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Valuation of Callable Convertible Bonds Using Binomial Trees Model with Default Risk, Convertible Hedging and Arbitrage, Duration and Convexity.

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Declaration

I hereby declare that this thesis has not been and will not be, submitted in whole or in part to another University for the award of any other degree.

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FAHAD ALDOSSARY, SUBMITTED FOR DOCTOR OF PHILOSOPHY

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Abstract

In this thesis, I develop a valuation model to price convertible bonds with call provision. Convertible bonds are hybrid instruments that possess both equity and debt characteristics. The purpose of this study is to build a pricing model for convertible and callable bonds and to compare the mathematical results of the model with real world market performance. I construct a two-factor valuation model, in which both the interest rate and the stock price are stochastic. I derive the partial differential equation of two stochastic variables and state the final and boundary conditions of the convertible bond using the mean reversion model on interest rate. Because it is difficult to obtain a closed solution for the American convertible bond due to its structural complexity, I use the binomial tree model to value the convertible bond by constructing the interest rate tree and stock price tree. As a convertible bond is a hybrid security of debt and equity, I combine the interest rate tree and stock price tree into one single tree. Default risk is added to the valuation tree to represent the event of a default. The model is then tested and compared with the performance of the Canadian convertible bond market. Moreover, I study the duration, convexity and Greeks of convertible bonds. These are important risk metrics in the portfolio management of the convertible bond to measure risks linked to interest rate, equity, volatility and other market factors. I investigate the partial derivative of the value of the convertible bond with respect to various parameters, such as the interest rate, stock price, volatility of the interest rate, volatility of the stock price, mean reversion of the interest rate and dividend yield of the underlying stock. A

convertible bond arbitrage portfolio is constructed to capture the abnormal returns from the Delta hedging strategy and I describe the risks associated with these returns. The portfolio is created by matching long positions in convertible bonds, with short positions in the underlying stock to create a Delta hedged convertible bond position, which captures income and volatility.

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CONTENTS

List of symbols and abbreviations

V	Convertible bond value
r	Risk-free interest rate
S	Stock price
k	Speed of mean reversion
θ	Central tendency or the long run value of the short-term rate
W	Standard Wiener process
K	Exercise price
r_∞	Long-term value
λ	Risk premium
σ_r	Volatility of risk- free interest rate
σ_s	Volatility of underlying stock
$\hat{\sigma}$	Term volatility
t	Present time
π	Up node Probability at t_1
p	Up node Probability at t_2 for upper nodes
q	Up node Probability at t_2 for lower nodes
P	Vasicek zero-coupon bond
τ	Time to maturity
F	Face value
c	Coupon
y	Coupon yield
$N(.)$	Cumulative standard distribution function
α	Conversion ratio
CP	Call price
B	Value of the straight bond
V^C	Callable convertible bond
D	Duration
Cx	Convexity
Δr	Change in interest rate
Δ	Delta
Γ	Gamma
q_s	Dividend yield of the underlying stock
Abbreviations	
SDE	Stochastic differential equation
PDE	Partial differential equation
Eqn	Equation
CRR	Cox, Ross, and Rubinstein model

List of Figures

List of Tables

1 Introduction

1.1 Introduction to the convertible bond valuation model

Convertible bonds are a developing segment of the corporate bond market. Convertible bonds are hybrid instruments that possess both equity and debt characteristics. Similar to straight bonds, convertible bonds are entitled to receive interest payments (coupons) and the full principal at maturity. However, convertible bonds typically pay lower interest than straight corporate debt because of the value of the convert option that is embedded in this derivative security. Convertible bondholders have the option to convert their bonds into common shares of the underlying stock at a pre-specified rate, which is called the conversion ratio. The conversion ratio is usually specified at the time of bond issuance. This ratio indicates the number of underlying shares into which the convertible bond can be converted. The conversion value is the market value of the underlying asset into which a convertible security may be exchanged. The conversion value is calculated by multiplying the current share price by the conversion ratio. A convertible bond can be converted only when the underlying equity is trading at the conversion value or higher. A convertible bond is called 'in-the-money' if the share price is higher than the conversion value.

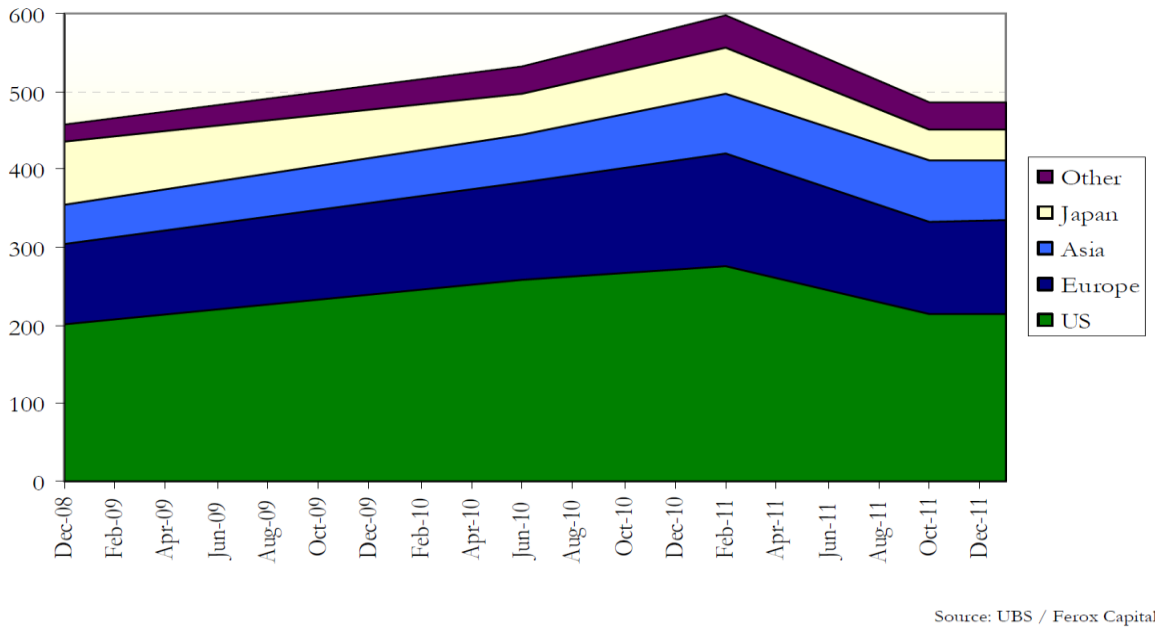
Often, convertible bonds contain embedded call options that provide the issuer with the right to redeem the bond at a specified price before maturity. Less frequently, the bonds may include embedded put options that allow the holder to sell the bond back to the issuer at a predetermined price.

A hard call feature allows the issuer of a convertible bond to redeem the convertible bond before maturity by paying the call price to the investor. The issuer may need to pay the accrued interest to the investor in addition to the call

price. The issuer can also exercise the hard call feature after the call date. The period during which the issuer may not redeem a convertible bond under any circumstance is called the hard non-call period. During the non-call period, the issuer is prohibited from redeeming the bond without the consent of the bondholders.

On the other hand, a convertible bond may have a soft call period that allows the issue to be called but provides bondholders with a capital gain to offset the loss of interest income. The most common soft call stipulates that the underlying equity instrument must trade for a specified period of time above a certain price level in order for the bond to be called (Navin,1999).

The convertible bond market has progressed substantially in recent years. The global convertible bond market is worth approximately US$500 billion in size and contains around 2,500 issues, according to Ferox's Capital Report in 2012 (Ferox Capital LLP, 2012). Figure 1 shows the size of the global convertible bond market in dollars. Evidently, the overall market capitalisation of the convertible bond market has shown a gradual increase since the financial crisis in 2008.



Source: UBS / Ferox Capital

Figure 1: Global convertible bond market - Ferox Capital

As of 31 December 2014, the US convertibles market had a market capitalisation of $268 billion from 2,346 issuers, composing about 10% of all debt in the US, according to Janney Montgomery Scott LLC (Janney Montgomery Scott LLC, 2014).

The convertible bond markets are expanding rapidly in some developing countries. For example, China issued more than 60 trillion yuan of new convertible bonds in 2010, which is almost three times the level from four years earlier. Hong Kong is an international financial centre that issued about 70 trillion yuan of convertible bonds in 2012. (Zhang, 2014).

In Canada, 143 convertible debenture issues were listed on the Toronto Stock Exchange (TSX) in 2014/15. According to a report by Deloitte, the estimated size of the Canadian convertible debenture market was approximately $14 billion in size and was issued by 92 separate issuers (Deloitte LLP, 2014). Canadian mid-cap companies and real estate investment trusts have been particularly active users of convertible debt in the past decade as an opportunity to finance acquisitions of new projects, enter new markets, or fund continuing operations. In this thesis, we will use the Canadian convertible bond market data for validation and data testing purposes.

A convertible bond is a hybrid instrument comprised of two components (debt and equity). Therefore, the convertible bond valuation is a two-factor valuation model that includes the interest rate and the stock price. Since the numerical valuation of a two-factor model is quite complex, most convertible bond pricing models assume a non-stochastic interest rate. As I believe that a stochastic interest rate will provide more accurate and efficient results, I develop a two-factor valuation model for convertible bonds with call provisions that is subject to the default risk of two stochastic variables using the mean reversion model for interest rate. I also derive the PDE for a European convertible bond with two stochastic variables and

state the final and boundary conditions of this convertible bond. Because of the difficulties of deriving the PDE for an American convertible bond due to the structural complexity of this style of bond, I use the binomial tree as numerical model to value the convertible bond.

Valuation of a convertible bond is not responsive to exact solution in closed form due to the security's optionality which is complex, according to Finnerty (2015). In convertible bonds, the owner has an option (American) to substitute them for common stocks. This is done by converting them to a specified number of equities at any period before they are redeemed. Most of the time, the entity has a call option (American), which prior to bondholders converting voluntarily, it can be used in forcing conversion. Bondholders can have several put options (European) with which redemption can be forced prematurely if the option of conversion is that of in-the-money.

This project will study a valuation model for convertible bonds with provisions, such as call options. The binomial tree model is one common way to value and forecast the prices of convertible callable bonds with options. According to Huang (2013), the pricing of convertible bonds cannot get closed-form solution; in most conditions, numerical methods should be adopted such as binary tree method, Monte Carlo method, finite difference method. As for Monte Carlo method, firstly it uses different stochastic differential equations to describe the pricing factor models in the market for simulation, then it makes pricing based on the characteristic of convertible bonds, for example, the boundary conditions acquired by all kinds of provisions. Because of the complexity of convertibles, the resulting pricing equation can be solved only numerically. The binomial tree method of two stochastic variables is presented to solve the price of convertible bond.

Some valuation models have been developed to provide estimations for valuing complex convertible bond. However, it has been widely argued that most of these models include some assumptions that are not predicted by historical information or real world data, particularly for the interest rates and volatility.

To value a convertible bond, we need to construct two trees: an interest rate tree and a stock price tree. We adopt the Vasicek model tree to value the debt through the interest rate tree, which will lead to calculating the price of the straight bond. For the equity part, we will adopt the Cox, Ross, & Rubinstein (CRR) approach, with some modifications, that allows us to assume non-constant volatility.

Kwon and Chiarella (2007) claimed that various previous models of interest rates put emphasis and built on Vasicek's model in several ways. In Vasicek's model, the assumption was that spot rate followed a process of mean reversion which had a volatility which was constant and a degree of mean reversion that was also constant. Rapid spot interest rate was the quantity in control of the present group of models. The spot rate is a quantity that is non-traded and hence the models normally take to account market price associated with risk in the interest rate. Since the risk's market price is a quantity that cannot be observed, assumptions were made and they relied on mathematical convenience as the basis and not economic aspects for the purpose of getting pricing PDE which permits several solution techniques to be applied. Heath-Jarrow-Morton (1992) established a model that led to a large departure from this broad and regular theme. They took into consideration the various quantities that drove the model that is a series of instant forward rates that are related directly to the traded bond's prices. Techniques derived from stochastic calculus were used to come up with a general framework relating to the interest rate's evolution which had an important characteristic stating the model had natural calibration to yield curve which is observed currently. Volatility processes of the forward rates form the major inputs

to HJM framework, which depicts a special model known as, Cox-Ingersoll-Ross model as being a case and part of one-factor HJM framework which is general and it corresponds to specific volatility process choice. Despite HJM model becoming largely accepted as most consistent and general framework by which derivatives relating to interest rates can be studied, added complexity, as well as lack of enough numerical techniques in the HJM, meant that models that had been formulated earlier maintained popularity specifically among their users. Nevertheless, owing to high developments relating to technology by computers, models of HJM are gradually becoming practical and practitioners are currently adopting several of these model forms for the purpose of hedging and pricing in interest rate instruments according to Kwon and Chiarella (2007).

Our model will discuss the valuation of convertible bonds with call provisions using a binomial tree model that is subject to default risk through data analysis of the Canadian bond market. One of this thesis's main aims is to build a pricing model for convertible and callable bonds and compare the results with the market price. A convertible bond is a bond that can be converted into a predetermined amount of the company's equity at certain times during its life; therefore, convertible bond valuation can be divided into two components:

- Equity – (the risk-free rate used to discount equity).
- Debt – (the risky bond rate used to discount the straight bond to reflect the future default probabilities).

The Vasicek model is adopted to construct the interest rate tree. The Vasicek model is a mean-reverting stochastic process for short-term interest rate valuation where the interest rate *r* is supposed to follow the Ornstein-Uhlenbeck process. For the underlying stock price tree, we adopt the Cox, Ross, and Rubinstein (CRR) model with some modifications that allow for a non-constant volatility in different intervals but a constant volatility within the same time interval.

Once the interest rate tree and the stock price tree are constructed, we combine the two trees into one single tree to find the price of the convertible non-callable bond as well as that of the callable bond. Each node of the combined tree will have four branches in the next period, t_{i+1}. These branches represent the combination of the two stochastic factors, the interest rate and the stock price. We use the single central node in the interest rate tree at $t \geq 2$ for both central nodes in the stock tree at $t \geq 2$. The same methodology is applicable for the nodes in other time intervals.

Since convertible bonds have a straight bond element in them, default risk can happen any time in the bond's life. As a result, it is essential that default risk is considered through the process of valuation. Choice concerning the rate of discount is the challenge faced when dealing with matters of credit risk. If the convertible element of the bond does not change, then cash payoff is exposed to credit risks and the necessary rate of interest takes to account the credit spread that matches with the issuer's credit rating. Besides that, if there is certainty in the convertibles conversion, shares can always be issued by the entity and ensure that the invested proceeds are risk-free. The discounting rate appropriate is now risk-free rate. Hull (2003) and Goldman Sachs (1994) put into consideration the likelihood of conversion in every node and also considered the rate of discount as being a properly weighted arithmetic average of risky rate and risk-free rate (it is obtained by summing up the credit spread of the issuer). The conversion probability for the last tree layer is either 0 or 1, based on whether or not the convertible has been converted. At the former nodes, the mean for conversion probabilities of all successor nodes is used to determine conversion probability. Suppose the convertible is put (or converted) at a node, resetting of the conversion probability to 0 or 1 has to be done.

Various models of valuation of convertible bonds are reviewed financial literature as seen in arguments by Carayannopoulos (2003). These models have to show both the option of exchanging bonds to equities as the features of conversion depicts and also show the credit risk associated with the bond. These authors examine two examples of models relating to credit risk: structural models which are needed in showing the value of a company in a given period of time to assess the probability of default, (2) reduced-form models which view default as a process that is exogenous and statistical. Structural models have limitations of data and this is the reason as to why these authors pick reduced-form model (the model of credit risk by Singleton and Duffie). In this model, the bonds default probability is always the function of both stock price of the issuer and time. The valuation model of convertible bonds combines option values to trade off bonds for equity plus risky straight bond's values. The tree model is an example of a numerical method they use in determining the prices of convertible bonds which they use to make comparisons with market prices. Takahashi (2001) shows arguments by others whereby jump process should represent the probability of default this is because the processes of diffusion are not in a position to discuss in detail empirical observations in that even prior to maturity there exists huge credit spreads. Turning attention to the problems, Jarrow Turnbell (1994) and Duffie-Singeton (1999) did not endogenously discuss in detail the probability of default by exogenously identified it through a jump process and obtained the securities prices which are arbitrage-free subject to the usual and ordinary default risk. Because it is difficult to derive the PDE for the American convertible bond due to its structural complexity such as conversion option, call option and default risk, I deploy the tree model for this project to determine the convertible bond price considering default risk while making the assumption that stock price and interest rate process are stochastic.

In this model, A new branch is added to the tree model to represent the default event and the probability of default λ. Therefore, the combined tree will have five branches for each node instead of four to reflect the default event. In the case of default, the stock price jumps to zero in the convertible bond valuation model, as the bond is no longer converted, and the bondholders will receive a portion of the bond's principal according to its recovery rate δ.

Then, we will test our convertible bond valuation against the real market prices of selected convertible bonds from the Toronto Stock Exchange (TSX). We will investigate the option-free convertible bond valuation model (convertible bonds that do not allow call provision) against the real market price of these convertible bonds. Moreover, we will examine the callable convertible bonds valuation model subject to default risk relative to the market price.

We also study the duration, convexity, and Greeks of convertible bonds. The duration, convexity and Greeks are important risk metrics in the portfolio management of convertible bonds to measure the risk associated with interest rates, equity, volatility, and other economic factors. Duration is a measure of the approximate price sensitivity of a bond to interest rate changes. More specifically, it is the approximate percentage change in bond price for a 100-basis point change in rates (Fabozzi, 2005). Convexity is a measure of the curvature of the value of a security or portfolio as a function of interest rates. It indicates how the duration changes as interest rates change.

Since the convertible bond is a hybrid of a bond and an underlying equity component, we investigate the partial derivative of the value of the convertible bond to various parameters, such as the interest rate, stock price, volatility of the interest rate, volatility of the stock price, mean reversion of the interest rate, and

dividend yield of the underlying stock. I also investigate the effects of these factors on the price of the convertible bond.

In addition, we will examine the importance of the Greeks of the convertible bond, such as Delta and Gamma. The Delta of a convertible bond measures the convertible equity's sensitivity to any stock price changes. Delta is used as an estimation tool in a hedging strategy that determines the number of equity shares to short against the convertible bond's long position.

Convertible bond arbitrage entails purchasing a convertible bond and selling short the underlying stock, creating a Delta hedge ratio. The Delta hedge strategy aims to benefit from the undervalued convertible bonds by going long for the convertible and short for the underlying stock. If the underlying stock price falls, the hedge fund will exploit its short position. It is also likely that convertible bonds decline less than their underlying stock because they are protected by their value as fixed-income instruments. Moreover, any discrepancies or mispricing in the relationships among the single components and additional features of the convertible bond will, therefore, lead to arbitrage opportunities that attract the attention of hedge fund managers (Werner, 2010).

Henderson and Zhao (2013) argue that convertible bonds are typically under priced relative to their fair values. Arbitrageurs, typically hedge funds, attempt to profit from this under pricing by establishing a long position in the convertible bonds and simultaneously shorting the issuer's stock to hedge their exposure to the issuer's stock price and default risk.

Moreover, we study convertible bond Delta arbitrage by producing the daily convertible bond arbitrage returns of 44 convertible bonds that were listed on the TSX for the period from 2009 to 2016.

The Delta hedging strategy is designed to generate returns from, firstly, the convertible bond yield income and short interest, and secondly from long volatility exposure from the option component of the convertible bond. We will create a convertible bond arbitrage portfolio to capture the abnormal returns from the Delta hedging strategy and describe the risks associated with these returns. The portfolio is created by matching long positions in convertible bonds, with short positions in the underlying stock to create a Delta hedged convertible bond position, that captures income and volatility. The Delta strategy is implemented by constructing an equally weighted portfolio of 44 hedged convertible bonds from 2009 to 2016. To obtain Delta, we present two calculation methods; Delta with the binomial tree model and Delta with the Black-Scholes model.

The thesis is organised as follows. In Chapter 2, we identify the interest rate model as the first stochastic factor in the convertible bond valuation model. We will explain the construction process of the Vasicek interest rate tree. In Chapter 3, we describe the CRR stock price model as the second stochastic factor that represents the equity component of the convertible bond. In Chapter 4, we derive the PDE of the two stochastic factors and state the boundary conditions of the European convertible bond. We then combine the two constructed trees into one single tree using the binomial model. We also compare the results of the numerical example with the real market prices of Canadian convertible bonds. In Chapter 5, we present the duration, convexity and Greeks and other risk metrics associated with convertible bond investment. We also show the sensitivity of the convertible bond value and duration to various parameters, such as the interest rate, stock price, volatility of the interest rate, volatility of the stock price, mean reversion of the interest rate, and dividend yield of the underlying stock. In Chapter 6, we illustrate the Delta hedging strategy and construct the convertible arbitrage portfolio. Chapter 7 concludes the thesis.

1.2 Literature review

Since convertible bonds are sophisticated financial instruments that play a major role in financial markets, some important valuation models have been developed in the past century. Because of the complexity of convertible bonds, most of these models are one-factor models that assume non-stochastic interest rates. A contingent claims approach to the valuation of convertible bonds was initially proposed by Ingersoll (1977) and Brennan and Schwartz (1977). They proposed that the value of a convertible bond is based on one underlying variable: the value of the firm. The price of a convertible bond is obtained by solving a PDE under a non-stochastic interest rate, in which case a convertible bond can be decomposed into a straight bond plus a warrant with an exercise price equal to the par value (i.e., $V = K + \max(\omega\, D_T - k, 0)$, where D_T is the value of the firm at T and ω is the fraction of the equity that the bond holders receive if the bond was converted) (Li,2005). Some numerical models have focused on finite difference schemes that also assume non-stochastic interest rates, such as, for example, Brennan and Schwartz (1980), McConnel and Schwartz (1986), Tsiveriotis and Fernandes (1998), Nyborg (1996), and Xingwen (2005). However, other studies have proposed that the value of a convertible bond is based on the equity rather than the value of the firm. The equity value model includes those proposed by Ho and Pfeffers (1996), Tsiveriotis and Fernandes (1998), and Hull (2003).

Some valuation models have been developed to price convertible bonds under the assumption of stochastic interest rates. Initially, Brennan and Schwartz (1980) extended their previous model and introduced a short-term, risk-free interest rate as an additional stochastic variable to capture the stochastic nature of the interest rate. Carayannopoulos (1996) extended the result suggested by King (1986) and provided an empirical investigation to test the contingent claims approach to the valuation of corporate convertible bonds under the assumption of a stochastic interest rate. Giovanni, Ana, and John (2003) solved a two-factor convertible

bonds model under a stochastic interest rate process that is assumed to follow Hull and White's (1990) framework. Kim (2006) discussed deriving the PDE of convertible bonds with a stochastic interest rate using Black-Scholes analysis (1973). In Chapter 4, I derive the PDE of convertible bonds with a stochastic interest rate by adopting the mean-reverting process suggested by the Vasicek model (1977).

Huang, Liu, and Rao (2013) claimed that the pricing of convertible bonds did not have a closed-form solution; under most conditions, a numerical method, such as the binary tree method, the Monte Carlo method, or the finite difference method, must be adopted. Mezofi (2015) introduced that closed-form solutions can only be used with a restricted set of assumptions, therefore numerical solutions are frequently used in practice. Numerical solutions such as lattice methods, finite difference methods or Monte Carlo simulations can include path-dependent payoff structure allowing more realistic implementation of convertible bond features.

The research did previously on how convertible bonds are valued by Monte Carlo or probability simulation is not sufficient enough (Kind and Amman 2008). Buchan (1997,1998) discusses in detail how Bossaert's technique relating to parametric optimization is applied to all these convertible bonds, this is done by using the value of the firm as the state variable underlying as well as giving an allowance of senior debt. Nevertheless, the assumption she made during the empirical implementation is that of European option as the conversion option and not American. This group of methods of pricing convertible bonds employs probability simulation and it overcomes most of the limitations of numerical methods for partial differential equations. Kind and Amman (2008) initiated an empirical and theoretical contribution. Initially, we suggest the convertible bond's pricing method which is the stock value- based building on the already developed Monte-

Carlo approach according to Garcia (2003). This method has two stages which are developed to deal with bias associated with the approach of Monte Carlo which is attributable to methods with only one stage. These two-step methods of simulation can be described as it being a parametric approach since it employs representations which are parametric of the initial and primary exercise decisions. An important foundation of numerical computation for prices of convertible bonds was laid by Ayache et al (2003) describing how recent finite-difference techniques of computing can take place of the computationally complex trinomial trees and sub-optimal binomial trees which fill and are everywhere in the literature. Anderson and Buffum in 2002 formulated computing and theory methods on a basis which was reasonably solid though the method of how the convertible bonds models can be parameterized was not determined. In the literature, several particular parameterizations have come up and they include Gregory and Arvanitis of 2001, Miralles and Bloch (2002) and Muromachi (1999), they all emanate from empirical observations. Normally, they do not come up with a price model that will bring any specific instrument near the market. As a matter of fact, when used in simple instruments for example coupon bonds and stock options, models of convertible bonds which are parameterized carelessly can result in huge price biases. In a case of market setting, where attention is in relative values or maybe or you are in need of hedging against all convertible bond prices using credit derivatives (straight debt) and options, obviously this would not be the ideal situation.

The complexity of numerical computation in financial theory has increased significantly in recent years, which has created more demand on the speed and efficiency of computer systems. Numerical methods are used to value convertible bonds, estimate their sensitivities, as well as carry out risk analysis. The American option contains early exercise characteristics making it increasingly complex. The complexity of the American option has necessitated the adoption of new methods

based on Monte Carlo Simulation, finite difference, as well as the binomial process. Relative to other numerical methods, Monte Carlo method has been employed more often to solve more complex problems (Boyle et al, 1997). For instance, analyse function $f(x)$ over d-dimensional unit hypercube. The simple integral approximation using Monte Carlo is equal to the mean value of the function f over n points that have been randomly selected from a hypercube unit. In essence, the n points are not random per se in a standard Monte Carlo application; rather, they are produced by a deterministic algorithm and later on defined by pseudorandom numbers. Based on the law of large numbers, the estimation meets the true value of the integrand as n inclines to infinity. Moreover, the central limit theorem clearly illustrates that the estimation of the standard error tends to zero as $1/\sqrt{n}$. Hence, the convergence rate error is not dependent on the problem's dimension and this is what gives this method its edge as compared to other methods of classical numerical integration. The only limitation to this approach, which is mild, is that function f should be square integrable. Moreover, the adoption of this method has been faster due to an increase in the availability of powerful computers.

One disadvantage of this approach is that for it to get exact results in complex problems, it requires many replications. Various methods of variance reduction have been developed to enhance precision. In Geske and Shastri's (1985) view, step sizes aren't zero and that they don't have to be equal. This notwithstanding, step sizes ought to be selected in such a way as to make sure that there is stable, efficient, and accurate solution convergence. The prices of the stock as well as time solution space are put together in a put and call option valuation problems. Based on time dimension, the date of expiry T establishes the optimum time which is allowed. The lower absolute bound is established by limited liability in stock-price space, $SMIN = 0$. Moreover, the derivative condition is used to establish the upper bound SMAX. In directly estimating primary stochastic process, there is a

possibility of not reaching upper and lower stock-price bounds. According to Monte Carlo simulation, reaching the bounds will be dependent on the number of jumps, and for the binomial process this would depend on the size of the up and down jumps and on the number of jumps. Stock price range is dependent on size of up and down jumps which is in turn dependent on the approximation of changes of the primary stock prices. According to the binomial process, the net is a cone and the number of stock price in a given time step is determined by the selection of the time steps and subsequently the stock price-time net. As regards finite difference estimations, time step is equally defined by expiration. The size of the stock price is described as $(SMAX - SMIN)/n$. The shape of the finite difference is rectangular and choosing the mesh size is crucial for a stable and accurate convergence to the solution. It is worth noting that critical mesh ratio tends to be sensitive to the model of differencing technique used.

Since convertible bonds may have other option features that significantly affect their valuations, such as call and put options, Tsiveriotis and Fernandes (1998) proposed a pricing approach that values convertible bonds numerically using lattice-based methods. This approach splits the value of a convertible bond into an equity component and a debt component. The binomial trees method for valuing convertible bonds was initially proposed by Cheung and Nelken (1994) as a one-factor model. This model assumes a constant interest rate and does not allow for provisions such as call and put options. Goldman and Sachs (1994) introduced the one-factor valuation model using a binomial tree that assumes a constant interest rate and allows for call and put options.

Hung and Wang (2002) presented a binomial tree method for convertible bonds in a two-factor model in which the stock price and the interest rate are stochastic. This model was extended by Chambers and Lu (2007) to include the correlation

between the equity price and the interest rate. These models suggested that Interest rates could be modeled using the Ho-Lee (1986) lognormal model.

Finnerty (2015) established a closed-form model for valuing contingent claims for the convertible bonds which calculates the exchange options value when the following aspects are stochastic; the share prices, the credit spread of the firm and the risk-free rate in the short term. Interest rate process is said to be in line with a framework by white and hull (1990). His model displays scientific evidence that does the comparison of market prices and the model for 148 corporate bonds samples of which issuing was done from 2006 to 2010. Respectively, the standard mean and median errors of pricing were 0.21% and -0.18%. This model evaluates the disruptive effect on short selling prohibition on prices of convertible bonds at the time of the financial crisis which occurred more recently.

An integrated framework for pricing of convertible bonds was initiated by Kyriakou and Ballotta (2015) and it consists of the value of the firm emerging as a reliable numerical scheme of pricing, movements in interest rates that are correlated stochastically and a jump diffusion (exponential). The stochastic model proposed fits jump diffusion (affine) framework by construction. Their model also incorporated stochastic rates of interests as well as a correlation structure (non-zero) relating to interest rates and value of the firm.

A recent model developed by Zhang and Zhao (2016) discusses the pricing of convertible bond with call provision based on the traditional Black-Scholes formula. By applying the principle of no arbitrage, the partial differential equation for the bond is established with identified boundary conditions, which solution results in the closed form of the pricing formula. Their model is one-factor model where the interest rate is assumed to be constant.

The literature which displays the framework of valuation of convertible bonds using the binomial tree as the basis based on credit risk starts with strategies that are quantitative from a research note by Goldman Sachs in 1994. This structure is built on Brownian motion (geometric) also, as an equity model (stochastic) and the risk-free rate of interest which is constant. It has been developed further by Fernandes and Tsiveriotis (1998) and by Pfeffer and Ho (1996). Wang and Hung (2002) employs a model which is in a reduced form and which uses Turnbull and Jarrow framework (1995) as its basis. They employ stock-price tree (stochastic) in this model that belongs to them and mix risky and risk-free rates in a single tree. As a result of Singleton and Duffie framework, Kalimipalli and Carayannopoulos examine trinomial tree (reduced-form) approach. This model here employs a stock price (stochastic) also hazard rate relies upon stock price movements. By using interest rate and stock price trees correlation Wang and Hung's model is expanded by Lu and Chambers (2007). A trinomial tree approach taking into account counterparty credit and market risks present in CB pricing structure are proposed by Xu (2011). Generalization of the reduced-form technique is done to incorporate the process of (CEV) constant elasticity of variance relating to equity price just before default.

The aim of my study is to present a new extension to these models in which the interest rate follows a mean-reversion model and the stock price follows a modified version of the CRR model that allows non-constant volatility in different intervals but keeps volatility constant within each time interval. Unlike previous models, I combine two trees with different structures into one single tree. The mean-reverting interest rate and the non-constant volatility of equity have a significant impact on the valuation process. It is also possible, under my approach, to include a default risk adjustment, which I will introduce in Section 4.5.

Only a few studies have raised the issue of duration and convexity in terms of convertible bonds, given their complexity. Brooks and Attinger (1992) provided a theoretical definition of the duration and convexity of convertible bonds. They expressed the duration of convertible bonds in terms of the straight bond duration, the equity duration, the rho of the conversion option, and the sensitivity of the equity return to the yield. Although their arguments form derivation is a correct definition, they do not explain how the conversion option is valued, nor does it provide values for Rho and Delta, as no valuation model was provided. Other studies, such as those by Calamos (1988), Gepts (1987), and Ferguson et al. (1995), discussed the approximations of empirical examples; however, these studies did not illustrate the derivation of the duration and convexity of convertible bonds. Sarkar (1999) stated that very little work has been done on convertible bond duration or convexity. On one hand, studies have attempted to estimate the duration and convexity of convertible bonds with ad-hoc measures but without the benefit of a fully explained valuation model. Sarkar (1999) provided a closed-form expression for the duration and convexity of a zero-coupon convertible bond. This model followed the contingent-claim approach of Ingersoll (1977) and Brennan and Schwartz (1980). Sarkar (1999) adopted one-factor valuation models that assume a constant risk-free interest rate.

In my analysis, I use a binomial model to study the duration and convexity of an American-style convertible bond. I provide an example of a closed-form expression for the duration and convexity of a European zero-coupon convertible bond based on a two-factor model that adopts a stochastic interest rate. I also study the duration that expresses the approximate change in the convertible value for any change in the interest rate. Moreover, I investigate the sensitivity of the convertible bond value and duration to various parameters, such as the interest rate, stock price, volatility of the interest rate, volatility of the stock price, mean reversion of the interest rate, and dividend yield of the underlying stock.

Several convertible bond arbitrage studies have identified pricing inefficiencies in some convertible bond markets due to their complex structures. Amman, Kind, and Wilde (2003) demonstrated that 21 convertible bonds listed on the French market were underpriced by 3% compared to their theoretical values between February 1999 and September 2000. This finding is consistent with those of other studies by King (1986), Kang and Lee (1996), Henderson (2005), and Chan and Chen (2007).

In this thesis, I will create a convertible bond arbitrage portfolio to capture the abnormal returns from the Delta hedging strategy and describe the risks associated with these returns. The portfolio is created by matching long positions in convertible bonds with short positions in the underlying stock to create a Delta hedged convertible bond position, which captures income and volatility. This portfolio also demonstrates, in a sense, that the underpricing of convertible bonds existed in the Canadian market in seven out of the eight years or all the years depending on the choice of Delta, in the observed period.

1.3 Data

The Toronto Stock Exchange (TSX) is one of the global markets on which convertible bonds are widely traded. A selection of 44 Canadian convertible bonds and their corresponding stocks are used in my valuation model. Data such as bond and stock symbols and prices, maturities, coupons, yields to maturity, and conversion ratios were collected from Financialpost.com and Google Finance.

Because there is a maturity date for each bond, MATLAB code is used to collect data automatically at different time intervals, such as daily, weekly, or monthly. Then, the data can be converted to an Excel spreadsheet according to the researcher's preference.

I also set up a contract with Stockwatch, which is one of the common data providers specializing in Canadian financial markets, in order to obtain any necessary historical data.

Interest rates and bond yields were collected from the Bank of Canada. A number of interest rates and yields were used in this thesis, and they are listed in Table 1. The data used in the duration and convexity section were collected from the Stockwatch and TSX databases. To study the sensitivities of the convertible bond values, durations, and convexities to several parameters, data needed to be obtained from the bonds and underlying stocks, such as the interest rate, stock price, volatility of the interest rate, volatility of the stock price, mean reversion of the interest rate, and dividend yield of the underlying stock.

Interest Rate - Yields	Time Period
Government of Canada Benchmark Bond Yields	3 Years
Government of Canada Benchmark Bond Yields	5 Years
Canadian Three-Year Zero-Coupon Bond Yield	3 Years
Canadian Five-Year Zero-Coupon Bond Yield	5 Years
Canadian Ten-Year Zero-Coupon Bond Yield	10 Years
Real Return Bond Yield	5 Years

Table 1: Interest rates and yields summary

For the analysis in Chapter 6, data from 44 Canadian convertible issues were collected for the period from 2009 to 2016. The required data were obtained from the Stockwatch and TSX databases. Data such as the conversion ratios, dividend yields, start dates, maturity dates, and face values of the convertible bonds were obtained from CIBC annual reports.

2 Interest rate model

2.1 The Vasicek model

The prices of all bonds and bond options are affected by changes in the term structures of interest rates. In this chapter, I will describe the Vasicek model, which is known as one of the equilibrium interest rate models. The Vasicek model is a mean-reverting stochastic process for short-term interest rate valuations.

Gupta and Zepta (2007) claimed that Cox-Ingersoll-Rox and Vasicek models are examples of two essential short-rate models. They have closed-form solutions to the several instruments of interest rates and also they are controllable and hence their importance. These two models have the same reaction to parameters changes according to the comparative studies done. Nevertheless, because of square-root term multiplication, a specific change in sigma does not influence bond prices as it does in Vasicek model. Additionally, the sigma value present in CIR model is often high and at times, this can be deceptive. It was concluded that because of the volatility parameter which was very stable the performance of the Vasicek model appeared to be better. In addition, Vasicek model is seen to have an edge over Cox-Ingersoll-Rox model because of model tractability as well as closed-form solutions availability in more complicated interest rate financial instruments. The difficulty of adopting Vasicek model is that a high sigma in the Vasicek model could result in negative interest rates which is not observable in the reality.

In the Vasicek model, the interest rate, *r*, is supposed to follow the Ornstein-Uhlenbeck process and has the following expression under the risk-neutral measure:

$$dr_t = k(\theta - r_t)dt + \sigma_r dW_r(t).$$

The constant θ denotes the central tendency or the long-run value of the short-term rate in the risk-neutral process. The positive constant k denotes the speed of mean reversion. The parameter σ_r is the volatility of the short-term rate, and W_r is a standard Wiener process.

The above process is sometimes called the elastic random walk or the mean reversion process. The instantaneous drift $k(\theta - r_t)$ represents the effect of pulling the process towards its long-term mean θ with a magnitude proportional to the deviation of the process from the mean. When the short-term rate is above its long-run equilibrium value, the drift is negative, driving the rate down towards this long-run value. When the rate is below its equilibrium value, the drift is positive, driving the rate up, towards this long-run value (Tuckman, 2011).

As in a risk-neutral process, the drift combines both interest rate expectations and the risk premium. The risk premium λ can be written separately and enters into the risk-neutral process as a constant drift and r_∞ is the long-term value. The Vasicek model is then written as follows:

$$dr_t = k(r_\infty - r_t)dt + \lambda dt + \sigma_r dW_r(t).$$

$$= k(\left[r_\infty + \frac{\lambda}{k}\right] - r_t)dt + \sigma_r dW_r(t).$$

where

$$\theta = \left[r_\infty + \frac{\lambda}{k}\right].$$

In this section, I represent the process with a Vasicek binomial tree for interest rate model valuation. The Vasicek interest rate tree is constructed as presented in Tuckman Model (2011). The Vasicek interest rate tree is shown in Figure2.

In Figure 2, π is the probability of an increase in the interest rate at $t = 1$, and 1- π represents the probability of a decrease in the interest rate at $t = 1$, whereas p is the probability of an increased interest rate at $t = 2$ at node r^{uu}, and 1- p represents the probability of a decreased interest rates at $t = 2$ at node r^{ud}. Similarly, q is the probability of an increased interest rates at $t = 2$ at node r^{ud}, and 1- q represents the probability of a decreased interest rate at $t = 2$ at node r^{dd}.

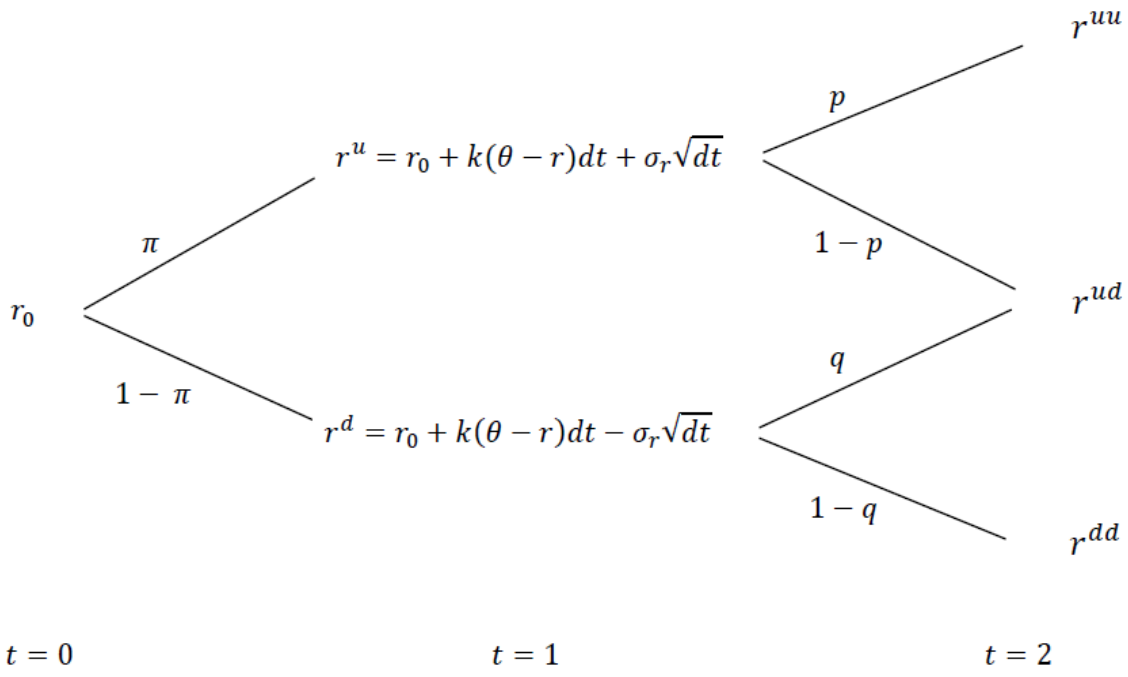


Figure 2: Vasicek interest rate tree

The tree-pricing model goes through a number of processes to value each node in the tree and the corresponding probabilities. As r_0 is a known value and denotes the current short-term rate, at $t = 1$, the up and down nodes can be calculated as $r^u = r_0 + k(\theta - r_t)dt + \sigma_r\sqrt{dt}$ and $r^d = r_0 + k(\theta - r_t)dt - \sigma_r\sqrt{dt}$, where the probability π is assumed to be 0.5.

Figure 3 shows that, on a monthly basis, there are 12 nodes from r_0 to r_1. Thus, when $t = \frac{1}{12}$, the first monthly node becomes:

$$r_{m1}^{u} = r_0 + \frac{k(\theta - r_t)}{12} + \frac{\sigma_r}{\sqrt{12}}.$$

and

$$r_{m1}^{d} = r_0 + \frac{k(\theta - r_t)}{12} - \frac{\sigma_r}{\sqrt{12}}.$$

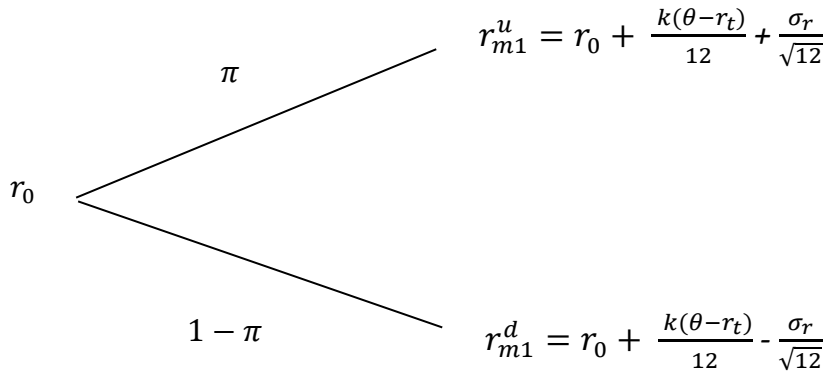


Figure 3: Interest rate pricing for the first month

The nodes at the next step, can be obtained as follows.

For the central node, the drift determines the expected value of the process after each time step. To find r^{ud}, we need to find the expected value of the interest rate at $t = 1$:

$$E(r_1) = r_0 + k(\theta - r_0).$$

Then we find r^{ud} as the expected value of interest rate at $t = 2$:

$$r^{ud} = E(r_1) + k(\theta - r_1).$$

Then, from the definition of the expected rate value, $E(r^u)$, and the definition of the standard deviation, r^{uu} node can be found by solving the following equations.

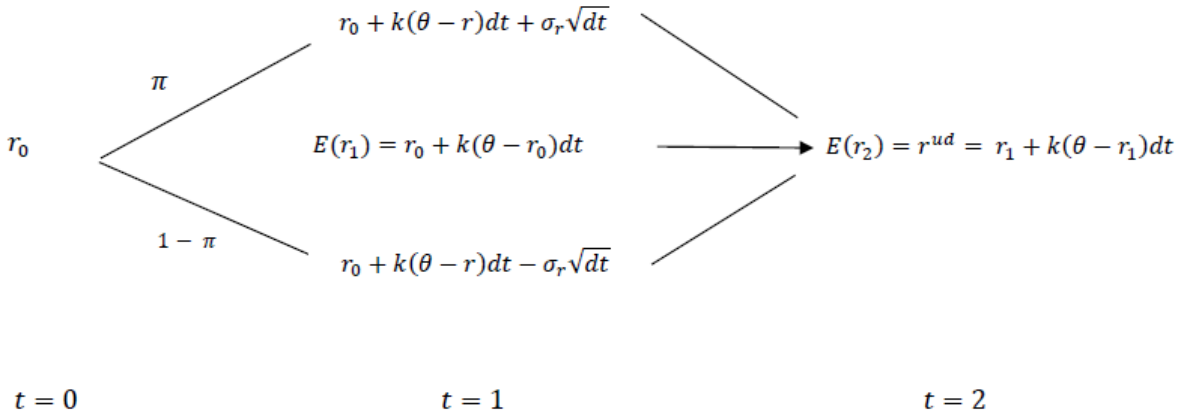


Figure 4: Valuation of r^{ud} as the expected value of the interest rate in step 2

From the expected value of the interest rate, $E(r^u)$, we know that:

$$p \times r^{uu} + (1 - p) \times r^{ud} = E(r^u). \qquad (1)$$

From the standard deviation definition:

$$\sqrt{p(r^{uu} - E(r^u)^2 + (1 - p)(r^{ud} - E(r^u)^2} = \sigma. \qquad (2)$$

By Solving (1) and (2), we obtain r^{uu} and p.

Similarly, r^{dd} node can be obtained by solving the definition of the expected rate value $E(r^d)$ and the definition of standard deviation.

From the expected rate value $E(r^d)$, we know that:

$$q \times r^{ud} + (1 - q) \times r^{dd} = E(r^d). \qquad (3)$$

From the standard deviation:

$$\sqrt{q(r^{ud}\text{-}E(r^d)^2 + (1 - q)(r^{dd}\text{-}E(r^d)^2} = \sigma. \qquad (4)$$

By Solving (3) and (4), we get r^{dd} and q.

For more time periods, the same methodology for interest rate tree pricing can be extended.

We run back data testing for Canadian 5-year zero coupon bond yield as a risk-free rate benchmark with mean reverting parameters from the date 04/01/2012. We construct a 3-year Vasicek model tree as shown in Table 2 below. Figure 5 shows the historical data of Canadian 5-year zero coupon bond yield from 2000 to 2015. The parameters are estimated using the likelihood function suggested by James and Webber (2000). MATLAB is used for tree construction and the associated codes are provided below.

	Jan 2012	Jan 2013	Jan 2014	Jan 2015
Rates	r_0	r_1	r_2	r_3
	0.0138	0.0162	0.0189	0.0218
	—	0.0104	0.0130	0.0156
	—	—	0.0071	0.0098
	—	—	—	0.0037
Mean	0.0138	0.0133	0.0130	0.0127
Real Market Data	0.0138	0.0147	0.019	0.0132
Parameters	$k = 0.167$	$\theta = 0.0112$	$\sigma = 0.0029$	$\Delta t = 1$

Table 2: Back testing for Canadian 5-year zero coupon bond yield

Figure 5: Canadian Five-Year Zero-Coupon Bond Yields from 2000 to 2015

In Figure 6, we estimate the Vasicek model parameters using 15 years of daily historical data of Canadian 5-year zero- coupon bond yield using the exact form of the likelihood function suggested by James and Webber (2000). Figure 6 presents 10 random paths generated for the short rate and construct the yield curve based on the parameters.

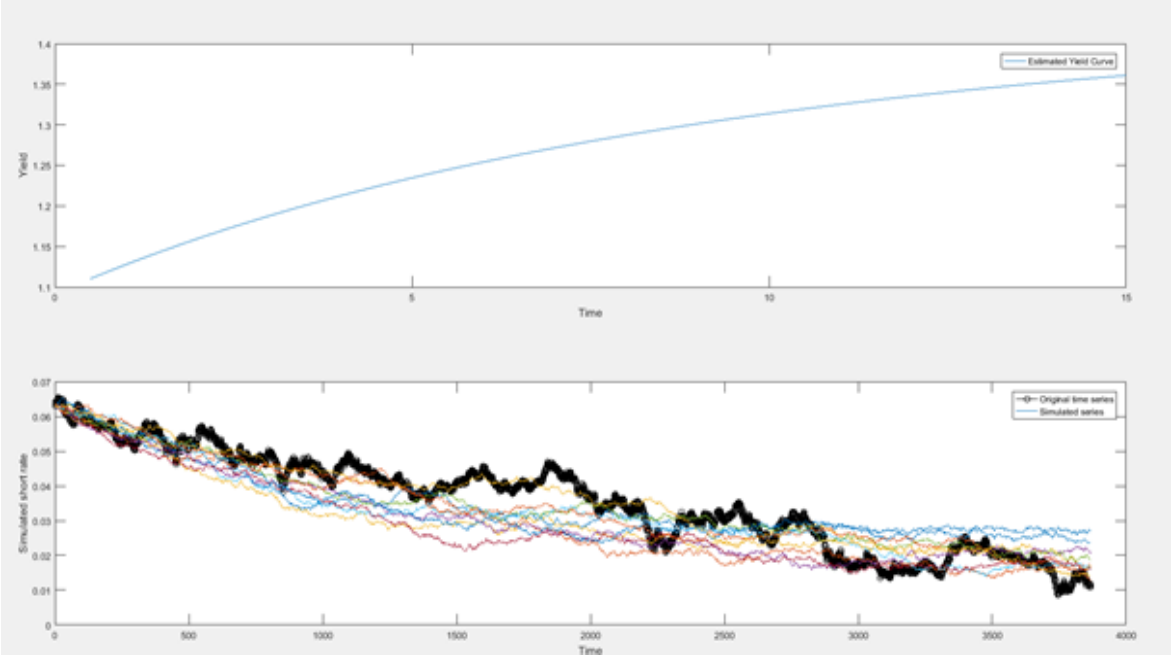


Figure 6: Simulation of Canadian 5-Year zero coupon bond yield from 2000 to 2015

It can be seen in Figure 6 that the black thick line is the original short rate with 10 simulated random paths in thin lines of various colours. The number of the observations in one year is 252, so $dt = 1/252$.

The Yield curve generated for a Vasicek zero-coupon bond $P(r,t)$ along with the random paths described earlier is expressed as

$$P(r,t) = A(t,T)e^{-rB(t,T)}$$

and the yield curve $r(t,T)$ is given by

$$r(t,T) = -log(P(t,T))/(T-t).$$

2.2 Vasicek zero-coupon bond pricing

Vasicek (1977) showed that the value at time t of a zero coupon bond that pays $\$1$ at time T is given by:

$$P(r,t) = A(t,T)e^{-rB(t,T)}$$

where

$$A(t,T) = exp\left\{\left(\theta - \frac{1}{2}\frac{{\sigma_r}^2}{k^2}\right)(B(t,T)-\tau) - \frac{{\sigma_r}^2}{4k}B^2(t,T)\right\},$$

$$B(t,T) = \frac{1-\ e^{-k\tau}}{k}.$$

The governing SDE for the bond price can be expressed as:

$$\frac{dP}{P} = r\ dt + \frac{\sigma_r}{k}(1-\ e^{-k\tau})dW_p.$$

where $\tau = T - t$, is time to maturity and k denotes the speed of mean reversion.

The volatility of the instantaneous rate of return of the zero coupon bond is given by:

$$\sigma_P = \frac{\sigma_r}{k}\,(1 - \,e^{-k\tau}).$$

When $T \to \infty$, the coefficient $(1 - \,e^{-k\tau})$ approaches 1 (Vasicek, 1977).

Thus, the PDE of the Vasicek zero coupon bond pricing is given by:

$$\frac{\partial P}{\partial t} + \mu(r,t)\frac{\partial P}{\partial r} + \frac{1}{2}{\sigma_r}^2\frac{\partial^2 P}{\partial r^2} - rP = 0,$$

where the drift term $\mu(r,t)$ is

$$\mu(r,t) = k(\theta - r).$$

Subject to the final condition that:

$$P(r,T) = 1.$$

3 Stock Price Model

3.1 CRR model

For the equity price tree, we adopt the Cox, Ross and Rubinstein (CRR) model with some modifications. The CRR model assumes a constant volatility σ_s over the periods of the pricing tree. In this study, I assume that the equity volatility σ_s is changeable (non- constant) in different intervals, but remains constant within the same time interval. This assumption will lead to having two central nodes for every advancing step from $t = 2$ onwards as illustrated in Figure 7, instead of one node, as in the normal CRR model. In every individual advancing step, the stock price S process will follow:

$$dS_t = rS_t dt + \sigma_s S_t dW_s(t).$$

If the underlying pays dividends q_s then the process becomes:

$$dS_t = (r - q_s)S_t dt + \sigma_s S_t dW_s(t).$$

In this model, there are two possible states in the market, up or down for each node within the time interval $[t_i, t_{i+1}]$. Suppose that S_O indicates the current stock price. Then after one period of time, the stock price can move up to $S_0 u$ with probability $p_u$or down to $S_0 d$ with probability $p_d = (1 - p_u)$, where $u > 1$ and $0 < d < 1$; and u and d are the magnitude of up and down respectively (Cox, 1979). The parameters u, d, p_u and p_d are stated in the following relations:

$$u = e^{\sigma_s \sqrt{dt}}$$

$$d = e^{-\sigma_s \sqrt{dt}} = \frac{1}{u}$$

In the CRR model, the up node probability p_u when the price is likely to increase is

$$p_u = \frac{(e^{rdt} - d)}{u - d}.$$

If the underlying asset pays dividends q_s then the process becomes:

$$p_u = \frac{(e^{(r-q_s)dt} - d)}{u - d}.$$

The probability p_d if the price decreases at t_1 is

$$p_d = 1 - p_u$$

where r is the risk-free interest rate.

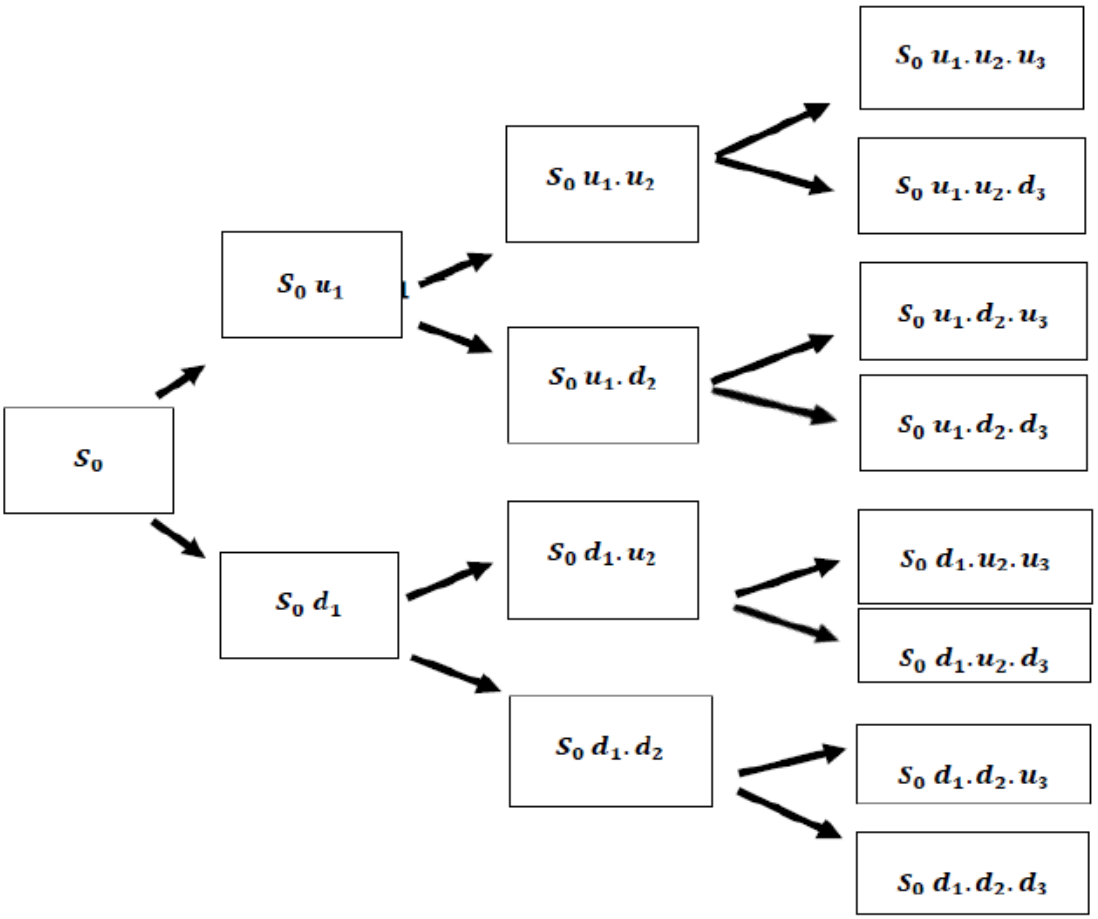


Figure 7: Modified CRR stock price tree

To find the price of the stock for any time fraction within the time intervals $[t_i, t_{i+1}]$ in the pricing tree, we need to reset u, d and Δt according to the new time fraction, and the parameters u and d becomes:

$$u = e^{\sigma_s \sqrt{dt * time\ fraction}}$$

$$d = \frac{1}{u}$$

For example, given a tree in which one step interval represents one year, the expected change in the stock price over the next month when $t = \frac{1}{12}$ is

$$u = e^{\sigma_s \sqrt{\frac{dt}{12}}} \quad \text{and} \quad d = \frac{1}{u}$$

We also run back data testing on the underlying asset of one of the selected convertible bond from Toronto Stock Exchange (TSX). Table 3 shows the primary data of the selected asset.

Issuer	Symbol	Coupon	Start date	Maturity	Last Price	Parity	Yield to Maturity	Premium	Conversion Ratio	Conversion Price	Symbol	Share Price
Advantage Energy	AAV.DB.H	5.00%	04-Jan-10	30-Jan-15	103.13	82.91	0.65%	22.77%	11.63	8.6	AAV	7.18

Table 3: Data for a convertible bond issued by Advantage Energy (AAV) and its underlying asset

We construct a 3-step CRR model tree from 04/01/2012 as shown in Table 4 below. Figure 8 shows the historical data chart of convertible bond and share price movements from the start date of the convertible bond of 04/01/2010 to the maturity date of 30/01/2015 as declared in Table 3.

	Jan 2012	Jan 2013	Jan 2014	Jan 2015
	S_0	S_1	S_2	S_3
	4.3600	5.9445	8.6061	12.7111
	—	3.1978	4.1061	5.8268
	—	—	4.6296	6.0646
	—	—	2.2089	2.7801
	—	—	—	6.8378
	—	—	—	3.1345
	—	—	—	3.2624
	—	—	—	1.4955
Mean	4.3600	4.5711	4.8876	5.2641
Real Market Data	4.3600	3.2	4.63	5.56

Table 4: 3- year CRR model tree for the Advantage Energy (AAV) share price

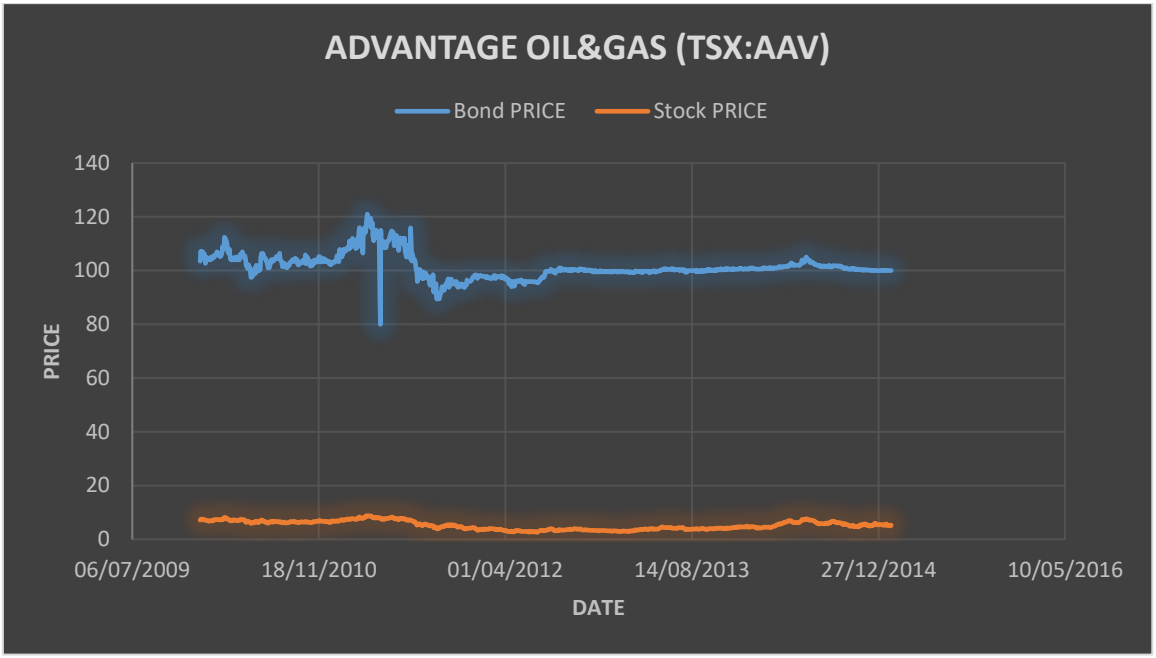


Figure 8: Historical prices of the convertible bond and its underlying stock price for Advantage Energy (AAV)

4 Convertible bonds pricing model involving two-stochastic factors

4.1 Deriving the PDE for the convertible bond option

In this section, we adopt similar ideas that were widely used in other research papers in deriving the Cox, Ingersoll, and Ross (1985) equation and Hull and White's (1990) (HW) framework, for example in Carayannopoulos (1996) and Barone, Bermudez and Hatgioannides (2003). We assume that the interest rate r follows the mean-reversion process suggested by Vasicek (1977).

When interest rates r and stock prices S are stochastic, the option price has a value of the form

$$C = C(S, r, t).$$

The value of the option is now a function of both S and r. We assume that the stock price process is governed by the CRR model

$$dS = rSdt + \sigma_s S dW_s(t).$$

and that the interest rate is modeled by the Vasicek model

$$dr = k(\theta - r)dt + \sigma_r dW_r(t).$$

Here $\{W_s(t),\ t \geq 0\}$ and $\{W_r(t),\ t \geq 0\}$ are two standard Brownian motions with zero-correlation between the interest rate and the stock price, that is,

$$(dW_s(t), dW_r(t)) = 0.$$

Similar to Li et al. (2008), the Itô's formula for the two random variables governed by dS and dr leads to

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}(rS_t + \sigma_s S_t dW_1) + \frac{\partial C}{\partial r}(k(\theta - r_t)dt + \sigma_r dW_2) + \frac{1}{2}\left(S^2 {\sigma_s}^2 \frac{\partial^2 C}{\partial S^2} + {\sigma_r}^2 \frac{\partial^2 C}{\partial r^2}\right)dt.$$

For the option price, following similar idea of Ugur (2008) for deriving Black-Scholes differential equation, consider a portfolio that consists of a short sell of a European option and long $\Delta_1$units of the underlying asset and long Δ_2 units of the zero-coupon bond price. The portfolio Π has the value

$$\Pi = \Delta_1 S_t + \Delta_2 P(t,T) - C(S,r,t).$$

Differentiating Π gives that

$$d\Pi = \Delta_1 dS_t + \Delta_2 dP(t,T) - dC(S,r,t).$$

From the Vasicek model, we know that

$$dP = \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial r}dr + \frac{1}{2}{\sigma_r}^2 \frac{\partial^2 P}{\partial r^2}dt.$$

Then

$$d\Pi = -\left(\frac{\partial C}{\partial t} + \frac{1}{2}S^2 {\sigma_s}^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2}{\sigma_r}^2 \frac{\partial^2 C}{\partial r^2}\right)\mathrm{dt} + \left(\Delta_1 - \frac{\partial C}{\partial S}\right)dS + \left(\Delta_2 \frac{\partial P}{\partial r} - \frac{\partial C}{\partial r}\right)dr + \Delta_2\left(\frac{\partial P}{\partial t} + \frac{1}{2}{\sigma_r}^2 \frac{\partial^2 P}{\partial r^2}\right)dt.$$

We can choose Δ_1 and Δ_2 that eliminate risk from the portfolio

$$\Delta_1 = \frac{dC}{dS}.$$

$$\Delta_2 = \frac{dC/dr}{dP/dr}.$$

Thus,

$$d\Pi = -\left(\frac{\partial C}{\partial t} + \frac{1}{2}S^2{\sigma_s}^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}{\sigma_r}^2\frac{\partial^2 C}{\partial r^2}\right)dt + \Delta_2\left(\frac{\partial P}{\partial t} + \frac{1}{2}{\sigma_r}^2\frac{\partial^2 P}{\partial r^2}\right)dt.$$

From the PDE of the Vasicek zero- coupon bond

$$\frac{\partial P}{\partial t} + \mu(r,t)\frac{\partial P}{\partial r} + \frac{1}{2}{\sigma_r}^2\frac{\partial^2 P}{\partial r^2} - rP = 0.$$

We can rewrite the formula, so that

$$\frac{\partial P}{\partial t} + \frac{1}{2}{\sigma_r}^2\frac{\partial^2 P}{\partial r^2} = rP - \mu(r,t)\frac{\partial P}{\partial r}.$$

The term of the drift $\mu(r,t)$ becomes

$$\mu(r,t) = k(\theta - r).$$

We can rewrite $d\Pi$, so that

$$d\Pi = -\left(\frac{\partial C}{\partial t} + \frac{1}{2}S^2{\sigma_s}^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}{\sigma_r}^2\frac{\partial^2 C}{\partial r^2}\right)dt + \Delta_2\left(rP - k(\theta - r)\frac{\partial P}{\partial r}\right)dt. \qquad (a)$$

In a non-arbitrage market, the condition that the portfolio value earns the risk free rate r implies that the change in the portfolio is:

$$d\Pi = r\Pi dt.$$

$$d\Pi = r\big(\big(\Delta_1 S_t + \Delta_2 P(t,T) - C(S,r,t)\big)dt.$$

$$d\Pi = (\Delta_1 r S_t + \Delta_2 rP - rC)dt. \qquad (b)$$

From Eqns (a) and (b), we obtain:

$$(\Delta_1 rS_t + \Delta_2 rP - rC)dt = -\left(\frac{\partial C}{\partial t} + \frac{1}{2}S^2{\sigma_s}^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}{\sigma_r}^2\frac{\partial^2 C}{\partial r^2}\right)dt$$
$$+\Delta_2\left(rP - k(\theta - r)\,\frac{\partial P}{\partial r}\right)dt.$$

This equation can be rearranged so that the PDE of the Vasicek model becomes

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S} + \frac{1}{2}S^2{\sigma_s}^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}{\sigma_r}^2\frac{\partial^2 C}{\partial r^2}$$
$$+k(\theta - r)\,\frac{\partial C}{\partial r} - rC = 0.$$

If the underlying asset pays dividend q_s and we assume that the risk premium λ enters into a risk – neutral process, then

$$\frac{\partial C}{\partial t} + (r - q_s)S_t\frac{\partial C}{\partial S} + \frac{1}{2}S^2{\sigma_s}^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}\,{\sigma_r}^2\frac{\partial^2 C}{\partial r^2}$$
$$+k(\left[r_\infty + \frac{\lambda}{k}\right] - r)\,\frac{\partial C}{\partial r} - rC = 0.$$

Recall that

$$\theta = \left[r_\infty + \frac{\lambda}{k}\right].$$

This is the PDE for the option with stochastic interest rate and asset. The right side is $= 0$ which indicates that the option has a European style.

4.2 Conditions and solutions

Following Li et al. (2008), consider boundary conditions for a call option (European). Note, interest rates which are negative are not considered.

At the maturity time T, the price of the call option becomes payoff function

$$C\,(S, r, T)\;=\;max(S - K, 0),$$

In the above equation, the stock price is represented by S, t is the time in terms of years and strike price is given by K.

At $S = 0$, the option is worthless:

$$C\ (0, r, t) = 0$$

At $S = S_{max}$, having S_{max} which is sufficiently large to show also solution behaviours as $S \to \infty$, hence we obtain a payoff $S(T) - K$ during time(T) the expiration time. Value present at t needs the exercise price K to be discounted back and taking to account that $S_{max}$is the time t price of underlying asset, tan hen the boundary condition appropriate is

$$C\ (S_{max}, r, t) = S_{max} - \ KP(r, t),$$

where $P(r, t)$ is the zero-coupon bond with $P(r, T)$= 1.

At $r \to \ \infty$, the price S and the value of the option are assumed to have a linear relationship with each other since the value of the bond diminishes down to zero and hence the discounting part is not present. Hence the price of the option becomes only the price of underlying stock:

$$C\ (S, r_{max}, t) = S.$$

For $r = 0$, certain PDE terms disappear and others presume simpler forms and hence according to Li et al boundary value problem is noted as;

$$\frac{\partial C}{\partial t} + \frac{1}{2} S^2 {\sigma_s}^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} {\sigma_r}^2 \frac{\partial^2 C}{\partial r^2}$$
$$+ k(\theta - r)\ \frac{\partial C}{\partial r} = 0.$$

The price of convertible bonds which are converted to stocks only at expiration may be noted as portfolio incorporating a long position for a single call share with

an exercise price K and principal F zero-coupon bond which at maturity they have $1 payoff. Hence convertible bond value is;

$$V(S,r,t) = C(S,r,t) + FP(r,t),$$

According to Otto (2000), the value of the European call option can be written as

$$C(S,r,t) = SN(d_1) - KP(r,t)N(d_2)$$

Therefore, the value of the convertible bond can be expressed as

$$V(S,r,t) = SN(d_1) - KP(r,t)N(d_2) + FP(r,t), \qquad (c)$$

where

$$d_1 = \frac{ln(S/K) - ln\,P(r,t) + (r - q_s)\frac{1}{2}\hat{\sigma}^2\tau}{\hat{\sigma}\sqrt{\tau}},$$

$$d_2 = d_1 - \hat{\sigma}\sqrt{\tau}.$$

where $\hat{\sigma}$ is the term volatility of the convertible bond in terms of risk-free bond price and expressed as

$$\hat{\sigma}^2 = {\sigma_s}^2 + {\sigma_P}^2.$$

From the Vasicek (1977) model that introduced in Section 2.2, we have

$$\sigma_P = \frac{\sigma_r}{k}(1 - e^{-k\tau}),$$

Therefore

$$\hat{\sigma}^2 = {\sigma_s}^2 + \frac{{\sigma_r}^2}{k^2}(1 - e^{-k\tau})^2.$$

4.3 American convertible bond

Since the location of the boundary the American option is not known in advance, this situation creates a free boundary problem. As there is no obvious explicit solution for the American convertible bond, we use the binomial tree model as a numerical method to value the convertible bond in the next section.

4.4 Two-stochastic-factor tree model

In this section, the binomial tree is used as a numerical model to value the convertible bond. Once the interest rate tree and the stock price tree are constructed, we combine the two trees into one single tree to find the price of the convertible non-callable bond as well as that of a callable bond. It is important to mention that we are combining two trees with different structures. As the interest rate tree has one central node and the stock price tree has two central nodes, I will use the single central node in the interest rate tree at $t \geq 2$ for both central nodes in the stock tree at $t \geq 2$. The same methodology is applicable for the nodes in other time intervals. As I assume no default risk for now, each node should have four branches for the movements of both the interest rate tree and the stock price factors, as seen in Figure 9 below.

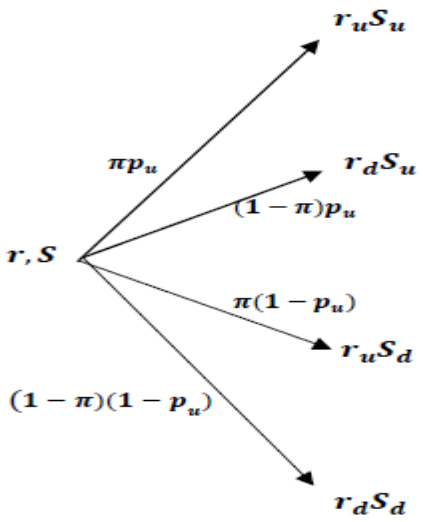


Figure 9: One-period two-stochastic factor tree model with no default risk

The probability for each pathway should involve both the interest rate and equity probabilities, as shown in Table 5 below.

Pathway	$r_u S_u$	$r_d S_u$	$r_u S_d$	$r_d S_d$
Probability	πp_u	$(1-\pi)p_u$	$\pi(1-p_u)$	$(1-\pi)(1-p_u)$

Table 5: Probabilities of one-period two-stochastic factor tree model with no default risk

After constructing the combined tree, we use the backward induction method to price the convertible bond. Therefore, we start the process from the maturity date backwards to the initial point. At the maturity date, the price of the convertible bond should satisfy

$$V = max[CV,\ F] + c$$

CV is the conversion value, F is the face value and c is the coupon value.

When α is the conversion ratio, which is defined as the number of shares into which each convertible bond can be converted, the conversion right can be represented by the *conversion value,* the price at which the bonds can be converted into common stocks. The conversion value can be expressed in terms of the conversion ratio, as follows:

$$CV = \alpha S$$

At any time t_i for each node of the tree, the price of the convertible bond should satisfy

$$V = max[CV,\ B]$$

where B is the value of the straight bond.

As we mentioned earlier, the value of the straight bond component at the maturity T is the combination of the face value and the coupon interest that is paid off at maturity, whereas the value of the straight bond component at any time t_i is calculated as the average weighted present value of the up and down nodes at t_{i+1}.

Assuming the probability of default is set to be zero, the value of the straight bond component at any time t_i is

$$B = \pi\left(\frac{B_u + c}{1 + r_*}\right) + (1 - \pi)\left(\frac{B_d + c}{1 + r_*}\right)$$

where r_* is the interest rate of the node that has already been determined by the interest rate tree.

The tree in Figure 10 shows an example of the nodes pricing methodology of the convertible bond valuation model. In this tree we price the node $r_{uu}S_{uu}$ at t_2 of 3-period two-factor tree model where, the t_3 nodes represent the maturity T of the convertible bond. At the maturity date $T = t_3$, the bondholder has the right to convert the bond to common stocks at the conversion ratio α or receive the principal F plus the final coupon c. The targeted node that to be priced in this example $r_{uu}S_{uu}$ is equal to the maximum of either the present value of the bond's expected value at maturity discounted on r_{uu} or its conversion value CV.

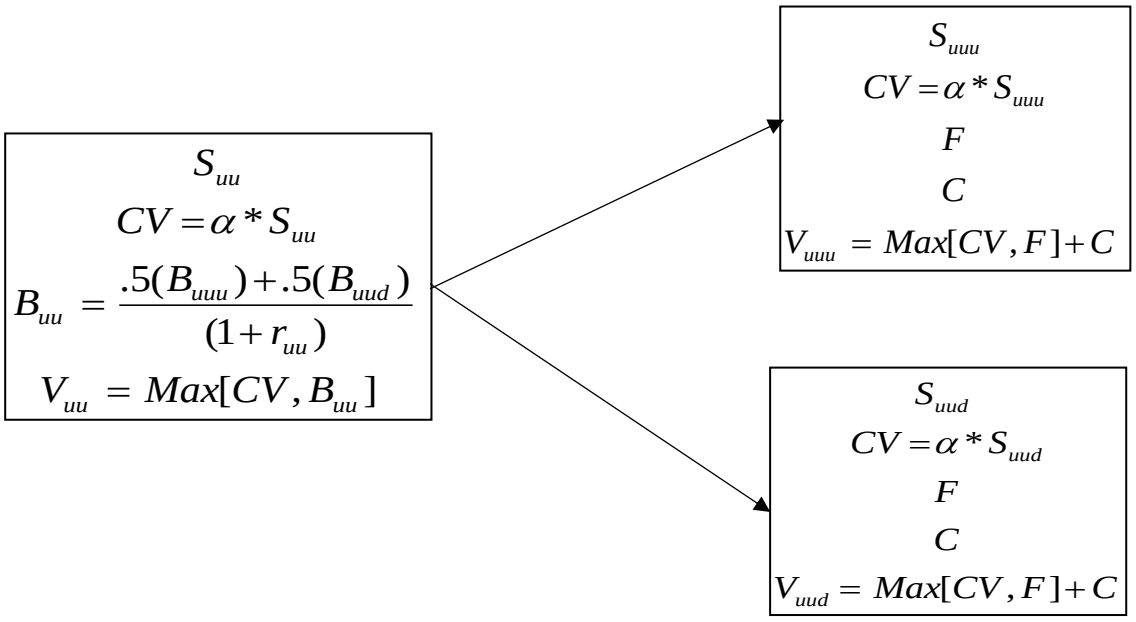


Figure 10: $r_{uu}S_{uu}$ node pricing with the two-stochastic-factor tree model from maturity

When the convertible bond has some advanced features, such as call and put options (callable and puttable convertible bonds), the issuer will find it profitable to call the convertible prior to maturity whenever the price of the convertible is greater than the call price. When the convertible bondholder is faced with a call, he usually has the choice to either redeem the bond at the call price or convert the bond to common stocks.

If the call price is below the bond value but above the conversion value, the issuer would call the bond, and the bondholders would take the call instead of converting.

At t_i when the call option is applicable, the callable convertible bond value satsify

$$V^C = Max[CV,\ Min[B, CP]],$$

where CV is the conversion value, B is the bond price, and CP is the call price.

In Figure 11, when a call provision is applied, the targeted node that is to be priced in this example, $r_{uu}S_{uu}$, rolling back from maturity, is equal to the maximum of its conversion value CV and the minimum of the straight bond component and the call price CP discounted on r_{uu}.

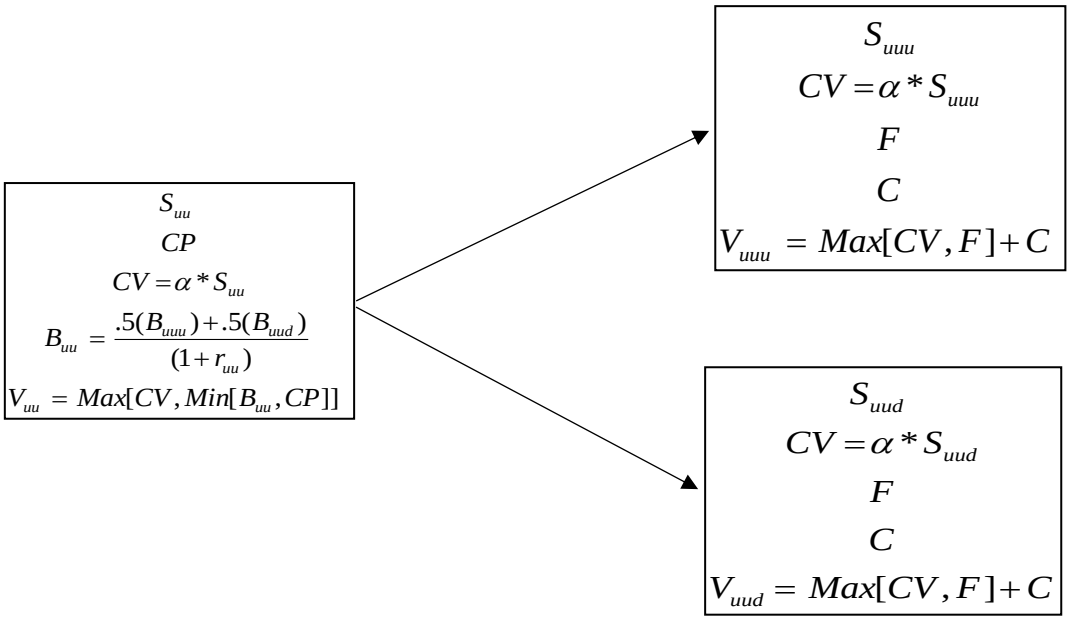


Figure 11: $r_{uu}S_{uu}$ node pricing of the callable convertible bond tree model from maturity

The tree in Figure 12 shows the complete tree for three periods of the two-stochastic-factor tree model.

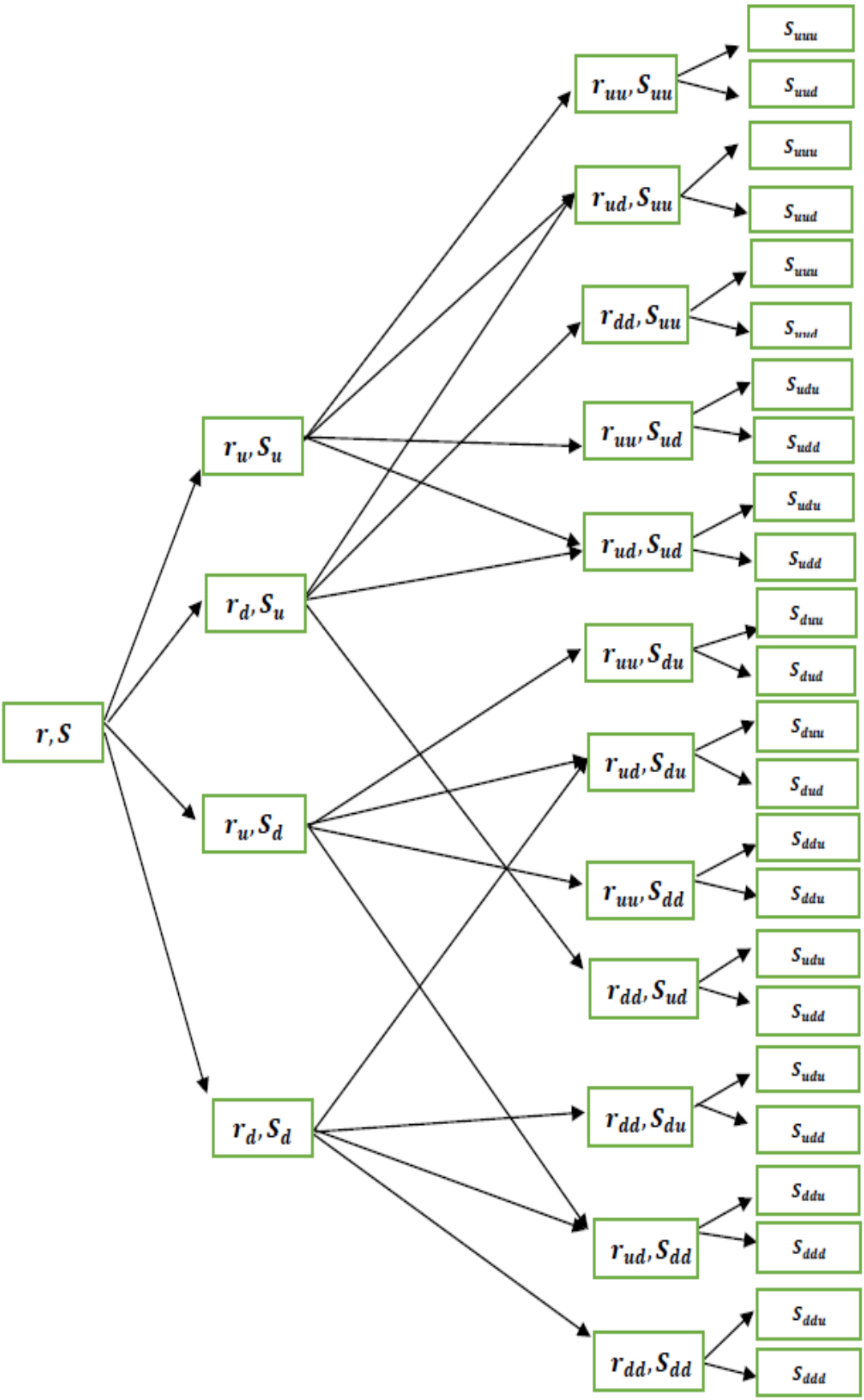


Figure 12: Three-period two-stochastic-factor tree model

4.4.1 Soft call and hard Call

A callable bond allows the issuer to redeem the bond from the bondholders at a known exercise date before its maturity date. The convertible bond's call provision usually has two features: soft-call and hard-call protection. Soft-call protection for a convertible bond V usually means that the bond can be recalled by the issuer only if the stock price has previously closed above a specified trigger price for any 20 of the 30 consecutive trading days prior the exercise date (Navin, 1999). On the other hand, the hard-call feature allows the issuer of a convertible bond to redeem the convertible bond before maturity by paying the call price to the bondholders. Hard-call protection restricts the issuer from exercising the bond's call provision prior to the hard-call date. However, the bond can be exercised or redeemed by the issuer at any time between the hard-call date and its maturity date. It is worth noting that not all callable bonds necessarily have these two features in the same issue. As can be seen in Table 6, some convertibles have both hard and soft calls. Other convertibles have only soft or hard calls. Few convertibles include no call provision features at issue. Table 6 shows the CIBC report of Canadian convertible bonds, including soft- and hard-call dates. We will use both soft and hard call in the numerical examples.

Convertible Debenture Report

As at market close on Monday, 11 July 2016 — Retail Trading: 416 594 8557 — Institutional Trading: 416 594 8553

CIBC

Convertible Debenture							Underlying Security				Conversion					Yields		Payment	Call Feature		
Size ($M)	Symbol	Issue	Coupon (%)	Maturity	Bid	Ask	Symbol	Bid	Div	Yield	Price	Shares / $100	Parity	Premium (%)	Cashflow Payback	Current (%)	YTM (%)	Dates	Soft Call	Trigger / Conv. Price	Hard Call
Oil & Gas																					
50.00	CJ.DB	Cardinal Energy	5.50	12/31/2020	110.00	115.00	CJ	9.36	0.42	4.49%	$10.50	9.524	89.14	29.0	17.2	4.78	3.21	31-Dec 30-Jun	31-Dec-18	13.125	31-Dec-20
40.00	EPS.DB	Epsilon Energy Ltd/Canada	7.75	3/31/2017	99.51	100.00	EPS	3.25	-	-	$4.45	22.472	73.03	36.9	3.5	7.75	7.75	30-Sep 31-Mar	31-Mar-15	5.56	None
23.75	GNF.DB	Greenfields Petroleum	9.00	5/31/2017	11.35	39.95	GNF	0.25	-	-	$8.55	11.700	2.87	1293.7	4.1	22.53	35.78	30-Nov 30-May	01-Jun-15	10.68	None
150.00	PGF.DB.B	Pengrowth Energy	6.25	3/31/2017	96.60	97.00	PGF	2.29	-	-	$11.51	8.687	19.89	387.6	12.3	6.44	7.20	31-Mar 30-Sep	31-Mar-15	14.390	None
97.75	TGL.DB	TransGlobe Energy Corp	6.00	3/31/2017	94.51	95.25	TGL	2.21	-	-	$13.63	7.337	16.21	487.4	13.2	6.30	7.51	30-Sep 31-Mar	31-Mar-15	17.038	31-Mar-17
85.00	TBE.DB	Twin Butte	6.25	12/31/2018	15.55	16.00	TBE	0.05	-	-	$3.05	32.787	1.64	876.0	2.3	39.06	37.94	30-Jun 31-Dec	31-Dec-16	3.812	31-Dec-17
57.50	ZAR.DB	Zargon Oil & Gas	6.00	6/30/2017	49.00	53.90	ZAR	0.44	-	-	$18.80	5.319	2.34	2203.0	8.6	11.13	23.45	31-Dec 30-Jun	30-Jun-15	23.500	None
Pipelines																					
172.50	PPL.DB.F	Pembina Pipeline Corp	5.75	12/31/2018	90.02	137.50	PPL	39.54	1.92	4.86%	$29.53	3.386	133.90	2.7	(4.8)	4.18	(1.08)	31-Dec 30-Jun	31-Dec-14	36.912	31-Dec-16
Gas/Propane/Electricity Distribution																					
330.00	JE.DB	Just Energy Group Inc. [1] [4] [5] [6]	6.00	6/30/2017	99.15	99.50	JE	7.93	0.50	6.31%	$18.00	5.556	44.06	125.9	17.2	6.03	6.15	30-Jun 31-Dec	30-Jun-13	22.500	30-Jun-15
100.00	JE.DB.B	Just Energy Group Inc. [1] [4] [5] [6] [9]	5.75	9/30/2018	99.00	99.64	JE	7.93	0.50	6.31%	$17.85	5.602	44.43	124.3	18.7	5.77	5.83	31-Mar 30-Sept	30-Sep-14	22.313	30-Sep-16
150.00	SPB.DB.F	Superior Plus Corp [1] [4] [7] [9] [23]	6.00	6/30/2018	101.75	102.00	SPB	11.10	0.72	6.49%	$15.10	6.623	73.51	38.8	23.1	5.88	5.53	30-Jun 31-Dec	30-Jun-14	18.875	30-Jun-16
Power																					
130.00	ATP.DB.U	Atlantic Power [19]	5.75	6/30/2019	96.00	96.65	ATP	3.23	-	-	$17.25	5.797	18.72	416.2	13.6	5.95	6.43	30-Dec 30-Jun	None	21.563	None
100.00	ATP.DB.D	Atlantic Power	6.00	12/31/2019	97.75	98.45	ATP	3.23	-	-	$14.50	6.897	22.28	342.0	12.7	6.09	6.29	30-Jun 31-Dec	None	18.125	31-Dec-17
143.75	BLX.DB.A	Boralex Inc [1] [4] [5] [7] [9]	4.50	6/30/2020	105.05	117.40	BLX	20.00	0.56	2.80%	$19.60	5.102	102.04	15.1	9.3	3.83	1.80	31-Dec 30-Jun	30-Jun-18	24.500	30-Jun-19
2185.00	EMA.IR	Emera Inc	4.00	9/29/2025	50.68	50.91	EMA	49.40	2.09	4.23%	$41.85	2.389	118.04	(56.9)	67.5	7.86	10.69	16-Nov 31-May	None	52.312	None
75.00	NPI.DB.B	Northland Power	5.00	6/30/2019	110.00	112.00	NPI	22.54	1.08	4.79%	$21.60	4.630	104.35	7.3	(19120349.4)	4.46	2.77	30-Jun 31-Dec	30-Jun-17	27.000	30-Jun-18
150.00	NPI.DB.C	Northland Power	4.75	6/30/2020	110.00	113.00	NPI	22.54	1.08	4.79%	$21.60	4.630	104.35	8.3	(34.6)	4.20	2.67	30-Jun 31-Dec	30-Jun-19	27.000	30-Jun-20
100.00	INE.DB.A	Innergex Renewable Energy [1] [4] [5] [7]	4.25	8/31/2020	107.75	109.00	INE	14.62	0.64	4.38%	$15.00	6.667	97.47	11.8	(691.9)	3.90	2.84	28-Feb 31-Aug	31-Aug-18	18.750	31-Aug-19
Mining & Materials																					
22.00	EFR.DB	Energy Fuels Inc	8.50	6/30/2017	99.00	101.00	EFR	2.93	-	-	$15.00	6.666	19.53	417.1	9.6	8.42	8.20	31-Dec 30-Jun	24-Jul-15	18.752	None
50.00	KGI.DB	Kirkland Lake Gold Inc	6.00	6/30/2017	104.75	107.00	KGI	11.90	-	-	$15.00	6.667	79.33	34.9	4.6	5.61	4.03	31-Dec 30-Jun	30-Jun-14	19.500	None
69.00	KGI.DB.A	Kirkland Lake Gold Inc	7.50	12/31/2017	105.25	114.28	KGI	11.90	-	-	$13.70	7.299	86.86	31.6	3.7	6.56	3.92	30-Jun 31-Dec	31-Dec-15	17.810	None

Table 6: CIBC report of Canadian convertible debentures

4.4.2 Call and convert conditions

Toronto Stock Exchange (TSX) applies regulations and conditions in the event that the issue is called or converted. According to a CIBC report, one or more of the following conditions might be applied (CIBC Wood Gundy, 2016).

At maturity or redemption:

i. Cash or stock valued at 95% of the weighted average trading price of shares.
ii. Cash or stock valued at the weighted average or at the market price on the date fixed for redemption.
iii. Interest may be paid in stock or paid from the proceeds of stock sales.
iv. Accrued interest is paid if converted.
v. Accrued interest is paid if converted, but only if there has been a notice of redemption.
vi. Upon change of control, an issuer may purchase debentures at 101.00% plus accrued.
vii. Upon change of control, an issuer may purchase debentures at 100.00% plus accrued.
viii. Upon change of control, an issuer may purchase debentures at 105.00% plus accrued.
ix. Upon change of control and under certain conditions, such as cash consideration, the holder may convert at an adjusted conversion price.
x. The redemption price is valued at 105.00% from the first call date to the second call date. The redemption price is valued at 102.50 % from the second call date maturity.
xi. Accrued interest is paid up to the last record date for distributions on the underlying units.
xii. Assets are convertible to common shares (or units) plus another security, such as notes or contingency value receipts.
xiii. Upon conversion, the issuer may elect to deliver cash instead of stock.

4.5 Two-stochastic-factor tree model with default risk

Although a convertible bond consists a straight bond component, default risk may occur during the life of the bond. If the issuer of a bond defaults, the bondholders do not receive the full principal but rather receive a portion of the face value F that is called the recovery rate δ. Therefore, a new branch will be added to the tree model to represent the case of default and the probability of default λ. Note that the default is assumed to occur over the same time interval as that of the pricing tree.

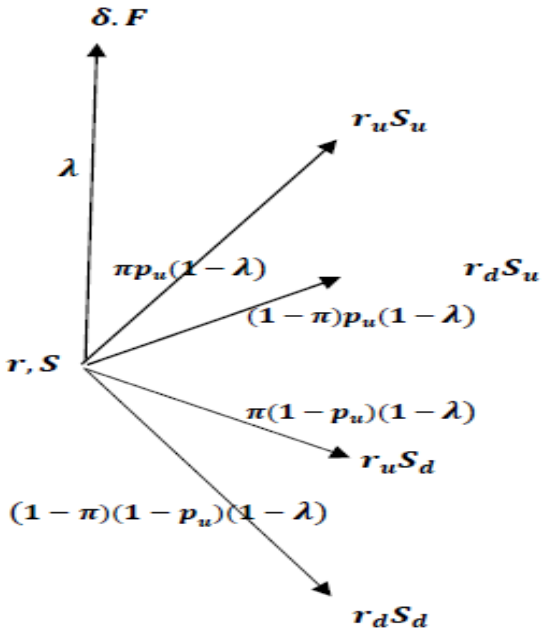


Figure 13: One-period two-stochastic-factor tree model with default risk

The probability of default is added to each pathway of the tree, and the total probability of each node is denoted as the average of the probabilities of both factors, as shown in Table 7.

Pathway	δ	$r_u S_u$	$r_d S_u$	$r_u S_d$	$r_d S_d$
Probability	λ	$\pi p_u(1-\lambda)$	$(1-\pi)p_u(1-\lambda)$	$\pi(1-p_u)(1-\lambda)$	$(1-\pi)(1-p_u)$ $(1-\lambda)$

Table 7: Probabilities of one-period two-stochastic-factor tree model with default risk

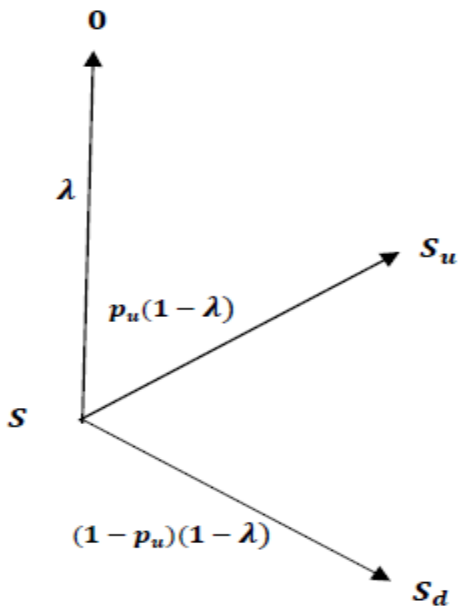


Figure 14: One-period stock price model with default risk

As discussed previously, in the CRR model, the stock price moves from start date $t_0$to the next interval $t_1$by multiplying S_0 by u and d for the up and down nodes, respectively. When a default occurs and the corporation goes to bankruptcy, the stock price jumps to zero in the convertible bond valuation model, as the bond is no longer converted. In the risk-neutral world, the value of a derivative security equals the present value of the expected payoff, so the default probability should be now taken into account to meet the no-arbitrage condition

$$Se^{(r-q_s)dt} = p_u(1-\lambda)S_u + (1-p_u)(1-\lambda)S_d + 0 \cdot \lambda,$$

where the expected yield rate is $r - q_s$, r is the risk-free interest rate, and q_s is the stock's continuous dividends yield.

In the case of a default, p_u becomes

$$p_u = \frac{\left(\frac{e^{(r-q_s)dt}}{(1-\lambda)-d}\right)}{u-d},$$

$$p_d = 1 - p_u.$$

Following Tsiveriotis and Fernandes (1998), Hull (2003) suggested that the value of a convertible bond is a combination of two components: a risk-free component and a risky component. The risk-free component represents the value of the convertible bond if it converts to equity, whereas the risky component represents the value of the convertible bond if it ends up as a bond.

Following Jarrow and Turnbull's (1995) approach, in order to find the probability of default λ, I need to find the risky bond price given the risky interest rate $\hat{r}$. This approach implies that the debt component should be discounted using an interest rate that reflects the credit risk of the issuer. The risky interest rate $\hat{r}$ can be determined by adding a credit spread r_c to the risk-free interest rate r. This spread is a representation of the credit spread implied by non-convertible bonds from the same issuer with maturities similar to that of the convertible bond. On the other hand, the credit spread is often obtained from the credit rating given to a defaultable corporate bond by credit rating agencies like Standard & Poor's, Fitch Ratings, and Moody's.

4.5.1 Credit spread (r_c)

The component of the risk premium or yield spread attributable to default risk is called the credit spread. The credit spread is the difference between the yield of a default-free bond and that of a defaultable bond of similar maturity with a different credit quality. The credit-spread risk is the risk that an issuer's debt obligation will decline due to an increase in the credit spread.

$$spread = (yield\ on\ the\ security) - (risk - free\ yield)$$

Therefore, the risky interest rate $\hat{r}$ can be expressed as

$$\hat{r} = r + r_c.$$

4.5.2 Default risk probability (λ)

Suppose that I price a four-year convertible bond with default risk and the probabilities of the default parameters are $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$. Then, the risky bond price is at $t_1$is

$$e^{-\hat{r}_1+r_0} = (1-\boldsymbol{\lambda_1}) + \delta\boldsymbol{\lambda_1}.$$

where $\hat{r}_1$ is the one-year risky interest rate.

Then, at t_1, the default probability λ_1 becomes

$$\lambda_1 = (1-e^{r_0-\hat{r}_1})/(1-\delta).$$

At t_2,

$$e^{-2\hat{r}_2+r_0} = \pi(1-\lambda_1)(1-\lambda_2+\delta\lambda_2)\cdot(e^{-r_{u1}}+e^{-r_{d1}}) + \delta\lambda_1.$$

At t_3,

$$\begin{aligned} e^{-3\hat{r}_3+r_0} = {} & \pi^2(1-\lambda_1)(1-\lambda_2)(1-\lambda_3+\delta\lambda_3) \\ & \cdot(e^{-r_{u1}}(e^{-r_{uu2}}+e^{-r_{ud2}}) + e^{-r_{d1}}(e^{-r_{ud2}}+e^{-r_{dd2}}) \\ & + \pi(1-\lambda_1)\delta\lambda_2\cdot(e^{-r_{u1}}+e^{-r_{d1}}) + \delta\lambda_1. \end{aligned}$$

At t_4,

$$\begin{aligned} e^{-4\hat{r}_4+r_0} = {} & \pi^3(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)(1-\lambda_4+\delta\lambda_4) \\ & \cdot\{e^{-r_{u1}}[e^{-r_{uu2}}(e^{-r_{uuu3}}+e^{-r_{uud3}}) \\ & + (e^{-r_{ud2}}(e^{-r_{uud3}}+e^{-r_{udd3}})] + e^{-r_{d1}}[e^{-r_{ud2}}(e^{-r_{uud3}}+e^{-r_{udd3}}) \\ & + e^{-r_{dd2}}(e^{-r_{udd3}}+e^{-r_{ddd3}})]\} + \pi^2(1-\lambda_1)(1-\lambda_2)\delta\lambda_3 \\ & \cdot[e^{-r_{u1}}(e^{-r_{uu2}}+e^{-r_{ud2}}) + e^{-r_{d1}}(e^{-r_{ud2}}+e^{-r_{dd2}})] \\ & + \pi(1-\lambda_1)\delta\lambda_2\cdot(e^{-r_{u1}}+e^{-r_{d1}}) + \delta\lambda_1. \end{aligned}$$

In Figure 15, when the call provision and default risk are applied, the targeted node to be priced in this example, $r_{uu}S_{uu}$, rolling back from maturity, is equal to the maximum of its conversion value CV and the minimum of the straight bond component and the call price CP discounted on r_{uu}. It can be seen that the value of the straight bond is the average weighted value of the up and down nodes, including the recovery value that represents the default event. The value of recovery is obtained by multiplying the recovery rate δ as a percentage of the face value or the principal F.

At the strike date of the call option, the callable convertible bond value is

$$V^C = Max[CV,\ Min[B, CP]].$$

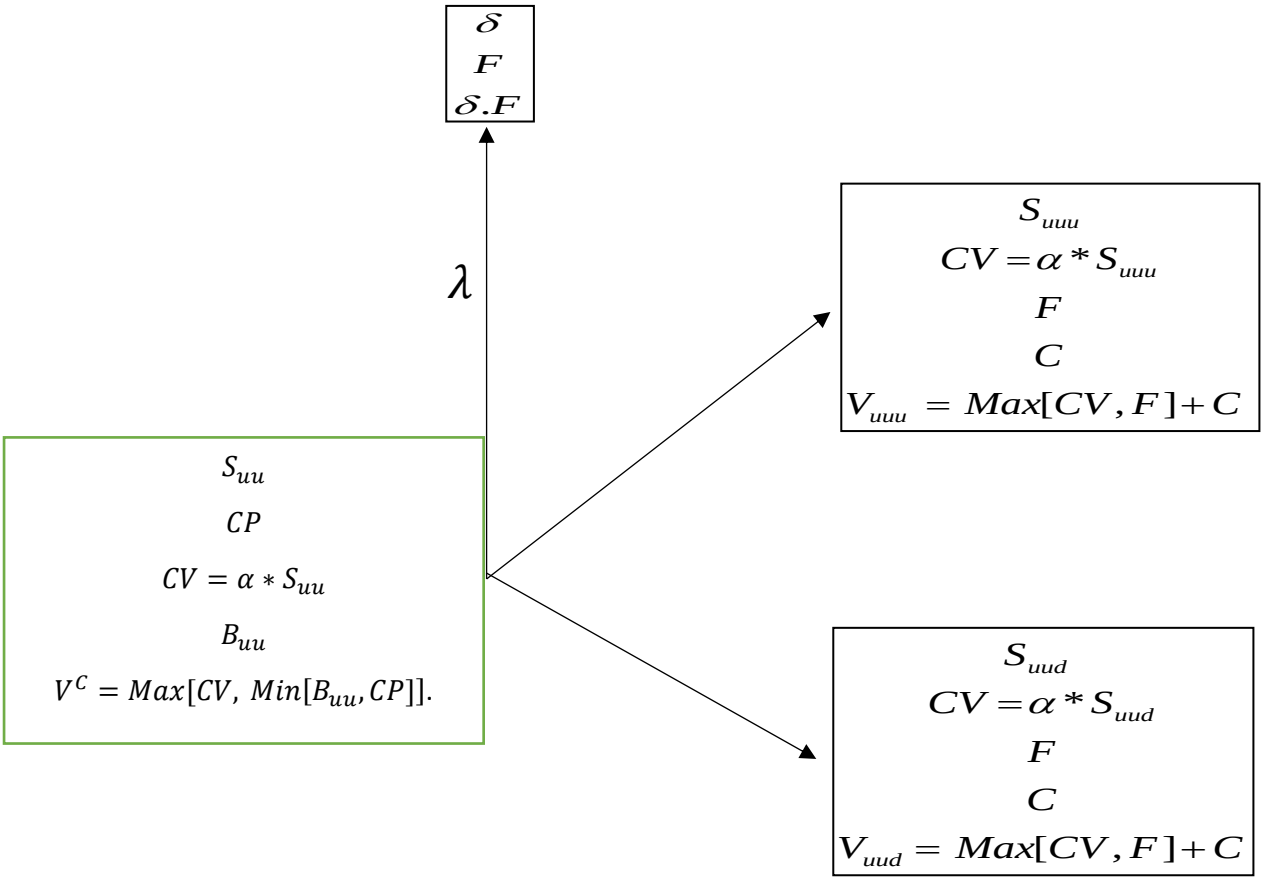


Figure 15: $r_{uu}S_{uu}$ node pricing of the callable convertible bond tree model with the default risk from maturity

Figure 16 shows the complete tree for three periods of two-stochastic-factor tree model with default risk.

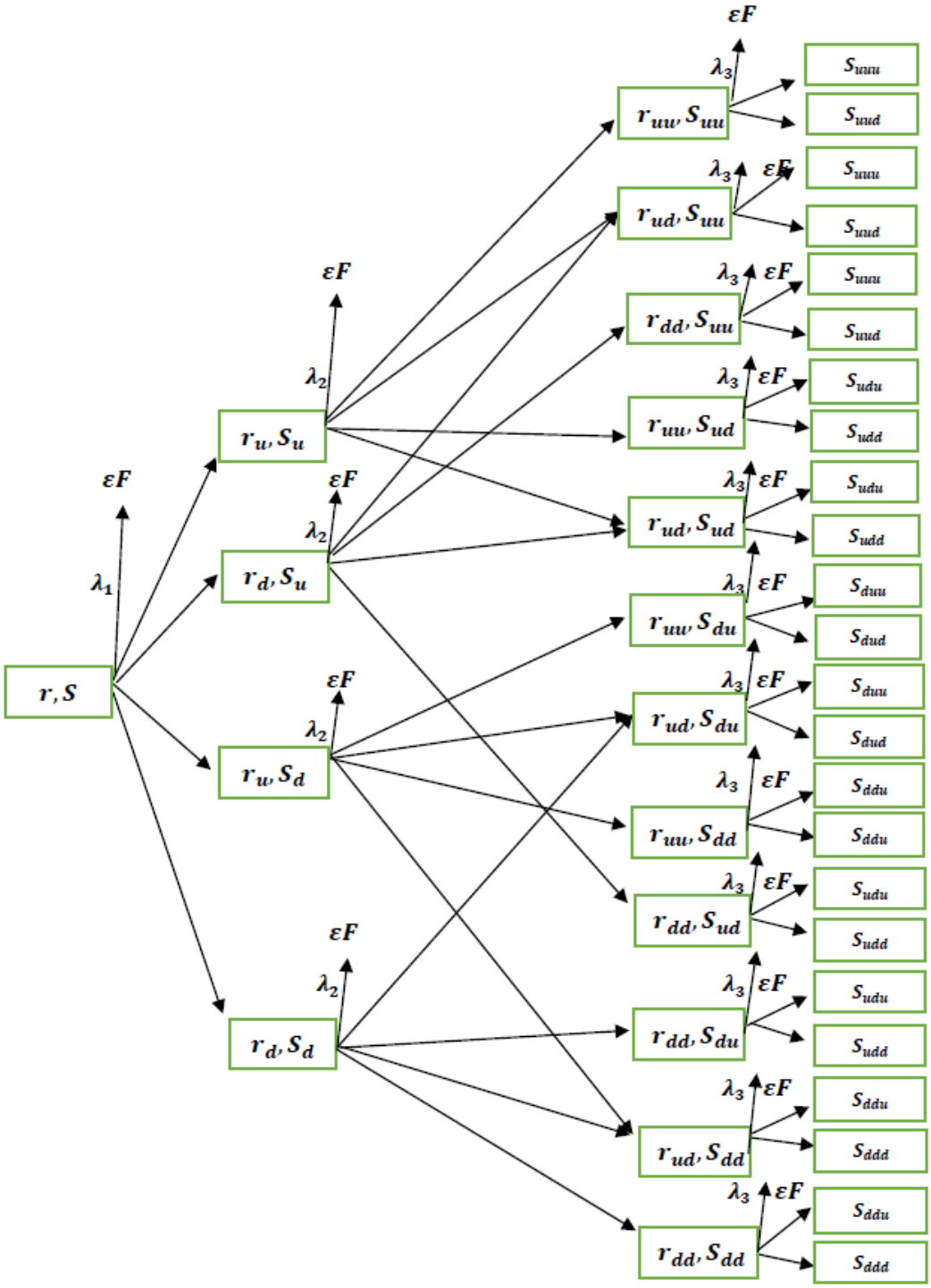


Figure 16: Three-period two-stochastic-factor tree model with credit risk

4.6 Numerical example

In this section, we will use a number of numerical examples to illustrate the valuation model, including an option-free convertible bond and a callable convertible bond with default risk. As discussed in the data section, this study will investigate Canadian convertible bonds issued on the TSX. I will use the numerical examples to compare this model to real market prices in targeted periods. I will use the five-year zero-coupon yields of government bonds as the risk-free interest rate, and I will provide the parameters of the Vasicek interest rate tree. The CRR model is used to price the equity component of the convertible bond, but stock price volatility is not assumed to be constant. Instead, the volatility σ_s is changeable (i.e., non-constant) in different intervals, but it is constant within the same time interval. The option-free convertible bond tree has four branches for each node to represent the two stochastic factors of the interest rate and equity. At maturity, the interest rate is set to be zero, so the maturity nodes have two rather than four branches. When the default (credit) risk is considered, the convertible bonds are discounted using the risky interest rate $\hat{r}$ by adding a credit spread r_c to the risk-free interest rate r. The recovery rate is usually published in the bond's original issue handbook at the time of going to market. The probability of default λ is calculated using the Tsiveriotis and Fernandes model. Bloomberg and Stockwatch are the sources of convertible bond credit ratings. The defaultable convertible bond tree has five rather than four branches going forward in one time step to represent the default event.

4.6.1 Option-free convertible bond

In this section, I value the Advantage Energy (AAV) 5% convertible bond (AAV.DB.H) listed on the TSX market. This convertible bond does not include a call option feature and can be converted to maturity at any time. This bond is a five-year convertible bond with a start date of 04/01/2010 and matured on

30/01/2015. The face value of this issue is 100 with a conversion ratio of $\alpha = 11.63$ at a conversion price of equity $(AVV) = 8.6$ at the date of issuing.

First, I construct the interest rate tree of the five-year zero coupon bond yield at the risk-free rate adopting the Vasicek model with the parameters given in Table 8 for the dates from 04/01/2012 to 03/01/2015. These are daily yields data for zero-coupon bonds, generated using pricing data for Government of Canada bonds and treasury bills. The number of nodes in interest rate tree after n time periods is $(n + 1)$. The interest rate tree is constructed in Figure 17 with the corresponding probabilities. The parameters are estimated using the likelihood function suggested by James and Webber (2000). The interest rate volatility σ_r is obtained from historical data.

parameters	r_0	k	θ	σ_r	dt
	1.375%	**0.167**	**0.0112**	**0.0029**	**1**

Table 8: Vasicek interest rate tree model parameters

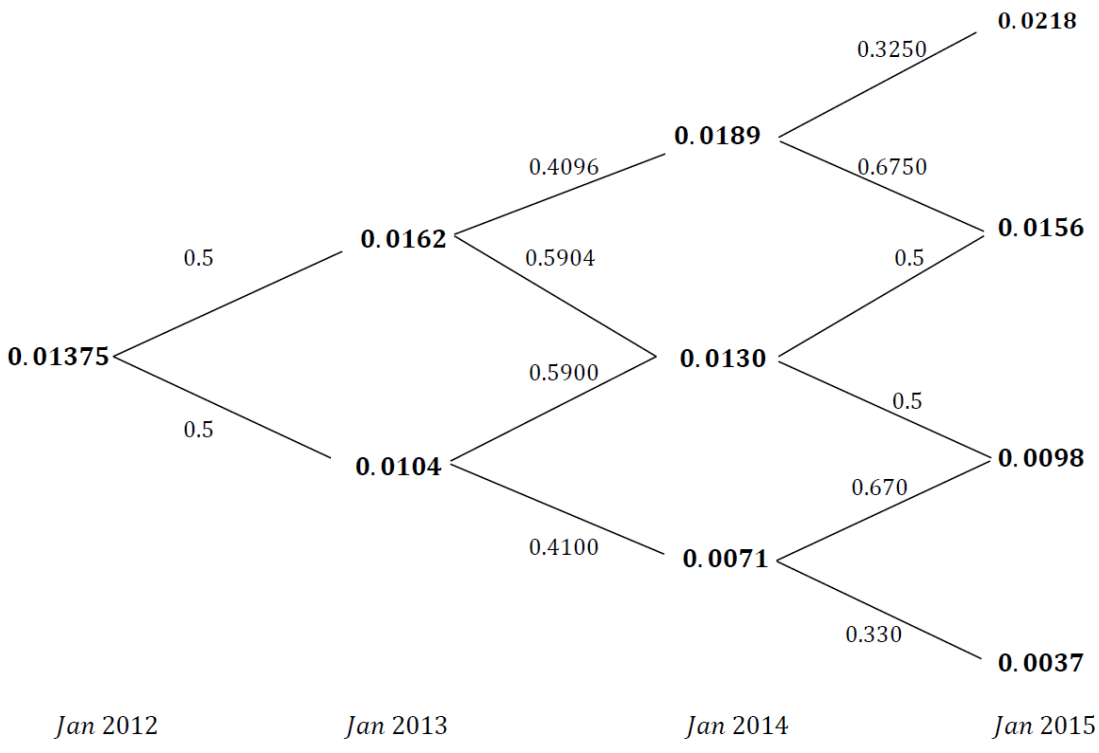


Figure 17: Vasicek interest rate tree

Second, I build the stock price tree for AAV for the same selected period used in the interest rate model. The equity volatility σ_s is assumed to be changeable (i.e., non-constant) in different intervals, but it is constant within the same time interval. This assumption leads to two central nodes at $t = 2$, four central nodes at $t = 3$, and so on for the following periods. Thus, the number of nodes after n time periods is (2^n). The parameters of the AAV stock price tree are shown in Table 9.

parameters	S_0	σ_{S1}	σ_{S2}	σ_{S3}	dt	q_S
	4.36	**31%**	**37%**	**39%**	**1**	**0**

Table 9: AAV stock price tree model parameters

The equity volatility σ_s is calculated from historical prices. As the parameters u and d are dependent on the volatility σ_s, u_n and d_n differ in each time interval of the tree. For three-period tree,

$$u_1 = e^{\sigma_{s1}\sqrt{dt}} = 1.363 \qquad d_1 = \frac{1}{u_1} = 0.733$$

$$u_2 = e^{\sigma_{s2}\sqrt{dt}} = 1.447 \qquad d_2 = \frac{1}{u_2} = 0.690$$

$$u_3 = e^{\sigma_{s3}\sqrt{dt}} = 1.476 \qquad d_3 = \frac{1}{u_3} = 0.677$$

The stock price 3-period tree and the corresponding probabilities of the stock tree are shown in Figure 18.

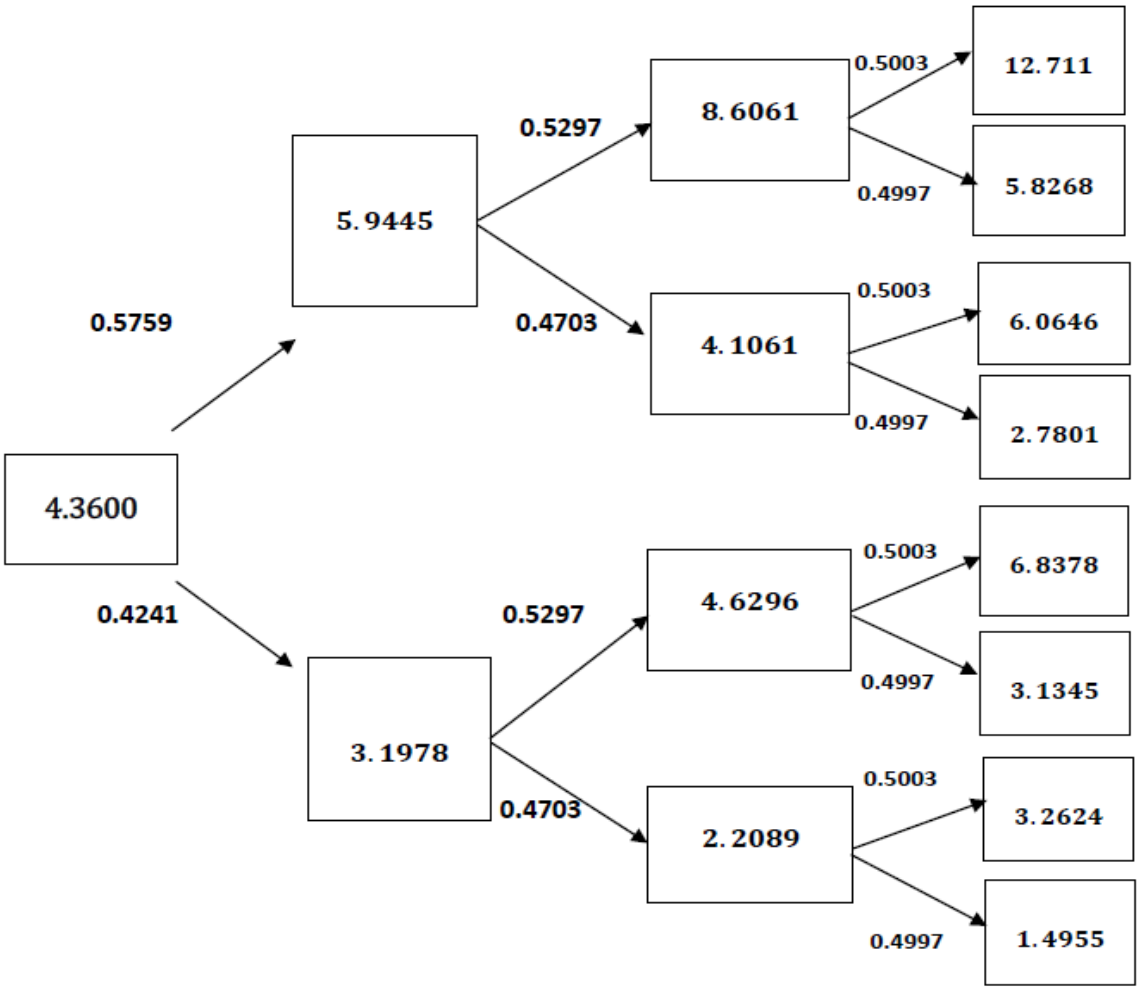


Figure 18: AAV stock price tree model

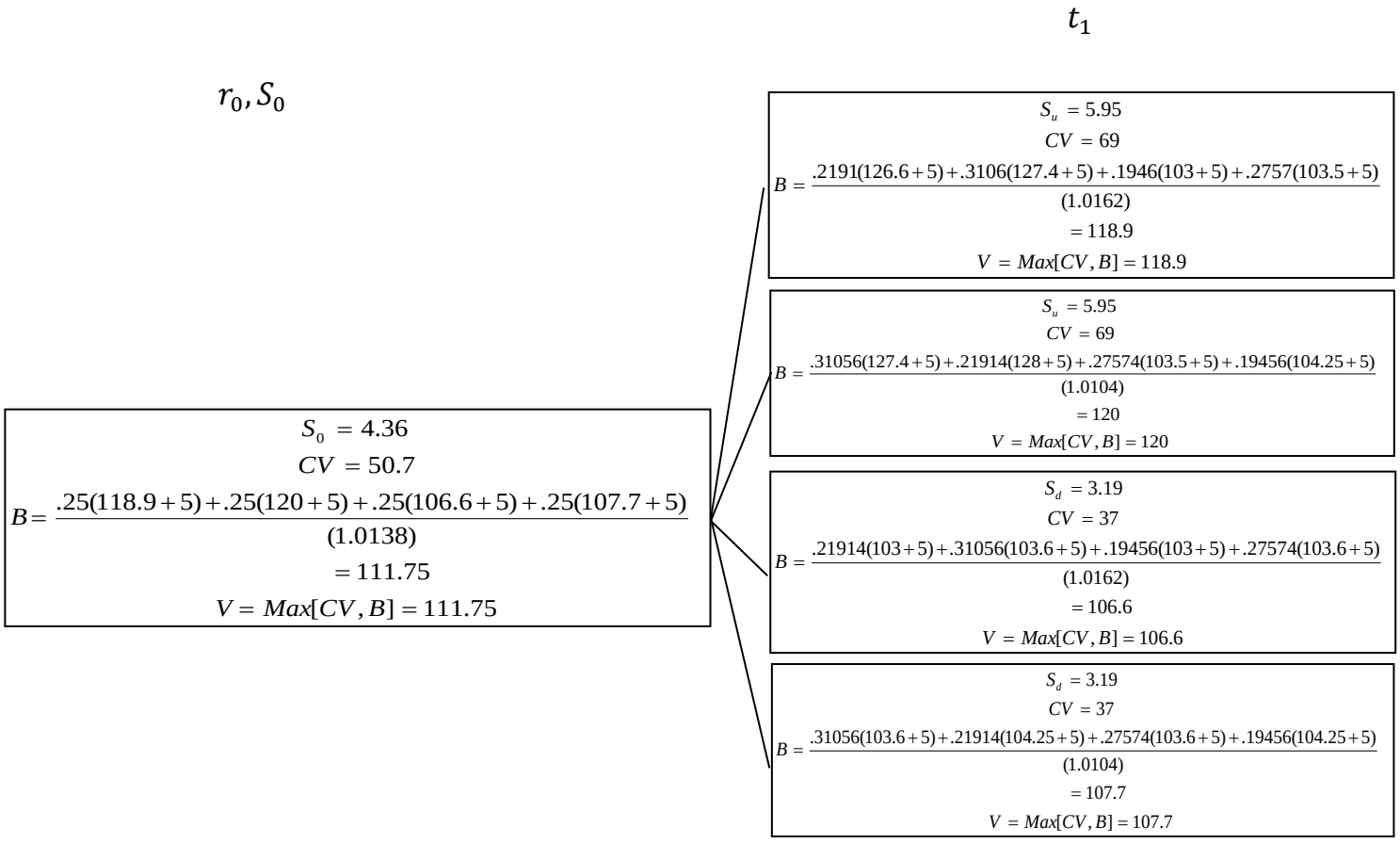
Next, I combine the interest rate tree and the equity tree into one single tree, the so-called two-factor tree. I need to use the backward induction method to solve the two-factor tree, starting from the bond's terminal value at the maturity date T. I also need to find the probabilities of each node of the tree, which, as explained earlier, are the average of the interest rate and equity probabilities, as shown in Table 5.

The value of convertible bond therefore consists of two components: the conversion value, which represents the equity component, and the present value of the straight bond. When using backward induction, the value of the convertible bond at maturity T is the maximum of the conversion value and the sum of the

face value F and the coupon interest. Note that the conversion value is denoted as the product of the conversion ratio α and the equity price at the given node.

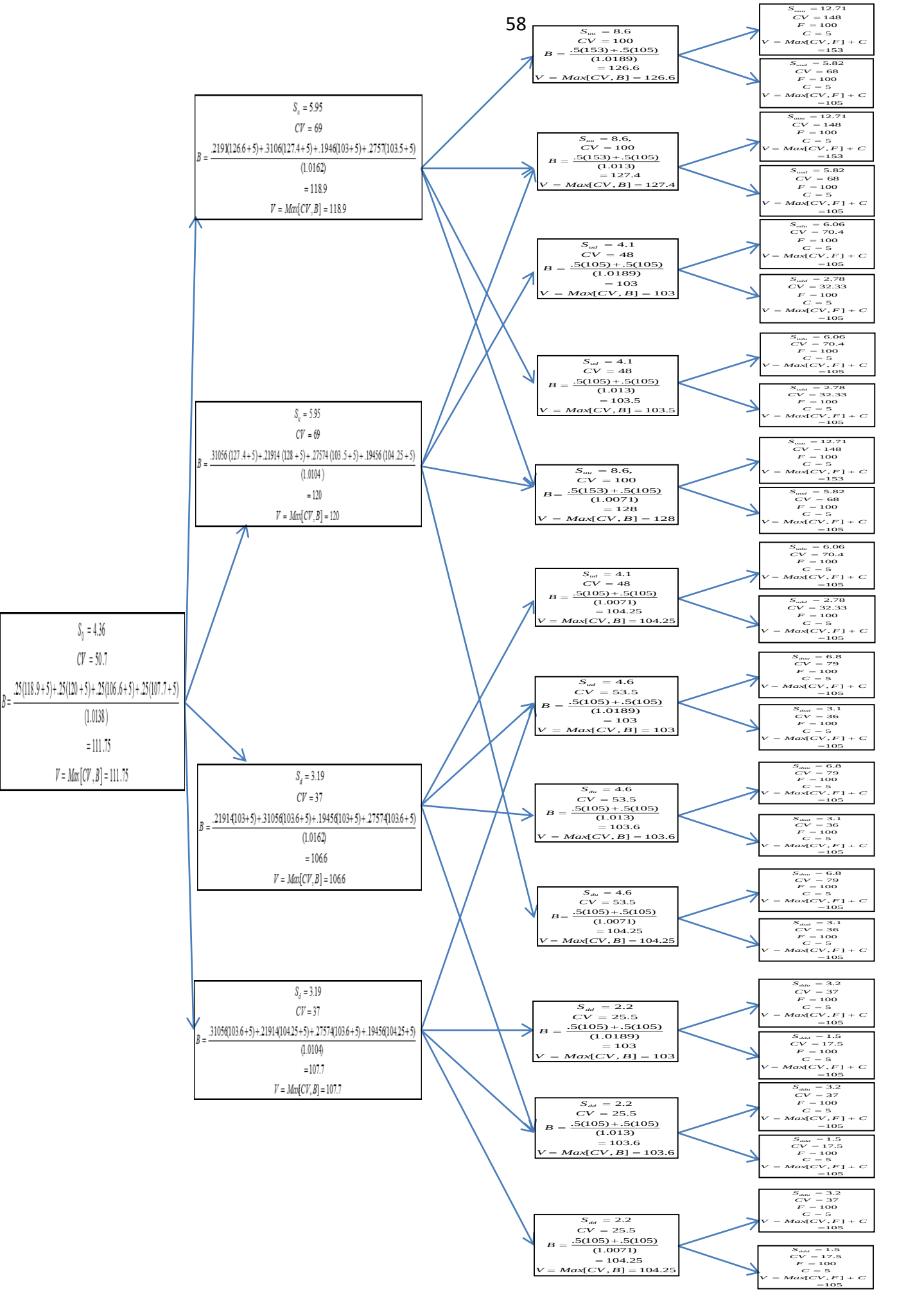
At any time t_i for each node of the tree, the price of the convertible bond satisfies

$$B_{cv} = max[CV,\ B].$$



In Period 3 at maturity T, the node S_{uuu} indicates that the convertible bondholder would exercise the conversion option, converting the bond to α shares of stock. The value of the convertible bond, V_{uuu}, would therefore be equal to its conversion value of 153 for the condition

$$V(S,r,T) = \begin{Bmatrix} F + c & if\ \ F + c \geq \alpha S \\ \alpha S & if\ \ F + c < \alpha S \end{Bmatrix}.$$


$S_0 = 4.36$
$CV = 50.7$
$B = \frac{.25(118.9+5)+.25(120+5)+.25(106.6+5)+.25(107.7+5)}{(1.0138)}$
$= 111.75$
$V = Max[CV, B] = 111.75$
$S_u = 5.95$
$CV = 69$
$B = \frac{.2191(126.6+5)+.3106(127.4+5)+.1946(103+5)+.2757(103.5+5)}{(1.0162)}$
$= 118.9$
$V = Max[CV, B] = 118.9$
$S_u = 5.95$
$CV = 69$
$B = \frac{.31056(127.4+5)+.21914(128+5)+.27574(103.5+5)+.19456(104.25+5)}{(1.0104)}$
$= 120$
$V = Max[CV, B] = 120$
$S_d = 3.19$
$CV = 37$
$B = \frac{.2191(103+5)+.31056(103.6+5)+.19456(103+5)+.27574(103.6+5)}{(1.0162)}$
$= 106.6$
$V = Max[CV, B] = 106.6$
$S_d = 3.19$
$CV = 37$
$B = \frac{.31056(103.6+5)+.21914(104.25+5)+.27574(103.6+5)+.19456(104.25+5)}{(1.0104)}$
$= 107.7$
$V = Max[CV, B] = 107.7$
$S_{uu} = 8.6$
$CV = 100$
$B = \frac{.5(153)+.5(105)}{(1.0189)}$
$= 126.6$
$V = Max[CV, B] = 126.6$
$S_{uu} = 8.6$
$CV = 100$
$B = \frac{.5(153)+.5(105)}{(1.013)}$
$= 127.4$
$V = Max[CV, B] = 127.4$
$S_{ud} = 4.1$
$CV = 48$
$B = \frac{.5(105)+.5(105)}{(1.0189)}$
$= 103$
$V = Max[CV, B] = 103$
$S_{ud} = 4.1$
$CV = 48$
$B = \frac{.5(105)+.5(105)}{(1.013)}$
$= 103.5$
$V = Max[CV, B] = 103.5$
$S_{uu} = 8.6$
$CV = 100$
$B = \frac{.5(153)+.5(105)}{(1.0071)}$
$= 128$
$V = Max[CV, B] = 128$
$S_{ud} = 4.1$
$CV = 48$
$B = \frac{.5(105)+.5(105)}{(1.0071)}$
$= 104.25$
$V = Max[CV, B] = 104.25$
$S_{ud} = 4.6$
$CV = 53.5$
$B = \frac{.5(105)+.5(105)}{(1.0189)}$
$= 103$
$V = Max[CV, B] = 103$
$S_{du} = 4.6$
$CV = 53.5$
$B = \frac{.5(105)+.5(105)}{(1.013)}$
$= 103.6$
$V = Max[CV, B] = 103.6$
$S_{du} = 4.6$
$CV = 53.5$
$B = \frac{.5(105)+.5(105)}{(1.0071)}$
$= 104.25$
$V = Max[CV, B] = 104.25$
$S_{dd} = 2.2$
$CV = 25.5$
$B = \frac{.5(105)+.5(105)}{(1.0189)}$
$= 103$
$V = Max[CV, B] = 103$
$S_{dd} = 2.2$
$CV = 25.5$
$B = \frac{.5(105)+.5(105)}{(1.013)}$
$= 103.6$
$V = Max[CV, B] = 103.6$
$S_{dd} = 2.2$
$CV = 25.5$
$B = \frac{.5(105)+.5(105)}{(1.0071)}$
$= 104.25$
$V = Max[CV, B] = 104.25$
$S_{uuu} = 12.71$
$CV = 148$
$F = 100$
$C = 5$
$V = Max[CV, F] + C$
$= 153$
$S_{uud} = 5.82$
$CV = 68$
$F = 100$
$C = 5$
$V = Max[CV, F] + C$
$= 105$
$S_{udu} = 6.06$
$CV = 70.4$
$F = 100$
$C = 5$
$V = Max[CV, F] + C$
$= 105$
$S_{udd} = 2.78$
$CV = 32.33$
$F = 100$
$C = 5$
$V = Max[CV, F] + C$
$= 105$
$S_{duu} = 6.8$
$CV = 79$
$F = 100$
$C = 5$
$V = Max[CV, F] + C$
$= 105$
$S_{dud} = 3.1$
$CV = 36$
$F = 100$
$C = 5$
$V = Max[CV, F] + C$
$= 105$
$S_{ddu} = 3.2$
$CV = 37$
$F = 100$
$C = 5$
$V = Max[CV, F] + C$
$= 105$
$S_{ddd} = 1.5$
$CV = 17.5$
$F = 100$
$C = 5$
$V = Max[CV, F] + C$
$= 105$

For the other terminal nodes, the value of the convertible bond is equal to the face value plus the coupon interest, as the conversion strategy is worthless. Rolling back to the initial node at t_0 which values the current period, we obtain a convertible bond value V of 111.75.

The AAV convertible bond was traded at a real market price between 100 and 111.99 for the period of 30/06/2011 – 01/01/2012.

Figure 19 illustrates the movement of the convertible bond and its underlying asset in the real market during the life of the issue.

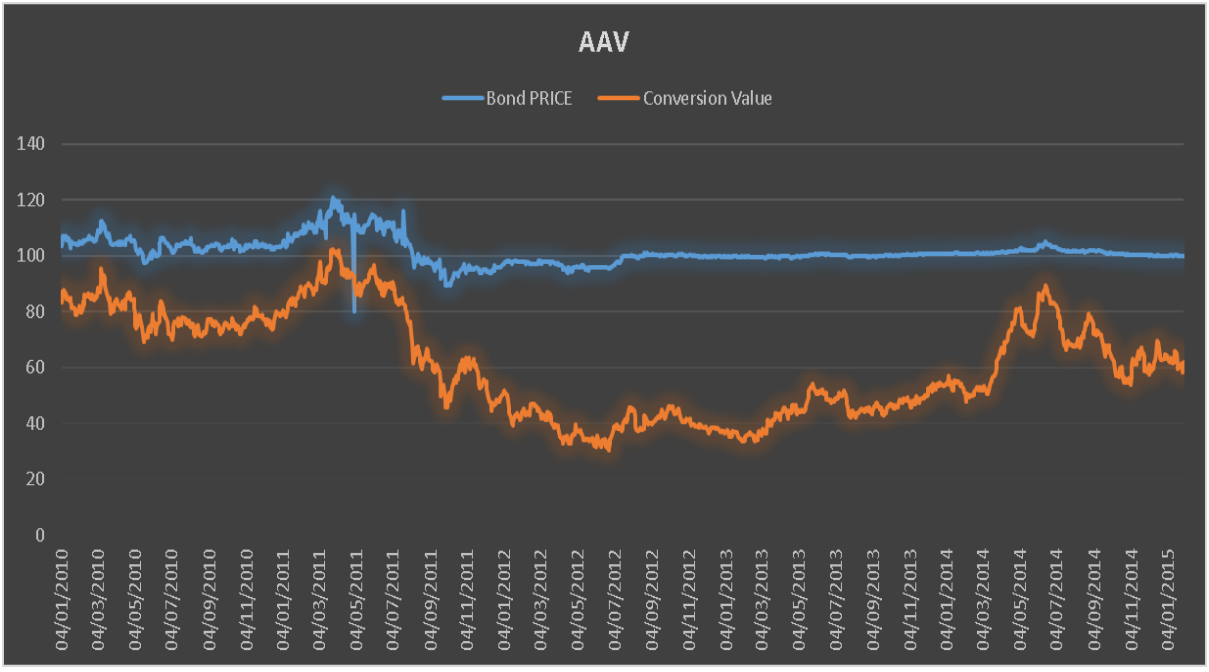


Figure 19: Relationship between the bond price and the conversion price of AAV

Figure 20 shows the 3- period pricing tree of the AAV convertible bond with a model price of 111.75.

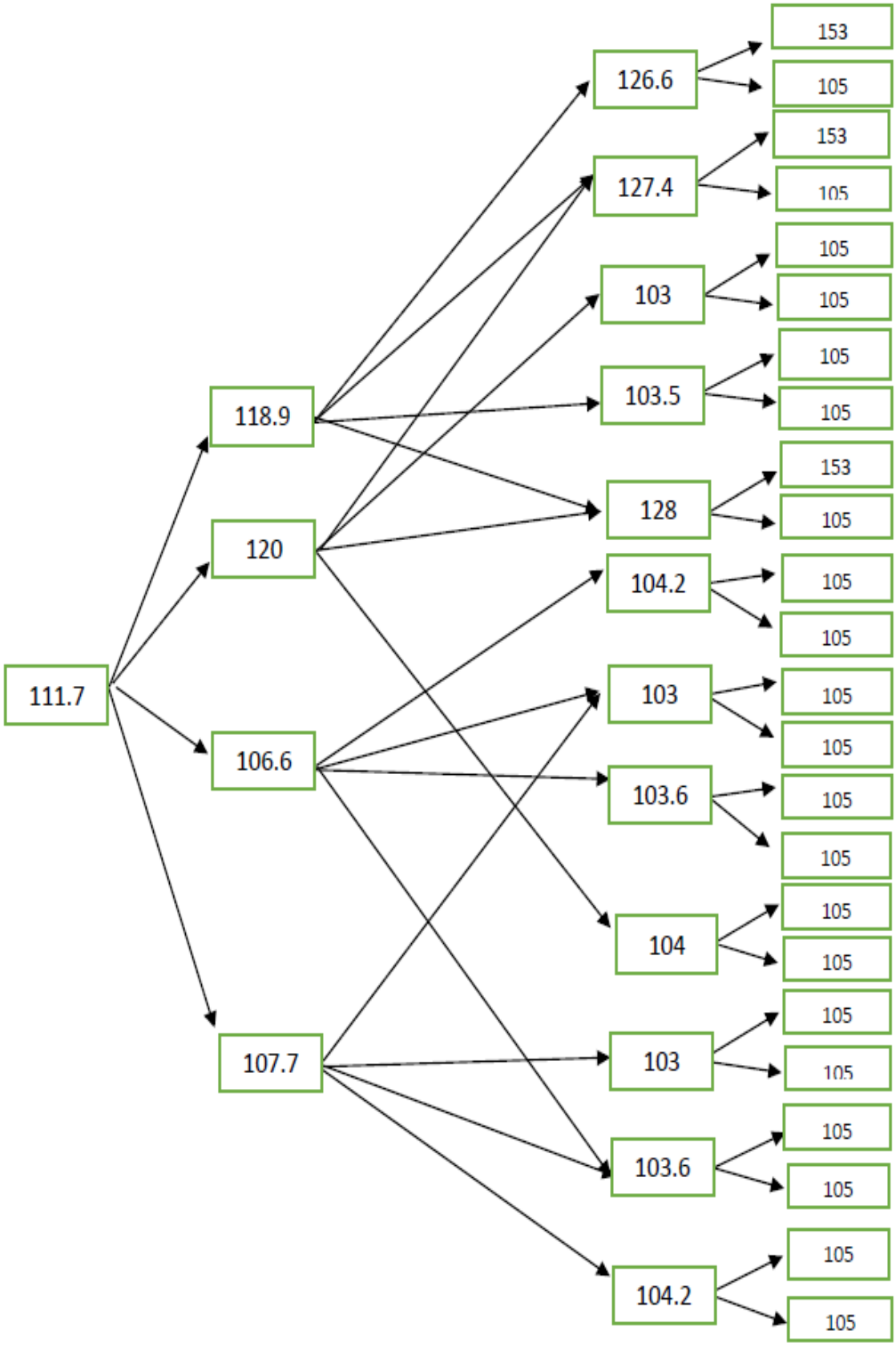


Figure 20: AAV convertible bond 3- period pricing tree

4.6.2 Callable convertible bond subject to default risk

In this section, we value the Calloway REIT (CWT) 5.75% convertible bond (CWT.DB.B) listed on the TSX market assuming that the convertible bond is featured with a call option and can be converted at any time to maturity. This is a 7-year convertible bond with a start date 05/01/2010 and matured on 30/06/2017. The face value F of this issue is 100, with a conversion ratio of $\alpha = 3.88$ at a conversion price of equity $(CWT) = 25.75$ at the date of issuing. The convertible bond has a strike call price (trigger) of $(CWT) = 32.188$, which gives a conversion price of $\alpha \times S = 3.88 \times 32.188 = 125$. The convertible can be called in a 5-year period from maturity with a trigger date of 30/06/2014.

First, I construct the interest rate tree used in the previous example of the five-year zero-coupon bond yield as the risk-free rate, adopting the Vasicek model with the following parameters for the dates from 04/01/2012 to 03/01/2015 in Table 10. These are daily yields data for zero-coupon bonds, generated using pricing data for Government of Canada bonds and treasury bills. The number of nodes in the interest rate tree after n time periods is $(n + 1)$. The interest rate tree is constructed in Figure 21 with the corresponding probabilities. The parameters are estimated using the likelihood function suggested by James and Webber (2000). The interest rate volatility σ_r is obtained from historical data.

parameters	r_0	k	θ	σ_r	dt
	1.375%	0.167	0.0112	0.0029	1

Table 10: Vasicek interest rate tree model parameters

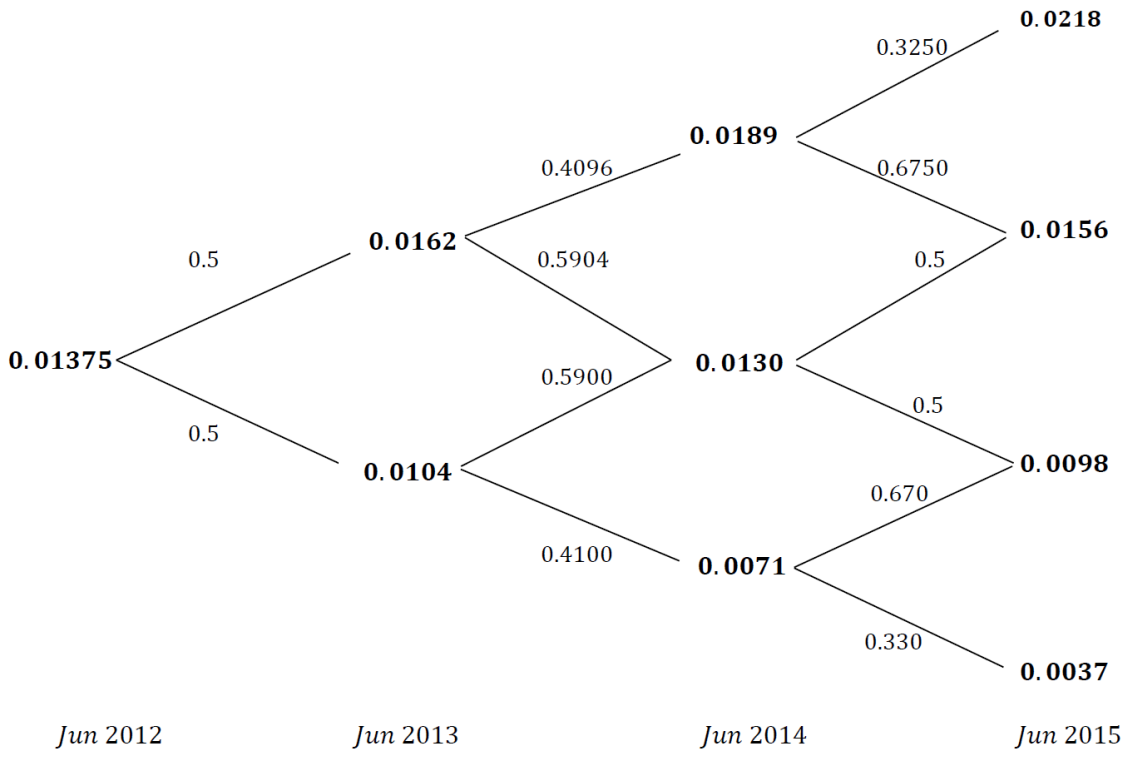


Figure 21: Vasicek interest rate tree

Second, I build the stock price tree of CWT for the same selected period used in the interest rate model. The equity volatility σ_s is assumed to be changeable (i.e., non-constant) in different intervals, but it is constant within the same time interval. This assumption leads to two central nodes at $t = 2$, four central nodes at $t = 3$, and so on for the following periods. Therefore, the number of nodes after n time periods is (2^n). The parameters of the CWT stock price tree are shown in Table 11.

parameters	S_0	σ_{S1}	σ_{S2}	σ_{S3}	dt	q_S
	28	**15%**	**14.5%**	**16%**	**1**	**0**

Table 11: CWT stock price tree model parameters

The stock price volatility σ_s is calculated from historical prices. As the parameters u and d are dependent on the volatility σ_s, therefore, u_n and $d_n$differ in each period interval of the tree. For a three-period tree,

$$u_1 = e^{\sigma_{s1}\sqrt{dt}} = 1.161 \qquad d_1 = \frac{1}{u_1} = 0.860$$

$$u_2 = e^{\sigma_{s2}\sqrt{dt}} = 1.156 \qquad d_2 = \frac{1}{u_2} = 0.865$$

$$u_3 = e^{\sigma_{s3}\sqrt{dt}} = 1.173 \qquad d_3 = \frac{1}{u_3} = 0.852$$

The stock price 3-period tree and the corresponding probabilities are shown in Figure 22.

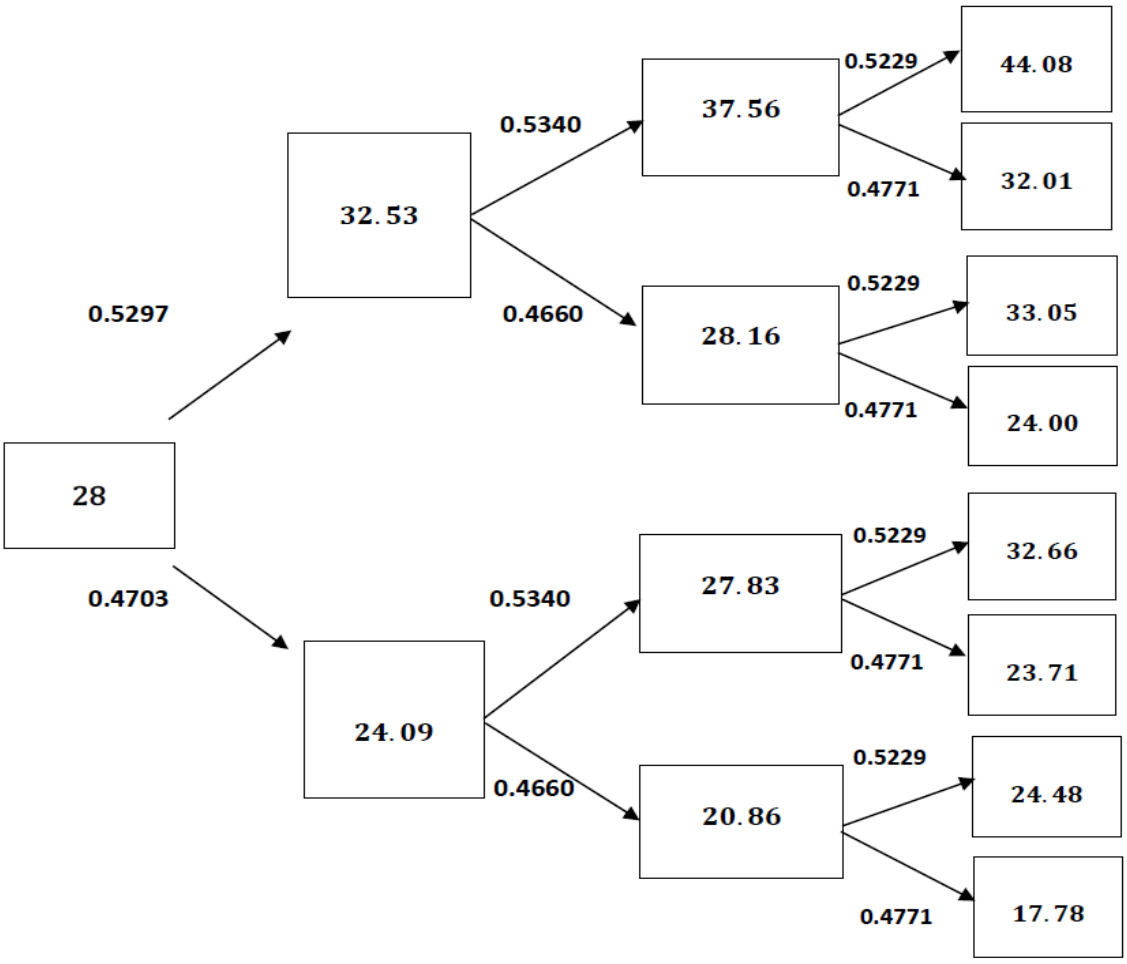


Figure 22: CWT stock price tree model

Default risk

As a convertible bond is divided into two components, an equity component and a debt component, we use the risk-free interest rate to discount equity and the risky bond rate to discount debt, which reflects the future default probabilities. As explained in Section 4.5, the risky interest rate can be determined by adding a credit spread r_c to the risk-free interest rate r. This spread is a representation of the credit spread implied by non-convertible bonds from the same issuer for maturities similar to that of the convertible bond. On the other hand, the credit spread is often obtained from the credit rating given to a defaultable corporate bond by credit rating agencies.

Suppose that the three-period risky yields are $1.97\%, 2.01\%, \text{and } 2.22\%$, respectively, with a constant recovery rate $\delta = 30\%$. Then, the three-period probabilities of default, $[\lambda_1, \lambda_2, \lambda_3]$, are given by

$$e^{-0.0197+0.01375} = (1 - \boldsymbol{\lambda_1}) + (0.3)\boldsymbol{\lambda_1}$$

$$\lambda_1 = 0.0034$$

$$\lambda_2 = 0.0155$$

$$\lambda_3 = 0.0221.$$

Therefore, the credit spread for each period t_i will be added to the risk-free interest rate to discount the defaultable bond, and it can be expressed as

$$r_{c1} = 0.0197 - 0.01375$$

$$= 0.00595$$

$$r_{c2} = 0.0201 - 0.01375$$

$$= 0.00635$$

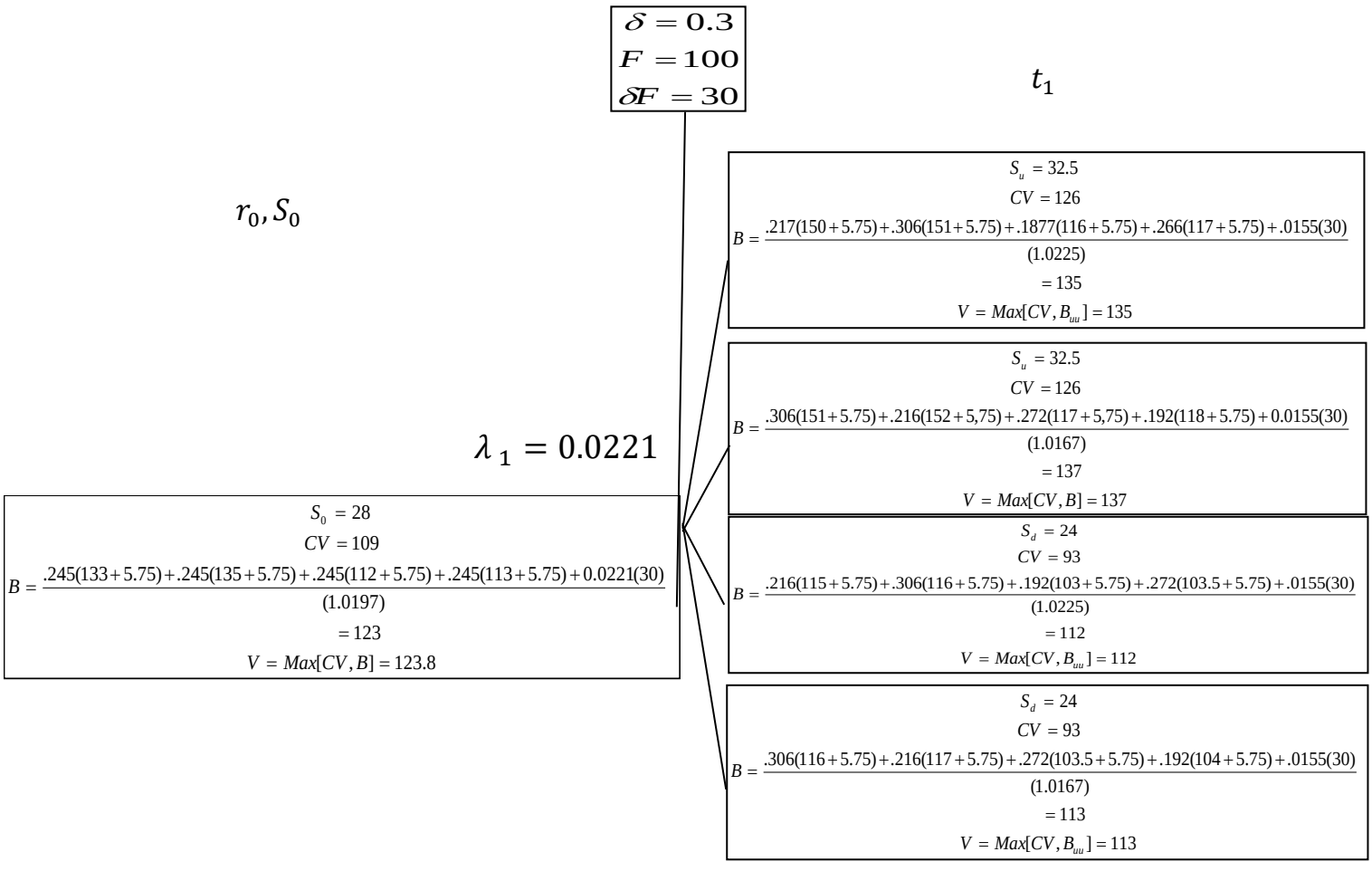
$$r_{c3} = 0.0222 - 0.01375$$

$$= 0.00845.$$

We use the methodology from the first example to combine the interest rate tree and the equity tree into one single tree to form the so-called two-factor tree. The single tree in this example will have five branches for each node instead of four to reflect the default event. The value of the convertible bond therefore consists of two components: the conversion value, which represents the equity component, and the present value of the straight bond. The debt component is obtained as the weighted average of three components: the up node, the down node, and the default component, which is represented as the recovery value paid when the corporation goes to default.

As the CWT convertible has a call provision with a determined strike price and date, the convertible bond price when the call option is applicable is

$$V^C = Max\left[CV,\ Min[B, CP]\right].$$



$\delta = 0.3$
$F = 100$
$\delta F = 30$

$\lambda_1 = 0.0221$

$S_0 = 28$
$CV = 109$
$B = \frac{.245(133+5.75)+.245(135+5.75)+.245(112+5.75)+.245(113+5.75)+0.0221(30)}{(1.0197)} = 123$
$V = Max[CV, B] = 1238$

$\lambda_2 = 0.0155$

$S_u = 32.5$
$CV = 126$
$B = \frac{.217(146+5.75)+.306(146+5.75)+.1877(116+5.75)+.266(117+5.75)+.0155(30)}{(1.0225)} = 135$
$V = Max[CV, B] = 133$

$\lambda_2 = 0.0155$

$S_u = 32.5$
$CV = 126$
$B = \frac{.306(146+5.75)+.216(146+5.75)+.272(117+5.75)+.192(118+5.75)+0.0155(30)}{(1.0167)} = 137$
$V = Max[CV, B] = 135$

$\lambda_2 = 0.0155$

$S_d = 24$
$CV = 93$
$B = \frac{.216(115+5.75)+.306(116+5.75)+.192(103+5.75)+.272(103.5+5.75)+.0155(30)}{(1.0225)} = 112$
$V = Max[CV, B_{uu}] = 112$

$\lambda_2 = 0.0155$

$S_d = 24$
$CV = 93$
$B = \frac{.306(116+5.75)+.216(117+5.75)+.272(103.5+5.75)+.192(104+5.75)+.0155(30)}{(1.0167)} = 113$
$V = Max[CV, B] = 113$

$\lambda_3 = 0.0034$

$S_{uu} = 37.6$
$CV = 146$
$B = \frac{.4955(176.75)+.4955(132.25)+.0034(30)}{(1.025)} = 151$
$V = Max[CV, Min(B, CP)] = 146$

$S_{uuu} = 44.08$, $CV = 171$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 176.75$

$S_{uud} = 32.6$, $CV = 126.5$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 132.25$

$\lambda_3 = 0.0034$

$S_{uu} = 37.6$
$CV = 146$
$B = \frac{.4955(176.75)+.4955(132.25)+.0034(30)}{(1.019)} = 152$
$V = Max[CV, Min(B, CP)] = 146$

$S_{uuu} = 44.08$, $CV = 171$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 176.75$

$S_{uud} = 32.6$, $CV = 126.5$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 132.25$

$\lambda_3 = 0.0034$

$S_{uu} = 28$
$CV = 108$
$B = \frac{.4955(133.75)+.4955(105.75)+.0034(30)}{(1.025)} = 116$
$V = Max[CV, Min(B, CP)] = 116$

$S_{udu} = 33$, $CV = 128$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 133.75$

$S_{udd} = 24$, $CV = 93$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$\lambda_3 = 0.0034$

$S_{uu} = 28$
$CV = 108$
$B = \frac{.4955(133.75)+.4955(105.75)+0.0034(30)}{(1.019)} = 117$
$V = Max[CV, Min(B, CP)] = 117$

$S_{udu} = 33$, $CV = 128$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 133.75$

$S_{udd} = 24$, $CV = 93$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$\lambda_3 = 0.0034$

$S_{uu} = 37.6$
$CV = 146$
$B = \frac{.4955(176.75)+.4955(132.25)+0.0034(30)}{(1.013)} = 152$
$V = Max[CV, Min(B, CP)] = 146$

$S_{uuu} = 44.08$, $CV = 171$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 176.75$

$S_{uud} = 32.6$, $CV = 126.5$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 132.25$

$\lambda_3 = 0.0034$

$S_{uu} = 28$
$CV = 108$
$B = \frac{.4955(133.75)+.4955(105.75)+.0034(30}{(1.013)} = 118$
$V = Max[CV, Min(B, CP)] = 118$

$S_{udu} = 33$, $CV = 128$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 133.75$

$S_{udd} = 24$, $CV = 93$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$\lambda_3 = 0.0034$

$S_{uu} = 27.6$
$CV = 107.8$
$B = \frac{.4955(131.75)+.4955(105.75)+.0034(30)}{(1.025)} = 115$
$V = Max[CV, Min(B, CP)] = 115$

$S_{duu} = 32.5$, $CV = 126$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 131.75$

$S_{dud} = 23.5$, $CV = 97$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$\lambda_3 = 0.0034$

$S_{uu} = 27.6$
$CV = 107.8$
$B = \frac{.4955(131.75)+.4955(105.75)+.0034(30)}{(1.019)} = 116$
$V = Max[CV, Min(B, CP)] = 116$

$S_{duu} = 32.5$, $CV = 126$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 131.75$

$S_{dud} = 23.5$, $CV = 97$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$\lambda_3 = 0.0034$

$S_{uu} = 27.6$
$CV = 107.8$
$B = \frac{.4955(131.75)+.4955(105.75)+.0034(30)}{(1.013)} = 117$
$V = Max[CV, Min(B, CP)] = 117$

$S_{duu} = 32.5$, $CV = 126$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 131.75$

$S_{dud} = 23.5$, $CV = 97$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$\lambda_3 = 0.0034$

$S_{uu} = 21$
$CV = 81$
$B = \frac{.4955(105.75)+.4955(105.75)+.0034(30)}{(1.025)} = 103$
$V = Max[CV, Min(B, CP)] = 103$

$S_{ddu} = 24.5$, $CV = 95$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$S_{ddd} = 18$, $CV = 70$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$\lambda_3 = 0.0034$

$S_{uu} = 21$
$CV = 81$
$B = \frac{.4955(105.75)+.4955(105.75)+.0034(30)}{(1.019)} = 103.5$
$V = Max[CV, Min(B, CP)] = 103.5$

$S_{ddu} = 24.5$, $CV = 95$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$S_{ddd} = 18$, $CV = 70$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$\lambda_3 = 0.0034$

$S_{uu} = 21$
$CV = 81$
$B = \frac{.4955(105.75)+.4955(105.75)+0.0034(30)}{(1.013)} = 104$
$V = Max[CV, Min(B, CP)] = 104$

$S_{ddu} = 24.5$, $CV = 95$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

$S_{ddd} = 18$, $CV = 70$, $F = 100$, $C = 5.75$, $V = Max[CV, F] + C = 105.75$

In Period 3 at maturity T, the nodes S_{uuu}, S_{uud}, S_{udu} and S_{duu} indicate that the convertible bondholder would exercise the conversion option, converting the bond to α shares of stock. The value of the convertible bond, V_{uuu}, would therefore be equal to its conversion values of 176.75, 132.25, 133.75 and 131.75 respectively, subject to the condition

$$V(S,r,T) = \begin{cases} F + C & if\ \ F + C \geq \alpha S \\ \alpha S & if\ \ F + C < \alpha S \end{cases}$$

As the strike date of the call option is set to be t_2, the convertible bond value at t_2 is

$$V^C = Max[CV,\ Min[B, CP]]$$

The nodes at t_2, such as $r_{uu}S_{uu}$, $r_{ud}S_{uu}$, and $r_{dd}S_{uu}$, are converted to common stocks even when bond prices are higher. The reason is that the bond price is eliminated by the call strike price at a trigger of 32.188, which gives a conversion price of 125. Therefore, the convertible callable bond price is equivalent to the maximum price of the conversion value or the minimum of the straight bond price and the call price.

For the other terminal nodes, the value of the convertible bond is equal to the face value plus the coupon interest, as the conversion strategy is worthless. Rolling back to the initial node at t_0, which values the current period, I obtain a convertible bond value V of 123.8.

The AAV convertible bond was traded at a real market price between 112.5 and 124 for the period of 30/06/2012 – 01/09/2012.

Figure 23 illustrates the movement of the convertible bond and its underlying asset in the real market during the life of the issue.

Figure 24 shows 3- period pricing tree of CWT callable convertible bond with model price of 123.

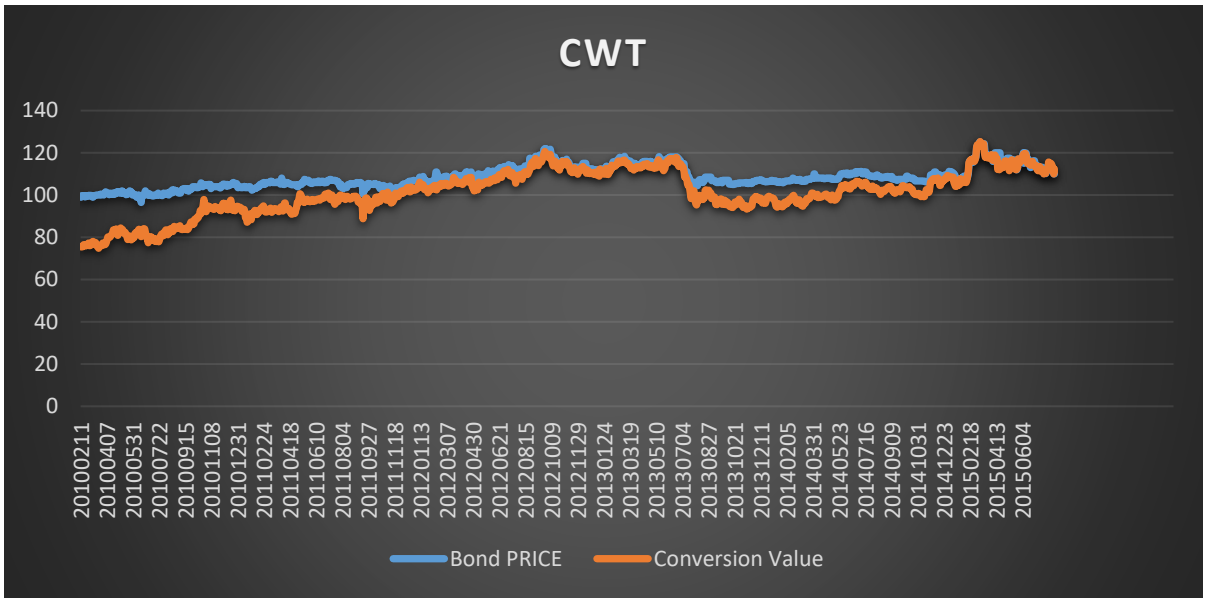


Figure 23: Relationship between bond price and conversion price of CWT

4.6.3 Conclusion of the numerical examples

Our tree model for pricing convertible bonds is based on two-factor models using modified CRR modeling of stock prices and Vasicek tree of interest rates. The first numerical example of option-free convertible bond indicates that our model produces a moderately different convertible bond price than that found in the real market price in TSX. For the investigated period, our model shows a price difference of 0.21% with the market price.

In the second numerical example, we use the approach of Jarrow and Turnbull (1995) framework to model the default probability. For this example, we price the callable convertible bond using the risky discount rate to reflect the future default probability. The model price is not far from the real market price and shows a better valuation model than the default- free pricing model. For the investigated period, our model shows a price difference of 0.16% with the market price. It is worth noting that the model can be used to price convertible bonds with complex provisions and other financial derivatives such as bond's call and put options.

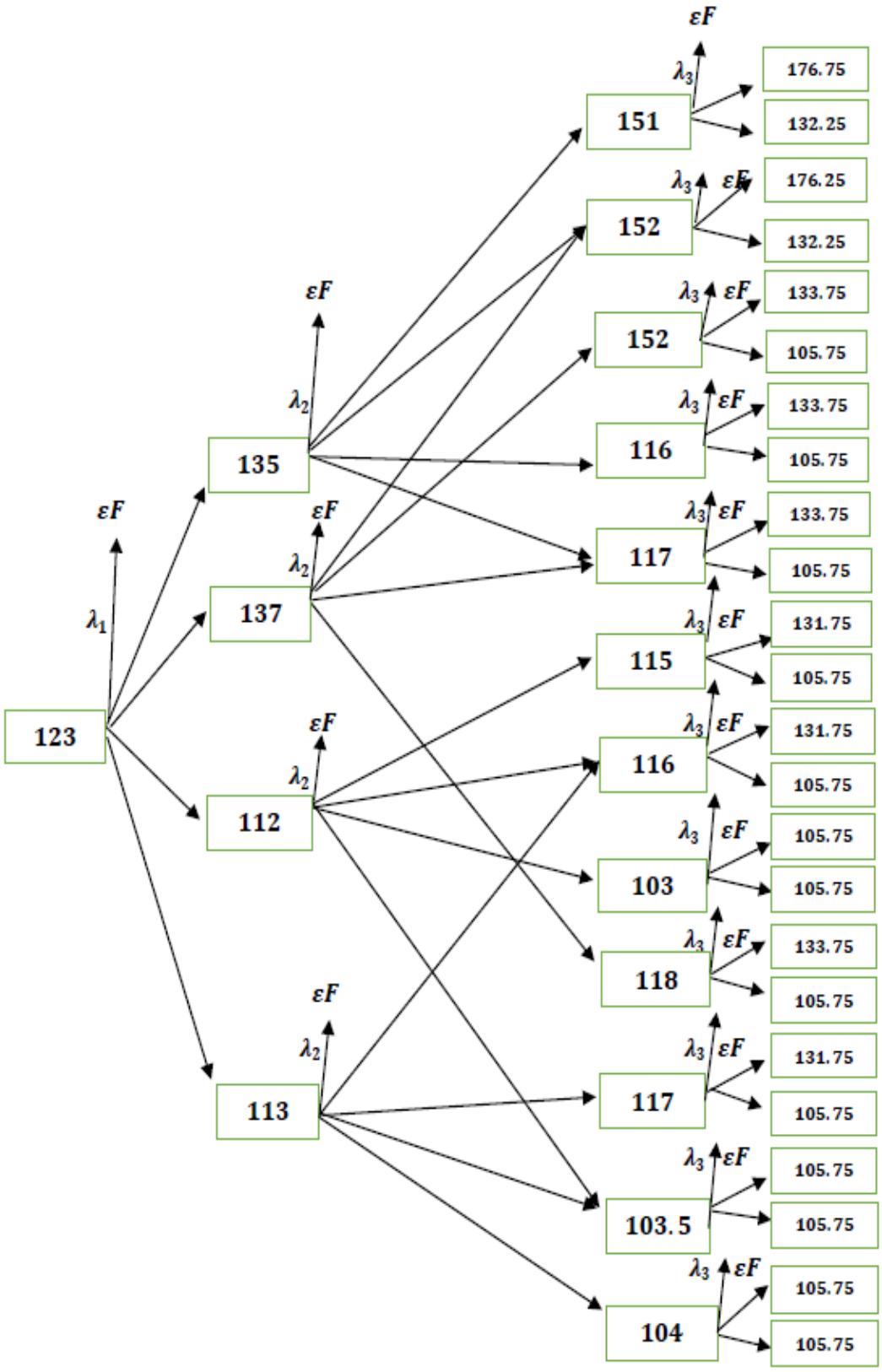


Figure 24: CWT callable convertible bond 3- period pricing tree with default risk

4.6.4 Further numerical examples

In this section, we extend the numerical examples to conclude 6 convertible bonds that traded in TSX market. We then compare the model price to the real market price in TSX.

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (31 Oct 2012)	Model price (31 Oct 2012)
ARE.DB.A	31-Oct-2015	6.25%	5.263	105	104.19

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (30 Jun 2012)	Model price (30 Jun 2012)
CNE.DB	30-Jun-15	8%	9.5	101.5	101.96

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (31 Oct 2012)	Model price (31 Oct 2012)
CAM.DB	31-Oct-2015	6.25%	8.333	104.8	105.43

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (31 Jul 2012)	Model price (31 Jul 2012)
GH.DB	31-Jul-2015	5.25%	9.39	120.49	119.72

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (31 Dec 2012)	Model price (31 Dec 2012)
CUS.DB.A	31-Dec-2015	5.75%	12.048	103.11	105.62

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (31 Oct 2012)	Model price (31 Oct 2012)
PBH.DB.A	31-Oct-2015	5.75%	4.464	104	105.68

The numerical examples of the annual spacing indicate that our model produces a moderately different convertible bond prices than that found in the real market price in TSX. For the investigated period, our model shows that ARE.DB.A convertible bond has a value of 104.19 compared to the market price of 105.

Our valuation model shows that CNE.DB was underpriced by 0.45% with a value of 101.96 compared to a market price of 101.5. it can be seen that other convertible bonds were also underpriced by various percentages such as CAM.DB, CUS.DB.A and PBH.DB.A.

For the investigated period, GH.DB convertible bond has a model price of 119.72 compared to a market price of 120.49.

In the periods under consideration, we can conclude that our convertible bond valuation model price is more or less close to the real market prices.

4.6.5 Monthly spacing numerical examples

In this section, we provide numerical examples of a finer spacing binomial tree where steps are generated on a monthly basis. As the American convertible bond may be exercised in between coupon payments, we need to calculate the accrued interest which is defined as the amount of interest that has accrued on a bond between coupon payments. To construct the monthly tree model, we use a similar method that was used in the previous annual numerical examples.

For the interest rate tree, there are 12 monthly nodes from r_0 to r_1. Thus, when $t = \frac{1}{12}$, the first monthly nodes become:

$$r_{m1}^{u} = r_0 + \frac{k(\theta - r_t)}{12} + \frac{\sigma_r}{\sqrt{12}}.$$

And

$$r_{m1}^{d} = r_0 + \frac{k(\theta - r_t)}{12} - \frac{\sigma_r}{\sqrt{12}}.$$

To find the price of the stock for any time fraction within the time intervals $[t_i, t_{i+1}]$ in the pricing tree, we need to reset u, d and Δt according to the new time fraction. For example, the stock price process over one-month interval $t = \frac{1}{12}$ would be

$$u = e^{\sigma_s \sqrt{\frac{dt}{12}}} \text{ and } \quad d = \frac{1}{u}$$

The accrued interest between coupon payments is obtained as

$$Accrued\ interest = C \times \frac{days\ since\ last\ payment}{total\ days\ in\ coupon\ payment}$$

For the monthly numerical examples, we valued the prices of 5 convertible bonds traded in TSX stock market. We then compare the model price to the real market price in TSX. The numerical examples show that our monthly model produces a small price difference between the prices of convertible bonds obtained from our monthly model techniques and those from TSX market. In the period under consideration, the model we used depicts that the value ARE.DB.A convertible bond is 101.11 while that of the market is 101.15.

According to our model, CNE.DB is under-priced by close to 2% with a valuation of 102.68 against a market price of 100. Moreover, it shows that other convertible bonds are also under-priced by a number of percentages like CUS.DB.A and GH.DB.

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (31 Jul 2015)	Model price (31 Oct 2015)
ARE.DB.A	31-Oct-2015	6.25%	5.263	101.15	101.11

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (30 Mar 2015)	Model price (30 Jun 2015)
CNE.DB	30-Jun-15	8%	9.5	100	102.68

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (30 Jul 2015)	Model price (31 Oct 2015)
CAM.DB	31-Oct-2015	6.25%	8.333	118.73	117.79

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (30 Apr 2015)	Model price (31 Jul 2015)
GH.DB	31-Jul-2015	5.25%	9.39	117.05	118.26

Convertible bond symbol	Maturity	Coupon	Conversion ratio	Market price (31 Sep 2015)	Model price (31 Dec 2015)
CUS.DB.A	31-Dec-2015	5.75%	12.048	100	100.33

5 Duration and convexity of convertible bonds

5.1 Introduction to duration and convexity

Duration and convexity are important risk measures in fixed income securities management. These measures can help to predict the change in the price of a bond with a given change in the interest rate. Duration measures the percentage change in the market value of a cash flow for a given change in the yield and comes close to the true value when the rate changes are small. With larger changes in rates, however, it is necessary to consider convexity, which is the curvature of the price-yield relationship (Dunetz and Mahoney, 1988). As described in Section 1.2, Brooks and Attinger (1992) initially provided the theoretical definition of the duration and convexity of convertibles. However, they did not describe how the conversion option was valued, nor did they provide values for Rho (ρ) and Delta (Δ), as no valuation model was provided. Sarkar (1999) provided a closed-form expression for the duration and convexity of a zero-coupon convertible bond. This model followed the contingent-claim approach of Ingersoll (1977) and Brennan and Schwartz (1980) and adopted a one-factor valuation model that assumed a constant risk-free interest rate.

In this chapter, I study duration and convexity numerically, as it is difficult to obtain a closed-form expression for an American convertible bond that can be converted at any time during its life. I use the present value method and the binomial tree method as numerical examples to illustrate the duration and convexity of American convertible bonds. Moreover, I provide an example for the duration and convexity of European zero-coupon convertible bonds based on a two-factor model that adopts a stochastic interest rate. I also study the duration that expresses the approximate change in the convertible value for any change in the interest rate. Moreover, I investigate the sensitivity of the convertible bond value and duration to various parameters, such as the interest rate, stock price, volatility

of the interest rate, volatility of the stock price, mean reversion of the interest rate, and dividend yield of the underlying stock. It can be seen that the results differ from the one factor cases considerably.

5.2 Duration

Duration is a measure of the approximate price sensitivity of a bond to interest rate changes. More specifically, it is the approximate percentage change in the bond price for a 100 basis point change in rates. The numerical duration is defined as

$$Duration = \frac{price\ if\ yields\ decline\ - price\ if\ yields\ rise}{(initial\ price)(2 \times change\ in\ interest\ rate)}$$

$$Duration = \frac{V_- - V_+}{(V_0)(2\Delta r)}$$

where

$V_- = price\ if\ interest\ rate\ declines\ by\ \Delta r$

$V_+ = price\ if\ interest\ rate\ increases\ by\ \Delta r$

$V_0 = initial\ price$

$\Delta r = change\ in\ interest\ rate$

Duration is used to approximate the percentage price change for a given change in the interest rate and a given duration.

$$Approximate\ percentage\ price\ change = -duration \times \Delta r \times 100$$

The negative sign on the right side of the equation shows the inverse relationship between a price change and a yield change (Fabozzi, 2005).

Figure 25 illustrates the approximate error in price between the straight line (duration) and the curved line (convexity). The formula for the duration of a bond shows that the duration —the price sensitivity or elasticity— depends on the maturity of the bond, the coupon level, and the yield to maturity (interest or discount rate). Holding other factors constant, the longer the time to maturity is, the greater the duration is, and the greater the bond's interest or coupon is, the smaller the duration is (Martellini, 2003).

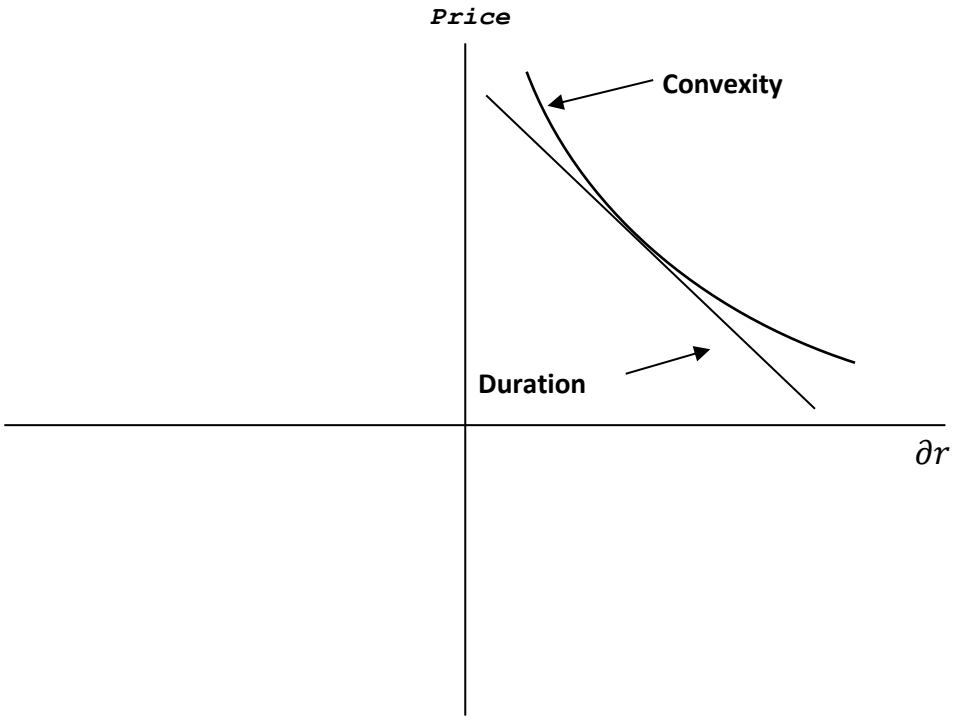


Figure 25: Relationship between duration and convexity

In this section, we conclude with two methods to value the duration of a convertible bond: the present value method and the binomial tree method.

For the present value method, I examine the outcomes for the AAV convertible bond price over a period when interest rate is raised. Figure 26 shows that the AAV convertible bond price falls as the interest rate increases. The duration is represented as a straight line that shows approximate estimation of the relationship between the bond value and the interest rate.

The change in price ΔV is expressed as

$$\Delta V = -D * V_0 * \Delta r.$$

Therefore, I find that the duration of the convertible bond is $D = 3.39$ for the date of 01/04/2011. This result means that at the time concerned, the approximate change in price for this bond is 3.39% given a 100-basis point or 1% change in the interest rate. Similarly, if the interest rate changes by 50 basis points, or 0.5%, up or down, the convertible bond price is likely to react to the interest rate change by shifting 1.695% in the opposite direction. The calculations of the duration can be seen in Table 12.

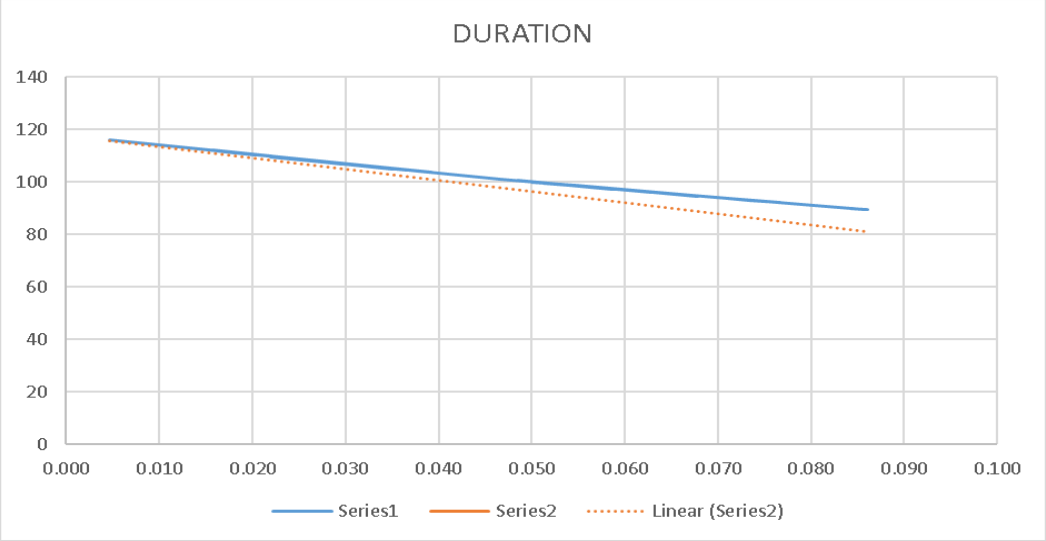


Figure 26: AAV convertible bond price (Series1) and interest rate (Series2) for one factor case

Present value													
				if r up					if r down				
1.005411	0.994619			1.009029	0.991052				1.00180117	0.998202			
	0.005381	99.46186			0.008948	99.10515				0.001798	99.82020681		
	3.587615				3.579396					3.595864			
	17.93807				17.89698					17.97932			
	V0	117.3999			V+	117.0021				V-	117.7995261		
						0.797394	0.2348	D	Delta p y inc	New price		Delta p y dic	New price
								3.396058831	-0.398697073	117.0012		0.398697073	117.798628

Table 12: Duration calculation sheet for the AAV convertible bond

For the binomial tree method, I first construct the tree for the convertible bond value V_+, which represents an increase in yield. Second, I construct the tree for the convertible bond value V_-, which represents a decline in yield. Δr is expressed as the change in the interest rate that should be added to the interest rate tree (Fabozzi,1999).

For illustration, I use the numerical example used in Section 4.6.1 for the AAV.DB.H option-free convertible bond. I assume that $\Delta r = 0.0025$ or 25 basis points for this example.

First, I construct the new interest rate tree for V_+ after shifting the interest rate up by Δr.

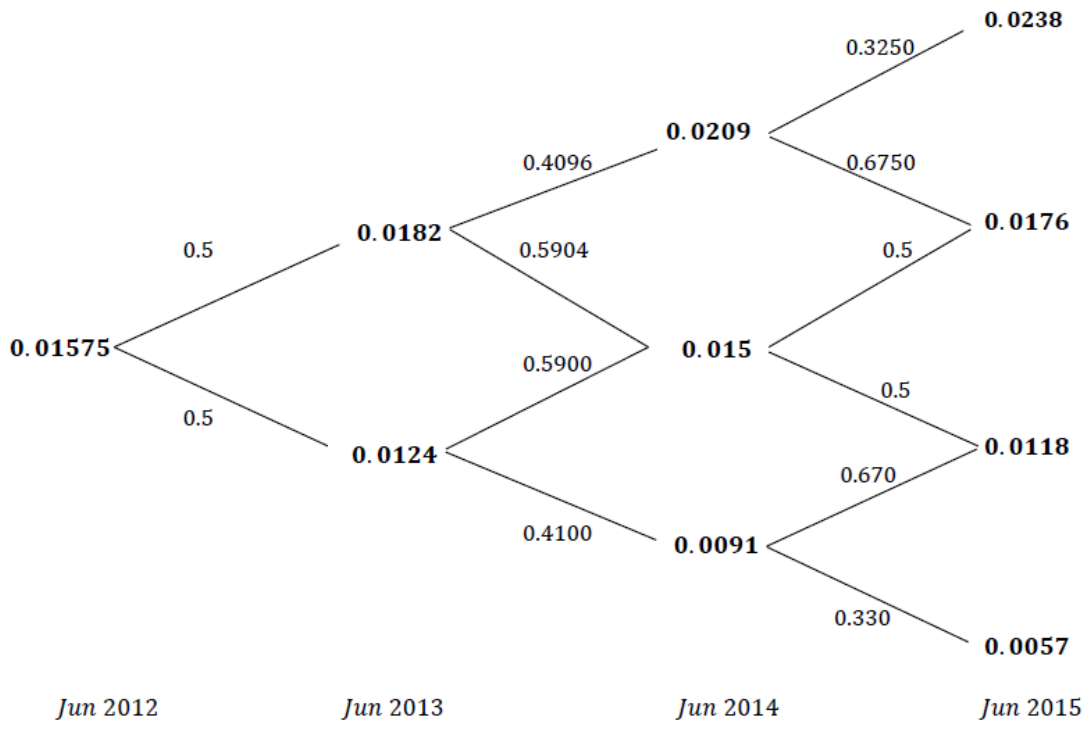


Figure 27: V_+ interest rate tree after shifting the rate up by + 25 basis points

The value of the convertible bond V_+ is 110.25.

Second, I construct the new interest rate tree for $V\_$ after shifting the interest rate down by Δr.

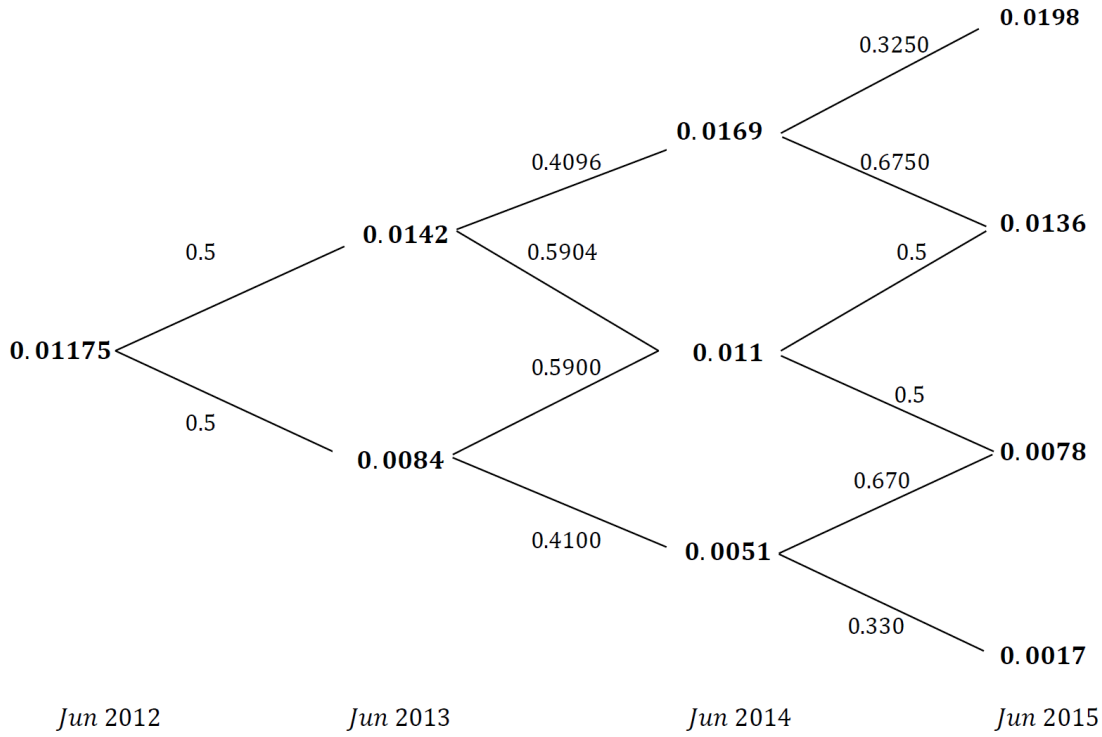


Figure 28: V_ interest rate tree after shifting the rate down by - 25 basis point

The value of the convertible bond $V\_$ is 113.12.

The duration is then

$$Duration = \frac{113.12 - 110.25}{(100)(2 \times 0.0025)}$$

$$Duration = 7.175$$

For the callable convertible bond duration, we use the numerical example from Section 4.6.2 for the CWT.DB.B convertible bond. The convertible bond has a strike call price of 125 with a trigger date of 30/06/2014. At the strike date of the call option, the callable convertible bond value is

$$V^C = Max[CV,\ Min[B, CP]].$$

We use the same procedure that used in the previous example. However, we assume that $\Delta r = 0.0035$, or 35 basis points, for this example.

In the first step, we construct the new interest rate tree for V_+ after shifting the rate up by $+35$ basis points.
The convertible bond value V_+ is 121.9.

In the next step, we construct the new interest rate tree for V_- after shifting the rate down by -35 basis points.

The convertible bond value V_- is 125.2.

The duration is then

$$Duration = \frac{125.2 - 121.9}{(100)(2 \times 0.0035)}$$

$$Duration = 4.715$$

Duration of call option

The duration of the call option of the convertible bond measures the sensitivity of the option value to changes in the interest rate (Fabozzi, 1999). The duration of the call option is directly related to the convertible bond and is expressed as

$$D_{option} = D \times Delta_{option} \times \frac{V}{Call\ option\ value},$$

where

$$Delta_{option} = \frac{Change\ in\ option\ value}{Change\ in\ convertible\ bond\ value}.$$

5.3 Convexity

Convexity is a measure of the curvature of the value of a security or portfolio as a function of the interest rate. It indicates how the duration changes as interest rates change. The numerical convexity is defined as

$$convexity\ (Cx) = \frac{V_+ + \ V_- - 2V_0}{2(V_0)(\Delta r)^2}$$

Convexity measures the curvature of the price-yield relationship, and the convex line shows that the bond price is a nonlinear function of the yield to maturity. As the duration relationship does not fully capture the true relationship between bond prices and yields, convexity contributes to a more accurate estimation of the price-yield relationship by using higher-order differentiation.

The maturity and the coupon rate are the main characteristics that show a strong relationship with convexity. Holding the other factors constant, convexity has an inverse relationship with the coupon rate; the greater the coupon rate is, the lower the convexity is. However, maturity is positively correlated with convexity; the greater the maturity is, the higher the convexity is (Martellini, 2003).

Convexity can be used in association with duration to obtain an accurate estimation of the percentage change in the convertible bond price. The percentage change in the convertible bond price is given as:

$$\Delta V = -D \times \Delta r + Cx \times \ (\Delta r)^2.$$

The convexity adjustment is the change required to be made to the convexity to obtain a better estimation. The convexity adjustment is given as

$$convexity\ \ adjustment\ = Cx \times \ (\Delta r)^2.$$

For clarification, we provide an example of the convexity and the convexity adjustment of the AAV convertible bond.

We will use the same previous example data for the AAV convertible bond where the initial bond price is 117.399 and $\Delta r = 0.001$.

When r increases by Δr, the present value of the convertible bond is 117.0021, and the percentage change in the convertible bond price is

$$\Delta V^{+} = \frac{V_{+} - V_{0}}{V_{0}} = -0.3388\%.$$

When r decreases by Δr, the present value of the convertible bond is 117.799, and the percentage change in the convertible bond price is

$$\Delta V^{-} = \frac{V_{-} - V_{0}}{V_{0}} = 0.3404\%.$$

The convexity of the convertible bond is

$$convexity\ (Cx) = \frac{V_{+} + V_{-} - 2V_{0}}{2(V_{0})(\Delta r)^{2}}.$$

$$convexity\ (Cx) = 7.647.$$

The adjusted convexity of the convertible bond is

$$convexity\ adjustment = Cx \times (\Delta r)^{2}$$
$$= 0.00076\%.$$

5.4 Duration and convexity of the European zero-coupon convertible bond

Mathematically, the duration of the convertible bond V is defined as the first derivative of V with respect to r, and it can be written as

$$Duration\ (D) = -\frac{1}{V}.\frac{\partial V}{\partial r}.$$

As there is no closed solution for the American convertible bond in the case we considered here, we differentiate the European convertible bond Value shown in Section 4.2.

As discussed in Section 4.2, the value of the European convertible bond is written as:

$$V(S,r,t) = SN(d_1) - KP(r,t)N(d_2) + FP(r,t),$$

By differentiating V with respect to r, the duration (D) of the convertible bond is obtained as

$$D = -\frac{1}{V}.\frac{\partial\big(SN(d_1) - KP(r,t)N(d_2) + FP(r,t)\big)}{\partial r}.$$

$$D = -\frac{1}{V}.\left[SN'(d_1)\frac{\partial d_1}{\partial r} - K\frac{\partial P}{\partial r}N(d_2) - KP(r,t)N'(d_2)\frac{\partial d_2}{\partial r} + F\frac{\partial P}{\partial r}\right].$$

Mathematically, the convexity of the convertible bond is defined as the second derivatives of V with respect to r.

$$convexity(Cx) = \frac{1}{V}.\frac{\partial^2 V}{\partial r^2}.$$

By differentiating D with respect to r, the convexity (Cx) of convertible bond is obtained as

$$Cx = \frac{1}{V}.\frac{\partial^2\left(SN(d_1) - KP(r,t)N(d_2) + FP(r,t)\right)}{\partial r^2}$$

$$\begin{aligned} Cx = \frac{1}{V}.\Bigg[& SN''(d_1)\left(\frac{\partial d_1}{\partial r}\right)^2 + SN'(d_1)\frac{\partial^2 d_1}{\partial r^2} - K\frac{\partial^2 P}{\partial r^2}N(d_2) \\ & - 2K\frac{\partial P}{\partial r}N'(d_2)\frac{\partial d_2}{\partial r} - KP(r,t)N''(d_2)\left(\frac{\partial d_2}{\partial r}\right)^2 \\ & - K\frac{\partial P}{\partial r}N'(d_2)\frac{\partial^2 d_2}{\partial r^2} + F\frac{\partial^2 P}{\partial r^2}\Bigg], \end{aligned}$$

where

$$d_1 = \frac{ln\left({}^{S}/_{K}\right) - ln\,P(r,t) + (r - q_s)\frac{1}{2}\hat{\sigma}^2\tau}{\hat{\sigma}\sqrt{\tau}},$$

$$d_2 = d_1 - \hat{\sigma}\sqrt{\tau}.$$

In the next section, I investigate the sensitivity of the duration of the convertible bond with respect to various parameters such as the short rate, the stock price, the volatilities of both factors, the long run, the mean reversion rate and the dividend yield of the underlying asset. We use the solution of the European convertible bond to study the partial derivatives of these parameters as there is no a closed-form expression for an American convertible bond. The findings are consistent with other studies, such as Choi (2004). However, it is possible to use binomial tree method to calculate approximate values of these derivatives, since they are tedious, I omit the details. In the Delta arbitrage section in the following, we illustrate how to use binomial methods to compute Delta, similar ideas apply to all quantities discussed here.

5.4.1 The sensitivity of the zero-coupon bond price and duration to interest rate changes

Here, I investigate the effect of interest rate movements on the values of the convertible bond V through the zero-coupon bond component:

$$\frac{\partial V}{\partial P}\frac{\partial P}{\partial r}$$

Recall that,

$$V(S,r,t) = SN(d_1) - KP(r,t)N(d_2) + FP(r,t)$$

and

$$P(r,t) = A(t,T)e^{-rB(t,T)}$$

We have

$$\frac{\partial V}{\partial P}\frac{\partial P}{\partial r} = (-KN(d_2) + F)\frac{\partial P}{\partial r}$$

The partial derivative of duration with respect to r is

$$\frac{\partial D}{\partial r} = \frac{1}{V^2}\left(\frac{\partial V}{\partial r}\right)^2 - \frac{1}{V}.\frac{\partial^2 V}{\partial r^2},$$

where

$$D = -\frac{1}{V}.\frac{\partial V}{\partial r}.$$

5.4.2 The sensitivity to the interest rate volatility σ_r

Changes in σ_r should affect the convertible bond price in the following form

$$\frac{\partial V}{\partial \sigma_r} = \frac{\partial(SN(d_1) - KP(r,t)N(d_2))}{\partial \sigma_r} + F\frac{\partial P}{\partial \sigma_r},$$

Use Chain rule, we obtain

$$\frac{\partial(SN(d_1) - KP(r,t)N(d_2))}{\partial \sigma_r} = \frac{\partial(SN(d_1) - KP(r,t)N(d_2))}{\partial \hat{\sigma}} . \frac{\partial \hat{\sigma}}{\partial \sigma_r},$$

and

$$\frac{\partial P}{\partial \sigma_r} = \frac{\partial P}{\partial r} \frac{\partial r}{\partial \sigma_r}$$

where

$$\hat{\sigma}^2 = {\sigma_s}^2 + \frac{{\sigma_r}^2}{k^2} (1 - e^{-k\tau})^2.$$

Therefore

$$\frac{\partial V}{\partial \sigma_r} = \frac{\partial(SN(d_1) - KP(r,t)N(d_2))}{\partial \hat{\sigma}} . \frac{\partial \hat{\sigma}}{\partial \sigma_r} + F \frac{\partial P}{\partial r} \frac{\partial r}{\partial \sigma_r}.$$

The partial derivative of duration with respect to σ_r is

$$\frac{\partial D}{\partial \sigma_r} = \frac{1}{V^2} \frac{\partial V}{\partial r} \frac{\partial V}{\partial \sigma_r} - \frac{1}{V} \frac{\partial^2 V}{\partial r \partial \sigma_r},$$

where

$$D = -\frac{1}{V} . \frac{\partial V}{\partial r}.$$

Figure 29 shows the real-life data testing of AAV convertible bond price with respect to σ_r.

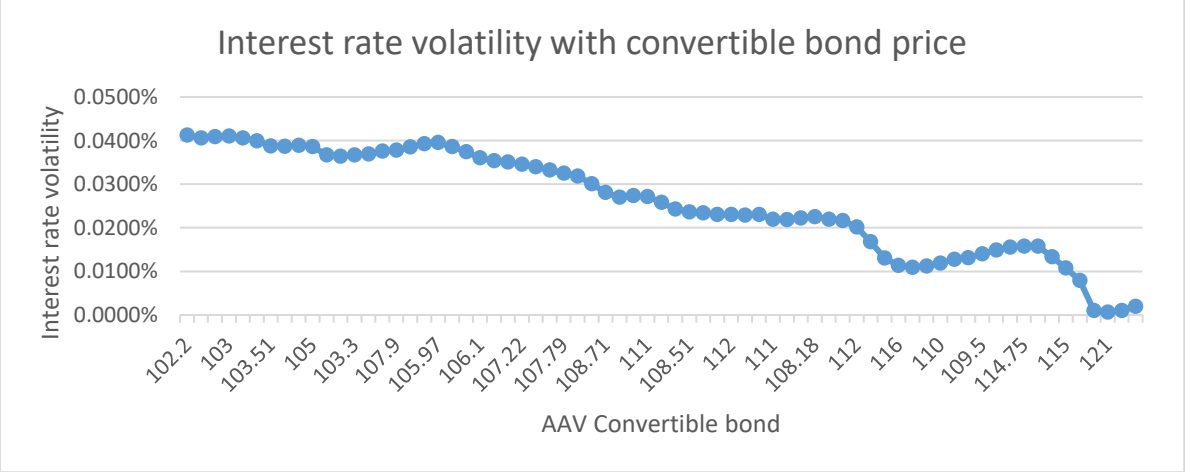


Figure 29: Relationship between the AAV convertible bond price and interest rate volatility

5.4.3 The impact of stock price changes on the duration and convertible bonds

Changes in S should affect the convertible bond price in the following form

$$\frac{\partial V}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - KP(r,t)N'(d_2)\frac{\partial d_2}{\partial S}$$

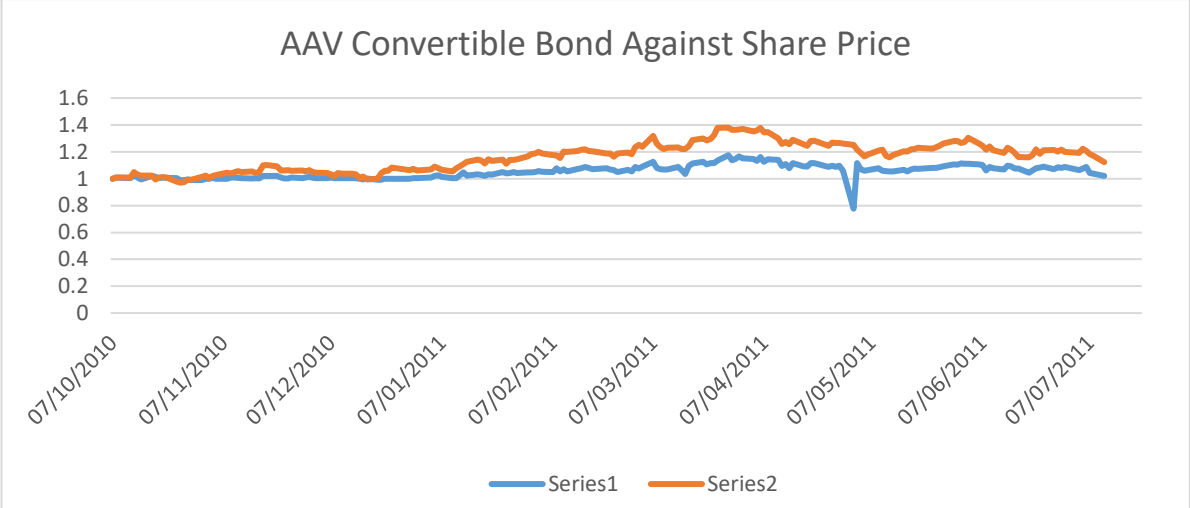


Figure 30: AAV convertible bond price (Series1) and share price (Series2)

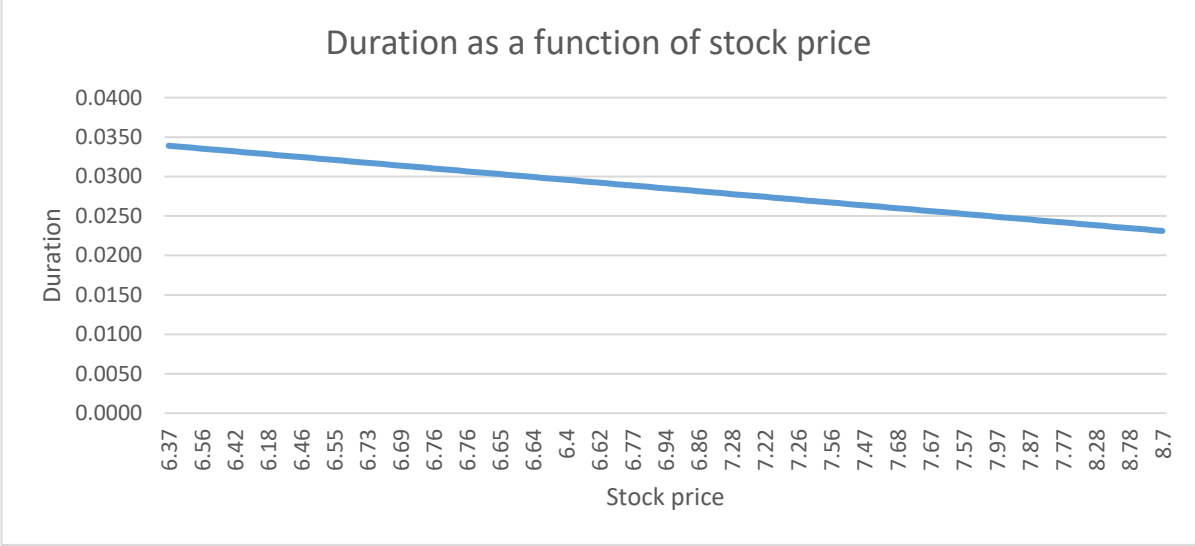


Figure 31: AAV Duration as a function of stock price for one factor case

5.4.4 The sensitivity to the stock price volatility σ_s

Changes in σ_S should affect the convertible bond price through the term volatility $\hat{\sigma}$.

The partial derivative of the value of the convertible bond with respect to σ_S is

$$\frac{\partial V}{\partial \sigma_S} = \frac{\partial V}{\partial \hat{\sigma}} \frac{\partial \hat{\sigma}}{\partial \sigma_S}$$

where

$$\hat{\sigma}^2 = {\sigma_s}^2 + \frac{{\sigma_r}^2}{k^2}(1 - e^{-k\tau})^2.$$

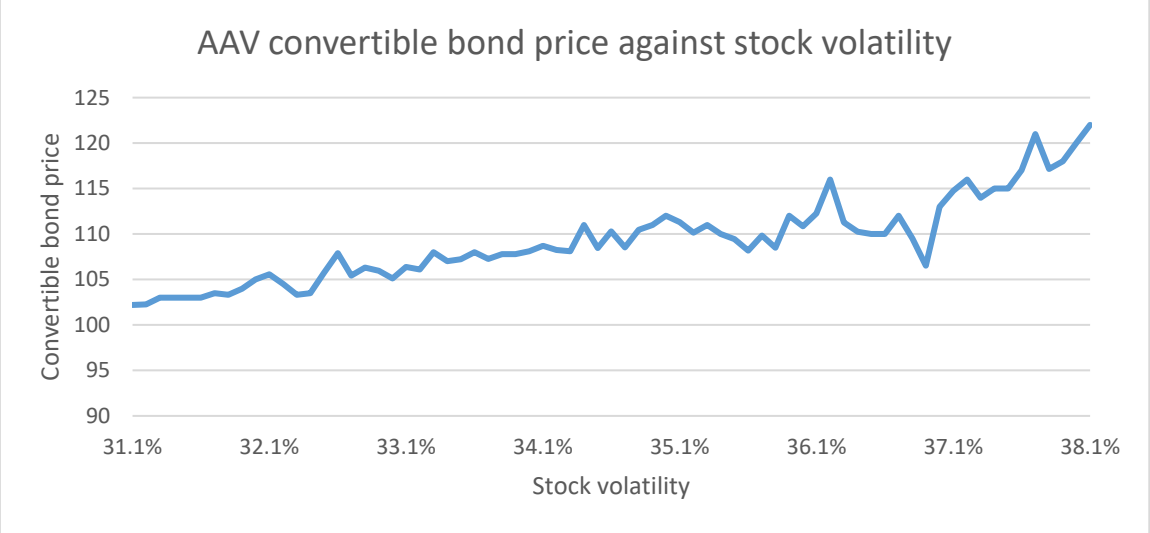


Figure 32: Relation between convertible bond price and stock volatility of AAV

5.4.5 The sensitivity to the long-run rate θ

The long-run rate θ should affect the convertible bond price through changes in the bond price component, as the long-run rate θ is an effective parameter in the interest rate process.

The partial derivative of the value of the convertible bond with respect to θ is

$$\frac{\partial V}{\partial \theta}.$$

Thus, the partial derivative of duration with respect to θ becomes

$$\frac{\partial D}{\partial \theta} = \frac{1}{V^2}\frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta} - \frac{1}{V}\frac{\partial^2 V}{\partial r \partial \theta},$$

where

$$D = -\frac{1}{V}.\frac{\partial V}{\partial r}.$$

5.4.6 The sensitivity to the mean reversion rate k

The mean reversion rate k should affect the convertible bond price through the change in the interest rate, which is reflected in the straight bond component. The partial derivative of the value of the convertible bond with respect to the mean reverting rate k is

$$\frac{\partial V}{\partial k}.$$

5.4.7 Dividend yield

The dividend yield should affect the convertible bond price through the stock price or the conversion component. Therefore, the partial derivative of the value of convertible bond with respect to the dividend yield is

$$\frac{\partial V}{\partial q_s} = SN'(d_1)\frac{\partial d_1}{\partial q_s} - KFe^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial q_s}$$

The partial derivative of duration with respect to the dividend yield q_s becomes

$$\frac{\partial D}{\partial q_s} = \frac{1}{V^2}\frac{\partial V}{\partial r}\frac{\partial V}{\partial q_s} - \frac{1}{V}\frac{\partial^2 V}{\partial r \partial q_s}.$$

where

$$D = -\frac{1}{V}.\frac{\partial V}{\partial r}.$$

Figure 33 shows the real-life data testing of Just Energy convertible bond price with respect to q_s.

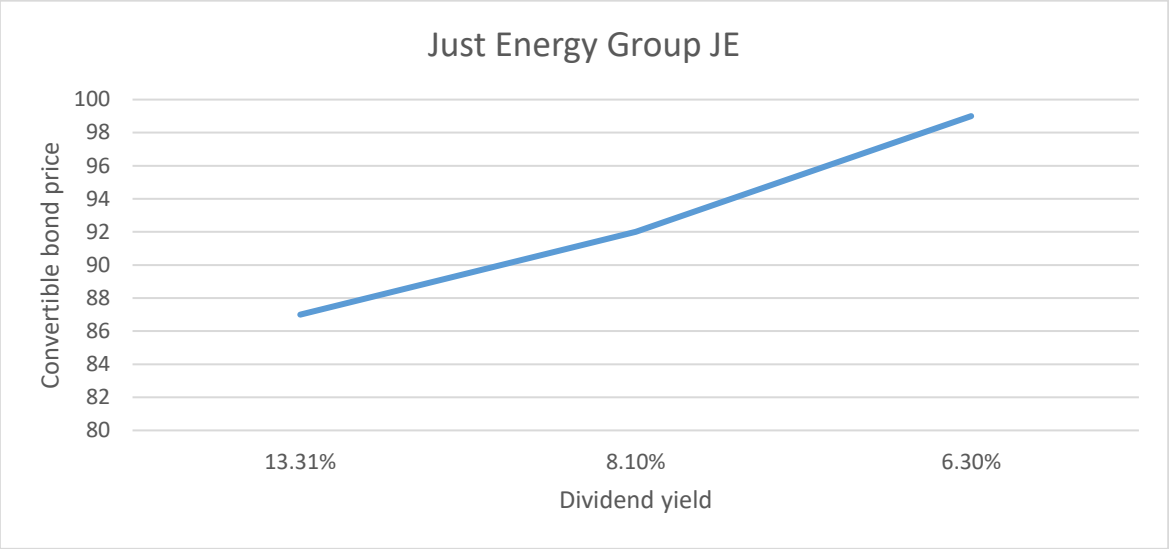


Figure 33: JE convertible bond as a function of the stock dividend yield

5.5 Convexity

In this section, I investigate the sensitivity of the convexity of the convertible bond to the various parameters, such as the short rate, the stock price, the volatilities of these factors, and the coupon rate using numerical computation on real life data example.

As described in Section 5.3, the convexity measure is an approximation of the (convex) curvature, as shown in Figure 25, which is expressed as the second derivative of V with respect to r.

$$convexity(Cx) = \frac{1}{V}.\frac{\partial^2 V}{\partial r^2}.$$

5.5.1 The sensitivity of convexity to stock prices.

Stock prices have a significant impact on the convexity of a convertible bond through the conversion option component (i.e., the option to convert to equity). The relationship between stock price movements and the equity option component of a convertible bond should always be positive. This property indicates that if the stock price increases, the equity option component should increase. As a result, the convexity of a convertible bond should decrease with stock price increases because convertible bondholders are likely to exercise the conversion option with an increase in the stock price, which indicates that the expected life of the bond should be shorter, and, therefore, the convexity should be lower.

In simple one-factor case all above statements are easy to verify. But in our multifactor case, the verification is not trivial.

Figure 34 shows that the convexity of the AAV convertible bond decreases with an increase in the corresponding stock price.

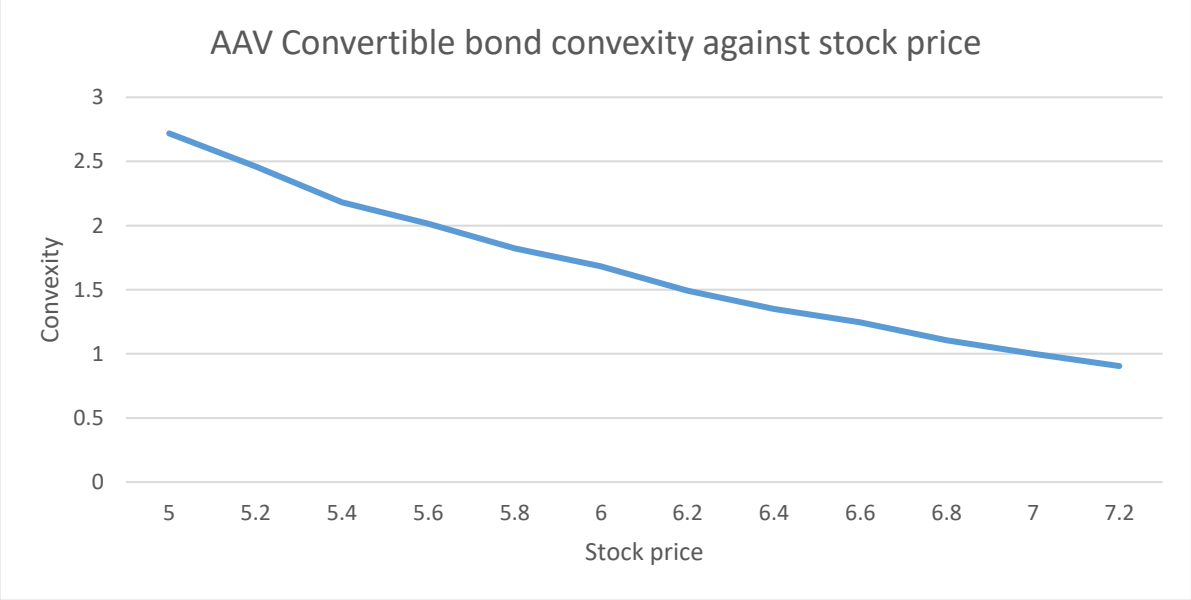


Figure 34: AAV convertible bond convexity as a function of stock price one factor case

5.5.2 The sensitivity of convexity to the interest rate.

The interest rate affects the convexity of a convertible bond mainly through the straight bond component of the convertible bond. When the interest rate increases, the convexity of the convertible bond should decrease, whereas a lower interest rate should increase the bond component, indicating a higher convexity of the convertible bond. Moreover, a lower interest rate should increase the present value of the coupon payment until the end of bond's life, which should increase the convexity of the bond component.

In simple one-factor case all above statements are easy to verify. But in our multifactor case, the verification is not trivial.

Figure 35 shows that the convexity of the AAV convertible bond decreases when the interest rate rises.

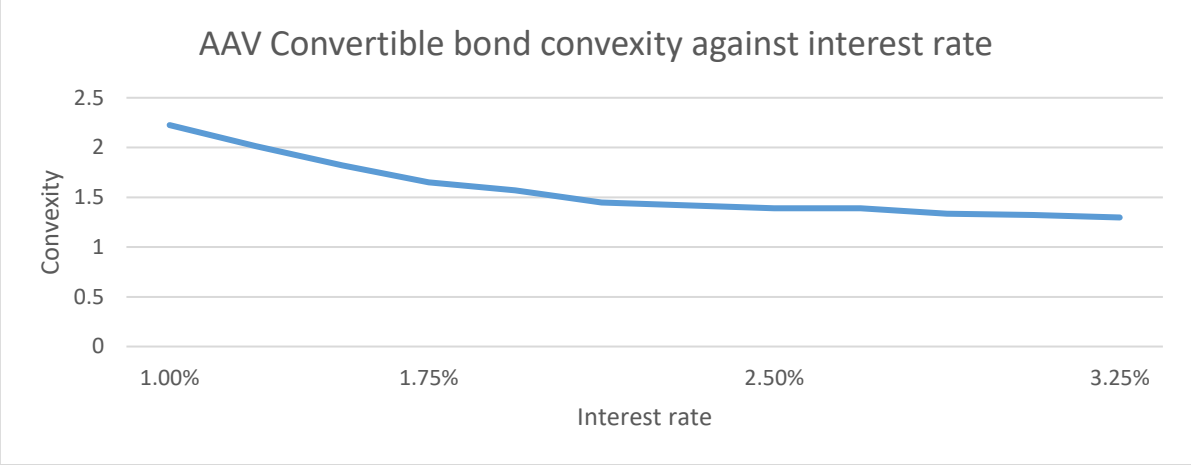


Figure 35: AAV convertible bond convexity as a function of interest rate one factor case

5.6 The Greeks of the convertible bond

In convertible arbitrage, traders rely on models that help to determine the fair value of a convertible bond. Several variables affect the mechanism between the value of a convertible bond and its components, such as the underlying stock

price, the volatility of the underlying stock, the credit spread, the risk-free rate, the time to maturity, and the dividend of the underlying stock. Each of these variables has Greeks that measure the rate of change of the convertible fair value when the given variable moves by one unit. In this section, I examine various Greeks that play an important role in convertible arbitrage strategies, namely, Delta and Gamma.

5.6.1 Delta of the convertible bond price

Delta measures the change in the convertible bond price V with respect to the change in the underlying common stock price S. The Delta under the single factor European option-pricing model of Black and Scholes is defined as

$$Delta\ (\Delta) = \frac{\partial V}{\partial S}.$$

$$\Delta = \ e^{-q_s(T-t)}N(d_1),$$

where

$$d_1 = \ \frac{ln\left(S/K\right) - ln\ F + (r - q_s)\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}},$$

where S is the current underlying stock price, K is the conversion price, r is the continuously compounded yield of a risk-free bond, q_s is the dividend yield, σ is the annualized stock return volatility, τ is the time to maturity in years, and $N(\ .\ )$ is the cumulative standard normal distribution function.

For the American convertible bond, it is also possible to carry out an approximate binomial calculation for Delta. For the binomial tree method, the convertible bond Delta Δ is

$$\Delta = \frac{V_u - V_d}{S_u - S_d}.$$

For further details of Delta, see Chapter 6.

The Delta of a convertible bond measures the convertible equity's sensitivity to any stock price changes. Delta is used as an estimation tool in the so-called Delta hedging strategy that determines the number of equity shares to short against the convertible bond's long position. Figure 36 and 37 show the slope of the Delta of the AAV convertible bond versus its underlying stock price.

The convertible bond Delta has a value between zero and one, so that

$$0 \leq \Delta \leq 1.$$

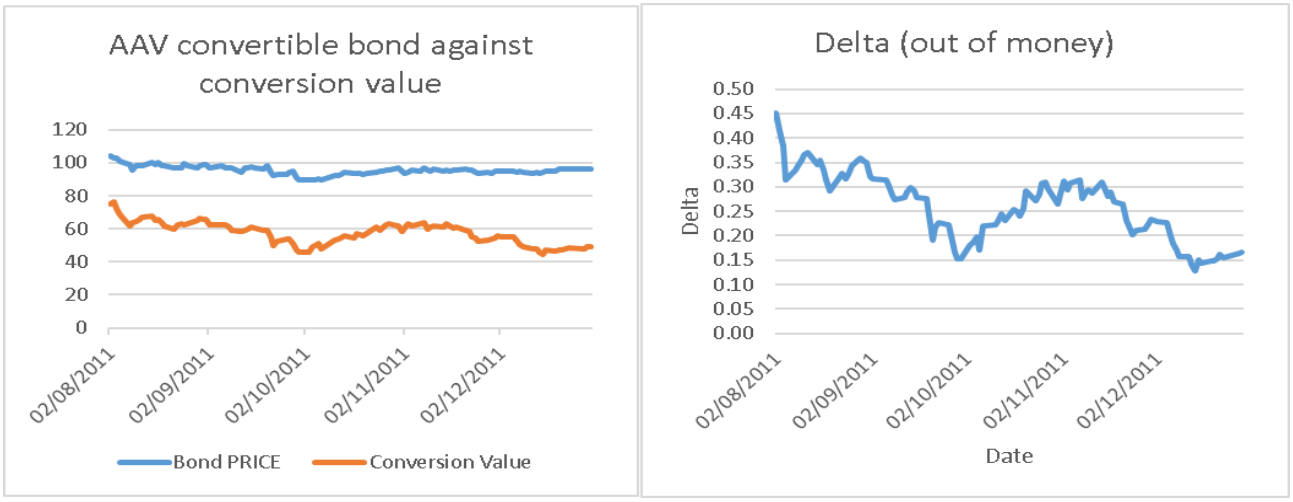


Figure 36: AAV convertible bond Delta (out of the money)

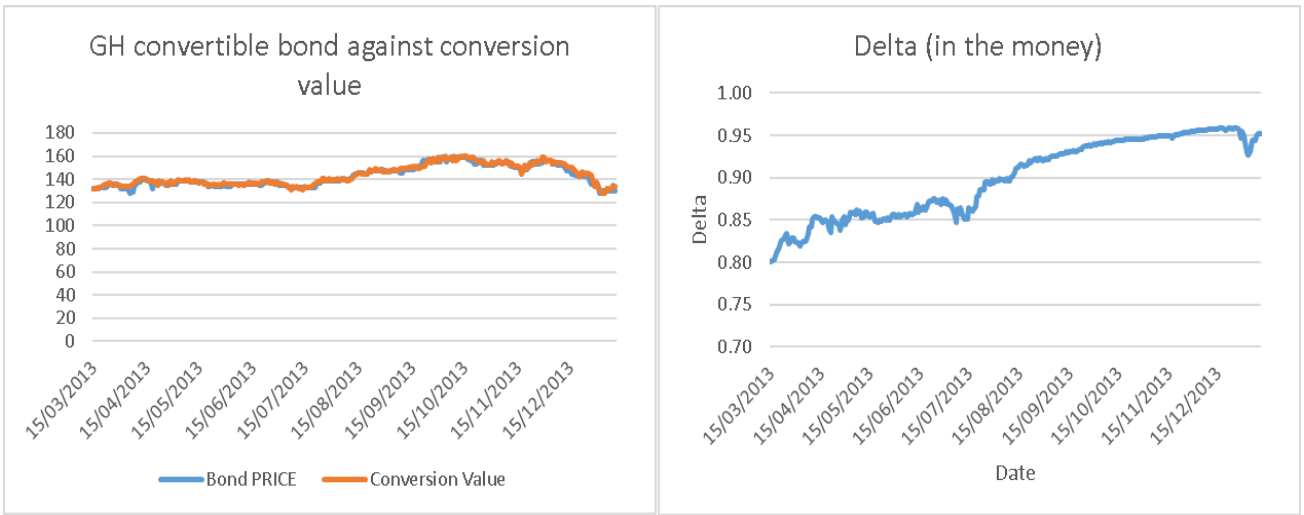


Figure 37: GH convertible bond Delta (in the money)

The convertible bond Delta reaches one when the convertible moves deep in the money, because the convertible behaves like the underlying stock price as the stock price far exceeds the conversion value. A Delta of one indicates that the convertible bond moves equivalently to the underlying stock price (Calamos, 2003).

On the other side, the convertible bond Delta reaches zero when the convertible moves far out of the money, which indicates that the convertible bond no longer has the opportunity to be converted to common stocks. The convertible bond should then behave like a fixed income security.

I will analyze the convertible bond Delta and arbitrage in Section 6.

5.6.2 Gamma of the convertible bond price

Gamma is the change in Delta with respect to the change in the underlying stock price. Gamma is expressed as the second derivative of the convertible bond price with respect to the underlying stock price. The Gamma under the single factor European option-pricing model of Black and Scholes is defined as

$$Gamma\ (\Gamma) = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}$$

$$\Gamma = N'(d_1)e^{-q_s(T-t)}/S\sigma\sqrt{T-t},$$

where

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-(d_1)^2/2}.$$

Convertibles that are deep in the money or far out of the money have low gamma values, whereas convertible bonds that are at the money have relatively higher

gamma values. Moreover, gamma is at its highest when the stock price equals the conversion value CV. In convertible arbitrage, a higher gamma indicates that the price of the long position of the convertible is more likely to increase than that of the short stock position is on the way up, and it is likely to decrease less than that of the short stock position is on the way down.

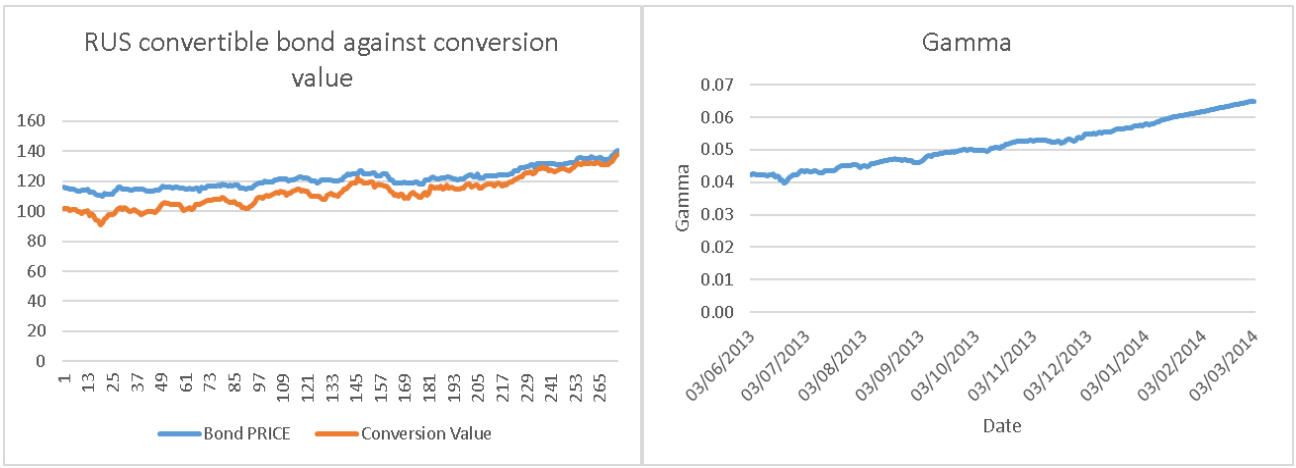


Figure 38: RUS convertible bond Gamma

6 Convertible Delta arbitrage

6.1 Introduction to convertible bond arbitrage

The Delta hedging that is associated with convertible bonds investment is called convertible arbitrage. Convertible arbitrage is an investment strategy that involves purchasing convertible securities and short selling the issuer's common stock. The number of shares sold short usually reflects a delta-neutral or market-neutral ratio. Under normal market conditions, the arbitrageur expects the combined position to be insensitive to fluctuations in the price of the underlying stock.

Convertible bond arbitrage strategy aims to benefit from undervalued convertible bonds by going long on the convertible and going short on the underlying stock. If the underlying stock price falls, the hedging fund will exploit its short position. It is also likely that the convertible bond will decline less than its underlying stock does because it is protected by its value as a fixed-income instrument.

Several convertible bond arbitrage studies have shown that there are pricing inefficiencies in some convertible bond markets due to their structural complexity. Amman, Kind, and Wilde (2003) showed that 21 convertible bonds listed on the French market were underpriced by 3% compared to their theoretical values between February 1999 and September 2000. This finding is consistent with those of other studies, such as King (1986), Kang and Lee (1996), Hutchinson (2004), Chan and Chen (2005), and Henderson (2005).

In this study, I analyze convertible bond Delta arbitrage by producing daily convertible bond arbitrage returns for 44 convertible bonds listed on the TSX for the period from 2009 to 2016.

The Delta hedging strategy is designed to generate returns from

1) the convertible bond yield income and short interest and
2) the long volatility exposure from the option component of the convertible bond.

In this section, I create a convertible bond arbitrage portfolio to capture the abnormal returns from the Delta hedging strategy and describe the risks associated with these returns.

The portfolio is created by matching long positions in convertible bonds with short positions in the underlying stock to create a Delta hedged convertible bond position that captures income and volatility. The Delta strategy is implemented by constructing an equally weighted portfolio of 44 hedged convertible bonds from 2009 to 2016.

The Delta hedging ratio of each convertible bond represents the number of short sold units of the underlying stock relative to one unit of the long convertible bond. It also measures the sensitivity of price movements between the convertible bond and its underlying stock. For instance, a Delta of 0.56 indicates that if the underlying stock price increases by 1%, the convertible bond price is likely to increase by 0.56%. Therefore, the hedging may be rebalanced as the stock price and/or the convertible price moves to capture the long volatility exposure.

In Section 6.2, I describe a typical convertible bond arbitrage position, provide a description of how this portfolio is constructed, and conclude with an explanation of how the return is captured from the convertible bond hybrid feature. In Section 6.3, I provide a brief summary of the sample that includes 44 convertible bonds listed on the TSX. In Section 6.4, I illustrate the convertible arbitrage hedging strategy results and present the buy and hold equity portfolio returns. In Section 6.5, I list the tables of yearly returns associated with the convertible arbitrage hedging strategy portfolio.

6.2 Convertible bond arbitrage and portfolio construction

A convertible bond arbitrage strategy is implemented by purchasing a convertible bond and selling the underlying stock short, creating a Delta hedged long volatility position. The short position is taken at the current Delta of the convertible bond. The return of the position is captured from the convertible bond coupon and the return of the short sale of the underlying stock, including the cost of borrowing the underlying stock for the short sale.

Moreover, the short sale minimizes the risk of the convertible bond portfolio, as the arbitrager will benefit from the short position in the underlying stock if the convertible bond price declines.

To create a Delta hedging for each convertible bond, I estimate the Delta for each convertible bond from the first trading day of the issuing. The Delta estimated Δ is then multiplied by conversion ratio of the convertible bond to calculate the number of shares or units of the underlying stock to be sold short. The Delta ratio is initiated for each trading day of the convertible bond. The Delta is obtained using two calculation methods: the binomial tree model and the Black-Scholes model with constant interest rate.

6.2.1 Delta of the binomial tree

The Delta is expressed as the ratio that estimates the change in the convertible price V with respect the change in the equity price S.

$$\Delta = \frac{\partial V}{\partial S}$$

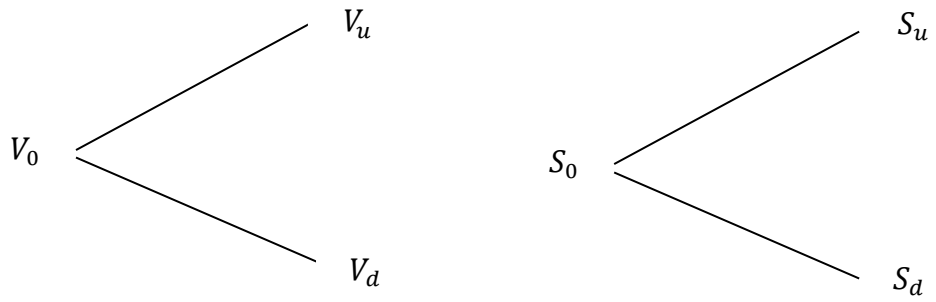


Figure 39: One-step binomial tree of the convertible bond and underlying stock

Let S be the value of the underlying stock at t_0. Then, its terminal value is S_u in the up state and S_d in the down state. Let V_0 be the convertible bond price at t_0, so V_u is the price of the convertible in the up state and V_d is the price of the convertible in the down state. Within the binomial-tree framework, the convertible bond Delta Δ is

$$\Delta = \frac{V_u - V_d}{S_u - S_d}.$$

For illustration, I use the numerical example presented in Section 4.6.2, which describes the CWT convertible bond. The hedging strategy involves purchasing the convertible bond on 01/12/2011 at the price of 108. The convertible bond is hedged against changes in the underlying stock price over one month. The number of short sale units relative to long convertible bonds is calculated to be 2.5 short sale shares for every unit of the convertible bond.

As the Delta measures the convertible bond's sensitivity to changes in the stock price or conversion value, the convertible Delta can be determined from the tree conversion value.

Note that

$$CV = \alpha S,$$

where α is the conversion ratio and S is the underlying stock price.

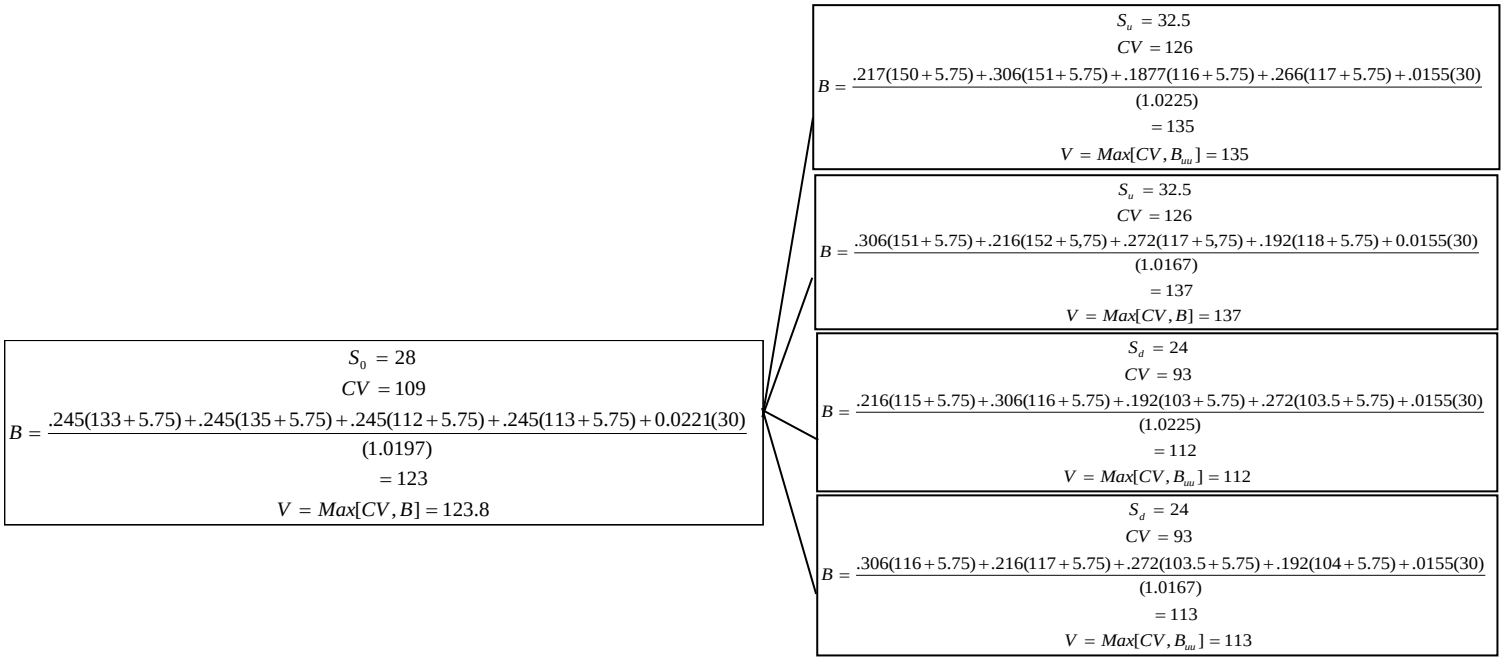


Figure 40: CWT pricing node

To calculate the Δ of the four nodes, I need to take the average Delta of the upper and lower nodes.

$$Upper\ Delta = (135 - 112)/(126 - 93)$$
$$= 0.69$$

$$Lower\ Delta = (137 - 113)/(126 - 93)$$
$$= 0.73$$
$$average\ delta = 0.71$$

Therefore, for a long investment in convertible bond V, I short the underlying stock S by an equivalent hedging value based on a Delta of 0.71. Table 13 shows an example of implementing the binomial Delta method over one month.

Date	Convertible bond price V	Stock price S	Change in V	Change in short position	Net change
20111201	108	26.97			
20111202	108.1	26.74	0.0009255	0.006392144	0.007317641
20111205	108.21	26.91	0.00101706	0.007810322	0.008827381
20111206	107.89	26.98	-0.0029616	0.010700441	0.007738847
20111207	108.8	26.99	0.00839914	0.010484379	0.018883524
20111208	107.5	26.71	-0.0120205	0.011238631	-0.000781856
20111209	107.5	26.42	0	0.007249617	0.007249617
20111212	107	26.47	-0.004662	0.009731069	0.005069056
20111213	106.85	26.55	-0.0014029	0.009247232	0.00784438
20111214	106.25	26.53	-0.0056312	0.001252324	-0.00437885
20111215	105.75	26.42	-0.004717	0.002023423	-0.002693567
20111216	106.25	26.03	0.00471699	-0.001547587	0.003169403
20111219	105.75	26.1	-0.004717	-0.017153933	-0.021870923
20111220	105.76	26.5	9.4558E-05	-0.011907757	-0.011813199
20111221	106.3	26.45	0.00509291	0.000253811	0.005346721
20111222	106.3	26.48	0	-0.008071553	-0.008071553
20111223	106.55	26.7	0.00234907	-0.011006937	-0.008657864
20111228	106.55	26.57	0	0.004514252	0.004514252
20111229	106.55	26.49	0	0.002515492	0.002515492
20111230	106.55	26.77	0	0.002240263	0.002240263
Total changes			-0.013516869	0.035965634	0.022448765

Table 13: Hedging strategy over one month

This example illustrates the convertible hedging strategy over one month. Although the convertible bond return declined by -1.35% over the month, the short position has a positive return of +3.59%, and the total return of the portfolio is +2.24%.

6.2.2 Delta of the Black-Scholes model

For a European-style convertible bond, the Black-Scholes model of constant interest rate can be used to determine the value of Delta Δ. The value of the convertible bond Delta Δ is

$$\Delta = e^{-q_s(\tau)} N(d_1),$$

where

$$d_1 = \frac{ln(S/K) - ln\, F + (r - q_s)\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

The convertible bond Delta has a value between zero and one, so that

$$0 \leq \Delta \leq 1,$$

where S is the current underlying stock price, K is the conversion price, r is the continuously compounded yield of the risk-free bond, q_s is the dividend yield, σ is the annualized stock return volatility, τ is the time to maturity in years, and $N(\,.\,)$ is the cumulative standard normal distribution function.

The number of short units against investing in a single convertible bond V is therefore

$$short\ units = \frac{V * \Delta}{S}.$$

The mean reverting volatility is used to estimate the amount of fluctuation in the underlying stock as it relates to the convertible bond and the conversion option. The historical volatility is calculated in a number of steps. First, I find the day-to-day price change (R_t) (the daily return) from the current and previous day (S_{t-1}) by

$$R_t = ln\left(\frac{S_t}{S_{t-1}}\right).$$

Then, I find the average price change over the estimation period (n) by the sum of the changes and by calculating R_m.

$$R_m = \sum R_t/n.$$

Therefore, the variance σ^2 from the mean is

$$\sigma^2 = \sqrt{\sum}(R_t - R_m)^2/N - 1.$$

For the annualized volatility, I multiply the variance by 252, the number of approximate trading days in a year.

The daily return is calculated for each convertible bond on each trading day at the current Delta from the start date of the security to its maturity date. The convertible may have removed from the portfolio if the expiry date has passed or if the convertible has been called by the issuer.

The returns for a position i on day t are calculated as follows

$$R_{it} = \frac{(V_{it} - V_{it-1}) + c_{it} + (-\Delta_{it-1})(S_t - S_{t-1})}{V_{it-1} + \Delta_{it-1}S_{t-1}},$$

where R_{it} is the return on position i at time t; V_{it} is the convertible bond closing price at time t; V_{it-1} is is the convertible bond closing price on the previous day $t-1$; c_{it} is the coupon payable between $t-1$ and t; Δ_{it-1} is the Delta hedging

ratio for position i at time $t-1$, where the negative sign represents the short sell position; S_t is the underlying equity closing price at time t; and S_{t-1} is the underlying equity closing price on the previous day $t-1$.

In order to obtain the coupon payable between $t-1$ and t, I need to calculate the accrued coupon on each trading day of the convertible bond. Accrued interest is calculated based on the number of trading days in the coupon period, the number of days in the accrued interest period, and the amount of coupon payments that are payable annually or semi-annually.

$$c_{it} = c \times \frac{1}{total\ days\ in\ coupon\ payment}$$

Then, a weighted average portfolio return can be calculated for the 44 convertible bonds from 2009 to 2016.

$$R_p = \sum_{i=1}^{n} w_{it} R_{it}$$

where R_{it} is the return on position i at time t; w_{it} is the weighting of position i on day t, n is the total number of position on day t.

6.3 Data

As mentioned in Section 6.2, I investigate 44 Canadian convertible issues between 2009 and 2016. Data regarding the issues and their characteristics, including stock and convertible bond prices, were obtained from the Stockwatch and TSX databases. Interest rates and government bond yields were collected from the Bank of Canada. Data such as the conversion ratios, dividend yields, start dates, maturity dates, and face values of the convertible bonds were obtained from CIBC reports. Table 14 shows the data sorting of the Delta hedging portfolio.

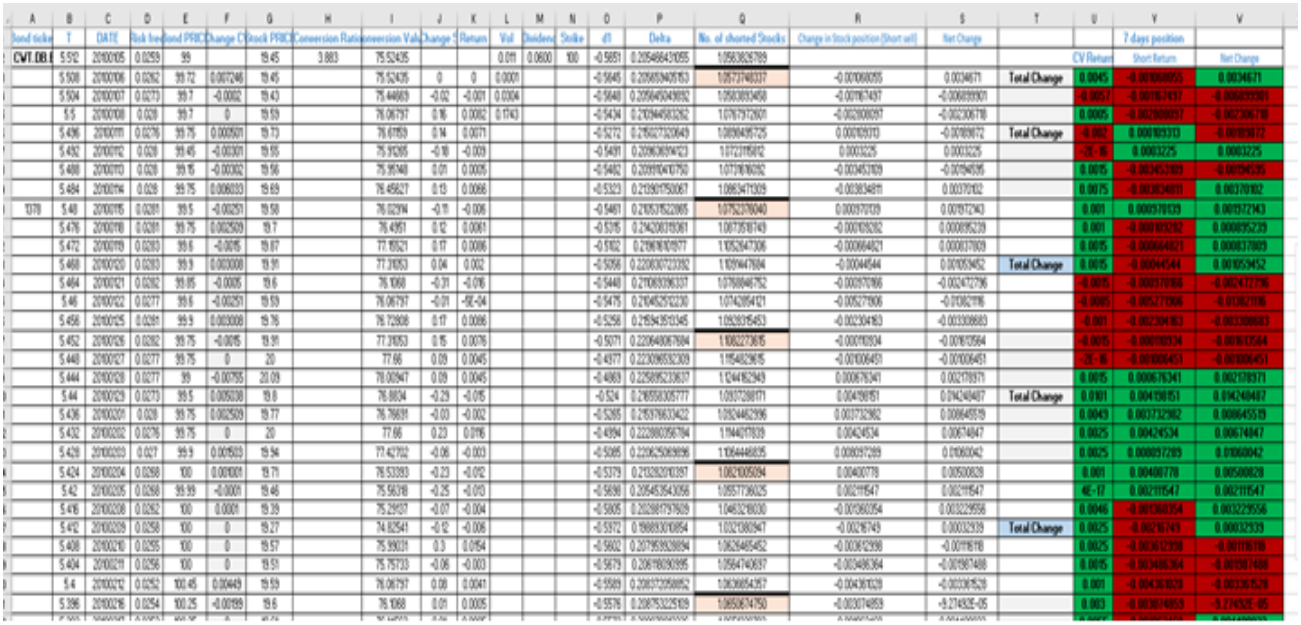


Table 14: Delta hedging portfolio data sorting

6.4 Results

6.4.1 Results of the binomial tree method

Table 15 illustrates a summary of the annual convertible bond's arbitrage return series based on the binomial method. In 2010, 16 new convertible bond positions were added to three positions that had already started in 2009, with an average position duration of 5.5 years. The majority of the listed positions were captured in 2013 and 2014, with 44 convertible bonds positions with average position durations of 2.5 and 1.8, respectively. By the end of 2015, 23 positions were closed out due to expiration or a call by the issuer.

The maximum average annual return on hedged positions was 25.72% in 2013, and the minimum position return was 9.32% in 2016. The maximum return on an individual position was 106.35%, and the minimum position return was -154%.

Year	Number of Positions	Average Position Duration (Yrs)	Max Position Return %	Min Position Return %	Daily Average Position Return %	Annualized Average Position Return %	Number of Positions Closed Out	Individual Positivity - Negativity of Return
31/12/2009	3	5.5	8.72%	-7.99%	0.04%	12.65%		2/-1
31/12/2010	19	4.3	45.06%	-14.08%	0.19%	9.58%		10/-9
31/12/2011	34	3.8	58.23%	-52.58%	0.02%	17.18%		12/-22
31/12/2012	42	3	58.23%	-46.52%	0.03%	10.19%		22/-20
31/12/2013	44	2.5	62.01%	-141.68%	0.10%	25.72%	1	22/-22
31/12/2014	44	1.8	85.73%	-81.98%	0.38%	21.04%	1	24/-20
31/12/2015	43	1	63.00%	-33.94%	0.45%	17.67%	23	24/-19
31/02/2016	20	0.3	106.35%	-154.03%	0.30%	9.32%	20	10/-14
Complete sample	44						44	

Table 15: Binomial tree Delta hedging summery

Figure 41 shows the average annual return of the Delta hedges of convertibles between 2009 and 2016.

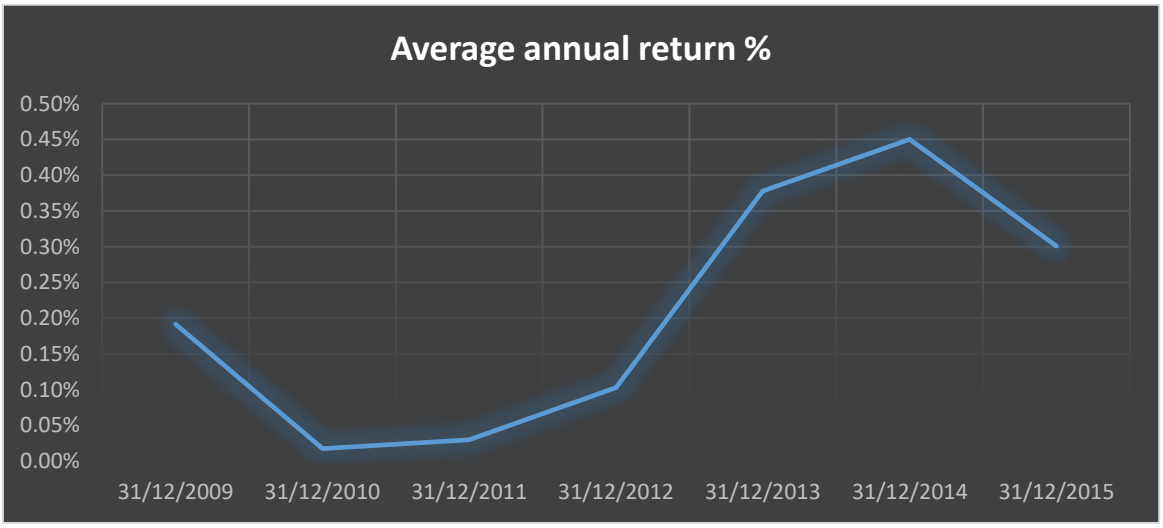


Figure 41: Average annual Delta -hedging return – binomial method

The histograms in Figure 42 illustrate the return distributions of the long convertible bonds and the short sell positions in the underlying stocks. These frequency figures show an example of the mean returns of the 44 convertible

bonds in 2013, when the maximum average annual return on the hedging positions was captured.

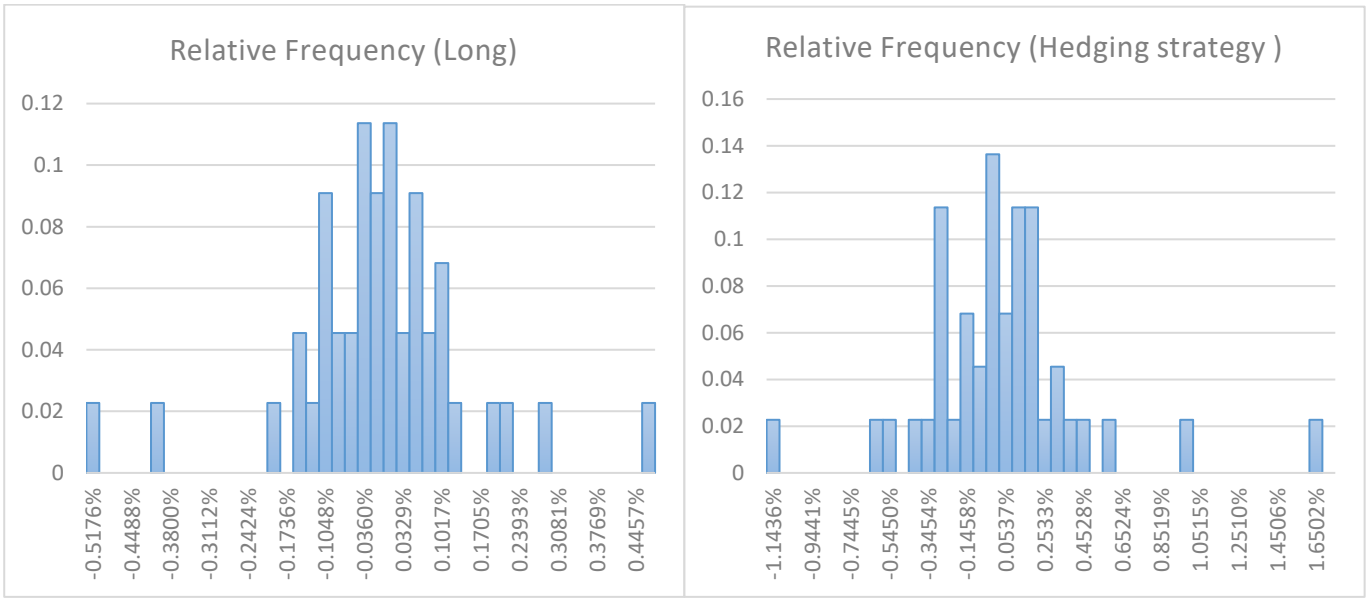
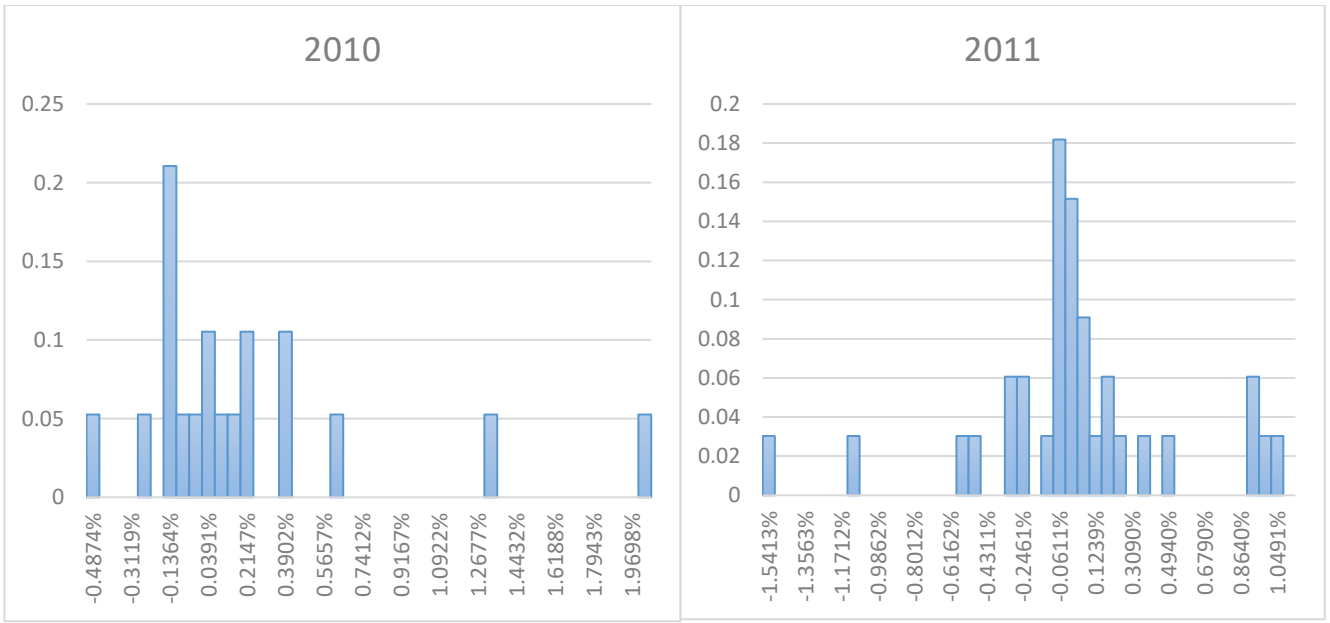
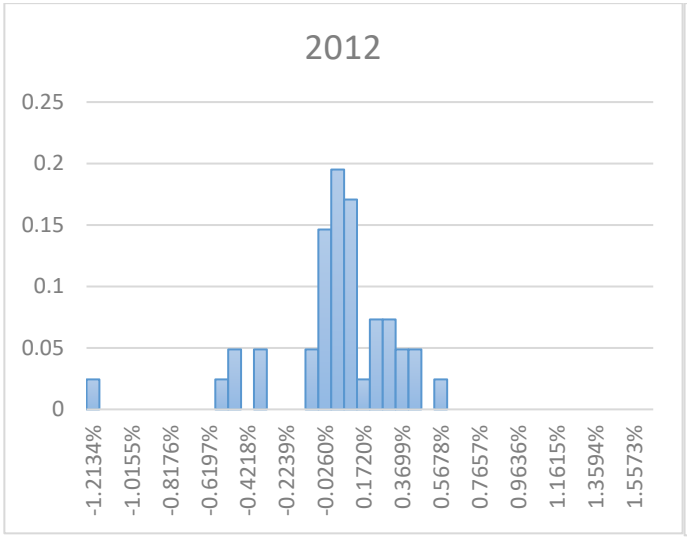
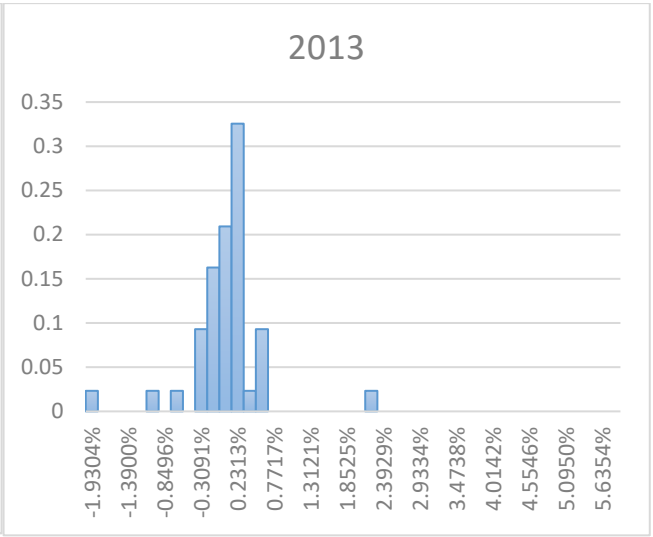


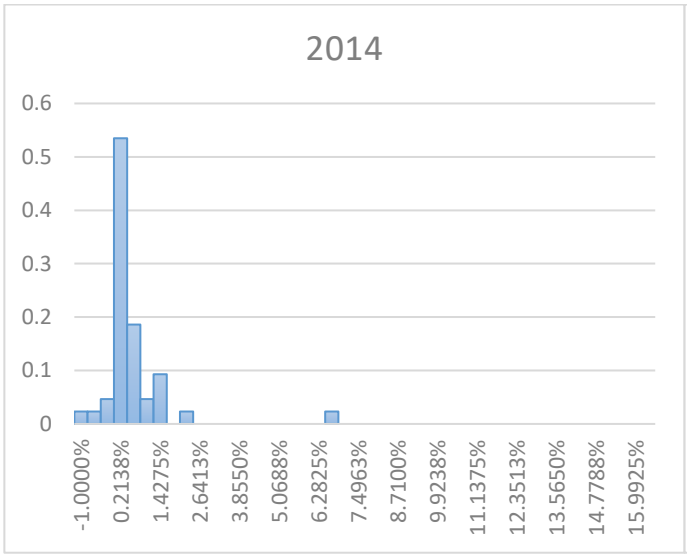
Figure 42: Return distributions of long convertibles positions and the hedging strategy

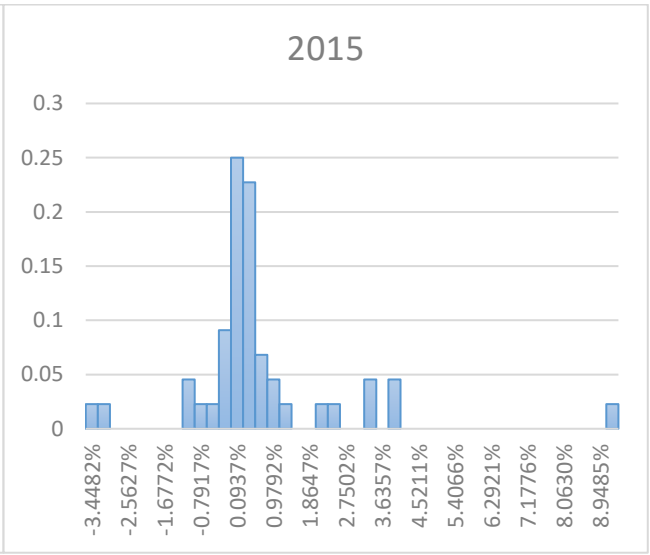
Return distributions

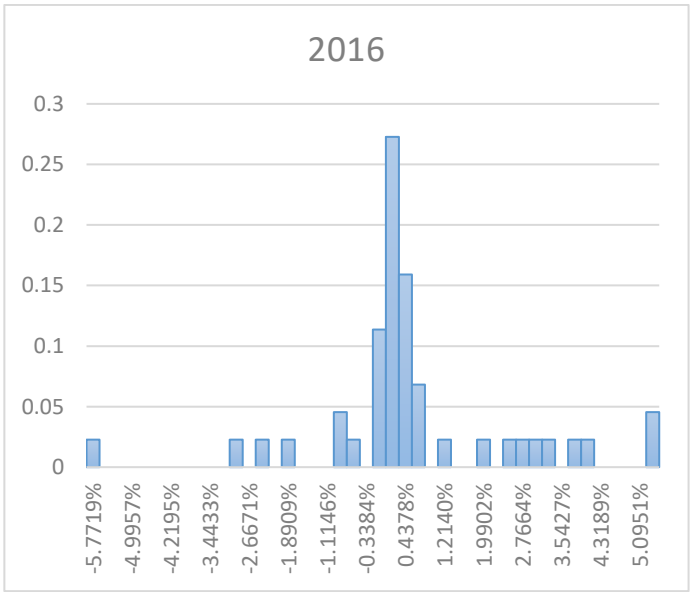



2012
0.25
0.2
0.15
0.1
0.05
0
-1.2134%
-1.0155%
-0.8176%
-0.6197%
-0.4218%
-0.2239%
-0.0260%
0.1720%
0.3699%
0.5678%
0.7657%
0.9636%
1.1615%
1.3594%
1.5573%


2013
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-1.9304%
-1.3900%
-0.8496%
-0.3091%
0.2313%
0.7717%
1.3121%
1.8525%
2.3929%
2.9334%
3.4738%
4.0142%
4.5546%
5.0950%
5.6354%


2014
0.6
0.5
0.4
0.3
0.2
0.1
0
-1.0000%
0.2138%
1.4275%
2.6413%
3.8550%
5.0688%
6.2825%
7.4963%
8.7100%
9.9238%
11.1375%
12.3513%
13.5650%
14.7788%
15.9925%


2015
0.3
0.25
0.2
0.15
0.1
0.05
0
-3.4482%
-2.5627%
-1.6772%
-0.7917%
0.0937%
0.9792%
1.8647%
2.7502%
3.6357%
4.5211%
5.4066%
6.2921%
7.1776%
8.0630%
8.9485%


2016
0.3
0.25
0.2
0.15
0.1
0.05
0
-5.7719%
-4.9957%
-4.2195%
-3.4433%
-2.6671%
-1.8909%
-1.1146%
-0.3384%
0.4378%
1.2140%
1.9902%
2.7664%
3.5427%
4.3189%
5.0951%

6.4.2 Results of the Black-Scholes model

Table 16 presents a summary of the annual convertible bond's arbitrage return series based on the Black-Scholes model. In 2010, 16 new convertible bond positions were added to three positions that had already started in 2009, with an average position duration of 5.5 years. The majority of listed positions were captured in 2013 and 2014, with 44 convertible bonds positions with average position durations of 2.5 and 1.8, respectively. By the end of 2015, 23 positions were closed out due to expiration or a call by the issuer.

Year	Number of Positions	Average Position Duration (Yrs)	Max Position Return %	Min Position Return %	Daily Average Position Return %	Annualized Average Position Return %	Number of Positions Closed Out	Individual Positivity - Negativity of Return
31/12/2009	3	5.5	2.98%	-2.41%	0.1608%	11.95%		2/-1
31/12/2010	19	4.3	26.62%	-14.08%	0.2127%	22.81%		18/-1
31/12/2011	34	3.8	35.28%	-34.04%	0.0676%	16.89%		28/-6
31/12/2012	42	3	14.59%	-18.09%	0.1142%	28.53%		36/-6
31/12/2013	44	2.5	56.99%	-64.99%	0.1286%	32.15%	1	23/-21
31/12/2014	44	1.8	22.45%	-20.47%	0.0629%	15.71%	1	25/-19
31/12/2015	43	1	43.07%	-28.76%	0.0491%	13.83%	23	17/-26
31/02/2016	20	0.3	11.02%	-19.70%	-0.2401%	-7.44%	20	6/-14
Complete sample	44						44	

Table 16: Black-Scholes Delta hedging summery

The maximum average annual return on hedging positions was 32% in 2013, and the minimum position return was -7.4% in 2016. The maximum return on an individual position was 57%, and the minimum position return was -64%.

Figure 43 shows the average annual return of the Delta hedging of the convertible bonds between 2009 and 2016.

The histograms in Figure 44 illustrate the return distributions of long convertible bonds and short sell positions in the underlying stock. These frequency figures show an example of the mean returns of 44 convertible bonds in 2013, when the maximum average annual return on hedging positions was captured.

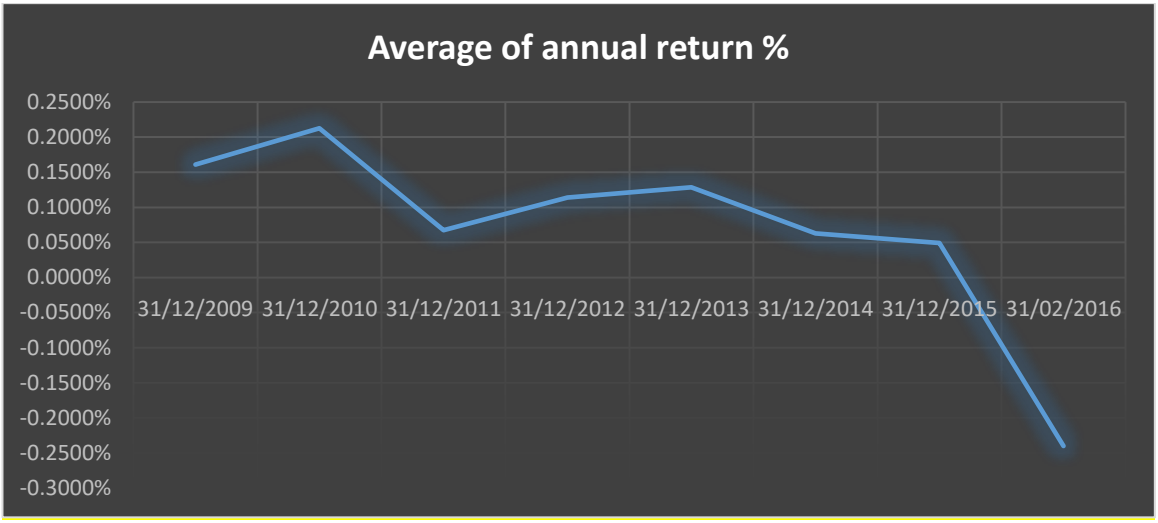


Figure 43: Average annual Delta -hedge return

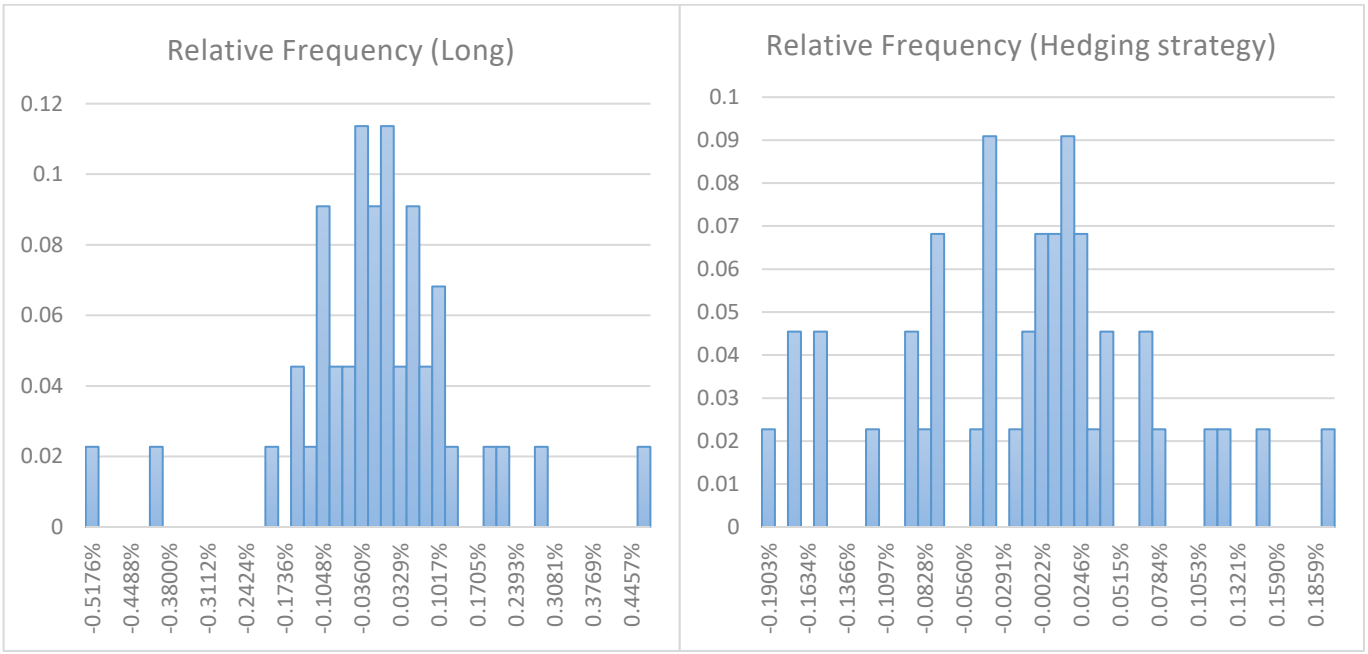
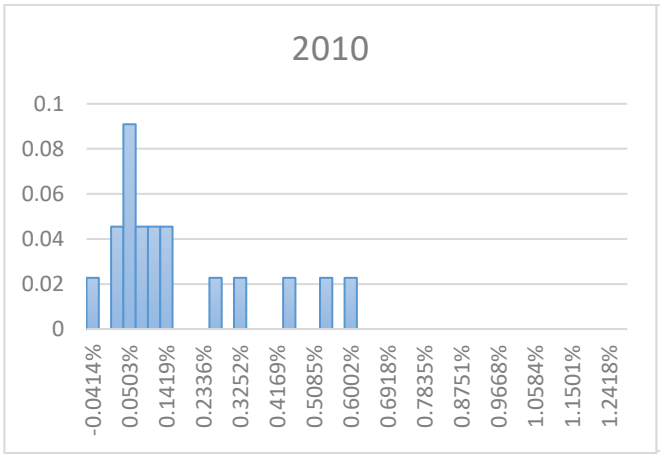
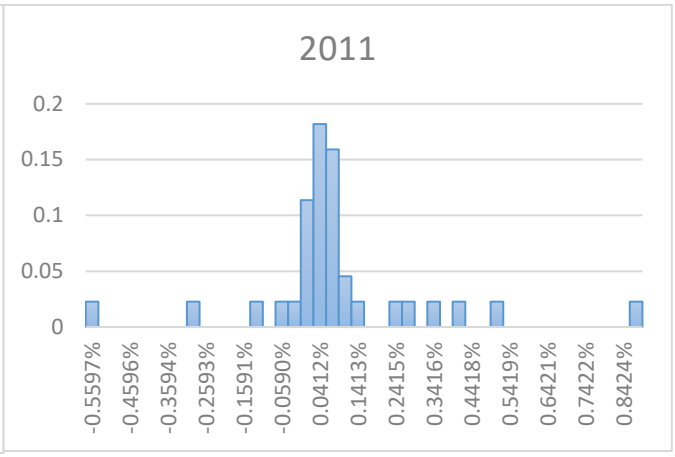
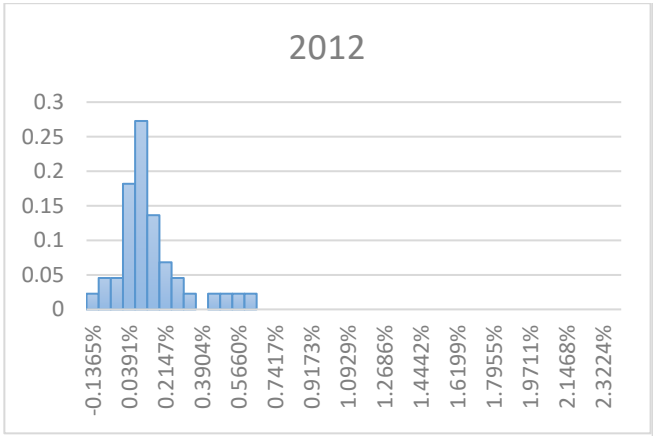


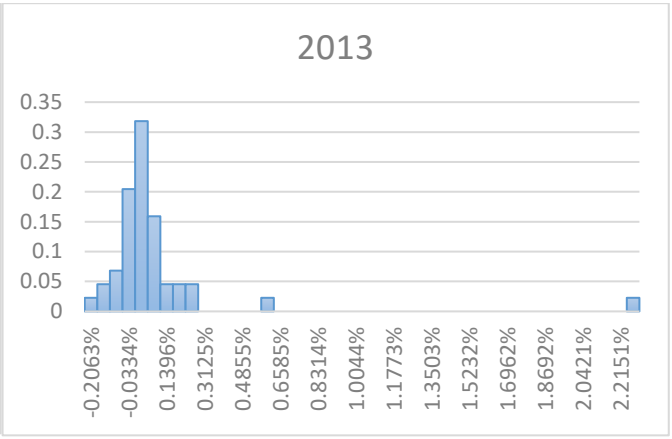
Figure 44: Return distributions of long convertibles positions and the hedging strategy

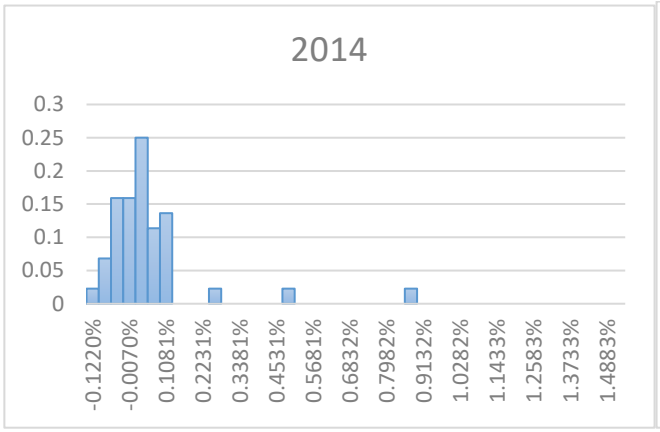
Return distributions

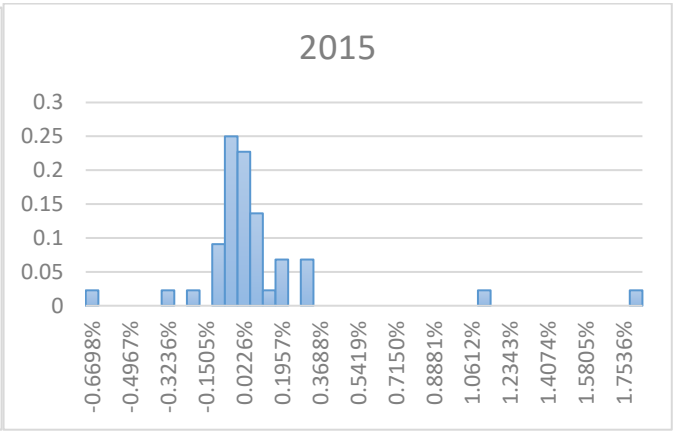


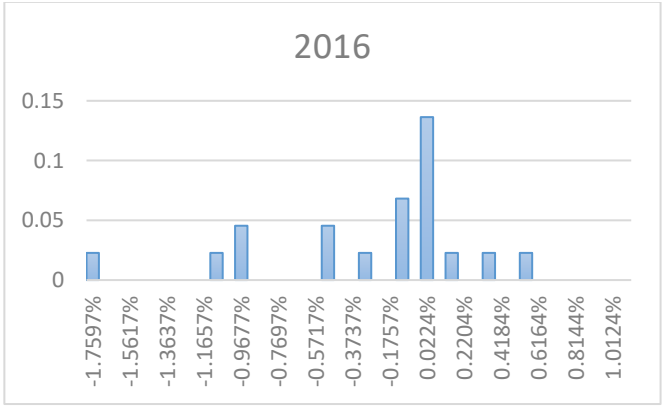












6.4.3 Summary of the results

The Delta strategy was implemented by constructing an equally weighted portfolio of 44 hedged convertible bonds from 2009 to 2016. The strategy aimed to produce deltas and returns for 44 convertible bonds that were listed on the TSX between 2009 and 2016. The return of the position is captured from the long convertible bond, the coupon interest of the convertible bond and the return of the short sale of the underlying stock, including the cost of borrowing the underlying stock for the short sale.

For the Black-Scholes model, our example indicated that annual average return was positive for most of the periods with a maximum average annual return of 32.15% in 2013. The worst returns were generated by positions added in 2016, with average annual returns of -7.4%. The maximum return on an individual position was 57%, and the minimum return on an individual position was -64%.

For the binomial method, the annual average return was positive for all periods with a maximum average annual return of 25.72% in 2013 and a minimum position return of 9.32% in 2016. The maximum return on an individual position was 106%, and the minimum return on an individual position was -154%. For both models, the majority of new positions were added in 2013 and 2014 with 44 positions. The lowest number of positions were captured at the opening of portfolios in 2009 with 3 positions.

Table 17 shows the summary of the annual average return between 2009 and 2016.

Year	Annualized Average Position Return % (Black-Scholes Model)	Annualized Average Position Return % (Binomial Method)
31/12/2009	**11.95%**	**12.65%**
31/12/2010	**22.81%**	**9.58%**
31/12/2011	**16.89%**	**17.18%**
31/12/2012	**28.53%**	**10.19%**
31/12/2013	**32.15%**	**25.72%**
31/12/2014	**15.71%**	**21.04%**
31/12/2015	**13.83%**	**17.67%**
31/02/2016	**-7.44%**	**9.32%**
Total sample	44	44

Table 17: Average annual returns of the Delta strategy

It can be seen that the average position returns of the Black-Scholes model and the binomial method were -7.44% and 9.32%, respectively. The average position duration was 0.3 years. The number of positions was only 20 when the majority of the positions were closed out.

The reason for the differences between the results of the two models is that the binomial method generated a higher Delta than the Black-Scholes model in most cases. The difference in Delta estimation can affect the determination of the number of shares selling short against the long convertible bond position. Therefore, the return of the short sale of the underlying stock is higher for the binomial tree method. Also, in the binomial tree mothed, the interest rate is assumed to be stochastic rather than constant as in the Black-Scholes model. Moreover, in 2016, when the majority of positions were closed out, some of the 20 active positions were out of money where Delta was almost zero. This is because the conversion value of the bond or the underlying stock fell far below the conversion price and the equity option component became nearly worthless. Therefore, any difference in the Delta estimation between the two models may result in a contrasted outcome, which was only seen in 2016.

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7 Conclusion

A convertible bond is a hybrid security of debt and equity. This type of bond provides its bondholders with the right to convert the issue to equity during the life of the bond. The conversion ratio determines the number of shares into which the bond can be converted. The call feature allows the bond to be purchased back by the issuer in the future at a pre-determined price and specified date.

In this thesis, I introduced a two-factor model for convertible bond valuation with default risk. I derived the interest rate and the stock price as two stochastic variables. The interest rate represents the debt component of the asset, and the stock price reflects the equity component. For interest rate modeling, I adopted a Vasicek model that captures mean reversion, where the drift $k(\theta - r_t)$ represents the expected instantaneous change in the interest rate at time t. I investigated the Vasicek model tree with data by back testing the Canadian five-year zero-coupon bond yield for the period between 2012 and 2015. The investigation shows a significant prediction for the Canadian interest rate, with a mean of 1.27% in 2015, whereas the actual rate was 1.316%.

For the underlying stock, I used the CRR model with some modifications. The model suggested that the equity volatility σ_s is non-constant in different intervals but remains constant within each time interval. This modification allows for two central nodes in the equity tree instead of one.

Then, I derived the PDE of the European convertible bond with respect to two stochastic variables, the interest rate and the underlying stock price. Because it was difficult to find a closed solution for the American convertible bond due to the complexity of its features, such as the option to convert and its callability, I used a binomial tree to find a numerical solution for the convertible bond price.

As a convertible bond is a hybrid of debt and equity, I combined the two trees into a single tree to value the convertible bond. I provided two numerical examples for my valuation model. In the first example, I priced an option-free convertible bond that does not allow a call provision and has no default risk. The second example provided a valuation model for a callable convertible bond with default risk. Table 18 summarizes the outcome of the model with respect to the market price.

Example 1: AAV option-free convertible bond	Model price	Market price (Range)
	111.75	100-111.99
Example 2: CWT callable convertible bond	123.8	112-124

Table 17: Model examples compared to market prices

The duration and convexity are significant elements for the study of the sensitivity of the convertible bond price to changes in the interest rate. The duration is defined as the first derivative of V with respect to the interest rate r. The convexity is expressed as the second derivative of V with respect to r. Because it is difficult to obtain a closed-form expression for the American convertible bond, I studied the duration and convexity numerically by two methods: the present value and the binomial tree method. I also provided an example of duration and convexity under the assumption of a European convertible bond. I studied the partial derivatives and sensitivities of convertible bond parameters, such as the short-run rate, the stock price, the volatilities of both factors, the long-run rate, the mean reversion rate, the dividend yield, and the coupon rate.

Convertible arbitrage strategy aims to manage the convertible bond investment risk by purchasing convertible securities and short selling the underlying common stock. The Delta ratio determines the number of shares to sell short against the

long convertible bond position. The Delta strategy was implemented by constructing an equally weighted portfolio of 44 hedged convertible bonds from 2009 to 2016. The strategy aimed to produce deltas and returns for 44 convertible bonds that were listed on the TSX between 2009 and 2016. For the Black-Scholes model, our example indicated that annual average return was positive for most of the periods with a maximum average annual return of 32.15% in 2013 and a minimum position return of -7.4% in 2016. The maximum return on an individual position was 57% and the minimum return on an individual position was -64%. For the binomial method, the annual average return was positive for all periods with a maximum average annual return of 25.72% in 2013 and a minimum position return of 9.32% in 2016. The maximum return on an individual position was 106% and the minimum return on an individual position was -154%. Table 18 shows the summary of the annual average return of both methods between 2009 and 2016. This confirms that one way or another, there is a systematic average undervaluation of convertible bonds in the Canadian market as observed mainly by Amman, Kind, and Wilde (2003) for the French market.

8 Appendix

8.1 MATLAB code for construction Vasicek tree

```
function [R,P,Mean]= Vasicek(n)
R=zeros(n+1,n+1);
P=zeros(2*n,n);
r0=0.0138;
alpha=0.167;
sigma=0.0029;
dt=1;
gama=0.112;
R(1,1)=r0;
digits(5)
for i=2:n+1   % colomn
    if mod(i,2)==0
        % central path with p=q=1/2;
        R(i/2,i)=R(i/2,i-1)+ alpha*(gama-R(i/2,i-1))*dt + sigma*sqrt(dt);    %
ru,ruud,ruuudd...
        R(i/2+1,i)=R(i/2,i-1)+ alpha*(gama-R(i/2,i-1))*dt - sigma*sqrt(dt);  %
rd,rudd,ruuddd...
        P(i-1,i-1)=0.5;
        P(i,i-1)=0.5;
        if i>2
            for  j=1:i/2-1
                if R(i/2-j,i)==0
                    Eru=R(i/2-j,i-1)+alpha*(gama-R(i/2-j,i-1))*dt;
                    R(i/2-j,i)=sigma^2*dt/(Eru-R(i/2+1-j,i))+Eru;
                    P((i/2-j)*2-1,i-1)=(Eru-R(i/2+1-j,i))/(R(i/2-j,i)-R(i/2+1-j,i));
                    P((i/2-j)*2,i-1)=1- P((i/2-j)*2-1,i-1);

                    Erd=R(i/2+j,i-1)+alpha*(gama-R(i/2+j,i-1))*dt;
                    R(i/2+1+j,i)=Erd-sigma^2*dt/(R(i/2+j,i)-Erd);
                    P((i/2+j)*2-1,i-1)=(Erd-R(i/2+1+j,i))/(R(i/2+j,i)-R(i/2+1+j,i));
                    P((i/2+j)*2,i-1)=1-P((i/2+j)*2-1,i-1);
                end
            end
        end
    else
        EXP=vpa(R((i-1)/2,i-2)+alpha*(gama-R((i-1)/2,i-2))*dt);
        R((i+1)/2,i)=EXP+alpha*(gama-EXP)*dt;
        if i>1
            for j=1:(i-1)/2
                if R((i+1)/2-j,i)==0
                    Eru=R((i+1)/2-j,i-1)+alpha*(gama-R((i+1)/2-j,i-1))*dt;
                    R((i+1)/2-j,i)=sigma^2*dt/(Eru-R((i+1)/2-j+1,i))+Eru;
                    P(i-2*j,i-1)=(Eru-R((i+1)/2-j+1,i))/(R((i+1)/2-j,i)-R((i+1)/2-
j+1,i));
                    P(i+1-2*j,i-1)=1- P(i+1-2*j-1,i-1);

                    Erd=vpa(R((i+1)/2-1+j,i-1)+alpha*(gama-R((i+1)/2-1+j,i-1))*dt);
                    R((i+1)/2+j,i)=Erd-sigma^2*dt/(R((i+1)/2-1+j,i)-Erd);
                    P(i+2*j-2,i-1)=(Erd-R((i+1)/2+j,i))/(R((i+1)/2-1+j,i)-
R((i+1)/2+j,i));
                    P(i+2*j-1,i-1)=1-P(i+2*j-2,i-1);
                end
            end
        end
    end
end

U=R(1,:);
L=[];
for k=1:length(U)
    m=R(k,k);
    L=[L m];
end
Mean=0.5.*(L+U);
%plot(U,'g-')
%plot(L,'b.')
%plot(Mean,'ro') % [R,P,~]= Vasicek(5)   %Interest Rate and Probabilities
% [R,~,Mean]= Vasicek(5); %Mean value
```

8.2 MATLAB code for generating Vasicek model parameters

```
% Parameter estimation for the Vasicek model.
% Uses exact form for the likelihood function
% Vasicek model is dr = alpha(mu - r)dt + sigma*W
%   alpha = mean reversion speed,
%   mu    = mean reversion level,
%   sigma = volatility.
clc; clear;

% Input the 5- year bond yield.
[r, date] = xlsread('Vasicek_short_rate_data.xls', 'Data', 'A3:B4000');
r = r/100;

% Number of observations, observations are daily.
N = length(r);
dt = 1/252;

% Find the OLS estimates for alpha, mu, and sigma.
% These estimates are used as starting values for the exact likelihood.
% From "Maximum Likelihood Estimation of the Vasicek Process: The
Matlab
% Implementation".
y = (r(2:N) - r(1:N-1))./sqrt(r(1:N-1));
x1 = dt./sqrt(r(1:N-1));
x2 = sqrt(r(1:N-1)).*dt;
b = regress(y, [x1 x2]);
alpha = -b(2);
mu = b(1)/alpha;
res = y - b(1).*x1 - b(2).*x2;
sigma = std(res)./sqrt(dt);

% Estimate alpha, mu, and sigma using the exact likelihood.

start = [alpha mu sigma];
params = fminsearch(@(b) CIR_LL(b,r,dt), start);
alpha  = params(1);
mu     = params(2);
sigma  = params(3);

% Generate the yield curve R(t,T) based on the parameters.
% Inline function for A(t,T).
A = inline('(2*gam*exp((alpha+gam)*(T-t)/2)/(2*gam + (alpha +
gam)*(exp(gam*(T-t))-1)))^(2*alpha*mu/sigma^2)',...
    'alpha','mu','sigma','gam','t','T');

% Inline function for B(t,T).
B = inline('2*(exp(gam*(T-t))-1) / (2*gam + (alpha+gam)*(exp(gam*(T-
t))-1))',...
    'alpha','mu','gam','t','T');

% Gamma parameter.
gam = sqrt(alpha^2 + 2*sigma^2);

% Define the required settings for the yield curve.
LastTenor = N/252;          % Last tenor is about 10 years.
t = 0;                      % Time zero t=0.
Inc = 1/2;                  % Increment for the yield curve tenor points.
```

```
T = [Inc:Inc:LastTenor];  % Tenor points.
CurrentRate = r(end);     % Current rate r(t) is most recent rate.

% Zero coupon bond: P(t,T) = A(t,T)*exp(B(t,T)*CurrentRate).
% Yield curve : R(t,T) = -log(P(t,T))/(T-t).
for i=1:length(T)
    P(i) = A(alpha,mu,sigma,gam,t,T(i))*exp(-
B(alpha,mu,gam,t,T(i))*CurrentRate);
    R(i) = -log(P(i))/(T(i)-t)*100;
end
% Simulate 10 paths for the short rate.
Nsims = 10;
for k=1:Nsims;
    f(1,k) = r(1);
    for t=2:N
        f(t-1,k) = max(0,f(t-1,k));
        f(t,k) = f(t-1,k) + alpha*(mu - f(t-1,k))*dt +
sigma*randn(1)*(dt);
    end
end
% Plot the results.
subplot(2,1,1)
plot(T,R)
legend('Estimated Yield Curve')
xlabel('Time');
ylabel('Yield');

subplot(2,1,2)
plot((1:N), r, 'k-o',(1:N), f)
legend('Original time series', 'Simulated series')
xlabel('Time')
ylabel('Simulated short rate')
```

8.3 CRR stock price tree – Matlab code

```
function [S,P]= StockPrice(~,n)

S0=4.36;
dt=1;
q=0;   %yield
r=0.0138;
% initial value of non-constant sigma1-sigma3
sigma=[0.305; 0.33; 0.31];
% calculate u1,d1 - u3,d3
u=exp(sigma*sqrt(dt))';
d=1./u;
pu=(exp((r-q)*dt)-d)./(u-d);
pd=1-pu;
% matrix of the rate stock price goes up or goes down
ud=zeros(2^n,n+1);
ud(1,1)=1;
ud(1:2,2)=[u(1);d(1)];
P=zeros(2^n,n);
P(1:2,1)=[pu(1);pd(1)];
for i=3:n+1
```

```
    for j=1:2:2^(i-1)-1
        ud(j:j+1,i)=ud((j+1)/2,i-1).*[u(i-1);d(i-1)];
        P(j,i-1)=(exp((r-q)*dt)-ud(j+1,i)/ud((j+1)/2,i-
1))./(ud(j,i)/ud((j+1)/2,i-1)-ud(j+1,i)/ud((j+1)/2,i-1));
        P(j+1,i-1)=1-P(j,i-1);
    end
end

% Stock price matrix
S=S0*ud;
end
```

8.4 AAV convertible bond – Matlab code

```
function [Cb,Int_R,S_Price,Int_P,S_Prob]=C_B(n)
%S0=4.36;
F=100;
c=0.05;
C=c*F;
CR=8.6;
S0=4.36;
%CB=zeros((2^(n-1))*n*2,n+1);
Cb=zeros(2^(2*n-1),n+1);

%----- reset Stock Price
%
[S_Price,S_Prob]= StockPrice(S0,n); % Stock Price
Prob_S=[zeros(size(S_Prob,1),1) S_Prob];
S_price=zeros(2^(2*n-1),n+1);
S_prob=zeros(2^(2*n-1),n+1);
for j=n:-1:1
    S=S_Price(1:2^j,j+1);
    P_S=Prob_S(1:2^j,j+1);
    for i=1:j
        s=reshape(S,2^(2*i-1),[]);
        s=repmat(s,2,1);
        S=reshape(s,[],1);

        p_S=reshape(P_S,2^(2*i-1),[]);
        p_S=repmat(p_S,2,1);
        P_S=reshape(p_S,[],1);
    end
    if j==n
        A=reshape(S,[],length(S)/2);
        AA=reshape(P_S,[],length(P_S)/2);
        l=length(A);
        B=[];
        BB=[];
        for i=1:2:l
            b=A(:,i);
            bb=AA(:,i);
            B=[B;b];
            BB=[BB;bb];
        end
        S_price(:,end)=B;
        S_prob(:,end)=BB;
    else
        S_price(1:length(S),j+1)=S;
        S_prob(1:length(P_S),j+1)=P_S;
    end
end
S_price(1,1)=S_Price(1,1);
S_probend=S_prob(1:2,end);
S_prob=S_prob(1:length(S_prob)/2,2:end-1);

%---------------------------------
```

```
%----- reset interest rate
[Int_R,Int_P]= Vasicek(n-1);
Int_rate=zeros(4^(n-1),n);
Int_rate(1,1)=Int_R(1,1);

for j=2:n
    r=Int_R(1:j,j)';
    for i=1:j-1
        k=repmat(r,2,1);
        [~,l]=size(k);
        j1=k(:,1:l-1);
        j2=k(:,2:l);
        J=[j1;j2];
        r=J;
    end
    Int_rate(1:4^(j-1),j)=r;
end

Prob_I=zeros(4^(n-1),n-1);
for j=1:n-1
    p=Int_P(1:2*j,j)';
    for i=1:j
        k=repmat(p,2,1);
        [~,l]=size(k);
        if l>2
            j1=k(:,1:l-2);
            j2=k(:,3:l);
        else
            j1=k(:,1);
            j2=k(:,2:l);
        end
        J=[j1;j2];
        p=J;
    end
    Prob_I(1:4^j,j)=p;
end

Cb(:,n+1)=max(S_price(:,n+1)*CR,F)+C;
for i=1:4^(n-1)
    Cb(i,n)=max(S_price(i,n)*CR, ((((S_probend(2)*Cb(2*i-1,end))+
(S_probend(1)*Cb(2*i,end))))/(1+Int_rate(i,n))));
end
for i=n-1:-1:1
    for j=1:4^(i-1)
        Cb(j,i)=max(S_price(j,i)*CR,(S_prob(4*j-3,i).*Prob_I(4*j-3,i)*(Cb(4*j-
3,i+1)+C)+S_prob(4*j-2,i).*Prob_I(4*j-2,i)*(Cb(4*j-2,i+1)+C)+S_prob(4*j-1,i).*Prob_I(4*j-
1,i)*(Cb(4*j-1,i+1)+C)+S_prob(4*j,i).*Prob_I(4*j,i)*(Cb(4*j,i+1)+C))/(1+Int_rate(j,i)));
    end
end

Cb_value=Cb(1,1);
%Prob
%Int_rate
%S_price

end

function [R,P,Mean]= Vasicek(n)

R=zeros(n+1,n+1);
P=zeros(2*n,n);
r0=0.0137;
alpha=0.167;
sigma=0.0029;
dt=1;
gama=0.0112;
R(1,1)=r0;

%digits(5)
for i=2:n+1   % colomn
    if mod(i,2)==0
        % central path with p=q=1/2;
        R(i/2,i)=R(i/2,i-1)+ alpha*(gama-R(i/2,i-1))*dt + sigma*sqrt(dt);    %
ru,ruud,ruuudd...
        R(i/2+1,i)=R(i/2,i-1)+ alpha*(gama-R(i/2,i-1))*dt - sigma*sqrt(dt);  %
rd,rudd,ruuddd...
        P(i-1,i-1)=0.5;
```

```
        P(i,i-1)=0.5;
        if i>2
            for  j=1:i/2-1
                if R(i/2-j,i)==0
                    Eru=R(i/2-j,i-1)+alpha*(gama-R(i/2-j,i-1))*dt;
                    R(i/2-j,i)=sigma^2*dt/(Eru-R(i/2+1-j,i))+Eru;
                    P((i/2-j)*2-1,i-1)=(Eru-R(i/2+1-j,i))/(R(i/2-j,i)-R(i/2+1-j,i));
                    P((i/2-j)*2,i-1)=1- P((i/2-j)*2-1,i-1);

                    Erd=round1(R(i/2+j,i-1)+alpha*(gama-R(i/2+j,i-1))*dt,6);
                    R(i/2+1+j,i)=Erd-sigma^2*dt/(R(i/2+j,i)-Erd);
                    P((i/2+j)*2-1,i-1)=round1((Erd-R(i/2+1+j,i))/(R(i/2+j,i)-
R(i/2+1+j,i)),6);
                    P((i/2+j)*2,i-1)=1-P((i/2+j)*2-1,i-1);
                end
            end
        end
    else
        EXP=(R((i-1)/2,i-2)+alpha*(gama-R((i-1)/2,i-2))*dt);
        R((i+1)/2,i)=EXP+alpha*(gama-EXP)*dt;
        if i>1
            for j=1:(i-1)/2
                if R((i+1)/2-j,i)==0
                    Eru=R((i+1)/2-j,i-1)+alpha*(gama-R((i+1)/2-j,i-1))*dt;
                    R((i+1)/2-j,i)=sigma^2*dt/(Eru-R((i+1)/2-j+1,i))+Eru;
                    P(i-2*j,i-1)=(Eru-R((i+1)/2-j+1,i))/(R((i+1)/2-j,i)-R((i+1)/2-
j+1,i));
                    P(i+1-2*j,i-1)=1- P(i+1-2*j-1,i-1);

                    Erd=round1((R((i+1)/2-1+j,i-1)+alpha*(gama-R((i+1)/2-1+j,i-
1))*dt),6);
                    R((i+1)/2+j,i)=Erd-sigma^2*dt/(R((i+1)/2-1+j,i)-Erd);
                    P(i+2*j-2,i-1)=round1((Erd-R((i+1)/2+j,i))/(R((i+1)/2-1+j,i)-
R((i+1)/2+j,i)),6);
                    P(i+2*j-1,i-1)=1-P(i+2*j-2,i-1);
                end
            end
        end
    end
end

U=R(1,:);
L=[];

for k=1:length(U)
    m=R(k,k);
    L=[L m];
end
Mean=0.5.*(L+U);
end


function [S,P]= StockPrice(S0,n)
dt=1;
q=0;   %yield
r=0.02;
% initial value of non-constant sigma1-sigma3
sigma=[0.31; 0.37; 0.39];

% calculate u1,d1 - u3,d3
u=exp(sigma*sqrt(dt))';
d=1./u;
pu=(exp((r-q)*dt)-d)./(u-d);
pd=1-pu;
% matrix of the rate stock price goes up or goes down
ud=zeros(2^n,n+1);
ud(1,1)=1;
ud(1:2,2)=[u(1);d(1)];
P=zeros(2^n,n);
P(1:2,1)=[pu(1);pd(1)];


for i=3:n+1
    for j=1:2:2^(i-1)-1
        ud(j:j+1,i)=ud((j+1)/2,i-1).*[u(i-1);d(i-1)];
        P(j,i-1)=(exp((r-q)*dt)-ud(j+1,i)/ud((j+1)/2,i-1))./(ud(j,i)/ud((j+1)/2,i-1)-
ud(j+1,i)/ud((j+1)/2,i-1));
```

```
        P(j+1,i-1)=1-P(j,i-1);
    end
end

% Stock price matrix
S=S0*ud;
end


function a= round1(a,n)
a=a*(10^n);
a=round(a);
a=a/(10^n);
end
% INPUT
% [Cb,Int_R,S_Price,Int_P,S_Prob]=C_B(4)
```

8.5 CWT convertible bond with default risk– Matlab code

```
function [Cb_callable,Int_R,S_Price]=CB_callable(n)
%CWT
%S0=28;
F=100;
c=0.0575;
C=c*F;
CR=3.88;
S0=28;
CP=125;
lambda_1=0.0034;
lambda_2=0.0155;
lambda_3=0.0221;
delta=30;

%CB=zeros((2^(n-1))*n*2,n+1);
Cb_callable=zeros(2^(2*n-1),n+1);

%----- reset Stock Price
%
S_Price= StockPrice(S0,n); % Stock Price

S_price=zeros(2^(2*n-1),n+1);
for j=n:-1:1
    S=S_Price(1:2^j,j+1);
    for i=1:j
        s=reshape(S,2^(2*i-1),[]);
        s=repmat(s,2,1);
        S=reshape(s,[],1);
    end
    if j==n
        A=reshape(S,[],length(S)/2);
        l=length(A);
        B=[];
        for i=1:2:l
            b=A(:,i);
            B=[B;b];
        end
        S_price(:,end)=B;
    else
        S_price(1:length(S),j+1)=S;
    end
end
S_price(1,1)=S_Price(1,1);

%----- reset interest rate
[Int_R,P]= Vasicek(n-1);
Int_rate=zeros(4^(n-1),n);
Int_rate(1,1)=Int_R(1,1);

for j=2:n
    r=Int_R(1:j,j)';
    for i=1:j-1
```

```
        k=repmat(r,2,1);
        [~,l]=size(k);
        j1=k(:,1:l-1);
        j2=k(:,2:l);
        J=[j1;j2];
        r=J;
    end
    Int_rate(1:4^(j-1),j)=r;
end
%-----------------------------

Prob=zeros(4^(n-1),n-1);
for j=1:n-1
    p=P(1:2*j,j)';
    for i=1:j
        k=repmat(p,2,1);
        [~,l]=size(k);
        if l>2
            j1=k(:,1:l-2);
            j2=k(:,3:l);
        else
            j1=k(:,1);
            j2=k(:,2:l);
        end
        J=[j1;j2];
        p=J;
    end
    Prob(1:4^j,j)=p;
end
Cb_callable(:,n+1)=max(S_price(:,n+1)*CR,F)+C;
for i=1:4^(n-1)
    Cb_callable(i,n)=max(S_price(i,n)*CR,min(((0.5*(1- lambda_3))*Cb_callable(2*i-1,end)+
(0.5*(1- lambda_3))*Cb_callable(2*i,end)+lambda_3*delta)+/(1+Int_rate(i,n)),CP));
end
for i=n-1:-1:2
    for j=1:4^(i-1)
        Cb_callable(j,i)=max(S_price(j,i)*CR,min(CP,(0.5*Prob(4*j-3,i)*(1-
lambda_2)*(Cb_callable(4*j-3,i+1)+C)+0.5*Prob(4*j-2,i)*(1- lambda_2)*(Cb_callable(4*j-
2,i+1)+C)+0.5*Prob(4*j-1,i)*(1- lambda_2)*(Cb_callable(4*j-1,i+1)+C)+0.5*Prob(4*j,i)*(1-
lambda_2)*(Cb_callable(4*j,i+1)+C))+lambda_2*delta /(1+Int_rate(j,i))));
    end
end
Cb_callable(1,1)=max(S_price(j,i)*CR,(0.5*Prob(4*j-3,i)*(1- lambda_1)*(Cb_callable(4*j-
3,i+1)+C)+0.5*Prob(4*j-2,i)*(1- lambda_1)*(Cb_callable(4*j-2,i+1)+C)+0.5*Prob(4*j-
1,i)*(1- lambda_1)*(Cb_callable(4*j-1,i+1)+C)+0.5*Prob(4*j,i)*(1-
lambda_1)*(Cb_callable(4*j,i+1)+C))+lambda_1*delta /(1+Int_rate(j,i)));

%Prob
%Int_rate
%S_price

end

function [R,P,Mean]= Vasicek(n)

R=zeros(n+1,n+1);
P=zeros(2*n,n);
r0=0.01375;
alpha=0.167;
sigma=0.0029;
dt=1;
gama=0.0112;
R(1,1)=r0;

for i=2:n+1   % colomn
    if mod(i,2)==0
        % central path with p=q=1/2;
        R(i/2,i)=R(i/2,i-1)+ alpha*(gama-R(i/2,i-1))*dt + sigma*sqrt(dt);    %
ru,ruud,ruuudd...
        R(i/2+1,i)=R(i/2,i-1)+ alpha*(gama-R(i/2,i-1))*dt - sigma*sqrt(dt);  %
rd,rudd,ruuddd...
        P(i-1,i-1)=0.5;
        P(i,i-1)=0.5;
        if i>2
            for  j=1:i/2-1
                if R(i/2-j,i)==0
                    Eru=R(i/2-j,i-1)+alpha*(gama-R(i/2-j,i-1))*dt;
```

```
                    R(i/2-j,i)=sigma^2*dt/(Eru-R(i/2+1-j,i))+Eru;
                    P((i/2-j)*2-1,i-1)=(Eru-R(i/2+1-j,i))/(R(i/2-j,i)-R(i/2+1-j,i));
                    P((i/2-j)*2,i-1)=1- P((i/2-j)*2-1,i-1);

                    Erd=R(i/2+j,i-1)+alpha*(gama-R(i/2+j,i-1))*dt;
                    R(i/2+1+j,i)=Erd-sigma^2*dt/(R(i/2+j,i)-Erd);
                    P((i/2+j)*2-1,i-1)=(Erd-R(i/2+1+j,i))/(R(i/2+j,i)-R(i/2+1+j,i));
                    P((i/2+j)*2,i-1)=1-P((i/2+j)*2-1,i-1);
                end
            end
        end
    else
        EXP=R((i-1)/2,i-2)+alpha*(gama-R((i-1)/2,i-2))*dt;
        R((i+1)/2,i)=EXP+alpha*(gama-EXP)*dt;
        if i>1
            for j=1:(i-1)/2
                if R((i+1)/2-j,i)==0
                    Eru=R((i+1)/2-j,i-1)+alpha*(gama-R((i+1)/2-j,i-1))*dt;
                    R((i+1)/2-j,i)=sigma^2*dt/(Eru-R((i+1)/2-j+1,i))+Eru;
                    P(i+1-2*j-1,i-1)=(Eru-R((i+1)/2-j+1,i))/(R((i+1)/2-j,i)-R((i+1)/2-
j+1,i));
                    P(i+1-2*j,i-1)=1- P(i+1-2*j-1,i-1);

                    Erd=R((i+1)/2-1+j,i-1)+alpha*(gama-R((i+1)/2-1+j,i-1))*dt;
                    R((i+1)/2+j,i)=Erd-sigma^2*dt/(R((i+1)/2-1+j,i)-Erd);
                    P(i+2*j-2,i-1)=(Erd-R((i+1)/2+j,i))/(R((i+1)/2-1+j,i)-
R((i+1)/2+j,i));
                    P(i+2*j-1,i-1)=1-P(i+2*j-2,i-1);
                end
            end
        end
    end
end

U=R(1,:);
L=[];

for k=1:length(U)
    m=R(k,k);
    L=[L m];
end
Mean=0.5.*(L+U);
end

function S= StockPrice(S0,n)
dt=1;
% initial value of non-constant sigma1-sigma3
sigma=[0.1; 0.11; 0.12 ];

% calculate u1,d1 - u3,d3
u=exp(sigma*sqrt(dt))';
d=1./u;

% matrix of the rate stock price goes up or goes down
ud=zeros(2^n,n+1);
ud(1,1)=1;
ud(1:2,2)=[u(1);d(1)];
for i=3:n+1
    for j=1:2:2^(i-1)-1
        ud(j:j+1,i)=ud((j+1)/2,i-1).*[u(i-1);d(i-1)];
    end
end

% Stock price matrix
S=S0*ud;
end
% INPUT
% [Cb_callable]=CB_callable(3)



%Default risk


%Default risk
```

```
function [lambda_1, lambda_2, lambda_3] = lambda_calc(r0, r0_star, delta, ksai, r2_star,
r1, u, d, r3_star, r2, uu, ud, dd)

lambda_1 = (1-exp(r0)-r0_star)/(1-delta);



lambda_2 = (1-(exp(-2*r2_star+r0)-ksai*lambda_1)/((exp(-r1*u)+exp(-r1*d))*pi*(1-
lambda_1)))/(1-ksai);

lambda_3 = 1-((exp(-3*r3_star+r0)-ksai*lambda_1-pi*(1-lambda_1)*ksai*lambda_2*(exp(-
r1*u)+exp(-r1*d))).*
    /pi^2*(1-lambda_1)*(1-lambda_2)*(exp(-r1*u)*(exp(-r2*uu)+exp(-r2*ud))+exp(-
r1)*d*(exp(-r2*ud)+exp(-r2*dd))))/(1-ksai);


end
```

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