, provided that the representations $(\pi_i, \mathscr{V}_i)$ of G obtained by restricting $\pi(x)$ to $\mathscr{V}_i$ are irreducible for all $i = 1, \cdots, k$. We say that $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ is the direct sum of the π_i.

Definition 1.4. *We call a function f defined on G a class function if $f(yxy^{-1}) = f(x)$ for all $x, y \in G$. We also say that f is central.*

Definition 1.5. *The character χ_π of a finite dimensional representation π of G is given by*

$$\chi_\pi(x) = tr(\pi(x)), \qquad x \in G.$$

Definition 1.6. *We say that two representations $(\pi_1, \mathscr{V}_1)$ and $(\pi_2, \mathscr{V}_2)$ are equivalent if there exists a linear isomorphism $S : \mathscr{V}_1 \to \mathscr{V}_1$ such that $S(\pi_1(x)v) = \pi_2(x)(S(v))$ for all $x \in G$ and $v \in \mathscr{V}_1$ (or simply $S\pi_1 = \pi_2 S$); and we write $\pi_1 \cong \pi_2$.*

Since $\mathscr{V}$ is finite-dimensional,

$$\operatorname{tr} \pi(f) = \int_G (\operatorname{tr} \pi(x)) f(x)\, dx = \int_G \chi_\pi(x) f(x)\, dx, \;\; f \in L^1(G).$$

Let $(\pi_j, \mathscr{V})$ be a finite-dimensional irreducible unitary representation of G, and $C^\infty(K\backslash G/K)$ the space of smooth K-biinvariant functions on G, i.e., $f(k_1 x k_2) = f(x)$ for all $x \in G$, $k_1, k_2 \in K$.

Definition 1.7. *A function $\Phi_j^{\mathscr{X}} \in C^\infty(K\backslash G/K)$ is spherical. In addition*

- $\Delta_{\mathscr{X}} \Phi_j^{\mathscr{X}}(x) = \lambda_j \Phi_j^{\mathscr{X}}(x)$,
- $\Phi_j^{\mathscr{X}}(e) = I$, $e =$ *identity element of* G,
- $\int_K \Phi_j^{\mathscr{X}}(xky)\, dk = \Phi_j^{\mathscr{X}}(x)\Phi_j^{\mathscr{X}}(y), \;\; x, y \in G.$

Definition 1.8. *A unitary finite-dimensional irreducible representation $(\pi_j, \mathscr{V})$ of G is called spherical if there exists a vector $e \in \mathscr{V}$ such that*

$$\pi_j(k)e = e \quad \textit{for all } k \in K.$$

Recall the heat kernel on a compact Riemannian manifold M:

$$K_M(t, x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y),$$

where the eigenfunctions $(\phi_k : k \geq 0)$ form an orthonormal basis of $L^2(M)$, with associated eigenvalues $(\lambda_k : k \geq 0)$.

Let $(\pi_k, \mathscr{V})$ be a spherical unitary finite-dimensional irreducible representation of G. Consider the function

$$\phi_k(x) = \left(\frac{d_k}{\mathrm{Vol}(\mathscr{X})}\right)^{\frac{1}{2}} (e, \pi_k(x^{-1})e), \ x \in G, \tag{1.3}$$

where $\mathrm{Vol}(\mathscr{X})$ is the volume of $\mathscr{X}$. By the symmetric properties of G, the function ϕ_k given by (1.3) is spherical (see e.g. Helgason [83]). Thus the heat kernel on a compact symmetric space $\mathscr{X}$ takes the form

$$K_{\mathscr{X}}(t, x, y) = \frac{1}{\mathrm{Vol}(\mathscr{X})} \sum_{k=0}^{\infty} d_k (e, \pi_k(x^{-1}y)e) e^{-\lambda_k t}.$$

By the unitary nature of the representation π_k the function

$$\Phi_k^{\mathscr{X}}(x, y) := (e, \pi_k(x^{-1}y)e)$$

is spherical, observing that the vectors $e \in \mathscr{V}$ can be normalised such that $(e, e) = 1$. Hence, $\Phi_k^{\mathscr{X}}(\theta)$ are spherical functions (i.e., radial eigenfunctions satisfying $\Phi_k^{\mathscr{X}}(0) = 1$), and we conclude that

$$K_{\mathscr{X}}(t, \rho) = \frac{1}{\mathrm{Vol}(\mathscr{X})} \sum_{k=0}^{\infty} d_k \Phi_k^{\mathscr{X}}(\rho) e^{-\lambda_k t}. \tag{1.4}$$

It follows from the G-invariance of the Laplacian $\Delta_{\mathscr{X}}$ that the heat kernel $K_{\mathscr{X}}$ is a two-point invariant function, i.e., it satisfies

$$K_{\mathscr{X}}(t, gx, gy) = K_{\mathscr{X}}(t, x, y) \quad \text{for all} \quad g \in G.$$

Remark 1.2. *When the symmetric space $\mathscr{X} = G/K$ is noncompact, the sum in (1.4) is replaced with an integral over $\mathscr{X}$, while $d_k/\mathrm{Vol}(\mathscr{X})$ is replaced with the Harish-Chandra Plancherel measure; as discussed in Chapter 6.*

1.4 Eigenfunctions of the Laplacian on the Sphere

The standard model for the n-dimensional spherical geometry is the unit sphere $\mathbf{S}^{n-1}$ of $\mathbf{R}^n$ defined by

$$\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n \,:\, |x| = 1\}.$$

The Euclidean metric $d_{\mathbf{R}^n}$ on $\mathbf{S}^{n-1}$ is defined by the formula

$$d_{\mathbf{R}^n}(x, y) = |x - y|.$$

Let x, y be vectors in $\mathbf{S}^{n-1}$ and let $\theta(x, y)$ be the Euclidean angle between x and y. The *spherical distance* between x and y is defined by

$$d_{\mathbf{S}^{n-1}}(x, y) = \theta(x, y),$$

and so

$$\cos\theta(x, y) = (x \cdot y).$$

The Laplacian on the sphere $\mathbf{S}^{n-1}$. An important function in the study of harmonic analysis on the sphere is the *spherical harmonic*. Spherical harmonics are the eigenfunctions of the

Laplacian on the sphere (the Laplacian on the sphere is also called *spherical Laplacian*). They are the analogues of exponentials for Fourier analysis on the sphere. These functions were introduced by Laplace and Legendre in the 1780's when studying gravitational theory.

Definition 1.9. *A function $f(x)$ defined on a domain in $\mathbf{R}^n$ is called harmonic if it satisfies the differential equation (Laplace equation)*

$$\mathbb{D}_n f = -\frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_2^2} - \cdots - \frac{\partial^2 f}{\partial x_n^2} = 0. \tag{1.5}$$

Definition 1.10. *A function $f(x)$ is homogeneous of degree k (or k-homogeneous) if*

$$f(tx) = t^k f(x)$$

for any $t > 0$.

The sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$ is a compact Riemannian manifold with a constant positive sectional curvature 1. The equation for the Cartesian coordinates $(x_1, x_2, \cdots, x_n)$ of a point on the sphere $\mathbf{S}^{n-1}$ in terms of the angular coordinates $(\theta_1, \theta_2, \cdots, \theta_{n-1})$ are

$$\begin{aligned} x_1 &= r \sin\theta_{n-1} \cdots \sin\theta_2 \sin\theta_1, \\ x_2 &= r \sin\theta_{n-1} \cdots \sin\theta_2 \cos\theta_1, \\ &\cdots \\ x_{n-1} &= r \sin\theta_{n-1} \cos\theta_{n-2}, \\ x_n &= r \cos\theta_{n-1}, \end{aligned}$$

where $r \geq 0$, $0 \leq \theta_1 \leq 2\pi$, $0 \leq \theta_i \leq \pi$ for $i = 2, ..., n-1$. When $r = 1$, these are the coordinates for the unit sphere $\mathbf{S}^{n-1}$. For simplicity, throughout this thesis we shall consider the case where a function $u(r\cos\theta_1, r\sin\theta_1) = u(r\cos\theta, r\sin\theta)$ is independent of $\theta_2, \theta_3, ..., \theta_{n-1}$.

The Laplace equation in $\mathbf{R}^n$ is given in polar coordinates by

$$\mathbb{D}_n u = -\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} \right) u - \frac{1}{r^2} \Delta_{\mathbf{S}^{n-1}} u = 0, \tag{1.6}$$

where

$$\Delta_{\mathbf{S}^{n-1}} = -\frac{\partial^2}{\partial \theta^2} - (n-2) \cot\theta \frac{\partial}{\partial \theta} \tag{1.7}$$

is the Laplacian on the sphere $\mathbf{S}^{n-1}$. To solve (1.6), we look for the homogeneous harmonic function

$$u(r, \theta) = R(r)\Theta(\theta) = r^k \Theta(\theta). \tag{1.8}$$

Thus,

$$\begin{aligned} \mathbb{D}_n u &= -\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} r^k \Theta \right) u - \frac{1}{r^2} \Delta_{\mathbf{S}^{n-1}} r^k \Theta \\ &= -r^{k-2} \left(k(k+n-2)\Theta + \Delta_{\mathbf{S}^{n-1}} \Theta \right) = 0. \end{aligned} \tag{1.9}$$

So u is harmonic in $\mathbf{R}^n$ (i.e., its Laplacian vanishes identically) if and only if Θ is an eigenfunction of $\Delta_{\mathbf{S}^{n-1}}$ with associated eigenvalues

$$\mu^2 = k(k+n-2), \quad k = 0, 1, 2, \cdots . \tag{1.10}$$

That is, the harmonics on the sphere $\mathbf{S}^{n-1}$ are the eigenfunctions Θ of the equation

$$\Delta_{\mathbf{S}^{n-1}}\Theta = k(k+n-2)\Theta.$$

Spherical Harmonics. Let $\mathfrak{P}_k^{n-1}$ denote the space of k-homogeneous polynomials of n variables, and

$$\mathfrak{H}_k^{n-1} = \left\{Y \in \mathfrak{P}_k^{n-1} \,:\, \mathbb{D}_n Y = 0\right\}$$

the space of homogeneous harmonic polynomials of degree k in n dimensions. By definition, spherical harmonics are the restrictions of elements in $\mathfrak{H}_k^{n-1}$ to the unit sphere $\mathbf{S}^{n-1}$. If $Y \in \mathfrak{H}_k^{n-1}$, then

$$Y(x) = |x|^k Y(\xi), \tag{1.11}$$

where $x = |x|\xi \in \mathbf{R}^n$, $\xi \in \mathbf{S}^{n-1}$. We shall call

$$\mathfrak{h}_k^{n-1} = \left\{S \in \mathfrak{P}_k^{n-1} \,:\, \Delta_{\mathbf{S}^{n-1}} S = 0\right\}$$

the space of spherical harmonics of degree k in n dimensions.

Remark 1.3. *We shall make no distinction between the two spaces* $\mathfrak{H}_k^{n-1}$ *and* $\mathfrak{h}_k^{n-1}$.

Let $x = r\xi \in \mathbf{R}^n$, $\xi \in \mathbf{S}^{n-1}$. Since $Y \in \mathfrak{H}_k^{n-1}$ is homogeneous, $Y(x) = r^k Y(\xi)$, and by (1.8) and (1.9) it follows that

Theorem 1.11. *The spherical harmonics* Y_k *of degree* k *are eigenfunctions of the Laplacian* $\Delta_{\mathbf{S}^{n-1}}$ *on the unit sphere* $\mathbf{S}^{n-1}$ *with associated eigenvalues* $\mu_k^2 = k(k+n-2)$.

Thus a (surface) spherical harmonic Y_k of degree k is the restriction to the unit sphere of a polynomial from $\mathfrak{H}_k^{n-1}$.

Let $L^2\left(\mathbf{S}^{n-1}\right)$ denote the space of square integrable functions on $\mathbf{S}^{n-1}$. Then it is a classical fact that $L^2\left(\mathbf{S}^{n-1}\right)$ decomposes into the orthogonal sum of spaces $\mathfrak{H}_k^{n-1}$ of degree k and thus each $f \in L^2\left(\mathbf{S}^{n-1}\right)$ has an expansion

$$f(\xi) = \sum_{k=0}^{\infty} (\pi_k f)(\xi), \;\; \xi \in \mathbf{S}^{n-1},$$

where π_k is the orthogonal projection onto the space $\mathfrak{H}_k^{n-1}$. Let $f, g \in L^2\left(\mathbf{S}^{n-1}\right)$ be two functions defined on the sphere $\mathbf{S}^{n-1}$. We define the (integral) inner product of f and g by

$$(f,g)_{L^2(\mathbf{S}^{n-1})} = \frac{1}{\nu_{n-1}} \int_{\mathbf{S}^{n-1}} f(x)g(x)\, d\nu_{n-1}(x),$$

where $d\nu_{n-1}$ is the surface area measure and ν_{n-1} the surface area of $\mathbf{S}^{n-1}$:

$$\nu_{n-1} = \int_{\mathbf{S}^{n-1}} d\nu_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{1.12}$$

Throughout this thesis, we shall use, for $\xi \in \mathbf{S}^{n-1}$, $0 \le \theta \le \pi$,

$$d\nu_{n-1}(\xi) = \nu_{n-2} \sin^{n-2}\theta \, d\theta.$$

Remark 1.4. *The use of ν as a parameter and ν_{n-1} as the volume of the unit sphere $\mathbf{S}^{n-1}$ will not cause any confusion.*

The dimension of the space $\mathfrak{P}_k^{n-1}$ is given by

$$\dim \mathfrak{P}_k^{n-1} = \binom{k+n-1}{n-1} = \binom{k+n-1}{k} = \frac{(n+k-1)!}{(n-1)!k!},$$

and thus

$$\begin{aligned} M_k^{n-1} := \dim \mathfrak{H}_k^{n-1} &= \dim \mathfrak{P}_k^{n-1} - \dim \mathfrak{P}_{k-2}^{n-1} && (1.13) \\ &= \binom{k+n-1}{k} - \binom{k+n-3}{k-2} \\ &= \frac{(2k+n-2)(k+n-3)!}{k!(n-2)!}, && (1.14) \end{aligned}$$

where M_k^{n-1} is also the multiplicity of the eigenvalues $\mu_k^2 = k(k+n-2)$ of the Laplacian $\Delta_{\mathbf{S}^{n-1}}$.

Theorem 1.12. *If $Y_k, Y_l \in \mathfrak{H}_k^{n-1}$, with $k \neq l$, then*

$$(Y_k, Y_l)_{L^2(\mathbf{S}^{n-1})} = \frac{1}{\nu_{n-1}} \int_{\mathbf{S}^{n-1}} Y_k(\zeta) Y_l(\zeta) \, d\nu_{n-1}(\zeta) = 0.$$

Proof. For $x = |x|\xi \in \mathbf{R}^n$, $\xi \in \mathbf{S}^{n-1}$, let $P_k(x)$ and $P_l(x)$ be two homogeneous harmonic polynomials of degrees k and l respectively. Then noting (1.11), i.e.,

$$Y_k(\xi) = \frac{1}{r^k} P_k(r\xi) = P_k(\xi),$$

we have by Green's theorem

$$\begin{aligned} 0 &= \int_{|x|\le 1} \left(P_k \mathbb{D}_n P_l - P_l \mathbb{D}_n P_k \right) \, dx \\ &= \int_{\mathbf{S}^{n-1}} \left(P_k(\xi) \frac{\partial}{\partial r} P_l(r\xi)\Big|_{r=1} - P_l(\xi) \frac{\partial}{\partial r} P_k(r\xi)\Big|_{r=1} \right) \, d\nu_{n-1}(\xi) \\ &= (l-k) \int_{\mathbf{S}^{n-1}} P_k(\xi) P_l(\xi) \, d\nu_{n-1}(\xi). \end{aligned}$$

Since $l \neq k$, we can divide both sides by $l-k$ to obtain the result. □

Zonal Harmonics. Let

$$\pi_k : L^2\left(\mathbf{S}^{n-1}\right) \to \mathfrak{H}_k^{n-1}$$

denote the orthogonal projection from the Hilbert space $L^2\left(\mathbf{S}^{n-1}\right)$ onto the subspace $\mathfrak{H}_k^{n-1} \subset L^2\left(\mathbf{S}^{n-1}\right)$. Let $\left(Y_j : 1 \le j \le M_k^{n-1}\right)$ be an orthonormal basis of $\mathfrak{H}_k^{n-1} \subset L^2\left(\mathbf{S}^{n-1}\right)$ and let $p \in \mathfrak{H}_k^{n-1}$ be arbitrary. Then from the theory of Fourier series, every $p \in \mathfrak{H}_k^{n-1}$ can be expanded in an absolutely and uniformly convergent series of spherical harmonics

$$p_k(\zeta) = \sum_{j=1}^{M_k^{n-1}} \widetilde{p}_j Y_j(\zeta),$$

where

$$\widetilde{p}_j = (p, Y_j)_{L^2(\mathbf{S}^{n-1})} = \frac{1}{\nu_{n-1}} \int_{\mathbf{S}^{n-1}} p(\xi)\overline{Y_j(\xi)}\, d\nu_{n-1}(\xi).$$

Thus,

$$\begin{aligned} p_k(\zeta) &= \frac{1}{\nu_{n-1}} \int_{\mathbf{S}^{n-1}} p(\xi) \sum_{j=1}^{M_k^{n-1}} \overline{Y_j(\xi)} Y_j(\zeta)\, d\nu_{n-1}(\xi) \\ &= \frac{1}{\nu_{n-1}} \int_{\mathbf{S}^{n-1}} p(\xi) \mathcal{Z}_j(\zeta, \xi)\, d\nu_{n-1}(\xi). \end{aligned}$$

Definition 1.13. *The kernel function*

$$\mathcal{Z}_k(\zeta, \xi) = \sum_{j=1}^{M_k^{n-1}} \overline{Y_j(\zeta)} Y_j(\xi) \tag{1.15}$$

is called the zonal harmonic, where $\left(Y_j : 1 \leq j \leq M_k^{n-1}\right)$ *is an orthonormal basis of* $\mathfrak{H}_k^{n-1}$.

Since $\pi_k \in \mathfrak{H}_k^{n-1}$, it can be expanded in terms of the orthonormal basis $\left(Y_j : 1 \leq j \leq M_k^{n-1}\right)$ of $\mathfrak{H}_k^{n-1}$:

$$\begin{aligned} \pi_k f(\zeta) &= \sum_{j=1}^{M_k^{n-1}} (f, Y_j)_{L^2(\mathbf{S}^{n-1})} Y_j(\zeta) = \left(f, \sum_{j=1}^{M_k^{n-1}} Y_j \overline{Y_j(\zeta)} \right)_{L^2(\mathbf{S}^{n-1})} \\ &= \frac{1}{\nu_{n-1}} \int_{\mathbf{S}^{n-1}} \mathcal{Z}_j(\zeta, \xi) f(\xi)\, d\nu_{n-1}(\xi). \end{aligned} \tag{1.16}$$

It follows from (1.16) the following addition formula for spherical harmonics (see also Dai and Xu [47, Chapter 1]):

$$\begin{aligned} \mathcal{Z}_k(\zeta, \xi) &= \sum_{j=1}^{M_k^{n-1}} \overline{Y_j(\zeta)} Y_j(\xi) = \frac{k+\nu}{\nu} C_k^{\nu}((\zeta \cdot \xi)) \\ &= \frac{M_k^{n-1} C_k^{\nu}((\zeta \cdot \xi))}{C_k^{\nu}(1)} := M_k^{n-1} \mathscr{C}_k^{\nu}((\zeta \cdot \xi)), \end{aligned} \tag{1.17}$$

where $C_k^{\nu}((\zeta \cdot \zeta'))$ $(\nu = (n-2)/2, n \geq 3)$ is the *Gegenbauer* polynomial (see Appendix B.6). Moreover,

$$\lim_{\nu \searrow 0} \left(\frac{k+\nu}{\nu} \right) C_k^{\nu}(\cos \vartheta) = \begin{cases} 1, & k = 0, \\ 2\cos k\vartheta, & k \geq 1. \end{cases} \tag{1.18}$$

Now setting $\zeta = \xi$ in (1.15), we have

$$\mathcal{Z}_k(\zeta, \zeta) = M_k^{n-1} \mathscr{C}_k^{\nu}(1) = \sum_{j=1}^{M_k^{n-1}} \left| Y_j(\zeta) \right|^2,$$

which is a constant for all $\zeta \in \mathbf{S}^{n-1}$. On integrating over $\mathbf{S}^{n-1}$, we obtain

$$M_k^{n-1}\mathscr{C}_k^{\nu}(1) = \frac{1}{\nu_{n-1}}\int_{\mathbf{S}^{n-1}} \mathcal{Z}_k(\zeta,\zeta)\, d\nu_{n-1}(\zeta) = \frac{1}{\nu_{n-1}}\int_{\mathbf{S}^{n-1}} \sum_{j=1}^{M_k^{n-1}} |Y_j(\zeta)|^2\, d\nu_{n-1}(\zeta) = M_k^{n-1}.$$

It follows that

$$C_k^{\nu}(1) = \frac{\nu}{k+\nu} M_k^{n-1}. \tag{1.19}$$

Theorem 1.14 (**Funk-Hecke Identity** (Morimoto [115, Theorem 2.39, p. 32])). *Let f be a complex-valued continuous function on the interval $[-1,1]$. Then for every spherical harmonic $Y_k(\zeta), \zeta \in \mathbf{S}^{n-1}$, the following integral identity holds:*

$$\frac{1}{\nu_{n-1}}\int_{\mathbf{S}^{n-1}} f(\zeta\cdot\zeta')Y_k(\zeta')\, d\nu_{n-1}(\zeta') = \frac{\nu_{n-2}}{\nu_{n-1}} Y_k(\zeta)\int_{-1}^{1} f(t)\mathscr{C}_k^{\nu}(t)\left(1-t^2\right)^{\nu-\frac{1}{2}}\, dt, \tag{1.20}$$

where $\mathscr{C}_k^{\nu}(t) := \frac{C_k^{\nu}(t)}{C_k^{\nu}(1)}$, $\nu = \frac{n-2}{2}$.

The main statement of this section is the following.

Proposition 1.15. *The functions*

$$\Phi_k^{\mathbf{S}^{n-1}}(\theta) = \mathscr{C}_k^{\frac{n-2}{2}}(\cos\theta) \tag{1.21}$$

are called the spherical functions on $\mathbf{S}^{n-1}$. They satisfy the eigenvalue problem

$$\Delta_{\mathbf{S}^{n-1}}\Phi_k^{\mathbf{S}^{n-1}}(\theta) = k(k+n-2)\Phi_k^{\mathbf{S}^{n-1}}(\theta)$$

with $\Phi_k^{\mathbf{S}^{n-1}}(0) = 1$, where $\Delta_{\mathbf{S}^{n-1}}$ is the radial part of the Laplacian on the unit sphere $\mathbf{S}^{n-1}$ given by (1.7).

Proof. We shall use the substitution method. Consider the eigenvalue problem

$$\frac{d^2\Theta}{d\theta^2} + (n-1)\frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \lambda\Theta = 0. \tag{1.22}$$

Making the substitution

$$\Theta(\theta) = \sin^{1-\frac{n}{2}}\theta\,\eta(\theta)$$

with

$$\begin{aligned}
\frac{d\Theta}{d\theta} &= \cos\theta\sin^{-\frac{n}{2}}\theta\eta - \frac{n}{2}\cos\theta\sin^{-\frac{n}{2}}\theta\eta + \sin^{1-\frac{n}{2}}\theta\frac{d\eta}{d\theta}\\
\frac{d^2\Theta}{d\theta^2} &= \sin^{1-\frac{n}{2}}\theta\frac{d^2\eta}{d\theta^2} + \left[2\cos\theta\sin^{-\frac{n}{2}}\theta - n\cos\theta\sin^{-\frac{n}{2}}\theta\right]\frac{d\eta}{d\theta}\\
&\quad + \left[\cos^2\theta\sin^{-\frac{n}{2}-1}\theta\left(-\frac{n}{2}+\frac{n^2}{4}\right) - \sin^{1-\frac{n}{2}}\theta\left(1-\frac{n}{2}\right)\right]\eta
\end{aligned}$$

in (1.22) and multiplying the resulting equation by $\sin^{\frac{n}{2}-1}\theta$, we obtain after some rearrangements

$$\begin{aligned}
&\frac{d^2\eta}{d\theta^2} + \left[2\frac{\cos\theta}{\sin\theta} - \frac{n\cos\theta}{\sin\theta} + (n-1)\frac{\cos\theta}{\sin\theta}\right]\frac{d\eta}{d\theta} \\
&+ \left[\cos^2\theta\sin^{-2}\theta\left(-\frac{n}{2} + \frac{n^2}{4}\right) - \left(1 - \frac{n}{2}\right) + (n-1)\cos^2\theta\sin^{-2}\theta\right]\eta \\
&+ \left[(n-1)\cos^2\theta\sin^{-2}\theta\left(1 - \frac{n}{2}\right) + \lambda\right]\eta = 0.
\end{aligned}$$

Further simplification gives

$$\frac{d^2\eta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\eta}{d\theta} + \left\{\lambda + \frac{n}{2} - 1 - \left(\frac{n}{2} - 1\right)^2\cos^2\theta\sin^{-2}\theta\right\}\eta = 0. \tag{1.23}$$

Again substituting

$$y(x) = \eta(\theta), \quad x = \cos\theta \tag{1.24}$$

with

$$\frac{d\eta}{d\theta} = -\frac{dy}{dx}\sin\theta, \quad \frac{d^2\eta}{d\theta^2} = (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$$

in (1.23) gives

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{\lambda + \frac{n}{2} - 1 - \frac{\left(\frac{n}{2}-1\right)^2 x^2}{1-x^2}\right\}y = 0. \tag{1.25}$$

We simplify the expression inside the braces:

$$\begin{aligned}
&\lambda + \frac{n}{2} - 1 - \frac{\left(\frac{n}{2}-1\right)^2 x^2}{1-x^2} \\
&= \frac{\lambda + \frac{n}{2} - 1 - \lambda x^2 - \frac{nx^2}{2} - \frac{n^2x^2}{4} + nx^2}{1-x^2} \\
&= \frac{\left(-\frac{1}{4} + \lambda + \frac{(n-1)^2}{4}\right)(1-x^2) - \left(\frac{n}{2}-1\right)^2}{1-x^2}.
\end{aligned}$$

If we set

$$\alpha(\alpha+1) = -\frac{1}{4} + \lambda + \frac{(n-1)^2}{4},$$

then we obtain a quadratic equation in α :

$$\alpha^2 + \alpha + \frac{1}{4} - \lambda - \frac{(n-1)^2}{4} = 0$$

whose solution is

$$\alpha = -\frac{1}{2} \pm \sqrt{\lambda + \frac{(n-1)^2}{4}} = -\frac{1}{2} \pm \frac{n-1+2k}{2}. \tag{1.26}$$

So, we obtain the associated Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{\alpha(\alpha+1) - \frac{\sigma^2}{1-x^2}\right\}y = 0 \tag{1.27}$$

of degree α given by (1.26) and order

$$\sigma = \pm\sqrt{\frac{(n-2)^2}{4}} = \pm\frac{n-2}{2}. \tag{1.28}$$

Hence,

$$\Theta_k^n(\theta) = \sin^{1-\frac{n}{2}}\theta P^{\frac{2-n}{2}}_{k+\frac{n-2}{2}}(\cos\theta), \tag{1.29}$$

where $P_\nu^\mu(z)$ is the associated Legendre function of the first kind with degree ν and order μ and argument z (see Appendix B.5). The expression (1.29) can be expressed in terms of Gegenbauer polynomials $C_k^{\frac{n-1}{2}}(\cos\theta)$, we just apply the formula (B.114). □

The hyperbolic version of Proposition 1.15 is Proposition 4.1 in Section 4.1.

The Special Case $\mathbf{S}^1$. We specialise to the case $n = 2$, $M_k^1 = 2$, $\mu_k^2 = k^2$. An orthogonal basis of $\mathfrak{H}_k^1$ is given by the real and imaginary parts of

$$z^k = (x_1 + ix_2)^k = (r\cos\theta, r\sin\theta)^k = r^k\cos k\theta + ir^k\sin k\theta,$$

for $z = (x_1, x_2) \in \mathbf{R}^2$. That is,

$$Y_{k,1}(\zeta) = r^k\cos k\theta, \quad Y_{k,2}(\zeta) = r^k\sin k\theta.$$

Hence, the spherical harmonics (eigenfunctions of the Laplacian $\Delta_{\mathbf{S}^1} = -\frac{d^2}{d\theta^2}$ on the circle $\mathbf{S}^1$) are precisely the cosine and sine functions. In terms of the Chebychev polynomials $T_k(\xi)$ and $U_k(\xi)$ (Gradshtejn and Ryzhik [66, pp. 993-996]), we have

$$T_k(\cos\theta) = \cos k\theta = \frac{k}{2}\lim_{\nu\searrow 0}\frac{1}{\nu}C_k^\nu(\cos\theta),\ U_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta} = C_k^1(\cos\theta),$$

and the orthogonal basis becomes

$$\begin{aligned} Y_{k,1}(\zeta) &= r^k T_k\left(\frac{x_1}{r}\right) = r^k\frac{k}{2}\lim_{\nu\to 0}\frac{1}{\nu}C_k^\nu\left(\frac{x_1}{r}\right), \\ Y_{k,2}(\zeta) &= r^{k-1}x_2U_{k-1}\left(\frac{x_1}{r}\right) = r^{k-1}x_2C_{k-1}^1\left(\frac{x_1}{r}\right). \end{aligned}$$

In this case the addition formula (zonal harmonic) (1.17) reduces to the addition formula

$$\begin{aligned} \mathcal{Z}_k(\zeta,\xi) &= \mathcal{C}_k^0(\zeta\cdot\xi) = T_k(\zeta\cdot\xi) = Y_1(\zeta)Y_1(\xi) + Y_2(\zeta)Y_2(\xi) \\ &= \cos k\theta\cos k\vartheta + \sin k\theta\sin k\vartheta = \cos k(\theta - \vartheta). \end{aligned}$$

It follows from Proposition 1.15 that

Proposition 1.16. *The function*

$$\Theta(\theta) = \Phi_k^{\mathbf{S}^2}(\theta) = \mathscr{C}_k^{\frac{1}{2}}(\cos\theta) = P_k(\cos\theta) = \frac{1}{2\pi}\int_0^{2\pi}(\cos\theta + i\sin\theta\cos\vartheta)^k\,d\vartheta,$$

satisfying $\Phi_k^{\mathbf{S}^2}(0) = 1$ *is the spherical function on* $\mathbf{S}^2$.

1.5 The Upper Half-Space Model of the Hyperbolic Space

We first consider the Poincaré upper half-plane model of the hyperbolic plane. On the upper half-plane

$$\mathbf{H} = \{z = x + iy \in \mathbf{C} : \operatorname{Im} z = y > 0\}$$

the Laplace-Beltrami operator is simply the Laplacian given by

$$\widetilde{\Delta} := \Delta_{\mathbf{H}} = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \tag{1.30}$$

with the associated Riemannian metric

$$g = ds^2 = \frac{dx^2+dy^2}{y^2},$$

and the hyperbolic area element

$$d\mu_{\mathbf{H}}(z) = d\mu(z) = \frac{dxdy}{y^2}.$$

For $s \in \mathbf{C}$, the function $f(z) = (\operatorname{Im} z)^s = y^s$, which we call the power function satisfies an eigenvalue equation

$$\widetilde{\Delta} y^s = s(1-s)y^s, \tag{1.31}$$

where $\lambda = s(1-s)$ is the eigenvalue corresponding to the eigenfunction f of $\widetilde{\Delta}$. The distance in this upper half-plane is given by the Poincaré distance

$$\cosh d(z,z') = 1 + \frac{|z-z'|^2}{2\operatorname{Im} z \operatorname{Im} z'}, \quad z, z' \in \mathbf{H}, \tag{1.32}$$

where $\cosh x$ is the hyperbolic cosine function of x, and $\operatorname{Im} z$ denotes the imaginary part of the complex number z.

In geodesic polar coordinates (r,θ), we have

$$\begin{aligned}
ds^2 &= d\rho^2 + \sinh^2\rho\, d\theta^2 \\
d\mu &= \frac{dxdy}{y^2} = \sinh\rho\, d\rho\, d\theta \\
\widetilde{\Delta} &= -\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\left(\sinh\rho\frac{\partial}{\partial\rho}\right) - \frac{1}{\sinh^2\rho}\frac{\partial^2}{\partial\theta^2},
\end{aligned}$$

where $\rho = d(z,z')$ denotes the distance between two points $z, z' \in \mathbf{H}$. The boundary of $\mathbf{H}$ is $\partial\mathbf{H} = \mathbf{R} \cup \{\infty\}$. Any point in $\partial\mathbf{H}$ is called a *point at infinity.* The spectrum of $\widetilde{\Delta}$ is absolutely continuous and equal to $\left[\frac{1}{4},\infty\right)$ (Lax and Phillips [99]), while the spectrum of the standard Laplace operator (Euclidean Laplacian)

$$\mathbb{D} = \mathbb{D}_2 = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

acting in $L^2\left(\mathbf{R}^2, dxdy\right)$ is the whole right half-line $[0,\infty)$, since $\mathbb{D}$ is positive and self-adjoint.

We now present the general upper half-space $\mathbf{H}^n$ model of the hyperbolic space. For $n \geq 2$,

$$\mathbf{H}^n = \left\{ w = (x, y) \,:\, x \in \mathbf{R}^{n-1}, y > 0 \right\}; \tag{1.33}$$

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \cdots + dx_{n-1}^2 + dy^2}{y^2}, \quad d\mu_{\mathbf{H}^n}(x) = \frac{dx_1 dx_2 \cdots dx_{n-1}\, dy}{y^n}; \tag{1.34}$$

$$\Delta_{\mathbf{H}^n} = -y^2 \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i \partial x_i} - y^2 \frac{\partial^2}{\partial y^2} - (2-n) y \frac{\partial}{\partial y}; \tag{1.35}$$

$$\cosh d(w, w') = 1 + \frac{|w - w'|^2}{2yy'}, \;\; w = (x, y), w' = (x', y') \in \mathbf{H}^n, x, x' \in \mathbf{R}^{n-1}, y, y' > 0. \tag{1.36}$$

In geodesic polar coordinates,

$$ds^2 = d\rho^2 + \sinh^2 \rho |d\xi|^2, \quad d\mu_{\mathbf{H}^n} = \sinh^{n-1} \rho \, d\rho d\nu_{n-1}; \tag{1.37}$$

$$\Delta_{\mathbf{H}^n} = -\frac{\partial^2}{\partial \rho^2} - (n-1) \frac{\cosh \rho}{\sinh \rho} \frac{\partial}{\partial \rho} - \frac{1}{\sinh^2 \rho} \Delta_{\mathbf{S}^{n-1}}, \tag{1.38}$$

where $|d\xi|^2$ is the Riemannian metric on $\mathbf{S}^{n-1}$. The radial part of the Laplacian in $\mathbf{H}^n$ is also denoted by $\Delta_{\mathbf{H}^n}$ and is given by

$$\Delta_{\mathbf{H}^n} = -\frac{\partial^2}{\partial \rho^2} - (n-1) \frac{\cosh \rho}{\sinh \rho} \frac{\partial}{\partial \rho}. \tag{1.39}$$

Remark 1.5. *The use of μ as a parameter and $\mu_{\mathbf{H}^n}$ as the volume measure on the hyperbolic space will not cause any confusion.*

1.5.1 Eigenfunctions of the Laplacian in Cartesian Coordinates

Consider the eigenvalue problem

$$-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = s(1-s) f \tag{1.40}$$

on $\mathbf{H}$. To solve (1.40), we assume a product solution of the form

$$f(x, y) = X(x) Y(y),$$

which when inserting into (1.40) gives two ODEs

$$Y'' - y^{-2} s(s-1) Y - \alpha Y = 0, \tag{1.41}$$

$$X'' + \alpha X = 0. \tag{1.42}$$

Clearly, for $\alpha > 0$,

$$X(x) = e^{2\pi i m x} \quad \text{with } \alpha = 4\pi^2 m^2, \;\; m \in \mathbf{Z},$$

solves (4.37). Setting

$$Y(y) = y^{\frac{1}{2}} u(y)$$

in (1.41), we have

$$y^2 \frac{d^2 u}{dy^2} + y \frac{du}{dy} - \left(\left(s - \frac{1}{2} \right)^2 + 4\pi^2 m^2 y^2 \right) u(y) = 0,$$

which is a modified Bessel equation of the form (B.42). The general solution, for arbitrary ν, of (B.42) can be written in the form

$$f(z) = C_1 K_\nu(z) + C_2 I_\nu(z),$$

where $I_\nu(z)$ is the modified Bessel function of the first kind given by (B.43) and $K_\nu(z)$ the modified Bessel function of the second kind given by (B.44). Hence,

$$f(x,y) = \begin{cases} e^{2\pi i m x} y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|m|y), & m \neq 0, \\ y^s, & m = 0, \end{cases}$$

where we have dropped the solution

$$f(x,y) = e^{2\pi i m x} y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi|m|y), \qquad m \neq 0,$$

because we expect $f(z)$ $(z = x + iy \in \mathbf{H})$ to have at most polynomial growth at infinity (see (B.50)), i.e., we expect f to satisfy the inequality

$$|f(z)| \leq C y^k, \quad \text{as } y \nearrow \infty$$

for constants $C > 0$ and k.

1.5.2 Eigenfunctions of the Laplacian in Geodesic Polar Coordinates

The eigenvalue problem

$$-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) g = s(1-s)g$$

can be written in geodesic polar coordinates (ρ, ψ) as

$$\frac{\partial^2 g}{\partial \rho^2} + \coth\rho \frac{\partial g}{\partial \rho} + \frac{1}{\sinh^2\rho} \frac{\partial^2 g}{\partial^2 \psi} - s(s-1)g = 0. \tag{1.43}$$

Inserting a product solution of the form

$$g(\rho, \psi) = R(\rho)\Psi(\psi)$$

in (1.43), we obtain two ordinary differential equations

$$\sinh^2\rho \frac{\partial^2 R}{\partial \rho^2} + \cosh\rho \sinh\rho \frac{\partial R}{\partial \rho} - s(s-1)R\sinh^2\rho - \alpha R = 0, \tag{1.44}$$

$$\frac{\partial^2 \Psi}{\partial^2 \psi} + \alpha\Psi = 0, \ \ \alpha = a^2, \qquad a^2 \text{ is a constant.} \tag{1.45}$$

Equation (1.45) has the general solution

$$\Psi(\psi) = e^{ia\psi} + e^{-ia\psi}.$$

As $\Psi(\psi)$ must obey the periodic boundary condition $\Psi(\psi) = \Psi(\psi + 2\pi)$, a must be an integer, namely $a = 0, \pm 1, \pm 2, \cdots$. Hence

$$\Psi(\psi) = e^{ia\psi}. \tag{1.46}$$

Making the substitutions

$$R(\rho) = v(x), \quad x = \cosh \rho$$

in (1.44) we have

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + \left[s(s-1) - \frac{a^2}{1-x^2}\right] v = 0, \tag{1.47}$$

which is the associated Legendre's equation of the form (B.74) and its solution is called the associated Legendre function of the first kind of degree μ and order ν. Hence,

$$g_{a,s}(z) = g_{a,s}(\rho,\psi) = R(\rho)\Psi(\psi) = \begin{cases} e^{ia\psi} P^a_{-s}(\cosh \rho), & a \neq 0, \\ P_{-s}(\cosh \rho), & a = 0, \end{cases}$$

where P^μ_ν is the associated Legendre function of the first kind, namely (see (B.77), see (B.88)),

$$P^a_{-s}(\cosh r) = \frac{\Gamma(a-s+1)}{2\pi\Gamma(1-s)} \int_0^{2\pi} \frac{e^{ia\psi}\, d\psi}{(\cosh \rho + \sinh \rho \cos \psi)^s}. \tag{1.48}$$

1.5.3 The Heat Kernel

In this subsection we construct the heat kernel in the Poincaré upper half-plane $\mathbf{H}$. The heat kernel in $\mathbf{H}$ is the fundamental solution of the non-Euclidean Cauchy problem

$$\begin{aligned} \frac{\partial u(t,z)}{\partial t} &= -\widetilde{\Delta} u(t,z), \qquad z \in \mathbf{H}, t > 0, \\ u(0,z) &= f(z), \qquad z \in \mathbf{H},\ f \in C^\infty(\mathbf{H}). \end{aligned} \tag{1.49}$$

Our approach involves the application of the Mehler-Fock transform and its inversion formula.

Definition 1.17. *For any $f \in C_0^\infty([0,\infty))$, the Mehler-Fock transform (of order zero) of $f(\rho)$ is defined by*

$$\mathcal{M}[f(\rho)](r) = \overline{f}(r) = \int_0^\infty f(\rho) F\left(\frac{1}{2} + ir, \frac{1}{2} - ir; 1; -\sinh^2\left(\frac{\rho}{2}\right)\right) \sinh \rho \, d\rho. \tag{1.50}$$

Theorem 1.18. *The Mehler-Fock inversion formula (of order zero) is given by*

$$\mathcal{M}^{-1}[\overline{f}(r)](\rho) = f(\rho) = \frac{1}{2\pi} \int_0^\infty \overline{f}(r) F\left(\frac{1}{2} + ir, \frac{1}{2} - ir; 1; -\sinh^2\left(\frac{\rho}{2}\right)\right) r \tanh \pi r \, dr. \tag{1.51}$$

Proof. The proof of Theorem 1.18 and the higher order version will be given in Section 6.3. □

We now apply the Mehler-Fock transform (1.50) to the initial value problem (1.49), to obtain

$$\begin{aligned} \frac{\partial \overline{u}(t,r)}{\partial t} &= -\overline{\widetilde{\Delta} u}(t,r) = s(s-1)\overline{u}(t,r) = -\left(\frac{1}{4} + r^2\right)\overline{u}(t,r), \\ \overline{u}(0,r) &= \overline{f}(r). \end{aligned} \tag{1.52}$$

Solving (1.52), we obtain

$$\overline{u}(t,r) = \overline{f}(r) e^{-\left(\frac{1}{4}+r^2\right)t}. \tag{1.53}$$

Applying the Mehler-Fock inversion formula (1.51) to both sides of (1.53) to get

$$\begin{aligned} u(t,z) &= \left(f * \mathcal{M}^{-1}\left[e^{-\left(\frac{1}{4}+r^2\right)t}\right]\right)(\rho) \\ &= \int_{\mathbf{H}} f(z')\widetilde{K}(t,z,z')\,d\mu(z'),\ \ z=(\rho,\theta), z'=(\rho',\theta'), d\mu(z')=d\mu(\rho',\theta'), \end{aligned}$$

where

$$\widetilde{K}(t,z,z') := K_{\mathbf{H}}(t,z,z') = \frac{1}{2\pi}\int_0^\infty e^{-\left(\frac{1}{4}+r^2\right)t} F\left(\frac{1}{2}+ir, \frac{1}{2}-ir; 1; -\sinh^2\left(\frac{\rho}{2}\right)\right) r\tanh \pi r\, dr. \tag{1.54}$$

Using the formula (B.86) with $\nu = -\frac{1}{2}+ir$, $\alpha=\rho$ and noting that $\coth x = i\cot ix, \sinh x = -i\sin ix$, we obtain

$$F\left(\frac{1}{2}+ir, \frac{1}{2}-ir; 1; -\sinh^2\left(\frac{\rho}{2}\right)\right) = \frac{\sqrt{2}}{\pi}\coth \pi r \int_\rho^\infty \frac{\sin ru}{\sqrt{\cosh u - \cosh\rho}}\,du. \tag{1.55}$$

Equation (1.54) becomes

$$\widetilde{K}(t,z,z') = \frac{1}{\sqrt{2}\pi^2}\int_\rho^\infty \frac{1}{\sqrt{\cosh u-\cosh\rho}}\int_0^\infty e^{-\left(\frac{1}{4}+r^2\right)t} r\sin ru\, dr du. \tag{1.56}$$

For the last integral on the right hand side of (1.56), we use (Gradshtejn and Ryzhik [66, p. 502, eq. 3.902(1)])

$$\int_0^\infty \beta e^{-b^2\beta^2}\sin a\beta\, d\beta = \frac{a\sqrt{\pi}}{4b^3}e^{-\frac{a^2}{4b^2}} \tag{1.57}$$

to obtain

$$\widetilde{K}(t,z,z') = \frac{\sqrt{2}}{(4\pi t)^{\frac{3}{2}}}e^{-\frac{t}{4}}\int_\rho^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sqrt{\cosh u-\cosh\rho}}\,du, \tag{1.58}$$

which is the heat kernel on the hyperbolic plane $\mathbf{H}$.

1.5.4 The Resolvent Kernel

By definition, the *spectrum* $\sigma(\widetilde{\Delta})$ of the Laplacian $\widetilde{\Delta}$ consists of those values of λ for which the resolvent

$$\mathcal{R}_\lambda = \left(\widetilde{\Delta}-\lambda\right)^{-1}$$

fails to exist as a bounded operator (see Appendix A.5). In other words, the *resolvent* of a self-adjoint positive operator $\widetilde{\Delta}$ is the bounded operator $\mathcal{R}_\lambda$ defined for $\lambda \notin [0,\infty)$. The aim of this subsection is the construction of the integral kernel $\widetilde{\mathcal{G}}_s(z,z')$ $(z,z' \in \mathbf{H}, s\in\mathbf{C})$ of the resolvent operator

$$\mathcal{R}_s = (\widetilde{\Delta}-s(1-s))^{-1},$$

or what is the same, the Green function of $\widetilde{\Delta}$. The Green function $\widetilde{\mathcal{G}}_s(z,z')$ of $\widetilde{\Delta}$ is related to the heat kernel $\widetilde{K}(t,z,z')$, $t>0$, by the formula

$$\widetilde{\mathcal{G}}_s(z,z') = \mathfrak{L}\left[\widetilde{K}(t,z,z')\right](\lambda = s(1-s)) = \int_0^\infty e^{-s(s-1)t}\widetilde{K}(t,z,z')\,dt,$$

where $\mathfrak{L}f$ is the Laplace transform of the function f. By inserting the heat kernel (1.58) we have

$$\begin{aligned}\widetilde{\mathcal{G}}_s(z,z) &= \frac{\sqrt{2}}{4\pi}\int_{d(z,z')}^{\infty}\left(\frac{1}{\sqrt{4\pi}}\int_0^{\infty} e^{-\left(s-\frac{1}{2}\right)^2 t} t^{-\frac{3}{2}} e^{-\frac{u^2}{4t}}\,dt\right)\frac{udu}{\sqrt{\cosh\,u-\cosh d(z,z')}}\\ &= \frac{1}{\sqrt{2}2\pi}\int_{d(z,z')}^{\infty}\frac{e^{-(s-\frac{1}{2})u}}{\sqrt{\cosh u-\cosh d(z,z')}}\,du = \frac{1}{2\pi}Q_{s-1}(\cosh d(z,z')),\end{aligned} \tag{1.59}$$

where we have used the integral formula (see (B.47))

$$\frac{1}{\sqrt{4\pi}}\int_0^{\infty} e^{-\left(s-\frac{1}{2}\right)^2 t} t^{-\frac{3}{2}} e^{-\frac{u^2}{4t}}\,dt = \frac{1}{u}e^{-(s-\frac{1}{2})u}.$$

Setting

$$\cosh\rho = \cosh^2\left(\frac{\rho}{2}\right)+\sinh^2\left(\frac{\rho}{2}\right) = 2\tau-1,\ \tau=\cosh^2\frac{d(z,z')}{2},$$

and applying (B.97) and (B.96) we have

$$\begin{aligned}Q_{s-1}(2\tau-1) &= \sqrt{\pi}\frac{\Gamma(s)}{2^{2s}\Gamma\left(s+\frac{1}{2}\right)}\tau^{-s}F\left(s,s;2s;\tau^{-1}\right)\\ &= \frac{1}{2}\frac{\Gamma(s)^2}{\Gamma(2s)}\tau^{-s}F\left(s,s;2s;\tau^{-1}\right)\\ &= Q_{s-1}(\cosh\rho).\end{aligned} \tag{1.60}$$

Hence,

$$\widetilde{\mathcal{G}}_s(z,z') = \frac{1}{4\pi}\frac{\Gamma(s)^2}{\Gamma(2s)}\tau^{-s}F\left(s,s;2s;\tau^{-1}\right), \tag{1.61}$$

which is the Green's function in $\mathbf{H}$.

1.5.5 The Wave Kernel

In this subsection we give an explicit formula for the wave kernel on the upper half-plane $\mathbf{H}$ by first obtaining the easier three dimensional wave kernel and then using the method of descent to construct the two-dimensional non-Euclidean wave kernel.

Let $\mathbf{H}^3$ be the 3-dimensional upper half-space, given by

$$\mathbf{H}^3 = \{w = z+jy\ :\ z\in\mathbf{C}, y>0\},$$

where

$$z = x_1+ix_2,\ \ i^2=j^2=-1.$$

It is known that (see the upper half-space $\mathbf{H}^n$ model on p. 16) in $\mathbf{H}^3$

$$\begin{aligned}\Delta_{\mathbf{H}^3} &= -y^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial y^2}\right)+y\frac{\partial}{\partial y},\\ ds^2 &= \frac{dx_1^2+dx_2^2+dy^2}{y^2},\ \ w=z+jy, w'=z'+jy'\in\mathbb{H},\\ \cosh d_{\mathbf{H}^3}(w,w') &= \frac{|z-z'|^2+y^2+y'^2}{2yy'},\\ d_{\mathbf{H}^3}(w) &= \frac{dx_1dx_2dy}{y^3}.\end{aligned}$$

We consider the initial value problem for the wave equation in $\mathbf{H}^3$ given by

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2} u\left(t,w\right) = -\Delta_{\mathbf{H}^3} u(t,w) + u(t,w), \quad t>0, w \in \mathbf{H}^3 \\
&u\left(0,w\right) = 0, \quad \frac{\partial}{\partial t} u\left(0,w\right) = f\left(w\right), \quad w \in \mathbf{H}^3, \; f \in C^\infty\left(\mathbf{H}^3\right).
\end{aligned} \tag{1.62}$$

It is well known that the solution operator of the wave equation (1.62) is

$$u(t,w) = \frac{\sin\left(t\sqrt{\Delta_{\mathbf{H}^3} - 1}\right)}{\sqrt{\Delta_{\mathbf{H}^3} - 1}} f(w). \tag{1.63}$$

It is not difficult to see that

$$W_{\mathbf{H}^3}(t,w,w') = \frac{\sin\left(t\sqrt{\Delta_{\mathbf{H}^3} - 1}\right)}{\sqrt{\Delta_{\mathbf{H}^3} - 1}}(w,w') = \frac{\delta\left(\rho - t\right)}{4\pi \sinh\, t}$$

is the wave kernel in $\mathbf{H}^3$, where $\rho = d(w,w')$ is the distance between $w, w' \in \mathbf{H}^3$, and δ is the Dirac delta function. Indeed noting that

$$\frac{\delta\left(\rho - t\right)}{4\pi \sinh t} = \frac{\delta\left(\rho - t\right)}{4\pi \sinh \rho},$$

and using the radial part $\Delta_{\mathbf{H}^3}$, we see that

$$\begin{aligned}
\left(\Delta_{\mathbf{H}^3} - 1\right) \frac{\delta\left(\rho - t\right)}{\sinh \rho} &= -\frac{1}{\sinh^2 \rho} \frac{\partial}{\partial \rho} \left[\delta'(\rho - t) \sinh \rho - \delta(\rho - t)\cosh \rho\right] - \frac{\delta\left(\rho - t\right)}{\sinh \rho} \\
&= -\frac{\delta''\left(\rho - t\right)}{\sinh \rho} = -\frac{\partial^2}{\partial t^2} \frac{\delta\left(\rho - t\right)}{\sinh \rho}.
\end{aligned}$$

Thus, we obtain the solution of the wave equation in $\mathbf{H}^3$, namely

$$u(t,w) = \frac{1}{4\pi \sinh\, t} \int_{\rho = t} f(w')\, d\mu_{\mathbf{H}^3}(w'). \tag{1.64}$$

We now proceed to the solution of the wave equation in $\mathbf{H}$:

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2} v\left(t,w\right) = -\left(\widetilde{\Delta} - \frac{1}{4}\right) v\left(t,w\right) \quad (0,\infty) \times \mathbf{H} \\
&v\left(0,w\right) = 0, \quad \frac{\partial}{\partial t} v\left(0,w\right) = f\left(w\right), \quad f \in C^\infty(\mathbf{H}).
\end{aligned} \tag{1.65}$$

As we have mentioned earlier we use the method of descent to deduce the solution of the wave equation in $\mathbf{H}$ from the solution of the wave equation in $\mathbf{H}^3$. The idea is that we regard $\mathbf{H}^3$ as having the coordinates $w = (x_1, x_2, y)$, $y > 0$, and at the same time $\mathbf{H}$ can also be regarded as having the coordinates $w = (x_1, x_2, y)$, $y > 0$ with the set $\{x_2 = 0\}$. Thus the Laplacian in $\mathbf{H}$ in this coordinate becomes

$$\widetilde{\Delta} = -y^2 \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x_1^2}\right).$$

Making the substitution

$$u\left(x_1, x_2, y\right) = \sqrt{y} v\left(x_1, y\right),$$

where v is a function independent of x_2, we see that

$$\begin{aligned}
\left(\Delta_{\mathbf{H}^3}-1\right)\sqrt{y}v\left(x_1,y\right)=&\Delta_{\mathbf{H}^3}\sqrt{y}v\left(x_1,y\right)-\sqrt{y}v\left(x_1,y\right)\\
&=-y^2\frac{\partial^2}{\partial x_1^2}\sqrt{y}v\left(x_1,y\right)-y^2\frac{\partial^2}{\partial x_2^2}\sqrt{y}v\left(x_1,y\right)\\
&\quad-y^2\frac{\partial^2}{\partial y^2}\sqrt{y}v\left(x_1,y\right)+y\frac{\partial}{\partial y}\sqrt{y}v\left(x_1,y\right)-\sqrt{y}v\left(x_1,y\right)\\
&\quad+y\left[\sqrt{y}\frac{\partial v}{\partial y}+\frac{1}{2}y^{-\frac{1}{2}}v\right]-\sqrt{y}v\left(x_1,y\right)\\
&\quad+y^{\frac{3}{2}}\frac{\partial v}{\partial y}+\frac{1}{2}y^{\frac{1}{2}}v-\sqrt{y}v\\
&=\sqrt{y}\left[-y^2\frac{\partial^2 v}{\partial x_1^2}-y^2\frac{\partial^2 v}{\partial y^2}\right]-\frac{1}{4}y^{\frac{1}{2}}v\\
&=\left(\widetilde{\Delta}-\frac{1}{4}\right)\sqrt{y}v\left(x_1,y\right).
\end{aligned}$$

With this substitution, formula (1.64), which is the solution of the wave equation in $\mathbf{H}^3$ reduces to the solution of the wave equation (1.65) in $\mathbf{H}$, namely

$$v(t,w)=\frac{1}{4\pi\sqrt{y}\sinh t}\int_{d_{\mathbf{H}^3}(w,w')=t}\sqrt{y'}f(w')\,d\mu_{\mathbf{H}^3}(w'), \tag{1.66}$$

where $\rho=d_{\mathbf{H}^3}(w,w')$ is the distance between two points $w,w'\in\mathbf{H}^3$. It remains to find an explicit value for the volume element $d\mu_{\mathbf{H}^3}(w')$. By definition,

$$\begin{aligned}
\cosh\, d_{\mathbf{H}^3}(w,w')&=\frac{\left(x_1-x_1'\right)^2+\left(x_2-x_2'\right)^2+y^2+y'^2}{2yy'}\\
&=\frac{\left(x_1-x_1'\right)^2+x_2'^2+y^2+y'^2}{2yy'},\quad x_2=0;
\end{aligned}$$

$$\left(x_1-x_1'\right)^2+x_2'^2+y^2+y'^2-2yy'\cosh t=0.$$

Setting $x_1=0$ for convenience, we obtain

$$x_2'=\sqrt{2yy'\cosh\, t-x_1'^2-y^2-y'^2}.$$

From the hyperbolic metric in $\mathbf{H}^3$, we obtain

$$\begin{aligned}
d\mu_{\mathbf{H}^3}(x_1',y')&=\frac{dx_1'\,\frac{d}{dt}\left(\sqrt{2yy'\cosh t-x_1'^2-y^2-y'^2}\right)dy'}{y'^3}\\
&=\frac{dx_1'yy'\sinh t dy'}{\sqrt{2yy'\cosh t-x_1'^2-y^2-y'^2}y'^3}\\
&=\frac{y\sinh t}{\sqrt{2yy'\cosh t-x_1'^2-y^2-y'^2}}\frac{dx_1'dy'}{y'^2}.
\end{aligned}$$

Using the hyperbolic trigonometric identities

$$\cosh t=1+2\sinh^2(t/2),\qquad 2yy'\cosh t=2yy'+4yy'\sinh^2(t/2),$$

and the relation

$$\sinh^2\left(\frac{d(z,z')}{2}\right)=\frac{|z-z'|^2}{4\,\mathrm{Im}\,z\,\mathrm{Im}\,z'}=\frac{x'^2+y^2-2yy'+y'^2}{4yy'},x=0,$$

we see that

$$\begin{aligned}4yy'\sinh^2\left(\frac{t}{2}\right)-4yy'\sinh^2\frac{d(z,z')}{2}&=2yy'\cosh\,t-2yy'-x'^2-y^2+2yy'-y'^2\\&=2yy'\cosh\,t-x'^2-y^2-y'^2.\end{aligned}$$

Hence,

$$d\mu_{\mathbf{H}^3}(x_1',y')=\frac{y\sinh\,t}{\sqrt{4yy'\left(\sinh^2\left(\frac{t}{2}\right)-\sinh^2\left(\frac{d(z,z')}{2}\right)\right)}}d\mu(z').$$

We now replace the integral over a sphere in $d\mu_{\mathbf{H}^3}$ in (1.66) with twice the integral over the projection of this sphere to $\mathbf{H}$ to obtain

$$v(t,z)=\frac{1}{4\pi}\int_{d_{\mathbf{H}}(z,z')\le t}\frac{f(z')}{\sqrt{\sinh^2\left(\frac{t}{2}\right)-\sinh^2\left(\frac{d(z,z')}{2}\right)}}d\mu(z'),\tag{1.67}$$

for $t>0$. We conclude that the wave kernel in $\mathbf{H}$ is given by

$$\widetilde{W}(t,\rho):=W_{\mathbf{H}}(t,\rho)=\frac{\sin\left(t\sqrt{\widetilde{\Delta}-\frac{1}{4}}\right)}{\sqrt{\widetilde{\Delta}-\frac{1}{4}}}(z,z')=\frac{1}{4\pi}\frac{1}{\sqrt{\sinh^2\left(\frac{t}{2}\right)-\sinh^2\left(\frac{\rho}{2}\right)}},\quad t>0.\tag{1.68}$$

1.6 The Poincaré Unit Ball Model of the Hyperbolic Space

For $n\ge 2$, the Poincaré unit ball $\mathbf{D}^n$ is the set

$$\mathbf{D}^n=\{x\in\mathbf{R}^n\,:\,|x|<1\},$$

with the Poincaré metric

$$ds^2=4\left(1-|x|^2\right)^{-2}|dx|^2.$$

The volume element on $\mathbf{D}^n$ is

$$d\mu_{\mathbf{D}^n}(x)=2^n\left(1-|x|^2\right)^{-n}dx_1\cdots dx_n.$$

The unit ball $\mathbf{D}^n$ has a boundary

$$\partial\mathbf{D}^n=\mathbf{S}^{n-1}=\{x\in\mathbf{R}^n\,:\,|x|=1\}.$$

One refers to points $\omega\in\partial\mathbf{D}^n$ as points at infinity. The Laplace-Beltrami operator in this model is given by

$$\Delta_{\mathbf{D}^n}=-\frac{\left(1-|x|^2\right)^2}{4}\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}-2(n-2)\left(1-|x|^2\right)\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}.\tag{1.69}$$

The non-Euclidean (hyperbolic) distance $d(x,x')$, $x,x' \in \mathbf{D}^n$, generated by the metric ds^2 on $\mathbf{D}^n$ has the form

$$\cosh d(x,x') = 1 + \frac{2|x-x'|^2}{(1-|x|^2)\,(1-|x'|^2)}.$$

Noting that

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} = r\frac{\partial}{\partial r},$$

we have, in geodesic polar coordinates,

$$\Delta_{\mathbf{D}^n} = -\frac{1}{4}\frac{\left(1-r^2\right)^n}{r^{n-1}}\frac{\partial}{\partial r}\left[\frac{r^{n-1}}{\left(1-r^2\right)^{n-2}}\frac{\partial}{\partial r}\right] - \frac{1}{4}\frac{\left(1-r^2\right)^2}{r^2}\left(\frac{\partial^2}{\partial\vartheta^2} + (n-2)\cot\vartheta\frac{\partial^2}{\partial\vartheta}\right) \tag{1.70}$$

and

$$d\mu_{\mathbf{D}^n} = \frac{2^n r^{n-1}}{\left(1-r^2\right)^{n-1}}\,drd\nu_{n-1}.$$

1.7 The Poisson Summation Formula as a Trace Formula

Prior to the explicit computation of the Selberg trace formula, we recall the Poisson summation formula which provided the basic concept that Selberg initially sought to generalise.

Now, let $\mathcal{S}(\mathbf{R}^n)$ be the Schwartz space of all infinitely differentiable functions f on $\mathbf{R}^n$ such that for any integer $N \geq 0$ and multiindex $\alpha = (\alpha_1, ..., \alpha_n)$, we have

$$\left(1+|x|^2\right)^N (D^\alpha f)(x) < \infty,$$

where $D^\beta = (-i)^{\alpha_1+\ldots+\alpha_n}\left(\partial_{x_1}\right)^{\alpha_1}\cdots\left(\partial_{x_n}\right)^{\alpha_n}$.

Suppose $f \in \mathcal{S}(\mathbf{R}^n)$. Let $L^2(\mathbf{R}^n/\mathbf{Z}^n)$ be the space of square integrable functions on the torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. Define an integral operator $\mathscr{T}_g$ acting on $g \in L^2(\mathbf{R}^n/\mathbf{Z}^n)$ by the convolution

$$\left(\mathscr{T}_f g\right)(x) = (f*g)(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy.$$

Since the exponential $\phi_m(x) = e^{2\pi i(x\cdot m)}$, $m \in \mathbf{Z}^n$, forms a complete orthonormal set of eigenfunctions of the Laplacian

$$\Delta_{\mathbf{T}^n} = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$$

on $L^2(\mathbf{R}^n/\mathbf{Z}^n)$ with the corresponding eigenvalue $\lambda_m = 4\pi^2|m|^2$, $\phi_m(x)$ are also eigenfunctions of the integral operator $\mathscr{T}_f$ with eigenvalue $\widehat{f}(m)$:

$$\mathscr{T}_f\phi_m(x) = \left(f*\phi_m\right)(x) = \int_{\mathbf{R}^n} f(x-y)\phi_m(y)\,dy = \widehat{f}(m)\phi_m(x),$$

where

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i(x\cdot\xi)}\,dx, \qquad \xi \in \mathbf{R}^n, \tag{1.71}$$

is the Euclidean Fourier transform of f, with the Fourier inversion formula

$$f(x) = \int_{\mathbf{R}^n} \widehat{f}(\xi)e^{2\pi i(x\cdot\xi)}\,dx, \qquad x \in \mathbf{R}^n. \tag{1.72}$$

Thus, if $\mathscr{T}_f$ is considered as an integral operator on the infinite dimensional Hilbert space $L^2(\mathbf{R}^n/\mathbf{Z}^n)$, its trace must be the infinite sum

$$\operatorname{tr} \mathscr{T}_f = \sum_{m \in \mathbf{Z}^n} \widehat{f}(m).$$

By Mercer's Theorem A.10, we can write the operator $\mathscr{T}_f$ as an integral operator over $\mathbf{R}^n/\mathbf{Z}^n$ with the kernel

$$K_f(x,y) = \sum_{m \in \mathbf{Z}^n} f(x-y-m).$$

Then, since

$$\mathbf{R}^n = \bigcup_{m \in \mathbf{Z}^n} (m + [0,1)^n) = \text{ a disjoint union,}$$

we have

$$(\mathscr{T}_f g)(x) = \int_{\mathbf{R}^n/\mathbf{Z}^n} K_f(x,y) g(y)\, dy.$$

Again by Mercer's Theorem A.10,

$$\operatorname{tr} \mathscr{T}_f = \int_{\mathbf{R}^n/\mathbf{Z}^n} K_f(x,x)\, dx = \int_{\mathbf{R}^n/\mathbf{Z}^n} \sum_{m \in \mathbf{Z}^n} f(x-x-m)\, dx = \sum_{m \in \mathbf{Z}^n} f(m).$$

In summary we have the following statement.

Theorem 1.19 (**Poisson summation formula**).

$$\sum_{m \in \mathbf{Z}^n} \widehat{f}(m) = \sum_{m \in \mathbf{Z}^n} f(m), \tag{1.73}$$

for a Schwartz function $f : \mathbf{R}^n \to \mathbf{C}$.

The basic idea of computing the trace of an appropriately-chosen integral operator in two different ways shall prove quite fruitful in the development of more sophisticated versions of 'the trace formula' - the Selberg trace formula.

1.8 The Selberg Trace Formula

Selberg theory is that area of spectral theory whose foundation consists of (Venkov [172, 173])

- theorems on expansion in automorphic eigenfunctions of the Laplacians defined on symmetric Riemann spaces and studying their spectra;
- the theory of Eisenstein series;
- the Selberg trace formula; and
- the theory of the Selberg zeta function.

The starting point in this direction is the celebrated Selberg trace formula. The central result in the spectral theory of hyperbolic surfaces is the Selberg trace formula. It is a formula that shows the equality of the spectrum of the Laplacian on a hyperbolic surface and the length spectrum of the surface. In physical terms, it can be viewed as a connection between quantum and classical mechanics, with the quantum side representing the Laplace spectrum (spectral theory) and classical mechanics representing the length spectrum (length of closed geodesics) on the surface. The Selberg trace formula (Selberg [147]) was originally introduced by Atle Selberg as an arithmetical relation, being a noncommutative generalisation of the Poisson summation formula, and it is as such used in number theory and harmonic analysis (see e.g. McKean [108], Hejhal [74, 75, 76], Terras [167], Buser [32], Iwaniec [87], Deitmar and Echterhoff [48], Müller [119, 120]). In general, the Laplace spectrum and the length spectrum are both defined on a configuration space of the form $\Gamma\backslash G/K$, where G is a noncompact semisimple Lie group, Γ a cofinite discrete subgroup of G and K a maximal compact subgroup of G (Faraut [58]). When Γ is a cocompact subgroup $\mathbf{Z}$ of the real number $G = \mathbf{R}$, the Selberg trace formula is essentially the Poisson summation formula. The case when $\Gamma\backslash G/K$ is not compact is harder because the spectrum is no longer purely discrete, there is a continuous spectrum which is described by Eisenstein series (see Section 2.2). The quotient $\mathscr{X} = G/K$ is a covering (symmetric) space of the Lie group G.

In this thesis we consider both compact and noncompact hyperbolic surfaces $\mathscr{M} = \Gamma\backslash G/K$, where Γ is the Fuchsian group of the first kind (i.e., a discrete subgroup of G), $G = SL(2,\mathbf{R})$ and $K = \mathbf{SO}(2)$. By the Iwasawa decomposition (Subsection 2.1.2) the Poincaré upper half-plane $\mathbf{H}$ can be identified with the quotient $SL(2,\mathbf{R})/\mathbf{SO}(2)$, and we say that $SL(2,\mathbf{R})/\{\pm I\}$ is the group of orientation-preserving isometries of $\mathbf{H}$; this identification realises the upper half-plane $\mathbf{H}$ as a symmetric space of noncompact type. In this case the Selberg trace formula describes the spectrum of the Laplacian on $\mathscr{M}$ in terms of geometric data involving the lengths of closed geodesics on $\mathscr{M}$, and also the trace formula is similar to the explicit formula relating the zeros of the Riemann zeta function to prime numbers (Titchmarsh [170]), with the zeta zeros corresponding to eigenvalues of the Laplacian, and the primes corresponding to closed geodesics (Hejhal [74], Randol [134]). Motivated by the analogy, Selberg introduced the Selberg zeta function of a hyperbolic surface whose analytic properties are encoded by the Selberg trace formula (Subsection 3.4.2).

The Selberg zeta function (see also Selberg [147]) is analogous to the famous Riemann zeta function (Titchmarsh [170])

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p\in\mathbf{P}} \frac{1}{1-p^{-s}}, \quad \operatorname{Re} s > 1,$$

where $\mathbf{P}$ is the set of prime numbers. The Selberg zeta function uses the lengths of closed geodesics instead of the prime numbers. If Γ is a subgroup of $SL(2,\mathbf{R})$, then the Selberg zeta function which is a meromorphic function defined in the complex plane by

$$Z(s) = \prod_{p} \left(1 - N(p)^{-s}\right)^{-1}, \quad \text{or} \quad Z(s) = \prod_{p}\prod_{n=0}^{\infty} \left(1 - N(p)^{-s-n}\right), \quad \operatorname{Re} s > 1,$$

where p runs over the prime congruent class and $N(p)$ is the norm of the congruent class p. For any hyperbolic surface of finite area there is an associated Selberg zeta function. The Selberg zeta function is defined in terms of the spectra of the surface.

The special case of the modular surface $\mathscr{M} = SL(2,\mathbf{Z})\backslash\mathbf{H}$ is of special interest in number theory. The spectral theory of the modular surface $SL(2,\mathbf{Z})\backslash\mathbf{H}$ is deeply important and remains exceedingly mysterious; the main tool used to study this theory is the Selberg trace formula.

Trace formulae are also crucial for understanding finer points about the decomposition of the spectrum of the Laplacian on hyperbolic surfaces, such as Weyl's law. For the special case $SL(2,\mathbf{Z})\backslash\mathbf{H}$, the Selberg zeta function is intimately connected to the Riemann zeta function. In this case the determinant of the scattering matrix is given by (see Section 2.2)

$$\varphi(s) = \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)},$$

which is a consequence of the presence of the continuous spectrum of the Laplacian on $SL(2,\mathbf{Z})\backslash\mathbf{H}$. The determinant of the scattering matrix $\varphi(s)$ is a function appearing in the constant term of the Fourier expansion of Eisenstein series. In particular, we see that if the Riemann zeta function has a zero at s_0, then the determinant of the scattering matrix has a pole at $s_0/2$, and hence the Selberg zeta function has a zero at $s_0/2$.

1.9 Outline and Organisation of the Thesis

This thesis consists of six chapters including this introductory chapter. Chapters 2 and 3 discuss the spectral theory of the Laplacian on compact and noncompact hyperbolic surfaces $\mathscr{M} = \mathit{\Gamma}\backslash\mathbf{H}$ (where $\mathit{\Gamma}$ is a discrete subgroup of $SL(2,\mathbf{R})$ and $\mathbf{H}$ the Poincaré upper half-plane), while Chapters 4, 5, and 6 are devoted to spectral functions of the Laplacians on n-dimensional non-Euclidean spaces. We outline in detail the plan and organisation of the thesis.

Chapter 2. We present some basic concepts in the study of spectral theory of hyperbolic surfaces which are needed for the computations of the trace formulae in Chapter 3 and in the remaining chapters. To be precise, in Section 2.1 we discuss the basics of the action of the group $\widetilde{G} = PSL(2,\mathbf{R}) = SL(2,\mathbf{R})/\{\pm I\}$ (the group of orientation-preserving isometries of the upper half-plane $\mathbf{H}$) on $\mathbf{H}$. It turns out that the quotient $\mathscr{M} = \mathit{\Gamma}\backslash\mathbf{H}$ is a Riemann surface, and when it is endowed with negative Gaussian curvature -1, then it is a hyperbolic surface. We also discuss an important discrete subgroup of $SL(2,\mathbf{R})$, the modular group $SL(2,\mathbf{Z})$ as well as the modular surface $SL(2,\mathbf{Z})\backslash\mathbf{H}$. Since the modular surface $\mathscr{M} = SL(2,\mathbf{Z})\backslash\mathbf{H}$ is noncompact, the continous spectrum of the Laplaian on $\mathscr{M}$ is well studied in terms of Eisenstein series; we discuss these Eisenstein series and establish their relevant properties in Section 2.2.

Chapter 3. Chapter 3 is devoted to the trace formulae for hyperbolic surfaces $\mathscr{M}$ and their applications. Let $K(z,z')$, $z,z'\in\mathbf{H}$, be a $\mathit{\Gamma}$-automorphic kernel defined by

$$K(z,z') = \sum_{\gamma\in\mathit{\Gamma}} k(z,\gamma z') = \widetilde{k}\left[u(z,\gamma z')\right], \qquad u(z,z') = \frac{|z-z'|^2}{\operatorname{Im} z \operatorname{Im} z'}, \tag{1.74}$$

and let $f\in L^2(\mathscr{M})$, f is automorphic, i.e., $f(\gamma z) = f(z)$ for all $\gamma\in\mathit{\Gamma}$, $z\in\mathbf{H}$. The kernel $k(z,z')$ is called a *point-pair invariant* (see Definition (3.1)). In Section 3.1 we derive the trace formula for a compact hyperbolic surface (see Theorem 3.8) by decomposing $\mathit{\Gamma}$ into conjugacy classes and then taking the trace of the automorphic kernel defined by (1.74) (which includes the identity and hyperbolic elements of $\mathit{\Gamma}$) in two ways. The left-hand side of the formula is the contribution of the Laplace spectrum, a spectral quantity, while the right-hand side is the contribution of the length spectrum, a geometric quantity, which is the sequence of the lengths of closed geodesics (see Definitions 2.20 and 2.21). A consequence of the trace formula for a compact hyperbolic surface is that the length spectrum and Laplace (eigenvalue) spectrum are equivalent geometric quantities (see Buser [32, p. 229]). In order to derive the trace formula for a noncompact hyperbolic surface $\mathscr{M}$ (see Theorems 3.16-3.18), we must compute the spectral expansion of $f \in L^2(\mathscr{M})$, and this is known as the *Selberg spectral (expansion) decomposition* of automorphic functions $f \in L^2(\mathscr{M})$, which decomposes $f \in L^2(\mathscr{M})$ into a sum involving the contribution of the discrete spectrum plus an integral involving the contribution of continous spectrum of $\mathscr{M}$ (see Subsection 3.2.1, (3.24)). The trace formula then arises by computing the trace of an automorphic kernel $K(z, z')$, with the corresponding discrete subgroup $\mathit{\Gamma}$ having parabolic elements since $\mathscr{M}$ is noncompact; and we take care of the contribution of the continuous spectrum (in terms of the Eisenstein series) to obtain the trace formula for a noncompact hyperbolic surface $\mathscr{M}$. We follow Selberg [148] in evaluating the trace, without using the Poisson summation formula or the Euler-Maclaurin summation formula of Kubota [97]. In Section 3.3, we compute the general case of the Selberg spectral expansion formula for functions $f \in \mathscr{M}$, namely the Parseval inner product formula for nonholomorphic Eisenstein series; this is given in Theorem 3.19.

An important application of the trace formula is in the computation of the determinant of the Laplacian. The determinant of the Laplacian on a compact Riemann surface is computed in Blau and Clements [29], D'Hoker and Phong [49], Sarnak [143], Steiner [156] and Voros [175], while Randol [135, 136, 137] discuss some other applications of the trace formula for a compact surface $\mathscr{M}$. The much more difficult noncompact case, in which the Laplacian consists of both discrete and continuous spectrum is considered by Efrat [56], Koyama [95], and Momeni and Venkov [114].

Let $(\lambda_k : k \geq 0)$ be the discrete spectrum of the Laplacian $\widetilde{\Delta}$ on a hyperbolic surface $\mathscr{M}$ satisfying

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots ; \, \lambda_k \nearrow \infty.$$

Then for $\operatorname{Re} w > 1$ we define the Minakshisundaram-Pleijel zeta function $\zeta_{\mathscr{M}}(w)$ by

$$\zeta_{\mathscr{M}}(w) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^w}. \tag{1.75}$$

By the Weyl's asymptotic law (2.15), $\zeta_{\mathscr{M}}(w)$ can be meromorphically continued to all $w \in \mathbf{C}$ with a simple pole at $w = 1$ (Randol [135]). In particular, $\zeta_{\mathscr{M}}(w)$ is holomorphic at $w = 0$.

Now differentiating (1.75) with respect to w we have, for $\operatorname{Re} w > 1$,

$$\frac{d}{dw}\zeta_{\mathscr{M}}(w) = \frac{d}{dw}\sum_{k=1}^{\infty} e^{-w \log \lambda_k} = -\sum_{k=1}^{\infty} \frac{\log \lambda_k}{\lambda_k^w},$$

and in paticular

$$\left.\frac{d}{dw}\zeta_{\mathscr{M}}(w)\right|_{w=0} = -\sum_{k=1}^{\infty} \log \lambda_k = -\log \prod_{k=1}^{\infty} \lambda_k.$$

Formally,

$$\det{}'\widetilde{\Delta} = \prod_{k=1}^{\infty} \lambda_k,$$

where the prime $'$ means that $\lambda_0 = 0$ has been omitted. Since $\zeta_{\mathscr{M}}(w)$ is holomorphic at $w = 0$, we can define the determinant of the Laplacian $\widetilde{\Delta}$ by

$$\det{}'\widetilde{\Delta} = \exp\left(-\frac{d}{dw}\zeta_{\mathscr{M}}(w)\Big|_{w=0}\right). \tag{1.76}$$

Remark 1.6. *The infinite product*

$$\prod_{k=1}^{\infty}\lambda_k = \exp\left(-\frac{d}{dw}\zeta_{\mathscr{M}}(w)\Big|_{w=0}\right)$$

does not converge, but $\zeta_{\mathscr{M}}(s)$ has an analytic continuation to a neighbourhood of $s = 0$. Therefore, definition (1.76) makes sense.

We now define for $\operatorname{Re} w > 1$, $s \in \mathbf{R}$, $s > 1$, the generalised Minakshisundaram-Pleijel zeta function $\zeta_{\mathscr{M}}(w; s)$ by

$$\zeta_{\mathscr{M}}(w; s) = \sum_{k=0}^{\infty}\frac{1}{(\lambda_k - s(1-s))^w}.$$

As before,

$$\frac{d}{dw}\zeta_{\mathscr{M}}(w; s)\Big|_{w=0} = -\sum_{k=1}^{\infty}\log\left(\lambda_k - s(1-s)\right),$$

and

$$\det\left(\widetilde{\Delta} - s(1-s)\right) = \prod_{k=0}^{\infty}\left(\lambda_k - s(1-s)\right). \tag{1.77}$$

Thus

$$\det\left(\widetilde{\Delta} - s(1-s)\right) = \exp\left(-\frac{d}{dw}\zeta_{\mathscr{M}}(w; s)\Big|_{w=0}\right).$$

As we have earlier mentioned the determinant of the Laplacian on a compact hyperbolic surface has been discussed by Blau and Clements [29], D'Hoker and Phong [49], Sarnak [143], Steiner [156], and Voros [175]. Their basic result using different approaches is the following statement.

Theorem 1.20. *Let Γ be a discrete subgroup of $SL(2, \mathbf{R})$ such that $\mathscr{M} = \Gamma\backslash\mathbf{H}$ is compact, and $\widetilde{\Delta}$ the Laplacian on $\mathscr{M} = \Gamma\backslash\mathbf{H}$. Then*

$$\det\left(\widetilde{\Delta} - s(1-s)\right) = \widetilde{Z}(s)Z_{\mathcal{I}}(s),$$

where

$$\widetilde{Z}(s) = e^{\widetilde{c}}Z(s), \quad \widetilde{c} = \frac{\mu(\Gamma\backslash\mathbf{H})}{2\pi}\left(-\frac{1}{2}\log 2\pi - \frac{1}{4} + 2\zeta'(-1) - s(s-1)\right).$$

In particular,

$$\det{}'\left(\widetilde{\Delta}\right) = e^{\widehat{c}}Z'(1), \quad \widehat{c} = \frac{\mu(\Gamma\backslash\mathbf{H})}{2\pi}\left(\frac{1}{2}\log 2\pi - \frac{1}{4} + 2\zeta'(-1)\right),$$

where $Z(s)$ is the Selberg zeta function given by (3.103) and $Z_{\mathcal{I}}(s)$ is the zeta function associated to the identity contribution in the trace formula (3.20) and is given by (3.119).

The noncompact version of Theorem 1.20 is given by Efrat [56] (see also Momeni and Venkov [114]), and their result is the following statement.

Theorem 1.21. *Let Γ be a Fuchsian group of the first kind such that $\mathscr{M} = \Gamma\backslash\mathbf{H}$ is noncompact, and $\widetilde{\Delta}$ the Laplacian on $\mathscr{M} = \Gamma\backslash\mathbf{H}$. Then*

$$det(\widetilde{\Delta} - s(1-s)) = \Phi(s)Z(s)^2 Z_{\mathcal{I}}(s)^2 \Gamma\left(s+\frac{1}{2}\right)^{-2}(2s-1)^{1-\Phi(\frac{1}{2})}e^{B(2s-1)^2-(2s-1)\log 2+D},$$

where $\Phi(s)$ is the scattering determinant and B, D are computable constants.

The case of a congruence subgroup of $SL(2,\mathbf{Z})$ is given by Koyama [95], namely

Theorem 1.22. *Let Γ be a congruence subgroup of $SL(2,\mathbf{Z})$ and let $\widetilde{\Delta}$ be the Laplacian on $\mathscr{M} = \Gamma\backslash\mathbf{H}$. Then*

$$det(\widetilde{\Delta}, s) = det_D(\widetilde{\Delta} - s(1-s))det_C(\widetilde{\Delta}, s),$$

where

$$det_C(\widetilde{\Delta}, s) = \left(s - \frac{1}{2}\right)^{-\frac{K_0}{2}} A^s \pi^{-s}\Gamma(s)\prod_{\chi} L(2s,\chi) \tag{1.78}$$

is the regularized determinant related to the continuous spectrum of the Laplacian $\widetilde{\Delta}$, and

$$det_D(\widetilde{\Delta} - s(1-s)) = \exp\left(-\frac{\partial}{\partial w}\zeta_{\mathscr{M}}(w;s)\Big|_{w=0}\right)$$

is the regularized determinant related to the discrete spectrum of $\widetilde{\Delta}$.

The notation in (1.78) is as follows: A is a positive integer composed of the primes dividing the positive integer N, $K_0 = \varphi\left(\frac{1}{2}\right)$ and $L(s,\chi)$ is the Dirichlet L-function (see e.g. Bump [31], Terras [167]). In this case, the determinant of the scattering matrix $\varphi(s)$ is given by

$$\varphi(s) = (-1)^{(1-K_0)/2}\frac{\Gamma(1-s)}{\Gamma(s)}\left(\frac{A}{\pi}\right)^{1-2s}\prod_{\chi}\frac{L(2-2s,\overline{\chi})}{L(2s,\chi)}.$$

It is the purpose of Subsection 3.4.3 to apply the trace formula for the modular surface $SL(2,\mathbf{Z})\backslash\mathbf{H}$ to prove Theorem 3.27, which is a special case of Efrat [56], Momeni and Venkov [114].

Chapter 4. From this chapter on we concentrate strictly on symmetric Riemannian manifolds of the form $\mathscr{X} = G/K$, where G is a connected semisimple Lie group and K a maximal compact subgroup of G. We give explicit expressions for the Poisson kernels on the unit sphere $\mathbf{S}^n$, the Euclidean ball $\mathbf{B}^n$; the real upper half-space $\mathbf{H}^n$; and the hyperbolic unit ball $\mathbf{D}^n$. Section 4.1 computes eigenfunctions of the Laplacian in the hyperbolic space $\mathbf{H}^n$. In Section 4.2 we determine the Poisson kernel on the real upper half-space $\mathbf{H}^n$, while Section 4.3 is devoted to the Poisson integral formula for the unit sphere $\mathbf{S}^n$. Since a Riemannian manifold is locally Euclidean, we obtain by asymptotics the Poisson kernel on the Euclidean ball $\mathbf{B}^n$ from the non-Euclidean one, this is done in Section 4.4; in the same section various useful identities in the context of special functions are established. Section 4.5 computes the Poisson integral formula for the hyperbolic unit ball $\mathbf{D}^n$.

In summary, in Sections 4.2-4.5 we obtain the following statement.

Theorem 1.23. *Let $\mathscr{X} = \mathbf{B}^n$, $\mathbf{D}^n$, $\mathbf{S}^n$ or $\mathbf{H}^n$. Let $\Delta_{\mathscr{X}}$ be the Laplacian on $\mathscr{X}$, and* `P` *a harmonic function of class C^2 on $\mathscr{X}$. The spherical harmonic expansion of* `P` *is given by the* *spectral sum*

$$\mathtt{P}_{\mathscr{X}}(r\zeta,\zeta') = \sum_{k=0}^{\infty} \mathbb{F}_k(r) M_k^{n-1} r^k \mathscr{C}_k^{\frac{n-2}{2}}(\zeta\cdot\zeta'), \qquad \zeta,\zeta' \in \mathbf{S}^{n-1}, r>0,$$

- $(\mathscr{X} = \mathbf{B}^n)$

$$\mathbb{F}_k(r) = 1;$$

- $(\mathscr{X} = \mathbf{D}^n)$

$$\mathbb{F}_k(r) = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(k+n-1)}{\Gamma\left(k+\frac{n}{2}\right)\Gamma(n-1)} F\left(k,-\frac{n}{2}+1;k+\frac{n}{2};r^2\right)$$

satisfying

$$\mathbb{F}_k(1) = 1;$$

- $(\mathscr{X} = \mathbf{S}^n)$

$$\mathbb{F}_k(r) := \mathbb{F}_k(\varrho,\widetilde{\varrho}) = \frac{\tan^k\left(\frac{\widetilde{\varrho}}{2}\right) F\left(k,1-\frac{n}{2};k+\frac{n}{2};-\tan^2\left(\frac{\widetilde{\varrho}}{2}\right)\right)}{\tan^k\left(\frac{\varrho}{2}\right) F\left(k,1-\frac{n}{2};k+\frac{n}{2};-\tan^2\left(\frac{\varrho}{2}\right)\right)},$$

with $0<\widetilde{\varrho}<\varrho<\infty$, $\widetilde{\varrho} = d(\zeta,\zeta')$;

- $(\mathscr{X} = \mathbf{H}^n)$

$$\mathbb{F}_k(r) : = \mathbb{F}_k(\rho,\widetilde{\rho}) = \frac{\tanh^k\left(\frac{\widetilde{\rho}}{2}\right) F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\widetilde{\rho}}{2}\right)\right)}{\tanh^k\left(\frac{\rho}{2}\right) F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\rho}{2}\right)\right)},$$

with $0<\widetilde{\rho}<\rho<\infty$, $\widetilde{\rho} = d(w,w')$.

The Poisson kernels which are expressed in terms of infinite series involving the hypergeometric function, Legendre and Gegenbauer polynomials, lead to different identities in the context of special functions. In fact letting the hyperbolic distance tend to infinity in the Poisson kernel on $\mathbf{H}^n$ gives the Poisson kernel on $\mathbf{D}^n$ (see Theorem 4.12). The hyperbolic Poisson kernel is also considered in Symeonidis [161, 162] using different methods. See also Cammarota and Orsingher [36], Byczkowski and Małecki [34], Byczkowski et al. [33] for the hyperbolic Poisson kernels in the context of Brownian motion.

Chapter 5. This chapter discusses fractional and integral representations of the heat kernels on compact symmetric spaces $\mathbf{S}^n$, $\mathbf{RP}^n$ and $\mathbf{CP}^n$, as well as explicit computations of the Minakshisundaram-Pleijel heat coefficients on $\mathbf{S}^n$. The Gegenbauer transform approach (Section 5.1) is used to compute a series representation formula (involving the Gegenbauer polynomial) for the heat kernel on $\mathbf{S}^n$. The series representation is then transformed into fractional and integral representations, using the Riemann-Liouville fractional derivative formula; this is contained in Section 5.2 (see Theorem 5.3). According to Minakshisundaram and Pleijel [111], the asymptotic expansion (as $t \searrow 0$) of the heat kernel K_M on an arbitrary n-dimensional compact Riemannian manifold M satisfies the asymptotic expansion

$$\int_M K_M(t,x,x)\,\mathrm{dVol}(x) = \operatorname{tr} e^{-t\Delta_M} = \frac{1}{(4\pi t)^{\frac{n}{2}}}\left(a_0^n + a_1^n t + a_2^n t^2 + \cdots + a_k^n t^k + O\left(t^{k+1}\right)\right), \tag{1.79}$$

where a_k^n are the Minakshisundaram-Pleijel heat coefficients which give geometric information about the manifold. For the computations of the Minakshisundaram-Pleijel heat coefficients a_k^n on an arbitrary compact Riemannian manifold see e.g., Berger et al. [26], Rosenberg [142], Chavel [41], Gilkey [63]), Craioveanu et al. [45, Sec. 4.1], Rosenberg [142, p. 107]. Here using Jacobi's theta functions we give a precise and relatively simple description of these coefficients for the Euclidean spheres $\mathbf{S}^n$ with $n \geq 1$. Apart from giving a description of the heat trace coefficients in this case, we follow Cahn and Wolf [35] to purely expand the trace of the heat operators on the spheres $\mathbf{S}^n$ $(n \geq 1)$ in terms of Jacobi's theta functions and their higher order derivatives; these are contained in Sections 5.3 (see Theorem 5.5). In section 5.4 using integral representations of spectral functions on the complex projective spaces $\mathbf{CP}^n$ $(n \geq 1)$ we present integral representations of $\mathbf{CP}^n$ (see Theorems 5.7 and 5.8). Finally in Section 5.5 we express the traces of the heat kernels on $\mathbf{S}^n$, $\mathbf{RP}^n$ and $\mathbf{CP}^n$ in terms of the Euclidean Poisson kernel.

Chapter 6. Having computed eigenfunctions of the Laplacian in $\mathbf{H}^n$ in Section 4.1, we give integral representation formulae for the generalised spherical functions in $\mathbf{H}^n$ in Section 6.1; a spherical function is a radial and normalised eigenfunction of the Laplacian which has the value one at the origin. We also obtain integral representation formulae for the general eigenfunctions in $\mathbf{H}^n$, which can simultaneously be viewed as integral transforms of harmonic functions defined in the Euclidean unit ball (see Theorem 6.3). In Section 6.2 we obtain the integral heat kernel in $\mathbf{H}^n$ using the hyperbolic wave equation; the main tool that relates the solution of the wave equation to that of the heat equation is the Euclidean Fourier transform. Section 6.3 is devoted to Green function in $\mathbf{H}^n$. Lastly, using the Green function and the spectral theory for a self-adjoint operator we establish the generalisation of the classical Mehler-Fock integral formula. We thereafter extend the Mehler-Fock formula to the heat kernel in $\mathbf{H}^n$ by appropriately choosing a spectral (test) function (see Theorem 6.7). We also establish the Mehler-Fock inversion formula via the hyperbolic Poisson kernel (see Theorem 6.8).

Appendix. Appendix A discusses the basic elements of Riemannian geometry, namely the Laplacian on Riemannian manifolds, the heat kernel on Riemannian manifolds and the spectral theorem, while Appendix B contains special functions that are needed in this thesis.

Other material that is related to the topic of this thesis but is not presented here, is contained in Awonusika and Taheri [8, 15] (see also Awonusika and Taheri [7, 9, 10, 11, 12, 13, 14, 16], Awonusika [5, 6]).

Chapter 2

The Spectrum and Geometry of Hyperbolic Surfaces

In this chapter we present some basic concepts in the study of non-Euclidean (hyperbolic) harmonic analysis, which are needed to understand the rest of the topics in this thesis. We shall discuss the basics of the action of the group $\widetilde{G} = PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{\pm I\}$, where $G = SL(2, \mathbf{R})$ is a group of 2×2 real matrices with determinant 1, and I is a 2×2 identity matrix. We shall also construct the upper half-plane $\mathbf{H}$ as a symmetric space $\mathbf{H} = G/K$, where $K = \mathbf{SO}(2)$ is the maximal compact subgroup of G; this construction is known as the *Iwasawa decomposition* (Subsection 2.1.2).

Let $\Gamma \subset G$ be a Fuchsian group of the first kind (Subsection 2.1.3). The quotient $\mathscr{M} = \Gamma\backslash G/K$ is a locally symmetric space. Let $\mathcal{D}_\Gamma \subset \mathbf{H}$ be a fundamental domain of Γ. Then

$$\mu(\mathscr{M}) = \int_{\mathcal{D}_\Gamma} d\mu < \infty,$$

where $d\mu$ is the area form attached to the metric ds^2 on G/K. So $\mathscr{M}$ is a locally symmetric space of finite volume. The G-invariant Riemannian metric ds^2 on G/K induces a canonical Riemannian metric ds^2 on $\mathscr{M}$. The area of $\mathscr{M}$ with respect to this metric is finite. Then the quotient $\mathscr{M} = \Gamma\backslash G/K$ is a two-dimensional Riemannian manifold (called Riemann surface), whose simply connected covering manifold is the symmetric space $\mathbf{H} = G/K$. Since the upper half-plane $\mathbf{H}$ is equipped with a Riemannian metric of constant negative curvature, the Riemann surface $\Gamma\backslash\mathbf{H}$ is called a *hyperbolic surface*.

Finally we discuss the nonholomorphic Eisenstein series as an example of automorphic forms for the modular group $SL(2, \mathbf{Z})$, the group of 2×2 matrices with entries in $\mathbf{Z}$ and determinant 1, or what is the same, we treat the nonholomorphic Eisenstein series $E(z, s)$, $z \in \mathbf{H}$, $s \in \mathbf{C}$, as an $SL(2, \mathbf{Z})$-invariant eigenfunction of the Laplacian on the hyperbolic surface (modular surface) $SL(2, \mathbf{Z})\backslash\mathbf{H}$. We shall present the Fourier expansion of the nonholomorphic Eisenstein series in terms of the K-Bessel function. Then one can show through the Fourier expansion that $E(z, s)$ is meromorphic in the whole complex s-plane and satisfies a particular functional equation; all these properties are discussed in Section 2.2. A new material on this subject is Bergeron [27].

2.1 The Upper Half-Plane and the Group $SL(2, \mathbf{R})$

In this section we describe the action of the group $SL(2, \mathbf{R})$ on the universal covering manifold of a Riemann surface, namely the hyperbolic upper half-plane $\mathbf{H}$.

The Poincaré upper half-plane is the Riemannian manifold

$$\mathbf{H} = \{z = x + iy \in \mathbf{C} \,:\, \operatorname{Im} z = y > 0\},$$

and

$$PSL(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R}) / \{\pm I\} = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,:\, a, b, c, d \in \mathbf{R}\,; ad - bc = 1 \right\} / \{\pm I\}$$

is the projective special linear group of all 2×2 real matrices with determinant 1; it is a topological group with respect to the metric induced by the norm

$$\|\gamma\| = \left(a^2 + b^2 + c^2 + d^2\right)^{1/2}.$$

The element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{G}$$

operates on $\mathbf{H}$ by means of the fractional linear transformation

$$z \to \gamma z = \frac{az + b}{cz + d}.$$

Ordinarily we will make no distinction between the matrix γ and the transformation γz. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{G},$$

we write

$$\gamma z = \frac{az + b}{cz + d}, \;\; z \in \mathbf{H}.$$

Let $\gamma z = \omega$. For any $z \in \mathbf{H}$ and $\gamma \in \mathit{\Gamma}$,

$$\omega = \frac{az + b}{cz + d} = \frac{(az + b)(c\overline{z} + d)}{(cz + d)(c\overline{z} + d)} = \frac{acx^2 + acy^2 + bd + bcx + adx + iy}{(cx + d)^2 + c^2 y^2},$$

and so

$$\operatorname{Im} \omega = \frac{y}{|cz + d|^2}. \tag{2.1}$$

This shows that if $z \in \mathbf{H}$, then $\omega = \gamma z \in \mathbf{H}$. Moreover, the action of the group G on $\mathbf{H}$ is transitive since for any $z, i \in \mathbf{H}$, $\gamma i = z$, with

$$\gamma = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}.$$

Also,

$$\frac{d\omega}{dz} = \frac{(cz+d)a-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}. \tag{2.2}$$

Next we show that the hyperbolic area and distance in $\mathbf{H}$ are $PSL(2,\mathbf{R})$-invariant, that is, they do not change under the action of $PSL(2,\mathbf{R})$. Indeed, if A is a set in $\mathbf{H}$ and $\gamma \in PSL(2,\mathbf{R})$, then $\mu(A)$ exists and

$$\mu(\gamma(A)) = \mu(A)$$

(see [8, Proposition 2.1], and also Anderson [2, Section 5.4]). So the hyperbolic area in $\mathbf{H}$ is $PSL(2,\mathbf{R})$-invariant. We also recall that the hyperbolic plane $\mathbf{H}$ is endowed with the

$$\text{metric } ds = \frac{|dz|}{\operatorname{Im} z} \quad \text{or} \quad ds^2 = \frac{dx^2+dy^2}{y^2},\ z = x+iy \in \mathbf{H} \tag{2.3}$$

and the

$$\text{Poincaré distance } d(z_1,z_2) = \cosh^{-1}\left[1 + \frac{|z_1-z_2|^2}{2\operatorname{Im} z_1 \operatorname{Im} z_2}\right], \tag{2.4}$$

$z_1 = x_1+iy_1, z_2 = x_2+iy_2 \in \mathbf{H}$. To see that (2.3) is $\widetilde{G}$- invariant, we use (2.2) to see that

$$\frac{|d(\gamma z)|}{\operatorname{Im}\ \gamma z} = \frac{|cz+d|^2}{y}|(\gamma z)' dz| = \frac{|cz+d|^2}{y}\frac{|dz|}{|cz+d|^2} = \frac{|dz|}{y}.$$

For (2.4) we use (2.1) to get

$$\begin{aligned} d(\gamma z_1, \gamma z_2) &= \cosh^{-1}\left[1 + \frac{|\gamma z_1 - \gamma z_2|^2}{2\operatorname{Im}\gamma z_1 \operatorname{Im}\gamma z_2}\right] = \cosh^{-1}\left[1 + \frac{\left|\frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}\right|^2}{2\frac{y_1y_2}{|cz_1+d|^2|cz_2+d|^2}}\right] \\ &= \cosh^{-1}\left[1 + \frac{|z_1(ad-bc)-z_2(ad-bc)|^2}{2y_1y_2}\right] = \cosh^{-1}\left[1+\frac{|z_1-z_2|^2}{2y_1y_2}\right] \\ &= d(z_1,z_2). \end{aligned}$$

A consequence of (2.4) is the following (Beardon [22, pp. 130-132]).

Corollary 2.1. *For $z_1, z_2 \in \mathbf{H}$,*

(*i*) $d(z_1,z_2) = \ln \frac{|z_1-\overline{z_2}|+|z_1-z_2|}{|z_1-\overline{z_2}|-|z_1-z_2|}$;

(*ii*) $\sinh^2 \frac{d(z_1,z_2)}{2} = \frac{|z_1-z_2|^2}{4 Im\, z_2\ Im\, z_2}$;

(*iii*) $\cosh^2 \frac{d(z_1,z_2)}{2} = \frac{|z-\overline{z_2}|^2}{4 Im\, z_1\ Im\, z_2}$;

(*iv*) $\tanh^2 \frac{d(z_1,z_2)}{2} = \left|\frac{z_1-z_2}{z_1-\overline{z}_2}\right|^2$.

2.1.1 Classifications of Isometries of the Upper Half-Plane

The transformation $\gamma \in \widetilde{G}$ can be classified according to their fixed points and traces. We do this as follows. For $\gamma \in \widetilde{G}$, the fixed point equation $z = \gamma z$ is quadratic:

$$z = \frac{az+b}{cz+d} \Rightarrow cz^2 + (d-a)z - b = 0.$$

Solving, we have

$$\begin{aligned} z &= \frac{-(d-a) \pm \sqrt{(d-a)^2+4cb}}{2c} \\ &= \frac{a-d \pm \sqrt{a^2+2ad-4ad+d^2+4cb}}{2c} \\ &= \frac{a-d \pm \sqrt{(\operatorname{tr}\gamma)^2-4}}{2c}. \end{aligned}$$

If $\operatorname{tr}\gamma < 2$, then $\sqrt{(\operatorname{tr}\gamma)^2-4}$ is imaginary and $c \neq 0$, so only one of the fixed points lies in the upper half-plane $\mathbf{H}$. In other words, γ has two fixed points in $\mathbf{C}$ which are complex conjugate, and therefore, one fixed point in $\mathbf{H}$.

If $\operatorname{tr}\gamma = 2$, then $\sqrt{(\operatorname{tr}\gamma)^2-4}$ vanishes and so γ has one fixed point in $\mathbf{R} \cup \{\infty\}$.

If $\operatorname{tr}\gamma > 2$, then $\sqrt{(\operatorname{tr}\gamma)^2-4}$ is positive, so we have two separate real or infinite fixed points in $\mathbf{R} \cup \{\infty\}$, one repulsive and one attractive. We say that a fixed point z_0 of a function f is *attractive* if for any value of z in the domain that is close enough to z_0, the iterated function sequence $z, f(z), f(f(z)), f(f(f(z))), \cdots$ converges to z_0, otherwise it is *repulsive.*

In summary,

Definition 2.2. *A transformation $\gamma \in \widetilde{G}$, $\gamma \neq I$, is*

(*i*) *hyperbolic if tr$\gamma > 2$;*

(*ii*) *elliptic if tr$\gamma < 2$;*

(*iii*) *parabolic if tr$\gamma = 2$.*

We expatiate further on these classifications of symmetries of $\mathbf{H}$. Let Γ be a subgroup of $SL(2, \mathbf{R})$.

Definition 2.3. *Let $a, b \in \Gamma$. We say that a is conjugate to b if for every $\gamma \in SL(2, \mathbf{R})$,*

$$\gamma a \gamma^{-1} = b.$$

Definition 2.4. *For τ in Γ, the conjugacy class of τ in Γ is*

$$Q = \{\tau\} = \left\{\gamma\tau\gamma^{-1} \,:\, \gamma \in \Gamma\right\}.$$

Definition 2.5. *For $\tau \in \Gamma$, the centralizer of τ in Γ is*

$$\Gamma_\tau = \{\gamma \in \Gamma \,:\, \gamma\tau = \tau\gamma\}.$$

- A hyperbolic transformation $\gamma_{\text{hyp}} \in G$ has two fixed points 0 and ∞ if and only if it is a diagonal matrix:
$$\gamma_{\text{hyp}} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \; a > 1.$$
Hence, every hyperbolic transformation γ_{hyp} can be conjugated to a dilation $z \mapsto \gamma_{\text{hyp}} z = a^2 z$, $a > 1$.

- An elliptic transformation $\gamma_{\text{ell}} \in G$ is conjugate to a rotation of the form $z \mapsto \gamma_{\text{ell}} z = e^{i\theta} z$, i.e.,

$$\gamma_{\text{ell}} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in \mathbf{SO}(2), \quad \theta \in \mathbf{R}.$$

  Remark 2.1. *Let $\mathbf{D} = \{\varsigma \in \mathbf{C} : |\varsigma| < 1\}$ be the Poincaré unit disk and $\hat{\gamma} : \mathbf{H} \to \mathbf{D}$. Let $z_0 \in \mathbf{H}$ be a fixed point of γ_{ell} such that $\hat{\gamma}(z_0) = 0$. Then $\hat{\gamma}\gamma_{ell}\hat{\gamma}^{-1}$ has a fixed point 0 (the origin) and so must be a rotation of the form $z \mapsto \gamma_{ell} z = az$ for some $a \in \mathbf{C}$ with $|a| = 1$. In our case, $a = e^{i\theta}$.*

- A parabolic transformation $\gamma_{\text{par}} \in G$ can be conjugated to a horizontal translation $z \mapsto \gamma_{\text{par}} z = z + b$, i.e.,

$$\gamma_{\text{par}} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in \mathbf{R}.$$

Remark 2.2. *The horizontal translations $z \mapsto z+b$, $b \in \mathbf{R}$, are the only maps whose fixed points are infinity.*

Definition 2.6. *A parabolic fixed point of Γ is also called a cusp of Γ.*

Definition 2.7. *A geodesic is a curve which is locally length minimising within the class of piecewise smooth curves.*

Definition 2.8. *A unique geodesic in $\mathbf{H}$ joining the two fixed points of a hyperbolic transformation γ is called the axis of γ, and is denoted h_γ.*

We shall see that for the upper half-plane $\mathbf{H}$, the role of geodesics is played by straight lines and semicircles orthogonal to the real axis $\mathbf{R} = \{z \in \mathbf{C} : \operatorname{Im} z = 0\}$. Any two points in $\mathbf{H}$ can be joined by a unique geodesic, and the distance between those points is measured along this geodesic.

Let $\widetilde{\gamma} : [0,1] \to \mathbf{H}$ be a *piecewise differentiable curve*

$$\widetilde{\gamma} = \{z(t) = x(t) + iy(t) \in \mathbf{H} : t \in [0,1]\}.$$

Then its hyperbolic length $\ell(\widetilde{\gamma})$ is given by

$$\ell(\widetilde{\gamma}) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)}\, dt = \int_0^1 \frac{\left|\frac{dz}{dt}\right|}{y(t)}\, dt.$$

For $z, z' \in \mathbf{H}$, the function $d(z, z')$ is now defined by

$$d(z, z') = \inf \ell(\widetilde{\gamma}),$$

where the infimum is taken over all $\widetilde{\gamma}$ which joins z to z' in $\mathbf{H}$.

Proposition 2.9. *If $\widetilde{\gamma} : [0,1] \to \mathbf{H}$ is a piecewise differentiable curve in $\mathbf{H}$, then for any $\gamma \in PSL(2, \mathbf{R})$*

$$\ell(\gamma(\widetilde{\gamma})) = \ell(\widetilde{\gamma}),$$

where $\widetilde{\gamma}$ is given by

$$\widetilde{\gamma} = \{z(t) = x(t) + iy(t) \in \mathbf{H} : t \in [0,1]\}.$$

Proof. Suppose the differentiable curve $\omega(t)$ is given by

$$\omega(t) = \gamma(z(t)) = u(t) + iv(t).$$

In view of (2.1) and (2.2) we have $\left|\frac{d\omega}{dz}\right| = \frac{v}{y}$. Hence,

$$\ell(\gamma(\widetilde{\gamma})) = \int_0^1 \frac{\left|\frac{d\omega}{dt}\right|}{v(t)}\, dt = \int_0^1 \frac{\left|\frac{d\omega}{dz}\frac{dz}{dt}\right|}{v(t)}\, dt = \int_0^1 \frac{\left|\frac{dz}{dt}\right|}{y(t)}\, dt = \ell(\widetilde{\gamma}).$$

□

By Proposition 2.9 and the $\widetilde{G}$-invariance of both the hyperbolic distance $d(z, z')$ and the area element $\mu(z)$, $z, z' \in \mathbf{H}$, and the fact that $\pm I$ acts on $\mathbf{H}$ the same way, we have the following

Proposition 2.10. *The group $PSL(2, \mathbf{R})$ of fractional linear transformations is the group of orientation-preserving isometries of $\mathbf{H}$.*

Proposition 2.11. *The geodesics, or curves minimizing the Poincaré metric in $\mathbf{H}$, are straight lines or circles orthogonal to the real axis.*

Proof. Consider any piecewise differentiable path $\widetilde{\gamma} : [a, b] \to \mathbf{H}$ joining $z = ai \in \mathbf{H}$ and $z' = bi \in \mathbf{H}$ $(b > a)$ on the imaginary axis $i\mathbf{R}$ with $\widetilde{\gamma}(t) = z(t) = x(t) + iy(t)$. Then

$$\begin{aligned}\ell(\widetilde{\gamma}) &= \int_a^b \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)}\, dt \geq \int_a^b \frac{\sqrt{\left(\frac{dy}{dt}\right)^2}}{y(t)}\, dt \\ &\geq \int_a^b \frac{\left|\frac{dy}{dt}\right|}{y(t)}\, dt \geq \int_a^b \frac{\frac{dy}{dt}}{y(t)}\, dt = \int_a^b \frac{dy}{y} = \log\left(\frac{b}{a}\right),\end{aligned}$$

and we obtain that

$$d(ai, bi) = \log\left(\frac{b}{a}\right), \quad 0 < a < b,$$

is the hyperbolic length of the segment of the y-axis joining ai to bi. Hence the y-axis is the geodesic joining ai and bi. We have also shown that

$$\ell(\widetilde{\gamma}) = d(ia, ib),$$

that is, the minimum is achieved (or what is the same $\ell(\widetilde{\gamma})$ is minimal) if and only if $x'(t) = 0$ (which implies $x(t) = 0$) and $y'(t) > 0$ for all $t \in [a, b]$. □

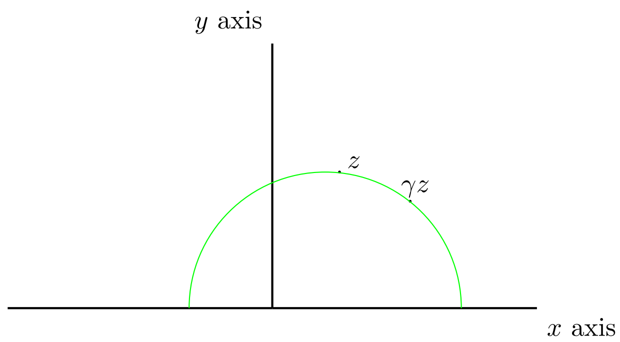

FIGURE 2.1: Geodesics on the upper half-plane $\mathbf{H}$.

Let $z \in \mathbf{H}$ be a point on h_γ. Then z is moved a distance $\ell(\gamma)$ by γ, i.e., the distance between z and γz on h_γ is $\ell(\gamma)$, where $\ell(\gamma)$ is the translation (displacement) length of γ. Now since $\gamma \in G$ is hyperbolic, γ acts on $\mathbf{H}$ via the transformation

$$\gamma z = a^2 z, \quad a > 1.$$

If y is real, then $\gamma(iy) = ia^2 y$ so that

$$d(iy, \gamma(iy)) = d\left(iy, ia^2 y\right) = \log a^2,$$

which is independent of y. This shows that we must have

$$\ell(\gamma) = \left|\log a^2\right| > 0,$$

i.e.,

$$a = e^{\frac{\ell(\gamma)}{2}}.$$

Hence, every hyperbolic matrix $\gamma \in G$ is conjugate to the diagonal matrix

$$\gamma_{\text{hyp}} = \begin{pmatrix} e^{\frac{\ell(\gamma)}{2}} & 0 \\ 0 & e^{-\frac{\ell(\gamma)}{2}} \end{pmatrix} : \quad z \mapsto \gamma_{\text{hyp}} z = e^{\ell(\gamma)} z, \ell(\gamma) > 0. \tag{2.5}$$

It follows from (2.5) that the following relation between the trace of γ and the displacement length of γ holds:

$$\operatorname{tr} \gamma = 2 \cosh \frac{\ell(\gamma)}{2}.$$

In general, the displacement length $\ell(\gamma)$ satisfies

$$d(z, \gamma z) = \ell \quad \text{for all } z \in h_\gamma, \quad d(z, \gamma z) > \ell \quad \text{for all } z \in \mathbf{H} - h_\gamma.$$

2.1.2 The Iwasawa Decomposition

In this subsection we shall discuss the realisation of the upper half-plane $\mathbf{H}$ as a homogeneous space $\mathbf{H} = G/K = SL(2, \mathbf{R})/\mathbf{SO}(2)$. Note that $\mathbf{SO}_0(2,1) \approx PSL(2, \mathbf{R})$.

Let $M(2, \mathbf{R})$ be the group of 2×2 matrices with entries in $\mathbf{R}$. Since the entries of a variable 2×2 real matrix are 4 free parameters, $M(2, \mathbf{R})$ is a 4-dimensional manifold.

In this subsection, we will construct a concrete image of $SL(2, \mathbf{R})$ by deriving a product decomposition for it. For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}),$$

the group $SL(2, \mathbf{R})$ which naturally lies in $M(2, \mathbf{R})$ and defined by the single equation $ad - bc = 1$ inside $M(2, \mathbf{R})$, is a $4 - 1 = 3$ dimensional connected Lie group.

Let N, A, K given by

$$N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathbf{R}, x \neq 0 \right\}; \tag{2.6}$$

$$A = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} : a \in \mathbf{R}, a \neq \pm 1, 0 \right\}; \tag{2.7}$$

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \leq \theta < 2\pi \right\}, \tag{2.8}$$

be subgroups of $G = SL(2, \mathbf{R})$. The Iwasawa decomposition (Iwasawa [88]) asserts that $G = SL(2, \mathbf{R})$ decomposes into $G = NAK$, or what is the same, every $g \in G$ can be written as $g = nak$, where $n \in N$, $a \in A$ and $k \in K$. These three subgroups are each one-dimensional with $N \cong \mathbf{R}$, $A \in \mathbf{R}^+ = (0, \infty)$, $K \cong \mathbf{S}^1$. A consequence of this decomposition is that the subgroups N, A and K fully account for the 3 dimensions of $SL(2, \mathbf{R})$.

Now, we recall that, for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

the equality

$$\mathrm{Im}\,(gz) = \frac{\mathrm{Im}\,(z)}{|cz + d|^2}$$

shows that if $g \in G$ and $z \in \mathbf{H}$, then so is gz, and hence $gz = \frac{az+b}{cz+d}$ is an action of the group $SL(2, \mathbf{R})$ on $\mathbf{H}$. The upper half-plane $\mathbf{H}$ can be identified with the quotient G/K so that a point $z \in \mathbf{H}$ corresponds to the coset gk, $g \in G$, $k \in K$, of all motions which send i to z. In such a realisation of $\mathbf{H}$ the group G acts on itself by matrix multiplication. If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

stabilises the point $i \in \mathbf{H}$ (i.e., $gi = i$), then $(ai + b)/(ci + d) = i$, or $ai + b = -c + di$, which implies $a = d$ and $b = -c$. We can therefore write $a = \cos\theta$ and $c = \sin\theta$. So the stabiliser $G_i = \{i \in \mathbf{H} : gi = i\}$ of the point $i \in \mathbf{H}$ is the set of matrices

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};$$

because of this, the special orthogonal group of 2×2 matrices of determinant 1

$$K = \mathbf{SO}(2) = \left\{ g \in SL(2, \mathbf{R}) : gg^t = I \right\} \tag{2.9}$$

is called the *stability group of* i.

If we consider the unique decomposition of $g \in G$ in the form

$$g = n(x)a(y)k(\theta)$$

with

$$n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad a(y) = \begin{pmatrix} \sqrt{y} & \\ & (\sqrt{y})^{-1} \end{pmatrix}, \quad k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

for $x \in \mathbf{R}$, $y > 0$, $0 \leq \theta < 2\pi$, then we have

$$z = n(x)a(y)i = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} i = x + iy.$$

It follows that via the map

$$\begin{aligned} G/K = SL(2,\mathbf{R})/\mathbf{SO}(2) &\to \mathbf{H} \\ gK &\mapsto gi, \end{aligned}$$

the upper half-plane $\mathbf{H}$ can be identified with the quotient G/K. That is, the action of $SL(2,\mathbf{R})$ on $\mathbf{H}$ is transitive, since for $y > 0$,

$$\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$$

sends $i \in \mathbf{H}$ to $z = x+iy \in \mathbf{H}$. In fact, one has $gi = nai$ and so $g^{-1}na$ lies in K, which means that there exists $k \in K$ with $g = nak$. Thus $G = NAK$. By applying inversion to this decomposition, we also have $G = KAN$. Hence, the existence of the Iwasawa decomposition is established. To prove uniqueness, we assume $nak = n'a'k'$, and by the existence of the decomposition, we see that k and k' fix i, and so $nai = n'a'i$. Clearly $nai = x + iy$. In particular, knowing that nai tells us the parameters determining n and a, we have $n = n'$, $a = a'$, and so $k = k'$.

In summary we have the following statement.

Theorem 2.12 (**Iwasawa decomposition of $SL(2,\mathbf{R})$**)**.** *Let N, A and K be given by (2.6), (2.7), and (2.8) respectively. Then $SL(2,R) = NAK$, or equivalently, every $g \in SL(2,\mathbf{R})$ can be uniquely written as $g = nak$, where $n \in N$, $a \in A$ and $k \in K$.*

The Lie algebra $\mathfrak{sl}(2,\mathbf{R})$ consists of all 2×2 real matrices of trace zero, i.e.,

$$\mathfrak{sl}(2,\mathbf{R}) = \{\mathfrak{p} \in M(2,\mathbf{R}) \,:\, \operatorname{tr}\mathfrak{p} = 0\},$$

where $M(n,\mathbf{R})$ is a set of all $n \times n$ matrices with coefficients in $\mathbf{R}$. The matrices

$$\mathfrak{p}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{p}_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathfrak{p}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

form a basis of $\mathfrak{sl}(2,\mathbf{R})$.

Let $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$ be as given above. Set

$$n(x) = e^{\mathfrak{p}_1^x}, \quad a(t) = e^{\mathfrak{p}_2^t}, \quad k(\theta) = e^{\mathfrak{p}_3^\theta},$$

where θ, t, x are arbitrary real numbers with $\mathfrak{p}_1^x = x\mathfrak{p}_1, \mathfrak{p}_2^t = t\mathfrak{p}_2, \mathfrak{p}_3^\theta = \theta\mathfrak{p}_3$. Then it is not difficult to see that

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(t) = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}, \quad k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where $G = SL(2, \mathbf{R}) = NAK$, with

$$N = \{n(x) \,:\, x \in \mathbf{R}\}, \quad A = \{a(t) \,:\, t \in \mathbf{R}\}, \quad K = \{k(\theta) \,:\, \theta \in \mathbf{R}\}.$$

In summary, we have

Corollary 2.13. *Any one-dimensional subgroup of $SL(2, \mathbf{R})$ is conjugate to one of the following subgroups:*

$$N = \left\{ n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \exp\begin{pmatrix} 0 & x \\ & 0 \end{pmatrix} \,:\, x \in \mathbf{R} \right\};$$
$$A = \left\{ a(t) = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} = \exp\begin{pmatrix} \frac{t}{2} & 0 \\ 0 & -\frac{t}{2} \end{pmatrix} \,:\, t \in \mathbf{R}, \right\};$$
$$K = \left\{ k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \,:\, 0 \le \theta < 2\pi \right\}.$$

For more on Iwasawa decomposition, see Iwasawa [88], Sugiura [160, Chapter V, Section 1], Lang [98], Taylor [165].

Furthermore, in geodesic polar coordinates $z = (\rho, \theta)$ we have (see e.g. Terras [167, p. 141])

$$z = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} e^{-\rho/2} & 0 \\ 0 & e^{\rho/2} \end{pmatrix} i, \quad 0 \le \theta < 2\pi, 0 \le \rho < \infty, \tag{2.10}$$

where

$$x = \frac{\sinh\rho\sin\theta}{\cosh\rho + \sinh\rho\cos\theta}, \quad y = \frac{1}{\cosh\rho + \sinh\rho\cos\theta}. \tag{2.11}$$

2.1.3 Hyperbolic Riemann Surfaces

By a Riemann surface, we mean a one-dimensional complex manifold, i.e., a two-dimensional analytic Riemannian manifold. The uniformisation theorem for Riemann surfaces tells us that a Riemann surface $\mathscr{M}$ can be obtained as either the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ or a quotient of $\mathbf{C}$ or $\mathbf{H}$ by a discrete group of isometries acting without fixed points on $\mathbf{C}$ or on $\mathbf{H}$.

A subgroup Γ of $PSL(2, \mathbf{R})$ is *discrete* if for every $z \in \mathbf{H}$, the orbit $\Gamma z = \{\gamma z \,:\, \gamma \in \Gamma\}$ has no accumulation points in $\mathbf{H}$. If Γ is any discrete group (where $\gamma \in \Gamma$ has no fixed points in $\mathbf{C}$ or $\mathbf{H}$) acting on the complex plane $\mathbf{C}$ or on the upper half-plane $\mathbf{H}$, then the quotient $\Gamma\backslash\mathbf{C}$ or $\Gamma\backslash\mathbf{H}$ is a Riemann surface, in fact a smooth Riemann surface. The condition that Γ acts on $\mathbf{C}$ or $\mathbf{H}$ without fixed points in $\mathbf{C}$ or $\mathbf{H}$ is equivalent to saying that Γ has no elliptic elements, bearing in mind that only elliptic transformations have fixed points in $\mathbf{H}$. If the discrete group Γ has elliptic elements, then the resulting quotient is called an *orbifold*. We classify the only Riemann surfaces of the form $\Gamma\backslash\mathbf{C}$ as the plane $\mathbf{C}$; the punctured disc $\mathbf{C}\backslash\{0\}$; and the tori ($\mathbf{T}^2 = \Gamma\backslash\mathbf{C}$=$\mathbf{Z}^2\backslash\mathbf{R}^2$).

In order for the quotient $\Gamma\backslash\mathbf{H}$ to give a reasonable (well defined) topological space, the action of the discrete subgroup $\Gamma \subset SL(2,\mathbf{R})$ must be *properly discontinuous.* We say that a discrete subgroup Γ of $SL(2,\mathbf{R})$ acts *properly discontinuously* on $\mathbf{H}$ if any compact subset of $\mathbf{H}$ contains only finitely many orbit points γz, $\gamma \in \Gamma$, $z \in \mathbf{H}$ (i.e., the orbits Γz, $z \in \mathbf{H}$, are locally finite); such a group Γ is called a *Fuchsian group.*

If we equip the Riemann surface $\Gamma\backslash\mathbf{H}$ with a complete Riemannian metric of constant Gaussian curvature -1, then $\mathscr{M} = \Gamma\backslash\mathbf{H}$ becomes a *hyperbolic surface.*

Example 2.1. *The group $\Gamma = PSL(2,\mathbf{Z}) = SL(2,\mathbf{Z})/\{\pm I\}$ consists of all transformations of the form*

$$z \mapsto \gamma z = \frac{az+b}{cz+d} \ : \ a,b,c,d \in \mathbf{Z}, ad-bc = 1,$$

and it is called the modular group. This group is obviously a Fuchsian group; that is, it is a discrete subgroup $PSL(2,\mathbf{Z}) \subset PSL(2,\mathbf{R})$ that acts properly discontinuously on $\mathbf{H}$ and the corresponding quotient $PSL(2,\mathbf{Z})\backslash H$ is called a modular surface. The modular group $\Gamma = SL(2,\mathbf{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ : \ Sz = z+1, \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ : \ Tz = -\frac{1}{z}$$

(*see Apostol [3, pp. 28-30]*).

Example 2.2. *For $N \in \mathbf{N}$, $N \geq 1$, the principal congruence subgroup $\widetilde{\Gamma}(N) \subseteq PSL(2,\mathbf{Z})$ of level N defined by*

$$\widetilde{\Gamma}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbf{Z}) : a \equiv d \equiv \pm 1, b \equiv c \equiv 0 \mod N \right\} \tag{2.12}$$

is a subgroup of $PSL(2,\mathbf{Z})$ of finite index (*see Example 2.4 below*).

We introduce some terminologies. If the area of $\Gamma\backslash\mathbf{H}$ is finite, that is,

$$\mu(\Gamma\backslash\mathbf{H}) = \int_{\Gamma\backslash\mathbf{H}} \frac{dxdy}{y^2} < \infty,$$

then the Fuchsian group Γ is called *cofinite.*

Let Γ be a Fuchsian group, the *limit set* $\Pi_\Gamma \subseteq \partial\mathbf{H} = \mathbf{R} \cup \{\infty\}$ of Γ is the set of all limit points of the orbits Γz, $z \in \mathbf{H}$.

Definition 2.14. *A subgroup $\Gamma \subset PSL(2,\mathbf{R})$ is a Fuchsian group if and only if Γ acts discontinuously on $\mathbf{H}$.*

Definition 2.15. *Let $\Gamma \subset PSL(2,\mathbf{R})$ be a Fuchsian group. An open set $\mathcal{D}_\Gamma \subset \mathbf{H}$ is called a fundamental domain of Γ if for $\gamma, \sigma \in \Gamma$,*

(*i*) $\gamma\mathcal{D}_\Gamma \cap \sigma\mathcal{D}_\Gamma = \emptyset$,

(*ii*) *the closure of* $\bigcup_{\gamma\in\Gamma} \gamma\mathcal{D}_\Gamma$ *coincides with* $\mathbf{H}$.

Any Fuchsian group $\Gamma \subset PSL(2,\mathbf{R})$ admits a fundamental domain $\mathcal{D}_\Gamma$. The fundamental domain $\mathcal{D}_\Gamma$ is not unique, but all fundamental domains $\mathcal{D}_\Gamma$ of Γ have the same positive hyperbolic area

$$\mu\left(\mathcal{D}_{\Gamma}\right) = \int_{\mathcal{D}_{\Gamma}} d\mu(z) = \int_{\mathcal{D}_{\Gamma}} \frac{dxdy}{y^2}.$$

Indeed, let the identity element $\pm I$ fix the point $z' \in \mathbf{H}$. Then the set

$$\mathcal{D}_{\Gamma} = \{z \in \mathbf{H} \,:\, d(z, z') \leq d(z, \gamma z') \text{ for all } \gamma \in \Gamma, \gamma \neq \pm I\}$$

is a fundamental domain for Γ called a *Dirichlet domain.*

Definition 2.16. *A Fuchsian group $\Gamma \subset PSL(2, \mathbf{R})$ is called the Fuchsian group of the first kind if every point on the boundary $\partial\mathbf{H} = \mathbf{R} \cup \{\infty\}$ is a limit point of an orbit Γz for some $z \in \mathbf{H}$.*

Fuchsian groups $\Gamma \subset PSL(2, \mathbf{R})$ of the first kind have the property that

$$\mu\left(\mathcal{D}_{\Gamma}\right) = \int_{\mathcal{D}_{\Gamma}} d\mu(z) < \infty;$$

and we say that Γ is cofinite (see, e.g., Katok [91, Section 4.5]).

Definition 2.17. *A cofinite Fuchsian group is called cocompact if the quotient $\Gamma\backslash\mathbf{H}$ is a compact hyperbolic surface.*

Example 2.3. *A fundamental domain for the modular group $\Gamma = PSL(2, \mathbf{Z}) \subset PSL(2, \mathbf{R})$ is an open subset $\mathcal{D}_{\Gamma} \subset \mathbf{H}$ given by*

$$\mathcal{D}_{\Gamma} = \left\{ z \in \mathbf{H} \,:\, |Re\, z| \leq \frac{1}{2},\, |z| \geq 1, \textit{and if } \; |z| = 1, \textit{then } \; Re\, z \geq 0 \right\}.$$

Indeed, by definition,

$$\begin{aligned} \mu(\Gamma\backslash\mathbf{H}) &= \int_{\mathcal{D}_{\Gamma}} d\mu(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} \right) dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) \,|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi}{3}. \end{aligned}$$

Therefore $\Gamma = PSL(2, \mathbf{Z}) \subset PSL(2, \mathbf{R})$ is a Fuchsian group of the first kind.

Example 2.4. *From formulae for the finite index $[\widetilde{\Gamma}(1) : \widetilde{\Gamma}(N)]$ of the principal congruence subgroup of level N (see Shimura [151, p. 22]), we deduce that*

$$\mu\left(\mathcal{D}_{\widetilde{\Gamma}(N)}\right) = \begin{cases} \frac{\pi}{6} \cdot N^3 \displaystyle\prod_{\substack{p \mid N \\ p \textit{ prime}}} \left(1 - p^{-2}\right), & N > 2, \\ 2\pi, & N = 2, \end{cases}$$

and therefore $\widetilde{\Gamma}(N)$ is a Fuchsian group of the first kind.

For more examples of Fuchsian groups of the first kind, see Bergeron [27, Subsec. 2.3.2].

Throughout this thesis, Γ is a Fuchsian group of the first kind.

A Fuchsian group of the first kind $\Gamma \subset PSL(2, \mathbf{R})$ is cocompact if and only if Γ has no parabolic elements. In other words, the hyperbolic Riemann surface $\Gamma\backslash\mathbf{H}$ is not compact if Γ possesses parabolic elements (see Katok [91, p. 90]). If there are no elliptic and parabolic elements, i.e., if Γ has only hyperbolic elements, then the group Γ is said to be *strictly hyperbolic*

Let f satisfy the condition

$$(f,f) = \int_{\Gamma\backslash\mathbf{H}} |f(z)|^2\,d\mu(z) < \infty.$$

We define the Hilbert space

$$L^2(\Gamma\backslash\mathbf{H}) = \{f : f(\gamma z) = f(z),\ \text{ for all } \gamma\in\Gamma,\, z\in\mathbf{H}\}$$

as the vector space of all smooth automorphic functions f which are square integrable on $\Gamma\backslash\mathbf{H}$ with the inner product given by

$$(f,g) = \int_{\Gamma\backslash\mathbf{H}} f(z)\overline{g(z)}\,d\mu(z), \tag{2.13}$$

for all complex-valued functions $f, g \in L^2(\Gamma\backslash\mathbf{H})$.

Now let $\widetilde{\Delta}$ be the Laplacian in $\mathbf{H}$, which is also the Laplacian on the hyperbolic surface $\mathscr{M} = \Gamma\backslash\mathbf{H}$. Then $\widetilde{\Delta}$ commutes with the action of G on $\mathbf{H}$. Therefore, it descends to an operator in $C^\infty(\mathscr{M})$. The Laplacian $\widetilde{\Delta}$ regarded as a linear operator

$$\widetilde{\Delta} : C_0^\infty(\mathscr{M}) \to L^2(\mathscr{M}),$$

is a symmetric, nonnegative operator on $L^2(\mathscr{M})$, i.e., it satisfies

$$(\widetilde{\Delta}f, g) = (f, \widetilde{\Delta}g), \qquad f, g \in C_0^\infty(\mathscr{M})$$

and

$$(\widetilde{\Delta}f, f) \geq 0, \qquad f \in C_0^\infty(\mathscr{M}).$$

It follows that $\widetilde{\Delta}$ is essentially self-adjoint, and the closure of $\widetilde{\Delta}$, which we also denote by $\widetilde{\Delta}$ is self-adjoint. If $f \in C^\infty(\mathscr{M})$ viewed as a C^∞ Γ-automorphic on $\mathbf{H}$, then the function $\widetilde{\Delta}f$ is again Γ-automorphic on $\mathbf{H}$.

If $\mathscr{M} = \Gamma\backslash\mathbf{H}$ is compact, then the spectrum of $\widetilde{\Delta}$ on $\Gamma\backslash\mathbf{H}$ is discrete and real, with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots; \qquad \lim_{n\nearrow\infty} \lambda_k = \infty, \tag{2.14}$$

and satisfies the Weyl's asymptotic formula (see Subsection 3.4.1)

$$\lambda_k \sim \frac{4\pi k}{\mu(\Gamma\backslash\mathbf{H})} \qquad \text{as } k \nearrow \infty. \tag{2.15}$$

Since $\widetilde{\Delta}$ is self-adjoint and positive, we always have $(\lambda_k : k \geq 0) \subset [0,\infty)$.

Next we discuss the concept of *small eigenvalues* of the Laplacian on $\mathscr{M}$. Let s_k be chosen so that $\lambda_k = s_k(1-s_k)$, where $(\lambda_k : k \geq 0)$ are the eigenvalues of the Laplacian on $\mathscr{M}$ corresponding to the eigenfunctions $(\varphi_k : k \geq 0)$, which form an orthonormal basis of $L^2(\mathscr{M})$. Define r_k by setting $s_k = \frac{1}{2} + ir_k$. Then $\lambda_k = \frac{1}{4} + r_k^2$, $k \geq 0$. Thus there are two r_k's for each λ_k except when $\lambda_k = \frac{1}{4}$ which corresponds to $r_k = 0$. Indeed, if $\lambda_k < \frac{1}{4}$, then $r_k \in \mathbf{C}$ and if $\lambda_k > \frac{1}{4}$, then $r_k \in \mathbf{R}$. For instance, if $\lambda_0 = 0$, then $r_0 = \pm\frac{i}{2}$. The eigenvalues $0 \leq \lambda_k < \frac{1}{4}$ (r_k imaginary) are called *small or exceptional eigenvalues.* The question of "small" eigenvalues is of great importance in number theory. In Maass [105], the eigenvalues of the Laplacian are written in the form $\lambda = r^2 + \frac{1}{4}$ in order to explain the Γ-factors $\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)$ and $\Gamma\left(\frac{s+1+ir}{2}\right)\Gamma\left(\frac{s+1-ir}{2}\right)$ occurring in the functional equation of certain Dirichlet series. The above notation for the eigenvalues was later developed by Selberg [147] and some other notable number theorists and lead to various number theoretic and geometric results, which depend on the eigenvalues in the interval $\left[0, \frac{1}{4}\right]$ (see e.g., Lax and Phillips [99], Wolfe [178], Patterson [126], Levitan [102], Phillips and Rudnick [127], and Hill and Parnovski [85] for such results).

If $\mathscr{M} = \mathit{\Gamma}\backslash\mathbf{H}$ is not compact, then $\widetilde{\Delta}$ has both discrete and continuous spectrum (see Section 3.2). We lift the eigenfunctions $(\varphi_k : k \geq 0)$ of the Laplacian $\widetilde{\Delta}$ on $\mathscr{M}$ to $\mathbf{H}$ in order to obtain automorphic eigenfunctions $(\phi_k : k \geq 0)$ on $\mathbf{H}$. Recall that the eigenfunction ϕ is automorphic if $\phi(\gamma z) = \phi(z)$, $\forall \gamma \in \mathit{\Gamma}, z \in \mathbf{H}$.

By a classical application of the Gauss-Bonnet theorem (see Anderson [2, Section 5.5]), we have

$$\kappa\mu\left(\mathit{\Gamma}\backslash\mathbf{H}\right) = 2\pi\chi\left(\mathit{\Gamma}\backslash\mathbf{H}\right), \ \ \kappa = -1, \tag{2.16}$$

where $\chi\left(\mathit{\Gamma}\backslash\mathbf{H}\right)$ is the Euler characteristic of $\mathit{\Gamma}\backslash\mathbf{H}$ and κ is the Gaussian curvature. The Euler characteristic of the g-holed torus is $2 - 2g$. Then

$$\mu\left(\mathit{\Gamma}\backslash\mathbf{H}\right) = -2\pi\chi\left(\mathit{\Gamma}\backslash\mathbf{H}\right) = -2\pi(2-2g) = 4\pi(g-1).$$

We again introduce some terminologies. If $\nu = \gamma^m$, $m \in \mathbf{Z} - \{0\}$, we shall write $\tilde{\gamma}_\nu = \tilde{\gamma}_\gamma^m$ and say that $\tilde{\gamma}_\nu$ is the m-fold iterate of $\tilde{\gamma}_\gamma^m$.

Definition 2.18. *An element $\nu \in \mathit{\Gamma} - \{0\}$ is primitive if it cannot be written in the form $\nu = \gamma^m$ with $\gamma \in \mathit{\Gamma}$ and $m \geq 2$; correspondingly we say that the conjugacy class $\{\nu\}$ is primitive and denoted $\{\nu\}_p$. Similarly, a closed geodesic $\tilde{\gamma}_\nu$ on $\mathscr{M}$ is primitive if it is not the m-fold iterate (with $m \geq 2$) of another closed geodesic $\tilde{\gamma}_\gamma$ on $\mathscr{M}$; correspondingly we call the set of lengths $\ell(\widetilde{\gamma})$ of primitive $\widetilde{\gamma}$ the primitive length spectrum.*

Proposition 2.19. *For every $\gamma \in \mathit{\Gamma} - \{I\}$ there exits a unique primitive $\nu \in \mathit{\Gamma}$ such that $\gamma = \nu^m$ for some $m \geq 1$. Similarly, for every nontrivial closed geodesic $\tilde{\gamma}$ on $\mathscr{M}$, there exists a unique primitive closed geodesic $\tilde{\gamma}_0$ such that $\tilde{\gamma} = \tilde{\gamma}_0^m$ for some $m \geq 1$. The centraliser of $\gamma \in \mathit{\Gamma}$ is*

$$\mathit{\Gamma}_\gamma = \left\{\nu^n \,:\, n \in \mathbf{Z}\right\}.$$

Proof. A proof can be found in Buser [32, p. 128]. See also Borthwick [30, pp. 32-33]. □

We recall that points in the quotient (topological space)

$$\mathit{\Gamma}\backslash\mathbf{H} = \{\mathit{\Gamma} z \,:\, z \in \mathbf{H}\}$$

correspond to orbits of $\mathit{\Gamma}$ with the topology in which the natural projection

$$\pi \,:\, \mathbf{H} \to \mathit{\Gamma}\backslash\mathbf{H} \tag{2.17}$$

given by

$$\pi(z) = \mathit{\Gamma} z$$

is continuous. A function on $\mathscr{M}$ can be thought of being a function on $\mathbf{H}$ that is Γ-automorphic. If $\gamma \in \Gamma$ is hyperbolic, then the axis h_γ connecting two fixed points of γ is preserved by γ and so projects to a closed geodesic under (2.17). This gives a *1-1 correspondence between the closed geodesics of $\Gamma\backslash\mathbf{H}$ and the conjugacy classes of the hyperbolic elements $\gamma \in \Gamma$* (see Borthwick [30, p. 32]). Thus

$$\ell(\gamma) = \ell\left(\pi\left(h_\gamma\right)\right) = \ell(\widetilde{\gamma}).$$

Next we define the concept of free homotopy class and the length spectrum on a compact hyperbolic surface $\mathscr{M}$. Let $\mathfrak{X} = \mathbf{R}/[t \mapsto t+1]$ and $\mathscr{M}$ be a compact hyperbolic surface.

Definition 2.20. *Two closed curves $\alpha, \beta : \mathfrak{X} \to \mathscr{M}$ are called freely homotopic, if there exists a continuous map $q : [0,1] \times \mathfrak{X} \to \mathfrak{X}$ such that*

$$q(0,t) = \alpha(t), \qquad q(1,t) = \beta(t), \quad t \in \mathfrak{X}.$$

In each free homotopy class $[\widetilde{\gamma}]$ of closed curves on $\mathscr{M}$, there exists a geodesic $\widetilde{\gamma}$ whose length is minimal among the curves in $[\widetilde{\gamma}]$. Thus on compact hyperbolic surfaces or, more generally, on compact manifolds with negative curvature, the free homotopy classes and the closed geodesics are in one-to-one correspondence.

Definition 2.21. *The length spectrum of $\mathscr{M}$ is the collection $\mathscr{L}_{\mathscr{M}}$ of lengths of closed geodesics $\ell(\widetilde{\gamma})$ given by*

$$\mathscr{L}_{\mathscr{M}} = \{\ell(\widetilde{\gamma}) : \widetilde{\gamma} \text{ is a closed geodesic on } \mathscr{M}\}$$

repeated according to multiplicity. The multiplicity of the length is the number of free homotopy classes of closed curves containing a geodesic of the given length. The corresponding length counting function is given by

$$\mathscr{N}_{\mathscr{M}}(t) = \#\left\{\ell \in \mathscr{L}_{\mathscr{M}} : \ell \leq t\right\}. \tag{2.18}$$

2.2 Automorphic Forms for the Modular Group $SL(2,\mathbf{Z})$

When a group acts discontinuously on a topological space, it is natural to study a function defined on the topological space and that are invariant under the group. In this section we study functions $f : \mathbf{H} \to \mathbf{C}$ that are invariant under the discrete subgroup $SL(2,\mathbf{Z}) \subset SL(2,\mathbf{R})$; that is we study functions that satisfy $f(\gamma z) = f(z)$ for all $\gamma \in SL(2,\mathbf{Z})$, $z \in \mathbf{H}$. We shall investigate an important example of functions of this type, the nonholomorphic Eisenstein series. Before going to the details of the nonholomorphic Eisenstein series as an essential example of automorphic forms, it becomes pertinent to first briefly discuss, in general, automorphic forms for $SL(2,\mathbf{Z})$, which are eigenfunctions of the Laplacian on $SL(2,\mathbf{Z})\backslash\mathbf{H}$; specifically, we give the Fourier expansion of automorphic forms for $SL(2,\mathbf{Z})$. We also call these eigenfunctions Maass waveforms because they were first systematically considered by Maass [105]. The modular surface $SL(2,\mathbf{Z})\backslash\mathbf{H}$ is not compact, and therefore the spectrum of the Laplace-Beltrami operator admits a continuous part spanned by Eisenstein series; as discussed in Section 3.2. The discrete part of the spectral decomposition of $L^2\left(\Gamma\backslash\mathbf{H}\right)$ is spanned by Maass cusp forms; as discussed in Subsection 3.2.1.

Through out this section, $\Gamma = SL(2,\mathbf{Z})$ is a discrete subgroup of $SL(2,\mathbf{R})$.

Definition 2.22. *Let $f : \mathbf{H} \longrightarrow \mathbf{C}$ be a smooth function. We say that f is of polynomial growth at infinity if for fixed $x \in \mathbf{R}$ and $z = x + iy \in \mathbf{H}$, $f(z)$ is bounded by a fixed polynomial in y as $y \nearrow \infty$. We say f is of rapid decay if for any fixed $N > 1$, $\left|y^N f(z)\right| \searrow 0$ as $y \nearrow \infty$. We say f is of rapid growth if for any fixed $N > 1$, $\left|y^{-N} f(z)\right| \nearrow \infty$ as $y \nearrow \infty$.*

Definition 2.23. *A function $f : \mathbf{H} \longrightarrow \mathbf{C}$ is a nonholomorphic modular (or automorphic) form or Maass waveform if*

(i) f is an eigenfunction of the hyperbolic Laplacian, i.e.,

$$\widetilde{\Delta} f = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = s(1-s) f, \quad s \in \mathbf{C};$$

(ii) f is invariant under the modular group, i.e.,

$$f(\gamma z) = f(z) \qquad \text{for all } \gamma \in \Gamma = SL(2, \mathbf{Z}) \text{ and all } z \in \mathbf{H};$$

(iii) f has at most polynomial growth at infinity.

Furthermore, if

$$\int_{x=0}^{1} f(x+iy)\, dx = 0,$$

then we call f a Maass cusp form.

The vector space of all automorphic forms with respect to $\Gamma = SL(2, \mathbf{Z})$ is denoted by $\mathfrak{C}\left(SL(2, \mathbf{Z}), s(1-s)\right) = \mathfrak{C}$.

Theorem 2.24 (Fourier Expansion of Maass Waveform). *An automorphic form $f \in \mathfrak{C}$ has a Fourier series expansion given by*

$$f(z) = a_0 y^s + \widetilde{a}_0 y^{1-s} + \sum_{n \neq 0} a_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}, \tag{2.19}$$

where $a_0, \widetilde{a}_0, a_n \in \mathbf{C}$, and $K_s(z)$ is the Bessel function of imaginary argument (see Appendix B.3).

Proof. Let $\mathbf{H}$ be the upper half-plane and $f \in \mathfrak{C}$. Since the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is in $SL(2, \mathbf{Z})$ it follows that $f(z)$ satisfies

$$f(z) = f(\gamma z) = f(z+1), \quad \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbf{Z}), z \in \mathbf{H}.$$

Thus $f(z)$ is a periodic function of x and therefore admits a Fourier expansion of the form

$$f(z) = \sum_{n \in \mathbf{Z}} c_n(y) e^{2\pi i n x}, \quad z = x + iy,$$

where

$$c_n(y) = \int_0^1 f(z) e^{-2\pi i n x}\, dx, \quad z = x + iy.$$

Since

$$-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f - s(1-s) f = 0,$$

it follows that
$$\frac{\partial^2}{\partial y^2}f = -\frac{\partial^2}{\partial x^2}f - s(1-s)fy^{-2}.$$

Thus
$$\begin{aligned} c_n''(y) &= \int_0^1 \frac{\partial^2}{\partial y^2} f e^{-2\pi i n x}\,dx = \int_0^1 \left[-\frac{\partial^2}{\partial x^2}f - s(1-s)fy^{-2}\right] e^{-2\pi i n x}\,dx \\ &= -\frac{s(1-s)}{y^2}c_n(y) - \int_0^1 \frac{\partial^2}{\partial x^2} f e^{-2\pi i n x}\,dx. \end{aligned}$$

By the periodicity of f, it follows that $\frac{\partial}{\partial x}f$ is also periodic and by integration by parts, we obtain
$$\begin{aligned} c_n''(y) + \frac{s(1-s)}{y^2}c_n(y) &= -\int_0^1 \frac{\partial^2}{\partial x^2} f e^{-2\pi i n x}\,dx \\ &= 4\pi^2 n^2 c_n(y). \end{aligned}$$

Hence, $c_n(y)$ satisfies the differential equation
$$c_n''(y) + \left[\frac{s(1-s)}{y^2} - 4\pi^2 n^2\right] c_n(y) = 0. \tag{2.20}$$

Now setting
$$c_n(y) = y^{\frac{1}{2}} u_n(y), \tag{2.21}$$

equation (2.20) becomes
$$y^2 \frac{du_n}{dy} + y\frac{du}{dy} - \left[\left(s - \frac{1}{2}\right)^2 + 4\pi^2 n^2\right] u_n = 0. \tag{2.22}$$

Equation (2.22) is a modified Bessel's equation with the general solution
$$u_n(y) = a_n K_{s-\frac{1}{2}}(2\pi|n|y) + b_n I_{s-\frac{1}{2}}(2\pi|n|y), \tag{2.23}$$

where $K_s(y)$ and $I_s(y)$ are the Bessel functions of imaginary argument (see Appendix B.3). We rule out the second solution $I_{s-\frac{1}{2}}(2\pi|n|y)$, by the polynomial growth of $f(z)$ as $y \nearrow \infty$, i.e., we set $b_n = 0$. Thus, by (2.21) we have
$$c_n(y) = a_n y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y).$$

For $n = 0$, equation (2.22) reduces to
$$y^2 \frac{du_0}{dy} + y\frac{du_0}{dy} - \left(s - \frac{1}{2}\right)^2 u_0 = 0. \tag{2.24}$$

Thus, the solution of (2.24) is
$$u_0(y) = a_0 y^{s-\frac{1}{2}} + \widetilde{a}_0 y^{\frac{1}{2}-s},$$

and we obtain
$$c_0(y) = y^{\frac{1}{2}}\left(a_0 y^{s-\frac{1}{2}} + \widetilde{a}_0 y^{\frac{1}{2}-s}\right) = a_0 y^s + \widetilde{a}_0 y^{1-s}.$$

Therefore,

$$\begin{aligned} f(z) &= c_0(y) + \sum_{n\neq 0} a_n y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y) \\ &= a_0 y^s + \widetilde{a}_0 y^{1-s} + \sum_{n\neq 0} a_n y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y). \end{aligned}$$

This completes the proof of the theorem. □

We next discuss an important example of automorphic forms for the modular group $SL(2, \mathbf{Z})$, namely the nonholomorphic Eisenstein series.

2.2.1 The Fourier Expansion of Nonholomorphic Eisenstein Series

Eisenstein series, first studied by Selberg [147, 148], are an essential ingredient in the theory of automorphic functions and automorphic forms with numerous applications to number theory and arithmetic geometry.

We recall from Section 1.5 that the power function $F_s(z) = f(z) = (\operatorname{Im} z)^s = y^s$, $s \in \mathbf{C}$, is an eigenfunction of the Laplacian $\widetilde{\Delta}$ on $\mathbf{H}$, and from Subsection 1.5.2, the averaged integral (see (1.48))

$$F_s(r) = P_{-s}(\cosh r) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{\cosh r + \sinh r \cos\psi}\right)^s d\psi, \quad y = \cosh r + \sinh r \cos\psi,$$

is also an eigenfunction on $\mathbf{H}$; this is an integration over the appropriate subgroup of $SL(2, \mathbf{R})$, bearing in mind that by the Iwasawa decomposition, $y = y(r, \psi) = \cosh r + \sinh r \cos\psi$. In the present situation, the integration must be replaced with summation because $SL(2, \mathbf{Z})$ is a discrete subgroup of $SL(2, \mathbf{R})$. Towards this end we note that

$$\operatorname{Im}(\gamma z) = \frac{y}{|cz+d|^2} \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}), z \in \mathbf{H},$$

and we make the following definition.

Definition 2.25. *The nonholomorphic Eisenstein series for the modular group $SL(2, \mathbf{Z})$ is defined, for $s \in \mathbf{C}$, $Re\, s > 1$, by*

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} Im\,(\gamma z)^s = \sum_{\substack{c,d \in \mathbf{Z} \\ (c,d)=1}}^{\infty} \frac{y^s}{|cz+d|^{2s}}, \tag{2.25}$$

and has meromorphic continuation to the whole complex plane, with a simple pole at $s = 1$ as its only singularity (as we shall see in Subsection 2.2.2).

We shall see shortly (in Subsection 2.2.2) that this analytic continuation and the functional equation for $E(z, s)$ can be obtained through the Fourier expansion of the Eisenstein series. For a complex variable $s \in \mathbf{C}$, the nonholomorphic Eisenstein series $E(z, s)$ in Definition 2.25 was first systematically studied by Maass [105].

The notation in (2.25) means that the sum runs over the representatives $\gamma \in \Gamma$ for the quotient

$$\Gamma_\infty \backslash \Gamma = \left\{ \begin{pmatrix} * & * \\ c & d \end{pmatrix} : (c,d) = 1 \right\}, \Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbf{Z} \right\}, \tag{2.26}$$

where (m,n) is the greatest common divisor of c and d. Since $y(z)$, the imaginary part of $z = x+iy \in \mathbf{C}$ is $SL(2,\mathbf{Z})$-invariant, i.e., for $s \in \mathbf{C}$

$$y(\gamma z)^s = y(z)^s \quad \text{for all } \gamma \in \Gamma_\infty, z \in \mathbf{H}, \tag{2.27}$$

the sum in (2.25) is well-defined and is automorphic with respect to $SL(2,\mathbf{Z})$. Since on the other hand, y^s, $s \in \mathbf{C}$, satisfies

$$-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) y^s = s(1-s) y^s,$$

and since the hyperbolic Laplacian

$$\widetilde{\Delta} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is $SL(2,\mathbf{Z})$-invariant in the sense

$$\widetilde{\Delta}(f(\gamma z)) = (\widetilde{\Delta} f)(\gamma z) \quad \text{for every } \gamma \in SL(2,\mathbf{Z}),$$

we see at once that

$$-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E(z,s) = s(1-s) E(z,s), \quad z = x+iy \in \mathbf{H}.$$

That

$$E(\gamma z, s) = E(z,s), \ \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$$

follows easily from the fact that for every $\gamma \in SL(2,\mathbf{Z})$, we have $\gamma\,(SL(2,\mathbf{Z})\backslash\mathbf{H}) = SL(2,\mathbf{Z})\backslash\mathbf{H}$.

The sum in (2.25) is absolutely and uniformly convergent on compact subsets of $\mathbf{H}$ (Sarnak [145]). In contrast to the holomorphic Eisenstein series of weight k (Sarnak [145], Bergeron [27, p. 100]), the Eisenstein series $E(z,s)$ is not a complex holomorphic function with respect to z, but real holomorphic for $\operatorname{Re} s > 1$. The most significant property of $E(z,s)$ is its analytic continuation and functional equation (Subsection 2.2.2).

If we multiply both sides of (2.25) by $\zeta(2s) = \sum_{k=1}^{\infty} k^{-2s}$, and write $m = kc$, $n = kd$, the Eisenstein series $E(z,s)$ becomes

$$\zeta(2s) E(z,s) = \frac{1}{2} \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{y^s}{|mz+n|^{2s}}.$$

Theorem 2.26. *For $z = x+iy \in \mathbf{H}$, $\operatorname{Re} s > 1$, $E(z,s)$ defined by (2.25) has the Fourier expansion*

$$E(z,s) = y^s + \varphi(s) y^{1-s} + \frac{2\sqrt{y}\pi^s}{\zeta(2s)\Gamma(s)} \sum_{n\neq 0} \sigma_{1-2s}(|n|)|n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x}, \tag{2.28}$$

where

$$\varphi(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}, \quad \sigma_k = \sum_{d|n} d^k = \textit{the sum of the kth powers of divisors } n. \tag{2.29}$$

Remark 2.3. *The use of* $(\varphi_k : k \geq 0)$ *as eigenfunctions of the Laplacian on* $\mathscr{M} = \Gamma\backslash\mathbf{H}$ *and* φ *as scattering determinant* (2.29) *will not cause any confusion.*

Proof of Theorem 2.26. We shall proceed by using the equality

$$E(z,s) = Z_Q(s)/2\zeta(2s),$$

where

$$Z_Q(s) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (Q_z(m,n))^{-s} = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} y^s|mz+n|^{-2s}, \quad z = x+iy \in \mathbf{H},$$

is the Epstein zeta function (Bateman and Grosswald [21], Selberg and Chowla [149]). Indeed,

$$\begin{aligned} Z_Q(s) &= 2\sum_{n=1}^{\infty}(Q(0,n))^{-s} + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(Q_z(m,n))^{-s} \\ &= 2y^s\sum_{n=1}^{\infty}n^{-2s} + 2y^s\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}|mz+n|^{-2s} \\ &= 2y^s\zeta(2s) + 2y^s\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}|mz+n|^{-2s}, \quad \operatorname{Re}s>1. \end{aligned} \tag{2.30}$$

By the Poisson summation formula (1.73), we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty}\frac{1}{|z+n|^{2s}} &= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-2\pi inu}}{|u+z|^{2s}}\,du = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-2\pi inu}}{[(u+x)^2+y^2]^s}\,du \\ &= 2y^{1-2s}\int_0^{\infty}\frac{du}{(1+u^2)^s} + 4y^{1-2s}\sum_{n=1}^{\infty}e^{2\pi inx}\int_0^{\infty}\frac{e^{-2\pi inuy}}{(1+u^2)^s}\,du. \end{aligned} \tag{2.31}$$

We next evaluate the two integrals on the right-hand side of (2.31). To this end, we use (B.48) to obtain

$$\sum_{n=-\infty}^{\infty}\frac{1}{|z+n|^{2s}} = y^{1-2s}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + 4y^{\frac{1}{2}-s}\frac{\pi^s}{\Gamma(s)}\sum_{n=1}^{\infty}e^{2\pi inx}n^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi ny).$$

Thus,

$$\begin{aligned} Z_Q(s) &= 2y^s\zeta(2s) + 2y^{1-s}\zeta(2s-1)\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \\ &\quad + 8y^{\frac{1}{2}}\frac{\pi^s}{\Gamma(s)}\sum_{m=1}^{\infty}m^{1-2s}\sum_{n=1}^{\infty}e^{2\pi imnx}(mn)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi mny) \\ &= 2y^s\zeta(2s) + 2y^{1-s}\zeta(2s-1)\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \\ &\quad + 8y^{\frac{1}{2}}\frac{\pi^s}{\Gamma(s)}\sum_{k=1}^{\infty}\sigma_{1-2s}(k)e^{2\pi ikx}k^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi ky). \end{aligned}$$

$\square$

Definition 2.27. *Any $f \in L^2\left(SL(2,\mathbf{Z})\backslash\mathbf{H}\right)$ that is in $\mathfrak{C}$ is said to be a nonholomorphic cusp form (Maass waveform) for $SL(2,\mathbf{Z})$ if*

$$f(z) = \sqrt{y}\sum_{n\neq 0} a_n K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}. \tag{2.32}$$

That is $f(z)$ is a cusp form if the constant term in the Fourier expansion in (2.19) vanishes:

$$\int_{x=0}^{1} f(x+iy)\,dx = 0 = a_0 y^s + \widetilde{a}_0 y^{1-s} \qquad y > 0.$$

2.2.2 Analytic Continuation and Functional Equation of $E(z,s)$

We now discuss the meromorphic continuation and functional equation of $E(z,s)$ through the Fourier expansion of $E(z,s)$. To this end, we write

$$\begin{aligned} E^*(z,s) := Z_Q(s) = 2\zeta(2s)E(z,s) &= 2\zeta(2s)y^s + 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)y^{1-s} \\ &+ 2\cdot 4\sqrt{y}\frac{\pi^s}{\Gamma(s)}\sum_{n=1}^{\infty}\sigma_{1-2s}(n)n^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi ny)e^{2\pi inx}. \end{aligned} \tag{2.33}$$

The function $\zeta(2s)E(z,s)$ has better analytic properties than $E(z,s)$; in particular, it has a meromorphic continuation to all s except for a simple pole at $s=1$. The precise statement is the following

Theorem 2.28. *The Eisenstein series $E^*(z,s)$ can be meromorphically continued to the whole complex plane, with a simple pole at $s=1$ as its only singularity. Moreover,*

$$Res\, E^*(z,s)\Big|_{s=1} = \pi. \tag{2.34}$$

Proof. The sum of the first two terms in (2.33) has a removable singularity at $s=\frac{1}{2}$, we show this as follows. By (B.21) and by setting $z=2s$, we have

$$\lim_{s\to\frac{1}{2}} 2\left(s-\frac{1}{2}\right)\zeta(2s)y^s = 2\cdot\frac{1}{2}\lim_{z\to 1}(z-1)\zeta(z)y^{\frac{z}{2}} = y^{\frac{1}{2}}. \tag{2.35}$$

Similarly,

$$\begin{aligned} \lim_{s\to\frac{1}{2}} 2\left(s-\frac{1}{2}\right)\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(s\right)}\zeta(2s-1)y^{1-s} &= 2\lim_{z\to 1}\frac{\sqrt{\pi}}{\Gamma\left(\frac{z}{2}\right)}\left(\frac{z-1}{2}\right)\Gamma\left(\frac{z-1}{2}\right)\zeta(z-1)y^{1-\frac{z}{2}} \\ &= 2\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}\right)}\Gamma\left(\frac{2}{2}\right)\zeta(0)y^{\frac{1}{2}} \\ &= -2\frac{1}{2}y^{\frac{1}{2}} = -y^{\frac{1}{2}}. \end{aligned} \tag{2.36}$$

We easily see that the sum of (2.35) and (2.36) vanishes, as advertised. Therefore, $s=1$ is the only singularity of $E^*(z,s)$, which arises as a simple pole from the second term $2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)y^{1-s}$ of (2.33) due to the factor $\zeta(2s-1)$. Since $H(s)$ is an entire function of s (Bateman and Grosswald [21]), it follows that $E^*(z,s)$ has a meromorphic continuation in the whole finite complex plane except for a simple pole at $s=1$.

The residue of $E^*(z,s)$ at $s=1$ can be evaluated as follows. Setting $z = 2s-1$, we have

$$\begin{aligned}\lim_{s\to 1}(s-1)2\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(s\right)}\zeta(2s-1)y^{1-s} &= \lim_{z\to 1}\left(\frac{z-1}{2}\right)2\sqrt{\pi}\frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{z+1}{2}\right)}\zeta(z)y^{\frac{1}{2}-\frac{z}{2}}\\ &= (z-1)\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{2}\right)}\left(\frac{1}{z-1}\right)y^0\\ &= \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{\Gamma(1)} = \pi.\end{aligned}$$

□

We turn now to the functional equation of $E(z,s)$.

Theorem 2.29. *The Eisenstein series $E(z,s)$ satisfies the functional equation*

$$\Lambda(2s)E(z,s) = \Lambda(2-2s)E(z,1-s), \quad \Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Proof. We can write (2.33) as

$$\pi^{-s}\Gamma\left(s\right)E^*(z,s) = 2y^s\pi^{-s}\Gamma\left(s\right)\zeta(2s) + 2y^{1-s}\pi^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) + 2\sqrt{y}H(s), \tag{2.37}$$

where

$$H(s) = 4\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)K_{s-\frac{1}{2}}(2\pi ny)e^{2\pi inx}. \tag{2.38}$$

Since K_ν is an even function of ν and

$$k^{-\frac{\nu}{2}}\sigma_\nu(k) = k^{-\frac{\nu}{2}}\sum_{d|k} d^\nu = k^{-\frac{\nu}{2}}\sum_{d|k}\left(\frac{k}{d}\right)^\nu = k^{\frac{\nu}{2}}\sigma_{-\nu}(k),$$

each term of the sum in (2.38) is unchanged if we replace s by $1-s$. Hence,

$$H(s) = H(1-s). \tag{2.39}$$

Replacing s with $2s-1$ in the Riemann's functional equation (B.19), we have

$$\zeta(2s-1) = 2(2\pi)^{2s-2}\sin\left(s-\frac{1}{2}\right)\pi\Gamma(2-2s)\zeta(2-2s). \tag{2.40}$$

Also, replacing s with $1-s$ in the Legendre duplication formula (B.3) and using (B.4) gives

$$\Gamma(2-2s) = \frac{\Gamma(1-s)\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}2^{2s-1}} = \frac{\Gamma(1-s)}{\sqrt{\pi}2^{2s-1}}\cdot\frac{\pi}{\Gamma\left(s-\frac{1}{2}\right)\sin\left(s-\frac{1}{2}\right)\pi}. \tag{2.41}$$

Using (2.41) in (2.40) we have

$$\zeta(2s-1) = \frac{\pi^{2s-\frac{3}{2}}\Gamma(1-s)}{\Gamma\left(s-\frac{1}{2}\right)}\zeta(2-2s). \tag{2.42}$$

Substituting (2.42) in (2.37) gives

$$\pi^{-s}\Gamma\left(s\right)E^*(z,s) = 2\left(\frac{y}{\pi}\right)^s\zeta(2s)\Gamma(s) + 2\left(\frac{y}{\pi}\right)^{1-s}\zeta(2-2s)\Gamma\left(1-s\right) + 2y^{\frac{1}{2}}H(s). \tag{2.43}$$

If we now write

$$\xi(s) = \pi^{-s}\Gamma(s)\, E^*(z,s),$$

we see from (2.39) and (2.43) that

$$\xi(1-s) = \pi^{s-1}\Gamma(1-s)E^*(z,1-s) = 2\left(\frac{y}{\pi}\right)^{1-s}\zeta(2-2s)\Gamma(1-s) + 2\left(\frac{y}{\pi}\right)^{s}\zeta(2s)\Gamma(s) + 2y^{\frac{1}{2}}H(s) = \xi(s).$$

Hence

$$\frac{\pi^{1-2s}E^*(z,s)}{\Gamma(1-s)} = \frac{E^*(z,1-s)}{\Gamma(s)} \quad \text{or } \pi^{-s}\Gamma(s)\, E^*(z,s) = \pi^{-(1-s)}\Gamma(1-s)\, E^*(z,1-s). \tag{2.44}$$

□

The generalisation of the material in this section to the general linear group $SL(n,\mathbf{Z})$ is considered in Awonusika and Taheri [8].

Chapter 3

Trace Formulae for Hyperbolic Surfaces and Applications

As we have earlier pointed out in Section 1.8 that the main tool for studying the structure of the spectrum of the Laplacian on (compact and noncompact) hyperbolic surfaces is the Selberg trace formula. In this chapter we shall develop the trace formulae for compact and noncompact hyperbolic surfaces $\mathscr{M} = \Gamma\backslash\mathbf{H}$ (Γ is a Fuchsian group of the first kind), and present some of its interesting applications, ranging from heat and eigenvalue asymptotics (Weyl's law) to the determinants of the Laplacian on $\mathscr{M}$.

In Section 3.1, we derive explicitly, the trace formula for a compact hyperbolic surface $\mathscr{M}$ by decomposing Γ into conjugacy classes. Section 3.2 accounts for a complete derivation of the trace formulae for a noncompact hyperbolic surface $\mathscr{M}$. The main task in the derivation of the trace formula is the evaluation of integrals involving the Γ-automorphic kernel which contains the identity, hyperbolic and parabolic elements of Γ.

When the hyperbolic surface $\mathscr{M}$ is compact we need to evaluate only the integral defining the trace of the automorphic kernel which contains the identity and hyperbolic elements of Γ since the spectrum of the Laplacian in this case is discrete. The resulting trace formula then relates the spectrum of the Laplacian on $\mathscr{M}$ to the length of closed geodesics on $\mathscr{M}$.

Computations are complicated when $\mathscr{M}$ is noncompact, because the spectrum of the Laplacian is no longer purely discrete but also continuous and this has significant spectral and geometric effects on the trace formula. The starting point in the process of presenting explicit trace formula for a noncompact hyperbolic surface $\mathscr{M}$ is the Selberg spectral decomposition formula for automorphic functions in $L^2(\mathscr{M})$; this spectral formula decomposes automorphic functions $f \in L^2(\mathscr{M})$ into the discrete and continuous parts. The discrete part involves summation over the discrete spectrum while the continuous part contains integral involving Eisenstein series; the continuous spectrum is understood and studied in terms of Eisenstein series.

The Parseval inner product formula is the generalisation of the Selberg spectral expansion formula for automorphic functions $f \in L^2(SL(2,\mathbf{Z})\backslash\mathbf{H})$, namely the Parseval inner product formula is the spectral expansion formula for the L^2-inner product (f, g), where f and g are automorphic functions in $L^2(SL(2,\mathbf{Z})\backslash\mathbf{H})$. In Section 3.3 we develop an explicit Parseval inner product formula for the nonholomorphic Eisenstein series $E(z, s)$.

Having successfully established the trace formula for a noncompact $\mathscr{M}$; in particular for the modular surface $\mathscr{M} = SL(2, \mathbf{Z})\backslash\mathbf{H}$, we next apply the trace formula to determine the determinant of the Laplacian $\widetilde{\Delta} - s(1-s)$ $(s \in \mathbf{C})$ on $SL(2, \mathbf{Z})\backslash\mathbf{H}$. The determinant obtained is a special case of Efrat [56], Momeni and Venkov [114]. The computations of this determinant are in two stages - the first step is to compute the trace and heat trace asymptotics on $SL(2, \mathbf{Z})\backslash\mathbf{H}$. The second stage involves explicit determinantion of the (Selberg) zeta functions associated to the identity, hyperbolic and parabolic elements of $SL(2, \mathbf{Z})$, and we then express the determinant in terms of these zeta functions. We determine the determinant for special values of $s \in \mathbf{R}$, $s > 0$.

3.1 The Trace Formula for a Compact Hyperbolic Surface

Throughout this section, Γ is a Fuchsian group of the first kind such that $\mathscr{M} = \Gamma\backslash\mathbf{H}$ is smooth and compact, i.e., the fundamental domain $\mathcal{D}_\Gamma$ of Γ has finite non-Euclidean area. Thus the Laplacian on $\mathscr{M}$ has purely discrete spectrum $(\lambda_k : k \geq 0)$ and the corresponding eigenfunctions form an orthonormal basis $(\varphi_k : k \geq 0)$ of $L^2(\mathscr{M})$.

Let $\widetilde{k} : [0, \infty) \longrightarrow \mathbf{C}$ be an even smooth function.

Definition 3.1. *For all $\gamma \in \Gamma$, $z = x + iy, z' = x' + iy' \in \mathbf{H}$, a smooth function*

$$k(z, z') = \widetilde{k}\left[u(z, z')\right], \qquad u(z, z') = \frac{|z - z'|^2}{yy'}, \tag{3.1}$$

satisfying

(*i*) $k(z, z') = k(\gamma z, \gamma z')$

(*ii*) $k(z, z') = k(z', z)$

is called a point-pair invariant.

Definition 3.2. *Functions f which satisfy*

$$f(\gamma z) = f(z) \qquad \textit{for all } \gamma \in \Gamma, z \in \mathbf{H}$$

are called automorphic functions. Hence, an automorphic function with respect to Γ defines a function on the hyperbolic surface $\Gamma\backslash\mathbf{H}$.

Let ϕ be an eigenfunction of the Laplacian on $\mathbf{H}$, with

$$\widetilde{\Delta}\phi = \lambda\phi = \left(\frac{1}{4} + r^2\right)\phi.$$

Then ϕ is also an eigenfunction for all of the invariant integral operators, say $\mathbb{L}$, corresponding to any point-pair invariant, and that the eigenvalue of $\mathbb{L}$ depends only on λ, and not on ϕ; that is, there exists a function h defined on the set of all possible eigenvalues such that

$$\int_{\mathbf{H}} k(z, z')\phi(z')\, d\mu(z') = h(\lambda)\phi(z). \tag{3.2}$$

In particular,

$$\mathbb{L}\phi = h(\lambda)\phi.$$

The precise statement is the following.

Theorem 3.3. *Let ϕ be a Γ-automorphic lift of the eigenfunction φ with respect to the covering $\mathbf{H} \to \mathscr{M} = \Gamma\backslash\mathbf{H}$. Then the equality (3.2) holds, where $k(z, z')$ is defined by (3.1) and $h(\lambda)$ depends only on the eigenvalue $\lambda = \frac{1}{4} + r^2$ and the point-pair invariant k.*

Proof. The integrand is absolutely integrable since ϕ is a bounded function on $\mathbf{H}$. Now, consider the geodesic polar coordinates $z = (\widetilde{\rho}, \widetilde{\theta})$, $0 \le \widetilde{\rho} < \infty$, $0 \le \widetilde{\theta} \le \pi$, where $\widetilde{\rho} = d(z, i)$ and $\widetilde{\theta} = \widetilde{\theta}(z, i)$ is the angle between the positive y-axis and the tangent at the unique geodesic passing through i and z at the point i. In fact, since $s(1-s)$, $s \in \mathbf{C}$, takes on all complex values as s ranges over the complex plane, we can regard h as a function of a complex parameter r, related to s by the equation $s = \frac{1}{2} + ir$. We note that in terms of r, $\widetilde{\Delta} y^{\frac{1}{2}+ir} = \left(\frac{1}{4} + r^2\right) y^{\frac{1}{2}+ir}$. Since the Laplacian $\widetilde{\Delta}$ is an isometry invariant and since the unit circle $\mathbf{S}^1$ acts on $\mathbf{H}$ by rotations with centre i, the averaged function

$$\psi(\widetilde{\rho}) = \frac{1}{\pi} \int_0^{\pi} \phi(\widetilde{\rho}, \widetilde{\theta})\, d\widetilde{\theta} \tag{3.3}$$

is again a solution (not Γ-automorphic) of the initial value problem

$$\begin{aligned} &\psi''(\widetilde{\rho}) + \coth \psi'(\widetilde{\rho}) + \left(\frac{1}{4} + r^2\right) \psi(\widetilde{\rho}) = 0 \\ &\psi(0) = \phi(i), \quad \psi'(0) = 0. \end{aligned}$$

By the uniqueness of the initial value problem we have

$$\psi(\widetilde{\rho}) = \phi(i) \frac{1}{\pi} \int_0^{\pi} \Psi(\widetilde{\rho}, \widetilde{\theta})\, d\widetilde{\theta}, \tag{3.4}$$

where

$$\Psi(z) = y^{\frac{1}{2}+ir}, \quad \Psi(\widetilde{\rho}, \widetilde{\theta}) = \left(\cosh \widetilde{\rho} + \sinh \widetilde{\rho} \cos \widetilde{\theta}\right)^{-\frac{1}{2}-ir}.$$

Multiplying (3.4) by $\widetilde{k}(2\cosh \widetilde{\rho} - 2) \sinh \widetilde{\rho}$ and integrating from 0 to ∞, and using the decomposition

$$\int_{\mathbf{H}} d\mu = \left(\int_0^{\infty} \sinh \widetilde{\rho}\, d\widetilde{\rho}\right) \left(\frac{1}{\pi} \int_{\mathbf{S}^1} d\nu_1\right) = \left(\int_0^{\infty} \sinh \widetilde{\rho}\, d\widetilde{\rho}\right) \left(\frac{1}{\pi} \int_0^{\pi} d\widetilde{\theta}\right)$$

together with (3.3), we obtain

$$\int_{\mathbf{H}} k(z, i) \phi(z)\, d\mu(z) = \phi(i) \int_{\mathbf{H}} k(z, i) y^{\frac{1}{2}+ir}\, d\mu(z) = h(r) \phi(i).$$

□

Next we state and prove a set of transforms which plays a key role in Selberg theory in particular and in harmonic analysis on Lie groups in general.

Theorem 3.4 (Abel-Selberg-Harish-Chandra integral transform). *The compactly supported smooth function $\widetilde{k} : [0, \infty) \longrightarrow \mathbf{C}$ and the real-valued C^{∞}-function $h\left(\frac{1}{4} + r^2\right)$ are connected by the relations*

$$Q(x) = \int_x^{\infty} \frac{\widetilde{k}(t)}{\sqrt{t - x}}\, dt, \tag{3.5}$$

where

$$\widetilde{k}(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(x)}{\sqrt{x-t}}, \tag{3.6}$$

and

$$Q\left(e^u + e^{-u} - 2\right) = g\left(u\right), \tag{3.7}$$

with

$$h\left(\frac{1}{4}+r^2\right) = \int_{-\infty}^{\infty} e^{iru} g(u)\,du, \quad g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru} h\left(\frac{1}{4}+r^2\right) dr. \tag{3.8}$$

Proof. Let the functions $Q(x)$ and $g(u)$ be as given in the theorem. We then show that

$$\widetilde{k}(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(x)}{\sqrt{x-t}},\ h\left(\frac{1}{4}+r^2\right) = \int_{-\infty}^{\infty} e^{iru} g(u)\,du.$$

To this end, let $|\operatorname{supp}\widetilde{k}| < C$, $0 \le w \le C$, we have, by integration by parts

$$Q'(w) = \int_w^\infty \frac{\widetilde{k}'(x)}{\sqrt{x-w}}\,dx = \int_w^C \frac{\widetilde{k}'(x)}{\sqrt{x-w}}\,dx = -2\int_w^C (x-w)^{\frac{1}{2}}\widetilde{k}''(x)\,dx.$$

Thus as $\epsilon \searrow 0$,

$$\begin{aligned}
-\frac{1}{\pi}\int_{t+\epsilon}^C \frac{Q'(w)\,dw}{\sqrt{w-t}}\,dx &= \frac{2}{\pi}\int_{t+\epsilon}^C \left(\int_w^C (x-w)^{\frac{1}{2}}\widetilde{k}''(x)\,dx\right)(w-t)^{-\frac{1}{2}}\,dw \\
&= \frac{2}{\pi}\int_{t+\epsilon}^C \int_{t+\epsilon}^x (x-w)^{\frac{1}{2}}\widetilde{k}''(x)(w-t)^{-\frac{1}{2}}\,dwdx \\
&= \frac{2}{\pi}\int_{t+\epsilon}^C \widetilde{k}''(x)\left(\int_{t+\epsilon}^x (x-w)^{\frac{1}{2}}(w-t)^{-\frac{1}{2}}\,dw\right)dx.
\end{aligned}$$

Making a substitution $u = (x-w)/(x-t)$, we see that

$$\int_t^x (x-w)^{\frac{1}{2}}(w-t)^{-\frac{1}{2}}\,dw = (x-t)B\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\pi}{2}(x-t). \tag{3.9}$$

By letting $\epsilon \to 0$, we have again by integration by parts,

$$\begin{aligned}
-\frac{1}{\pi}\int_t^C \frac{Q'(w)\,dw}{\sqrt{w-t}}\,d &= \frac{2}{\pi}\int_t^C \widetilde{k}''(x)\left(\frac{\pi}{2}(x-t)\right)dx = \int_t^C (x-t)\,d\widetilde{k}'(x)dx \\
&= -\int_t^C \widetilde{k}'(x)\,dx = \widetilde{k}(t) - \widetilde{k}(C) = \widetilde{k}(t).
\end{aligned}$$

This proves the first part. For the second part, let $\phi(z) = y^s$, $s = \frac{1}{2} + ir$, $r \in \mathbf{R}$. Then by Theorem 3.3 we have

$$\begin{aligned}
h(\lambda) = \int_{\mathbf{H}} k(i,z)y^s\,d\mu(z) &= \int_0^\infty \int_{-\infty}^\infty \widetilde{k}\left(\frac{x^2+(y-1)^2}{y}\right) y^s\,\frac{dxdy}{y^2} \\
&= \int_0^\infty \left[\int_{\frac{(y-1)^2}{y}}^\infty \frac{\widetilde{k}(u)}{\sqrt{u-\frac{(y-1)^2}{y}}}\,du\right] y^{s-\frac{3}{2}}\,dy,\ u = \frac{x^2+(y-1)^2}{y}.
\end{aligned}$$

Setting

$$Q\left(\frac{(y-1)^2}{y}\right) = \int_{\frac{(y-1)^2}{y}}^\infty \frac{\widetilde{k}(u)}{\sqrt{u-\frac{(y-1)^2}{y}}}\,du = Q\left(y+y^{-1}-2\right)$$

with $y = e^u$, we obtain

$$\begin{aligned} h(\lambda) &= \int_0^{\infty} Q\left(y + y^{-1} - 2\right) y^{s-\frac{3}{2}}\, dy \\ &= \int_{-\infty}^{\infty} g(u) e^{iru}\, du = h\left(\frac{1}{4} + r^2\right) = h(r). \end{aligned}$$

The Fourier inversion formula (1.72) completes the proof of the theorem. $\square$

Next we define an integral operator on a compact hyperbolic surface $\mathscr{M} = \Gamma\backslash\mathbf{H}$ whose kernel is Γ-automorphic. Towards this end, let the function $\widetilde{k} : [0, \infty) \to \mathbf{C}$ satisfy the decay condition

$$|\widetilde{k}(\rho)| = |\widetilde{k}(2\cosh\rho - 2)| = O\left(e^{-\rho(1+\varepsilon)}\right), \quad \rho = d(z, z')$$

for some $\varepsilon > 0$. Then the sum

$$K(z, z') = \sum_{\gamma\in\Gamma} k(z, \gamma z') \tag{3.10}$$

converges uniformly on compact sets (Patterson [125]). The function (kernel) $K(z, z')$ given by (3.10) is Γ-biinvariant (i.e., $K(\gamma' z, \gamma'' z') = K(z, z')$ for all $\gamma', \gamma'' \in \Gamma$) and symmetric (i.e., $K(z, z') = K(z', z)$).

By Definition (A.8), we can regard $K(z, z')$ as the kernel of the Hilbert-Schmidt operator $\mathbb{L}$ defined by (see Appendix A.5)

$$(\mathbb{L}f)(z) = \int_{\Gamma\backslash\mathbf{H}} K(z, z') f(z')\, d\mu(z'), \quad f \in L^2\left(\Gamma\backslash\mathbf{H}\right). \tag{3.11}$$

Selberg trace formula now arises from the computation of the trace

$$\operatorname{tr}\mathbb{L} = \int_{\Gamma\backslash\mathbf{H}} K(z, z) d\mu(z) \tag{3.12}$$

in two different ways, namely the spectral side and the geometric side. We shall consider first the contribution of the spectral side of the trace formula (3.12).

The Spectral Side. For the spectral side of the trace formula (3.12) we have the following statement.

Theorem 3.5 (Pre-trace formula). *Let $(\varphi_k : k \geq 0)$ be automorphic eigenfunctions of the Laplacian $\widetilde{\Delta}$ on a compact surface $\mathscr{M}$ with corresponding eigenvalues $(\lambda_k : k \geq 0)$, satisfying $0 = \lambda_0 < \lambda_1 \leq \cdots$. Then the following holds:*

(i) $K(z, z') = \sum\limits_{k=0}^{\infty} h(\lambda_k)\varphi_k(z)\varphi_k(z')$;

(ii) $\sum\limits_{k=0}^{\infty} h(\lambda_k)$ *is finite.*

In particular

$$\int_{\Gamma\backslash\mathbf{H}} K(z, z)\, d\mu(z) = \sum_{k=0}^{\infty} h(\lambda_k).$$

Proof. By (3.2), (3.10) and since the translates $\gamma(\mathcal{D}_\Gamma)$ by $\gamma \in \Gamma$ cover the upper half-plane $\mathbf{H}$, we have

$$\begin{aligned}\int_{\mathcal{D}_\Gamma} K(z,z')\phi_k(z)\,d\mu(z) &= \sum_{\gamma\in\Gamma}\int_{\mathcal{D}_\Gamma} k(z,\gamma z')\phi_k(z')\,d\mu(z') \\ &= \int_{\gamma(\mathcal{D}_\Gamma)} k(z,\gamma z')\phi_k(z')\,d\mu(z') \\ &= \int_{\mathbf{H}} k(z,z')\phi_k(z')\,d\mu(z') \\ &= h(\lambda_k)\phi_k(z),\end{aligned} \tag{3.13}$$

where we have used the fact that the eigenfunction $\phi(z)$ is Γ-invariant and that the hyperbolic area $d\mu(z)$ is $PSL(2,\mathbf{R})$-invariant. By viewing the functions φ_k as Γ-automorphic eigenfunctions of the Laplacian $\widetilde{\Delta}$ on $\mathbf{H}$, we have by Theorem 3.3 (see also (3.13)),

$$\int_{\mathbf{H}} k(z,z')\varphi_k(z)\,d\mu(z) = h(\lambda_k)\varphi_k(z'),$$

which can also be written as

$$\int_{\Gamma\backslash\mathbf{H}} K(z,z')\varphi_k(z)\,d\mu(z) = h(\lambda_k)\varphi_k(z').$$

In other words, φ_k are the eigenfunctions of the Hilbert-Schmidt operator $\mathbb{L}$ corresponding to the kernel $K(z,z')$, and since they are complete, they form a complete orthonormal set of eigenfunctions for this operator. It therefore follows from the Hilbert-Schmidt theorem (Theorem A.9) that

$$K(z,z') = \sum_{k=0}^{\infty} h(\lambda_k)\varphi_k(z)\varphi_k(z'), \tag{3.14}$$

where the convergence is in $L^2(\mathscr{M}\times\mathscr{M})$. This proves part (i). Since the sum in part (i) converges, part (ii) follows by setting $z = z'$ and integrating over $\Gamma\backslash\mathbf{H}$ bearing in mind that φ_k has norm 1. □

Equation (3.14) is called the *pre-trace formula* for a compact hyperbolic surface $\mathscr{M}$.

Remark 3.1. *If we regard the ϕ_k's as automorphic eigenfunctions of the Laplacian on* $\mathbf{H}$*, then the pre-trace formula* (3.14) *becomes*

$$K(z,z') = \sum_{k=0}^{\infty} h(\lambda_k)\phi_k(z)\phi_k(z'). \tag{3.15}$$

The Geometric Side. The geometric side of the trace formula (3.12) comes from an explicit computation of the trace formula

$$\operatorname{tr}\mathbb{L} = \int_{\Gamma\backslash\mathbf{H}} K(z,z)\,d\mu(z) = \sum_{\gamma\in\Gamma}\int_{\Gamma\backslash\mathbf{H}} k(z,\gamma z)\,d\mu(z).$$

Since any element $\beta \in \Gamma - \{I\}$ can be uniquely written as

$$\beta = \sigma^{-1}\gamma^m\sigma, \qquad \sigma \in \Gamma_\gamma\backslash\Gamma,\ m \geq 1,$$

we have

$$\begin{aligned}
\operatorname{tr}\mathbb{L} &= \int_{\Gamma\backslash\mathbf{H}} k(0)\,d\mu(z) + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\sum_{\sigma\in\Gamma_\gamma\backslash\Gamma}\int_{\Gamma\backslash\mathbf{H}} k\left(z,\sigma^{-1}\gamma^m\sigma z\right)\,d\mu(z)\\
&= \int_{\Gamma\backslash\mathbf{H}} k(0)\,d\mu(z) + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\sum_{\sigma\in\Gamma_\gamma\backslash\Gamma}\int_{\sigma(\Gamma\backslash\mathbf{H})} k\left(z,\gamma^m z\right)\,d\mu(z),
\end{aligned}$$

where $\{\gamma\}_p$ is the primitive conjugacy class of γ in Γ. Let $\mathcal{D}_\gamma = \bigcup\limits_{\sigma\in\Gamma_\gamma\backslash\Gamma}\sigma\left(\Gamma\backslash\mathbf{H}\right)$ be a fundamental domain for the centraliser Γ_γ when $\Gamma\backslash\mathbf{H}$ is a fundamental domain for Γ. Then

$$\begin{aligned}
\operatorname{tr}\mathbb{L} &= \int_{\Gamma\backslash\mathbf{H}} k(0)\,d\mu(z) + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\int_{\mathcal{D}_\gamma} k\left(z,\gamma^m z\right)\,d\mu(z)\\
&= \int_{\Gamma\backslash\mathbf{H}} k(0)\,d\mu(z) + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\int_{\Gamma_\gamma\backslash\mathbf{H}} k\left(z,\gamma^m z\right)\,d\mu(z). \qquad (3.16)
\end{aligned}$$

Next we compute according to when $\gamma = I$ and $\gamma \neq I$. We shall call the first term on the right-hand side of (3.16) the identity contribution and the second term the hyperbolic contribution.

3.1.1 Computation of the Trace for the Identity Element

Let $c(\mathcal{I})$ denote the contribution of the identity transformation in Γ. Then

$$c(\mathcal{I}) = \int_{\Gamma\backslash\mathbf{H}} k(0)\,d\mu(z) = k(0)\mu\left(\Gamma\backslash\mathbf{H}\right), \qquad (3.17)$$

where by Theorem 3.4

$$\begin{aligned}
k(0) = \widetilde{k}(0) &= -\frac{1}{\pi}\int_0^{\infty}\frac{dQ(x)}{\sqrt{x}} = -\frac{1}{\pi}\int_0^{\infty}\frac{dg(u)}{\sqrt{e^u+e^{-u}-2}}\\
&= -\frac{1}{\pi}\int_0^{\infty}\frac{g'(u)du}{e^{\frac{u}{2}}-e^{-\frac{u}{2}}} = \frac{1}{2\pi^2}\int_0^{\infty} rh(r)\left(\int_0^{\infty}\frac{\sin\left(ru\right)}{\sinh(u/2)}\,du\right)dr\\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\,dr,
\end{aligned}$$

where we have used (Bateman et al. [20, p. 88])

$$\int_0^{\infty}\frac{\sin\left(ru\right)}{\sinh(u/2)}\,du = \pi\tanh(\pi r).$$

We therefore obtain

Proposition 3.6.

$$c(\mathcal{I}) = \frac{\mu\left(\Gamma\backslash\mathbf{H}\right)}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\,dr. \qquad (3.18)$$

3.1.2 Computation of the Trace for the Hyperbolic Element

Here we present the contribution of the hyperbolic element of $\gamma \in \Gamma$ (i.e., the second term on the right-hand side of (3.16)). Noting that every hyperbolic element $\gamma \in \Gamma$ is conjugate to the dilation

$$\gamma z\,:\, z \to e^{\ell(\gamma)}z \quad \text{with} \quad \mathcal{D}_\gamma = \left\{ z = x+iy \in \mathbf{H}\,:\, x\in\mathbf{R},\, y\in \left[1, e^{\ell(\gamma)}\right]\right\},$$

we have

$$\sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\int_{\Gamma_\gamma\backslash\mathbf{H}} k\left(z,\gamma^m z\right)\, d\mu(z) = \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\left(\int_1^{e^{\ell(\gamma)}} k\left(z, e^{m\ell(\gamma)}z\right)\frac{dy}{y^2}\right) dx. \tag{3.19}$$

Let $c(\mathcal{H})$ denote the contribution of the hyperbolic transformation in Γ. Then by the definition of the point-part invariant $k(z,z')$, we have

$$\begin{aligned} c(\mathcal{H}) &= \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\int_1^{e^{\ell(\gamma)}}\int_{-\infty}^{\infty} k\left[\frac{|e^{m\ell(\gamma)}z - z|^2}{e^{m\ell(\gamma)}y^2}\right] d\mu(z) \\ &= 2\sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\ell(\gamma)\int_0^{\infty} k\left[\frac{\left(e^{m\ell(\gamma)}-1\right)^2}{e^{m\ell(\gamma)}}\left(1+\alpha^2\right)\right] d\alpha, \end{aligned}$$

where we have made the substitution $x = \alpha y$. Again making the substitution

$$v = N^{m\ell(\gamma)}\left(1+\alpha^2\right), N^{m\ell(\gamma)} = \frac{\left(e^{m\ell(\gamma)}-1\right)^2}{e^{m\ell(\gamma)}},$$

we obtain

$$\begin{aligned} c(\mathcal{H}) &= 2\sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\ell(\gamma)\int_{N^{m\ell(\gamma)}}^{\infty}\frac{k(v)\,dv}{2N^{m\ell(\gamma)}\sqrt{N^{-m\ell(\gamma)}v-1}} \\ &= \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\ell(\gamma)N^{-\frac{m\ell(\gamma)}{2}}\int_{N^{m\ell(\gamma)}}^{\infty}\frac{k(v)\,dv}{\sqrt{v-N^{m\ell(\gamma)}}}. \end{aligned}$$

By noting that

$$N^{m\ell(\gamma)} = e^{m\ell(\gamma)} + e^{-m\ell(\gamma)} - 2,$$

and applying Theorem 3.4, we therefore obtain

Proposition 3.7.

$$\begin{aligned} c(\mathcal{H}) &= \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\ell(\gamma)\frac{e^{\frac{m\ell(\gamma)}{2}}}{e^{m\ell(\gamma)}-1}Q\left(N^{m\ell(\gamma)}\right), \\ &= \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\frac{\ell(\gamma)g\left(m\ell(\gamma)\right)}{e^{\frac{m\ell(\gamma)}{2}} - e^{\frac{-m\ell(\gamma)}{2}}} = \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\frac{\ell(\gamma)g\left(m\ell(\gamma)\right)}{2\sinh\frac{m\ell(\gamma)}{2}}. \end{aligned}$$

Since the traces in both sides involve integrals of the automorphic kernel at the diagonal, it follows immediately that the spectral side is equal to the geometric side.

The precise statement is given in the following theorem.

Theorem 3.8. *Let $\mathscr{M} = \Gamma\backslash\mathbf{H}$ be a compact hyperbolic surface and let h satisfy the conditions*

(S.I) $h(z)$ is an even function, i.e., $h(z) = h(-z)$;

(S.II) $h(z)$ is analytic on $|Im(z)| < \frac{1}{2}+\epsilon, \epsilon > 0$;

(S.III) $h(z) \leq A(1+|z|^2)^{-1-\epsilon}, A > 0, \epsilon > 0.$

Then the following identity holds:

$$\sum_{k \geq 0} h(\lambda_k) = c(\mathcal{I}) + c(\mathcal{H}), \tag{3.20}$$

where $(\lambda_k : k \geq 0)$ *are the purely discrete spectrum of the Laplacian* $\widetilde{\Delta}$ *on* $\mathscr{M}$ *corresponding to the eigenfunctions* $(\varphi_k : k \geq 0)$ *which forms an orthonormal basis of* $L^2(\mathscr{M})$*. The function* $c(\mathcal{I})$ *which is the contribution of the identity element of* Γ *is given by*

$$c(\mathcal{I}) = \frac{\mu\left(\Gamma \backslash \mathbf{H}\right)}{4\pi} \int_{-\infty}^{\infty} rh\left(\frac{1}{4} + r^2\right) \tanh \pi r \, dr.$$

The term $c(\mathcal{H})$ *denotes the contribution of the hyperbolic element* γ *of* Γ *and is given by*

$$c(\mathcal{H}) = \sum_{\{\gamma\}_p} \sum_{m=1}^{\infty} \frac{\ell(\gamma)}{2\sinh \frac{m\ell(\gamma)}{2}} g(m\ell(\gamma)), \tag{3.21}$$

where $\{\gamma\}_p$ *denotes the hyperbolic conjugacy class of* γ *in* Γ*,* $\ell(\gamma)$ *is the primitive length of* γ *and the function* g *is given by*

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iru} h\left(r^2 + \frac{1}{4}\right) dr.$$

The conditions $(S.I) - (S.III)$ *above are such that all the series and integrals are absolutely convergent.*

The Selberg trace formula for a general locally symmetric space $\Gamma \backslash G/K$ is discussed in Awonusika and Taheri [8].

3.2 The Trace Formula for a Noncompact Hyperbolic Surface

Let $\Gamma \subset SL(2,\mathbf{R})$ be a Fuchsian group of the first kind such that $\mathscr{M} = \Gamma \backslash \mathbf{H}$ is noncompact, that is, Γ is noncocompact but cofinite, and so $\mathscr{M}$ has finitely many cusps. We recall that if $\mathscr{M} = \Gamma \backslash \mathbf{H}$ is noncompact or what is the same, if Γ has a noncompact fundamental domain $\mathcal{D}_\Gamma$, then $\mathcal{D}_\Gamma$ has a finite (nonzero) numbers of zero interior angles; the vertices of those angles are fixed points of the parabolic transformations in Γ and are called *cusps* (or *parabolic vertices*) of $\mathcal{D}_\Gamma$. In this case the spectrum of $\widetilde{\Delta}$ is more complicated, no longer purely discrete.

In this section, we shall obtain the trace formula for a noncompact hyperbolic surface $\mathscr{M} = \Gamma \backslash \mathbf{H}$. Since $\mathscr{M} = \Gamma \backslash \mathbf{H}$ is noncompact, there is a finite number of primitive parabolic conjugacy classes in Γ. Before computing the trace formula, it is necessary to study the spectral decomposition of $\widetilde{\Delta}$ on the noncompact surface $\mathscr{M} = \Gamma \backslash \mathbf{H}$.

Let $\mathscr{M} = \Gamma \backslash \mathbf{H}$ be a noncompact, finite area hyperbolic surface. The conditions on $\mathscr{M}$ mean that it consists of a relatively compact part together with finitely many regions stretch out to infinity, i.e.,

$$\mathscr{M} = \mathscr{M}_0 \cup Z_1 \cup \cdots \cup Z_{n_c},$$

where $\mathscr{M}_0$ is a compact Riemann surface with smooth boundary and

$$Z_i = [a_i, \infty) \times \mathbf{S}^1, \qquad i = 1, \cdots, n_c,$$

are cusps, n_c is the number of cusps of $\mathscr{M}$, $a_i > 0$. In this case, the spectrum of the Laplacian on $\mathscr{M}$ is both discrete and continuous. Indeed, the Laplacian on $\mathscr{M}$ has absolutely continuous spectrum in $\left[\frac{1}{4}, \infty\right)$ with multiplicity equal to the number of cusps of $\mathscr{M}$, while the discrete spectrum consists of finitely many eigenvalues in $\left[0, \frac{1}{4}\right)$, where each eigenvalue has finite multiplicity (Lax and Phillips [99]). The spectral decomposition of the absolutely continuous part of $\widetilde{\Delta}$ is described by generalized eigenfunctions $E_i(z, s)$, $i = 1, \cdots, n_c$. These generalized eigenfunctions are constructed as Eisenstein series.

3.2.1 Selberg Spectral Expansion of Automorphic Functions

Let $\Gamma = SL(2, \mathbf{Z})$ and $\mathscr{M} = SL(2, \mathbf{Z})\backslash\mathbf{H}$. The Selberg spectral decomposition of $L^2(\Gamma\backslash\mathbf{H})$ states that

$$L^2(\Gamma\backslash\mathbf{H}) = \mathbf{C} \oplus L^2_{\text{cusp}}(\Gamma\backslash\mathbf{H}) \oplus L^2_{\text{cont}}(\Gamma\backslash\mathbf{H}), \tag{3.22}$$

where $\mathbf{C}$ is the one dimensional space of constant functions, $L^2_{\text{cusp}}(\Gamma\backslash\mathbf{H})$ is the Hilbert space of square integrable functions on $\mathbf{H}$ whose constant term is zero, and $L^2_{\text{cont}}(\Gamma\backslash\mathbf{H})$ represents all square integrable functions on $\mathbf{H}$ which are representable as integrals of the Eisenstein series (Eisenstein transform) $E(z, s) : C_0^\infty(\mathbf{R}) \to L^2(\Gamma\backslash\mathbf{H})$ defined by

$$(f, E(*, s)) = \frac{1}{4\pi}\int_{SL(2,\mathbf{Z})\backslash\mathbf{H}} f(z)\overline{E(z, s)}d\mu(z). \tag{3.23}$$

The reason for the introduction of terminologies $L^2_{\text{cusp}}(\Gamma\backslash\mathbf{H}), L^2_{\text{cont}}(\Gamma\backslash\mathbf{H})$ is because the classical definition of cusp form requires that the constant term in the Fourier expansion around any cusp (a real number or infinity) be zero, and also because the Eisenstein series is in the continuous spectrum of the Laplacian. The spectrum of $\widetilde{\Delta}$ in $L^2_{\text{cusp}}(\Gamma\backslash\mathbf{H})$ is discrete, whereas the spectrum of $\widetilde{\Delta}$ in $L^2_{\text{cont}}(\Gamma\backslash\mathbf{H})$ is absolutely continuous.

Since the quotient $SL(2, \mathbf{Z})\backslash\mathbf{H}$ is not compact, the spectral expansion of a square-integrable function on $\Gamma\backslash\mathbf{H}$ is a sum of two terms - the first term is Fourier series which contains eigenfunctions associated to the discrete spectrum and the second term is an integral which contains eigenfunctions associated to the continuous spectrum of the hyperbolic Laplacian, which are the Eisenstein series.

The Selberg spectral expansion of a function $f \in L^2(SL(2, \mathbf{Z})\backslash\mathbf{H})$ says that every $f \in L^2(\Gamma\backslash\mathbf{H})$ admits the spectral expansion (see Terras [167, pp. 254-257])

$$f(z) = \sum_{j=0}^{\infty} (f, \phi_j)\, \phi_j(z) + \frac{1}{4\pi}\int_{-\infty}^{\infty} \left(f, E\left(z, \frac{1}{2} + ir\right)\right) E\left(z, \frac{1}{2} + ir\right) dr, \tag{3.24}$$

where $(\phi_j : j \geq 0)$ are automorphic eigenfunctions which form an orthonormal basis of $L^2(\mathscr{M})$. For any $f \in L^2(\Gamma\backslash\mathbf{H})$, for which f and $\widetilde{\Delta} f$ are smooth and bounded, the series and integral in (3.24) converge absolutely and uniformly for z ranging over compact sets in $\mathbf{H}$. In particular, let $K(z,z')$ be the automorphic kernel defined by (3.10). Then

$$K(z,z') = \sum_{k=0}^{\infty} h(r_k)\, \phi_k(z)\phi_k(z') + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) E\left(z, \frac{1}{2}+ir\right) \overline{E\left(z', \frac{1}{2}+ir\right)}\, dr, \tag{3.25}$$

with the function $h(r)$ satisfying the conditions $(S.I) - (S.III)$ in Theorem 3.8.

In the subsections that follow we deal with the right-hand side of the trace formula

$$\sum_{k=0}^{\infty} h(r_k) = \int_{\mathscr{M}} K(z,z)\, d\mu(z) - \frac{1}{4\pi}\int_{\mathscr{M}}\int_{-\infty}^{\infty} h(r)\left|E\left(z, \frac{1}{2}+ir\right)\right|^2 dr d\mu(z), \tag{3.26}$$

where

$$K(z,z') = \sum_{\gamma\in\Gamma} k(u(z,\gamma z')), \quad u(z,z') = \frac{|z-z'|^2}{yy'},\ z, z' \in \mathbf{H},$$

is an automorphic kernel with respect to Γ, and

$$H(z,z') := \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) E\left(z, \frac{1}{2}+ir\right) \overline{E\left(z', \frac{1}{2}+ir\right)}\, dr \tag{3.27}$$

is in a certain sense the noncompact part or the principal part of the kernel K.

We shall obtain, via the function $H(z,z')$ a quite elegant, explicit expression for the continuous spectrum of $\widetilde{\Delta}$. Since $\mathscr{M}$ is noncompact we have more work to do here; we need to take care of the contribution of the parabolic element in the series (3.2.1) and the contribution of the continuous spectrum given by (3.27). The next subsection does the preliminary work in the explicit computation of the trace formula for noncompact $\mathscr{M}$, namely it presents the formula we need to attack the integral (3.27); this formula is known as the *Maass-Selberg relation.*

3.2.2 The Maass-Selberg Relation

This subsection obtains the tool needed to deal with the continuous spectrum of $\mathscr{M}$. Let $\mathscr{M} = \Gamma\backslash\mathbf{H}$ be noncompact. We assume straight away that Γ has a simple cusp, which is equivalent to saying that $\Gamma = SL(2,\mathbf{Z})$. The function $\varphi(s)$ given in (2.29) satisfies the following properties:

$$E(z,1-s) = \varphi(1-s)E(z,s), \ \text{ or } \quad E(z,s) = \varphi(s)E(z,1-s); \tag{3.28}$$

$$E(z,s)\varphi(s)^{-1} = \varphi(1-s)E(z,s), \quad \varphi(s)\varphi(1-s) = 1; \tag{3.29}$$

$$\varphi\left(\frac{1}{2}-ir\right) = \overline{\varphi\left(\frac{1}{2}+ir\right)}, \quad \left|\varphi\left(\frac{1}{2}+ir\right)\right|^2 = 1. \tag{3.30}$$

For analytic properties of $\varphi(s)$ and various analytic continuations of $E(s,z)$, see Kubota [97], Müller [118].

Now, for $s\in\mathbf{C}$, we define the truncated Eisenstein series $E^Y(z,s)$ by

$$E^Y(z,s) = \begin{cases} E(z,s), & y \le Y, \\ E(z,s) - y^s - \varphi(s)y^{1-s}, & y > Y \end{cases} \tag{3.31}$$

for a sufficiently large positive number Y with the corresponding fundamental domain $\mathcal{D}_\Gamma^Y$ defined by

$$\mathcal{D}_\Gamma^Y = \{z \in \mathcal{D}_\Gamma \,:\, \operatorname{Im} z = y \le Y\}. \tag{3.32}$$

The function $E^Y(z,s)$ is also automorphic with respect to Γ and it is called the *compact part of* $E(z,s)$. The word 'truncation' here describes the Γ-automorphic function $E^Y(z,s)$ as the function obtained by cutting off the constant term $y^s+\varphi(s)y^{1-s}$ of the Fourier expansion of $E(z,s)$. Since for $\operatorname{Re} s>1$, $E(z,s)-y^s$ is bounded as $y\to\infty$ and the fact that $\mathscr{M}$ has finite invariant measure, it follows that $E^Y(z,s)\in L^2(\mathscr{M})$.

The starting point is to develop a formula involving two Eisenstein series $E(z,s)$ and $E(z,s')$ for $z\in\mathcal{D}_\Gamma$, $s,s'\in\mathbf{C}$. Noting that

$$\widetilde{\Delta}E^Y(z,s)=s(1-s)E^Y(z,s), \qquad \widetilde{\Delta}E^Y(z,s')=s'(1-s')E^Y(z,s'), \tag{3.33}$$

we have

$$\begin{aligned}&(s'-s)(s+s'-1)\int_{\mathcal{D}_\Gamma}E^Y(z,s)E^Y(z,s')\,d\mu(z)\\&=\int_{\mathcal{D}_\Gamma}\Big(E^Y(z,s)\widetilde{\Delta}E^Y(z,s')-E^Y(z,s')\widetilde{\Delta}E^Y(z,s)\Big)\,d\mu(z).\end{aligned}\tag{3.34}$$

Let the fundamental domain $\mathcal{D}_\Gamma=\mathscr{M}$ be fixed and denote by D the subdomain of $\mathcal{D}_\Gamma$ consisting of all points $z\in\mathscr{M}$ such that $y\le Y$. Then by Green's theorem

$$\begin{aligned}&\int_D\Big(E^Y(z,s)\widetilde{\Delta}E^Y(z,s')-E^Y(z,s')\widetilde{\Delta}E^Y(z,s)\Big)\,d\mu(z)\\&=\int_{\partial D}\left(E^Y(z,s)y\frac{\partial E^Y(z,s')}{\partial\nu}-E^Y(z,s')y\frac{\partial E^Y(z,s)}{\partial\nu}\right)\frac{dl}{y},\end{aligned}\tag{3.35}$$

where $\frac{\partial}{\partial\nu}$ is the outer normal derivative and dl is Euclidean arc length, or what is the same, $y\frac{\partial}{\partial\nu}$ is the invariant outer normal derivative on ∂D and $\frac{dl}{y}$ is the invariant length in $\mathbf{H}$. Taking into account the discontinuity along the line $y=Y$, and observing that the integrals over the boundary of $\mathcal{D}_\Gamma$ cancel out and that the contribution coming from the line segment $y=Y$, $|x|\le\frac12$ is

$$\begin{aligned}&\left(Y^s+\varphi(s)Y^{1-s}\right)\frac{d}{dy}\left(Y^{s'}+\varphi(s')Y^{1-s'}\right)-\left(Y^{s'}+\varphi(s')Y^{1-s'}\right)\frac{d}{dy}\left(Y^s+\varphi(s)Y^{1-s}\right)\\&=(s'-s)\left(Y^{s+s'-1}-\varphi(s)\varphi(s')Y^{1-s-s'}\right)+(s'+s-1)\left(\varphi(s)Y^{s'-s}-\varphi(s')Y^{s-s'}\right),\end{aligned}\tag{3.36}$$

we obtain, for $s\neq s'$, $s+s'\neq1$,

$$\int_{\mathcal{D}_\Gamma}E^Y(z,s)E^Y(z,s')d\mu(z)=\frac{Y^{s+s'-1}-\varphi(s)\varphi(s')Y^{1-s-s'}}{s+s'-1}+\frac{Y^{s-s'}\varphi(s')-Y^{s'-s}(s)}{s-s'},\tag{3.37}$$

or by setting $s'=\overline{s}$,

$$\int_{\mathcal{D}_\Gamma}\left|E^Y(z,s)\right|^2d\mu(z)=\frac{Y^{s+\overline{s}-1}-\varphi(s)\varphi(\overline{s})Y^{1-s-\overline{s}}}{s+\overline{s}-1}+\frac{Y^{s-\overline{s}}\varphi(\overline{s})-Y^{\overline{s}-s}(s)}{s-\overline{s}}.\tag{3.38}$$

Setting $s=\sigma+ir$, $r\in\mathbf{R}$, $r\neq0$, $\sigma\neq1/2$ in (3.38), we have

$$\int_{\mathcal{D}_\Gamma}\left|E^Y(z,\sigma+ir)\right|^2d\mu(z)=\frac{Y^{2\sigma-1}-Y^{1-2\sigma}|\varphi(\sigma+ir)|^2}{2\sigma-1}+\frac{Y^{2ir}\overline{\varphi(\sigma+ir)}-Y^{-2ir}\varphi(\sigma+ir)}{2ir}.\tag{3.39}$$

Taking the limit as $\sigma \to \frac{1}{2}$ in (3.39), the first term on the right hand side of (3.39) becomes

$$\begin{aligned}&\lim_{\sigma \to \frac{1}{2}} \left(\frac{Y^{2\sigma-1} - Y^{1-2\sigma}|\varphi(\sigma+ir)|^2}{2\sigma - 1} \right) \\ &= 2\log Y - \frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right),\end{aligned}$$

where we have applied the following approximations as $\sigma \to \frac{1}{2}$:

$$\begin{aligned} Y^{2\sigma-1} &= 1 + (2\sigma - 1)\log Y + \cdots, \\ Y^{1-2\sigma} &= 1 - (2\sigma - 1)\log Y + \cdots, \\ \varphi(\sigma + ir) &= \varphi\left(\frac{1}{2}+ir\right) + \left(\sigma - \frac{1}{2}\right)\varphi'\left(\frac{1}{2}+ir\right) + \cdots, \\ \varphi(\sigma+ir)\,\varphi(\sigma-ir) &= 1 + (2\sigma-1)\varphi'\left(\frac{1}{2}+ir\right)\varphi\left(\frac{1}{2}+ir\right)^{-1} + \cdots, \end{aligned}$$

where, in the last approximation, we have applied (3.29). Hence as $\sigma \to \frac{1}{2}$ in (3.39), we have

Proposition 3.9 (**Maass-Selberg relation**).

$$\begin{aligned}\int_{\mathcal{D}_\Gamma} \left| E^Y\left(z, \frac{1}{2}+ir\right)\right|^2 d\mu(z) = 2\log Y - \frac{\varphi'\left(\frac{1}{2}+ir\right)}{\varphi\left(\frac{1}{2}+ir\right)} \\ + \frac{Y^{2ir}\overline{\varphi\left(\frac{1}{2}+ir\right)} - Y^{-2ir}\varphi\left(\frac{1}{2}+ir\right)}{2ir}.\end{aligned} \tag{3.40}$$

3.2.3 Computation of the Spectral Trace: The Continuous Spectrum

Having obtained the tool we need to attack the contribution of the continuous spectrum, we now evaluate completely the integral (3.27). Towards this end, we first note that by the spectral decomposition given by (3.24) and (3.25),

$$\int_{\mathcal{D}_\Gamma} |P(z,z')|^2\, d\mu(z') = \int_{\mathcal{D}_\Gamma} |\left(K(z,z') - H(z,z')\right)|^2\, d\mu(z') < \infty,$$

and that for $f \in L^2_{\text{cusp}}(\Gamma\backslash\mathbf{H})$,

$$\int_{\mathcal{D}_\Gamma} H(z,z') f(z')\, d\mu(z') = 0. \tag{3.41}$$

Furthermore, the integral operator whose kernel is $P(z,z')$ is a Hilbert-Schmidt operator and so in particular is a compact operator. So we can now proceed to compute explicit formula for the trace

$$\sum_{k=0}^{\infty} h(r_k) = \int_{\mathcal{D}_\Gamma} P(z,z)\, d\mu(z) = \int_{\mathcal{D}_\Gamma} \left(K(z,z) - H(z,z)\right)\, d\mu(z).$$

We have seen in Section 3.1 that when γ is not a parabolic transformation, the component

$$c(\gamma) = \int_{D_\gamma} k(z, \gamma z)\, d\mu(z)$$

of the trace is expressed in terms of the function $h(r)$. But for a parabolic transformation γ the individual $c(\gamma)$ do not exist in general, so we consider a modified component

$$\begin{aligned} c(\infty) &= \int_{\mathcal{D}_\Gamma} P(z,z)d\mu(z) = \lim_{Y\nearrow\infty} \int_{\mathcal{D}_\Gamma^Y} P(z,z)d\mu(z) \\ &= \lim_{Y\nearrow\infty} \int_{\mathcal{D}_\Gamma^Y} \left(K(z,z) - H(z,z)\right) d\mu(z), \end{aligned} \tag{3.42}$$

where $\mathcal{D}_\Gamma^Y$ is the fundamental domain (3.32) corresponding to the truncated Eisenstein series $E^Y(z,s)$. We consider first the limit

$$\lim_{Y\nearrow\infty} \int_{\mathcal{D}_\Gamma^Y} H(z,z)\, d\mu(z). \tag{3.43}$$

Indeed, from (3.43) and (3.27), we have as $Y \nearrow \infty$

$$\begin{aligned} \int_{\mathcal{D}_\Gamma^Y} H(z,z)\, d\mu(z) &= \frac{1}{4\pi} \int_{\mathcal{D}_\Gamma^Y} \int_{-\infty}^{\infty} h(r) \left| E\left(z, \frac{1}{2} + ir\right)\right|^2 dr d\mu(z) \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \left(\int_{\mathcal{D}_\Gamma^Y} \left| E\left(z, \frac{1}{2} + ir\right)\right|^2 d\mu(z)\right) dr, \end{aligned}$$

where the change of the order of integration is justified since the integrand is nonnegative. Thus by (3.31) and (3.40), we obtain

$$\begin{aligned} &\int_{\mathcal{D}_\Gamma^Y} H(z,z)\, d\mu(z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \times \\ &\quad \times \left(\int_{\mathcal{D}_\Gamma^Y} \left| E\left(z, \frac{1}{2} + ir\right) - y^{\frac{1}{2}+ir} - y^{\frac{1}{2}-ir} \sqrt{\pi} \frac{\Gamma(ir)}{\Gamma\left(\frac{1}{2}+ir\right)} L_0\left(\frac{1}{2}+ir\right)\right|^2 d\mu(z)\right) dr, \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \int_{\mathcal{D}_\Gamma^Y} \left| E\left(z, \frac{1}{2}+ir\right)\right|^2 d\mu(z) dr + o(1) \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \left[2\log Y - \frac{\varphi'\left(\frac{1}{2}+ir\right)}{\varphi\left(\frac{1}{2}+ir\right)} + \frac{Y^{2ir}\overline{\varphi\left(\frac{1}{2}+ir\right)} - Y^{-2ir}\varphi\left(\frac{1}{2}+ir\right)}{2ir}\right] dr + o(1) \\ &= \log Y \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)\, dr - \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi'\left(\frac{1}{2}+ir\right)}{\varphi\left(\frac{1}{2}+ir\right)} dr \\ &\quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{Y^{2ir}\overline{\varphi\left(\frac{1}{2}+ir\right)} - Y^{-2ir}\varphi\left(\frac{1}{2}+ir\right)}{2ir} dr + o(1) \end{aligned} \tag{3.44}$$

as $Y \nearrow \infty$, where $L_0(s)$ is some Dirichlet series. The first term on the right-hand side of (3.44) is

$$\log Y \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)\, dr = g(0) \log Y.$$

Since $h(r) = h(-r)$, we see that the last term on the right of (3.44) becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty} h(r) \frac{Y^{2ir}\overline{\varphi\left(\frac{1}{2}+ir\right)} - Y^{-2ir}\varphi\left(\frac{1}{2}+ir\right)}{2ir} dr \\
&= \int_{-\infty}^{\infty} h(-r) \frac{Y^{-2ir}\varphi\left(\frac{1}{2}+ir\right) - Y^{2ir}\varphi\left(\frac{1}{2}-ir\right)}{-2ir} dr \\
&= \int_{-\infty}^{\infty} h(r) \frac{Y^{2ir}\varphi\left(\frac{1}{2}-ir\right) + Y^{-2ir}\varphi\left(\frac{1}{2}-ir\right)}{2ir} dr \\
&= \int_{-\infty}^{\infty} h(r) \frac{Y^{2ir}\varphi\left(\frac{1}{2}-ir\right)}{ir} dr = \int_{-\infty}^{\infty} h(r) \frac{\varphi\left(\frac{1}{2}-ir\right)}{ir} e^{2ir\log Y} dr.
\end{aligned}$$

The integral

$$\int_{-\infty}^{\infty} h(r) \frac{\varphi\left(\frac{1}{2}-ir\right)}{ir} e^{2ir\log Y} dr \tag{3.45}$$

is equivalent to the integral

$$\int_{-\infty}^{\infty} h(r) \frac{\operatorname{Re}\varphi\left(\frac{1}{2}-ir\right)}{r} \sin(2r\log Y) dr. \tag{3.46}$$

In other words, the integrals (3.45) and (3.46) have the same limits. Indeed, using the limit formula

$$f(0) = \lim_{\alpha \nearrow \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \alpha t}{t} dt,$$

we see that the integral (3.46) has the limit

$$\lim_{Y \nearrow \infty} \int_{-\infty}^{\infty} h(r) \frac{\operatorname{Re}\varphi\left(\frac{1}{2}-ir\right)}{r} \sin(2r\log Y) dr = \pi h(0)\varphi\left(\frac{1}{2}\right).$$

Hence, as $Y \nearrow \infty$, we obtain

Proposition 3.10.

$$\int_{\mathcal{D}_\Gamma^Y} H(z,z)\, dA(z) = g(0)\log Y - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) \frac{\varphi'\left(\frac{1}{2}+ir\right)}{\varphi\left(\frac{1}{2}+ir\right)} dr + \frac{1}{4} h(0)\varphi\left(\frac{1}{2}\right) + o(1).$$

3.2.4 Computation of the Trace for the Parabolic Elements

That the explicit computation of the first integral on the right-hand side of (3.26) is the most complicated task in our effort to completely compute the explicit trace formula for a noncompact hyperbolic surface $\mathscr{M}$ is not an overstatement as we shall see in the sequel. To be precise we want to compute an explicit formula for the

$$\int_{\mathcal{D}_\Gamma^Y} K(z,z)\, d\mu(z) = \sum_{\gamma \in \Gamma_\infty} \int_{\mathcal{D}_\Gamma^Y} k(z, \gamma z)\, d\mu(z) \tag{3.47}$$

as $Y \nearrow \infty$. Towards this end, we recall that any parabolic transformation is conjugate to the translation $\Gamma_\infty : z \to \gamma z = z+1$, $z \in \mathbf{H}$, with the fundamental domain $\mathcal{D}_\Gamma^Y = [0,1] \times (0,\infty)$.

Indeed,

$$\begin{aligned}\lim_{Y\nearrow\infty}\int_{\mathcal{D}_\Gamma^Y} K(z,z)\,d\mu(z) &= \lim_{Y\nearrow\infty}\sum_{l\neq 0, l\in\mathbf{Z}}\int_{\mathcal{D}_\Gamma^Y} k(u(z,z+l))\,d\mu(z)\\ &= \lim_{Y\nearrow\infty} 2\sum_{l=1}^{\infty}\int_0^Y\int_0^1 k\left(\frac{l^2}{y^2}\right)\frac{dxdy}{y^2}\\ &= \lim_{Y\nearrow\infty} 2\sum_{l=1}^{\infty}\int_0^Y k\left(\frac{l^2}{y^2}\right)\frac{dy}{y^2}.\end{aligned}\tag{3.48}$$

The main task now is the evaluation of the integral in (3.48). For this purpose we make a substitution $u = l/y$ and incorporate the sum into the integral to get

$$2\sum_{l=1}^{\infty}\int_0^Y k\left(\frac{l^2}{y^2}\right)\frac{dy}{y^2} = 2\sum_{l=1}^{\infty}\int_{\frac{l}{Y}}^{\infty} k\left(u^2\right)\frac{1}{l}\,du = 2\int_0^{\infty} k\left(u^2\right)\left(\sum_{1\le l\le Yu}^{\infty}\frac{1}{l}\right)du. \tag{3.49}$$

Noting that for $u > 0$,

$$\sum_{1\le k\le Au}\frac{1}{k} = \log(Au) + C + O\left(\frac{1}{\sqrt{Au}}\right),$$

where C is the Euler-Mascheroni constant, we obtain

$$\begin{aligned}2\sum_{l=1}^{\infty}\int_0^Y k\left(\frac{l^2}{y^2}\right)\frac{dy}{y^2} &= 2\int_0^{\infty} k\left(u^2\right)\left(\log(Yu)+C\right)\,du + O\left(\frac{1}{\sqrt{Y}}\right)\\ &= 2\left(\log Y + C\right)\int_0^{\infty} k\left(u^2\right)\,du\\ &\quad + 2\int_0^{\infty}(\log u)k\left(u^2\right)\,du + O\left(\frac{1}{\sqrt{Y}}\right)\end{aligned}\tag{3.50}$$

as $Y \nearrow \infty$. Setting $u^2 = t$, and noting that $Q\left(e^u + e^{-u} - 2\right) = g(u)$ and $e^u + e^{-u} - 2 = 0$ implies $u = 0$, the first integral on the right-hand side of (3.50) becomes

Lemma 3.11.

$$\int_0^{\infty} k\left(u^2\right)\,du = \frac{1}{2}\int_0^{\infty}\frac{k(t)}{\sqrt{t}}\,dt = \frac{1}{2}Q(0) = \frac{1}{2}g(0). \tag{3.51}$$

It remains to evaluate the second integral on the right-hand side of (3.50). By the same substitution as in the first integral, we obtain

$$\begin{aligned}2\int_0^{\infty}(\log u)k\left(u^2\right)\,du &= \frac{1}{2}\int_0^{\infty}(\log t)\frac{k(t)}{\sqrt{t}}\,dt = -\frac{1}{2}\int_0^{\infty}\left(\frac{\log t}{\sqrt{t}}\right)\frac{1}{\pi}\int_0^w\frac{dQ(w)}{\sqrt{w-t}}\,dt\\ &= -\frac{1}{2\pi}\int_0^{\infty}\int_0^w\frac{\log t}{\sqrt{t(w-t)}}\,dtdQ(w).\end{aligned}\tag{3.52}$$

The second integral on the right-hand side of (3.52) yields

$$\begin{aligned}\int_0^w\frac{\log t}{\sqrt{t(w-t)}}\,dt &= \int_0^w\frac{\log t}{\sqrt{w}\sqrt{t\left(1-\frac{t}{w}\right)}}\,dt = \int_0^1\frac{\log uw}{\sqrt{u\left(1-u\right)}}\,du \qquad \left(\frac{t}{w}=u\right)\\ &= \int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}\log u\,du + \log w\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}\,du.\end{aligned}\tag{3.53}$$

Using the identity (Gradshtejn and Ryzhik [66, p. 540, eq. 4.253.1])

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\log t\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left[\psi(\alpha)-\psi(\alpha+\beta)\right], \tag{3.54}$$

and the special values (B.12) of the diagamma function $\psi(s)$, the first integral on the right-hand side of (3.53) becomes

$$\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}\log u\,du = \pi\left[\psi\left(\frac{1}{2}\right)-\psi(1)\right] = -2\pi\log 2, \tag{3.55}$$

while the second integral on the right-hand side of (3.53) is the beta function

$$\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}\,du = B\left(\frac{1}{2},\frac{1}{2}\right) = \pi.$$

Thus

$$\int_0^w \frac{\log t}{\sqrt{t(w-t)}}\,dt = \pi(\log w - 2\log 2). \tag{3.56}$$

By (3.52), (3.53), (3.55) and (3.56), we obtain

$$2\int_0^\infty (\log u)k\left(u^2\right)\,du = -\frac{1}{2}\int_0^\infty (\log w - 2\log 2)dQ(w), \tag{3.57}$$

and by (3.7) we have

$$\begin{aligned}2\int_0^\infty (\log u)k\left(u^2\right)\,du &= -\frac{1}{2}\int_0^\infty \log\left(e^u+e^{-u}-2\right)\,dg(u) + \log 2\int_0^\infty dQ(w)\\ &= -\frac{1}{2}\int_0^\infty \log\left(e^u+e^{-u}-2\right)\,dg(u) - \log 2Q(0)\end{aligned} \tag{3.58}$$

By applying elementary laws of logarithm, we obtain

$$\begin{aligned}2\int_0^\infty (\log u)k\left(u^2\right)\,du &= -\frac{1}{2}\int_0^\infty \log\left(1-e^{-u}\right)^2 dg(u) - \frac{1}{2}\int_0^\infty u\,dg(u) - \log 2g(0)\\ &= -\int_0^\infty \log\left(1-e^{-u}\right)dg(u) + \frac{1}{4}\int_{-\infty}^\infty g(u)\,du - \log 2g(0)\\ &= -\int_0^\infty \log\left(1-e^{-u}\right)dg(u) + \frac{1}{4}h(0) - \log 2g(0).\end{aligned} \tag{3.59}$$

Noting that

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-iru}h(r)\,dr, \qquad g'(u) = -\frac{1}{2\pi}\int_{-\infty}^\infty ire^{-iru}h(r)\,dr,$$

the integral on the right-hand side of (3.59) gives

$$\begin{aligned}-\int_0^\infty \log\left(1-e^{-u}\right)dg(u) &= -\int_0^\infty \log\left(1-e^{-u}\right)g'(u)\,du\\ &= \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\left\{\int_0^\infty ire^{-iru}\log\left(1-e^{-u}\right)\,du\right\}\,dr.\end{aligned} \tag{3.60}$$

Lemma 3.12.

$$\int_0^\infty ire^{-iru}\log\left(1-e^{-u}\right)\,du = -C - \frac{\Gamma'(1+ir)}{\Gamma(1+ir)},$$

where C is the Euler-Mascheroni constant.

Proof. Making the substitution $x = 1 - e^{-u}$, $e^{-u} = 1 - x$, we have

$$\int_0^\infty ire^{-iru} \log\left(1 - e^{-u}\right) du = \int_0^1 ir(1-x)^{-1+ir} \log x \, dx.$$

Again, using (3.54) with $\alpha = 1$ and (B.12), we have

$$\begin{aligned}\int_0^\infty ire^{-iru} \log\left(1 - e^{-u}\right) du &= \int_0^1 ir(1-x)^{-1+ir} \log x \, dx \\ &= ir\frac{\Gamma(ir)}{\Gamma(1+ir)} \left[\psi(1) - \psi(1+ir)\right] \\ &= -C - \frac{\Gamma'\left(1+ir\right)}{\Gamma\left(1+ir\right)}.\end{aligned}$$

□

Thus,

$$\begin{aligned}-\int_0^\infty \log\left(1 - e^{-u}\right) dg(u) &= -C\frac{1}{2\pi}\int_{-\infty}^\infty h(r)\, dr - \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\frac{\Gamma'\left(1+ir\right)}{\Gamma\left(1+ir\right)}\, dr \\ &= -Cg(0) - \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\frac{\Gamma'\left(1+ir\right)}{\Gamma\left(1+ir\right)}\, dr.\end{aligned}$$

Hence, from (3.59), (3.60) and Lemma 3.12 we obtain

Lemma 3.13.

$$2\int_0^\infty (\log u) k\left(u^2\right) du = -Cg(0) - \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\frac{\Gamma'\left(1+ir\right)}{\Gamma\left(1+ir\right)}\, dr + \frac{1}{4}h(0) - \log 2g(0).$$

Therefore, from (3.48), (3.50), Lemmas 3.11 and (3.13), we obtain

Proposition 3.14.

$$\int_{\mathcal{D}_\Gamma^Y} K(z,z) d\mu(z) = g(0)\log Y - \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\frac{\Gamma'\left(1+ir\right)}{\Gamma\left(1+ir\right)}\, dr + \frac{1}{4}h(0) - \log 2g(0) + o(1)$$

as $Y \nearrow \infty$.

In summary we have found explicit formulae for the contribution of the continuous spectrum (Proposition 3.10) and parabolic elements (Proposition 3.14) and by adding both formulae we obtain

Proposition 3.15.

$$\begin{aligned}c(\infty) = \frac{1}{4\pi}\int_{-\infty}^\infty h(r)\frac{\varphi'\left(\frac{1}{2}+ir\right)}{\varphi\left(\frac{1}{2}+ir\right)}\, dr - \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\frac{\Gamma'\left(1+ir\right)}{\Gamma\left(1+ir\right)}\, dr \\ - g(0)\log 2 + \frac{1}{4}\left(1 - \varphi\left(\frac{1}{2}\right)\right) h(0).\end{aligned}$$

The trace formula (3.26) in general says that the spectral side (which is the left-hand side) is equal to the right-hand side (the geometric side) which is the sum of the contribution of the identity, hyperbolic, parabolic elements and continuous spectrum. We recall that the contributions of the identity and hyperbolic elements have been taken care of in Subsections 3.1.1 and 3.1.2 respectively. Adding Propositions 3.6, 3.7 and 3.15, we obtain the following statement.

Theorem 3.16.

$$\begin{aligned}\sum_{k=0}^{\infty} h(r_k) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} h\left(\frac{1}{4}+r^2\right)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr - \frac{\varphi\left(\frac{1}{2}\right)}{4}h\left(\frac{1}{4}\right)\\ &+ \frac{\mu\left(\Gamma\backslash\mathbf{H}\right)}{4\pi}\int_{-\infty}^{\infty} rh\left(\frac{1}{4}+r^2\right)\tanh\pi r\,dr + \sum_{\{\gamma\}_p}\sum_{n=1}^{\infty}\frac{\ell(\gamma)}{2\sinh\frac{n\ell(\gamma)}{2}}g(n\ell(\gamma))\\ &- g(0)\log 2 + \frac{1}{4}h\left(\frac{1}{4}\right) - \frac{1}{2\pi}\int_{-\infty}^{\infty} h\left(r^2+\frac{1}{4}\right)\frac{\Gamma'}{\Gamma}\left(1+ir\right)\,dr.\end{aligned} \tag{3.61}$$

We can simplify further since $\varphi(s)$ is given explicitly. To this end, inserting $s \to 2s-1$ in the functional equation (B.22) for $\zeta(s)$, and then using (2.29), we obtain

$$\varphi(s) = \pi^{2s-1}\frac{\Gamma\left(1-s\right)}{\Gamma(s)}\frac{\zeta(2-2s)}{\zeta(2s)}. \tag{3.62}$$

Taking the logarithmic derivative of (3.62) we obtain

$$\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right) = 2\log\pi - \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right) - \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-ir\right) - 2\frac{\zeta'}{\zeta}\left(1+2ir\right) - 2\frac{\zeta'}{\zeta}\left(1-2ir\right). \tag{3.63}$$

Thus from Theorem (3.16), we have

$$\begin{aligned}&\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr\\ &= \frac{\log\pi}{2\pi}\int_{-\infty}^{\infty} h(r)\,dr - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-ir\right)\right]\,dr\\ &\quad - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\left[\frac{\zeta'}{\zeta}\left(1+2ir\right)+\frac{1}{2ir}\right]\,dr - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\left[\frac{\zeta'}{\zeta}\left(1-2ir\right)-\frac{1}{2ir}\right]\,dr.\end{aligned}$$

Since $h(r) = h(-r)$, we have

$$\begin{aligned}&\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr\\ &= g(0)\log\pi - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)\,dr - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\left[\frac{\zeta'}{\zeta}\left(1+2ir\right)+\frac{1}{2ir}\right]\,dr.\end{aligned}$$

Using the formula

$$\int_{\mathrm{Im}\, r=-\varepsilon}\frac{h(r)}{r}\,dr = \pi i h(0)$$

which follows from Cauchy integral formula, and the identity (B.29), we obtain

$$
\begin{aligned}
&\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr \\
&= g(0)\log\pi - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)\,dr \\
&\quad + \frac{1}{\pi}\int_{\operatorname{Im} r=-\varepsilon} h(r)\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+2ir}}\right)dr - \frac{1}{2\pi i}\int\limits_{\operatorname{Im} r=-\varepsilon}\frac{h(r)}{r}\,dr \\
&= g(0)\log\pi - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)\,dr \\
&\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\int\limits_{\operatorname{Im} r=-\varepsilon} h(r)e^{-2ir\log n}\,dr - \frac{1}{2\pi i}\int\limits_{\operatorname{Im} r=-\varepsilon}\frac{h(r)}{r}\,dr \\
&= g(0)\log\pi - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)\,dr + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\log n) - \frac{1}{2}h(0),
\end{aligned}
$$

Hence, by noting that $\varphi\left(\frac{1}{2}\right)=1$, we have shown that the contributions of the parabolic elements of $SL(2,\mathbf{Z})$ and continuous spectrum of $\widetilde{\Delta}$ is given by

Proposition 3.17.

$$
c(\infty) = g(0)\log\left(\frac{\pi}{2}\right) - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\left[\frac{\Gamma'}{\Gamma}(1+ir) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)\right]dr + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\log n).
$$

In summary we have the following statement.

Theorem 3.18. *Let $\mathscr{M} = SL(2,\mathbf{Z})\backslash\mathbf{H}$ be a noncompact, finite-area hyperbolic surface and let $h(r)$ satisfy the conditions (S.I)-(S.III) in Theorem 3.8. Then the following equality holds:*

$$
\sum_{k\geq 0} h(r_k) = c(\mathcal{I}) + c(\mathcal{H}) + c(\mathcal{P}_1) + c(\mathcal{P}_2) + c(\mathcal{P}_3) + c(\mathcal{P}_4), \tag{3.64}
$$

where $\left(\lambda_k = \frac{1}{4}+r_k^2 \,:\, k\geq 0\right)$ is the discrete spectrum of the Laplacian $\widetilde{\Delta}$ on $\mathscr{M}$. The term $c(\mathcal{I})$ which is the contribution of the identity element of the group $SL(2,\mathbf{Z})$ is given by

$$
c(\mathcal{I}) = \frac{1}{12}\int_{-\infty}^{\infty} rh\left(\frac{1}{4}+r^2\right)\tanh\pi r\,dr. \tag{3.65}
$$

The term $c(\mathcal{H})$ which is the contribution of the hyperbolic elements is given by

$$
c(\mathcal{H}) = \sum_{\{\gamma\}_p}\sum_{n=1}^{\infty}\frac{\ell(\gamma)}{2\sinh\frac{n\ell(\gamma)}{2}}g(n\ell(\gamma)), \tag{3.66}
$$

where $\{\gamma\}_p$ denotes the hyperbolic conjugacy class of γ in $SL(2,\mathbf{Z})$ and the function g is given by

$$
g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru}h\left(r^2+\frac{1}{4}\right)\,dr.
$$

Finally, the last four terms are the contributions of the parabolic elements of $SL(2,\mathbf{Z})$ and the continuous spectrum of $\mathscr{M}$ and are given by

$$\begin{aligned} c(\mathcal{P}_1) &= g(0)\log\frac{\pi}{2}, \\ c(\mathcal{P}_2) &= 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\ln n), \\ c(\mathcal{P}_3) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} h\left(r^2+\frac{1}{4}\right)\frac{\Gamma'}{\Gamma}(1+ir)\,dr, \\ c(\mathcal{P}_4) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} h\left(r^2+\frac{1}{4}\right)\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr, \end{aligned} \tag{3.67}$$

where $\Lambda(n)$ is the Mangoldt function defined by (B.30). All the series and integrals are absolutely convergent.

3.3 The Parseval Inner Product Formula

In this section we present a result on the Parseval formula for the inner product of two non-holomorphic Eisenstein series for the modular surface $SL(2,\mathbf{Z})\backslash\mathbf{H}$. The Parseval inner product formula for automorphic functions is a generalisation of the Selberg spectral decomposition of automorphic functions in $L^2(\mathscr{M})$ discussed in Subsection 3.2.1, namely formula (3.24). The Parseval inner product formula is the spectral expansion of the inner product of two automorphic functions in $L^2(\mathscr{M})$, where as Selberg spectral expansion is the spectral expansion of a single automorphic function in $L^2(\mathscr{M})$. In other words the Parseval inner product formula is a complicated version of the Selberg spectral expansion of automorphic functions $f\in L^2(\mathscr{M})$.

The Rankin-Selberg transform. For $z\in\mathbf{H}$, $\gamma\in SL(2,\mathbf{Z})$, the function $g(z)$ satisfies $g(\gamma z)=g(z+1)=g(z)$, and so it admits the Fourier expansion

$$g(z)=\sum_{m=-\infty}^{\infty}a_m(y)e^{2\pi m i x},\quad z=x+iy\in\mathbf{H}.$$

Let $\mathscr{R}g$ denote the *Rankin-Selberg transform* of g. The typical "unfolding" trick is the following

$$\begin{aligned} (\mathscr{R}g)(s) &= \int_{SL(2,\mathbf{Z})\backslash\mathbf{H}} g(z)E(z,s)\,d\mu(z) = \int_{SL(2,\mathbf{Z})\backslash\mathbf{H}}\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbf{Z})} g(\gamma z)\mathrm{Im}\,(\gamma z)^s\,d\mu(z) \\ &= \int_{\Gamma_\infty\backslash\mathbf{H}} g(z)\mathrm{Im}\,(z)^s\,d\mu(z) = \int_0^\infty y^s\int_0^1 g(x+iy)\frac{dxdy}{y^2} \\ &= \int_0^\infty a_0(y)y^{s-2}\,dy, \end{aligned} \tag{3.68}$$

where $\mathrm{Re}\,s>1$ and $a_0(y)$ is the constant term in the Fourier expansion of g. Here we have used the fundamental domain $\mathcal{D}_\Gamma=[0,1]\times(0,\infty)$ for $\Gamma_\infty\backslash\mathbf{H}$, or what is the same, $\mathcal{D}_\infty=\{0\le\mathrm{Re}\,z\le1\}\subset\mathbf{H}$ for the fundamental domain for Γ_∞. Therefore the properties of $E(z,s)$ given in Subsection 2.2.2 imply the corresponding properties of $(\mathscr{R}g)(s)$, i.e., $(\mathscr{R}g)(s)$ can be meromorphically continued to the whole complex plane with a simple pole at $s=1$ with the following properties (Zagier [179]):

$$\operatorname{Res}\,(\mathscr{R}g)(s)\Big|_{s=1} = \frac{3}{\pi}\int_{SL(2,\mathbf{Z})\backslash\mathbf{H}} g(z)\,d\mu(z),$$
$$(\widetilde{\mathscr{R}g})(s) = \Lambda(2s)(\mathscr{R}g)(s) = \pi^{-s}\Gamma(s)\zeta(2s)(\mathscr{R}g)(s),$$
$$(\widetilde{\mathscr{R}g})(s) = (\widetilde{\mathscr{R}g})(1-s),$$

where

$$\operatorname{Res} E(z,s)\Big|_{s=1} = \frac{3}{\pi} \quad \text{for all } z \in \mathbf{H}.$$

The idea of integrating an $SL(2,\mathbf{Z})$-invariant function $g(z)$ against an Eisenstein series was introduced independently by Rankin [138] and Selberg [147], who observed that in the region of absolute convergence of the Eisenstein series this integral equals the Mellin transform of the constant term in the Fourier expansion of g.

For $f_1, f_2 \in L^2(SL(2,\mathbf{Z})\backslash\mathbf{H})$, the Parseval inner product formula (Motohashi [116, p. 14, eq. 1.1.47])

$$(f_1,f_2) = \sum_{j=1}^{\infty}(f_1,\phi_j)\,\overline{(f_2,\phi_j)} + \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(f_1, E\left(z,\frac{1}{2}+ir\right)\right)\overline{\left(f_2, E\left(z,\frac{1}{2}+ir\right)\right)}\,dr \tag{3.69}$$

is a general case of the Selberg spectral decomposition (3.24), where $(\phi_j : j \geq 1)$ is a cusp form. The Selberg spectral expansion (3.24) and the Parseval inner produt formula (3.69) have been computed for nonholomorphic automorphic forms by notable authors, namely Goldfeld [65] and Motohashi [116]. Goldfeld [65] considers an $L^2\,(SL(2,\mathbf{Z})\backslash\mathbf{H})$-integrable product of automorphic form associated with a congruence subgroup of $SL(2,\mathbf{Z})$ using (3.24). The Parseval formula (3.69) for the inner product $\langle f, P_m(\cdot,\overline{s})\rangle$ is computed by Motohashi ([116]), where $P_m(z,s)$ is the nonholomorphic Poincaré series given by

$$P_m(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbf{Z})}(\operatorname{Im}\gamma z)^s e^{2\pi i m\gamma z}, \qquad z\in\mathbf{H}, \operatorname{Re}s>1, m\geq 0, \tag{3.70}$$

and $\overline{s}$ is the complex conjugate of s. To be precise Motohashi [116] computes the following inner product expansion

$$\begin{aligned}(P_m(\cdot,s_1),P_m(\cdot,\overline{s})) &= \sum_{j=0}^{\infty}(P_m(\cdot,s_1),\psi_j)\,\overline{(P_m(\cdot,\overline{s}),\psi_j)} \\ &\quad + \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(P_m(\cdot,s_1),E\left(z,\frac{1}{2}+ir\right)\right)\overline{\left(P_m(\cdot,\overline{s}),E\left(z,\frac{1}{2}+ir\right)\right)}\,dr.\end{aligned} \tag{3.71}$$

Unlike Goldfeld [65] that applies formula (3.24), we shall use the complicated version of (3.24), i.e., the Parseval inner product formula (3.69); it is a complicated version in the sense that we have two inner products as the integrand to work with, while (3.24) has only one inner product as the integrand. Also, [116] considers a single Poincaré series, we shall replace the Poincaré series in (3.71) with the product

$$F(z,s_1,s_2) = \widetilde{E}(z,s_1)\widetilde{E}(z,s_2) \in L^2\left(SL(2,\mathbf{Z})\backslash\mathbf{H}\right), \qquad z\in\mathbf{H}, s\in\mathbf{C},$$

where

$$\widetilde{E}(z,s) = \Lambda(2s)E(z,s) = \frac{1}{2}\pi^s\Gamma(s)E^*(z,s),$$

with $\Lambda(s)$ given by (B.23). The functional equation (2.44) then implies

$$\widetilde{E}(z,s) = \widetilde{E}(z,1-s).$$

The precise statement is the following (see also Awonusika [5]).

Theorem 3.19. *Let $f_1, f_2 \in L^2\left(SL(2,\mathbf{Z})\backslash\mathbf{H}\right)$ be nonholomorphic automorphic forms whose inner product admits the Parseval inner product formula (3.69). If*

$$f_1(z) = F\left(z,s_1,s_2\right) = \widetilde{E}\left(z,s_1\right)\widetilde{E}\left(z,s_2\right) \in L^2\left(SL(2,\mathbf{Z})\backslash\mathbf{H}\right)$$

and

$$f_2(z) = F\left(z,\overline{s}_1',\overline{s}_2'\right) = \widetilde{E}\left(z,\overline{s}_1'\right)\widetilde{E}\left(z,\overline{s}_2'\right) \in L^2\left(SL(2,\mathbf{Z})\backslash\mathbf{H}\right),$$

then

$$\begin{aligned}&(F(\cdot,s_1,s_2),F(\cdot,\overline{s}_1',\overline{s}_2'))\\ &= \frac{1}{4\pi^{2(s_2+s_2')}}\sum_{j=1}^{\infty}\frac{\mathit{\Theta}\left(s_1,s_2,r_j\right)\mathit{\Theta}\left(s_1',s_2',r_j\right)}{\left|a_j(1)\right|^2} + \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\mathit{\Xi}(s_1,s_2,r)\mathit{\Xi}(s_1',s_2',r)}{|\Lambda(1+2ir)|^2}\,dr,\end{aligned} \tag{3.72}$$

where

$$\begin{aligned}\mathit{\Theta}\left(s_1,s_2,r_j\right) &= \Gamma\left(s_1,s_2,r_j\right)L_{\overline{\phi_j}}\left(s_2-s_1+\frac{1}{2}\right)L_{\overline{\phi_j}}\left(s_2+s_1-\frac{1}{2}\right),\\ \mathit{\Theta}\left(s_1',s_2',r_j\right) &= \Gamma\left(s_1',s_2',r_j\right)L_{\phi_j}\left(s_2'-s_1'+\frac{1}{2}\right)L_{\phi_j}\left(s_2'+s_1'-\frac{1}{2}\right),\\ \Gamma\left(s_1,s_2,r_j\right) &= \Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(c\right)\Gamma\left(d\right),\\ \Gamma\left(s_1',s_2',r_j\right) &= \Gamma\left(a'\right)\Gamma\left(b'\right)\Gamma\left(c'\right)\Gamma\left(d'\right),\end{aligned}$$

with

$$\begin{aligned}a &= \frac{s_2+s_1-\frac{1}{2}-ir_j}{2},\ b = \frac{s_2-s_1+\frac{1}{2}-ir_j}{2},\ c = \frac{s_2+s_1-\frac{1}{2}+ir_j}{2},\ d = \frac{s_2-s_1+\frac{1}{2}+ir_j}{2},\\ a' &= \frac{s_2'+s_1'-\frac{1}{2}+ir_j}{2},\ b' = \frac{s_2'-s_1'+\frac{1}{2}+ir_j}{2},\ c' = \frac{s_2'+s_1'-\frac{1}{2}-ir_j}{2},\ d' = \frac{s_2'-s_1'+\frac{1}{2}-ir_j}{2},\end{aligned}$$

and

$$\begin{aligned}\mathit{\Xi}(s_1,s_2,r) &= \Lambda\left(s_1+s_2-\frac{1}{2}-ir\right)\Lambda\left(s_2-s_1+\frac{1}{2}-ir\right)\\ &\quad\times\Lambda\left(s_1-s_2+\frac{1}{2}-ir\right)\Lambda\left(-s_1-s_2+\frac{3}{2}-ir\right),\\ \mathit{\Xi}(s_1',s_2',r) &= \Lambda\left(s_1'+s_2'-\frac{1}{2}+ir\right)\Lambda\left(s_2'-s_1'+\frac{1}{2}+ir\right)\\ &\quad\times\Lambda\left(s_1'-s_2'+\frac{1}{2}+ir\right)\Lambda\left(-s_1'-s_2'+\frac{3}{2}+ir\right)\end{aligned}$$

with the Dirichlet series $L_\phi(s)$ *given by*

$$L_\phi(s) = \sum_{m=1}^{\infty} \frac{a_j(m)}{m^s}, \quad L_{\overline{\phi}}(s) = \sum_{m=1}^{\infty} \frac{\overline{a_j(m)}}{m^s},$$

which converge for $Re\, s > 1$ *since* ϕ *is a cusp form. In particular, for* $s_1 = s_2' = \frac{1}{2} + ir$, $s_2 = s_1' = \frac{1}{2} - ir$,

$$\begin{aligned}
&\left(F\left(\cdot, \frac{1}{2}+ir, \frac{1}{2}-ir\right), F\left(\cdot, \frac{1}{2}+ir, \frac{1}{2}-ir\right)\right) \\
&= \frac{1}{4\pi^2} \sum_{j=1}^{\infty} \frac{\widetilde{\Theta}\,(r_j)}{|a_j(1)|^2} + \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\widetilde{\Xi}(r)}{|\Lambda(1+2ir)|^2}\, dr \\
&= \frac{1}{4\pi^2} \left[\sum_{j=1}^{\infty} \frac{\mathsf{N}\,(r_j)\,\mathsf{L}\,(r_j)}{|a_j(1)|^2} + \pi \int_{-\infty}^{\infty} \frac{\mathsf{N}(r)\mathsf{Q}(r)\cosh \pi r}{|\zeta(1+2ir)|^2}\, dr \right],
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\Theta}\,(r) &= \Theta^2\left(\frac{1}{2}+ir, \frac{1}{2}-ir, r\right) \\
&= \left|\Gamma\left(\frac{1}{4}+\frac{ir}{2}\right)\right|^6 \left|\Gamma\left(\frac{1}{4}+\frac{3ir}{2}\right)\right|^2 \left|L_\phi\left(\frac{1}{2}\right)\right|^2 \left|L_{\overline{\phi}}\left(\frac{1}{2}+2ir\right)\right|^2, \\
\widetilde{\Xi}\,(r) &= \Xi^2\left(\frac{1}{2}+ir, \frac{1}{2}-ir, r\right) = \left|\Lambda\left(\frac{1}{2}+ir\right)\right|^6 \left|\Lambda\left(\frac{1}{2}+3ir\right)\right|^2 \\
\mathsf{N}(r) &= \left|\Gamma\left(\frac{1}{4}+\frac{ir}{2}\right)\right|^2 \left|\Gamma\left(\frac{1}{4}+\frac{3ir}{2}\right)\right|^2, \\
\mathsf{L}(r) &= \left|L_\phi\left(\frac{1}{2}\right)\right|^2 \left|L_{\overline{\phi}}\left(\frac{1}{2}+2ir\right)\right|^2, \quad \mathsf{Q}(r) = \left|\zeta\left(\frac{1}{2}+ir\right)\right|^2 \left|\zeta\left(\frac{1}{2}+3ir\right)\right|^2.
\end{aligned}$$

Proof. By the inner product formula (3.69) it suffices to compute explicit formula for the inner product

$$\begin{aligned}
&(F(\cdot, s_1, s_2), F(\cdot, \overline{s}_1', \overline{s}_2')) \\
&= \sum_{j=1}^{\infty} (F(\cdot, s_1, s_2), \phi_j)\, \overline{(F(\cdot, \overline{s}_1', \overline{s}_2'), \phi_j)} \\
&\quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(F(\cdot, s_1, s_2), E\left(z, \frac{1}{2}+ir\right)\right) \overline{\left(F(\cdot, \overline{s}_1', \overline{s}_2'), E\left(z, \frac{1}{2}+ir\right)\right)}\, dr.
\end{aligned} \tag{3.73}$$

Each inner product on the right-hand side of (3.73) will be handled separately. Towards this end we set

$$\mathbb{F}_{\phi_j}(s_1, s_2, s_1', s_2') = (F(\cdot, s_1, s_2), \phi_j)\, \overline{(F(\cdot, \overline{s}_1', \overline{s}_2'), \phi_j)} \tag{3.74}$$

$$\mathbb{F}_E(s_1, s_2, s_1', s_2', r) = \left(F(\cdot, s_1, s_2), E\left(z, \frac{1}{2}+ir\right)\right) \overline{\left(F(\cdot, \overline{s}_1', \overline{s}_2'), E\left(z, \frac{1}{2}+ir\right)\right)}. \tag{3.75}$$

The main tool is the Rankin-Selberg transform. Clearly the Rankin-Selberg method (3.68) can be viewed as the inner product $(\mathscr{R}g)(s) = (E(\cdot, s), g)$. With this inner product notation and noting that ϕ is $SL(2,\mathbf{Z})$-automorphic we have

$$\begin{aligned}
(F(\cdot,s_1,s_2),\phi_j) &= \int_{SL(2,\mathbf{Z})\backslash\mathbf{H}} \widetilde{E}(z,s_1)\overline{\phi_j(z)}\widetilde{E}(z,s_2)\,\frac{dxdy}{y^2} \\
&= \Lambda(2s_2)\int_{SL(2,\mathbf{Z})\backslash\mathbf{H}}\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbf{Z})} \widetilde{E}(z,s_1)\overline{\phi_j(z)}\mathrm{Im}\,(\gamma z)^{s_2}\,\frac{dxdy}{y^2} \\
&= \Lambda(2s_2)\int_{\Gamma_\infty\backslash\mathbf{H}} \widetilde{E}(z,s_1)\overline{\phi_j(z)}\mathrm{Im}\,(z)^{s_2}\,\frac{dxdy}{y^2} \\
&= \Lambda(2s_2)\int_0^\infty\int_0^1 \widetilde{E}(z,s_1)\overline{\phi_j(z)}y^{s_2-2}\,dxdy.
\end{aligned}$$

Noting that the constant term of the product $\widetilde{E}(z,s_1)\overline{\phi_j(z)}$ is (see e.g. Zagier [180])

$$\int_0^1 \widetilde{E}(x+iy,s_1)\overline{\phi_j(x+iy)}\,dx = 2y\sum_{m\neq 0}\overline{a_j(m)}\sigma_{2s_1-1}(m)|m|^{\frac{1}{2}-s_1}K_{s_1-\frac{1}{2}}(2\pi|m|y)K_{-ir_j}(2\pi|m|y),$$

we obtain

$$\begin{aligned}
(F(\cdot,s_1,s_2),\phi_j) = 2(2\pi)^{-s_2}\Lambda(2s_2)&\sum_{m\neq 0}\overline{a_j(m)}\sigma_{2s_1-1}(m)|m|^{\frac{1}{2}-2s_1-s_2} \\
&\times\int_0^\infty \eta^{s_2-1}K_{s_1-\frac{1}{2}}(\eta)K_{-ir_j}(\eta)\,d\eta.
\end{aligned}$$

By using the identity (B.51) with

$$\sigma = 1-s_2,\ \mu = s_1-\frac{1}{2},\ \nu=-ir_j,\ a=b=1,$$

we have

$$\begin{aligned}
\int_0^\infty \eta^{s_2-1}K_{s_1-\frac{1}{2}}(\eta)K_{-ir_j}(\eta)\,d\eta = \frac{2^{s_2}}{8\Gamma(s_2)}&\Gamma\left(\frac{s_2+s_1-\frac{1}{2}-ir_j}{2}\right)\Gamma\left(\frac{s_2-s_1+\frac{1}{2}-ir_j}{2}\right) \\
&\times\Gamma\left(\frac{s_2+s_1-\frac{1}{2}+ir_j}{2}\right)\Gamma\left(\frac{s_2-s_1+\frac{1}{2}+ir_j}{2}\right).
\end{aligned} \tag{3.76}$$

Thus,

$$(F(\cdot,s_1,s_2),\phi_j) = \frac{1}{4\pi^{2s_2}}\zeta\left(2s_2\right)\Gamma\left(s_1,s_2,r_j\right)\sum_{m\neq 0}\overline{a_j(m)}\sigma_{2s_1-1}(m)|m|^{\frac{1}{2}-2s_1-s_2},$$

where

$$\Gamma\left(s_1,s_2,r_j\right) = \Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(c\right)\Gamma\left(d\right),$$

with

$$a=\frac{s_2+s_1-\frac{1}{2}-ir_j}{2},\ b=\frac{s_2-s_1+\frac{1}{2}-ir_j}{2},\ c=\frac{s_2+s_1-\frac{1}{2}+ir_j}{2},\ d=\frac{s_2-s_1+\frac{1}{2}+ir_j}{2}. \tag{3.77}$$

The following equality holds (see e.g. Goldfeld [65]):

$$\sum_{m\neq 0} \overline{a_j(m)}\sigma_{2s_1-1}|m|^{\frac{1}{2}-2s_1-s_2} = 2\frac{L_{\overline{\phi_j}}\left(s_2-s_1+\frac{1}{2}\right)L_{\overline{\phi_j}}\left(s_2+s_1-\frac{1}{2}\right)}{\overline{a_j(1)}\zeta\left(2s_2\right)}.$$

Hence,

$$(F(\cdot,s_1,s_2),\phi_j) = \frac{1}{2\pi^{2s_2}\overline{a_j(1)}}\Gamma\left(s_1,s_2,r_j\right)L_{\overline{\phi_j}}\left(s_2-s_1+\frac{1}{2}\right)L_{\overline{\phi_j}}\left(s_2+s_1-\frac{1}{2}\right). \tag{3.78}$$

Similarly, following the same procedure as in the case of $(F(\cdot,\overline{s}_1',\overline{s}_2'),\phi_j)$, we obtain

$$\begin{aligned}\overline{(F(\cdot,\overline{s}_1',\overline{s}_2'),\phi_j)} &= \left(F(\cdot,s_1',s_2'),\overline{\phi_j}\right)\\ &= \frac{1}{2\pi^{2s_2'}a_j(1)}\Gamma\left(s_1',s_2',r_j\right)L_{\phi_j}\left(s_2'-s_1'+\frac{1}{2}\right)L_{\phi_j}\left(s_2'+s_1'-\frac{1}{2}\right),\end{aligned} \tag{3.79}$$

where

$$\Gamma\left(s_1',s_2',r_j\right) = \Gamma\left(a'\right)\Gamma\left(b'\right)\Gamma\left(c'\right)\Gamma\left(d'\right),$$

with

$$a' = \frac{s_2'+s_1'-\frac{1}{2}+ir_j}{2},\ b' = \frac{s_2'-s_1'+\frac{1}{2}+ir_j}{2},\ c' = \frac{s_2'+s_1'-\frac{1}{2}-ir_j}{2},\ d' = \frac{s_2'-s_1'+\frac{1}{2}-ir_j}{2}.$$

From (3.74), (3.78) and (3.79), the first term on the right-hand side of (3.73) gives

$$\begin{aligned}\sum_{j=1}^{\infty}\mathbb{F}_{\phi_j}(s_1,s_2,s_1',s_2') &= \sum_{j=1}^{\infty}\left(F(\cdot,s_1,s_2),\phi_j\right)\overline{(F(\cdot,\overline{s}_1',\overline{s}_2'),\phi_j)}\\ &= \frac{1}{4\pi^{2(s_2+s_2')}}\sum_{j=1}^{\infty}\frac{\varTheta\left(s_1,s_2,r_j\right)\varTheta\left(s_1',s_2',r_j\right)}{\left|a_j(1)\right|^2},\end{aligned} \tag{3.80}$$

where

$$\begin{aligned}\varTheta\left(s_1,s_2,r_j\right) &= \Gamma\left(s_1,s_2,r_j\right)L_{\overline{\phi_j}}\left(s_2-s_1+\frac{1}{2}\right)L_{\overline{\phi_j}}\left(s_2+s_1-\frac{1}{2}\right)\\ \varTheta\left(s_1',s_2',r_j\right) &= \Gamma\left(s_1',s_2',r_j\right)L_{\phi_j}\left(s_2'-s_1'+\frac{1}{2}\right)L_{\phi_j}\left(s_2'+s_1'-\frac{1}{2}\right).\end{aligned}$$

Before expanding the remaining inner products, namely the integrand in (3.73) we need to give an expression for the Fourier expansion of $F(z,s_1,s_2)$. Since $F(z,s_1,s_2)$ is a product of $SL(2,\mathbf{Z})$-automorphic functions, it is itself $SL(2,\mathbf{Z})$-automorphic, i.e., $F(z+1,s_1,s_2) = F(z,s_1,s_2)$. It therefore admits a Fourier expansion

$$F(z,s_1,s_2) = \sum_{k\in\mathbf{Z}} c_k(y,s_1,s_2)e^{2\pi ikx},$$

whose constant term $c_0(y, s_1, s_2)$ is given by

$$\begin{aligned} c_0(y, s_1, s_2) &= 8y \sum_{m=1}^{\infty} \sigma_{1-2s_1}(m) m^{s_1-\frac{1}{2}} \sigma_{1-2s_2}(m) m^{s_2-\frac{1}{2}} K_{s_1-\frac{1}{2}}(2\pi my) K_{s_2-\frac{1}{2}}(2\pi my) \\ &= \int_{x=0}^{1} F(x+iy, s_1, s_2)\, dx. \end{aligned}$$

So, by the Rankin-Selberg method (3.68), we have

$$\begin{aligned} \left(F(\cdot, s_1, s_2), E\left(z, \frac{1}{2}+ir\right)\right) &= \int_{SL(2,\mathbf{Z})\backslash \mathbf{H}} F(z, s_1, s_2) E\left(z, \frac{1}{2}-ir\right) \frac{dxdy}{y^2} \\ &= \int_0^{\infty} c_0(y, s_1, s_2) y^{-\frac{3}{2}-ir}\, dy \\ &= 8(2\pi)^{-\frac{1}{2}+ir} \sum_{m=1}^{\infty} \frac{\sigma_{2s_1-1}(m)\sigma_{2s_2-1}(m)}{m^{s_1+s_2-\frac{1}{2}-ir}} \\ &\quad \times \int_0^{\infty} y^{-\frac{1}{2}-ir} K_{s_1-\frac{1}{2}}(\eta) K_{s_2-\frac{1}{2}}(\eta)\, d\eta. \end{aligned} \tag{3.81}$$

We shall give a closed form formula for the series in (3.81). Towards this end we use the Dirichlet series (Ramanujan's formula)

$$\sum_{k=1}^{\infty} \frac{\sigma_p(k)\sigma_q(k)}{k^s} = \frac{\zeta(s)\zeta(s-p)\zeta(s-q)\zeta(s-p-q)}{\zeta(2s-p-q)},$$

to see that the sum on the right-hand side of (3.81) becomes

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{\sigma_{2s_1-1}(m)\sigma_{2s_2-1}(m)}{m^{s_1+s_2-\frac{1}{2}-ir}} \\ &= \frac{\zeta\left(s_1+s_2-\frac{1}{2}-ir\right)\zeta\left(s_2-s_1+\frac{1}{2}-ir\right)\zeta\left(s_1-s_2+\frac{1}{2}-ir\right)\zeta\left(-s_1-s_2+\frac{3}{2}-ir\right)}{\zeta(1-2ir)}. \end{aligned} \tag{3.82}$$

The integral on the right-hand side of (3.81) is

$$\begin{aligned} \int_0^{\infty} y^{-\frac{1}{2}-ir} K_{s_1-\frac{1}{2}}(\eta) K_{s_2-\frac{1}{2}}(\eta)\, d\eta &= \frac{2^{-3} 2^{\frac{1}{2}-ir}}{\Gamma\left(\frac{1}{2}-ir\right)} \Gamma\left(\frac{s_1+s_2-\frac{1}{2}-ir}{2}\right) \Gamma\left(\frac{s_2-s_1+\frac{1}{2}-ir}{2}\right) \\ &\quad \times \Gamma\left(\frac{s_1-s_2+\frac{1}{2}-ir}{2}\right) \Gamma\left(\frac{-s_1-s_2+\frac{3}{2}-ir}{2}\right). \end{aligned} \tag{3.83}$$

Hence, from (3.81), (3.82) and (3.83) we obtain

$$\begin{aligned} &\left(F(\cdot, s_1, s_2), E\left(z, \frac{1}{2}+ir\right)\right) \\ &= \frac{\Lambda\left(s_1+s_2-\frac{1}{2}-ir\right)\Lambda\left(s_2-s_1+\frac{1}{2}-ir\right)\Lambda\left(s_1-s_2+\frac{1}{2}-ir\right)\Lambda\left(-s_1-s_2+\frac{3}{2}-ir\right)}{\Lambda(1-2ir)}. \end{aligned} \tag{3.84}$$

Similarly,

$$\begin{aligned} &\overline{\left(F(\cdot, \overline{s}_1', \overline{s}_2'), E\left(z, \frac{1}{2}+ir\right)\right)} = \left(F(\cdot, s_1', s_2'), E\left(z, \frac{1}{2}-ir\right)\right) \\ &= \frac{\Lambda\left(s_1'+s_2'-\frac{1}{2}+ir\right)\Lambda\left(s_2'-s_1'+\frac{1}{2}+ir\right)\Lambda\left(s_1'-s_2'+\frac{1}{2}+ir\right)\Lambda\left(-s_1'-s_2'+\frac{3}{2}+ir\right)}{\Lambda(1+2ir)}. \end{aligned} \tag{3.85}$$

Combining (3.75), (3.84) and (3.85), the second term on the right-hand side of (3.73) gives

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\mathbb{F}_E(s_1,s_2,s_1',s_2',r)\,dr=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Xi(s_1,s_2,r)\Xi(s_1',s_2',r)}{|\Lambda(1+2ir)|^2}\,dr, \tag{3.86}$$

where

$$\begin{aligned}\Xi(s_1,s_2,r)=&\,\Lambda\left(s_1+s_2-\frac{1}{2}-ir\right)\Lambda\left(s_2-s_1+\frac{1}{2}-ir\right)\\ &\times\Lambda\left(s_1-s_2+\frac{1}{2}-ir\right)\Lambda\left(-s_1-s_2+\frac{3}{2}-ir\right)\end{aligned}$$

and

$$\begin{aligned}\Xi(s_1',s_2',r)=&\,\Lambda\left(s_1'+s_2'-\frac{1}{2}+ir\right)\Lambda\left(s_2'-s_1'+\frac{1}{2}+ir\right)\\ &\times\Lambda\left(s_1'-s_2'+\frac{1}{2}+ir\right)\Lambda\left(-s_1'-s_2'+\frac{3}{2}+ir\right).\end{aligned}$$

This proves the first part of the theorem. To see the second part (i.e., the particular case), we use $\Lambda(s)$ given by (B.23) and the identity (B.5) with $s_1=\frac{1}{2}+ir$, $s_2=\frac{1}{2}-ir$. $\square$

3.4 Zeta Functions and Determinant of the Laplacian

Having computed explicitly the trace formula for a noncompact hyperbolic surface $\mathscr{M}=\Gamma\backslash\mathbf{H}$, we are going to put the formula (i.e., Theorem 3.18) into action. Here our Γ is the discrete subgroup $SL(2,\mathbf{Z})$. This section is involved as we are going to set necessary machineries in motion for the explicit computation of the determinant of the shifted Laplacian $\widetilde{\Delta}-s(1-s)$, $s\in\mathbf{C}$, on $SL(2,\mathbf{Z})\backslash\mathbf{H}$. Before getting to the determinant of the Laplacian we have to compute all what we need to put together - these include the heat trace, the resolvent of the Laplacian, the Selberg zeta function (or what is the same, the zeta function associated to the hyperbolic contribution), the zeta function associated to the identity contribution and the zeta function associated to the parabolic contribution. Thereafter we express the determinant in terms of all these zeta functions. It is remarkable that all the spectral zeta functions mentioned can be realised using the trace formula. In particular we compute the determinant of the Laplacian for some particular values of $s\in\mathbf{R}$, $s>0$, and express the determinant with higher s, $s>0$, in terms of the determinant with lower s, $s>0$. Our starting point is the construction of the trace of the heat kernel on $SL(2,\mathbf{Z})\backslash\mathbf{H}$, which is the first application of the trace formula.

3.4.1 The Heat Trace and Eigenvalue Asymptotics

The heat trace asymptotics on $\Gamma\backslash\mathbf{H}$ plays a crucial role in the computation of the determinant of the Laplacian on $\Gamma\backslash\mathbf{H}$; as we shall see in Subsection 3.4.3.

By the method of images, we can obtain the heat kernel $K_{\mathscr{M}}(t,z,z')$ of a hyperbolic surface $\mathscr{M}=\Gamma\backslash\mathbf{H}$ from the heat kernel $\widetilde{K}(t,z,z')$ of the hyperbolic plane $\mathbf{H}$. The method of images says that for $z,z'\in\mathbf{H}$ and $\gamma\in\Gamma$,

$$K_{\mathscr{M}}(t,z,z')=\sum_{\gamma\in\Gamma}\widetilde{K}(t,z,\gamma z'), \tag{3.87}$$

where $\widetilde{K}(t,z,z')$ is given by (1.58). Let $\widetilde{\Delta}$ be the Laplacian on $\mathscr{M}$. The trace of the heat operator $e^{-t\widetilde{\Delta}}$ on $\mathscr{M}$ is given by

$$\operatorname{tr} e^{-t\widetilde{\Delta}} = \sum_{k=0}^{\infty} e^{-t\lambda_k} = \int_{\Gamma\backslash\mathbf{H}} K_{\mathscr{M}}(t,z,z)\,d\mu(z) = \sum_{\gamma\in\Gamma}\int_{\Gamma\backslash\mathbf{H}} \widetilde{K}(t,z,\gamma z)\,d\mu(z), \tag{3.88}$$

with the main property that $\widetilde{K}(t,z,z') = \widetilde{K}(t,d(\gamma z,\gamma z')) = \widetilde{K}(t,d(z,z'))$ for all $\gamma\in\Gamma$. Since any element $\beta\in\Gamma-\{I\}=\{\gamma\}_p$ can be uniquely written as

$$\beta = \sigma^{-1}\gamma^n\sigma, \qquad \sigma\in\Gamma/\Gamma_\gamma,\ n\geq 1,$$

we have

$$\begin{aligned}\sum_{k=0}^{\infty} e^{-t\lambda_k} &= \int_{\Gamma\backslash\mathbf{H}} \widetilde{K}(t,d(z,z))\,d\mu(z) + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\sum_{\sigma\in\Gamma/\Gamma_\gamma}\int_{\Gamma\backslash\mathbf{H}} \widetilde{K}\left(t,d\left(z,\sigma^{-1}\gamma^m\sigma z\right)\right)\,d\mu(z)\\ &= \int_{\Gamma\backslash\mathbf{H}} \widetilde{K}(t,d(z,z))\,d\mu(z) + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\sum_{\sigma\in\Gamma/\Gamma_\gamma}\int_{\sigma(\Gamma\backslash\mathbf{H})} \widetilde{K}\left(t,d\left(z,\gamma^m z\right)\right)\,d\mu(z).\end{aligned}$$

Let $\mathcal{D}_\gamma = \bigcup_{\sigma\in\Gamma/\Gamma_\gamma}\sigma\left(\Gamma\backslash\mathbf{H}\right)$ be a fundamental domain for the centraliser Γ_σ when $\Gamma\backslash\mathbf{H}$ is a fundamental domain for Γ. Then

$$\begin{aligned}\operatorname{tr} e^{-t\widetilde{\Delta}} &= \int_{\Gamma\backslash\mathbf{H}} \widetilde{K}(t,d(z,z))\,d\mu(z) + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\int_{\mathcal{D}_\gamma} \widetilde{K}\left(t,d\left(z,\gamma^m z\right)\right)\,d\mu(z)\\ &= \int_{\Gamma\backslash\mathbf{H}} \widetilde{K}(t,d(z,z))\,d\mu(z) + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\int_{\Gamma_\gamma\backslash\mathbf{H}} \widetilde{K}\left(t,d\left(z,\gamma^m z\right)\right)\,d\mu(z).\end{aligned}$$

Since every hyperbolic element $\gamma\in\Gamma$ is conjugate to the dilation

$$\gamma z\,:\,z\to e^{\ell(\gamma)}z \quad\text{with}\quad \mathcal{D}_\gamma = \left\{x+iy\in\mathbf{C}\,:\,x\in\mathbf{R},\,y\in\left[1,e^{\ell(\gamma)}\right]\right\},$$

we have

$$\operatorname{tr} e^{-t\widetilde{\Delta}} = \int_{\Gamma\backslash\mathbf{H}} \widetilde{K}(t,d(z,z))\,d\mu(z) + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\left(\int_1^{e^{\ell(\gamma)}} \widetilde{K}\left(t,d\left(z,e^{m\ell(\gamma)}z\right)\right)\frac{dy}{y^2}\right)dx. \tag{3.89}$$

Noting that

$$d(z,e^{m\ell(\gamma)}z) = \cosh^{-1}\left[1+\frac{|z-e^{m\ell(\gamma)}z|^2}{2\operatorname{Im} z\operatorname{Im} e^{m\ell(\gamma)}z}\right] \quad = \cosh^{-1}\left[1+2\sinh^2\left(\frac{m\ell(\gamma)}{2}\right)\left(1+\frac{x^2}{y^2}\right)\right],$$

with

$$\eta = \cosh^{-1}\left[1+2\sinh^2\left(\frac{m\ell(\gamma)}{2}\right)\left(1+\frac{x^2}{y^2}\right)\right],\ x=yu,\quad \sinh\eta\,d\eta = 4u\sinh^2\left(\frac{m\ell(\gamma)}{2}\right)du,$$

the first integral on the right-hand side of (3.89) is independent of z. Thus

$$\int_{\Gamma\backslash\mathbf{H}} \widetilde{K}(t,d(z,z))\,d\mu(z) = \mu(\Gamma\backslash\mathbf{H})\widetilde{K}(t,0) = \frac{\mu(\Gamma\backslash\mathbf{H})e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}}\int_0^\infty \frac{be^{-\frac{b^2}{4t}}}{\sinh(b/2)}\,db.$$

The double integral on the right hand side of (3.89) becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\int_1^{e^{\ell(\gamma)}} \widetilde{K}\left(t,d\left(z,e^{m\ell(\gamma)}z\right)\right)\frac{dy}{y^2}\right)dx = \ell(\gamma)\int_{-\infty}^{\infty}\widetilde{K}(t,\eta)\,du\\
&= 2\ell(\gamma)\int_{m\ell(\gamma)}^{\infty}\widetilde{K}(t,\eta)\frac{\sinh\eta}{4u\sinh^2\left(\frac{m\ell(\gamma)}{2}\right)}\,d\eta\\
&= \frac{\ell(\gamma)}{\sinh\left(\frac{m\ell(\gamma)}{2}\right)}\int_{m\ell(\gamma)}^{\infty}\frac{e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}}\int_\eta^\infty\frac{be^{-\frac{b^2}{4t}}}{\sqrt{\cosh b-\cosh\eta}}\,db\frac{\sinh\eta\,d\eta}{\sqrt{\cosh\eta-\cosh m\ell(\gamma)}}\\
&= \frac{\ell(\gamma)}{\sinh\left(\frac{m\ell(\gamma)}{2}\right)}\frac{e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}}\int_{m\ell(\gamma)}^{\infty}be^{-\frac{b^2}{4t}}\,db\int_{m\ell(\gamma)}^{b}\frac{\sinh\eta\,d\eta}{\sqrt{\cosh b-\cosh\eta}\sqrt{\cosh\eta-\cosh m\ell(\gamma)}}\\
&= \frac{\ell(\gamma)}{\sinh\left(\frac{m\ell(\gamma)}{2}\right)}\frac{e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}}\int_{m\ell(\gamma)}^{\infty}be^{-\frac{b^2}{4t}}\,db\int_c^d\frac{d\xi}{\sqrt{d-\xi}\sqrt{\xi-c}}\\
&= \frac{\ell(\gamma)}{2\sinh\left(\frac{m\ell(\gamma)}{2}\right)}\frac{e^{-\frac{t}{4}}}{\sqrt{4\pi t}}e^{-\frac{m^2\ell(\gamma)^2}{4t}}.
\end{aligned}$$

Hence, we obtain

Proposition 3.20.

$$tr\,e^{-t\widetilde{\Delta}} = \frac{\mu(\Gamma\backslash\mathbf{H})e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}}\int_0^\infty\frac{be^{-\frac{b^2}{4t}}}{\sinh(b/2)}\,db + \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\frac{\ell(\gamma)}{2\sinh\left(\frac{m\ell(\gamma)}{2}\right)}\frac{e^{-\frac{t}{4}}}{\sqrt{4\pi t}}e^{-\frac{m^2\ell(\gamma)^2}{4t}}, \tag{3.90}$$

which is the trace of the heat operator on $\Gamma\backslash\mathbf{H}$.

The heat trace formula (3.90) can also be obtained directly from the trace formula (3.20). Towards this end, we use a pair of function

$$\begin{aligned}
h(r) &= e^{-(r^2+\frac{1}{4})t}, t>0,\\
g(u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-(r^2+\frac{1}{4})t}e^{-iru}\,dr = \frac{e^{-\frac{t}{4}}}{(4\pi t)^{\frac{1}{2}}}e^{-\frac{u^2}{4t}}
\end{aligned}$$

and using

$$\frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\int_{-\infty}^{\infty}re^{-(r^2+\frac{1}{4})t}\tanh(\pi r)\,dr = \frac{\mu(\Gamma\backslash\mathbf{H})}{(4\pi t)^{\frac{3}{2}}}e^{-\frac{t}{4}}\int_0^\infty\frac{ue^{-\frac{u^2}{4t}}}{\sinh(u/2)}\,du,$$

to obtain (3.90).

The small-t behaviour of the heat trace is determined by the integral term in (3.90) (see Lemma 3.21 below). The right-hand side of (3.90) provides at once the full asymptotic expansion of the trace of the heat operator for $t \searrow 0$.

Proposition 3.21. *The trace of the heat operator on a compact hyperbolic surface $\mathscr{M} = \Gamma\backslash\mathbf{H}$ given by Proposition 3.20 admits the Minakshisundaram-Pleijel asymptotic expansion (see (A.8))*

$$\operatorname{tr} e^{-t\widetilde{\Delta}} = \sum_{k=0}^{\infty} e^{-\lambda_k t} \sim \frac{1}{4\pi t}\sum_{k=0}^{\infty} a_k t^k \qquad \text{as} \quad t \searrow 0, \tag{3.91}$$

where the Minakshisundaram-Pleijel heat coefficients a_k are given by

$$a_0 = \mu(\Gamma\backslash\mathbf{H}), \quad a_k = \mu(\Gamma\backslash\mathbf{H})\sum_{j=0}^{k}(-1)^{k-j}\frac{\left(\frac{1}{4}\right)^{k-j}}{(k-j)!}\frac{B_{2j}\left(2^{1-2j}-1\right)}{j!}, \quad k \geq 1. \tag{3.92}$$

In particular,

$$\operatorname{tr} e^{-t\widetilde{\Delta}} \sim \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi t} \qquad \text{as } t \searrow 0.$$

Proof. We examine the asymptotic behaviour of each term of the right-hand side of the heat trace formula in 3.90 as $t \searrow 0$.

Contribution of the identity element. To determine the asymptotic expansion of the first term on the right-hand side of (3.90) we use the series representation

$$\frac{u}{2\sinh(u/2)} = \sum_{m=0}^{\infty}\left(2^{1-2m}-1\right)B_{2m}\frac{u^{2m}}{(2m)!},$$

where B_m is the mth Bernoulli number (see (B.27)). So we have

$$\begin{aligned}
\frac{\mu(\Gamma\backslash\mathbf{H})e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\frac{ue^{-\frac{u^2}{4t}}}{2\sinh(u/2)}\,du &= \frac{\mu(\Gamma\backslash\mathbf{H})e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}}\sum_{m=0}^{\infty}\left(2^{1-2m}-1\right)B_{2m}e^{t}\int_{-\infty}^{\infty}e^{-\frac{1}{4t}y^2}\,dy \\
&= \frac{\mu(\Gamma\backslash\mathbf{H})e^{-\frac{t}{4}}}{4\pi t}\sum_{m=0}^{\infty}\left(2^{1-2m}-1\right)B_{2m}\frac{t^m}{m!} \\
&= \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi t}\left[1-\frac{t}{3}+\frac{t^2}{15}+\cdots\right] \\
&\sim \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi t} \qquad \text{as} \quad t \searrow 0.
\end{aligned} \tag{3.93}$$

Contribution of the hyperbolic elements. The last term on the right-hand side of (3.90) decays exponentially as $t \searrow 0$. □

Theorem 3.22 (Weyl asymptotic distribution of eigenvalues). *Let $\mathscr{N}(\lambda)$ denote the number of eigenvalues λ_k that are less than or equal to λ (called the eigenvalue counting function):*

$$\mathscr{N}(\lambda) = \#\left\{k : \lambda_k \leq \lambda\right\}.$$

Then to the leading order

$$\mathscr{N}(\lambda) = \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\lambda + o(\sqrt{\lambda}) \qquad \text{as} \quad \lambda \nearrow \infty, \tag{3.94}$$

or equivalently

$$\lambda_k \sim \frac{4\pi k}{\mu(\Gamma\backslash\mathbf{H})} \qquad \text{as} \quad k \nearrow \infty. \tag{3.95}$$

Remark 3.2. *Lemma 3.21 clearly implies that*

$$\int_0^\infty e^{-\lambda t} d\mathscr{N}(\lambda) \sim \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi t} \qquad \text{as} \;\; t \searrow 0. \tag{3.96}$$

To prove Theorem 3.22, we need the following Tauberian theorem whose proof can be found in Widder [177, p. 192], Taylor [166, Prop. 3.2].

Theorem 3.23 (Karamata's Tauberian Theorem). *Let $(\lambda_m : m \geq 0)$ be a nondecreasing sequence of nonnegative real numbers such that the series*

$$\sum_{m=0}^{\infty} e^{-\lambda_m t}$$

converges for every $t > 0$. If

$$\sum_{m=0}^{\infty} e^{-\lambda_m t} \sim \frac{A}{t^\alpha}$$

as $t \searrow 0$ for some $A, \alpha > 0$, then the associated counting function $\mathscr{N}(\lambda) = \#\{k : \lambda_k \leq \lambda\}$ satisfies

$$\mathscr{N}(\lambda) \sim \frac{A\lambda^\alpha}{\Gamma(\alpha+1)} \qquad \text{as} \quad \lambda \nearrow \infty. \tag{3.97}$$

Proof of Theorem 3.22. Since the eigenvalues of $\widetilde{\Delta}$ form a nondecreasing sequence of nonnegative real numbers, the counting function $\mathscr{N}(\lambda)$ is nondecreasing. Therefore, Weyl asymptotic formula (3.94) follows from (3.97) with

$$A = \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}, \quad \alpha = 1.$$

□

3.4.2 The Trace of the Resolvent and Selberg Zeta Functions

Another operator of great importance in spectral theory is the resolvent operator. In order to calculate the trace of the resolvent of $\widetilde{\Delta}$ on $\mathscr{M}$, or what is the same, the trace of the resolvent operator

$$\mathcal{R}_s = \left(\widetilde{\Delta} - \lambda\right)^{-1}$$

on $\mathscr{M}$ we choose the function

$$g(u) = \frac{1}{2\left(s-\frac{1}{2}\right)} e^{-\left(s-\frac{1}{2}\right)|u|} - \frac{1}{2\left(a-\frac{1}{2}\right)} e^{-\left(a-\frac{1}{2}\right)|u|}, \quad u \in \mathbf{R}, 1 < \operatorname{Re} s < a, \tag{3.98}$$

whose Fourier (Selberg) transform

$$h(r) = \int_{-\infty}^{\infty} g(u) e^{iru}\, du = \frac{1}{\left(s-\frac{1}{2}\right)^2 + r^2} - \frac{1}{\left(a-\frac{1}{2}\right)^2 + r^2} \tag{3.99}$$

satisfies conditions $(S.I)-(S.III)$ in Theorem 3.8. Putting (3.98) and (3.99) into the trace formula (3.20), we have

$$\begin{aligned}\sum_{k=0}^{\infty}\left[\frac{1}{r_k^2+\left(s-\frac{1}{2}\right)^2}-\frac{1}{r_k^2+\left(a-\frac{1}{2}\right)^2}\right] &= \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\int_{-\infty}^{\infty} r\left[\frac{\tanh(\pi r)}{r_k^2+\left(s-\frac{1}{2}\right)^2}-\frac{\tanh(\pi r)}{r_k^2+\left(a-\frac{1}{2}\right)^2}\right]dr\\ &+\frac{1}{2s-1}\sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\frac{\ell(\gamma)e^{-\left(s-\frac{1}{2}\right)m\ell(\gamma)}}{e^{\frac{m\ell(\gamma)}{2}}-e^{-\frac{m\ell(\gamma)}{2}}}\\ &-\frac{1}{2a-1}\sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\frac{\ell(\gamma)e^{-\left(a-\frac{1}{2}\right)m\ell(\gamma)}}{e^{\frac{m\ell(\gamma)}{2}}-e^{-\frac{m\ell(\gamma)}{2}}}. \qquad (3.100)\end{aligned}$$

Next we give closed forms for the series on the right-hand side of (3.100); the closed form obtained is expressed in terms of the hyperbolic analogue of the Riemann zeta function, known as the *Selberg zeta function*. Towards this end we set $\alpha = s-\frac{1}{2}$ to obtain

$$\begin{aligned}\ell(\gamma)\sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\frac{e^{-\alpha m\ell(\gamma)}}{2\sinh\frac{m\ell(\gamma)}{2}} &= \ell(\gamma)\sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}e^{-m\ell(\gamma)\left(\alpha+k+\frac{1}{2}\right)}\\ &= \ell(\gamma)\sum_{\{\gamma\}_p}\sum_{k=0}^{\infty}\frac{e^{-\ell(\gamma)\left(\alpha+k+\frac{1}{2}\right)}}{1-e^{-\ell(\gamma)\left(\alpha+k+\frac{1}{2}\right)}}\\ &= \sum_{\{\gamma\}_p}\sum_{k=0}^{\infty}\frac{d}{d\alpha}\log\left(1-e^{-\ell(\gamma)\left(\alpha+k+\frac{1}{2}\right)}\right)\\ &= \frac{d}{d\alpha}\log\prod_{\{\gamma\}_p}\prod_{k=0}^{\infty}\left(1-e^{-\ell(\gamma)\left(\alpha+k+\frac{1}{2}\right)}\right)=\frac{Z'\left(\alpha+\frac{1}{2}\right)}{Z\left(\alpha+\frac{1}{2}\right)}. \qquad (3.101)\end{aligned}$$

In summary, the closed form representation of the series on the right-hand side of (3.100) is given by

Proposition 3.24.

$$\sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\frac{\ell(\gamma)e^{-\left(s-\frac{1}{2}\right)m\ell(\gamma)}}{\left(e^{\frac{m\ell(\gamma)}{2}}-e^{-\frac{m\ell(\gamma)}{2}}\right)}=\frac{Z'(s)}{Z(s)}, \qquad (3.102)$$

where

$$Z(s)=\prod_{\{\gamma\}_p}\prod_{k=0}^{\infty}\left(1-e^{-\ell(\gamma)(s+k)}\right), \qquad Re\, s>1, \qquad (3.103)$$

is called the Selberg zeta function. The outer product is over the primitive hyperbolic connjugacy classes of γ in Γ and $\ell(\gamma)$ is the length of the closed geodesic associated to $\{\gamma\}_p$. The Selberg zeta function $Z(s)$ converges absolutely in the half-plane $Re\, s>1$ and admits an analytic continuation to the whole complex plane.

From equation (3.100) and Proposition 3.24 we obtain

Proposition 3.25.

$$\begin{aligned}\frac{1}{2s-1}\frac{Z'(s)}{Z(s)}-\frac{1}{2a-1}\frac{Z'(a)}{Z(a)} &= \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\int_{-\infty}^{\infty} r\left[\frac{\tanh(\pi r)}{r^2+\left(a-\frac{1}{2}\right)^2}-\frac{\tanh(\pi r)}{r^2+\left(s-\frac{1}{2}\right)^2}\right]dr\\ &+\sum_{k=0}^{\infty}\left[\frac{1}{r_k^2+\left(s-\frac{1}{2}\right)^2}-\frac{1}{r_k^2+\left(a-\frac{1}{2}\right)^2}\right]. \qquad (3.104)\end{aligned}$$

We can further give an explicit expression for the integral term on the right-hand side of (3.104) in terms of the digamma function. To do this we employ the calculus of residue to evaluate the contribution of the identity element given by

$$\mathcal{I}(s) = \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\int_{-\infty}^{\infty} r\left(\frac{1}{r^2+\beta^2} - \frac{1}{\left(s-\frac{1}{2}\right)^2+r^2}\right)\tanh(\pi r)\,dr. \tag{3.105}$$

Towards this end we define, for $N\in\mathbf{N}$,

$$\mathcal{J}_N(s) = \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\int_{C_N} z\left(\frac{1}{z^2+\beta^2} - \frac{1}{\left(s-\frac{1}{2}\right)^2+z^2}\right)\tanh(\pi z)\,dz, \tag{3.106}$$

where the integration is performed counterclockwise along the path C_N, consisting of the interval $[-N,N]$ and the semicircle C_N' of radius N with centre at the origin, lying in the upper half-plane $\operatorname{Re} z>0$. In the domain $\operatorname{Re} s>\frac{1}{2}$, and in the upper half-plane $\operatorname{Re} z>0$, the integrand

$$f(z) = z\left(\frac{1}{z^2+\beta^2} - \frac{1}{\left(s-\frac{1}{2}\right)^2+z^2}\right)\tanh(\pi z) \tag{3.107}$$

has only simple poles at $z_k = i\left(k+\frac{1}{2}\right)$, $k=\mathbf{N}\cup\{0\}$, coming from $\tanh\pi z$ and simple poles at $z=i\left(s-\frac{1}{2}\right)$ and $z=i\beta$ coming from the denominators $\left(s-\frac{1}{2}\right)^2+z^2$ and $z^2+\beta^2$ respectively. Thus by Cauchy's integral formula, we have

$$\mathcal{J}_N(s) = -\frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}2\pi i\left[\operatorname{Res} f(z)\Big|_{z=i\left(s-\frac{1}{2}\right)} + \operatorname{Res} f(z)\Big|_{z=i\beta} + \sum_{k=0}^{N-1}\operatorname{Res} f(z)\Big|_{z=i\left(k+\frac{1}{2}\right)}\right], \tag{3.108}$$

where

$$2\pi i\operatorname{Res}_{z=i\left(s-\frac{1}{2}\right)} f(z) = 2\pi i\left[\frac{1}{2}\tanh\pi i\left(s-\frac{1}{2}\right)\right] = -\pi\tan\pi\left(s-\frac{1}{2}\right), \tag{3.109}$$

$$2\pi i\operatorname{Res}_{z=i\beta} f(z) = 2\pi i\left[\frac{1}{2}\tanh\pi i\beta\right] = -\pi\tan\pi\beta, \tag{3.110}$$

$$2\pi i\operatorname{Res}_{z=i\left(k+\frac{1}{2}\right)} f(z) = -2\left[\frac{k+\frac{1}{2}}{-\left(k+\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2} + \frac{k+\frac{1}{2}}{-\left(k+\frac{1}{2}\right)^2+\beta^2}\right]. \tag{3.111}$$

So as $N\nearrow\infty$, we have

$$\mathcal{I}(s) = B(s)+E(s), \tag{3.112}$$

where

$$B(s) = \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\pi\tan\pi\left(s-\frac{1}{2}\right) - \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\pi\tan\pi\beta, \tag{3.113}$$

$$E(s) = -2\frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\sum_{k=0}^{\infty}\left[\frac{k+\frac{1}{2}}{-\left(k+\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2} + \frac{k+\frac{1}{2}}{-\left(k+\frac{1}{2}\right)^2+\beta^2}\right]. \tag{3.114}$$

Using the identities (B.13), (B.11) and by resolving the expression inside the square bracket in (3.114) into partial fractions we have (see Awonusika and Taheri [8])

$$B(s) = \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\sum_{k=0}^{\infty}\left[\frac{1}{k+\frac{1}{2}-\alpha} - \frac{1}{k+\frac{1}{2}+\alpha} - \frac{1}{k+\frac{1}{2}-\beta} + \frac{1}{k+\frac{1}{2}+\beta}\right], \tag{3.115}$$

$$E(s) = \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi}\sum_{k=0}^{\infty}\left[\frac{1}{\beta+\frac{1}{2}+k} - \frac{1}{\beta-\frac{1}{2}-k} + \frac{1}{\alpha-\frac{1}{2}-k} - \frac{1}{\alpha+\frac{1}{2}+k}\right]. \tag{3.116}$$

Hence, we obtain

$$\mathcal{I}(s) = \frac{\mu(\Gamma\backslash\mathbf{H})}{2\pi}\sum_{k=0}^{\infty}\left[\frac{1}{a+k} - \frac{1}{s+k}\right] = \frac{\mu(\Gamma\backslash\mathbf{H})}{2\pi}(\psi(s)-\psi(a)),$$

where $\psi(s)$ is the digamma function (Appendix B.1).

Therefore, we obtain the following formula for the logarithmic derivative of the Selberg zeta function (see also Awonusika and Taheri [8]).

Theorem 3.26. *For $1 < Re\, s < a$, let $Z(s)$ be the Selberg zeta function defined by* (3.103)*. The following formula for the logarithmic derivative of the Selberg zeta function holds:*

$$\frac{1}{2s-1}\frac{Z'(s)}{Z(s)} - \frac{1}{2a-1}\frac{Z'(a)}{Z(a)} = \sum_{k=0}^{\infty}\left[\frac{1}{\lambda_k - s(1-s)} - \frac{1}{\lambda_k - a(1-a)}\right] + \frac{\mu(\Gamma\backslash\mathbf{H})}{2\pi}\psi(s) - \psi(a). \tag{3.117}$$

The sum on the right-hand side of (3.117) is absolutely convergent for $s \in \mathbf{C}$, except at the poles where $\lambda_k = s(1-s)$ or $s = -m$, $m \geq 0$. The poles are simple with unit residues. Moreover, the Selberg zeta function $Z(s)$ can be analytically continued to an entire function of s whose zeros (trivial and nontrivial) are characterised as follows.

(i) The nontrivial zeros of $Z(s)$ are located at $s = 1$ and $s = \frac{1}{2} \pm ir_k$, $k = 1, 2, 3, \cdots$. The zero at $s = 1$ (corresponding to $k = 0$) has multiplicity 1.

(ii) The trivial zeros are located at $s = -m$, $m \geq 0$, and have multiplicity $2g-1$ for $m = 0$ and $\frac{(2m+1)\mu(\Gamma\backslash\mathbf{H})}{2\pi} = 2(g-1)(2m+1)$ for $m > 0$.

(iii) $Z(s)$ satisfies the functional equation

$$Z(s) = Z(1-s)\cdot\exp\left[\mu(\Gamma\backslash\mathbf{H})\int_0^{z=s-\frac{1}{2}} v\tan\pi v\,dv\right]. \tag{3.118}$$

Remark 3.3. *It follows from the formula* (3.117) *that for $s \in \mathbf{C}$*

$$Z'(s) \neq 0 \quad \text{for} \quad Re\, s < \tfrac{1}{2} \text{ and } Im\, s \neq 0.$$

(see Minamide [112]). See also Minamide [113], Luo [104], Jorgenson and Smajlovic [90].

The Zeta Function Associated to the Contribution of Identity Element. The zeta function $Z_{\mathcal{I}}(s)$ corresponding to the identity contribution $\mathcal{I}(s)$ in the trace formula (3.20) is given by (see e.g. Momeni and Venkov [114])

$$Z_{\mathcal{I}}(s) = \left(\frac{(2\pi)^s}{\Gamma(s)}\right)^{\frac{1}{6}} \frac{1}{G(s)^{\frac{1}{3}}}, \tag{3.119}$$

where $G(s) = \frac{1}{\Gamma_2(s)}$ denotes the Barnes G-function, $\Gamma_2(s)$ is the Barnes double gamma function (Barnes [18]) defined by the infinite product

$$G(s+1) = \frac{1}{\Gamma_2(s+1)} = (2\pi)^{\frac{s}{2}} e^{-\frac{s}{2}-(C+1)\frac{s^2}{2}} \prod_{k=1}^{\infty}\left(1+\frac{s}{k}\right)^k e^{-s+\frac{s^2}{2k}}$$

(C is the Euler-Mascheroni constant) with the following properties:

(i) $G(s+1) = \frac{1}{\Gamma_2(s+1)} = \frac{\Gamma(s)}{\Gamma_2(s)}$;

(ii) $G(1) = \frac{1}{\Gamma_2(1)} = 1 = \Gamma_2(1)$;

(iii) for $s \in \mathbf{R}$, $\log G(s+1) = \log\frac{1}{\Gamma_2(s+1)} = \frac{1}{2}\left(s^2-\frac{1}{6}\right)\log s - \frac{3}{4}s^2 + \frac{1}{2}s\log 2\pi + \zeta'(-1) + o(1)$ as $s \nearrow \infty$.

The Zeta Function Associated to the Contribution of Parabolic Elements. If we insert (3.99) into the trace formula in Theorem 3.18, we see that

$$\mathcal{P}_1(s) = \left(\frac{1}{2s-1} - \frac{1}{2\beta}\right)\log\left(\frac{\pi}{2}\right),$$

and by the method of obtaining $Z(s)$, we have

$$\begin{aligned}\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log Z_{\mathcal{P}_1}(s) &= \frac{d}{ds}\mathcal{P}_1(s) = \log\left(\frac{\pi}{2}\right)\frac{d}{ds}\frac{1}{2s-1}\\ &= \frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}(s)\log\left(\frac{\pi}{2}\right) = \frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log\left(\frac{\pi}{2}\right)^s,\end{aligned}$$

which implies

$$Z_{\mathcal{P}_1}(s) = \left(\frac{\pi}{2}\right)^s. \tag{3.120}$$

For $\mathcal{P}_2(s)$, we have

$$\mathcal{P}_2(s) = 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\left(\frac{1}{2s-1}n^{-(2s-1)} - \frac{1}{2\beta}n^{-(2\beta)}\right)\log\frac{\pi}{2},$$

so that

$$\frac{d}{ds}\mathcal{P}_2(s) = 2\frac{d}{ds}\frac{1}{2s-1}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{2s}} = \frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log\zeta(2s)^{-1},$$

which implies

$$Z_{\mathcal{P}_2}(s) = \zeta(2s)^{-1}, \tag{3.121}$$

where we have used (B.29). We next compute $Z_{\mathcal{P}_3}(s)$. Using Hejhal [76, p. 435] we obtain

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\psi(1+ir)\frac{1}{r^2+\frac{1}{4}+s(s-1)}\,dr = -\frac{1}{s-\frac{1}{2}}\frac{d}{ds}\log\Gamma\left(s+\frac{1}{2}\right).$$

Thus

$$\begin{aligned}\frac{d}{ds}\mathcal{P}_3(s) &= -\frac{d}{ds}\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(1+ir)\left[\frac{1}{r^2+\frac{1}{4}+s(s-1)}-\frac{1}{r^2+\beta^2}\right]dr\\ &= -\frac{1}{2}\frac{d}{ds}\frac{1}{s-\frac{1}{2}}\frac{d}{ds}\log\Gamma\left(s+\frac{1}{2}\right).\end{aligned}$$

So,

$$\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log Z_{\mathcal{P}_3}(s) = \frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log\Gamma\left(s+\frac{1}{2}\right)^{-1}$$

gives

$$Z_{\mathcal{P}_3}(s) = \Gamma\left(s+\frac{1}{2}\right)^{-1}. \tag{3.122}$$

Similarly,

$$\mathcal{P}_4(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi\left(\frac{1}{2}+ir\right)\frac{1}{r^2+\frac{1}{4}+s(s-1)}\,dr = -\frac{1}{2s-1}\frac{d}{ds}\log\Gamma\left(s\right)$$

implies

$$Z_{\mathcal{P}_4}(s) = \Gamma\left(s\right)^{-1}. \tag{3.123}$$

In summary, the zeta function associated to the parabolic contribution is given by

$$Z_{\mathcal{P}}(s) = Z_{\mathcal{P}_1}(s)Z_{\mathcal{P}_2}(s)Z_{\mathcal{P}_3}(s)Z_{\mathcal{P}_4}(s) = \left(\frac{\pi}{2}\right)^s\zeta(2s)^{-1}\Gamma\left(s+\frac{1}{2}\right)^{-1}\Gamma\left(s\right)^{-1}. \tag{3.124}$$

3.4.3 Zeta Regularised Determinant of the Laplacian

We are now set to give explicit computation of the determinant of the Laplacian $\widetilde{\Delta}-s(1-s)$ on $\mathscr{M} = SL(2,\mathbf{Z})\backslash\mathbf{H}$ in terms of the zeta functions we have obtained in Subsection 3.4.2. We also give determinant expression for the det $\widetilde{\Delta}-s(1-s)$ for special values of $s\in\mathbf{R}$, $s>0$.

The precise statement is the following.

Theorem 3.27. *Let $\widetilde{\Delta}$ be the Laplacian on the modular surface $SL(2,\mathbf{Z})\backslash\mathbf{H}$ with eigenvalues $\lambda = s(1-s)$, $s\in\mathbf{C}$. Then*

$$det(\widetilde{\Delta}-s(1-s)) = e^{c_1 s(s-1)+c_2}Z(s)Z_{\mathcal{I}}(s)Z_{\mathcal{P}}(s), \qquad \text{for some constants } c_1,\ c_2, \tag{3.125}$$

where

(i) $Z(s)$ is the zeta function associated to the contribution of the hyperbolic elements (or simply the Selberg zeta function) and it is given by

$$Z(s) = \prod_{\{\gamma\}_p}\prod_{m=1}^{\infty}\left(1-e^{-\ell(\gamma)(s+m)}\right), \qquad Re\,s>1; \tag{3.126}$$

(ii) $Z_{\mathcal{I}}(s)$ is the zeta function associated to the contribution of the identity element given by

$$Z_{\mathcal{I}}(s) = \left(\frac{(2\pi)^s}{\Gamma(s)G(s)^2}\right)^{1/6}; \tag{3.127}$$

and

(iii) $Z_{\mathcal{P}}(s)$ *is the zeta function associated to the contribution of the parabolic elements and it is given by*

$$Z_{\mathcal{P}}(s) = \left[2^s \Lambda(2s)\Gamma\left(s+\frac{1}{2}\right)\right]^{-1}, \quad \Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

In particular, we have the following special values.

(i) $det'(\widetilde{\Delta}) = 2^{\frac{7}{6}} \cdot 3 \cdot \pi^{-\frac{4}{3}} e^{c_2} Z'(1)$.

(ii) $det\left(\widetilde{\Delta}+2\right) = 2^{\frac{4}{3}} \cdot 3 \cdot 5 \cdot \pi^{-\frac{13}{6}} Z(2) e^{2c_1+c_2}$.

(iii) $det\left(\widetilde{\Delta}+6\right) = 2^{-2} \cdot 3 \cdot 5^{-1} \cdot 7 \cdot \pi^{-\frac{5}{6}} e^{4c_1} \frac{Z(3)}{Z(2)} det\left(\widetilde{\Delta}+2\right)$.

(iv) $det\left(\widetilde{\Delta}+12\right) = 2^{\frac{5}{6}} \cdot 3^{-\frac{7}{6}} \cdot 5 \cdot 7^{-1} \pi^{-\frac{5}{6}} e^{6c_1} \frac{Z(4)}{Z(3)} det\left(\widetilde{\Delta}+6\right)$.

(v) $det\left(\widetilde{\Delta}+20\right) = 2^{-\frac{11}{6}} \cdot 3^{-\frac{1}{3}} \cdot 5^{-1} \cdot 11 \pi^{-\frac{5}{6}} e^{8c_1} \frac{Z(5)}{Z(4)} det\left(\widetilde{\Delta}+12\right)$.

(vi) $det\left(\widetilde{\Delta}+\frac{3}{4}\right) = -2^{-\frac{151}{72}} \pi^{-\frac{11}{12}} \zeta'(-2)^{-1} Z\left(\frac{3}{2}\right) \exp\left(-\frac{\zeta'(-1)}{2} + \frac{3}{4}c_1 + c_2\right)$,

(vii) $det\left(\widetilde{\Delta}+\frac{15}{4}\right) = -2^{\frac{2}{3}} 3^{-\frac{1}{6}} \pi^{-1} \frac{\zeta'(-2)}{\zeta'(-4)} e^{3c_1} \frac{Z\left(\frac{5}{2}\right)}{Z\left(\frac{3}{2}\right)} det\left(\widetilde{\Delta}+\frac{3}{4}\right)$

(viii) $det\left(\widetilde{\Delta}+\frac{35}{4}\right) = -2^{-\frac{1}{3}} 3^{-\frac{1}{3}} 5^{-\frac{1}{6}} \frac{\zeta'(-4)}{\zeta'(-6)} \frac{Z\left(\frac{7}{2}\right)}{Z\left(\frac{5}{2}\right)} e^{c_1} det\left(\widetilde{\Delta}+\frac{15}{4}\right)$.

(ix) $det\left(\widetilde{\Delta}+\frac{63}{4}\right) = -2^{-\frac{4}{3}} 3^{-\frac{1}{3}} 5^{-\frac{1}{3}} 7^{-\frac{1}{6}} \pi^{-10} e^{11c_1} \frac{\zeta'(-6)}{\zeta'(-8)} \frac{Z\left(\frac{9}{2}\right)}{Z\left(\frac{7}{2}\right)} det\left(\widetilde{\Delta}+\frac{35}{4}\right)$.

Proof of Theorem 3.27

We start with the asymptotics of the heat kernel function, and we use Theorem 3.18.

Proposition 3.28 (Minakshisundaram's formula). *Let the trace formula for the modular surface $\mathscr{M} = SL(2,\mathbf{Z})\backslash\mathbf{H}$ be given by Theorem 3.18. Then for some constants A, B, C', D, the trace of the heat operator on $\mathscr{M}$ given by*

$$\Theta(t) := tr\, e^{-t\widetilde{\Delta}} = \sum_{k=0}^{\infty} e^{-\left(r^2+\frac{1}{4}\right)t} \tag{3.128}$$

has the asymptotic expansion

$$\Theta(t) = \frac{A}{t} + B\frac{\log t}{\sqrt{t}} + \frac{C'}{\sqrt{t}} + D + O(\sqrt{t}\log t) \qquad as\ t \searrow 0. \tag{3.129}$$

Proof. As usual we insert the spectral function

$$h(r) = e^{-\left(r^2+\frac{1}{4}\right)t}, \quad t > 0,$$

into the trace formula in Theorem 3.18, and we obtain the equality

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} = \mathcal{I}(t) + \mathcal{H}(t) + \mathcal{P}_1(t) + \mathcal{P}_2(t) + \mathcal{P}_3(t) + \mathcal{P}_4(t), \tag{3.130}$$

where

$$\mathcal{I}(t) = \frac{1}{12}\int_{-\infty}^{\infty} re^{-\left(r^2+\frac{1}{4}\right)t}\tanh(\pi r)\,dr, \;\; \mathcal{H}(t) = \sum_{\{\gamma\}_p}\sum_{m=1}^{\infty}\frac{\ell(\gamma)}{2\sinh\frac{m\ell(\gamma)}{2}}\frac{e^{-\frac{t}{4}}}{(4\pi t)^{\frac{1}{2}}}e^{-\frac{m^2\ell(\gamma)^2}{4t}}$$

$$\mathcal{P}_1(t) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{t}{4}}\log\frac{\pi}{2}, \;\; \mathcal{P}_2(t) = 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\frac{1}{2\sqrt{\pi t}}e^{-\left(\frac{t}{4}+\frac{\log^2 n}{t}\right)}$$

$$\mathcal{P}_3(t) = -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\left(r^2+\frac{1}{4}\right)t}\frac{\Gamma'}{\Gamma}(1+ir)\,dr, \;\; \mathcal{P}_4(t) = -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\left(r^2+\frac{1}{4}\right)t}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr.$$

It suffices to compute the asymptotics as $t\searrow 0$ of each term on the right-hand side of the trace formula (3.130). The asymptotics as $t\searrow 0$ of $\mathcal{I}(t)$ and $\mathcal{H}(t)$ have been computed earlier in Subsection 3.4.1. It remains to find the asymptotic behaviours as $t\searrow 0$ of $\mathcal{P}_1(t),\mathcal{P}_2(t),\mathcal{P}_3(t),\mathcal{P}_4(t)$. Indeed, by integration by parts, we have

$$\mathcal{P}_3(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-\left(r^2+\frac{1}{4}\right)t}\frac{d}{dr}\ln\Gamma\left(1+ir\right)\,dr = \frac{2tie^{-\frac{t}{4}}}{\pi}\int_{-\infty}^{\infty} re^{-r^2t}\ln\Gamma\left(1+ir\right)\,dr.$$

In summary, we have (as $t\searrow 0$)

(i) $\mathcal{P}_1(t) = \frac{e^{-\frac{t}{4}}}{\sqrt{t}}\frac{\log\left(\frac{\pi}{2}\right)}{2\sqrt{\pi}} = \frac{C_1}{\sqrt{t}} + O\left(\sqrt{t}\right)$;

(ii) $\mathcal{P}_2(t) = o(1)$;

(iii) $\mathcal{P}_3(t) = B_1\frac{\ln t}{\sqrt{t}} + \frac{C_2}{\sqrt{t}} + D_1 + O\left(t\ln t\right)$ (Efrat [56]);

(iv) $\mathcal{P}_4(t) = B_2\frac{\ln t}{\sqrt{t}} + \frac{C_3}{\sqrt{t}} + D_2 + O\left(t\ln t\right)$ (Venkov [171]).

The combination of all these asymptotics proves the proposition. □

Now let $\widetilde{\Delta}$ be the Laplacian on the modular surface $\mathscr{M} = SL(2,Z)\backslash\mathbf{H}$. The regularized determinant $\det(\widetilde{\Delta}-s(1-s))$ of $\widetilde{\Delta}$ is done through the analytic continuation of the spectral zeta function

$$\zeta_{\mathscr{M}}(w;s) = \sum_{k=0}^{\infty}\frac{1}{\left(\lambda_k - s(1-s)\right)^w}, \qquad s\gg 0, w\gg 0, \tag{3.131}$$

which was first studied by Minakshisundaram and Pleijel [111] for Riemannian manifolds M, where they show that $\zeta_{\mathscr{M}}(w;s)$ can be analytically continued to a domain in the complex w-plane including $w=0$. We shall see in a moment that $\zeta_{\mathscr{M}}(w;s)$ is regular at $w=0$.

In view of the formal relations

$$\left.\frac{d}{dw}\zeta_{\mathscr{M}}(w;s)\right|_{w=0} = -\sum_{k=0}^{\infty}\log\left(\lambda_k - s(1-s)\right) = -\log\prod_{k=0}^{\infty}\left(\lambda_k - s(1-s)\right)$$

and

$$\det(\widetilde{\Delta}-s(1-s)) = \prod_{k=0}^{\infty}\left(\lambda_k - s(1-s)\right), \tag{3.132}$$

we define the determinant of the Laplacian $\widetilde{\Delta}$ on $SL(2,\mathbf{Z})\backslash\mathbf{H}$ by

$$\det(\widetilde{\Delta}-s(1-s)) = \exp\left(-\frac{\partial}{\partial w}\zeta_{\mathscr{M}}(w;s)\bigg|_{w=0}\right). \tag{3.133}$$

An application of the Mellin transform gives

$$\zeta_{\mathscr{M}}(w;s) = \frac{1}{\Gamma(w)} \int_0^\infty \Theta(t) e^{-s(s-1)t} t^{w-1}\, dt. \tag{3.134}$$

Since $\frac{1}{\Gamma(w)}$ vanishes at $w = 0$ we conclude that for $s \gg 0$, $\zeta_{\mathscr{M}}(w;s)$ is regular at $w = 0$.

Proposition 3.29.

$$\begin{aligned}
\frac{\partial}{\partial w} \zeta_{\mathscr{M}}(w;s)\bigg|_{w=0} &= -\log\det(\widetilde{\Delta} - \mathrm{s}(1-\mathrm{s})) \\
&\sim As(s-1)\log s(s-1) - As(s-1) \\
&\quad + 2\sqrt{\pi}B\,(s(s-1))^{\frac{1}{2}}\,[\log s(s-1) + (C + \log 4 - 2)] \\
&\quad - 2\sqrt{\pi}C'\,(s(s-1))^{\frac{1}{2}} - D\log s(s-1) \quad \textit{as } s \nearrow \infty,
\end{aligned} \tag{3.135}$$

where C is the Euler-Mascheroni constant.

Proof. By (3.129) and (3.134) we have

$$\zeta_{\mathscr{M}}(w;s) = \zeta_{\mathscr{M},1}(w;s) + \zeta_{\mathscr{M},2}(w;s) + \zeta_{\mathscr{M},3}(w;s) + \zeta_{\mathscr{M},4}(w;s) + \zeta_{\mathscr{M},5}(w;s),$$

where

$$\begin{aligned}
\zeta_{\mathscr{M},1}(w;s) &= \frac{A}{\Gamma(w)} \int_0^\infty e^{-s(s-1)t} t^{w-2}\, dt = A(w-1)^{-1}(s(s-1))^{-(w-1)} \\
\zeta_{\mathscr{M},2}(w;s) &= \frac{B}{\Gamma(w)} \int_0^\infty e^{-s(s-1)t} t^{w-\frac{3}{2}} \log t\, dt \\
\zeta_{\mathscr{M},3}(w;s) &= \frac{C'}{\Gamma(w)} \int_0^\infty e^{-s(s-1)t} t^{w-\frac{3}{2}}\, dt = \frac{C'}{\Gamma(w)}(s(s-1))^{-(w-\frac{1}{2})}\Gamma\left(w - \frac{1}{2}\right) \\
\zeta_{\mathscr{M},4}(w;s) &= \frac{D}{\Gamma(w)} \int_0^\infty e^{-s(s-1)t} t^{w-1}\, dt = D(s(s-1))^{-w} \\
\zeta_{\mathscr{M},5}(w;s) &= \frac{1}{\Gamma(w)} \int_0^\infty e^{-s(s-1)t} O(\sqrt{t}\log t) t^{w-1}\, dt.
\end{aligned}$$

To simplify $\zeta_{\mathscr{M},2}(w;s)$ further, we use the identity (Gradshtejn and Ryzhik [66, p. 573, eq. 4.352(1)])

$$\int_0^\infty x^{\nu-1} e^{-\mu x} \log x\, dx = \frac{1}{\mu^\nu}\Gamma(\nu)[\psi(\nu) - \log\mu]$$

to obtain

$$\zeta_{\mathscr{M},2}(w;s) = \frac{B}{\Gamma(w)}(s(s-1))^{-(w-\frac{1}{2})}\Gamma\left(w-\frac{1}{2}\right)\left[\psi\left(w-\frac{1}{2}\right) - \log s(s-1)\right],$$

so that

$$\begin{aligned}
\frac{\partial}{\partial w}\zeta_{\mathscr{M},2}(w;s)\bigg|_{w=0} &= B\Gamma\left(-\frac{1}{2}\right)(s(s-1))^{\frac{1}{2}}\left[\psi\left(-\frac{1}{2}\right) - \log s(s-1)\right] \\
&= 2B\sqrt{\pi}(s(s-1))^{\frac{1}{2}}(C + \log 4 - 2) + 2B\sqrt{\pi}(s(s-1))^{\frac{1}{2}}\log s(s-1),
\end{aligned}$$

where we have used (B.15) and (B.16). Also,

$$\begin{aligned}
\frac{\partial}{\partial w}\zeta_{\mathscr{M},1}(w;s)\Big|_{w=0} &= -As(s-1)+As(s-1)\log s(s-1),\\
\frac{\partial}{\partial w}\zeta_{\mathscr{M},3}(w;s)\Big|_{w=0} &= -2\sqrt{\pi}C'\left(s(s-1)\right)^{\frac{1}{2}},\\
\frac{\partial}{\partial w}\zeta_{\mathscr{M},4}(w;s)\Big|_{w=0} &= -D\log s(s-1),\\
\zeta_{\mathscr{M},5}(w;s)\searrow 0 \quad &\text{as} \quad s\nearrow\infty.
\end{aligned}$$

□

We proceed now to compute the determinant of the Laplacian using the trace formula (3.64) with the spectral function (3.99). With this spectral function the spectral side of the trace formula (3.64) has the derivative

$$\frac{d}{ds}\sum_{k=0}^{\infty}\left[\frac{1}{[\lambda_k-s(1-s)]}-\frac{1}{[\lambda_k-a(1-a)]}\right]=-\sum_{k=0}^{\infty}\frac{2s-1}{[\lambda_k-s(1-s)]^2}. \tag{3.136}$$

We also observe that

$$\begin{aligned}
\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log\det(\widetilde{\Delta}+s(s-1)) &= \frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\left[-\frac{d}{dw}\zeta_{\mathscr{M}}(w;s)\Big|_{w=0}\right]\\
&= \frac{d}{dw}\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\left[-\sum_{k=0}^{\infty}\frac{1}{(\lambda_k+s(s-1))^{w}}\Big|_{w=0}\right]\\
&= \frac{d}{dw}\frac{d}{ds}\left[\sum_{k=0}^{\infty}\frac{w}{(\lambda_k+s(s-1))^{w+1}}\Big|_{w=0}\right]\\
&= -\sum_{k=0}^{\infty}\frac{(2s-1)}{(\lambda_k+s(s-1))^{2}}.
\end{aligned} \tag{3.137}$$

Let $\mathcal{I}(s)$ denote the term $c(\mathcal{I})$ with the spectral function (3.99). Then from (3.64), (3.136) and (3.137) we have

$$\begin{aligned}
\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log\det(\widetilde{\Delta}-s(1-s)) &= \frac{d}{ds}\mathcal{I}(s)+\frac{d}{ds}\mathcal{H}(s)+\frac{d}{ds}\mathcal{P}_1(s)+\frac{d}{ds}\mathcal{P}_2(s)\\
&\quad+\frac{d}{ds}\mathcal{P}_3(s)+\frac{d}{ds}\mathcal{P}_4(s)\\
&= \frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log Z_{\mathcal{I}}(s)+\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log Z(s)\\
&\quad+\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log Z_{\mathcal{P}_1}(s)+\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log Z_{\mathcal{P}_2}(s)\\
&\quad+\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log Z_{\mathcal{P}_3}(s)+\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log Z_{\mathcal{P}_4}(s).
\end{aligned}$$

Integrating both sides of the last equation with respect to s, we obtain

$$\det(\widetilde{\Delta}-s(1-s))=e^{c_1 s(s-1)+c_2}Z_{\mathcal{I}}(s)Z(s)Z_{\mathcal{P}}(s), \tag{3.138}$$

where c_1 and c_2 are constants of integration.

Remark 3.4. *The constants c_1 and c_2 can be obtained by comparing the asymptotic behaviours of the logarithm of both sides of* (3.138)*. The asymptotics of* $\log$ *det*$(\widetilde{\Delta} - s(1-s))$ *is given by Proposition* (3.29)*.*

Next we compute explicitly $\det(\widetilde{\Delta} - s(1-s))$ for some special values of $s \in \mathbf{R}$, $s > 0$.

First we observe that

$$\det(\widetilde{\Delta} - s(1-s)) = s(s-1)\prod_{k=1}^{\infty}(\lambda_k - s(1-s))\,, \tag{3.139}$$

and so

$$\frac{d}{ds}\det(\widetilde{\Delta} - s(1-s))\Big|_{s=1} = \prod_{k=1}^{\infty}\lambda_k = \det{}'(\widetilde{\Delta}). \tag{3.140}$$

Since $Z(s)$ has a simple zero at $s=1$, $Z_{\mathcal{I}}(1) = (2\pi)^{\frac{1}{6}}$ (since $\Gamma_2(1) = 1$) and $\Lambda(2) = \frac{\pi}{6}$, we obtain

$$\det{}'(\widetilde{\Delta}) = \frac{d}{ds}e^{c_1 s(s-1)+c_2}Z_{\mathcal{I}}(s)Z(s)Z_{\mathcal{P}}(s)\Big|_{s=1} = 2^{\frac{7}{6}}\cdot 3\cdot \pi^{-\frac{4}{3}}e^{c_2}Z'(1).$$

For the other special values of $s \in \mathbf{R}$, $s > 1$, explicit computations show that

- $(s=2)$

$$\begin{aligned}\det\left(\widetilde{\Delta}+2\right) &= e^{2c_1+c_2}Z_{\mathcal{I}}(2)\,Z_{\mathcal{P}}(2)\,Z(2)\\ &= 2^{\frac{4}{3}}\cdot 3\cdot 5\cdot \pi^{-\frac{13}{6}}Z(2)e^{2c_1+c_2}.\end{aligned}$$

- $(s=3)$

$$\begin{aligned}\det\left(\widetilde{\Delta}+6\right) &= e^{6c_1+c_2}Z_{\mathcal{I}}(3)\,Z_{\mathcal{P}}(3)\,Z(3)\\ &= 2^{-\frac{2}{3}}\cdot 3^2\cdot 7\cdot \pi^{-3}Z(3)e^{6c_1+c_2}\\ &= 2^{-2}\cdot 3\cdot 5^{-1}\cdot 7\cdot \pi^{-\frac{5}{6}}e^{4c_1}\frac{Z(3)}{Z(2)}\det\left(\widetilde{\Delta}+2\right).\end{aligned}$$

- $(s=4)$

$$\begin{aligned}\det\left(\widetilde{\Delta}+12\right) &= e^{12c_1+c_2}Z_{\mathcal{I}}(4)\,Z_{\mathcal{P}}(4)\,Z(4)\\ &= 2^{\frac{1}{6}}\cdot 3^{\frac{5}{6}}\cdot 5\cdot \pi^{-\frac{23}{6}}Z(4)e^{12c_1+c_2}\\ &= 2^{-\frac{7}{6}}\cdot 3^{-\frac{1}{6}}\pi^{-\frac{5}{3}}e^{10c_1}\frac{Z(4)}{Z(2)}\det\left(\widetilde{\Delta}+2\right)\\ &= 2^{\frac{5}{6}}\cdot 3^{-\frac{7}{6}}\cdot 5\cdot 7^{-1}\pi^{-\frac{5}{6}}e^{6c_1}\frac{Z(4)}{Z(3)}\det\left(\widetilde{\Delta}+6\right).\end{aligned}$$

- $(s = 5)$

$$\begin{aligned}
\det\left(\widetilde{\Delta} + 20\right) &= e^{20c_1+c_2} Z_{\mathcal{I}}(5)\, Z_{\mathcal{P}}(5)\, Z(5) \\
&= 2^{-\frac{5}{3}} \cdot 3^{\frac{1}{2}} \cdot 11 \cdot \pi^{-\frac{14}{3}} Z(5) e^{20c_1+c_2} \\
&= 2^{-3} \cdot 3^{-\frac{1}{2}} \cdot 5^{-1} \cdot 11\pi^{-\frac{5}{2}} e^{18c_1} \frac{Z(5)}{Z(2)} \det\left(\widetilde{\Delta} + 2\right) \\
&= 2^{-1} \cdot 3^{-\frac{3}{2}} \cdot 7^{-1} \cdot 11\pi^{-\frac{5}{3}} e^{14c_1} \frac{Z(5)}{Z(3)} \det\left(\widetilde{\Delta} + 6\right) \\
&= 2^{-\frac{11}{6}} \cdot 3^{-\frac{1}{3}} \cdot 5^{-1} \cdot 11\pi^{-\frac{5}{6}} e^{8c_1} \frac{Z(5)}{Z(4)} \det\left(\widetilde{\Delta} + 12\right).
\end{aligned}$$

- $(s = \frac{3}{2})$

$$\begin{aligned}
\det\left(\widetilde{\Delta} + \frac{3}{4}\right) &= e^{\frac{3}{4}c_1+c_2} Z_{\mathcal{I}}\left(\frac{3}{2}\right) Z_{\mathcal{P}}\left(\frac{3}{2}\right) Z\left(\frac{3}{2}\right) \\
&= 2^{\frac{29}{72}} \pi^{\frac{1}{12}} e^{-\frac{1}{24}} \mathsf{A}^{\frac{1}{2}} \cdot 2^{-\frac{1}{2}} \pi \zeta(3)^{-1} Z\left(\frac{3}{2}\right) e^{\frac{3}{4}c_1+c_2} \\
&= -2^{-\frac{151}{72}} \pi^{-\frac{11}{12}} \zeta'(-2)^{-1} Z\left(\frac{3}{2}\right) \exp\left(-\frac{\zeta'(-1)}{2} + \frac{3}{4}c_1 + c_2\right).
\end{aligned}$$

- $(s = \frac{5}{2})$

$$\begin{aligned}
\det\left(\widetilde{\Delta} + \frac{15}{4}\right) &= e^{\frac{15}{4}c_1+c_2} Z_{\mathcal{I}}\left(\frac{5}{2}\right) Z_{\mathcal{P}}\left(\frac{5}{2}\right) Z\left(\frac{5}{2}\right) \\
&= 2^{\frac{77}{72}} 3^{-\frac{1}{6}} \pi^{\frac{1}{12}} e^{-\frac{1}{24}} \mathsf{A}^{\frac{1}{2}} \cdot 2^{-\frac{1}{2}} \pi^2 3^{-1} \zeta(5)^{-1} Z\left(\frac{5}{2}\right) e^{\frac{15}{4}c_1+c_2} \\
&= 2^{-\frac{103}{72}} 3^{-\frac{1}{6}} \pi^{-\frac{23}{12}} \zeta'(-4)^{-1} Z\left(\frac{5}{2}\right) \exp\left(-\frac{\zeta'(-1)}{2} + \frac{15}{4}c_1 + c_2\right) \\
&= -2^{\frac{2}{3}} 3^{-\frac{1}{6}} \pi^{-1} \frac{\zeta'(-2)}{\zeta'(-4)} e^{3c_1} \frac{Z\left(\frac{5}{2}\right)}{Z\left(\frac{3}{2}\right)} \det\left(\widetilde{\Delta} + \frac{3}{4}\right).
\end{aligned}$$

- $(s = \frac{7}{2})$

$$\begin{aligned}
\det\left(\widetilde{\Delta} + \frac{35}{4}\right) &= e^{\frac{35}{4}c_1+c_2} Z_{\mathcal{I}}\left(\frac{7}{2}\right) Z_{\mathcal{P}}\left(\frac{7}{2}\right) Z\left(\frac{7}{2}\right) \\
&= 2^{\frac{149}{72}} 3^{-\frac{1}{2}} 5^{-\frac{1}{6}} \pi^{\frac{1}{12}} e^{-\frac{1}{24}} \mathsf{A}^{\frac{1}{2}} \cdot 2^{-\frac{3}{2}} \pi^4 3^{-2} 5^{-1} \zeta(7)^{-1} Z\left(\frac{7}{2}\right) e^{\frac{35}{4}c_1+c_2} \\
&= -2^{-\frac{175}{72}} 3^{-\frac{1}{2}} 5^{-\frac{1}{6}} \pi^{-\frac{23}{12}} \zeta'(-6)^{-1} Z\left(\frac{7}{2}\right) \exp\left(-\frac{\zeta'(-1)}{2} + \frac{35}{4}c_1 + c_2\right) \\
&= 2^{\frac{1}{3}} 3^{-\frac{1}{2}} 5^{-\frac{1}{6}} \pi^{-1} \frac{\zeta'(-2)}{\zeta'(-6)} \frac{Z\left(\frac{7}{2}\right)}{Z\left(\frac{3}{2}\right)} e^{4c_1} \det\left(\widetilde{\Delta} + \frac{3}{4}\right) \\
&= -2^{-\frac{1}{3}} 3^{-\frac{1}{3}} 5^{-\frac{1}{6}} \frac{\zeta'(-4)}{\zeta'(-6)} \frac{Z\left(\frac{7}{2}\right)}{Z\left(\frac{5}{2}\right)} e^{c_1} \det\left(\widetilde{\Delta} + \frac{15}{4}\right).
\end{aligned}$$

- $\left(s=\frac{9}{2}\right)$

$$\begin{aligned}
\det\left(\widetilde{\Delta}+\frac{63}{4}\right) &= e^{\frac{63}{4}c_1+c_2} Z_{\mathcal{I}}\left(\frac{9}{2}\right) Z_{\mathcal{P}}\left(\frac{9}{2}\right) Z\left(\frac{9}{2}\right)\\
&= 2^{\frac{11}{12}} 3^{-\frac{11}{6}} 5^{-\frac{3}{2}} 7^{-\frac{7}{6}} \pi^{-4} e^{-\frac{1}{24}} \mathsf{A}^{\frac{1}{2}} \cdot 2^{-\frac{1}{72}} \pi^{\frac{1}{12}} \zeta(9)^{-1} Z\left(\frac{9}{2}\right) e^{\frac{63}{4}c_1+c_2}\\
&= 2^{-\frac{223}{72}} 3^{-\frac{5}{6}} 5^{-\frac{1}{2}} 7^{-\frac{1}{6}} \pi^{-\frac{143}{12}} \zeta'(-8)^{-1} Z\left(\frac{9}{2}\right) \exp\left(-\frac{\zeta'(-1)}{2}+\frac{63}{4}c_1+c_2\right)\\
&= -2^{-1} 3^{-\frac{5}{6}} 5^{-\frac{1}{2}} 7^{-\frac{1}{6}} \pi^{-11} e^{15c_1} \frac{\zeta'(-2)}{\zeta'(-8)} \frac{Z\left(\frac{9}{2}\right)}{Z\left(\frac{3}{2}\right)} \det\left(\widetilde{\Delta}+\frac{3}{4}\right)\\
&= 2^{-\frac{5}{3}} 3^{-\frac{2}{3}} 5^{-\frac{1}{2}} 7^{-\frac{1}{6}} \pi^{-10} \frac{\zeta'(-4)}{\zeta'(-8)} \frac{Z\left(\frac{9}{2}\right)}{Z\left(\frac{5}{2}\right)} e^{12c_1} \det\left(\widetilde{\Delta}+\frac{15}{4}\right)\\
&= -2^{-\frac{4}{3}} 3^{-\frac{1}{3}} 5^{-\frac{1}{3}} 7^{-\frac{1}{6}} \pi^{-10} e^{11c_1} \frac{\zeta'(-6)}{\zeta'(-8)} \frac{Z\left(\frac{9}{2}\right)}{Z\left(\frac{7}{2}\right)} \det\left(\widetilde{\Delta}+\frac{35}{4}\right).
\end{aligned}$$

In Awonusika and Taheri [15] the zeta regularised determinants of the Laplacians on the real and complex hyperbolic manifolds $\mathscr{M}=\Gamma\backslash\mathbf{H}^n$ and $\mathscr{M}=\Gamma\backslash\mathbf{CH}^n$ are discussed and identified.

Furthermore, let (M,g) be a compact Riemannian manifold. Following Osgood et al. [124], Sarnak [144], the height function of a Riemannian manifold M is defined by

$$h(g)=-\log\det{}'\Delta_M.$$

The height function $h(g)$ admits a Polyakov formula (Polyakov [132, 133]) which describes the variation of the determinant under conformal deformations of the metric. To conformally deform a metric we mean to change the distances between points while maintaining the angles between vectors. Polyakov's formula for the determinant is the key tool in the analysis of extrema of determinants on manifolds. The authors Sarnak [144, 146], Osgood et al. [124] show that for a closed Riemann surface there exists a unique metric of constant curvature at which the regularized determinant attains a maximum. They also prove the corresponding statement for the case of a compact Riemann surface with boundary, with appropriate boundary conditions (see Chang [37], Chang and Yang [38, 39, 40], Okikiolu [121] for functional determinants in higher dimensions). A consequence of the Polyakov formula for a closed Riemann surface is the Onofri's inequality (Onofri [122], Onofri and Virasoro [123]), which implies that the determinant of the Laplacian on $\mathbf{S}^2$ under the conformal change of the metric is maximised by the standard metric. The generalisation of Onofri's inequality to the n-dimensional unit sphere $\mathbf{S}^n$ was considered by Beckner [23, 24], Dolbeault et al. [53]) using symmetrisation arguments in terms of the Poisson semigroup, ultraspherical and spherical harmonic polynomials.

Chapter 4

Poisson Integral Representations in Euclidean and Non-Euclidean spaces

Of fundamental importance in the theory of harmonic functions and especially for solving the Dirichlet problem in the unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$, is the so-called Poisson kernel. In the classical situation of the Laplacian in $\mathbf{R}^n$, the exact formula for the kernel leads to many important results concerning behaviours of harmonic functions, and also leads to various identities in the context of special functions. One of the aims of this chapter is to derive a formula, in a closed form, for the Euclidean Poisson kernel and then compute a series representation involving the Gengebauer polynomial. The said formula for the Euclidean Poisson kernel will be deduced from the explicitly computed series representation for Poisson kernel in a ball in the n-dimensional real hyperbolic space $\mathbf{H}^n$, involving the Gegenbauer polynomial and the Gauss hypergeometric functions. We also give integral representations of the Euclidean Poisson kernel involving special functions. The case of the unit sphere $\mathbf{S}^n$ is also considered. In Section 4.1, we compute explicitly the eigenfunctions of the Laplacian in $\mathbf{H}^n$ using the method of separation of variables. In Sections 4.2 and 4.3, we respectively compute the Poisson kernels on the upper half-space $\mathbf{H}^n$ and the unit sphere $\mathbf{S}^n$. In Section 4.4 we deduce the n-dimensional Euclidean Poisson kernel from the non-Euclidean one, by considering a small hyperbolic distance, bearing in mind that every Riemannian manifold is locally Euclidean. Finally, in Section 4.5 we present various series representations for the Poisson kernel in the Poincaré (hyperbolic) unit ball $\mathbf{D}^n$ in terms of special functions, from which the Poisson integral formula for $\mathbf{D}^n$ follows. The Poisson kernels on symmetric spaces are also considered in Symeonidis [162] using different methods. See also Cammarota and Orsingher [36], and Byczkowski and Małecki [34] for Poisson kernels in the context of hyperbolic Brownian motion.

We shall see in Chapter 5, Section 5.5 that the traces of the heat kernels on compact symmetric spaces can be purely expressed in terms of the Euclidean Poisson kernel and in Chapter 6, Section 6.1 that the generalised eigenfunctions of the Laplacian in the upper half-space $\mathbf{H}^n$ can be given as an integral transform of harmonic functions in the Euclidean ball $\mathbf{B}^n$; this justifies the choice of the contents of this chapter before considering other spectral functions in the remaining chapters.

4.1 Eigenfunctions of the Laplacian in the Hyperbolic Space

In this section we solve explicitly the eigenvalue problem in the upper half-space $\mathbf{H}^n$ so as to obtain the eigenfunctions of the Laplacian in $\mathbf{H}^n$ which will be useful in the sequel.

The eigenvalue problem on the upper half-space $\mathbf{H}^n$ is the equation

$$-\frac{\partial^2}{\partial\rho^2}-(n-1)\frac{\cosh\rho}{\sinh\rho}\frac{\partial}{\partial\rho}-\frac{1}{\sinh^2\rho}\Delta_{\mathbf{S}^{n-1}}G=\lambda G, \tag{4.1}$$

where $G=G(\rho,\varphi)$ $(0<\rho<\infty, 0\le\varphi\le\pi)$ are the eigenfunctions of the Laplacian $\Delta_{\mathbf{H}^n}$ with eigenvalues $\lambda=s(n-1-s)$ $(s=(n-1)/2+ir, r\in\mathbf{R})$. To solve (4.1), we assume a product solution of the form

$$G(\rho,\varphi)=\Theta(\rho)\Psi(\varphi).$$

Differentiating and substituting in (4.1), we have

$$\Psi\frac{\partial^2\Theta}{\partial\rho^2}+(n-1)\Psi\frac{\cosh\rho}{\sinh\rho}\frac{\partial\Theta}{\partial\rho}+\frac{\Theta}{\sinh^2\rho}\Delta_{\mathbf{S}^{n-1}}\Psi=-\lambda\Theta\Psi.$$

Multiplying both sides of the last equation by $\frac{\sinh^2\rho}{\Theta\Psi}$, we obtain

$$\frac{\sinh^2\rho}{\Theta}\frac{\partial^2\Theta}{\partial\rho^2}+\frac{(n-1)}{\Theta}\cosh\rho\sinh\rho\frac{\partial\Theta}{\partial\rho}+\lambda\sinh^2\rho=-\frac{1}{\Psi}\Delta_{\mathbf{S}^{n-1}}\Psi.$$

Since the left hand side depends only on ρ and the right hand side depends only on φ, we can equate each side to a constant, say, μ^2. Thus, we obtain a pair of ODE and PDE, namely

$$\frac{\sinh^2\rho}{\Theta}\frac{d^2\Theta}{d\rho^2}+\frac{(n-1)}{\Theta}\cosh\rho\sinh\rho\frac{d\Theta}{d\rho}+\lambda\sinh^2\rho=\mu^2 \tag{4.2}$$

$$\Delta_{\mathbf{S}^{n-1}}\Psi=-\mu^2\Psi, \tag{4.3}$$

where

$$\mu^2=m(m+n-2),\qquad m=0,1,2,\cdots,$$

are the eigenvalues of $\Delta_{\mathbf{S}^{n-1}}$ corresponding to the eigenfunctions Ψ. We are interested in the radial part (4.2). Multiplying equation (4.2) by $\frac{\Theta}{\sinh^2\rho}$ we have

$$\frac{d^2\Theta}{d\rho^2}+(n-1)\frac{\cosh\rho}{\sinh\rho}\frac{d\Theta}{d\rho}+\left\{\lambda-\frac{\mu^2}{\sinh^2\rho}\right\}\Theta=0. \tag{4.4}$$

Substituting

$$\Theta(\rho)=\sinh^{1-\frac{n}{2}}\rho\,\eta(\rho),$$

with

$$\begin{aligned}\frac{d\Theta}{d\rho}&=\cosh\rho\sinh^{-\frac{n}{2}}\rho\eta-\frac{n}{2}\cosh\rho\sinh^{-\frac{n}{2}}\rho\eta+\sinh^{1-\frac{n}{2}}\rho\frac{d\eta}{d\rho},\\ \frac{d^2\Theta}{d\rho^2}&=\sinh^{1-\frac{n}{2}}\rho\frac{d^2\eta}{d\rho^2}+\left[2\cosh\rho\sinh^{-\frac{n}{2}}\rho-n\cosh\rho\sinh^{-\frac{n}{2}}\rho\right]\frac{d\eta}{d\rho}\\ &\quad+\left[\cosh^2\rho\sinh^{-\frac{n}{2}-1}\rho\left(-\frac{n}{2}+\frac{n^2}{4}\right)+\sinh^{1-\frac{n}{2}}\rho\left(1-\frac{n}{2}\right)\right]\eta,\end{aligned}$$

in (4.4) and then multiplying the resulting equation by $\sinh^{\frac{n}{2}-1}\rho$, we obtain after some rearrangements

$$\begin{aligned}&\frac{d^2\eta}{d\rho^2}+\left[2\frac{\cosh\rho}{\sinh\rho}-\frac{n\cosh\rho}{\sinh\rho}+(n-1)\frac{\cosh\rho}{\sinh\rho}\right]\frac{d\eta}{d\rho}\\&+\left[\cosh^2\rho\sinh^{-2}\rho\left(-\frac{n}{2}+\frac{n^2}{4}\right)+\left(1-\frac{n}{2}\right)+(n-1)\cosh^2\rho\sinh^{-2}\rho\right]\eta\\&+\left[(n-1)\cosh^2\rho\sinh^{-2}\rho\left(1-\frac{n}{2}\right)+\lambda-\mu^2\sinh^{-2}\rho\right]\eta=0.\end{aligned}$$

Further simplification gives

$$\frac{d^2\eta}{d\rho^2}+\frac{\cosh\rho}{\sinh\rho}\frac{d\eta}{d\rho}+\left\{\lambda-\frac{n}{2}+1-\left[\left(\frac{n}{2}-1\right)^2\cosh^2\rho+\mu^2\right]\sinh^{-2}\rho\right\}\eta=0. \tag{4.5}$$

Setting

$$y(x)=\eta(\rho),\ x=\cosh\rho,\ \frac{d\eta}{d\rho}=\frac{dy}{dx}\sinh\rho,\ \frac{d^2\eta}{d\rho^2}=\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx} \tag{4.6}$$

gives

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left\{-\lambda+\frac{n}{2}-1-\left[\frac{\left(\frac{n}{2}-1\right)^2x^2}{1-x^2}+\frac{\mu^2}{1-x^2}\right]\right\}y=0.$$

Denote the expression inside the braces by $\mathtt{I}_n(\lambda,\mu;x)$. Then

$$\begin{aligned}\mathtt{I}_n(\lambda,\mu;x)&=\frac{-\lambda+\frac{n}{2}-1+\lambda x^2-\frac{nx^2}{2}-\frac{n^2x^2}{4}+nx^2-\mu^2}{1-x^2}\\&=\frac{\left(-\frac{1}{4}-\lambda+\frac{(n-1)^2}{4}\right)\left(1-x^2\right)-\left[\left(\frac{n}{2}-1\right)^2+\mu^2\right]}{1-x^2}.\end{aligned}$$

If we set

$$\alpha(\alpha+1)=-\frac{1}{4}-\lambda+\frac{(n-1)^2}{4},$$

then we obtain a quadratic equation in α :

$$\alpha^2+\alpha+\frac{1}{4}+\lambda-\frac{(n-1)^2}{4}=0$$

whose solution is

$$\alpha=-\frac{1}{2}\pm\sqrt{-\lambda+\frac{(n-1)^2}{4}}=-\frac{1}{2}\pm ir,\ s=\frac{n-1}{2}+ir. \tag{4.7}$$

Hence, we obtain the associated Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left\{\alpha(\alpha+1)-\frac{\sigma^2}{1-x^2}\right\}y=0 \tag{4.8}$$

of degree α given by (4.7) and order

$$\sigma=\pm\sqrt{\frac{(n-2)^2}{4}+\mu^2}=\pm\frac{n-2+2m}{2}. \tag{4.9}$$

In summary we have the following statement.

Proposition 4.1. *For $s = \frac{n-1}{2} + ir$, $r \in \mathbf{R}$, a solution of the n-dimensional hyperbolic eigenvalue problem*

$$\frac{d^2\Theta}{d\rho^2} + (n-1)\frac{\cosh\rho}{\sinh\rho}\frac{d\Theta}{d\rho} + \left\{s(n-1-s) - \frac{m(m+n-2)}{\sinh^2\rho}\right\}\Theta = 0 \tag{4.10}$$

is given by

$$\Theta^n_{s,m}(\rho) = \sinh^{1-\frac{n}{2}}\rho P^{-\frac{n-2+2m}{2}}_{-\frac{1}{2}+ir}(\cosh\rho). \tag{4.11}$$

In particular, the function

$$\Theta^n_{s,0}(\rho) = \sinh^{1-\frac{n}{2}}\rho P^{-\frac{n-2}{2}}_{-\frac{1}{2}+ir}(\cosh\rho), \tag{4.12}$$

solves the eigenvalue problem

$$\frac{d^2\Theta}{d\rho^2} + (n-1)\frac{\cosh\rho}{\sinh\rho}\frac{d\Theta}{d\rho} + s(n-1-s)\Theta = 0; \tag{4.13}$$

and

$$\Theta^n_{0,m}(\rho) = \sinh^{1-\frac{n}{2}}\rho P^{-\frac{n-2+2m}{2}}_{\frac{n-2}{2}}(\cosh\rho), \tag{4.14}$$

solves the eigenvalue problem

$$\frac{d^2\Theta}{d\rho^2} + (n-1)\frac{\cosh\rho}{\sinh\rho}\frac{d\Theta}{d\rho} - \frac{m(m+n-2)}{\sinh^2\rho}\Theta = 0, \tag{4.15}$$

where $P^\mu_\nu(z)$ is the associated Legendre function of the first kind with degree ν and order μ, and argument z (see Appendix B.5).

4.2 Special Functions Representation of the Poisson Kernel

We now compute the Poisson kernel in $\mathbf{H}^n$, or what is the same we solve the Laplace equation in $\mathbf{H}^n$ involving the radial part of the Laplacian in $\mathbf{H}^n$ using Proposition 4.1.

Let $\mathbf{H}^n$ be the n-dimensional upper half-space with origin $o = (0, ..., 0, 1) \in \mathbf{H}^n$. As usual $\widetilde{\rho} = d(w, w')$, $w = (x, y), w' = o = (0, ..., 0, 1) \in \mathbf{H}^n$. Consider the n-dimensional hyperbolic Dirichlet problem

$$\begin{aligned}&\left[\frac{\partial^2}{\partial\widetilde{\rho}^2} + (n-1)\frac{\cosh\widetilde{\rho}}{\sinh\widetilde{\rho}}\frac{\partial}{\partial\widetilde{\rho}} + \frac{1}{\sinh^2\widetilde{\rho}}\left(\frac{\partial^2}{\partial\widetilde{\theta}^2} + (n-2)\frac{\cos\widetilde{\theta}}{\sin\widetilde{\theta}}\frac{\partial}{\partial\widetilde{\theta}}\right)\right]P_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta) = 0\\ &P_{\widetilde{\rho},\rho}\left(\widetilde{\theta},\theta\right) = \delta\left(\widetilde{\theta}-\theta\right),\end{aligned} \tag{4.16}$$

where $0 < \widetilde{\rho} < \rho < \infty, \widetilde{\theta} - \theta \in (0, \pi]$, and δ is the Dirac delta function. To solve the initial value problem (4.16), we assume a product solution of the form

$$P_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta) = R(\widetilde{\rho})\Theta(\widetilde{\theta}). \tag{4.17}$$

Differentiating (4.17) and then substituting in (4.16) we obtain two ordinary differential equations

$$\Theta''(\widetilde{\theta}) + (n-2)\cot\widetilde{\theta}\,\Theta'(\widetilde{\theta}) + \mu^2\Theta(\widetilde{\theta}) = 0 \tag{4.18}$$

$$\sinh^2\widetilde{\rho}\,R''(\widetilde{\rho}) + (n-1)\cosh\widetilde{\rho}\sinh\widetilde{\rho}\,R'(\widetilde{\rho}) - \mu^2 R(\widetilde{\rho}) = 0, \tag{4.19}$$

where $\mu^2 = k(k+n-2)$, $k \geq 0$, is the eigenvalue of the Laplacian on the unit sphere $\mathbf{S}^{n-1}$, i.e., (4.18) is the eigenvalue problem on the unit sphere $\mathbf{S}^{n-1}$. To solve (4.18) we make a substitution

$$\Theta(\widetilde{\theta}) = y(\xi), \quad \xi = \cos\widetilde{\theta},$$

to obtain

$$\left(1-\xi^2\right)y''(\xi) - (n-1)\xi y(\xi) + \mu^2 y(\xi) = 0, \tag{4.20}$$

which can be called the $(n-1)$-dimensional Legendre's equation. When $n = 3$, (4.20) reduces to the classical Legendre's equation (see Appendix B.5). However, in the study of special functions, equation (4.20) is called the *Gegenbauer equation* (see (B.112) with $\nu = \frac{n-2}{2}$); its solution $C_k^{\nu}(t)$ is called the *Gegenbauer polynomial* (see Appendix B.6). Thus

$$y(\xi) = C_k^{\frac{n-2}{2}}(\xi), \;\; \xi = \cos\widetilde{\theta},$$

solves (4.20). Hence,

$$\Theta(\widetilde{\theta}) = AC_k^{\frac{n-2}{2}}(\cos\widetilde{\theta}). \tag{4.21}$$

We observe that (4.19) is precisely the equation (4.15) in Proposition 4.1, with the solution

$$R(\widetilde{\rho}) = \sinh^{\frac{2-n}{2}}\widetilde{\rho}\,P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\widetilde{\rho}), \qquad k \geq 0. \tag{4.22}$$

So by (4.17), (4.21) and (4.22) we obtain

$$\begin{aligned} P_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta) &= \sum_{k=0}^{\infty} A_k C_k^{\frac{n-2}{2}}(\cos\widetilde{\theta})R_k(\widetilde{\rho}) \\ &= \sum_{k=0}^{\infty} A_k \sinh^{\frac{2-n}{2}}\widetilde{\rho}\,P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\widetilde{\rho})C_k^{\frac{n-2}{2}}(\cos\widetilde{\theta}). \end{aligned} \tag{4.23}$$

By the initial condition (4.16) we have

$$P_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta) = \delta(\widetilde{\theta}-\theta) = \sum_{k=0}^{\infty} A_k \sinh^{\frac{2-n}{2}}\rho\,P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\rho)C_k^{\frac{n-2}{2}}(\cos\widetilde{\theta}). \tag{4.24}$$

To determine the constant A_k we multiply both sides of the equation (4.24) by $\sin^{n-2}\widetilde{\theta}C_m^{\frac{n-2}{2}}(\cos\widetilde{\theta})$ and integrate from 0 to π with respect to $\widetilde{\theta}$ to get

$$\begin{aligned} \int_0^{\pi}\delta(\widetilde{\theta}-\theta)\sin^{n-2}\widetilde{\theta}C_m^{\frac{n-2}{2}}(\cos\widetilde{\theta})\,d\widetilde{\theta} &= \sum_{k=0}^{\infty} A_k \sinh^{\frac{2-n}{2}}\rho\,P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\rho) \\ &\quad \times \int_0^{\pi}\sin^{n-2}\widetilde{\theta}C_k^{\frac{n-2}{2}}(\cos\widetilde{\theta})C_m^{\frac{n-2}{2}}(\cos\widetilde{\theta})d\widetilde{\theta} \\ &= A_m \sinh^{\frac{2-n}{2}}\rho\,P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-m}(\cosh\rho)\frac{\pi 2^{3-n}\Gamma(m+n-2)}{m!\left(m+\frac{n-2}{2}\right)\Gamma\left(\frac{n-2}{2}\right)^2}, \end{aligned}$$

where we have used the orthogonality property (B.116) and (B.117), for $0 < \widetilde{\theta} \le \pi$. Thus,

$$A_m = \frac{\sin^{n-2}\theta C_m^{\frac{n-2}{2}}(\cos\theta)}{\sinh^{\frac{2-n}{2}}\rho P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-m}(\cosh\rho)} \frac{m!\left(m+\frac{n-2}{2}\right)\Gamma\left(\frac{n-2}{2}\right)^2}{\pi 2^{3-n}\Gamma(m+n-2)}. \tag{4.25}$$

Hence, from (4.23) and (4.25) we obtain

$$P_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta) = \frac{\Gamma\left(\frac{n-2}{2}\right)^2}{2^{3-n}\pi}\sin^{n-2}\theta \sum_{k=0}^{\infty} \frac{k!\left(k+\frac{n-2}{2}\right)}{\Gamma(k+n-2)} \frac{\sinh^{\frac{2-n}{2}}\widetilde{\rho} P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\widetilde{\rho})}{\sinh^{\frac{2-n}{2}}\rho P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\rho)}$$
$$\times C_k^{\frac{n-2}{2}}(\cos\widetilde{\theta}) C_k^{\frac{n-2}{2}}(\cos\theta). \tag{4.26}$$

By applying the addition formula (B.115) with $\eta = 0$, $m = 0$, we have

$$C_k^{\frac{n-2}{2}}(\cos\widetilde{\theta}) C_k^{\frac{n-2}{2}}(\cos\theta) = C_k^{\frac{n-2}{2}}(\cos(\widetilde{\theta}-\theta)) C_k^{\frac{n-2}{2}}(1) \tag{4.27}$$

and by the identity (B.104) (the last identity), namely

$$C_k^{\frac{n-2}{2}}(1) = \frac{\Gamma(n+k-2)}{k!(n-3)!}, \tag{4.28}$$

equation (4.26) becomes

$$P_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta) = \frac{\Gamma\left(\frac{n-2}{2}\right)^2 \sin^{n-2}\theta}{2^{3-n}\pi(n-3)!} \sum_{k=0}^{\infty}\left(k+\frac{n-2}{2}\right) \frac{\sinh^{\frac{2-n}{2}}\widetilde{\rho} P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\widetilde{\rho})}{\sinh^{\frac{2-n}{2}}\rho P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\rho)}$$
$$\times C_k^{\frac{n-2}{2}}(\cos(\widetilde{\theta}-\theta)). \tag{4.29}$$

Proposition 4.2.

$$\frac{\Gamma\left(\frac{n-2}{2}\right)^2}{2^{3-n}\pi(n-3)!} = \frac{2}{n-2}\frac{\nu_{n-2}}{\nu_{n-1}},$$

where ν_{n-1} is the surface area of the $(n-1)$-dimensional sphere (see (1.12)).

Proof. Indeed, by the Legendre duplication formula (B.3), namely

$$2^{2-n}\sqrt{\pi}\Gamma(n-1) = \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right), \tag{4.30}$$

we have

$$\frac{\Gamma\left(\frac{n-2}{2}\right)^2}{2^{3-n}\pi(n-3)!} = \frac{\left(\frac{n}{2}-1\right)^2\Gamma\left(\frac{n-2}{2}\right)^2}{\left(\frac{n}{2}-1\right)^2 2^{3-n}\pi(n-3)!} = \frac{2\Gamma\left(\frac{n}{2}\right)^2}{2^{2-n}\pi\Gamma(n-1)(n-2)}$$
$$= \frac{2}{n-2}\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} = \frac{2}{n-2}\frac{\nu_{n-2}}{\nu_{n-1}}. \tag{4.31}$$

□

In summary we have the following statement (see also Awonusika [6]).

Theorem 4.3. *For $0<\widetilde{\rho}<\rho<\infty, \widetilde{\theta}-\theta\in(0,\pi]$, the solution to the n-dimensional hyperbolic Dirichlet problem (Laplace equation)*

$$\left[\frac{1}{\sinh^{n-1}\widetilde{\rho}}\frac{\partial}{\partial\widetilde{\rho}}\left(\sinh^{n-1}\widetilde{\rho}\frac{\partial}{\partial\widetilde{\rho}}\right)+\frac{1}{\sinh^2\widetilde{\rho}}\left(\frac{1}{\sin^{n-2}\widetilde{\theta}}\frac{\partial}{\partial\widetilde{\theta}}\left(\sin^{n-2}\widetilde{\theta}\frac{\partial}{\partial\widetilde{\theta}}\right)\right)\right]P_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta)=0$$
$$P_{\widetilde{\rho},\rho}\left(\widetilde{\theta},\theta\right)=\delta\left(\widetilde{\theta}-\theta\right),$$

is given by

$$P_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta)=\frac{\nu_{n-2}}{\nu_{n-1}}\sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)\frac{\sinh^{\frac{2-n}{2}}\widetilde{\rho}P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\widetilde{\rho})}{\sinh^{\frac{2-n}{2}}\rho P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\rho)}C_k^{\frac{n-2}{2}}(\cos(\widetilde{\theta}-\theta))\sin^{n-2}\theta.$$

In particular, the Poisson kernel in $\mathbf{H}^n$ is given by

$$P_{\widetilde{\rho},\rho}^{\mathbf{H}^n}(\widetilde{\theta},\theta)=\frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)\frac{\sinh^{\frac{2-n}{2}}\widetilde{\rho}P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\widetilde{\rho})}{\sinh^{\frac{2-n}{2}}\rho P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\rho)}C_k^{\frac{n-2}{2}}(\cos(\widetilde{\theta}-\theta));$$

and the corresponding Poisson integral formula for $\mathbf{H}^n$ is

$$P_{\widetilde{\rho},\rho}^{\mathbf{H}^n}[\phi](\theta)=\nu_{n-2}\int_0^{\pi}P_{\widetilde{\rho},\rho}^{\mathbf{H}^n}(\widetilde{\theta},\theta)\phi(\widetilde{\theta})\sin^{n-2}\widetilde{\theta}\,d\widetilde{\theta}=\int_0^{\pi}P_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta)\phi(\widetilde{\theta})\,d\widetilde{\theta}$$

for a continuous function ϕ on $[0,\pi]$.

Remark 4.1. *It is not difficult to write the associated Legendre function $P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\widetilde{\rho})$ in terms of the Gauss hypergeometric function. Indeed, using* (B.70) *and* (B.80) *we obtain*

$$P_{\frac{n-2}{2}}^{-\frac{n-2}{2}-k}(\cosh\widetilde{\rho})=\frac{2^{\frac{2-n}{2}}}{\Gamma\left(k+\frac{n}{2}\right)}\tanh^k\left(\frac{\widetilde{\rho}}{2}\right)\sinh^{\frac{n-2}{2}}\widetilde{\rho}F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\widetilde{\rho}}{2}\right)\right). \tag{4.32}$$

Hence,

$$P_{\widetilde{\rho},\rho}^{\mathbf{H}^n}(\widetilde{\theta},\theta)=\frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)\frac{\tanh^k\left(\frac{\widetilde{\rho}}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\widetilde{\rho}}{2}\right)\right)}{\tanh^k\left(\frac{\rho}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\rho}{2}\right)\right)}$$
$$\times C_k^{\frac{n-2}{2}}(\cos(\widetilde{\theta}-\theta)). \tag{4.33}$$

Therefore, our results agree with those of Symeonidis [162], Cammarota and Orsingher [36], Byczkowski and Małecki [34] who use different methods.

An important property of the Poisson kernel is the following

Remark 4.2.

$$\nu_{n-2}\int_0^{\pi}P_{\widetilde{\rho},\rho}^{\mathbf{H}^n}(\vartheta)\sin^{n-2}\vartheta\,d\vartheta=\int_0^{\pi}P_{\widetilde{\rho},\rho}(\vartheta)\,d\vartheta=1.$$

Proof. We use the hypergeometric function expression given by (4.33) to see that

$$\begin{aligned}
\int_0^\pi P_{\widetilde{\rho},\rho}(\theta)\,d\theta &= \nu_{n-2}\int_0^\pi P^{\mathbf{H}^n}_{\widetilde{\rho},\rho}(\theta)\sin^{n-2}\theta\,d\theta \\
&= \frac{\nu_{n-2}}{\nu_{n-1}}\int_0^\pi \sin^{n-2}\theta\,d\theta \\
&\quad + \frac{\nu_{n-2}}{\nu_{n-1}}\sum_{k=1}^\infty\left(\frac{2k}{n-2}+1\right)\frac{\tanh^k\left(\frac{\widetilde{\rho}}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\widetilde{\rho}}{2}\right)\right)}{\tanh^k\left(\frac{\rho}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\rho}{2}\right)\right)} \\
&\quad \times\int_0^\pi C_k^{\frac{n-2}{2}}(\cos\theta)\sin^{n-2}\theta\,d\theta, \qquad (4.34)
\end{aligned}$$

where we have used (B.103) (the first identity) and (B.59). To evaluate the first integral on the right-hand side of (4.34), we use the beta function (B.33) with $x=\frac{1}{2}$, $y=\frac{n-1}{2}$, $t=\cos\theta$, to obtain

$$B\left(\frac{1}{2},\frac{n-1}{2}\right)=\int_0^\pi \sin^{n-2}\theta\,d\theta=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}=\frac{\nu_{n-1}}{\nu_{n-2}}. \qquad (4.35)$$

The proof is complete by noting that for $k>0$

$$\int_0^\pi C_k^{\frac{n-2}{2}}(\cos\theta)\sin^{n-2}\theta\,d\theta=0.$$

(see (B.117)). □

4.3 The Poisson Integral Formula for $\mathbf{S}^n$

The result in Theorem 4.3 for the upper half-space $\mathbf{H}^n$ can be extended to that for the unit sphere $\mathbf{S}^n$ by analytic continuation, namely we set $\widetilde{\rho}\to ir$ and $\rho\to i\varrho$ to obtain

Theorem 4.4. *Let*

$$\mathbf{S}^n=\left\{x\in\mathbf{R}^{n+1}\,:\,|x|=1\right\}$$

be the n-dimensional unit sphere and denote by $d(x,x')$ the spherical distance between two points $x,x'\in\mathbf{S}^n$. Let $o\in\mathbf{S}^n$ and

$$\mathbf{S}^n_\varrho(o)=\{x\in\mathbf{S}^n\,:\,r=d(o,x)<\varrho<\pi\}\,.$$

Then the solution to the n-dimensional spherical Dirichlet problem (Laplace equation)

$$\begin{aligned}
&\left[\frac{1}{\sin^{n-1}r}\frac{\partial}{\partial r}\left(\sin^{n-1}r\frac{\partial}{\partial r}\right)+\frac{1}{\sin^2 r}\left(\frac{1}{\sin^{n-2}\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin^{n-2}\vartheta\frac{\partial}{\partial\vartheta}\right)\right)\right]P_{r,\varrho}(\vartheta,\vartheta')=0 \\
&P_{r,\varrho}\left(\vartheta,\vartheta'\right)=\delta\left(\vartheta-\vartheta'\right),\ \ \vartheta-\vartheta'\in(0,\pi]
\end{aligned}$$

is given by

$$\begin{aligned}
P_{r,\varrho}(\vartheta,\vartheta') &= \frac{\nu_{n-2}}{\nu_{n-1}}\sum_{k=0}^\infty\left(\frac{2k}{n-2}+1\right)\frac{\tan^k\left(\frac{r}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};-\tan^2\left(\frac{r}{2}\right)\right)}{\tan^k\left(\frac{\varrho}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};-\tan^2\left(\frac{\varrho}{2}\right)\right)} \\
&\quad\times C_k^{\frac{n-2}{2}}(\cos(\vartheta-\vartheta'))\sin^{n-2}\vartheta'.
\end{aligned}$$

In particular, the Poisson kernel on $\mathbf{S}^n$ *is*

$$P^{\mathbf{S}^n}_{r,\varrho}(\vartheta,\vartheta') = \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)\frac{\tan^k\left(\frac{r}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};-\tan^2\left(\frac{r}{2}\right)\right)}{\tan^k\left(\frac{\varrho}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};-\tan^2\left(\frac{\varrho}{2}\right)\right)}$$
$$\times C_k^{\frac{n-2}{2}}(\cos(\vartheta-\vartheta'));$$

and the corresponding Poisson integral formula is

$$P^{\mathbf{S}^n}_{r,\varrho}[\phi](\vartheta) = \nu_{n-2}\int_0^{\pi} P^{\mathbf{S}^n}_{r,\varrho}(\vartheta,\vartheta')\phi(\vartheta')\sin^{n-2}\vartheta'\,d\vartheta'$$

for a continuous function ϕ *on* $[0,\pi]$.

Remark 4.3.

$$\nu_{n-2}\int_0^{\pi} P^{\mathbf{S}^n}_{r,\varrho}(\vartheta)\sin^{n-2}\vartheta\,d\vartheta = 1.$$

Proof. As in the case of $\mathbf{H}^n$. □

4.4 Integral Representations of the Euclidean Poisson Kernel

It is a topic of particular interest in the theory of harmonic functions to solve the Dirichlet problem (see Theorem 4.8 below) in the unit ball $\mathbf{B}^n$ of $\mathbf{R}^n$, and of fundamental importance in solving this problem is the Poisson kernel $P_{\mathbf{B}^n}(x,y)$, given by (4.42) below.

The upper half-space $\mathbf{H}^n$, being a Riemannian manifold, must behave locally like Euclidean space $\mathbf{R}^n$. Let

$$\mathbf{B}^n = \{x\in\mathbf{R}^n \,:\, |x|<1\}$$

be the unit ball in $\mathbf{R}^n$ with boundary $\partial\mathbf{B}^n = \mathbf{S}^{n-1}$. It is interesting to see that for a small value of the hyperbolic distance, the n-dimensional hyperbolic Poisson kernel reduces to the Poisson kernel on the unit ball.

The precise statement is the following.

Theorem 4.5. *For small values of* $\widetilde{\rho}$ *and* ρ*, the following holds:*

$$P^{\mathbf{H}^n}_{\widetilde{\rho},\rho}(\theta,\widetilde{\theta}) = \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)\frac{\tanh^k\left(\frac{\widetilde{\rho}}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\widetilde{\rho}}{2}\right)\right)}{\tanh^k\left(\frac{\rho}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\rho}{2}\right)\right)}$$
$$\times C_k^{\frac{n-2}{2}}(\cos(\widetilde{\theta}-\theta))$$
$$\sim \frac{1}{n\omega_n}\frac{1-r^2}{(1-2r\cos\vartheta+r^2)^{\frac{n}{2}}},\quad r=\frac{\widetilde{\rho}}{\rho},\ \ \vartheta=\widetilde{\theta}-\theta, 0\le r<1,$$

where

$$P_{\mathbf{B}^n}(r,\vartheta) = \frac{1}{n\omega_n}\frac{1-r^2}{(1-2r\cos\vartheta+r^2)^{\frac{n}{2}}} = \frac{1}{n\omega_n}\frac{1-r^2}{\left[(1-r)^2+4r\sin^2\frac{\vartheta}{2}\right]^{\frac{n}{2}}} \tag{4.36}$$

is the Poisson kernel on the Euclidean ball $\mathbf{B}^n$ *with* $x\in\mathbf{B}^n$, $y\in\mathbf{S}^{n-1}$ *and*

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\nu_{n-1}}{n} \tag{4.37}$$

is the volume of the unit ball $\mathbf{B}^n$ *in* $\mathbf{R}^n$. *Furthermore,* $P_{\mathbf{B}^n}(r,\vartheta)$ *admits the following series representations:*

$$\frac{1-r^2}{(1-2r\cos\vartheta+r^2)^{\frac{n}{2}}} = \sum_{k=0}^{\infty}\frac{2k+n-2}{n-2}C_k^{\frac{n-2}{2}}(\cos\vartheta)r^k \tag{4.38}$$

$$= \sum_{k=0}^{\infty} M_k^{n-1}F\left(n-2+k,-k;\frac{n-1}{2};\sin^2\frac{\vartheta}{2}\right)r^k \tag{4.39}$$

$$= \frac{e^{i\pi(1-n)/2}\left(1-r^2\right)^{\frac{3}{2}-\frac{n}{2}}}{\sqrt{\pi r}\,\Gamma\left(\frac{n}{2}\right)}\sum_{k=-\infty}^{\infty} e^{-ik\vartheta}Q_{k-\frac{1}{2}}^{\frac{n}{2}-\frac{1}{2}}\left(\frac{1+r^2}{2r}\right). \tag{4.40}$$

The integral

$$P_{\mathbf{B}^n}[\phi](r,\theta) = \nu_{n-2}\int_0^{\pi} P_{\mathbf{B}^n}(r,\theta,\theta')\phi(\theta')\sin^{n-2}\theta' d\theta' \tag{4.41}$$

is called the *Poisson integral formula for the unit ball* $\mathbf{B}^n$ *in* $\mathbf{R}^n$.

Remark 4.4. *(i) In Cartesian coordinates we have*

$$P_{\mathbf{B}^n}(x,y) = \frac{1}{n\omega_n}\frac{1-|x|^2}{|x-y|^n} \tag{4.42}$$

(*see Taheri [163, Chs. 2, 5 & 8], Krantz [96]*).

(ii) If $n=2$*, then we obtain the Poisson kernel on the unit disk:*

$$P_{\mathbf{D}}(r,\vartheta) = P(r,\vartheta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\vartheta+r^2} = \frac{1}{2\pi}\left(1+2\sum_{k=1}^{\infty}r^k\cos k\vartheta\right). \tag{4.43}$$

The last term on the right-hand side of (4.43) *follows by applying the limit formula* (1.18).

Proof of Theorem 4.5. In this proof we make use of some properties of the Gegenbauer polynomial in Appendix B.6. Indeed noting that $\tanh\frac{\rho}{2}\sim\frac{\rho}{2}$, and using the identities (B.103) (the first and second), we have

$$\begin{aligned}
\mathbb{I} &= \sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)\frac{\tanh^k\left(\frac{\tilde{\rho}}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\tilde{\rho}}{2}\right)\right)}{\tanh^k\left(\frac{\rho}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\rho}{2}\right)\right)}\\
&\quad\times C_k^{\frac{n-2}{2}}(\cos\vartheta)\\
&\sim \sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)r^kC_k^{\frac{n-2}{2}}(\cos\vartheta)\\
&= \frac{2}{n-2}\left[\frac{n-2}{2}\sum_{k=0}^{\infty}r^kC_k^{\frac{n-2}{2}}(\cos\vartheta)+(n-2)r\cos\vartheta+\sum_{k=2}^{\infty}kr^kC_k^{\frac{n-2}{2}}(\cos\vartheta)\right].
\end{aligned}$$

Using the functional equation (B.108), we obtain

$$\begin{aligned}
\mathbb{I} &\sim \sum_{k=0}^{\infty} r^k C_k^{\frac{n-2}{2}}(\cos\vartheta) + 2r\cos\vartheta + \frac{2}{n-2}\sum_{k=2}^{\infty}\left[r^k(n-2)\left\{\cos\vartheta C_{k-1}^{\frac{n}{2}}(\cos\vartheta) - C_{k-2}^{\frac{n}{2}}(\cos\vartheta)\right\}\right] \\
&= 2\left[\frac{1}{2}\sum_{k=0}^{\infty} r^k C_k^{\frac{n-2}{2}}(\cos\vartheta) + r\cos\vartheta + \sum_{k=2}^{\infty} r^k\cos\vartheta C_{k-1}^{\frac{n}{2}}(\cos\vartheta) - \sum_{k=2}^{\infty} r^k C_{k-2}^{\frac{n}{2}}(\cos\vartheta)\right] \\
&= 2\left[\frac{1}{2}\sum_{k=0}^{\infty} r^k C_k^{\frac{n-2}{2}}(\cos\vartheta) + r\sum_{k=0}^{\infty} r^k\cos\vartheta C_k^{\frac{n}{2}}(\cos\vartheta) - r^2\sum_{k=0}^{\infty} r^k C_k^{\frac{n}{2}}(\cos\vartheta)\right].
\end{aligned}$$

By the generating function of $C_k^{\nu}(x)$ (see (B.102)), we have

$$\begin{aligned}
\mathbb{I} &\sim 2\left[\frac{1}{2}\left(1-2r\cos\vartheta+r^2\right)^{-\frac{n-2}{2}} + r\cos\vartheta\left(1-2r\cos\vartheta+r^2\right)^{-\frac{n}{2}} - r^2\left(1-2r\cos\vartheta+r^2\right)^{-\frac{n}{2}}\right] \\
&= 2\left[\frac{1}{2}\left(1-2r\cos\vartheta+r^2\right)^{-\frac{n-2}{2}} + \left(1-2r\cos\vartheta+r^2\right)^{-\frac{n}{2}}\left(r\cos\vartheta-r^2\right)\right] \\
&= 2\left(1-2r\cos\vartheta+r^2\right)^{-\frac{n}{2}}\left[r\cos\vartheta-r^2+\frac{1}{2}\left(1-2r\cos\vartheta+r^2\right)\right] \\
&= \left(1-2r\cos\vartheta+r^2\right)^{-\frac{n}{2}}\left[1-r^2\right].
\end{aligned}$$

We have therefore shown that the following identity holds:

$$\sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right) r^k C_k^{\frac{n-2}{2}}(\cos\vartheta) = \frac{1-r^2}{(1-2r\cos\vartheta+r^2)^{\frac{n}{2}}}; \tag{4.44}$$

this establishes the fact that $\mathbf{H}^n$ being a Riemannian manifold, is locally Euclidean.

It is interesting to see that we can as well recover the left-hand side of (4.44) from the right-hand side using the definition of the Gegenbauer polynomial. Towards this end, by the definition (B.102) and the identity (B.118), we have

$$\begin{aligned}
\frac{1-r^2}{(1-2r\cos\vartheta+r^2)^{\frac{n}{2}}} &= \left(1-r^2\right)\sum_{m=0}^{\infty} C_m^{\frac{n}{2}}(\cos\vartheta) r^m \\
&= \left(1-r^2\right)\frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{n}{2}\right)}\sum_{m=0}^{\infty}\sum_{0\le l\le\frac{m}{2}} a_l C_{m-2l}^{\frac{n-2}{2}}(\cos\vartheta) r^m,
\end{aligned}$$

where $a_l = m-2l+\frac{n-2}{2}$. Letting $k = m-2l$ gives

$$\begin{aligned}
\frac{1-r^2}{(1-2r\cos\vartheta+r^2)^{\frac{n}{2}}} &= \left(1-r^2\right)\sum_{k=0}^{\infty}\frac{k+\frac{n}{2}-1}{\frac{n}{2}-1} C_k^{\frac{n-2}{2}}(\cos\vartheta) r^k \sum_{l=0}^{\infty} r^{2l} \\
&= \sum_{k=0}^{\infty}\frac{2k+n-2}{n-2} C_k^{\frac{n-2}{2}}(\cos\vartheta) r^k, \qquad (4.45)
\end{aligned}$$

where we have used the first identity in (B.107). To prove (4.40), we write

$$P_{\mathbf{B}^n}(r,\vartheta) = \frac{1}{\omega_n}\frac{1-r^2}{(2r)^{\frac{n}{2}}}\frac{1}{\left[\frac{1+r^2}{2r}-\cos\vartheta\right]^{\frac{n}{2}}}. \tag{4.46}$$

Using the generalised Heine identity (Cohl and Dominici [43])

$$\frac{1}{[z-\cos\vartheta]^{\mu}} = \sqrt{\frac{2}{\pi}}\frac{e^{-i\pi\left(\mu-\frac{1}{2}\right)}\left(z^2-1\right)^{-\frac{\mu}{2}+\frac{1}{4}}}{\Gamma(\mu)}\sum_{k=-\infty}^{\infty} e^{ik\psi}Q_{k-\frac{1}{2}}^{\mu-\frac{1}{2}}(z),$$

with $\mu = n/2$ and $z = (1+r^2)/2r$ proves (4.40). The fact that

$$|C_k^{\nu}(t)| = O\left(k^{2\nu-1}\right)$$

follows from

$$M_k^n = O\left(k^{n-1}\right) \tag{4.47}$$

(see e.g. Sogge [153, p. 57]); thus the series on the right-hand side of (4.38) converges absolutely when $0 \le r < 1$. This completes the proof of the theorem. □

As we have earlier mentioned in the beginning of this chapter that the trace of the heat operators on rank one compact symmetric spaces can be expressed in terms of the Euclidean Poisson kernel, we now, prior to Chapter 5 obtain some identities involving the Euclidean Poisson kernel that will suit our needs in Section 5.5.

If we set $\vartheta = 0$ in (4.36), then we have

$$P_{\mathbf{B}^n}(r,0) = \frac{1}{\nu_{n-1}}\frac{1+r}{(1-r)^{n-1}} = \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty} M_k^{n-1}r^k. \tag{4.48}$$

Setting $r = e^{-t}$ in (4.48), we see that

$$\frac{1+r}{(1-r)^{n-1}} = \frac{1+e^{-t}}{(1-e^{-t})^{n-1}} = \frac{2^{2-n}\cosh\frac{t}{2}}{e^{-\frac{n-2}{2}t}\left(\sinh\frac{t}{2}\right)^{n-1}}.$$

Hence,

$$P_{\mathbf{B}^n}(e^{-t},0) = \frac{1}{\nu_{n-1}}\frac{2^{2-n}\cosh\frac{t}{2}}{e^{-\frac{n-2}{2}t}\left(\sinh\frac{t}{2}\right)^{n-1}} = \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty} M_k^{n-1}e^{-kt}. \tag{4.49}$$

Also,

$$P_{\mathbf{B}^n}(e^{-t},\vartheta) = \frac{1}{\nu_{n-1}}\frac{2^{\frac{2-n}{2}}e^{-\frac{2-n}{2}t}\sinh t}{(\cosh t-\cos\vartheta)^{\frac{n}{2}}},\quad P_{\mathbf{B}^n}(e^{-t},0) = \frac{1}{\nu_{n-1}}\frac{2^{\frac{2-n}{2}}e^{-\frac{2-n}{2}t}\sinh t}{(\cosh t-1)^{\frac{n}{2}}}. \tag{4.50}$$

That is, from (4.49) and (4.50) we obtain the identities

Proposition 4.6.

$$\frac{\cosh\frac{t}{2}}{\left(\sinh\frac{t}{2}\right)^{n-1}} = \frac{2^{\frac{n-2}{2}}\sinh t}{(\cosh t-1)^{\frac{n}{2}}} = \sum_{k=0}^{\infty} M_k^{n-1}e^{-kt}. \tag{4.51}$$

We also have the special case

$$\frac{1-r^2}{(1-2r\cos\vartheta+r^2)^{\frac{3}{2}}} = \sum_{k=0}^{\infty}(2k+1)P_k(\cos\vartheta)r^k. \tag{4.52}$$

A good reference for the n-dimensional Euclidean Poisson kernel is a new 2-volume book by Taheri [163, 164].

The following property of the Euclidean Poisson kernel holds:

Remark 4.5.

$$\frac{\nu_{n-2}}{\nu_{n-1}}\int_0^\pi \sum_{k=0}^\infty M_k^{n-1}\frac{C_k^{\frac{n-2}{2}}(\cos\vartheta)}{C_k^{\frac{n-2}{2}}(1)} r^k \sin^{n-2}\vartheta\, d\vartheta = \frac{\nu_{n-2}}{\nu_{n-1}}\int_0^\pi \frac{(1-r^2)\sin^{n-2}\vartheta}{(1-2r\cos\vartheta+r^2)^{\frac{n}{2}}}\, d\vartheta = 1. \quad (4.53)$$

Proof. We follow the proof of Remark 4.2, and use the formulae (B.64), (4.35) and (B.59) to see that

$$(1-r^2)\frac{\nu_{n-2}}{\nu_{n-1}}\int_0^\pi \frac{\sin^{n-2}\vartheta\, d\vartheta}{(1-2r\cos\vartheta+r^2)^{\frac{n}{2}}} = (1-r^2)\frac{\nu_{n-2}}{\nu_{n-1}} B\left(\frac{n-1}{2},\frac{1}{2}\right) F\left(1,\frac{n}{2};\frac{n}{2};r^2\right) = 1.$$

□

The following proposition says that the Euclidean Poisson kernel is an eigenfunction of the Euclidean Laplacian.

Proposition 4.7. *The function*

$$P_{\mathbf{B}^n}(r,\vartheta) = \frac{1}{\nu_{n-1}}\sum_{k=0}^\infty M_k^{n-1}\mathscr{C}_k^{\frac{n-2}{2}}(\cos\vartheta) r^k,$$

is harmonic in the unit ball $\mathbf{B}^n$.

Proof. It suffices to show that the following equality is satisfied:

$$\mathbb{D}_n P_{\mathbf{B}^n}(r,\vartheta) = 0,$$

where $\mathbb{D}_n$ is the Euclidean Laplacian in polar coordinates given by (1.6). Indeed,

$$\begin{aligned}
\mathbb{D}_n P_{\mathbf{B}^n}(r,\theta) &= -\frac{1}{\nu_{n-1}}\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial}{\partial r}\left(\sum_{k=0}^\infty \frac{2k+n-2}{n-2} C_k^{\frac{n-2}{2}}(\cos\vartheta) r^k\right)\right] \\
&\quad - \frac{1}{\nu_{n-1}}\frac{1}{r^2}\Delta_{\mathbf{S}^{n-1}}\left[\sum_{k=0}^\infty \frac{2k+n-2}{n-2} C_k^{\frac{n-2}{2}}(\cos\vartheta) r^k\right] \\
&= -\frac{1}{\nu_{n-1}}\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left[r^{n-1}\left(\sum_{k=0}^\infty \frac{2k+n-2}{n-2} k C_k^{\frac{n-2}{2}}(\cos\vartheta) r^{k-1}\right)\right] \\
&\quad + \frac{1}{\nu_{n-1}}\frac{1}{r^2}\left[\sum_{k=0}^\infty \frac{2k+n-2}{n-2} k(k+n-2) C_k^{\frac{n-2}{2}}(\cos\vartheta) r^k\right] \\
&= -\frac{1}{\nu_{n-1}}\frac{1}{r^{n-1}}\left[\left(\sum_{k=0}^\infty \frac{2k+n-2}{n-2} k(k+n-2) C_k^{\frac{n-2}{2}}(\cos\vartheta)\right) r^{k+n-3}\right] \\
&\quad + \frac{1}{\nu_{n-1}}\frac{1}{r^2}\left[\sum_{k=0}^\infty \frac{2k+n-2}{n-2} k(k+n-2) C_k^{\frac{n-2}{2}}(\cos\vartheta) r^k\right] = 0.
\end{aligned}$$

□

See Taheri [163, Sec. 8.1] for a different proof of Proposition 4.7.

In summary we have the following statement.

Theorem 4.8. *Let* $x = r\zeta \in \mathbf{B}^n$, $\zeta \in \mathbf{S}^{n-1}$, *and let* $\vartheta(\zeta,\zeta')$ *denote the geodesic distance on the sphere* $\mathbf{S}^{n-1}$ *from* ζ *to* ζ', *with* $\cos\vartheta(\zeta,\zeta') = (\zeta\cdot\zeta')$. *If* $\phi \in C(\mathbf{S}^{n-1})$, *then the Euclidean*

Dirichlet problem

$$\begin{aligned} \mathbb{D}_n u = 0 \qquad & in \quad \mathbf{B}^n, \\ u = \phi \qquad & on \quad \partial\mathbf{B}^n = \mathbf{S}^{n-1}, \end{aligned}$$

has the solution $u \in C\left(\overline{\mathbf{B}^n}\right)$ *given by*

$$u(x) = u(r\zeta) = P_{\mathbf{B}^n}[\phi](r\zeta) = \int_{\mathbf{S}^{n-1}} P_{\mathbf{B}^n}(r\zeta,\zeta')\phi(\zeta')d\nu_{n-1}(\zeta'),$$

where

$$P_{\mathbf{B}^n}(r\zeta,\zeta') = \frac{1}{\nu_{n-1}}\frac{1-r^2}{(1-2r\cos(\zeta,\zeta')+r^2)^{\frac{n}{2}}} = \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty} M_k^{n-1}\frac{C_k^{\frac{n-2}{2}}(\zeta\cdot\zeta')}{C_k^{\frac{n-2}{2}}(1)}r^k.$$

Moreover,

$$\lim_{r\to 1} P_{\mathbf{B}^n}[\phi](r\zeta) = \phi(\zeta).$$

Remark 4.6. *What remains in the proof of Theorem 4.8 is to show that* $u \in C\left(\overline{\mathbf{B}^n}\right)$*, and for this it suffices to show that* u *is continuous at every point* $\zeta \in \partial\mathbf{B}^n = \mathbf{S}^{n-1}$ *(see Taheri [163, pp. 277-278]).*

We next give integral representations of the Poisson kernel on the unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$.

Theorem 4.9 (Integral representations of the Euclidean Poisson Kernel). *The Poisson kernel* $P_{\mathbf{B}^n}$ *defined by*

$$P_{\mathbf{B}^n}(r,\theta,\theta') = \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty}\frac{2k+n-2}{n-2}C_k^{\frac{n-2}{2}}(\cos\alpha)r^k$$

has the following integral representations:

$$P_{\mathbf{B}^n}(r,\theta,\theta') = \frac{1}{\nu_{n-1}}\frac{\nu_{n-3}}{\nu_{n-2}}\int_0^{\pi}\frac{\left(1-r^2\right)\sin^{n-3}\omega\, d\omega}{\left[1-2r\cos\alpha+r^2\right]^{\frac{n}{2}}}, \tag{4.54}$$

where

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\omega;$$

and

$$P_{\mathbf{B}^n}(r,\theta,\theta') = \frac{2^{3(3-n)}}{\nu_{n-2}}\left(\frac{\cos\theta+\cos\theta'}{2}\right)^{2-n}\left(r\frac{d}{dr}+\frac{n-2}{2}\right)\left[\frac{\mathtt{I}_{n-3,n-3}(-r,\theta,\theta')}{(1+r)^{n-2}}\right], \tag{4.55}$$

where

$$\mathtt{I}_{\mu,\nu}(r,\theta,\theta') = \int_0^{\infty}\widetilde{I}_{\frac{\mu}{2}}\left(\frac{r\eta}{1-r}\right)\widetilde{K}_{\frac{\nu}{2}}\left(\frac{\eta}{1-r}\right)\widetilde{J}_{\frac{\mu}{2}}(a\eta)\,\widetilde{J}_{\frac{\nu}{2}}(b\eta)\,\eta^{\mu+\nu+1}\,d\eta, \tag{4.56}$$

with

$$a = \frac{\sin\theta}{\cos\theta+\cos\theta'},\quad b = \frac{\sin\theta'}{\cos\theta+\cos\theta'},\quad \widetilde{H}_{\alpha}(\xi) = \left(\frac{\xi}{2}\right)^{-\alpha}H_{\alpha}(\xi);$$

here J_α, I_α and K_α are the Bessel functions defined in Appendix B.3. In particular,

$$\begin{aligned}
P_{\mathbf{B}^n}(r,\theta,\theta') = {} & \frac{2^{3-n}}{\nu_{n-2}\Gamma\left(\frac{n-1}{2}\right)^2}\left(\frac{\cos\theta+\cos\theta'}{2}\right)^{2-n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\sin\theta}{\cos\theta+\cos\theta'}\right)^{2k} \\
& \times\Gamma(k+n-1)F\left(-k,-\frac{n-3}{2}-k;\frac{n-5}{2};\frac{\sin^2\theta'}{\sin^2\theta}\right) \\
& \times\left(r\frac{d}{dr}+\frac{n-2}{2}\right)(1+r)^{2k+n-2}F\left(k+n-1,k+\frac{n-1}{2};\frac{n-1}{2};r^2\right)
\end{aligned} \tag{4.57}$$

and

$$\begin{aligned}
\frac{\nu_{n-3}}{\nu_{n-2}}\int_0^\pi P_{\mathbf{B}^n}(r,\omega)\sin^{n-3}\omega\,d\omega &= \frac{1}{\nu_{n-1}}\frac{\nu_{n-3}}{\nu_{n-2}}\int_0^\pi\frac{(1-r^2)\sin^{n-3}\omega}{(1-2r\cos\omega+r^2)^{\frac{n}{2}}}\,d\omega \\
&= \frac{1}{\nu_{n-1}}(1-r^2)\,F\left(\frac{n}{2},\frac{3}{2};\frac{n-1}{2};r^2\right).
\end{aligned} \tag{4.58}$$

Proof. By the addition formula (B.115), we have

$$C_k^{\frac{n-2}{2}}(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\omega) = \frac{k!(n-3)!}{\Gamma(k+n-2)}C_k^{\frac{n-2}{2}}(\cos\theta)C_k^{\frac{n-2}{2}}(\cos\theta').$$

Thus,

$$\int_0^\pi C_k^{\frac{n-2}{2}}(\cos\alpha)\sin^{n-3}\omega\,d\omega = \frac{k!(n-3)!}{\Gamma(k+n-2)}C_k^{\frac{n-2}{2}}(\cos\theta)C_k^{\frac{n-2}{2}}(\cos\theta')\frac{\nu_{n-2}}{\nu_{n-3}}. \tag{4.59}$$

Hence, for $n\geq 3$,

$$\begin{aligned}
P_{\mathbf{B}^n}(r,\theta,\theta') &= \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty}\frac{2k+n-2}{n-2}C_k^{\frac{n-2}{2}}(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\omega)r^k \\
&= \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty}\frac{2k+n-2}{n-2}\frac{k!(n-3)!}{\Gamma(k+n-2)}C_k^{\frac{n-2}{2}}(\cos\theta)C_k^{\frac{n-2}{2}}(\cos\theta')r^k \\
&= \frac{2}{n-2}\frac{\nu_{n-3}}{\nu_{n-1}\nu_{n-2}}\int_0^\pi\sin^{n-3}\omega\sum_{k=0}^{\infty}\left(k+\frac{n-2}{2}\right)r^kC_k^{\frac{n-2}{2}}(\cos\alpha)\,d\omega \\
&= \frac{1}{\nu_{n-1}}\frac{2}{n-2}\frac{\nu_{n-3}}{\nu_{n-2}}\left(r\frac{d}{dr}+\frac{n-2}{2}\right)\int_0^\pi\sin^{n-3}\omega\left[\sum_{k=0}^{\infty}r^kC_k^{\frac{n-2}{2}}(\cos\alpha)\right]d\omega \\
&= \frac{1}{\nu_{n-1}}\frac{2}{n-2}\frac{\nu_{n-3}}{\nu_{n-2}}\left(r\frac{d}{dr}+\frac{n-2}{2}\right)\int_0^\pi\frac{\sin^{n-3}\omega\,d\omega}{(1-2r\cos\alpha+r^2)^{\frac{n-2}{2}}} \qquad (4.60) \\
&= \frac{1}{\nu_{n-1}}\frac{\nu_{n-3}}{\nu_{n-2}}\int_0^\pi\sin^{n-3}\omega\left[\frac{1-r^2}{(1-2r\cos\alpha+r^2)^{\frac{n}{2}}}\right]d\omega, \qquad (4.61)
\end{aligned}$$

which proves (4.54). For the second part (4.55), we use the equality (Kobayashi and Möllers [93])

$$\begin{aligned}
&(1-r)^{-2\sigma}\left(\frac{\cos\theta+\cos\theta'}{2}\right)^{-2\sigma}\mathrm{I}_{2\sigma-1,2\sigma-1}(r,\theta,\theta') \\
&= \frac{2^{8\sigma-4}}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(k+1)}{\Gamma(k+2\sigma)}\Gamma(\sigma)^2C_k^\sigma(\cos\theta)C_k^\sigma(\cos\theta')r^k,
\end{aligned}$$

where $\mathtt{I}_{\mu,\nu}$ is the integral given by (4.56), with $\sigma = (n-2)/2$, to see that

$$\begin{aligned}
&\frac{\pi}{2^{4(n-3)}}\left(\frac{\cos\theta+\cos\theta'}{2}\right)^{2-n}\left(r\frac{d}{dr}+\frac{n-2}{2}\right)(1+r)^{2-n}\mathtt{I}_{n-3,n-3}(-r,\theta,\theta')\\
&=\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+n-2)}\Gamma\left(\frac{n-2}{2}\right)^2 C_k^{\frac{n-2}{2}}(\cos\theta)C_k^{\frac{n-2}{2}}(\cos\theta')\left(r\frac{d}{dr}+\frac{n-2}{2}\right)r^k\\
&=\sum_{k=0}^{\infty}\frac{k!\left(k+\frac{n-2}{2}\right)}{\Gamma(k+n-2)}\Gamma\left(\frac{n-2}{2}\right)^2 C_k^{\frac{n-2}{2}}(\cos\theta)C_k^{\frac{n-2}{2}}(\cos\theta')r^k\\
&=\nu_{n-2}2^{3-n}\pi P_{\mathbf{B}^n}(r,\theta,\theta'),
\end{aligned}$$

which proves (4.55). To obtain (4.57) we evaluate the integral $\mathtt{I}_{n-3,n-3}(-r,\theta,\theta')$ explicitly. We have

$$\begin{aligned}
\mathtt{I}_{n-3,n-3}(-r,\theta,\theta') &= (-1)^{-\frac{n-3}{2}}\,2^{2n-6}\,(1+r)^{n-3}\,r^{-\frac{n-3}{2}}(ab)^{-\frac{n-3}{2}}\\
&\quad\times\int_0^\infty \eta I_{\frac{n-3}{2}}\left(-\frac{r\eta}{1+r}\right)K_{\frac{n-3}{2}}\left(\frac{\eta}{1+r}\right)J_{\frac{n-3}{2}}(a\eta)\,J_{\frac{n-3}{2}}(b\eta)\,d\eta.
\end{aligned}\tag{4.62}$$

Using (B.37), we get

$$\begin{aligned}
\mathtt{I}_{n-3,n-3}(-r,\theta,\theta') &= (-1)^{-\frac{n-3}{2}}\,2^{2n-6}\,(1+r)^{n-3}\,r^{-\frac{n-3}{2}}(ab)^{-\frac{n-3}{2}}\\
&\quad\times\frac{\left(\frac{ab}{4}\right)^{\frac{n-3}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{a}{2}\right)^{2k}}{k!\Gamma\left(k+\frac{n-1}{2}\right)}F\left(-k,-\frac{n-3}{2}-k;\frac{n-5}{2};\frac{b^2}{a^2}\right)\\
&\quad\times\int_0^\infty I_{\frac{n-3}{2}}\left(-\frac{r\eta}{1+r}\right)K_{\frac{n-3}{2}}\left(\frac{\eta}{1+r}\right)\eta^{2k+n-2}\,d\eta;
\end{aligned}$$

and again using (B.52), we obtain

$$\begin{aligned}
\mathtt{I}_{n-3,n-3}(-r,\theta,\theta') &= (-1)^{-\frac{n-3}{2}}\,2^{2n-6}\,(1+r)^{n-3}\,r^{-\frac{n-3}{2}}(ab)^{-\frac{n-3}{2}}\\
&\quad\times\frac{\left(\frac{ab}{4}\right)^{\frac{n-3}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{a}{2}\right)^{2k}}{k!\Gamma\left(k+\frac{n-1}{2}\right)}F\left(-k,-\frac{n-3}{2}-k;\frac{n-5}{2};\frac{b^2}{a^2}\right)\\
&\quad\times\frac{(-1)^{\frac{n-3}{2}}r^{\frac{n-3}{2}}(1+r)^{n+2k-1}}{2^{-2k-n+3}}\frac{\Gamma(k+n-1)\Gamma\left(k+\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)};\\
&=\frac{2^{2n-6}\,(1+r)^{2n-4}}{\Gamma\left(\frac{n-1}{2}\right)^2}\sum_{k=0}^{\infty}\frac{(-1)^k a^{2k}}{k!}F\left(-k,-\frac{n-3}{2}-k;\frac{n-5}{2};\frac{b^2}{a^2}\right)\\
&\quad\times\Gamma(k+n-1)(1+r)^{2k}F\left(k+n-1,k+\frac{n-1}{2};\frac{n-1}{2};r^2\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
P_{\mathbf{B}^n}(r,\theta,\theta') &= \frac{2^{3-n}}{\nu_{n-2}\Gamma\left(\frac{n-1}{2}\right)^2}\left(\frac{\cos\theta+\cos\theta'}{2}\right)^{2-n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\sin\theta}{\cos\theta+\cos\theta'}\right)^{2k}\\
&\quad\times\Gamma(k+n-1)F\left(-k,-\frac{n-3}{2}-k;\frac{n-5}{2};\frac{\sin^2\theta'}{\sin^2\theta}\right)\\
&\quad\times\left(r\frac{d}{dr}+\frac{n-2}{2}\right)(1+r)^{2k+n-2}F\left(k+n-1,k+\frac{n-1}{2};\frac{n-1}{2};r^2\right),
\end{aligned}$$

which gives (4.57). We can also use (B.38) in (4.62) to obtain

$$\begin{aligned}
\mathtt{I}_{n-3,n-3}(-r,\theta,\theta') &= (-abr)^{-\frac{n-3}{2}}\, 2^{2n-6}\,(1+r)^{n-3}\left(\frac{1}{2}\frac{ab}{\sqrt{a^2+b^2}}\right)^{\frac{n-3}{2}} \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\frac{ab}{\sqrt{a^2+b^2}}\right)^{2k}}{k!\Gamma(\frac{n-1}{2}+k)} \\
&\quad\times \int_0^{\infty} \eta^{\frac{n-1}{2}+2k} I_{\frac{n-3}{2}}\left(-\frac{r\eta}{1+r}\right) K_{\frac{n-3}{2}}\left(\frac{\eta}{1+r}\right) J_{\frac{n-3}{2}+2k}\left(\eta\sqrt{a^2+b^2}\right)\, d\eta \\
&= (-1)^{-\frac{n-3}{2}}\, 2^{2n-6}\left(\frac{1}{2}\frac{1}{\sqrt{a^2+b^2}}\right)^{\frac{n-3}{2}} \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\frac{ab}{\sqrt{a^2+b^2}}\right)^{2k}}{k!\Gamma(\frac{n-1}{2}+k)} \\
&\quad\times \frac{r^{\frac{-n-2k+2}{2}}(1+r^2)^{\frac{n-3}{4}+k}(1+r)^{\frac{3n+4k-5}{2}}\, e^{-\left(\frac{n-2}{2}+2k\right)\pi i} Q_{\frac{n-4}{2}}^{\frac{n-2}{2}+2k}\left(-\frac{1+r^2}{r}\right)}{\sqrt{2\pi}\,[1+r^2+r^4]^{\frac{n-2}{4}+k}}.
\end{aligned} \tag{4.63}$$

Hence,

$$\begin{aligned}
P_{\mathbf{B}^n}(r,\theta,\theta') &= \frac{(-1)^{-\frac{n-3}{2}}}{\sqrt{2\pi}\nu_{n-2}}\left(\frac{\cos\theta+\cos\theta'}{2}\right)^{2-n}\left(r\frac{d}{dr}+\frac{n-2}{2}\right) \\
&\quad\times\left(\frac{1}{\sqrt{a^2+b^2}}\right)^{\frac{n-3}{2}} \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\frac{ab}{\sqrt{a^2+b^2}}\right)^{2k}}{k!\Gamma(\frac{n-1}{2}+k)} r^{-\frac{n+2k-2}{2}}(1+r^2)^{\frac{n-3}{4}+k} \\
&\quad\times\frac{(1+r)^{\frac{n+4k-1}{2}}\, e^{-\left(\frac{n-2}{2}+2k\right)\pi i} Q_{\frac{n-4}{2}}^{\frac{n-2}{2}+2k}\left(-\frac{1+r^2}{r}\right)}{[1+r^2+r^4]^{\frac{n-2}{4}+k}}.
\end{aligned}$$

For the particular case (4.58), we put $\theta=\theta'=\pi/2$ in (4.61) and apply (B.64) to get

$$\begin{aligned}
&\frac{1}{\nu_{n-1}}\frac{\nu_{n-3}}{\nu_{n-2}}\int_0^{\pi}\frac{(1-r^2)\sin^{n-3}\omega}{(1-2r\cos\omega+r^2)^{\frac{n}{2}}}\,d\omega \\
&= \frac{1}{\nu_{n-1}}\frac{\nu_{n-3}}{\nu_{n-2}}(1-r^2)\, B\left(\frac{n-2}{2},\frac{1}{2}\right) F\left(\frac{n}{2},\frac{3}{2};\frac{n-1}{2};r^2\right) \\
&= \frac{1}{\nu_{n-1}}(1-r^2)\, F\left(\frac{n}{2},\frac{3}{2};\frac{n-1}{2};r^2\right) = P_{\mathbf{B}^n}\left(r,\frac{\pi}{2},\frac{\pi}{2}\right),
\end{aligned} \tag{4.64}$$

which proves (4.58), and hence completes the proof of the theorem. □

An interesting identity involving the Gauss hypergeometric function can be established by setting $\theta=\theta'=\pi/2$ in (4.60) and applying (B.39) and (B.41) to get

$$\begin{aligned}
&\frac{1}{\nu_{n-1}}\frac{2}{n-2}\frac{\nu_{n-3}}{\nu_{n-2}}\left(r\frac{d}{dr}+\frac{n-2}{2}\right)\int_0^{\pi}\left[\frac{\sin^{n-3}\omega}{(1-2r\cos\omega+r^2)^{\frac{n-2}{2}}}\right]d\omega \\
&= \frac{1}{\nu_{n-1}}\frac{2}{n-2}\left(r\frac{d}{dr}+\frac{n-2}{2}\right) F\left(\frac{n-2}{2},\frac{1}{2};\frac{n-1}{2};r^2\right).
\end{aligned} \tag{4.65}$$

Using (B.61) in (4.65) and comparing the resulting equation with (4.64), we obtain the identity

Proposition 4.10.

$$(1-r^2)\, F\left(\frac{n}{2},\frac{3}{2};\frac{n-1}{2};r^2\right) = \frac{2r^2}{(n-1)} F\left(\frac{n}{2},\frac{3}{2};\frac{n+1}{2};r^2\right) + F\left(\frac{n-2}{2},\frac{1}{2};\frac{n-1}{2};r^2\right). \tag{4.66}$$

For the special case $n = 3$, we have from (4.66)

Proposition 4.11.

$$\left(1 - r^2\right) F\left(\frac{3}{2}, \frac{3}{2}; 1; r^2\right) = r^2 F\left(\frac{3}{2}, \frac{3}{2}; 2; r^2\right) + F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right). \tag{4.67}$$

4.5 Series Representations of the Poisson Kernel on $\mathbf{D}^n$

This section presents different representations for the Poisson kernel on the Poincaré (hyperbolic) unit ball $\mathbf{D}^n$. We transform the Poisson kernel expressed in terms of Cartesian coordinates into different representations in terms of special functions. Different identities in the context of special functions are obtained.

Consider the hyperbolic Poisson kernel on the unit ball $\mathbf{D}^n$ given in Cartesian coordinates by (Helgason [78, 83] (see also Jaming [89]))

$$P_{\mathbf{D}^n}(z, \zeta) = \frac{1}{\nu_{n-1}} \left(\frac{1 - |z|^2}{|z - \zeta|^2}\right)^{n-1} \tag{4.68}$$

for $z \in \mathbf{D}^n, \zeta \in \partial\mathbf{D}^n = \mathbf{S}^{n-1}$. Our starting point is to write (4.68) in polar coordinates $(z, \zeta) = (r, \theta, \theta') = (r, \vartheta)$. Towards this end we set $\zeta = e^{i\theta'}$ and $z = re^{i\theta}$, $r < 1$, $0 \leq \theta \leq 2\pi$, $0 \leq \theta' \leq 2\pi$, to obtain a closed form

$$P_{\mathbf{D}^n}(r, \vartheta) = \frac{1}{\nu_{n-1}} \left(\frac{1 - r^2}{1 - 2r\cos\vartheta + r^2}\right)^{n-1}, \quad \vartheta = \theta - \theta'. \tag{4.69}$$

We shall see that the Poisson kernel (4.69) can be expressed in different forms involving special functions. We first compute series representations for $P_{\mathbf{D}^n}(r, \vartheta)$. The precise statement is the following.

Theorem 4.12. *For $n \geq 3$, $r < 1$, $0 \leq \vartheta \leq 2\pi$, we have*

$$P_{\mathbf{D}^n}(r, \vartheta) = \frac{1}{\nu_{n-1}} \sum_{k=0}^{\infty} M_k^{n-1} \mathscr{C}_k^{\frac{n-2}{2}}(\cos\vartheta) \mathbb{F}_k(r) r^k, \tag{4.70}$$

where

$$\begin{aligned} \mathbb{F}_k(r) &= \frac{(n-1)_k}{\left(\frac{n}{2}\right)_k} F\left(k, -\frac{n}{2} + 1; k + \frac{n}{2}; r^2\right) \\ &= \frac{C_k^{\frac{n-1}{2}}(x)}{P_k^{\left(\frac{n-2}{2}, \frac{n-2}{2}\right)}(x)} F\left(k, -\frac{n}{2} + 1; k + \frac{n}{2}; r^2\right), \quad -1 \leq x \leq 1, \end{aligned} \tag{4.71}$$

satisfying

$$\mathbb{F}_k(1) = 1, \tag{4.72}$$

and $P_k^{\nu,\mu}(x)$ *is the Jacobi polynomial (see Appendix B.6). Moreover,*

$$P_{\mathbf{D}^n}(r,\vartheta) = \frac{1}{\nu_{n-1}}\frac{2\Gamma\left(\frac{n}{2}\right)}{(n-2)}\frac{\left[\frac{(1-r^2)e^{i\pi}}{r}\right]^{\frac{n-1}{2}}}{\sqrt{\pi}\Gamma(n-1)} \times\sum_{k=1}^{\infty}\frac{\Gamma(k+n-1)}{\Gamma(k)}\left(k+\frac{n-2}{2}\right)Q_{k+\frac{n}{2}-\frac{3}{2}}^{-\frac{n}{2}+\frac{1}{2}}\left(\frac{1+r^2}{2r}\right)C_k^{\frac{n-2}{2}}(\cos\vartheta); \tag{4.73}$$

$$P_{\mathbf{D}^n}(r,\vartheta) = \frac{1}{\nu_{n-1}}\frac{2\Gamma\left(\frac{n}{2}\right)(r-1)^{\frac{n}{2}-1}}{(n-2)\Gamma(n-1)}\left(\frac{2}{r}\right)^{\frac{n}{4}-\frac{1}{2}} \times\sum_{k=0}^{\infty}\Gamma(k+n-1)\left(k+\frac{n-2}{2}\right)(2r)^{\frac{k}{2}}P_{-\frac{n}{2}}^{-\frac{n}{2}-k+1}\left(\frac{\sqrt{1+r^2}}{r-1}\right)C_k^{\frac{n-2}{2}}(\cos\vartheta); \tag{4.74}$$

and

$$P_{\mathbf{D}^n}(\widetilde{\rho},\vartheta) = \frac{1}{2\nu_{n-1}}\sum_{k=0}^{\infty}M_k^{n-1}\frac{(n-1)_k}{\left(\frac{n}{2}\right)_k}\tanh^k\widetilde{\rho}F\left(\frac{k}{2},\frac{k}{2}+\frac{1}{2};k+\frac{n}{2};\tanh\widetilde{\rho}\right)\mathscr{C}_k^{\frac{n-2}{2}}(\cos\vartheta). \tag{4.75}$$

In particular, $P_{\mathbf{D}^n}(\widetilde{\rho},\vartheta)$ *and* $P_{\widetilde{\rho},\rho}^{\mathbf{H}^n}(\vartheta)$ *are related to one-another by*

$$P_{\mathbf{D}^n}(\widetilde{\rho},\vartheta) = \frac{1}{\nu_{n-1}}\lim_{\rho\nearrow\infty}P_{\widetilde{\rho},\rho}^{\mathbf{H}^n}(\widetilde{\theta},\theta). \tag{4.76}$$

Proof. By the definition (B.102) and the identity (B.118), we have

$$\begin{aligned}\left(\frac{1-r^2}{1-2r\cos\vartheta+r^2}\right)^{n-1} &= (1-r^2)^{n-1}\sum_{m=0}^{\infty}C_m^{n-1}(\cos\vartheta)r^m\\ &= (1-r^2)^{n-1}\frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma(n-1)}\sum_{m=0}^{\infty}\sum_{0\le l\le\frac{m}{2}}a_lC_{m-2l}^{\frac{n-2}{2}}(\cos\vartheta)r^m,\end{aligned}$$

where

$$a_l = \frac{m-2l+\frac{n-2}{2}}{l!}\frac{\Gamma\left(l+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma(m+n-1-l)}{\Gamma\left(m+\frac{n}{2}-l\right)}.$$

Letting $k=m-2l$ gives

$$a_k = \frac{k+\frac{n-2}{2}}{l!}\frac{\Gamma\left(l+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma(k+n-1+l)}{\Gamma\left(k+\frac{n}{2}+l\right)}.$$

So,

$$\begin{aligned}\left(\frac{1-r^2}{1-2r\cos\vartheta+r^2}\right)^{n-1} &= \left(1-r^2\right)^{n-1}\frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma(n-1)}\sum_{k=0}^{\infty} r^k C_k^{\frac{n-2}{2}}(\cos\vartheta)\\ &\quad\times\sum_{l=0}^{\infty}\frac{k+\frac{n-2}{2}}{l!}\frac{\Gamma\left(l+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma(m+n-1-l)}{\Gamma\left(m+\frac{n}{2}-l\right)}r^{2l}\\ &= \left(1-r^2\right)^{n-1}\frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma(n-1)}\sum_{k=0}^{\infty} r^k C_k^{\frac{n-2}{2}}(\cos\vartheta)\\ &\quad\times\sum_{l=0}^{\infty}\frac{\Gamma(k+n-1)}{\Gamma\left(k+\frac{n}{2}-1\right)}\frac{\Gamma\left(l+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma(k+n-1+l)\Gamma\left(k+\frac{n}{2}\right)}{\Gamma(k+n-l)\,\Gamma\left(k+\frac{n}{2}+l\right)}\frac{r^{2l}}{l!}\\ &= \frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{\Gamma(k+n-1)}{\Gamma\left(k+\frac{n}{2}-1\right)}r^k F\left(k,-\frac{n}{2}+1;k+\frac{n}{2};r^2\right)C_k^{\frac{n-2}{2}}(\cos\vartheta),\end{aligned}$$

where we have used (B.55) and (B.65). Expanding further, we have

$$\begin{aligned}\left(\frac{1-r^2}{1-2r\cos\vartheta+r^2}\right)^{n-1} &= \Gamma\left(\frac{n}{2}-1\right)\sum_{k=0}^{\infty}\frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(k+\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}\right)}\mathbb{F}_k(r)r^k C_k^{\frac{n-2}{2}}(\cos\vartheta)\\ &= \sum_{k=0}^{\infty}\frac{2k+n-2}{n-2}\mathbb{F}_k(r)r^k C_k^{\frac{n-2}{2}}(\cos\vartheta),\end{aligned}$$

where

$$\mathbb{F}_k(r) = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(k+n-1)}{\Gamma\left(k+\frac{n}{2}\right)\Gamma(n-1)}F\left(k,-\frac{n}{2}+1;k+\frac{n}{2};r^2\right).$$

By the identity (B.73), we obtain (4.72). To obtain (4.73), it suffices to show that

$$F\left(k,1-\frac{n}{2};k+\frac{n}{2};r^2\right) = \frac{\Gamma\left(\frac{n}{2}\right)e^{i\pi\left(\frac{n-1}{2}\right)}}{\sqrt{\pi}\Gamma(k)r^{\frac{n-1}{2}}\left(1-r^2\right)^{\frac{1-n}{2}}}Q_{k+\frac{n}{2}-\frac{3}{2}}^{-\frac{n}{2}+\frac{1}{2}}\left(\frac{1+r^2}{2r}\right). \tag{4.77}$$

Towards this end, we use (B.72) to obtain

$$F\left(k,1-\frac{n}{2};k+\frac{n}{2};r^2\right) = \left(1+r^2\right)^{-k}F\left(\frac{k}{2},\frac{k}{2}+\frac{1}{2};k+\frac{n}{2};\left(\frac{2r}{1+r^2}\right)^2\right), \tag{4.78}$$

and then apply (B.95) to get (4.77). To obtain (4.74), it suffices to show that

$$F\left(k,1-\frac{n}{2};k+\frac{n}{2};r^2\right) = \frac{\left(1+r^2\right)^{-\frac{k}{2}}}{(r-1)^{1-\frac{n}{2}}}\left(\frac{2}{r}\right)^{\frac{k}{2}+\frac{n}{4}-\frac{1}{2}}\Gamma\left(k+\frac{n}{2}\right)P_{-\frac{n}{2}}^{-k-\frac{n}{2}+1}\left(\frac{\sqrt{1+r^2}}{r-1}\right). \tag{4.79}$$

To see this, we apply (B.82) to (4.78) and we get (4.79). The assertion (4.76) follows by noting that as $\rho\nearrow\infty$, $\tanh\left(\frac{\rho}{2}\right)\to 1$ and then using (B.73) to see that

$$\begin{aligned}
P_{\mathbf{D}^n}(\widetilde{\rho},\vartheta) &= \frac{1}{\nu_{n-1}}\lim_{\rho\nearrow\infty} P^{\mathbf{H}^n}_{\widetilde{\rho},\rho}(\widetilde{\theta},\theta) \\
&= \frac{1}{\nu_{n-1}}\lim_{\rho\nearrow\infty}\sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)\frac{\tanh^k\left(\frac{\widetilde{\rho}}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\widetilde{\rho}}{2}\right)\right)}{\tanh^k\left(\frac{\rho}{2}\right)F\left(k,1-\frac{n}{2};k+\frac{n}{2};\tanh^2\left(\frac{\rho}{2}\right)\right)}C_k^{\frac{n-2}{2}}(\cos\vartheta) \\
&= \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty}\frac{2k+n-2}{n-2}r^k\frac{F\left(k,1-\frac{n}{2};k+\frac{n}{2};r^2\right)}{F\left(k,1-\frac{n}{2};k+\frac{n}{2};1\right)}C_k^{\frac{n-2}{2}}(\cos\vartheta) \\
&= \frac{1}{\nu_{n-1}}\sum_{k=0}^{\infty}M_k^{n-1}\mathscr{C}_k^{\frac{n-2}{2}}(\cos\vartheta)\mathbb{F}_k(r)r^k,\ \ r=\tanh\left(\frac{\widetilde{\rho}}{2}\right).
\end{aligned} \tag{4.80}$$

In terms of the associated Legendre polynomial we obtain

$$P_{\mathbf{D}^n}(\widetilde{\rho},\vartheta) = \frac{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}{\nu_{n-1}}\sum_{k=0}^{\infty}M_k^{n-1}\mathscr{C}_k^{\frac{n-2}{2}}(\cos\vartheta)\frac{\Gamma(k+n-1)}{\Gamma(n-1)}\sinh^{\frac{2-n}{2}}\widetilde{\rho}P^{\frac{2-n}{2}-k}_{\frac{n-2}{2}}(\cosh\widetilde{\rho}). \tag{4.81}$$

This completes the proof of the theorem. □

Remark 4.7. *If* $n=2$ *in* (4.73)*, then we obtain the Poisson kernel on the unit disk (see* (4.43)*):*

$$P_{\mathbf{D}}(r,\vartheta) = P(r,\vartheta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\vartheta+r^2} = \frac{1}{2\pi}\left(1+2\sum_{k=1}^{\infty}r^k\cos k\vartheta\right). \tag{4.82}$$

Proof. Since the limit (1.18) exists, it suffices to show that

$$\frac{1}{\sqrt{\pi}}\left[\frac{(1-r^2)}{r}e^{i\pi}\right]^{\frac{1}{2}}\frac{\Gamma(k+1)}{\Gamma(k)}Q^{-\frac{1}{2}}_{k-\frac{1}{2}}\left(\frac{1+r^2}{2r}\right) = r^k.$$

Thanks to (B.100a). □

Remark 4.8. *In view of* (4.77) *and* (4.79)*, we have the following identity in the context of special functions*

$$P^{-k-\frac{n}{2}+1}_{-\frac{n}{2}}\left(\frac{\sqrt{1+r^2}}{r-1}\right) = \frac{i2^{-\frac{k}{2}-\frac{n}{4}+\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma(k)\Gamma\left(k+\frac{n}{2}\right)r^{\frac{n}{4}-\frac{k}{2}}}\frac{(1+r^2)^{\frac{k}{2}}}{(r+1)^{1-\frac{n}{2}}}Q^{-\frac{n}{2}+\frac{1}{2}}_{k+\frac{n}{2}-\frac{3}{2}}\left(\frac{1+r^2}{2r}\right).$$

In particular, if $k=1$*, then*

$$P^{-\frac{n}{2}}_{-\frac{n}{2}}\left(\frac{\sqrt{1+r^2}}{r-1}\right) = \frac{i2^{1-\frac{n}{4}}}{n\sqrt{\pi}r^{\frac{n}{4}-\frac{1}{2}}}\frac{(1+r^2)^{\frac{1}{2}}}{(r+1)^{1-\frac{n}{2}}}Q^{-\frac{n}{2}+\frac{1}{2}}_{-\frac{1}{2}+\frac{n}{2}}\left(\frac{1+r^2}{2r}\right).$$

Putting $k=1$, $n=4$ in (4.77), and using (B.99a), (B.100a) and (B.100b), we obtain

Proposition 4.13.

$$F\left(1,-1;3;r^2\right) = \frac{-i\left(1-r^2\right)^{\frac{3}{2}}}{\sqrt{\pi}r^{\frac{3}{2}}}Q^{-\frac{3}{2}}_{\frac{3}{2}}\left(\frac{1+r^2}{2r}\right) = \frac{i}{r}\left(1-\frac{1}{3r^2}\right). \tag{4.83}$$

Next we establish an important property of the Poisson kernel.

Remark 4.9.

$$\nu_{n-2}\int_0^{\pi}P_r^{\mathbf{D}^n}(\vartheta)\sin^{n-2}\vartheta\,d\vartheta = 1.$$

Proof. Clearly, for $n = 2$, we have by (B.63)

$$\int_0^{2\pi} P_{\mathbf{D}}(r,\vartheta)\, d\vartheta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos\vartheta + r^2}\, d\vartheta = \left(1-r^2\right) F\left(1,1;1;r^2\right) = 1.$$

For $n \geq 2$,

$$\nu_{n-2}\int_0^{\pi} P_r^{\mathbf{D}^n}(\vartheta)\sin^{n-2}\vartheta\, d\vartheta = \frac{\left(1-r^2\right)^{n-1}\nu_{n-2}}{\nu_{n-1}}\int_0^{\pi}\frac{\sin^{n-2}\vartheta\, d\vartheta}{\left(1-2r\cos\vartheta + r^2\right)^{n-1}}.$$

Using (B.64), (4.35) and (B.59), we find that

$$\nu_{n-2}\int_0^{\pi} P_r^{\mathbf{D}^n}(\vartheta)\sin^{n-2}\vartheta\, d\vartheta = \frac{\left(1-r^2\right)^{n-1}\nu_{n-2}}{\nu_{n-1}} B\left(\frac{n-1}{2},\frac{1}{2}\right) F\left(n-1,\frac{n}{2};\frac{n}{2};r^2\right) = 1.$$

□

The integral

$$P_r^{\mathbf{D}^n}[\phi](\theta) = \nu_{n-2}\int_0^{\pi} P_r^{\mathbf{D}^n}(\vartheta,\vartheta')\phi(\vartheta')\sin^{n-2}\vartheta'\, d\vartheta' \tag{4.84}$$

is called the Poisson integral formula for the hyperbolic unit ball $\mathbf{D}^n$, for a continuous function ϕ on $[0,\pi]$.

4.6 Summation of Certain Series Involving Legendre Polynomials

In some occasions in computations we find closed forms of certain infinite series useful, especially when we are interested in only the exact expressions of some series involving special functions. In this section, as a consequence of the series representations of the Poisson kernel we deduce closed forms for certain infinite series involving the Legendre polynomial.

To start with, consider the generating function for the Legendre polynomial

$$(1-2rt+r^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} r^k P_k(t), \quad t=\cos\theta,\ r<1. \tag{4.85}$$

Differentiating both sides of (4.85) l times w.r.t. t, we have

$$\begin{aligned}
&\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left[-\frac{1}{2}(2l-1)\right](-2r)^l(1-2rt+r^2)^{-\frac{1}{2}-l}\\
&= 1\cdot 3\cdot 5\cdot 7\cdots(2l-1)r^l(1-2rt+r^2)^{-\frac{1}{2}-l} = (2l-1)!!r^l(1-2rt+r^2)^{-\frac{1}{2}-l}\\
&= \sum_{k=0}^{\infty} r^k \frac{d^l}{dt^l}P_k(t),
\end{aligned}$$

where

$$k!! = \begin{cases} 1\cdot 3\cdot 5\cdot \ldots\cdot k & \text{if } k \text{ is odd},\\ 2\cdot 4\cdot 6\cdot \ldots\cdot k & \text{if } k \text{ is even},\\ 1 & \text{if } k\leq 0.\end{cases} \tag{4.86}$$

That is,

$$(1-2rt+r^2)^{-\nu} = \frac{1}{(2\nu-2)!!r^{\nu-\frac{1}{2}}}\sum_{k=0}^{\infty} r^k \frac{d^{\nu-\frac{1}{2}}}{dt^{\nu-\frac{1}{2}}}P_k(t) = \frac{\left(1-t^2\right)^{\frac{\nu}{2}-\frac{1}{4}}}{(2\nu-2)!!r^{\nu-\frac{1}{2}}}\sum_{k=0}^{\infty} r^k P_k^{\nu-\frac{1}{2}}(t), \quad (4.87)$$

where we have used the identity (B.93).

For the special case $\nu = 1$, we see that

$$\begin{aligned}\left(1-2rt+r^2\right)^{-1} &= r^{-\frac{1}{2}}\left(1-t^2\right)^{\frac{1}{4}}\sum_{k=0}^{\infty} r^k P_k^{\frac{1}{2}}(t) \qquad (4.88)\\ &= \sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty} r^{k-\frac{1}{2}}\cos\left(k+\frac{1}{2}\right)\theta, \quad t=\cos\theta,\end{aligned}$$

where we have used (B.84d). On the other hand (see Dixon and Lacroix [50])

$$\left(1-2r\cos\theta+r^2\right)^{-1} = \sum_{k=0}^{\infty}\frac{2k+1}{2r}Q_k\left(\frac{1+r^2}{2r}\right)P_k(\cos\theta). \quad (4.89)$$

Equating (4.88) and (4.89) we obtain the identity

Proposition 4.14.

$$\sum_{k=0}^{\infty}(2k+1)Q_k\left(\frac{1+r^2}{2r}\right)P_k(\cos\theta) = 2\sqrt{\frac{2r}{\pi}}\sum_{k=0}^{\infty} r^k\cos\left(k+\frac{1}{2}\right)\theta. \quad (4.90)$$

To obtain another identity, we put $\nu = 3/2$ in (4.87) to get

$$(1-2rt+r^2)^{-\frac{3}{2}} = \frac{1}{r}\sum_{k=0}^{\infty} r^k\frac{d}{dt}P_k(t),$$

and by applying (B.91), we have

$$\begin{aligned}(1-2rt+r^2)^{-\frac{3}{2}} &= \frac{1}{r}\sum_{k=0}^{\infty} r^k\frac{P_k(t)-(k+1)P_{k+1}(t)}{1-t^2}\\ &= \frac{1}{r\left(1-t^2\right)}\left[\sum_{k=0}^{\infty} r^k P_k(t) - \sum_{k=0}^{\infty} r^k(k+1)P_{k+1}(t)\right]. \qquad (4.91)\end{aligned}$$

Using (4.85) in (4.91) we obtain

Proposition 4.15.

$$\sum_{k=0}^{\infty} r^k(k+1)P_{k+1}(t) = \frac{1-2rt+r^2-r+rt^2}{(1-2rt+r^2)^{\frac{3}{2}}}. \quad (4.92)$$

Again, formally we have

$$\sum_{k=0}^{\infty}\frac{1}{k+1}\rho^{k+1}P_k(t) = \sum_{k=0}^{\infty}P_k(t)\int_0^{\rho} r^k\,dr = \int_0^{\rho}\frac{dr}{\sqrt{1-2rt+r^2}} = \ln\frac{\rho-t+\sqrt{1-2\rho t+\rho^2}}{1-t}. \quad (4.93)$$

The identity (4.93) looks interesting, we therefore summarise it as

Proposition 4.16.

$$\sum_{k=0}^{\infty} \frac{1}{k+1} \rho^{k+1} P_k(t) = \ln \frac{\rho - t + \sqrt{1 - 2\rho t + \rho^2}}{1-t}.$$

In particular,

$$\sum_{k=0}^{\infty} \frac{1}{k+1} P_k(t) = \ln \left(1 + \sqrt{\frac{2}{1-t}}\right).$$

Next we consider infinite series involving products of the Legendre polynomial of the first kind, namely we find closed forms for the two infinite series

$$\sum_{k=0}^{\infty} P_k(\cos\theta) P_k(\cos\theta') r^k, \qquad \sum_{k=0}^{\infty} (2k+1) P_k(\cos\theta) P_k(\cos\theta') r^k$$

in terms of elliptic integrals, by means of the addition formula

$$P_k(\cos\alpha) = P_k(\cos\theta) P_k(\cos\theta') + 2 \sum_{m=1}^{k} \frac{(k-m)!}{(k+m)!} P_k^m(\cos\theta) P_k^m(\cos\theta') \cos m\omega, \tag{4.94}$$

where

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\omega.$$

Integrating both sides of (4.94) with respect to ω, $0 \le \omega \le \pi$, we have

$$P_k(\cos\theta) P_k(\cos\theta') = \frac{1}{\pi} \int_0^{\pi} P_k(\cos\alpha)\, d\omega.$$

Thus,

$$\begin{aligned} \sum_{k=0}^{\infty} P_k(\cos\theta) P_k(\cos\theta') r^k &= \frac{1}{\pi} \int_0^{\pi} \sum_{k=0}^{\infty} P_k(\cos\alpha) r^k \, d\omega \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{d\omega}{\sqrt{1 - 2r\cos\alpha + r^2}}, \end{aligned} \tag{4.95}$$

where we have used (B.105); and by (4.95)

$$\begin{aligned} \sum_{k=0}^{\infty} (2k+1) P_k(\cos\theta) P_k(\cos\theta') r^k &= \left(1 + 2r\frac{d}{dr}\right) \sum_{k=0}^{\infty} P_k(\cos\theta) P_k(\cos\theta') r^k \\ &= \frac{1}{\pi} \int_0^{\pi} \sum_{k=0}^{\infty} (2k+1) P_k(\cos\alpha) r^k \, d\omega \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{\left(1 - r^2\right) d\omega}{(1 - 2r\cos\alpha + r^2)^{\frac{3}{2}}}, \end{aligned} \tag{4.96}$$

where we have used (4.52). In particular, setting $\theta = \theta' = \frac{\pi}{2}$ in (4.95) and using (B.85) gives

$$\begin{aligned}
&\frac{1}{\pi}\sum_{k=0}^{\infty}\cos^2\left(\frac{k\pi}{2}\right)\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)^2}{\Gamma\left(\frac{k}{2}+1\right)^2}r^k \\
&=\left\{1+\left(\frac{1}{2}\right)^2 r^2+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2 r^4+\cdots+\left(\frac{(2n-1)!!}{2^n n!}\right)^2 r^{2n}+\cdots\right\} \\
&=\frac{1}{\pi}\int_0^{\pi}\frac{d\omega}{\sqrt{1-2r\cos\omega+r^2}}=F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) \\
&=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{1-r^2\sin^2\omega}}=\frac{2}{\pi}\mathbf{K}(r),
\end{aligned} \tag{4.97}$$

where $\mathbf{K}(r)$ is the (complete) elliptic integral of the first kind (see Gradshtejn and Ryzhik [66, Section 8.11]). Similarly, from (4.96) we have

$$\begin{aligned}
\frac{1}{\pi}\sum_{k=0}^{\infty}(2k+1)\cos^2\left(\frac{k\pi}{2}\right)\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)^2}{\Gamma\left(\frac{k}{2}+1\right)^2}r^k &= \frac{1}{\pi}\int_0^{\pi}\frac{\left(1-r^2\right)d\omega}{\left(1-2r\cos\omega+r^2\right)^{\frac{3}{2}}} \\
&= \left(1-r^2\right)F\left(\frac{3}{2},\frac{3}{2};1;r^2\right),
\end{aligned} \tag{4.98}$$

where we have used (B.63). On the other hand

$$\begin{aligned}
\int_0^{\pi}\frac{\left(1-r^2\right)d\omega}{\left(1-2r\cos\omega+r^2\right)^{\frac{3}{2}}} &= \left(1+2r\frac{d}{dr}\right)\int_0^{\pi}\frac{d\omega}{\sqrt{1-2r\cos\omega+r^2}} \\
&= 2\left(1+2r\frac{d}{dr}\right)\mathbf{K}(r) \\
&= \frac{2}{r'^2}\left(2\mathbf{E}(r)-\mathbf{K}(r)r'^2\right), \quad r'^2 = 1-r^2,
\end{aligned} \tag{4.99}$$

where

$$\mathbf{E}(r) = \int_0^{\frac{\pi}{2}}\sqrt{1-r^2\sin^2\theta}\,d\theta = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right), \quad \frac{d\mathbf{K}(r)}{dr} = \frac{\mathbf{E}(r)}{rr'} - \frac{\mathbf{K}(r)}{r}. \tag{4.100}$$

The function $\mathbf{E}(r)$ is the (complete) elliptic integral of the second kind (see Gradshtejn and Ryzhik [66, Section 8.11]).

In summary, we obtain the following identities.

Proposition 4.17. *The following identities hold in the context of special functions:*

(i) $\frac{1}{\pi}\sum\limits_{k=0}^{\infty}\cos^2\left(\frac{k\pi}{2}\right)\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)^2}{\Gamma\left(\frac{k}{2}+1\right)^2}r^k = F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = \frac{2}{\pi}\mathbf{K}(r)$,

(ii) $\frac{1}{\pi}\sum\limits_{k=0}^{\infty}(2k+1)\cos^2\left(\frac{k\pi}{2}\right)\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)^2}{\Gamma\left(\frac{k}{2}+1\right)^2}r^k = \left(1-r^2\right)F\left(\frac{3}{2},\frac{3}{2};1;r^2\right) = \frac{2}{\pi r'^2}\left(2\mathbf{E}(r)-\mathbf{K}(r)(1-r^2)\right)$.

In view of (4.98)-(4.100), we obtain the identity

Proposition 4.18.

$$\left(1-r^2\right)F\left(\frac{3}{2},\frac{3}{2};1;r^2\right) = \frac{2}{1-r^2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right) - F\left(\frac{1}{2},\frac{1}{2};1;r^2\right).$$

We can also establish another interesting identity by replacing r^k with $J_{2k}(r)$ in (4.98) and then using

$$\sum_{k=0}^{\infty}(2k+1)J_{2k}(a)P_k(x) = \frac{a}{2}J_0\left(a\sqrt{\frac{1-x}{2}}\right)$$

to get

$$\begin{aligned}\sum_{k=0}^{\infty}(2k+1)\cos^2\left(\frac{k\pi}{2}\right)\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)^2}{\Gamma\left(\frac{k}{2}+1\right)^2}J_{2k}(r) &= \frac{r}{2}\int_0^{\pi} J_0\left(r\sqrt{\frac{1-\cos\omega}{2}}\right)\,d\omega\\ &= \frac{r}{2}\int_0^{\pi} J_0\left(r\sin\frac{\omega}{2}\right)\,d\omega.\end{aligned}$$

Hence, we obtain the identity

Proposition 4.19.

$$\sum_{k=0}^{\infty}(2k+1)\cos^2\left(\frac{k\pi}{2}\right)\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)^2}{\Gamma\left(\frac{k}{2}+1\right)^2}J_{2k}(r) = \frac{\pi r}{2}\left[J_0\left(\frac{r}{2}\right)\right]^2.$$

Again, considering the equality (Dougall [54])

$$\frac{2^{\nu}\Gamma(\nu+1)}{\Gamma(2\nu+1)}\sum_{k=0}^{\infty}\frac{\Gamma(2\nu+k+1)}{\Gamma(k+1)}r^{\nu+k}P_{\nu+k}^{\nu}(\cos\theta) = \frac{r^{\nu}\sin^{\nu}\theta}{(1-2r\cos\theta+r^2)^{\nu+\frac{1}{2}}}, \tag{4.101}$$

for $r<1, \operatorname{Re}\nu\geq 0$. Integrating both sides of (4.101) from 0 to π with respect to θ and then setting $x=\cos\theta$, $-1\leq x\leq 1$, we have

$$\frac{2^{\nu}\Gamma(\nu+1)}{\Gamma(2\nu+1)}\sum_{k=0}^{\infty}\frac{\Gamma(2\nu+k+1)}{\Gamma(k+1)}r^k\int_{-1}^{1}P_{\nu+k}^{\nu}(x)\left(1-x^2\right)^{-\frac{1}{2}}\,dx = \int_{-1}^{1}\frac{\left(1-x^2\right)^{\frac{\nu}{2}-\frac{1}{2}}}{(1-2rx+r^2)^{\nu+\frac{1}{2}}}\,dx.$$

On applying (B.90) and using (B.64), we obtain the identity

Proposition 4.20.

$$\begin{aligned}\frac{\pi 2^{2\nu}\Gamma(\nu+1)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)}{\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)^{-1}}\sum_{k=0}^{\infty}r^k\frac{\Gamma(2\nu+k+1)}{\Gamma(k+1)}\frac{\Gamma\left(\frac{k}{2}+1\right)^{-1}\Gamma\left(-\nu-\frac{k}{2}+\frac{1}{2}\right)^{-1}}{\Gamma\left(1+\frac{\nu}{2}+\frac{k}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}-\frac{k}{2}\right)}&\\ = B\left(\frac{\nu+1}{2},\frac{1}{2}\right)F\left(\nu+\frac{1}{2},\frac{\nu+1}{2};\frac{\nu}{2}+1;r^2\right).&\end{aligned} \tag{4.102}$$

Equation (4.102) is the generalisation of (4.97). By setting $\nu=0$ in (4.102) and comparing with (4.97) we obtain the equality

Corollary 4.21.

$$\cos\left(\frac{k\pi}{2}\right) = \frac{\pi}{\Gamma\left(\frac{1}{2}-\frac{k}{2}\right)\Gamma\left(\frac{1}{2}+\frac{k}{2}\right)},\quad k\geq 0.$$

Chapter 5

The Gegenbauer Transform and Heat Kernels on $\mathbf{S}^n$ and $\mathbf{CP}^n$

In this chapter we discuss the heat kernels on the unit sphere $\mathbf{S}^n$ and the complex projective space $\mathbf{CP}^n$. We also calculate explicitly the heat kernel coefficients on $\mathbf{S}^n$, $n \geq 1$. Since the real projective space $\mathbf{RP}^n$ is the sphere with antipodal points identified (see pp. 133-134), we concentrate mainly here on $\mathbf{S}^n$ and $\mathbf{CP}^n$. The sphere $\mathbf{S}^n$, the real projective space $\mathbf{RP}^n$ and the complex projective space $\mathbf{CP}^n$ are important examples of n-dimensional rank one compact Riemannian symmetric spaces of the form $\mathscr{X} = G/K$, where G is a connected, semisimple compact Lie group and K is a maximal compact subgroup of G (see Section 1.2). We use the Gegenbauer transform to derive the integral heat kernel on the unit sphere $\mathbf{S}^n$; this gives a series representation of the heat kernels in terms of the Gegenbauer polynomial; we then apply the Riemann-Liouville fractional derivative formula to express the heat kernels in closed forms. By expressing the trace of the heat operator on $\mathbf{S}^n$ in terms of Jacobi's theta functions and their higher order derivatives we compute the heat kernel coefficients a_k^n in the Minakshisundaram-Pleijel asymptotic expansion (1.79) for the special case $M = \mathbf{S}^n$. The explicit fractional and integral representations of the heat kernels on the complex projective spaces are also presented. Finally we express the traces of the heat kernels on $\mathbf{S}^n$, $\mathbf{RP}^n$ and $\mathbf{CP}^n$, in terms of the Euclidean Poisson kernel.

5.1 The Gegenbauer Transform and its Inversion Formula

In this section we discuss the Gegenbauer transform and its inversion formula. The Gegenbauer transform developed here will be used to compute the heat kernel on the unit sphere $\mathbf{S}^n$ in the next section. The Gegenbauer transform approach for obtaining the solution of the heat equation and consequently the heat kernel on $\mathbf{S}^n$ is appearing here for the first time in the Literature.

Let $f(x)$ be a function defined on the closed interval $-1 \leq x \leq 1$. In this section and the next section we temporarily denote the Gegenbauer transform of a function f by $\widetilde{f}$.

Definition 5.1. *The Gegenbauer transform $\mathscr{G}[f] = \widetilde{f}$ of a function f is given by*

$$\mathscr{G}[f(x)](k;\nu) = \widetilde{f}(k;\nu) = \int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} C_k^{\nu}(x) f(x)\, dx, \quad k \geq 0, \tag{5.1}$$

or equivalently,

$$\mathscr{G}[f(\cos\varphi)](k;\nu) = \widetilde{f}(k;\nu) = \int_0^\pi \sin^{2\nu}\varphi C_k^\nu(\cos\varphi) f(\cos\varphi)\, d\varphi, \quad k \geq 0.$$

Suppose $f(x)$ admits a Gegenbauer-Fourier expansion of the form

$$f(x) = \sum_{k=0}^{\infty} c(k;\nu) C_k^\nu(x), \tag{5.2}$$

where, because of the orthogonality property (B.116), $c(k;\nu)$ is given by

$$c(k;\nu) = \mathsf{a}_{k,\nu} \int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} C_k^\nu(x)\, f(x)\, dx = \mathsf{a}_{k,\nu} \widetilde{f}(k;\nu), \tag{5.3}$$

where

$$\mathsf{a}_{k,\nu} = \frac{k!(k+\nu)\Gamma(\nu)^2}{\pi 2^{1-2\nu}\Gamma(2\nu+k)}.$$

Substituting $c(k;\nu)$ given by the integral (5.3) in (5.2), we see at once that the Gegenbauer inversion formula is given by the series representation

$$\mathscr{G}^{-1}[\widetilde{f}(k;\nu)](x) = f(x) = \sum_{k=0}^{\infty} \mathsf{a}_{k,\nu} \widetilde{f}(k;\nu) C_k^\nu(x). \tag{5.4}$$

Theorem 5.2. *Given the Gegenbauer differential equation*

$$\left(1-x^2\right) y''(x) - (2\nu+1)xy'(x) + k(k+2\nu)y(x) = 0, \tag{5.5}$$

one has

$$\mathscr{G}\left[\left(1-x^2\right) f''(x) - (2\nu+1)xf'(x)\right](k;\nu) = -k(k+2\nu)\widetilde{f}(k;\nu). \tag{5.6}$$

Proof. In the form of the self-adjoint (Sturm-Liouville) form, the differential equation satisfied by $C_k^\nu(x)$ (i.e., (5.5)) can be written

$$\frac{d}{dx}\left[\left(1-x^2\right)^{\nu+\frac{1}{2}} y'(x)\right] + k(k+2\nu)\left(1-x^2\right)^{\nu-\frac{1}{2}} y = 0. \tag{5.7}$$

Also,

$$\left(1-x^2\right) f''(x) - (2\nu+1)xf' = \frac{d}{dx}\left[\left(1-x^2\right)^{\nu+\frac{1}{2}} f'(x)\right]\left(1-x^2\right)^{-\nu+\frac{1}{2}}. \tag{5.8}$$

Applying the Gegenbauer transform to the right-hand side of (5.8) gives

$$\int_{-1}^{1} f(x) \frac{d}{dx}\left[\left(1-x^2\right)^{\nu+\frac{1}{2}} \frac{dC_k^\nu(x)}{dx}\right] dx.$$

By (5.5) and (5.7) with $y(x) = C_k^\nu(x)$, we see that the last term reduces to

$$-k(k+2\nu)\int_{-1}^{1} f(x)\left(1-x^2\right)^{\nu-\frac{1}{2}} C_k^\nu(x)\, dx = -k(k+2\nu)\widetilde{f}(x).$$

□

The Gegenbauer transform is also a tool for establishing the Euclidean Poisson kernel and thereby giving several identities concerning special functions (see Awonusika and Taheri [8] for details).

5.2 The Heat Kernel on S^n via the Gegenbauer Transform

The unit sphere $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$ is a compact smooth n-dimensional manifold and a symmetric space of rank one in virtue of the identification

$$\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n), \tag{5.9}$$

where $\mathbf{SO}(n)$ is the special orthogonal group of real $n \times n$ matrices with determinant one (see, e.g., Berger et al. [26]).

The heat kernel on the sphere $\mathbf{S}^n$ has been obtained by notable authors, namely Terras [167] (for $n = 2$), Faraut [58] and Taylor [166, pp. 113-116]. In Terras [167], the heat kernel on the sphere $\mathbf{S}^2$ is obtained by using the method of separation of variables and the addition formula for spherical harmonics. In Taylor [166, pp. 113-116] the heat kernel on the sphere $\mathbf{S}^{2k+1}$, $k = 0, 1, 2, \ldots$, is computed via the Poisson integral formula for the Euclidean ball; it is worth noting that the heat kernel on the even dimensional sphere is not given in Taylor [166]. Faraut [58, pp. 220-227] computes the heat kernels on the sphere $\mathbf{S}^n$, $n = 2, 3$ using an integral transform defined on the Lie group $\mathbf{SU}(2)$ (see also Faraut [58, pp. 176-182]). In this section we use the Gegenbauer transform developed in Section 5.1 (see also Conte [44]) to compute explicit formulae for the heat kernels on the sphere $\mathbf{S}^n$, $n = 1, 2, \ldots$. The Gegenbauer transform presents the heat kernel on $\mathbf{S}^n$ as a spectral sum involving the Gegenbauer polynomial; we later transform the series representation into fractional and integral representations using the Riemann-Liouville fractional derivative. Our result in the odd-dimensional case agrees with that of Taylor [166, 113-116] who uses a different method.

Now consider the Cauchy problem for the heat equation on the sphere $\mathbf{S}^n$:

$$\begin{aligned} &\frac{\partial}{\partial t} u(t,\theta) = \left[\frac{\partial^2}{\partial \theta^2} + (n-1)\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right] u(t,\theta), \quad t > 0, \\ &u(0,\theta) = u_0(\theta). \end{aligned} \tag{5.10}$$

In order to reduce the second-order partial differential equation in (5.10) to a second-order ordinary differential equation we make the substitution

$$u(t,\theta) = v(t,\zeta),\ \zeta = \cos\theta,$$

in (5.10) to obtain

$$\begin{aligned} \frac{\partial}{\partial t} v(t,\zeta) &= \left[\left(1-\zeta^2\right)\frac{\partial^2}{\partial\zeta^2} - n\zeta\frac{\partial}{\partial\zeta}\right] v(t,\zeta) \\ v(0,\zeta) &= v_0(\zeta). \end{aligned} \tag{5.11}$$

The Gegenbauer transform of both sides of equation (5.11) now gives

$$\begin{aligned} \frac{d}{dt}\widetilde{v}(t,k;n) &= -k(k+n-1)\widetilde{v}(t,k;n) \\ \widetilde{v}(0,k;n) &= \widetilde{v}_0(k;n). \end{aligned} \tag{5.12}$$

The unique solution to the initial value problem (5.12) is given by

$$\widetilde{v}(t,k;n) = \widetilde{v}_0(k;n)e^{-k(k+n-1)t}.$$

By the Gegenbauer transform and its inversion formula, we have

$$\begin{aligned} v(t,\zeta) &= \sum_{k=0}^{\infty} \mathsf{a}_{k,\nu} e^{-k(k+n-1)t} \widetilde{v}_0(k;n) C_k^{\frac{n-1}{2}}(\zeta) \\ &= \sum_{k=0}^{\infty} \mathsf{a}_{k,\nu} e^{-k(k+n-1)t} \int_{-1}^{1} \left(1-\zeta'^2\right)^{\frac{n-2}{2}} C_k^{\frac{n-1}{2}}(\zeta') v_0(\zeta') C_k^{\frac{n-1}{2}}(\zeta)\, d\zeta' \\ &= \int_0^{\pi} U(t,\theta,\theta') v_0(\theta') \sin^{n-1}\theta'\, d\theta' \\ &= u(t,\theta), \end{aligned} \tag{5.13}$$

where

$$U(t,\theta,\theta') = \sum_{k=0}^{\infty} \mathsf{a}_{k,\nu} e^{-k(k+n-1)t} C_k^{\frac{n-1}{2}}(\cos\theta') C_k^{\frac{n-1}{2}}(\cos\theta) \tag{5.14}$$

and

$$\mathsf{a}_{k,\nu} = \frac{k!\left(k+\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)^2}{\pi 2^{2-n}\Gamma(k+n-1)}.$$

Using (4.27), (4.28) and (4.31) in (5.14), we obtain

$$U(t,\theta,\theta') = \frac{\nu_{n-1}}{\nu_n} \sum_{k=0}^{\infty} \left(\frac{2k}{n-1}+1\right) e^{-k(k+n-1)t} C_k^{\frac{n-1}{2}}(\cos(\theta-\theta')).$$

Hence,

$$u(t,\theta) = \nu_{n-1} \int_0^{\pi} K_{\mathbf{S}^n}(t,\theta,\theta') u_0(\theta') \sin^{n-1}\theta'\, d\theta',$$

where

$$\begin{aligned} K_{\mathbf{S}^n}(t,\theta) &= \frac{1}{\nu_n} \sum_{k=0}^{\infty} \left(\frac{2k}{n-1}+1\right) e^{-k(k+n-1)t} C_k^{\frac{n-1}{2}}(\cos\theta) \\ &= \frac{1}{\nu_n} \sum_{k=0}^{\infty} M_k^n e^{-k(k+n-1)t} \mathscr{C}_k^{\frac{n-1}{2}}(\cos\theta) \end{aligned} \tag{5.15}$$

is the heat kernel on the unit sphere $\mathbf{S}^n$.

For future purposes we write out the following special cases.

- $(n=1)$

$$\begin{aligned} \widetilde{\Theta}_1(t,\theta) := K_{\mathbf{S}^1}(t,\theta) &= \frac{1}{2\pi}\left(1+2\sum_{k=1}^{\infty} e^{-k^2 t}\cos k\theta\right) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-k^2 t} e^{ik\theta} \\ &= \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(\theta+2k\pi)^2}{4t}}, \end{aligned} \tag{5.16}$$

which is the heat kernel on the unit circle $\mathbf{S}^1$. Equalities in (5.16) are a consequence of the limit (1.18) and the Poisson summation formula. The theta function $\widetilde{\Theta}_1(t,\theta)$ and the classical Jacobi's theta function $\Theta_3(t,\theta)$ are related to one-another by

$$\widetilde{\Theta}_1(t,\theta) = \frac{1}{2\pi}\Theta_3\left(\frac{it}{\pi},\frac{\theta}{2}\right).$$

For these classical Jacobi's theta functions and more, see Whittaker and Watson [176, Ch. 21]. The theta function $\widetilde{\Theta}_1(t,\theta)$ is periodic, i.e.,

$$\widetilde{\Theta}_1(t,\theta+2k\pi)=\widetilde{\Theta}_1(t,\theta).$$

- $(n=2)$

$$K_{\mathbf{S}^2}(t,\theta)=\frac{e^{\frac{t}{4}}}{2\pi}\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)e^{-\left(k+\frac{1}{2}\right)^2t}P_k(\cos\theta), \tag{5.17}$$

  which is the heat kernel on the unit sphere $\mathbf{S}^2$, where $M_k^2=2k+1$ is the multiplicity of eigenvalues $(k(k+1):k\geq 0)$ of the Laplacian on $\mathbf{S}^2$ and $\nu_2=4\pi$ is the area of the sphere $\mathbf{S}^2$.

The following remark is an important property of the heat kernel, and it can be established the same way as we did for the Poisson kernel in $\mathbf{H}^n$ (see Remark 4.2, p. 106).

Remark 5.1. *The heat kernel on a compact manifold integrates to one (see e.g. Chavel [41], see also Li [103] for the noncompact case):*

$$\nu_{n-1}\int_0^{\pi}K_{\mathbf{S}^n}(t,\theta)\sin^{n-1}\theta\,d\theta=1.$$

Next we transform the heat kernels $K_{\mathbf{S}^n}(t,\theta)$ given by the spectral sum (5.15) according to whether n is odd or even. In other words we use the Riemann-Liouville fractional derivative formula to show that the heat kernel on the odd-dimensional sphere $\mathbf{S}^n$ $(n\geq 1)$ can be expressed in terms of the theta function $\widetilde{\Theta}_1(t,\theta)$ and that the heat kernel on the even-dimensional sphere $\mathbf{S}^n$ $(n\geq 2)$ can be expressed in terms of the corresponding theta function, denoted $\widetilde{\Theta}_2(t,\theta)$ (we shall see the series defining this theta function in a moment). To actualise this goal we use the fractional representations (B.120)-(B.123) in (5.15) to obtain

$$K_{\mathbf{S}^n}(t,\theta)=\frac{e^{\frac{(n-1)^2t}{4}}}{\nu_n}\frac{2^{\frac{n+1}{2}}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma(n)}\frac{\partial^{\frac{n-1}{2}}}{\partial(\cos\theta+1)^{\frac{n-1}{2}}}\sum_{k=0}^{\infty}e^{-\left(k+\frac{n-1}{2}\right)^2t}\cos\left(k+\frac{n-1}{2}\right)\theta, \tag{5.18}$$

where

$$\frac{\partial^{\frac{1}{2}}f}{\partial(\cos\theta+1)^{\frac{1}{2}}}=\frac{1}{\sqrt{\pi}}\int_{\pi}^{\theta}\frac{\frac{\partial f}{\partial\vartheta}}{\sqrt{\cos\theta-\cos\vartheta}}d\vartheta,\quad \frac{\partial^{\frac{1}{2}}h}{\partial(1-\cos\theta)^{\frac{1}{2}}}=\frac{1}{\sqrt{\pi}}\int_0^{\theta}\frac{\frac{\partial h}{\partial\vartheta}}{\sqrt{\cos\vartheta-\cos\theta}}d\vartheta,$$

provided $f(\pi)=0$, $h(0)=0$. A consequence of the fractional representation formula for the Gegenbauer polynomial C_k^{ν} is the following integral representation for the Legendre polynomial P_k:

$$\begin{aligned}C_k^{\frac{1}{2}}(\cos\theta)=P_k(\cos\theta)&=\frac{2^{\frac{3}{2}}}{\sqrt{\pi}(2k+1)}\frac{\partial^{\frac{1}{2}}}{\partial(\cos\theta+1)^{\frac{1}{2}}}\cos\left(k+\frac{1}{2}\right)\theta\\&=\frac{\sqrt{2}}{\pi}\int_{\theta}^{\pi}\frac{\sin\left(k+\frac{1}{2}\right)\vartheta\,d\vartheta}{\sqrt{\cos\theta-\cos\vartheta}}.\end{aligned} \tag{5.19}$$

For future purposes we also write out the heat kernels for the special cases $n=2,3$.

- $(n = 2)$

$$K_{\mathbf{S}^2}(t,\theta) = \frac{e^{\frac{1}{4}t}}{\sqrt{2\pi}} \left(\frac{\partial}{\partial(\cos\theta + 1)} \right)^{\frac{1}{2}} \widetilde{\Theta}_2(t,\theta), \tag{5.20}$$

where the theta function $\widetilde{\Theta}_2(t,\theta)$ is given by

$$\begin{aligned} \widetilde{\Theta}_2(t,\theta) &:= \frac{1}{\pi} \sum_{k=0}^{\infty} e^{-\left(k+\frac{1}{2}\right)^2 t} \cos\left(k + \frac{1}{2}\right)\theta = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-\left(k+\frac{1}{2}\right)^2 t} e^{i\left(k+\frac{1}{2}\right)\theta} \\ &= \frac{1}{\sqrt{4\pi t}} \sum_{m=-\infty}^{\infty} (-1)^m e^{-\frac{(\theta+2m\pi)^2}{4t}}, \end{aligned} \tag{5.21}$$

which is antiperiodic, i.e.,

$$\widetilde{\Theta}_2(t,\theta + 2k\pi) = (-1)^k \widetilde{\Theta}_2(t,\theta).$$

The theta function $\widetilde{\Theta}_2(t,\theta)$ and the classical Jacobi's theta function $\Theta_2(t,\theta)$ are related to one-another by

$$\widetilde{\Theta}_2(t,\theta) = \frac{1}{2\pi}\Theta_2\left(\frac{it}{\pi}, \frac{\theta}{2}\right).$$

So, from (5.20) and (5.21) we have

$$\begin{aligned} K_{\mathbf{S}^2}(t,\theta) &= \frac{e^{\frac{1}{4}t}\sqrt{2}}{(4\pi t)^{\frac{3}{2}}} \int_\theta^\pi \sum_{k=-\infty}^{\infty} (-1)^k \frac{(\vartheta + 2k\pi)e^{-\frac{(\vartheta+2k\pi)^2}{4t}}}{\sqrt{\cos\theta - \cos\vartheta}}\, d\vartheta \\ &= \frac{e^{\frac{t}{4}}}{\sqrt{2\pi}} \int_\theta^\pi \frac{\left(-\frac{\partial}{\partial\vartheta}\right)\widetilde{\Theta}_2(t,\vartheta)}{\sqrt{\cos\theta - \cos\vartheta}}\, d\vartheta, \end{aligned} \tag{5.22}$$

since

$$f(\pi) = \widetilde{\Theta}_2(t,\pi) = 0.$$

- $(n = 3)$

$$\begin{aligned} K_{\mathbf{S}^3}(t,\theta) &= \frac{e^{\frac{t}{4}}}{2\pi^2} \left(-\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \right) \sum_{k=0}^{\infty} e^{-(k+1)^2 t} \cos(k+1)\theta \\ &= \frac{e^{\frac{t}{4}}}{2\pi} \left(-\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \right) \widetilde{\Theta}_1(t,\theta) \\ &= \frac{e^{\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}} \sum_{k=-\infty}^{\infty} \frac{(\theta + 2k\pi)}{\sin\theta} e^{-\frac{(\theta+2k\pi)^2}{4t}}. \end{aligned}$$

Continuing in this way and noting that

$$\frac{\pi}{\nu_n} \frac{2^{\frac{n+1}{2}}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma(n)} = \frac{1}{(2\pi)^{\frac{n-1}{2}}},$$

we have the following statement (see also Awonusika and Taheri [10]).

Theorem 5.3. *For* $t > 0$, $0 \le \theta < \pi$, *the heat kernel* $K_{\mathbf{S}^n}(t,\theta)$ *associated to the Laplacian on the unit sphere* $\mathbf{S}^n$ *is given by the series representation*

$$K_{\mathbf{S}^n}(t,\theta) = \frac{e^{\frac{(n-1)^2 t}{4}}}{\nu_n} \sum_{k=0}^{\infty} M_k^n e^{-\left(k+\frac{n-1}{2}\right)^2 t} \mathscr{C}_k^{\frac{n-1}{2}}(\cos\theta), \qquad \text{for all } n \ge 1, \tag{5.23}$$

where

$$\mathscr{C}_k^{\frac{n-1}{2}}(\cos\theta) := \frac{C_k^{\frac{n-1}{2}}(\cos\theta)}{C_k^{\frac{n-1}{2}}(1)}$$

is the spherical function on $\mathbf{S}^n$ *satisfying the eigenvalue equation*

$$\left(\frac{\partial^2}{\partial\theta^2} + (n-1)\cot\theta \frac{\partial}{\partial\theta}\right)\Phi_k^{\mathbf{S}^n}(\theta) = -k(k+n-1)\Phi_k^{\mathbf{S}^n}(\theta), \tag{5.24}$$

$$\Phi_k^{\mathbf{S}^n}(0) = \mathscr{C}_k^{\frac{n-1}{2}}(1) = 1. \tag{5.25}$$

Moreover, $K_{\mathbf{S}^n}(t,\theta)$ *is given by the following fractional representations:*

(a) n *odd,* $n \ge 1$,

$$\begin{aligned} K_{\mathbf{S}^n}(t,\theta) &= e^{\frac{(n-1)^2}{4}t}\left(\frac{1}{2\pi}\frac{\partial}{\partial(\cos\theta+1)}\right)^{\frac{n-1}{2}} \widetilde{\Theta}_1(t,\theta) \\ &= e^{\frac{(n-1)^2}{4}t}\left(-\frac{1}{2\pi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{\frac{n-1}{2}} \widetilde{\Theta}_1(t,\theta), \end{aligned} \tag{5.26}$$

where $\widetilde{\Theta}_1(t,\theta)$ *is the heat kernel on the circle* $\mathbf{S}^1$ *given by* (5.16)*;*

(b) n *even,* $n \ge 2$,

$$\begin{aligned} K_{\mathbf{S}^n}(t,\theta) &= e^{\frac{(n-1)^2}{4}t}\left(\frac{1}{2\pi}\frac{\partial}{\partial(\cos\theta+1)}\right)^{\frac{n-1}{2}} \widetilde{\Theta}_2(t,\theta) \\ &= e^{\frac{n(n-2)}{4}t}\left(-\frac{1}{2\pi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{\frac{n-2}{2}} K_{\mathbf{S}^2}(t,\theta), \end{aligned} \tag{5.27}$$

where $\widetilde{\Theta}_2(t,\theta)$ *is given by* (5.21).

Remark 5.2. *The noncompact analogue of Theorem 5.3, i.e., the case of the real hyperbolic case* $\mathbf{H}^n$ *will be discussed in Chapter 6* (*precisely Theorem 6.5*).

The Green Function on the Sphere $\mathbf{S}^n$. The Green function is an important spectral function for studying the spectrum of a Riemannian manifold. The Green function $G_{\mathbf{S}^n}(\theta;k)$ of $\Delta_{\mathbf{S}^n}$ on $\mathbf{S}^n$ is the kernel of the resolvent operator $(\Delta_{\mathbf{S}^n} - k(k+n-1))^{-1}$, $k \ge 0$, for which $\lambda = k(k+n-1)$ is not the eigenvalue of $\Delta_{\mathbf{S}^n}$. Here we compute the Green function on $\mathbf{S}^n$ using the Laplace transform approach. That is we obtain the Green function on $\mathbf{S}^n$ by taking the Laplace transform of the heat kernel on $\mathbf{S}^n$; this gives fractional and integral representations of the Green function on $\mathbf{S}^n$ in the form of the heat kernel on $\mathbf{S}^n$ given by Theorem 5.3. We compute according to whether n is odd or even.

- n odd, $n \geq 1$. Indeed from (5.26) we have

$$\begin{aligned} G_{\mathbf{S}^n}(\theta; k) &= \int_0^\infty e^{-\lambda t} K_{\mathbf{S}^n}(t,\theta)\, dt \\ &= \frac{1}{\sqrt{4\pi}} \left(-\frac{1}{2\pi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{\frac{n-1}{2}} \sum_{m=-\infty}^{\infty} \int_0^\infty e^{-\left(k+\frac{n-1}{2}\right)^2 t} e^{-\frac{(\theta+2m\pi)^2}{4t}} t^{-1/2}\, dt \\ &= \frac{1}{2\left(k+\frac{n-1}{2}\right)} \left(-\frac{1}{2\pi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{\frac{n-1}{2}} \eta_1(\theta; k), \end{aligned} \tag{5.28}$$

  where

$$\eta_1(\theta; k) = \sum_{m=-\infty}^{\infty} e^{-(2m\pi+\theta)\left(k+\frac{n-1}{2}\right)}.$$

  Here we have used (B.49) and (B.47).

- n even, $n \geq 2$. From (5.27), (B.49) and (B.47) we obtain

$$\begin{aligned} G_{\mathbf{S}^n}(\theta; k) &= \int_0^\infty e^{-\lambda t} K_{\mathbf{S}^n}(t,\theta)\, dt \\ &= \left(-\frac{1}{2\pi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{\frac{n-2}{2}} \int_0^\infty e^{-k(k+n-1)t} e^{-\frac{n(n-2)t}{4}} K_{\mathbf{S}^2}(t,\theta)\, dt \\ &\times \int_0^\infty t^{-3/2} e^{-\left(k+\frac{n-1}{2}\right)^2 t} e^{-\frac{(\theta+2m\pi)^2}{4t}}\, dt d\vartheta \\ &= \frac{1}{2\pi} \left(-\frac{1}{2\pi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{\frac{n-2}{2}} \frac{1}{\sqrt{2}} \int_\theta^\pi \frac{\eta_2(\vartheta; k)}{\sqrt{\cos\theta - \cos\vartheta}} d\vartheta, \end{aligned} \tag{5.29}$$

  where

$$\eta_2(\theta; k) = \sum_{m=-\infty}^{\infty} (-1)^m e^{-(2m\pi+\theta)\left(k+\frac{n-1}{2}\right)}.$$

The Heat Kernel on the Real Projective Space $\mathbf{RP}^n$. The real projective space $\mathbf{RP}^n$ is a compact smooth n-dimensional manifold and again a symmetric space of rank one in view of the characterisation

$$\mathbf{RP}^n = \mathbf{S}^n / \{\pm I\} = \mathbf{SO}(n+1)/\mathbf{O}(n), \tag{5.30}$$

where $\mathbf{O}(n)$ denotes the orthogonal groups of real $n \times n$ matrices (see, e.g., Berger et al. [26]). As we have mentioned earlier that since $\mathbf{RP}^n$ is obtained from $\mathbf{S}^n$ by identifying the antipodal points, we will briefly discuss the heat kernel on $\mathbf{RP}^n$. If $n = 1$, $\mathbf{RP}^n$ reduces to the circle $\mathbf{S}^1$. In view of the double covering description of $\mathbf{RP}^n$ and using the theory of Riemannian coverings it is not difficult to see that the eigenfunctions of $\mathbf{RP}^n$ are only those descending from the cover $\mathbf{S}^n$ whose degree of homogeneity is even (see, e.g., Berger et al. [26]). As a consequence the eigenvalues of $\Delta_{\mathbf{RP}^n}$ are in turn given by $\lambda_{2k} = 2k(2k+n-1)$, $k \geq 0$, with the multiplicity

$$M_{2k}^n = \frac{(2k+n-2)!}{(n-1)!(2k)!}(n+4k-1), \qquad k \geq 0. \tag{5.31}$$

Now since $\mathbf{RP}^n$ is obtained from $\mathbf{S}^n$ by identifying the antipodal points, its diameter is $\frac{\pi}{2}$ and its volume is

$$\mathrm{Vol}(\mathbf{RP}^n) = \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} = \frac{\nu_n}{2}. \tag{5.32}$$

Moreover the radial part of the Laplacian on $\mathbf{RP}^n$ coincides with that on $\mathbf{S}^n$ and subsequently the heat kernels on $\mathbf{S}^n$ and $\mathbf{RP}^n$ are related to one-another by

$$K_{\mathbf{RP}^n}(t,\theta) = K_{\mathbf{S}^n}(t,\theta) + K_{\mathbf{S}^n}(t,\pi-\theta). \tag{5.33}$$

Using the identity (B.131) the spherical function on $\mathbf{S}^n$ takes the form

$$\Phi_k^{\mathbf{S}^n}(\theta) = \frac{k!\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(k+\frac{n}{2}\right)} P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(\cos\theta),$$

and upon noting that

$$\Phi_k^{\mathbf{S}^n}(\pi-\theta) = \frac{k!\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(k+\frac{n}{2}\right)} P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(\cos(\pi-\theta)) = \frac{k!\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(k+\frac{n}{2}\right)} P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(-\cos\theta),$$

and using the identity (B.133) of the Jacobi polynomials we have

$$\Phi_k^{\mathbf{S}^n}(\pi-\theta) = (-1)^k \Phi_k^{\mathbf{S}^n}(\theta).$$

As a result from (5.33) we obtain

$$\begin{aligned} K_{\mathbf{RP}^n}(t,\theta) &= \frac{1}{\nu_n}\sum_{k=0}^{\infty} M_k^n [1+(-1)^k]\Phi_k^{\mathbf{S}^n}(\theta) e^{-k(k+n-1)t} \\ &= \frac{e^{\frac{(n-1)^2}{4}t}}{\mathrm{Vol}(\mathbf{RP}^n)} \sum_{k=0}^{\infty} M_{2k}^n \Phi_{2k}^{\mathbf{S}^n}(\theta) e^{-\left(2k+\frac{n-1}{2}\right)^2 t}. \end{aligned} \tag{5.34}$$

In summary we have the following statement.

Proposition 5.4. *For $t>0$, $0\le\theta<\pi$. The heat kernel $K_{\mathbf{RP}^n}(t,\theta)$ associated to the Laplacian on the real projective space $\mathbf{RP}^n$ is given by the following series representation:*

$$K_{\mathbf{RP}^n}(t,\theta) = \frac{2e^{\frac{(n-1)^2 t}{4}}}{\nu_n}\sum_{k=0}^{\infty} M_{2k}^n e^{-\left(2k+\frac{n-1}{2}\right)^2 t} \mathscr{C}_{2k}^{\frac{n-1}{2}}(\cos\theta), \tag{5.35}$$

with the spherical functions $\Phi_{2k}^{\mathbf{S}^n}(\theta) = \mathscr{C}_{2k}^{\frac{n-1}{2}}(\theta)$ satisfying

$$\left(\frac{\partial^2}{\partial\theta^2} + (n-1)\cot\theta\frac{\partial}{\partial\theta}\right)\Phi_{2k}^{\mathbf{S}^n}(\theta) = -2k(2k+n-1)\Phi_{2k}^{\mathbf{S}^n}(\theta) \tag{5.36}$$

$$\Phi_{2k}^{\mathbf{S}^n}(0) = 1. \tag{5.37}$$

5.3 Minakshisundaram-Pleijel Heat Coefficients

We calculate the heat coefficients a_k^n appearing in the Minakshisundaram-Pleijel asymptotic expansion (A.8) for the special case of the sphere $M=\mathbf{S}^n$. We first express the trace of the heat operator on $\mathbf{S}^n$ purely in terms of Jacobi's theta functions and their higher order derivatives, and then take the asymptotic of the Jacobi function as $t\searrow 0$; thereafter the heat coefficients follow after some mathematical rearrangements.

Indeed, from Theorem 5.3 we have

$$\operatorname{tr} e^{-t\Delta_{\mathbf{S}^n}} = e^{\frac{(n-1)^2 t}{4}} \sum_{k=0}^{\infty} \frac{(k+n-2)!}{(n-1)!k!}(n+2k-1)e^{-\left(k+\frac{n-1}{2}\right)^2 t}, \ t>0, \tag{5.38}$$

where we have taken the integral of the heat kernel $K_{\mathbf{S}^n}(t,\theta)$ at $\theta = 0$. As a starting point of the computation of the Minakshisundaram-Pleijel heat coefficients a_k^n for $\mathbf{S}^n$, we shall write the multiplicity M_k^n as a product of the form

$$M_k^n = \frac{(k+n-2)!}{(n-1)!k!}(n+2k-1) = \frac{2k+n-1}{(n-1)!}\prod_{j=1}^{n-2}(k+j). \tag{5.39}$$

We shall compute according to whether n is odd or even. We first consider the case of n odd, $n \geq 3$.

(a) Odd $n \geq 3$. Writing the multiplicity M_k^n in a polynomial form gives

$$\begin{aligned} M_k^n &= \frac{(k+n-2)!}{(n-1)!k!}(n+2k-1) = \frac{2k+n-1}{(n-1)!}\prod_{j=1}^{n-2}(k+j) \\ &= \frac{2}{(n-1)!}\prod_{j=0}^{r}\left[\left(k+\frac{n-1}{2}\right)^2 - j^2\right], \qquad r=(n-3)/2 \\ &= \frac{2}{(n-1)!}\sum_{m=0}^{r}\mathtt{C}_{m,n}\left(k+\frac{n-1}{2}\right)^{2m+2}, \end{aligned} \tag{5.40}$$

where the integers $(\mathtt{C}_{m,n})$ are the coefficients of the polynomial

$$\prod_{j=0}^{r}(x^2-j^2) = \sum_{m=0}^{r}\mathtt{C}_{m,n}x^{2m+2}. \tag{5.41}$$

We note that the first few coefficients $(\mathtt{C}_{m,n} : 0 \leq m \leq r)$ are given in Table 5.1 below.

Table 5.1: The coefficients $(\mathtt{C}_{m,n} : 0 \leq m \leq r)$.

$\mathtt{C}_{0,3}$	$\mathtt{C}_{0,5}$	$\mathtt{C}_{1,5}$	$\mathtt{C}_{0,7}$	$\mathtt{C}_{1,7}$	$\mathtt{C}_{2,7}$	$\mathtt{C}_{0,9}$	$\mathtt{C}_{1,9}$	$\mathtt{C}_{2,9}$	$\mathtt{C}_{3,9}$
1	-1	1	4	-5	1	-36	49	-14	1

Inserting the multiplicity M_k^n given by (5.40) into the heat trace formula (5.38) we obtain another form of the heat trace formula, namely,

$$\begin{aligned} \operatorname{tr} e^{-t\Delta_{\mathbf{S}^n}} &= \frac{2e^{\frac{(n-1)^2}{4}t}}{(n-1)!}\sum_{k=0}^{\infty}\sum_{m=0}^{r}\mathtt{C}_{m,n}\left(k+\frac{n-1}{2}\right)^{2m+2} e^{-\left(k+\frac{n-1}{2}\right)^2 t} \\ &= \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!}\sum_{m=0}^{r}\mathtt{C}_{m,n}\sum_{p=(n-1)/2}^{\infty} 2p^{2m+2}e^{-p^2 t} \qquad \left(p = k+\frac{n-1}{2}\right) \\ &= \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!}\sum_{m=0}^{r}(-1)^{m+1}\mathtt{C}_{m,n}\vartheta_1^{(m+1)}(t). \end{aligned} \tag{5.42}$$

Here $\vartheta_1^{(m+1)}$ (with $m \geq 0$) are the derivatives of the Jacobi theta function $\vartheta_1 = \vartheta_1(t)$ defined for $t > 0$ by

$$\vartheta_1(t) = \sum_{j=-\infty}^{\infty} e^{-j^2 t}. \tag{5.43}$$

In the calculations below we need the asymptotics of $\vartheta_1(t)$ and its derivatives $\vartheta_1^{(m+1)}(t)$ as $t \searrow 0$. Towards this end let us first note that as a result of the Poisson summation formula the theta function satisfies the inversion formula

$$\vartheta_1(t) = \sum_{j=-\infty}^{\infty} e^{-j^2 t} = \sqrt{\frac{\pi}{t}} \sum_{j=-\infty}^{\infty} e^{-j^2 \frac{\pi^2}{t}}, \tag{5.44}$$

and hence the asymptotics

$$\vartheta_1(t) \sim \sqrt{\frac{\pi}{t}} \quad \text{as} \quad t \searrow 0; \qquad \vartheta_1'(t) \sim -\frac{1}{2}\pi^{\frac{1}{2}} t^{-\frac{3}{2}}, \quad \text{as} \quad t \searrow 0;$$

$$\vartheta_1(t) \sim 1 \quad \text{as} \quad t \nearrow \infty; \qquad \vartheta_1'(t) \sim 0, \quad \text{as} \quad t \nearrow \infty.$$

The asymptotics of the higher order derivatives of ϑ_1 are analogous and will be given below. Now we return to the heat trace (5.38) and its formulation via the theta function given above. In order to motivate the discussion and to illustrate the main ideas it is helpful to start by considering the cases $3 \leq n \leq 7$ before moving to the general case. Notice that the case $n = 1$ reduces to what is already given in (5.43).

- $(n = 3)$ Here by referring to (5.38) we can write

$$\begin{aligned} \operatorname{tr} e^{-t\Delta_{\mathbf{S}^3}} &= \sum_{k=0}^{\infty} (k+1)^2 e^{-k(k+2)t} = \sum_{p=1}^{\infty} p^2 e^{-(p^2-1)t} = e^t \sum_{p=1}^{\infty} p^2 e^{-p^2 t} \\ &= -\frac{e^t}{2} \frac{d}{dt} \sum_{p=0}^{\infty} 2e^{-p^2 t} = -\frac{e^t}{2} \vartheta_1'(t), \end{aligned} \tag{5.45}$$

where a straightforward differentiation gives

$$\vartheta_1'(t) = -\frac{1}{2}\pi^{\frac{1}{2}} t^{-\frac{3}{2}} \sum_{j=-\infty}^{\infty} e^{-j^2 \frac{\pi^2}{t}} + \pi^{\frac{5}{2}} t^{-\frac{5}{2}} \sum_{j=-\infty}^{\infty} j^2 e^{-j^2 \frac{\pi^2}{t}} \sim -\frac{1}{2}\pi^{\frac{1}{2}} t^{-\frac{3}{2}}.$$

Thus as $t \searrow 0$ we obtain

$$\operatorname{tr} e^{-t\Delta_{\mathbf{S}^3}} \sim 2^{-2} \sqrt{\pi} t^{-\frac{3}{2}} e^t = \frac{2\pi^2 e^t}{(4\pi t)^{\frac{3}{2}}} \implies a_k^3 = \frac{2\pi^2}{k!} \qquad k \geq 0. \tag{5.46}$$

On the other hand,

$$K_{\mathbf{S}^3}(t, \theta) = -\frac{1}{\sqrt{4\pi t}} \frac{e^t}{2\pi} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sum_{m=-\infty}^{\infty} e^{-\frac{(\theta - 2\pi m)^2}{4t}}. \tag{5.47}$$

Thus,

$$\begin{aligned} \operatorname{tr} e^{-t\Delta_{\mathbf{S}^3}} &= -\frac{2\pi^2}{\sqrt{4\pi t}} e^t \frac{1}{2\pi} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} e^{-\frac{\theta^2}{4t}} |_{\theta=0} + O\left(t^\infty\right) \\ &= \frac{2\pi^2}{\sqrt{4\pi t} 2\pi} \frac{e^t}{\sin\theta} \frac{\theta}{2t} e^{-\frac{\theta^2}{4t}} |_{\theta=0} + O\left(t^\infty\right) \\ &\sim \frac{2\pi^2}{(4\pi t)^{\frac{3}{2}}} e^t \qquad \text{as } t \searrow 0 \end{aligned} \tag{5.48}$$

(compare with (5.46)).

- ($n = 5$) Indeed from the trace formula (5.38) we have

$$\begin{aligned} \operatorname{tr} e^{-t\Delta_{\mathbf{S}^5}} &= \frac{1}{12} \sum_{k=0}^{\infty} (k+1)(k+2)^2(k+3) e^{-k(k+4)t} \\ &= \frac{e^{4t}}{4!} \left[\frac{d^2}{dt^2} \sum_{p=0}^{\infty} 2e^{-p^2 t} + \frac{d}{dt} \sum_{p=0}^{\infty} 2e^{-p^2 t} \right] \quad (p = k+2) \\ &= \frac{e^{4t}}{4!} \left[\vartheta_1''(t) + \vartheta_1'(t) \right], \end{aligned} \tag{5.49}$$

where a further differentiation of ϑ_1 gives

$$\vartheta_1''(t) \sim \frac{3}{4} \pi^{\frac{1}{2}} t^{-\frac{5}{2}}.$$

Thus it is plain that as $t \searrow 0$ we have

$$\begin{aligned} \operatorname{tr} e^{-t\Delta_{\mathbf{S}^5}} &\sim \frac{e^{4t}}{24} \left[\frac{3}{4} \pi^{\frac{1}{2}} t^{-\frac{5}{2}} - \frac{1}{2} \pi^{\frac{1}{2}} t^{-\frac{3}{2}} \right] \\ &= \frac{\pi^3}{(4\pi t)^{\frac{5}{2}}} \left[1 + \sum_{k=1}^{\infty} \left(\frac{4^k}{k!} - \frac{2}{3} \frac{4^{k-1}}{(k-1)!} \right) t^k \right] \\ \Longrightarrow a_0^5 = \pi^3, &\qquad a_k^5 = \pi^3 \left(\frac{4^k}{k!} - \frac{2}{3} \frac{4^{k-1}}{(k-1)!} \right) \qquad k \geq 1. \end{aligned} \tag{5.50}$$

- ($n = 7$) Here the heat trace can be written as

$$\begin{aligned} \operatorname{tr} e^{-t\Delta_{\mathbf{S}^7}} &= \frac{2}{6!} \sum_{k=0}^{\infty} (k+1)(k+2)(k+3)^2(k+4)(k+5) e^{-k(k+6)t} \\ &= \frac{e^{9t}}{6!} \left[-\frac{d^3}{dt^3} \sum_{p=0}^{\infty} 2e^{-p^2 t} - 5\frac{d^2}{dt^2} \sum_{p=0}^{\infty} 2e^{-p^2 t} - 4\frac{d^2}{dt^2} \sum_{p=0}^{\infty} 2e^{-p^2 t} \right] \\ &\qquad (p = k+3) \\ &= \frac{e^{9t}}{6!} \left[-\vartheta_1'''(t) - 5\vartheta_1''(t) - 4\vartheta_1'(t) \right]. \end{aligned} \tag{5.51}$$

Now similar to what was seen before a further differentiation of ϑ_1 gives

$$\vartheta_1'''(t) \sim -\frac{1 \cdot 3 \cdot 5}{8} \pi^{\frac{1}{2}} t^{-\frac{7}{2}} \qquad \text{as} \qquad t \searrow 0. \tag{5.52}$$

Thus as $t \searrow 0$ we obtain

$$\operatorname{tr} e^{-t\Delta_{\mathbf{S}^7}} \sim \frac{e^{9t}}{360}\left[\frac{15}{16}\pi^{\frac{1}{2}}t^{-\frac{7}{2}} - \frac{15}{8}\pi^{\frac{1}{2}}t^{-\frac{5}{2}} + \pi^{\frac{1}{2}}t^{-\frac{3}{2}}\right] =$$
$$= \frac{\pi^4/3}{(4\pi t)^{\frac{7}{2}}}\left[1 + 7t + \sum_{k=2}^{\infty}\left(\frac{9^k}{k!} - \frac{2\cdot 9^{k-1}}{(k-1)!} + \frac{16}{15}\frac{9^{k-2}}{(k-2)!}\right)t^k\right]$$
$$\Longrightarrow a_0^7 = \frac{\pi^4}{3},\ a_1^7 = \frac{7\pi^4}{3},\ a_k^7 = \frac{\pi^4}{3}\left(\frac{9^k}{k!} - \frac{2\cdot 9^{k-1}}{(k-1)!} + \frac{16}{15}\frac{9^{k-2}}{(k-2)!}\right)\ k \geq 2. \quad (5.53)$$

Having computed the asymptotics of the heat trace on $\mathbf{S}^n$ for the special cases $3 \leq n \leq 7$ we are now ready to move on to the general case (5.42) for n odd, $n \geq 3$. First by invoking the Leibniz rule and a basic induction argument the asymptotics of the higher order derivatives of ϑ_1 as $t \searrow 0$ are seen to be

$$\vartheta_1^{(m+1)}(t) \sim (-1)^{m+1}\frac{(2m+1)!!\pi^{\frac{1}{2}}t^{-m-\frac{3}{2}}}{2^{m+1}}, \qquad m \geq 0, \quad (5.54)$$

where the double factorial $k!!$ is as defined in (4.86). Substituting the asymptotic formula (5.54) into (5.42) we obtain

$$\operatorname{tr} e^{-t\Delta_{\mathbf{S}^n}} \sim \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!}\frac{(n-2)!!\sqrt{2\pi}}{(2t)^{\frac{n}{2}}}\left[1 + \frac{1}{(n-2)!!}\sum_{m=0}^{r-1}\mathtt{C}_{m,n}\frac{(2m+1)!!}{(2t)^{m-\frac{n-3}{2}}}\right]. \quad (5.55)$$

Using the identities

$$(n-2)!! = \frac{2^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}}, \qquad \frac{1}{(n-1)!} = \frac{\nu_n}{\Gamma\left(\frac{n}{2}\right)(4\pi)^{\frac{n}{2}}}$$

we see that

$$\frac{(n-2)!!\sqrt{2\pi}}{(2t)^{\frac{n}{2}}}\frac{2^{\frac{n-1}{2}}\nu_n}{\sqrt{\pi}(4\pi)^{\frac{n}{2}}} = \frac{\nu_n}{(4\pi t)^{\frac{n}{2}}}.$$

Hence, as $t \searrow 0$ we obtain the heat trace asymptotic formula

$$\operatorname{tr} e^{-t\Delta_{\mathbf{S}^n}} \sim \frac{\nu_n e^{\frac{(n-1)^2}{4}t}}{(4\pi t)^{\frac{n}{2}}}\left[1 + \frac{1}{(n-2)!!}\sum_{m=0}^{r-1}\mathtt{C}_{m,n}\frac{(2m+1)!!}{(2t)^{m-r}}\right]. \quad (5.56)$$

The explicit heat coefficients a_k^n can then be derived from the heat trace asymptotic formula (5.56). Thus we obtain

$$a_k^n = \frac{\nu_n}{(n-2)!!}\sum_{j=0}^{r}\frac{\left(\frac{n-1}{2}\right)^{2k-2j}}{(k-j)!}\mathtt{C}_{r-j,n}(n-2-2j)!!2^j, \ \ k \geq 0, \quad (5.57)$$

with $1/(-m!) = 0$ for $m \in \mathbf{N}$.

(b) Even $n \geq 2$. We next consider the case of the even dimensional sphere $\mathbf{S}^n$, $n \geq 2$. The computations of spectral invariants on even dimensional Riemannian manifolds are complicated compared with the case of the odd dimensional ones (see e.g., Taylor [166], Polterovich [131], Awonusika and Taheri [8, 15]). We shall see shortly that the heat coefficients associated with the even dimensional sphere $\mathbf{S}^n$ are complicated to handle because of the Jacobi's theta function involved, which requires one to work with summation containing the Bernoulli numbers before the asymptotics for small t can be obtained.

Towards this end we note that the multiplicity M_k^n admits the polynomial representation

$$\begin{aligned} M_k^n &= \frac{(k+n-2)!}{(n-1)!k!}(n+2k-1) = \frac{2k+n-1}{n-1}\prod_{j=1}^{n-2}\frac{k+j}{j} \\ &= \frac{2\left(k+\frac{n-1}{2}\right)}{(n-1)!}\prod_{j=1/2}^{\frac{n-3}{2}}\left[\left(k+\frac{n-1}{2}\right)^2 - j^2\right] \\ &= \frac{2}{(n-1)!}\sum_{m=0}^{r}\mathtt{D}_{m,n}\left(k+\frac{n-1}{2}\right)^{2m+1}, \qquad r=(n-2)/2, \end{aligned} \tag{5.58}$$

where the coefficients $(\mathtt{D}_{m,n} : 0 \le m \le r)$ are defined by the polynomial

$$\prod_{j=1/2}^{\frac{n-3}{2}}\left(x^2 - j^2\right) = \sum_{m=0}^{r}\mathtt{D}_{m,n}x^{2m}.$$

Here again the first few coefficients $(\mathtt{D}_{m,n} : 0 \le m \le r)$ are given in Table 5.2 below.

TABLE 5.2: The coefficients $(\mathtt{D}_{m,n} : 0 \le m \le r)$.

$\mathtt{D}_{0,2}$	$\mathtt{D}_{0,4}$	$\mathtt{D}_{1,4}$	$\mathtt{D}_{0,6}$	$\mathtt{D}_{1,6}$	$\mathtt{D}_{2,6}$	$\mathtt{D}_{0,8}$	$\mathtt{D}_{1,8}$	$\mathtt{D}_{2,8}$	$\mathtt{D}_{3,8}$
1	$-1/4$	1	$9/16$	$-5/2$	1	$-225/64$	$259/16$	$-34/5$	1

Thus we obtain

$$\begin{aligned} \operatorname{tr} e^{-t\Delta_{\mathbf{S}^n}} &= \frac{2e^{\frac{(n-1)^2}{4}t}}{(n-1)!}\sum_{k=0}^{\infty}\sum_{m=0}^{r}\mathtt{D}_{m,n}\left(k+\frac{n-1}{2}\right)^{2m+1}e^{-\left(k+\frac{n-1}{2}\right)^2 t} \\ &= \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!}\sum_{m=0}^{r}\mathtt{D}_{m,n}2\sum_{p=(n-1)/2}^{\infty}p^{2m+1}e^{-p^2 t} \qquad \left(p=k+\frac{n-1}{2}\right) \\ &= \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!}\sum_{m=0}^{r}(-1)^m\mathtt{D}_{m,n}\vartheta_2^{(m)}(t), \end{aligned} \tag{5.59}$$

where

$$\vartheta_2(t) = \sum_{m=0}^{\infty}(2m+1)e^{-\left(m+\frac{1}{2}\right)^2 t} \backsimeq \frac{1}{t} + \sum_{m=0}^{\infty}\mathtt{B}_m\frac{t^m}{m!} \qquad \text{(see e.g., Mulholland [117]).} \tag{5.60}$$

Here $\mathtt{B}_m$ is expressed in terms of the mth Bernoulli number B_m (see Appendix B.2) and it is given by

$$\mathtt{B}_m = \frac{(-1)^m}{(m+1)}B_{2m+2}\left(1-2^{-2m-1}\right) \qquad m \ge 0. \tag{5.61}$$

Similar to what was done in the case of odd n, here we also examine the asymptotics of the heat trace (5.38) for the special cases $2 \le n \le 6$ before moving on to the general case $n \ge 2$.

- ($n = 2$) We have

$$
\begin{aligned}
\operatorname{tr} e^{-t\Delta_{\mathbf{S}^2}} &= e^{\frac{t}{4}} \sum_{k=0}^{\infty} (2k+1) e^{-\left(k+\frac{1}{2}\right)^2 t} = e^{\frac{t}{4}} \vartheta_2(t) \\
&= \left(1 + \frac{t}{4} + \frac{t^2}{16 \cdot 2!} + \cdots\right)\left(\frac{1}{t} + \frac{1}{12} + \frac{7}{480} t + \cdots\right) \\
&= \frac{\nu_2}{4\pi t}\left(1 + \frac{t}{3} + \frac{t^2}{15} + \frac{t^3}{160} + O\left(t^4\right)\right) \quad \text{as } t \searrow 0.
\end{aligned} \tag{5.62}
$$

Thus in terms of the mth Bernoulli number B_m we have

$$
\operatorname{tr} e^{-t\Delta_{\mathbf{S}^2}} \backsimeq \frac{\nu_2 e^{\frac{t}{4}}}{4\pi t}\left[1 + \sum_{m=1}^{\infty} \frac{\mathtt{B}_{m-1} t^m}{(m-1)!}\right], \tag{5.63}
$$

where the associated heat coefficients a_k^2 are given by

$$
a_k^2 = 4\pi \sum_{j=0}^{k} \frac{\left(\frac{1}{4}\right)^{k-j}}{(k-j)!} \frac{\mathtt{B}_{j-1}}{(j-1)!}, \quad k \geq 0. \tag{5.64}
$$

On the other hand the trace of the heat kernel on $\mathbf{S}^2$ admits the integral representation over closed geodesics (see e.g., Taylor [166, pp. 113-116]), and by considering its principal value we have

$$
\begin{aligned}
\operatorname{tr} e^{-t\Delta_{\mathbf{S}^2}} &\backsimeq \frac{\nu_2 e^{\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}} \int_0^{\pi} \frac{\varphi e^{-\frac{\varphi^2}{4t}}}{\sin\frac{\varphi}{2}}\, d\varphi \\
&= \frac{\nu_2 e^{\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}} \int_0^{\pi} \sum_{m=0}^{\infty} (-1)^m B_{2m}\left(2^{1-2m} - 1\right) \frac{\varphi^{2m}}{(2m)!} e^{-\frac{\varphi^2}{4t}}\, d\varphi \\
&= t^{-1} e^{\frac{t}{4}} \sum_{m=0}^{\infty} (-1)^m B_{2m}\left(2^{1-2m} - 1\right) \frac{t^m}{m!} \\
&= \frac{\nu_2}{4\pi t}\left(1 + \frac{t}{3} + \frac{t^2}{15} + O\left(t^3\right)\right) \quad \text{as } t \searrow 0
\end{aligned} \tag{5.65}
$$

(compare with (5.62)).

- ($n = 4$) Let us start by noting that in general for $n \geq 4$ even we write the series $\vartheta_2(t)$ in (5.60) in the more suggestive form

$$
\vartheta_2(t) = 2 \sum_{p=1/2}^{\infty} p e^{-p^2 t} \qquad \left(m + \frac{1}{2} = p\right). \tag{5.66}
$$

Indeed from the heat trace (5.38) we can write

$$
\begin{aligned}
\operatorname{tr} e^{-t\Delta_{\mathbf{S}^4}} &= \frac{e^{\frac{9t}{4}}}{3} \sum_{k=0}^{\infty} (k+1)(k+2)\left(k + \frac{3}{2}\right) e^{-\left(k+\frac{3}{2}\right)^2 t} \\
&= \frac{e^{\frac{9t}{4}}}{6}\left[-\frac{d}{dt} 2 \sum_{p=1/2}^{\infty} p e^{-p^2 t} - \frac{1}{4} 2 \sum_{p=1/2}^{\infty} p e^{-p^2 t}\right] \quad \left(p = k + \frac{3}{2}\right) \\
&= \frac{e^{\frac{9t}{4}}}{6}\left[-\vartheta_2'(t) - \frac{1}{4}\vartheta_2(t)\right],
\end{aligned} \tag{5.67}
$$

where

$$\vartheta_2'(t) \backsimeq -\frac{1}{t^2} + \sum_{m=1}^{\infty} \mathtt{B}_m \frac{t^{m-1}}{(m-1)!}. \tag{5.68}$$

Inserting the theta functions (5.60) and (5.68) into (5.67) we have

$$\operatorname{tr} e^{-t\Delta_{\mathbf{S}^4}} = \frac{\nu_4 e^{\frac{9t}{4}}}{(4\pi t)^2}\left[1 - \frac{t}{4} - \sum_{m=2}^{\infty} u_m^4 t^m\right], \tag{5.69}$$

where

$$u_0^4 = 1,\ u_1^4 = -\frac{1}{4},\ u_k^4 = -\frac{1}{(k-2)!}\left(\mathtt{B}_{k-1} + \frac{1}{4}\mathtt{B}_{k-2}\right),\ k \geq 2. \tag{5.70}$$

Thus we obtain the Minakshisundaram-Pleijel heat coefficients $a_k^4 = a_k^4(\mathbf{S}^4)$ given by

$$a_k^4 = \nu_4 \sum_{j=0}^{k} \frac{\left(\frac{9}{4}\right)^{k-j}}{(k-j)!} u_j^4, \qquad k \geq 0. \tag{5.71}$$

- $(n = 6)$ Here we can write

$$\begin{aligned}
\operatorname{tr} e^{-t\Delta_{\mathbf{S}^6}} &= \frac{e^{\frac{25t}{4}}}{60} \sum_{k=0}^{\infty} (k+1)(k+2)(k+3)(k+4)\left(k+\frac{5}{2}\right) e^{-\left(k+\frac{5}{2}\right)^2 t} \\
&= \frac{e^{\frac{25t}{4}}}{60}\left[\sum_{p=1/2}^{\infty} p^5 e^{-p^2 t} - \frac{5}{2}\sum_{p=1/2}^{\infty} p^3 e^{-p^2 t} + \frac{9}{16}\sum_{p=1/2}^{\infty} p e^{-p^2 t}\right] \\
&\qquad \left(p = k + \frac{5}{2}\right) \\
&= \frac{e^{\frac{25t}{4}}}{120}\left[\vartheta_2''(t) + \frac{5}{2}\vartheta_2'(t) + \frac{9}{16}\vartheta_2(t)\right],
\end{aligned} \tag{5.72}$$

where

$$\vartheta_2''(t) \backsimeq \frac{2}{t^3} + \sum_{m=2}^{\infty} \frac{\mathtt{B}_m t^{m-2}}{(m-2)!}.$$

With these asymptotic expansions of the Jacobi's theta function ϑ_2 we obtain the heat trace asymptotics

$$\operatorname{tr} e^{-t\Delta_{\mathbf{S}^6}} = \frac{\nu_6 e^{\frac{25t}{4}}}{(4\pi t)^3}\left[1 - \frac{5t}{4} + \frac{9t^2}{32} + \sum_{m=3}^{\infty} u_m^6 t^m\right], \tag{5.73}$$

where

$$u_0^6 = 1,\ u_1^6 = -\frac{5}{4},\ u_2^6 = \frac{9}{32},\ u_k^6 = \frac{1/2}{(k-3)!}\left(\mathtt{B}_{k-1} + \frac{5}{2}\mathtt{B}_{k-2} + \frac{9}{16}\mathtt{B}_{k-3}\right),\ k \geq 3. \tag{5.74}$$

Thus the Minakshisundaram-Pleijel heat coefficients $a_k^6 = a_k^6(\mathbf{S}^6)$ are given by

$$a_k^6 = \nu_6 \sum_{j=0}^{k} \frac{\left(\frac{25}{4}\right)^{k-j}}{(k-j)!} u_j^6, \qquad k \geq 0. \tag{5.75}$$

Motivated by the asymptotics of the derivatives $\vartheta_2^{(\ell)}(t)$, $(\ell \geq 1)$ of the theta function $\vartheta_2(t)$ we give an asymptotic formula for the derivatives $\vartheta_2^{(\ell)}(t)$, $\ell \geq 1$, namely,

$$\vartheta_2^{(\ell)}(t) \backsimeq \frac{(-1)^\ell \ell!}{t^{1+\ell}} + \sum_{m=\ell}^{\infty} \frac{\mathtt{B}_m t^{m-\ell}}{(m-\ell)!}. \tag{5.76}$$

So from the asymptotic formulae (5.59) and (5.76) we obtain the following asymptotic expansion for the heat trace formula (5.59) associated with the even dimensional sphere $\mathbf{S}^n$:

$$\begin{aligned}
\operatorname{tr} e^{-t\Delta_{\mathbf{S}^n}} &\backsimeq \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!} \frac{r!}{t^{\frac{n}{2}}} \left\{ 1 + \frac{1}{r!} \left[\sum_{m=0}^{r-1} \mathtt{D}_{m,n} m! t^{r-m} \right.\right. \\
&\quad \left.\left. + \sum_{m=0}^{r} \widetilde{\mathtt{D}}_{m,n} \sum_{\ell=m}^{\infty} \frac{\mathtt{B}_\ell t^{\frac{n}{2}+\ell-m}}{\Gamma(\ell-m+1)} \right] \right\} \\
&= \frac{e^{\frac{(n-1)^2}{4}t}}{(4\pi t)^{\frac{n}{2}}} \left[1 + \frac{1}{r!} \left[\sum_{m=0}^{r-1} \mathtt{D}_{m,n} m! t^{r-m} + \sum_{m=0}^{r} \widetilde{\mathtt{D}}_{m,n} \sum_{\ell=m}^{\infty} \frac{\mathtt{B}_\ell t^{\frac{n}{2}+\ell-m}}{\Gamma(\ell-m+1)} \right] \right],
\end{aligned} \tag{5.77}$$

where we have set $\widetilde{\mathtt{D}}_{m,n} = (-1)^m \mathtt{D}_{m,n}$. By the series expansion of the exponential function we finally obtain

$$\begin{aligned}
a_k^n &= \frac{\nu_n}{\Gamma(r+1)} \sum_{j=0}^{k} \frac{(2r+1)^{2k-2j}}{4^{k-j}(k-j)!} \mathtt{D}_{r-j,n} \Gamma(r-j+1), \quad 0 \leq k \leq r; \\
a_k^n &= \frac{\nu_n}{\Gamma(r+1)} \sum_{j=0}^{k} \frac{(2r+1)^{2k-2j}}{4^{k-j}(k-j)!} \sum_{\ell=0}^{r} \frac{(-1)^\ell \mathtt{D}_{\ell,n}}{(j-r-1)!} \mathtt{B}_{j+\ell-r-1}, \quad k \geq r+1.
\end{aligned} \tag{5.78}$$

Theorem 5.5. *For $t > 0$, the heat trace*

$$\operatorname{tr} e^{-t\Delta_{\mathbf{S}^n}} = \int_{\mathbf{S}^n} K_{\mathbf{S}^n}(t,0)\, d\nu_n(\zeta) = e^{\frac{(n-1)^4 t}{2}} \sum_{k=0}^{\infty} \frac{(k+n-2)!}{(n-1)!k!} (n+2k-1) e^{-\left(k+\frac{n-1}{2}\right)^2 t} \tag{5.79}$$

satisfies the Minakshisundaram-Pleijel asymptotic expansion (A.8) with the heat coefficients $a_k^n = a_k^n(\mathbf{S}^n)$ given by (5.57) and (5.78) according to whether n is odd or even.

Remark 5.3. *It follows from Theorem 5.5 that the first few heat trace coefficients for the sphere are given by*

$$a_0^n = \mathrm{Vol}(M) = \nu_n, \qquad a_1^n = \frac{1}{6} \int_M \mathrm{Scal} = \frac{n(n-1)}{6} \nu_n, \tag{5.80}$$

$$a_2^n = \frac{1}{360} \int_M \left(5\mathrm{Scal}^2 - 2|\mathrm{Ric}|^2 + 2|\mathrm{Rm}|^2 \right) = \frac{5n^2(n-1)^2 - 2n(n-1)^2 + 4n(n-1)}{360} \nu_n, \tag{5.81}$$

in agreement with the general formulae expressed using polynomials in the curvatures tensor and its derivatives (see e.g. Chavel [41, pp. 154-155]).

Remark 5.4. *In his paper Polterovich [130] gives explicit computations of the heat invariants on arbitrary Riemannian manifolds using the asymptotics of the derivatives of the resolvent approach; this allows the heat invariants to be written in terms of powers of the Laplacian and distance function on the Riemannian manifold. In Polterovich [131], the particular case of the sphere $\mathbf{S}^n$ is considered, where the asymptotics of the trace of the heat operator on $\mathbf{S}^n$ is expressed* *as*

$$\sum_{k=0}^{\infty} M_k^n e^{-\lambda_k t} \sim t^{-\frac{n}{2}} \sum_{k=0}^{\infty} a_{k,n} t^k \qquad \text{as } t \searrow 0.$$

The heat coefficient $a_{k,n}$ *of Polterovich [131] differs from ours by a constant* $(4\pi)^{-\frac{n}{2}}$*, i.e.,* $a_{k,n} = (4\pi)^{-\frac{n}{2}} a_k^n$*. For instance it is obtained that*

$$a_{k,2} = \frac{1}{k!4^{k-1}} \sum_{l=0}^{k} (-1)^l \frac{\Gamma(k+1)}{\Gamma(l+1)\Gamma(k-l+1)} B_{2l} \left(1 - 2^{2l-1}\right)$$
$$a_{k,5} = \frac{4^{k-3}(6-k)\sqrt{\pi}}{3 \cdot k!}, \quad a_{k,7} = \frac{3^{2k-6}\left(16k^2 - 286k + 1215\right)\sqrt{\pi}}{640 \cdot k!}.$$

By inspection these coefficients satisfy the equalities

$$a_{k,2} = (4\pi)^{-1} a_k^2, \quad a_{k,5} = (4\pi)^{-\frac{5}{2}} a_k^5, \quad a_{k,7} = (4\pi)^{-\frac{7}{2}} a_k^7.$$

For the particular case of $\mathbf{S}^n$*, the powers of the distance* $d(\theta) = 2 - 2\cos(\theta)$ *are expressed in terms of the Gegenbauer polynomial* $C_k^{\frac{n-1}{2}}(\cos\theta)$*, namely*

$$d(\theta)^m = 2^m (1 - \cos\theta)^m = 2^m \sum_{l=0}^{m} c_{ml} \frac{C_l^{\frac{n-1}{2}}(\cos\theta)}{C_l^{\frac{n-1}{2}}(1)},$$

where by the orthogonality of $C_l^{\frac{n-1}{2}}(\cos\theta)$*, the constant* c_{ml} *is given by*

$$c_{ml} = \frac{(-1)^l 2^m \Gamma\left(m + \frac{n}{2}\right) m!}{(m-l)!(m+l+n-1)!} \frac{(4\pi)^{\frac{n}{2}} M_l^n}{\nu_n}.$$

All these together with the result of Polterovich [130] on arbitrary compact Riemannian manifolds give

$$a_{k,n} = \sum_{m=1}^{j} \frac{2(-1)^k \Gamma\left(j + \frac{n}{2} + 1\right)}{(j-m)!(m+k)!(2m+n)!} \sum_{l=1}^{m} (-1)^l \frac{(2m+n-1)!}{(m-l)!(m+n+l-1)!} M_l^n \lambda_l^{m+k},$$

for $k \geq 1, j \geq 2k$*. After some mathematical substitutions and rearrangements with the use of combinatorial identities and some facts about Gegenbauer polynomials, he obtains the following heat coefficients on* $\mathbf{S}^n$*, according to whether* n *is odd or even:*

$$a_{k,2m+1} = \sum_{l=1}^{m} \frac{m^{2k-2m+2l}\Gamma\left(l + \frac{1}{2}\right) K_l^m}{(k-m+l)!(2m)!}, \quad m \geq 1, \tag{5.82}$$

where the coefficients K_l^m *are defined by*

$$\prod_{j=0}^{m-1} \left(y^2 - j^2\right) = \sum_{l=1}^{m} K_l^m y^{2l};$$

and

$$a_{k,2m} = \frac{1}{\Gamma(2m)} \left[\sum_{l=0}^{m-1} \frac{\Gamma(m-l)}{\Gamma(k-l+1)} \left(m - \frac{1}{2}\right)^{2k-2l} L_l^m + \right.$$
$$\left. + \sum_{l=0}^{m-1} L_l^m \sum_{q=m-l}^{k-l} (-1)^{q+m-l-1} \frac{\left(m-\frac{1}{2}\right)^{2k-2l-2q} B_{2q}}{q\Gamma(k-l-q+1)\Gamma(q-m+l+1)} \left(\frac{1-2^{2q-1}}{2^{2q-1}}\right) \right], \tag{5.83}$$

where the coefficients L_l^m *are defined by*

$$\prod_{j=1/2}^{m-3/2} (y^2 - j^2) = \sum_{l=0}^{m-1} L_l^m y^{2m-2l-2}.$$

It is worth noting that the second sum in (5.83) *vanishes for* $m > k$.

Details of spectral invariants such as the heat trace, Minakshisundaram-Pleijel asymptotics of the heat trace, Minakshisundaram-Pleijel heat coefficients, spectral zeta functions, and zeta regularised functional determinants of the Laplacians on $\mathbf{S}^n$, $\mathbf{RP}^n$ and $\mathbf{CP}^n$ are discussed in Awonusika and Taheri [8, 15].

Furthermore, in Awonusika and Taheri [10] we express the Maclaurin asymptotic expansion of the heat kernel on $\mathbf{S}^n$ as a spectral series involving the trace of the associated heat operator, and consequently in terms of the Minakshisundaram-Pleijel heat coefficients on $\mathbf{S}^n$; and present an explicit spectral formula for the full Maclaurin expansion of the heat kernel on $\mathbf{S}^n$ (see also Awonusika and Taheri [7, 12]).

5.4 Integral Representations of the Heat Kernels on $\mathbf{CP}^n$

In this section we give explicit series, fractional and integral representations of the heat kernels on the complex projective spaces $\mathbf{CP}^n$, $n \geq 1$. To obtain fractional formulae for the heat kernels on $\mathbf{CP}^n$ we use the Riemann-Liouville fractional derivative formula and to establish integral representations we use integral formulae for the spherial functions on $\mathbf{CP}^n$.

The complex projective space $\mathbf{CP}^n$ is the set of all complex 1-dimensional subspaces through the origin in $\mathbf{C}^{n+1}$. It is a compact smooth complex n-dimensional manifold (real dimension $2n$) and a rank one symmetric space in virtue of

$$\mathbf{CP}^n = \mathbf{SU}(n+1)/S(\mathbf{U}(n) \times \mathbf{U}(1)), \qquad n \geq 1, \tag{5.84}$$

where $\mathbf{U}(n)$ is the group of complex $n \times n$ unitary matrices and $\mathbf{SU}(n)$ the group of $n \times n$ special unitary matrices of determinant 1 (see, e.g., Berger et al. [26]).

The volume of $\mathbf{CP}^n$ is given by the formula

$$\mathrm{Vol}\,(\mathbf{CP}^n) = \frac{4^n \pi^n}{n!}, \qquad n \geq 1, \tag{5.85}$$

and it is easily seen that when $n = 1$ the space $\mathbf{CP}^1$ reduces to the 2-sphere $\mathbf{S}^2$. The radial part of the Laplacian on $\mathbf{CP}^n$ is given by

$$\Delta_{\mathbf{CP}^n} = -\left[\frac{\partial^2}{\partial\theta^2} + \left(\cot\theta + (n-1)\cot\frac{\theta}{2}\right)\frac{\partial}{\partial\theta}\right], \tag{5.86}$$

where $\theta = \theta(o, x)$ is the complex projective distance between point $x \in \mathbf{CP}^n$ and the origin $o \in \mathbf{CP}^n$; the associated eigenvalues are given by $(\lambda_k : k \geq 0) = (k(k+n) : k \geq 0)$ with the multiplicity

$$(mc)_k^n = \frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{k!\Gamma(n)}\right]^2, \qquad k \geq 0. \tag{5.87}$$

Now using the basic identity $2\cot x = \cot x/2 - \tan x/2$ we have

$$\Delta_{\mathbf{CP}^n} = -\frac{\partial^2}{\partial\theta^2} - \left[\left(n - \frac{1}{2}\right)\cot\frac{\theta}{2} - \frac{1}{2}\tan\frac{\theta}{2}\right]\frac{\partial}{\partial\theta}. \tag{5.88}$$

By substituting $t = \theta/2$ we obtain the Jacobi operator (see (B.125))

$$\mathcal{L}_{(\alpha,\beta)} = 4\Delta_{\mathbf{CP}^n} = -\frac{\partial^2}{\partial t^2} - \left[(2\alpha+1)\cot t - (2\beta+1)\tan t\right]\frac{\partial}{\partial t}, \tag{5.89}$$

where $\alpha = n-1$, $\beta = 0$. So here the spherical functions on $\mathbf{CP}^n$ are the Jacobi polynomials

$$\begin{aligned}\Phi_k^{\mathbf{CP}^n}(\theta) &= \frac{k!\Gamma(n)}{\Gamma(k+n)}P_k^{(n-1,0)}(\cos\theta)\\ &= F\left(k+n, -k; n; \sin^2\frac{\theta}{2}\right).\end{aligned} \tag{5.90}$$

In other words, the function $\Phi_k^{\mathbf{CP}^n}(\theta)$ solves the complex projective eigenvalue equation

$$\begin{aligned}&\Delta_{\mathbf{CP}^n}\Phi_k^{\mathbf{CP}^n}(\theta) = k(k+n)\Phi_k^{\mathbf{CP}^n}(\theta)\\ &\Phi_k^{\mathbf{CP}^n}(0) = 1.\end{aligned}$$

Thus the heat kernel on $\mathbf{CP}^n$ is

$$K_{\mathbf{CP}^n}(t,\theta) = \frac{1}{\mathrm{Vol}(\mathbf{CP}^n)}\sum_{k=0}^{\infty}(mc)_k^n\Phi_k^{\mathbf{CP}^n}(\theta)e^{-k(k+n)t}, \tag{5.91}$$

where $\Phi_k^{\mathbf{CP}^n}$ is the spherical function on $\mathbf{CP}^n$.

Similar to what we did in the case of $\mathbf{S}^n$, we give fractional representations for the heat kernels on $\mathbf{CP}^n$ by using the formula (B.135), namely

$$P_k^{(a,0)}(\cos\theta) = \sqrt{\frac{2}{\pi}}\frac{k!\Gamma(a)}{\Gamma(a+k+1)}\frac{\partial^{\frac{1}{2}}}{\partial(\cos\theta+1)^{\frac{1}{2}}}C_{2k+1}^{a}\left(\cos(\theta/2)\right),$$

to obtain

$$\begin{aligned}K_{\mathbf{CP}^n}(t,\theta) &= \frac{e^{\frac{n^2}{4}t}}{\mathrm{Vol}(\mathbf{CP}^n)}\sum_{k=0}^{\infty}\frac{2k+n}{n}\frac{\Gamma(k+n)}{k!\Gamma(n)}P_k^{(n-1,0)}(\cos\theta)e^{-\left(k+\frac{n}{2}\right)^2 t}\\ &= \sqrt{\frac{2}{\pi}}\frac{e^{\frac{n^2}{4}t}\Gamma(n-1)}{\Gamma(n+1)\mathrm{Vol}(\mathbf{CP}^n)}\frac{\partial^{\frac{1}{2}}}{\partial(\cos\theta+1)^{\frac{1}{2}}}\sum_{k=0}^{\infty}(2k+n)e^{-\left(k+\frac{n}{2}\right)^2 t}C_{2k+1}^{n-1}\left(\cos(\theta/2)\right)\\ &= \frac{2\sqrt{2}e^{\frac{n^2}{4}t}}{n\pi\mathrm{Vol}(\mathbf{CP}^n)}\int_\theta^\pi\frac{\omega_1(t,\vartheta)\sin\frac{\vartheta}{2}}{\sqrt{\cos\theta-\cos\vartheta}}d\vartheta,\end{aligned} \tag{5.92}$$

where

$$\omega_1(t,\theta) = \sum_{k=0}^{\infty}\left(k+\frac{n}{2}\right)e^{-\left(k+\frac{n}{2}\right)^2 t}C_{2k}^{n}\left(\cos(\theta/2)\right).$$

Here we have used (B.110) and the fact that $C^{a}_{2k+1}(\cos(\theta/2))$ is an odd polynomial in $\cos(\theta/2)$ and so $C^{n-1}_{2k+1}(\cos(\pi/2)) = 0$.

We write out the heat kernel on $\mathbf{CP}^2$; this will lead to explicit expansions of the Gegenbauer polynomial for the special case $n = 5$, thereby give identities in the context of special functions. Indeed,

$$\begin{aligned}
K_{\mathbf{CP}^2}(t,\theta) &= \frac{1}{\mathrm{Vol}(\mathbf{CP}^2)} \sum_{k=0}^{\infty}(k+1)P_k^{(1,0)}(\cos\theta)e^{-k(k+2)t} \\
&= \frac{e^t\sqrt{\frac{2}{\pi}}}{\mathrm{Vol}(\mathbf{CP}^2)} \sum_{k=0}^{\infty}(k+1)e^{-(k+1)^2 t}\frac{\partial^{\frac{1}{2}}}{\partial(\cos\theta+1)^{\frac{1}{2}}}C^{1}_{2k+1}(\cos(\theta/2)) \\
&= \frac{e^t\sqrt{\frac{2}{\pi}}}{\mathrm{Vol}(\mathbf{CP}^2)} \sum_{k=0}^{\infty}(k+1)e^{-(k+1)^2 t}\frac{\partial^{\frac{1}{2}}}{\partial(\cos\theta+1)^{\frac{1}{2}}}\frac{\sin(k+1)\theta}{\sin(\theta/2)} \\
&= \frac{e^t\sqrt{\frac{2}{\pi}}}{\mathrm{Vol}(\mathbf{CP}^2)} \sum_{k=0}^{\infty}(k+1)e^{-(k+1)^2 t} \\
&\quad \times \frac{1}{2\sqrt{\pi}}\int_{\theta}^{\pi}\frac{\cot(\vartheta/2)\sin(k+1)\vartheta - 2(k+1)\cos(k+1)\vartheta}{\sin(\vartheta/2)\sqrt{\cos\theta-\cos\vartheta}}\,d\vartheta.
\end{aligned} \tag{5.93}$$

Also, by setting $n = 2$ in (5.92) and comparing with (5.93) we obtain the following expansions of the Gegenbauer polynomial $C^2_k(t)$:

Corollary 5.6.

$$C^2_{2k}\left(\cos\frac{\vartheta}{2}\right) = \frac{\frac{1}{2}\cot\frac{\vartheta}{2}\sin(k+1)\vartheta - (k+1)\cos(k+1)\vartheta}{\sin^2\frac{\vartheta}{2}};$$

and

$$C^2_k(\cos\theta) = \frac{\cos\theta\sin(k+2)\theta - (k+2)\sin\theta\cos(k+2)\theta}{2\sin^3\theta},$$

which clearly satisfy the property $C^2_0(\cos\theta) = 1$.

Again, we use the fractional representation (B.134), namely

$$P_k^{(n-1,0)}(\cos\theta) = \frac{2^{\frac{5}{2}-n}\Gamma(k+1)}{\sqrt{\pi}\Gamma(k+n)}\partial^{\frac{1}{2}}_{\cos\theta+1}\partial^{n-1}_{\cos(\theta/2)+1}\frac{\cos((k+\frac{n}{2})\theta)}{2k+n}$$

in (5.91) to obtain

$$K_{\mathbf{CP}^n}(t,\theta) = \frac{2^{\frac{5}{2}-n}e^{\frac{n^2}{4}t}}{\sqrt{\pi}\mathrm{Vol}(\mathbf{CP}^n)\Gamma(n+1)}\partial^{\frac{1}{2}}_{\cos\theta+1}\partial^{n-1}_{\cos(\theta/2)+1}\sum_{k=0}^{\infty}e^{-\left(k+\frac{n}{2}\right)^2 t}\cos\left(k+\frac{n}{2}\right)\theta. \tag{5.94}$$

Setting $n = 2$ in (5.94) we see that

$$\begin{aligned} K_{\mathbf{CP}^2}(t,\theta) &= \frac{e^t}{\sqrt{2\pi}\mathrm{Vol}(\mathbf{CP}^2)} \partial^{\frac{1}{2}}_{\cos\theta+1}\partial_{\cos(\theta/2)+1}\sum_{k=0}^{\infty} e^{-(k+1)^2 t}\cos(k+1)\,\theta \\ &= e^t\left(\frac{1}{2\pi}\frac{\partial}{\partial(\cos\theta+1)}\right)^{1/2}\left(\frac{1}{8\pi}\frac{\partial}{\partial\cos(\theta/2)}\right)\widetilde{\Theta}_1(t,\theta) \\ &= \frac{e^t}{2^{5/2}\pi^2}\int_\theta^\pi \frac{\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin(\vartheta/2)}\right)\left(-\frac{\partial}{\partial\vartheta}\right)\widetilde{\Theta}_1(t,\vartheta)}{\sqrt{\cos\theta-\cos\vartheta}}\,d\vartheta. \end{aligned} \tag{5.95}$$

Continuing in this way, we obtain the following fractional formula for the heat kernels on $\mathbf{CP}^n$:

$$K_{\mathbf{CP}^n}(t,\theta) = e^{\frac{n^2}{4}t}\left(\frac{1}{2\pi}\frac{\partial}{\partial(\cos\theta+1)}\right)^{1/2}\left(\frac{1}{8\pi}\frac{\partial}{\partial\cos(\theta/2)}\right)^{n-1}\Theta_1^{\pm}(t,\theta), \tag{5.96}$$

where $\Theta_1^{+}(t,\theta) = \widetilde{\Theta}_1(t,\theta)$ is the theta function corresponding to the case n even, $n \geq 2$; and $\Theta_1^{-}(t,\theta) = \widetilde{\Theta}_2(t,\theta)$ is the theta function corresponding to the case n odd, $n \geq 1$.

In summary we have the following statement (see also Awonusika and Taheri [9]).

Theorem 5.7. *For $t > 0$, $0 \leq \theta < \pi$, the heat kernel $K_{\mathbf{CP}^n}(t,\theta)$ associated to the Laplacian on the complex projective space $\mathbf{CP}^n$ is given by the following series representation:*

$$K_{\mathbf{CP}^n}(t,\theta) = \frac{e^{\frac{n^2}{4}t}}{Vol(\mathbf{CP}^n)}\sum_{k=0}^{\infty}\frac{2k+n}{n}\frac{\Gamma(k+n)}{k!\Gamma(n)}P_k^{(n-1,0)}(\cos\theta)e^{-\left(k+\frac{n}{2}\right)^2 t}, \quad n \geq 1. \tag{5.97}$$

Moreover, $K_{\mathbf{CP}^n}(t,\theta)$ admits the following fractional and integral formulae:

$$K_{\mathbf{CP}^n}(t,\theta) = \frac{(n-1)!e^{\frac{n^2}{4}t}}{2^{2n-3/2}\pi^{n+1}}\int_\theta^\pi \frac{\omega_1(t,\vartheta)\sin\frac{\vartheta}{2}}{\sqrt{\cos\theta-\cos\vartheta}}\,d\vartheta;$$

and according to whether n is odd or even,

(a) n odd, $n \geq 1$,

$$K_{\mathbf{CP}^n}(t,\theta) = \frac{e^{\frac{n^2}{4}t}}{2^{2n-3/2}\pi^n}\int_\theta^\pi \frac{\left(-\frac{\partial}{\partial\vartheta}\right)\left(-\frac{1}{\sin(\vartheta/2)}\frac{\partial}{\partial\vartheta}\right)^{n-1}\widetilde{\Theta}_2(t,\vartheta)}{\sqrt{\cos\theta-\cos\vartheta}}\,d\vartheta;$$

(b) n even, $n \geq 2$,

$$K_{\mathbf{CP}^n}(t,\theta) = \frac{e^{\frac{n^2}{4}t}}{2^{2n-3/2}\pi^n}\int_\theta^\pi \frac{\left(-\frac{\partial}{\partial\vartheta}\right)\left(-\frac{1}{\sin(\vartheta/2)}\frac{\partial}{\partial\vartheta}\right)^{n-1}\widetilde{\Theta}_1(t,\vartheta)}{\sqrt{\cos\theta-\cos\vartheta}}\,d\vartheta.$$

We can also write the spherical function $\Phi_k^{\mathbf{CP}^n}(\theta)$ in terms of the Jacobi function $\phi_r^{(\alpha,\beta)}(x)$ (see (B.131)-(B.129)), namely

$$\Phi_k^{\mathbf{CP}^n}(\theta) = F\left(k+n,-k;n;\sin^2(\theta/2)\right) = \phi^{(n-1,0)}_{(2k+n)/i}(i\theta/2), \tag{5.98}$$

and then compute integral representations for $K_{\mathbf{CP}^n}(t,\theta)$ involving the Gauss hypergeometric function. Towards this end, we use the integral representation (B.130) for the Jacobi function

$\phi_r^{(\alpha,\beta)}(x)$ to give an explicit integral formula for the spherical function $\Phi_k^{\mathbf{CP}^n}(\theta)$, namely

$$\Phi_k^{\mathbf{CP}^n}(\theta) = \frac{\Gamma(n)}{\sqrt{\pi}\Gamma\left(n-\frac{1}{2}\right)} \frac{2^{n-3/2}}{\sin^{2n-2}(\theta/2)\cos^{1/2}(\theta/2)} \times \int_0^\theta \cos\left(k+\frac{n}{2}\right)\vartheta\, F\left(\frac{1}{2},\frac{1}{2};n-\frac{1}{2};\frac{\cos(\theta/2)-\cos(\vartheta/2)}{2\cos(\theta/2)}\right) d\vartheta. \tag{5.99}$$

If we now insert this into the heat kernel (5.91) we obtain

$$K_{\mathbf{CP}^n}(t,\theta) = \frac{e^{\frac{n^2}{4}t}2^{n-1/2}}{\mathrm{Vol}(\mathbf{CP}^n)} \frac{\sin^{2-2n}(\theta/2)\cos^{-1/2}(\theta/2)}{\sqrt{\pi}\Gamma(n+1)\Gamma\left(n-\frac{1}{2}\right)} \times \int_0^\theta \omega_2(t,\vartheta) F\left(\frac{1}{2},\frac{1}{2};n-\frac{1}{2};\frac{\cos(\theta/2)-\cos(\vartheta/2)}{2\cos(\theta/2)}\right) d\vartheta, \tag{5.100}$$

where

$$\omega_2(t,\theta) = \sum_{k=0}^{\infty}\left(k+\frac{n}{2}\right)\left[\frac{\Gamma(k+n)}{k!}\right]^2 \cos\left(k+\frac{n}{2}\right)\theta e^{-\left(k+\frac{n}{2}\right)^2 t}. \tag{5.101}$$

For the special case $n=2$, we have

$$K_{\mathbf{CP}^2}(t,\theta) = \frac{e^t}{2^{3/2}\pi^3}\sin^{-2}(\theta/2)\cos^{-1/2}(\theta/2) \times \int_0^\theta \omega_2(t,\vartheta) F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{\cos(\theta/2)-\cos(\vartheta/2)}{2\cos(\theta/2)}\right) d\vartheta, \tag{5.102}$$

where

$$\omega_2(t,\theta) = \sum_{k=0}^{\infty}(k+1)^3\cos(k+1)\,\theta e^{-(k+1)^2 t}. \tag{5.103}$$

Using the identity (B.60) we obtain another statement for the heat kernel $K_{\mathbf{CP}^n}(t,\theta)$.

Theorem 5.8. *For $t>0$, $0\le\theta<\pi$, the heat kernel $K_{\mathbf{CP}^n}(t,\theta)$ associated to the Laplacian on the complex projective space $\mathbf{CP}^n$ is given by the following integral representation (5.100). In particular,*

$$\begin{aligned} K_{\mathbf{CP}^2}(t,\theta) &= \frac{e^t}{2^{3/2}\pi^3}\sin^{-2}(\theta/2)\cos^{-1/2}(\theta/2) \\ &\quad\times \int_0^\theta \omega_2(t,\vartheta) F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{\cos(\theta/2)-\cos(\vartheta/2)}{2\cos(\theta/2)}\right) d\vartheta \\ &= \frac{e^t}{2\pi^3}\sin^{-2}(\theta/2)\int_0^\theta \omega_2(t,\vartheta)\frac{\arcsin\left(\frac{\cos(\theta/2)-\cos(\vartheta/2)}{2\cos(\theta/2)}\right)^{1/2}}{\sqrt{\cos(\theta/2)-\cos(\vartheta/2)}}\, d\vartheta, \end{aligned} \tag{5.104}$$

where

$$\omega_2(t,\theta) = \sum_{m=1}^{\infty} m^3\cos(m\theta)e^{-m^2 t}. \tag{5.105}$$

Remark 5.5. *In Hafoud and Intissar [72] the integral representation (B.136) is constructed for the Jacobi polynomial $P_k^{a,b}(\cos\theta)$ to obtain the following integral formula for the heat kernel on*

$\mathbf{CP}^n$.

$$K_{\mathbf{CP}^n}(t,\theta) = \frac{e^{\frac{n^2}{4}t}2^{2-n}}{4^n\pi^{n+1}} \int_{\theta/2}^{\pi/2} \frac{\left(-\frac{\partial}{\partial\vartheta}\right)\cos\vartheta\left(-\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\right)^{n-1}\omega_2(t,\vartheta)}{\sqrt{\cos^2\theta/2-\cos^2\vartheta}}\,d\vartheta, \tag{5.106}$$

where

$$\omega_2(t,\theta) = \sum_{k=0}^{\infty}(2k+n)\frac{\sin(2k+n)\theta}{\sin\theta}e^{-\left(k+\frac{n}{2}\right)^2 t}.$$

Furthermore, in Awonusika and Taheri [9] we express the Maclaurin asymptotic expansion of the heat kernel on $\mathbf{CP}^n$ as a spectral series involving the trace of the associated heat operator, and consequently in terms of the Minakshisundaram-Pleijel heat coefficients on $\mathbf{CP}^n$; and present an explicit spectral formula for the full Maclaurin expansion of the heat kernel on $\mathbf{CP}^n$ (see also Awonusika and Taheri [16]).

5.5 The Heat Trace Formulae via the Euclidean Poisson Kernel

As we have earlier mentioned in Section 4.4 that the trace of the heat operator on a compact symmetric space can be expressed in terms of the Euclidean Poisson kernel, we now present the traces of the heat operators on the sphere $\mathbf{S}^n$, the real projective space $\mathbf{RP}^n$ and the complex projective space $\mathbf{CP}^n$ in terms of the Euclidean Poisson kernel.

Towards this end we consider the Laplacian $\sqrt{\Delta'_{\mathscr{X}}}$ on $\mathscr{X}$, where $\Delta'_{\mathscr{X}}$ is the shifted Laplacian given by

$$\Delta'_{\mathscr{X}} = \Delta_{\mathscr{X}} + \varrho^2$$

whose eigenvalues are $\lambda'_k = \lambda_k + \varrho^2$, $\varrho > 0$. We then compute explicit formulae for the heat trace

$$\operatorname{tr} e^{-t\sqrt{\Delta'_{\mathscr{X}}}} = \sum_{k=0}^{\infty} d_k e^{-t\sqrt{\lambda'_k}},$$

where $\mathscr{X} = \mathbf{S}^n, \mathbf{RP}^n, \mathbf{CP}^n$.

- $\mathscr{X} = \mathbf{S}^n$. For the sphere $\mathbf{S}^n$, $d_k = M_k^n, \lambda_k = k(k+n-1)$, $\varrho = (n-1)/2$; and we have

$$\begin{aligned}\operatorname{tr} e^{-t\sqrt{\Delta'_{\mathbf{S}^n}}} &= \operatorname{tr} e^{-t\sqrt{\Delta_{\mathbf{S}^n}+\frac{(n-1)^2}{4}}} = \sum_{k=0}^{\infty} M_k^n e^{-t\sqrt{\lambda'_k}} \\ &= e^{-t\frac{n-1}{2}}\sum_{k=0}^{\infty} M_k^n e^{-tk} = \nu_n e^{-t\frac{n-1}{2}} P_{\mathbf{B}^{n+1}}(e^{-t},0) \\ &= \frac{2^{1-n}\cosh\frac{t}{2}}{\left(\sinh\frac{t}{2}\right)^n} = \frac{1}{2^{\frac{n-1}{2}}}\frac{\sinh t}{(\cosh t-1)^{\frac{n+1}{2}}}.\end{aligned}$$

- $\mathscr{X} = \mathbf{RP}^n$. In this case, $d_k = M^n_{2k}$, $\lambda_{2k} = 2k(2k+n-1)$, $\varrho = (n-1)/2$; and we obtain

$$\begin{aligned}
\operatorname{tr} e^{-t\sqrt{\Delta'_{\mathbf{RP}^n}}} &= \operatorname{tr} e^{-t\sqrt{\Delta_{\mathbf{RP}^n}+\frac{(n-1)^2}{4}}} = \sum_{k=0}^{\infty} M^n_{2k} e^{-t\sqrt{\lambda'_{2k}}} \\
&= e^{-t\frac{n-1}{2}} \sum_{k=0}^{\infty} M^n_{2k} e^{-2tk} = \frac{1}{2} e^{-t\frac{n-1}{2}} \sum_{m\in 2\mathbf{Z}} M^n_m e^{-tm} \\
&= \frac{1}{4}\nu_n e^{-t\frac{n-1}{2}} P_{\mathbf{B}^{n+1}}(e^{-t},0) \\
&= \frac{2^{-1-n}\cosh\frac{t}{2}}{\left(\sinh\frac{t}{2}\right)^n} = \frac{1}{2^{\frac{n+3}{2}}} \frac{\sinh t}{(\cosh t-1)^{\frac{n+1}{2}}}.
\end{aligned}$$

- $\mathscr{X} = \mathbf{CP}^n$. Here $d_k = (mc)^n_k$, $\lambda_k = k(k+n)$, $\varrho = n/2$; we have

$$\begin{aligned}
\operatorname{tr} e^{-t\sqrt{\Delta'_{\mathbf{CP}^n}}} &= \operatorname{tr} e^{-t\sqrt{\Delta_{\mathbf{CP}^n}+\frac{n^2}{4}}} = \sum_{k=0}^{\infty} (mc)^n_k e^{-t\left(k+\frac{n}{2}\right)} \\
&= \frac{2^{-n}\cosh\frac{t}{2}}{\left(\sinh\frac{t}{2}\right)^{n+1}} = \frac{1}{2^{\frac{n}{2}}} \frac{\sinh t}{(\cosh t-1)^{\frac{n+2}{2}}}.
\end{aligned}$$

Chapter 6

Integral Representations in the Real Hyperbolic Space $\mathbf{H}^n$

This chapter presents integral formulae for spectral functions in the n-dimensional upper half-space $\mathbf{H}^n$. Section 6.1 is devoted to the development of integral formulae for the generalised spherical functions and general eigenfunctions of the Laplacian in $\mathbf{H}^n$; the integral representations of general eigenfunctions can also be interpreted as integral transforms of harmonic functions on the hyperbolic unit ball. In Section 6.2 as a result of the beautiful relation between the solutions of the heat and wave equations, we obtain the heat kernel in $\mathbf{H}^n$ from the wave equation in $\mathbf{H}^n$; this is possible using the Euclidean Fourier transform. Section 6.3 is devoted to the Green function of the Laplacian in $\mathbf{H}^n$. Using the Green function of the Laplacian in the upper half-space $\mathbf{H}^n$, we derive explicitly, in Section 6.4, the generalisation of the Mehler-Fock integral formula, from which the heat kernels in $\mathbf{H}^n$ are deduced by appropriately choosing a spectral function. We also derive the Mehler-Fock inversion formula via the Poisson kernel in $\mathbf{H}^n$. Finally we establish a Millson-type recursion formula for the Mehler-Fock integral transform.

6.1 Generalised Spherical Functions

In harmonic analysis on hyperbolic spaces, the theory of (zonal) spherical functions plays a crucial role. The spherical functions are normalised eigenfunctions of the Laplacian that take the value one at the origin. The asymptotic behaviour of the spherical function at ∞ was studied by Harish-Chandra [73] for general symmetric spaces of rank one.

In this section we compute eigenfunctions of the Laplacian in the real hyperbolic space $\mathbf{H}^n$ in terms of an orthonormal basis $\left(Y_{k,j} : k \geq 0, 1 \leq j \leq M_k^{n-1}\right)$ of $\mathfrak{H}_k^{n-1} \subset L^2\left(\mathbf{S}^{n-1}\right)$, which consists of spherical harmonic functions on $\mathbf{S}^{n-1}$; such eigenfunctions are called *general eigenfunctions*. We also give integral representations for spherical functions of the Laplacian in $\mathbf{H}^n$.

We start the construction of spherical functions from what we already know in Section 1.5 (see also the beginning of Subsection 2.2.1). The function y^s, $s = \frac{n-1}{2} + ir$, $r \in \mathbf{R}$, is an eigenfunction of the Laplacian $\Delta_{\mathbf{H}^n}$ in $\mathbf{H}^n$ with eigenvalues $\lambda = s(n-1-s) = \frac{(n-1)^2}{4} + r^2$. The aim of this section is to develop the corresponding spherical functions. We recall that in geodesic polar coordinates $w = (\rho, \theta)$, we have

$$f(w) = y = \frac{1}{\cosh\rho + \sinh\rho\cos\theta},$$

where $\rho = d(w, o)$, $w = (x, y) \in \mathbf{H}^n$, and $w' = o = (0, 0, ..., 0, 1) \in \mathbf{H}^n$, the origin, and $\theta = \theta(\zeta, \zeta'), \zeta, \zeta' \in \mathbf{S}^{n-1}$. Since the Laplacian $\Delta_{\mathbf{H}^n}$ is an isometry invariant and since the unit sphere $\mathbf{S}^{n-1}$ acts on $\mathbf{H}^n$ by rotations with centre $w' = o = (0, 0, ..., 0, 1)$, the averaged function

$$\Phi_r^{\mathbf{H}^n}(\rho, \zeta) = \frac{1}{\nu_{n-1}} \int_{\mathbf{S}^{n-1}} \frac{d\nu_{n-1}(\zeta')}{[\cosh\rho + \sinh\rho\,(\zeta\cdot\zeta')]^{\frac{n-1}{2}+ir}},$$

or what is the same

$$\Phi_r^{\mathbf{H}^n}(\rho) = \frac{\nu_{n-2}}{\nu_{n-1}} \int_0^{\pi} \frac{\sin^{n-2}\theta\, d\theta}{[\cosh\rho + \sinh\rho\cos\theta]^{\frac{n-1}{2}+ir}}, \tag{6.1}$$

is also an eigenfunction of $\Delta_{\mathbf{H}^n}$, which is the generalisation of (1.48). The functions $\Phi_r^{\mathbf{H}^n}(\rho)$ are called *spherical functions* in $\mathbf{H}^n$, which are radial eigenfunctions of the Laplacian $\Delta_{\mathbf{H}^n}$ satisfying

$$\Phi_{-r}^{\mathbf{H}^n}(\rho) = \Phi_r^{\mathbf{H}^n}(\rho) = \Phi_r^{\mathbf{H}^n}(-\rho),\ \Phi_r^{\mathbf{H}^n}(0) = 1.$$

More generally, for a continuous function $f(\zeta)$ on $\mathbf{S}^{n-1}$,

$$\left(\Phi_r^{\mathbf{H}^n} f\right)(\rho, \zeta) = \frac{1}{\nu_{n-1}} \int_{\mathbf{S}^{n-1}} \frac{f(\zeta')\, d\nu_{n-1}(\zeta')}{[\cosh\rho + \sinh\rho\,(\zeta\cdot\zeta')]^{\frac{n-1}{2}+ir}}. \tag{6.2}$$

Using the radial part (1.39) of $\Delta_{\mathbf{H}^n}$ and differentiating under the integral sign, and noting that

$$\Delta_{\mathbf{H}^n} y^{\frac{n-1}{2}+ir} = \left(\frac{(n-1)^2}{4} + r^2\right) y^{\frac{n-1}{2}+ir}, \tag{6.3}$$

we see that $\left(\Phi_r^{\mathbf{H}^n} f\right)(\zeta)$ is an eigenfunction of the Laplacian $\Delta_{\mathbf{H}^n}$ in $\mathbf{H}^n$. Now putting $\mu = \frac{2-n}{2}$, $\nu = -\frac{1}{2} + ir$ in the formula (B.87) or putting $t = \cos\theta$, $\mu = \frac{n-2}{2}$, $\nu = -\frac{1}{2} - ir$, $q = \cosh\rho$ in the equality (B.89), we obtain

$$\Phi_r^{\mathbf{H}^n}(\rho) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\sinh^{\frac{2-n}{2}}\rho P^{\frac{2-n}{2}}_{-\frac{1}{2}+ir}(\cosh\rho) \tag{6.4}$$

(compare with Gruet [70, p. 1019] who uses different methods, see also Proposition 4.1). In terms of other special functions we have

$$\begin{aligned}\Phi_r^{\mathbf{H}^n}(\rho) &= \mathscr{C}^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(\cosh\rho) = \frac{C^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(\cosh\rho)}{C^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(1)} \\ &= \frac{\Gamma\left(-\frac{n}{2}+\frac{3}{2}+ir\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}+ir\right)} P^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}_{-\frac{n-1}{2}+ir}(\cosh\rho),\end{aligned}$$

where $P_k^{\alpha,\beta}(t)$ is the Jacobi polynomial (see Appendix B.6), and we have used the formula (B.114), the last identity in (B.104), (B.80) (see also (B.119)) and (B.125)-(B.129).

An important function in the theory of symmetric spaces is the Harish-Chandra c-function (Harish-Chandra [73]). The Harish-Chandra c-function was originally defined as a meromorphic function in terms of the asymptotic behaviour of a zonal spherical function of a noncompact semisimple Lie group G. The Harish-Chandra c-function determines the Plancherel measure for the spherical transform on G. It turns out that this c-function plays a key role in every aspect of harmonic analysis on G and various symmetric spaces of G (see e.g. Helgason [77, 80, 83, 84]).

We now want to see that the Harish-Chandra c-function on a noncompact symmetric space $G/K = \mathbf{H}^n = \mathbf{SO}_0(n,1)/\mathbf{SO}(n)$ can be described in terms of the asymptotic behaviour of the spherical function $\Phi_r^{\mathbf{H}^n}(\rho)$ as $\rho \nearrow \infty$.

Theorem 6.1. *Let* $Re\, ir > 0$. *Then*

$$\lim_{\rho \nearrow \infty} e^{\left(\frac{n-1}{2}+ir\right)\rho} \Phi_r^{\mathbf{H}^n}(\rho) = \frac{2^{n-2}\Gamma\left(\frac{n}{2}\right)\Gamma(ir)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}+ir\right)} = c(r), \tag{6.5}$$

where $c(r)$ *is the Harish-Chandra c-function in* $\mathbf{H}^n$.

Proof. Making the substitution

$$u = \tan\left(\frac{\theta}{2}\right), \; du = \frac{1}{2}\left(\sec^2\left(\frac{\theta}{2}\right)\right) d\theta = \frac{1}{2}\left(1+u^2\right) d\theta$$

in (6.1) and noting the trigonometric identities

$$\cos\theta = \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-u^2}{1+u^2}, \qquad \sin\theta = \frac{2u}{1+u^2},$$

we have (after some rearrangements)

$$\Phi_r^{\mathbf{H}^n}(\rho) = 2^{n-1}\frac{\nu_{n-2}}{\nu_{n-1}} e^{-\left(\frac{n-1}{2}+ir\right)\rho} \int_0^\infty \left(1+u^2e^{-2\rho}\right)^{-\left(\frac{n-1}{2}+ir\right)} \left(1+u^2\right)^{-\left(\frac{n-1}{2}+ir\right)} u^{n-2}\, du.$$

By following the same argument as in the two-dimensional case (see Helgason [83, pp. 39-40]), we see that

$$\lim_{\rho \nearrow \infty} e^{\left(\frac{n-1}{2}+ir\right)\rho} \Phi_r^{\mathbf{H}^n}(\rho) = 2^{n-1}\frac{\nu_{n-2}}{\nu_{n-1}} \int_0^\infty \left(1+u^2\right)^{-\left(\frac{n-1}{2}+ir\right)} u^{n-2}\, du.$$

Let

$$\xi = \left(1+u^2\right)^{-1}, \; u = (1-\xi)^{\frac{1}{2}}\xi^{-\frac{1}{2}}.$$

Then

$$\lim_{\rho \nearrow \infty} e^{\left(\frac{n-1}{2}+ir\right)\rho} \Phi_r^{\mathbf{H}^n}(\rho) = 2^{n-2}\frac{\nu_{n-2}}{\nu_{n-1}} \int_0^1 \xi^{ir-1}(1-\xi)^{\frac{n-3}{2}}\, d\xi = \frac{2^{n-2}\Gamma\left(\frac{n}{2}\right)\Gamma(ir)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}+ir\right)},$$

where we have expressed the beta function in terms of the gamma function (see (B.33)). □

We shall see later in this chapter that the Harish-Chandra c-function (6.5) determines the Plancherel formula in $\mathbf{H}^n$, and hence the heat kernel in $\mathbf{H}^n$.

In summary we have the following statement for the eigenvalue problem in the real hyperbolic space $\mathbf{H}^n$.

Proposition 6.2. *The functions*

$$\Phi_r^{\mathbf{H}^n}(\rho) = \mathscr{C}^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(\cosh\rho) = 2^{-\frac{2-n}{2}}\Gamma\left(\frac{n}{2}\right)\sinh^{\frac{2-n}{2}}\rho P^{\frac{2-n}{2}}_{-\frac{1}{2}+ir}(\cosh\rho) \tag{6.6}$$

are called spherical functions in $\mathbf{H}^n$*, satisfying the eigenvalue problem*

$$\left[\frac{\partial^2}{\partial\rho^2}+(n-1)\coth\rho\frac{\partial}{\partial\rho}\right]\Phi_r^{\mathbf{H}^n}(\rho)=s(s-n+1)\Phi_r^{\mathbf{H}^n}(\rho) \tag{6.7}$$

and the limit formula

$$\lim_{\rho\nearrow\infty}e^{\left(\frac{n-1}{2}+ir\right)\rho}\Phi_r^{\mathbf{H}^n}(\rho)=\frac{2^{n-2}\Gamma\left(\frac{n}{2}\right)\Gamma(ir)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}+ir\right)}=c(r),$$

where $c(r)$ *is the Harish-Chandra c-function in* $\mathbf{H}^n$*, with* $\Phi_r^{\mathbf{H}^n}(0)=1$.

Next we construct a more general eigenfunction of the Laplacian in $\mathbf{H}^n$ that involves the spherical harmonic function discussed in Section 1.4; we do this by replacing the continuous functions f defined on $\mathbf{S}^{n-1}$ in (6.2) with spherical harmonics. Towards this end, let $\left(Y_{k,j}:k\geq 0,1\leq j\leq M_k^{n-1}\right)$ be an orthonormal basis of $\mathfrak{H}_k^{n-1}\subset L^2\left(\mathbf{S}^{n-1}\right)$, which consists of spherical harmonic functions on $\mathbf{S}^{n-1}$. Consider the function

$$\left(\Phi_r^{\mathbf{H}^n}\left[Y_{k,j}\right]\right)(\rho,\zeta)=\frac{1}{\nu_{n-1}}\int_{\mathbf{S}^{n-1}}\frac{Y_{k,j}(\zeta')\,d\nu_{n-1}(\zeta')}{\left[\cosh\rho-\sinh\rho\,(\zeta\cdot\zeta')\right]^{\frac{n-1}{2}+ir}}, \tag{6.8}$$

which by the reason stated above is also an eigenfunction of the Laplacian $\Delta_{\mathbf{H}^n}$ in $\mathbf{H}^n$. The choice of the spherical harmonic function allows us to apply the Funk-Hecke Theorem 1.14 to (6.8). Thus we obtain

$$\left(\Phi_r^{\mathbf{H}^n}\left[Y_{k,j}\right]\right)(\rho,\zeta)=\frac{\nu_{n-2}}{\nu_{n-1}}Y_{k,j}(\zeta)\int_0^\pi\frac{\mathscr{C}_k^{\frac{n-2}{2}}(\cos\theta)\sin^{n-2}\theta\,d\theta}{\left[\cosh\rho-\sinh\rho\cos\theta\right]^{\frac{n-1}{2}+ir}}. \tag{6.9}$$

Substituting the k-degree zonal spherical harmonics $\mathscr{C}_k^{\frac{n-2}{2}}$ given by the Rodrigue's formula (see (B.104) and (B.113))

$$\mathscr{C}_k^{\frac{n-2}{2}}(t):=\frac{C_k^{\frac{n-2}{2}}(t)}{C_k^{\frac{n-2}{2}}(1)}=\left(-\frac{1}{2}\right)^k\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{2k+n-1}{2}\right)}\left(1-t^2\right)^{\frac{3-n}{2}}\frac{d^k}{dt^k}\left(1-t^2\right)^{\frac{2k+n-3}{2}} \tag{6.10}$$

into (6.9) and integrating by parts, we have

$$\begin{aligned}\left(\Phi_r^{\mathbf{H}^n}\left[Y_{k,j}\right]\right)(\rho,\zeta)&=\frac{\nu_{n-2}Y_{k,j}(\zeta)}{\nu_{n-1}}\left(-\frac{1}{2}\right)^k\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{2k+n-1}{2}\right)}\int_{-1}^1\frac{\frac{d^k}{dt^k}\left(1-t^2\right)^{\frac{2k+n-3}{2}}\,dt}{\left[\cosh\rho-t\sinh\rho\right]^{\frac{n-1}{2}+ir}}\\&=c_{k,n}Y_{k,j}(\zeta)\int_{-1}^1\frac{d^k}{dt^k}\left(\left[\cosh\rho-t\sinh\rho\right]^{-\frac{n-1}{2}-ir}\right)\left(1-t^2\right)^{\frac{2k+n-3}{2}}\,dt\\&=c_{k,n}Y_{k,j}(\zeta)\prod_{l=0}^{k-1}\left(-\frac{n-1}{2}-ir-l\right)\sinh^k\rho\frac{\nu_{n+2k-1}}{\nu_{n+2k-2}}\Phi_r^{\mathbf{H}^{n+2k}}(\rho)\\&=Y_{k,j}(\zeta)\Psi_{r,k,n}(\rho),\end{aligned}$$

where $\Phi_r^{\mathbf{H}^{n+2k}}(\rho)$ is the spherical function of the Laplacian $\Delta_{\mathbf{H}^{n+2k}}$ in $\mathbf{H}^{n+2k}$ and the constant $c_{k,n}$ is given by

$$c_{k,n}=\frac{\nu_{n-2}}{\nu_{n-1}}\left(\frac{1}{2}\right)^k\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{2k+n-1}{2}\right)}=\left(\frac{\pi}{2}\right)^k\frac{\nu_{2k+n-2}}{\nu_{n-2}}.$$

The function

$$\Psi_{r,k,n}(\rho) = c_{k,n} \prod_{l=0}^{k-1} \left(-\frac{n-1}{2} - ir - l \right) \sinh^k \rho \frac{\nu_{n+2k-1}}{\nu_{n+2k-2}} \Phi_r^{\mathbf{H}^{n+2k}}(\rho) \tag{6.11}$$

is called the *generalised spherical function* (also called *spherical function of type* k) in the real upper half-space $\mathbf{H}^n$, satisfying

$$\Psi_{r,k,n}(\rho)/\mathscr{N}_{k,n}(r) = \Psi_{-r,k,n}(\rho)/\mathscr{N}_{k,n}(-r), \quad \mathscr{N}_{k,n}(r) = \prod_{l=0}^{k-1} \left(-\frac{n-1}{2} - ir - l \right).$$

In terms of other special functions, we have

$$\begin{aligned} \Psi_{r,k,n}(\rho) &= c_{k,n,r,l} 2^{\frac{n+2k-2}{2}} \Gamma \left(\frac{n+2k}{2} \right) \sinh^{\frac{2-n-k}{2}} \rho P^{\frac{2-n-2k}{2}}_{-\frac{1}{2}+ir} (\cosh \rho) \\ &= c_{k,n,r,l} \sinh^k \rho F \left(\frac{n-1}{2} + k + ir, \frac{n-1}{2} + k - ir; \frac{n}{2} + k; -\sinh^2 \left(\frac{\rho}{2} \right) \right), \end{aligned} \tag{6.12}$$

where

$$c_{k,n,r,l} = c_{k,n} \frac{\nu_{n+2k-1}}{\nu_{n+2k-2}} \prod_{l=0}^{k-1} \left(-\frac{n-1}{2} - ir - l \right);$$

here we have applied (B.80).

Now that we have constructed the generalised spherical functions in $\mathbf{H}^n$, before giving the integral representations of the generalised spherical functions we compute the general eigenfunctions in $\mathbf{H}^n$, as well as thier integral representations. Towards this end, we define, for $r \in \mathbf{R}$, the eigenspace $\mathscr{E}_r$ by

$$\mathscr{E}_r \left(\mathbf{H}^n \right) = \left\{ f \in C^\infty \left(\mathbf{H}^n \right) \,:\, \Delta_{\mathbf{H}^n} f = \left(\frac{(n-1)^2}{4} + r^2 \right) f \right\}.$$

A result of Strichartz [159] says that *if* $\Psi_{r,k,j}$ *are the generalised spherical functions in* $\mathbf{H}^n$ *given by* (6.11)*, and* $Y_k(\zeta)$ *a spherical harmonic function of degree* k *in* n *dimensions which forms an orthonormal basis* $\left(Y_{k,j} : k \geq 0, 1 \leq j \leq M_k^{n-1} \right)$ *of* $\mathfrak{H}_k^{n-1}$*, then any* $H \in \mathscr{E}_r \left(\mathbf{H}^n \right)$ *can be expanded in an absolutely and uniformly convergent series of the form (spherical harmonic expansion)*

$$H(\rho, \zeta) = \sum_{k=0}^{\infty} \Psi_{r,k,n}(\rho) \sum_{j=1}^{M_k^{n-1}} a_{k,j} Y_{k,j}(\zeta), \quad \rho \geq 0, \quad \zeta \in \mathbf{S}^{n-1}, \tag{6.13}$$

for suitable coefficients $a_{k,j}$*.*

A function H with the spherical harmonic expansion (6.13) is called the *general eigenfunction of* $\Delta_{\mathbf{H}^n}$.

Remark 6.1. *Since every Riemannian manifold is locally Euclidean, the hyperbolic spherical function* $\Psi_{r,k,n}(\rho)$ *behaves locally like* ρ^k *for* $0 \leq \rho < 1$*:*

$$H(\rho, \zeta) \sim \sum_{k=0}^{\infty} \rho^k \sum_{j=1}^{M_k^{n-1}} a_{k,j} Y_{k,j}(\zeta),$$

for suitable coefficients $a_{k,j}$*. Thus* $H(\rho, \zeta)$ *becomes a Euclidean harmonic function in the unit ball* $\mathbf{B}^n$ *in* $\mathbf{R}^n$*. If in particular,* $a_{k,j} = \overline{Y_{k,j}(\zeta')}, \zeta' \in \mathbf{S}^{n-1}$*, then the hyperbolic general eigenfunction*

$H(\rho,\zeta)$ behaves locally like the Euclidean Poisson kernel in $\mathbf{B}^n$ (see Section 4.4):

$$H(\rho,\zeta,\zeta')=\sum_{k=0}^{\infty}\rho^k\sum_{j=1}^{M_k^{n-1}}\overline{Y_{k,j}(\zeta')}Y_{k,j}(\zeta)=\sum_{k=0}^{\infty}\rho^k M_k^{n-1}\mathscr{C}_k^{\frac{n-2}{2}}((\zeta\cdot\zeta'))=\frac{1-\rho^2}{(1-2\rho(\zeta\cdot\zeta')+\rho^2)^{\frac{n}{2}}}.$$

We are now set to give integral representation formulae for the generalised spherical functions and general eigenfunctions of the Laplacian in $\mathbf{H}^n$. We start with the generalised spherical functions $\Psi_{r,k,n}(\rho)$. Towards this end we use the transformation formula (B.70) in (6.12) to obtain

$$\Psi_{r,k,n}(\rho)=c_{k,n,r,l}\sinh^k\rho\left(\cosh^2\frac{\rho}{2}\right)^{-s_k^+}F\left(s_k^+,\frac{1}{2}+ir;\frac{n}{2}+k;\tanh^2\left(\frac{\rho}{2}\right)\right) \tag{6.14}$$

$$=c_{k,n,r,l}(-i\varrho)^k\left(1-\varrho^2\right)^{s_k^+}F\left(s_k^+,\frac{1}{2}+ir;\frac{n}{2}+k;\frac{\varrho^2}{\varrho^2-1}\right), \tag{6.15}$$

where $s_k^+=\frac{n-1}{2}+k+ir$, $\varrho^2=-\sinh^2(\rho/2)$. Replacing this Gauss hypergeometric function by its integral formula (B.62) gives

$$\Psi_{r,k,n}(\varrho)=c_{k,n,r,l}\left(1-\varrho^2\right)^{\frac{n}{2}+k}\frac{(-i)^k\varrho^k\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(s_k^+\right)\Gamma\left(\frac{1}{2}-ir\right)}\int_0^1\frac{t^{\frac{n-3}{2}+k+ir}(1-t)^{-\frac{1}{2}-ir}}{(1-\varrho^2(1-t))^{\frac{1}{2}-ir}}\,dt. \tag{6.16}$$

For general eigenfunctions $H(\varrho,\zeta)$, we use (6.16) in (6.13) to get

$$H(\varrho,\zeta)=\frac{1}{\Gamma\left(\frac{1}{2}-ir\right)}\int_0^1F_{r,n}^{(1)}(\varrho t\zeta)\frac{t^{\frac{n-3}{2}+ir}(1-t)^{-\frac{1}{2}-ir}}{(1-\varrho^2(1-t))^{\frac{1}{2}-ir}}\,dt,$$

where

$$F_{r,n}^{(1)}(\varrho t\zeta)=\sum_{k=0}^{\infty}\frac{c_{k,n,r,l}(-i)^k\Gamma\left(k+\frac{n}{2}\right)}{(1-\varrho^2)^{-\frac{n}{2}-k}\Gamma\left(s_k^+\right)}(\varrho t)^k\sum_{j=1}^{M_k^{n-1}}a_{k,j}Y_{k,j}(\zeta),\quad \rho\geq 0,\quad \zeta\in\mathbf{S}^{n-1}.$$

Again, if we set $\upsilon=\tanh^2(\varrho/2)$ in (6.14), we obtain

$$\Psi_{r,k,n}(\upsilon)=c_{k,n,r,l}2^k\left(1-\upsilon^2\right)^{\frac{n-1}{2}+ir}\frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(s_k^+\right)\Gamma\left(\frac{1}{2}-ir\right)}\int_0^1(\upsilon t)^k\frac{t^{\frac{n-3}{2}+ir}(1-t)^{-\frac{1}{2}-ir}}{(1-\upsilon^2 t)^{\frac{1}{2}-ir}}\,dt. \tag{6.17}$$

Using the integral formula (6.17) in (6.13) we obtain another integral representation for the general eigenfunction H, namely,

$$H(\upsilon,\zeta)=\frac{\left(1-\upsilon^2\right)^{\frac{n-1}{2}+ir}}{\Gamma\left(\frac{1}{2}-ir\right)}\int_0^1F_{r,n}^{(2)}(\upsilon t)\frac{t^{\frac{n-3}{2}+ir}(1-t)^{-\frac{1}{2}-ir}}{(1-\upsilon^2 t)^{\frac{1}{2}-ir}}\,dt, \tag{6.18}$$

where

$$F_{r,n}^{(2)}(\upsilon t\zeta)=\sum_{k=0}^{\infty}\frac{c_{k,n,r,l}2^k\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}+k+ir\right)}(\upsilon t)^k\sum_{j=1}^{M_k^{n-1}}a_{k,j}Y_{k,j}(\zeta)$$

and $\upsilon\geq 0$, $\zeta\in\mathbf{S}^{n-1}$.

We recall that any harmonic function F in the Euclidean unit ball $\mathbf{B}^n$ can be written

$$F(\rho\zeta)=\sum_{k=0}^{\infty}\rho^k\sum_{j=1}^{M_k^{n-1}}\widetilde{a}_{k,j}Y_{k,j}(\zeta),\quad 0\leq\rho<1,$$

for suitable coefficients $\widetilde{a}_{k,j}$.

Thus for $0 \le \rho < 1$, $\varrho = -\sinh^2(\rho/2)$ and $\upsilon = \tanh^2(\rho/2)$, the general eigenfunctions $H(\varrho, \zeta)$ and $H(\upsilon, \zeta)$ turn out to be integral transforms of Euclidean harmonic functions in the Euclidean unit ball.

In summary we have the following statement.

Theorem 6.3. *Let $\Delta_{\mathbf{H}^n}$ be the Laplacian in the upper half-space $\mathbf{H}^n$ and let $\Psi_{r,k,n}$ be the generalised spherical functions of $\Delta_{\mathbf{H}^n}$ given by (6.11). Let $Y_k(\zeta)$ be spherical harmonic functions of degree k in n dimensions which form an orthonormal basis $\left(Y_{k,j} : k \ge 0, 1 \le j \le M_k^{n-1}\right)$ of $\mathfrak{H}^{n-1}$. Then the generalised spherical function $\Psi_{r,k,n}(\rho)$ in $\mathbf{H}^n$ has explicit integral representations*

$$\Psi_{r,k,n}(\varrho) = c_{k,n,r,l} \frac{(-i)^k \varrho^k \left(1-\varrho^2\right)^{\frac{n}{2}+k} \Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}+k+ir\right)\Gamma\left(\frac{1}{2}-ir\right)} \int_0^1 \frac{t^{\frac{n-3}{2}+k+ir}(1-t)^{-\frac{1}{2}-ir}}{\left(1-\varrho^2(1-t)\right)^{\frac{1}{2}-ir}}\, dt,$$

where $\varrho = -\sinh^2(\rho/2)$; and

$$\Psi_{r,k,n}(\upsilon) = \left(1-\upsilon^2\right)^{\frac{n-1}{2}+ir} \frac{2^k c_{k,n,r,l} \Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(s_k^+\right)\Gamma\left(\frac{1}{2}-ir\right)} \int_0^1 (\upsilon t)^k \frac{t^{\frac{n-3}{2}+ir}(1-t)^{-\frac{1}{2}-ir}}{\left(1-\upsilon^2 t\right)^{\frac{1}{2}-ir}}\, dt,$$

where $\upsilon = \tanh^2(\rho/2)$. Moreover, the general eigenfunction $H \in \mathscr{E}_r(\mathbf{H}^n)$ has integral representations

$$H(\varrho, \zeta) = \frac{1}{\Gamma\left(\frac{1}{2}-ir\right)} \int_0^1 F_{r,n}^{(1)}(\varrho t \zeta) \frac{t^{\frac{n-3}{2}+ir}(1-t)^{-\frac{1}{2}-ir}}{\left(1-\varrho^2(1-t)\right)^{\frac{1}{2}-ir}}\, dt,$$

where

$$F_{r,n}^{(1)}(\varrho t \zeta) = \Gamma\left(\frac{n-1}{2}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{i}{2}\right)^k}{\left(1-\varrho^2\right)^{-\frac{n}{2}-k}} \frac{\pi^{2k+\frac{1}{2}} \mathscr{N}_{k,n}(r)}{\Gamma\left(\frac{n-1}{2}+k+ir\right)} (\varrho t)^k \sum_{j=1}^{M_k^{n-1}} a_{k,j} Y_{k,j}(\zeta);$$

and

$$H(\upsilon, \zeta) = \frac{\left(1-\upsilon^2\right)^{\frac{n-1}{2}+ir}}{\Gamma\left(\frac{1}{2}-ir\right)} \int_0^1 F_{r,n}^{(2)}(\upsilon t \zeta) \frac{t^{\frac{n-3}{2}+ir}(1-t)^{-\frac{1}{2}-ir}}{\left(1-\upsilon^2 t\right)^{\frac{1}{2}-ir}}\, dt,$$

where

$$F_{r,n}^{(2)}(\upsilon t \zeta) = \Gamma\left(\frac{n-1}{2}\right) \sum_{k=0}^{\infty} \frac{\pi^{2k+\frac{1}{2}} \mathscr{N}_{k,n}(r)}{\Gamma\left(\frac{n-1}{2}+k+ir\right)} (\upsilon t)^k \sum_{j=1}^{M_k^{n-1}} a_{k,j} Y_{k,j}(\zeta).$$

Remark 6.2. *The spherical functions $\Phi_k^{\mathbf{S}^n}(\theta)$ on the unit sphere $\mathbf{S}^n$ given by Proposition 1.15 with associated eigenvalues $(k(k+n-1) : k \ge 0)$ can be obtained from the spherical function $\Phi_r^{\mathbf{H}^n}(\rho)$ in the hyperbolic upper half-space $\mathbf{H}^n$ with associated eigenvalues $\left((n-1)^2/4 + r^2 : r \in \mathbf{R}\right)$ by analytic continuation, i.e., by letting $r \to i\,(k+(n-1)/2)$ and $\rho \to i\theta$ in $\Phi_r^{\mathbf{H}^n}(\rho)$:*

$$\begin{aligned}
\Phi_k^{\mathbf{S}^n}(\theta) = \mathscr{C}_k^{\frac{n-1}{2}}(\cos\theta) &= i^{\frac{n-2}{2}} 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \sin^{\frac{2-n}{2}}\theta P_{k+\frac{n-2}{2}}^{\frac{2-n}{2}}(\cos\theta) \\
&= \frac{C_k^{\frac{n-1}{2}}(\cos\theta)}{C_k^{\frac{n-1}{2}}(1)} = F\left(k+n-1, -k; \frac{n}{2}; \sin^2\frac{\theta}{2}\right) \\
&= \frac{k!\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(k+\frac{n}{2}\right)} P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(\cos\theta).
\end{aligned}$$

(*See also Section 1.4 and* (B.119).)

Before ending this section we shall discuss some integral transforms that will be useful in the sequel. As we have mentioned earlier in this section we shall see the role of the Harish-Chandra c-function obtained above in some of the computations we will be dealing with in the subsequent sections. It turns out that the inverse of the square of the absolute value of the product of the Harish-Chandra c-function is *Plancherel measure* which plays a central role in every aspect of harmonic analysis on symmetric spaces (as we shall see shortly for the particular case of the hyperbolic space $\mathbf{H}^n$). Of a particular importance is the inversion formula for the spherical transform which gives the spectral distribution of the eigenvalues of the Laplacian in the upper half-space $\mathbf{H}^n$.

The spherical function $\Phi_r^{\mathbf{H}^n}(\rho)$ is the non-Euclidean analogue of e^{ir} in the Euclidean harmonic analysis. In other words, the spherical function $\Phi_r^{\mathbf{H}^n}(\rho)$ plays the same role for Non-Euclidean spaces as the function e^{ir} plays in the Euclidean space.

We now introduce the spherical Fourier transform which is the non-Euclidean version of the classical (Euclidean) Fourier transform (1.71).

Definition 6.4. *Let $f \in C_0^\infty(\mathbf{H}^n)$. The Fourier transform of f is given by*

$$\widehat{f}(r,\zeta) = \int_{\mathbf{H}^n} f(w) y^{\frac{n-1}{2}+ir}\, d\mu_{\mathbf{H}^n}(w), \quad y = (\cosh\rho + \sinh\rho\cos\theta(\zeta,\zeta'))^{-1}. \tag{6.19}$$

If we now use the decomposition

$$\int_{\mathbf{H}^n} d\mu_{\mathbf{H}^n} = \frac{1}{\nu_{n-1}} \int_0^\infty \int_{\mathbf{S}^{n-1}} d\nu_{n-1} d\rho = \frac{\nu_{n-2}}{\nu_{n-1}} \int_0^\infty \int_0^\pi \sin^{n-2}\theta\, d\theta \sinh^{n-1}\rho\, d\rho, \tag{6.20}$$

with $w = (\rho,\theta) = \rho$, $y = (\cosh\rho + \sinh\rho\cos\theta)^{-1}$, we obtain

$$\begin{aligned}\widetilde{f}(r) &= \int_0^\infty \left(\frac{\nu_{n-2}}{\nu_{n-1}} \int_0^\pi \frac{\sin^{n-2}\theta\, d\theta}{(\cosh\rho + \sinh\rho\cos\theta)^{\frac{n-1}{2}+ir}} \right) f(\rho) \sinh^{n-1}\rho\, d\rho \\ &= \int_0^\infty \Phi_r^{\mathbf{H}^n}(\rho) f(\rho) \sinh^{n-1}\rho\, d\rho. \end{aligned} \tag{6.21}$$

Integral (6.21) is the generalisation of the Mehler-Fock transform (1.50). We call $\widetilde{f}(r)$ given by (6.21) the spherical transform of f. Moreover, the inversion formula for the spherical transform

$$f(\rho) = \frac{2^{n-1}}{2\pi\nu_{n-1}} \int_0^\infty \Phi_r^{\mathbf{H}^n}(\rho) \widetilde{f}(r) |c(r)|^{-2}\, dr \tag{6.22}$$

and the Plancherel formula

$$\|f(\rho)\|_2^2 = \frac{2^{n-1}}{2\pi\nu_{n-1}} \int_0^\infty |\widetilde{f}(r)|^2 |c(r)|^{-2}\, dr \tag{6.23}$$

hold (see Section 6.3), where

$$\sigma_{\mathbf{H}^n}(r) := |c(r)|^{-2} = [c(r)c(-r)]^{-1} = \frac{\pi \left|\Gamma\left(\frac{n-1}{2} + ir\right)\right|^2}{2^{2n-4}\Gamma\left(\frac{n}{2}\right)^2 |\Gamma(ir)|^2}, \tag{6.24}$$

and it is called the Plancherel measure in $\mathbf{H}^n$.

The name "Plancherel measure" is not surprising because it arises in the Plancherel inversion formula (6.22) and in the Plancherel norm (6.23). Recall that the function $c(r)$ is obtained as the asymptotic function at infinity of the spherical functions $\Phi_r^{\mathbf{H}^n}(\rho)$. The integral (6.22) is the generalisation of the Mehler-Fock inversion formula (1.51). It follows from (6.24) the asymptotic behaviour of $\sigma_{\mathbf{H}^n}(r)$ for $\mathbf{R} \ni r \nearrow \infty$:

$$\sigma_{\mathbf{H}^n}(r) \sim r^{n-1}, \qquad r \nearrow \infty \tag{6.25}$$

(compare with the asymptotics (4.47) of the multiplicity M_k^n of the sphere $\mathbf{S}^n$). The Plancherel measure $\sigma_{\mathbf{H}^n}(r)$ can further be written explicitly as (see Section 6.3)

$$\begin{aligned}
\sigma_{\mathbf{H}^n}(r) &= \frac{\pi}{2^{2n-4}\Gamma\left(\frac{n}{2}\right)^2} \prod_{j=0}^{(n-3)/2} \left(r^2+j^2\right), \quad n \text{ odd}, n \geq 3; \\
\sigma_{\mathbf{H}^n}(r) &= \frac{\pi r \tanh \pi r}{2^{2n-4}\Gamma\left(\frac{n}{2}\right)^2} \prod_{j=1/2,3/2}^{(n-3)/2} \left(r^2+j^2\right) \quad n \text{ even}, n \geq 4
\end{aligned} \tag{6.26}$$

(for the case $n = 2$ the product is omitted).

We shall see how we moved from (6.24) to (6.26) in Section 6.3 when establishing the generalisation of the Mehler-Fock integral formula (6.22) via the spectral properties of the Lalacian $\Delta_{\mathbf{H}}$. It follows that

$$2^{2n-4}\Gamma\left(\frac{n}{2}\right)^2 \pi^{-1} \sigma_{\mathbf{H}^n}(r) = \frac{\left|\Gamma\left(\frac{n-1}{2}+ir\right)\right|^2}{|\Gamma(ir)|^2}.$$

Furthermore, $\sigma_{\mathbf{H}^n}(r)$, n odd, is analytic, and in fact a polynomial in r^2; and $\sigma_{\mathbf{H}^n}(r)$, n even, is a meromorphic function with simple poles on the imaginary axis.

Example 6.1. *For the special case $n = 2$, i.e., in $\mathbf{H}$,*

$$\sigma_{\mathbf{H}}(r) = \sigma(r) = \frac{\pi\left|\Gamma\left(\frac{1}{2}+ir\right)\right|^2}{|\Gamma(ir)|^2} = \pi r \tanh \pi r,$$

where we have used (B.4) and (B.5). Thus the spherical transform of the function $f \in C_0^\infty(\mathbf{H})$ becomes

$$\widetilde{f}(r) = \int_0^\infty \Phi_r^{\mathbf{H}}(\rho) f(\rho) \sinh \rho \, d\rho; \tag{6.27}$$

and the corresponding inversion formula is

$$f(\rho) = \frac{1}{2\pi} \int_0^\infty \Phi_r^{\mathbf{H}}(\rho) \widetilde{f}(r) r \tanh \pi r \, dr$$

(see (1.50) and (1.51)).

6.2 The Heat Kernel via the Wave Equation

As we have pointed out earlier that the heat kernel plays a central role in harmonic analysis on Riemannian manifolds, of a particular importance is the wave equation. The wave equation is not only useful in its own right as an equation of mathematical physics, but also in number theory. For instance, Weyl's asymptotic distribution of eigenvalues with remainder term is proved using the wave equation (see Hörmander [86], Duistermaat and Guillemin [55], Bérard [25], Chazarain [42], Sogge [153], see also Pinsky and Taylor [128]); wave equation method is also used in the computation of the asymptotics expansion of hyperbolic lattice point counting function in Lax and Phillips [99], Levitan [102]. In this section we shall use the relation between the solutions of the heat and wave equations to compute the heat kernel in the real hyperbolic space $\mathbf{H}^n$. The required appropriate solution of the wave equation in $\mathbf{H}^n$ is given by Helgason [83, pp. 574-577] (see also Lax and Phillips [99]). This is the approach of Grigor'yan and Noguchi [69], but for completion we shall revisit the discussion in details. The main tool is the Euclidean Fourier transform.

Consider the initial value problem for the wave equation in the upper half-space $\mathbf{H}^n$, given by

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \left(\frac{\partial^2}{\partial \rho^2} + (n-1)\frac{\cosh\rho}{\sinh\rho}\frac{\partial}{\partial\rho}\right)u + \frac{(n-1)^2 u}{4}, \\ u(0,w) = u_0(w), &\quad \frac{\partial u(0,w)}{\partial t} = 0, \end{aligned} \tag{6.28}$$

for $t > 0$, $\rho = d(w,w') > 0$, $w = (x,y), w' = (x',y') \in \mathbf{H}^n$, $u_0 \in C^\infty(\mathbf{H}^n)$.

Let

$$\mathbf{S}_\rho(w) = \{w' = (x',y') \in \mathbf{H}^n \,:\, d(w,w') = \rho\}$$

be the sphere centred at $w \in \mathbf{H}^n$ with radius ρ.

According to Helgason [83, pp. 574-577] the solution u of the wave equation (6.28) is given by

(a) n odd, $n \geq 3$,

$$\begin{aligned} u(t,w) &= \cos\left(t\sqrt{\Delta_{\mathbf{H}^n} - \frac{(n-1)^2}{4}}\right)u_0(w) \\ &= c_n \frac{\partial}{\partial t}\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(\frac{1}{\sinh t}\int_{\mathbf{S}_t(w)} u_0(w')\,d\nu_{n-1}(w')\right), \end{aligned} \tag{6.29}$$

where

$$c_n = \frac{1}{2(n-3)!!\nu_{n-2}};$$

and

(b) n even, $n \geq 2$,

$$\begin{aligned} u(t,w) &= \cos\left(t\sqrt{\Delta_{\mathbf{H}^n} - \frac{(n-1)^2}{4}}\right)u_0(w) \\ &= \widetilde{c}_n \frac{\partial}{\partial t}\int_0^t \frac{\sinh\rho}{(\cosh t - \cosh\rho)^{1/2}}\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}}\left(\frac{1}{\sinh\rho}\int_{\mathbf{S}_\rho(w)} u_0(w')\,d\nu_{n-1}(w')\right)d\rho, \end{aligned} \tag{6.30}$$

where

$$\widetilde{c}_n = \frac{1}{\sqrt{2}\pi(n-3)!!\nu_{n-2}}.$$

In this section we shall obtain the solution of the heat equation in the upper half-space $\mathbf{H}^n$ via the solution (6.29)-(6.30) of the wave equation (6.28). The main tool that relates these two equations is the Euclidean Fourier transform. We shall consider two heat equations in $\mathbf{H}^n$; the first one involves the radial part of the classical Laplacian $\Delta_{\mathbf{H}^n}$ and the second is the heat equation associated to the shifted Laplacian $-\Delta'_{\mathbf{H}^n} = -\Delta_{\mathbf{H}^n} + \left(\frac{n-1}{2}\right)^2$, namely

$$\begin{aligned}\frac{\partial v}{\partial \tau} &= \left(\frac{\partial^2}{\partial \rho^2} + (n-1)\frac{\cosh\rho}{\sinh\rho}\frac{\partial}{\partial\rho}\right) v,\\ v(0,w) &= v_0(w);\end{aligned} \tag{6.31}$$

and

$$\begin{aligned}\frac{\partial v}{\partial \tau} &= \left(\frac{\partial^2}{\partial \rho^2} + (n-1)\frac{\cosh\rho}{\sinh\rho}\frac{\partial}{\partial\rho}\right) v + \frac{(n-1)^2 v}{4},\\ v(0,w) &= v_0(w),\end{aligned} \tag{6.32}$$

for $t>0$, $\rho = d(w,w')>0$, $w=(x,y), w'=(x',y')\in\mathbf{H}^n$, $v_0\in C^\infty(\mathbf{H}^n)$.

We recall that in Chapter 5 we used the Gegenbauer transform to solve the heat equation on $\mathbf{S}^n$, and after applying the Riemann-Liouville fractional derivative formula we obtained fractional formulae for the heat kernel on $\mathbf{S}^n$; here we use the wave equation in $\mathbf{H}^n$ to obtain fractional formulae for the heat kernel in $\mathbf{H}^n$.

It is known that the solution operator to the heat equation (6.31) is given by

$$v(\tau,w) = \exp\left(-\tau\Delta_{\mathbf{H}^n}\right) v_0(w), \tag{6.33}$$

and the solution operator to the heat equation (6.32) is given by

$$v(\tau,w) = \exp\left(-\tau\Delta'_{\mathbf{H}^n}\right) v_0(w) = \exp\left(-\tau\Delta_{\mathbf{H}^n} + \frac{(n-1)^2}{4}\right) v_0(w). \tag{6.34}$$

Next we shall relate the heat operator $e^{-\tau\Delta'_{\mathbf{H}^n}}$ to the wave operator $\cos\left(t\sqrt{\Delta_{\mathbf{H}^n} - \frac{(n-1)^2}{4}}\right)$ using the Euclidean Fourier transform, namely

$$\exp\left(-\tau\Delta_{\mathbf{H}^n} + \frac{(n-1)^2}{4}\right) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{4\tau}}\cos\left(t\sqrt{\Delta_{\mathbf{H}^n} - \frac{(n-1)^2}{4}}\right) dt; \tag{6.35}$$

and for a function $u_0 = v_0\in C^\infty(\mathbf{H}^n)$, we have

$$\begin{aligned} v(\tau,w) = \left(e^{-\tau\Delta'_{\mathbf{H}^n}} v_0\right)(w) &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{4\tau}}\cos\left(t\sqrt{\Delta'_{\mathbf{H}^n}}\right) u_0(w)\,dt\\ &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{4\tau}} u(t,w)\,dt,\end{aligned} \tag{6.36}$$

where $u(t,w)$ is the solution of the wave equation (6.28) given by (6.29)-(6.30). We now insert the solution of the wave equation which we already know into (6.36) to obtain the solution of the heat equation (6.32), from which we can then deduce the solution to the heat equation (6.31) using the relation

$$\exp\left(-\tau\Delta'_{\mathbf{H}^n}\right) = \exp\left(-\tau\Delta_{\mathbf{H}^n} + \frac{(n-1)^2}{4}\right). \tag{6.37}$$

We shall do this according to whether n is odd or even. We first consider the case n odd.

(a) n odd, $n \geq 3$. Substituting (6.29) into (6.36) gives

$$v(\tau, w) = \frac{c_n}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{4\tau}} \frac{\partial}{\partial t} \left(\frac{1}{\sinh t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(\frac{1}{\sinh t} \int_{\mathbf{S}_t(w)} u_0(w')\, d\nu_{n-1}(w') \right) dt.$$

What we do next is to integrate by parts. Towards this end we apply the formula

$$\int_0^{\infty} f(t) \frac{\partial}{\partial t} \left(\frac{1}{\sinh t} \frac{\partial}{\partial t} \right)^{\mu-1} [h(t)u(t)]\, dt = (-1)^{\mu} \int_0^{\infty} \left\{ h(t) \left(\frac{1}{\sinh t} \frac{\partial}{\partial t} \right)^{\mu} f(t) \right\} u(t)\, dt, \tag{6.38}$$

with

$$f(t) = e^{-\frac{t^2}{4\tau}}, \quad h(t) = \frac{1}{\sinh t}, \quad u(t) = \int_{\mathbf{S}_t(w)} u_0(w')\, d\nu_{n-1}(w'),$$

to get

$$v(\tau, w) = \frac{2c_n}{\sqrt{4\pi\tau}} \int_0^{\infty} \int_{\mathbf{S}_t(w)} \left(-\frac{1}{\sinh t} \frac{\partial}{\partial t} \right)^{\frac{n-1}{2}} e^{-\frac{t^2}{4\tau}} u_0(w')\, d\nu_{n-1}(w')\, dt.$$

By the decomposition (6.20), we have

$$v(\tau, w) = \frac{2c_n}{\sqrt{4\pi\tau}} \int_{\mathbf{H}^n} \left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-1}{2}} e^{-\frac{\rho^2}{4\tau}} u_0(w')\, d\mu_{\mathbf{H}^n}(w'), \tag{6.39}$$

bearing in mind that $\mathbf{S}_t(w)$ is the sphere in $\mathbf{H}^n$ with the centre $w \in \mathbf{H}^n$ and radius $t > 0$. It follows from (A.5) that the integral kernel of the heat operator $e^{-\tau\Delta'_{\mathbf{H}^n}}$ associated to the shifted Laplacian $\Delta'_{\mathbf{H}^n}$ is given by

$$\mathsf{K}_{\mathbf{H}^n}(\tau, w, w') = \frac{2c_n}{\sqrt{4\pi\tau}} \left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-1}{2}} e^{-\frac{\rho^2}{4\tau}};$$

and by the relation (6.37) we obtain the heat kernel in $\mathbf{H}^n$ associated to the classical Laplacian $\Delta_{\mathbf{H}^n}$:

$$K_{\mathbf{H}^n}(\tau, w, w') = \frac{1}{\sqrt{4\pi\tau}} \left(-\frac{1}{2\pi \sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-1}{2}} e^{-\frac{(n-1)^2\tau}{4}} e^{-\frac{\rho^2}{4\tau}},$$

where we have used the identity

$$2c_n = \frac{1}{(n-3)!!\nu_{n-2}} = (2\pi)^{-\frac{n-1}{2}}.$$

(b) n even, $n \geq 2$: We now compute the solution of the heat equation in $\mathbf{H}^n$ for the case n even. As we did for the case n odd we substitute the corresponding solution of the wave equation in $\mathbf{H}^n$, namely equation (6.30) in (6.36) to obtain

$$v(\tau, w) = \frac{\widetilde{c}_n}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{4\tau}} \left[\frac{\partial}{\partial t} \int_0^t \frac{\sinh \rho}{\sqrt{\cosh t - \cosh \rho}} \left(\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-2}{2}} \left(\frac{\mathfrak{u}(\rho)}{\sinh \rho} \right) d\rho \right] dt.$$

By one-time integration by parts we get

$$v(\tau, w) = \frac{\widetilde{c}_n}{\sqrt{4\pi}\tau^{\frac{3}{2}}} \int_0^{\infty} t e^{-\frac{t^2}{4\tau}} \left[\int_0^t \frac{\sinh \rho}{\sqrt{\cosh t - \cosh \rho}} \left(\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-2}{2}} \frac{\mathfrak{u}(\rho)}{\sinh \rho}\, d\rho \right] dt,$$

where

$$\mathtt{u}(\rho) = \int_{\mathbf{S}_\rho(w)} u_0(w')\, d\nu_{n-1}(w').$$

Changing the order of integration, we obtain

$$v(\tau, w) = \frac{\widetilde{c}_n}{\sqrt{4\pi}\tau^{\frac{3}{2}}} \int_0^\infty \left(\int_\rho^\infty \frac{te^{-\frac{t^2}{4\tau}}}{\sqrt{\cosh t - \cosh \rho}}\, dt \right) \sinh \rho \left(\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-2}{2}} \left(\frac{\mathtt{u}(\rho)}{\sinh \rho} \right) d\rho.$$

By applying the formula (6.38) we get,

$$v(\tau, w) = \frac{\widetilde{c}_n}{\sqrt{4\pi}\tau^{\frac{3}{2}}} \int_0^\infty \left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-2}{2}} \left(\int_\rho^\infty \frac{te^{-\frac{t^2}{4\tau}}}{\sqrt{\cosh t - \cosh \rho}}\, dt \right) \mathtt{u}(\rho)\, d\rho,$$

which on using the decomposition (6.20) becomes

$$v(\tau, w) = \frac{\widetilde{c}_n}{\sqrt{4\pi}\tau^{\frac{3}{2}}} \int_{\mathbf{H}^n} \left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-2}{2}} \left(\int_\rho^\infty \frac{te^{-\frac{t^2}{4\tau}}}{\sqrt{\cosh t - \cosh \rho}}\, dt \right) u_0(w')\, d_{\mathbf{H}^n}\mu(w').$$

Thus the integral kernel of the heat operator $e^{-\tau \Delta'_{\mathbf{H}^n}}$ associated to the shifted Laplacian $\Delta'_{\mathbf{H}^n}$ is given by

$$\mathsf{K}_{\mathbf{H}^n}(\tau, w, w') = \frac{\widetilde{c}_n}{\sqrt{4\pi}\tau^{\frac{3}{2}}} \left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-2}{2}} \left(\int_\rho^\infty \frac{te^{-\frac{t^2}{4\tau}}}{\sqrt{\cosh t - \cosh \rho}}\, dt \right),$$

and by the relation (6.37) we obtain the heat kernel of the heat operator $e^{-\tau \Delta_{\mathbf{H}^n}}$ associated to the classical Laplacian $\Delta_{\mathbf{H}^n}$:

$$K_{\mathbf{H}^n}(\tau, w, w') = \frac{e^{-\frac{(n-1)^2 \tau}{4}}}{2^{\frac{n+3}{2}} \pi^{\frac{n+1}{2}} \tau^{\frac{3}{2}}} \left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-2}{2}} \left(\int_\rho^\infty \frac{te^{-\frac{t^2}{4\tau}}}{\sqrt{\cosh t - \cosh \rho}}\, dt \right),$$

where we have used the identity

$$\frac{\widetilde{c}_n}{2\sqrt{\pi}} = \frac{1}{2^{\frac{n+3}{2}} \pi^{\frac{n+1}{2}}}.$$

In summary we have the following statement which should be compared with Theorem 5.3, p. 131.

Theorem 6.5. *For $t > 0$, $0 < \rho < \infty$, the heat kernel $K_{\mathbf{H}^n}(t, \rho)$ associated to the Laplacian $\Delta_{\mathbf{H}^n}$ in the hyperbolic space $\mathbf{H}^n$ is given by the following fractional and integral representations:*

(a) n odd $n \geq 3$,

$$K_{\mathbf{H}^n}(t, w, w') = \left(-\frac{1}{2\pi} \frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\frac{n-1}{2}} e^{-\frac{(n-1)^2 t}{4}} K_{\mathbf{H}^1}(t, w, w'), \tag{6.40}$$

where

$$K_{\mathbf{H}^1}(t, w, w') = \frac{1}{\sqrt{4\pi t}} e^{-\frac{\rho^2}{4t}}$$

coincides with the heat kernel in $\mathbf{R}$; and (*b*) n *even* $n \geq 2$,

$$K_{\mathbf{H}^n}(t,w,w') = \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} e^{-\frac{n(n-2)t}{4}} K_{\mathbf{H}}(t,w,w'), \tag{6.41}$$

where

$$K_{\mathbf{H}}(t,w,w') = \widetilde{K}(t,w,w') = \frac{\sqrt{2}}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{t}{4}} \int_\rho^\infty \frac{u e^{-\frac{u^2}{4t}}}{\sqrt{\cosh u - \cosh\rho}}\, du \tag{6.42}$$

is the heat kernel in $\mathbf{H}$ (*see Subsection 1.5.3*).

6.3 Fractional and Integral Representations of the Green Function

As we have mentioned in Chapter 5 an important spectral function for studying the spectrum of a Riemannian manifold in general and the hyperbolic space in particular is the Green function. Let $\Delta_{\mathbf{H}^n}$ be the Laplacian in $\mathbf{H}^n$. The Green function $G_{\mathbf{H}^n}(\cdot,\cdot;s)$ of $\Delta_{\mathbf{H}^n}$ in $\mathbf{H}^n$ is the kernel of the resolvent operator $(\Delta_{\mathbf{H}^n} - s(n-1-s))^{-1}$, $s \in \mathbf{C}$, for which $\lambda = s(n-1-s)$ is not an eigenvalue of $\Delta_{\mathbf{H}^n}$. In this section we shall compute the Green function in $\mathbf{H}^n$ using two different approaches. The first approach takes the Laplace transform of the heat kernel in $\mathbf{H}^n$ to obtain integral representations of the Green function in $\mathbf{H}^n$, in the form of the heat kernel in $\mathbf{H}^n$. The second approach is for future purposes; it involves solving an eigenvalue problem in $\mathbf{H}^n$ by substitution method and the Green function is expressed in terms of the Gauss hypergeometric function. This hypergeometric function representation of the Green function will be used in the next section to derive the Mehler-Fock inversion formula of higher order, which we later use to establish the heat kernel in $\mathbf{H}^n$.

We start with the Laplace transform approach. Let $K_{\mathbf{H}^n}(t,w,w')$ be the heat kernel in $\mathbf{H}^n$. In our usual way, we calculate according to whether n is odd or even. We start with the case n odd.

- n odd, $n \geq 1$. Indeed from (6.40) we have

$$\begin{aligned} G_{\mathbf{H}^n}(w,w';s) &= \int_0^\infty e^{\lambda t} K_{\mathbf{H}^n}(t,w,w')\, dt \\ &= \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-1}{2}} \int_0^\infty e^{-s(s-n+1)t} e^{-\frac{(n-1)^2 t}{4}} K_{\mathbf{H}^1}(t,w,w')\, dt \\ &= \frac{1}{2\left(s-\frac{n-1}{2}\right)} \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-1}{2}} e^{-\rho\left(s-\frac{n-1}{2}\right)}, \end{aligned} \tag{6.43}$$

where we have used (B.49) and (B.47). Setting $s = (n-1)/2 + ir$, we obtain

$$G_{\mathbf{H}^n}(w,w';r) = \frac{1}{2ir}\left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-1}{2}} e^{-i\rho r}. \tag{6.44}$$

- n even, $n \geq 2$. From (6.41) we obtain

$$
\begin{aligned}
G_{\mathbf{H}^n}(w,w';s) &= \int_0^\infty e^{\lambda t} K_{\mathbf{H}^n}(t,w,w')\,dt \\
&= \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \int_0^\infty e^{-s(s-n+1)t} e^{-\frac{n(n-2)t}{4}} K_{\mathbf{H}}(t,w,w')\,dt \\
&= \frac{1}{2\pi}\left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \frac{1}{\sqrt{2}} \int_\rho^\infty \frac{e^{-u\left(s-\frac{n-1}{2}\right)}\,du}{\sqrt{\cosh u - \cosh\rho}},
\end{aligned}
\tag{6.45}
$$

where we have used (B.49) and (B.47). We can simplify further. Towards this end, we apply (B.97), (B.96) and write

$$
\cosh\rho = \cosh^2(\rho/2) + \sinh^2(\rho/2) = 2\zeta - 1, \quad \zeta = \cosh^2(\rho/2),
$$

to get

$$
\begin{aligned}
G_{\mathbf{H}^n}(w,w';s) &= \frac{1}{2\pi}\left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} Q_{s-\frac{n}{2}}(\cosh\rho) \\
&= \frac{1}{2\pi}\left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \sqrt{\pi}\frac{\Gamma\left(s-\frac{n}{2}+1\right)\zeta^{-s+\frac{n}{2}-1}}{2^{2s-n+2}\Gamma\left(s-\frac{n}{2}+\frac{3}{2}\right)} \\
&\quad \times F\left(s-\frac{n}{2}+1, s-\frac{n}{2}+1; 2s-n+2; \zeta^{-1}\right) \\
&= \frac{1}{4\pi}\left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \frac{\Gamma\left(s-\frac{n}{2}+1\right)^2}{\Gamma(2s-n+2)}\zeta^{-s+\frac{n}{2}-1} \\
&\quad \times F\left(s-\frac{n}{2}+1, s-\frac{n}{2}+1; 2s-n+2; \zeta^{-1}\right),
\end{aligned}
\tag{6.46}
$$

where we have used (B.3). If we now set $s = (n-1)/2 + ir$, we obtain

$$
\begin{aligned}
G_{\mathbf{H}^n}(w,w';r) &= \frac{1}{4\pi}\left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \frac{\Gamma\left(\frac{1}{2}+ir\right)^2}{\Gamma(1+2ir)}\zeta^{-\frac{1}{2}-ir} \\
&\quad \times F\left(\frac{1}{2}+ir, \frac{1}{2}+ir; 1+2ir; \zeta^{-1}\right) \\
&= \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \widetilde{\mathcal{G}}_r(w,w'),
\end{aligned}
$$

where $\widetilde{\mathcal{G}}_r$ is the Green function on $\mathbf{H}$.

For the second approach we solve the following eigenvalue problem in $\mathbf{H}^n$:

$$
R''(\widetilde{\rho}) + (n-1)\coth(\widetilde{\rho})R'(\widetilde{\rho}) + s(n-s-1)R(\widetilde{\rho}) = 0. \tag{6.47}
$$

Making the substitution

$$
R(\widetilde{\rho}) = G_{\mathbf{H}^n}(\zeta;s), \quad \zeta = \cosh^2\left(\frac{\widetilde{\rho}}{2}\right) = \frac{|x-x'|^2 + (y+y')^2}{4yy'}, \; w = x+iy, w' = x'+iy' \in \mathbf{H}^n,
$$

with

$$\frac{\partial\zeta}{\partial\widetilde{\rho}} = \sinh\left(\frac{\widetilde{\rho}}{2}\right)\cosh\left(\frac{\widetilde{\rho}}{2}\right), \quad \frac{\partial}{\partial\widetilde{\rho}}R(\widetilde{\rho}) = \frac{\partial G_{\mathbf{H}^n}(\zeta;s)}{\partial\zeta}\frac{\partial\zeta}{\partial\widetilde{\rho}} = G'_{\mathbf{H}^n}(\zeta;s)\sinh\left(\frac{\widetilde{\rho}}{2}\right)\cosh\left(\frac{\widetilde{\rho}}{2}\right)$$

$$\begin{aligned}
\frac{\partial^2}{\partial\widetilde{\rho}^2}R(\widetilde{\rho}) &= \frac{\partial}{\partial\widetilde{\rho}}\left(G'_{\mathbf{H}^n}(\zeta;s)\sinh\left(\frac{\widetilde{\rho}}{2}\right)\cosh\left(\frac{\widetilde{\rho}}{2}\right)\right)\\
&= \frac{\partial}{\partial\zeta}G'_{\mathbf{H}^n}(\zeta;s)\frac{\partial\zeta}{\partial\widetilde{\rho}}\sinh\left(\frac{\widetilde{\rho}}{2}\right)\cosh\left(\frac{\widetilde{\rho}}{2}\right) + \frac{1}{2}G'_{\mathbf{H}^n}(\zeta;s)\sinh^2\left(\frac{\widetilde{\rho}}{2}\right) + \frac{1}{2}G'_{\mathbf{H}^n}(\zeta;s)\cosh^2\left(\frac{\widetilde{\rho}}{2}\right)\\
&= G''_{\mathbf{H}^n}(\zeta;s)\sinh^2\left(\frac{\widetilde{\rho}}{2}\right)\cosh^2\left(\frac{\widetilde{\rho}}{2}\right) + \frac{1}{2}G'_{\mathbf{H}^n}(\zeta;s)\sinh^2\left(\frac{\widetilde{\rho}}{2}\right) + \frac{1}{2}G'_{\mathbf{H}^n}(\zeta;s)\cosh^2\left(\frac{\widetilde{\rho}}{2}\right),
\end{aligned}$$

in (6.47) to obtain

$$\zeta(1-\zeta)G''_{\mathbf{H}^n}(\zeta;s) + \left(\frac{n}{2} - n\zeta\right)G'_{\mathbf{H}^n}(\zeta;s) - s(n-1-s)G_{\mathbf{H}^n}(\zeta;s) = 0.$$

This is the Gauss hypergeometric equation (B.54) with $a = s$, $b = n-1-s$, $c = n/2$. The solution $f(z) = F(a,b;c;z)$ is regular at $z = 0$, whereas we require a solution regular at $z = \infty$ since $\zeta \in [1,\infty)$. We therefore choose the Kummer solution (B.69) to obtain

$$G_{\mathbf{H}^n}(\zeta;s) = C_n(s)\zeta^{-s}F\left(s, s-\frac{n-2}{2}; 2s-(n-2); \zeta^{-1}\right), \tag{6.48}$$

where

$$C_n(s) = \frac{1}{2^n\pi^{\frac{n}{2}}}\frac{\Gamma(s)\Gamma\left(s-\frac{n}{2}+1\right)}{\Gamma(2s-(n-2))}.$$

The Green function $G_{\mathbf{H}^n}(\zeta;s)$ has an integral representation (see (B.62))

$$G_{\mathbf{H}^n}(\zeta;s) = \frac{1}{2^n\pi^{\frac{n}{2}}}\frac{\Gamma(s)}{\Gamma\left(s-\frac{n}{2}+1\right)}\int_0^1 [t(1-t)]^{s-\frac{n}{2}}(\zeta-t)^{-s}\,dt.$$

In summary we have the following statement.

Theorem 6.6. *For* $s = \frac{n-1}{2} + ir$, $r \in \mathbf{R}$, $0 < \rho < \infty$, $\rho = d(w,w')$, *the Green function* $G_{\mathbf{H}^n}(\rho;s)$ *associated to the Laplacian* $\Delta_{\mathbf{H}^n}$ *in the hyperbolic space* $\mathbf{H}^n$ *is given by the following fractional and hypergeometric function representations:*

(a) n *odd,* $n \geq 1$,

$$G_{\mathbf{H}^n}(w,w';s) = \frac{1}{2\left(s-\frac{n-1}{2}\right)}\left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-1}{2}} e^{-\rho\left(s-\frac{n-1}{2}\right)}; \tag{6.49}$$

(b) n *even,* $n \geq 2$,

$$G_{\mathbf{H}^n}(w,w';s) = \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \widetilde{\mathcal{G}}_s(w,w'), \tag{6.50}$$

where $\widetilde{\mathcal{G}}_s$ *is the Green function in* $\mathbf{H}$ *(see* (1.61)*); and*

(c) general $n \geq 2$,

$$G_{\mathbf{H}^n}(\zeta;r) = \frac{1}{2^n\pi^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n-1}{2}+ir\right)\Gamma\left(\frac{1}{2}+ir\right)}{\Gamma(1+2ir)}\zeta^{-\left(\frac{n-1}{2}+ir\right)}F\left(\frac{n-1}{2}+ir, \frac{1}{2}+ir; 1+2ir; \zeta^{-1}\right).$$

6.4 The Generalised Mehler-Fock Integral Formula

As we have earlier promised in Subsection 1.5.3 that the proof of Theorem 1.18 would be given for $n \geq 2$ in this section; we shall now present the explicit proof of the generalised Mehler-Fock integral formula using the Green function and the spectral resolution of self-adjoint operators (Birman and Solomjak [28], Guseinov [71]). The Mehler-Fock inversion formula for $\mathbf{H}^n$ is an integral transform involving the spherical functions in $\mathbf{H}^n$, namely the Legendre function, associated Legendre function and the Gegenbauer polynomial. The kernel of the classical Mehler-Fock inversion formula (see Theorem 1.18, see also Mehler [109], Fock [61]) is the Legendre function P_ν, $\nu = \frac{1}{2} + ir$, $r \in \mathbf{R}$, which is the associated Legendre function P_ν^μ of order $\mu = 0$. The order of the kernel is the order of the formula. In this section we compute the Mehler-Fock integral formula of order μ, $\mu \geq 0$, or what is the same the Mehler-Fock integral formula whose kernel is the spherical function $\mathscr{C}_\nu^\mu$ discussed in Section 6.1, namely

$$\mathscr{C}^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(\cosh\rho) := \frac{C^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(\cosh\rho)}{C^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(1)}) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\sinh^{\frac{2-n}{2}}\rho P^{\frac{2-n}{2}}_{-\frac{1}{2}+ir}(\cosh\rho). \tag{6.51}$$

In particular we extend the formula to the heat kernel in $\mathbf{H}^n$ by appropriately choosing a spectral test function. We also compute the generalised Mehler-Fock inversion formula via the Poisson kernel in $\mathbf{H}^n$. Some recursion formulae for the integral transform are established.

The precise statement is the following

Theorem 6.7. *Let E_ω, $\omega \in \mathbf{R}$, be the spectral projection for the shifted hyperbolic Laplacian $\Delta'_{\mathbf{H}^n} = \Delta_{\mathbf{H}^n} + \frac{(n-1)^2}{4}$. For $f \in L^2(\mathbf{H}^n)$ define the operator $\Delta'_{\mathbf{H}^n}$ by*

$$\Delta'_{\mathbf{H}^n} f = \int_0^\infty \omega\, dE_\omega f. \tag{6.52}$$

Then for a suitable function h,

$$h\left(\sqrt{\Delta'_{\mathbf{H}^n}}\right) f(w) = \int_{\mathbf{H}^n} K_{\mathbf{H}^n}(w,w') f(w')\, d\mu_{\mathbf{H}^n}(w'), \tag{6.53}$$

where the spectral kernel $K_{\mathbf{H}^n}(w,w')$ given by

$$\begin{aligned} K_{\mathbf{H}^n}(\rho) = K_{\mathbf{H}^n}(w,w') &= \frac{1}{\pi i}\int_0^\infty r h(r)\left(G_{\mathbf{H}^n}\left(\zeta;\frac{n-1}{2} - ir\right) - G_{\mathbf{H}^n}\left(\zeta;\frac{n-1}{2}+ir\right)\right) dr \\ &= \frac{2^{n-3}\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n+2}{2}}}\int_0^\infty h(r)\sigma_{\mathbf{H}^n}(r)\mathscr{C}^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(\cosh\rho)\, dr, \end{aligned} \tag{6.54}$$

is the generalised Mehler-Fock integral formula for $\mathbf{H}^n$, satisfying the Millson formula

$$K_{\mathbf{H}^{n+2}}(\rho) = -\frac{1}{2\pi\sinh\rho}\frac{\partial}{\partial\rho}K_{\mathbf{H}^n}(\rho), \tag{6.55}$$

and $\sigma_{\mathbf{H}^n}(r)$ is the Plancherel measure (6.26). Moreover, for an appropriate spectral function $h_t(r)$, (*a*) *n odd, $n \geq 1$,*

$$K_{\mathbf{H}^n}(t,w,w') = e^{-\frac{(n-1)^2}{4}t} \frac{(-1)^{\frac{n-1}{2}}}{\sqrt{2}(2\pi)^{\frac{n}{2}}\sqrt{t}} \left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-1}{2}} e^{-\frac{\rho^2}{4t}};$$

(*b*) *n even, $n \geq 2$,*

$$K_{\mathbf{H}^n}(t,w,w') = \frac{e^{-\frac{n(n-2)}{4}t}e^{-\frac{t}{4}}\sqrt{2}}{8\pi^{\frac{3}{2}}t^{\frac{3}{2}}} \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \int_\rho^\infty \frac{ue^{-\frac{u^2}{4t}}\,du}{\sqrt{\cosh u - \cosh\rho}}.$$

Proof. The proof of the first part of the theorem uses the Green function discussed in the previous section and the spectral theory of self-adjoint operator. We start with the first part that uses the Green function in $\mathbf{H}^n$. Indeed, from (6.48) we have

$$G_{\mathbf{H}^n}\left(\zeta;\frac{n-1}{2}+ir\right) = C_n\left(\frac{n-1}{2}+ir\right)\zeta^{-\left(\frac{n-1}{2}+ir\right)}F\left(\frac{n-1}{2}+ir,\frac{1}{2}+ir;1+2ir;\zeta^{-1}\right);$$
$$G_{\mathbf{H}^n}\left(\zeta;\frac{n-1}{2}-ir\right) = C_n\left(\frac{n-1}{2}-ir\right)\zeta^{-\left(\frac{n-1}{2}-ir\right)}F\left(\frac{n-1}{2}-ir,\frac{1}{2}-ir;1-2ir;\zeta^{-1}\right),$$

with

$$C_n\left(\frac{n-1}{2}+ir\right) = \frac{1}{2^n\pi^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n-1}{2}+ir\right)\Gamma\left(\frac{1}{2}+ir\right)}{\Gamma(1+2ir)};$$
$$C_n\left(\frac{n-1}{2}-ir\right) = \frac{1}{2^n\pi^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n-1}{2}-ir\right)\Gamma\left(\frac{1}{2}-ir\right)}{\Gamma(1-2ir)}.$$

By using the transformation formula (B.66), and the identities

$$\Gamma(1-\beta+\alpha) = (\alpha-\beta)\Gamma(\alpha-\beta),\ \ \Gamma(1-\alpha+\beta) = (\beta-\alpha)\Gamma(\beta-\alpha),$$

we obtain the formula

$$G_{\mathbf{H}^n}\left(\zeta;\frac{n-1}{2}-ir\right) - G_{\mathbf{H}^n}\left(\zeta;\frac{n-1}{2}+ir\right) = \frac{\Gamma\left(\frac{n-1}{2}-ir\right)\Gamma\left(\frac{n-1}{2}+ir\right)\Gamma\left(\frac{1}{2}+ir\right)\Gamma\left(\frac{1}{2}-ir\right)}{2^n\pi^{\frac{n}{2}}(-2ir)\Gamma(-2ir)\Gamma(2ir)\Gamma\left(\frac{n}{2}\right)} \times F\left(\frac{n-1}{2}+ir,\frac{n-1}{2}-ir;\frac{n}{2};1-\zeta\right). \tag{6.56}$$

Using (B.5) and the identity

$$\Gamma\left(\frac{n-1}{2}-ir\right)\Gamma\left(\frac{n-1}{2}+ir\right) = \left|\Gamma\left(\frac{n-1}{2}+ir\right)\right|^2 = \frac{2^{2n-4}\Gamma\left(\frac{n}{2}\right)^2\sigma_{\mathbf{H}^n}(r)}{r\sinh\pi r}, \tag{6.57}$$

we get

$$G_{\mathbf{H}^n}\left(\cosh^2\left(\frac{\rho}{2}\right);\frac{n-1}{2}-ir\right) - G_{\mathbf{H}^n}\left(\cosh^2\left(\frac{\rho}{2}\right);\frac{n-1}{2}+ir\right)$$
$$= \frac{i}{\pi^{\frac{n}{2}}r}2^{n-3}\Gamma\left(\frac{n}{2}\right)\sigma_{\mathbf{H}^n}(r)\mathscr{C}^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(\cosh\rho).$$

Writing

$$\mathscr{Q}_n(r) := 2^{2n-4}\pi^{-1}\Gamma\left(\frac{n}{2}\right)^2\sigma_{\mathbf{H}^n}(r), \tag{6.58}$$

it then follows from (6.57) that

$$\mathscr{Q}_{n+2}(r) = \left[\left(\frac{n-1}{2}\right)^2 + r^2\right]\mathscr{Q}_n(r),\ n \geq 1. \tag{6.59}$$

The recursion relation (6.59) will be useful later for establishing the Millson recursion formula (6.55).

We complete the proof of the first part of the theorem by applying the spectral properties of self-adjoint operators. Towards this end, let E_ω, $\omega \in \mathbf{R}$, be the spectral projection for the self-adjoint Laplacian $\Delta'_{\mathbf{H}^n}$:

$$\Delta'_{\mathbf{H}^n} f = \int_0^\infty \omega\, dE_\omega f, \quad f \in L^2\left(\mathbf{H}^n\right).$$

Using Stone's formula A.15, we have

$$dE_\omega f(w) = \frac{1}{2\pi i}\left[\mathcal{R}_{\omega+i0} - \mathcal{R}_{\omega-i0}\right] f(w)\, d\omega, \tag{6.60}$$

where $\mathcal{R}_\omega$ is the resolvent operator in $\mathbf{H}^n$ defined by

$$\left(\mathcal{R}_\omega f\right)(w) = \int_{\mathbf{H}^n} G'_{\mathbf{H}^n}(w, w'; \omega) f(w')\, d\mu_{\mathbf{H}^n}(w'),\ \ f \in L^2\left(\mathbf{H}^n\right),$$

with

$$G'_{\mathbf{H}^n}(w, w'; \omega) = \begin{cases} G_{\mathbf{H}^n}(\zeta; \frac{n-1}{2} + i\sqrt{\omega}), & \operatorname{Im}\omega < 0, \\ G_{\mathbf{H}^n}(\zeta; \frac{n-1}{2} - i\sqrt{\omega}), & \operatorname{Im}\omega > 0. \end{cases} \tag{6.61}$$

Putting (6.61) in (6.60), we obtain

$$\left(E_\omega f\right)(w) = \int_{\mathbf{H}^n} \pi_\omega(w, w') f(w')\, d\mu_{\mathbf{H}^n}(w'), \tag{6.62}$$

where $\pi_\omega(w, w')$ is the spectral function of $\Delta'_{\mathbf{H}^n}$, which of course, is the spectral kernel of the projection operator E_ω in $\mathbf{H}^n$ given by

$$\begin{aligned}
\pi_\omega(w, w') &= \frac{1}{2\pi i}\int_0^\omega \left[G'_{\mathbf{H}^n}(w, w'; r + i0) - G'_{\mathbf{H}^n}(w, w'; r - i0)\right] dr \\
&= \frac{1}{2\pi i}\int_0^\omega \left[G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} - i\sqrt{r}\right) - G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} + i\sqrt{r}\right)\right] dr \\
&= \frac{1}{\pi i}\int_0^{\sqrt{\omega}} r\left[G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} - ir\right) - G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} + ir\right)\right] dr \\
&= \frac{2^{n-3}\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}+1}}\int_0^{\sqrt{\omega}} \sigma_{\mathbf{H}^n}(r)\mathscr{C}^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(\cosh\rho)\, dr,
\end{aligned}$$

for $\omega \geq 0$, and $\pi_\omega(w, w') = 0$ if $\omega < 0$, since the spectrum of $\Delta'_{\mathbf{H}^n}$ fills the interval $[0, \infty)$.

Next, we establish the integral (6.53). Indeed, by Theorem 3.3 (see also the statement leading to Theorem 3.3) and the integral formula (3.11), we have

$$h\left(\sqrt{\Delta'_{\mathbf{H}^n}}\right) f(w) = \int_{\mathbf{H}^n} K_{\mathbf{H}^n}(w, w') f(w')\, d\mu_{\mathbf{H}^n}(w'), \quad f \in L^2\left(\mathbf{H}^n\right). \tag{6.63}$$

On the other hand, by the spectral theory of self-adjoint operator, we have

$$\begin{aligned} h\left(\sqrt{\Delta'_{\mathbf{H}^n}}\right) f(w) &= \int_0^\infty h(\sqrt{\omega})\, dE_\omega f(w) \\ &= \frac{1}{2\pi i}\int_0^\infty h(\sqrt{\omega})\left[\int_{\mathbf{H}^n} G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} - ir\right) f(w')\, d\mu_{\mathbf{H}^n}(w')\right. \\ &\qquad \left. - G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} + ir\right) f(w')\, d\mu_{\mathbf{H}^n}(w')\right] d\omega. \end{aligned} \tag{6.64}$$

Comparing (6.64) with (6.63), we see that

$$\begin{aligned} K_{\mathbf{H}^n}(\rho) = K_{\mathbf{H}^n}(w, w') &= \frac{1}{2\pi i}\int_0^\infty h(\sqrt{\omega})\left(G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} - ir\right) - G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} + ir\right)\right) d\omega \\ &= \frac{1}{\pi i}\int_0^\infty r h(r)\left(G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} - ir\right) - G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2} + ir\right)\right) dr \\ &= \frac{2^{n-3}\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n+2}{2}}}\int_0^\infty h(r)\sigma_{\mathbf{H}^n}(r)\mathscr{C}^{\frac{n-1}{2}}_{-\frac{n-1}{2}-ir}(\cosh\rho)\, dr. \end{aligned} \tag{6.65}$$

This establishes the generalised Mehler-Fock integral formula (6.54). For the recursion formula (6.55) we differentiate both sides of (6.65) with respect to ρ and apply the formula

$$\left.\frac{dP^{\frac{2-n}{2}}_{-\frac{1}{2}+ir}(x)}{dx}\right|_{x=\cosh\rho} = -\frac{\left(\frac{n-1}{2}\right)^2 + r^2}{\sinh\rho} P^{-\frac{n}{2}}_{-\frac{1}{2}+ir}(\cosh\rho) + \left(\frac{n-2}{2}\right)\frac{\cosh\rho}{\sinh^2\rho} P^{\frac{2-n}{2}}_{-\frac{1}{2}+ir}(\cosh\rho), \tag{6.66}$$

and the recursion formula (6.59) to obtain (6.55). This completes the proof of the first part of the theorem.

Before proceeding to the proof of the second part of the theorem, let us write out the Mehler-Fock integral formulae for the special cases $n = 1, 2$.

- $(n = 1)$
$$K_{\mathbf{H}^1}(\rho) = \frac{1}{\sqrt{2\pi}}\sinh^{1/2}\rho\int_0^\infty h(r) P^{\frac{1}{2}}_{-\frac{1}{2}+ir}(\cosh\rho)\, dr,$$
which on using (B.84c) gives
$$K_{\mathbf{H}^1}(\rho) = \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\cosh\rho r\, dr = g(\rho) = Q\left(e^\rho + e^{-\rho} - 2\right),$$
which is (or coincides with) the classical Euclidean Fourier transform of h (see Theorem 3.4); this is not surprising in view of the fact that $\mathbf{H}^1$ coincides with $\mathbf{R}$.

- $(n = 2)$
$$K_{\mathbf{H}}(\rho) = \frac{1}{2\pi}\int_0^\infty h(r) P_{-\frac{1}{2}+ir}(\cosh\rho) r\tanh\pi r\, dr, \tag{6.67}$$
which is the Mehler-Fock inversion formula of order zero (or simply the classical Mehler-Fock integral formula; see Theorem 1.18).

What we do next is the transformation of the integral formula (6.54) into the heat kernel $K_{\mathbf{H}^n}(t, \rho)$ in $\mathbf{H}^n$, $t > 0$. Towards this end, we use our appropriate spectral function $h_t(r)$ in $\mathbf{H}^n$, namely

$$h_t(r) = e^{-r^2 t} e^{-\frac{(n-1)^2}{4}t}, \quad t > 0,$$

in (6.54) to obtain

$$K_{\mathbf{H}^n}(t,w,w') = \frac{e^{-\frac{(n-1)^2}{4}t}}{(2\pi)^{\frac{n}{2}}} \sinh^{\frac{2-n}{2}}\rho \int_0^\infty e^{-r^2 t} \mathscr{Q}_n(r) P^{\frac{2-n}{2}}_{-\frac{1}{2}+ir}(\cosh\rho)\, dr. \tag{6.68}$$

The Millson formula (6.55) for the heat kernel now takes the form

$$K_{\mathbf{H}^n}(t,w,w') = -\frac{e^{-nt}}{2\pi \sinh\rho} \frac{\partial}{\partial\rho} K_{\mathbf{H}^{n-2}}(t,w,w'),\ n \geq 3. \tag{6.69}$$

Next we give explicit expressions for (6.68). We do this according to whether n is odd or even.

(a) n odd, $n \geq 1$. We use the following formula for the Plancherel measure to give explicit expression for (6.68) (see e.g. Strichartz [159]):

$$\sinh^{\frac{2-n}{2}}\rho \mathscr{Q}_n(r) P^{\frac{2-n}{2}}_{-\frac{1}{2}+ir}(\cosh\rho) = \frac{(2\pi)^{\frac{n}{2}}}{\pi} \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-1}{2}} \cos\rho r.$$

Using this identity in (6.68) we obtain

$$K_{\mathbf{H}^n}(t,w,w') = \frac{e^{-\frac{(n-1)^2}{4}t}}{2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}} \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-1}{2}} \int_0^\infty e^{-r^2 t} \cos\rho r\, dr. \tag{6.70}$$

On applying the equality

$$\int_0^\infty e^{-\beta^2\eta^2} \cos\alpha\eta\, d\eta = \frac{\sqrt{\pi}}{2\beta} e^{-\frac{\alpha^2}{4\beta^2}},\ \ \alpha > 0,$$

we get

$$K_{\mathbf{H}^n}(t,w,w') = \frac{e^{-\frac{(n-1)^2}{4}t}}{\sqrt{2}(2\pi)^{\frac{n}{2}}\sqrt{t}} \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-1}{2}} e^{-\frac{\rho^2}{4t}},$$

which agrees with (6.40).

(b) n even, $n \geq 2$. Similarly, for n even, we use the identity (see e.g. Strichartz [158])

$$\sinh^{\frac{2-n}{2}}\rho \mathscr{Q}_n(r) P^{\frac{2-n}{2}}_{-\frac{1}{2}+ir}(\cosh\rho) = r\tanh\pi r \frac{(2\pi)^{\frac{n}{2}}}{2\pi} \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} P_{-\frac{1}{2}+ir}(\cosh\rho).$$

By replacing the Legendre function with its integral representation we have

$$\sinh^{\frac{2-n}{2}}\rho \mathscr{Q}_n(r) P^{\frac{2-n}{2}}_{-\frac{1}{2}+ir}(\cosh\rho) = \frac{(2\pi)^{\frac{n}{2}}\sqrt{2}}{2\pi^2} \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \int_\rho^\infty \frac{r\sin ru\, du}{\sqrt{\cosh u - \cosh\rho}}. \tag{6.71}$$

Now using (6.71) and (1.57) in (6.68) we obtain

$$K_{\mathbf{H}^n}(t,w,w') = \frac{e^{-\frac{n(n-2)}{4}t} e^{-\frac{t}{4}}\sqrt{2}}{8\pi^{\frac{3}{2}}t^{\frac{3}{2}}} \left(-\frac{1}{2\pi}\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}} \int_\rho^\infty \frac{ue^{-\frac{u^2}{4t}}\, du}{\sqrt{\cosh u - \cosh\rho}}, \tag{6.72}$$

which agrees with (6.41). This completes the proof of the theorem.

∎

We have derived the generalised Mehler-Fock integral formula via the Green function and spectral projection of the Laplacian in $\mathbf{H}^n$, next we approach this integral formula via the Poisson kernel in $\mathbf{H}^n$. The precise statement is the following

Theorem 6.8. *Let* $\zeta = \cosh^2(\rho/2)$, $\rho = d(w, w')$, $w, w' \in \mathbf{H}^n$. *Then*

$$\begin{aligned} K_{\mathbf{H}^n}(w,w') &= \frac{1}{4\pi^n i} \int\limits_{\frac{n-1}{2}-i\infty}^{\frac{n-1}{2}+i\infty} \int\limits_{\mathbf{R}^{n-1}} \widetilde{P}^s_{\mathbf{H}^n}(w,\xi)\widetilde{P}^{n-1-s}_{\mathbf{H}^n}(w',\xi) \frac{\Gamma(n-1-s)\Gamma(s)}{\Gamma\left(\frac{n-1}{2}-s\right)\Gamma\left(s-\frac{n-1}{2}\right)} h(s)\, d\xi ds \\ &= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_0^\infty h(r)\mathscr{Q}_n(r) F\left(\frac{n-1}{2}+ir, \frac{n-1}{2}-ir; \frac{n}{2}; 1-\zeta\right) dr, \end{aligned} \tag{6.73}$$

satisfying the Millson formula

$$K_{\mathbf{H}^{n+2}}(w,w') = -\frac{1}{4\pi}\frac{\partial}{\partial\zeta} K_{\mathbf{H}^n}(w,w'), \tag{6.74}$$

where

$$\widetilde{P}_{\mathbf{H}^n}(w,\xi) = \frac{y}{|x-\xi|^2+y^2} \tag{6.75}$$

with $x, \xi \in \mathbf{R}^{n-1}$, $y > 0$, $w = (x, y) \in \mathbf{H}^n$.

Remark 6.3. *The Poisson kernel in the upper half-space* $\mathbf{H}^n$ *in Cartesian coordinates as given by Byczkowski et al. [33] is*

$$P_{\mathbf{H}^n}(w,\xi) = \frac{\Gamma(n-1)}{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)} \left(\frac{y}{|x-\xi|^2+y^2}\right)^{n-1} \tag{6.76}$$

with $x, \xi \in \mathbf{R}^{n-1}$, $y > 0$, $w = (x, y) \in \mathbf{H}^n$. *It is worth mentioning that the Poisson kernel in* $\mathbf{R}^2$ *coincides with that in* $\mathbf{H}$ *(in view of* (6.76) *with* $n = 2$*); see Appendix A.4 for the Euclidean Poisson kernel.*

Proof of Theorem 6.8 . The proof uses an integral formula involving the product of the Poisson kernel $P_{\mathbf{H}^n}$, namely (see e.g. Mandouvalos [107])

$$\int\limits_{\mathbf{R}^{n-1}} \widetilde{P}^s_{\mathbf{H}^n}(w,\xi)\widetilde{P}^{n-1-s}_{\mathbf{H}^n}(w',\xi)\, d\xi = \pi^{\frac{n-1}{2}} \frac{\Gamma\left(\frac{n-1}{2}-s\right)}{\Gamma\left(n-1-s\right)} \mathsf{f}\left(\zeta; s\right) + \pi^{\frac{n-1}{2}} \frac{\Gamma\left(s-\frac{n-1}{2}\right)}{\Gamma\left(s\right)} \mathsf{f}(\zeta; n-1-s),$$

where

$$\mathsf{f}(\zeta; s) = 4^{-s}\zeta^{-s} F\left(s, s+1-\frac{n}{2}; 2s-n+2; \zeta^{-1}\right).$$

Making the substitution $s = ((n-1)/2) + ir$, we have

$$\begin{aligned} &\frac{1}{4\pi^n i} \int\limits_{\frac{n-1}{2}-i\infty}^{\frac{n-1}{2}+i\infty} \int\limits_{\mathbf{R}^{n-1}} \widetilde{P}^s_{\mathbf{H}^n}(w,\xi)\widetilde{P}^{n-1-s}_{\mathbf{H}^n}(w',\xi) \frac{\Gamma(n-1-s)\Gamma(s)}{\Gamma\left(\frac{n-1}{2}-s\right)\Gamma\left(s-\frac{n-1}{2}\right)} h(s)\, d\xi ds \\ &= \frac{\pi^{\frac{n-1}{2}}}{4\pi^n} \int\limits_{-\infty}^{\infty} \left[\frac{\Gamma\left(\frac{n-1}{2}+ir\right)}{\Gamma(ir)} \mathsf{f}\left(\zeta; \frac{n-1}{2}+ir\right) + \frac{\Gamma\left(\frac{n-1}{2}-ir\right)}{\Gamma(-ir)} \mathsf{f}\left(\zeta; \frac{n-1}{2}-ir\right) \right] h(r)\, dr. \end{aligned} \tag{6.77}$$

Interestingly we can express the hypergeometric function appearing in the integral in terms of the Green function in $\mathbf{H}^n$, namely

$$F\left(\frac{n-1}{2}+ir, \frac{1}{2}+ir; 1+2ir; \zeta^{-1}\right) = \frac{2^n \pi^{\frac{n}{2}} \Gamma(1+2ir) \zeta^{\frac{n-1}{2}+ir}}{\Gamma\left(\frac{n-1}{2}+ir\right)\Gamma\left(\frac{1}{2}+ir\right)} G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2}+ir\right).$$

Using this equality in (6.77), we have

$$\begin{aligned} K_{\mathbf{H}^n}(w,w') &= 2i \int_{-\infty}^{\infty} \frac{r\Gamma(2ir)\Gamma(-2ir)\left[G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2}+ir\right) - G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2}-ir\right)\right]}{\Gamma(-ir)\Gamma(ir)\Gamma\left(\frac{1}{2}+ir\right)\Gamma\left(\frac{1}{2}-ir\right)} h(r)\, dr \\ &= \frac{1}{\pi i} \int_0^\infty r h(r) \left[G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2}-ir\right) - G_{\mathbf{H}^n}\left(\zeta; \frac{n-1}{2}+ir\right)\right] dr. \end{aligned} \tag{6.78}$$

On using (6.56) we obtain (6.73). For the recursion formula (6.74), we use the integral representation (B.62) of the Gauss hypergeometric function to obtain

$$K_{\mathbf{H}^n}(w,w') = \frac{2^{1-n}}{\pi^{\frac{n}{2}}} \int_0^\infty \frac{h(r)\mathscr{Q}_n(r)}{\Gamma\left(\frac{n-1}{2}-ir\right)\Gamma(ir)} \int_0^1 \frac{\eta^{\frac{n-3}{2}-ir}(1-\eta)^{ir-1}}{(1-\eta(1-\zeta))^{\frac{n-1}{2}+ir}}\, d\eta dr.$$

Differentiating both sides with respect to ζ and using the recursion formula (6.59), we get

$$\begin{aligned} \frac{\partial}{\partial\zeta} K_{\mathbf{H}^n}(w,w') &= \frac{2^{1-n}}{\pi^{\frac{n}{2}}} \int_0^\infty h(r)\mathscr{Q}_n(r) \frac{\left(-\left(\frac{n-1}{2}\right)-ir\right)\left(\frac{n-1}{2}-ir\right)}{\left(\frac{n-1}{2}-ir\right)\Gamma\left(\frac{n-1}{2}-ir\right)\Gamma(ir)} \\ &\quad \times \int_0^1 \eta^{\frac{n-1}{2}-ir}(1-\eta)^{ir-1}(1-\eta(1-\zeta))^{-\frac{n+1}{2}-ir}\, d\eta dr \\ &= \frac{2^{1-n}}{\pi^{\frac{n}{2}}} \int_0^\infty h(r) \frac{\left[-\left(\frac{n-1}{2}\right)^2 - r^2\right]\mathscr{Q}_n(r)}{\Gamma\left(\frac{n+1}{2}-ir\right)\Gamma(ir)} \\ &\quad \times \int_0^1 \eta^{\frac{n-1}{2}-ir}(1-\eta)^{ir-1}(1-\eta(1-\zeta))^{-\frac{n+1}{2}-ir}\, d\eta dr \\ &= -\frac{4\pi}{2^{n+1}\pi^{\frac{n+2}{2}}} \int_0^\infty h(r) \frac{\mathscr{Q}_{n+2}(r)}{\Gamma\left(\frac{n+1}{2}-ir\right)\Gamma(ir)} \\ &\quad \times \int_0^1 \eta^{\frac{n-1}{2}-ir}(1-\eta)^{ir-1}(1-\eta(1-\zeta))^{-\frac{n+1}{2}-ir}\, d\eta dr \\ &= -4\pi K_{\mathbf{H}^{n+2}}(w,w'). \end{aligned}$$

This completes the proof of the theorem. □

6.5 Concluding Remarks

The generalisation of the trace formulae discussed in Chapter 3 and their associated applications are considered in Awonusika and Taheri [8], namely, we study the Selberg spectral theory of the quotient space $\mathtt{S}_\Gamma = \Gamma\backslash G/K$, where G is a connected semisimple Lie Group of noncompact type with finite centre, Γ a discrete torsion-free subgroup of G, and K a maximal compact subgroup of G. We construct the Selberg trace formula and the Selberg zeta function attached to the manifold $\mathtt{S}_\Gamma = \Gamma\backslash\mathscr{X} = \Gamma\backslash G/K$. Also discussed are the McKean integral formula for the Selberg zeta function $Z_\Gamma(s)$ and zeta regularised determinants of the Laplacians on $\mathtt{S}_\Gamma$. In this case the space $\mathscr{X} = G/K$ is a symmetric space of noncompact type and we specialise to the special cases of real hyperbolic manifolds $\mathscr{M} = \Gamma\backslash\mathbf{H}^n$ and the complex hyperbolic manifolds $\mathscr{M} = \Gamma\backslash\mathbf{CH}^n$. We also generalise the material in Section 2.2 to the general linear group $\mathrm{GL}(n, \mathbf{R})$, namely we give explicit computations of the Fourier expansion of Eisenstein series for the subgroup $\mathrm{GL}(n, \mathbf{Z})$, and we specialise to the case $\mathrm{SL}(3, \mathbf{Z})$ where and when necessary, i.e., we restrict ourselves to forms which are automorphic under the action of the modular group $\mathrm{SL}(3, \mathbf{Z})$. The explicit computation of Eisenstein series for the modular group $\mathrm{SL}(3, \mathbf{Z})$ is presented. For the higher rank group $\mathrm{GL}(n, \mathbf{R})$, the K-Bessel function appearing in the Fourier expansion of nonholomorphic Eisenstein series for the modular group $\mathrm{SL}(2, \mathbf{Z})$ (see (2.28)) is replaced with the Whittaker function, precisely Jacquet's the Whittaker function.

In Awonusika and Taheri [8] (see also Awonusika and Taheri [15]) we include discussions of spectral functions and spectral invariants on the real hyperbolic spaces $\mathscr{X} = \mathbf{H}^n$ and the complex hyperbolic spaces $\mathscr{X} = \mathbf{CH}^n$. We present explicit constructions of spherical functions of the Laplacian on $\mathbf{CH}^n$, in terms of special functions, namely Jacobi polynomials and hypergeometric functions. The Minakshisundaram-Pleijel asymptotics (as $t \searrow 0$) of the heat kernel at the diagonal and the Minakshisundaram-Pleijel zeta functions on noncompact symmetric spaces $\mathscr{X} = \mathbf{H}^n, \mathbf{CH}^n$ are treated explicitly. The Minakshisundaram-Pleijel zeta function in this case of a noncompact symmetric space is simply the Mellin transform of the heat kernel at the diagonal; and we express the associated heat coefficients in relation to the heat coefficients for the corresponding compact symmetric spaces. The heat kernel and Green function on $\mathbf{CH}^n$ is obtained in a similar way as in the case of the real hyperbolic space $\mathbf{H}^n$. As in the cases of the sphere $\mathbf{S}^n$ and the real hyperbolic space $\mathbf{H}^n$, the heat kernel on the complex hyperbolic space $\mathbf{CH}^n$ is obtained from the heat kernel on the complex projective space $\mathbf{CP}^n$ by analytic continuation, with the Riemann-Liouville fractional integral replaced with the Weyl fractional integral. Similar to what we did in the case of the real hyperbolic space $\mathbf{H}^n$, we establish the complex hyperbolic Mehler-Fock integral formula for the complex hyperbolic space $\mathbf{CH}^n$ via the Green function and spectral resolution of self-adjoint operators.

Appendix A

The Laplacian on a Riemannian Manifold

A.1 The Laplace-Beltami Operator

Smooth Manifold: An n-dimensional ($n \geq 1$) smooth manifold M is a topological space that is locally homeomorphic to the n-dimensional Euclidean space $\mathbf{R}^n$ by smooth transformations. This homeomorphism permits differentiation to be defined. Formally, an n-dimensional *smooth manifold* is a set M together with a collection of *local charts* $\{(U_i, \vartheta_i)\}$, where $U_i \subset M$ with $\bigcup_i U_i = M$, and $\vartheta_i : U_i \subset M \longrightarrow \mathbf{R}^n$ is a bijection. For each pair of local charts (U_i, ϑ_i) and (U_j, ϑ_j), it is required that $\vartheta_j(U_i \cap U_j)$ is open and $\vartheta_{ij} = \vartheta_i \circ \vartheta_j^{-1}$ are diffeomorphisms, called the transition maps.

Riemannian manifold. A *Riemannian manifold* (M, g) is a smooth manifold M with a family of smoothly varying positive definite inner products $g = g_p$ on T_pM for each $p \in M$. The family g is called a *Riemannian metric*, which in local coordinates is given by

$$g = \sum_{i,j} g_{ij} dx^i dx^j.$$

Some examples of Riemannian manifolds of relevance in this thesis include the Euclidean space $\mathbf{R}^n$, the Euclidean sphere $\mathbf{S}^n$, the real projective space $\mathbf{RP}^n$, the complex projective space $\mathbf{CP}^n$, the quaternionic projective space $\mathbf{PQ}^{2n}$, the Cayley projective plane $\mathbf{P}(\mathbf{Cay})^8$, the real hyperbolic space $\mathbf{H}^n$, the complex hyperbolic space $\mathbf{CH}^n$, the quaternionic hyperbolic space $\mathbf{QH}^{2n}$ and the Cayley hyperbolic plane $\mathbf{CayH}^8$; and their quotients.

One of the basic differential operators in Riemannian geometry is the Laplace-Beltrami operator. Indeed recall that for a function $f : M \longrightarrow \mathbf{R}$, the *gradient* of f, denoted $\operatorname{grad} f$, is the vector field defined by

$$(\operatorname{grad} f(p), X) = X(f).$$

In local coordinates, the gradient of f is given by

$$(\operatorname{grad} f)_i = \sum_j g^{ij} \frac{\partial f}{\partial x_j},$$

where g^{ij} is the inverse of g_{ij}. The *divergence operator* is defined to be the adjoint of the gradient, allowing 'integration by parts' on manifolds with special structure. If a Riemannian manifold M is orientable, we define the *volume form* of a Riemannian metric to be the top dimensional form (i.e., a differential form of top degree) dV which in local coordinates is given by

$$\mathrm{dV} = \sqrt{\det g} dx^1 \cdots dx^n,$$

whenever $\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right)$ is a positively oriented basis of $T_x M$. We set the volume of M to be

$$\mathrm{V}(M) = \int_M \mathrm{dV}(x).$$

A volume form, in turn, enables the definition of the *divergence of a vector field* on the manifold. In local coordinates, the divergence of a vector field X is given by

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left(\sqrt{\det g} X_i\right),$$

where $\det g$ denotes the determinant of the matrix g_{ij}. Hence, the *Laplace-Beltrami operator* on functions defined on M with respect to the metric g is defined by

$$\Delta_{(M,g)} = \Delta_M = \Delta = -\operatorname{div}\operatorname{grad},$$

which in local coordinates is given by

$$\Delta = -\frac{1}{\sqrt{\det g}} \sum_j \frac{\partial}{\partial x_j} \left(\sum_i g^{ij} \sqrt{\det g} \frac{\partial}{\partial x_i} \right). \tag{A.1}$$

The Euclidean space $\mathbf{R}^n$ is a Riemannian manifold with its metric tensor given as the Euclidean metric

$$g = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. In Cartesian coordinates,

$$ds^2 = \sum_i \left(dx^i\right)^2 = |dx|^2,$$

and the Laplace-Beltrami operator is reduced to the familiar n-dimensional Euclidean Laplacian

$$\Delta = \mathbb{D}_n = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

The three spaces that are the only complete, simply connected Riemannian manifolds of given constant sectional curvature κ are the Euclidean space $\mathbf{R}^n$ of curvature $\kappa = 0$, the n-dimensional sphere of curvature $\kappa = 1$ and the n-dimensional hyperbolic space of curvature $\kappa = -1$ (Do Carmo [51]).

For details of Riemannian geometry and Riemannian manifolds, we refer the reader to Do Carmo [51], Lee [101].

A.2 The Heat Kernel on a Riemannian Manifold

Definition A.1. *A continuous function $K_M(t, x, y) : (0, \infty) \times M \times M \to \mathbf{R}$ is called a fundamental solution of the heat equation*

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= -\Delta_M u(t,x), \qquad x \in M, t > 0, \\ u(0,x) &= u_0(x), \qquad x \in M, \end{aligned} \tag{A.2}$$

or the heat kernel on M if it belongs to $C^{1,2}((0, \infty) \times M \times M)$ and satisfies the following conditions:

(i) $\frac{\partial K_M(t,x,y)}{\partial t} = -\Delta_M K_M(t, x, y)$*;*

(ii) $K_M(t, x, y) = K_M(t, y, x)$*;*

(iii) $\lim_{t \searrow 0} K_M(t, x, y) = \delta(x - y)$*, where $\delta(x, y) = \delta(x - y)$ is the Dirac measure, i.e.,*

$$\lim_{t \searrow 0} \int_M K_M(t, x, y) u_0(y)\, dV(y) = u_0(x), \quad u_0 \in C^\infty(M),\ x \in M;$$

(iv) if M is compact, then

$$K_M(t, x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y), \tag{A.3}$$

where $(\phi_k : k \geq 0)$ is a complete orthonormal basis of the Hilbert space $L^2(M)$ consisting of eigenfunctions of Δ_M, with associated eigenvalues $(\lambda_k : k \geq 0)$ satisfying

$$\Delta_M \phi_k = \lambda_k \phi_k \tag{A.4}$$

with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty.$$

In particular, each eigenvalue has finite multiplicity.

It follows from Definition A.1 that the solution $u(t, x)$ of the heat equation (A.2) is given by

$$u(t, x) = \left(e^{-t\Delta_M} f\right)(x) = \int_M K(t, x, y) f(y)\, dy, \qquad f \in C^\infty(M). \tag{A.5}$$

If $M = \mathbf{R}^n$, then the classical heat kernel in $\mathbf{R}^n$ is given by

$$K_{\mathbf{R}^n}(t, x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}, \ \ x, y \in \mathbf{R}^n, t > 0. \tag{A.6}$$

In 1948 by construction of a parametrix for the heat equation Minakshisundaram and Pleijel [111] showed that on a compact Riemannian manifold M without boundary the heat kernel $K_M(t, x, y)$ has the asymptotic expansion

$$K_M(t, x, x) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} u_k^n(x, x) t^k \quad \text{ as } \ t \searrow 0, \tag{A.7}$$

and as a result the heat trace satisfies the expansion

$$
\begin{aligned}
\Theta_M(t) = \operatorname{Tr} e^{-t\Delta_M} &= \int_M K_M(t,x,x)\,\mathrm{dVol}(x) \\
&= \sum_{k=0}^{\infty} e^{-\lambda_k t} \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} a_k^n t^k \quad \text{as} \;\; t \searrow 0,
\end{aligned} \tag{A.8}
$$

where the coefficients a_k^n (with $k \geq 0$) are given by

$$
a_0^n = \mathrm{Vol}(M), \qquad a_k^n = \int_M u_k^n(x,x)\,\mathrm{dVol}(x), \qquad k \geq 1. \tag{A.9}
$$

For an account on heat kernels in Riemannian geometry the reader is referred to the monographs Chavel [41], Grigor'yan [68] or Li [103] and the extensive list of references therein.

A.3 The Wave Kernel on a Riemannian Manifold

Let M be a compact Riemannian manifold and Δ_M the Laplace-Beltrami operator on M. Let $(\phi_k : k \geq 0)$ be a complete orthonormal basis of $L^2(M)$ consisting of eigenfunctions of Δ_M such that the corresponding eigenvalues $(\lambda_k : k \geq 0)$ satisfy

$$
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots.
$$

Consider the initial value problem for the wave equation on M:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = -\Delta_M u \quad \text{on} \quad (0,\infty) \times M \\
&u(0,x) = f(x), \quad \frac{\partial}{\partial t} u(0,x) = g(x), \quad x \in M.
\end{aligned} \tag{A.10}
$$

It is known that the solution operator to the wave equation (A.10) is given by

$$
\begin{aligned}
u(t,x) &= \left(\cos(t\sqrt{\Delta_M}) f\right)(x) + \left(\frac{\sin(t\sqrt{\Delta_M})}{\sqrt{\Delta_M}} g\right)(x) \\
&= \int_M W_M^{(1)}(t,x,y) f(y)\,dy + \int_M W_M^{(2)}(t,x,y) f(y)\,dy,
\end{aligned}
$$

where

$$
W_M^{(1)}(t,x,y) = \sum_{k=1}^{\infty} \cos\left(t\sqrt{\lambda_k}\right) \phi_k(x)\phi_k(y), \;\; W_M^{(2)}(t,x,y) = \sum_{k=1}^{\infty} \frac{\sin(t\sqrt{\lambda_k})}{\sqrt{\lambda_k}} \phi_k(x)\phi_k(y)
$$

are the wave kernels on M.

The half-wave operator

$$
U(t) = e^{it\sqrt{\Delta_M}} \tag{A.11}
$$

solves the half-wave equation

$$
\begin{aligned}
&\frac{1}{i}\frac{\partial u}{\partial t} = \sqrt{\Delta_M} u \quad \text{on} \quad \mathbf{R} \times M, \\
&u(0,x) = f(x), \quad x \in M,
\end{aligned} \tag{A.12}
$$

where $u \in C^\infty(\mathbf{R} \times M)$, $f \in C^\infty(M)$. The operator $U(t)$ is defined by

$$(U(t)f)(x) = \left(e^{it\sqrt{\Delta_M}} f\right)(x) = \int_M U(t,x,y) f(y)\, dV(y),$$

where the half-wave kernel is

$$U(t,x,y) = \sum_{k=1}^{\infty} e^{i\lambda_k t} \phi_k(x)\phi_k(y) \tag{A.13}$$

on $\mathbf{R} \times M \times M$, which converges in the sense of distributions.

For example the solution of the wave equation on $M = \mathbf{R}^n$,

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = -\mathbb{D}_n u \quad \text{on} \quad (0,\infty) \times \mathbf{R}^n, \\ &u(0,x) = f(x), \quad \frac{\partial}{\partial t} u(0,x) = g(x), \quad x \in \mathbf{R}^n, \end{aligned} \tag{A.14}$$

where $f, g \in C^\infty(\mathbf{R}^n)$, is given explicitly, via spherical means, by (see e.g. Taheri [163, pp. 343-344], Folland [62, Ch. 5])

$$u(t,x) = \frac{1}{1 \cdot 3 \cdots (n-2)\nu_{n-1}} \left[\mathbb{N}_f(t,x) + \mathbb{N}_g(t,x)\right],$$

where

(a) n odd, $n \geq 3$,

$$\begin{aligned} \mathbb{N}_f(t,x) &= \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2} \int_{\mathbf{S}_t(x)} f(y)\, d\nu_{n-1}(y)\right), \\ \mathbb{N}_g(t,x) &= \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2} \int_{\mathbf{S}_t(x)} g(y)\, d\nu_{n-1}(y)\right); \end{aligned}$$

and

(b) n even, $n \geq 2$,

$$u(t,x) = \frac{2}{1 \cdot 3 \cdots (n-1)\nu_n} \left[\mathbb{N}'_f(t,x) + \mathbb{N}'_g(t,x)\right],$$

where

$$\begin{aligned} \mathbb{N}'_f(t,x) &= \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(\int_{\mathbf{B}_t(x)} \frac{f(y)\, dy}{\sqrt{t^2 - |y-x|^2}}\right), \\ \mathbb{N}'_g(t,x) &= \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(\int_{\mathbf{B}_t(x)} \frac{g(y)\, dy}{\sqrt{t^2 - |y-x|^2}}\right). \end{aligned}$$

Here $d\nu_{n-1}$ is the volume measure of the sphere $\mathbf{S}_t(x) = \{y \in \mathbf{R}^n : |y-x| = t\}$ and $\mathbf{B}_t(x)$ is the closed ball in $\mathbf{R}^n : \mathbf{B}_t(x) = \{y \in \mathbf{R}^n : |y-x| \leq t\}$.

A.4 The Poisson, Heat and Wave Kernels in $\mathbf{R}^n$

It is known by Fourier transform that the solution of the wave equation (A.14) in $\mathbf{R}^n$ is given by (see e.g. Stein and Shakarchi [154, Ch. 6])

$$\begin{aligned} u(t,x) &= \left(\cos\left(t\sqrt{\mathbb{D}_n}\right) f\right)(x) + \left(\frac{\sin\left(t\sqrt{\mathbb{D}_n}\right)}{\sqrt{\mathbb{D}_n}} g\right)(x) \\ &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \widehat{f}(\xi)\cos(t|\xi|)e^{ix\cdot\xi}\,d\xi + \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \widehat{g}(\xi)\frac{\sin(t|\xi|)e^{ix\cdot\xi}}{|\xi|}\,d\xi \\ &= \left(\frac{\partial}{\partial t}W_t\right) * f + W_t * g, \end{aligned}$$

for $f, g \in \mathcal{S}(\mathbf{R}^n)$, where

$$W_t(x) = W^{(2)}_{\mathbf{R}^n}(t,x,y) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}\frac{\sin(t|\xi|)}{|\xi|}\,d\xi, \quad x \in \mathbf{R}^n,\ t>0,$$

and

$$\frac{\partial}{\partial t}W_t(x) = W^{(1)}_{\mathbf{R}^n}(t,x,y) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}\cos(t|\xi|)\,d\xi, \quad x \in \mathbf{R}^n,\ t>0.$$

By analytic continuation, we have for $t > 0$ (Sogge [153, pp. 6-13]),

$$\begin{aligned} W_t(x) = W^{(2)}_{\mathbf{R}^n}(t,x) &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i(x)\cdot\xi}\frac{\sin(t|\xi|)}{|\xi|}\,d\xi \\ &= c'_n \lim_{\epsilon\to 0} \operatorname{Im}\left(|x|^2 - (t-i\epsilon)^2\right)^{-\frac{n-1}{2}}, \end{aligned}$$

where

$$c'_n = \frac{1}{2}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\pi^{\frac{n+1}{2}}}$$

and

$$H(t,x) = t^2 - r^2(x,y)$$

is a natural quadratic term associated with the wave operator $\frac{\partial^2}{\partial t^2} + \Delta$. Here $r^2(x,y) = d^2(x,y)$ is the Riemannian distance between points x and y in M. The half-wave kernel on $\mathbf{R}^n$ is

$$U_{\mathbf{R}^n}(t,x,y) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{it|\xi|}\,d\xi, \quad x \in \mathbf{R}^n.$$

Now, if $t = i\tau$, $\tau > 0$, the half-wave kernel $U_{\mathbf{R}^n}(t,x,y)$ becomes

$$U_{\mathbf{R}^n}(i\tau,x,y) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-\tau|\xi|}\,d\xi, \quad x \in \mathbf{R}^n.$$

Note that the heat kernel on $\mathbf{R}^n$ is

$$K(t,x,y) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{-t|\xi|^2}e^{i(x-y)\cdot\xi}\,d\xi = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4t}}, \quad x,y \in \mathbf{R}^n,\ t>0.$$

Using the well known (subordination) identity (see e.g. Stein and Weiss [155, p. 6(i)])

$$e^{-\eta A} = \frac{\eta}{2\sqrt{\pi}}\int_0^\infty e^{-\frac{\eta^2}{4t}}e^{-tA^2}t^{-\frac{3}{2}}\,dt, \quad \eta>0,\ A>0,$$

with $A = |\xi|$, we have

$$\begin{aligned} U_{\mathbf{R}^n}(i\tau, x, y) &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} e^{-\tau|\xi|}\, d\xi \\ &= \frac{\tau}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{\tau^2}{4t}} \left\{ \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} \right\} t^{-\frac{3}{2}}\, dt \\ &= \frac{\tau}{(4\pi)^{\frac{n+1}{2}}} \int_0^\infty e^{-\frac{\tau^2+|x-y|^2}{4t}} t^{-\frac{n+3}{2}}\, dt. \end{aligned}$$

Changing variables $s = 1/t$, we obtain

$$\begin{aligned} U_{\mathbf{R}^n}(i\tau, x, y) &= \frac{\tau}{(4\pi)^{\frac{n+1}{2}}} \Gamma\left(\frac{n+1}{2}\right) \left(\frac{\tau^2 + |x-y|^2}{4}\right)^{-\frac{n+1}{2}} \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{\tau}{(\tau^2 + |x-y|^2)^{\frac{n+1}{2}}}, \quad x, y \in \mathbf{R}^n, \tau > 0 \\ &= P_{\mathbf{R}^n}(\tau, x, y), \end{aligned}$$

where $P_{\mathbf{R}^n}(\tau, x, y)$ is the Poisson kernel on $((0,\infty) \times \mathbf{R}^n \times \mathbf{R}^n)$.

A.5 The Hilbert-Schmidt Spectral Theorem

This section discusses the aspect of spectral theory of self-adjoint operators relevant to the content of this thesis.

Let T be a linear operator on a Hilbert space $\mathscr{H}$.

Definition A.2. *An operator $T : L^2(M) \to L^2(M)$ on $\mathscr{H}$ is said to be self-adjoint if*

$$(Tf, g) = (f, Tg) \qquad \text{for all } f, g \in \mathscr{H}.$$

Definition A.3. *The eigenvalues of T are the complex numbers λ such that the determinant of $T - \lambda I$ is equal to zero. The set of such λ is called the spectrum. It can consist at most k points since $\det(T - \lambda I)$ is a polynomial of degree k. If λ is not an eigenvalue, then $T - \lambda$ has an inverse since $\det(T - \lambda I) \neq 0$.*

Definition A.4. *A number $\lambda \in \mathbf{C}$ is said to be in the resolvent set $\rho(T)$ of T if $T - \lambda I$ is a bijection with a bounded inverse. The operator*

$$\mathcal{R}_\lambda = (T - \lambda I)^{-1}$$

is then called the resolvent of T at λ. If $\lambda \notin \rho(T)$, then λ is said to be in the spectrum $\sigma(T)$ of T.

Definition A.5. *Let $\lambda \in \mathbf{C}$, $\lambda \neq 0$. Then $x \in \mathscr{H}$, $x \neq 0$, is called an eigenvector of T if it satisfies the eigenvalue problem $Tx = \lambda x$. The number λ is called the corresponding eigenvalue of T. If λ is an eigenvalue, then $T - \lambda I$ is not injective so $\lambda \in \sigma(T)$. The set of all eigenvalues is called the point spectrum of T.*

Given an interval $I \subset \mathbf{R}$ (or more generally a Borel subset), the associated spectral projector is defined by

$$E_I = \chi_I(T),$$

where χ_I denotes the characteristic function.

Stone's Formula. Let T be a self-adjoint operator on $\mathscr{H}$. Then

$$\frac{1}{2}\left[E_{[a,b]} + E_{(a,b)}\right] = \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_a^b \left[(T-z-i\varepsilon)^{-1} - (T-z+i\varepsilon)^{-1}\right] dz. \tag{A.15}$$

Definition A.6. *Let $\sigma_{ess}(T)$ denotes the essential spectrum of T. We say that $\lambda \in \sigma_{ess}(T)$ if and only if $E_{(\lambda-\varepsilon,\lambda+\varepsilon)}$ is infinite dimensional for all $\varepsilon > 0$. If $\lambda \in \sigma(T)$, but $E_{(\lambda-\varepsilon,\lambda+\varepsilon)}$ is finite dimensional for some $\varepsilon > 0$, we say $\lambda \in \sigma_{dis}$, the discrete spectrum of T, in other words the complement of the essential spectrum is the discrete spectrum. An eigenvalue of T could be contained in $\sigma_{ess}(T)$, in which case it is called an embedded eigenvalue.*

Definition A.7. *A point $\lambda \in \sigma(T)$ is in the continuous spectrum $\sigma_{con}(T)$ if $E_{\{\lambda\}} = 0$, and in the point spectrum otherwise. The continuous spectrum is called absolutely continuous if $E_{\mathcal{A}} = 0$ for any Borel subset $\mathcal{A} \subset \sigma_{con}(T)$ with zero Lebesgue measure.*

Let $\mathscr{H}$ be a Hilbert space. An unbounded operator T defined on some dense domain $\mathscr{D}(T) \subset \mathscr{H}$ is *self-adjoint* if $T = T^*$, with the same domain. The operator T is *essentially self-adjoint* on $\mathscr{D}(T) \subset \mathscr{D}(T^*)$, $T = T^*$ on $\mathscr{D}(T)$, and the closure of T is self-adjoint. This implies that the extension of T from the domain $\mathscr{D}(T)$ is uniquely determined. If $(M,g) = M$ is a complete Riemannian manifold, then the Laplace-Beltrami operator on M is essentially self-adjoint on $C_0^\infty(M) \subset L^2(M)$.

Definition A.8. *Let M be a compact Riemannian manifold and $K \in L^2(M\times M)$. The integral operator T defined by*

$$(Tf)(x) = \int_M K(x,y)f(y)\,dy \tag{A.16}$$

is called a Hilbert-Schmidt operator and K is the Hilbert-Schmidt kernel, satisfying

$$K(x,y) = \overline{K(y,x)} \qquad \text{for all } x,y \in M.$$

Theorem A.9 (Hilbert-Schmidt Theorem). *Let M be a compact Riemannian manifold and let T be the integral operator defined by*

$$(Tf)(x) = \int_M K(x,y)f(y)\,dV(y), \qquad f \in L^2(M),$$

where $K : M\times M \to (-\infty,\infty)$ is the Hilbert-Schmidt kernel. Then the eigenvalue problem

$$T\psi = \eta\psi$$

has a complete orthonormal system of eigenfunctions $(\psi_k : k \geq 0)$ in $L^2(M)$ with corresponding eigenvalues $(\eta_k : k \geq 0)$, where $\eta_k \searrow 0$ as $k \nearrow \infty$. Moreover, the kernel K has the following expansion in the L^2 sense:

$$K(x,y) = \sum_{k=0}^{\infty} \eta_k\psi_k(x)\psi_k(y).$$

Proof. The proof can be found in Riesz and Nagy [141, pp. 242-246] and Dodziuk [52]. □

Theorem A.10 (Mercer's Theorem). *Let M, K be as in Theorem A.9. Assume in addition that almost all eigenvalues $(\eta_k : k \geq 0)$ are nonnegative. Then K has the expansion*

$$K(x,y) = \sum_{k=0}^{\infty} \eta_k \psi_k(x) \psi_k(y),$$

where the convergence of the series is uniform on $M \times M$.

Proof. See Riesz and Nagy [141, pp. 242-246], Dodziuk [52]. □

A.6 Classical Trace Formulae

Let M be a compact Riemannian manifold. Let $\mathscr{H}$ be a Hilbert space and T a bounded operator in $\mathscr{H}$. Assume that there is an orthonormal basis $(\psi_k : k \geq 0)$ of $\mathscr{H}$ consisting of eigenfunctions of T with corresponding eigenvalues $(\eta_k : k \geq 0)$ such that

$$T\psi_k = \eta_k \psi_k, \qquad \eta_k \in \mathbf{C}.$$

Definition A.11. *A bounded operator T is said to be of trace class if and only if for some orthonormal basis $(\psi_k : k \geq 0)$ one has*

$$\sum_{k=0}^{\infty} (|T|\psi_k, \psi_k) < \infty,$$

where $|T| = \sqrt{TT^*}$; T^* is the conjugate transpose of T. Then

$$tr(T) = \sum_{k=0}^{\infty} (T\psi_k, \psi_k)$$

is a finite number, independent of the orthonormal basis.

Equivalently,

Definition A.12. *A bounded operator T is said to be of trace class if and only if the series*

$$\sum_{k=0}^{\infty} \eta_k$$

is absolutely convergent; one then has that

$$tr(T) = \sum_{k=0}^{\infty} \eta_k$$

is a finite number, independent of the orthonormal basis.

An illustration of Definition A.12 is that if we assume that $\mathscr{H} = L^2(M)$ so that $(\psi_k : k \geq 0)$ is an orthonormal basis of $L^2(M)$, then the Schwartz kernel K of T is defined by

$$K(x,y) = \sum_{k=0}^{\infty} \eta_k \psi_k(x) \overline{\psi_k(y)} \tag{A.17}$$

on $M \times M$, and thus

$$\int_M K(x,x)\, dV(x) = \sum_k \eta_k, \tag{A.18}$$

given that the series converges absolutely and uniformly. Hence, for all $f \in \mathscr{H}$

$$(Tf)(x) = \int_M K(x,y) f(y)\, dV(x)$$

and we write

$$\operatorname{tr}(T) = \int_M K(x,x)\, dV(x).$$

For example the trace of the heat kernel on M is given by

$$\Theta_M(t) = \operatorname{tr} e^{-t\Delta_M} = \int\limits_M K(t,x,x)\, dV(x) = \sum_{k=0}^{\infty} e^{-\lambda_k t}. \tag{A.19}$$

Similarly, the traces of the wave kernels on M are

$$\cos\left(t\sqrt{\Delta_M}\right) = \operatorname{Re} e^{it\sqrt{\Delta_M}} = \sum_{k=1}^{\infty} \cos\left(t\sqrt{\lambda_k}\right) = \operatorname{Re} \sum_{k=0}^{\infty} e^{i\sqrt{\lambda_k}t} \tag{A.20}$$

$$\frac{\sin\left(t\sqrt{\Delta_M}\right)}{\sqrt{\Delta_M}} = \sum_{k=1}^{\infty} \frac{\sin(t\sqrt{\lambda_k})}{\sqrt{\lambda_k}}, \tag{A.21}$$

where

$$U(t) := e^{it\sqrt{\Delta_M}} \tag{A.22}$$

is the half-wave operator with the trace

$$\operatorname{tr} U(t) = \int_M U(t,x,x)\, dV(x) = \sum_{k=1}^{\infty} e^{i\sqrt{\lambda_k}t}.$$

Appendix B

Special Functions and Integral Formulae

We include in this appendix some special functions that we used in this thesis. These include the gamma function, digamma function (also called psi function), Riemann zeta function, Bessel functions, hypergeometric functions, Legendre functions, associated Legendre functions, Gegenbauer polynomials and Jacobi Polynomials. Our main references include Gradshtejn and Ryzhik [66], Magnus et al. [106], Abramowitz and Stegun [1].

B.1 The Gamma Function

The gamma function (Gradshtejn and Ryzhik [66, Sec. 8.31]) is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt, \quad \operatorname{Re} s > 0. \tag{B.1}$$

It has a meromorphic continuation to the whole complex s-plane given by

$$\begin{aligned}\Gamma(s) &= \int_0^1 t^{s-1}\, dt \sum_{n=0}^\infty (-1)^n \frac{t^n}{n!} = \sum_{n=0}^\infty (-1)^n \frac{1}{n!} \int_0^1 t^{n+s-1}\, dt \\ &= \sum_{n=0}^\infty (-1)^n \frac{1}{n!} \frac{1}{s+n}.\end{aligned}$$

Hence, $\Gamma(s)$ admits a simple pole at $s = -n$, $n \in \mathbf{N}_0$, with residue

$$\operatorname{Re}_{s=-n} \Gamma(s) = (-1)^n \frac{1}{n!}; \tag{B.2}$$

and satisfies the following properties.

$$\sqrt{\pi}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) \quad \text{(Legendre duplication formula);} \tag{B.3}$$

$$\Gamma(s+1) = s\Gamma(s), \;\; \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}; \tag{B.4}$$

$$\Gamma\left(\frac{1}{2}+ia\right)\Gamma\left(\frac{1}{2}-ia\right)=\left|\Gamma\left(\frac{1}{2}+ia\right)\right|^2=\frac{\pi}{\cosh\pi a},\qquad a\in\mathbf{R}; \tag{B.5}$$

$$\Gamma(1+ia)\,\Gamma(1-ia)=\frac{\pi a}{\sinh\pi a},\qquad a\in\mathbf{R}. \tag{B.6}$$

The asymptotics of $\log\Gamma(s)$ is given by (Abramowitz and Stegun [1, eq. 6.1.41])

$$\ln\Gamma(z)\sim\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln 2\pi+\frac{1}{12}\frac{1}{z+1}+O\left(\frac{1}{z^2}\right)\quad\text{as}\quad z\nearrow\infty. \tag{B.7}$$

The following Stirling's formulae hold (Magnus et al. [106, p. 12]):

$$|\Gamma(\nu+ix)\Gamma(\nu-ix)|=O\left(x^{2\nu-1}e^{-\pi x}\right)\quad\text{for large } x, \tag{B.8}$$

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\sim z^{\alpha-\beta}\left[1+\frac{1}{2z}(\alpha-\beta)(\alpha+\beta-1)+O\left(z^{-2}\right)\right],\quad|\arg z|<\pi. \tag{B.9}$$

For $m\in\mathbf{N}$, the Pochhammer symbol $(s)_m$ is defined by

$$(s)_m=\frac{\Gamma(s+m)}{\Gamma(s)},\quad(-s)_m=(-1)^m\prod_{k=0}^{m-1}(s-k). \tag{B.10}$$

The digamma function (Gradshtejn and Ryzhik [66, Sec. 8.36], Magnus et al. [106]) is defined by

$$\psi(z)=\frac{d}{dz}\log\Gamma(z)=\frac{\Gamma'(z)}{\Gamma(z)},$$

with the series representation

$$\psi(z)-\psi(w)=\sum_{n=0}^{\infty}\left[\frac{1}{n+w}-\frac{1}{n+z}\right], \tag{B.11}$$

and has the following properties and special values.

$$\psi(1)=-C \tag{B.12}$$

$$\pi\tan\pi z=\psi\left(\frac{1}{2}+z\right)-\psi\left(\frac{1}{2}-z\right) \tag{B.13}$$

$$\psi(-z)=\psi(z+1)+\pi\cot\pi z \tag{B.14}$$

$$\psi\left(\frac{1}{2}\right)=-C-2\log 2 \tag{B.15}$$

$$\psi(z+1)=\psi(z)+\frac{1}{z} \tag{B.16}$$

$$\psi\left(\frac{1}{2}\pm n\right)=-C+2\left[\sum_{k=1}^{n}\frac{1}{2k-1}-\log 2\right]. \tag{B.17}$$

B.2 The Riemann Zeta Function

The *Riemann zeta function* $\zeta(s)$ (Titchmarsh [170]) is a function of a complex variable $s\in\mathbf{C}$ defined for $\operatorname{Re}s>1$ by

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}. \tag{B.18}$$

The series converges absolutely for $\operatorname{Re} s > 1$ and uniformly in every half plane $\operatorname{Re} s > 1 + \epsilon$ ($\epsilon > 0$). Riemann proved that ζ possesses an analytic continuation into the whole s-plane which is regular except for a simple pole at $s = 1$ and satisfies the Riemann's functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \tag{B.19}$$

or equivalently,

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s)\zeta(s). \tag{B.20}$$

The Riemann zeta function $\zeta(s)$ can be analytically continued into the half plane $\operatorname{Re} s > 0$ and the continuation is regular for $\operatorname{Re} s > 0$, except for a simple pole at $s = 1$ with residue 1. Further, at $s = 1$, $\zeta(s)$ has the expansion (see e.g. Titchmarsh [170], Siegel [152])

$$\zeta(s) = \frac{1}{s-1} + C + c_1(s-1) + c_2(s-1)^2 + \cdots, \tag{B.21}$$

and that $\zeta(s)$ satisfies the functional equation

$$\Lambda(s) = \Lambda(1-s), \tag{B.22}$$

where

$$\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s), \tag{B.23}$$

and $C = 0.57721566 \cdots$ is the Euler-Mascheroni constant.

Taking $s = 2n+1$, $n = 1, 2, \cdots$ in (B.20), the factor $\cos\left(\frac{\pi s}{2}\right)$ vanishes and we have

$$\zeta(-2n) = 0, \qquad n = 1, 2, \cdots, \tag{B.24}$$

which are often referred to as the *trivial zeros of* $\zeta(s)$. Setting $s = 2n$, $n = 1, 2, \cdots$ in (B.20) and applying

$$\zeta(-n) = \begin{cases} -\frac{1}{2}, & n = 0, \\ -\frac{Bn+1}{n+1}, & n = 1, 2, \cdots, \end{cases} \tag{B.25}$$

we obtain the identity

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \qquad n = 0, 1, 2, \cdots, \tag{B.26}$$

where B_n = the nth Bernoulli number defined by

$$\frac{s}{e^s - 1} = \sum_{n=0}^{\infty} B_n \frac{s^n}{n!}, \qquad (|s| < 2\pi). \tag{B.27}$$

The first few Bernoulli numbers are illustrated in Table B.1 below.

TABLE B.1: The mth Bernoulli Numbers B_m.

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
1	1/2	1/6	0	−1/30	0	1/42	0	−1/30	0	5/66

By (B.26) and Table B.1 we obtain the following special values of the Riemann zeta functions:

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}, \cdots. \tag{B.28}$$

The following logarithmic derivative of $\zeta(s)$ (Titchmarsh [170, eq. (1.1.8)]) holds:

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n}, \tag{B.29}$$

where $\Lambda(n)$ is the Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \ln p, & \text{if } n \text{ is an integer power } p^k,\ k \geq 1, \text{ of a prime number } p, \\ 0 & \text{for other natural number } n. \end{cases} \tag{B.30}$$

B.3 The Bessel Functions

The *Bessel's equation of order n* (Abramowitz and Stegun [1, Ch. 9]) is

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + \left(z^2 - n^2\right) f = 0, \quad n = 0, 1, 2, ..., \tag{B.31}$$

and has a solution

$$J_n(z) = f(z) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{z}{2}\right)^{2k+n} \frac{1}{k!\Gamma(n+k+1)}, \tag{B.32}$$

called the *Bessel function of the first kind of order n*. By substituting the beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \operatorname{Re} x > 0, \operatorname{Re} y > 0, \tag{B.33}$$

with $x = k + \frac{1}{2}$, $y = \nu + \frac{1}{2}$, into (B.32), we have

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu + \frac{1}{2}\right)} \int_{-1}^{1} \left(1 - t^2\right)^{\nu - 1/2} e^{izt}\, dt, \qquad \operatorname{Re} \nu > -\frac{1}{2}. \tag{B.34}$$

According to Abramowitz and Stegun [1, eq. 11.4.38],

$$\int_0^\infty J_0(a\eta) \cos \eta\alpha \, d\eta = \frac{1}{\sqrt{a^2 - \alpha^2}}, \qquad 0 \leq \alpha < a, \tag{B.35}$$

where $J_0(z)$ is the Bessel function of the first kind of order 0 .

For arbitrary parameter ν, which can be real or complex, the Bessel function J_ν is defined by (Abramowitz and Stegun [1, Ch. 10])

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \qquad |z| < \infty, \tag{B.36}$$

and it is known as the *Bessel function of the first kind of order ν*. The following formula for the product of two Bessel functions holds (Gradshtejn and Ryzhik [66, p. 918, eq. 8.442(2)], Glasser and Montaldi [64]):

$$J_\mu(az)J_\nu(bz) = \frac{\left(\frac{az}{2}\right)^\mu \left(\frac{bz}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{az}{2}\right)^{2k}}{k!\Gamma(\mu+k+1)} F\left(-k, -\mu-k; \nu-1; \frac{b^2}{a^2}\right), \tag{B.37}$$

$$J_\nu(az)J_\nu(bz) = \left(\frac{1}{2}\frac{abz}{\sqrt{a^2+b^2}}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\frac{abz}{\sqrt{a^2+b^2}}\right)^{2k}}{k!\Gamma(\nu+k+1)} J_{\nu+2k}\left(z\sqrt{a^2+b^2}\right). \tag{B.38}$$

In terms of the elliptic integral (Dougall [54, eq. (23)]),

$$\int_0^\infty e^{-x\mu} J_\nu(\varrho x)J_\nu(ax)\,dx = \frac{1}{\pi}\varrho^\nu a^\nu \int_0^\pi \frac{\sin^{2\nu}\varphi\,d\varphi}{(\mu^2+a^2-2a\varrho\cos\varphi+\varrho^2)^{\nu+\frac{1}{2}}}, \tag{B.39}$$

for $\varrho > 0$, $a > 0$, $\varrho < a$, $\nu > -\frac{1}{2}$, $\mu > 0$. In particular,

$$\int_0^\infty e^{-x\mu} J_0(\varrho x)J_0(ax)\,dx = \frac{1}{\pi}\int_0^\pi \frac{d\varphi}{\sqrt{\mu^2+a^2-2a\varrho\cos\varphi+\varrho^2}}. \tag{B.40}$$

Also, in terms of the Gauss hypergeometric function (Gradshtejn and Ryzhik [66, p. 660, eq. 6.512(1)])

$$\int_0^\infty J_\mu(\varrho x)J_\nu(ax)\,dx = \varrho^\nu a^{-\nu-1}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu-\nu+1}{2}\right)} F\left(\frac{\mu+\nu+1}{2}, \frac{\mu-\nu+1}{2}; \nu+1; \frac{\varrho^2}{a^2}\right), \tag{B.41}$$

for $\varrho > 0$, $a > 0$, $\varrho < a$, $\mathrm{Re}\,(\mu+\nu) > -1$, $\mu > 0$; for $a < \varrho$, we interchange a and ϱ. See Appendix B.4 for the Gauss hypergeometric function.

The differential equation

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} - \left(z^2+\nu^2\right)f = 0, \quad n = 0, 1, 2, ..., \tag{B.42}$$

is called a *modified Bessel's equation* (Abramowitz and Stegun [1, Sec. 9.6]). The general solution, for arbitrary ν, of (B.42) can be written in the form

$$h(z) = C_1K_\nu(z) + C_2I_\nu(z),$$

where

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \qquad |z| < \infty, \quad |\arg z| < \pi, \tag{B.43}$$

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi}, \qquad |\arg z| < \pi, \quad \nu \neq 0, \pm 1, \pm 2, ...; \tag{B.44}$$

$I_\nu(z)$ is the *modified Bessel function of the first kind* and $K_\nu(z)$ the *modified Bessel function of the second kind*, having integral representations

$$K_\nu(z) = \frac{1}{2}\int_0^\infty e^{-\frac{z}{2}\left(t+\frac{1}{t}\right)} t^{\nu-1}\,dt, \qquad \mathrm{Re}\,z > 0, \quad \nu \text{ arbitrary}; \tag{B.45}$$

$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{\pm zt}(1-t^2)^{\nu-\frac{1}{2}}\,dt, \ \mathrm{Re}\left(\nu+\frac{1}{2}\right) > 0. \tag{B.46}$$

It follows from (B.45) that

$$\int_0^\infty e^{-\sigma t} e^{-\frac{\alpha^2}{4t}} t^{-\nu}\, dt = 2\left(\frac{\alpha}{2\sqrt{\sigma}}\right)^{-\nu+1} K_{\nu-1}\left(\sqrt{\sigma}\alpha\right). \tag{B.47}$$

By the definition of the gamma function and a simple change of variables, we have

$$\int\limits_{-\infty}^{\infty} \frac{e^{-2\pi i m t}}{\left(t^2+y^2\right)^s}\, dt = \begin{cases} y^{1-2s}\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}, & m=0, \\ 2y^{\frac{1}{2}-s}\frac{\pi^s}{\Gamma(s)}|m|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|m|y), & m\neq 0. \end{cases} \tag{B.48}$$

For the special value $\nu = \frac{1}{2}$, we have (Lebedev [100, p. 136])

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}, \quad I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{2z}}\sinh z. \tag{B.49}$$

In particular, the following asymptotics hold (Lebedev [100, p. 136]):

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z} \quad \text{as } z \nearrow \infty, \quad I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \quad \text{as } z \nearrow \infty. \tag{B.50}$$

The Mellin transform of a product of modified Bessel functions are given by

$$\begin{aligned} \int_0^\infty t^{-\sigma} K_\mu(at)K_\nu(bt)\, dt = {}& \frac{2^{-2-\sigma}a^{-\nu+\sigma-1}b^\nu}{\Gamma(1-\sigma)}\Gamma\left(\frac{1-\sigma+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\sigma-\mu+\nu}{2}\right) \\ & \times\Gamma\left(\frac{1-\sigma+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\sigma-\mu-\nu}{2}\right) \\ & \times F\left(\frac{1-\sigma+\mu+\nu}{2}, \frac{1-\sigma-\mu+\nu}{2}; 1-\sigma; 1-\frac{b^2}{a^2}\right), \\ & \operatorname{Re}(a+b)>0,\ \operatorname{Re}\sigma < 1-\operatorname{Re}|\mu| < \operatorname{Re}|\nu|; \end{aligned} \tag{B.51}$$

$$\begin{aligned} \int_0^\infty t^{-\sigma} K_\mu(at)I_\nu(bt)\, dt = {}& \frac{b^\nu\Gamma\left(\frac{1}{2}-\frac{\sigma}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\sigma}{2}-\frac{\mu}{2}+\frac{\nu}{2}\right)}{2^{\sigma+1}\Gamma(\nu+1)a^{-\sigma+\nu+1}} \\ & \times F\left(\frac{1}{2}-\frac{\sigma}{2}+\frac{\mu}{2}+\frac{\nu}{2}, \frac{1}{2}-\frac{\sigma}{2}-\frac{\mu}{2}+\frac{\nu}{2}; \nu+1; \frac{b^2}{a^2}\right), \\ & a>b,\ \operatorname{Re}(\nu+1-\sigma\pm\mu)>0; \end{aligned} \tag{B.52}$$

$$\begin{aligned} \int_0^\infty t^{\nu+1}K_\mu(at)I_\mu(bt)J_\nu(ct)\, dt = {}& \frac{(ab)^{-\nu-1}c^\nu e^{-\left(\nu+\frac{1}{2}\right)\pi i} Q^{\nu+\frac{1}{2}}_{\mu-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2ab}\right)}{\sqrt{2\pi}\left[\left(\frac{a^2+b^2+c^2}{2ab}\right)^2-1\right]^{\frac{\nu}{2}+\frac{1}{4}}} \\ & \operatorname{Re}a > |\operatorname{Re}b| + |\operatorname{Im}c|,\ \operatorname{Re}\nu>-1, \operatorname{Re}(\nu+\mu)>-1. \end{aligned} \tag{B.53}$$

B.4 The Gauss Hypergeometric Function

A particular solution of the hypergeometric equation (Abramowitz and Stegun [1, Ch. 15])

$$z(1-z)\frac{d^2f}{dz^2} + (c-(a+b+1)z)\frac{df}{dz} - abf = 0 \tag{B.54}$$

(where z is a complex variable, a, b and c are parameters which can take arbitrary real or complex values) is

$$h(z) = F(a, b; c; z) = {}_2F_1 = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n, \qquad |z| < 1, \tag{B.55}$$

$c \neq 0, -1, -2, ...,$ where

$$(\beta)_n = \beta(\beta+1)\cdots(\beta+n-1) = \frac{\Gamma(\beta+n)}{\Gamma(\beta)}, n = 1, 2, ...; \quad (\beta)_0 = 1, \tag{B.56}$$

is the Pochhammer symbol. The series on the right-hand side of (B.55) is known as the *hypergeometric series* (hypergeometric function). If either a or b is zero or a negative integer, the series terminates after a finite number of terms, and its sum is then a polynomial in z. It follows from (B.55) and (B.56) that

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

If $b = c$, then we have

$$F(a, b; b; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!}. \tag{B.57}$$

On the other hand

$$\frac{1}{(1-z)^a} = \sum_{n=0}^{\infty} \binom{a+n-1}{n} z^n = \sum_{n=0}^{\infty} \binom{a+n-1}{a-1} z^n = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!}. \tag{B.58}$$

By (B.57) and (B.58), we obtain (Abramowitz and Stegun [1, eq. 15.1.8])

$$F(a, b; b; -z) = \frac{1}{(1+z)^a} = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{(-z)^n}{n!}, \tag{B.59}$$

with the identity (Gradshtejn and Ryzhik [66, 9. 121(26)])

$$F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\arcsin z}{z}. \tag{B.60}$$

The following differentiation formula holds (Abramowitz and Stegun [1, p. 557, eq. 15.2.1]):

$$\frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a+1, b+1; c+1; z). \tag{B.61}$$

For $\operatorname{Re} c > \operatorname{Re} b > 0$, the hypergeometric function $F(a, b; c; z)$ admits the integral representation (Gradshtejn and Ryzhik [66, p. 1005, eq. 9.111])

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt. \tag{B.62}$$

In terms of the elliptic integral, we have (Gradshtejn and Ryzhik [66, p. 407, eq. 3.665(2)])

$$a^{-2\nu}F\left(\nu,\nu;1;\frac{b^2}{a^2}\right)=\frac{1}{\pi}\int_0^\pi\frac{d\omega}{(a^2+b^2-2ab\cos\omega)^\nu},\quad |b|<|a|, \tag{B.63}$$

$$\int_0^\pi\frac{\sin^{2\mu-1}\psi\,d\psi}{(1-2a\cos\psi+a^2)^{\nu+\frac{1}{2}}}=B\left(\mu,\frac{1}{2}\right)F\left(\nu+\frac{1}{2},\nu-\mu+1;\mu+\frac{1}{2};a^2\right),\ \mathrm{Re}\,\mu>0,|a|<1. \tag{B.64}$$

The hypergeometric function $F(a,b;c;z)$ satisfies various transformation rules. For example (Gradshtejn and Ryzhik [66, p. 1008, eqs 9.1311, 9.131(1)-(2), 9.132(1)-(2)])

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z), \tag{B.65}$$

$$\begin{aligned}F(a,b;c;z)=&\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-z)^{-a}F\left(a,c-b;a+1-b;\frac{1}{1-z}\right),\\&+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1-z)^{-b}F\left(b,c-a;b-a+1;\frac{1}{1-z}\right),\end{aligned} \tag{B.66}$$

$$\begin{aligned}F(a,b;c;z)=&\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F\left(a,a+1-c;a+1-b;\frac{1}{z}\right),\\&+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F\left(b,b+1-c;b+1-a;\frac{1}{z}\right),\\&a-b\neq\pm m,m=0,1,2,\cdots,\end{aligned} \tag{B.67}$$

$$\begin{aligned}F(a,b;c;z)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F\left(a,b;a+b-c+1;1-z\right),\\&+(1-z)^{c-a-b}\frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}F\left(c-a,c-b;c-a-b+1;1-z\right).\end{aligned} \tag{B.68}$$

Also (Abramowitz and Stegun [1, eqs 15.5.7, 15.4.4], Gradshtejn and Ryzhik [66, p. 1008, eq. 9.131(1)])

$$F(a,b;c;z)=z^{-a}F(a,a-c+1;a-b+1;z^{-1}), \tag{B.69}$$

$$F(a,b;c;z)=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right), \tag{B.70}$$

$$F\left(-a,a+1;1;z\right)=P_a(1-2z), \tag{B.71}$$

$$F(2a,2a+1-c;c,z)=(1+z)^{-2a}F\left(a,a+\frac{1}{2};c;\frac{4z}{(1+z)^2}\right). \tag{B.72}$$

We also have the special value (Gradshtejn and Ryzhik [66, p. 1008, eq. 9.122(1)])

$$F(a,b,;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \mathrm{Re}\,c>\mathrm{Re}\,(a+b). \tag{B.73}$$

B.5 The Legendre Functions

The differential equation

$$(1-z^2)\frac{d^2f}{dz^2}-2z\frac{df}{dz}+\left[\nu(\nu+1)-\frac{\mu^2}{1-z^2}\right]f=0 \tag{B.74}$$

is called the *associated Legendre equation of degree ν and order μ* and its solution is called the *associated Legendre function of the first kind of degree ν and order μ* (Abramowitz and Stegun [1, Ch. 8]). If $\mu = 0$, then (B.74) reduces to the Legendre equation of degree ν.

Now if we set

$$t = \frac{1}{2}(1-z) \Rightarrow z = 1-2t$$

in the Legendre equation

$$(1-z^2)\frac{d^2 f}{dz^2} - 2z\frac{df}{dz} + \nu(\nu+1)f = 0, \tag{B.75}$$

then we obtain the hypergeometric equation

$$t(1-t)\frac{d^2 u}{dt^2} + (1-2t)\frac{du}{dt} + [\nu(\nu+1)]\, u = 0, \tag{B.76}$$

with $a = -\nu, \quad b = \nu+1, c = 1$. Hence,

$$P_\nu(z) = F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right), \tag{B.77}$$

and it is called the Legendre function of the first kind of degree ν and it is a solution of (B.75). Because of the property

$$P_{-\nu-1}(z) = P_\nu(z), \tag{B.78}$$

the Legendre function

$$P_\nu(z) = P_{-s}(x),$$

where $P_{-s}(x)$, $x = \cos r$, is a solution of

$$(1-x^2)\frac{d^2 v}{dx^2} - 2x\frac{dv}{dx} + s(s-1)v = 0. \tag{B.79}$$

In general for $\operatorname{Re}\mu \geq 0$ (Bateman et al. [19, p. 122, eq. 7], Abramowitz and Stegun [1, p. 562, eq. 15.4.10])

$$P_\nu^\mu(z) = \frac{2^\mu}{\Gamma(1-\mu)}\left(z^2-1\right)^{-\frac{\mu}{2}} F\left(1-\mu+\nu, -\mu-\nu; 1-\mu; \frac{1-z}{2}\right), \tag{B.80}$$

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right), \tag{B.81}$$

$$P_{2a-c}^{1-c}\left[(1-z)^{-\frac{1}{2}}\right] = 2^{1-c}\Gamma(c)^{-1} z^{\frac{c}{2}-\frac{1}{2}}(1-z)^{\frac{1}{2}-\frac{c}{2}+a} F\left(a, a+\frac{1}{2}; c; z\right). \tag{B.82}$$

The associated Legendre function of the first kind satisfies the following recurrence relations (Abramowitz and Stegun [1, p. 333, eqs 8.5.1, 8.5.5])

$$P_\nu^{\mu+1}(z) = \left(z^2-1\right)^{-\frac{1}{2}}\left[(\nu-\mu)zP_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)\right], \tag{B.83a}$$

$$P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu+1)\left(z^2-1\right)^{\frac{1}{2}} P_\nu^{\mu-1}(z). \tag{B.83b}$$

From (B.83) we can easily deduce the following special values (Gradshtejn and Ryzhik [66, p. 967, eqs 8.753(1), 8.754(1-2)])

$$P_{-s}(1) = 1, \quad P_0(\cosh\alpha) = P_{-1}(\cosh\alpha) = 1 \tag{B.84a}$$

$$P_0^{\mu}(\cos\vartheta) = \frac{1}{\Gamma(1-\mu)}\cot^{\mu}\frac{\vartheta}{2} \tag{B.84b}$$

$$P_{\nu-\frac{1}{2}}^{\frac{1}{2}}(\cosh\alpha) = \sqrt{\frac{2}{\pi\sinh\alpha}}\cosh\nu\alpha. \tag{B.84c}$$

$$P_{\nu-\frac{1}{2}}^{\frac{1}{2}}(\cos\vartheta) = \sqrt{\frac{2}{\pi\sin\vartheta}}\cos\nu\vartheta. \tag{B.84d}$$

Also for the argument zero, we have (Abramowitz and Stegun [1, p. 334, eq. 8.6.1])

$$P_{\nu}^{\mu}(0) = 2^{\mu}\frac{\cos\left[\frac{\pi}{2}(\nu+\mu)\right]\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}-\frac{\mu}{2}+1\right)}. \tag{B.85}$$

The associated Legendre function P_{ν}^{μ} has the following integral representations:

$$P_{\nu}(\cosh\alpha) = \frac{2}{\pi}\cot\left(\nu+\frac{1}{2}\right)\pi\int_{\alpha}^{\infty}\frac{\sinh\left(\nu+\frac{1}{2}\right)\theta}{\sqrt{2\cosh\theta-2\cosh\alpha}}\,d\theta \tag{B.86}$$

$$P_{\nu}^{\mu}(\cosh\alpha) = \sqrt{\frac{2}{\pi}}\frac{\sinh^{\mu}\alpha}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_0^{\alpha}\frac{\cosh\left(\nu+\frac{1}{2}\right)t\,dt}{(\cosh\alpha-\cosh t)^{\mu+\frac{1}{2}}} \tag{B.87}$$

$$\mathrm{Re}\,\mu < \frac{1}{2},\ \alpha > 0$$

$$P_{\nu}^{m}(z) = (-1)^m\frac{\Gamma(\nu+1)}{\pi\Gamma(\nu-m+1)}\int_0^{\pi}\frac{\cos mt\,dt}{\left(z+\sqrt{z^2-1}\cos t\right)^{\nu+1}} \tag{B.88}$$

$$P_{\nu}^{-\mu}(q) = \frac{\left(q^2-1\right)^{\frac{\mu}{2}}}{2^{\mu}\sqrt{\pi}\Gamma\left(\frac{1}{2}+\mu\right)}\int_{-1}^{1}\frac{\left(1-t^2\right)^{\mu-\frac{1}{2}}\,dt}{\left[q+t\sqrt{q^2-1}\right]^{\mu-\nu}},\ \mathrm{Re}\,\mu > -\frac{1}{2}. \tag{B.89}$$

According to Gradshtejn and Ryzhik [66, p. 772, eq. 7.132(1)], the following equality holds:

$$\int_{-1}^{1}\left(1-t^2\right)^{\sigma-1}P_{\nu}^{\mu}(t)\,dt = \frac{\pi 2^{\mu}\Gamma\left(\sigma+\frac{\mu}{2}\right)\Gamma\left(\sigma-\frac{\mu}{2}\right)}{\Gamma\left(\sigma+\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\sigma-\frac{\nu}{2}\right)\Gamma\left(-\frac{\mu}{2}+\frac{\nu}{2}+1\right)\Gamma\left(-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}\right)}, \tag{B.90}$$

for $2\mathrm{Re}\,\sigma > \mathrm{Re}\,\mu$. The following derivatives hold (Gradshtejn and Ryzhik [66, p. 965, eq. 8.733(1)], Sergo [150, p. 41]):

$$\frac{dP_{\nu}^{\mu}(x)}{dx} = \frac{P_{\nu}^{\mu}(x)-(\nu-\mu+1)P_{\nu+1}^{\mu}(x)}{1-x^2}, \tag{B.91}$$

$$= \frac{(\nu+\mu)(\nu-\mu+1)}{\sqrt{x^2-1}}P_{\nu}^{\mu-1}(x) - \frac{\nu\mu}{x^2-1}P_{\nu}^{\mu}(x), \tag{B.92}$$

$$P_k^m(\cos\theta) = \sin^m\theta\frac{d^m P_k(\cos\theta)}{(d\cos\theta)^m}, \tag{B.93}$$

$$\lim_{\nu\searrow 0}\frac{\partial}{\partial\nu}P_{\nu}(z) = \frac{z-1}{2}F\left(1,1;2;\frac{1-z}{2}\right). \tag{B.94}$$

The second solution of (B.74) is $Q_{\nu}^{\mu}(z)$ and it is called the *associated Legendre function of the second kind* of degree ν and order μ. The function $Q_{\nu}(z)$ is the *Legendre function of the second kind of degree* ν (Abramowitz and Stegun [1, Ch. 8]). It can also be written in terms of the hypergeometric function, namely (Gradshtejn and Ryzhik [66, P. 970, eq. 8.771(2)])

$$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\sqrt{\pi}\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}\left(z^2-1\right)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right), \tag{B.95}$$

for $\mu+\nu \neq -m,\ m \geq 1$; and (Lebedev [100, p. 200])

$$Q_\nu(z) = \sqrt{\pi}\frac{\Gamma(\nu+1)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}(z+1)^{-\nu-1} F\left(\nu+1, \nu+1; 2\nu+2; \frac{2}{1+z}\right). \tag{B.96}$$

The function $Q_\nu(z)$ has an integral representation (Lebedev [100, eq. 7.4.8])

$$Q_\nu(\cosh\rho) = \frac{1}{\sqrt{2}}\int_\rho^\infty \frac{e^{-(\nu+\frac{1}{2})\theta}}{\sqrt{\cosh\theta-\cosh\rho}}\, d\theta. \tag{B.97}$$

The following integral involving the product of Legendre functions holds (Gradshtejn and Ryzhik [66, p. 770, eq. 7.114(1)]):

$$\int_1^\infty P_\nu(x)Q_\sigma(x)\, dx = \frac{1}{(\sigma-\nu)(\sigma+\nu+1)}. \tag{B.98}$$

Also, the associated Legendre function of the second kind satisfies the following recurrence relations (Gradshtejn and Ryzhik [66, p. 967, eqs 8.732(3-4)])

$$Q_\nu^{\mu+2}(z) = -2(\mu+1)z\left(z^2-1\right)^{-\frac{1}{2}} Q_\nu^{\mu+1}(z) + (\nu-\mu)(\nu+\mu+1)Q_\nu^\mu(z), \tag{B.99a}$$

$$Q_{\nu-1}^\mu(z) = Q_{\nu+1}^\mu(z) - (2\nu+1)\left(z^2-1\right)^{\frac{1}{2}} Q_\nu^{\mu-1}(z). \tag{B.99b}$$

From (B.99) we can easily deduce the following special values (Abramowitz and Stegun [1, p. 334, eqs 8.6.11, 8.6.10, 8.6.13])

$$Q_\nu^{-\frac{1}{2}}(z) = \frac{-i(2\pi)^{\frac{1}{2}}\left(z^2-1\right)^{-\frac{1}{4}}}{(2\nu+1)}\left[z+\left(z^2-1\right)^{\frac{1}{2}}\right]^{-\nu-\frac{1}{2}}, \tag{B.100a}$$

$$Q_\nu^{\frac{1}{2}}(z) = i\sqrt{\frac{\pi}{2}}\left(z^2-1\right)^{-\frac{1}{4}}\left[z+\left(z^2-1\right)^{\frac{1}{2}}\right]^{-\nu-\frac{1}{2}}, \tag{B.100b}$$

$$Q_{\nu-\frac{1}{2}}^{\frac{1}{2}}(\cos\vartheta) = -\sqrt{\frac{\pi}{2\sin\vartheta}}\sin\nu\vartheta. \tag{B.100c}$$

The Legendre polynomials P_ν^μ and Q_ν^μ satisfy the relation (Abramowitz and Stegun [1, eq. 8.2.7])

$$P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}\left(\frac{z}{\left(z^2-1\right)^{1/2}}\right) = \sqrt{\frac{2}{\pi}}\frac{\left(z^2-1\right)^{1/4} e^{-\mu i\pi}Q_\nu^\mu(z)}{\Gamma(\nu+\mu+1)}. \tag{B.101}$$

B.6 The Gegenbauer Polynomials

The Gegenbauer polynomial $C_k^\nu(t)$ (Gradshtejn and Ryzhik [66, Sec. 8.93]) is the generalisation of the Legendre polynomial $P_k(t)$ (Gradshtejn and Ryzhik [66, Sec. 8.91]) and is defined by the coefficient of α^k in the power series expansion of the function

$$\left(1-2t\alpha+\alpha^2\right)^{-\nu} = \sum_{k=0}^\infty C_k^\nu(t)\alpha^k, \tag{B.102}$$

with

$$C_0^{\nu}(t)=1,\ C_1^{\nu}(t)=2\nu t,\ C_2^{\nu}(t)=2\nu(\nu+1)t^2-\nu,\cdots, \tag{B.103}$$

and the special cases

$$C_0^0(\cos\theta)=1,\quad C_k^1(\cos\theta)=\frac{\sin(k+1)\theta}{\sin\theta},\quad C_k^{\nu}(1)=\begin{pmatrix}2\nu+k-1\\ k\end{pmatrix}. \tag{B.104}$$

A special case of (B.102) is the Legendre polynomial $P_k(t)$ defined by

$$\left(1-2t\alpha+\alpha^2\right)^{-\frac{1}{2}}=\sum_{k=0}^{\infty}P_k(t)\alpha^k. \tag{B.105}$$

For $\nu>-\frac{1}{2}$, $C_k^{\nu}(t)$ has a series representation (Bateman et al. [20, p. 175, eq. (18)])

$$C_k^{\nu}(t)=\sum_{0\le l\le\frac{k}{2}}(-1)^l\frac{\Gamma(k-l+\nu)}{\Gamma(\nu)l!(k-2l)!}(2t)^{k-2l}. \tag{B.106}$$

In terms of the Gauss hypergeometric function it is given by

$$\begin{aligned}C_k^{\nu}(t)&=\frac{\Gamma(2\nu+k)}{k!\Gamma(2\nu)}F\left(-k,k+2\nu;\nu+\frac{1}{2};\frac{1-t}{2}\right), \\ &=\frac{2^k\Gamma(\nu+k)}{k!\Gamma(\nu)}t^kF\left(-\frac{k}{2},\frac{1-k}{2};1-\nu-k;\frac{1}{t^2}\right),\end{aligned} \tag{B.107}$$

with the recursion formulae

$$kC_k^{\nu}(t)=2\nu\left[tC_{k-1}^{\nu+1}(x)-C_{k-2}^{\nu+1}(t)\right] \tag{B.108}$$

$$kC_k^{\nu}(t)=(k+2\nu-1)tC_{k-1}^{\nu}(t)-2\nu\left(1-t^2\right)C_{k-2}^{\nu+1}(t). \tag{B.109}$$

Differentiating (B.106) gives

$$\begin{aligned}\frac{d}{dt}C_k^{\nu}(t)&=2\sum_{0\le l\le\frac{k-1}{2}}(-1)^l\frac{\Gamma(k-l+\nu)}{\Gamma(\nu)l!(k-2l-1)!}(2t)^{k-2l-1}\\ &=2\nu\sum_{0\le l\le\frac{k-1}{2}}(-1)^l\frac{\Gamma(k-1-l+\nu+1)}{\Gamma(\nu+1)l!(k-2l-1)!}(2t)^{k-2l-1}.\end{aligned}$$

Therefore, we obtain the differential recursion formula

$$\frac{d}{dt}C_k^{\nu}(t)=2\nu C_{k-1}^{\nu+1}(t); \tag{B.110}$$

and in general,

$$\frac{d^m}{dt^m}C_k^{\nu}(t)=2^m\frac{\Gamma(\nu+m)}{\Gamma(\nu)}C_{k-m}^{\nu+m}(t). \tag{B.111}$$

It is not difficult to see that using (B.110) and (B.111), the Gegenbauer polynomial $y(t)=C_k^{\nu}(t)$ satisfies the second-order differential equation

$$\left(1-t^2\right)y''-(2\nu+1)ty'+k(k+2\nu)y=0. \tag{B.112}$$

Expanding the right-hand side of the identity (B.109), we obtain the Rodrigues' formula (Gradshtejn and Ryzhik [66, p. 993, eq. 8.939(7)])

$$C_k^\nu(t) = \frac{(-1)^k \Gamma(\frac{1+2\nu}{2}) \Gamma(k+2\nu) \left(1-t^2\right)^{\frac{1}{2}-\nu}}{2^k k! \Gamma(2\nu) \Gamma\left(\frac{2\nu+1}{2}+k\right)} \frac{d^k}{dt^k} \left(1-t^2\right)^{k+\nu-\frac{1}{2}}. \tag{B.113}$$

The Gegenbauer polynomial $C_k^\nu(t)$ and the associated Legendre polynomial $P_k^\nu(t)$ are related to one-another by

$$C_k^\nu(t) = \frac{\Gamma(2\nu+k)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(2\nu)\Gamma(k+1)} \left\{\frac{1}{4}\left(t^2-1\right)\right\}^{\frac{1}{4}-\frac{\nu}{2}} P^{\frac{1}{2}-\nu}_{\nu+k-\frac{1}{2}}(t), \tag{B.114}$$

with

$$C_k^{\frac{1}{2}}(t) = P_k(t).$$

The following addition formula also holds (Gradshtejn and Ryzhik [66, p. 992, eq. 8.934(3)]):

$$\begin{aligned} &C_k^\nu(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\eta) \\ &= \frac{\Gamma(2\nu-1)}{[\Gamma(\nu)]^2} \sum_{m=0}^{k} \frac{2^{2m}(k-m)![\Gamma(\nu+m)]^2}{\Gamma(2\nu+k+m)} (2\nu+2m-1)\sin^m\alpha\sin^m\beta \\ &\quad \times C_{k-m}^{\nu+m}(\cos\alpha) C_{k-m}^{\nu+m}(\cos\beta) C_m^{\nu-\frac{1}{2}}(\cos\eta), \quad \alpha,\beta,\eta \in \mathbf{R},\ \nu \neq \frac{1}{2}. \end{aligned} \tag{B.115}$$

As an orthogonal polynomial, $C_k^\nu(t)$ satisfies the orthogonality properties

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} \left[C_m^\nu(x)\right]^2 dx = \frac{\pi 2^{1-2\nu}\Gamma(2\nu+m)}{m!(m+\nu)\Gamma(\nu)^2}, \tag{B.116}$$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} C_k^\nu(x)\, dx = 0, \quad k>0,\ \nu > -\frac{1}{2}. \tag{B.117}$$

The polynomial $C_k^\nu(t)$ satisfies the recursive formula

$$C_m^\mu(t) = \frac{\Gamma(\nu)}{\Gamma(\mu)} \sum_{0\le l\le \frac{m}{2}} a_l C_{m-2l}^\nu(t), \ \ \mu > \nu > -\frac{1}{2}, \tag{B.118}$$

where

$$a_l = \frac{m-2l+\nu}{l!} \frac{\Gamma(l+\mu-\nu)}{\Gamma(\mu-\nu)} \frac{\Gamma(m+\mu-l)}{\Gamma(m+\nu+1-l)}.$$

It has the Laplace integral representation

$$\mathscr{C}_\nu^\mu(x) = \frac{C_\nu^\mu(x)}{C_\nu^\mu(1)} = \frac{\nu_{n-2}}{\nu_{n-1}} \int_0^\pi \left[x + \left(x^2-1\right)^{\frac{1}{2}}\cos\theta\right]^\nu \sin^{2\mu-1}\theta\, d\theta, \ \ \mu > 0. \tag{B.119}$$

In terms of the fractional representation we have

$$C_k^{\frac{n-1}{2}}(\cos\theta) = \frac{2^{\frac{n+1}{2}}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}(2k+n-1)\Gamma(n-1)} \frac{\partial^{\frac{n-1}{2}}}{\partial(\cos\theta+1)^{\frac{n-1}{2}}} \cos\left(k+\frac{n-1}{2}\right)\theta, \tag{B.120}$$

where

$$\frac{\partial^{\frac{1}{2}}}{\partial(y-q)^{\frac{1}{2}}}h(y) = \frac{\partial}{\partial y}\frac{\partial^{-\frac{1}{2}}}{\partial(y-q)^{-\frac{1}{2}}}h(y) = \frac{1}{\sqrt{\pi}}\frac{\partial}{\partial y}\int_q^y (y-x)^{-\frac{1}{2}}h(x)\,dx$$
$$= \frac{1}{\sqrt{\pi}}\int_q^y \frac{h'(x)}{\sqrt{y-x}}\,dx + \frac{1}{\sqrt{\pi}}\frac{h(q)}{\sqrt{y-q}}, \tag{B.121}$$

provided that $h(q)$ exists; and moreover, for a function g,

$$\frac{\partial^{\beta}}{\partial(g(y)-g(q))^{\beta}}h(y) = \frac{\partial^{\beta-k}}{\partial(g(y)-g(q))^{\beta-k}}\frac{\partial^k}{\partial y^k}h(y) + \sum_{k=0}^{m-1}\frac{(g(y)-g(q))^{k-\beta}h^{(k)}(q)}{\Gamma(k-\beta+1)}, \tag{B.122}$$

and

$$\frac{\partial^{\frac{1}{2}}}{\partial(g(y)-g(q))^{\frac{1}{2}}}h(y) = \frac{1}{\sqrt{\pi}}\int_q^y \frac{h(x)}{\sqrt{g(y)-g(x)}}\,dx + \frac{1}{\sqrt{\pi}}\frac{h(q)}{\sqrt{g(y)-g(q)}}. \tag{B.123}$$

In particular, if m is not an integer, the Riemann-Liouville fractional derivative is given by (Kilbas et al. [92, Sec 2.1])

$$\frac{d^m}{dy^m}h(y) = \frac{d^k}{dy^k}\frac{1}{\Gamma(k-m)}\int_0^y (y-x)^{k-m-1}h(x)\,dx, \tag{B.124}$$

where k is the smallest integer not less than m.

B.7 The Jacobi Polynomials

The Jacobi polynomial $P_k^{(\alpha,\beta)}(t)$ satisfies the second-order differential equation

$$\left(1-t^2\right)y'' - (\alpha-\beta+(\alpha+\beta+2)t)y' + k(k+\alpha+\beta+1)y = 0. \tag{B.125}$$

Writing

$$R_k^{(\alpha,\beta)}(t) = \frac{k!}{(\alpha+1)_k}P_k^{(\alpha,\beta)}(t) = F\left(-k, k+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right) \tag{B.126}$$

and setting $k \to \frac{ir-\alpha-\beta-1}{2}$, with $t = \cosh 2x$, we obtain the Jacobi function

$$R^{(\alpha,\beta)}_{\frac{ir-\alpha-\beta-1}{2}}(\cosh 2x) = \frac{\Gamma\left(\frac{ir-\alpha-\beta+1}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\frac{ir-\alpha-\beta-1}{2}\right)}P^{(\alpha,\beta)}_{\frac{ir+\alpha-\beta+1}{2}}(\cosh 2x)$$
$$= F\left(\frac{\alpha+\beta+1-ir}{2}, \frac{ir+\alpha+\beta+1}{2}; \alpha+1; -\sinh^2 x\right) \tag{B.127}$$
$$= \phi_r^{(\alpha,\beta)}(x);$$

and the associated Jacobi operator is

$$-\mathcal{L}_{(\alpha,\beta)} = \frac{d^2}{dx^2} + [(2\alpha+1)\coth x + (2\beta+1)\tanh x]\frac{d}{dx}, \tag{B.128}$$

or what is the same, $\phi_r^{(\alpha,\beta)}(x)$ solves the differential equation

$$\left(\frac{d^2}{dx^2}+[(2\alpha+1)\coth x+(2\beta+1)\tanh x]\frac{d}{dx}\right)\phi_r^{(\alpha,\beta)}(x)=-\left(\rho^2+r^2\right)\phi_r^{(\alpha,\beta)}(x), \tag{B.129}$$

such that $\phi_r^{(\alpha,\beta)}(0)=1$. Here $\rho=\alpha+\beta+1$.

The Jacobi function $\phi_r^{(\alpha,\beta)}(x)$ admits the following integral representation (see Koornwinder [94]):

$$\begin{aligned}\phi_r^{(\alpha,\beta)}(x)=&\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\frac{2^{3/2-\alpha}}{\sinh^{2\alpha}x\cosh^{\alpha+\beta}x}\\&\times\int_0^x\cos ry\frac{F\left(\alpha+\beta,\alpha-\beta;\alpha+\frac{1}{2};\frac{\cosh x-\cosh y}{2\cosh x}\right)}{(\cosh 2x-\cosh 2y)^{1/2-\alpha}}\,dy,\quad \operatorname{Re}\alpha>-1/2.\end{aligned} \tag{B.130}$$

Furthermore, the Gegenbauer polynomial $C_k^\nu(t)$ is related to the Jacobi polynomial $P_k^{(\alpha,\beta)}(t)$ by

$$C_k^\nu(t)=\frac{(2\nu)_k}{\left(\nu+\frac{1}{2}\right)_k}P_k^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(t), \tag{B.131}$$

where

$$P_k^{(\alpha,\beta)}(t)=\frac{(\alpha+1)_k}{k!}F\left(-k,k+\alpha+\beta+1;\alpha+1;\frac{1-t}{2}\right),\quad k\geq 0,\ \alpha,\beta>-1, \tag{B.132}$$

with

$$P_k^{(\alpha,\beta)}(1)=\frac{(\alpha+1)_k}{k!},\quad P_k^{(\alpha,\beta)}(-t)=(-1)^kP_k^{(\beta,\alpha)}(t). \tag{B.133}$$

The following fractional representations hold:

$$P_k^{(\alpha,\beta)}(\cos\varrho)=\frac{2^{2\beta-\alpha+\frac{3}{2}}\Gamma(k+\beta+1)}{\sqrt{\pi}\Gamma(k+2\sigma)}\partial_{\cos\varrho+1}^{\beta+\frac{1}{2}}\partial_{\cos\frac{\varrho}{2}+1}^{\alpha-\beta}\frac{\cos((k+\sigma)\varrho)}{2k+2\sigma} \tag{B.134}$$

for α and β positive integer or half-integer, $\alpha>\beta$, $\sigma=(\alpha+\beta+1)/2$;

$$P_k^{(\alpha,\beta)}(\cos\varrho)=\frac{2^{\beta+\frac{1}{2}}\Gamma(k+\beta+1)\Gamma(\alpha-\beta)}{\sqrt{\pi}\Gamma(k+\alpha+\beta+1)}\partial_{\cos\varrho+1}^{\beta+\frac{1}{2}}C_{2k+2\beta+1}^{\alpha-\beta}\left(\cos\frac{\varrho}{2}\right) \tag{B.135}$$

for α arbitrary and β integer or half-integer, $\alpha-\beta$ not a negative integer or zero;

$$P_k^{a,0}(\cos\theta)=\frac{k!}{2^{a-1}\pi\Gamma(k+a+1)}\int_{\frac{\varrho}{2}}^{\frac{\pi}{2}}\frac{\left(-\frac{\partial}{\partial\vartheta}\right)\cos\vartheta\left(-\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\right)^a\left(\frac{\sin(2k+a+1)\vartheta}{\sin\vartheta}\right)}{\sqrt{\cos^2(\theta/2)-\cos^2\vartheta}}\,d\vartheta. \tag{B.136}$$

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