Theory of Generalised Biquandles and its Applications to Generalised Knots

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Declaration

I hereby declare that this thesis has not been and will not be, submitted in whole or in part to another University for the award of any other degree.

I also declare that this thesis was composed by myself and that the work contained therein is my own, except where stated otherwise, such as citations.

Signature:

(Ansgar Wenzel)

Acknowledgements

This has been quite some journey but in the end, I arrived.

First of all, let me thank my supervisor Roger Fenn for support and motivation. I would also like to thank James Hirschfeld for help and support, particularly when teaching.

I would also like to thank my family: My parents for their constant support, my brothers and their girlfriends for support and proofreading. Danke!

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I would also like to thank my employers: Thank you to both IPD and Hildebrand Technology for their support and understanding.

Finally, I want to say thank you to my wife for her enduring support during my studies and particularly during my writing period for her enduring support.

Thank you!

London, the 6th of November, 2016.

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Abstract

In this thesis we present a range of different knot theories and then generalise them. Working with this, we focus on biquandles with linear and quadratic biquandle functions (in the quadratic case we restrict ourselves to functions with commutative coefficients). In particular, we show that if a biquandle is commutative, the biquandle function must have non-commutative coefficients, which ties in with the Alexander biquandle in the linear case.

We then describe some computational work used to calculate rack and birack homology.

Chapter 1

Introduction

The study of quandles and racks is a very fruitful and interesting field, with many applications, particularly to knot theory. Biquandles and biracks, while introduced in 1992/1995 in [FRS93; FRS95], have only recently begun to be taken seriously as a field of research and promise rich pickings in the future. We hope to have added some interesting results and ideas, particularly on biquandles with polynomial functions (Chapter 5) and the study of the homology of biquandles (Chapters 6 and 7).

1.1 History and Literature Review

We are now giving a short history of the subject, including some important literature for the field, which is heavily influenced by [Car12; EN15].

Arguably, the first time quandle-like structures were described was in a paper from 1927 by Burstin and Mayer on *Finite Distributive Groups* ([BM29]; translation by the author of this thesis in [BM14b]). *Keis*, which we now know as *involutive quandles*, were defined in 1943 in [Mit43] by Takasaki. The more general *racks*, 1959 defined as *wracks* were discussed in an unpublished correspondence between John Conway and Gavin Wraith. In 1982, the term *quandles* was first coined by David Joyce in [Joy82], while at the same time Matveev discussed them as *distributive groupoids* ([Mat82], translation: [Mat84]). Shortly afterwards, Brieskorn in [Bri88] knew them as *automorphic sets*.

In 1992, Colin Rourke and Roger Fenn then re-introduced the quandle idea in [FR92], introducing the term *racks* based on Conway and Wraiths term wracks and their application to knot theory. They further generalised this concept with Brian Sanderson to *biracks* and *biquandles* in 1993/1995 in [FRS93; FRS95]. Biquandles in particular were further researched by Louis Kauffman in [KR03]. In [Kau99], Louis Kauffman in 1999 introduced virtual knots and connected virtual knots and biquandles in 2005/2007 in [KM05; HK07].

The first mention of *quandle homology* we found was in [Car+01] from 1999. Two papers that we found helpful in our research on quandle homology are [NP09; Nie09]. A particularly interesting result is that the quandle chain complex from Equation (6.1) splits, as was shown in [LN03].

Birack and biquandle homology are more recent and very much in flux with the first inkling probably in [Car+09]. We do give a definition in Definition 6.6 but apart from that, not much is known about it. There is some mention in [Fen+14, Section 6.2] and there is a definition for augmented biracks in [Cen+14] and it was mentioned in passing in [Fen12b, Chapter 6]. However, this is still a very new area of research and not much is known[1].

1.2 Structure

This thesis is divided into three main parts: Chapters 2 and 3 should be read as a general introduction to classical and other knot theories and quandles, racks, biquandles and biracks, respectively. Additionally, we calculate biquandle generators and relations for some welded knots that were found via a nontrivial welded knot search in [BF11]. The second part, comprising Chapters 4 and 5, is a closer look at biquandles with linear and quadratic functions, respectively. In Chapter 5, we present some results for commutative biquandles with both commutative and non-commutative coefficients. The third part, consisting of Chapters 6 and 7, concerns itself with the new field of biquandle homology. In Chapter 6, we describe the theory and give a new closed form definition of biquandle homology before presenting our programme that calculates said biquandle homology in Chapter 7. As this programme is adaptable for quandle homologies as well, we hope that it may see fruitful use in this area of research.

In Appendix A, we provide the code used for some of the calculations in Chapter 5. Appendix B presents drawings of the welded knots in Chapter 3 as closed braids while the paper [FW16] on which Chapter 7 is based, can be found in Appendix C.

[1] For example, at the time of writing Google scholar has three hits for "biquandle homology" and six for "birack homology".

Part I

Theory

Chapter 2

Knots and other Prerequisites

This chapter introduces the necessary prerequisites for understanding (bi)racks and (bi)quandles. In particular, knot theory and the labelling of knots is discussed. The material on classical knot theory is based on the work presented in [Cro04] and [Man04], with some additional material from [Fen12a]. The general knot theory section is based on [Fen15].

2.1 Classical Knot Theory

Classical knot theory as we understand it has been around for a bit longer than 100 years, although its origins go back to Gauss and Euler. There are a number of good resources available online and in print, for example [Prz07; TG96], so we will refrain from providing more details.

Definition 2.1 (Geotopy)**.** *Two maps $f, g : X \to Y$ are* geotopic *if there are homeomorphisms $s : X \to X$ and $t : Y \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"] & Y \arrow[d, "t"'] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Definition 2.2 (Knots; Knot Diagram)**.** *A* classical knot *(or link) is the geotopy class of an embedding of circles into $\mathbb{R}^3$,*

$$f : S^1 \sqcup \cdots \sqcup S^1 \hookrightarrow \mathbb{R}^3. \tag{2.1}$$

If two knots K_1 and K_2 are geotopic, we write $K_1 \cong K_2$ and call them equivalent.

If we have a knot with l components and each component spans a disk disjoint from all the others, we call this the unknot *and represent it by l flat circles in space, separated by planes.*

In addition, if we can represent a class of knots with a knot which has continuous non-zero tangents at every point, we call this class tame. *Otherwise, we call it* wild.

We then define a diagram *of a knot as follows: Suppose a classical knot is represented by an embedding f as in Equation* (2.1)*, then the composition*

$$S^1 \sqcup \cdots \sqcup S^1 \xhookrightarrow{f} \mathbb{R}^3 \xrightarrow{\pi} \mathbb{R}^2, \tag{2.2}$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is a projection, is called a knot diagram *if the following holds:*

1. *The composition is an immersion into the plane in general position, that is the singularities are all double points (which we call* crossings*).*

2. *At the double points the over and under information is given as in Figure 2.1, where the break indicates the under part of the double point.*

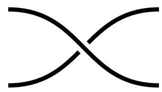
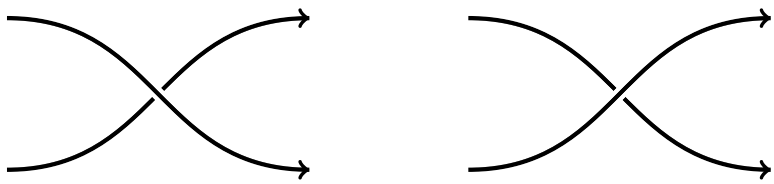


Figure 2.1: Crossing for a Classical Knot Diagram

In particular, we have positive and negative crossings as in Figure 2.2.



(a) Positive Crossing (b) Negative Crossing

Figure 2.2: Positive and negative Crossing for a Classical Knot

2.1.1 Reidemeister Moves

Consider the well-known diagram of the figure 8 as presented in Figure 2.3. One can take one end and twist it, thereby getting an unknot.

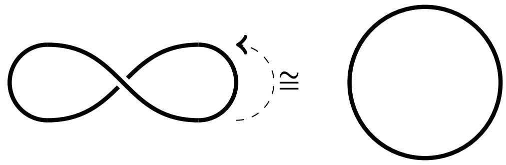


Figure 2.3: The Figure 8 is the Unknot

In order to better classify knots, we need to generalise and formalise such moves. This results in the four *Reidemeister moves*.

Definition 2.3 (Classical Reidemeister Moves)**.**

R_0 : *This is the topological equivalence of the original immersion in the surface, but can also be interpreted as geotopy.*

R_1 : *This is a twist or untwist in either direction, see Figure 2.4.*

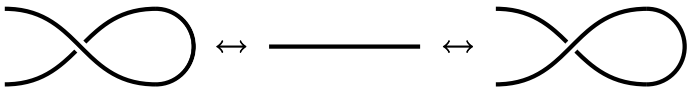


Figure 2.4: Reidemeister 1 Move

R_2 : *This moves a loop over another, creating or removing it, see Figure 2.5.*

Figure 2.5: Reidemeister 2 Move

R_3 : *This moves a string along, over or under a crossing, see Figure 2.6.*

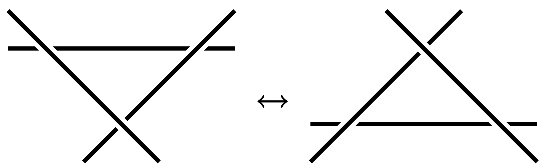


Figure 2.6: Reidemeister 3 Move

Theorem 2.4 (Reidemeister Moves)**.** *Any two knot diagrams of the same knot can be related via a sequence of the four Reidemeister moves.*

A proof of Theorem 2.4 can be found in [Rei26].

Combining R_2 and R_3 moves only, the knot-theory equivalent of regular homotopy is obtained. There is a continuous non-zero tangent vector at every point, something that does not hold for R_1 moves. By changing the parametrisation we can assume all tangent vectors to be unit vectors and we therefore get a map from S^1 to itself. The homotopy number of this map then defines the so called *turning number* which is an invariant of regular homotopy, see [Whi37]. In particular, the theorem in [Whi37] states that regular homotopy classes of maps of a circle into the plane are classified by their turning numbers.

Definition 2.5 (Standard Immersed Curve)**.** *Using the turning number, we can define the* standard immersed curve*, C_N. Starting with an anticlockwise oriented circle, if $N > 0$, we attach $N-1$ monogons to the inside. If, on the other hand, $N \leq 0$ we attach $1-N$ monogons to the outside.*

Remark. *Note that this immersed curve is regularly homotopic to any curve with turning number N.*

2.2 Knot Codes

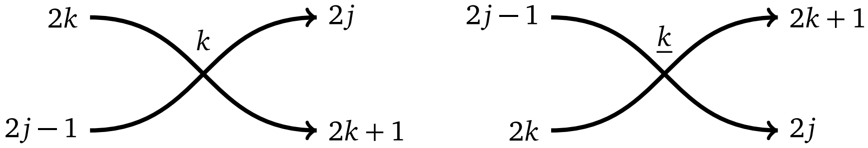
In order to distinguish knots, we associate a *code* to their diagrams. Initially, the *Gauss Code* is associated to just the immersion of the circle in the plane; the over/under information at crossings being omitted.

Definition 2.6 (Gauss Code)**.** *Label all n crossings $1, 2, \ldots, n$. Then start at any point of the knot and go around following the orientation and noting the crossing labels. Eventually, one ends up with a sequence $i_1, i_2, \ldots, i_{2n}$. This sequence is called the* Gauss code.

Remark. *Of course, this is not unique. If there are k components, the Gauss code will have k such sequences as above, one for each component.*

Unfortunately, this is a rather long code, especially for knots with several components and a large number of crossings. In order to simplify this, the *DT-code* (see e.g. [Fen12a]) was developed. This labels the edges of a diagram sequentially, changing at each crossing. We can start anywhere on the knot, labelling the edge we start on 1 and following the knot along its orientation, labelling the edges 2, 3, etc. up to $2n$ as we progress (for n crossings, if $n \geq 2$). This gives every crossing a unique incoming edge labelled with an even integer, $2k$, say. We name this crossing k. In addition, there will also be an odd labelled incoming edge labelled $2j-1$, say, see Figure 2.7.

We thus associate $k \mapsto j$ as a permutation and we name the crossings Type I if the even edge comes in from the top (northwest), see Figure 2.7 (a), and Type II if it come from the bottom (southwest), see Figure 2.7 (b). For Type II crossings, we underline the label.



(a) Type I Crossing (b) Type II Crossing

Figure 2.7: Type I and Type II Crossing with Permutation $k \mapsto j$

2.2.1 DT Code Examples

The Trefoil

Consider the immersion associated with the trefoil, where the crossings are all of Type II. We have,

$$1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2$$
$$\iff \qquad (\underline{132})$$

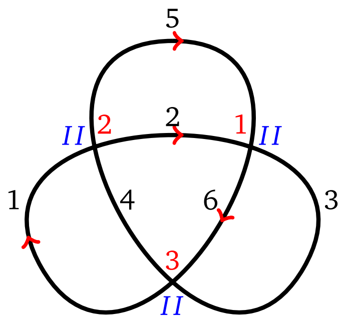


Figure 2.8: The Trefoil with its DT Code, $\underline{132}$

The Figure 8 knot

Let us consider a more interesting example, the Figure 8 knot as in Figure 2.9, as it has crossings of both types, I and II. As can be seen from the diagram, we have two permutation or conjugacy classes,

$$
\begin{aligned}
\text{Type } II: &\quad 1 \mapsto 3 \mapsto 1 \\
\text{Type } I: &\quad 2 \mapsto 4 \mapsto 2 \\
\Leftrightarrow &\quad (\underline{13})(24).
\end{aligned}
$$

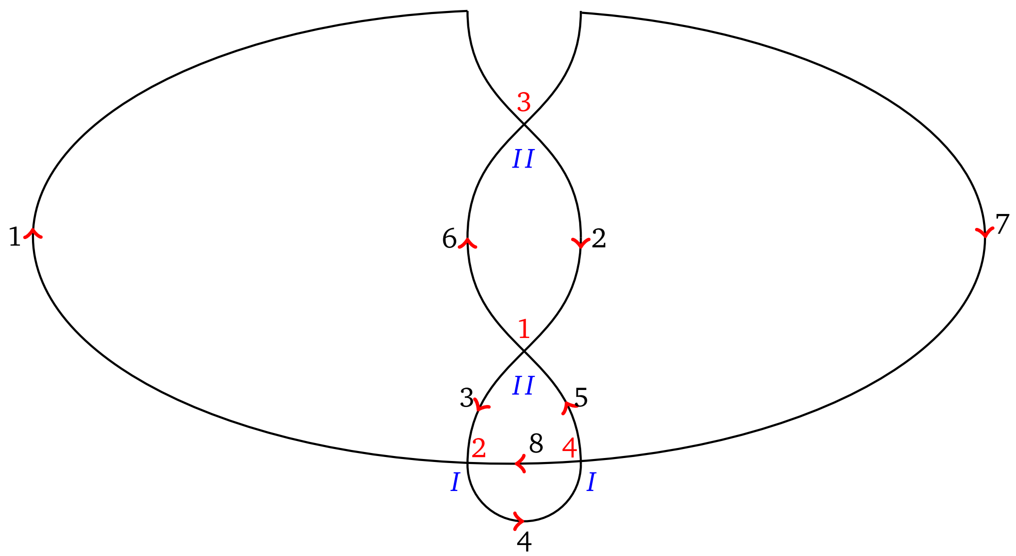


Figure 2.9: The Figure 8 Knot with its DT Code, *($\underline{13}$)(24)*

We can further enhance the DT code by adding information about the sign of the crossing, e.g. positive or negative. Thus, the trefoil in Figure 2.8 has the enhanced code $\underline{132}^{+++}$ with three positive crossings, see Figure 2.10 (a). The enhanced code $\underline{132}^{---}$ gives us the left handed trefoil, see Figure 2.10 (b). The Figure 8 knot in Figure 2.9 thus has the enhanced code $(\underline{13})(24)^{--++}$.

2.2.2 Non-classical Knots arising from DT Codes

In this section, we consider DT codes that do not correspond to immersions such as those present in Equations (2.1) and (2.2). As a simple example, consider $1\underline{3}2$, which gives rise to the knot diagram in Figure 2.11.

(a) The right-handed Trefoil (b) The left-handed Trefoil

Figure 2.10: Right- and left-handed Trefoil with their enhanced DT Codes,$\underline{132}^{\pm\pm\pm}$

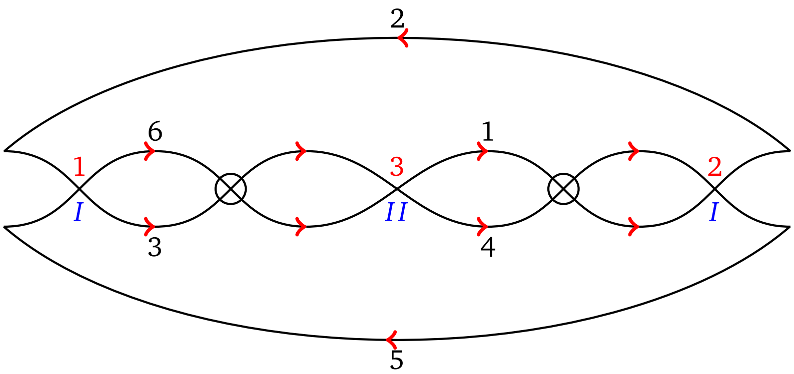


Figure 2.11: Knot Diagram for DT code $1\underline{3}2$

2.3 Virtual Knot Theory

This knot diagram is impossible to realise without the introduction of virtual crossings, shown in the diagram with small enclosing circles. They will be considered next. Virtual knots were introduced in 1999 by Louis Kauffman in [Kau99].

2.3.1 Virtual Crossing

By changing the immersion in Equation (2.2) from $\mathbb{R}^3$ to other surfaces embedded in $\mathbb{R}^3$, for example a torus ($\mathbb{T}^2$), other crossing types arise. For ease of drawing, we then project into $\mathbb{R}^2$. Then, the knot may not have any crossing in the new surface, but when projecting the diagram into $\mathbb{R}^2$, we get a *virtual crossing*, as in Figure 2.12 and, in closer detail, in Figure 2.13 where the virtual crossings are indicated by $\otimes$. Note that there is no inverse (or negative) virtual crossing.

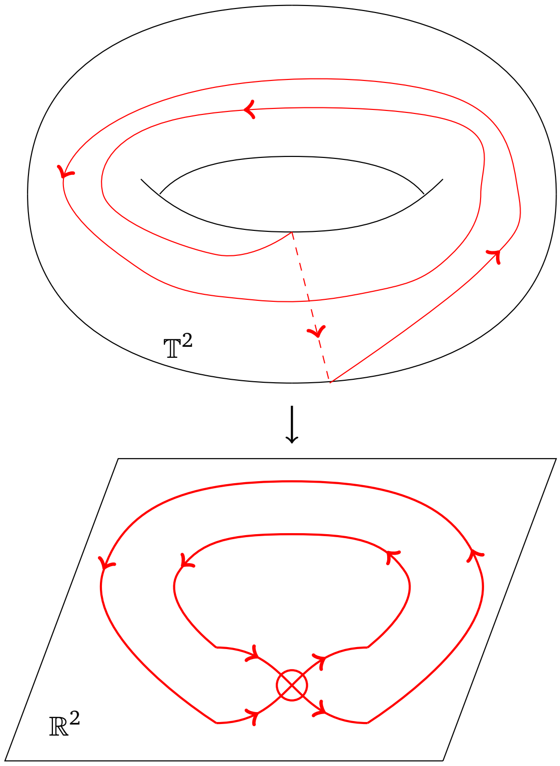


Figure 2.12: Simple Torusunknot projected into $\mathbb{R}^2$

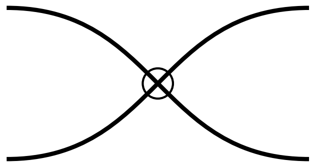


Figure 2.13: Virtual Crossing

Note that both classical and virtual crossings can appear in the same knot diagram. As in the classical case, they are related by Reidemeister moves.

2.3.2 Virtual Reidemeister Moves

The virtual Reidemeister moves include all of the classical moves, as listed in Definition 2.3. Additionally, they also include the virtual crossings as in Definition 2.7:

Definition 2.7 (Virtual Reidemeister Moves)**.** VR_1 : *This move is a twist or untwist in either direction. See Figure 2.14.*

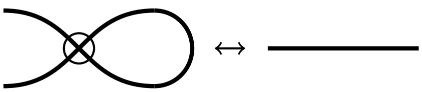


Figure 2.14: Virtual Reidemeister 1 Move

VR_2 : This moves a loop over another, creating or removing two points. See Figure 2.15.

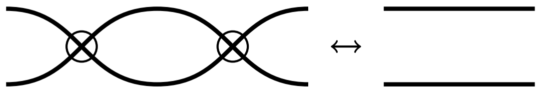


Figure 2.15: Virtual Reidemeister 2 Move

VR_3 : This moves a string with 2 crossings past a third virtual crossing. See Figure 2.16.

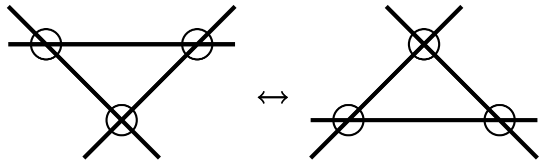


Figure 2.16: Virtual Reidemeister 3 Move

In addition, we have the following mixed Reidemeister move, involving both classical and virtual crossings.

VR_4 : This moves two virtual crossings over a classical crossing. See Figure 2.17.

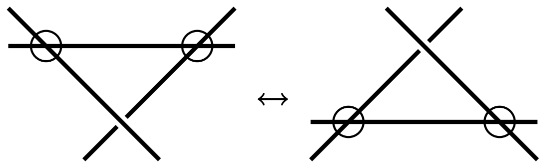


Figure 2.17: Virtual Reidemeister 4 Move

Forbidden Moves

Note that moving two classical crossings over or under a virtual crossing is not permitted. See Figures 2.18 and 2.19 for relevant examples. If we allow the first forbidden move, this results in the theory of Welded Knots*, which was first introduced in [FRR97]. If both are allowed, the theory collapses in the sense that all knots are the unknot, see [Nel01].*

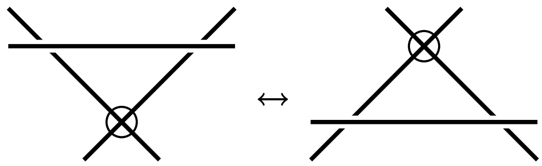


Figure 2.18: First forbidden Move in Virtual Knot Theory

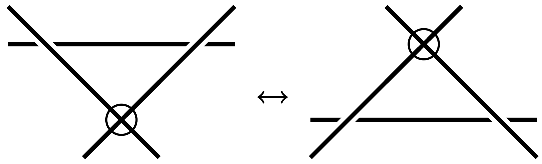


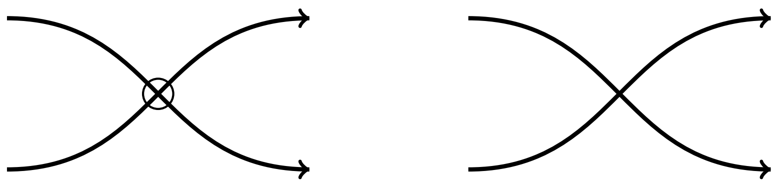
Figure 2.19: Second forbidden Move in Virtual Knot Theory

2.4 Further Knot Theories

There exist other crossing types, for example singular or doodle crossings, which give rise to other knot theories.

2.4.1 Doodle Crossing

Should we have no information about the crossing, we could also have a *doodle*, see Figure 2.20 (b). This type of crossing can for example be generated by "doodling", for example when one is on the phone. Doodle crossings were introduced by Fenn and Taylor in 1979 in [FT79]. Note that Doodles do not allow an R_3 move (as the theory then collapses everything to unknots).



(a) Virtual Crossing (b) Doodle Crossing

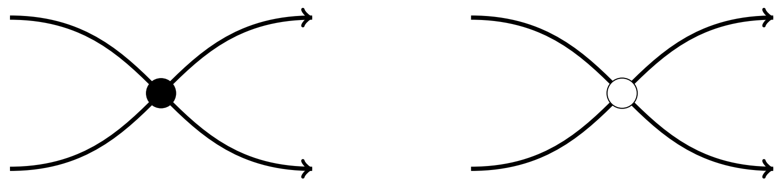
Figure 2.20: Virtual and Doodle Crossing

2.4.2 Virtual Doodles and Flat Virtual Knot Theory

These two theories are very similar. The Virtual Doodles arise from a virtualisation of doodles, whereas the Flat Virtual Knot Theory arises from a flattening of Virtual Knot Theory. These differences manifest themselves in the permissible R_3 moves who will be considered in greater detail in Section 2.5.4.

2.4.3 Singular Crossing

Singular crossings were first introduced in [Vas90; Vas92].



(a) Positive singular Crossing (b) Negative singular Crossing

Figure 2.21: Positive and negative singular Crossing

2.4.4 Flat Singular Knot Theory

Combining doodle crossings (Figure 2.20(b)) and singular crossings (Figure 2.21) results in the Flat Singular Knot Theory. Its Reidemeister moves are discussed in more detail in Section 2.5.4.

2.5 General Knot Theory

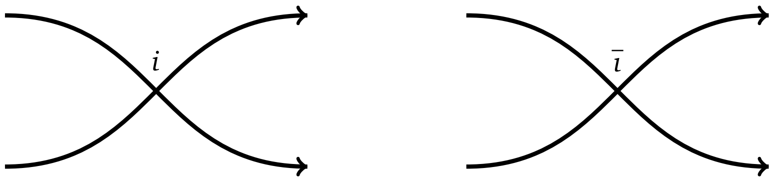
We can combine all of the previously discussed knot theories into one unified knot theory, the *general knot theory*. This will be introduced in this section, which is heavily based on [Fen15; KM15].

Definition 2.8 (Generalised Knots)**.** *We consider a knot diagram, K, as in Definition 2.2 with an underlying graph, $\Gamma(K)$. This graph consists of edges, which we call* arcs *and vertices, which are the* crossings *of the knot. In the classical case, the crossings would be the singularities (or double points). We note that the arcs may or may not be oriented.*

The vertices are equipped with crossing type information*. In particular, there is a* positive *and a* negative *crossing which may or may not be distinct. For example, in the classical case they are distinct (see Figure 2.2) and in the virtual and doodle case, they are the same (see Figure 2.20).*

As in the classical case, a general knot is the equivalence class of its diagram, which is defined as in the classical case.

In general knot theory, we denote a crossing of type C_i by i if it is a positive crossing and by $\bar{\imath}$ if it is a negative crossing, see Figure 2.22.



(a) Positive Crossing of Type C_i (b) Negative Crossing of Type C_i

Figure 2.22: Positive and negative Crossing of Type C_i

2.5.1 General Reidemeister Moves

In the case of the general knot theory, we extend the four Reidemeister moves as follows:

Definition 2.9 (Generalised Reidemeister Moves)**.** R_0 : *This is the same as in the classical case (see Definition 2.3).*

R_1 : *This is the same as in the classical case, just for general crossing types. See Figure 2.23.*

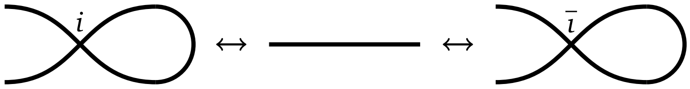


Figure 2.23: Generalised Reidemeister 1 Move

R_2 : *This introduces or removes two parallel opposite crossings as well as the bigon defined between them. See Figure 2.24.*

Figure 2.24: Generalised Reidemeister 2 Move

R_3 : *This passes two crossings of type i through a* dominated *crossing of type j. See Figure 2.25.*

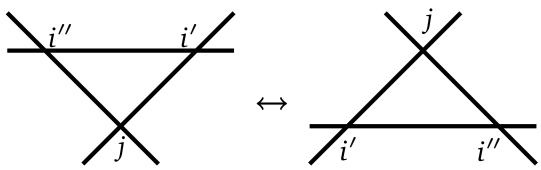


Figure 2.25: Generalised Reidemeister 3 Move (C_i dominates C_j)

In addition, we have the following, new, fourth Reidemeister move, which exists in two different versions, commuting *and* anti-commuting.

$R_4{}^a$: *This exchanges two parallel copies of type C_i and C_j. See Figure 2.26.*

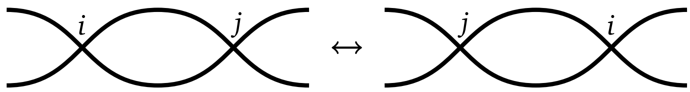


Figure 2.26: Generalised Reidemeister 4 Move (Commuting)

$R_4{}^b$: *This exchanges two parallel copies of type C_i and C_j while changing the sign of one crossing. See Figure 2.27.*

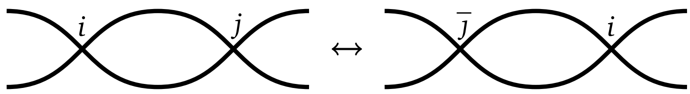


Figure 2.27: Generalised Reidemeister 4 Move (Anti-Commuting)

We say that $R_4{}^a$ is commuting *and that $R_4{}^b$ is* anti-commuting.

In addition, the different moves may or may not be always permissible, except for R_2 which we assume to be always possible.

Remark. *Note that the theoretical $R_4{}^c$ move, where we alter the sign of both crossings, is equivalent to $R_4{}^a$, by virtue of changing the orientation via an R_0 move.*

2.5.2 Reidemeister Moves in different Knot Theories

Both classical and virtual knot theories have been investigated before and are thus omitted.

Doodles

Doodles have, as mentioned before, no allowed R_3 move. They do admit an R_0 move, an R_1 move and an R_2 move. The R_4 move for doodles is not considered as there is only one type of crossing.

Flat Virtual Knot Theory

There exists an R_0 move, two R_1 moves (one for each crossing type) and two R_2 moves. In addition, there are three R_3 moves (see Figure 2.37). One interesting further avenue of research would be to see what happens if we do a flat virtualization and introduce a *commuting* R_4 move, similar as to what was done for classical knots in [Kau99].

Virtual Doodles

In this case, we have the same moves as in the case of the Flat Virtual Knot Theory minus one R_3 move (see Figure 2.37, the now-forbidden R_3 move is Figure 2.37 (c)).

Singular Knot Theory

For Singular Knot Theory, we have all the Reidemeister moves from the classical case, see Definition 2.3. In addition, there are two R_2 moves, four R_3 moves and four commuting R_4 moves. However, there is no R_1 move for singular crossings.

Flat Singular Knot Theory

In this case we have one R_1 move, two R_2 moves, three R_3 moves and two R_2 moves, which are discussed in more detail in Section 2.5.4.

2.5.3 Impact of Reidemeister Moves on Orientation

In this part, we are going to research how the four generalised Reidemeister moves are affected by the orientation of the crossings and the arcs of a generalised knot diagram.

R_0: An orientation reversing homeomorphism of the plane, changing the sign of the crossing types, is called a *mirror* and the resulting diagram is called the *mirror diagram*. The *reverse* of a diagram is created by changing the orientation of some or all of the immersed spheres. Changing the signs of all the crossings results in the *inverse* of the diagram.

Should a diagram be equivalent to its mirror image then we call it *amphicheiral*. It is *reversible* if equivalent to its reverse and *invertible* if equivalent to its inverse.

R_1: As this is depending on the orientation of the monogon and the crossing, there are four cases in total.

R_2: Similarly to R_1, there are four cases.

R_3: A naive counting results in a total of 32 options (two possibilities each for the orientation of the three crossings as well as the inner and outer faces). However, after careful consideration, it turns out that there are only 8 cases. This will be explored in detail next, see Theorem 2.10.

R_4: First, we consider a commuting R_4 move. In this case, we can have an oriented bigon or a non-oriented bigon, resulting in a total of three cases. Should we have an anti-commuting R_4 move, we have six cases, the three from the commuting move plus the mirror images.

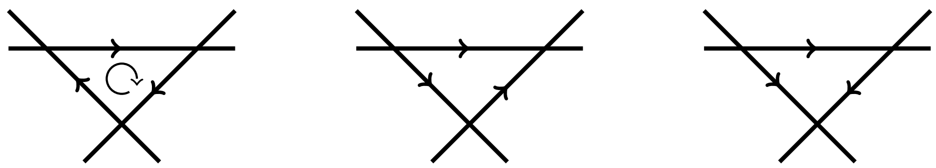
Orientation effects on R_3 moves

First, we will assume that the sign of the crossing labelled i'' in Figure 2.25 is unchanged by the move. We then have the following three possibilities for orientation.

1. If the triangle formed by the three arcs is oriented, we call the move *oriented*. In particular, the orientation of the triangle is reversed after the move, see Figure 2.28 (a).

2. All three arcs are oriented from left to right. Then we call the move *braid-like*, see Figure 2.28 (b).

3. If the move is neither braid-like nor oriented, we call it *mixed*, see Figure 2.28 (c).



(a) Oriented (b) Braid-like (c) Mixed

Figure 2.28: Examples for oriented, braid-like and mixed Orientation before R_3 Move

Theorem 2.10 (General R_3 Moves)**.** *Because R_2 moves are always permitted, there are at most 8 cases, i.e. we only need to consider* braid-like *moves.*

In order to prove this theorem, we need both the following notation and Lemma 2.11.

Notation. *We can always rotate the diagram so that the arc connecting the two crossings of the same type is oriented from left to right and we then denote by $R(u, v; \epsilon, \eta; \xi)$ the most general R_3 move as in Figure 2.29.*

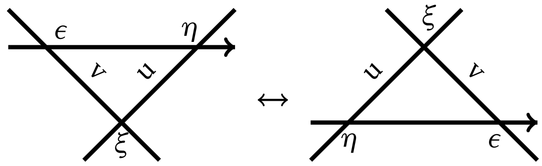


Figure 2.29: Generalised Reidemeister 3 Move Notation

Here, u and v denote the orientation of the two arcs. The third arc is always oriented from left to right. The orientations, or signs of the three crossings are given by ϵ, η for the C_i's and ξ for C_j. As each of those has two possibilities, there are $2^5 = 32$ possibilities in total.

Lemma 2.11 (Equivalent R_3 Moves). *The following R_3 moves are equivalent:*

$$R_3(u, v; \epsilon, \eta; \xi) \cong R_3(v, \bar{u}; \eta, \bar{\epsilon}; \xi) \cong R_3(\bar{u}, \bar{v}; \bar{\epsilon}, \bar{\eta}; \xi) \cong R_3(\bar{v}, u; \bar{\eta}, \epsilon; \xi)$$

This Lemma will be proved using a trick invented by Turaev. The proof is taken from [Fen15].

Proof. The three diagrams below, Figures 2.30 to 2.32, are a pictorial proof. Each figure shows an R_2 move which creates a bigon, after which an R_3 move is made. Then the bigon is removed by another R_2 move, which completes the original R_3 move. In particular, this sequence of moves can be undone, that is, it is reversible.

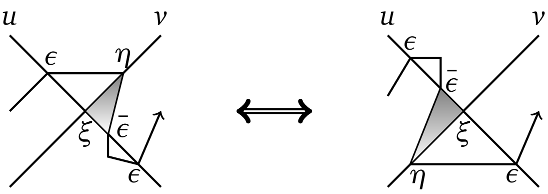


Figure 2.30: $R_3(u, v; \epsilon, \eta; \xi) \cong R_3(v, \bar{u}; \eta, \bar{\epsilon}; \xi)$

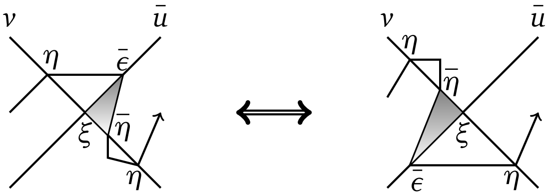


Figure 2.31: $R_3(v, \bar{u}; \eta, \bar{\epsilon}; \xi) \cong R_3(\bar{u}, \bar{v}; \bar{\epsilon}, \bar{\eta}; \xi)$

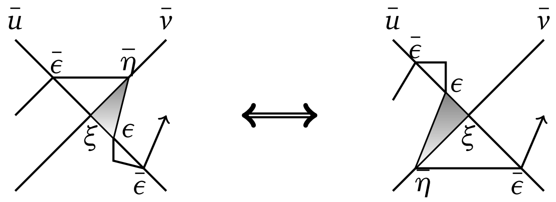


Figure 2.32: $R_3(\bar{u}, \bar{v}; \bar{\epsilon}, \bar{\eta}; \xi) \cong R_3(\bar{v}, u; \bar{\eta}, \epsilon; \xi)$

q.e.d.

From Lemma 2.11 it follows that the orientations of the two other arcs, u and v can be assumed to be *left-to-right*. This means it is sufficient to consider only the braid-like moves as in Figure 2.28 (b).

Corollary 2.12 (Braid-like R_3 Moves are sufficient)**.** *It is sufficient to only consider* braid-like *R_3 moves.*

Thus, we can shorten our notation to $R_3(\epsilon, \eta; \xi)$. For this, see Figure 2.33.

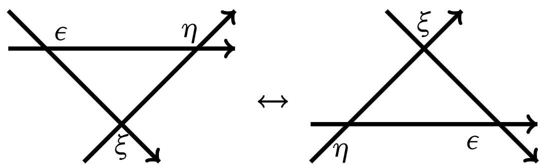


Figure 2.33: Generalised Reidemeister 3 Move Notation

Proof of Theorem 2.10. By Lemma 2.11, we can assume that the only R_3 moves are $R_3(\epsilon, \eta; \xi)$. This leaves $2^3 = 8$ possible orientation combinations, 2 each for the three crossings. *q.e.d.*

Further Reductions

Lemma 2.13 (Reduction of R_3 Moves for one Crossing Type)**.** *If there is only one type of crossing involved then the following moves are equivalent:*

(a) $R_3(\epsilon, \eta; \xi)$ *and* $R_3(\bar{\epsilon}, \xi; \eta)$.

(b) $R_3(\epsilon, \eta; \xi)$ *and* $R_3(\xi, \bar{\eta}; \epsilon)$.

Proof. Consider the following Figure 2.34 where we add an $\bar{\epsilon}$ crossing on both sides, at the upper left and lower right.

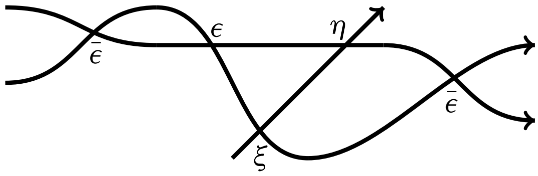


Figure 2.34: Generalised Reidemeister 3 Move

Cancelling the $\epsilon,\bar{\epsilon}$ bigon on the left, we arrive at the finish of an $R_3(\bar{\epsilon},\xi;\eta)$ move. Cancelling the $\epsilon,\bar{\epsilon}$ bigon on the right, we arrive at the beginning of an $R_3(\bar{\epsilon},\xi;\eta)$ move.

Together, this proves (a).

Adding two $\bar{\eta}$ crossings instead of the $\bar{\epsilon}$ crossings and repeating the argument proves (b). *q.e.d.*

Theorem 2.14 (Equivalent R_3 Moves; one Crossing)**.** *If only one type of crossing is involved in an R_3 move, then the only possible R_3 moves are $R_3(+,+;+)$ and $R_3(+,+;-)$.*

Proof. By Lemma 2.13 (a), we can always assume the first entry, ϵ, to be positive. This leaves four possibilities, namely $R_3(+,\pm;\pm)$. By Lemma 2.13 (b), $R_3(+,+;+)$ is equivalent to $R_3(+,-;+)$ By repeated application of (a), (b) and (a), $R_3(+,-;+)$ is equivalent to $R_3(+,-;-)$. *q.e.d.*

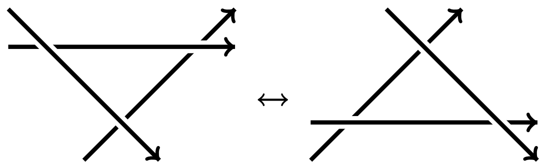
Remark. *Should the positive and the negative crossing be equivalent (such as in the virtual or the doodle case, for example), there is at most one possible R_3 move.*

2.5.4 Examples of Reidemeister Moves in different Knot Theories

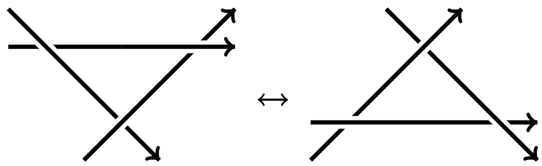
In this section, we are going to apply Theorems 2.10 and 2.14 to different knot theories. Note that neither doodles nor virtual doodles are considered, as these theories do not admit any R_3 moves.

Classical Case

In the classical case, the eight possible R_3 moves are reduced by Theorem 2.14 to two. These two R_3 moves are shown in Figure 2.35. Note that the second move (b) is forbidden and that the allowed move has some form of hierarchy. As in the virtual case (Definition 2.7), if the forbidden move is allowed then the theory collapses, at least for knots. In the case of links (that is, more than one embedded circle), the theory does not collapse, see e.g. [Nel01]. The orientation of a classical crossing has been defined earlier in Figure 2.2.



(a) $R_3(+,+;+)$



(b) $R_3(+,+;-)$ *(forbidden)*

Figure 2.35: Reidemeister 3 Moves in Classical Knot Theory

Virtual Knot Theory

In the virtual case, we have seven different R_3 moves as in Figure 2.36 where only (a) to (c) are allowed. We also have the allowed R_3 moves from the classical case, see Figure 2.35.

Virtual Doodles and Flat Virtual Knot Theory

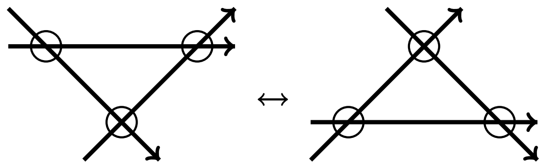
For both these theories, there are four R_3 moves in Figure 2.37. In Flat Virtual Knot Theory, Figure 2.37 (a) to (c) hold, whereas for virtual doodles Figure 2.37 (c) and (d) are forbidden and (a) and (b) hold.

Singular Knot Theory

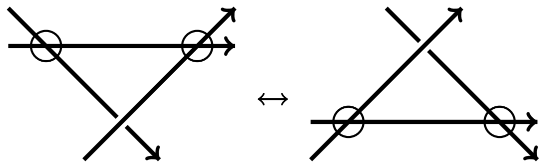
In Singular Knot Theory, we have all the classical R_3 moves, see Figure 2.35. In addition, there are two purely singular ones with only positive or negative singular crossings, all of which are forbidden, see Figure 2.38. Furthermore, there are 16 mixed R_3 moves, of which a further four are allowed (Figures 2.39 and 2.40). In total, of the 26 possible R_3 moves, only 12 are allowed.

Flat Singular Knot Theory

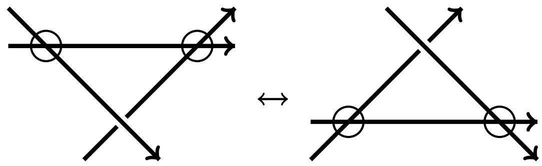
R_1 : Flat Singular Knot Theory has one R_1 move, for the doodle crossing (Figure 2.41).



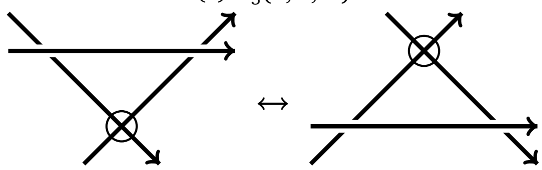
(a) $R_3(v, v; v)$



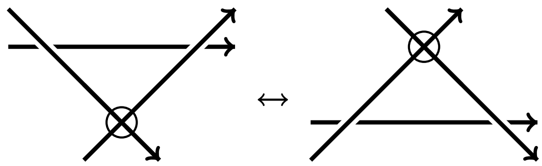
(b) $R_3(v, v; -)$



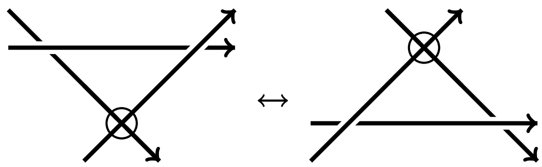
(c) $R_3(v, v; +)$



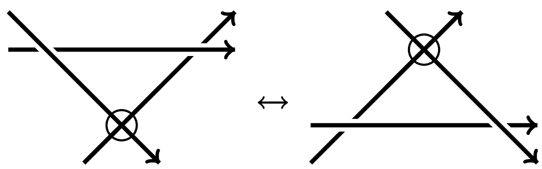
(d) $R_3(-, +; v)$ *(1^{st} forbidden move, allowed in welded knots)*



(e) $R_3(+, -; v)$ *(2^{nd} forbidden move)*



(f) $R_3(-, -; v)$ *(forbidden)*



(g) $R_3(+, +; v)$ *(forbidden)*

Figure 2.36: Virtual Reidemeister 3 Moves in Virtual Knot Theory

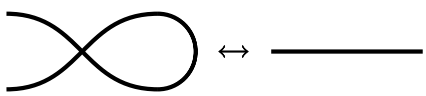
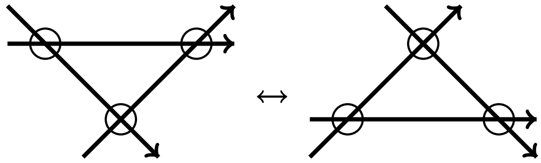
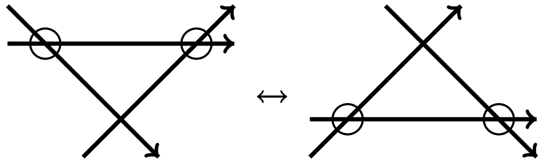


Figure 2.41: Flat Singular: Reidemeister 1 Move

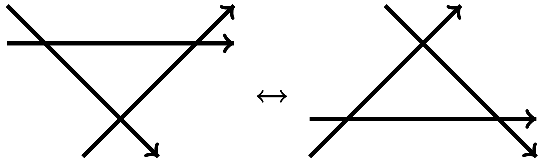
R_2 : There is one R_2 move for the doodle only, one for the singular crossings and no combin-



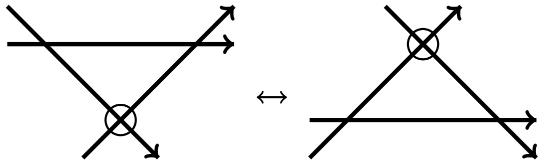
(a) $R_3(v, v; v)$



(b) $R_3(v, v; d)$

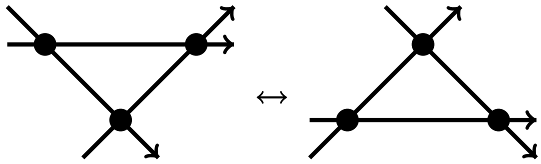


(c) $R_3(d, d; d)$ *(forbidden for Virtual Doodles only)*

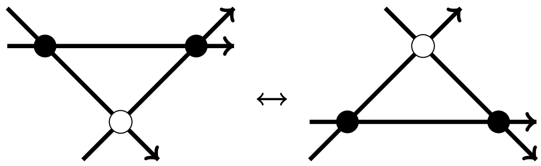


(d) $R_3(d, d; v)$ *(forbidden)*

Figure 2.37: Reidemeister 3 Moves in Flat Virtual Knot Theory and Virtual Doodles



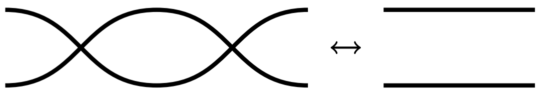
(a) $R_3(s_+, s_+; s_+)$ *(forbidden)*



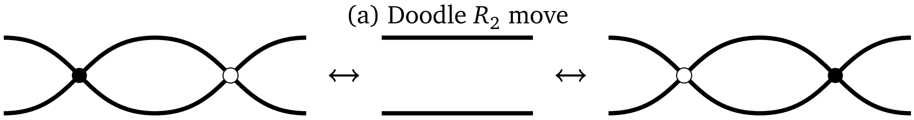
(b) $R_3(s_+, s_+; s_-)$ *(forbidden)*

Figure 2.38: Singular only Reidemeister 3 Moves in Singular Knot Theory *(all forbidden)*

ations (Figure 2.42).

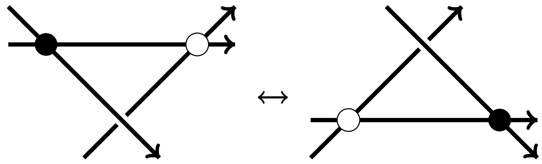


(a) Doodle R_2 move

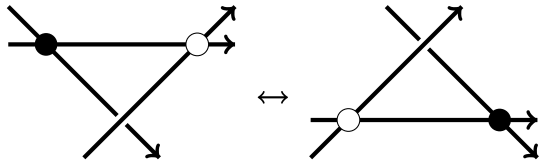


(b) Singular R_2 move

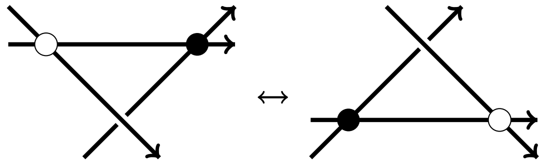
Figure 2.42: Flat Virtual: Reidemeister 2 Moves



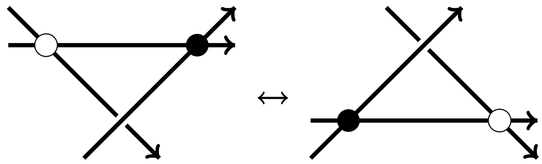
(a) $R_3(s_+, s_-; +)$ (*forbidden*)



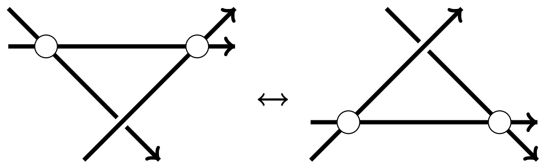
(b) $R_3(s_+, s_-; -)$ (*forbidden*)



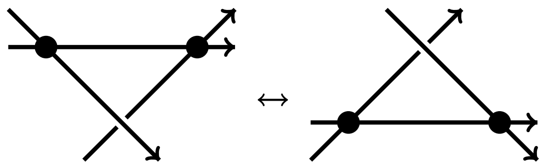
(c) $R_3(s_-, s_+; +)$ (*forbidden*)



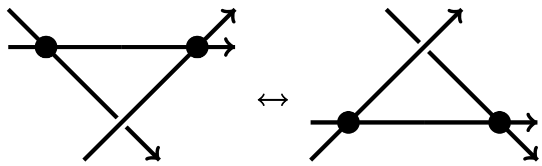
(d) $R_3(s_-, s_+; -)$ (*forbidden*)



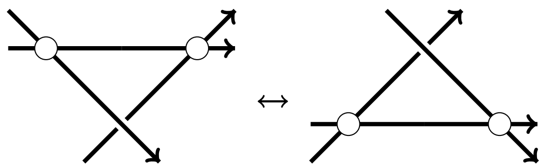
(e) $R_3(s_-, s_-; -)$ (*forbidden*)



(f) $R_3(s_+, s_+; +)$ (*forbidden*)

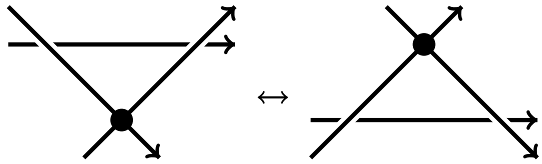


(g) $R_3(s_+, s_+; -)$ (*forbidden)*

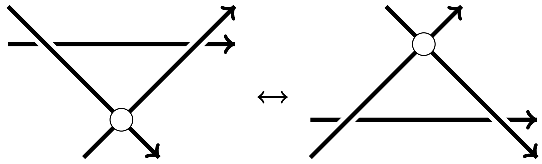


(h) $R_3(s_-, s_-; +)$ (*forbidden)*

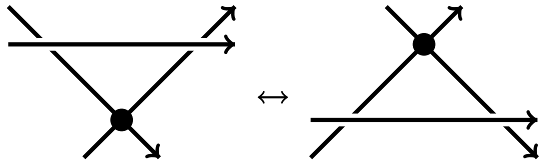
Figure 2.39: $R_3(.,.;\pm)$ in Singular Knot Theory *(all forbidden)*



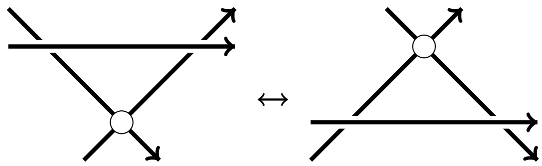
(a) $R_3(+,-;s_+)$



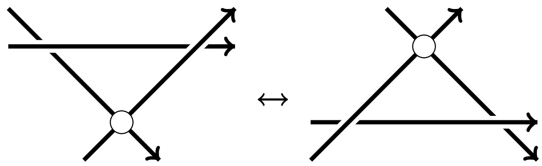
(b) $R_3(+,-;s_-)$



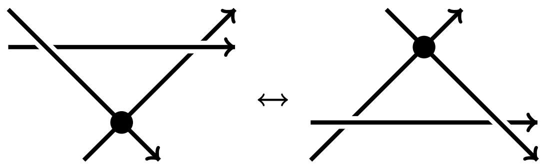
(c) $R_3(-,+;s_+)$



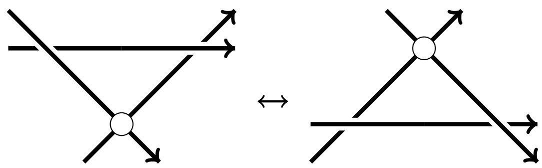
(d) $R_3(-,+;s_-)$



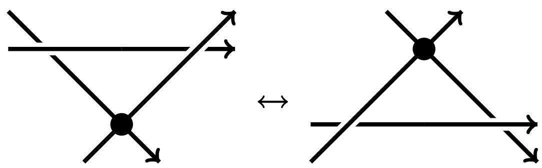
(e) $R_3(-,-;s_-)$ *(forbidden)*



(f) $R_3(+,+;s_+)$ *(forbidden)*



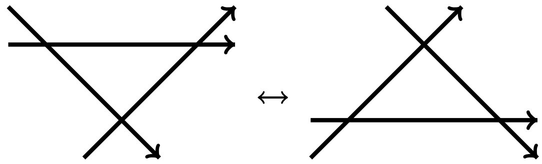
(g) $R_3(+,+;s_-)$ *(forbidden)*



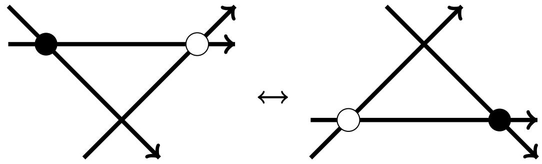
(h) $R_3(-,-;s_+)$ *(forbidden)*

Figure 2.40: $R_3(.,.;s_\pm)$ in Singular Knot Theory

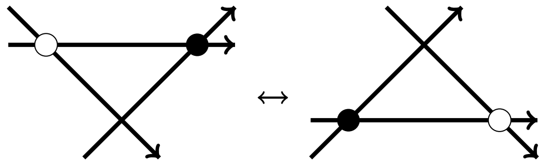
R_3 : There are a total of 9 different R_3 moves, six of which are forbidden (Figures 2.43 and 2.44). In the diagrams, d denotes a doodle crossing and $s_\pm$ a positive or negative singular crossing, respectively.



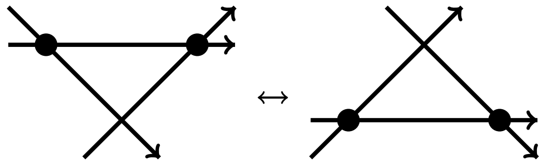
(a) $R_3(d,d;d)$



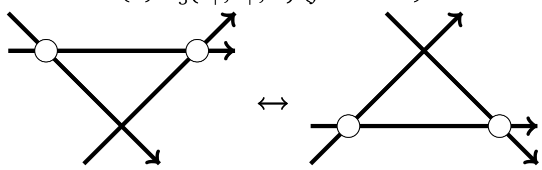
(b) $R_3(s_+,s_-;d)$ *(forbidden)*



(c) $R_3(s_-,s_+;d)$ *(forbidden)*



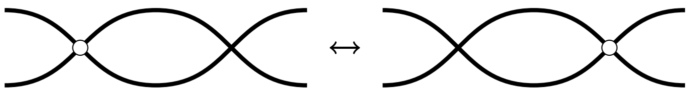
(d) $R_3(s_+,s_+;d)$ *(forbidden)*



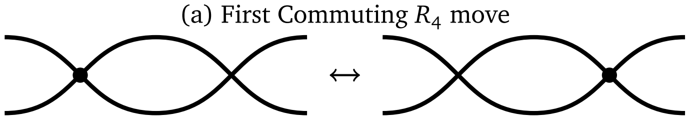
(e) $R_3(s_-,s_-;d)$ *(forbidden)*

Figure 2.43: Flat Singular: Reidemeister 3 Moves $R_3(.,.;d)$

R_4 : There are two commuting R_4 moves (Figure 2.45).

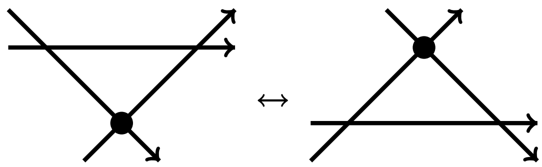


(a) First Commuting R_4 move

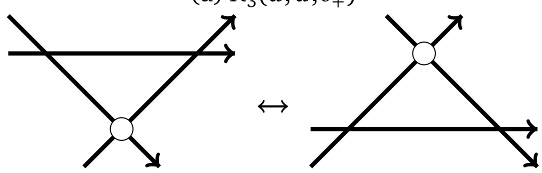


(b) Second Commuting R_4 move

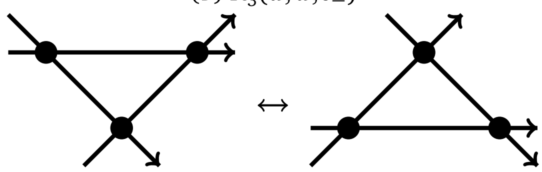
Figure 2.45: Flat Singular: Reidemeister 4 Moves



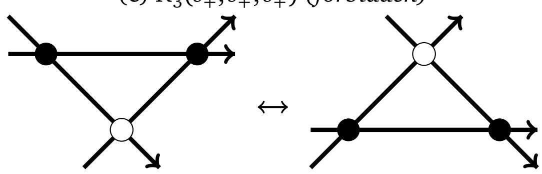
(a) $R_3(d, d; s_+)$



(b) $R_3(d, d; s_-)$



(c) $R_3(s_+, s_+; s_+)$ *(forbidden)*



(d) $R_3(s_+, s_+; s_-)$ *(forbidden)*

Figure 2.44: Flat Singular: Reidemeister 3 Moves $R_3(., .; s\pm)$

Chapter 3

Quandles, Biquandles and all that

This chapter introduces several algebraic structures, quandles, biquandles, racks and biracks. First, the maps of those structures are introduced before the effect of Reidemeister moves on those maps is investigated. We are going to give some examples of the maps and the effects on them. After this, we are defining quandles, biquandles, racks and biracks before calculating the biquandle generators and relations of some welded knots.

Most of this chapter is based on [Fen15] with some additional information from [EN15; FR92; Fen12b]. The welded knots in the last section were found in [BF11].

The intention of this chapter is to develop a framework to show how to start from a knot diagram and construct a map to a biquandle that is independent of the allowed Reidemeister moves for that type of knot.

3.1 Sideways Map and Switch

In this section, we are defining the sideways map and the switch and how they relate to a knot crossing.

Definition 3.1 (Sideways Map; Switch). *We first define the* sideways map, $F : X \times X \to X \times X$ *from one algebraic structure X to itself, preserving structure on X if there exists one. F is defined by two other functions $f_x, f^x : X \to X$, written $a \mapsto a_x, a^x$ for all $a, x \in X$ via $F(a,b) = (f_a(b), f^b(a))$. We require both F and f_x, f^x to be bijective. In the expressions a_x and a^x the element $a \in X$ is well defined since for $a_1, a_2, x \in X$,*

$$a_1^x = a_2^x \quad \Rightarrow f^x(a_1) = f^x(a_2) \quad \Rightarrow a_1 = a_2$$

$$(a_1)_x = (a_2)_x \quad \Rightarrow f_x(a_1) = f_x(a_2) \quad \Rightarrow a_1 = a_2.$$

Furthermore, let $T(x, y) = (y, x)$. Then we can define the dual of F as

$$F^*(a, b) \quad = T \circ F \circ T(a, b) \quad = (b^a, a_b)$$

and this makes clear that F^* is bijective as well. Additionally, we define the switch *corresponding to F by $S : X \times X \to X \times X : (b_a, a) \mapsto (a^b, b)$ or $(f_a(b), a) \mapsto (f^b(a), b)$ and its inverse $S^{-1} : X \times X \to X \times X : (b^a, a) \mapsto (a_b, b)$.*

Notation. *We denote multiple up and down operations as follows:*

$$f_c(f_b(a)) = (a_b)_c = a_{bc}$$

and similarly for up operations or combinations of up and down operations.

Futhermore,

$$f^c(f^b(a)) = \left(a^b\right)^c = a^{bc}$$

Inverse operations are denoted as

$$\left(f^b\right)^{-1}(a)$$

and

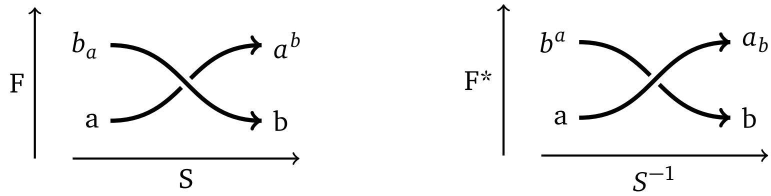
$$a^{b^{-1}} = \left(f^b\right)^{-1}(a)$$

and can be combined as well.

Proposition 3.2 (The Switch is bijective)**.** *The switch is bijective.*

The proof will follow later.

Remark. *To understand the definition of the sideways and the switch maps better, consider Figure 3.1.*



(a) Sideways and Switch Map Diagram for positive crossing

(b) Sideways and Switch Map Diagram for negative crossing

Figure 3.1: Sideways and Switch Map Diagram for positive and negative crossings

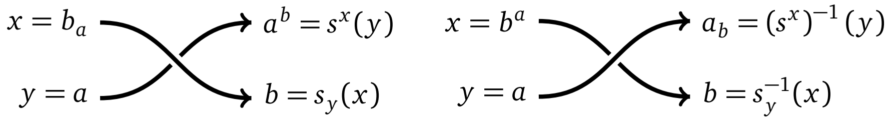
We can now define s_x, s^y by

$$S(x,y) = \left(s^x(y), s_y(x)\right).$$

S gives rise to the following relations between f_a, f^b and s^x, s_y,

$$\begin{aligned} s_y(x) &= b = f_a^{-1}(b_a) &&= f_y^{-1}(x) \quad (3.1)\\ s^x(y) &= f^b(a) = f^b(y) &&= f^{f_y^{-1}(x)}(y). \quad (3.2)\end{aligned}$$

These functions can also be defined diagrammatically, see Figure 3.2.



(a) Sideways and Switch Map Diagram for Relations, positive crossing

(b) Sideways and Switch Map Diagram for Relations, negative crossing

Figure 3.2: Sideways and Switch Map Diagram for Relations

This leads to the following Lemma 3.3.

Lemma 3.3 (Switch Functions are bijective)**.** *The switch functions s^x, s_y as defined earlier are bijective.*

Proof. This follows from Equations (3.1), (3.2) and the fact that f_a, f^a are bijective (see Definition 3.1). *q.e.d.*

Proof of Proposition 3.2. This follows from Lemma 3.3. *q.e.d.*

Corollary 3.4. *It follows from Lemma 3.3 that S^{-1} is bijective.*

3.2 Relation of Sideways Map and Switch to the generalised Reidemeister Moves

In this chapter we consider the effect of the Reidemeister moves on the switch and sideways maps.

3.2.1 R_1

Let us consider the following Figure 3.3.

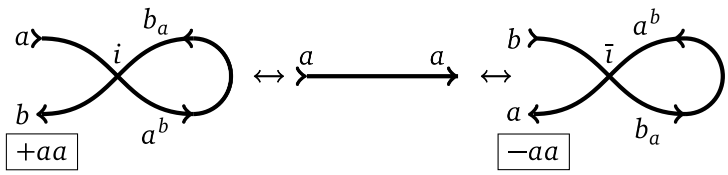
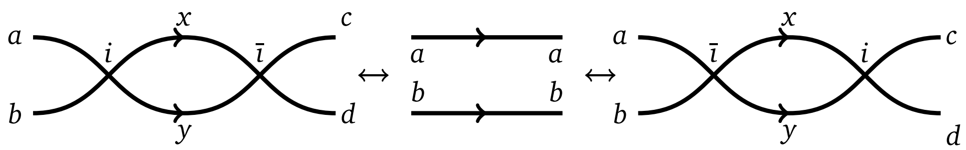


Figure 3.3: Generalised Reidemeister 1 Move, Switch and Sideways Maps

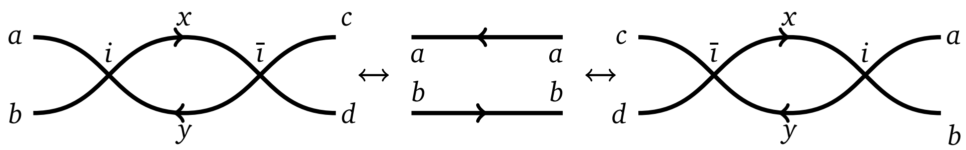
Thus, $a = b$ and $a_a = a^a$. We label the left crossing $+aa$ and the right one $-aa$.

3.2.2 R_2

Next, examine the following Figures 3.4(a) and 3.4(b).



(a) Generalised Reidemeister 2 Move, same Orientation



(b) Generalised Reidemeister 2 Move, opposite Orientation

Figure 3.4: Generalised Reidemeister 2 Moves

If we go from inside out, i.e. that the bigon is about to disappear, labels x, y are given and we must show that $a = c$ and $b = d$. If the bigon is just born, we need to find unique labels x, y.

In the left case of Figure 3.4(a), from the i crossing we have that $a = y_b$ and $x = b^y$. From the $\bar{\imath}$ crossing, we get $c = y_d$ and $x = d^y$. If the bigon is about to disappear, then it holds $d = x^{y^{-1}} = b$ and $c = y_d = y_b = a$. If the bigon has just been created, then $y = a_{b^{-1}}$ and $x = b^{a_{b^{-1}}}$.

In the right case of Figure 3.4(a), from the $\bar{\imath}$ crossing we have $x = d_y$ and $c = y^d$. From the i crossing, we get that $x = b_y$ and $a = y^b$. If the bigon is about to disappear, we have that $d = x_{y^{-1}} = b$ as well as $c = y^d = y^b = a$. If it has just been created, then the labels are $y = a^{b^{-1}}$ and $x = b_{a^{b^{-1}}}$.

In the left case of Figure 3.4(b), if the bigon is about to disappear, we have $F^{-1}(y, x) = (b, a) = (d, c)$ which is possible as F is invertible. If it has just been created we have $(y, x) = F(b, a)$.

In the right case of Figure 3.4(b), if the bigon is about to disappear, then $F(x, y) = (a, b) = (c, d)$ holds. If it has just been created, then $(x, y) = F^{-1}(a, b)$ as F is invertible.

3.2.3 R_3

As shown in Theorem 2.10, only braid-like moves need to be considered. Thus, for $R_3(\epsilon, \eta; \xi)$, we need the set-theoretic *Yang-Baxter-Equation*[1] for the switch S of the C_i crossing and the switch T of the C_j crossing, according to the notation given in Figure 2.33,

$$(S^\epsilon \times 1)(1 \times T^\xi)(S^\eta \times 1) = (1 \times S^\eta)(T^\xi \times 1)(1 \times S^\epsilon) \tag{3.3}$$

where $\epsilon, \eta, \xi = \pm 1$.

In terms of diagrams, see the following Figure 3.5. One thinks about this similar as to a braid.

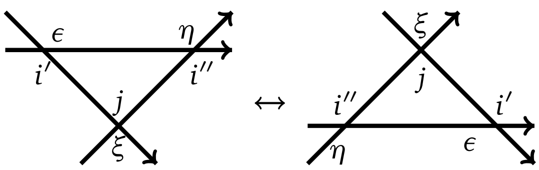


Figure 3.5: Generalised Reidemeister 3 Move and Yang-Baxter

[1]For an introduction to this in terms of knots, braids et al. see for example [YG89; YG94; Eis05] or for a more general introduction that is not focused on knots or braids, see [Jim94] or [Nic12].

3.2.4 R_4

If S_1 and S_2 are the corresponding switches and F_1 and F_2 the corresponding sideways maps, then the fourth generalised Reidemeister moves requires them to commute in pairs, that is $S_1S_2 = S_2S_1$ and $F_1F_2 = F_2F_1$.

3.3 Yang-Baxter in different Knot Theories

In this section, we present some examples of different cases of the Yang-Baxter Equations for different knot theories.

3.3.1 Involutive Crossings

If a crossing is the same as its inverse, it holds that $S^2 = \text{id}$ and $F = F^*$ for the switch and the sideways map, respectively. This leads, in particular, to the following theorem.

Theorem 3.5 (Involutive Crossings; up & down Operations). *If the crossing is involutive, then the up and down operations satisfy $a^b = a_b$ for all a, b.*

3.3.2 R_3 Move with one Type of Crossing

Let us first consider those R_3 moves with only one type of crossing. As there is only one type of crossing, there is only one type of switch. This gives us the following Yang-Baxter Equations from Equation (3.3):

$$(S \times 1)(1 \times S)(S \times 1) = (1 \times S)(S \times 1)(1 \times S) \tag{3.4}$$

$$(S \times 1)(1 \times S^{-1})(S \times 1) = (1 \times S)(S^{-1} \times 1)(1 \times S). \tag{3.5}$$

Thus, for doodles, none of these should hold as there is no permissible R_3 move. In the classical case, Equation (3.4) must be true and Equation (3.5) must fail.

The condition in Equation (3.4) implies the following relations,

$$a^{bc_b} = a^{cb^c}, \; c_{ba^b} = c_{ab_a}, \; {b_a}^{c_a} = {b^c}_{a^c}. \tag{3.6}$$

These can be visualised in Figure 3.6, where the crossings appear to lie on the surface of a cube.

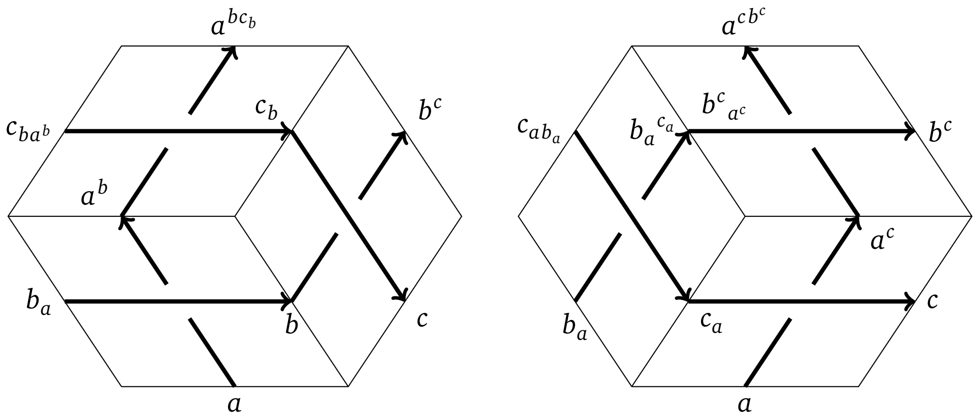


Figure 3.6: Relations (3.6) arising from Equation (3.4), visualised

3.3.3 R_3 Move with two Crossings

In this section, we investigate which relations arise from R_3 moves with two types of crossing for zero, one or two involutive crossing types. In particular, we only investigate crossings with both crossing types present, as the case with only one crossing has already been considered earlier in Section 3.3.2. Note that we always have some version of the Yang-Baxter Equation with only one crossing type for both crossing types as well, as in Equations (3.4) and (3.5).

Two involutive Crossings

As both types are involutive, we have the following version of the Yang-Baxter Equation (3.3):

$$(S\times 1)(1\times T)(S\times 1) = (1\times S)(T\times 1)(1\times S), \tag{3.7}$$

and interchanging of S and T is possible.

Example. *Examples for this kind of knot theory with two involutive crossings are the Flat Virtual Knot Theory and the Virtual Doodles, see Section 2.5.4. As a reminder, the R_3 moves are described in Figure 2.37, Chapter 2. These give rise to the following four versions of the Yang Baxter Equation* (3.3),

$$\begin{aligned}
(T\times 1)(1\times T)(T\times 1) &= (1\times T)(T\times 1)(1\times T) && (3.8)\\
(T\times 1)(1\times S)(T\times 1) &= (1\times T)(S\times 1)(1\times T) && (3.9)\\
(S\times 1)(1\times S)(S\times 1) &= (1\times S)(S\times 1)(1\times S) && (3.10)\\
(S\times 1)(1\times T)(S\times 1) &= (1\times S)(T\times 1)(1\times S) && (3.11)
\end{aligned}$$

where S stands for the doodle and T for the virtual crossing. Note that in the case of Flat Virtual Knot Theory, Equations (3.8) *to* (3.10) *hold while Equation* (3.11) *is not fulfilled. In the case of Virtual Doodles, Equations* (3.8) *and* (3.9) *hold, but Equations* (3.10) *and* (3.11) *do not.*

One involutive Crossing

In this case, we have to consider two cases as exactly one of the crossings is involutive. Let us first assume that the i-type crossings are involutive. The notation for this is the same as in Figure 3.5. This gives the following versions of the Yang-Baxter Equation (3.3):

$$\begin{aligned} (S\times 1)(1\times T)(S\times 1) &= (1\times S)(T\times 1)(1\times S) \\ (S\times 1)(1\times T^{-1})(S\times 1) &= (1\times S)(T^{-1}\times 1)(1\times S) \end{aligned}$$

In the second case, assume the j-type crossing is involutive. This gives the following versions of the Yang-Baxter Equation (3.3):

$$\begin{aligned} (S\times 1)(1\times T)(S\times 1) &= (1\times S)(T\times 1)(1\times S) \\ (S^{-1}\times 1)(1\times T)(S\times 1) &= (1\times S)(T\times 1)(1\times S^{-1}) \\ (S\times 1)(1\times T)(S^{-1}\times 1) &= (1\times S^{-1})(T\times 1)(1\times S) \\ (S^{-1}\times 1)(1\times T)(S^{-1}\times 1) &= (1\times S^{-1})(T\times 1)(1\times S^{-1}) \end{aligned}$$

Example. *Examples for this type of knot theory are Virtual Knot Theory or Flat Singular Knot Theory, see Sections 2.3 and 2.4.4. Their respective R_3 moves are presented in Figures 2.16 and 2.44. The Virtual Knot Theory gives rise to the following versions of the Yang-Baxter Equation* (3.3)*, all of which must hold. Here, S and S^{-1} denote a classical positive and negative crossing, respectively, and T a virtual crossing.*

$$\begin{aligned} (S\times 1)(1\times S)(S\times 1) &= (1\times S)(S\times 1)(1\times S) \\ (T\times 1)(1\times T)(T\times 1) &= (1\times T)(T\times 1)(1\times T) \\ (T\times 1)(1\times S)(T\times 1) &= (1\times T)(S\times 1)(1\times T) \\ (T\times 1)(1\times S^{-1})(T\times 1) &= (1\times T)(S^{-1}\times 1)(1\times T) \end{aligned}$$

The following virtual knot theory Yang-Baxter Equations must not hold:

$$(S \times 1)(1 \times S^{-1})(S \times 1) = (1 \times S)(S^{-1} \times 1)(1 \times S)$$

$$(S \times 1)(1 \times T)(S^{-1} \times 1) = (1 \times S^{-1})(T \times 1)(1 \times S) \tag{3.12}$$

$$(S^{-1} \times 1)(1 \times T)(S \times 1) = (1 \times S)(T \times 1)(1 \times S^{-1}) \tag{3.13}$$

$$(S^{-1} \times 1)(1 \times T)(S^{-1} \times 1) = (1 \times S^{-1})(T \times 1)(1 \times S^{-1})$$

$$(S \times 1)(1 \times T)(S \times 1) = (1 \times S)(T \times 1)(1 \times S)$$

Note that Equation (3.12) *must hold for welded knots as this equation arises from the first forbidden move, see Figure 2.36(d). If additionally Equation* (3.13) *holds, the theory collapses as this arises from the second forbidden move, see Figure 2.36(e).*

For the Flat Singular Knot Theory, the following Yang-Baxter Equations must hold, where S and S^{-1} are the positive and negative singular crossings, respectively and T is the doodle crossing:

$$(T \times 1)(1 \times T)(T \times 1) = (1 \times T)(T \times 1)(1 \times T)$$

$$(T \times 1)(1 \times S)(T \times 1) = (1 \times T)(S \times 1)(1 \times T)$$

$$(T \times 1)(1 \times S^{-1})(T \times 1) = (1 \times T)(S^{-1} \times 1)(1 \times T)$$

The following must not hold:

$$(S \times 1)(1 \times T)(S^{-1} \times 1) = (1 \times S^{-1})(T \times 1)(1 \times S)$$

$$(S^{-1} \times 1)(1 \times T)(S \times 1) = (1 \times S)(T \times 1)(1 \times S^{-1})$$

$$(S \times 1)(1 \times T)(S \times 1) = (1 \times S)(T \times 1)(1 \times S)$$

$$(S^{-1} \times 1)(1 \times T)(S^{-1} \times 1) = (1 \times S^{-1})(T \times 1)(1 \times S^{-1})$$

$$(S \times 1)(1 \times S)(S \times 1) = (1 \times S)(S \times 1)(1 \times S)$$

$$(S \times 1)(1 \times S^{-1})(S \times 1) = (1 \times S)(S^{-1} \times 1)(1 \times S)$$

No involutive Crossings

In this case of no involutive crossings, we then have the following versions of the Yang-Baxter Equation (3.3):

$$
\begin{aligned}
(S\times 1)(1\times T)(S\times 1) &= (1\times S)(T\times 1)(1\times S)\\
(S^{-1}\times 1)(1\times T)(S\times 1) &= (1\times S)(T\times 1)(1\times S^{-1})\\
(S\times 1)(1\times T)(S^{-1}\times 1) &= (1\times S^{-1})(T\times 1)(1\times S)\\
(S^{-1}\times 1)(1\times T)(S^{-1}\times 1) &= (1\times S^{-1})(T\times 1)(1\times S^{-1})\\
(S\times 1)(1\times T^{-1})(S\times 1) &= (1\times S)(T^{-1}\times 1)(1\times S)\\
(S^{-1}\times 1)(1\times T^{-1})(S\times 1) &= (1\times S)(T^{-1}\times 1)(1\times S^{-1})\\
(S\times 1)(1\times T^{-1})(S^{-1}\times 1) &= (1\times S^{-1})(T^{-1}\times 1)(1\times S)\\
(S^{-1}\times 1)(1\times T^{-1})(S^{-1}\times 1) &= (1\times S^{-1})(T^{-1}\times 1)(1\times S^{-1})
\end{aligned}
$$

Example. *A good example for this type of knot theory is Singular Knot Theory, see Section 2.4.3. Its R_3 moves are presented in Figures 2.38 to 2.40. The allowed moves give rise to the following Yang-Baxter Equations, where S and S^{-1} are the positive and negative classical knot crossing, and T and T^{-1} the positive and negative singular crossing, respectively:*

$$
\begin{aligned}
(S\times 1)(1\times T)(S^{-1}\times 1) &= (1\times S^{-1})(T\times 1)(1\times S)\\
(S\times 1)(1\times T^{-1})(S^{-1}\times 1) &= (1\times S^{-1})(T^{-1}\times 1)(1\times S)\\
(S^{-1}\times 1)(1\times T)(S\times 1) &= (1\times S)(T\times 1)(1\times S^{-1})\\
(S^{-1}\times 1)(1\times T^{-1})(S\times 1) &= (1\times S)(T^{-1}\times 1)(1\times S^{-1})\\
(S\times 1)(1\times S)(S\times 1) &= (1\times S)(S\times 1)(1\times S)
\end{aligned}
$$

The forbidden moves give rise to the following Yang-Baxter Equations, which therefore must not hold:

$$
\begin{aligned}
(T\times 1)(1\times T)(T\times 1) &= (1\times T)(T\times 1)(1\times T)\\
(T\times 1)(1\times T^{-1})(T\times 1) &= (1\times T)(T^{-1}\times 1)(1\times T)\\
(S^{-1}\times 1)(1\times T^{-1})(S^{-1}\times 1) &= (1\times S^{-1})(T^{-1}\times 1)(1\times S^{-1})\\
(S\times 1)(1\times T)(S\times 1) &= (1\times S)(T\times 1)(1\times S)\\
(S\times 1)(1\times T^{-1})(S\times 1) &= (1\times S)(T^{-1}\times 1)(1\times S)\\
(S^{-1}\times 1)(1\times T)(S^{-1}\times 1) &= (1\times S^{-1})(T\times 1)(1\times S^{-1})\\
(T\times 1)(1\times S)(T^{-1}\times 1) &= (1\times T^{-1})(S\times 1)(1\times T)\\
(T\times 1)(1\times S^{-1})(T^{-1}\times 1) &= (1\times T^{-1})(S^{-1}\times 1)(1\times T)\\
(T^{-1}\times 1)(1\times S)(T\times 1) &= (1\times T)(S\times 1)(1\times T^{-1})\\
(T^{-1}\times 1)(1\times S^{-1})(T\times 1) &= (1\times T)(S^{-1}\times 1)(1\times T^{-1})\\
(T^{-1}\times 1)(1\times S^{-1})(T^{-1}\times 1) &= (1\times T^{-1})(S^{-1}\times 1)(1\times T^{-1})\\
(T\times 1)(1\times S)(T\times 1) &= (1\times T)(S\times 1)(1\times T)\\
(T\times 1)(1\times S^{-1})(T\times 1) &= (1\times T)(S^{-1}\times 1)(1\times T)\\
(T^{-1}\times 1)(1\times S)(T^{-1}\times 1) &= (1\times T^{-1})(S\times 1)(1\times T^{-1})
\end{aligned}
$$

3.3.4 R_4 Move

Let S_1, S_2 and F_1, F_2 be the corresponding switches and sideways maps, respectively, then for the general R_4 move, they must commute in pairs,

$$S_1S_2 = S_2S_1 \text{ and } F_1F_2 = F_2F_1$$

3.4 Definition Quandle, Biquandle and Others

In this section we give definitions of quandles, racks, biquandles and biracks which are the structures we study in the rest of this thesis.

This part is inspired by [EN15] but reformulated to take into account the results of the previous sections in this chapter.

Definition 3.6 (Definition Birack, Biquandle)**.** *Let X be a set with two binary functions*

$$f_a, f^a : X \times X \to X.$$

(X, f_a, f^a) *is a* birack *if the two functions define a switch map (as defined in Definition 3.1) on X and they fulfil the requirements arising from the R_3 move, see Section 3.2.3 and particularly Equation* (3.3)*. Thus, the following requirements must hold:*

- $a^{b c_b} = a^{c b^c}$, $c_{b a^b} = c_{a b_a}$, $b_a{}^{c_a} = b^c{}_{a^c}$ *(From Equation* (3.6)*, Section 3.3.2)*
- f_a, f^b *must be invertible (From Section 3.1)*

X *is a* biquandle *if additionally the following requirement hold:*

- $a^a = a_a$ *(From Section 3.2.1)*

Definition 3.7 (Definition Rack, Quandle)**.** *Let X be a set with one binary function $f^a : X \times X \to X$. Then (X, f^a) is a* rack *if and only if $(X, f^a, f_a = id)$ is a birack. Similarly, (X, f^a) is a* quandle *if and only if $(X, f^a, f_a = id)$ is a biquandle.*

Remark. *There are other structures arising when relaxing or removing some of the requirements, for example keis, shelves or spindles. These structures are not discussed in this thesis, but an overview can be found in [Cra04].*

3.5 Application to Welded Knots

In this chapter we calculate generators and relations of the knot biquandles for those welded knots found in the table in [BF11, p. 13]. For ease of reference, we have reproduced the relevant parts of the table here, Table 3.1.

knot	braid word
w3.1	$\sigma_1\tau_2\sigma_3\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}\tau_2\sigma_3^{-1}\sigma_2$
w3.2	$\tau_1\sigma_2^{-1}\tau_1\sigma_1^{-1}\sigma_1^{-1}\tau_2$
w4.1	$\sigma_1\tau_1\sigma_1^{-1}\sigma_2\sigma_1\tau_1\sigma_1^{-1}\sigma_2^{-1}$
w4.2	$\sigma_1^{-1}\sigma_2^{-1}\sigma_3\tau_2\sigma_1\sigma_4^{-1}\sigma_3\tau_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_2^{-1}$
w4.3	$\sigma_1^{-1}\sigma_2\sigma_3\tau_2\sigma_1\sigma_4^{-1}\sigma_3\tau_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_2$
w4.4	$\sigma_1^{-1}\sigma_2\sigma_3\tau_2\sigma_1\sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_3\sigma_4\sigma_3^{-1}\tau_2$
w4.5	$\tau_1\sigma_2\sigma_1^{-1}\tau_1\sigma_1\sigma_2$
w4.6	$\sigma_1^{-1}\sigma_2^{-1}\tau_3\sigma_2^{-1}\sigma_1\sigma_4^{-1}\tau_3\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_3^{-1}\sigma_2$
w6.1	$\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3\tau_2$

Table 3.1: Braid words for welded braids

We can turn these braids into knots by connecting the respective end points. We have reproduced them in closed braid form as knots in Appendix B.

The calculation of a knot biquandle works similarly as to that of a knot quandle, see for example [EN15].

In the following Table 3.2, we present the generators and relations for the different welded knots. The calculation of biquandles from this is highly nontrivial and goes beyond the scope of this thesis. While the number of generators could conceivably be lower, in case of doubt we made the decision to go for easier readability at the expense of more generators, except in the case of w3.1 and w4.4.

knot	generators	relations
w3.1	a, b, c, d	${{d_a}^{b_a}}_d = a^{bd}$, $c^{b_a} = a^{bd^{{d_a}^{b_a}}}$,
		${b_a}^c = {d^{{d_a}^{b_a}}}_{a^b}$, ${b_a}^{d_{a^b}} = c$, ${b_a}^c = {d^{{d_a}^{b_a}}}_{a^b}$
w3.2	a, b, c	$c_a = b_a$, $b_c = a^c$, $c^b = a^b$
w4.1	a, b, x, y	$y_{{xa^b}_x} = {x^y}_{ab_a}$, $b = x^{a^b}$, $y = a^{x^y}$, ${{a^b}_x}^{y_x} = {b_a}^{{x^y}_a}$
w4.2	a, b, c, e, x, z	$x_z = a_{ba^{b_c}}$, ${e_c}^{a^{bcd}} = b^{axz^x}$, $b_a = z^e$, $c_{a^b} = e_{b_a}$,
		$a_{ba^{b_c}} = x_z, x_{b^a} = c^e$, ${z^x}_{b^{ax}} = a^{bcae_c}$
w4.3	a, b, c, d, x	$x_{a^b} = c^d$, $d^{a^b} = c_{b_a}$, $x_{a^b d} = b_{a{b_a}^c}$,
		$a^{bx{a^b}_{d^x}} = {d_c}^{{b_a}^{cb_a}}$, ${{b_a}^{cb_a}}_{d_c} = {a^b}_{d^x a^{bx}}$
w4.4	a, b, c, x	$x_{a^b} = b_{a{b_a}^c}$, $c^{{c_{b_a}}^{a^b}} = x_{{a^b}_{c_{b_a}}}$,
		${{{c_{b_a}}^{a^b}}_c}^{{b_a}^{cb_a}} = {{a^b}_{c_{b_a}}}^{xa^{bx}}$, ${{b_a}^{cb_a}}_{{{c_{b_a}}^{a^b}}_c} = {a^{bx}}_{{{a^b}_{c_{b_a}}}^x}$
w4.5	a, b, c, y	$c^{c_{y^b}} = b^a$, $b^{a^c} = a_b$, $y^c = a$, $b_{c_y} = y$
w4.6	a, b, c, e, x, y	$c_{a^b} = e^{b_a}$, ${a_b}^c = x_{{b_a}^e}$, $e^{b^a a^{bc}} = y^{b_a e^x}$,
		${a^{bc}}_{e^{ba}} = {{b_a}^{ex}}_y$, $x^{b^{ae}} = c_{a_b}$, $y = {b^{ae}}_x$
w6.1	a, b, c, y, z	$c^y = b_{ac}$, $c_y = {b_a}^y$, $z^y = a_y$, $y_z = c^{{b_a}^z}$,
		$z^{c^{b_a} a^b} = y^{ay_{b_a}}$, $y_{b_a y^a} = {a^b}_{z^{c^{b_a}}}$

Table 3.2: Biquandle generators and relations for welded braids

Part II

Biquandle Functions

Chapter 4

Relations, Quaternions and More

In this part we are considering relations and restrictions on different types of biquandle maps. In this chapter, we consider linear operations in more detail and derive some extra requirements where the coefficients are quaternions. In the next chapter, we consider quadratic biquandle operations. This is done by plugging the different types in the usual biquandle axioms:

Definition 4.1 (Biquandle Axioms)**.** *The biquandle axioms are as follows for up and down operations notation as defined in Definitions 3.1 and 3.6.*

$$a^a = a_a \tag{4.1}$$

$$a_{b c_b} = a_{c b^c} \tag{4.2}$$

$$a_b{}^{c_b} = a^c{}_{b^c} \tag{4.3}$$

$$a^{b c_b} = a^{c b^c} \tag{4.4}$$

We also investigate the inverse functions with Maple, [Map15]. The code for this can be found in Appendix A.

We also calculate some restrictions via a graphical approach, where we colour the R_3 move. First, we are going to introduce quaternions.

4.1 Quaternions

Quaternions were first described by Hamilton in 1843 in a letter to his friend John Graves and it was first published in [Ham44]. However, we are going to use their generalisation, general quaternions. Most of the theory in this chapter is based on [BF04; BF08b; Fen08; Fen01] while the calculations are loosely based on [BF08a]. Some points have been added from [FT07]. The following Definition 4.2 is taken from [Fen08].

Definition 4.2 (Quaternions). *Let $\mathbb{F}$ be a field of characteristic not equal to two. Take any two non-zero $\lambda, \mu \in \mathbb{F}$. Denote the algebra of dimension four over $\mathbb{F}$ with basis $\{1, i, j, k\}$ and relations $i^2 = \lambda, j^2 = \mu, ij = -ji = k$ by $\left(\frac{\lambda,\mu}{\mathbb{F}}\right)$ or by $\mathcal{Q}$. The multiplication table of this algebra is given in Table 4.1:*

	i	j	k
i	λ	k	λj
j	$-k$	μ	$-\mu i$
k	$-\lambda j$	μi	$-\lambda\mu$

Table 4.1: Multiplication Table of $\left(\frac{\lambda,\mu}{\mathbb{F}}\right)$

We will call the elements of $\mathcal{Q}$ (generalised) quaternions. $\mathbb{F}$ *is called the* underlying field *and the* parameters *of the quaternion algebra are λ and μ. The quaternions are denoted by upper-case letters $(A, B, \ldots)$.*

Generally, a quaternion has the form $A = a_0 + a_1 i + a_2 j + a_3 k$ with $a_0, a_1, a_2, a_3 \in \mathbb{F}$. The coordinate a_0 is called the scalar *part of A and the vector $\boldsymbol{a} = a_1 i + a_2 j + a_3 k$ is called the* pure *part of A. This allows us to rewrite $A = a_0 + \boldsymbol{a}$. Equivalently, a quaternion is* scalar *if its pure part is zero and* pure *if its scalar part is zero. We will denote the scalar element by lower case letters $(a, b, \ldots)$, and the pure element by bold lower case letters $(\boldsymbol{a}, \boldsymbol{b}, \ldots)$.*

We can retrieve the classical quaternions with this approach as $\left(\frac{-1,-1}{\mathbb{R}}\right)$. The algebra of 2×2 *matrices with entries in $\mathbb{F}$ can be written as $M_2(\mathbb{F}) = Mat_{2\times 2} = \left(\frac{-1,1}{\mathbb{F}}\right)$.*

Definition 4.3 (Conjugation, Norm & Trace of Quaternions). *The* conjugate *of a quaternion A is $\bar{A} = a_0 - \boldsymbol{a}$, its* norm *is given by $N(A) = A\bar{A}$ and the* trace *of A is defined as $tr(A) = A + \bar{A} = a_0 + \boldsymbol{a} + a_0 - \boldsymbol{a} = 2a_0$.*
Conjugation is an anti-isomorphism of order 2, that is,

$$\overline{A+B} = \bar{A} + \bar{B},\ \overline{AB} = \bar{A}\bar{B},\ \overline{aA} = a\bar{A},\ \bar{\bar{A}} = A.$$

In addition, $A = \bar{A}$ if and only if A is scalar and $A = -\bar{A}$ if and only if A is pure.

Furthermore, $N(AB) = N(A)N(B)$ and we note that this is a scalar and we denote the set of values of the norm function by $\mathcal{N}$. In particular, $\mathcal{N} \subset \mathbb{F}$ and it is multiplicatively closed.

In addition, $\mathcal{N}^* = \mathcal{N} - \{0\}$ *is a multiplicative subgroup of* $\mathbb{F}^*$.

Similar as for complex numbers, we define the inverse of a quaternion A *as* $A^{-1} = \frac{\bar{A}}{N(A)}$, *so the inverse exists if and only if* $N(A) \neq 0$.

Definition 4.4 (Multiplication of Quaternions)**.** *Let* A, B *be two quaternions. Then there is a bilinear form given by*

$$\begin{aligned} A \cdot B &= \frac{1}{2}\left(A\bar{B} + B\bar{A}\right) = \frac{1}{2}\left(\bar{A}B + \bar{B}A\right) = \frac{1}{2} tr\left(A\bar{B}\right) \\ &= a_0 b_0 - \lambda a_1 b_1 - \mu a_2 b_2 + \lambda\mu a_3 b_3. \end{aligned} \tag{4.5}$$

Since $\lambda, \mu \neq 0$, *this is non-degenerate. The corresponding quadratic form is given by*

$$N(A) = a_0^2 - \lambda a_1^2 - \mu a_2^2 + \lambda\mu a_3^2.$$

We can also define a cross product symbolically for pure quaternions as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} -\mu i & -\lambda j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

This cross product has the usual rules as the standard cross product of bilinearity and skew symmetry.
Additionally, it holds that

$$AB = ab - \boldsymbol{a} \cdot \boldsymbol{b} + b\boldsymbol{a} + a\boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}. \tag{4.6}$$

Now let $\boldsymbol{a}, \boldsymbol{b}$ *be pure quaternions. Then we can restrict Equation* (4.6) *to*

$$\boldsymbol{ab} = -\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}$$

where for pure quaternions

$$\boldsymbol{a} \cdot \boldsymbol{b} = -\lambda a_1 b_1 - \mu a_2 b_2 + \lambda\mu a_3 b_3.$$

Furthermore,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{c} \cdot \boldsymbol{a})\,\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{a})\,\boldsymbol{c}.$$

The equivalent scalar triple product *is*

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \lambda\mu \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Definition 4.5 (Commutator)**.** *We define the following two commutators of quaternions*

$$\begin{aligned} [A, B] &= AB - BA &&= 2\boldsymbol{a} \times \boldsymbol{b} \qquad (4.7)\\ (A, B) &= A^{-1}B^{-1}AB &&= |A|^{-2}\,|B|^{-2}\bar{A}\bar{B}AB. \end{aligned}$$

Remark. *Note that the first Commutator in* (4.7) *is also more generally known as the Lie Bracket for vector fields. For further reading, see [War71] or [EW06].*

4.2 Relations for Linear Operations

In this section we are developing restrictions on linear biquandle operations arising from the Biquandle axioms and invertibility of the operations. The results will rely on comparing coefficients. This gives at the least sufficient conditions and, in our opinion, necessary conditions for biquandles.

Remark. *Let X be a biquandle with an algebraic structure, then it should have "dimension" at least 3 in the sense that there are 3 elements $a, b, c \in X$ such that if $Pa + Qb + Rc = 0$, then $P = Q = R = 0$. In other words, they should be at least 3 linearly independet elements.*

This also ensures that coefficient comparisons give necessary conditions for biquandles.

First, let us assume that the up and down operations are linear. In particular, let $A, B, C, D \in R$, a possibly non-commutative, associative ring of some sort. This gives us the general form

$$\begin{aligned} f^b(a) &= a^b &&= C * a + D * b \qquad (4.8)\\ f_a(b) &= b_a &&= A * a + B * b. \qquad (4.9) \end{aligned}$$

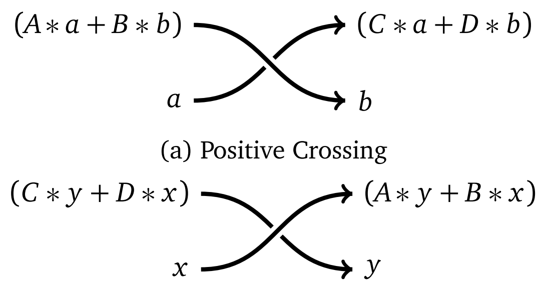
Equivalently, this can be written in matrix notation:

$$\begin{pmatrix} f_a(b) \\ f^b(a) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{4.10}$$

Furthermore, we assume that the biquandle X has the structure of an R-module, i.e. $X = R[x_1, \ldots, x_n]$, ring variables with coefficients in R, generated by monomials in $x_1, \ldots, x_n$ which may or may not commute.

For our purposes, translation by λ does not change any results and in fact, by $a_a = a^a$, they would have to be translated by the same λ. This follows from easily from Definition 4.1. One can think of λ as the constant that gets added (and usually ignored) when integrating a function.

The graphical equivalent is shown in Figure 4.1.



(a) Positive Crossing

(b) Negative Crossing

Figure 4.1: Positive and negative Crossing for a Classical Knot with linear Functions

4.2.1 Relations arising from Biquandle Axioms

Theorem 4.6 (Linear biquandle relations)**.** *For a linear biquandle X over a ring R with operations of the form* (4.8)*,* (4.9)*, the following equations hold:*

$$A + B = C + D, \ [B, C] = 0, \ [A, B] = AD \tag{4.11}$$

where $AD = 0$ in the commutative case, i.e. either one of them is zero or a zero-divisor.

Proof. First, we are considering the Biquandle Axioms. Using Equations (4.8), (4.9) in Axioms (5.5) to (5.8) gives us the following relations:

$$
\begin{aligned}
C+D &= A+B && (4.12)\\
CB &= BC && (4.13)\\
AC &= DA+CA && (4.14)\\
DB &= AD+BD && (4.15)\\
AC &= BA+AA && (4.16)\\
AB &= AD+BA && (4.17)\\
DC &= CD+DA && (4.18)\\
DB &= CD+DD && (4.19)
\end{aligned}
$$

which is our desired result. *q.e.d.*

Combining (4.15) and (4.19) gives

$$(A+B)D = (C+D)D.$$

From (4.14) and (4.16), we get

$$(D+C)A = (B+A)A.$$

The following relation arises from (4.16) and (4.17)

$$AC = AB - AD + AA$$

and (4.14) and (4.18) give

$$(A-D)C = C(A-D).$$

In particular, we can combine Equations (4.12), (4.13) and (4.14) (where we assume A to be invertible) to get the following:

$$\begin{aligned} D &= A+B-C \\ AC &= A^2+BA \\ C &= A+A^{-1}BA \\ AB+A^{-1}BAB &= BA+BA^{-1}BA \end{aligned}$$

4.2.2 Relations from Invertibility

After studying the relations arising from Reidemeister diagrams and biquandle axioms, we are now considering invertibility.

Furthermore, as we require the up and down operations in (4.8) and (4.9) to be invertible, we get the following relations:

Define

$$f_a(b) = A*a+B*b =: x.$$

Hence,

$$f_a^{-1}(x) = X*x+Y*a = XA*a+XB*b+Y*a = b$$

and from this we get,

$$XB = \mathbb{I} \Leftrightarrow X = B^{-1}$$

as well as

$$XA+Y = 0 \Leftrightarrow Y = -XA = -B^{-1}A.$$

Thus,

$$f_a^{-1}(x) = B^{-1}(x-A*a). \tag{4.20}$$

Furthermore, define

$$f^b(a) = C * a + D * b := y.$$

Hence,

$$\left(f^b\right)^{-1}(y) = Z * y + W * b = ZC * a + ZD * b + W * b = a$$

and from this we get,

$$ZC = \mathbb{I} \Leftrightarrow Z = C^{-1}$$

as well as

$$ZD + W = 0 \Leftrightarrow W = -ZD = -C^{-1}D.$$

Thus,

$$\left(f^b\right)^{-1}(y) = C^{-1}(y - D * b). \tag{4.21}$$

Example. *In particular, in [Fen12b] it has been shown that for doodles the up action is equal to the down action, that is*

$$a^b = b_a = A(a + b)$$

with A invertible and $A \neq 0$. This gives a useful invariant for doodles, as defined in [Fen08].

Alternatively, we can write this in terms of matrix requirements as defined in Equation (4.10). In particular, we require both B and C to be invertible, i.e. both B^{-1} and C^{-1} exist. In order to calculate the coefficients of the inverse functions, we can solve the following equation:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We note that for this to be solvable, we require the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ to be invertible. This, then, requires that both B^{-1} and C^{-1} exist. In addition we require the following two invariants to be a unit,

$$\begin{aligned} \Delta &= 1 - B^{-1}AC^{-1}D \\ \Delta' &= 1 - C^{-1}DB^{-1}A \end{aligned}$$

as this is a matrix over a non-commutative ring. Finally, the result is given as follows,

$$X = -C^{-1}D\Delta^{-1}B^{-1} \tag{4.22}$$

$$Y = \Delta'^{-1}C^{-1} \tag{4.23}$$

$$Z = \Delta^{-1}B^{-1} \tag{4.24}$$

$$W = -B^{-1}A\Delta'^{-1}C^{-1} \tag{4.25}$$

which is the same for both approaches.

4.2.3 Back to Quaternions

In this section, we are applying the results of this chapter to the quaternions introduced earlier. This is based on [BF04; BF08a; BF08b; Fen08].

In general, the following is also a switch:

$$S = \begin{pmatrix} A & tB \\ t^{-1}C & D \end{pmatrix}$$

where $t \in Z(R)$, the center of the ring with $A, B, C, D \in R$. Furthermore, both S^{-1} and S^T are switches.

Let us now consider these results in the context of the switch as introduced in Definition 3.1, in particular the case with only one type of switch as in Equation (3.4),

$$(S \times \mathbb{I})(\mathbb{I} \times S)(S \times \mathbb{I}) = (\mathbb{I} \times S)(S \times \mathbb{I})(\mathbb{I} \times S)$$

where, as in Equation (4.10),

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A, B, C, D \in \mathcal{Q}$, the quaternions and B, C are invertible. In this case, both of the following are also valid:

$$S^* = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \text{ and } S^{T*} = \begin{pmatrix} \bar{D} & \bar{B} \\ \bar{C} & \bar{A} \end{pmatrix}$$

Another example is the so-called Budapest Switch with

$$S = \begin{pmatrix} 1+i & -j \\ j & 1+i \end{pmatrix}.$$

We can generalise this to

$$S = \begin{pmatrix} 1+U & -V \\ V & 1+U \end{pmatrix}$$

with $U, V \in S^2$, the set of pure unit quaternions and $U \perp V$ and then, from Lemma 8.2 in [BF08a] we know that in fact, all switches with entries in the ring of quaternions with integer coefficients are of Budapest type with $U, V \in \{\pm i, \pm j, \pm k\}$.

In particular, from Lemma 3.2 in [BF08a], it holds that both A, B and $|A-1|$ are units. Graphically, this looks like in Figure 4.2:

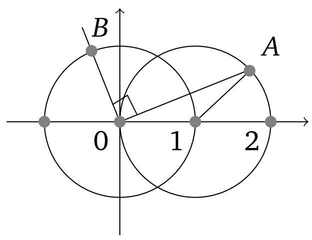


Figure 4.2: Quaternions for linear Biquandles

Chapter 5

Relations arising from quadratic up and down Operations

In this chapter, we are extending the results and methods from the previous Chapter 4 to the quadratic case. This chapter is based entirely on our own research.

Let us assume that the up and down operations are quadratic. In particular, let $A_i, B_i, C_i, D_i \in R$, an associative, commutative ring of $i = 1, 2, \emptyset$. We also assume $a * A = A * a$ for $A \in R$ and $a \in BQ$, a biquandle, which may be commutative or not. As in the previous chapter, we assume the biquandle is a ring of variables $R[x_1, \ldots, x_n]$ in $x_1, \ldots, x_n$ with coefficients in R, generated by monomials in $x_1, \ldots, x_n$, which may or may not commute. Furthermore, we assume the biquandle admits some multiplication for $x, y \in BQ, r \in R$ such that $r * (x \cdot y) = (r * x)y = x(r * y)$.

The general form is thus given by

$$\begin{aligned} f^b(a) &= a^b = C_0 * a + C_1 * a^2 + C * a \cdot b + D_0 * b + D_1 * b^2 + D * b \cdot a && (5.1) \\ f_a(b) &= b_a = A_0 * a + A_1 * a^2 + A * a \cdot b + B_0 * b + B_1 * b^2 + B * b \cdot a. && (5.2) \end{aligned}$$

It holds that any translation by a scalar (λ) has to be the same for both equations and does not give rise to any new relations. In particular, as the functions have to be bijective, $A_1 = B_1 = C_1 = D_1 = 0$ if BQ is commutative.

As in the linear case, we can write this in matrix form,

$$\begin{pmatrix} f^b(a) \\ f_a(b) \end{pmatrix} = \begin{pmatrix} A_0 & A_1 & A_2 & B_0 & B_1 & B_2 \\ C_0 & C_1 & C_2 & D_0 & D_1 & D_2 \end{pmatrix} \begin{pmatrix} a & a^2 & a \cdot b & b & b^2 & b \cdot a \end{pmatrix}^T$$

or, for just one of the functions in the commutative case,

$$f_a(b) = \begin{pmatrix} b^2 & b & 1 \end{pmatrix} \begin{array}{c} \\ b^2 \\ b \\ b \end{array} \begin{array}{c} \begin{array}{ccc} a^2 & a & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & A & A_0 \\ 0 & B_0 & 0 \end{bmatrix} \end{array} \begin{pmatrix} a^2 \\ a \\ 1 \end{pmatrix}$$

and similarly for $f^b(a)$.

In particular, we write $A * a$ as Aa and $a \cdot b$ as ab for simplicity where it is unambigious.

As in the linear case, linear independence holds. Additionally, quadratic independence is necessary.

5.1 Commutative Biquandles

We first consider the case where the biquandle is commutative. As in this case $A_1 = B_1 = C_1 = D_1 = 0$, we have biquandle functions of the general form

$$f^b(a) = a^b = C_0 a + Cab + D_0 b \tag{5.3}$$

$$f_a(b) = b_a = A_0 a + Aab + B_0 b \tag{5.4}$$

Furthermore, as in the linear case, we require B_0, C_0 to be invertible.

5.1.1 Relations arising from Axioms

We now consider relations arising from the biquandle axioms. As a reminder, we now list the biquandle axioms as discussed in Chapter 3.

Definition 5.1 (Biquandle Axioms)**.** *The biquandle axioms are as follows for up and down operations notation as defined in Definitions 3.1 and 3.6.*

$$
\begin{aligned}
a^{a} &= a_{a} && (5.5)\\
a_{b c_b} &= a_{c b^c} && (5.6)\\
{a_b}^{c_b} &= {a^c}_{b^c} && (5.7)\\
a^{b c_b} &= a^{c b^c} && (5.8)
\end{aligned}
$$

The four axioms in Equations (5.5) to (5.8) give rise to the following Theorem 5.2.

Theorem 5.2 (Commutative quadratic Biquandle with commutative Coefficients; Axioms)**.** *Let BQ be a commutative biquandle with a structure as described before. If the up and down functions are of quadratic form and have commutative coefficients, they are always of the form*

$$
\begin{aligned}
f_a(b) &= A_0 * a + B_0 * b + \lambda * a \cdot b && (5.9)\\
f^{b}(a) &= C_0 * a + D_0 * b + \lambda * a \cdot b && (5.10)
\end{aligned}
$$

and the following relations

$$
\begin{aligned}
A_0 D_0 &= 0\\
\lambda^2 A_0 &= 0\\
\lambda A_0^2 &= 0\\
\lambda^3 &= 0\\
\lambda^2 D_0 &= 0\\
\lambda D_0^2 &= 0
\end{aligned}
\qquad
\begin{aligned}
\lambda A_0 B_0 &= \lambda A_0 C_0\\
\lambda^2 B_0 &= \lambda^2 C_0\\
\lambda B_0 D_0 &= \lambda C_0 D_0\\
\lambda B_0^2 &= \lambda B_0 D_0 + \lambda B_0\\
\lambda C_0^2 &= \lambda A_0 B_0 + \lambda C_0\\
\lambda A_0 B_0 + \lambda D_0 &= \lambda C_0 D_0 + \lambda A_0\\
\lambda B_0 C_0 &= \lambda C_0 D_0 + \lambda C_0\\
&= \lambda A_0 B_0 + \lambda B_0
\end{aligned}
$$

hold.

Proof. This follows from plugging in the equations into the axioms and compare coefficients via the code in Appendix A. Additionally, we used (5.12) and the fact that B_0, C_0 have to be invertible).

In particular, when considering Axiom (5.5), we get

$$A_0a + B_0a + Aa^2 = C_0a + D_0a + Ca^2.$$

Thus we have the two relations

$$A_0 + B_0 = C_0 + D_0 \tag{5.11}$$

and

$$A = C. \tag{5.12}$$

If either $A, C = 0$ this collapses to the linear case. *q.e.d.*

5.1.2 Relations from Invertibility

From Definitions 3.1 and 3.6, we require the biquandle functions to not only be bijective but also explicitedly invertible. The relations arising from this requirement are calculated in this section and lead to Theorem 5.3.

Theorem 5.3 (Commutative quadratic Biquandle with commutative Coefficients; Invertibility)**.** *For a commutative biquandle, where the biquandle functions have commutative coefficients and the biquandle functions are of general quadratic form as in Theorem 5.2, we have the following inverses and conditions on the coefficients:*

$$\begin{aligned} f_a^{-1}(b) &= B_0^{-1}x - B_0^{-1}A_0a - \left(B_0^{-1}\right)^2 \lambda x a \\ (f^b)^{-1}(a) &= C_0^{-1}y - C_0^{-1}D_0b - (C_02{-}1)^2 \lambda y b \end{aligned}$$

where $x = f_a(b), y = f^b(a)$ *and, additionally,* $\lambda^2 = \lambda A_0 = \lambda D_0 = 0$ *as well as* $B_0, C_0 \neq 0$.

Proof. First, let the functions be as in Theorem 5.2. We define $f_a(b) =: x, f^b(a) =: y$. Furthermore, let us assume that the inverse is of similar quadratic form, i.e.

$$\begin{aligned} f_a^{-1}(x) &= X_0x + Y_0a + \mu x a &= b \\ (f^b)^{-1}(y) &= Z_0y + W_0b + \nu y b &= a. \end{aligned}$$

Considering first f_a and its inverse, we get the following set of relations by plugging x into f_a^{-1}.

$$\begin{aligned} X_0 A_0 + Y_0 &= 0 \\ X_0 B_0 &= \mathbb{I} \end{aligned} \qquad \begin{aligned} X_0 \lambda + \mu B_0 &= 0 \\ \mu A_0 &= 0 \\ \mu \lambda &= 0 \end{aligned}$$

From this after simplification the result as detailed in Theorem 5.3 follows.

The proof for $(f^b)^{-1})(a)$ works analogously. *q.e.d.*

Corollary 5.4. *We can simplify the list relations in Theorem 5.2 with the results of Theorem 5.3 to*

$$\begin{aligned} A_0 D_0 &= 0 \\ \lambda^2 &= 0 \\ \lambda A_0 &= 0 \end{aligned} \qquad \begin{aligned} \lambda D_0 &= 0 \\ \lambda B_0^2 &= \lambda B_0 \\ \lambda C_0^2 &= \lambda C_0 \\ \lambda B_0 C_0 &= \lambda C_0 \\ &= \lambda B_0 \end{aligned}$$

and so $\lambda B_0 = \lambda C_0 = \lambda = \lambda B_0^2 = \lambda C_0^2$.

Remark. *For the linear case there is only one possible case, namely*

$$a^b = \lambda a + (1 - \lambda\mu)\, b,\ a_b = \mu a$$

where $a, b \in BQ$, *a* $\mathbb{Z}\left[\lambda^{\pm 1}, \mu^{\pm 1}\right]$ *module, see [Saw99].*

Theorem 5.5. *Similarly to the linear case, in the quadratic case we have a biquandle with*

$$a^b = Aa + Bb + \lambda ab,\ b_a = Ca + Db + \lambda ab$$

with $A + B = C + D, \lambda^2 = AD = \lambda A = \lambda D = 0$ *as well as* $\lambda B = \lambda C = \lambda$. *As a special case of this general form, we have*

$$a^b = b_a = a + \lambda (1 + a)\, b$$

where $\lambda \neq 0, \lambda^2 = 0$.

Proof. Follows from Theorems 5.2 and 5.3 and corollary 5.4. *q.e.d.*

5.2 Non-Commutative Biquandles

Next, we consider non-commutative quadratic biquandles with commutative coefficients and quadratic inverses. Let us start by considering invertibility instead of the axioms.

5.2.1 Relations arising from Invertibility

As in the commutative case in Section 5.1 we require the up and down operations in (5.1) and (5.2) to be invertible. In order to simplify matters, we restrict ourselves quadratic inverse functions. In this case we get the following relations:

Let

$$f_a(b) = A_0a + A_1a^2 + Aab + B_0b + B_1b^2 + Bba =: x.$$

We can then define

$$f_a^{-1}(x) = X_0x + X_1x^2 + Xxa + Y_0a + Y_1a^2 + Yax.$$

Furthermore, let

$$f^b(a) = C_0a + C_1a^2 + Cab + D_0b + D_1b + Dba := y.$$

Thence we can define

$$f^{-1^b}(y) = Z_0y + Z_1y^2 + Zyb + W_0b + W_1b^2 + Wby$$

from which Theorem 5.6 follows.

Theorem 5.6 (Non-commutative quadratic Biquandle with commutative Coefficients; Invertibility)**.** *For a non-commutative biquandle BQ with quadratic up and down functions with commutative coefficients of the form,*

$$\begin{aligned} f_a(b) &= A_0a + A_1a^2 + Aab + B_0b + B_1b^2 + Bba &&=: x \\ f^b(a) &= C_0a + C_1a^2 + Cab + D_0b + D_1b^2 + Dba &&=: y \end{aligned}$$

we first consider the case where it holds that $A_1 = B_1 = C_1 = D_1 = A^2 = B^2 = C^2 = D^2 = 0.$

Furthermore, the quadratic inverses of f_a, f^b are of one of the following forms

$$\begin{aligned} f_a^{-1}(x) &= Aax + A_0B_0a - (AA_0 + BA_0)a^2 - B_0x + Bxa \\ &= (Aa - B_0)(x - A_0a) - BA_0a^2 \\ \left(f^b\right)^{-1}(y) &= Cby + D_0C_0b - (CD_0 + DD_0)b^2 - C_0y + Dyb \\ &= (Cb - C_0)(y - D_0b) - DD_0b^2 \end{aligned}$$

where $a, b \in BQ$, BQ a non-commutative biquandle and $A, B, C, D, A_0, B_0, C_0, D_0 \in R$, a commutative, associative ring. Special cases of inverses for either of $A, B, C, D = 0$ ater given by

$$\begin{aligned} f_a^{-1}(x) &= A_0B_0a - B_0x - BA_0a^2 + Bxa &&= (Ba - B_0)(x - A_0a) \\ f_a^{-1}(x) &= Aax + A_0B_0a - AA_0a^2 - B_0x &&= (Aa - B_0)(x - A_0a) \\ \left(f^b\right)^{-1}(y) &= D_0C_0b - C_0y - DD_0b^2 + Dyb &&= (Db - C_0)(y - D_0b) \\ \left(f^b\right)^{-1}(y) &= Cby + D_0C_0b - CD_0b^2 - C_0y &&= (Cb - C_0)(y - D_0b). \end{aligned}$$

Another form is as follows, where A_1, B_1, C_1, D_1 are not necessarily zero. Let $B_1, C_1 \neq 0$ be a zero divisor. Then we have the following two special cases for inverses:

$$\begin{aligned} f_a^{-1}(x) &= A_0B_0^{-1}x + B_0^{-1}A_1x^2 + B_0^{-1}a \\ f_a^{-1}(x) &= A_0B_0^{-1}x + A_0(B_0^{-1})^2(A+B)x^2 + B_0^{-1}a - A(B_0^{-1})^2xa - B(B_0^{-1})^2ax \\ \left(f^b\right)^{-1}(y) &= D_0C_0^{-1}y + C_0^{-1}D_1y^2 + C_0^{-1}b \\ \left(f^b\right)^{-1}(y) &= D_0C_0^{-1}y + D_0(C_0^{-1})^2(D+C)y^2 + C_0^{-1}b - D(C_0^{-1})^2yb - C(C_0^{-1})^2by. \end{aligned}$$

In the first case, $(B_0^{-1})^2 = (C_0^{-1})^2 = 0$. In the second case, all of A_1, D_1 are zero divisors as well and $AB = A^2 = B^2 = CD = C^2 = D^2 = 0 \neq (B_0^{-1})^2, (C_0^{-1})^2$.

In the proof we use Lemma 5.7.

Lemma 5.7 (Restriction of non-commutative quadratic down Function; Invertibility)**.** *Let f_a, f^b be as in Theorem 5.6. Then the following holds:*

(i) The inverse of the down function is as in Theorem 5.6.

(ii) The inverse of the up function is as in Theorem 5.6.

Proof. We only show (i); (ii) follows similarly.

In case (i), we define

$$f_a(b) = A_0 a + A_1 a^2 + Aab + B_0 b + B_1 b^2 + Bba =: x.$$

We can then define

$$f_a^{-1}(x) = Xax + X_0 a + X_1 a^2 + Y_0 x + Y_1 x^2 + Yxa = b,$$

which after expanding becomes

$$\begin{aligned} f_a^{-1}(x) &= Xa\left(Aa^2b + A_1a^3 + B_1ab^2 + A_0a^2 + B_0ab + Bba\right) + X_1a^2 + \\ &\quad + Y_1\left(Aab + A_1a^2 + B_1b^2 + A_0a + B_0b + Bba\right)^2 + X_0a + \\ &\quad + Y_0\left(Aab + A_1a^2 + B_1n^2 + A_0a + B_0b + Bba\right) + \\ &\quad + Y\left(Aab + A_1a^2 + B_1n^2 + A_0a + B_0b + Bba\right)a \\ &= b. \end{aligned}$$

After further expansion and coefficient comparison, we get the following relations:

$$a^4: \quad 0 = Y_1A_1^2 \tag{5.13}$$

$$a^3: \quad 0 = XA_1 + YA_1 + 2Y_1A_0A_1 \tag{5.14}$$

$$a^3b: \quad 0 = YA_1A \tag{5.15}$$

$$a^2: \quad 0 = X_1 + XA_0 + Y_0A_1 + YA_0 + Y_1A_0^2 \tag{5.16}$$

$$a^2b: \quad 0 = XA + Y_1A_0A + Y_1A_1B_0 \tag{5.17}$$

$$a^2b^2: \quad 0 = Y_1A_1B_1 \tag{5.18}$$

$$a^2ba: \quad 0 = Y_1A_1B \tag{5.19}$$

$$a: \quad 0 = X_0 + Y_0A_0 \tag{5.20}$$

$$ab: \quad 0 = XB_0 + Y_0A + Y_1A_0B_0 \tag{5.21}$$

$$ab^2: \quad 0 = XB_1 + Y_1A_0B_1 + Y_1AB_0 \tag{5.22}$$

$$ab^3: \quad 0 = Y_1AB_1 \tag{5.23}$$

$$aba: \quad 0 = XB + YA + Y_1A_0B + Y_1AA_0 \tag{5.24}$$

$$aba^2: \quad 0 = Y_1AA_1 \tag{5.25}$$

$$abab: \quad 0 = Y_1A^2 \tag{5.26}$$

$$ab^2a: \quad 0 = Y_1AB \tag{5.27}$$

$$b^4: \quad 0 = Y_1B_1^2 \tag{5.28}$$

$$b^3: \quad 0 = Y_1B_0B_1 \tag{5.29}$$

$$b^3a: \quad 0 = Y_1B_1B \tag{5.30}$$

$$b^2: \quad 0 = Y_0B_1 + Y_1B_0^2 \tag{5.31}$$

$$b^2a: \quad 0 = YB_1 + Y_1B_0B + Y_1B_1A_0 \tag{5.32}$$

$$b^2a^2: \quad 0 = Y_1B_1A_1 \tag{5.33}$$

$$b^2ab: \quad 0 = Y_1B_1A \tag{5.34}$$

$$ba: \quad 0 = Y_0B + YB_0 + Y_1B_0A_0 \tag{5.35}$$

$$ba^2: \quad 0 = YB + Y_1B_0A_1 + Y_1BA_0 \tag{5.36}$$

$$ba^3: \quad 0 = Y_1BA_1 \tag{5.37}$$

$$bab: \quad 0 = Y_1B_0A + Y_1BB_0 \tag{5.38}$$

$$bab^2: \quad 0 = Y_1BB_1 \tag{5.39}$$

$$baba: \quad 0 = Y_1B_1^2 \tag{5.40}$$

$$ba^2b: \quad 0 = Y_1BA \tag{5.41}$$

$$b: \quad \mathbb{I} = Y_0B_0 \tag{5.42}$$

Thus, by Equations (5.42) and(5.20), we have

$$Y_0 = B_0^{-1} \tag{5.43}$$

$$B_0 \neq 0 \tag{5.44}$$

$$X_0 = B_0^{-1}A_0.$$

Next, let us consider Equations (5.13) and (5.28) from which follows

$$0 = Y_1A_1^2 = Y_1B_1^2.$$

Thus, we have the following 5 cases:

$$\begin{cases} A_1 = B_1 = 0, & Y_1 \neq 0 \quad (5.45) \\ Y_1 = 0, & A_1, B_1 \neq 0 \quad (5.46) \\ A_1^2 = B_1^2 = Y_1 = 0, & A_1, B_1 \neq 0 \quad (5.47) \\ A_1^2 = B_1^2 = 0, & A_1, B_1, Y_1 \neq 0 \quad (5.48) \\ A_1 = B_1 = Y_1 = 0 & \quad (5.49) \end{cases}$$

Case 1: Let us first consider case (5.45), that is, $A_1 = B_1 = 0$ and $Y_1 \neq 0$.

By Equations (5.31), (5.32) and (5.42), we have

$$0 = \overbrace{Y_0B_1}^{=0} + Y_1B_0^2 = \overbrace{YB_1}^{=0} + Y_1B_0B + \overbrace{Y_1B_1A_0}^{=0} = Y_1BA$$

$$\Leftrightarrow \qquad Y_1B_0^2 = Y_1B_0B = Y_1BA.$$

Thus, as both $B_0, Y_1 \neq 0$, $B_0^2 = 0$ holds. Furthermore, it follows that $B = 0$, $B_0 = B = A$ or that $B \neq 0$ is a zero divisor for Y_1. Consider first $B = A = B_0$. Then Equations (5.35) and (5.36) become

$$\begin{aligned}
0 =& \; Y_0B + YB_0 + Y_1B_0A_0 = \overbrace{Y_0B_0}^{=\mathbb{I}} + YB + Y_1BA_0\\
0 =& \; YB + \underbrace{Y_1B_0A_1}_{=0} + Y_1BA_0 = \qquad YB + Y_1BA_0\\
\Leftrightarrow \qquad & \mathbb{I} + YB + Y_1BA_0 = \qquad YB + Y_1BA_0\\
& \mathbb{I} = \qquad 0,
\end{aligned}$$

↯

which is a contradiction. So, consider $B = 0$. Then Equation (5.22) can be written as

$$0 = \underbrace{XB_1}_{=0} + \underbrace{Y_1A_0B_1}_{=0} + Y_1AB_0 = Y_1AB_0.$$

Thus, either $A = 0$, which is the linear case, or $A = B_0$. In the latter case, it follows from Equation (5.35) that

$$0 = \overbrace{Y_0B}^{=0} + YB_0 + Y_1B_0A_0 \Leftrightarrow Y_1B_0A_0 = YB_0.$$

We then combine Equations (5.17) and (5.21) and get a contradiction with

$$\underbrace{XA}_{=XB_0} + \underbrace{Y_1A_0A}_{=Y_1A_0B_0} + \underbrace{Y_1A_1B_0}_{=0} = XB_0 + \underbrace{Y_0A}_{=Y_0B_0=\mathbb{I}} + Y_1A_0B_0. \Leftrightarrow 0 = \mathbb{I}.$$

↯

The case with $B \neq 0$ a zero divisor for Y_1 can be contradicted in a similar way by comparing Equations (5.17) and (5.21) in the case of $A = B_0$. ↯

Thus, Case (5.45) cannot hold.

Case 2: Next, consider $Y_1 = 0$ and $A_1, B_1 \neq 0$, i.e. Case (5.46).

By combining Equations (5.31) and (5.32), we get

$$Y_0B_1 + \underbrace{Y_1B_0^2}_{=0} = YB_1 + \underbrace{Y_1B_0B}_{=0} + \underbrace{Y_1B_1A_0}_{=0}$$

and thus $Y = Y_0 = B_0^{-1}$ as $B_1 \neq 0$. Alternatively, B_1 is a zero divisor. First, assume $Y = Y_0 = B_0^{-1}$- Following from Equations (5.35) and (5.36), we have

$$\begin{aligned}
\overbrace{Y_0}^{=Y} B + \underbrace{\overbrace{Y}^{=B_0^{-1}} B_0}_{=\mathbb{I}} + \overbrace{Y_1B_0A_0}^{=0} &= 0 \\
YB + \underbrace{Y_1B_0A_1}_{=0} + \underbrace{Y_1BA_0}_{=0} &= 0 \\
\Leftrightarrow \quad \mathbb{I} &= 0,
\end{aligned}$$

$\lightning$

which is a contradiction. Thus, Case (5.46) cannot hold either.

In the case of B_1 a zero divisor, the following additional relations hold:

$$\begin{aligned}
0 &= XA_1 + YA_1 \\
&= YA_1A \\
&= X_1 + XA_0 + Y_0A_1 + YA_0 \\
&= XA \\
&= YB \\
&= XB + YA \\
&= Y_0B + YB_0.
\end{aligned}$$

By rearranging we get $X = -A(B_0^{-1})^2, Y = -B(B_0^{-1})^2$. By $XA_1 + YA_1 = 0$, either $A_1 = 0$, which was considered earlier already, $(B_0^{-1})^2 = 0$, $B_0^{-1} = A_1$ or A_1 is a zero divisor.

If we have $(B_0^{-1})^2 = 0$, then both $X, Y = 0$ and $X_1 = B_0^{-1}A_1$ as in the theorem.

If $B_0^{-1} = A_1$, $X = Y = X_1 = 0$ which is the linear case, as $Y_1 = 0$.

So assume A_1 is a zero divisor, then $X_1 = A_0(B_0^{-1})^2(A+B)$, as in the theorem. Furthermore, by $XA = YB = 0, A^2 = B^2 = 0$ as $(B_0^{-1})^2 = 0$ would be the linear case. However, by $0 = XB + YA$, $AB = 0$ as well.

Case 3: In particular, with the same reasoning as in the previous case, Case (5.47) does not hold either.

Case 4: Next, let us consider $A_1^2 = B_1^2 = 0$ and $A_1, B_1, Y_1 \neq 0$, i.e. Case (5.48). First consider Equations (5.18) and (5.19). From this, we get $B_1 = B$. Furthermore, from (5.23), we get $A = A_1$ and so, by (5.27), it holds that $A = B$. Thus, from (5.29),$A = B = A_1 = B_1 = B_0$ follows. However, from (5.31),

$$0 = Y_0B_1 + Y_1 \underbrace{B_0^2}_{=0} = Y_0B_1 = B_0^1B_0 = \mathbb{I}.$$

$$\lightning$$

This is again a contradiction and thus Case (5.48) cannot hold.

Case 5: Finally, consider Case (5.49). In this case we assume $A_1 = B_1 = Y_1 = 0$.

Thus, Equation (5.17) can be written as

$$0 = XA + \underbrace{Y_1A_0A}_{=0} + \underbrace{Y_1A_1B_0}_{=0} = XA. \tag{5.50}$$

Hence, we have two cases, namely either at least one of A, X is equal to 0 or both $A, X \neq 0$ and $XA = 0$.

First, consider the case where at least one of A, X is zero. Then Equation (5.21) results in

$$0 = XB_0 + Y_0A + \underbrace{Y_1A_0B_0}_{=0} \Leftrightarrow XB_0 = -Y_0A.$$

As both $B_0, Y_0 \neq 0$ by Equations (5.43) and (5.44) we have $X = A = 0$, since at least one of them is zero by Equation (5.50).

Furthermore, Equation (5.36) can be written as

$$0 = YB + \underbrace{Y_1B_0A_1}_{=0} + \underbrace{Y_1BA_0}_{=0} = YB.$$

Thus, either at least one of Y, B is zero or both are nonzero and $YB = 0$. However, Equation (5.35) gives

$$0 = Y_0B + YB_0 + \underbrace{Y_1B_0A_0}_{=0} = Y_0B + YB_0.$$

Since at least one of B, Y is zero and both $B_0, Y_0 \neq 0$, we must have $Y = B = 0$ and thus the linear case follows.

Instead, consider $Y, B \neq 0$ and $YB = 0$. By multiplying Equation (5.35) with B on both sides, we arrive at

$$Y_0B^2 = 0$$

and thus $Y = B$. In this case, we then have

$$f_a^{-1}(x) = A_0B_0a - B_0x - BA_0a^2 + Bxa = (Ba - B_0)(x - A_0a). \tag{5.51}$$

Now let us consider $A, X \neq 0$ and $XA = 0$. Then, with a similar reasoning as before, $Y = B = 0$ or $YB = 0$ with $Y, B \neq 0$. This gives us the following two possibilities:

$$\begin{aligned} f_a^{-1}(x) &= Aax + A_0B_0a - AA_0a^2 - B_0x \\ &= (Aa - B_0)(x - A_0a) \end{aligned} \tag{5.52}$$

$$\begin{aligned} f_a^{-1}(x) &= Aax + A_0B_0a - (AA_0 + BA_0)a^2 - B_0x + Bxa \\ &= (Aa - B_0)(x - A_0a) - BA_0a^2. \end{aligned} \tag{5.53}$$

Here, (5.52) is the case for $Y = B = 0$ and (5.53) for $YB = 0$, $Y, B \neq 0$, respectively.

Thus, the only three possible cases for non-commutative biquandles with quadratic down functions with commutative coefficients are given in Equations (5.51), (5.52) and (5.53).

q.e.d.

Proof of Theorem 5.6. By Lemma 5.7 (i) and (ii), the equations stated in Theorem 5.6 are the only possible forms.

In addition, considering the proofs of both part (i) and (ii) of Lemma 5.7, we see that the additional restrictions on the coefficients in the theorem hold true and thus the theorem follows. *q.e.d.*

5.2.2 Relations arising from Axioms

Using the axioms stated in Definitions 3.1 and 3.6, in this section we derive further restrictions on biquandle functions as in Theorem 5.6.

Remark. *We restrict ourselves to the case $A_1, B_1, C_1, D_1 = 0$ due to space and time constraints, as this does not lead to nice close-form functions but rather to a whole new theory. We aim to explore this further in future research.*

We will first consider Axiom (5.5). Using the quadratic up and down functions as in Theorem 5.6, we get:

$$A_0 a + B_0 a + A a^2 + B a^2 = C_0 a + D_0 a + C a^2 + D a^2.$$

Thus we have the first two relations,

$$A_0 + B_0 = C_0 + D_0 \tag{5.54}$$

and

$$A + B = C + D, \tag{5.55}$$

where at least one of A, B, C, D is nonzero and, by (5.55) and Theorem 5.6, we must have at least two of A, B, C, D nonzero. Otherwise, the theory collapses to the linear case.

From the other three axioms in Equations (5.6) to (5.8) we get the following Theorem 5.8.

Theorem 5.8 (Non-commutative quadratic Biquandle with commutative Coefficients; Axioms)**.** *Let BQ be a non-commutative biquandle. If the up and down functions are of quadratic form with quadratic inverse and commutative coefficients, they always have one of the following forms:*

Case 1:

$$f_a(b) = A(ab+ba)+A_0a+B_0b$$
$$f^b(a) = A(ab+ba)+B_0a+A_0b$$

or

$$f_a(b) = A(ab+ba+a+b)+b$$
$$f^b(a) = A(ab+ba+a+b)+a$$

or

$$f_a(b) = A(ab+ba+a+b)+b$$
$$f^b(a) = A(ab+Bba)+a$$

or

$$f_a(b) = A(ab+ba)+b$$
$$f^b(a) = A(ab+ba+a+b)+a$$

or

$$f_a(b) = A(ab+ba)+b$$
$$f^b(a) = A(ab+ba)+a$$

or

$$f_a(b) = Aab+Bba+A_0(a+b)+b \text{ with } A_0^2 = 0$$
$$f^b(a) = Aab+Bba+A_0(a+b)+a$$

Case 2c:

$$f_a(b) = Aab+A_0a+b \text{ with } A_0 \in \{A,0\}$$
$$f^b(a) = Aab+a+A_0b$$

Case 3c:

$$f_a(b) = Bba+A_0a+b \text{ with } A_0 \in \{B,0\}$$
$$f^b(a) = Bba+a+A_0b$$

Case 4:

$$f_a(b) = A_0a+b \text{ with } A_0 \in \{C,0\}$$
$$f^b(a) = C(ab-ba)+a+A_0b$$

or

$$f_a(b) = Bba+A_0a+b \text{ with } A_0 \in \{B,0\}$$
$$f^b(a) = Bab+a+A_0b$$

or

$$f_a(b) = Aab+A_0a+b \text{ with } A_0 \in \{A,0\}$$
$$f^b(a) = Aab+a+A_0b$$

or

$$f_a(b) = A(ab-ba)+A_0a+b \text{ with } A_0 \in \{A,0\}$$
$$f^b(a) = a+A_0b$$

Case 5a:

$$f_a(b) = (C+D)ba+b \text{ with } C \in \{0,D\}$$
$$f^b(a) = Cab+Dba+a$$

or

$$f_a(b) = (C+D)ab+b \text{ with } D \in \{0,C\}$$
$$f^b(a) = Cab+Dba+a$$

or

$$f_a(b) = Aab+Bba+b \text{ with } A \in \{0,B\}$$
$$f^b(a) = (A+B)ba+a$$

or

$$f_a(b) = Aab+Bba+b \text{ with } B \in \{0,A\}$$
$$f^b(a) = (A+B)ab+a$$

or

$$f_a(b) = A(ab+ba)+b$$
$$f^b(a) = C(ab+ba)+a$$

where in all cases $A^2 = B^2 = C^2 = D^2 = 0$ and all coefficients are elements of a commutative, associative ring. Additionally, the coefficients of b (in $f_a(b)$) and a (in $f^b(a)$) are invertible.

Proof. From the other three Axioms (5.6), (5.7) and (5.8), we get the following relations via the code in Appendix A with (5.55) and the fact that $A^2 = B^2 = C^2 = D^2 = 0$ the following relations:

$$\begin{align}
0 &= BB_0D \tag{5.56}\\
0 &= AA_0B \tag{5.57}\\
0 &= AA_0C \tag{5.58}\\
0 &= AB_0C \tag{5.59}\\
0 &= ABC \notag\\
0 &= A_0BD \notag\\
0 &= ABD \notag\\
0 &= A_0BC \notag\\
0 &= BCD_0 \tag{5.60}\\
0 &= BCD \notag\\
0 &= AA_0D \notag\\
0 &= A_0D_0 \notag\\
0 &= ACD_0 \tag{5.61}\\
0 &= ACD \notag\\
0 &= BDD_0 \notag\\
0 &= BCD \notag\\
0 &= ADD_0 \notag\\
0 &= ACC_0 \tag{5.62}\\
0 &= CDD_0 \notag\\
0 &= BC_0D \tag{5.63}\\
0 &= 2A_0CD \notag
\end{align}$$

$$\begin{align}
0 &= 2ABA_0 \notag\\
0 &= 2ABD_0 \notag\\
0 &= A_0^2C + A_0^2D \notag\\
0 &= AD_0^2 + BD_0^2 \notag\\
0 &= AA_0^2 + A_0^2B \notag\\
0 &= CD_0^2 + DD_0^2 \notag\\
BB_0 + A_0BB_0 &= BB_0C_0 \tag{5.64}\\
ABB_0 &= BB_0C \tag{5.65}\\
BB_0^2 &= BB_0 + BB_0D_0 \tag{5.66}\\
A_0^2 + A_0B_0 &= A_0C_0 \tag{5.67}\\
AA_0 + A_0BB_0 &= A_0C + AA_0C_0 \tag{5.68}\\
AA_0B_0 + AB_0 &= AB_0C_0 \tag{5.69}\\
ABB_0 &= ABC_0 \tag{5.70}\\
A_0B + AA_0B_0 &= A_0D + A_0BC_0 \tag{5.71}\\
ABB_0 &= AB_0D \notag\\
AB_0^2 &= AB_0D_0 + AB_0 \tag{5.72}\\
BC_0 + A_0B_0C &= BC_0^2 \tag{5.73}\\
B_0^2C &= B_0C + BC_0D_0 \tag{5.74}\\
BB_0C &= BCC_0 \notag\\
AC_0 + A_0B_0D &= AC_0^2 \notag\\
A_0B_0C + AD_0 &= A_0C + AC_0D_0 \notag\\
AB_0D &= AC_0D \notag\\
B_0^2D &= AC_0D_0 + B_0D \tag{5.75}
\end{align}$$

$$\begin{aligned} BB_0D &= BC_0D & CC_0D &= B_0CD \\ BD_0+A_0B_0D &= A_0D+BC_0D_0 & CD_0+C_0DD_0 &= B_0CD_0+AD_0 \quad (5.79) \\ CC_0^2 &= CC_0+A_0CC_0 \quad (5.76) & CC_0D &= AC_0D \\ CC_0+CC_0D_0 &= B_0CC_0 \quad (5.77) & C_0D_0+D_0^2 &= B_0D_0 \\ CC_0D &= BCC_0 \quad (5.78) & C_0D+C_0DD_0 &= B_0C_0D \quad (5.80) \\ C_0^2D &= C_0D+A_0C_0D & CC_0D_0+DD_0 &= BD_0+B_0DD_0 \quad (5.81) \end{aligned}$$

We first notice that by Equations (5.64), (5.66), (5.72), (5.74), (5.76), (5.77) and (5.80), Equation (5.54) and the invertibility of B_0, C_0, we get

$$X = X\left(C_0 - A_0\right) = X\left(B_0 - D_0\right), \tag{5.82}$$

where $X = A, B, C, D$ and so, as $B_0 - D_0 = C_0 - A_0 = \mathbb{I}$ and $B_0, C_0 \neq 0$, either $C_0 = A_0 + \mathbb{I}$, $B_0 = D_0 + \mathbb{I}$ or $B_0 = \mathbb{I}, D_0 = 0$, $C_0 = \mathbb{I}, A_0 = 0$ or a combination of them. Furthermore, by Equations (5.65), (5.70) and (5.78) and the invertibility of B_0, C_0, it holds

$$AB = BC = AD = CD \tag{5.83}$$

and by comparing Equations (5.62) and (5.63), we note that $AC = BD$.

Using this in Equations (5.56), (5.58), (5.61) and (5.62), we get

$$ACX_0 = 0,$$

where $X_0 = A_0, B_0, C_0, D_0$. With Equations (5.57) and (5.60) it follows

$$ABA_0 = ABD_0 = 0.$$

However, since $B_0, C_0 \neq 0$, $AC = BD = 0$ holds. Thus, we have one of the following cases:

Case 1: $A = C$, $B = D$, all nonzero

Case 2: $A = C$ and

Case 2a: $B = 0, A, C, D \neq 0$

Case 2b: $D = 0, A, B, C \neq 0$

Case 2c: $B, D = 0, A, C \neq 0$

Case 3: $B = D$ and

Case 3a: $A = 0, B, C, D \neq 0$

Case 3b: $C = 0, A, B, D \neq 0$

Case 3c: $A, C = 0, B, D \neq 0$

Case 4: Case 4a: $A, B = 0, C, D \neq 0$

Case 4b: $A, D = 0, B, C \neq 0$

Case 4c: $B, C = 0, A, D \neq 0$

Case 4d: $C, D = 0, A, B \neq 0$

Case 5: Alternatively, due to Equation (5.54) we have the following two cases

Case 5a: $A_0 = D_0 = 0$, $B_0 = C_0$

Case 5b: $A_0 = B_0 = C_0 = D_0$.

We shall now discuss all of these cases:

Case 1: $A = C$, $B = D$, all nonzero:

Then by Equation (5.70), either $A = B$ or $B_0 = C_0$ as A, B, C, D are non-zero. First, assume $A = B$. Then, by Equations (5.68) and (5.81), either $B_0 = C_0$ or one of the following:

(i) $A_0 = D_0 = A$

(ii) $A_0 = A$, $D_0 = 0$

(iii) $D_0 = A$, $A_0 = 0$

(iv) $A_0 = D_0 = 0$

First, assume $B_0 = C_0$. Then, by Equation (5.54), $A_0 = D_0$, as required.

Thus, one of the following must hold:

(i) $A_0 = D_0 = A$.

Then, by Equations (5.54) and (5.82), $B_0 = C_0 = \mathbb{I} + A$, as required.

(ii) $A_0 = A$, $D_0 = 0$.

Then, by Equation (5.82), $B_0 = \mathbb{I}$ and $C_0 = A + \mathbb{I}$, as required.

(iii) $D_0 = A$, $A_0 = 0$.

Then, by Equation (5.82), $C_0 = \mathbb{I}$ and $B_0 = a + \mathbb{I}$, as required.

(iv) $A_0 = D_0 = 0$.

Then, by Equation (5.82), $B_0 = C_0 = \mathbb{I}$, as required.

Alternatively, let $B_0 = C_0$. Then, by Equation (5.54), $A_0 = D_0$. Thus, by Equation (5.67), $A_0^2 = 0$ and so $B_0^2 = B_0 + A_0$ (see Equation (5.82)), as required.

Case 2: $A = C$ and

Case 2a: $B = 0, A, C, D \neq 0$:

By Equation (5.55), we have $A = A + D$ and so $D = 0$, which is a contradiction. ↯

Case 2b: $D = 0, A, B, C \neq 0$:

By Equation (5.55), we have $A + B = A$ and so $B = 0$, which is a contradiction. ↯

Case 2c: $B, D = 0, A, C \neq 0$:

By Equation (5.71), $AA_0B_0 = 0$ and so, by Equation (5.69),

$$AB_0C_0 = AB_0 \Leftrightarrow A = AC_0$$

follows as B_0 is invertible. Hence, with Equation (5.68),

$$AA_0C_0 = 0 \Leftrightarrow AA_0 = 0.$$

Similarly, with Equations (5.75) and (5.77), we get $A = AB_0$ and by Equation (5.81), $AD_0 = 0$.

Now, as $A \neq 0$ (since otherwise we have the linear case), $B_0 = C_0 = \mathbb{I}$. Furthermore, either $A_0 = A = D_0$ or $A_0 = 0 = D_0$, as required.

Case 3: $B = D$

Case 3a: $A = 0$, $B, C, D \neq 0$:

By Equation (5.55), we have $B = B + C$ and so $C = 0$, which is a contradiction. ↯

Case 3b: $C = 0$, $A, B, D \neq 0$:

By Equation (5.55), we have $A + B = B$ and so $A = 0$, which is a contradiction. ↯

Case 3c: $A, C = 0$, $B, D \neq 0$:

Similarly to Case 2c, by Equations (5.71) and (5.73), $BC_0 = B$ and $BA_0 = 0$. By Equations (5.79) to (5.81), $BD_0 = 0$ and $BC_0 = 0$ and so, $B_0 = C_0 = \mathbb{I}$ and either $A_0 = D_0 = 0$ or $A_0 = D_0 = B$, as required.

Case 4: Case 4a: $A, B = 0$, $C, D \neq 0$:

By Equation (5.55), we have $C = -D$. Then, by Equation (5.75), $D = DB_0$ and so, as both are nonzero, $B_0 = \mathbb{I}$. Furthermore, by Equations (5.68) and (5.76), we have $DA_0 = 0$ and $DC_0 = D$ and so $C_0 = \mathbb{I}$. Hence, by Equation (5.82), $DD_0 = 0$. Thus, by Equation (5.54), either $D_0 = A_0 = 0$ or $A_0 = D_0 = D$, as required.

Case 4b: $A, D = 0$, $B, C \neq 0$:

By Equation (5.55), we have $C = B$. Similarly as in Case 4a, we have $B_0 = C_0 = \mathbb{I}$ and $A_0 = D_0$ with either $A_0 = C$ or $A_0 = 0$.

Case 4c: $B, C = 0$, $A, D \neq 0$:

By Equation (5.55), we have $A = D$. Similarly as in Case 4a, we have $B_0 = C_0 = \mathbb{I}$ and $A_0 = D_0$ with either $A_0 = D$ or $A_0 = 0$.

Case 4d: $C, D = 0$, $A, B \neq 0$:

By Equation (5.55), we have $A = -B$. Similarly as in Case 4a, we have $B_0 = C_0 = \mathbb{I}$ and $A_0 = D_0$ with either $A_0 = A$ or $A_0 = 0$.

Case 5: Case 5a: $A_0 = D_0 = 0$, $B_0 = C_0$:

In this case we have

$$XB_0 = X$$

for $X = A, B, C, D$ by Equation (5.82). Thus, $B_0 = \mathbb{I}$ as at least two of A, B, C, D are non-zero and by Equations (5.56) and (5.59), $BD = AC = 0$.

Assume without loss of generality that $A = 0$. Then $B = C + D$ and $BD = CD + DD = CD = 0$. Thus, either $C = 0$ or $C = D$. If $C = 0$, $B = D$. If $C = D$, $B = 2D$.

Similarly, if $B = 0$, $A = 2C$ or $A = C$, if $C = D$ or $D = 0$, respectively.

Similarly, if $C = 0$, $D = 2B$ or $D = B$, if $B = A$ or $A = 0$, respectively.

Similarly, if $D = 0$, $C = 2A$ or $C = A$, if $A = B$ or $B = 0$, respectively.

If all are non-zero, $B = D$ and $A = C$.

Case 5b: $A_0 = B_0 = C_0 = D_0$:

Then by Equation (5.82), $X = 0$ for all $X = A, B, C, D$, which is the linear case and hence a contradiction. $\lightning$

q.e.d.

Part III

Biquandle Homology

Chapter 6

Quandle and Biquandle Homology

In this chapter, we define quandle homology, as, for example, stated in [NP09], and extend this definition to biquandle homology.

6.1 Homology

In this section we state the definition of homology as given in [Bre93].

Definition 6.1 (p-simplex). *Let $\mathbb{R}^\infty$ have the standard basis $(e_i)_{i\leq 0}$. Then the* standard p-simplex *is given by* $\Delta_p = \left\{x = \sum_{i=0}^{p} \lambda_i e_i \mid \sum \lambda_i = 1, 0 \leq \lambda_i \leq 1\right\}$. *The λ_i are called* barycentric coordinates.

Definition 6.2 (Affine Singular n-simplex; Face Map). *Given points $v_o, \ldots, v_n \in \mathbb{R}^n$, let $[v_o, \ldots, v_n]$ denote the map $\Delta_n \to \mathbb{R}^n$, taking $\sum_i \lambda_i e_i \mapsto \sum_i \lambda_i v_i$, which is called an* affine singular n-simplex.

The affine singular simplex $\left[e_o, \ldots, \widehat{e_i}, \ldots, e_p\right] : \Delta_{p-1} \to \Delta_p$, where $e_1, \ldots, \widehat{e_i}, \ldots, e_n$ means every e_k except e_i, is called the ith face map *and is denoted by F_i^p.*

Definition 6.3 (Boundary Map). *If X is a topological space, then a* singular p-simplex *of X is a map $\sigma_p : \Delta_p \to X$. The* singular p-chain group $\Delta_p(X)$ *is the free abelian group based on the singular p-simplices.*

If $\sigma : \Delta_p \to X$ is a singular p-simplex, then the ith face *of σ is given by $\sigma^{(i)} = \sigma \circ F_i^p$. The* boundary *of σ is $\partial_p \sigma = \sum_{i=0}^{p} (-1)^i \sigma^{(i)}$, a (p-1)-chain. If $c = \sum_\sigma n_\sigma \sigma$ is a p-chain, then we put $\partial_p c = \partial_p \left(\sum_\sigma n_\sigma \sigma\right) = \sum_\sigma n_\sigma \partial_p \sigma$. That is, ∂_p is extended to $\Delta_p(X)$ so as to be a homomorphism,*

$$\partial_p : \Delta_p(X) \to \Delta_{p-1}(X).$$

In addition, note that $\partial_p \partial_{p+1} = 0$. *This is commonly also denoted as* $\partial^2 = 0$.

Definition 6.4 (Chain Complex; Homology Group). *We define the following chain complex*

$$\dots \longrightarrow \Delta_{p+1}(X) \xrightarrow{\partial_{p+1}} \Delta_p(X) \xrightarrow{\partial_p} \Delta_{p-1}(X) \longrightarrow \dots$$

Chains in the kernel of ∂_p *are called* p-cycles *or* $Z_p(X)$ *and chains in the image of* $\partial_{p+1}(X)$ *are called* p-boundaries *or* $B_p(X)$.

Then the pth singular homology group *of a space* X *is defined as*

$$H_p(X) = Z_p(X)/B_p(X) = (\ker \partial_p)/(im\, \partial_{p+1})$$

6.2 Quandle Homology

In this section we state the definition of the quandle homology, as given in [NP09] for example.

Definition 6.5 (Rack, Degenerate, Quandle Chain Complex; L.E.S of Quandle Homologies).

(i) *For a given rack* X, *let* $C_n^R(X)$ *be the free abelian group generated by n-tuples* $(x_1, x_2, \dots, x_n)$, $x_i \in X$, *i.e.* $C_n^R(X) = \mathbb{Z}X^n = (\mathbb{Z}X)^{\otimes n}$. *Define a* boundary homomorphism $\partial^n : C_n^R(X) \to C_{n-1}^R(X)$ *by*

$$\partial^n(x_1, x_2, \dots, x_n) = \sum_{i=2}^{n} (-1)^i \left((x_1, \dots, \widehat{x_i}, \dots, x_n) - \left(x_1^{x_i}, x_2^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n \right) \right).$$

Then $(C_*^R(X), \partial^n)$ *is called a* rack chain complex *of* X *and we usually write* ∂ *instead of* ∂^n.

(ii) *Let* X *be a quandle. Then there is a subchain complex,* $C_n^D(X) \subset C_n^R(X)$, *generated by n-tuples* $(x_1, \dots, x_n)$ *where* $x_i = x_{i+1}$ *for some* i. *This subchain complex,* $(C_n^D(X), \partial)$ *is called a* degenerate chain complex *of a quandle* X.

(iii) *The* quotient chain complex $C_n^Q(X) = C_n^R(X)/C_n^D(X)$ *is also called the* quandle chain complex. *We now have the short exact sequence of chain complexes:*

$$0 \longrightarrow C_n^D(X) \longrightarrow C_n^R X \longrightarrow C_n^Q(X) \longrightarrow 0 \tag{6.1}$$

As shown by Litherland and Nelson in [LN03], this short exact sequence splits via the splitting map $\alpha : C_n^Q(X) \to C_n^R(X)$ *where*

$$\alpha(x_1, \ldots, x_n) = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}).$$

In particular, α *is a chain complex monomorphism and* $H_n^R(X) = H_n^D(X) \oplus \alpha_*(H_n^Q(X))$.

(iv) *The homology of rack, degenerate and quandle chain complexes are called* rack, degenerate *and* quandle homology*, respectively. We then have the long exact sequence of the homology of quandles:*

$$\ldots \longrightarrow H_n^D(X) \longrightarrow H_n^R(X) \longrightarrow H_n^Q(X) \longrightarrow H_{n-1}^D(X) \longrightarrow \ldots$$

(v) *For an abelian group* G*, define the chain complex* $C_*^Q(X;G) = C_*^Q \otimes G$*, with* $\partial = \partial \otimes id$. *The groups of cycles and boundaries are denoted by* $\ker(\partial) = Z_n^Q(X;G) \subset C_n^Q(X;G)$ *and* $Im(\partial) = B_n^Q(X;G) \subset C_n^Q(X;G)$*, respectively. Then the nth* quandle homology group *of a quandle* X *with coefficient group* G *is defined as*

$$H_n^Q(X;G) = H_n(C_*^Q(X;G)) = Z_n^Q(X;G)/B_n^Q(X;G).$$

6.3 Biquandle Homology

The Definition of the Homology of a Quandle as in Definition 6.5 can be extended naturally to Biquandles.

Remark. *We are not aware of any complete definition in the literature. There is some mention in [Fen+14, Section 6.2] and there is a definition for Augmented Biracks in [Cen+14] but while Definition 6.6 is indeed the natural generalisation of rack/quandle homology, the only other mention we could find is in [Fen12b, Chapter 6]. There is also a definition in [Car+09, Section 3], but it is incomplete and thus not very usable for our purposes.*

Definition 6.6 (Birack, Degenerate, Biquandle Chain Complex; L.E.S of Biquandle Homologies).

(i) *For a given birack X, let $C_n^R(X)$ be the free abelian group generated by n-tuples $(x_1, x_2, \ldots, x_n)$, $x_i \in X$, i.e. $C_n^R(X) = \mathbb{Z}X^n = (\mathbb{Z}X)^{\otimes n}$. Define a* boundary homomorphism *$\partial : C_n^R(X) \to C_{n-1}^R(X)$ by*

$$\partial(x_1, x_2, \ldots, x_n) = \sum_{i=2}^{n} (-1)^i \left((x_1, \ldots, \widehat{x_i}, \ldots, x_n) - \left(x_1^{x_i}, x_2^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1 x_i}, \ldots, x_{n x_i}\right)\right).$$

Then $(C_*^R(X), \partial)$ is called a birack chain complex *of X.*

(ii) *Let X be a biquandle. Then there is a subchain complex, $C_n^D(X) \subset C_n^R(X)$, generated by n-tuples $(x_1, \ldots, x_n)$ where $x_i = x_{i+1}$ for some i. This subchain complex $(C_n^D(X), \partial)$ is called a* degenerate chain complex *of a biquandle X.*

(iii) *The* quotient chain complex *$C_n^Q(X) = C_n^R(X)/C_n^D(X)$ is also called the* biquandle chain complex. *We now have the short exact sequence of chain complexes:*

$$0 \longrightarrow C_n^D(X) \longrightarrow C_n^R X \longrightarrow C_n^Q(X) \longrightarrow 0 \tag{6.2}$$

We think this sequence splits usefully as in the quandle case; see Conjecture 6.7. It is clear that this sentence splits as every $x* \in C_n^R(X)$ is either in $C_n^Q(X)$ or $C_n^D(X)$, but never in both (or none).

(iv) *The homology of birack, degenerate and biquandle chain complexes are called* birack, degenerate *and* biquandle homology, *respectively. Thus, have the long exact sequence of the homology of biquandles:*

$$\ldots \longrightarrow H_n^D(X) \longrightarrow H_n^R(X) \longrightarrow H_n^Q(X) \longrightarrow H_{n-1}^D(X) \longrightarrow \ldots$$

(v) *For an abelian group G, define the chain complex $C_*^Q(X;G) = C_*^Q \otimes G$, with $\partial = \partial \otimes id$. The groups of cycles and boundaries are denoted by $\ker(\partial) = Z_n^Q(X;G) \subset C_n^Q(X;G)$ and $Im(\partial) = B_n^Q(X;G) \subset C_n^Q(X;G)$, respectively. Then the nth* biquandle homology group *of a biquandle X with coefficient group G is defined as*

$$H_n^Q(X;G) = H_n(C_*^Q(X;G)) = Z_n^Q(X;G)/B_n^Q(X;G).$$

6.3.1 Splitting Biquandle Homology Conjecture

We had a closer look at the proof in [LN03, Chapter 2] and think it adaptable enough to state the following conjecture.

Conjecture 6.7 (The Biquandle Chain Complexes split). *The short exact sequence of (biquandle) chain complexes*

$$0 \longrightarrow C_n^D(X) \longrightarrow C_n^R X \longrightarrow C_n^Q(X) \longrightarrow 0$$

splits usefully.

Chapter 7

Homology Computation

In this chapter we present an algorithm that calculates biquandle and quandle homologies. First, we discuss the theory on which the algorithm is based on. After this, we describe the algorithm used and give an example of this algorithm. We then present notes on the implementation before the chapter finishes with background on the Smith Normal Form of a matrix.

7.1 Theory

We introduce preliminaries and define the algorithm which is the basis of [FW16] afterwards.

7.1.1 Preliminaries

This section is based on [EH10] and conversations with Roger Fenn, see e.g. [Fen14]. A more gentle introduction can be found in [Zom05].

Let F, G be two linear boundary maps, where $F : U \to V$, $G : V \to W$ and $G \circ F = 0$. As we are only interested in Homology with coefficients in $\mathbb{Z}$, let $U, V, W = \mathbb{Z}^{\{p,q,r\}}$, respectively. Furthermore, assume that F, G are defined by $(p \times q)$ and $(q \times r)$ matrices, respectively. In other words, $F = (f_{ij})$ and $G = (g_{ij})$ with $f_{ij}, g_{ij} \in \mathbb{Z}$. Thus, we have the following:

$$\begin{array}{ccccc} \mathbb{Z}^p & \overset{F}{\to} & \mathbb{Z}^q & \overset{G}{\to} & \mathbb{Z}^r \\ 1 \times p & p \times q & 1 \times q & q \times r & 1 \times r \end{array}$$

Now let $B = \mathrm{im}(F)$ and $Z = \ker(G)$. Since $B \subset Z$, we can define $H := Z/B$, the homology group of this sequence.

Note that the notation n_{ij} and $n_{i,j}$ is used interchangeably in this chapter. Let $\boldsymbol{r}_i$ denote the ith row of F, $i \in \mathbb{Z}_p$. Then we can write

$$F = \left(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_p\right)^T$$

and since $\boldsymbol{e}_i F = \boldsymbol{r}_i$, we see that B is in the row space of F, i.e. $B \in span(\boldsymbol{r}_i)$. Similarly, let $\boldsymbol{c}_i^T$ denote the ith column of G, $i \in \mathbb{Z}_r$. Then

$$G = \left(c_1^T, c_2^T, \ldots, c_r^T\right).$$

From this, it follows that $Z = \ker(G) = \{\lambda | \lambda G = 0\}$, i.e. $Z^{\perp} = span(\boldsymbol{c}_i)$.

7.1.2 Algorithm

Next, we present the individual steps in the calculation of quandle and biquandle homology.

Let $X \in \mathbb{Z}^{q \times q}, Y \in \mathbb{Z}^{r \times r}$ be elementary matrices such that

$$XGY = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

where D is a $p \times p$ invertible matrix, p is the rank of G and all of the other matrix entries are zero. Let $z_i = e_i X$, then the last $q - p$ rows of X are a basis of Z. Since $\ker(G) \supset \operatorname{im}(F)$, we can find unique $n_{ij} \in \mathbb{Z}, i = 1, \ldots, p, j = 1, \ldots, q - p$ such that

$$\boldsymbol{r}_i = \sum_{j=1}^{q-p} n_{ij} \boldsymbol{z}_j$$

or, equivalently,

$$F = N * X_{(p)}^{(q)}$$

where $X_{(p)}^{(q)}$ denotes the last $q - p$ rows of X and $N = (n_{ij})$. Thus, the homology $H = Z/B$ has N as a presentation matrix as an abelian group.

Reducing N to its Smith Normal Form,

$$S_N = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}$$

where $\Delta = \text{diag}(1, \ldots, 1, d_1, \ldots, d_k)$ is a diagonal matrix of size $s \times s$, with $1 < d_1 | d_2 | \cdots | d_k$.

Then the homology is

$$H \cong \mathbb{Z}^{p-s} \oplus \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k}.$$

7.2 Quandle Homology Calculation Example

As an example, we will calculate the second quandle homology group of the Dihedral Quandle of order 3, namely $H_2^Q\{R_3\}$. The Dihedral Quandle of order 3 is defined over $\mathbb{Z}_3$ with the quandle operation $x^y = 2 * y - x \mod 3$. More generally, the Dihedral Quandle of order n is defined over $\mathbb{Z}_n$ with $x^y = 2 * y - x \mod n$.

The quandle homology has been defined in Definition 6.5 and we note that degenerate words are put equal to zero.

Here, we have $U = \{010, 012, 020, 021, 101, 102, 120, 121, 201, 202, 210, 212\}$, $V = \{01, 02, 10, 12, 20, 21\}$ and $W = \{0, 1, 2\}$, the words of length three, two and one, respectively. This leads to the following two maps,

$$G = \begin{array}{c} \\ 01 \\ 02 \\ 10 \\ 12 \\ 20 \\ 21 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \end{array}$$

and

$$F = \begin{array}{c} \\ 010 \\ 012 \\ 020 \\ 021 \\ 101 \\ 102 \\ 120 \\ 121 \\ 201 \\ 202 \\ 210 \\ 212 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 02 & 10 & 12 & 20 & 21 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

where we show the words as row names and the boundaries as column names. As an example,

$$\partial(01) = 1 - 0 - 1 + 2 = -1 \cdot 0 + 1 \cdot 2.$$

After Gaussian Elimination, this results in the following elementary matrix X. Here, the matrix D is not in the lower right but in the upper left corner.

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Y$$

Thus, as $\rho = 2$, the last 4 rows are the required basis.

In addition, we calculate a basis for the row space of F:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

plus 8 rows of zeros. Using standard calculation, we then find the matrix N:

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

plus 8 rows of zeros, which has the following Smith Normal Form:

$$S_N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with an additional 8 rows of zeros.

As there are only ones in the diagonal, it holds that $H_2^Q(R_3) = \emptyset$.

7.3 Implementation

We have programmed the algorithm from Section 7.1.2 in R [R C13] with the help of some third-party libraries [BM14a; VR02; Bor14]. The corresponding paper, [FW16], is accepted for publication in a peer reviewed Open Access journal, the Journal of Open Research Software (JORS). We reproduced the paper in Appendix C.

7.3.1 S Test

In addition to the above mentioned algorithm we have implemented S Test. This may be used to test whether a given set in combination with an up and a down action gives rise to a quandle or biquandle. For this one has to test if:

1. The Switch, the up and the down action (i.e. S, f_a, f^a) are all bijective.

2. The Yang-Baxter Equation $S_1S_2S_1 = S_2S_1S_2$ holds, where $S_1(a,b,c) = (S(a,b),c)$ and $S_2(a,b,c) = (a,S(b,c))$.

Implementation of the Tests

```
input : k, an integer
output: TRUE/FALSE
1 XSquare = allCombinations(k);
2 SX = XSquare[,2:1];
3 SX[,2] = UpAction(XSquare[,1],SX[,1],k);
4 XSquare[, 2] = DownAction(SX[, 1], XSquare[, 1], k);
5 permutationsS = checkpermutations(SX);
6 permutationsf = checkf(SX, k, XSquare);
7 permutationsg = checkf(XSquare, k, SX);
8 YangBaxter = checkYB(SX, k, XSquare);
9 print(paste0("The permutation checks hold that S is ", permutationsS, ", f is ",
  permutationsf, " and g is ", permutationsg, " and that the Yang-Baxter check holds ",
  YangBaxter, "."));
10 return(all(permutationsS,permutationsf,permutationsg,YangBaxter));
```

Figure 7.1: S Test Algorithm

Here k is the order of the Biquandle in question. *Upaction* and *downaction* calculate the up and down function for the two elements and the order of the biquandle. Note that in the package it is by default set up for the dihedral quandle. *checkpermutations* checks that the switch function as defined in Definition 3.1 is bijective. *Checkf* does the same for the up and down functions while *checkYB* checks if the Yang-Baxter Equation (3.3) holds.

XSquare can be thought of as the left side of the sideways map while **SX** is the right side. In other words, $XSquare = (b_a, a) \to (a^b, b) = SX$ and this is calculated in rows 2-4.

```
input : B, a matrix
C = unique(B);
if all(C != B) then
  | return(FALSE);
end
return(TRUE);
```

Figure 7.2: checkpermutations

The *checkpermutations* function checks if all rows of a matrix are unique.

```
input : SX, a matrix, XSquare, a matrix, k, an integer
S = unique(cbind(XSquare[, 1], SX[, 1:2]));
if nrow(S) != (k * k) then
    return(FALSE);
end
return(TRUE);
```

Figure 7.3: checkf

The *checkf* works similarly as the *checkpermutations* function. It tests the biquandle functions rather than the sideways map.

```
input : S, a matrix, X, a matrix, k, an integer
for i = 0,...,(k−1) do
    for j = 1,...,nrow(S) do
        LHS <- c(i, S[j, ]);
        RHS <- LHS;
        LHS[1:2] <- findSresult(LHS[1:2], X, S);
        LHS[2:3] <- findSresult(LHS[2:3], X, S);
        LHS[1:2] <- findSresult(LHS[1:2], X, S);
        RHS[2:3] <- findSresult(RHS[2:3], X, S);
        RHS[1:2] <- findSresult(RHS[1:2], X, S);
        RHS[2:3] <- findSresult(RHS[2:3], X, S);
        if !all(LHS == RHS) then
            return(FALSE);
        end
    end
end
return(TRUE);
```

Figure 7.4: checkYB

The *checkYB* function calculates both sides of the Yang Baxter equation and compares them. If they agree, it returns TRUE, otherwise FALSE.

input : x, a matrix with two columns, X, a matrix, k, an integer

```
for i = 1,...,nrow(X) do
    if all(X[i, ] == x) then
        match.id = i;
        break;
    end
end
return(S[match.id, ]);
```

Figure 7.5: findSresult

The sideways map for two given values is calculate here by the *findSresult* function.

7.4 Hermite and Smith Normal Form

For homology calculations, we need the Smith Normal Form, which is introduced in this section. The algorithm for its calculation is presented as well.

The Hermite Normal Form is named after *Charles Hermite* (1822 - 1901, [Her51]) and the Smith Normal Form is named after *Henry John Stephen Smith* (1826 - 1883, [Smi61]). They are both normal forms of a matrix.

7.4.1 Definition

This chapter is based on [Coh93] with some additional material from [JW09], where the authors use the lower triangular Hermite Normal Form.

Quadratic Matrices

Definition 7.1 (Hermite Normal Form; Cohen). *Let $M = \left(m_{i,j}\right)$ be a $n \times n$ matrix with integer coefficients. M is in* Hermite Normal Form *(HNF) if there exists an $r \leq n$ and a strictly increasing map $f : [r+q, n] \to [1, m]$ satisfying the following properties:*

(1) For $r+q \leq j \leq n$, $m_{f(j),j} \geq 1$, $m_{i,j} = 0$ if $i > f(j)$ and $0 \leq m_{f(k),j} < m_{f(k),k}$ if $k < j$.

(2) The first r columns of M are equal to 0.

Equivalently, a matrix is HNF if all $m_{i,j} = 0$ for $i > j$, the pseudo diagonal $m_{i,i}$ are positive and $m_{i,i} > m_{i,j}$ for $j > i$.

Definition 7.2 (Smith Normal Form; Cohen)**.** *An $n \times n$ matrix M is in* Smith Normal Form *(SNF) if M is a diagonal matrix with some non-negative integer coefficients such that $m_{i+1,i+1}|m_{i,i}$. In other words, it looks as follows:*

$$\begin{pmatrix} d_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & d_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & d_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & d_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_n \end{pmatrix}$$

where $d_{i+q}|d_i$ and if $d_k = 0$, $d_l = 0$ for $l > k$.

Rectangular Matrices

In [JW09], a more general definition for rectangular matrices with integer entries is given by:

Definition 7.3 (Hermite Normal Form; Jäger)**.** *A matrix $A = (A_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathbb{Z}^{m \times n}$ with rank r is in* Hermite Normal Form *if the following conditions hold:*

(i) $\exists i_1, \dots, i_r$ *with* $1 \leq i_1 < \dots < i_r \leq m$ *with* $A_{i_j,j} \in \mathbb{Z} - 0$ *for* $1 \leq j \leq r$. *We call those* $A_{i_j,j}$ pseudo diagonal elements.

(ii) $A_{i,j} = 0$ *for* $1 \leq i \leq i_j - 1, 1 \leq j \leq r$.

(iii) The columns $r+1, \dots, n$ are all zero.

(iv) $A_{i_j,l} = \psi(A_{i_j,l}, A_{i_j,j})$ *for* $1 \leq l < j \leq r$, *where for* $a, b \in \mathbb{Z},\ b \neq 0,\ \psi(a,b) := a - \left\lfloor \frac{a}{b} \right\rfloor b$.

The matrix A is in left Hermite Normal Form *if its transpose A^T is in Hermite Normal Form.*

Definition 7.4 (Smith Normal Form; Jäger). *A matrix $A = (A_{i,j})_{1\leq i\leq m, 1\leq j\leq n} \in \mathbb{Z}^{m\times n}$ with rank r is in* Smith Normal Form *if the following conditions hold:*

(i) A *is a diagonal matrix.*

(ii) $A_{i,i} \in \mathbb{Z} - 0$ *for* $1 \leq i \leq r$.

(iii) $A_{i,i} | A_{i+1,i+1}$ *for* $1 \leq i \leq r-1$.

(iv) $A_{i,i} = 0$ *for* $r+1 \leq i \leq \min(m,n)$.

Theorem 7.5 (Transformation Matrix for HNF). *Let $A \in \mathbb{Z}^{m\times n}$. Then there exists a matrix $V \in GL_n(\mathbb{Z})$ such that $H = A * V$ is in Hermite Normal Form, where H is uniquely determined. The matrix V is called the corresponding* transformation matrix *for the Hermite Normal Form.*

This theorem also holds for the left Hermite Normal Form with $U \in GL_m(\mathbb{Z})$. The proof for Theorem 7.5 can be found in [Her51].

7.4.2 Calculation - Algorithm

This section is again based on [JW09]. We use the Kannan-Bachem algorithm, see [KB79], for the calculation of the Smith Normal Form as described in the aforementioned paper.

Kannan-Bachem Algorithm

The Kannan-Bachem algorithm is given by:

input : $A \in \mathbb{Z}^{m,n}, U \in GL_m(\mathbb{Z}), V \in GL_n(\mathbb{Z})$
output: (A,U,V)

while *A is not in diagonal form* **do**
    (A,V)=HermiteNormalForm(A,V);
    (A^T,U)=HermiteNormalForm(A^T,U);
end
(A,U,V)=DiagonalMatrixToSmithNormalForm(A,U,V);

Figure 7.6: SmithNormalForm Algorithm

In short, the Kannan-Bachem Algorithm alternately computes the Hermite Normal Form and the left Hermite Normal Form, the Hermite Normal Form of the matrix transpose, until the matrix is in diagonal form. In the last step, the diagonal matrix is turned into Smith Normal Form via the DiagonalMatrixToSmithNormalForm algorithm.

In order to apply the Kannan-Bachem algorithm we first need to define the HermiteNormal-Form algorithm.

HermiteNormalForm Algorithm

```
   input  : A ∈ Z^{m,n}, V ∈ GL_n(Z)
   output: (A,V)
1  for t = 1,...,n do // Compute HNF of first t columns
2      r = 0;
3      for s = 1,...,m do
4          if A_{s,r+1} ≠ 0 ∨ A_{s,t} ≠ 0 then
5              r = r + 1;
6              i_r;
7              if t = r then
8                  if A_{s,t} ∉ Z then
9                      β = sign(A_{s,t});
10                     a_t = β * a_t;
11                     v_t = β * v_t;
12                 else
13                     RowOneGCD(A,V,s,r,t);
14                 end
15             end
16             for l = 1,...,r − 1 do // Here, ψ is as in Definition 7.3
17                 β = ψ(A_{s,l}, A_{s,r});
18                 a_l = a_l − β * a_r;
19                 v_l = v_l − β * v_r;
20             end
21             if t = r then
22                 t = t + 1;
23                 next;
24             end
25         end
26     end
27 end
```

Figure 7.7: HermiteNormalForm Algorithm

In Steps 4 - 13, the current pseudo diagonal element $A_{s,r}$ is computed. Rows 16 - 19 ensure that Condition (iv) from Definition 7.3 holds. When, after this, $t = r$, the new pseudo diagonal element is found and we go on to the next column.

For the sake of completeness and readability, we define the function RowToGCD separately.

RowOneGCD Algorithm

input : $A \in \mathbb{Z}^{m,n}$, $V = [v_1, \ldots, v_n] \in GL_n(\mathbb{Z}), i, j, l$ with $1 \leq i \leq m$, $1 \leq j < l \leq n$
output: (A,V)

1 **if** $A_{i,j} \neq 0 \vee A_{i,j} \neq 0$ **then**
2   **Compute** d :=gcd($A_{i,j}$,$A_{i,l}$) and u, v with $d = u * A_{i,j} + v * A_{i,l}$;
3     $\left[a_j, a_l\right] = \left[a_j, a_l\right] * \begin{pmatrix} u & -\frac{A_{i,l}}{d} \\ v & \frac{A_{i,j}}{d} \end{pmatrix}$;
4     $\left[v_j, v_l\right] = \left[v_j, v_l\right] * \begin{pmatrix} u & -\frac{A_{i,l}}{d} \\ v & \frac{A_{i,j}}{d} \end{pmatrix}$;
5 **end**

Figure 7.8: RowOneGCD Algorithm

In step 2, the algorithm computes gcd(x, y) and u, v with $gcd = ux + vy$ via the well-known Euclidean Algorithm. In steps 3 and 4, the jth and lth column in both the original and the transformation matrix are changed and switched in such a way that the conditions for $A_{i,j}$, $A_{i,l}$ in the HNF algorithm are fulfilled.

Finally the DiagonalMatrixToSmithNormalForm function needs to be defined in order to be able to execute the Kannan-Bachem algorithm.

DiagonalMatrixToSmithNormalForm Algorithm

input : $A \in \mathbb{Z}^{m,n}$ in diagonal form,
$U = [u_1, \dots, u_m]^T \in GL_m(\mathbb{Z}), V = [v_1, \dots, v_n] \in GL_n(\mathbb{Z})$
output: (A,U,V)

1 **for** $k = 1, \dots, \min(m,n) - 1$ **do**
2   **for** $l = \min(m,n) - 1, \dots, k$ **do**
3   **if** $A_{l,l} \nmid A_{l+1,l+1}$ **then**
4       $g = A_{l,l} * A_{l+1,l+1}$;
5       $A_{l,l} = \text{gcd}(A_{l,l}, A_{l+1,l+1})$;
6       $A_{l+1,l+1} = g / A_{l,l}$;
7   **Compute** $d := \text{gcd}(A_{l,l}, A_{l+1,l+1})$ and u, v with $d = u * A_{l,l} + v * A_{l+1,l+1}$;
8       $[u_l, u_{l+1}]^T = \begin{pmatrix} u & v \\ -\frac{A_{l+1,l+1}}{d} & \frac{A_{l,l}}{d} \end{pmatrix} * [u_l, u_{l+1}]^T$;
9       $[v_l, v_{l+1}] = [v_l, v_{l+1}] * \begin{pmatrix} 1 & -v * \frac{A_{l+1,l+1}}{d} \\ 1 & u * \frac{A_{l,l}}{d} \end{pmatrix}$;
10   **end**
11   **end**
12 **end**
13 **for** $l = 1 \dots, \min(m,n)$ **do**
14   **if** $A_{l,l} \neq 0$ **then**
15     $\beta = \text{sign}(A_{l,l})$;
16     $A_{l,l} = \beta * A_{l,l}$;
17     $v_t = \beta * v_t$;
18   **end**
19 **end**

Figure 7.9: DiagonalMatrixToSmithNormalForm Algorithm

In steps 4-6, the two neighbouring diagonal elements $A_{l,l}$, $A_{l+1,l+1}$ are replaced by their gcd and lcm, respectively. Steps 7-9 follow from the following equation from [Her51],

$$\begin{pmatrix} u & v \\ -\frac{A_{l+1,l+1}}{d} & \frac{A_{l,l}}{d} \end{pmatrix} \begin{pmatrix} A_{l,l} & 0 \\ 0 & A_{l+1,l+1} \end{pmatrix} \begin{pmatrix} 1 & -v * \frac{A_{l+1,l+1}}{d} \\ 1 & u * \frac{A_{l,l}}{d} \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & lcm(A_{l,l}, A_{l+1,l+1}) \end{pmatrix}.$$

These steps are then repeated until conditions (iii) and (iv) from Definition 7.4 hold true. After steps 13-17, (ii) holds as well and (i) is one of the requirements for this algorithm. According to [JW09], this algorithm does not need more than $\min(m,n)^2$ gcd computations.

Chapter 8

Conclusion

This thesis had two main foci, restrictions on biquandle functions in the linear and quadratic case and biquandle homology.

We first gave an introduction to Knot Theory in Chapter 2. This chapter started with an introduction to Classical Knot Theory in Section 2.1 which included the three different Reidemeister moves, see Section 2.1.1, Definition 2.3 and particularly Figures 2.4 to 2.6. Following this, we presented possibilities of describing knots with different codes in Section 2.2. This started with the simple Gauss Code, see Definition 2.6 which was gradually developed, including more and more information until culminating in the enhanced DT code, see e.g. Figure 2.10.

In Section 2.2.2, we considered knots arising from given DT codes and used this to introduce virtual crossings in Sections 2.3 and 2.3.1. After this, we enhanced the classical Reidemeister moves with virtual crossings in Section 2.3.2 and particularly Definition 2.7. These were in essence the classical Reidemeister moves as in Definition 2.3, but with virtual crossings instead. We also introduced a new Reidemeister 4 move, which combined both classical and virtual crossings in one move, see Figure 2.17.

The combinations introduced a new concept, forbidden Reidemeister moves in Definition 2.7 as moving a classical crossing over or under a virtual crossing is not permitted. Allowing the first forbidden move gives rise to a different knot theory, welded knots, allowing both forbidden moves causes everything to collapse to the unknot.

From this, in Section 2.4 we introduced other knot theories, incorporating doodle, classical, virtual and singular crossings in various combinations. We then generalised this to a generalised knot theory in Section 2.5, Definition 2.8. Reidemeister moves were similarly generalised in Section 2.5.1, Definition 2.9. In Section 2.5.2 this result was then applied to the different, previously introduced knot theories, see Section 2.4.

In Section 2.5.3, we considered the impact the different Reidemeister moves have on Orientation. A particularly rich example was the Reidemeister 3 move in Section 2.5.3 and for which we found in Theorem 2.10 that only braid-like moves need to be considered, a result which shortened the necessary notation, see Figure 2.33. For Reidemeister 3 moves with only one crossing type, this result was further improved:

Theorem 2.14 (Equivalent R_3 Moves; one Crossing)**.** *If only one type of crossing is involved in an R_3 move, then the only possible R_3 moves are $R_3(+,+;+)$ and $R_3(+,+;-)$.*

We finished Chapter 2 with examples of Reidemeister 3 moves in different knot theories, which can be found in Section 2.5.4.

After this we defined biquandles in Chapter 3. First, in Section 3.1 we defined the sideways map and switch:

Definition 3.1. *We first define the* sideways map*, $F : X \to X$ from one algebraic structure X to itself, preserving structure on X if there exists one. F is defined by two other functions $f_x, f^x : X \to X$, written $a \mapsto a_x, a^x$ for all $a, x \in X$ via $F(a,b) = (f_a(b), f^b(a))$. We require both F and f_x, f^x to be bijective. In the expressions a_x and a^x the element $a \in X$ is well defined since for $a_1, a_2, x \in X$,*

$$\begin{aligned} a_1^x = a_2^x \quad &\Rightarrow f^x(a_1) = f^x(a_2) \quad \Rightarrow a_1 = a_2 \\ (a_1)_x = (a_2)_x \quad &\Rightarrow f_x(a_1) = f_x(a_2) \quad \Rightarrow a_1 = a_2. \end{aligned}$$

Furthermore, we define the switch *corresponding to F by $S : X \to X : S(b_a, a) \to (a^b, b)$ or $(f_a(b), a) \mapsto (f^b(a), b)$.*

We also showed that both of the switch and sideways map as well as their functions, f_a, f^a, s_x, s^y, are all bijective and gave a graphical definition in Figure 3.1 where we defined $S(x,y) := (s^x(y), s_y(x))$.

Following this, we explored the relationship between the Reidemeister moves and the switch and sideways maps in Section 3.2, from which the Yang-Baxter Equation (3.3) naturally arose. We then considered examples of the Yang-Baxter Equation in Section 3.3 for knot theories with involutive crossings (Section 3.3.1) and only one type of crossing (Section 3.3.2).

We further examined those with two types of crossings (Section 3.3.3) of which both (Section 3.3.3), exactly one (Section 3.3.3) or no crossing was involutive (Section 3.3.3).

We were then in position to give the definition of biquandles and quandles in Section 3.4, where quandles are biquandles with one of the function being the identity.

Definition 3.6 (Definition Birack, Biquandle). *Let X be a set with two binary functions*

$$f_a, f^a : X \times X \to X.$$

(X, f_a, f^a) *is a* birack *if the two functions define a switch map (as defined in Definition 3.1) on X and they fulfil the requirements arising from the R_3 move, see Section 3.2.3 and particularly Equation* (3.3). *Thus, the following requirements must hold:*

- $a^{bc_b} = a^{cb^c}$, $c_{ba^b} = c_{ab_a}$, ${b_a}^{c_a} = {b^c}_{a^c}$ *(From Equation* (3.6)*, Section 3.3.2)*
- f_a, f^b *must be invertible (From Section 3.1)*

X *is a* biquandle *if additionally the following requirement hold:*

- $a^a = a_a$ *(From Section 3.2.1)*

The Chapter was finished with the calculation of generators and relations of biquandles of certain welded knots, see Table 3.1 in Section 3.5.

After two introductory chapters, the second part on restrictions on biquandle functions started with the linear case in Chapter 4. We first introduced general quaternions in Section 4.1, and here particularly Definition 4.2, before calculating restrictions or relations on biquandles with linear functions in Section 4.2. In particular, let $A, B, C, D \in R$, a non-commutative, associative ring of some sort. This gives us the general form

$$\begin{aligned} f^b(a) &= a^b &&= Ca + Db \\ f_a(b) &= b_a &&= Aa + Bb. \end{aligned}$$

Equivalently, this can be written in matrix notation:

$$\begin{pmatrix} f_a(b) \\ f^b(a) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

For these, we first calculated relations based on the biquandle axioms in Section 4.2.1. We also calculated relations arising from invertibility in Section 4.2.2, where we both inverted directly to a linear function (Equations (4.20) and (4.21)) and used matrix inversion (Equations (4.22) to (4.25)). Then the chapter finished with an application of the results to general quaternions in Section 4.2.3 and particularly Figure 4.2.

The quadratic case was considered next in Chapter 5. We first considered the case of commutative biquandles, where the biquandle functions have commutative coefficients in Section 5.1. Considering the biquandle axioms, we calculated restrictions as follows:

Theorem 5.2 (Commutative quadratic Biquandle with commutative Coefficients; Axioms)**.** *Let BQ be a commutative biquandle with a structure as described before. If the up and down functions are of quadratic form and have commutative coefficients, they are always of the form*

$$f_a(b) = A_0 * a + B_0 * b + \lambda * a \cdot b \tag{5.9}$$

$$f^b(a) = C_0 * a + D_0 * b + \lambda * a \cdot b \tag{5.10}$$

and the following relations

$$\begin{aligned} A_0 D_0 &= 0 \\ \lambda^2 A_0 &= 0 \\ \lambda A_0^2 &= 0 \\ \lambda^3 &= 0 \\ \lambda^2 D_0 &= 0 \\ \lambda D_0^2 &= 0 \end{aligned} \qquad \begin{aligned} \lambda A_0 B_0 &= \lambda A_0 C_0 \\ \lambda^2 B_0 &= \lambda^2 C_0 \\ \lambda B_0 D_0 &= \lambda C_0 D_0 \\ \lambda B_0^2 &= \lambda B_0 D_0 + \lambda B_0 \\ \lambda C_0^2 &= \lambda A_0 B_0 + \lambda C_0 \\ \lambda A_0 B_0 + \lambda D_0 &= \lambda C_0 D_0 + \lambda A_0 \\ \lambda B_0 C_0 &= \lambda C_0 D_0 + \lambda C_0 \\ &= \lambda A_0 B_0 + \lambda B_0 \end{aligned}$$

hold.

However, considering invertibility in Section 5.1.2, we arrived at one of the main results of this thesis:

Theorem 5.3 (Commutative quadratic Biquandle with commutative Coefficients; Invertibility)**.** *For a commutative biquandle, where the biquandle functions have commutative coefficients* *and the biquandle functions are of general quadratic form as in Theorem 5.2, we have the following inverses and conditions on the coefficients:*

$$
\begin{aligned}
f_a^{-1}(b) &= B_0^{-1}x - B_0^{-1}A_0 a - \left(B_0^{-1}\right)^2 \lambda x a \\
(f^b)^{-1}(a) &= C_0^{-1}y - C_0^{-1}D_0 b - (C_0 2{-}1)^2 \lambda y b
\end{aligned}
$$

where $x = f_a(b), y = f^b(a)$ *and, additionally,* $\lambda^2 = \lambda A_0 = \lambda D_0 = 0$ *as well as* $B_0, C_0 \neq 0$.

In the second part of this chapter, Section 5.2, we then considered non-commutative biquandles with commutative coefficients and quadratic inverses. We examined invertibility for which we got a nice result as well:

Theorem 5.6 (Non-commutative quadratic Biquandle with commutative Coefficients; Invertibility)**.** *For a non-commutative biquandle BQ with quadratic up and down functions with commutative coefficients of the form,*

$$
\begin{aligned}
f_a(b) &= A_0 a + A_1 a^2 + Aab + B_0 b + B_1 b^2 + Bba &&=: x \\
f^b(a) &= C_0 a + C_1 a^2 + Cab + D_0 b + D_1 b^2 + Dba &&=: y
\end{aligned}
$$

we first consider the case where it holds that $A_1 = B_1 = C_1 = D_1 = A^2 = B^2 = C^2 = D^2 = 0$.

Furthermore, the quadratic inverses of f_a, f^b are of one of the following forms

$$
\begin{aligned}
f_a^{-1}(x) &= Aax + A_0B_0a - (AA_0 + BA_0)\,a^2 - B_0x + Bxa \\
&= (Aa - B_0)(x - A_0a) - BA_0a^2 \\
\left(f^b\right)^{-1}(y) &= Cby + D_0C_0b - (CD_0 + DD_0)\,b^2 - C_0y + Dyb \\
&= (Cb - C_0)(y - D_0b) - DD_0b^2
\end{aligned}
$$

where $a, b \in BQ$, BQ a non-commutative biquandle and $A, B, C, D, A_0, B_0, C_0, D_0 \in R$, a commutative, associative ring. Special cases of inverses for either of $A, B, C, D = 0$ ater given by

$$
\begin{aligned}
f_a^{-1}(x) &= A_0B_0a - B_0x - BA_0a^2 + Bxa &&= (Ba - B_0)(x - A_0a) \\
f_a^{-1}(x) &= Aax + A_0B_0a - AA_0a^2 - B_0x &&= (Aa - B_0)(x - A_0a) \\
\left(f^b\right)^{-1}(y) &= D_0C_0b - C_0y - DD_0b^2 + Dyb &&= (Db - C_0)(y - D_0b) \\
\left(f^b\right)^{-1}(y) &= Cby + D_0C_0b - CD_0b^2 - C_0y &&= (Cb - C_0)(y - D_0b).
\end{aligned}
$$

Another form is as follows, where A_1, B_1, C_1, D_1 are not necessarily zero. Let $B_1, C_1 \neq 0$ be a zero divisor. Then we have the following two special cases for inverses:

$$
\begin{aligned}
f_a^{-1}(x) &= A_0B_0^{-1}x + B_0^{-1}A_1x^2 + B_0^{-1}a \\
f_a^{-1}(x) &= A_0B_0^{-1}x + A_0(B_0^{-1})^2(A+B)x^2 + B_0^{-1}a - A(B_0^{-1})^2xa - B(B_0^{-1})^2ax \\
\left(f^b\right)^{-1}(y) &= D_0C_0^{-1}y + C_0^{-1}D_1y^2 + C_0^{-1}b \\
\left(f^b\right)^{-1}(y) &= D_0C_0^{-1}y + D_0(C_0^{-1})^2(D+C)y^2 + C_0^{-1}b - D(C_0^{-1})^2yb - C(C_0^{-1})^2by.
\end{aligned}
$$

In the first case, $(B_0^{-1})^2 = (C_0^{-1})^2 = 0$. In the second case, all of A_1, D_1 are zero divisors as well and $AB = A^2 = B^2 = CD = C^2 = D^2 = 0 \neq (B_0^{-1})^2, (C_0^{-1})^2$.

Furthermore, we then applied part a of this result to the biquandle axioms which led to:

Theorem 5.8 (Non-commutative quadratic Biquandle with commutative Coefficients; Axioms)**.** *Let BQ be a non-commutative biquandle. If the up and down functions are of quadratic form with quadratic inverse and commutative coefficients, they always have one of the following forms:*

Case 1:

$$
\begin{aligned}
f_a(b) &= A(ab+ba)+A_0a+B_0b \\
f^b(a) &= A(ab+ba)+B_0a+A_0b \\
\text{or} & \\
f_a(b) &= A(ab+ba+a+b)+b \\
f^b(a) &= A(ab+ba+a+b)+a \\
\text{or} & \\
f_a(b) &= A(ab+ba+a+b)+b \\
f^b(a) &= A(ab+Bba)+a \\
\text{or} & \\
f_a(b) &= A(ab+ba)+b \\
f^b(a) &= A(ab+ba+a+b)+a \\
\text{or} & \\
f_a(b) &= A(ab+ba)+b \\
f^b(a) &= A(ab+ba)+a \\
\text{or} & \\
f_a(b) &= Aab+Bba+A_0(a+b)+b \text{ with } A_0^2=0 \\
f^b(a) &= Aab+Bba+A_0(a+b)+a
\end{aligned}
$$

Case 2c:

$$
\begin{aligned}
f_a(b) &= Aab+A_0a+b \text{ with } A_0\in\{A,0\} \\
f^b(a) &= Aab+a+A_0b
\end{aligned}
$$

Case 3c:

$$
\begin{aligned}
f_a(b) &= Bba+A_0a+b \text{ with } A_0\in\{B,0\} \\
f^b(a) &= Bba+a+A_0b
\end{aligned}
$$

Case 4:

$$
\begin{aligned}
f_a(b) &= A_0a+b \text{ with } A_0\in\{C,0\} \\
f^b(a) &= C(ab-ba)+a+A_0b \\
\text{or} & \\
f_a(b) &= Bba+A_0a+b \text{ with } A_0\in\{B,0\} \\
f^b(a) &= Bab+a+A_0b \\
\text{or} & \\
f_a(b) &= Aab+A_0a+b \text{ with } A_0\in\{A,0\} \\
f^b(a) &= Aab+a+A_0b \\
\text{or} & \\
f_a(b) &= A(ab-ba)+A_0a+b \text{ with } A_0\in\{A,0\} \\
f^b(a) &= a+A_0b
\end{aligned}
$$

Case 5a:

$$
\begin{aligned}
f_a(b) &= (C+D)ba+b \text{ with } C\in\{0,D\} \\
f^b(a) &= Cab+Dba+a \\
\text{or} & \\
f_a(b) &= (C+D)ab+b \text{ with } D\in\{0,C\} \\
f^b(a) &= Cab+Dba+a \\
\text{or} & \\
f_a(b) &= Aab+Bba+b \text{ with } A\in\{0,B\} \\
f^b(a) &= (A+B)ba+a \\
\text{or} & \\
f_a(b) &= Aab+Bba+b \text{ with } B\in\{0,A\} \\
f^b(a) &= (A+B)ab+a \\
\text{or} & \\
f_a(b) &= A(ab+ba)+b \\
f^b(a) &= C(ab+ba)+a
\end{aligned}
$$

where in all cases $A^2 = B^2 = C^2 = D^2 = 0$ and all coefficients are elements of a commutative, associative ring. Additionally, the coefficients of b (in $f_a(b)$) and a (in $f^b(a)$) are invertible.

In the third and last part of the main body of this thesis, the focus was on computational work around biquandle homology. First, in Chapter 6, we stated the definition of homology in general in Section 6.1. In Section 6.2 the definition of quandle homology was presented, see Definition 6.5. We then proceeded in Section 6.3 by giving a new definition of biquandle homology:

Definition 6.6 (Birack, Degenerate, Biquandle Chain Complex; L.E.S of Biquandle Homologies)**.**

(i) *For a given birack X, let $C_n^R(X)$ be the free abelian group generated by n-tuples $(x_1, x_2, \ldots, x_n)$, $x_i \in X$, i.e. $C_n^R(X) = \mathbb{Z}X^n = (\mathbb{Z}X)^{\otimes n}$. Define a* boundary homomorphism *$\partial : C_n^R(X) \to C_{n-1}^R(X)$ by*

$$\partial(x_1, x_2, \ldots, x_n) = \sum_{i=2}^{n} (-1)^i \left((x_1, \ldots, \widehat{x_i}, \ldots, x_n) - \left(x_1^{x_i}, x_2^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1\,x_i}, \ldots, x_{n\,x_i} \right) \right).$$

Then $(C_*^R(X), \partial)$ is called a birack chain complex *of X.*

(ii) *Let X be a biquandle. Then there is a subchain complex, $C_n^D(X) \subset C_n^R(X)$, generated by n-tuples $(x_1, \ldots, x_n)$ where $x_i = x_{i+1}$ for some i. This subchain complex $(C_n^D(X), \partial)$ is called a* degenerate chain complex *of a biquandle X.*

(iii) *The* quotient chain complex *$C_n^Q(X) = C_n^R(X)/C_n^D(X)$ is also called the* biquandle chain complex*. We now have the short exact sequence of chain complexes:*

$$0 \longrightarrow C_n^D(X) \longrightarrow C_n^R X \longrightarrow C_n^Q(X) \longrightarrow 0 \tag{6.2}$$

We think this sequence splits usefully as in the quandle case; see Conjecture 6.7. It is clear that this sentence splits as every $x* \in C_n^R(X)$ is either in $C_n^Q(X)$ or $C_n^D(X)$, but never in both (or none).

(iv) *The homology of birack, degenerate and biquandle chain complexes are called* birack*,* degenerate *and* biquandle homology*, respectively. Thus, have the long exact sequence of the*

homology of biquandles:

$$\cdots \longrightarrow H_n^D(X) \longrightarrow H_n^R(X) \longrightarrow H_n^Q(X) \longrightarrow H_{n-1}^D(X) \longrightarrow \cdots$$

(v) *For an abelian group G, define the chain complex* $C_*^Q(X;G) = C_*^Q \otimes G$*, with* $\partial = \partial \otimes id$*. The groups of cycles and boundaries are denoted by* $\ker(\partial) = Z_n^Q(X;G) \subset C_n^Q(X;G)$ *and* $Im(\partial) = B_n^Q(X;G) \subset C_n^Q(X;G)$*, respectively. Then the nth* biquandle homology group *of a biquandle X with coefficient group G is defined as*

$$H_n^Q(X;G) = H_n(C_*^Q(X;G)) = Z_n^Q(X;G)/B_n^Q(X;G).$$

From this, a conjecture arose, finishing this chapter:

Conjecture 6.7 (The Biquandle Chain Complexes split). *The short exact sequence of (biquandle) chain complexes*

$$0 \longrightarrow C_n^D(X) \longrightarrow C_n^R X \longrightarrow C_n^Q(X) \longrightarrow 0$$

splits usefully.

The last chapter of this thesis, Chapter 7, concerned itself entirely with the computation of biquandle homology. It started with an introduction to the algorithm used in Section 7.1, discussing first the theory behind the algorithm and then the algorithm itself. After this, an example calculation of the second quandle homology group of the dihedral quandle of order 3 was presented in Section 7.2 ($H_2^Q(R_3) = \emptyset$).

After this example calculation, remarks on the implementation were made in Section 7.3. An extra programme, testing whether a set of given order with two given functions gives rise to a biquandle was presented in Section 7.3.1 before the chapter finished with an introduction to Hermite and Smith Normal Forms of Matrices in Section 7.4, covering both definition and theory for both quadratic and rectangular matrices in Section 7.4.1. Finally, the Kannan-Bachem algorithm for the calculation of the Smith Normal Form was presented in Section 7.4.2.

The resulting paper, [FW16], is shown a submitted for publication in Appendix C. This software is another main result of this thesis.

8.1 Outlook

We now want to give an idea of further promising research related to this thesis. For further research, several possible avenues exist, for example the calculation of the biquandles of the welded knots in Section 3.5. This would probably require an extension of the calculation of quandles via the fundamental group of knots. If a way could be found to calculate the biquandle based on the knot generators, this would allow a much faster computational calculation of biquandles of any knot.

Another promising area would be higher order biquandle functions, which could be even or odd, commutative or not. While we made a start on quadratic biquandles with commutative coefficients in the biquandle functions, it would be interesting to see the results for non-commutative coefficients. It would also be promising to research biquandle functions of differing order. We think that two promising approaches are via Sylvester matrices and their extensions to multivariate polynomials via tensor and also via some of the techniques that arose from [Kul53]. In particular, automorphisms of polynomial rings in several variables or Cremona groups could be useful. We hope in particular that these methods would prove fruitful in treating any case with at least one of the coefficients and biquandle non-commutative.

During our research, we did not find much, if any, references to Flat Singular Knot Theory and this should be an interesting field to study. Similarly, Flat Virtual Knot Theory and the effects of a commuting move on it is presently unknown. Research in both of those areas could improve our understanding of knot theories with two types of crossings, for example.

Finally, more computational work is required. The algorithm used is memory hungry and preliminary work in adapting it to C++ shows approximately 60-70% reduction in memory requirements for the boundary matrix calculations. Additionally, the software as a whole could be optimised and parallelised, such that one could calculate higher order homology groups of larger biquandles.

Appendix A

Code used in Chapter 3 to calculate relations arising from biquandle conditions

This code has been written in MAPLE (see [Map15]) and calculates the coefficients for a_{bc_b}. It is adaptable adapted to the other relations as defined in Section 4.2, Axioms (5.5) to (5.8). It is also adaptable to the non-commutative case in which case matrices should be used.

> $restart; with(Physics):$

> $Setup(noncommutativeprefix = \{var, coeff\}):$

> $var2\_down\_var1 := coeff_1 var_1 + coeff_2 var_2;$

$$var2\_down\_var1 := coeff_1\, var_1 + coeff_2\, var_2 \tag{1}$$

> $var1\_up\_var2 := coeff_3 var_1 + coeff_4 var_2;$

$$var1\_up\_var2 := coeff_3\, var_1 + coeff_4\, var_2 \tag{2}$$

> $first\_left := eval(var2\_down\_var1, [var_1 = b, var_2 = a]);$

$$first\_left := a\, coeff_2 + b\, coeff_1 \tag{3}$$

> $second\_left := eval(var2\_down\_var1, [var_2 = c, var_1 = b]);$

$$second\_left := b\, coeff_1 + c\, coeff_2 \tag{4}$$

> $LHS := collect(expand(eval(var2\_down\_var1, [var_1 = second\_left, var_2 = first\_left])), [a, b, c], recursive);$

$$LHS := a\ \text{`\^{}`}(coeff_2, 2) + (coeff_2\, coeff_1 + \text{`\^{}`}(coeff_1, 2))\, b + c\, coeff_1\, coeff_2 \tag{5}$$

> $coeff(LHS, a);$

$$\text{`\^{}`}(coeff_2, 2) \tag{6}$$

> $coeff(LHS, b);$

$$coeff_2\, coeff_1 + \text{`\^{}`}(coeff_1, 2) \tag{7}$$

> $coeff(LHS, c);$

$$coeff_1\, coeff_2 \tag{8}$$

> $first\_right := eval(var2\_down\_var1, [var_1 = c, var_2 = a]);$

$$first\_right := a\, coeff_2 + c\, coeff_1 \tag{9}$$

> $second\_right := eval(var1\_up\_var2, [var_2 = c, var_1 = b]);$

$$second\_right := b\, coeff_3 + c\, coeff_4 \tag{10}$$

> $RHS := collect(expand(eval(var2\_down\_var1, [var_1 = second\_right, var_2 = first\_right])), [a, b, c], recursive);$

$$RHS := a\ \text{`\^{}`}(coeff_2, 2) + b\, coeff_1\, coeff_3 + (coeff_1\, coeff_4 + coeff_2\, coeff_1)\, c \tag{11}$$

> $coeff(RHS, a);$

$$\text{`\^{}`}(coeff_2, 2) \tag{12}$$

> $coeff(RHS, b);$

$$coeff_1\, coeff_3 \tag{13}$$

> $coeff(RHS, c);$

$$coeff_1\, coeff_4 + coeff_2\, coeff_1 \tag{14}$$

>

```
> restart;
> m_up_n := C0·m + C·m·n + D0·n + C1·m·m;
```
$$m\_up\_n := C\,m\,n + C1\,m^2 + C0\,m + D0\,n \quad (1)$$

```
> n_down_m := A0·m + A·m·n + B0·n + A1·m·n;
```
$$n\_down\_m := A\,m\,n + A1\,m\,n + A0\,m + B0\,n \quad (2)$$

```
> aupa := eval(n_down_m, [m = a, n = a]);
```
$$aupa := A\,a^2 + A1\,a^2 + A0\,a + B0\,a \quad (3)$$

```
> adowna := eval(m_up_n, [m = a, n = a]);
```
$$adowna := C\,a^2 + C1\,a^2 + C0\,a + D0\,a \quad (4)$$

```
> coeff(collect(aupa, a), a);
```
$$A0 + B0 \quad (5)$$

```
> coeff(collect(aupa, a^2), a^2);
```
$$A + A1 \quad (6)$$

```
> coeff(collect(adowna, a), a);
```
$$C0 + D0 \quad (7)$$

```
> coeff(collect(adowna, a^2), a^2);
```
$$C + C1 \quad (8)$$

```
> first_left1 := eval(n_down_m, [m = c, n = a]);
```
$$first\_left1 := A\,a\,c + A1\,a\,c + A0\,c + B0\,a \quad (9)$$

```
> second_left1 := eval(m_up_n, [m = b, n = c]);
```
$$second\_left1 := C\,b\,c + C1\,b^2 + C0\,b + D0\,c \quad (10)$$

```
> LHS1 := collect(expand(eval(n_down_m, [n = first_left1, m = second_left1])), [a, b, c],
    distributed);
```
$$LHS1 := \left(A^2\,C1 + 2\,A\,A1\,C1 + A1^2\,C1\right) a\,b^2\,c + \left(A\,B0\,C1 + A1\,B0\,C1\right) a\,b^2 + \left(A^2\,C + 2\,A\,A1\,C + A1^2\,C\right) a\,b\,c^2 + \left(A^2\,C0 + 2\,A\,A1\,C0 + A\,B0\,C + A1^2\,C0 + A1\,B0\,C\right) a\,b\,c + \left(A\,B0\,C0 + A1\,B0\,C0\right) a\,b + \left(A^2\,D0 + 2\,A\,A1\,D0 + A1^2\,D0\right) a\,c^2 + \left(A\,B0\,D0 + A1\,B0\,D0 + A\,B0 + A1\,B0\right) a\,c + B0^2\,a + \left(A\,A0\,C1 + A0\,A1\,C1\right) b^2\,c + A0\,C1\,b^2 + \left(A\,A0\,C + A0\,A1\,C\right) b\,c^2 + \left(A\,A0\,C0 + A0\,A1\,C0 + A0\,C\right) b\,c + A0\,C0\,b + \left(A\,A0\,D0 + A0\,A1\,D0\right) c^2 + \left(A0\,B0 + A0\,D0\right) c \quad (11)$$

```
> first_right1 := eval(n_down_m, [m = b, n = a]);
```
$$first\_right1 := A\,a\,b + A1\,a\,b + A0\,b + B0\,a \quad (12)$$

```
> second_right1 := eval(n_down_m, [m = b, n = c]);
```
$$second\_right1 := A\,b\,c + A1\,b\,c + A0\,b + B0\,c \quad (13)$$

```
> RHS1 := collect(expand(eval(n_down_m, [n = first_right1, m = second_right1])), [a, b, c],
    distributed);
```
$$RHS1 := \left(A^3 + 3\,A^2\,A1 + 3\,A\,A1^2 + A1^3\right) a\,b^2\,c + \left(A^2\,A0 + 2\,A\,A0\,A1 + A0\,A1^2\right) a\,b^2 + \left(2\,A^2\,B0 + 4\,A\,A1\,B0 + 2\,A1^2\,B0\right) a\,b\,c + \left(A\,A0\,B0 + A0\,A1\,B0 + A\,B0 + A1\,B0\right) a\,b + \left(A\,B0^2 + A1\,B0^2\right) a\,c + B0^2\,a + \left(A^2\,A0 + 2\,A\,A0\,A1 + A0\,A1^2\right) b^2\,c + \left(A\,A0^2 + A0^2\,A1\right) b^2 + \left(A\,A0\,B0 + A0\,A1\,B0 + A\,A0 + A0\,A1\right) b\,c + \left(A0^2 + A0\,B0\right) b + A0\,B0\,c \quad (14)$$

```
>
```

> sort(collect(LHS1 − RHS1, [a, b, c]));

$$A0\,D0\,c + \left(-A\,A0^{2} - A0^{2}\,A1 + A0\,C1 + \left(-A^{2}\,A0 - 2\,A\,A0\,A1 + A\,A0\,C1 - A0\,A1^{2} + A0\,A1\,C1\right)c\right)b^{2} + \left(A\,A0\,D0 + A0\,A1\,D0\right)c^{2} + \left(\left(-A^{2}\,A0 - 2\,A\,A0\,A1 + A\,B0\,C1 - A0\,A1^{2} + A1\,B0\,C1 + \left(-A^{3} - 3\,A^{2}\,A1 + A^{2}\,C1 - 3\,A\,A1^{2} + 2\,A\,A1\,C1 - A1^{3} + A1^{2}\,C1\right)c\right)b^{2} + \left(A^{2}\,D0 + 2\,A\,A1\,D0 + A1^{2}\,D0\right)c^{2} + \left(-A\,A0\,B0 + A\,B0\,C0 - A0\,A1\,B0 + A1\,B0\,C0 - A\,B0 - A1\,B0 + \left(A^{2}\,C + 2\,A\,A1\,C + A1^{2}\,C\right)c^{2} + \left(-2\,A^{2}\,B0 + A^{2}\,C0 - 4\,A\,A1\,B0 + 2\,A\,A1\,C0 + A\,B0\,C - 2\,A1^{2}\,B0 + A1^{2}\,C0 + A1\,B0\,C\right)c\right)b + \left(-A\,B0^{2} + A\,B0\,D0 - A1\,B0^{2} + A1\,B0\,D0 + A\,B0 + A1\,B0\right)c\right)a + \left(-A0^{2} - A0\,B0 + A0\,C0 + \left(A\,A0\,C + A0\,A1\,C\right)c^{2} + \left(-A\,A0\,B0 + A\,A0\,C0 - A0\,A1\,B0 + A0\,A1\,C0 - A\,A0 - A0\,A1 + A0\,C\right)c\right)b \tag{15}$$

> first_left2 := eval(n_down_m, [m = b, n = a]);

$$first\_left2 := A\,a\,b + A1\,a\,b + A0\,b + B0\,a \tag{16}$$

> second_left2 := eval(n_down_m, [m = b, n = c]);

$$second\_left2 := A\,b\,c + A1\,b\,c + A0\,b + B0\,c \tag{17}$$

> LHS2 := collect(expand(eval(m_up_n, [n = second_left2, m = first_left2])), [a, b, c], distributed);

$$LHS2 := \left(A^{2}\,C1 + 2\,A\,A1\,C1 + A1^{2}\,C1\right)a^{2}\,b^{2} + \left(2\,A\,B0\,C1 + 2\,A1\,B0\,C1\right)a^{2}\,b + B0^{2}\,C1\,a^{2} + \left(A^{2}\,C + 2\,A\,A1\,C + A1^{2}\,C\right)a\,b^{2}\,c + \left(A\,A0\,C + 2\,A\,A0\,C1 + A0\,A1\,C + 2\,A0\,A1\,C1\right)a\,b^{2} + \left(2\,A\,B0\,C + 2\,A1\,B0\,C\right)a\,b\,c + \left(A0\,B0\,C + 2\,A0\,B0\,C1 + A\,C0 + A1\,C0\right)a\,b + B0^{2}\,C\,a\,c + B0\,C0\,a + \left(A\,A0\,C + A0\,A1\,C\right)b^{2}\,c + \left(A0^{2}\,C + A0^{2}\,C1\right)b^{2} + \left(A0\,B0\,C + A\,D0 + A1\,D0\right)b\,c + \left(A0\,C0 + A0\,D0\right)b + B0\,D0\,c \tag{18}$$

> first_right2 := eval(m_up_n, [m = a, n = c]);

$$first\_right2 := C\,a\,c + C1\,a^{2} + C0\,a + D0\,c \tag{19}$$

> second_right2 := eval(m_up_n, [m = b, n = c]);

$$second\_right2 := C\,b\,c + C1\,b^{2} + C0\,b + D0\,c \tag{20}$$

> RHS2 := collect(expand(eval(n_down_m, [n = first_right2, m = second_right2])), [a, b, c], distributed);

$$RHS2 := \left(A\,C1^{2} + A1\,C1^{2}\right)a^{2}\,b^{2} + \left(A\,C\,C1 + A1\,C\,C1\right)a^{2}\,b\,c + \left(A\,C0\,C1 + A1\,C0\,C1\right)a^{2}\,b + \left(A\,C1\,D0 + A1\,C1\,D0\right)a^{2}\,c + B0\,C1\,a^{2} + \left(A\,C\,C1 + A1\,C\,C1\right)a\,b^{2}\,c + \left(A\,C0\,C1 + A1\,C0\,C1\right)a\,b^{2} + \left(A\,C^{2} + A1\,C^{2}\right)a\,b\,c^{2} + \left(2\,A\,C\,C0 + 2\,A1\,C\,C0\right)a\,b\,c + \left(A\,C0^{2} + A1\,C0^{2}\right)a\,b + \left(A\,C\,D0 + A1\,C\,D0\right)a\,c^{2} + \left(A\,C0\,D0 + A1\,C0\,D0 + B0\,C\right)a\,c + B0\,C0\,a + \left(A\,C1\,D0 + A1\,C1\,D0\right)b^{2}\,c + A0\,C1\,b^{2} + \left(A\,C\,D0 + A1\,C\,D0\right)b\,c^{2} + \left(A\,C0\,D0 + A1\,C0\,D0 + A0\,C\right)b\,c + A0\,C0\,b + \left(A\,D0^{2} + A1\,D0^{2}\right)c^{2} + \left(A0\,D0 + B0\,D0\right)c \tag{21}$$

>

> sort(collect(LHS2 − RHS2, [a, b, c]));

$$-A0\,D0\,c + \left(B0^{2}\,C1 - B0\,C1 + \left(A^{2}\,C1 + 2\,A\,A1\,C1 - A\,C1^{2} + A1^{2}\,C1 - A1\,C1^{2}\right)b^{2} + \left(2\,A\,B0\,C1 - A\,C0\,C1 + 2\,A1\,B0\,C1 - A1\,C0\,C1 + \left(-A\,C\,C1 - A1\,C\,C1\right)c\right)b + \left(-A\,C1\,D0 - A1\,C1\,D0\right)c\right)a^{2} + \left(A0^{2}\,C + A0^{2}\,C1 - A0\,C1 + \left(A\,A0\,C - A\,C1\,D0 \right.\right. \tag{22}$$

$+ A0\, A1\, C - A1\, C1\, D0)\, c)\, b^2 + (-A\, D0^2 - A1\, D0^2)\, c^2 + ((A\, A0\, C + 2\, A\, A0\, C1 - A\, C0\, C1 + A0\, A1\, C + 2\, A0\, A1\, C1 - A1\, C0\, C1 + (A^2\, C + 2\, A\, A1\, C - A\, C\, C1 + A1^2\, C - A1\, C\, C1)\, c)\, b^2 + (-A\, C\, D0 - A1\, C\, D0)\, c^2 + (-A\, C0^2 + A0\, B0\, C + 2\, A0\, B0\, C1 - A1\, C0^2 + A\, C0 + A1\, C0 + (-A\, C^2 - A1\, C^2)\, c^2 + (2\, A\, B0\, C - 2\, A\, C\, C0 + 2\, A1\, B0\, C - 2\, A1\, C\, C0)\, c)\, b + (-A\, C0\, D0 - A1\, C0\, D0 + B0^2\, C - B0\, C)\, c)\, a + (A0\, D0 + (-A\, C\, D0 - A1\, C\, D0)\, c^2 + (-A\, C0\, D0 + A0\, B0\, C - A1\, C0\, D0 + A\, D0 - A0\, C + A1\, D0)\, c)\, b$

> first_left3 := eval(m_up_n, [m = a, n = b]);

$$first\_left3 := C\, a\, b + C1\, a^2 + C0\, a + D0\, b \quad (23)$$

> second_left3 := eval(n_down_m, [m = b, n = c]);

$$second\_left3 := A\, b\, c + A1\, b\, c + A0\, b + B0\, c \quad (24)$$

> LHS3 := collect(expand(eval(m_up_n, [n = second_left3, m = first_left3])), [a, b, c], distributed);

$$LHS3 := C1^3\, a^4 + 2\, C\, C1^2\, a^3\, b + 2\, C0\, C1^2\, a^3 + C^2\, C1\, a^2\, b^2 + (A\, C\, C1 + A1\, C\, C1)\, a^2\, b\, c + (A0\, C\, C1 + 2\, C\, C0\, C1 + 2\, C1^2\, D0)\, a^2\, b + B0\, C\, C1\, a^2\, c + (C0^2\, C1 + C0\, C1)\, a^2 + (A\, C^2 + A1\, C^2)\, a\, b^2\, c + (A0\, C^2 + 2\, C\, C1\, D0)\, a\, b^2 + (A\, C\, C0 + A1\, C\, C0 + B0\, C^2)\, a\, b\, c + (A0\, C\, C0 + 2\, C0\, C1\, D0 + C\, C0)\, a\, b + B0\, C\, C0\, a\, c + C0^2\, a + (A\, C\, D0 + A1\, C\, D0)\, b^2\, c + (A0\, C\, D0 + C1\, D0^2)\, b^2 + (B0\, C\, D0 + A\, D0 + A1\, D0)\, b\, c + (A0\, D0 + C0\, D0)\, b + B0\, D0\, c \quad (25)$$

> first_right3 := eval(m_up_n, [m = a, n = c]);

$$first\_right3 := C\, a\, c + C1\, a^2 + C0\, a + D0\, c \quad (26)$$

> second_right3 := eval(m_up_n, [m = b, n = c]);

$$second\_right3 := C\, b\, c + C1\, b^2 + C0\, b + D0\, c \quad (27)$$

> RHS3 := collect(expand(eval(m_up_n, [n = second_right3, m = first_right3])), [a, b, c], distributed);

$$RHS3 := C1^3\, a^4 + 2\, C\, C1^2\, a^3\, c + 2\, C0\, C1^2\, a^3 + C\, C1^2\, a^2\, b^2 + C^2\, C1\, a^2\, b\, c + C\, C0\, C1\, a^2\, b + C^2\, C1\, a^2\, c^2 + (2\, C\, C0\, C1 + C\, C1\, D0 + 2\, C1^2\, D0)\, a^2\, c + (C0^2\, C1 + C0\, C1)\, a^2 + C^2\, C1\, a\, b^2\, c + C\, C0\, C1\, a\, b^2 + C^3\, a\, b\, c^2 + 2\, C^2\, C0\, a\, b\, c + C\, C0^2\, a\, b + (C^2\, D0 + 2\, C\, C1\, D0)\, a\, c^2 + (C\, C0\, D0 + 2\, C0\, C1\, D0 + C\, C0)\, a\, c + C0^2\, a + C\, C1\, D0\, b^2\, c + C1\, D0\, b^2 + C^2\, D0\, b\, c^2 + (C\, C0\, D0 + C\, D0)\, b\, c + C0\, D0\, b + (C\, D0^2 + C1\, D0^2)\, c^2 + (C0\, D0 + D0^2)\, c \quad (28)$$

>

> sort(collect(LHS3 - RHS3, [a, b, c]));

$$(2\, C\, C1^2\, b - 2\, C\, C1^2\, c)\, a^3 + (-C^2\, C1\, c^2 + (C^2\, C1 - C\, C1^2)\, b^2 + (A0\, C\, C1 + C\, C0\, C1 + 2\, C1^2\, D0 + (A\, C\, C1 + A1\, C\, C1 - C^2\, C1)\, c)\, b + (B0\, C\, C1 - 2\, C\, C0\, C1 - C\, C1\, D0 - 2\, C1^2\, D0)\, c)\, a^2 + (A0\, C\, D0 + C1\, D0^2 - C1\, D0 + (A\, C\, D0 + A1\, C\, D0 - C\, C1\, D0)\, c)\, b^2 + (-C\, D0^2 - C1\, D0^2)\, c^2 + ((A0\, C^2 - C\, C0\, C1 + 2\, C\, C1\, D0 + (A\, C^2 + A1\, C^2 - C^2\, C1)\, c)\, b^2 + (-C^2\, D0 - 2\, C\, C1\, D0)\, c^2 + (-C^3\, c^2 + A0\, C\, C0 - C\, C0^2 \quad (29)$$

$$+2\, C0\, C1\, D0 + C\, C0 + \left(A\, C\, C0 + A1\, C\, C0 + B0\, C^2 - 2\, C^2\, C0\right) c\right) b + \left(B0\, C\, C0 - C\, C0\, D0 - 2\, C0\, C1\, D0 - C\, C0\right) c\right) a + \left(-C^2\, D0\, c^2 + A0\, D0 + \left(B0\, C\, D0 - C\, C0\, D0 + A\, D0 + A1\, D0 - C\, D0\right) c\right) b + \left(B0\, D0 - C0\, D0 - D0^2\right) c$$

> $m\_up\_n\_invers := Z0 \cdot m + Z1 \cdot m^2 + W0 \cdot n + W1 \cdot n^2 + Z \cdot m \cdot n;$

$$m\_up\_n\_invers := W1\, n^2 + Z\, m\, n + Z1\, m^2 + W0\, n + Z0\, m \qquad \textbf{(30)}$$

> $n\_down\_m\_invers := X0 \cdot m + X1 \cdot m^2 + Y0 \cdot n + Y1 \cdot n^2 + X \cdot m \cdot n;$

$$n\_down\_m\_invers := X\, m\, n + X1\, m^2 + Y1\, n^2 + X0\, m + Y0\, n \qquad \textbf{(31)}$$

> $x := eval(n\_down\_m, [n = b, m = a]);$

$$x := A\, a\, b + A1\, a\, b + A0\, a + B0\, b \qquad \textbf{(32)}$$

> $y := eval(m\_up\_n, [m = a, n = b]);$

$$y := C\, a\, b + C1\, a^2 + C0\, a + D0\, b \qquad \textbf{(33)}$$

> $down\_invers := eval(n\_down\_m\_invers, [n = x, m = a]);$

$$down\_invers := X\, a\, (A\, a\, b + A1\, a\, b + A0\, a + B0\, b) + X1\, a^2 + Y1\, (A\, a\, b + A1\, a\, b + A0\, a + B0\, b)^2 + X0\, a + Y0\, (A\, a\, b + A1\, a\, b + A0\, a + B0\, b) \qquad \textbf{(34)}$$

> $test := sort(collect(down\_invers - b, [a, b]));$

$$test := B0^2\, Y1\, b^2 + \left(A0^2\, Y1 + (A + A1)^2\, Y1\, b^2 + A0\, X + X1 + (2\, (A + A1)\, A0\, Y1 + (A + A1)\, X)\, b\right) a^2 + \left(2\, (A + A1)\, B0\, Y1\, b^2 + A0\, Y0 + X0 + (2\, A0\, B0\, Y1 + B0\, X + (A + A1)\, Y0)\, b\right) a + (B0\, Y0 - 1)\, b \qquad \textbf{(35)}$$

> $up\_invers := eval(m\_up\_n\_invers, [m = y, n = b]);$

$$up\_invers := W1\, b^2 + Z\left(C\, a\, b + C1\, a^2 + C0\, a + D0\, b\right) b + Z1\left(C\, a\, b + C1\, a^2 + C0\, a + D0\, b\right)^2 + W0\, b + Z0\left(C\, a\, b + C1\, a^2 + C0\, a + D0\, b\right) \qquad \textbf{(36)}$$

> $sort(collect(up\_invers - a, [b, a]));$

$$C1^2\, Z1\, a^4 + 2\, C0\, C1\, Z1\, a^3 + \left(C0^2\, Z1 + C1\, Z0\right) a^2 + \left(C^2\, Z1\, a^2 + D0^2\, Z1 + D0\, Z + W1 + (2\, C\, D0\, Z1 + C\, Z)\, a\right) b^2 + (C0\, Z0 - 1)\, a + \left(2\, C\, C1\, Z1\, a^3 + D0\, Z0 + (2\, C\, C0\, Z1 + 2\, C1\, D0\, Z1 + C1\, Z)\, a^2 + W0 + (2\, C0\, D0\, Z1 + C\, Z0 + C0\, Z)\, a\right) b \qquad \textbf{(37)}$$

>

>

Appendix B

Welded Braids drawn as Knots

We have drawn the welded braids from Section 3.5 in the following pages. The diagram generators are presented in the following Table B.1.

knot	braid word	Diagram
w3.1	$\sigma_1\tau_2\sigma_3\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}\tau_2\sigma_3^{-1}\sigma_2$	Figure B.1
w3.2	$\tau_1\sigma_2^{-1}\tau_1\sigma_1^{-1}\sigma_1^{-1}\tau_2$	Figure B.2
w4.1	$\sigma_1\tau_1\sigma_1^{-1}\sigma_2\sigma_1\tau_1\sigma_1^{-1}\sigma_2^{-1}$	Figure B.3
w4.2	$\sigma_1^{-1}\sigma_2^{-1}\sigma_3\tau_2\sigma_1\sigma_4^{-1}\sigma_3\tau_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_2^{-1}$	Figure B.4
w4.3	$\sigma_1^{-1}\sigma_2\sigma_3\tau_2\sigma_1\sigma_4^{-1}\sigma_3\tau_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_2$	Figure B.5
w4.4	$\sigma_1^{-1}\sigma_2\sigma_3\tau_2\sigma_1\sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_3\sigma_4\sigma_3^{-1}\tau_2$	Figure B.6
w4.5	$\tau_1\sigma_2\sigma_1^{-1}\tau_1\sigma_1\sigma_2$	Figure B.7
w4.6	$\sigma_1^{-1}\sigma_2^{-1}\tau_3\sigma_2^{-1}\sigma_1\sigma_4^{-1}\tau_3\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_3^{-1}\sigma_2$	Figure B.8
w6.1	$\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3\tau_2$	Figure B.9

Table B.1: Braid words for welded braids

The biquandle relations and generators of the knots in Table B.1 can be found in Table 3.2 in Section 3.5.

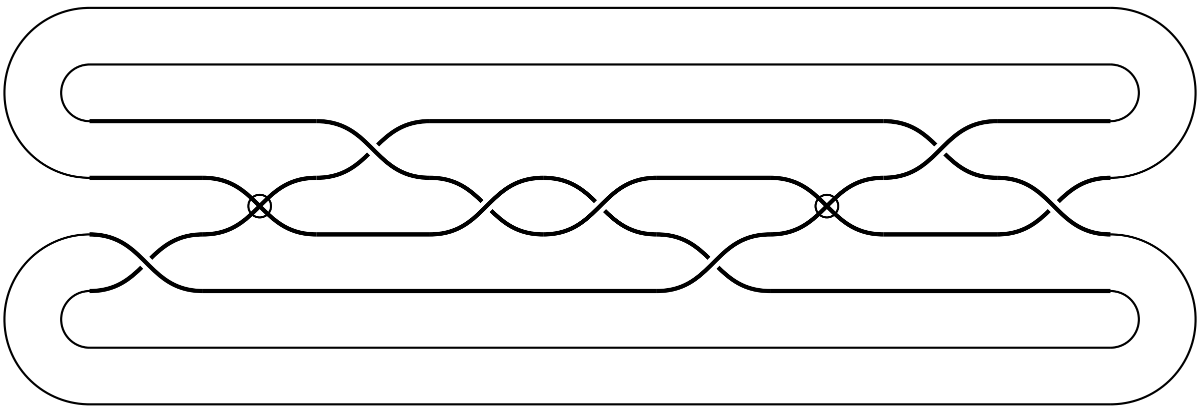


Figure B.1: Closed Braid w3.1

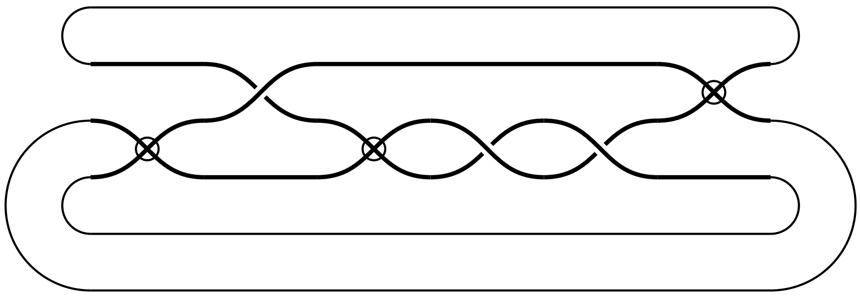


Figure B.2: Closed Braid w3.2

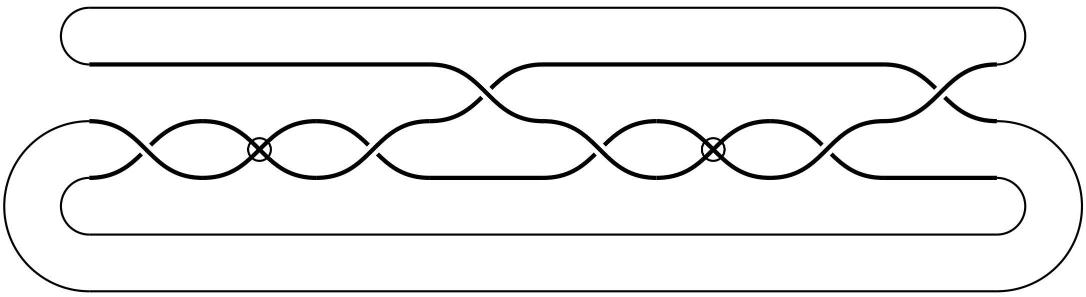


Figure B.3: Closed Braid w4.1

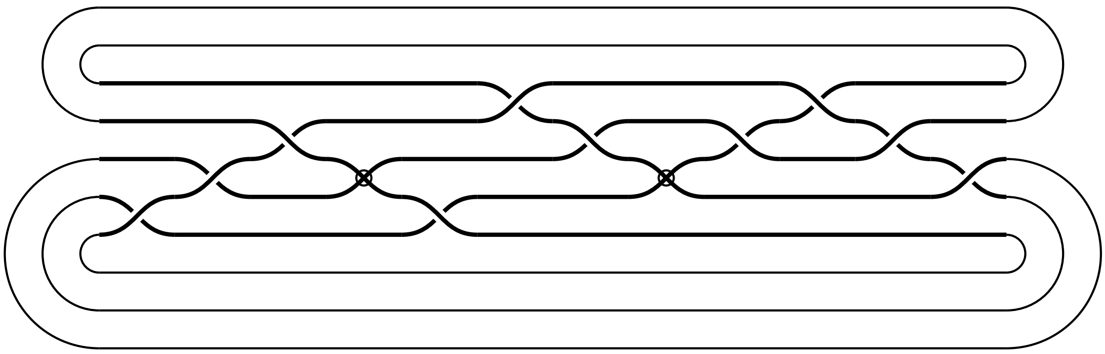


Figure B.4: Closed Braid w4.2

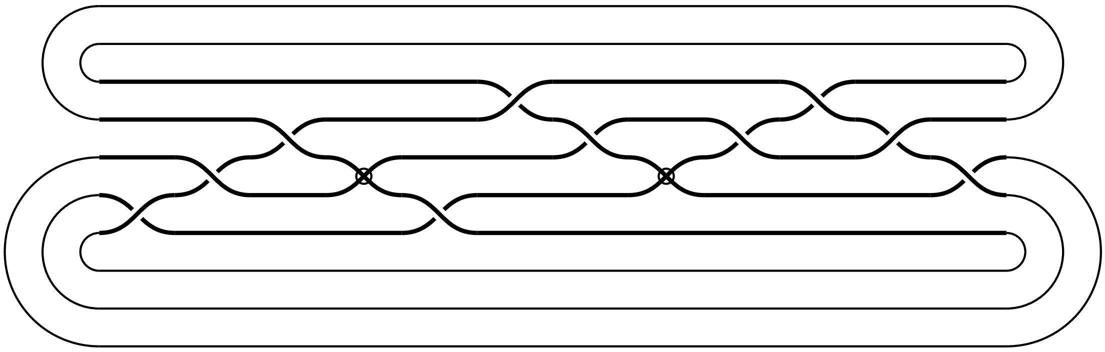


Figure B.5: Closed Braid w4.3

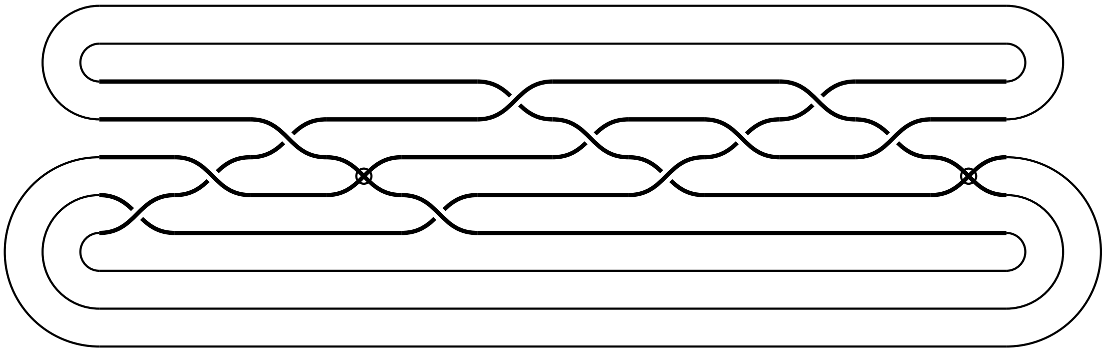


Figure B.6: Closed Braid w4.4

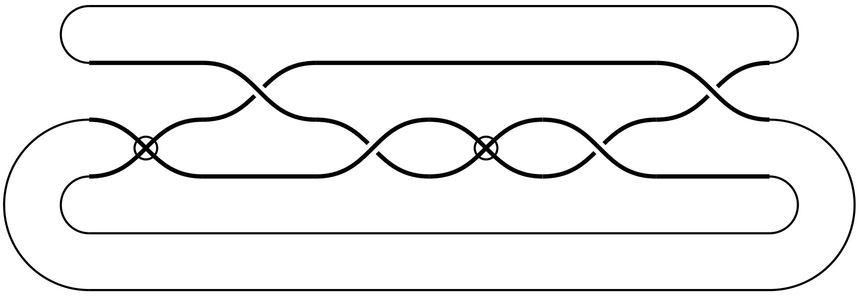


Figure B.7: Closed Braid w4.5

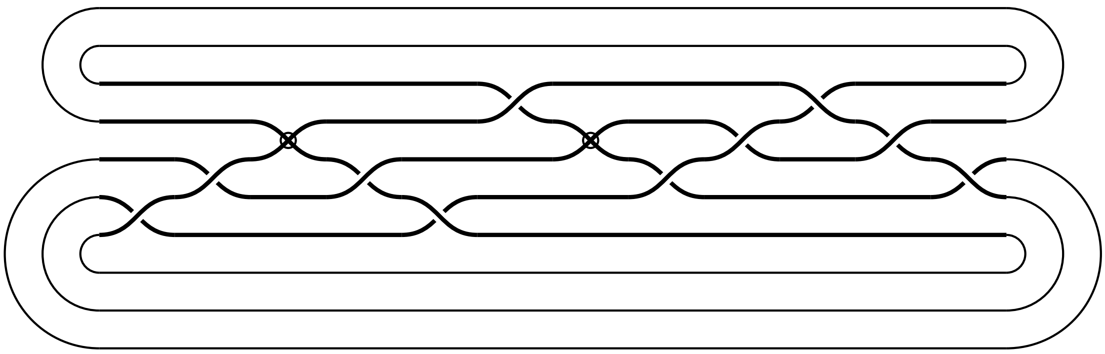


Figure B.8: Closed Braid w4.6

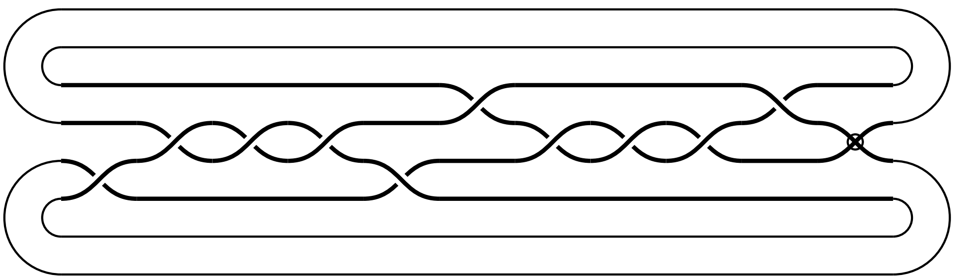


Figure B.9: Closed Braid w6.1

Appendix C

R Software Paper

Here we present the R software papers, [FW16]. It is reproduced 'as is'.

Software paper for submission to the Journal of Open Research Software

Please submit the completed paper to: editor.jors@ubiquitypress.com

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(1) Overview

Title Quandle and Biquandle Homology Calculation in R

Paper Authors
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Abstract
In knot theory several knot invariants have been found over the last decades. This paper concerns itself with invariants of several of those invariants, namely the Homology of racks, quandles, biracks and biquandles. The software described in this paper calculates the rack, quandle and degenerate homology groups of racks and biracks. It works for any rack/quandle with finite elements where there are homology coefficients in Z_k. The up and down actions can be given either as a function of the elements of Z_k or provided as a matrix. When calculating a rack, the down action should coincide with the identity map. We have provided actions for both the general dihedral quandle and the group quandle over S_3. We also provide a second function to test if a set with a given action (or with both actions) gives rise to a quandle or biquandle. The program is provided as an R package and can be found at https://github.com/ansgarwenzel/quhomology.

Keywords
Homology, Quandle, Biquandle, Rack, Birack, R, Knot Theory
AMS subject classification: 57M27, 57M25

Introduction
This introduction is divided into two parts: First, we are going to give some mathematical background before introducing the software itself.

Mathematical Background

Racks and Quandles were first described by John Conway and Gavin Wraith in 1959 in unpublished correspondence. In [2], David Joyce showed that racks are indeed a knot invariant. Subsequently, Fenn and Rourke introduced racks as proper knot invariants in [3]. It is instructive to consider racks as groups without their multiplicative elements. Formally, a rack is defined as a set of elements endowed with a binary operation which satisfies the following axioms (where the third axiom holds only for a quandle, not for a rack) for all a,b,c in the rack/quandle R:

1. $\forall a,b \in R \exists ! c \in R s.t. a^c = b$
2. b a $(c)^{(b^c)}$ $(c) = a^{\square}$
3. $a^a = a$ (only quandle)

Biracks and Biquandles are defined similarly, now with two operations, satisfying the following axioms:

1. $a^{bc_b} = a^{cb^c}$
2. $a_{bc_b} = a_{cb^c}$
3. $a_b{}^{c_b} = a^c{}_{b^c}$
4. $a^a = a_a$ (only biquandle)

In addition, both operations satisfy the rack axiom 1.

Here, the up and down actions allow us to introduce a switch map via $f_a(x) = x_a$, $f^a(x) = x^a$ as $S(a, f_a(b)) = S(b, f^b(a))$. The utility of this is clear when one observes that the axioms can be reformulated using the switch map:

- S, f^a and f_a are all bijective.
- Yang Baxter Equation: $S_1 S_2 S_1 = S_2 S_1 S_2$, where $S_1(a,b,c) = (S(a,b),c)$ and $S_2(a,b,c) = (a, S(b,c))$.

This reformulation is more amenable to computational work.

The homology of the biquandle is defined as follows:
Let X be a birack. Then $C_n^R(X)$ be the free abelian group generated by n-tuples $(x_1 x_2 \dots x_n), x_i \in X$, that is, $C_n^R(X) = Z X^n$.
We define a boundary homomorphism by

$$\partial(x_1 x_2 \dots x_n) = \sum_{i=2}^{n} (-1)^i \left[(x_1 \dots \hat{x}_i \dots x_n) - x_1^{x_i} \dots x_{i-1}^{x_i} (x_{i+1})_{x_i} (x_n)_{x_i} \right]$$

and $(C_{\square}^R(X), \partial)$ is called a chain complex of X.

Furthermore, we have a subchain complex $C_n^D(X) \subset C_n^R(X)$, generated by degenerate n-tuples $(x_1 x_2 \dots x_n), x_i \in X$ with $x_i = x_{i+1}$ for some i.

Together with the boundary homomorphism, $\left(C_{\square}^{D}(X),\partial\right)$ is is called the degenerate chain complex of X.
Using both of those chain complexes, we can define the biquandle chain complex via the quotient chain complex, $C_n^Q(X)=C_n^R(X)/C_n^D(X)$.
This gives the following short exact sequence of chain complexes,

$$0\rightarrow C_n^D(X)\rightarrow C_n^R(X)\rightarrow C_n^Q(X)\rightarrow 0 \quad .$$

We can then define the birack, biquandle and degenerate homology groups in the usual way. In addition, we have the following long exact sequence of homology groups

$$\ldots\rightarrow H_n^D(X)\rightarrow H_n^R(X)\rightarrow H_n^Q(X)\rightarrow H_{n-1}^D(X)\ldots$$

The algorithm for the homology calculation is described in [1] for this specific software.

Software

The software, which is provided as an R package, can be accessed on github. It provides two primary functions. One of these calculates the homology of racks and biracks, whilst the other verifies if a rack or birack is induced by a given set with one or two actions.

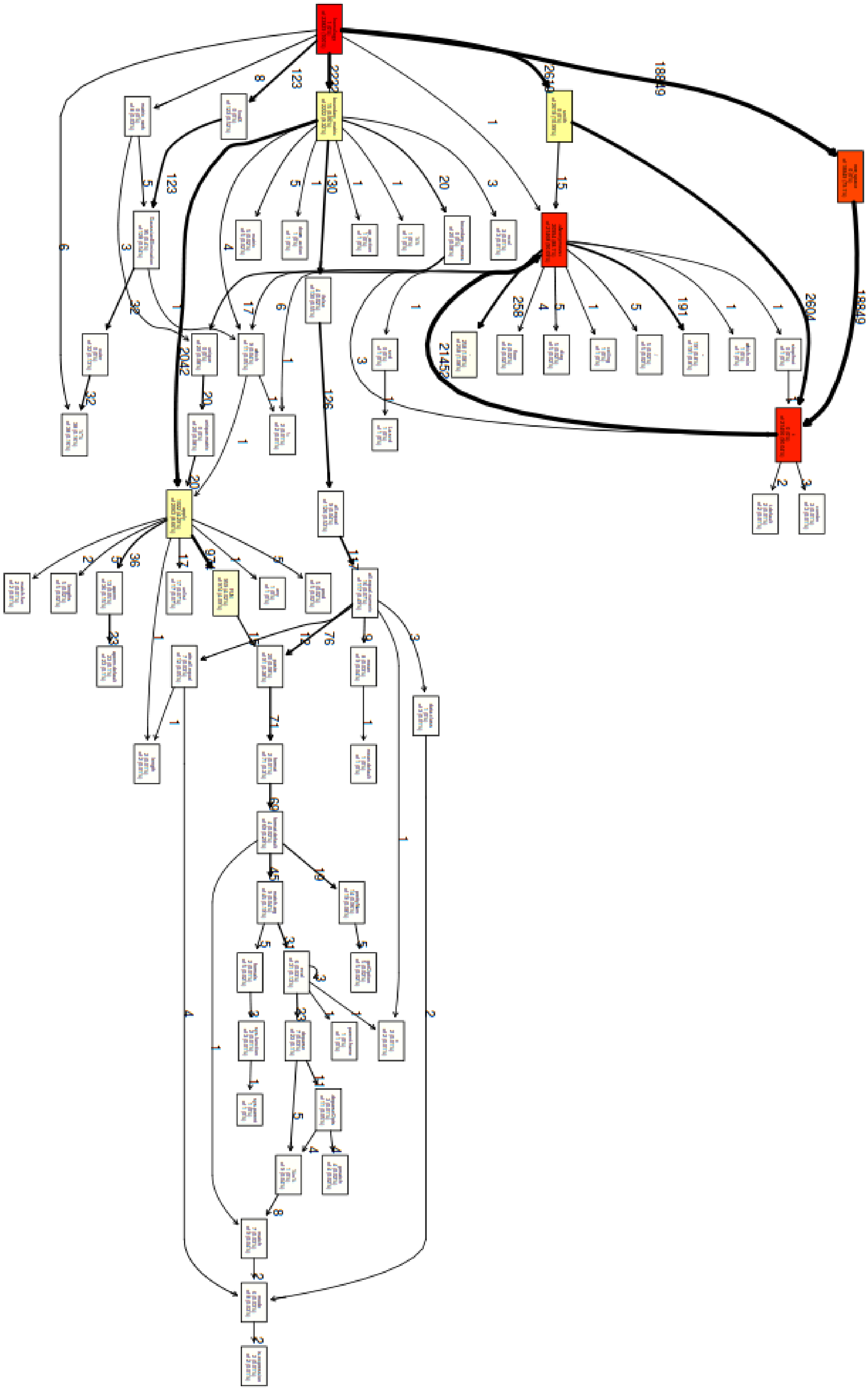
Implementation and architecture

This software is implemented in R. It has been tested on MacOS X, CentOS/Ubuntu and Windows without any problems.

The algorithm for the homology calculation is described in the paper [1].

The program is divided into the following two main parts: The calculation of the boundary matrix and the subsequent calculation of the Homology. For a graphical description see the following figure, which was created with the following code:

```
library(proftools)
Rprof(tmp <- tempfile(), line.profiling = T);
homology(4,5,F);Rprof(append=F); pd <- readProfileData(tmp)
plotProfileCallGraph(pd,style=google.style,score="total",nodeSizeScore
= "none",layout="dot",rankDir = "LR")
```



The boundary matrices are computed using the functions *boundary_matrix* (for quandle and rack boundary matrices) and *boundary_matrix_degenerate* (for degenerate boundary matrices), respectively. The methodology of both functions is similar, differing only in the manner in which degenerate or non-degenerate entries are removed where required. In particular, after creating a right-sized matrix, they call *boundary_names* or *boundary-names_degenerate* to create the row and column names of the boundary matrix. After this, they loop through the column names to calculate their boundaries and construct the matrix (for details see [1]). These boundary matrices represent the boundary maps of the simplicial complex of the rack/birack.

After both boundary matrices have been calculated, they are returned to the *homology* and *degenerate_homology* functions, respectively. As is the case for the boundary matrices functions, these two functions only differ in the boundary functions called and in their respective output texts.
As an aside, those two functions should be the only ones that would have to be called by the user in order to calculate a homology.
Following on with the algorithm described in [1], those two functions calculate the image and kernel of the boundary map representations (the boundary matrices) before finding a representation of the homology group. For this, they call various functions, namely *findX*, which "finds X" (this is defined in [1]), *row_space*, which calculates a basis of the row space of a matrix, *matrix_rank*, which calculates the rank of a matrix and *GaussianElimination*, a function written by Prof John Fox (see [4] for a source) which does a Gaussian Elimination on a matrix and returns its reduced row echelon form.
Using the function *smith,* the smith natural form is obtained from the representation matrix of the homology. This is done via repeated calculation of the hermite normal form of the matrix and its transpose, using the *hermiteNF* function from the *numbers* package, [8]. In addition, this function checks if the diagonal of the matrix is in the correct form via the function *check_more_push* and, if required, will call *push_down* to do what the name implies.

Finally. The homology group is obtained using the diagonal of the smith normal form.

The second function of this package, the testing if a give operation or operations give rise to a quandle or biquandle is done via the function *S_test*. This function receives as input the order of the underlying set and then proceeds to test the two requirements for a quandle/biquandle as described before.

Quality control

The results of the program have been compared to known results and in addition, *R CMD check* has been used to quality check the code itself. Additionally, a few tests have been provided in test/testthat.R.

(2) Availability

Operating system

Any OS that can install and run R (at least version 3.0.0).

Programming language

R 3.0.0+.

Additional system requirements

The more RAM, the higher the homology groups that can be calculated. Presently, output is on screen, but can be changed to a file, if necessary.
Required input devices: keyboard only.

Dependencies

The program requires the R standard installation, together with the packages *MASS* and *numbers*.

Software location:

Archive

Name: CRAN
Persistent identifier: https://cran.r-project.org/web/packages/quhomology/index.html
Licence: GNU GPL v3.0
Publisher: Ansgar Wenzel
Date published: 02/05/2016
Version: 1.1.0

Code Repository

Names: GitHub
Identifier: https://github.com/ansgarwenzel/quhomology
License: GNU GPL v3.0
Date published: 05/05/2016
Version: 1.1.0

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