Classification of Arcs in Finite Geometry and Applications to Operational Research

by

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Declaration of Authorship

I hereby declare that the work represented in this thesis is entirely my own unless otherwise stated, and has been not presented for examination, in whole or in part, to this or any other university.

Signed:

Date:

To The Souls Of My Parents

Abstract

In $\mathrm{PG}(2,q)$, the projective plane over the field $\mathbb{F}_q$ of q elements, a (k,n)-arc is a set $\mathcal{K}$ of k points with at most n points on any line of the plane. When $n=2$, a $(k,2)$-arc is called a k-arc. A fundamental question is to determine the values of k for which $\mathcal{K}$ is complete, that is, not contained in a $(k+1,n)$-arc. In particular, what is the largest value of k for a complete $\mathcal{K}$, denoted by $m_n(2,q)$?

This thesis focusses on using some algorithms in Fortran and GAP to find large complete (k,n)-arcs in $\mathrm{PG}(2,q)$. A blocking set $\mathcal{B}$ is a set of points such that each line contains at least t points of $\mathcal{B}$ and some line contains exactly t points of $\mathcal{B}$. Here, $\mathcal{B}$ is the complement of a (k,n)-arc $\mathcal{K}$ with $t=q+1-n$. Non-existence of some (k,n)-arcs is proved for $q=19,23,43$. Also, a new largest bound of complete (k,n)-arcs for prime q and $n>(q-3)/2$ is found. A new lower bound is proved for smallest size of complete (k,n)-arcs in $\mathrm{PG}(2,q)$. Five algorithms are explained and the classification of (k,n)-arcs is found for some values of n and q. High performance computing is an important part of this thesis, where Algorithm Five is used with OpenMP that reduces the time of implementation. Also, a (k,n)-arc $\mathcal{K}$ corresponds to a projective $[k,n,d]_q$-code of length k, dimension n, and minimum distance $d=k-n$. Some applications of finite geometry to operational research are also explained.

Introduction

A projective plane of order q consists of a set of q^2+q+1 points and a set of q^2+q+1 lines, where each line contains exactly $q+1$ points and two distinct points lie on exactly one line. It follows from the definition that each point is contained in exactly $q+1$ lines and two distinct lines have exactly one common point.

A (k,n)-arc is a set $\mathcal{K}$ of k points, such that there is some n but no $n+1$ are collinear, where $n \geq 2$; a (k,n)-arc is complete if there is no $(k+1,n)$-arc containing it.

Many studies have been done to find all complete (k,n)-arcs in $\mathrm{PG}(2,q)$ for some values of q. The size of the largest and the second largest complete (k,n)-arc are denoted by $m_n(2,q)$ and $m'_n(2,q)$; also the minimum size of a complete (k,n)-arc is denoted by $t_n(2,q)$. The value of $m_2(2,q)$ is $q+1$ for odd q and $q+2$ for even q.

In this thesis, a t-fold blocking set $\mathcal{B}$ in $\mathrm{PG}(2,q)$ is a set of points such that each line contains at least t points of $\mathcal{B}$ and some line contains exactly t points of $\mathcal{B}$. A 1-fold blocking set is called a blocking set. A 2-fold blocking set is called a double and a 3-fold blocking set is called a triple blocking set. A blocking set is the complement of a (k,n)-arc $\mathcal{K}$ in $\mathrm{PG}(2,q)$ with $t = q+1-n$. The smallest blocking sets are just the lines and any blocking sets containing a line will be called trivial. A blocking set is said to be minimal, when no proper subset of it is a blocking set. A new largest bound for a (k,n)-arc in $\mathrm{PG}(2,q)$, for $n > (q-3)/2$ and prime q, has been proved and applied to $\mathrm{PG}(2,47)$ in Chapter 2.

A new lower bound for the smallest complete (k,n)-arc in $\mathrm{PG}(2,q)$ has been found and applied for $t_2(2,q)$ and $t_3(2,q)$. Also, constructions of complete k-arcs from a quadrangle and the configuration of the union of two conics are found. The classification of (k,n)-arcs in $\mathrm{PG}(2,q)$ for some q and n is done using four different methodologies. Also, a comparison among these methodologies is done to show which method is best according to the time of implementation to get the final results. All these results are presented in Chapter 3. High Performance Computing (HPC) technique is used in Chapter 4 to accelerate the calculations without affecting the accuracy of the results. Here HPC depends on dividing the big problem into smaller problems and solve each of them individually.

The relationship between coding theory and finite projective spaces is presented in Chapter 5. A linear $[k, n, d]$-code is an n-dimensional subspace of the k-dimensional vector space $V(k, q)$ with non-zero vectors having weight at least d. An important problem in coding theory is that to optimise one of the parameters k, n, d for given values of the other two and fixed q. So, e errors can be corrected for a code with minimum distance at least $2e + 1$. Chapter 6 provides general definitions and the historical development of operational research. In addition, applications of finite geometry to operational research and a generalisation of Kirkman's problem and the golf problem are considered in Chapter 7.

Aim and Objectives

The aim of this research study is investigate structures in finite geometry and apply the results to operational research.

The objectives are the following.

1. To find the largest and the smallest complete (k, n)-arcs in $\mathrm{PG}(2, q)$.
2. To establish some good algorithms to find the classification of (k, n)-arcs in $\mathrm{PG}(2, q)$ in a short time.
3. To apply the results of (k, n)-arcs to coding theory.
4. To apply the results to operational research.

Thesis Structure

This thesis is organised into seven chapters.

Chapter 1: Background

This chapter includes the general definitions of finite geometry. It also includes some important theorems and lemmas which are used to prove new lower bounds of the smallest and the largest complete (k, n)-arcs in $\mathrm{PG}(2, q)$. The stabiliser groups and methods of the classification of the largest size of (k, n)-arcs are also given in this chapter.

Chapter 2: Blocking Sets

This chapter provides some basic equations about blocking sets, and includes an overview of previous studies in this area. A significant part of this chapter focusses on the proof of non-existence of some arcs in $\mathrm{PG}(2, q)$ and finding new largest bounds of (k, n)-arcs.

Chapter 3: Classification of (k, n)-arcs in $\mathrm{PG}(2, q)$

This chapter includes a new lower bound for the smallest complete (k, n)-arc in $\mathrm{PG}(2, q)$ with a comparison to previous bounds. Four algorithms have been used in this chapter to find large sizes of (k, n)-arcs with respect to the time of implementation.

Chapter 4: High Performance Computing (HPC)

This chapter includes some details of the High Performance Computing (HPC) techniques used to accelerate the calculations without affecting the accuracy of the results. In this chapter the paralleling computing systems can be classified as shared memory (OpenMP) and distributed memory (MPI).

Chapter 5: Coding Theory

This chapter presents the relationship between projective geometry and coding theory. MDS codes of dimension three and codes of dimension five are considered.

Chapter 6: Operational Research

This chapter provides general definitions and the historical development of operational research. Also, the concept and the approach of operational research have been explained.

Chapter 7: Application of Finite Geometry to Operational Research

This chapter highlights new applications of finite geometry to operational research. A generalisation of Kirkman's Problem is also provided.

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Chapter 1

Background

1.1 Group Theory

Definition 1.1. [40] A *group* is a set G together with an operation $\circ$ satisfying the following requirements:

(a) for each pair x, y of elements of G, $x \circ y$ is an element of G;

(b) for all elements x, y, z of G, $(x \circ y) \circ z = x \circ (y \circ z)$;

(c) there is an element $e \in G$ such that $e \circ g = g = g \circ e$ for all $g \in G$;

(d) given an element $g \in G$, there is an element $g^* \in G$ such that

$$g \circ g^* = e = g^* \circ g.$$

A group G is *abelian* if, for all $x, y \in G$, $x \circ y = y \circ x$. A *cyclic group* is a group generated by a single element; that is, a group consisting of all powers of one of its elements. A finite group G of order n is cyclic if and only if it contains an element of order n. Also, a group of prime order is cyclic.

Definition 1.2. [40] An *action* of a group G on a set Ω is a function $\mu : \Omega \times G \longrightarrow \Omega$ with the following properties:

(a) $((x, g)\mu, h)\mu = (x, g \circ h)\mu$ for all $x \in \Omega$ and $g, h \in G$;

(b) $(x,1)\mu = x$ for all $x \in \Omega$, where 1 is the identity of G;

(c) $((x,g)\mu, g^{-1})\mu = ((x,g^{-1})\mu, g)\mu = x$ for all $x \in \Omega, g \in G$.

A relationship $\sim$ on Ω by the rule that $x \sim y$ if there exist $g \in G$ with $(x,g)\mu = y$. So $\sim$ is an equivalence relation.

Definition 1.3. Let G be a group. A *homomorphism* $\theta : G \longrightarrow H$ is a function θ from G to H that satisfies the condition, for all $g_1, g_2 \in G$,

$$(g_1 g_2)\theta = (g_1\theta)(g_2\theta).$$

A homomorphism that is one-to-one and onto is an *isomorphism*; then G and H are *isomorphic*. A bijective homomorphism θ group to itself is an *automorphism*.

Definition 1.4. A bijection $f : X \longrightarrow X$ is a permutation on X, and the set $S(X)$ of permutations f is a group under composition of functions.

Definition 1.5. [18]

(i) A *permutation* ρ is κ-cycle if there exists a positive κ and an integer i such that

  (a) κ is the smallest positive integer such that $\rho^{\kappa}(i) = i$;

  (b) ρ fixes each j not in $\{i, \rho(i), \ldots, \rho^{\kappa-1}(i)\}$.

(ii) A *permutation group* is a group whose elements are permutations; that is, a subgroup of a symmetric group.

(iii) The order of a permutation is equal to the least common multiple of the lengths of its cycles.

Example 1.1. There are 27 possible maps from the set $X = \{1,2,3\}$ to itself. Most of these are not injective, only six of these maps are permutations denoted as follows:

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$
$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \phi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

where the values of each map are given in the second row. Thus the map β takes 1 to 3, $\beta(2) = 1$ and $\beta(3) = 2$. Since the maps are injective, the entries in the second row are all distinct. The five non-identity elements of $S(3)$ are *cycles*, and may be written as

$$(123), (132), (23), (13), (12).$$

Definition 1.6. A group G is a *semidirect product* of a subgroup N by a subgroup H if the following conditions are satisfied:

1. $G = NH$;

2. N is a normal subgroup of G;

3. $H \cap N = \{e\}$.

1.2 Finite Fields

A field is a set $\mathbb{F}$ closed under two operations $+, \times$ such that

(i) $(\mathbb{F}, +)$ is an abelian group with identity 0;

(ii) $(\mathbb{F} \setminus \{0\}, \times)$ is an abelian group with identity 1;

(iii) $a(b + c) = ab + ac, (a + b)c = ac + bc$, for all $a, b, c \in \mathbb{F}$.

Let $\mathbb{F}_q$ denoted a field of q elements. Note that q must always be an integral power p^h of a prime p. Here, p is *characteristic* of the finite field. For q prime, $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$. Here, for all $q = p^h$, there exists $\alpha \in \mathbb{F}_q$ such that $\mathbb{F}_q = \{0, 1, \alpha, \ldots, \alpha^{q-2} \mid \alpha^{q-1} = 1\}$. Here, α is the root of a primitive polynomial of degree h over $\mathbb{F}_p$. In Table 1.1, some primitive polynomials $f(x)$ are given for the fields of prime power order up to 17.

Table 1.1: Primitive cubic polynomials [37]

q	$f(x)$	q	$f(x)$
2	x^3+x+1	9	$x^3-\beta x^2-1$
3	x^3-2x^2-x-2	11	x^3-x^2-x-3
4	x^3-x^2-x-2	13	x^3-x^2-2
5	x^3-x^2-1	16	$x^3-\gamma x^2-\gamma$
7	x^3-x^2-2x-4	17	x^3-8x^2-1
8	$x^3-x-\alpha^4$		

There exists a primitive element s in $\mathbb{F}_q$ such that

$$\mathbb{F}_q = \{0, 1, s, \ldots, s^{q-2} \mid s^{q-1} = 1\}.$$

When q is not prime, the root α of $f(x)$ is such an element. If $q = p^h$, then $\mathbb{F}_p$ is the *prime subfield* of $\mathbb{F}_q$.

Definition 1.7. An automorphism σ of $\mathbb{F}_q$ is a permutation of $\mathbb{F}_q$ such that

$$(x+y)\sigma = x\sigma + y\sigma, (xy)\sigma = (x\sigma)(y\sigma) \quad \text{for all} \quad x, y \in \mathbb{F}_q.$$

The group $Aut(\mathbb{F}_q)$ of automorphisms of $\mathbb{F}_q, q = p^h$, is isomorphic to $\mathbb{Z}_h$. It is generated by the *Frobenius automorphism* ϕ, where $x\phi = x^p$ for all $x \in \mathbb{F}_q$; so $x\phi^i = x^{p^i}$. Equivalently, if σ is any automorphism of $\mathbb{F}_q$, then $\sigma = \phi^i$ for some i. It is occasionally convenient to write x^σ instead of $x\sigma$. Thus

$$Aut(\mathbb{F}_q) = \{1, \sigma, \sigma^2, \ldots, \sigma^{h-1}\}.$$

Since ϕ is an automorphism, $(x+y)^p = x^p + y^p$ for all $x, y \in \mathbb{F}_q$.

1.3 Projective Spaces Over a Finite Field

Let $V = V(n+1, q)$ be an $(n+1)-$dimensional vector space over a field $\mathbb{F}_q$ with zero element $\mathbf{0}$. Consider the equivalence relation on the elements of $V_0 = V \setminus \{0\}$ whose equivalence classes are the one-dimensional subspaces of V with zero removed. Thus, if $X, Y \in V_0$, then X is equivalent to Y if $Y = tX$ for some t in $\mathbb{F}_q \setminus \{0\}$; that is, $y_i = tx_i$ for all i. Then the set of equivalence classes is the *n-dimensional projective space* over $\mathbb{F}_q$ and is denoted by $\mathrm{PG}(n, q)$. The elements of $\mathrm{PG}(n, q)$ are *points*; the equivalence class of the vector X is the point $P(X)$. It will also be said that X is a *coordinate vector* for $P(X)$ or that X is a vector representing $P(X)$. In this case, tX with t in $\mathbb{F}_q \setminus \{0\}$ also represents $P(X)$; that is, by definition, $P(tX) = P(X)$. So, the points of $\mathrm{PG}(n, q)$ can be described in terms of coordinates as in Table 1.2, where $x_0, x_1, \ldots, x_{n-1} \in \mathbb{F}_q$. So

TABLE 1.2: Type of elements of $\mathrm{PG}(n, q)$

Type of elements	Number of elements
$P(x_0, \ldots, x_{n-1}, 1)$	q^n
$P(x_0, \ldots, x_{n-2}, 1, 0)$	q^{n-1}
$\vdots$	$\vdots$
$P(x_0, 1, 0, \ldots, 0)$	q
$P(1, 0, \ldots, 0)$	1
	$\theta(n, q)$

$$|\mathrm{PG}(n, q)| = \theta(n, q) = (q^{n+1} - 1)/(q - 1).$$

Definition 1.8. Let F be a subfield of the field K.

(i) The dimension of K as a vector space over F is the *degree* of K over F, and is denoted by $[K : F]$.

(ii) K is a *finite extension* of F if $[K : F]$ is finite.

(iii) When $[K : F] = 2, \ldots, n$, then K is a *quadratic*, *cubic*, $\ldots$, *n-ic* extension of F.

Definition 1.9. A collineation of projective plane Π is a map ψ of Π onto Π such that

1. ψ is bijection;
2. ψ maps points onto points and lines onto lines;
3. if P and ℓ are an incident point and line in Π, then $\psi(P)$ and $\psi(\ell)$ are incident.

Theorem 1.10. *The Fundamental Theorem of Projective Geometry* [37]

(i) If $\{P_0, \ldots, P_{n+1}\}, \{P'_0, \ldots, P'_{n+1}\}$ *are two sets of* $n+2$ *points of* $\mathrm{PG}(n, K)$ *such that no* $n+1$ *points chosen from the same set lie in a subspace of dimension* $n-1$*, then there exists a unique projectivity* T *such that* $P'_i = P_iT$*, for all* $i \in \{0, \ldots, n+1\}$.

(ii) Let $S = \mathrm{PG}(2, K)$*, and* $\psi : S \longrightarrow S$ *be a collineation, then* $\psi = \sigma T$*, where* σ *is an automorphism and* T *is a projectivity. This means that if* $K = \mathbb{F}_q$*, and* $P(X') = P(X)\psi$*, then there exists* $\varrho \in N_h, t_{ij} \in \mathbb{F}_q$ *for* $(i, j) \in N_h^2$ *and* $t_{ij} \in \mathbb{F}_q \setminus \{0\}$ *such that* $tX^* = X^{p^\varrho}T$*, where* $X^{p^\varrho} = (x_0^{p^\varrho}, \ldots, x_n^{p^\varrho})$ *and* $T = (t_{ij}), i, j \in N_h = \{0, 1, \ldots, h\}$.

Definition 1.11. For any positive integer n, the *general linear group*, $\mathrm{GL}(n, q)$, is the set of all invertible $n \times n$ matrices over $\mathbb{F}_q$ under matrix multiplication. The order of the group $\mathrm{GL}(n, q)$ is

$$(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}).$$

Definition 1.12. The *special linear group* $\mathrm{SL}(n, q)$ is the subgroup of the group $\mathrm{GL}(n, q)$ consisting of those matrices of determinant 1.

Definition 1.13. [37]

(i) The *projective general linear group* $\mathrm{PGL}(n, q)$ is the group of projectivities of $\mathrm{PG}(n-1, q)$ with respect to the operation of composition of maps.

(ii) The *collineation group* $\mathrm{P\Gamma L}(n, q)$ is the group of collineations of $\mathrm{PG}(n-1, q)$ with respect to the operation of composition of maps.

Definition 1.14. The *projective special linear group* $\mathrm{PSL}(n, q)$ is the quotient group $\mathrm{SL}(n, q)/\mathbb{Z}$, where $\mathbb{Z}$ is the subgroup of scalar matrices in $\mathrm{SL}(n, q)$.

1.4 Affine Spaces

If $\mathcal{H}_\infty$ is any hyperplane in $\mathrm{PG}(n,K)$ that is, a subspace of dimension $n-1$, then $\mathrm{AG}(n,K) = \mathrm{PG}(n,K)\backslash\mathcal{H}_\infty$ is an *affine space* of n dimensions over field K. When $K=\mathbb{F}_q$, write $\mathrm{AG}(n,K)=\mathrm{AG}(n,q)$. The *subspaces* of $\mathrm{AG}(n,K)$ are the subspaces of $\mathrm{PG}(n,K)$, apart from $\mathcal{H}_\infty$, with the points of $\mathcal{H}_\infty$ deleted in each case. Here $\mathcal{H}_\infty$ is referred to as the hyperplane at infinity of $\mathrm{AG}(n,q)$.

1.5 Projective Planes

Consider the projective plane $\mathrm{PG}(2,q)$ over $\mathbb{F}_q$. So, $\mathrm{PG}(2,q)$ contains q^2+q+1 points and q^2+q+1 lines. There are exactly $q+1$ points on each line, and $q+1$ lines through each point. The points and lines of $\mathrm{PG}(2,q)$ satisfy the axioms of a projective plane:

1. every two distinct points are on a unique common line;
2. every two distinct lines contain a unique common point;
3. there are four distinct points, no three of which are on a common line.

The constants k,l,m determine the line ℓ. A point $P(x,y,z)$ is incident with a line $\ell(k,l,m)$ if and only if

$$XU^t = (x \quad y \quad z)\begin{pmatrix} k \\ l \\ m \end{pmatrix} = kx+ly+mz=0.$$

With every non-singular matrix $T=(t_{ij})\in\mathbb{F}_q$ associate a bijection from the point $P(X)=P(x,y,z)$ to $P(X')=P(x',y',z')$ and the line $\ell(U)=\ell(k,l,m)$ to $\ell(U'^t)=\ell(k',l',m')^t$ with $X'=XT$ and $U'^t=T^{-1}U^T$. In other words,

$$(x' \quad y' \quad z') = (x \quad y \quad z)\begin{pmatrix} t_{00} & t_{01} & t_{02} \\ t_{10} & t_{11} & t_{12} \\ t_{20} & t_{21} & t_{22} \end{pmatrix},$$

$$\begin{pmatrix} k' \\ l' \\ m' \end{pmatrix} = \begin{pmatrix} t_{00} & t_{01} & t_{02} \\ t_{10} & t_{11} & t_{12} \\ t_{20} & t_{21} & t_{22} \end{pmatrix}^{-1} \begin{pmatrix} k \\ l \\ m \end{pmatrix}.$$

Such a bijection is a *projectivity* or *projective linear transformation.* A projectivity preserves the incidence between points and lines: a point $P(X)$ lies on a line $\ell(U^t)$ if and only if the image $P(X')$ of $P(X)$ lies on the image $\ell(U'^t)$ of $\ell(U^t)$. Indeed, $X'U'^t = XTT^{-1}U^t = 0$. As with coordinate triples, also the matrix T of a projectivity is only determined up to a scalar factor.

The group of all projectivities of $\mathrm{PG}(2,q)$ is the projective general linear group $\mathrm{PGL}(3,q)$ and has order $q^3(q^3-1)(q^2-1)$. From the Fundamental Theorem of Projective Geometry, a projectivity is uniquely determined by the four images of the vertices of a quadrangle. In other words, $\mathrm{PGL}(3,q)$ acts transitively on ordered quadrangles.

A *collineation* in $\mathrm{PG}(2,q)$ is a bijection mapping points to points and lines to lines, which preserves incidence. Each projectivity is a collineation. However, in general not all collineations are projectivities. The Frobenius automorphism mapping the point $P(x,y,z)$ to $P(x^p,y^p,z^p)$ is a collineation, but not a projectivity. From the Fundamental Theorem of Projective Geometry, let ψ be a collineation, then there exists a non-singular matrix T and a field automorphism σ, such that

$$\psi : (x \quad y \quad z) \mapsto (x \quad y \quad z)^{\sigma} \begin{pmatrix} t_{00} & t_{01} & t_{02} \\ t_{10} & t_{11} & t_{12} \\ t_{20} & t_{21} & t_{22} \end{pmatrix},$$

$$\psi : \begin{pmatrix} k \\ l \\ m \end{pmatrix} \mapsto \begin{pmatrix} t_{00} & t_{01} & t_{02} \\ t_{10} & t_{11} & t_{12} \\ t_{20} & t_{21} & t_{22} \end{pmatrix}^{-1} \begin{pmatrix} k \\ l \\ m \end{pmatrix}^{\sigma}.$$

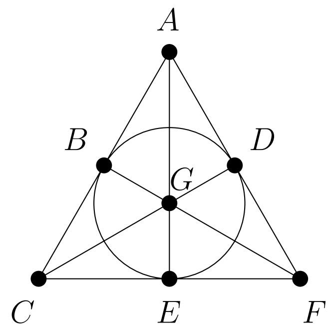
The group of all collineations of $\mathrm{PG}(2,q)$, $q=p^h$, is the collineation group $\mathrm{P\Gamma L}(3,q)$ and has order $hq^3(q^3-1)(q^2-1)$. So, $\mathrm{PGL}(3,q)$ is a subgroup of $\mathrm{P\Gamma L}(3,q)$ and

$$[\mathrm{P\Gamma L}(3,q) : \mathrm{PGL}(3,q)] = h.$$

When q is prime, then $\mathrm{PGL}(3,q) = \mathrm{P\Gamma L}(3,q)$.

Example 1.2. The simplest finite projective plane is PG$(2, 2)$. It consists of seven points and seven lines with three points on every line and three lines through every point. This projective plane is the *Fano plane* and may be illustrated as in Figure 1.1. The points are A, B, C, D, E, F, G and the lines are ABC, AGE, ADF, CEF, CDG, BDE, BFG.

Figure 1.1: Fano plane



In all known examples the order q of projective plane is a prime power. It is a fact that, for any given prime power, there exists at least one projective plane of that order, namely PG$(2, q)$. It is not known whether planes of non-prime power order exist, although it is commonly believed that any finite projective plane must have prime power order.

1.6 Affine Planes

An affine plane AG$(2, q)$ of order q is an incidence structure of points and lines with the following properties:

1. every two points are incident with a unique line;
2. given a point P and a line ℓ such that $P \notin \ell$ then there exist a unique line m such that $P \in m$ and $m \cap \ell = \emptyset$;
3. there are three points that are not collinear;
4. every line is incident with a constant q points and every point is incident with $q + 1$ lines;
5. an affine plane of order q has q^2 points and $q^2 + q$ lines.

1.7 Projective Lines

The projective line, $\mathrm{PG}(1,q)$, has $q+1$ points:

$$\mathrm{PG}(1,q) = \{(1,0)\} \cup \{(x,1) \mid x \in \mathbb{F}_q\}.$$

Each point $P(x,1)$ of $\mathrm{PG}(1,q)$ can be represented by the non-homogeneous coordinate x in $\mathbb{F}_q$ and the coordinate for $P(1,0)$ is ∞. These points are often referred as the parameters $t \in \mathbb{F}_q \cup \{\infty\}$.

A projectivity $\mathfrak{T}$ of $\mathrm{PG}(1,q)$ is given by $Y = XT$, where $X = (x_0, x_1)$, $Y = (y_0, y_1)$ and

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

From the Fundamental Theorem of Projective Geometry, any three points of $\mathrm{PG}(1,q)$ can be mapped to any other three points by a projectivity. A simple example of how a projectivity of the line maps points to points is presented below.

Example 1.3. In $\mathrm{PG}(1,5)$ there are six points $(0,1)$, $(1,1)$, $(2,1)$, $(3,1)$, $(4,1)$, $(1,0)$. Then the projectivity

$$T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

maps these points of $\mathrm{PG}(1,5)$ to the points $(1,1)$, $(4,1)$, $(0,4)$, $(3,1)$, $(1,0)$, $(2,1)$.

1.8 Conics

An *irreducible conic* of $\mathrm{PG}(2,q)$ is a set of points $\mathcal{C}$ whose coordinates (x,y,z) satisfy the following equation:

$$Q(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz,$$

with condition that $(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}) \neq (0,0,0)$ at any points of $\mathcal{C}$ and $a,b,c,d,e,f \in \mathbb{F}_q$.

Let $C = v(Q) = \{P(x,y,z) \mid Q(x,y,z) = 0\}$.

When q is odd,

$$Q(x,y,z) = XAX^T,$$

where $X = (x,y,z)$ and

$$A = \begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix}.$$

Consider a projectivity $\mathfrak{T} \in \mathrm{PGL}(3,q)$. Let $\mathcal{C}$ denote the conic represented by the matrix A. Then the image $\mathcal{C}'$ of $\mathcal{C}$ by $\mathfrak{T}$ is represented by the matrix $A' = T^{-1}A(T^{-1})^t$. Indeed,

$$\begin{aligned} X'A'X'^t &= (XT)(T^{-1}A(T^{-1})^t)(XT)^t \\ &= XTT^{-1}A(T^{-1})^tT^tX^t \\ &= XAX^t. \end{aligned}$$

An *oval* $\mathcal{O}$ is a set of $q+1$ points in $\mathrm{PG}(2,q)$, with the property that every line is incident with at most two points of $\mathcal{O}$. A conic is an example of an oval in $\mathrm{PG}(2,q)$. Here the classification of lines and points of the plane with respect to a conic is considered from a geometric viewpoint, and then an algebraic method for classifying points and lines with respect to conic is given. Since no three points of conic $\mathcal{C}$ are collinear, any line l in $\Pi = \mathrm{PG}(2,q)$ can intersect $\mathcal{C}$ in at most two points, Hence, l is geometrically classified in the following way:

1. l is a tangent line to $\mathcal{C}$ when $|l \cap \mathcal{C}| = 1$;

2. l is a secant line to $\mathcal{C}$ when $|l \cap \mathcal{C}| = 2$;

3. l is an exterior line to $\mathcal{C}$ when $|l \cap \mathcal{C}| = 0$.

Note 1.15. [37] For q odd, there are $q+1$ tangent lines, $\frac{1}{2}q(q+1)$ secant lines, and $\frac{1}{2}q(q-1)$ exterior lines.

1.8.1 Conics in Planes of Even Order

The points of the plane are geometrically classified with respect to a conic. If P is a point in a plane of even order, then P is classified in three ways with respect to $\mathcal{C}$:

1. P is a *conic point*, when P is on $\mathcal{C}$;
2. P is the *nucleus*, if P lies on all the tangent lines to $\mathcal{C}$;
3. P is a *regular point*, if P is not on $\mathcal{C}$ and not the nucleus.

Any conic point has one tangent and q secants through it, while the nucleus has $q+1$ tangents through it. The remaining points have exactly one tangent, $\frac{1}{2}q$ secants and $\frac{1}{2}q$ exterior lines through them. The set of $q+2$ points obtained from adding the nucleus to the oval is a *hyperoval*; it is a maximal arc of degree 2.

Lemma 1.16 ([37]). *In* $\mathrm{PG}(2,q)$ *for* q *even, the nucleus of the conic*

$$v(ax^2+by^2+cz^2+dxy+exz+fyz)$$

is the point $P(f,e,d)$.

Example 1.4. Let $\mathcal{C}$ be a conic in the projective plane $\mathrm{PG}(2,q)$, $q>4$ and even. Let N be the nucleus of $\mathcal{C}$, and let $\mathcal{H}$ denote the hyperoval which consist of the $q+1$ points of $\mathcal{C}$ and the nucleus of $\mathcal{C}$. So, $\mathcal{H}$ is a set of $q+2$ points with the property that no three are collinear. By construction, there are no tangent lines to $\mathcal{H}$. Any line in the plane must be either a secant line or an exterior line. Let $\mathcal{O}$ be the oval which consists of N and any q points of $\mathcal{C}$. Recall that five points, no three collinear, determine a unique conic. The sets $\mathcal{C}$ and $\mathcal{O}$ have q points in common and these q points satisfy a quadratic equation. But, the point N of $\mathcal{O}$ does not satisfy this quadratic equation. Hence, there is no quadratic equation which every point of $\mathcal{O}$ satisfies; so $\mathcal{O}$ is not a conic.

1.8.2 Conics in Planes of Odd Order

In planes of odd order, every oval is a conic. Here, every conic is an oval and an oval has $q+1$ points on it, every conic consist of $q+1$ points that satisfy some non-degenerate quadratic equation.

If P is a point in the plane, then P is classified in three ways with respect to $\mathcal{C}$:

1. when P is on $\mathcal{C}$, then P is a *conic point*;

2. when P is not on $\mathcal{C}$, but on a tangent line to $\mathcal{C}$, then P is *exterior point*;

3. when P is neither on $\mathcal{C}$ nor a tangent line to $\mathcal{C}$, then P is an *interior point.*

Any conic point has one tangent and q secants through it. An interior point has $\frac{1}{2}(q+1)$ exterior lines through it, an exterior point has exactly two tangents, $\frac{1}{2}(q-1)$ secant and $\frac{1}{2}(q-1)$ exterior lines through it.

Example 1.5. Consider the conic $\mathcal{C}$ whose points satisfy $-xy + y^2 + 2yz + z^2 = 0$. The lines $x = 0$ and $y = 0$ are tangent lines to $\mathcal{C}$. Since $(0, 0, 1)$ lies on both of these tangent lines, then $(0, 0, 1)$ is an external point with respect to $\mathcal{C}$. The lines through $(0, 0, 1)$ are $x = 0$ and $y = mx$, for $m \in \mathbb{F}_q$. Then $y = mx$ is a secant when m is non-zero square and is a tangent when m is a non-square element.

Lemma 1.17. *A conic in* $\mathrm{PG}(2, q)$ *has the following canonical forms:*

(i) $v(X_0^2 + X_1X_2)$, *all* q*;*

(ii) $v(X_0^2 - X_1X_2)$, *all* q*;*

(iii) $v(a_0X_0^2 + a_1X_1^2 + a_2X_2^2)$, *with* $a_0a_1a_2 \neq 0$, q *odd;*

(iv) $v(X_0^2 + X_1^2 + X_2^2)$, q *odd.*

Corollary 1.18 ([37]). *In* $\mathrm{PG}(2, q)$ *for* q *even, the* $q + 1$ *tangents to a conic are concurrent.*

Theorem 1.19 ([19]). *If* $F = \sum_{i\leq j} a_{ij}x_ix_j$, *then* $\mathcal{F} = v(F)$ *is singular if and only if* $\delta = 0$, *where*

$$\delta = 4a_{00}a_{11}a_{22} + a_{01}a_{02}a_{12} - a_{00}a_{12}^2 - a_{11}a_{02}^2 - a_{22}a_{01}^2.$$

Theorem 1.20 ([19]). *A conic of* $\mathrm{PG}(2,q)$ *is reducible if and only if it is singular.*

(1) *Suppose a conic* $\mathcal{C}$ *of* $\mathrm{PG}(2,q)$ *is reducible. If it reduces to two distinct lines* ℓ *and* m*, then the point* $\ell \cap m$ *is a singular point of the conic. If* $\mathcal{C}$ *reduces to two coincident lines, then every point of* $\mathcal{C}$ *is singular. In either case, the conic is singular.*

(2) *Let* $F = ax^2+by^2+cz^2+dxy+fxz+eyz$ *be the equation of a singular conic. Without loss of generality, let* $(1,0,0)$ *be a singular point of* $\mathcal{C}$*. Then* $(1,0,0)$ *satisfies* $F=0$ *and also*

$$\frac{\partial F}{\partial x} = 2ax + dy + fz = 0,$$
$$\frac{\partial F}{\partial y} = dx + 2by + ez = 0,$$
$$\frac{\partial F}{\partial z} = fx + ey + 2cz = 0.$$

Therefore, $a = d = f = 0$*. The equation of* $\mathcal{C}$ *becomes* $\psi = by^2 + cz^2 + eyz = 0$*. The quadratic* ψ *is reducible, and therefore the conic* $\mathcal{C}$ *is reducible.*

Theorem 1.21 (Segre's Theorem [57]). *If* q *is odd, every* $(q+1)$*-arc of* $\mathrm{PG}(2,q)$ *is an irreducible conic.*

Example 1.6 ([19]). Let $\mathcal{C}$ be a conic of $\mathrm{PG}(2,q)$ with equation

$$x^2 + 2xy + y^2 + 4yz + 4z^2 = 0.$$

(i) Find the equation of the tangent to $\mathcal{C}$ at $(1,-1,0)$.

(ii) Find the tangent of $\mathcal{C}$ passing through $(1,0,0)$.

Solution From the given equation of the conic, $a = 1$, $b = 1$, $c = 4$, $d = 2$, $e = 0$, $f = 4$. The matrix A of the given conic is

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}.$$

(i) The tangent at the point $(1, -1, 0)$ is

$$(1, -1, 0)\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0x + 0y - 2z = 0,$$

which is the line of equation $z = 0$.

(ii) For $P = (1, 0, 0)$, then $P^T AX = x + y$ and $P^T AP = 1$. So the equation of the tangents passing through $(1, 0, 0)$ is

$$(P^T AX)^2 - (P^T AP)(X^T AX) = 0,$$

that is $(x + y)^2 - (x^2 + 2xy + y^2 + 4yz + 4z^2) = -4(yz + z^2) = -4z(y + z) = 0$.

The two tangents passing through $(1, 0, 0)$ are the lines $z = 0$ and $y + z = 0$.

1.9 Arcs

Definition 1.22. [37] A (k, n)-arc in $\mathrm{PG}(2, q)$ is a set $\mathcal{K}$ of k points, no $n+1$ of which are collinear, but with at least one set of n points collinear. When $n = 2$, a $(k, 2)$-arc is called a k-arc.

Definition 1.23. [37] A (k, n)-arc is *complete* if it is not contained in a $(k+1, n)$-arc.

Definition 1.24. [37] A set of lines concurrent at a point P of a projective plane Π is a *pencil* of lines. A set of points incident with a fixed line ℓ in Π is a *range* of points.

Notation 1.25. For a (k, n)-arc $\mathcal{K}$ in $\mathrm{PG}(2, q)$, let

$$\begin{aligned}
\tau_i &= \text{the total number of } i\text{-secants of } \mathcal{K},\\
\rho_i = \rho_i(P) &= \text{the number of } i\text{-secants through a point } P \text{ of } \mathcal{K},\\
\sigma_i = \sigma_i(Q) &= \text{the number of } i\text{-secants through a point } Q \text{ of } \mathrm{PG}(2, q) \setminus \mathcal{K},\\
m_n(2, q) &= \text{the maximum size of a } (k, n)\text{-arc in } \mathrm{PG}(2, q),\\
t_n(2, q) &= \text{the minimum size of a } (k, n)\text{-arc in } \mathrm{PG}(2, q).
\end{aligned}$$

Definition 1.26. [37]

(i) The *type of a point* P on a (k, n)-arc $\mathcal{K}$ is the $(n + 1)$-tuple $(\rho_0, \rho_1, \dots, \rho_n)$.

(ii) The *i-secant distribution* of $\mathcal{K}$ is the $(n + 1)$-tuple $(\tau_n, \tau_{n-1}, \dots, \tau_0)$.

Theorem 1.27 ([37])**.**

$$m_2(2, q) = \begin{cases} q + 2, & \textit{for } q \textit{ even;} \\ q + 1, & \textit{for } q \textit{ odd.} \end{cases}$$

Theorem 1.28 ([37])**.** (1)

$$m_n(2, q) \begin{cases} = (n - 1)q + n, & \textit{for } q \textit{ even and } n \mid q; \\ < (n - 1)q + n, & \textit{for } q \textit{ odd.} \end{cases}$$

(2) *A (k, n)-arc $\mathcal{K}$ is maximal if and only if every line in* $\mathrm{PG}(2, q)$ *is either an n-secant or an external line.*

Lemma 1.29 ([37]). *For a (k,n)-arc $\mathcal{K}$, the following equations hold:*

$$\sum_{i=0}^{n} \tau_i = q^2+q+1; \tag{1.1}$$
$$\sum_{i=1}^{n} i\tau_i = k(q+1); \tag{1.2}$$
$$\sum_{i=2}^{n} \tfrac{1}{2}i(i-1)\tau_i = \tfrac{1}{2}k(k-1). \tag{1.3}$$

Lemma 1.30 ([37]). *For a (k,n)-arc $\mathcal{K}$, the following equations also hold:*

$$\sum_{i=1}^{n} \rho_i = q+1; \tag{1.4}$$
$$\sum_{i=2}^{n} (i-1)\rho_i = k-1; \tag{1.5}$$
$$\sum_{i=0}^{n} \sigma_i = q+1; \tag{1.6}$$
$$\sum_{i=1}^{n} i\sigma_i = k; \tag{1.7}$$
$$\sum_{P\in\mathcal{K}} \rho_i = i\tau_i; \tag{1.8}$$
$$\sum_{Q\in\Pi\setminus\mathcal{K}} \sigma_i = (q+1-i)\tau_i. \tag{1.9}$$

Notation 1.31. (i) Let the Equations (1.4) and (1.5) of Lemma 1.30 have M distinct solutions

$$B_j = (\rho_{1j},\ldots,\rho_{nj}),\ j=1,\ldots,M.$$

(ii) Let b_j be the number of points on the (k,n)-arc $\mathcal{K}$ with solution B_j.

(iii) Let the Equations (1.6) and (1.7) have L distinct solutions

$$M_j = (\sigma_{0j},\ldots,\sigma_{nj}),\ j=1,\ldots,L.$$

(iv) Let m_j be the number of points in $\mathrm{PG}(2,q)\setminus\mathcal{K}$ with solution M_j.

Lemma 1.32 ([36]). *For a (k,n)-arc $\mathcal{K}$ in $\mathrm{PG}(2,q)$, the following equations hold:*

$$\sum_{j=1}^{M} b_j\rho_{ij} = i\tau_i; \tag{1.10}$$

$$\sum_{j=1}^{M} b_j = k; \tag{1.11}$$

$$\sum_{j=1}^{L} m_j\sigma_{ij} = (q+1-i)\tau_i; \tag{1.12}$$

$$\sum_{j=1}^{L} m_j = q^2+q+1-k. \tag{1.13}$$

Lemma 1.33 ([37]). *If $\mathcal{K}$ is a complete (k,n)-arc, then $(q+1-n)\tau_n \geq q^2+q+1-k$, with equality if and only if $\sigma_n = 1$ for all Q in $\mathrm{PG}(2,q)\setminus\mathcal{K}$.*

1.10 Stabiliser Group

If Q is a point of $\mathrm{PG}(2,q)$, then Q^g denotes the image of Q through the right action of the non-singular matrix g that induces the projectivity $\mathfrak{T}$; that is, $\mathfrak{T} \in \mathrm{PGL}(3,q)$. Let S be a set of points of $\mathrm{PG}(2,q)$, possibly an (k,n)-arc. Then $S^g = \{Q^g \mid Q \in S\}$.

Definition 1.34 (Orbit). For $G \leq \mathrm{PGL}(3,q)$, if $\mathfrak{T} \in \mathrm{PGL}(3,q)$, then the *orbit* of a point $Q \in \mathrm{PG}(2,q)$ under G is the set $Q^G = \{Q^g \mid g \in G\}$. The *orbit* of $S \subset \mathrm{PG}(2,q)$ under G is the set $\{Q^G \mid Q \in S\}$.

Definition 1.35 (Stabiliser). A set $S \subseteq \mathrm{PG}(2,q)$ is said to be stabilised by the projectivity $\mathfrak{T} \in \mathrm{PGL}(3,q)$ if and only if $S^g = S$.

To determine the automorphism or stabiliser group of a (k,n)-arc S, first calculate every projectivity $\mathfrak{T} \in \mathrm{PGL}(3,q)$ that maps S to itself; that is $S^g = S$. This is achieved by finding every g that maps a 4-arc of S onto the frame points, where the frame points are $P(1,0,0), P(0,1,0), P(0,0,1), P(1,1,1)$ and then determining if $S^g = S$. The stabiliser group of S consists of all such projectivities. If this group has order $i = 1,2,3,5,7,11,13,15,17,19,23$, then it is isomorphic to $\mathbf{Z}_i$. If the group has order $i = 4,6,8,9,10,12,14,18,20,21,22,24$, then the group with which it is isomorphic is determined by the orders of its elements.

$\mathbf{Z}$	$=$	group of integers;
$\mathbf{Z}_n$	$=$	cyclic group of order n;
$\mathbf{S}_n$	$=$	symmetric group of degree n;
$\mathbf{A}_n$	$=$	alternating group of degree n;
$\mathbf{D}_n$	$=$	dihedral group of order $2n = \langle r, s \mid r^n = s^2 = (rs)^2 = 1\rangle$;
$\mathbf{Q}_n$	$=$	dicyclic group of order $2n = \langle r, s \mid r^n = 1, r^{n/2} = s^2 = (sr^{-1})^2\rangle$;
$\mathbf{V}_4$	$=$	Klein 4-group which is the direct product of two copies of the cyclic group of order 2;
$G \times H$	$=$	the direct product of G and H;
$G \rtimes H$	$=$	the semi-direct product of G with H, where G is normal subgroup.

In the tables, N is the number of elements of order O in the group.

Table 1.3: Groups of order 4

Groups	O	N	O	N	O	N
$\mathbf{Z}_4$	1	1	2	1	4	2
$\mathbf{Z}_2 \times \mathbf{Z}_2$	1	1	2	3		

Table 1.4: Groups of order 6

Groups	O	N	O	N	O	N	O	N
$\mathbf{Z}_6$	1	1	2	1	3	2	6	2
$\mathbf{S}_3$	1	1	2	3	3	2		

Table 1.5: Groups of order 8

Groups	O	N	O	N	O	N	O	N
$\mathbf{Z}_8$	1	1	2	1	4	2	8	4
$\mathbf{Z}_2 \times \mathbf{Z}_4$	1	1	2	3	4	4		
$(\mathbf{Z}_2)^3$	1	1	2	7				
$\mathbf{D}_4$	1	1	2	5	4	2		
$\mathbf{Q}_4$	1	1	2	1	4	6		

Table 1.6: Groups of order 9

Groups	O	N	O	N	O	N
$\mathbf{Z}_9$	1	1	3	2	9	6
$\mathbf{Z}_3 \times \mathbf{Z}_3$	1	1	3	8		

Table 1.7: Groups of order 10

Groups	O	N	O	N	O	N	O	N
$\mathbf{Z}_{10}$	1	1	2	1	5	4	10	4
$\mathbf{D}_5$	1	1	2	5	5	4		

Table 1.8: Groups of order 12

Groups	O	N	O	N	O	N	O	N	O	N	O	N
$\mathbf{Z}_{12}$	1	1	2	1	3	2	4	2	6	2	12	4
$\mathbf{Z}_6 \times \mathbf{Z}_2$	1	1	2	3	3	2	6	6				
$\mathbf{D}_6$	1	1	2	7	3	2	6	2				
$\mathbf{Q}_6$	1	1	2	1	3	2	4	6	6	2		
$\mathbf{A}_4$	1	1	2	3	3	8						

Table 1.9: Groups of order 14

Groups	O	N	O	N	O	N	O	N
$\mathbf{Z}_{14}$	1	1	2	1	7	6	14	6
$\mathbf{D}_7$	1	1	2	7	7	6		

Table 1.10: Groups of order 16

Groups	O	N	O	N	O	N	O	N	O	N	
$\mathbf{Z}_{16}$	1	1	2	1	4	2	8	4	16	8	
$\mathbf{Z}_8 \times \mathbf{Z}_2$	1	1	2	3	4	4	8	8			abelian
$\mathbf{Z}_4 \times \mathbf{Z}_4$	1	1	2	3	4	12					abelian
$\mathbf{Z}_4 \times (\mathbf{Z}_2)^2$	1	1	2	7	4	8					abelian
$(\mathbf{Z}_2)^4$	1	1	2	15							
$\mathbf{D}_8$	1	1	2	9	4	2	8	4			
$\mathbf{Q}_8$	1	1	2	1	4	10	8	4			
$\mathbf{D}_4 \times \mathbf{Z}_2$	1	1	2	11	4	4					
$\mathbf{Q}_4 \times \mathbf{Z}_2$	1	1	2	3	4	12					non-abelian
$\mathbf{Z}_8 \rtimes \mathbf{Z}_2, \mathbf{H}_1$	1	1	2	3	4	4	8	8			non-abelian
$\mathbf{Z}_8 \rtimes \mathbf{Z}_2, \mathbf{H}_2$	1	1	2	5	4	6	8	4			
$\mathbf{Z}_4 \rtimes \mathbf{Z}_2$	1	1	2	3	4	12					non-abelian
$(\mathbf{Z}_4 \times \mathbf{Z}_2) \rtimes \mathbf{Z}_2, \mathbf{H}_3$	1	1	2	7	4	8					non-abelian
$(\mathbf{Z}_4 \times \mathbf{Z}_2) \rtimes \mathbf{Z}_2, \mathbf{H}_4$	1	1	2	7	4	8					non-abelian

Table 1.11: Groups of order 18

Groups	O	N	O	N	O	N	O	N	O	N	O	N
$\mathbf{Z}_{18}$	1	1	2	1	3	2	6	2	9	6	18	6
$\mathbf{Z}_6 \times \mathbf{Z}_3$	1	1	2	1	3	8	6	8				
$\mathbf{D}_9$	1	1	2	9	3	2	9	6				
$\mathbf{S}_3 \times \mathbf{Z}_3$	1	1	2	3	3	8	6	6				
$(\mathbf{Z}_3 \times \mathbf{Z}_3) \rtimes \mathbf{Z}_2$	1		2	9	3	8						

Table 1.12: Groups of order 20

Groups	O	N	O	N	O	N	O	N	O	N	O	N
$\mathbf{Z}_{20}$	1	1	2	1	4	2	5	4	10	5	20	7
$\mathbf{Z}_{10} \times \mathbf{Z}_2$	1	1	2	3	5	4	10	12				
$\mathbf{D}_{10}$	1	1	2	11	5	4	10	4				
$\mathbf{Q}_{10}$	1	1	2	1	4	10	5	4	10	4		
$\mathbf{Z}_5 \rtimes \mathbf{Z}_4$	1	1	2	5	4	10	5	4				

Table 1.13: Groups of order 21

Groups	O	N	O	N	O	N	O	N
$\mathbf{Z}_{21}$	1	1	3	2	7	6	21	12
$\mathbf{Z}_7 \rtimes \mathbf{Z}_3$	1	1	3	14	7	6		

Table 1.14: Groups of order 22

Groups	O	N	O	N	O	N	O	N
$\mathbf{Z}_{22}$	1	1	2	1	11	10	22	10
$\mathbf{D}_{11}$	1	1	2	11	11	10		

Table 1.15: Groups of order 24

Groups	O	N	O	N	O	N	O	N	O	N	O	N	O	N	O	N
$\mathbf{Z}_{24}$	1	1	2	1	3	2	4	2	6	2	8	4	12	4	24	8
$\mathbf{Z}_{12} \times \mathbf{Z}_2$	1	1	2	3	3	2	4	4	6	6	12	8				
$\mathbf{Z}_6 \times (\mathbf{Z}_2)^2$	1	1	2	7	3	2	6	14								
$\mathbf{S}_4$	1	1	2	9	3	8	4	6								
$\mathbf{D}_{12}$	1	1	2	13	3	2	4	2	6	2	12	4				
$\mathbf{Q}_{12}$	1	1	2	1	3	2	4	14	6	2	12	4				
$\mathbf{D}_6 \times \mathbf{Z}_2$	1	1	2	15	3	2	6	6								
$\mathbf{A}_4 \times \mathbf{Z}_2$	1	1	2	7	3	8	6	8								
$\mathbf{Q}_6 \times \mathbf{Z}_3$	1	1	2	3	3	2	4	12	6	6						
$\mathbf{D}_4 \times \mathbf{Z}_3$	1	1	2	5	3	2	4	2	6	10	12	4				
$\mathbf{Q}_4 \times \mathbf{Z}_3$	1	1	2	1	3	2	4	6	6	2	12	12				
$\mathbf{S}_3 \times \mathbf{Z}_4$	1	1	2	7	3	2	4	8	6	2	12	4				
$\mathrm{SL}(2,3)$	1	1	2	1	3	8	4	6	6	8						
$\mathbf{Z}_3 \rtimes \mathbf{Z}_8$	1	1	2	1	3	2	4	2	6	2	8	12	12	4		
$\mathbf{Z}_4 \rtimes \mathbf{D}_4$	1	1	2	9	3	2	4	6	6	6						

Definition 1.36 (Companion matrix)**.** [18] Let $f(x)$ be a monic polynomial in $F[x]$:

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

The *companion matrix* $C(f)$ is the $n \times n$ matrix given by

$$\mathrm{C}(f) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

In $\mathrm{PG}(2, q)$, let

$$f(x) = x^3 + a_2 x^2 + a_1 x + a_0.$$

The *companion matrix* $C(f)$ is the 3×3 matrix given by

$$\mathrm{C}(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}.$$

1.11 Methodology to Find $m_n(2, q)$

There are many methods used to find the classification of (k, n)-arcs in $\mathrm{PG}(2, q)$. Some of these methods are now described.

1.11.1 Method 1 [60], [2]

For any (k, n)-arc $\mathcal{K}$, $k \geq n + 2$, there are at least four points in $\mathcal{K}$ no three of which are collinear. Let $A_1 = \{P_i \mid i = 1, 2, \ldots, k\}$ and $A_2 = \{P_i' \mid i = 1, 2, \ldots, k\}$ be two (k, n)-arcs in $\mathrm{PG}(2, q)$, where the coordinates of the points P_i and P_i' are the following:

$$P_i = P(x_i(1), x_i(2), x_i(3)) \quad \text{and} \quad P_i' = P(y_i(1), y_i(2), y_i(3)).$$

By the Fundamental Theorem of Projective Geometry, there exists a unique projectivity which takes any set of four points of the (k,n)-arc A_1 no three collinear to any set of four points of A_2 no three collinear.

Consider the (3×3) matrix $Z = (z_{i,j})$, $i,j = 1,2,3$. Map a fixed set of four points of A_1 no three collinear, say $\{P_1, P_2, P_3, P_4\}$, to any set of four points of A_2 no three collinear, say $\{P'_1, P'_2, P'_3, P'_4\}$. The (k,n)-arcs A_1 and A_2 are said to be projectively equivalent if $ZX_i = \lambda Y_j$, $i,j = n+2, \ldots, k$, where X_i and Y_j are the column vectors that represent the other points P_i of A_1 and P'_j of A_2.

To determine Z, fix a quadrangle Q of A_1. Then find Z such that Q maps to a quadrangle Q' of A_2. Do this for every Q'. Here $Z \in G$ if it takes the remaining points of A_1 to the remaining points of A_2.

1.11.2 Method 2

This algorithm depends on the type of i-secant distribution and is used to find the large complete (k,n)-arcs in $\mathrm{PG}(2,q)$. To explain this method, the classification of $(k,4)$-arcs in $\mathrm{PG}(2,8)$ is used. The Equations (1.1), (1.2) and (1.3) of Lemma 1.29 are used here.

1 **The construction of the distinct $(4,4)$-arcs**

Let $A = \{1,2,4,37\}$ be a $(4,4)$-arc in $\mathrm{PG}(2,8)$. A $(4,4)$-arc has the same type of i-secant distribution as A. Therefore there is only one $(4,4)$-arc in $\mathrm{PG}(2,8)$ with respect to the type of i-secant distribution. This can be calculated from the following equations:

$$\begin{aligned} \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 73, \\ \tau_1 + 2\tau_2 + 3\tau_3 + 4\tau_4 &= 36, \\ \tau_2 + 3\tau_3 + 6\tau_4 &= 6. \end{aligned}$$

Since $\tau_4 = 1, \tau_3 = 0, \tau_2 = 0$, so the only type of $(4,4)$-arc is $(1,0,0,32,40)$.

2 **The construction of the distinct $(5,4)$-arcs**

From Step 1, there is only one $(4,4)$-arc A, and there are 64 points of index zero which do not lie on 4-secant of A. So by adding one point of the points of index zero to A, then there is only one type of $(5,4)$-arc denoted by B, satisfying the following:

$$\begin{aligned}\tau_0+\tau_1+\tau_2+\tau_3+\tau_4 &= 73,\\ \tau_1+2\tau_2+3\tau_3+4\tau_4 &= 45,\\ \tau_2+3\tau_3+6\tau_4 &= 10.\end{aligned}$$

Since $\tau_4 = 1, \tau_3 = 0$, so the only type of $(5,4)$-arc is $(1,0,4,33,35)$.

3 **The construction of the distinct $(6,4)$-arcs**

From Step 2, there is only one $(5,4)$-arc B, and there are 63 points of index zero. So by adding one point of the points of index zero to B, two distinct $(6,4)$-arcs C_1 and C_2 are obtained. Then C_1 is of type $(1,0,9,14,7)$ and C_2 is of type $(1,1,6,17,6)$.

4 **The construction of the distinct $(k,4)$-arcs**

Table 1.16 shows the number of i-secant distribution of $(k,4)$-arcs in PG$(2,8)$. Here, ξ is the number of distinct $(k,4)$-arcs according to i-secant distribution.

TABLE 1.16: The i-secant distribution of $(k,4)$-arcs in PG$(2,8)$

ξ	$(k,4)-arcs$	ξ	$(k,4)-arcs$	ξ	$(k,4)-arcs$
9	$(7,4)-arcs$	128	$(15,4)-arcs$	15	$(23,4)-arcs$
20	$(8,4)-arcs$	130	$(16,4)-arcs$	6	$(24,4)-arcs$
32	$(9,4)-arcs$	127	$(17,4)-arcs$	2	$(25,4)-arcs$
52	$(10,4)-arcs$	119	$(18,4)-arcs$	1	$(26,4)-arcs$
75	$(11,4)-arcs$	94	$(19,4)-arcs$	1	$(27,4)-arcs$
95	$(12,4)-arcs$	71	$(20,4)-arcs$	1	$(28,4)-arcs$
108	$(13,4)-arcs$	45	$(21,4)-arcs$		
118	$(14,4)-arcs$	32	$(22,4)-arcs$		

Chapter 2

Blocking Sets

A t-fold blocking set $\mathcal{B}$ in $\mathrm{PG}(2,q)$ or $\mathrm{AG}(2,q)$ is a set of points such that each line contains at least t points of $\mathcal{B}$ and some lines contain exactly t points of $\mathcal{B}$. A 1-fold blocking set is called a blocking set. A blocking set in $\mathrm{PG}(2,q)$ should contain no line. A 2-fold and a 3-fold blocking sets are called a double and a triple blocking set, respectively.

However, a blocking set can be considered as the complement of a (k,n)-arc $\mathcal{K}$ in $\mathrm{PG}(2,q)$ with $t = q+1-n$. The smallest blocking sets are just the lines, and any blocking set containing a line will be called trivial. A blocking set is said to be minimal, when no proper subset of it is a blocking set, and some authors call them *irreducible* instead of minimal. A minimal blocking set in a projective plane is either a line, or does not contain a line. The terminology is not standard; sometimes it is supposed that a blocking set contains no line. For $\mathcal{K}$ with an external line l, so $\mathcal{K}$ is contained in the affine plane $\mathrm{AG}(2,q) = \mathrm{PG}(2,q) \setminus l$. The complement of $\mathcal{K}$ within $\mathrm{AG}(2,q)$ is a t-blocking set with $t = q-n$.

Blocking sets have been first studied in 1969 by Di Paola [51], where the author has calculated the minimum size of a non-trivial blocking set in $\mathrm{PG}(2,q)$, having orders of $4,5,7,8,9$. The major challenge was finding the minimum size of blocking set in $\mathrm{PG}(2,q)$. In 1970, Bruen [13, 14] proved that $|\mathcal{B}| \geq q+\sqrt{q}+1$ for any non-trivial blocking set. A *Baer subplane* of $\mathrm{PG}(2,q)$ is a $(q+\sqrt{q}+1)$-set $\mathcal{B}$ of type $(1,\sqrt{q}+1)$. An alternative approach is to consider a blocking set which does not contain a Baer subplane. Bruen and Thas showed in 1977 [15] that a blocking set with $q+\sqrt{q}+2$ points necessarily contains a Baer subplane. Therefore, for a non-trivial minimal blocking set which does not contain a Baer subplane, $\mathcal{B} \geq q+\sqrt{q}+3$ will hold. Bruen and Thas in [15] have introduced another term for these blocking sets: *Rédei type*. A blocking set of Rédei type is a blocking set of size $q+m$ that has an m-secant. In Theorem 13.4.1 and Theorem 13.4.2 [36], it is shown that, when q is odd, then there is a blocking set, the *projective triangle*, with size $3(q+1)/2$, while for even q there is a blocking set, the *projective triad*, with size $(3q+2)/2$. Hill and Mason studied multiple blocking sets in $\mathrm{PG}(2,q)$. In 1981 [35], it is shown that, for even q, there are 2-blocking sets of sizes $3q$ and 3-blocking sets of sizes $4q$. Brouwer and Wilbrink [12] have shown that the size of the smallest example is between $q+q/p^e+1$ and $q+(q-1)/(p^e-1)$ for a divisor e of n. Bruen and Silverman improved the latter bound to $q+\sqrt{2q}+1-a/(2q)$ in 1987 [16], and to $q+q^{2/3}+1$ for $p>3$. In the special case of $q=p^2$ the bound is improved to $q+q^{3/4}/\sqrt{2}+1$ by Blokhuis and Storme in 1995 [11]. After that it has been improved in 1996 to $q+2\sqrt{q}+1$ by Ball and Blokhuis [6]. One of the most important questions raised by Szőnyi in 1977, is the following: “What are the possible sizes of minimal blocking sets in the interval $(q+1, 3(q+1)/2)$?”

According to Ball in 1994 [3], the arcs $(78,8)$ and $(90,9)$ are the largest complete arcs in $\mathrm{PG}(2,11)$, while for $\mathrm{PG}(2,13)$, there exist no arcs of size $(106,9)$, $(110,10)$, $(134,11)$. Furthermore, for a triple blocking set in $\mathrm{PG}(2,q)$ a new lower bound has been found for $q<11$ by Ball [5] in 1996. For a double blocking set in $\mathrm{PG}(2,q)$, Ball and Blokhuis [6] have introduced a new lower bound for $q \geq 11$. In addition, Daskalov [30] investigated $\mathrm{PG}(2,17)$, and found the largest complete (k,n)-arc for $n=11,\ldots,16$ in 2004.

Here, a new upper bound is found for the size of a (k,n)-arc in $\mathrm{PG}(2,q)$, for all values of $n>(q+3)/2$ and prime q. It may be noted that a (k,n)-arc is the complement of a $\{q^2+q+1-k, q+1-n\}$-blocking set and conversely.

2.1 Some Basic Equations

According to Richardson [53], *"The name blocking set originates from game theory, where we have a set of individuals, and certain subsets called coalitions, with the property that a coalition can force a particular decision. A blocking set then is a subset that is not a coalition, but contains at least one member of each coalition, so that it can block any decision without being able to force one"*.

Theorem 2.1. *A t-blocking set in* $\mathrm{AG}(2,q)$ *has at least* $(t+1)(q-1)+1$ *points.*

Theorem 2.2. *A t-blocking set in* $\mathrm{AG}(2,q)$*,* $(t,q)=1$*, has at least* $(t+1)(q-1)+t$ *points.*

Theorem 2.3 (Ball [3])**.** *Let $\mathcal{K}$ be (k,r)-arc in* $\mathrm{PG}(2,p)$*, where p is prime.*

(1) *If $r \le (p+1)/2$, then $k \le (r-1)p+1$.*

(2) *If $r \ge (p+3)/2$, then $k \le (r-1)p+r-(p+1)/2$.*

Theorem 2.4 (Ball [4])**.** *Let $\mathcal{B}$ be t-fold blocking set in* $\mathrm{PG}(2,p)$*, p prime and $p>3$.*

(1) *If $t<p/2$, then $|\mathcal{B}| \ge (t+\frac{1}{2})(p+1)$.*

(2) *If $t>p/2$, then $|\mathcal{B}| \ge (t+1)p$.*

Theorem 2.5 (Ball [4])**.** *Let $\mathcal{B}$ be a t-fold blocking set in* $\mathrm{PG}(2,q)$ *that contains a line.*

(1) *If $(t-1,q)=1$, then $|\mathcal{B}| \ge q(t+1)$.*

(2) *If $(t-1,q)>1$ and $t \le q/2+1$, then $|\mathcal{B}| \ge tq+q-t+2$.*

(3) *If $(t-1,q)>1$ and $t \ge q/2+1$, then $|\mathcal{B}| \ge t(q+1)$.*

Definition 2.6. [3] A polynomial in $\mathbb{F}_q$ is *fully reducible* if it factors completely into linear factors over $\mathbb{F}_q$. If in the sequence of coefficients of a polynomial a long run of zeros occurs, this polynomial is *lacunary*.

Theorem 2.7 (Ball [4])**.** *Let $f \in \mathbb{F}_q[x]$ be fully reducible, and suppose that f has the form $f(x)=x^q v(x)+w(x)$, where v and w have no common factor. Let $m<q$ be the maximum of the degrees of v and w. Let e be maximal such that f and hence also v and w are a p^e-th power. Then one of the following holds:*

(1) $e = h$ *and* $m = 0$;

(2) $e \geq h/2$ *and* $m \geq p^e$;

(3) $e < h/2$ *and* $m \geq p^e[(p^{h-e} + 1)/(p^e + 1)]$;

(4) $e = 0,\ m = 1$ *and* $f(x) = a(x^q - x)$.

Theorem 2.8 gives a slight improvement of Theorem 2.7 as follows.

Theorem 2.8 (Daskalov [30]). *Let $\mathcal{B}$ be an $\{l, t\}$-blocking set in* $\mathrm{PG}(2, p)$, *p prime.*

(1) *If $t < p/2$, and $p > 3$, then $l \geq n(p + 1) + (p + 1)/2$.*

(2) *If $l = t(p + 1) + (p + 1)/2$, then*

(a) *through each point of $\mathcal{B}$ there are exactly $(p+3)/2$ lines that are not t-secants*;

(b) *through each point of $\mathcal{B}$ there are exactly $(p - 1)/2$ lines that are t-secants*;

(c) *the total number of t-secants is $\mu = l(p - 1)/(2t)$.*

Lemma 2.9 ([37], Chapter 12). *For any set of k points in* $\mathrm{PG}(2, q)$, *the following hold:*

$$\sum_{i=0}^{q+1} \tau_i = q^2 + q + 1; \tag{2.1}$$

$$\sum_{i=1}^{q+1} i\tau_i = |\mathcal{B}|(q + 1); \tag{2.2}$$

$$\sum_{i=2}^{q+1} i(i - 1)\tau_i = |\mathcal{B}|(|\mathcal{B}| - 1). \tag{2.3}$$

Here, τ_i is the number of i-secants to $\mathcal{B}$.

2.2 Known Lower Bounds for $\mathcal{B}$

According to [38], Table 2.1 gives lower bounds on the number of points in a t-blocking set $\mathcal{B}$ of $\mathrm{PG}(2, q)$ without conditions on the t-blocking sets. Also, Table 2.2 gives lower bounds on the number of points in a t-blocking set $\mathcal{B}$ of $\mathrm{PG}(2, q)$ with conditions on the t-blocking sets.

TABLE 2.1: Lower bounds for t-blocking sets without condition

q	t	$\|\mathcal{B}\| \geq$	Sharp for	References
$q = p$ prime, $p > 3$	$t < p/2$	$(2t+1)(p+1)/2$	$t = 1, (p-1)/2$	[4]
$q = p$ prime, $p > 3$	$t > p/2$	$(t+1)p$	$t = (p+1)/2$	[4]
$q < 9$	2	$3q$	$t = 2$	[9], [25], [42]
$q = 11, 13, 17, 19$	2	$(5q+7)/2$		[4], [6]
q square, $q > 4$	2	$2q + 2\sqrt{q} + 2$	$t = 2$	[4], [6]
$q = p^{2e+1}, q > 19$	2	$2q + p^e \lfloor \frac{p^{e+1}+1}{p^e+1} \rfloor + 2$		[4], [6]
$q = 5, 7, 9$	3	$4q$	$t = 3$	[5], [25], [42]
$q = 8$	3	31	$t = 3$	[35],
$q = 11, 13, 17$	3	$(7q+9)/2$		[4], [5]
q odd square, $q > 21$	3	$3q + 3\sqrt{q} + 3$	$t = 3$	[4], [5]
$q = p^{2e+1}, q > 17$	2	$3q + p^e \lfloor \frac{p^{e+1}+1}{p^e+1} \rfloor + 3$		[4], [5]
q even square, $q > 4$,	3	$3q + 2\sqrt{q} + 3$		[4], [5]
or $q \in \{25, 49, 81, 121\}$				

TABLE 2.2: Lower bounds for t-blocking sets with condition

q	t	Condition	$\|\mathcal{B}\| \geq$	References
q	t	$\mathcal{B}$ does not contain a line	$tq + \sqrt{tq} + 1$	[4]
q	t	$\mathcal{B}$ contains a line and $(t-1, q) = 1$	$q(t+1)$	[10]
q	t	$\mathcal{B}$ contains a line and $(t-1, q) > 1, t \leq q/2 + 1$	$tq + q - t + 2$	[17]
q	t	$\mathcal{B}$ contains a line and $(t-1, q) > 1, t \geq q/2 + 1$	$t(q+1)$	[4]

2.3 Non-existence of some (k, n)-Arcs in $\mathrm{PG}(2, q)$

In this section, the non-existence of (k, n)-arcs in $\mathrm{PG}(2, q)$ is proved for q=19, 23, 43 and $n > \frac{1}{2}(q + 3)$.

2.3.1 Non-existence of some arcs in $\mathrm{PG}(2, 19)$

Theorem 2.10. (1) *There exists no* $(211, 12)$*-arc in* $\mathrm{PG}(2, 19)$*; so* $m_{12}(2, 19) \leq 210$.

(2) *There exists no* $(231, 13)$*-arc in* $\mathrm{PG}(2, 19)$*; so* $m_{13}(2, 19) \leq 230$.

(3) *There exists no* $(291, 16)$*-arc in* $\mathrm{PG}(2, 19)$*; so* $m_{16}(2, 19) \leq 290$.

Proof. (1) Finding a maximum $(k, 12)$-arc in $\mathrm{PG}(2, 19)$ is equivalent to finding a minimum 8-fold blocking set. Theorem 2.4 implies that $\mathcal{B}$ must have at least 170 points. Theorem 2.8 gives that the total number of 8-secants is $(170 * 18)/16$ which is not an integer number. Therefore a $(211, 12)$-arc does not exist and

$$m_{12}(2, 19) \leq 210.$$

The other bounds are proved in the same way. □

Theorem 2.11. (1) *There exists no* $(251, 14)$*-arc in* $\mathrm{PG}(2, 19)$*; so* $m_{14}(2, 19) \leq 250$.

(2) *There exists no* $(271, 15)$*-arc in* $\mathrm{PG}(2, 19)$*; so* $m_{15}(2, 19) \leq 270$.

(3) *There exists no* $(311, 17)$*-arc in* $\mathrm{PG}(2, 19)$*; so* $m_{17}(2, 19) \leq 310$.

(4) *There exists no* $(331, 18)$*-arc in* $\mathrm{PG}(2, 19)$*; so* $m_{18}(2, 19) \leq 330$.

Proof. (1) Finding a maximum $(251, 14)$-arc is equivalent to finding a $\{130, 6\}$-blocking set $\mathcal{B}$. Theorem 2.8 implies that the total number of 6-secants is 195. Let r be the length of the longest secant. If $r = 20$, then $\mathcal{B}$ contains a line and Theorem 2.5 implies that $|\mathcal{B}| \geq 133$, a contradiction. If $16 < r \leq 19$ then considering lines through a point on the longest secant but not in $\mathcal{B}$, so $\mathcal{B}$ must have at least $6 * 19 + r$ points. This contradicts that $|\mathcal{B}| = 130$.

Consider the intersection of the 6-secants through $P \notin \mathcal{B}$ with the longest secant. So,

$$\tau_6 \geq 9r + (20 - r)(19 - i). \tag{2.4}$$

The values of τ_6 are calculated from (2.4) for $i \leq 8$, and give Table 2.3.

TABLE 2.3: The values of τ_6 for $i \leq 8$

r	16	15	14	13	12	11	10	9	8
i	0	1	2	3	4	5	6	7	8
$\tau_6 \geq$	220	225	228	229	228	225	220	213	204

This shows that all values of τ_6 for $r = 8, \ldots, 16$ give contradictions. This is because the total number of 6-secants is 195. For $r = 6, 7$, Lemma 2.9 gives the following:

$$\begin{aligned} \tau_6 + \tau_7 &= 381, \\ 6\tau_6 + 7\tau_7 &= 2600, \\ 3\tau_6 + 42\tau_7 &= 16770. \end{aligned}$$

There is no solution for this system. Therefore, no 130 point 6-blocking set exists and hence no $(251, 14)$-arc exists.

The remaining cases are proved similarly. $\square$

2.3.2 Non-existence of some arcs in $\mathrm{PG}(2, 23)$

Theorem 2.12. (1) *There exists no* $(301, 14)$*-arc in* $\mathrm{PG}(2, 23)$*; so* $m_{14}(2, 23) \leq 300$.

(2) *There exists no* $(325, 15)$*-arc in* $\mathrm{PG}(2, 23)$*; so* $m_{15}(2, 23) \leq 324$.

(3) *There exists no* $(349, 16)$*-arc in* $\mathrm{PG}(2, 23)$*; so* $m_{16}(2, 23) \leq 348$.

(4) *There exists no* $(373, 17)$*-arc in* $\mathrm{PG}(2, 23)$*; so* $m_{17}(2, 23) \leq 372$.

(5) *There exists no* $(421, 19)$*-arc in* $\mathrm{PG}(2, 23)$*; so* $m_{19}(2, 23) \leq 420$.

Proof. (1) Finding a maximum $(k, 14)$-arc in $\mathrm{PG}(2, 23)$ is equivalent to finding a minimum 10-fold blocking set. Theorem 2.4 implies, since 23 is prime, that such a set must have at least 252 points. Theorem 2.8 shows that the total number of 10-secants is $(252 * 22)/20$ which is not an integer. Therefore there exists no $(301, 14)$-arc in $\mathrm{PG}(2, 23)$ and $m_{14}(2, 23) \leq 300$. In the same way, the other bounds are shown. □

Theorem 2.13. (1) *There exists no* $(397, 18)$*-arc in* $\mathrm{PG}(2, 23)$*; so* $m_{18}(2, 23) \leq 396$.

(2) *There exists no* $(445, 20)$*-arc in* $\mathrm{PG}(2, 23)$*; so* $m_{20}(2, 23) \leq 444$.

(3) *There exists no* $(469, 21)$*-arc in* $\mathrm{PG}(2, 23)$*; so* $m_{21}(2, 23) \leq 468$.

(4) *There exists no* $(493, 22)$*-arc in* $\mathrm{PG}(2, 23)$*; so* $m_{22}(2, 23) \leq 492$.

Proof. (1) Finding a maximum $(397, 18)$-arc is equivalent to finding a $\{156, 6\}$-blocking set $\mathcal{B}$. Theorem 2.8 implies that the total number of 6-secants is 286. Let r be the length of the longest secant. If $r = 24$, then $\mathcal{B}$ contains a line and Theorem 2.5 can be applied. It follows from Theorem 2.5 that $|\mathcal{B}| \geq 161$, a contradiction. If $18 < r \leq 23$ then consider lines through a point on the longest secant but not in $\mathcal{B}$. Since $\mathcal{B}$ must have at least $6 * 23 + r$ points, then the values $18 < r \leq 23$ do not give $|\mathcal{B}|$.

Consider the intersection of the 6-secants through $P \notin \mathcal{B}$ with the longest secant. So,

$$\tau_6 \geq 11r + (24 - r)(23 - i). \tag{2.5}$$

The values of τ_6 are calculated from (2.5) for $i = 0, \ldots, 10$, and give Table 2.4.

TABLE 2.4: The values of τ_6 for $i \leq 10$

r	18	17	16	15	14	13	12	11	10	9	8
i	0	1	2	3	4	5	6	7	8	9	10
$\tau_6 \geq$	336	341	344	345	344	341	336	329	320	309	296

This shows that all values of τ_6 for $r = 8, \ldots, 18$ give contradictions. This is because the total number of 6-secants is 286. For $r = 6, 7$, Lemma 2.9 gives

$$\begin{aligned} \tau_6 + \tau_7 &= 553, \\ 6\tau_6 + 7\tau_7 &= 3744, \\ 3\tau_6 + 42\tau_7 &= 24180. \end{aligned}$$

There is no solution for this system. So, no 156 point 6-blocking set exists and hence no $(397, 18)$-arc exists.

The proof of the remaining cases is similar. □

2.3.3 Non-existence of some arcs in $\mathrm{PG}(2, 43)$

Theorem 2.14. *In* $\mathrm{PG}(2, 43)$ *there exists no* (k, n)*-arc for the following cases.*

Case I:

TABLE 2.5: The values of n that make μ a non-integer

k	991	1035	1079	1123	1167	1211	1299	1343	1431	1475
n	24	25	26	27	28	29	31	32	34	35
$m_n(2, 43) \leq$	990	1034	1078	1122	1166	1210	1298	1342	1430	1474
k	1519	1651	1695							
n	36	39	40							
$m_n(2, 43) \leq$	1518	1650	1694							

Case II:

TABLE 2.6: The values of n that make μ an integer

k	1255	1387	1563	1605	1739	1783
n	30	33	37	38	41	42
$m_n(2, 43) \leq$	1254	1386	1562	1606	1738	1782

Proof. (1) The largest $(k, 24)$-arc in PG$(2, 43)$ is equivalent to the smallest 20-blocking set. According to Theorem 2.4, a 20-blocking set should have at least 920 points. From Theorem 2.8, there are 23 lines not on 20-secants while there are 21 lines which are 20-secants. Then μ is not n integer. Therefore there exist no $(991, 24)$-arc in PG$(2, 43)$ and $m_{24}(2, 43) \leq 990$.

Similarly, the remaining cases are proved.

(2) Finding a maximum $(1255, 30)$-arc is equivalent to finding a $\{638, 14\}$-blocking set $\mathcal{B}$. Theorem 2.8 implies that the total number of 14-secants is 957. Let r represent the length of the longest secant. If $r = 44$, then $\mathcal{B}$ contains a line and Theorem 2.5 can be applied. It follows from Theorem 2.5 that $|\mathcal{B}| \geq 645$, a contradiction. If $37 \leq r \leq 43$ then consider lines through a point on the longest secant but not in $\mathcal{B}$. So, $\mathcal{B}$ must have at least $14 * 43 + r$ points, a contradiction.

Consider the intersection of the 14-secants through $P \notin \mathcal{B}$ with the longest secant. So,

$$\tau_{14} \geq 21r + (44 - r)(43 - i). \tag{2.6}$$

The values of τ_{14} are calculated according to (2.6) as shown in Table 2.7.

TABLE 2.7: The values of τ_{14} for $i \leq 22$

r	36	35	34	33	32	31	30	29	28	27	26	25
i	0	1	2	3	4	5	6	7	8	9	10	11
$\tau_{14} \geq$	1100	1113	1124	1133	1140	1145	1148	1149	1148	1145	1140	1133
r	24	23	22	21	20	19	18	17	16	15	14	
i	12	13	14	15	16	17	18	19	20	21	22	
$\tau_{14} \geq$	1124	1113	1100	1085	1068	1049	1028	1005	980	953	924	

This shows that all values of τ_{14} for $r = 16, ..., 36$ and $i = 0, \ldots, 20$ give a contradiction. This is because the total number of 14-secants is 957.

For $r = 14$ and $r = 15$ then from Lemma 2.9 the standard equations for the set B are the following:

$$\begin{aligned} \tau_{14} + \tau_{15} &= 1893, \\ 14\tau_{14} + 15\tau_{15} &= 28072, \\ 182\tau_{14} + 210\tau_{15} &= 406406. \end{aligned}$$

There is no solution of this system. So no 638-point 14-blocking set exists and hence no $(1255, 30)$-arc exists.

The remaining cases are proved similarly. □

2.4 New Largest Bound

Theorem 2.15. *For $\frac{1}{2}(q+3) < n < q$, with q prime,*

$$m_n(2, q) \leq \frac{(q+1)(2n-3)}{2}.$$

Proof. From Theorem 2.3, a (k, n)-arc satisfies

$$k \leq (q+1)(n - \tfrac{3}{2}) + 1.$$

Suppose that there exists a $((q+1)(n - \frac{3}{2}) + 1, n)$-arc $\mathcal{K}$.

Let $\mathcal{B}$ be an $\{l, t\}$-blocking set that is the complement of $\mathcal{K}$. Since $l = q^2 + q + 1 - k$ and $t = q + 1 - n$, so

$$l \geq (q+1)(q - n + \tfrac{3}{2}).$$

This implies that $\mathcal{B}$ is a $(q+1)(q - n + \frac{3}{2}), q + 1 - n$-blocking set, and

$$\begin{aligned} |\mathcal{B}| &= t(q+1) + \tfrac{1}{2}(q+1) \\ &= (q+1)(q - n + \tfrac{3}{2}). \end{aligned} \tag{2.7}$$

Let T be the total number of t-secants of $\mathcal{B}$. From Theorem 2.7, $f(x) = x^q v(x) + w(x)$. Since $|\mathcal{B}| = (q+1)(\frac{1}{2} + t)$, then the lacunary polynomial from a point of $\mathcal{B}$ is $xv(x) * qw(x)$

which satisfies $f(x) = (vx + w)(v_1 w - w_1 v)$. This implies that the number of different factors in $f(x)$ is precisely $(q + 3)/2$. So, each point of $\mathcal{B}$ is incident with precisely $(q - 1)/2$ t-secants. Then count $\{(x, L)\}$, where x is in $\mathcal{B}$ and L is a t-secant. So $Tt = |\mathcal{B}|(q - 1)/2$; this implies that $T = |\mathcal{B}|(q - 1)/(2t)$. Then

$$\begin{aligned} T &= (q+1)(t+\tfrac{1}{2})(q-1)/(2t) && (2.8)\\ &= (q^2-1)(2t+1)/(4t). && (2.9)\end{aligned}$$

If $\mu = (q^2 - 1)(2t + 1)/(4t)$ is not an integer, this implies that there exists no

$$((q+1)(n-\frac{3}{2})+1, n) - \text{arc}.$$

So

$$m_n(2, q) \leq (q+1)(n-\frac{3}{2}).$$

Suppose that μ is an integer. Let L be an r-secant and $P \in L \backslash \mathcal{B}$, where r is the largest number of points of $\mathcal{B}$ on any line through P. If there are s lines that are t-secants to $\mathcal{B}$, and s_i through P that are $(t + i)$-secants, for $1 \leq i \leq m$, then

$$\begin{aligned} |\mathcal{B}| &= st + s_1(t+1) + s_2(t+2) + \cdots + s_m(t+m) + r\\ &= st + s_1(t+1) + s_2(t+1) + s_2 + \cdots + s_m(t+1) + (m-1)s_m + r\\ &= st + (s_1 + s_2 + \cdots + s_m)(t+1) + s_2 + 2s_3 + \cdots + (m-1)s_m + r\\ &= st + s'(t+1) + s'' + r,\end{aligned}$$

where $s + s' = q$ and $s' = s_1 + s_2 + \cdots + s_m, s'' = s_2 + 2s_3 + \cdots + (m-1)s_m$. So,

$$st + (q-s)(t+1) + r \leq |\mathcal{B}|.$$

This implies that

$$s \geq q(t+1) + r - |\mathcal{B}|.$$

Since $|\mathcal{B}| = t(q+1) + (q+1)/2 = (t+\frac{1}{2})(q+1)$, so

$$\begin{aligned} s &\geq r + q(t+1) - (t+\tfrac{1}{2})(q+1) \\ &\geq (q-1)/2 + r - t. \end{aligned} \tag{2.10}$$

Now, the number of t-secants is at least $s(q+1-r) + r(q-1)/2$. So

$$l(q-1)/(2t) \geq r(q-1)/2 + (\tfrac{1}{2}(q-1) + r - t)(q+1-r) \tag{2.11}$$
$$\geq (q^2-1)/2 + (r-t)(q+1-r). \tag{2.12}$$

So, the Inequality (2.12) becomes

$$r^2 - r(q+1+t) + t(q+1) - (q^2-1)/(4t) \geq 0. \tag{2.13}$$

To solve (2.13) for $r = t, t+1, \ldots, q+1$, the values of r can be divided into the following cases.

(1) When $r = q+1$, then $\mathcal{B}$ contains a line, and Theorem 2.5 implies that $|\mathcal{B}| \geq q(t+1)$, a contradiction.

(2) When $\frac{1}{2}(q+1) + t < r \leq q$, then consider lines through a point on the longest secant but not on $\mathcal{B}$. So $\mathcal{B}$ must have at least $qt + r$ points. This contradicts that $|\mathcal{B}| = (t+\frac{1}{2})(q+1)$.

(3) When $t+2 \leq r \leq \frac{1}{2}(q+1)+t$, then, since $t = q+1-n$, $n > \frac{1}{2}(q+3)$, so $q > 2t+1$. Let $f(r) = r^2 - (q+1+t)r + t(q+1) + (q^2-1)/(4t)$, so

$$\begin{aligned} f(r) &> r^2 - (3t+2)r + (2t+1)(t+1), \\ &> r^2 - (3t+2)r + (\tfrac{1}{2}(3t+2))^2 - (\tfrac{1}{2}(3t+2))^2 + (2t+1)(t+1), \\ &> (r - \tfrac{1}{2}(3t+2))^2 - t^2/4. \end{aligned}$$

Here, $f(r)$ is positive for some values of t and negative for others.

Therefore (2.13) is not true for $t+2 \leq r \leq \frac{1}{2}(q+1)+t$.

(4) When $t \leq r \leq t+1$, according to Lemma 2.9,

$$\begin{aligned} \tau_t + \tau_{t+1} &= q^2+q+1; & (2.14)\\ t\tau_t + (t+1)\tau_{t+1} &= |\mathcal{B}|(q+1); & (2.15)\\ t(t-1)\tau_t + t(t+1)\tau_{t+1} &= |\mathcal{B}|(|\mathcal{B}|-1). & (2.16) \end{aligned}$$

Multiplying Equation (2.14) by t and subtracting Equation (2.15) gives

$$\begin{aligned} \tau_{t+1} &= |\mathcal{B}|(q+1) - t(q^2+q+1)\\ &= \tfrac{1}{2}(q^2+2q+2qt+1); & (2.17)\\ \tau_t &= q^2+q+1+t(q^2+q+1) - |\mathcal{B}|(q+1)\\ &= \tfrac{1}{2}(q^2-2qt+1). & (2.18) \end{aligned}$$

Substituting the values of τ_t and τ_{t+1} in (2.16) implies that

$$\begin{aligned} t(t-1)\tau_t + t(t+1)\tau + t + 1 &= tq + t^2(q^2+3q+1)\\ &\neq |\mathcal{B}|(|\mathcal{B}|-1). \end{aligned}$$

Therefore, there exists no $((q+1)(n-\frac{3}{2})+1, n)$-arc in $\mathrm{PG}(2,q)$ for $n > (q+3)/2$. Hence

$$m_n(2,q) \leq (q+1)(n-\tfrac{3}{2}) \quad \text{for } n > \tfrac{1}{2}(q+3).$$

□

2.5 Application of Theorem 2.15

Case I : Bounds for complete (k,n)-arcs when μ is a non-integer.

Theorem 2.16. *In* $\mathrm{PG}(2,47)$, *there exist no* (k,n)*-arc for the following values of* k, *giving corresponding upper bounds for* $m_n(2,47)$.

TABLE 2.8: The values of n when μ is not an integer

k	1177	1225	1273	1321	1369	1417	1465	1513
n	26	27	28	29	30	31	32	33
$m_n(2,47) \leq$	1176	1224	1272	1320	1368	1416	1464	1512
k	1561	1609	1705	1753	1801	1897	1993	
n	34	35	37	38	39	41	43	
$m_n(2,47) \leq$	1560	1608	1704	1752	1800	1896	1992	

Proof. For $k = 1177$ and $n = 26$, then l=1080, t=22;

$$\begin{aligned} |\mathcal{B}| &= t(q+1) + \tfrac{1}{2}(q+1) \\ &= 22 * 48 + 44 \\ &= 1080. \end{aligned}$$

This implies that $|\mathcal{B}| = l$.

Assume that the total number of t-secants is T. Then, from (2.9),

$$\begin{aligned} T &= |\mathcal{B}|(q-1)/2t \\ &= (1080 * 46)/44. \end{aligned}$$

As $\mu = (1080 * 46)/44$ is not an integer, then there exists no $(1177, 26)$-arc. So

$$m_{26}(2,47) \leq 1176.$$

The remaining cases are proved similarly. □

Case II: Bounds for complete (k,n)-arcs when T is integer.

Theorem 2.17. *In* PG$(2,47)$ *there exists no* (k,n)*-arc for the following values of* k*. Hence the upper bound for* $m_n(2,47)$ *is established in the corresponding cases.*

TABLE 2.9: The values of n when μ is integer

k	1657	1849	1945	2041	2089	2137
n	36	40	42	44	45	46
$m_n(2,47) \leq$	1656	1848	1944	2040	2088	2136

Proof. Finding a $(1657, 36)$-arc is equivalent to finding a $\{600, 12\}$-blocking set $\mathcal{B}$. The total number of 12-secants is 1150. Let r be the length of the longest secant. If $r = 48$, then $\mathcal{B}$ contains a line and $|\mathcal{B}| \geq 611$, a contradiction. If $35 \leq r \leq 47$, considering lines through a point on the longest secant but not in $\mathcal{B}$, then $\mathcal{B}$ must have at least $12 * 47 + r$ points. This contradicts that $|\mathcal{B}| = 600$.

Now, consider the intersection of the 12-secants through $P \notin \mathcal{B}$ with the longest secant. Then (2.13) becomes

$$\tau_{12} \geq (r - 12)(48 - r). \tag{2.19}$$

The lower bounds for τ_{12} are calculated according to (2.19) as shown in Table 2.10.

TABLE 2.10: The values of $14 \leq r \leq 36$

r	36	35	34	33	32	31	30	29	28	27	26	25
$\tau_{12} \geq$	288	299	308	315	320	323	324	323	320	315	308	299
r	24	23	22	21	20	19	18	17	16	15	14	
$\tau_{12} \geq$	288	275	260	243	224	203	180	155	128	99	68	

This shows that all values of r for $r = 14, \ldots, 36$ give a contradiction. This is because the total number of 12-secants is 46.

However, for $r = 12$ and $r = 13$, Equations (2.1), (2.2), (2.3) of Lemma 2.9 become the following:

$$\begin{aligned} \tau_{12} + \tau_{13} &= 2257, \\ 12\tau_{12} + 13\tau_{13} &= 28800, \\ 132\tau_{12} + 156\tau_{13} &= 359400. \end{aligned}$$

As there is no solution of this system, so no $\{600, 12\}$-blocking set exists and hence no $(1657, 36)$-arc exists.

The other bounds are established similarly. $\square$

Here, all of the largest bounds which were found by Theorem 2.15 are the same as known bounds as in [7].

2.6 Chapter Summary

This chapter explains some basic equations for (k, n)-arcs. Also, the non-existence of (k, n)-arcs in $\mathrm{PG}(2, q)$ is proved for $q = 19, 23, 43$ and $n > \frac{1}{2}(q + 3)$. Finally, a new upper bound for k is proved and applied to $\mathrm{PG}(2, 47)$.

Chapter 3

Classification of (k, n)-arcs in $\mathrm{PG}(2, q)$

3.1 Introduction

Many studies have been done to find all complete $(k, 2)$ and $(k, 3)$-arcs in $\mathrm{PG}(2, q)$ for some values of q. The size of the largest and the second largest complete $(k, 2)$-arc are denoted by $m(2, q)$ and $m'(2, q)$ as in Notation 1.25. Theorem 1.27 shows the value of $m(2, q)$, but the value of $m'(2, q)$ is not known in general.

The full classification of k-arcs in $\mathrm{PG}(2, q)$ for $q \leq 19$ is shown in [37]. Sticker [26], [27] obtained the full classification of k-arcs in $\mathrm{PG}(2, 23)$, $\mathrm{PG}(2, 25)$, $\mathrm{PG}(2, 27)$. Coolsaet [29] obtained the classification of k-arcs in $\mathrm{PG}(2, 31)$, in 2014. The values of $m'(2, q)$ in $\mathrm{PG}(2, q)$ have been shown by Chao and Kaneta [23] to be $21, 22, 24$ for $q = 25, 27, 29$. The classification of all arcs of size $k \geq q - 8$ has been calculated by Kéri [41] for values of $q \leq 32$.

The classification of $(k, 3)$-arcs in $\mathrm{PG}(2, q)$ for $q \leq 9$ has been done in 2001 by Marcugini et al. [45]. Here the largest size of complete arc has been found to be 21 in $\mathrm{PG}(2, 11)$ and 23 in $\mathrm{PG}(2, 13)$, while the smallest size is found to be 15 in $\mathrm{PG}(2, 13)$. They also found that there is a complete $(k, 3)$-arc for each k, where $15 \leq k \leq 23$, [44], [47]. Bartoli [8] showed that the smallest and largest $(k, 3)$-arc in $\mathrm{PG}(2, 16)$ have sizes 15 and 28. The size of the smallest complete (k, n)-arc in $\mathrm{PG}(2, q)$ is denoted by $t_n(2, q)$.

Many attempts have been done to find a general lower bound of $t_2(2, q)$ in $\mathrm{PG}(2, q)$. In 1959 Segre [56] showed that $t_2(2, q) < \sqrt{2q} + 1$. Furthermore, Ball [3] found that

$t_2(2,q) < \sqrt{3q} + \frac{1}{2}$. Regarding $(k,3)$-arcs, a bound for the smallest size $t_3(2,q)$ of a complete $(k,3)$-arc is

$$t_3(2,q) \leq \frac{q - 8 + \sqrt{24q^3 - 23q^2 - 40q + 16}}{2(q-2)},$$

as indicated by Marcugini et al. [47]. Hirschfeld and Pichanick [39] found that

$$t_n(2,q) \leq \sqrt{n(n-1)(q+1)}.$$

A new lower bound for the smallest complete (k,n)-arc in PG$(2,q)$ has been found in Theorem 3.3. This general bound can be applied for $t_2(2,q)$ and $t_3(2,q)$.

In this chapter, the classification of (k,n)-arcs in PG$(2,q)$ for some q has been done using four different methodologies. Also a comparison among these methodologies has been done to show which method is best according to the time of implementation to get the final result.

3.2 Known Results for $m_n(2,q)$ and $t_n(2,q)$ in PG$(2,q)$

Many studies have been done to find the largest and the smallest complete (k,n)-arcs in PG$(2,q)$. Some of these studies focused on the full classification of k-arcs in PG$(2,q)$.

3.2.1 The classification of k-arcs in PG$(2,q)$ for $q \leq 29$

The full classification of k-arcs in PG$(2,q)$ are given in Table 3.1, Table 3.2 and Table 3.3 for $q \leq 29$. For $q \leq 8$, see [37]. For $q = 9$, see [49]. For $q = 11$, see [54]. For $q = 13$, see [2]. For $q = 17, 19$, see [21]. For $q = 23, 25$, see [26]. For $q = 27$, see [27]. For $q = 29$, see [22]. For $q = 31$, see [29]. For $q = 32$, see [48]. Table 3.4 gives the values of $t_2(2,q)$ for $2 \leq q \leq 29$.

Here η is the number of projectively-distinct complete arcs of this size. In each table the entry gives the number of projectively distinct arcs for the corresponding k and q.

TABLE 3.1: The classification of k-arcs in $\mathrm{PG}(2,q)$ for $4 \leq k \leq 18$ and $5 \leq q \leq 17$

	$q=5$	$q=7$	$q=8$	$q=9$	$q=11$	$q=13$	$q=16$	$q=17$
$k=4$	1	1	1	1	1	1	1	1
$k=5$	1	1	1	2	2	3	4	4
$k=6$	1	3	5	7	15	26	61	74
$k=7$		1	2	4	21	80	454	733
$k=8$		1	2	2	21	181	2633	5441
$k=9$			2	1	5	110	6014	17633
$k=10$			1	1	2	27	4899	21064
$k=11$					1	2	1171	6814
$k=12$					1	2	587	629
$k=13$						1	260	15
$k=14$						1	100	4
$k=15$							30	1
$k=16$							9	1
$k=17$							3	1
$k=18$							2	1

TABLE 3.2: The classification of k-arcs in $\mathrm{PG}(2,q)$ for $4 \leq k \leq 7$ and $19 \leq q \leq 29$

	$q=19$	$q=23$	$q=25$	$q=27$	$q=29$
$k=4$	1	1	1	1	1
$k=5$	5	6	8	4	10
$k=6$	117	257	365	174	682
$k=7$	1768	7613	14114	8261	41301

TABLE 3.3: The classification of k-arcs in PG$(2,q)$ for $8 \le k \le 30$ and $19 \le q \le 29$

	$q=19$	$q=23$	$q=25$	$q=27$	$q=29$
$k=8$	20361	172416	419385	311313	1933469
$k=9$	115492	2235523	7490938	7348659	58423579
$k=10$	280104	15032508	74026338	101047498	1072049736
$k=11$	235320	46333282	366007216	744145433	11123944005
$k=12$	55708	56846595	806719354	2665334400	60140705285
$k=13$	2733	23362684	690593155	4145194407	153994534160
$k=14$	83	2634266	195308347	2452359922	167238862321
$k=15$	5	64773	15070303	472714330	67799467128
$k=16$	4	692	263843	24808360	8854773945
$k=17$	1	41	1492	290532	314349510
$k=18$	1	22	222	1431	2540088
$k=19$	1	6	58	183	7280
$k=20$	1	4	29	82	1477
$k=21$		1	9	32	646
$k=22$		1	5	15	293
$k=23$		1	1	4	98
$k=24$		1	1	3	43
$k=25$			1	1	10
$k=26$			1	1	5
$k=27$				1	1
$k=28$				1	1
$k=29$					1
$k=30$					1

TABLE 3.4: Size and number of the smallest complete k-arc, $q \leq 29$

q	2	3	4	5	7	8	9	11	13	16	17	19	23	25	27	29
$t_2(2, q)$	4	4	6	6	6	6	6	7	8	9	10	10	10	12	12	13
η	1	1	2	1	2	4	2	1	2	8	560	29	1	606	7	708

3.2.2 The classification of $(k, 3)$-arcs in PG$(2, q)$ for $q \leq 13$

The full classification of $(k, 3)$-arcs in PG$(2, q)$ are given in Table 3.5 and Table 3.6 for $q \leq 13$. For $q = 5$, see [28]; for $q = 7$, see [46]; for $q = 8$, see [43], [60]; for $q = 9$, see [45]; for $q = 11$, see [44]; for $q = 13$, see [47]. Table 3.7 gives the values of $t_3(2, q)$ for $q \leq 13$.

TABLE 3.5: The classification of $(k, 3)$-arcs in PG$(2, q)$ for $4 \leq k \leq 15$ and $5 \leq q \leq 13$

	$q = 5$	$q = 7$	$q = 8$	$q = 9$	$q = 11$	$q = 13$
$k = 4$	1	1	1	1	1	1
$k = 5$	2	3	2	3	3	4
$k = 6$	7	14	15	24	37	62
$k = 7$	13	53	98	188	543	1349
$k = 8$	13	180	505	1341	6743	25670
$k = 9$	16	526	2248	8231	70550	405813
$k = 10$	7	907	6680	36572	574775	5175900
$k = 11$	2	923	12664	111833	3520994	52242281
$k = 12$		395	12781	209172	15291647	403124641
$k = 13$		65	5822	211818	44020760	2282452774
$k = 14$		4	871	97050	76936027	9001288812
$k = 15$		1	43	16386	73157838	23188169036

TABLE 3.6: The classification of $(k, 3)$-arcs in $\mathrm{PG}(2, q)$ for $16 \leq k \leq 23$, $5 \leq q \leq 13$

	$q = 5$	$q = 7$	$q = 8$	$q = 9$	$q = 11$	$q = 13$
$k = 16$				734	32916332	36058738738
$k = 17$				6	5884405	30742092308
$k = 18$					333585	12779923892
$k = 19$					4467	2246238494
$k = 20$					17	140208097
$k = 21$					2	2507054
$k = 22$						9805
$k = 23$						7

TABLE 3.7: Number of $t_3(2, q), 5 \leq q \leq 13$

q	5	7	8	9	11	13	16
$t_3(2, q)$	9	9	11	12	13	15	15
η	2	1	2	4	5	33	1

Theorem 3.1 ([47])**.** *Let* $\mathcal{K}$ *be a complete* $(k, 3)$*-arc in* $\mathrm{PG}(2, q)$*. Then*

$$k \geq \frac{q - 8 + \sqrt{24q^3 - 23q^2 - 40q + 16}}{2(q - 2)}.$$

Theorem 3.2 ([39])**.** *Let* $\mathcal{K}$ *be a complete* (k, n)*-arc in* $\mathrm{PG}(2, q)$*, where* $n \geq 2$ *and* $q \geq n$*. Then*

$$k \geq \sqrt{n(n-1)(q+1)}.$$

3.3 New Lower Bounds

This section shows some new lower bounds for the smallest complete (k, n)-arcs $\mathcal{K}$ in $\mathrm{PG}(2, q)$. A comparison among Theorem 3.1, Theorem 3.2 and Theorem 3.3 is given.

Theorem 3.3. *In* PG$(2,q)$*, a complete* (k,n)*-arc does not exist for* $k \le k^*$*, where*

$$k^* = \frac{(q+1-n^2) + \sqrt{(q+1-n^2)^2 + 4(n^2-n)(q+1-n)(q^2+q+1)}}{2(q+1-n)}.$$

Proof. From Lemma 1.30, Equations (1.4) and (1.5) are as follows:

$$\begin{aligned} \rho_1 + \rho_2 + \cdots + \rho_n &= q+1, \\ \rho_2 + 2\rho_3 + \cdots + (n-1)\rho_n &= k-1. \end{aligned}$$

(1) With $\lfloor d \rfloor$ the integer part of d, let $m = \left\lfloor \frac{k-1}{n-1} \right\rfloor$, $n > 2$.

So the maximum value that ρ_n can have is m.

(2) Let $r_{n-i} = \left\lfloor \frac{(k-1) - \sum_{v=1}^{i}(n-v)\rho_{n-v+1}}{(n-i)-1} \right\rfloor$. Here r_{n-i} gives the possible values for ρ_{n-i}.

The points of the plane are of the following types:

(i) $r_{n-1} = \left\lfloor \frac{(k-1)-(n-1)\rho_n}{(n-2)} \right\rfloor$ gives the possible values for ρ_{n-1}; the points are of type

$$(\rho_0, \rho_1, 0, \ldots, 0, \rho_{n-1}, j),$$

where $j = 1, 2, \ldots, m$.

(ii) $r_{n-2} = \left\lfloor \frac{(k-1)-(n-1)\rho_n-(n-2)\rho_{n-1}}{(n-3)} \right\rfloor$ gives the possible values for ρ_{n-2}; the points are of type

$$(\rho_0, \rho_1, 0, \ldots, 0, \rho_{n-2}, \rho_{n-1}, j),$$

where $j = 1, 2, \ldots, m$.

(iii) $r_2 = (k-1) - (n-1)\rho_n - (n-2)\rho_{n-1}, \ldots, 2\rho_3$ gives the possible values for ρ_2; the points are of type

$$(\rho_0, \rho_1, \rho_2, \ldots, \rho_{n-1}, j),$$

where $j = 1, 2 \ldots, m$.

Suppose that α denotes the number of points of PG$(2,q)$ of type $(r_2, r_3, \ldots, r_{n-1})$.

From Lemma 1.32, Equations (1.10) and (1.11),

$$\sum_{j=1}^{m} j \left(\sum_{\rho_{n-1}=0}^{m} \sum_{\rho_{n-2}=0}^{r_{n-1}} \cdots \sum_{\rho_2=0}^{r_2} \alpha(r_2, r_3, \ldots, r_{n-1}, j) \right) = n\tau_n, \tag{3.1}$$

$$\sum_{\rho_{n-1}=0}^{m} \sum_{\rho_{n-2}=0}^{r_{n-1}} \cdots \sum_{\rho_2=0}^{r_2} \alpha(r_2, r_3, \ldots, r_{n-1}, j) = k, \tag{3.2}$$

where τ_n is the total number of n-secants of a (k,n)-arc in PG$(2,q)$.

Since $m > 0$ for $k \geq n$, so

$$m \left(\sum_{\rho_{n-1}=0}^{m} \sum_{\rho_{n-2}=0}^{r_{n-1}} \cdots \sum_{\rho_2=0}^{r_2} \alpha(r_2, r_3, \ldots, r_n - 1, j) \right)$$
$$\geq \sum_{j=1}^{m} j \left(\sum_{\rho_{n-1}=0}^{m} \sum_{\rho_{n-2}=0}^{r_{n-1}} \cdots \sum_{\rho_2=0}^{r_2} \alpha(r_2, r_3, \ldots, r_n - 1, j) \right).$$

This implies that

$$\tau_n \leq \frac{mk}{n}. \tag{3.3}$$

Since

$$m \leq \frac{(k-1)}{(n-1)},$$

so

$$\tau_n \leq \frac{k(k-1)}{n(n-1)}. \tag{3.4}$$

On the other hand, if the (k,n)-arc $\mathcal{K}$ is complete, Lemma 1.33 implies that

$$\tau_n \geq \frac{q^2+q+1-k}{q+1-n}. \tag{3.5}$$

Now, from Equations (3.4) and (3.5),

$$\frac{k^2-k}{n^2-n} > \frac{q^2+q+1-k}{q+1-n}.$$

Hence

$$(q+1-n)k^2-(q+1-n)k > (n^2-n)(q^2+q+1)-(n^2-n)k,$$
$$(q+1-n)k^2-(q+1-n-n^2+n)k-(n^2-n)(q^2+q+1) > 0,$$
$$(q+1-n)k^2-(q+1-n^2)k-(n^2-n)(q^2+q+1) > 0. \qquad (3.6)$$

Now, Inequality (3.6) implies that $k = k^* > 0$. □

This can be applied to k-arcs and $(k,3)$-arcs, as in Table 3.8 and Table 3.9, with the notation $n^* = b_n(2,q)$ and $n = 2,3$.

TABLE 3.8: Bounds for complete k-arcs for $4 \le q \le 23$

q	4	5	7	8	9	11	13	16	17	19	23
$b_2(2,q)$	5	5	5	5	6	6	6	7	7	7	8
$t_2(2,q)$	6	6	6	6	6	7	8	9	10	10	10
$m_2(2,q)$	6	6	8	10	10	12	14	18	20	20	25

TABLE 3.9: Bounds for complete $(k,3)$-arcs for $4 \le q \le 16$

q	4	5	7	8	9	11	13	16
$b_3(2,q)$	7	8	9	9	9	10	11	12
$t_3(2,q)$	7	9	9	11	12	13	15	15
$m_3(2,q)$	9	11	15	15	17	21	23	28

3.3.1 Comparison with known results

(i) Table 3.10 gives the comparison, for $(k,3)$-arcs, among Theorem 3.1, Theorem 3.2 and Theorem 3.3, for $4 \le q \le 16$.

TABLE 3.10: Lower bounds for complete $(k,3)$-arcs for $4 \le q \le 16$

q	Theorem 3.1 k	Theorem 3.2 k	Theorem 3.3 k	Exact value k
4	7	6	7	7
5	8	6	8	9
7	9	7	9	9
8	9	8	9	11
9	10	8	10	12
11	10	9	10	13
13	11	10	11	15
16	12	11	12	15

(ii) From Theorem 3.3, when $n = 3$, this implies the same result as Theorem 3.1.

Theorem 3.4. *Let* $\mathcal{K}$ *be a* (k,n)*-arc in* $\mathrm{PG}(2,q)$*. When* $\tau_n \le k$*, the following statements hold*:

(1) $k \le q + 2 + n(n-2)$;

(2) *if* $\mathcal{K}$ *is complete, then* $q + n - 1 \le k \le n(n-2) + q + 2$.

Proof. From Lemma 1.29, Equations (1.2) and (1.3) are as follows:

$$\tau_1 + n\tau_n + \sum_{i=2}^{n-1} i\tau_i = k(q+1); \tag{3.7}$$

$$n(n-1)\tau_n + \sum_{i=2}^{n-1} i(i-1)\tau_i = k(k-1). \tag{3.8}$$

By subtracting Equation (3.7) from (3.8),

$$n(n-2)\tau_n = k(k-q-2) + \tau_1 - \sum_{i=2}^{n-1} i(i-2)\tau_i. \tag{3.9}$$

Since $k \geq \tau_n$, so

$$n(n-2)k \geq k(k-q-2) + \tau_1 - \sum_{i=2}^{n-1} i(i-2)\tau_i. \tag{3.10}$$

If $\tau_1 - \sum_{i=2}^{n-1} i(i-2)\tau_i \geq 0$, then

$$n(n-2) \geq k-q-2. \tag{3.11}$$

Now, Equation (3.11) implies that

$$k \leq q+2+n(n-2). \tag{3.12}$$

On the other hand, if the (k,n)-arc $\mathcal{K}$ is complete, Lemma 1.33 implies that

$$(q+1-n)\tau_n \geq q^2+q+1-k. \tag{3.13}$$

Since $k \geq \tau_n$, so

$$(q+1-n)k \geq q^2+q+1-k, \tag{3.14}$$

$$(q+2-n)k \geq q^2+q+1, \tag{3.15}$$

$$k \geq \frac{q^2+q+1}{q+2-n}, \tag{3.16}$$

$$k \geq q+n-1+\frac{n^2-3n+3}{q+2-n}, \tag{3.17}$$

$$k \geq q+n-1. \tag{3.18}$$

Equations (3.12) and (3.18) imply that

$$q+n-1 \leq k \leq n(n-2)+q+2. \tag{3.19}$$

□

Proposition 3.5. *In* PG$(2,q)$*, if* $n \mid q$ *and* q *even, then*

(1) $\tau_n = (q+1)^2 - q(q+1)/n$.

(2) $\tau_0 = q(q+1-n)/n$.

Proof. (1) From Theorem 1.28, Equations (1.1) and (1.2) of Lemma 1.29 be

$$\tau_0 + \tau_n = q^2 + q + 1 \tag{3.20}$$

$$n\tau_n = k(q + 1) \tag{3.21}$$

Since $k = (n - 1)q + n$, Equation (3.21) implies that

$$\begin{aligned} \tau_n &= \frac{(q+1)[(n-1)q+n]}{n} \\ &= (q+1)^2 - \frac{q(q+1)}{n}. \end{aligned}$$

(2) From Equation (3.20),

$$\begin{aligned} \tau_0 &= q^2 + q + 1 - (q+1)^2 + \frac{q(q+1)}{n} \\ &= \frac{q(q+1-n)}{n}. \end{aligned}$$

□

Table 3.11 shows a special case.

TABLE 3.11: Application of Proposition (3.5) for some values of n

n	2	$q/2$	q
$\lvert k\rvert$	$q+2$	$q(q-1)/2$	q^2
τ_0	$q(q-1)/2$	$q+2$	1
τ_n	$(q+1)(q+2)/2$	q^2-1	$q(q+1)$

Example 3.1. (i) $m_2(2, 8) = 10$ and the i-secant distribution is $(45, 0, 28)$.

(ii) $m_2(2, 16) = 18$ and the i-secant distribution is $(153, 0, 120)$.

(iii) $m_4(2, 8) = 28$ and the i-secant distribution is $(63, 0, 0, 0, 10)$.

(iv) $m_4(2, 16) = 52$ and the i-secant distribution is $(255, 0, 0, 0, 18)$.

(v) $m_8(2, 8) = 64$ and the i-secant distribution is $(72, 0, 0, 0, 0, 0, 0, 0, 1)$.

(vi) $m_8(2, 16) = 256$ and the i-secant distribution is $(272, 0, 0, 0, 0, 0, 0, 0, 1)$.

3.4 Construction of Complete k-Arcs in PG$(2, q)$ from a Quadrangle

The equation of a conic $\mathcal{C}$ is $v(x_0, x_1, x_2) = 0$, where

$$v(x_0, x_1, x_2) = a_1x_0^2 + a_2x_1^2 + a_3x_2^2 + a_4x_0x_1 + a_5x_0x_2 + a_6x_1x_2, \tag{3.22}$$

$a_i \in \mathbb{F}_q$. With $U_0, U_1, U_2 \in \mathcal{C}$, where $U_0 = P(1, 0, 0), U_1 = P(0, 1, 0), U_2 = P(0, 0, 1)$ then

$$a_4x_0x_1 + a_5x_0x_2 + a_6x_1x_2 = 0. \tag{3.23}$$

This is non-singular for $a_4a_5a_6 \neq 0$. By dividing Equation (3.23) by a_4 then

$$x_0x_1 + \frac{a_5}{a_4}x_0x_2 + \frac{a_6}{a_4}x_1x_2 = 0. \tag{3.24}$$

Let $\alpha = \frac{a_5}{a_4}$ and $\beta = \frac{a_6}{a_4}$, then Equation (3.24) becomes

$$x_0x_1 + \alpha x_0x_2 + \beta x_1x_2 = 0. \tag{3.25}$$

Substitute $P(1, 1, 1)$ in Equation (3.25), then $1 + \alpha + \beta = 0$ implies

$$\beta = -(1 + \alpha). \tag{3.26}$$

Assume $\alpha = 0$, so $\beta = -1$ substitute these values in Equation (3.25), obtains

$$x_0x_1 - x_1x_2 = 0,$$

which implies Equation (3.25) is degenerate, a contradiction. So, $\alpha \in \mathbb{F}_q \setminus \{0, -1\}$. Then, for q odd, the form of a conic containing the frame points is

$$\mathcal{C}_i = v(x_0x_1 + \alpha x_0x_2 - (1 + \alpha)x_1x_2); \tag{3.27}$$

where $i = 1, \ldots, q - 2$.

Table 3.12 illustrates a simple application of Equation (3.27).

TABLE 3.12: Application of Equation (3.27) in PG$(2,5)$

i	conic equation
1	$x_0x_1 + x_0x_2 + 3x_1x_2$
2	$x_0x_1 + 2x_0x_2 + 2x_1x_2$
3	$x_0x_1 + 3x_0x_2 + 1x_1x_2$

If there are two conics $\mathcal{C}' = v(x_0x_1+\alpha_1x_0x_2+\beta_1x_1x_2)$, $\mathcal{C}'' = v(x_0x_1+\alpha_2x_0x_2+\beta_2x_1x_2)$, where $\alpha_1+\beta_1+1=0, \alpha_2+\beta_2+1=0$, then the general form of the union of them is

$$\begin{aligned}\mathcal{C}'\cup\mathcal{C}'' &= v(x_0^2x_1^2+\alpha_1\alpha_2x_0^2x_2^2+\beta_1\beta_2x_1^2x_2^2+(\alpha_1+\alpha_2)x_0^2x_1x_2+(\beta_1+\beta_2)x_0x_1^2x_2\\ &\quad +(\alpha_1\beta_2+\alpha_2\beta_1)x_0x_1x_2^2). \qquad (3.28)\end{aligned}$$

So, applying Equation (3.28) to the conics in Table 3.12,

$$\begin{aligned}\mathcal{C}_1\cup\mathcal{C}_2 &= v(x_0^2x_1^2+2x_0^2x_2^2+x_1^2x_2^2+3x_0^2x_1x_2+3x_0x_1x_2^2).\\ \mathcal{C}_1\cup\mathcal{C}_3 &= v(x_0^2x_1^2+3x_0^2x_2^2+3x_1^2x_2^2+4x_0^2x_1x_2+4x_0x_1^2x_2).\\ \mathcal{C}_2\cup\mathcal{C}_3 &= v(x_0^2x_1^2+x_0^2x_2^2+3x_1^2x_2^2+3x_0x_1^2x_2+3x_0x_1x_2^2).\end{aligned}$$

From the general form of conic and Corollary 1.16, the final form of hyperoval contained the frame points is the following.

$$\mathcal{H}_i = \{P(x_0,x_1,x_2) \mid v(x_0x_1+\alpha x_0x_2-(1+\alpha)x_1x_2)\}\cup\{P(-(\alpha+1),\alpha,1)\}, \qquad (3.29)$$

where $i=1,\dots,q-2$. So, the following proposition gives the number of i-secants for the union of two hyperovals.

Proposition 3.6. *In* PG$(2,q)$*, q even, if* $|\mathcal{H}\cap\mathcal{H}'|=4$*, then* $|\mathcal{H}\cup\mathcal{H}'|$ *has the following constants:*

$$\begin{aligned}\tau_4 &= (q^2-8q+12)/4,\\ \tau_3 &= 4(q-2),\\ \tau_2 &= (q^2-2q+12)/2,\\ \tau_1 &= 0,\\ \tau_0 &= q^2/4.\end{aligned}$$

3.5 Algorithms for the Classification of (k, n)-Arcs in $\mathrm{PG}(2, q)$

Several attempts have been made to find the largest complete (k, n)-arcs in $\mathrm{PG}(2, q)$. This section illustrates the algorithms which are used to find the classification of (k, n)-arcs in $\mathrm{PG}(2, q)$ and make a comparison among them depending on the time to get the results. Here, one of these methods is developed to reduce the time of implementation. So, Algorithm One 3.5.1 and Algorithm Three 3.5.3 are first applied to $\mathrm{PG}(2, 5)$.

3.5.1 Algorithm One

This algorithm is used to find the maximum size of (k, n)-arc $\mathcal{K}$ in $\mathrm{PG}(2, q)$ and depends on the Fundamental Theorem of Projective Geometry. There is a unique projectivity of $\mathrm{PG}(2, q)$ transforming four points no three are collinear to any other four points no three are collinear. Two (k, n)-arcs $\mathcal{K}_1$, $\mathcal{K}_2$ are equivalent if $\mathcal{K}_1 T = \mathcal{K}_2$ for a projectivity $\mathfrak{T}$.

To find a matrix T which transforms the frame points to any given 4-arc say $\{P(a_0, a_1, a_2)$, $P(b_0, b_1, b_2), P(c_0, c_1, c_2), P(d_0, d_1, d_2)\}$, let

$$(1, 0, 0)T = \lambda(a_0, a_1, a_2),\ (0, 1, 0)T = \gamma(b_0, b_1, b_2),\ (0, 0, 1)T = \nu(c_0, c_1, c_2),$$

where $\lambda, \gamma, \nu \in \mathbb{F}_q$. So

$$T = \begin{pmatrix} \lambda a_0 & \lambda a_1 & \lambda a_2 \\ \gamma b_0 & \gamma b_1 & \gamma b_2 \\ \nu c_0 & \nu c_1 & \nu c_2 \end{pmatrix}.$$

Also $(1, 1, 1)T = \rho(d_0, d_1, d_2), \rho \in \mathbb{F}_q$, which implies that

$$\begin{pmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \gamma \\ \nu \\ -\rho \end{pmatrix} = (0, 0, 0).$$

$$A = \begin{vmatrix} d_0 & b_0 & c_0 \\ d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \end{vmatrix}, B = \begin{vmatrix} a_0 & d_0 & c_0 \\ a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \end{vmatrix}, C = \begin{vmatrix} a_0 & b_0 & d_0 \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix}, D = \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix},$$

where $A, B, C, D \neq 0$. Then,

$$\mathfrak{T} = \begin{pmatrix} Aa_0 & Aa_1 & Aa_2 \\ Bb_0 & Bb_1 & Bb_2 \\ Cc_0 & Cc_1 & Cc_2 \end{pmatrix}.$$

The main steps of Algorithm One to classify $(k, 4)$-arcs in PG$(2, 5)$ are the following.

1. Start with the frame points

$$P(1, 0, 0), P(0, 1, 0), P(0, 0, 1), P(1, 1, 1).$$

2. The are $31 - 4 = 27$ remaining points. So some of these points are of index zero which do not lie on any 4-secant and the others are not of index zero and lie on a 4-secant. Add one point of index zero to the frame points; the number of $(5, 3)$-arcs obtained is 27.

3. This step finds which arcs are projectively distinct. When there is a projective transformation $\mathfrak{T}$ that maps the points of $\mathcal{K}_1$ to $\mathcal{K}_2$, $\mathcal{K}_1$ and $\mathcal{K}_2$ are projectively equivalent. Similarly, check if $\mathcal{K}_1$ is projectively equivalent to $\mathcal{K}_3, \ldots, \mathcal{K}_{27}$. Repeat this procedure for $\mathcal{K}_2$ with $\mathcal{K}_3, \ldots, \mathcal{K}_{27}$, and similarly for the remain arcs. Therefore there are $\sum_{i=1}^{27} i = 378$ checks to decide which arcs are equivalent or not equivalent.

4. From step 3, there are 4 projectively distinct $(5, 3)$-arcs. So, add the remaining points of index zero to these arcs; the number of $(6, 3)$-arcs and $(6, 4)$-arcs obtained is 98. Repeat step 3; then there are 8 projectively distinct $(6, 3)$-arcs and 2 projectively distinct $(6, 4)$-arcs. To classify $(7, 4)$-arcs start with the two projectively distinct $(6, 4)$-arcs.

5. Table 3.13 shows the result of the classification of $(k, 4)$-arcs in PG$(2, 5)$.

Here, ξ is the number of $(k, 4)$-arcs and η is the number of projectively distinct $(k, 4)$-arcs.

TABLE 3.13: The classification of $(k, 4)$-arcs in PG$(2, 5)$

k	7	8	9	10	11	12	13	14	15	16
ξ	46	120	406	891	1545	1849	1594	825	250	35
η	6	24	60	123	176	207	165	82	27	6

3.5.2 Algorithm Two

A new algorithm has been obtained by modifying Algorithm One using the stabiliser group of arcs. Algorithm Two is the following.

1. Start with an initial (k, n)-arc $\mathcal{K}$.

2. Add all points of index zero to obtain $(k + 1, n)$-arcs. If there is no added point to a (k, n)-arc $\mathcal{K}$, then $\mathcal{K}$ complete.

3. Find the stabiliser group for each $(k + 1, n)$-arc.

4. Separate $(k + 1, n)$-arcs into different sets A_i, according to the group of the arcs.

5. Find the projectively distinct arcs for each set A_i and separate them into sets B_{ij}. Here B_{ij} are the sets of projectively distinct $(k + 1, n)$-arcs.

6. Collect all projectively distinct arcs in B_{ij} and put them in M. Here M is set of all projectively distinct $(k + 1, n)$-arcs.

7. Repeat steps $2, \ldots, 5$, for all arcs of M.

3.5.3 Algorithm Three

This algorithm depends on the type of i-secant distribution $(\tau_n, \tau_{n-1}, \ldots, \tau_0)$ and used to find the largest complete (k, n)-arcs in PG$(2, q)$. Algorithm Three is the following.

1. Start with an initial (k, n)-arc $\mathcal{K}$.

2. Add all points of index zero to obtain $(k+1, n)$-arcs. If there is no added point to $\mathcal{K}$, then $\mathcal{K}$ is complete.

3. Find the i-secant distribution for each $(k+1, n)$-arc.

4. Separate $(k+1, n)$-arcs into different sets A_i, according to the type of n-secant distribution.

5. Choose only one $(k+1, n)$-arc of each A_i and put it in M. Here M is the set of distinct $(k+1, n)$-arcs according to n-secant distribution.

6. Repeat steps $2, \ldots, 4$, to all arcs of M.

As an application of Algorithm Three, $(k, 4)$-arcs in PG$(2, 5)$ will be classified.

1. **The construction of the distinct** $(4, 4)$**-arcs**

   Let $A = \{1, 2, 4, 21\}$ be a $(4, 4)$-arc in PG$(2, 5)$. Here, all $(4, 4)$-arcs have the same type of i-secant distribution. This can be calculated from Equations (1.1), (1.2) and (1.3) of Lemma 1.29 are as follows:

$$\begin{aligned} \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 31, \\ \tau_1 + 2\tau_2 + 3\tau_3 + 4\tau_4 &= 24, \\ \tau_2 + 3\tau_3 + 6\tau_4 &= 6. \end{aligned}$$

   Since $\tau_4 = 1, \tau_3 = 0, \tau_2 = 0$, so the type of $(4, 4)$-arc is $(1, 0, 0, 20, 10)$.

2. **The construction of the distinct** $(5, 4)$**-arcs**

   From step 1, there is only one $(4, 4)$-arc A, and there are 25 points of index zero for A. So by adding one point of the points of index zero to A, then there is only one $(5, 4)$-arc denoted by B, for which

$$\begin{aligned} \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4 &= 31, \\ \tau_1 + 2\tau_2 + 3\tau_3 + 4\tau_4 &= 30, \\ \tau_2 + 3\tau_3 + 6\tau_4 &= 10. \end{aligned}$$

   Since $\tau_4 = 1, \tau_3 = 0$, so the type of $(5, 4)$-arc is $(1, 0, 4, 18, 8)$.

3. **The construction of the distinct** $(6, 4)$**-arcs**

   From step 2, there is only one $(5, 4)$-arc B, and there are 24 points of index zero. So, by adding these points to B, two distinct $(6, 4)$-arcs C_1 and C_2 are obtained. Then C_1 is of type $(1, 0, 9, 14, 7)$ and C_2 is of type $(1, 1, 6, 17, 6)$.

4. **The construction of the distinct** $(k, 4)$**-arcs,** $k = 7, \ldots, 16$

   Tables 3.14, ..., 3.23 show the number of i-secant distribution of $(k, 4)$-arcs in PG$(2, 5)$. Here, ξ is the number of $(k, 4)$-arcs.

TABLE 3.14: The i-secant distribution of $(7, 4)$-arcs in PG$(2, 5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
3	2	0	9	16	4	24	1	2	9	14	5
6	1	3	6	17	4	13	1	1	12	11	6

TABLE 3.15: The i-secant distribution of $(8, 4)$-arcs in PG$(2, 5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
1	1	6	4	18	2	12	2	1	13	11	4
5	2	3	7	17	2	39	1	4	10	12	4
22	2	2	10	14	3	2	2	0	16	8	5
10	1	5	7	15	3	28	1	3	13	9	5
1	1	2	16	6	6						

TABLE 3.16: The i-secant distribution of $(9, 4)$-arcs in PG$(2, 5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
2	3	4	6	18	0	4	3	1	15	9	3
13	2	6	6	16	1	15	1	5	15	5	5
6	3	3	9	15	1	114	2	4	12	10	3
1	3	0	18	6	4	44	2	3	15	7	4
6	1	8	6	14	2	58	1	7	9	11	3
22	3	2	12	12	2	58	1	6	12	8	4
63	2	5	9	13	2						

TABLE 3.17: The i-secant distribution of $(10, 4)$-arcs in PG$(2, 5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
4	3	3	18	3	4	89	3	6	9	12	1
1	5	0	15	10	1	58	3	4	15	6	3
19	4	4	9	14	0	27	2	6	15	4	4
8	3	7	6	15	0	179	3	5	12	9	2
2	1	8	15	2	5	48	1	10	9	8	3
25	4	3	12	11	1	38	1	9	12	5	4
16	4	2	15	8	2	153	2	8	9	10	2
12	2	9	6	13	1	204	2	7	12	7	3
8	1	11	6	11	2						

TABLE 3.18: The i-secant distribution of $(11,4)$-arcs in PG$(2,5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
5	3	7	16	1	4	4	2	13	4	11	1
12	5	3	16	5	2	67	5	4	13	8	1
4	6	2	13	10	0	18	4	5	16	3	3
2	1	15	4	9	2	74	2	12	7	8	2
7	1	14	7	6	3	135	2	11	10	5	3
13	1	13	10	3	4	268	4	7	10	9	1
19	2	10	13	2	4	227	4	6	13	6	2
46	5	5	10	11	0	392	3	9	10	7	2
32	4	8	7	12	0	132	3	8	13	4	3
88	3	10	7	10	1						

TABLE 3.19: The i-secant distribution of $(12,4)$-arcs in PG$(2,5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
1	1	18	6	2	4	91	4	12	6	8	1
5	3	12	12	0	4	64	3	14	6	6	2
4	2	15	9	1	4	389	5	9	9	7	1
1	3	16	0	12	0	29	6	5	15	3	2
1	6	4	18	0	3	8	5	7	15	1	3
5	3	15	3	9	1	190	6	6	12	6	1
11	2	16	6	4	3	96	3	13	9	3	3
31	5	10	6	10	0	91	4	10	12	2	3
5	7	3	15	5	1	401	4	11	9	5	2
32	7	4	12	8	0	286	5	8	12	4	2
108	6	7	9	9	0						

Table 3.20: The i-secant distribution of $(13, 4)$-arcs in PG$(2, 5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
4	3	19	3	3	3	125	8	7	9	7	0
3	3	18	6	0	4	192	6	12	6	6	1
23	6	13	3	9	0	5	4	17	3	5	2
5	5	15	3	7	1	136	5	14	6	4	2
49	7	10	6	8	0	90	8	6	12	4	1
7	8	5	15	1	2	19	9	4	12	6	0
4	9	3	15	3	1	344	6	11	9	3	2
12	6	10	12	0	3	121	7	18	12	2	2
28	4	16	6	2	3	360	7	9	9	5	1
67	5	13	9	1	3						

Table 3.21: The i-secant distribution of $(14, 4)$-arcs in PG$(2, 5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
3	6	18	1	4	2	180	8	12	7	2	2
21	6	17	4	1	3	64	9	9	10	1	2
16	7	14	7	0	3	47	10	8	7	6	0
35	9	11	4	7	0	30	11	5	10	5	0
42	7	15	4	3	2	73	10	7	10	3	1
3	11	4	13	2	1	61	8	13	4	5	1
2	10	6	13	0	2	248	9	10	7	4	1

TABLE 3.22: The i-secant distribution of $(15, 4)$-arcs in PG$(2, 5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
6	9	16	3	0	3	11	12	10	3	6	0
30	10	14	3	2	2	10	13	6	9	2	1
9	12	8	9	0	2	23	11	12	3	4	1
59	11	11	6	1	2	15	13	7	6	5	0
2	15	1	12	3	0	82	12	9	6	3	1
3	10	15	0	5	1						

TABLE 3.23: The i-secant distribution of $(16, 4)$-arcs in PG$(2, 5)$

ξ	τ_4	τ_3	τ_2	τ_1	τ_0	ξ	τ_4	τ_3	τ_2	τ_1	τ_0
1	12	16	0	0	3	2	15	10	0	6	0
5	15	8	6	0	2	6	15	9	3	3	1
13	14	11	3	1	2	8	16	6	6	2	1

Table 3.24 shows the number of distinct type of i-secant distribution in PG$(2, 5)$ using Algorithm Three. Here, N is the number of distinct type of i-secant distribution.

TABLE 3.24: The classification of $(k, 4)$-arcs in PG$(2, 5)$ according to their i-secant distribution

k	7	8	9	10	11	12	13	14	15	16
N	4	9	13	17	19	21	19	14	11	6

3.5.4 Classification of (k, n)-arcs in PG$(2, q)$ using Algorithm Three

In this section (k, n)-arcs in PG$(2, q)$ are classified.

Case I: Classification of (k, n)-arcs in PG$(2, 5), n = 2, 3, 4$

(i) From Note 1.15 there exists one type of i-secant distribution of $(6, 2)$-arcs as shown in Table 3.25.

TABLE 3.25: $(6, 2)$-arcs in PG$(2, 5)$

Points							τ_2	τ_1	τ_0
1	2	3	5	7	20	15	10	1	5

(ii) There are two distinct $(11, 3)$-arcs according to i-secant distribution as shown in Table 3.26.

TABLE 3.26: $(11, 3)$-arcs in PG$(2, 5)$

Points											τ_3	τ_2	τ_1	τ_0
1	2	3	5	6	7	8	10	14	16	20	15	10	1	5
1	2	3	5	6	7	8	10	15	16	20	16	7	4	4

(iii) There are six distinct $(16, 4)$-arcs according to i-secant distribution as shown in Table 3.27, Table 3.28.

TABLE 3.27: $(16, 4)$-arcs in PG$(2, 5)$

Points									τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	4	5	6	7	8	10	15	10	0	6	0
14	15	16	18	20	21	23							
1	2	3	5	6	7	8	12	13	12	16	0	0	3
16	18	20	23	25	30	31							

TABLE 3.28: $(16,4)$-arcs in $\mathrm{PG}(2,5)$

Points									τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	5	6	7	8	12	13	15	8	6	0	2
14	16	18	20	22	25	30							
1	2	3	5	6	7	8	12	13	14	11	3	1	2
14	16	18	20	25	30	31							
1	2	3	4	5	6	7	8	9	15	9	3	3	1
10	11	14	15	18	20	25							
1	2	3	4	5	6	7	8	10	16	6	6	2	1
11	14	15	18	20	21	29							

Note that all results of (k,n)-arcs in $\mathrm{PG}(2,5)$ are calculated within 1 second.

Case II: Classification of (k,n)-arcs in $\mathrm{PG}(2,7), n = 2,3,4,5$

(i) From Note 1.15 there exists one type of i-secant distribution of $(8,2)$-arcs as shown in Table 3.29.

TABLE 3.29: $(8,2)$-arcs in $\mathrm{PG}(2,7)$

Points								τ_2	τ_1	τ_0
1	2	3	4	7	12	32	33	28	8	21

(ii) There is one type of $(15,3)$-arcs as shown in Table 3.30.

TABLE 3.30: $(15,3)$-arcs in $\mathrm{PG}(2,7)$

Points									τ_3	τ_2	τ_1	τ_0
1	2	3	4	5	7	8	12	13	30	15	0	12
21	32	33	34	38	52							

(iii) There are two distinct $(22, 4)$-arcs according to i-secant distribution as shown in Table 3.31.

TABLE 3.31: $(22, 4)$-arcs in PG$(2, 7)$

Points											τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	4	5	7	8	10	12	13	15	30	16	3	2	6
16	19	21	27	30	32	33	34	38	43	49					
1	2	3	4	5	6	7	8	12	13	16	28	21	0	1	7
20	21	26	28	32	33	34	38	39	46	52					

(iv) There are four distinct $(29, 5)$-arcs according to i-secant distribution as shown in Tables 3.32.

TABLE 3.32: $(29, 5)$-arcs in PG$(2, 7)$

Points										τ_5	τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	4	5	7	8	10	11	12	28	21	0	0	0	8
13	15	16	19	21	22	27	30	32	33						
34	38	40	43	49	50	54	55	56							
1	2	3	4	5	7	8	10	11	12	27	20	5	1	0	4
13	15	16	19	21	22	23	25	27	30						
32	33	34	37	38	43	45	49	52							
1	2	3	4	6	7	8	10	12	13	28	18	6	0	2	3
14	15	16	18	19	20	21	24	26	28						
32	33	35	38	40	44	46	48	53							
1	2	3	4	5	7	8	10	11	12	31	13	4	6	1	2
13	19	21	22	25	26	27	29	31	32						
33	35	37	38	42	43	44	51	57							

The results of (k, n)-arcs in PG$(2, 7)$ were obtained within 12 seconds.

Case III: Classification of (k, n)-arcs in $\mathrm{PG}(2, 8), n = 2, 3, 4, 5, 6$

(i) From Theorem 1.28 2 there exists one type of i-secant distribution of $(10, 2)$-arcs as shown in Table 3.33.

Table 3.33: $(10, 2)$-arcs in $\mathrm{PG}(2, 8)$

Points										τ_2	τ_1	τ_0
1	2	3	10	11	40	44	48	53	61	45	0	28

(ii) There are four distinct $(15, 3)$-arcs according to i-secant distribution as shown in Tables 3.34.

Table 3.34: $(15, 3)$-arcs in $\mathrm{PG}(2, 8)$

Points									τ_3	τ_2	τ_1	τ_0
1	2	3	4	10	11	12	20	38	25	30	0	18
40	44	48	50	53	61							
1	2	3	4	10	11	12	20	38	27	24	6	16
40	44	48	53	59	61							
1	2	3	4	10	11	12	13	21	29	18	12	14
38	40	44	53	60	61							
1	2	3	4	10	11	12	40	48	31	12	18	12
52	53	56	61	69	73							

(iii) There is one type of $(28, 4)$-arcs as shown in Table 3.35.

Table 3.35: $(28, 4)$-arcs in $\mathrm{PG}(2, 8)$

Points										τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	4	10	11	12	15	20	22	63	0	0	0	10
28	30	34	35	38	40	44	46	48	50					
53	59	61	63	64	65	68	71							

(iv) There are three distinct $(33, 5)$-arcs according to i-secant distribution as shown in Tables 3.36.

TABLE 3.36: $(33, 5)$-arcs in PG$(2, 8)$

Points											τ_5	τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	4	8	10	11	12	15	17	20	39	18	10	0	0	6
24	25	26	30	34	36	38	40	41	44	48						
50	51	52	53	55	58	61	62	66	70	72						
1	2	3	4	8	10	11	12	15	20	22	48	0	16	0	9	0
24	30	34	35	38	40	41	44	48	50	51						
52	53	58	59	61	63	64	66	69	71	73						
1	2	3	4	5	10	11	12	15	20	22	36	28	0	0	5	4
23	28	30	32	34	35	38	40	42	43	44						
46	48	50	53	59	61	63	64	65	68	71						

(v) There are two distinct $(42, 6)$-arcs according to i-secant distribution as shown in Table 3.37 and Table 3.38.

TABLE 3.37: $(42, 6)$-arcs in PG$(2, 8)$

Points											τ_6	τ_5	τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	5	6	7	8	9	10	11	15	50	0	18	0	3	0	2
16	20	22	23	24	26	27	28	29	30	31							
32	34	37	38	39	40	41	42	43	47	48							
51	52	53	59	60	65	66	72	73									

TABLE 3.38: $(42,6)$-arcs in PG$(2,8)$

Points											τ_6	τ_5	τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	4	7	8	9	10	12	13	15	39	24	6	0	0	0	4
18	20	21	22	26	28	30	33	34	37	40							
41	43	44	45	46	53	54	56	57	58	59							
61	62	64	65	67	68	69	72	73									

The results of (k,n)-arcs in PG$(2,8)$ were calculated within 62 seconds.

Case IV: Classification of (k,n)**-arcs in** PG$(2,9), n = 2,3,4,5$

(i) From Note 1.15 there exists one type of i-secant distribution of $(10,2)$-arcs as shown in Table 3.39.

TABLE 3.39: $(10,2)$-arcs in PG$(2,9)$

Points										τ_2	τ_1	τ_0
1	2	3	5	9	23	43	53	56	63	45	10	36

(ii) There are two distinct $(17,3)$-arcs according to i-secant distribution as shown in Table 3.40.

TABLE 3.40: $(17,3)$-arcs in PG$(2,9)$

Points											τ_3	τ_2	τ_1	τ_0
1	2	3	5	7	9	12	15	23	39	43	38	22	12	19
53	56	58	63	67	79									
1	2	3	5	6	7	11	26	29	30	49	39	19	15	18
52	57	61	63	86	90									

(iii) There is one type of $(28, 4)$-arcs as shown in Table 3.41.

TABLE 3.41: $(28, 4)$-arcs in PG$(2, 9)$

Points											τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	4	5	6	7	9	11	14	19	63	0	0	28	0
20	23	25	32	43	53	55	56	57	63	64					
67	76	82	85	89	90										

(iv) There are two distinct $(37, 5)$-arcs according to i-secant distribution as shown in Table 3.42.

TABLE 3.42: $(37, 5)$-arcs in PG$(2, 9)$

Points											τ_5	τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	4	6	9	10	16	17	18	20	45	36	0	0	1	9
22	25	27	30	36	43	44	47	51	53	55						
57	58	60	61	63	64	69	71	76	79	81						
83	87	90	91													
1	2	3	4	6	9	10	16	17	18	20	48	29	4	0	2	8
22	25	27	30	36	43	44	47	51	53	55						
57	58	60	61	63	64	69	71	79	81	83						
85	87	90	91													

Case V: Classification of (k, n)-arcs in PG$(2, 11), n = 2, 3, 4, 5$

(i) From Note 1.15 there exists one type of i-secant distribution of $(12, 2)$-arcs as shown in Table 3.43.

TABLE 3.43: $(12, 2)$-arcs in PG$(2, 11)$

Points												τ_2	τ_1	τ_0
1	2	3	5	8	17	83	84	89	98	113	128	66	12	55

(ii) There is one type of $(21, 3)$-arcs as shown in Tables 3.44.

TABLE 3.44: $(21, 3)$-arcs in PG$(2, 11)$

Points											τ_3	τ_2	τ_1	τ_0
1	2	3	5	7	12	21	22	23	28	33	63	21	21	28
55	70	79	88	96	102	115	117	121	129					

(iii) There is one type of $(32, 4)$-arcs as shown in Table 3.45.

TABLE 3.45: $(32, 4)$-arcs in PG$(2, 11)$

Points											τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	12	13	14	15	17	21	30	39	65	30	16	2	20
45	46	49	55	62	65	68	71	75	77	96					
105	108	111	113	115	117	118	125	126	132						

(iv) There is one type of $(43, 5)$-arcs as shown in Table 3.46.

TABLE 3.46: $(43, 5)$-arcs in PG$(2, 11)$

Points											τ_5	τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	5	8	12	13	14	15	17	21	60	45	10	3	0	15
22	23	28	30	39	45	46	49	55	62	65						
68	71	72	73	75	77	79	88	96	105	106						
108	111	113	115	117	118	125	126	129	132							

Case VI: Classification of (k, n)-arcs in $\mathrm{PG}(2, 13), n = 2, 3, 5$

(i) From Note 1.15 there exists one type of i-secant distribution of $(14, 2)$-arcs as shown in Table 3.47.

TABLE 3.47: $(14, 2)$-arcs in $\mathrm{PG}(2, 13)$

Points														τ_2	τ_1	τ_0
1	2	3	8	19	21	28	30	46	48	68	125	147	163	91	14	78

(ii) There are four distinct $(23, 3)$-arcs according to i-secant distribution as shown in Tables 3.48.

TABLE 3.48: $(23, 3)$-arcs in $\mathrm{PG}(2, 13)$

Points												τ_3	τ_2	τ_1	τ_0
1	2	3	7	14	37	39	46	51	55	66	69	71	40	29	43
84	98	100	122	125	126	143	163	171	172	180		71	40	29	43
1	2	3	12	23	25	34	35	36	39	41	46	70	43	26	44
51	56	82	85	87	96	118	142	170	171	181					
1	2	3	4	7	46	50	51	66	69	72	84	72	37	32	42
94	98	101	103	106	111	114	139	163	177	183					
1	2	3	13	30	46	51	66	84	86	92	94	73	34	35	41
101	103	111	126	129	137	143	163	166	172	174					

(iii) There are two distinct $(49, 5)$-arcs according to i-secant distribution as shown in Table 3.49.

TABLE 3.49: $(49, 5)$-arcs in PG$(2, 13)$

Points											τ_5	τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	4	5	7	8	9	18	19	22	82	49	16	14	4	18
25	28	29	34	35	37	38	43	46	48	50						
54	55	58	63	67	69	70	80	84	95	98						
100	101	103	112	113	126	127	148	152	153	156						
159	163	166	172	183												
1	2	3	4	5	7	8	9	18	19	22	82	47	22	8	6	18
25	28	29	34	35	37	38	43	46	48	50						
54	55	58	63	67	69	70	80	84	95	98						
101	102	103	112	113	126	127	148	152	153	156						
159	163	166	172	183												

Case VII: Classification of (k, n)-arcs in PG$(2, 16), n = 2, 4$

(i) From Theorem 1.28 2 there exists one type of i-secant distribution of $(18, 2)$-arcs as shown in Table 3.50.

TABLE 3.50: $(18, 2)$-arcs in PG$(2, 16)$

Points											τ_2	τ_1	τ_0
1	2	3	69	79	85	109	112	122	124	158	153	0	120
166	201	207	222	234	250	263							

(ii) There is one type of $(52, 4)$-arcs as shown in Table 3.51.

TABLE 3.51: $(52, 4)$-arcs in $\mathrm{PG}(2, 16)$

Points													τ_4	τ_3	τ_2	τ_1	τ_0
1	2	3	6	7	10	11	16	19	23	24	30	32	221	0	0	0	52
38	39	53	60	64	65	67	70	85	87	90	91	106					
119	120	125	132	153	156	165	169	174	186	188	192	206					
209	211	212	217	221	232	234	239	244	251	255	262	263					

Case VIII: Classification of (k, n)-arcs in $\mathrm{PG}(2, 17), n = 2, 3$

(i) From Note 1.15 there exists one type of i-secant distribution of $(18, 2)$-arcs as shown in Table 3.52.

TABLE 3.52: $(18, 2)$-arcs in $\mathrm{PG}(2, 17)$

Points											τ_2	τ_1	τ_0
1	2	3	100	7	21	29	36	44	73	91	153	18	136
132	137	141	163	173	262	271							

(ii) There are three distinct $(28, 3)$-arcs according to i-secant distribution as shown in Table 3.53 and Table 3.54.

TABLE 3.53: $(28, 3)$-arcs in $\mathrm{PG}(2, 17)$

Points													τ_3	τ_2	τ_1	τ_0
1	2	3	5	7	15	33	47	52	62	68	80	90	100	78	48	81
81	100	104	184	186	189	197	206	225	232	240	243	254				
287	288															

TABLE 3.54: $(28, 3)$-arcs in PG$(2, 17)$

Points													τ_3	τ_2	τ_1	τ_0
1	2	3	5	15	26	33	34	39	60	83	92	99	101	75	51	80
100	101	105	143	162	164	168	172	179	194	211	255	284				
289	296															
1	2	3	5	15	21	24	28	41	53	76	100	104	98	84	42	83
105	107	145	149	184	190	192	195	202	231	238	244	260				
271	296															

Here, all of the results which were found by Algorithm Three are the same as known results as in [7].

3.5.5 Algorithm Four to find the classification of (k, n)-arcs

This algorithm modifies the Algorithm Two using Algorithm Three. The main steps of Algorithm Four are the following.

1. Start with an initial (k, n)-arc $\mathcal{K}$.
2. Add all points of index zero to obtain $(k+1, n)$-arcs. If there is no added point to a (k, n)-arc $\mathcal{K}$, then $\mathcal{K}$ complete.
3. Find the n-secant distribution for each $(k+1, n)$-arc.
4. Separate $(k+1, n)$-arcs into different sets A_i, according to the type of n-secant distribution.
5. Find the stabiliser group of $(k+1, n)$-arc for each A_i and separate them into sets B_{ij}. Here B_{ij} are the sets of distinct stabiliser $(k+1, n)$-arcs.
6. Find the projectively distinct arcs for each set B_{ij} and put them in M. Here M is set of all projectively distinct $(k+1, n)$-arcs.
7. Repeat steps $2, \ldots, 5$, for all arcs of M.

From the classification of $(k, 4)$-arcs in PG$(2, 5)$, Algorithm Four gives the following results. Here δ represents the number of arcs with the same type of i-secant distribution and ζ represents the number of arcs with the same group.

TABLE 3.55: Stabilisers of $(7, 4)$-arcs in PG$(2, 5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
3	2	0	9	16	4	3	8	24	1	2	9	14	5	16	1
6	1	3	6	17	4	6	2							8	2
								13	1	1	12	11	6	10	2
														3	8

TABLE 3.56: Stabilisers of $(8, 4)$-arcs in PG$(2, 5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
1	1	6	4	18	2	1	8	12	2	1	13	11	4	8	2
5	2	3	7	17	2	5	4							4	4
22	2	2	10	14	3	14	1	39	1	4	10	12	4	24	1
						8	2							13	2
10	1	5	7	15	3	7	1							1	4
						3	2							2	8
1	1	2	16	6	6	1	8	2	2	0	16	8	5	2	32
								28	1	3	13	9	5	20	1
														8	2

TABLE 3.57: Stabilisers of $(9,4)$-arcs in PG$(2,5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
2	3	4	6	18	0	2	24	15	1	5	15	5	5	10	1
6	3	3	9	15	1	6	3							5	2
13	2	6	6	16	1	13	2	114	2	4	12	10	3	91	1
1	3	0	18	6	4	1	24							18	2
22	3	2	12	12	2	22	2							5	8
6	1	8	6	14	2	4	2	44	2	3	15	7	4	12	1
						2	4							29	2
63	2	5	9	13	2	32	1							3	4
						26	2	58	1	7	9	11	3	49	1
						4	5							9	2
4	3	1	15	9	3	4	3	58	1	6	12	8	4	40	1
														17	2
														1	8

Table 3.58: Stabilisers of $(10, 4)$-arcs in PG$(2, 5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
4	3	3	18	3	4	4	6	12	2	9	6	13	1	12	1
1	5	0	15	10	1	1	20	8	1	11	6	11	2	6	2
19	4	4	9	14	0	19	2							2	4
8	3	7	6	15	0	6	6	90	3	6	9	12	1	67	1
						2	12							16	2
2	1	8	15	2	5	2	4							4	3
25	4	3	12	11	1	15	1							3	6
						10	2	58	3	4	15	6	3	34	1
16	4	2	15	8	2	13	1							18	2
						3	8							3	3
26	2	6	15	4	4	10	1							3	6
						15	2	37	1	9	12	5	4	19	1
						1	4							17	2
179	3	5	12	9	2	141	1							1	4
						35	2	153	2	8	9	10	2	118	1
						3	4							29	2
46	1	10	9	8	3	39	1							3	4
						4	2							2	8
						3	4							1	16
								202	2	7	12	7	3	173	1
														29	2

Table 3.59: Stabilisers of $(11, 4)$-arcs in PG$(2, 5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
5	3	7	16	1	4	5	4	68	5	4	13	8	1	52	1
12	5	3	16	5	2	12	2							16	2
4	6	2	13	10	0	4	8	76	2	12	7	8	2	61	1
2	1	15	4	9	2	2	4							12	2
7	1	14	7	6	3	4	1							3	4
						3	2	18	4	5	16	3	3	11	1
13	1	13	10	3	4	13	2							7	2
19	2	10	13	2	4	11	1	135	2	11	10	5	3	101	1
						6	2							31	2
						2	8							3	4
46	5	5	10	11	0	15	1	265	4	7	10	9	1	225	1
						30	2							35	2
						1	20							5	4
88	3	10	7	10	1	69	1	224	4	6	13	6	2	178	1
						19	2							39	2
31	4	8	7	12	0	14	1							7	4
						9	2	393	3	9	10	7	2	320	1
						8	4							70	2
4	2	13	4	11	1	4	8							3	4
								132	3	8	13	4	3	119	1
														13	2

TABLE 3.60: Stabilisers of $(12,4)$-arcs in PG$(2,5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
4	3	15	3	9	1	4	3	1	6	4	18	0	3	1	24
1	1	18	6	2	4	1	8	12	2	16	6	4	3	9	1
5	3	12	12	0	4	4	6							3	8
						1	24	33	5	10	6	10	0	11	1
4	2	15	9	1	4	4	4							19	2
1	3	16	0	12	0	1	96							3	8
33	7	4	12	8	0	15	1	188	6	6	12	6	1	164	1
						11	2							12	2
						7	4							7	3
94	4	12	6	8	1	77	1							3	4
						17	2							2	12
68	3	14	6	6	2	47	1	408	4	11	9	5	2	345	1
						17	2							59	2
						4	4							4	4
107	6	7	9	9	0	63	1	91	4	10	12	2	3	61	1
						27	2							30	2
						8	3	98	3	13	9	3	3	68	1
						5	4							20	2
						4	6							4	3
391	5	9	9	7	1	335	1							6	6
						56	2	8	5	7	15	1	3	8	2
29	6	5	15	3	2	23	1	278	5	8	12	4	2	218	1
						6	2							49	2
														7	4
														4	8

TABLE 3.61: Stabilisers of $(13,4)$-arcs in PG$(2,5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
13	3	18	6	0	4	3	24	32	4	16	6	2	3	21	1
5	3	19	3	3	3	5	6							11	2
23	6	13	3	9	0	5	3	11	6	10	12	0	3	5	3
						18	6							6	6
4	4	17	3	5	2	4	4	67	5	13	9	1	3	33	1
6	5	15	3	7	1	6	2							34	2
48	7	10	6	8	0	34	1	88	8	6	12	4	1	74	1
						10	2							14	2
						4	8	145	5	4	6	4	2	113	1
7	8	5	15	1	2	7	4							32	2
4	9	3	15	3	1	4	3	362	7	9	9	5	1	310	1
122	8	7	9	7	0	76	1							52	2
						40	2	120	7	8	12	2	2	93	1
						6	4							18	2
195	6	12	6	6	1	170	1							6	4
						13	2							3	8
						6	3	356	6	11	9	3	2	293	1
						4	4							63	2
						2	12								
21	9	4	12	6	0	9	2								
						5	3								
						4	4								
						3	24								

Table 3.62: Stabilisers of $(14, 4)$-arcs in PG$(2, 5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
35	9	11	4	7	0	14	1	48	7	15	4	3	2	31	1
						11	2							17	2
						10	4	75	10	7	10	3	1	56	1
21	6	17	4	1	3	21	2							19	2
16	7	14	7	0	3	9	2	177	8	12	7	2	2	145	1
						7	4							32	2
6	6	18	1	4	2	6	8	63	9	9	10	1	2	37	1
50	10	8	7	6	0	38	1							26	2
						9	2	253	9	10	7	4	1	221	1
						3	8							29	2
61	8	13	4	5	1	47	1							3	4
						14	2	30	11	5	10	5	0	14	1
2	10	6	13	0	2	2	8							16	2
3	11	4	13	2	1	3	8	5	9	16	3	0	3	5	6

TABLE 3.63: Stabilisers of $(15,4)$-arcs in PG$(2,5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
3	10	15	0	5	1	3	20	9	13	6	9	2	1	9	1
11	12	10	3	6	0	6	3	25	11	12	3	4	1	16	1
						5	12							9	2
30	10	14	3	2	2	17	1	83	12	9	6	3	1	62	1
						13	2							13	2
9	12	8	9	0	2	9	2							5	3
58	11	11	6	1	2	44	1							3	6
						8	2	16	13	7	6	5	0	11	2
						6	4							5	4
						6	4	2	15	1	12	3	0	2	24

TABLE 3.64: Stabilisers of $(16,4)$-arcs in PG$(2,5)$

δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$	δ	τ_4	τ_3	τ_2	τ_1	τ_0	ζ	$\|G\|$
1	12	16	0	0	3	1	96	12	14	11	3	1	2	12	2
2	15	10	0	6	0	2	120	6	15	9	3	3	1	6	3
4	15	8	6	0	2	4	4	8	16	6	6	2	1	8	2

Figure 3.1 shows a comparison among Algorithm One, Algorithm Three and Algorithm Four to classify $(k, 4)$-arcs in PG$(2, 5)$.

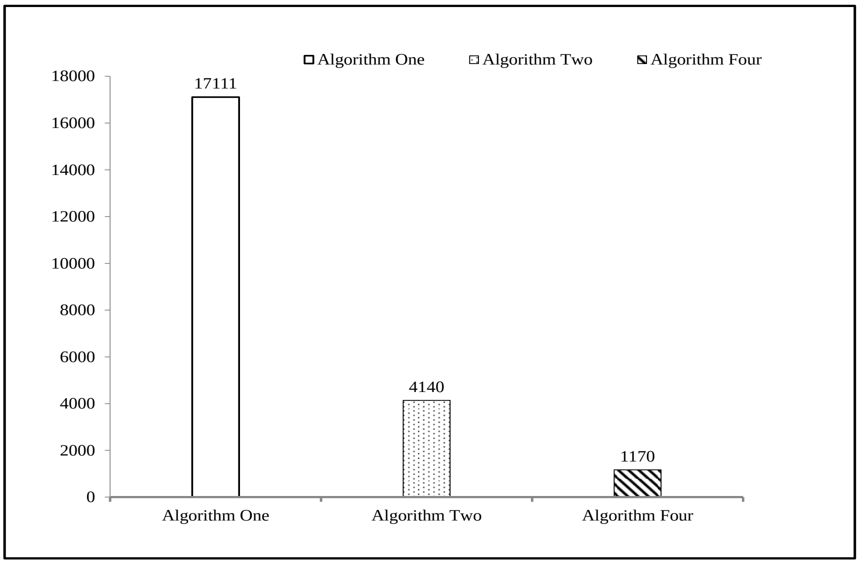


FIGURE 3.1: Time required for implementation of Algorithm Three and Algorithm Four

3.6 Chapter Summary

This chapter describes a survey for known results for maximum and minimum size of complete (k, n)-arcs in PG$(2, q)$ is proved for k-arcs and $(k, 3)$-arcs. Also, a new lower bound is proved for smallest size of complete (k, n)-arcs in PG$(2, q)$. A comparison with known results is shown. Furthermore, a construction of complete k-arcs in PG$(2, q)$ from quadrangle is proved and applied for PG$(2, 5)$. Four algorithms are explained, and the classification of (k, n)-arcs is shown for some values of n and q. Algorithm Two gives the same results for large complete (k, n)-arcs which are obtained using Algorithm One. Algorithm Two is faster than Algorithm One. Finally, the program which is written in Fortran as mentioned in Appendix C is faster than the program which is written in GAP as mentioned in Appendix B.

Chapter 4

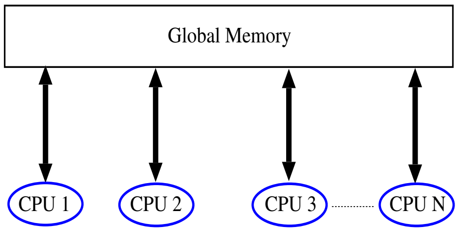
High Performance Computing (HPC)

Many complex mathematical problems deal with extremely large amounts of data that require to be processed. In such cases, a conventional program, known as a serial program, may require days, weeks, or even months before obtaining the results. This requires High Performance Computing (HPC) techniques to accelerate the calculations without affecting the accuracy of the results. The HPC relies on dividing a big problem into smaller problems and solving each of them individually, or more precisely in a parallel way [59]. Thus, parallel computing is the key role of the HPC. Based on accessing the memory by the processors, the paralleling computing systems can be divided into shared memory, distributed memory, or shared-distributed hybrid memory [55]. Besides these techniques, Graphical Processor Units (GPUs) are utilised as an effective tool to accelerate the calculations.

4.1 Parallel Computing on a Shared Memory Architecture - OpenMP

In the shared-memory architecture, processors share the physical memory, that is the processors read/write from/to any memory location, as depicted in Figure 4.1.

FIGURE 4.1: Shared memory architecture



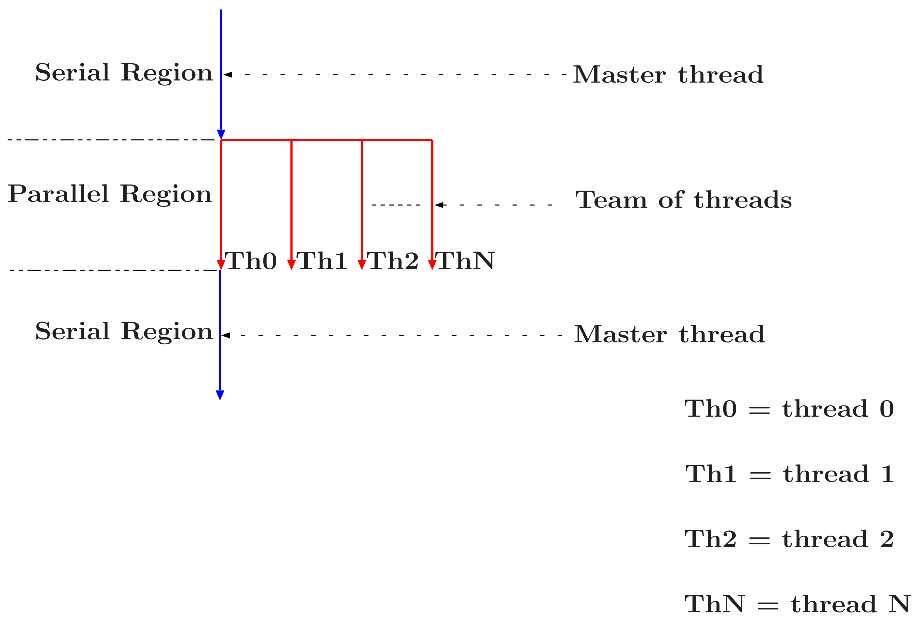
The CPUs have been developed during the last decade. They started with more than one processing unit (core). These cores are designed to work independently forming a multi-core processor. This increases the number of tasks that can be preformed at the same time. From the software point of view, OpenMP has been developed to provide a standard (API) that can be utilised to develop parallel programs using the shared-memory technique. The OpenMP matches the state of the hardware [20].

In sequential programs, the CPU provides one thread to execute the program instructions sequentially, whereas in OpenMP the program is executed serially by a master thread, usually denoted as Thread 0, until it reaches the parallel region, which is a group of instructions that need to be executed concurrently. In the parallel region, the master thread creates a group of threads; depending on the nature of the program, a master thread may or may not wait for the other threads to complete their jobs at the end of the parallel region. Figure 4.2 shows the sequence of OpenMP threads in a program.

The program instructions are executed sequentially in the serial programming by one thread which is dedicated by the CPU. In OpenMP, the program is divided into two regions: serial region and parallel region, which is a set of instructions that need to be executed concurrently. The serial region is executed by the master thread while the parallel region is executed by threads which are generated by the master thread, as shown in Figure 4.2.

At the end of the parallel region, the master thread may or may nor wait for other threads to complete executing their tasks.

FIGURE 4.2: Threads diagram in OpenMP program



The number of instructions executed in the parallel region equals the number of specified or available threads. The OpenMP program accelerates the calculations of the given problem; however, the acceleration factor depends on the size of the parallel region as compared with the size of the whole program. This has been formalised by Gene Amdahl [20] as

$$S = \frac{1}{(1-P) + \dfrac{P}{S_P}}, \tag{4.1}$$

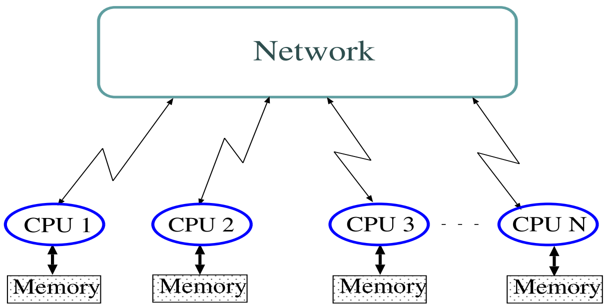
where S is the expected acceleration, P is the size of the program, and S_p is the acceleration obtained from the parallel region.

The OpenMP has feature of adaptation; that is, the program parallelised using OpenMP can be executed in a parallel or serial version. The adaptation feature allows the developer to switch between the serial and parallel versions without introducing any changes. This is achieved by changing the compilation option(s). For instance, the OpenMP directives in Fortran start with "!$" which are comments for non-OpenMP compilers.

4.2 Parallel Computing on a Distributed Memory Architecture - MPI

As shown in Figure 4.3, the CPUs in the distributed memory architecture are physically separated and a communication network is used to maintain the connection among the CPUs. Each CPU has its own local memory which is non-accessible by other CPUs. Thus, CPU2 in Figure 4.3 cannot directly fetch data in the local memory available in CPU1. Message passing is used to perform the communication between the CPUs. Therefore, the distributed memory architecture is known as the message passing architecture [33]. Since 1994, the Message Passing Interface (MPI) is used for message passing.

FIGURE 4.3: Distributed memory architecture



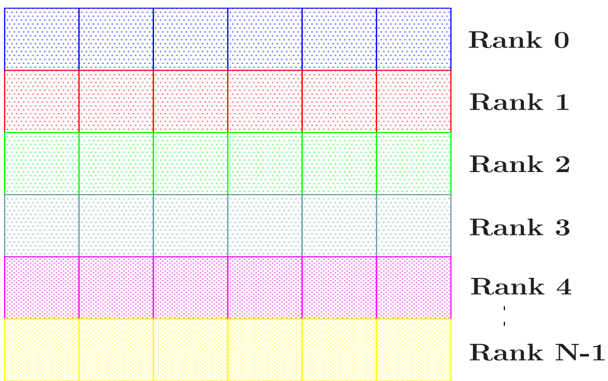
Each CPU in the MPI communication world has an identifier which is a unique number used to distinguish between the ranks. The ranks begin with 0 and end with number of CPUs -1; therefore, the ranks of the MPI communication world with 8 CPUs are Rank 0, ..., Rank 7. The problem data are divided by the developer using the ranks. Let us consider the 2-dimensional matrix shown in Figure 4.4; the matrix is divided over a number of CPUs and can be performed in many ways. In this case, the division is randomly chosen to be row by row. In practice, the division of data among ranks may affect the efficiency of the parallel program. The key feature of the MPI is that the developer can build an application-dependent topology, which is known as virtual topology [33]. This is achieved by the aid of ranks and the MPI communication world. The virtual topology concept overcomes the problem of selecting the efficient topology that matches the hardware of the HPC.

The communications among the CPUs, which are in the same MPI communication world, are unicast, multicast, or broadcast. The unicast is the point-to-point communication, whereas the multicast is the point to multi-points communications. The broadcast is the case when a CPU sends data to all other CPUs. Although it is essential, the communication between CPUs reduces the overall computational performance. Thus, the broadcast should be avoided as much as possible.

The MPI is a powerful tool for parallel programming. Nevertheless, the MPI library contains many functions which may be considered a restriction for developers with limited experience in parallel programming. Practically, this large number of functions does not add complexity to developing a parallel program using MPI. Many of the MPI functions are combinations of a small number of concepts [33]. Moreover, to implement an MPI program with a good acceleration of performance, only a small number of functions is required. Basically, the following steps are essential for any MPI program.

- Create the MPI communication world.
- Discover the CPUs which are participating in the MPI communication world.
- Discover the CPU's rank.
- Exchange the data with the CPU(s).
- Terminate the MPI communication world.

FIGURE 4.4: Example of MPI data fraction



4.3 Algorithm Five to Find the Classification of (k, n)-Arcs

This algorithm has been obtained by modifying the Algorithm Four by using parallel computing on a shared memory architecture-OpenMP, as mentioned in Appendix C.1.2. Here a laptop is used with the following details.

1. CPU Intel (R) core (TM) i7 – 4710 HQ CPU @ 2.5 GHz.
2. RAM 16 GB.

Figure 4.5 shows a comparison between Algorithm One, Algorithm Three, Algorithm Four and Algorithm Five.

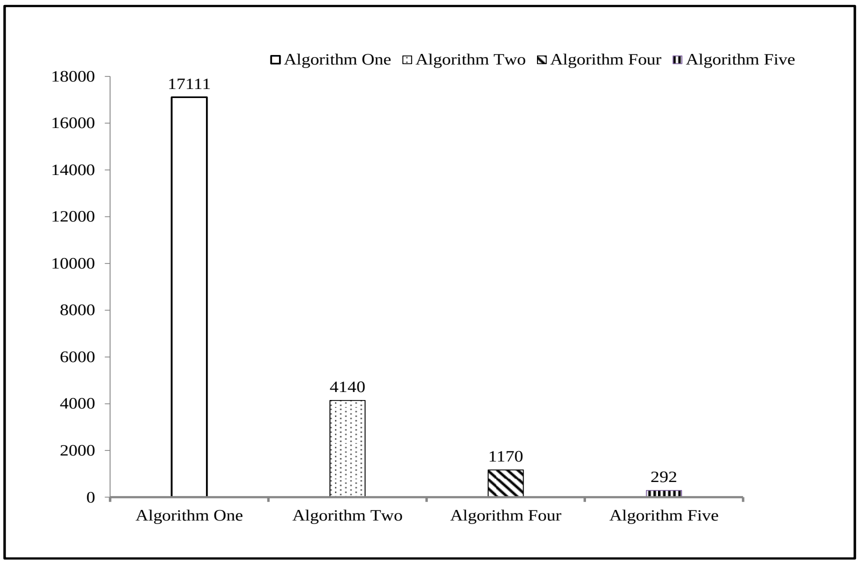


FIGURE 4.5: Time required for implementation of Algorithm One, Algorithm Three, Algorithm Four and Algorithm Five

4.3.1 Application of Algorithm Five Using OpenMP

Figure 4.6 shows a comparison between different threads to classify k-arcs in PG$(2, 13)$.

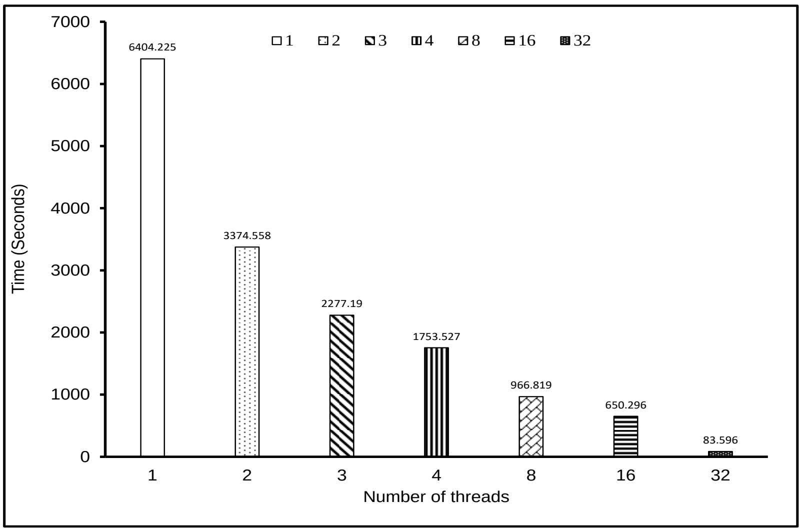


FIGURE 4.6: Time required for implementation of Algorithm Five

Here, the cluster of the University of Sussex is used to find the classification of k-arcs in PG$(2, 13)$ with different numbers of threads.

4.4 Chapter Summary

This chapter has showed some applications of HPC. A new algorithm with OpenMP technique is applied. Also, a comparison among all five algorithms is shown.

Chapter 5

Coding Theory

5.1 Introduction

Projective geometry has applications in modern information and communication science, more specifically, in coding theory. The aim of coding theory is to develop methods that enable the recipient of message to detect or even correct errors that occur while transmitting data. Many aspects of coding theory can be directly translated into geometry problems.

5.2 Linear Codes

A linear $[k, n, d]$ code C over $q = p^h$ is an n-dimensional subspace of the k-dimensional vector space $V = V(k, q)$ over $\mathbb{F}_q$. The minimum distance d of the code is the smallest number of positions in which two different elements of C differ. Equivalently, d is the smallest number of non-zero symbols in any non-zero vector of C. Here, C is MDS if and only if any n columns of G are linearly independent; this property is preserved under multiplication of the columns by non-zero scalars. So, consider the columns of G as points $P_1, P_2, \ldots, P_k$ of $\mathrm{PG}(n-1, q)$. It follows that C is MDS if and only if $\{P_1, \ldots, P_k\}$ is a $k-$arc of $\mathrm{PG}(n-1, q)$; so a $k-$arc corresponds to an MDS code. This gives the relation between linear MDS codes and arcs. One of the main problems of coding theory is to correct as many errors as possible.

Following [34], let $\mathbb{Z}^+$ denote the set of positive integers and $\mathbb{F}$ a finite set. This finite set constitutes our alphabet, and its elements are called letters. Then

$$V = \{x_1 \cdots x_k \mid x_i \in \mathbb{F}, 1 \leq i \leq k; k \in \mathbb{Z}^+\}$$

is the set of all possible n-tuples of letters (repetition allowed) from $\mathbb{F}$. A subset C of V is referred to as a *code*, and the elements of C are *codewords*. For a fixed $k \in \mathbb{Z}^+$, then refer to a subset C of V as a *block code* of length k if each of its elements is a k-tuple of elements of $\mathbb{F}$. Here take $\mathbb{F}$ to be a finite field. If C has n information symbols, where n is the dimension of C as a vector space over $\mathbb{F}$, then refer to C as a (k, n) code.

Let $C = \{00000, 11111\}$. If the encoder transmits 00000 but 00101 is received instead, then the decoder on realising that there are more 0's than 1's decodes the message as 00000. Such codes are *repetition codes*. This code detects up to 4 errors and corrects 2 of them.

In general, repetition codes can detect the presence of up to $d - 1$ errors and correct up to $\left\lfloor \dfrac{d-1}{2} \right\rfloor$ errors in each d-tuple.

The space is $V(k, q) = ((\mathbb{F}_q)^k, +, \times)$. For $x \in V(k, q)$, write

$$x = (x_1, x_2, \ldots, x_k) = x_1 x_2 \cdots x_k.$$

Definition 5.1. (i) A *linear code* is a subspace of $V(k, q)$.

(ii) If $\dim C = n$, then C is a $[k, n]$-code or $[k, n]_q$-code,

or, if $d(C) = d$, it is a $[k, n, d]$-code or $[k, n, d]_q$-code.

Definition 5.2. A *generator matrix* G of a $[k, n]$-code C is an $n \times k$ matrix whose rows form a basis for C. A *parity check matrix* for C is a $(k - n) \times k$ matrix H such that $c \in C$ if and only if $cH^T = 0$. Also, the *dual* code of C, denoted by $C^\perp$, is the set of vectors of $V(k, q)$ which are orthogonal to every codeword in C:

$$C^\perp = \{v \in V(k, q) \mid v.u = 0, \text{for all } u \in C\}.$$

5.3 The Main Coding Problem

The notation (k, M, d) is used to represent a code with length k, a total of M codewords, and minimum distance d. The main coding theory problem is to optimise a code (k, M, d) and make a balance among k for fast transmission, large M to enable transmission of a wide variety of messages, and large d to correct many errors.

Let $A_q(k, d)$ be the maximum value of M for which there exists a q-ary (k, M, d)-code.

Theorem 5.3.

(i) $A_q(k, 1) = q^k$.

(ii) $A_q(k, k) = q$.

Definition 5.4. An e-error-correcting code C in $(\mathbb{F}_q)^k$ is *perfect* if any vector in $(\mathbb{F}_q)^k$ is at distance at most e from exactly one codeword; that is, every received message is corrected.

Theorem 5.5. (i) *A code C can detect up to s errors in any codeword if*

$$d(C) \geq s + 1.$$

(ii) *A code C can correct up to t errors in any codeword if*

$$d(C) \geq 2t + 1.$$

From [8], the Singleton bound states that, if C is an $[k, n, d]_q$-code, then $d \leq k - n + 1$. The Singleton defect for linear code C is defined as $\Delta(C) = k + 1 - n - d$. Linear codes meeting the Singleton bound are *Maximum Distance Separable* (MDS). A linear code C having Singleton defect equal to 1 and such that also $C^{\perp}$ has Singleton defect 1 is *Near Maximum Distance Separable* (NMDS). In general, a linear code having Singleton defect equal to 1 is *Almost Maximum Distance Separable* (AMDS). Not all the AMDS-codes are NMDS-codes.

Theorem 5.6 ([31]). *If $k > n + q$, then every $[k, n, k - n]_q$-code is* NMDS.

Theorem 5.7 ([1]). *If $k > m_{n-2}(n - 2, q) + 2$, then every $[k, n, d]_q$*-AMDS code is NMDS.

Note 5.8. A q-ary $[k, n, d]$-code is a q-ary (k, q^n, d)-code.

Theorem 5.9. *There exists a projective $[k, n, d]_q$-code if and only if there exists an $(k, k - d)$-arc in* $\mathrm{PG}(n - 1, q)$.

5.4 MDS Codes of Dimension Three

From Theorem 5.9, if $q = 13, n = 3$ then in $\mathrm{PG}(2, 13)$ there exists a correspondence between $(k, 2)$-arcs and projective $[k, 3, k - 2]_{13}$-codes.

Table 5.1 shows the MDS codes corresponding to $(k, 2)$-arcs in $\mathrm{PG}(2, 13)$, where e is the number of errors corrected.

TABLE 5.1: MDS Codes over $\mathbb{F}_{13}$

$\mathcal{K}$	MDS code	e	$\mathcal{K}$	MDS code	e	$\mathcal{K}$	MDS code	e
$(5, 2)$-arc	$[5, 3, 3]$	1	$(12, 2)$-arc	$[12, 3, 10]$	4	$(19, 2)$-arc	$[19, 3, 17]$	8
$(6, 2)$-arc	$[6, 3, 4]$	1	$(13, 2)$-arc	$[13, 3, 11]$	5	$(20, 2)$-arc	$[20, 3, 18]$	8
$(7, 2)$-arc	$[7, 3, 5]$	2	$(14, 2)$-arc	$[14, 3, 12]$	5	$(21, 2)$-arc	$[21, 3, 19]$	9
$(8, 2)$-arc	$[8, 3, 6]$	2	$(15, 2)$-arc	$[15, 3, 13]$	6	$(22, 2)$-arc	$[22, 3, 20]$	9
$(9, 2)$-arc	$[9, 3, 7]$	3	$(16, 2)$-arc	$[16, 3, 14]$	6	$(23, 2)$-arc	$[23, 3, 21]$	10
$(10, 2)$-arc	$[10, 3, 8]$	3	$(17, 2)$-arc	$[17, 3, 15]$	7			
$(11, 2)$-arc	$[11, 3, 9]$	4	$(18, 2)$-arc	$[18, 3, 16]$	7			

5.5 Codes of Dimension Five

From Theorem 5.9, if $q = 13, n = 5$, then in $\mathrm{PG}(2, 13)$ there exists a correspondence between $(k, 5)$-arcs and projective $[k, 5, k - 5]_{13}$-codes.

Table 5.2 shows codes corresponding to $(k, 5)$-arcs in $\mathrm{PG}(2, 13)$ and e is the number of errors corrected.

TABLE 5.2: Codes of dimension 5 over $\mathbb{F}_{13}$

$\mathcal{K}$	Code	e	$\mathcal{K}$	Code	e	$\mathcal{K}$	Code	e
$(7, 5)$-arc	$[7, 5, 2]$	0	$(22, 5)$-arc	$[22, 5, 17]$	8	$(37, 5)$-arc	$[37, 5, 32]$	16
$(8, 5)$-arc	$[8, 5, 3]$	1	$(23, 5)$-arc	$[23, 5, 18]$	9	$(38, 5)$-arc	$[38, 5, 33]$	16
$(9, 5)$-arc	$[9, 5, 4]$	2	$(24, 5)$-arc	$[24, 5, 19]$	9	$(39, 5)$-arc	$[39, 5, 34]$	17
$(10, 5)$-arc	$[10, 5, 5]$	2	$(25, 5)$-arc	$[25, 5, 20]$	10	$(40, 5)$-arc	$[40, 5, 35]$	17
$(11, 5)$-arc	$[11, 5, 6]$	3	$(26, 5)$-arc	$[26, 5, 21]$	10	$(41, 5)$-arc	$[41, 5, 36]$	18
$(12, 5)$-arc	$[12, 5, 7]$	3	$(27, 5)$-arc	$[27, 5, 22]$	11	$(42, 5)$-arc	$[42, 5, 37]$	18
$(13, 5)$-arc	$[13, 5, 8]$	4	$(28, 5)$-arc	$[28, 5, 23]$	11	$(43, 5)$-arc	$[43, 5, 38]$	19
$(14, 5)$-arc	$[14, 5, 9]$	4	$(29, 5)$-arc	$[29, 5, 24]$	12	$(44, 5)$-arc	$[44, 5, 39]$	19
$(15, 5)$-arc	$[15, 5, 10]$	5	$(30, 5)$-arc	$[30, 5, 25]$	12	$(45, 5)$-arc	$[45, 5, 40]$	20
$(16, 5)$-arc	$[16, 5, 11]$	5	$(31, 5)$-arc	$[31, 5, 26]$	13	$(46, 5)$-arc	$[46, 5, 41]$	20
$(17, 5)$-arc	$[17, 5, 12]$	6	$(32, 5)$-arc	$[32, 5, 27]$	13	$(47, 5)$-arc	$[47, 5, 42]$	21
$(18, 5)$-arc	$[18, 5, 13]$	6	$(33, 5)$-arc	$[33, 5, 28]$	14	$(48, 5)$-arc	$[48, 5, 43]$	21
$(19, 5)$-arc	$[19, 5, 14]$	7	$(34, 5)$-arc	$[34, 5, 29]$	14	$(49, 5)$-arc	$[49, 5, 44]$	22
$(20, 5)$-arc	$[20, 5, 15]$	7	$(35, 5)$-arc	$[35, 5, 30]$	15			
$(21, 5)$-arc	$[21, 5, 16]$	8	$(36, 5)$-arc	$[36, 5, 31]$	15			

5.6 Chapter Summary

This chapter has shown that arcs have applications to coding theory. Each arc can be interpreted as a linear code.

Chapter 6

Operational Research

6.1 Introduction

This chapter aims at providing a snapshot of operational research (OR). It starts by giving a historical overview of the emergence of this discipline of science. A number of the definitions of operational research are presented with the aim of developing a better understanding of its concepts. Finally, a brief description of each stage of the operational research process is provided.

6.2 Development of Operational Research

The concept of operational research emerged in the late 1930s, in the UK, mainly in military and industrial operations. In fact, OR can be traced back to the period of the Second World War when it was introduced as a systematic method to deal with the scarcity of resources and other strategic problems in the military operations, mainly by the British army. The main motive for its development was the introduction of the radar defence system for the Royal Air Force, and the belief that radar system would solve many problems for fighter direction and control. Thus, OR has been used to incorporate radar data with ground-based viewer data for fighter interruption. Due to its success during the war, several scientists realised that the principles that they had used to solve military-related problems can be applied to various other issues in other fields. Therefore, OR has been introduced to other disciplines such as health care, manufacturing, and transportation. As an example, in 1947, George Dantzig developed the simplex algorithm for Linear Programming (LP). This helped the growth of OR, and it is one of the most widely used techniques of OR. Secondly, the computer revolution made the process of dealing with complex problems much easier, particularly the development of OR packages. As soon as the simplex method had been developed and used, the expansion of other methods followed rapidly. The last twenty years have witnessed the development of most of the OR techniques that are now used, such as computer simulation, game theory, scheduling algorithms, non-linear dynamic programming and integer dynamic programming. For more details see [58] and [32].

6.3 The Concept of Operational Research

The concept of operational research has commonly been misinterpreted by many as a combination of mathematical tools. Although, operational research is fundamentally based on a variety of mathematical models and techniques, however, in reality, its scope goes much beyond that operational research, according to the Institute for Operational Research and the Management Science (INFORMS) as cited by Monios and Bergqvist [50], 2017, is a *"discipline that deals with the application of advanced analytical methods to help make better decisions"*. Whereas Churchman, [24] in 1957, who is seen as one of the pioneers of operations research, gives a more comprehensive definition, and describes OR as *"the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to problems"*. Similarly, Rajgopal [52] in 2004 defines it as a *" systematic approach to solving problems, which uses one or more analytical tools in the process of analysis "*. In light of the aforementioned definitions, operational research can be viewed as an objective methodology or a complementary analytic approach that helps managers and decision-makers make the most appropriate decisions. In other words, it plays a consultative role through providing decision-makers with a set of robust, scientifically derived alternative solutions for a complex problem at hand. Nevertheless, the final decision regarding selecting the most appropriate solution is always left to be made by a human being based on their knowledge, perspective and experience.

6.4 The Operational Research Approach

Having considered operational research as a tool to facilitate making good decisions, it is now essential to have a better understanding of such a tool to secure its successful application to a general problem. Accordingly, this section is dedicated to presenting general guidelines for the implementation of operational research in practice. According to Rajgopal [52] 2004 the process of implementing operational research can be decomposed into seven sequential stages as briefly described here in below.

Orientation: This is the first stage in the operational research approach. The main objective of this stage is to establish the multifunctional team that will be responsible for addressing the problem at hand. Each multifunctional team has, typically, a leader, and involves members from different departments that will be influenced by or affect the problem.

The key aim of the orientation stage is to develop a clear understanding of the problem and its relation to various operational aspects of the system, and to reach a general agreement on what should be the most important focus of the project. This can be achieved through the regular meeting and discussions of the team members along with reviewing and studying documents and literature related to the problem.

Problem Definition: This stage can be seen as the most difficult stage of the operational research approach in most cases. It aims to provide a clear definition of the problem with regard to both the problem's scope and the results required. This can be met through defining three main components of the problem, namely:

(i) establishing a very clear objective of the study alongside determining its scope, so as to set limits for the analysis;

(ii) specifying factors that can influence the defined objective. This should also be accompanied by providing a set of all the decision alternatives that can be made by the decision maker and which can affect the objective;

(iii) defining the constraints on the decision alternatives and setting boundaries for the modelled system operates.

Data Collection: Having defined the problem clearly, the multifunctional team at this stage should work on collecting the data required to translating the problem into a model that can be objectively analysed. Typically, data can be gathered from two main sources, namely, observation and standard. In terms of observation, data is usually derived from observing the technology of the system in operation. Whereas, in regard to the second source of data, the company documents and records are used to provide standard information on the problem at hand. However, it is also common to employ other data collection techniques such as interview and questionnaires in collecting the required data.

Model Formulation: This stage aims at translating the defined problem into mathematical relationships using the data collected throughout stage three. Based on the complexity of the resulting model, a standard mathematical model such as linear programming can be used to solve a simple model and arrive at a solution. Whereas, for a very complex model, the team may prefer to simplify the mode and utilize a heuristic approach, or even use simulations. However, the option to use a combination of different mathematical, heuristic and simulation models still exists to resolve the decision problem in many cases.

Model Solution: This is considered as the simplest stage amongst all operational research stages. The main objective of this stage is to provide a solution to the constructed model of the decision problem by using one or a combination of problem-solving techniques. During this stage, a particular attention should be paid to the sensitivity analysis, which concerns with acquiring more information about the conduct of the most favourable solution when the model is subjected to some parameter changes. This is particular critical in case the parameters of the constructed model cannot be estimated precisely.

Validation and Analysis: This is considered an important stage that must be promptly undertaken after obtaining the optimum solution and before developing the final course of action for implementation. It aims to test and check whether or not the proposed solution model can do what it is expected to do, and whether it is able to adequately predict the behaviour of the system under study. Indeed, the operational research team has to be sure that the outputs of the proposed solution model do not include unexpected or surprising results. In other words, the team should secure that the proposed solution makes sense, and its results are intuitively acceptable. This can be achieved by comparing the output of the proposed model with historical output data under similar implementation conditions. Obtaining similar outputs gives an indication of the validity of the proposed model. Moreover, operational research team may also need to use simulation as a technique for verifying the results of the proposed mathematical model. This is particularly important when the proposed model represents a new system, and where the historical data and output is not available.

Implementation and Monitoring: This is the final stage of operational research approach. The key objective of this stage is to translate the results obtained from the solution of a validated model into operating instructions and procedures that are understandable by the people who will be responsible for implementing and administering the recommended system. The responsibility of undertaking this task lies mainly with the operations team. Nevertheless, having completed the implementation process, the responsibility for monitoring the system is typically shifted to an operating team.

6.5 Chapter Summary

This chapter provides a brief overview of operational research. Also, the historical development of operational research and its basic concepts have been explained.

Chapter 7

Application of Finite Geometry to Operational Research

7.1 Introduction

This chapter aims to discuss applications to operational research (OR). The results of finite geometry can be applied to game theory, probability, and scheduling problems, which are the some of the important aspects of operational research.

7.2 Application to Probability

An affine plane of order n has n^2 points, n^2+n lines and $n+1$ parallel classes. Then some problems of probability, scheduling, and game theory can be solved using affine planes.

Problem 1: There are nine teams to match up in a schedule of four days D_i, where each team plays once in group of 3.

Solution To design this schedule use $\mathrm{AG}(2,q)$. Since in $\mathrm{PG}(2,3)$ there are 13 points, 13 lines, each line has 4 points and 4 lines pass through each point, then Table 7.1 shows the points of $\mathrm{PG}(2,3)$ and Table 7.2 illustrates lines of $\mathrm{PG}(2,3)$.

TABLE 7.1: Points of PG(2, 3)

$P_1(1,0,0)$	$P_2(0,1,0)$	$P_3(0,0,1)$	$P_4(1,2,1)$	$P_5(1,1,2)$
$P_6(1,0,2)$	$P_7(1,0,1)$	$P_8(1,1,1)$	$P_9(1,1,0)$	$P_{10}(0,1,1)$
$P_{11}(1,2,0)$	$P_{12}(0,1,2)$	$P_{13}(1,2,2)$		

TABLE 7.2: Lines of PG(2, 3)

L_1	P_1	P_2	P_9	P_{11}	L_2	P_2	P_3	P_{10}	P_{12}	L_3	P_3	P_4	P_{11}	P_{13}
L_4	P_4	P_5	P_{12}	P_1	L_5	P_5	P_6	P_{13}	P_2	L_6	P_6	P_7	P_1	P_3
L_7	P_7	P_8	P_2	P_4	L_8	P_8	P_9	P_3	P_5	L_9	P_9	P_{10}	P_4	P_6
L_{10}	P_{10}	P_{11}	P_5	P_7	L_{11}	P_{11}	P_{12}	P_6	P_8	L_{12}	P_{12}	P_{13}	P_7	P_9
L_{13}	P_{13}	P_1	P_8	P_{10}										

Since, $\mathrm{AG}(2,3) = \mathrm{PG}(2,3) \setminus L_2$, then by deleting line 2 and its points from each line of PG(2, 3), so there are nine 3-secants. Table 7.3 and Figure 7.2 illustrate the 3-secants in AG(2, 3). Figure 7.1 shows the coordinates of the points in AG(2, 3).

TABLE 7.3: 3-secants in AG(2, 3)

S_1	P_1	P_9	P_{11}	S_2	P_4	P_{11}	P_{13}	S_3	P_1	P_4	P_5
S_4	P_5	P_6	P_{13}	S_5	P_1	P_6	P_7	S_6	P_4	P_7	P_8
S_7	P_5	P_8	P_9	S_8	P_4	P_6	P_9	S_9	P_5	P_7	P_{11}
S_{10}	P_6	P_8	P_{11}	S_{11}	P_7	P_9	P_{13}	S_{12}	P_1	P_8	P_{13}

line 1	
x	y
0	0
1	0
2	0

line 5	
x	y
1	2
0	2
2	2

line 8	
x	y
1	1
1	0
1	2

line 11	
x	y
2	0
0	2
1	1

line 3	
x	y
2	1
2	0
2	2

line 6	
x	y
0	2
0	1
0	0

line 9	
x	y
1	0
2	1
0	2

line 12	
x	y
2	2
0	1
1	0

line 4	
x	y
2	1
1	2
0	0

line 7	
x	y
0	1
1	1
2	1

line 10	
x	y
2	0
1	2
0	1

line 13	
x	y
2	2
0	0
1	1

FIGURE 7.1: Coordinates of the points in AG$(2, 3)$

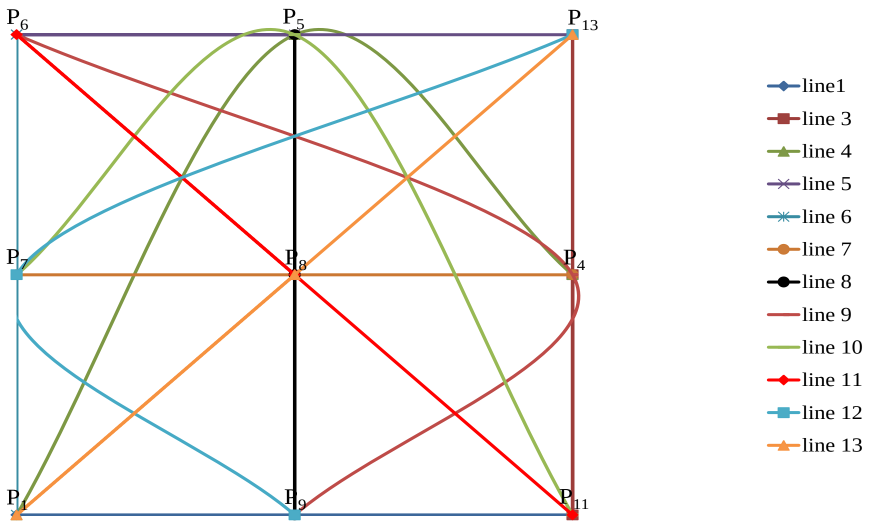


FIGURE 7.2: Lines in AG$(2, 3)$

Here, there are four parallel classes in AG$(2,3)$ as shown in Figures 7.3, 7.4, 7.5, 7.6.

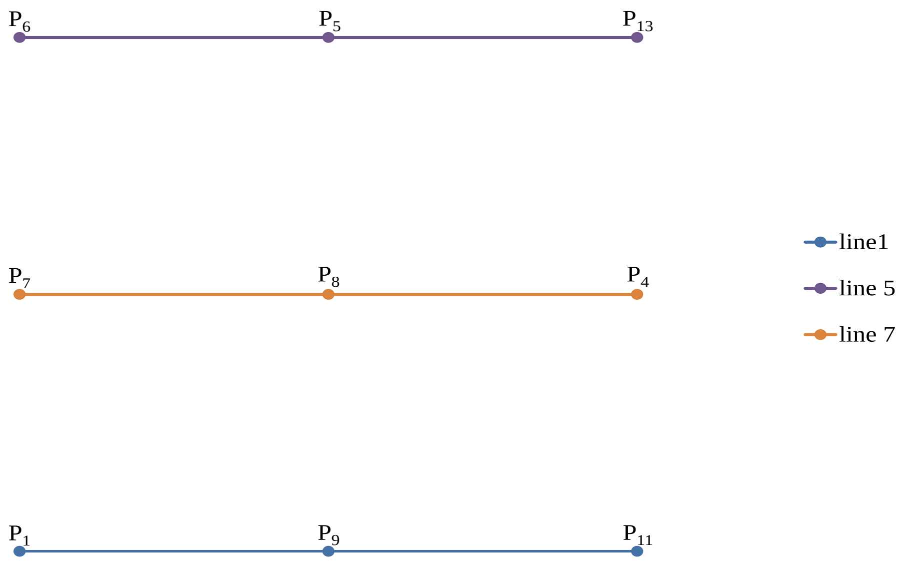


FIGURE 7.3: Class 1 of AG$(2,3)$

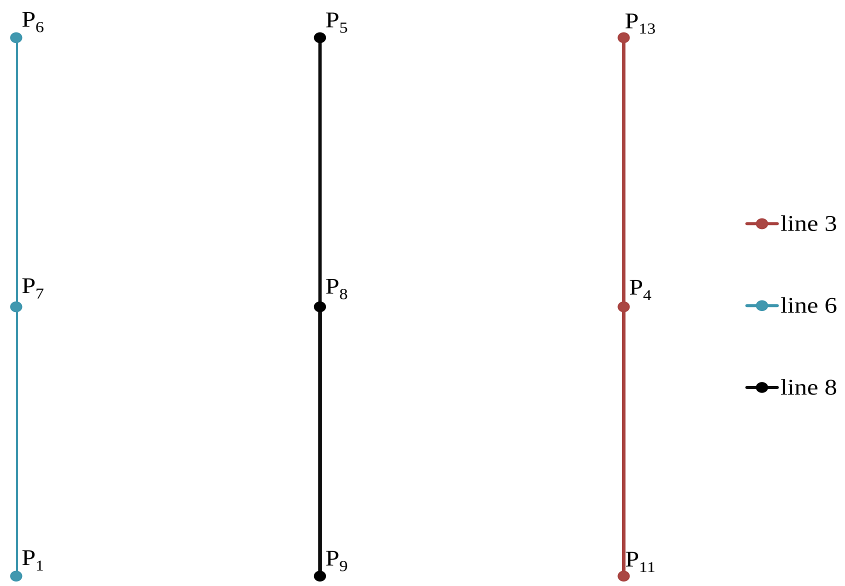


FIGURE 7.4: Class 2 of AG$(2,3)$

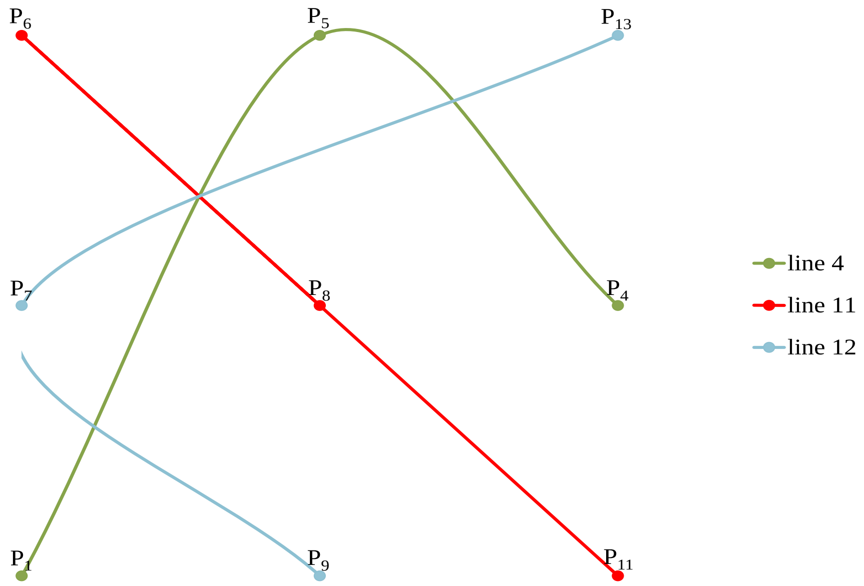


FIGURE 7.5: Class 3 of AG(2, 3)

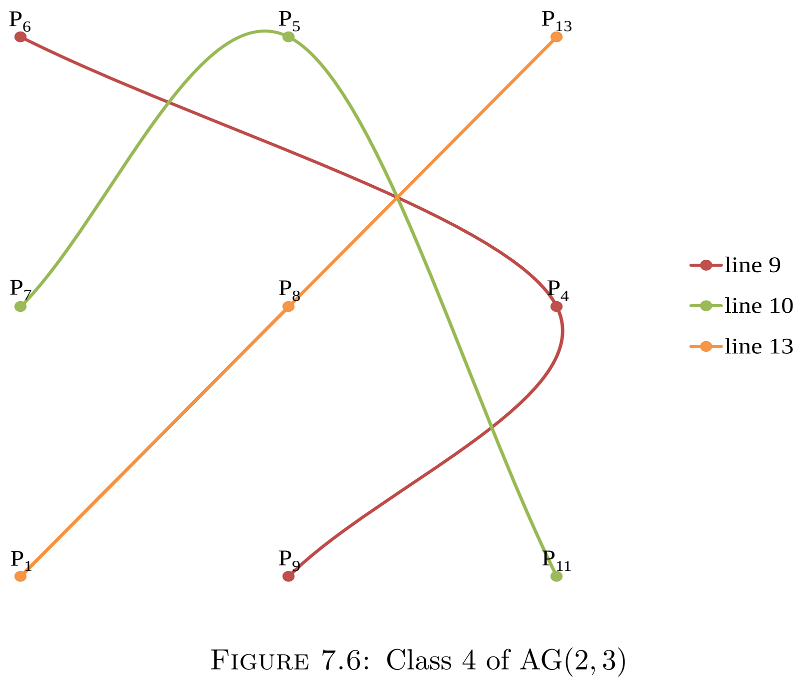


FIGURE 7.6: Class 4 of AG(2, 3)

So, each class represents a group of 3 players. Therefore,

$$\begin{array}{lllllllllllll}
D_1 & : & S_1 & \cup & S_4 & \cup & S_6 & = & (P_1, P_9, P_{11}) & (P_5, P_6, P_{13}) & (P_4, P_7, P_8) \\
D_2 & : & S_2 & \cup & S_5 & \cup & S_7 & = & (P_4, P_{11}, P_{13}) & (P_1, P_6, P_7) & (P_5, P_8, P_9) \\
D_3 & : & S_3 & \cup & S_{10} & \cup & S_{11} & = & (P_1, P_4, P_5) & (P_6, P_8, P_{11}) & (P_7, P_9, P_{13}) \\
D_4 & : & S_8 & \cup & S_9 & \cup & S_{12} & = & (P_4, P_6, P_9) & (P_5, P_7, P_{11}) & (P_1, P_8, P_{13})
\end{array}$$

7.3 Application of Projective Planes to Scheduling Problems

7.3.1 A simple Schedule Obtained via a Projective Plane

The projective plane $\mathrm{PG}(2, q)$ over the field $\mathbb{F}_q$ of q elements can be used as a model for scheduling problems given a certain number of teams and a certain number of weeks in the desired schedule. This representation can be used to assign the teams to multi-team meetings so that each team plays each other team exactly once. The model is designed as follows:

(a) each point represents a team;

(b) each line represents a meeting;

(c) the number of the meetings (competition) is $q + 1$.

In Problem 1, there are nine players to match up in a schedule of four days D_i. Here, using $\mathcal{K} = \mathrm{PG}(2, 3) \setminus L_2$. The nine points are $P_1, P_2, P_3, P_4, P_5, P_6, P_8, P_{10}, P_{11}$, which correspond to nine players as shown in Table 7.4.

Table 7.4

p = Player	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
Point in PG(2,3)	P_1	P_2	P_3	P_4	P_5	P_6	P_8	P_{10}	P_{11}

Also, let $S = \{S_i \mid i \leq 12\}$ represent the twelve 3-secants as illustrated in Table 7.5.

TABLE 7.5: 3-secants in PG(2, 3)

S_1	1	2	11	S_4	1	4	5	S_7	2	4	8	S_{10}	5	10	11
S_2	2	3	10	S_5	2	5	6	S_8	3	5	8	S_{11}	6	8	11
S_3	3	4	11	S_6	1	3	6	S_9	4	6	10	S_{12}	1	8	10

$$\begin{array}{ccccccccc}
S_1 & \cap & S_4 & \cap & S_6 & \cap & S_{12} & = & P_1 \\
S_1 & \cap & S_2 & \cap & S_5 & \cap & S_7 & = & P_2 \\
S_2 & \cap & S_3 & \cap & S_6 & \cap & S_8 & = & P_3 \\
S_3 & \cap & S_4 & \cap & S_7 & \cap & S_9 & = & P_4 \\
S_4 & \cap & S_5 & \cap & S_8 & \cap & S_{10} & = & P_5 \\
S_5 & \cap & S_6 & \cap & S_9 & \cap & S_{11} & = & P_6 \\
S_7 & \cap & S_8 & \cap & S_{11} & \cap & S_{12} & = & P_8 \\
S_2 & \cap & S_9 & \cap & S_{10} & \cap & S_{12} & = & P_{10} \\
S_1 & \cap & S_3 & \cap & S_{10} & \cap & S_{11} & = & P_{11}
\end{array}$$

Then, looking for sets of three lines which do not meet in $\mathcal{K}$.

$$\begin{array}{ccccccc}
S_1 & \cap & S_8 & \cap & S_9 & = & \emptyset \\
S_2 & \cap & S_4 & \cap & S_{11} & = & \emptyset \\
S_3 & \cap & S_5 & \cap & S_{12} & = & \emptyset \\
S_6 & \cap & S_7 & \cap & S_{10} & = & \emptyset
\end{array}$$

So, the solution of the problem is the following:

$$\begin{array}{ccccccccccc}
D_1 & : & S_1 & \cup & S_8 & \cup & S_9 & = & (P_1, P_2, P_{11}) & (P_3, P_5, P_8) & (P_4, P_6, P_{10}) \\
D_2 & : & S_2 & \cup & S_4 & \cup & S_{11} & = & (P_1, P_4, P_5) & (P_2, P_3, P_{10}) & (P_6, P_8, P_{11}) \\
D_3 & : & S_3 & \cup & S_5 & \cup & S_{12} & = & (P_1, P_8, P_{10}) & (P_2, P_5, P_6) & (P_3, P_4, P_{11}) \\
D_4 & : & S_6 & \cup & S_7 & \cup & S_{10} & = & (P_1, P_3, P_6) & (P_2, P_4, P_8) & (P_5, P_{10}, P_{11})
\end{array}$$

7.3.2 Constraint Satisfaction Problem (CSP)

Assume that

(1) there are m players,

(2) players can be divided into r groups,

(3) any two players play only one time in the same group,

(4) the number of weeks this can take place is w.

To solve this problem, first it is necessary to explain the following constraints:

(a) $P = \{P_1, \ldots, P_m\}$ is the set of all players;

(b) the size of each group of players is n;

(c) the number of groups of n players is $t = r \times w$;

(d) $S = \{S_1, \ldots, S_t\}$ is the set of all groups of players;

(e) $S_{ij} \cap S_{ik} = \emptyset$, where $1 \leq i \leq w$ and $1 \leq j < k \leq r$;

(f) $\mid S_{ij} \cap S_l k \mid \leq 1$, where $1 \leq i < l \leq w$ and $1 \leq j < k \leq r$.

Problem 2: In a local golf club, there are 28 social golfers, each of whom plays golf once every 9 days, and always in groups of 4. How can this be arranged?

Solution The target is to come up with a schedule for these golfers to last as many weeks as possible, such that no golfer plays in the same group as any other golfer on more than one occasion. In other words, according to Section 7.3.2 this problem can be described more explicitly by enumerating four constraints which must be satisfied. Then, $m = 28, n = 4, r = 7, w = 9, t = 63$. Also,

(a) $P = \{P_1, \ldots, P_{28}\}$ is the set of all players,

(b) the size of each group of players is 4,

(c) $S = \{S_1, \ldots, S_{63}\}$ is the set of all groups of players,

(d) $S_{ij} \cap S_{ik} = \emptyset$, where $1 \leq i \leq 9$ and $1 \leq j < k \leq 7$,

(e) $\mid S_{ij} \cap S_{lk} \mid \leq 1$, where $1 \leq i < l \leq 9$ and $1 \leq j < k \leq 7$.

Here, the problem is to divide 28 players into groups of 4 players. Therefore from the classification of $(k, 4)$-arcs in PG$(2, 8)$, the largest $(k, 4)$-arc $\mathcal{K}$ has $k = 28$ with i-secant distribution of type $(63, 0, 0, 0, 10)$ as mention in Table 3.26. Now, the sixty-three 4-secants represent the groups of 4 players, and the 28 players are represented by the 28 points. Table 7.6 shows the points of $\mathcal{K}$ which correspond to 28 players, and Table 7.7 illustrates the sixty-three 4-secants which correspond to the groups of 4 players.

TABLE 7.6

p = Player	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}
Point in PG(2,8)	P_1	P_2	P_3	P_4	P_{10}	P_{11}	P_{12}	P_{15}	P_{20}	P_{22}	P_{28}	P_{30}	P_{34}	P_{35}
p = Player	p_{15}	p_{16}	p_{17}	p_{18}	p_{19}	p_{20}	p_{21}	p_{22}	p_{23}	p_{24}	p_{25}	p_{26}	p_{27}	p_{28}
Point in PG(2,8)	P_{38}	P_{40}	P_{44}	P_{46}	P_{48}	P_{50}	P_{53}	P_{59}	P_{61}	P_{63}	P_{64}	P_{65}	P_{68}	P_{71}

TABLE 7.7: 4-secants in PG$(2, 8)$

S_1	p_1	p_2	p_4	p_{25}	S_2	p_{11}	p_{12}	p_{13}	p_{24}	S_3	p_{16}	p_{20}	p_{21}	p_{26}
S_4	p_2	p_3	p_{15}	p_{26}	S_5	p_{11}	p_{14}	p_{22}	p_{25}	S_6	p_5	p_8	p_{21}	p_{22}
S_7	p_3	p_4	p_5	p_{13}	S_8	p_5	p_{12}	p_{17}	p_{26}	S_9	p_6	p_{13}	p_{21}	p_{27}
S_{10}	p_4	p_6	p_{14}	p_{16}	S_{11}	p_6	p_9	p_{12}	p_{23}	S_{12}	p_7	p_{14}	p_{17}	p_{23}
S_{13}	p_7	p_9	p_{22}	p_{27}	S_{14}	p_7	p_{13}	p_{15}	p_{18}	S_{15}	p_{18}	p_{22}	p_{24}	p_{28}
S_{16}	p_5	p_{10}	p_{15}	p_{23}	S_{17}	p_{10}	p_{14}	p_{24}	p_{27}	S_{18}	p_8	p_9	p_{15}	p_{25}
S_{19}	p_6	p_8	p_{17}	p_{28}	S_{20}	p_{13}	p_{16}	p_{18}	p_{25}	S_{21}	p_{19}	p_{22}	p_{23}	p_{26}
S_{22}	p_5	p_7	p_{16}	p_{24}	S_{23}	p_8	p_{13}	p_{14}	p_{26}	S_{24}	p_1	p_{10}	p_{16}	p_{22}
S_{25}	p_5	p_6	p_{18}	p_{25}	S_{26}	p_{14}	p_{15}	p_{20}	p_{28}	S_{27}	p_2	p_{20}	p_{23}	p_{24}
S_{28}	p_1	p_6	p_7	p_{26}	S_{29}	p_{15}	p_{16}	p_{17}	p_{27}	S_{30}	p_3	p_{23}	p_{25}	p_{27}
S_{31}	p_2	p_7	p_8	p_{19}	S_{32}	p_1	p_{11}	p_{15}	p_{21}	S_{33}	p_4	p_9	p_{24}	p_{26}
S_{34}	p_3	p_9	p_{11}	p_{17}	S_{35}	p_2	p_9	p_{16}	p_{18}	S_{36}	p_{17}	p_{21}	p_{24}	p_{25}
S_{37}	p_4	p_8	p_{20}	p_{27}	S_{38}	p_3	p_{12}	p_{16}	p_{28}	S_{39}	p_{10}	p_{25}	p_{26}	p_{28}
S_{40}	p_8	p_{10}	p_{12}	p_{18}	S_{41}	p_4	p_{10}	p_{17}	p_{19}	S_{42}	p_{11}	p_{18}	p_{26}	p_{27}
S_{43}	p_9	p_{19}	p_{21}	p_{28}	S_{44}	p_1	p_{17}	p_{18}	p_{20}	S_{45}	p_1	p_{12}	p_{19}	p_{27}
S_{46}	p_9	p_{10}	p_{13}	p_{20}	S_{47}	p_2	p_{13}	p_{17}	p_{22}	S_{48}	p_2	p_5	p_{27}	p_{28}
S_{49}	p_1	p_5	p_9	p_{14}	S_{50}	p_3	p_{14}	p_{18}	p_{19}	S_{51}	p_3	p_6	p_{20}	p_{22}
S_{52}	p_2	p_6	p_{10}	p_{11}	S_{53}	p_4	p_{18}	p_{21}	p_{23}	S_{54}	p_4	p_7	p_{11}	p_{28}
S_{55}	p_3	p_7	p_{10}	p_{21}	S_{56}	p_5	p_{11}	p_{19}	p_{20}	S_{57}	p_1	p_{13}	p_{23}	p_{28}
S_{58}	p_4	p_{12}	p_{15}	p_{22}	S_{59}	p_6	p_{15}	p_{19}	p_{24}	S_{60}	p_2	p_{12}	p_{14}	p_{21}
S_{61}	p_8	p_{11}	p_{16}	p_{23}	S_{62}	p_7	p_{12}	p_{20}	p_{25}	S_{63}	p_1	p_3	p_8	p_{24}

Here each group meets thirty-two other groups in four different players, for instance, S_1 meets 8 other groups in each of p_1, p_2, p_4, p_{25}. Therefore, there remain 32 groups meeting S_1 at points other than those of S_1. This gives Table 7.8.

Table 7.8

$$
\begin{array}{ccccccccccccccccccc}
S_1 & \cap & S_{24} & \cap & S_{28} & \cap & S_{32} & \cap & S_{44} & \cap & S_{45} & \cap & S_{49} & \cap & S_{57} & \cap & S_{63} & = & p_1 \\
S_1 & \cap & S_4 & \cap & S_{27} & \cap & S_{31} & \cap & S_{35} & \cap & S_{47} & \cap & S_{48} & \cap & S_{52} & \cap & S_{60} & = & p_2 \\
S_4 & \cap & S_7 & \cap & S_{30} & \cap & S_{34} & \cap & S_{38} & \cap & S_{50} & \cap & S_{51} & \cap & S_{55} & \cap & S_{63} & = & p_3 \\
S_1 & \cap & S_7 & \cap & S_{10} & \cap & S_{33} & \cap & S_{37} & \cap & S_{41} & \cap & S_{53} & \cap & S_{54} & \cap & S_{58} & = & p_4 \\
S_6 & \cap & S_7 & \cap & S_8 & \cap & S_{16} & \cap & S_{22} & \cap & S_{25} & \cap & S_{48} & \cap & S_{49} & \cap & S_{56} & = & p_5 \\
S_9 & \cap & S_{10} & \cap & S_{11} & \cap & S_{19} & \cap & S_{25} & \cap & S_{28} & \cap & S_{51} & \cap & S_{52} & \cap & S_{59} & = & p_6 \\
S_{12} & \cap & S_{13} & \cap & S_{14} & \cap & S_{22} & \cap & S_{28} & \cap & S_{31} & \cap & S_{54} & \cap & S_{55} & \cap & S_{62} & = & p_7 \\
S_6 & \cap & S_{18} & \cap & S_{19} & \cap & S_{23} & \cap & S_{31} & \cap & S_{37} & \cap & S_{40} & \cap & S_{61} & \cap & S_{63} & = & p_8 \\
S_{11} & \cap & S_{13} & \cap & S_{18} & \cap & S_{33} & \cap & S_{34} & \cap & S_{35} & \cap & S_{43} & \cap & S_{46} & \cap & S_{49} & = & p_9 \\
S_{16} & \cap & S_{17} & \cap & S_{24} & \cap & S_{39} & \cap & S_{40} & \cap & S_{41} & \cap & S_{46} & \cap & S_{52} & \cap & S_{55} & = & p_{10} \\
S_2 & \cap & S_5 & \cap & S_{32} & \cap & S_{34} & \cap & S_{42} & \cap & S_{52} & \cap & S_{54} & \cap & S_{56} & \cap & S_{61} & = & p_{11} \\
S_2 & \cap & S_8 & \cap & S_{11} & \cap & S_{38} & \cap & S_{40} & \cap & S_{45} & \cap & S_{58} & \cap & S_{60} & \cap & S_{62} & = & p_{12} \\
S_2 & \cap & S_7 & \cap & S_9 & \cap & S_{14} & \cap & S_{20} & \cap & S_{23} & \cap & S_{46} & \cap & S_{47} & \cap & S_{57} & = & p_{13} \\
S_5 & \cap & S_{10} & \cap & S_{12} & \cap & S_{17} & \cap & S_{23} & \cap & S_{26} & \cap & S_{49} & \cap & S_{50} & \cap & S_{60} & = & p_{14} \\
S_4 & \cap & S_{14} & \cap & S_{16} & \cap & S_{18} & \cap & S_{26} & \cap & S_{29} & \cap & S_{32} & \cap & S_{58} & \cap & S_{59} & = & p_{15} \\
S_3 & \cap & S_{10} & \cap & S_{20} & \cap & S_{22} & \cap & S_{24} & \cap & S_{29} & \cap & S_{35} & \cap & S_{38} & \cap & S_{61} & = & p_{16} \\
S_8 & \cap & S_{12} & \cap & S_{19} & \cap & S_{29} & \cap & S_{34} & \cap & S_{36} & \cap & S_{41} & \cap & S_{44} & \cap & S_{47} & = & p_{17} \\
S_{14} & \cap & S_{15} & \cap & S_{25} & \cap & S_{35} & \cap & S_{40} & \cap & S_{42} & \cap & S_{44} & \cap & S_{50} & \cap & S_{53} & = & p_{18} \\
S_{20} & \cap & S_{21} & \cap & S_{31} & \cap & S_{41} & \cap & S_{43} & \cap & S_{45} & \cap & S_{50} & \cap & S_{56} & \cap & S_{59} & = & p_{19} \\
S_3 & \cap & S_{26} & \cap & S_{27} & \cap & S_{37} & \cap & S_{44} & \cap & S_{46} & \cap & S_{51} & \cap & S_{56} & \cap & S_{62} & = & p_{20} \\
S_3 & \cap & S_6 & \cap & S_9 & \cap & S_{32} & \cap & S_{36} & \cap & S_{43} & \cap & S_{53} & \cap & S_{55} & \cap & S_{60} & = & p_{21} \\
S_5 & \cap & S_6 & \cap & S_{13} & \cap & S_{15} & \cap & S_{21} & \cap & S_{24} & \cap & S_{47} & \cap & S_{51} & \cap & S_{58} & = & p_{22} \\
S_{11} & \cap & S_{12} & \cap & S_{16} & \cap & S_{21} & \cap & S_{27} & \cap & S_{30} & \cap & S_{53} & \cap & S_{58} & \cap & S_{61} & = & p_{23} \\
S_2 & \cap & S_{15} & \cap & S_{17} & \cap & S_{22} & \cap & S_{27} & \cap & S_{33} & \cap & S_{36} & \cap & S_{59} & \cap & S_{63} & = & p_{24} \\
S_1 & \cap & S_5 & \cap & S_{18} & \cap & S_{20} & \cap & S_{25} & \cap & S_{30} & \cap & S_{36} & \cap & S_{39} & \cap & S_{62} & = & p_{25} \\
S_3 & \cap & S_4 & \cap & S_8 & \cap & S_{21} & \cap & S_{23} & \cap & S_{28} & \cap & S_{33} & \cap & S_{39} & \cap & S_{42} & = & p_{26} \\
S_9 & \cap & S_{13} & \cap & S_{17} & \cap & S_{29} & \cap & S_{30} & \cap & S_{37} & \cap & S_{42} & \cap & S_{45} & \cap & S_{48} & = & p_{27} \\
S_{15} & \cap & S_{19} & \cap & S_{26} & \cap & S_{38} & \cap & S_{39} & \cap & S_{43} & \cap & S_{48} & \cap & S_{54} & \cap & S_{57} & = & p_{28}
\end{array}
$$

Also,

$$
\begin{array}{cccccccccccccc}
S_1 & \cap & S_8 & \cap & S_{14} & \cap & S_{17} & \cap & S_{43} & \cap & S_{51} & \cap & S_{61} & = \emptyset \\
S_4 & \cap & S_{10} & \cap & S_{13} & \cap & S_{30} & \cap & S_{36} & \cap & S_{40} & \cap & S_{56} & = \emptyset \\
S_3 & \cap & S_7 & \cap & S_{12} & \cap & S_{15} & \cap & S_{18} & \cap & S_{45} & \cap & S_{52} & = \emptyset \\
S_5 & \cap & S_9 & \cap & S_{16} & \cap & S_{31} & \cap & S_{33} & \cap & S_{38} & \cap & S_{44} & = \emptyset \\
S_{19} & \cap & S_{20} & \cap & S_{27} & \cap & S_{42} & \cap & S_{49} & \cap & S_{55} & \cap & S_{58} & = \emptyset \\
S_{11} & \cap & S_{22} & \cap & S_{32} & \cap & S_{37} & \cap & S_{39} & \cap & S_{47} & \cap & S_{50} & = \emptyset \\
S_{21} & \cap & S_{25} & \cap & S_{29} & \cap & S_{46} & \cap & S_{54} & \cap & S_{60} & \cap & S_{63} & = \emptyset \\
S_2 & \cap & S_6 & \cap & S_{26} & \cap & S_{28} & \cap & S_{35} & \cap & S_{41} & \cap & S_{57} & = \emptyset \\
S_{23} & \cap & S_{24} & \cap & S_{34} & \cap & S_{48} & \cap & S_{53} & \cap & S_{59} & \cap & S_{62} & = \emptyset
\end{array}
$$

Therefore, all the sixty-three groups of 4 players can be divided into nine days D_i, where there are seven 4 players in each day.

$$\begin{array}{lcccccccccccccccc}
D_1 & : & S_1 & \cup & S_8 & \cup & S_{14} & \cup & S_{17} & \cup & S_{43} & \cup & S_{51} & \cup & S_{61} \\
D_2 & : & S_4 & \cup & S_{10} & \cup & S_{13} & \cup & S_{30} & \cup & S_{36} & \cup & S_{40} & \cup & S_{56} \\
D_3 & : & S_3 & \cup & S_7 & \cup & S_{12} & \cup & S_{15} & \cup & S_{18} & \cup & S_{45} & \cup & S_{52} \\
D_4 & : & S_5 & \cup & S_9 & \cup & S_{16} & \cup & S_{31} & \cup & S_{33} & \cup & S_{38} & \cup & S_{44} \\
D_5 & : & S_{19} & \cup & S_{20} & \cup & S_{27} & \cup & S_{42} & \cup & S_{49} & \cup & S_{55} & \cup & S_{58} \\
D_6 & : & S_{11} & \cup & S_{22} & \cup & S_{32} & \cup & S_{37} & \cup & S_{39} & \cup & S_{47} & \cup & S_{50} \\
D_7 & : & S_{21} & \cup & S_{25} & \cup & S_{29} & \cup & S_{46} & \cup & S_{54} & \cup & S_{60} & \cup & S_{63} \\
D_8 & : & S_2 & \cup & S_6 & \cup & S_{26} & \cup & S_{28} & \cup & S_{35} & \cup & S_{41} & \cup & S_{57} \\
D_9 & : & S_{23} & \cup & S_{24} & \cup & S_{34} & \cup & S_{48} & \cup & S_{53} & \cup & S_{59} & \cup & S_{62}
\end{array}$$

Problem 3: Sixteen players need to be arranged in groups of four and three players to play for six days, where each player plays in the same number of groups.

Solution The target of this problem is to divide 16 players into groups of 3 and 4 players. From the classification of $(16, 4)$-arcs $\mathcal{K}$ in PG$(2, 5)$, with i-secant distribution of type $(12, 16, 0, 0, 3)$ as mention in Table 3.27. Then, the twelve 4-secants represent the groups of 4 players and the sixteen 3-secants represent the group of 3 players and the 16 players are represented by the 16 points. Table 7.9 shows the points of $\mathcal{K}$ which correspond to 16 players, Table 7.10 shows the groups of 4 players and Table 7.11 shows the groups of 3 players.

TABLE 7.9

p = Player	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
Point in PG(2,5)	P_1	P_2	P_3	P_5	P_6	P_7	P_8	P_{12}
p = Player	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
Point in PG(2,5)	P_{13}	P_{16}	P_{18}	P_{20}	P_{23}	P_{25}	P_{30}	P_{31}

TABLE 7.10

$(p_1, p_2, p_{25}, p_{30})$	$(p_2, p_3, p_{16}, p_{13})$	(p_3, p_5, p_6, p_{25})	$(p_6, p_7, p_{20}, p_{30})$
(p_5, p_7, p_8, p_{31})	$(p_1, p_5, p_{12}, p_{13})$	$(p_1, p_7, p_{16}, p_{18})$	$(p_2, p_8, p_{12}, p_{20})$
$(p_3, p_{13}, p_{18}, p_{20})$	$(p_6, p_{12}, p_{16}, p_{23})$	$(p_8, p_{18}, p_{23}, p_{25})$	$(p_{13}, p_{23}, p_{30}, p_{31})$

Table 7.11

(p_1, p_3, p_{23})	(p_2, p_5, p_{18})	(p_1, p_6, p_8)	(p_2, p_7, p_{23})
(p_3, p_8, p_{30})	(p_{12}, p_{25}, p_{31})	(p_2, p_6, p_{13})	(p_3, p_7, p_{12})
(p_8, p_{13}, p_{16})	(p_5, p_{16}, p_{30})	(p_6, p_{18}, p_{31})	(p_5, p_{20}, p_{23})
(p_7, p_{13}, p_{25})	(p_{16}, p_{20}, p_{25})	(p_{12}, p_{18}, p_{30})	(p_1, p_{20}, p_{31})

So, the solution of this problem is the following:

D_1 : (p_3, p_5, p_6, p_{25}) $(p_2, p_8, p_{12}, p_{20})$ $(p_1, p_7, p_{16}, p_{18})$ $(p_{13}, p_{23}, p_{30}, p_{31})$

D_2 : $(p_1, p_2, p_{25}, p_{30})$ (p_5, p_7, p_8, p_{31}) $(p_3, p_{13}, p_{18}, p_{20})$ $(p_6, p_{12}, p_{16}, p_{23})$

D_3 : $(p_6, p_7, p_{20}, p_{30})$ (p_1, p_3, p_{23}) $(p_2, p_5, p_{18})(p_{12}, p_{25}, p_{31})$ (p_8, p_{13}, p_{16})

D_4 : $(p_1, p_5, p_{12}, p_{13})$ (p_2, p_7, p_{23}) $(p_3, p_8, p_{30})(p_6, p_{18}, p_{31})$ (p_{16}, p_{20}, p_{25})

D_5 : $(p_2, p_3, p_{16}, p_{31})$ (p_1, p_6, p_8) $(p_5, p_{20}, p_{23})(p_7, p_{13}, p_{25})$ (p_{12}, p_{18}, p_{30})

D_6 : $(p_8, p_{18}, p_{23}, p_{25})$ (p_1, p_{20}, p_{31}) $(p_2, p_6, p_{13})(p_3, p_7, p_{12})$ (p_5, p_{16}, p_{30})

Here, each players participated in 3 groups of 4 players and 3 groups of 3 players.

7.4 Generalisation of Kirkman's Problem

Definition 7.1 (Kirkman' Problem)**.** A Kirkman triple system is a method of choosing 3-sets called blocks from a set of v objects, and of partitioning the set of blocks into subsets called rounds, so that each object occurs exactly once per round and each object-pair occurs in exactly one triple in the system.

A generalisation of Kirkman's problem as follow. If $m = q(n-1) + n$ schoolgirls go walking each day in $q + 1 - q/n$ groups of n, they can walk for $q + 1$ days so that each girl has walked in the same group as has every other girl and with no girl twice.

Problem 4: A school has 64 girls, who need to be arrange in 8 groups of 8 girls. How can they be arranged walk for nine days, such that no two girls walk together twice?

Solution Here, $m = 64, r = 8, w = 8, t = 64$. According to Section 7.3.2 this problem can be described more explicitly by enumerating four constraints which must be satisfied. Then,

(a) $P = \{P_1, \ldots, P_{64}\}$ is the set of all girls,

(b) the size of each group of girls is 8,

(c) $S = \{S_1, \ldots, S_{64}\}$ is the set of all groups of girls,

(d) $S_{ij} \cap S_{ik} = \emptyset$, where $1 \leq i \leq 9$ and $1 \leq j < k \leq 8$,

(e) $\mid S_{ij} \cap S_{lk} \mid \leq 1$, where $1 \leq i < l \leq 9$ and $1 \leq j < k \leq 8$.

The classification in Table 7.12 and Table 7.13 shows the solution of this problem.

Table 7.12: 8-secants in PG(2, 8)

D_1	1	2	4	8	16	32	37	55	63	66	70	5	21	26	44	53
	61	62	68	3	19	24	42	51	57	58	60	72	15	20	38	47
	49	50	52	56	7	12	30	39	33	34	36	40	48	69	14	23
	28	29	31	35	43	59	9	18	10	11	13	17	25	41	46	73
D_2	2	3	5	9	17	33	38	56	66	68	72	7	23	28	46	55
	62	63	69	4	20	25	43	52	58	59	61	73	16	21	39	39
	50	51	53	57	8	13	31	40	34	35	37	41	49	70	15	24
	29	30	32	36	44	60	10	19	11	12	14	18	26	42	47	1
D_3	3	4	10	18	34	39	57	66	72	73	2	14	30	35	53	62
	48	49	51	55	63	11	29	38	43	44	46	50	58	1	24	33
	25	26	28	32	40	56	61	15	16	17	19	23	31	47	52	70
	7	9	13	21	37	42	60	69	5	8	12	20	36	41	59	68
D_4	4	5	7	11	19	35	40	58	68	70	1	9	25	30	48	57
	66	69	73	8	24	29	47	56	60	61	63	2	18	23	41	50
	52	53	55	59	10	15	33	42	36	37	39	43	51	72	17	26
	31	32	34	38	46	62	12	21	13	14	16	20	28	44	49	3
D_5	7	8	10	14	38	43	61	70	59	60	62	66	1	17	40	49
	41	42	44	48	56	72	4	31	32	33	35	39	47	63	68	13
	23	25	29	37	53	58	3	12	21	24	28	36	52	57	2	11
	19	20	26	34	50	55	73	9	15	16	18	30	46	51	69	5
D_6	8	9	11	15	23	39	44	62	72	1	5	13	29	34	52	61
	70	73	4	12	28	33	51	60	68	69	2	10	26	31	49	58
	56	57	59	63	14	19	37	46	40	41	43	47	55	3	21	30
	35	36	38	42	50	66	16	25	17	18	20	24	32	48	53	7

TABLE 7.13: 8-secants in PG(2, 8)

	9	10	12	16	24	40	63	72	55	56	58	62	70	13	18	36
D_7	46	48	52	60	3	8	26	35	44	47	51	59	2	7	25	34
	42	43	49	57	73	5	23	32	38	39	41	53	69	1	19	28
	30	31	33	37	61	66	11	20	14	15	17	21	29	50	68	4
	12	13	15	19	43	48	66	2	69	70	72	3	11	32	50	59
D_8	46	47	49	53	61	4	9	36	37	38	40	44	52	68	73	18
	28	30	34	42	58	63	8	17	26	29	33	41	57	62	7	16
	24	25	31	39	55	60	5	14	20	21	23	35	51	56	1	10
	18	19	21	25	33	49	72	8	73	1	3	7	15	31	36	63
D_9	55	57	61	69	12	17	35	44	53	56	60	68	11	16	34	43
	51	52	58	66	9	14	32	41	47	48	50	62	5	10	28	37
	39	40	42	46	70	2	20	29	23	24	26	30	38	59	4	13

Problem 5: A school has ten girls. How can they arranged to walk for six days, such that no two girls walk together twice?

Solution The target of this problem is to divide 10 girls into groups of 2 and 3 girls. From the classification of $(10, 3)$-arcs $\mathcal{K}$ in PG$(2, 5)$, with i-secant distribution of type $(10, 15, 0, 0, 6)$. Then, the ten 3-secants represent the groups of 3 girls and the fifteen 2-secants represent the group of 2 girls, and the 10 girls are represented by the 10 points. Table 7.14 shows the points of $\mathcal{K}$ which correspond to 10 girls, Table 7.15 shows the groups of 3 girls and Table 7.16 shows the groups of 2 girls.

TABLE 7.14

g = girl	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
Point in PG(2,5)	P_1	P_2	P_3	P_5	P_6	P_7	P_9	P_{11}	P_{17}	P_{24}

TABLE 7.15

(g_1, g_3, g_9)	(g_2, g_4, g_{10})	(g_3, g_4, g_5)	(g_1, g_5, g_7)	(g_2, g_6, g_7)
(g_3, g_8, g_{10})	(g_2, g_5, g_8)	(g_4, g_7, g_9)	(g_1, g_6, g_8)	(g_6, g_9, g_{10})

Table 7.16

(g_1, g_2)	(g_2, g_3)	(g_5, g_6)	(g_4, g_6)	(g_7, g_8)
(g_1, g_4)	(g_3, g_6)	(g_5, g_9)	(g_2, g_9)	(g_3, g_7)
(g_4, g_8)	(g_5, g_{10})	(g_7, g_{10})	(g_8, g_9)	(g_1, g_{10})

So, the solution of this problem is the following:

$D_1 :\ (g_1, g_3, g_9)\ (g_2, g_5, g_8)\ (g_4, g_6)\ (g_7, g_{10})$

$D_2 :\ (g_2, g_4, g_{10})\ (g_1, g_6, g_8)\ (g_3, g_7)\ (g_5, g_9)$

$D_3 :\ (g_3, g_4, g_5)\ (g_2, g_6, g_7)\ (g_1, g_{10})\ (g_8, g_9)$

$D_4 :\ (g_1, g_5, g_7)\ (g_6, g_9, g_{10})\ (g_2, g_3)\ (g_4, g_8)$

$D_5 :\ (g_3, g_8, g_{10})\ (g_4, g_7, g_9)\ (g_1, g_2)\ (g_5, g_6)$

$D_6 :\ (g_1, g_4)\ (g_2, g_9)\ (g_3, g_6)\ (g_7, g_8)\ (g_5, g_{10})$

Here, each girl participated in 3 groups of 3 girls and 3 groups of 2 girls.

7.5 Chapter Summary

In this chapter some application of finite geometry to operational research, probability and scheduling problems are explained. Constraint satisfaction problems is considered an important application to (k, n)-arcs to $\mathrm{AG}(2, q)$, $\mathrm{PG}(2, q)$. Also, a generalisation of Kirkman's problem is showed. Finally, the chapter presents, in details, solutions of five problems to show the application of projective plane to operational research.

Appendix A

Points and Lines

A.1 Projective Plane of Order Five

Since $\mathrm{PG}(2,5)$ contains 31 points and 31 lines, every line contains 6 points and every point passes through it 6 lines. To find the points of $\mathrm{PG}(2,5)$. Let

$$f(x) = x^3 - x^2 - 1$$

be an irreducible polynomial over $\mathbb{F}_5$, then the matrix

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

gives a cyclic projectivity by right multiplication on the points of $\mathrm{PG}(2,5)$. Let P_0 be represented by vector $(1,0,0)$. Then $P_0T^i = P_i$, $i = 0,1,...,30$, are 31 points of $\mathrm{PG}(2,5)$. To find the lines of $\mathrm{PG}(2,5)$ Let L_1 be the line which contains the points $1,2,15,21,25,30$, then $L_1T^i = L_{i+1}$, $i = 1,2,...,30$ are the lines of $\mathrm{PG}(2,5)$. Table A.1 and Table A.2 show the points and lines of $\mathrm{PG}(2,5)$.

TABLE A.1: Points of PG$(2,5)$

$P_1(1,0,0)$	$P_7(1,3,4)$	$P_{13}(1,2,2)$	$P_{19}(1,1,3)$	$P_{25}(1,1,0)$	$P_{31}(0,1,4)$
$P_2(0,1,0)$	$P_8(1,4,3)$	$P_{14}(1,3,2)$	$P_{20}(1,2,3)$	$P_{26}(0,1,1)$	
$P_3(0,0,1)$	$P_9(1,2,4)$	$P_{15}(1,3,0)$	$P_{21}(1,2,0)$	$P_{27}(1,0,2)$	
$P_4(1,0,1)$	$P_{10}(1,4,4)$	$P_{16}(0,1,3)$	$P_{22}(0,1,2)$	$P_{28}(1,3,1)$	
$P_5(1,1,1)$	$P_{11}(1,4,2)$	$P_{17}(1,0,3)$	$P_{23}(1,0,4)$	$P_{29}(1,1,4)$	
$P_6(1,1,2)$	$P_{12}(1,3,3)$	$P_{18}(1,2,1)$	$P_{24}(1,4,1)$	$P_{30}(1,4,0)$	

TABLE A.2: Lines of PG$(2,5)$

L_1	P_1	P_2	P_{15}	P_{21}	P_{25}	P_{30}	L_{17}	P_{17}	P_{18}	P_{31}	P_6	P_{10}	P_{15}
L_2	P_2	P_3	P_{16}	P_{22}	P_{26}	P_{31}	L_{18}	P_{18}	P_{19}	P_1	P_7	P_{11}	P_{16}
L_3	P_3	P_4	P_{17}	P_{23}	P_{27}	P_1	L_{19}	P_{19}	P_{20}	P_2	P_8	P_{12}	P_{17}
L_4	P_4	P_5	P_{18}	P_{24}	P_{28}	P_2	L_{20}	P_{20}	P_{21}	P_3	P_9	P_{13}	P_{18}
L_5	P_5	P_6	P_{19}	P_{25}	P_{29}	P_3	L_{21}	P_{21}	P_{22}	P_4	P_{10}	P_{14}	P_{19}
L_6	P_6	P_7	P_{20}	P_{26}	P_{30}	P_4	L_{22}	P_{22}	P_{23}	P_5	P_{11}	P_{15}	P_{20}
L_7	P_7	P_8	P_{21}	P_{27}	P_{31}	P_5	L_{23}	P_{23}	P_{24}	P_6	P_{12}	P_{16}	P_{21}
L_8	P_8	P_9	P_{22}	P_{28}	P_1	P_6	L_{24}	P_{24}	P_{25}	P_7	P_{13}	P_{17}	P_{22}
L_9	P_9	P_{10}	P_{23}	P_{29}	P_2	P_7	L_{25}	P_{25}	P_{26}	P_8	P_{14}	P_{18}	P_{23}
L_{10}	P_{10}	P_{11}	P_{24}	P_{30}	P_3	P_8	L_{26}	P_{26}	P_{27}	P_9	P_{15}	P_{19}	P_{24}
L_{11}	P_{11}	P_{12}	P_{25}	P_{31}	P_4	P_9	L_{27}	P_{27}	P_{28}	P_{10}	P_{16}	P_{20}	P_{25}
L_{12}	P_{12}	P_{13}	P_{26}	P_1	P_5	P_{10}	L_{28}	P_{28}	P_{29}	P_{11}	P_{17}	P_{21}	P_{26}
L_{13}	P_{13}	P_{14}	P_{27}	P_2	P_6	P_{11}	L_{29}	P_{29}	P_{30}	P_{12}	P_{18}	P_{22}	P_{27}
L_{14}	P_{14}	P_{15}	P_{28}	P_3	P_7	P_{12}	L_{30}	P_{30}	P_{31}	P_{13}	P_{19}	P_{23}	P_{28}
L_{15}	P_{15}	P_{16}	P_{29}	P_4	P_8	P_{13}	L_{31}	P_{31}	P_1	P_{14}	P_{20}	P_{24}	P_{29}
L_{16}	P_{16}	P_{17}	P_{30}	P_5	P_9	P_{14}							

A.2 Projective Plane of Order Seven

Since PG(2,7) contains 57 points and 57 lines, every line contains 8 points and every point passes through it 8 lines. To find the points of PG(2,7). Let

$$f(x) = x^3 - x^2 - 2x - 4$$

be an irreducible polynomial over $\mathbb{F}_7$, then the matrix

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

gives a cyclic projectivity by right multiplication on the points of PG(2,7). Let P_0 be represented by vector $(1,0,0)$. Then $P_0T^i = P_i$, $i = 0,1,...,56$, are 57 points of PG(2,7). To find the lines of PG(2,7) Let L_1 be the line which contains the points $1,2,8,25,37,39,50,55$, then $L_1T^i = L_{i+1}$, $i = 1,2,...,56$ are the lines of PG(2,7). Table A.3 and Table A.4 show the points and lines of PG(2,7).

Table A.3: Points of PG(2,7)

$P_1(1,0,0)$	$P_{11}(1,3,5)$	$P_{21}(1,5,3)$	$P_{31}(1,5,4)$	$P_{41}(1,4,6)$	$P_{51}(0,1,1)$
$P_2(0,1,0)$	$P_{12}(1,3,6)$	$P_{22}(1,0,3)$	$P_{32}(1,1,1)$	$P_{42}(1,2,1)$	$P_{52}(1,4,4)$
$P_3(0,0,1)$	$P_{13}(1,2,3)$	$P_{23}(1,0,2)$	$P_{33}(1,6,4)$	$P_{43}(1,6,6)$	$P_{53}(1,1,4)$
$P_4(1,4,2)$	$P_{14}(1,0,1)$	$P_{24}(1,5,2)$	$P_{34}(1,1,5)$	$P_{44}(1,2,4)$	$P_{54}(1,1,6)$
$P_5(1,5,6)$	$P_{15}(1,6,2)$	$P_{25}(1,5,0)$	$P_{35}(1,3,1)$	$P_{45}(1,1,3)$	$P_{55}(1,2,0)$
$P_6(1,2,6)$	$P_{16}(1,5,1)$	$P_{26}(0,1,5)$	$P_{36}(1,6,1)$	$P_{46}(1,0,5)$	$P_{56}(0,1,2)$
$P_7(1,2,5)$	$P_{17}(1,6,5)$	$P_{27}(1,4,1)$	$P_{37}(1,6,0)$	$P_{47}(1,3,2)$	$P_{57}(1,4,3)$
$P_8(1,3,0)$	$P_{18}(1,3,3)$	$P_{28}(1,6,3)$	$P_{38}(0,1,6)$	$P_{48}(1,5,5)$	
$P_9(0,1,3)$	$P_{19}(1,0,4)$	$P_{29}(1,0,6)$	$P_{39}(1,4,0)$	$P_{49}(1,3,4)$	
$P_{10}(1,4,5)$	$P_{20}(1,1,2)$	$P_{30}(1,2,2)$	$P_{40}(0,1,4)$	$P_{50}(1,1,0)$	

TABLE A.4: Lines of PG$(2,7)$

L_{1}	P_{1}	P_{2}	P_{8}	P_{25}	P_{37}	P_{39}	P_{50}	P_{55}	L_{30}	P_{30}	P_{31}	P_{37}	P_{54}	P_{9}	P_{11}	P_{22}	P_{27}
L_{2}	P_{2}	P_{3}	P_{9}	P_{26}	P_{38}	P_{40}	P_{51}	P_{56}	L_{31}	P_{31}	P_{32}	P_{38}	P_{55}	P_{10}	P_{12}	P_{23}	P_{28}
L_{3}	P_{3}	P_{4}	P_{10}	P_{27}	P_{39}	P_{41}	P_{52}	P_{57}	L_{32}	P_{32}	P_{33}	P_{39}	P_{56}	P_{11}	P_{13}	P_{24}	P_{29}
L_{4}	P_{4}	P_{5}	P_{11}	P_{28}	P_{40}	P_{42}	P_{53}	P_{1}	L_{33}	P_{33}	P_{34}	P_{40}	P_{57}	P_{12}	P_{14}	P_{25}	P_{30}
L_{5}	P_{5}	P_{6}	P_{12}	P_{29}	P_{41}	P_{43}	P_{54}	P_{2}	L_{34}	P_{34}	P_{35}	P_{41}	P_{1}	P_{13}	P_{15}	P_{26}	P_{31}
L_{6}	P_{6}	P_{7}	P_{13}	P_{30}	P_{42}	P_{44}	P_{55}	P_{3}	L_{35}	P_{35}	P_{36}	P_{42}	P_{2}	P_{14}	P_{16}	P_{27}	P_{32}
L_{7}	P_{7}	P_{8}	P_{14}	P_{31}	P_{43}	P_{45}	P_{56}	P_{4}	L_{36}	P_{36}	P_{37}	P_{43}	P_{3}	P_{15}	P_{17}	P_{28}	P_{33}
L_{8}	P_{8}	P_{9}	P_{15}	P_{32}	P_{44}	P_{46}	P_{57}	P_{5}	L_{37}	P_{37}	P_{38}	P_{44}	P_{4}	P_{16}	P_{18}	P_{29}	P_{34}
L_{9}	P_{9}	P_{10}	P_{16}	P_{33}	P_{45}	P_{47}	P_{1}	P_{6}	L_{38}	P_{38}	P_{39}	P_{45}	P_{5}	P_{17}	P_{19}	P_{30}	P_{35}
L_{10}	P_{10}	P_{11}	P_{17}	P_{34}	P_{46}	P_{48}	P_{2}	P_{7}	L_{39}	P_{39}	P_{40}	P_{46}	P_{6}	P_{18}	P_{20}	P_{31}	P_{36}
L_{11}	P_{11}	P_{12}	P_{18}	P_{35}	P_{47}	P_{49}	P_{3}	P_{8}	L_{40}	P_{40}	P_{41}	P_{47}	P_{7}	P_{19}	P_{21}	P_{32}	P_{37}
L_{12}	P_{12}	P_{13}	P_{19}	P_{36}	P_{48}	P_{50}	P_{4}	P_{9}	L_{41}	P_{41}	P_{42}	P_{48}	P_{8}	P_{20}	P_{22}	P_{33}	P_{38}
L_{13}	P_{13}	P_{14}	P_{20}	P_{37}	P_{49}	P_{51}	P_{5}	P_{10}	L_{42}	P_{42}	P_{43}	P_{49}	P_{9}	P_{21}	P_{23}	P_{34}	P_{39}
L_{14}	P_{14}	P_{15}	P_{21}	P_{38}	P_{50}	P_{52}	P_{6}	P_{11}	L_{43}	P_{43}	P_{44}	P_{50}	P_{10}	P_{22}	P_{24}	P_{35}	P_{40}
L_{15}	P_{15}	P_{16}	P_{22}	P_{39}	P_{51}	P_{53}	P_{7}	P_{12}	L_{44}	P_{44}	P_{45}	P_{51}	P_{11}	P_{23}	P_{25}	P_{36}	P_{41}
L_{16}	P_{16}	P_{17}	P_{23}	P_{40}	P_{52}	P_{54}	P_{8}	P_{13}	L_{45}	P_{45}	P_{46}	P_{52}	P_{12}	P_{24}	P_{26}	P_{37}	P_{42}
L_{17}	P_{17}	P_{18}	P_{24}	P_{41}	P_{53}	P_{55}	P_{9}	P_{14}	L_{46}	P_{46}	P_{47}	P_{53}	P_{13}	P_{25}	P_{27}	P_{38}	P_{43}
L_{18}	P_{18}	P_{19}	P_{25}	P_{42}	P_{54}	P_{56}	P_{10}	P_{15}	L_{47}	P_{47}	P_{48}	P_{54}	P_{14}	P_{26}	P_{28}	P_{39}	P_{44}
L_{19}	P_{19}	P_{20}	P_{26}	P_{43}	P_{55}	P_{57}	P_{11}	P_{16}	L_{48}	P_{48}	P_{49}	P_{55}	P_{15}	P_{27}	P_{29}	P_{40}	P_{45}
L_{20}	P_{20}	P_{21}	P_{27}	P_{44}	P_{56}	P_{1}	P_{12}	P_{17}	L_{49}	P_{49}	P_{50}	P_{56}	P_{16}	P_{28}	P_{30}	P_{41}	P_{46}
L_{21}	P_{21}	P_{22}	P_{28}	P_{45}	P_{57}	P_{2}	P_{13}	P_{18}	L_{50}	P_{50}	P_{51}	P_{57}	P_{17}	P_{29}	P_{31}	P_{42}	P_{47}
L_{22}	P_{22}	P_{23}	P_{29}	P_{46}	P_{1}	P_{3}	P_{14}	P_{19}	L_{51}	P_{51}	P_{52}	P_{1}	P_{18}	P_{30}	P_{32}	P_{43}	P_{48}
L_{23}	P_{23}	P_{24}	P_{30}	P_{47}	P_{2}	P_{4}	P_{15}	P_{20}	L_{52}	P_{52}	P_{53}	P_{2}	P_{19}	P_{31}	P_{33}	P_{44}	P_{49}
L_{24}	P_{24}	P_{25}	P_{31}	P_{48}	P_{3}	P_{5}	P_{16}	P_{21}	L_{53}	P_{53}	P_{54}	P_{3}	P_{20}	P_{32}	P_{34}	P_{45}	P_{50}
L_{25}	P_{25}	P_{26}	P_{32}	P_{49}	P_{4}	P_{6}	P_{17}	P_{22}	L_{54}	P_{54}	P_{55}	P_{4}	P_{21}	P_{33}	P_{35}	P_{46}	P_{51}
L_{26}	P_{26}	P_{27}	P_{33}	P_{50}	P_{5}	P_{7}	P_{18}	P_{23}	L_{55}	P_{55}	P_{56}	P_{5}	P_{22}	P_{34}	P_{36}	P_{47}	P_{52}
L_{27}	P_{27}	P_{28}	P_{34}	P_{51}	P_{6}	P_{8}	P_{19}	P_{24}	L_{56}	P_{56}	P_{57}	P_{6}	P_{23}	P_{35}	P_{37}	P_{48}	P_{53}
L_{28}	P_{28}	P_{29}	P_{35}	P_{52}	P_{7}	P_{9}	P_{20}	P_{25}	L_{57}	P_{57}	P_{1}	P_{7}	P_{24}	P_{36}	P_{38}	P_{49}	P_{54}
L_{29}	P_{29}	P_{30}	P_{36}	P_{53}	P_{8}	P_{10}	P_{21}	P_{26}									

A.3 Projective Plane of Order Eight

$\mathbb{F}_8 = \{0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6 : \alpha^3 + \alpha^2 + 1 = 2 = 0\}$

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha^4 & 1 & 0 \end{bmatrix}.$$

Table A.5 and Table A.6 show the points and lines of PG$(2, 8)$.

TABLE A.5: Points of PG$(2, 8)$

$P_1(1,0,0)$	$P_2(0,1,0)$	$P_3(0,0,1)$	$P_4(1,\alpha^3,0)$	$P_5(0,1,\alpha^3)$	$P_6(1,\alpha^3,1)$
$P_7(1,0,\alpha^6)$	$P_8(1,\alpha^6,0)$	$P_9(0,1,\alpha^6)$	$P_{10}(1,\alpha^3,\alpha^4)$	$P_{11}(1,\alpha^4,\alpha^2)$	$P_{12}(1,1,\alpha^5)$
$P_{13}(1,\alpha^2,\alpha^5)$	$P_{14}(1,\alpha^2,1)$	$P_{15}(1,0,\alpha^5)$	$P_{16}(1,\alpha^2,0)$	$P_{17}(0,1,\alpha^2)$	$P_{18}(1,\alpha^3,\alpha)$
$P_{19}(1,\alpha^5,\alpha^5)$	$P_{20}(1,\alpha^2,\alpha^3)$	$P_{21}(1,\alpha,\alpha^2)$	$P_{22}(1,1,\alpha^2)$	$P_{23}(1,1,\alpha)$	$P_{24}(1,\alpha^5,\alpha^2)$
$P_{25}(1,1,\alpha^6)$	$P_{26}(1,\alpha^6,\alpha^4)$	$P_{27}(1,\alpha^4,\alpha^5)$	$P_{28}(1,\alpha^2,\alpha^2)$	$P_{29}(1,1,\alpha^3)$	$P_{30}(1,\alpha,1)$
$P_{31}(1,0,\alpha^4)$	$P_{32}(1,\alpha^4,0)$	$P_{33}(0,1,\alpha^4)$	$P_{34}(1,\alpha^3,\alpha^6)$	$P_{35}(1,\alpha^6,1)$	$P_{36}(1,0,\alpha^2)$
$P_{37}(1,1,0)$	$P_{38}(0,1,1)$	$P_{39}(1,\alpha^3,\alpha^3)$	$P_{40}(1,\alpha,\alpha^3)$	$P_{41}(1,\alpha,\alpha)$	$P_{42}(1,\alpha^5,\alpha^3)$
$P_{43}(1,\alpha,\alpha^5)$	$P_{44}(1,\alpha^2,\alpha^6)$	$P_{45}(1,\alpha^6,\alpha^6)$	$P_{46}(1,\alpha^6,\alpha^3)$	$P_{47}(1,\alpha,\alpha^6)$	$P_{48}(1,\alpha^6,\alpha^5)$
$P_{49}(1,\alpha^2,\alpha^4)$	$P_{50}(1,\alpha^4,\alpha)$	$P_{51}(1,\alpha^5,\alpha^6)$	$P_{52}(1,\alpha^6,\alpha^2)$	$P_{53}(1,1,1)$	$P_{54}(1,0,\alpha^3)$
$P_{55}(1,\alpha,0)$	$P_{56}(0,1,\alpha)$	$P_{57}(1,\alpha^3,\alpha^2)$	$P_{58}(1,1,\alpha^4)$	$P_{59}(1,\alpha^4,\alpha^6)$	$P_{60}(1,\alpha^6,\alpha)$
$P_{61}(1,\alpha^5,\alpha)$	$P_{62}(1,\alpha^5,1)$	$P_{63}(1,0,\alpha)$	$P_{64}(1,\alpha^5,0)$	$P_{65}(0,1,\alpha^5)$	$P_{66}(1,\alpha^3,\alpha^5)$
$P_{67}(1,\alpha^2,\alpha)$	$P_{68}(1,\alpha^5,\alpha^4)$	$P_{69}(1,\alpha^4,\alpha^4)$	$P_{70}(1,\alpha^4,\alpha^3)$	$P_{71}(1,\alpha,\alpha^4)$	$P_{72}(1,\alpha^4,1)$
$P_{73}(1,0,1)$					

TABLE A.6: Lines of PG$(2, 8)$

L_1	P_1	P_2	P_4	P_8	P_{16}	P_{32}	P_{37}	P_{55}	P_{64}
L_2	P_2	P_3	P_5	P_9	P_{17}	P_{33}	P_{38}	P_{56}	P_{65}
$\vdots$									
L_{73}	P_{73}	P_1	P_3	P_7	P_{15}	P_{31}	P_{36}	P_{54}	P_{63}

A.4 Projective Plane of Order Nine

$\mathbb{F}_9 = \{0, 1, 2, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \beta^6 : \beta^2 - \beta - 1 = 3 = 0\}$

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & \beta \end{bmatrix}.$$

Table A.7 and Table A.8 show the points and lines of PG$(2, 9)$.

TABLE A.7: Points of PG$(2, 9)$

$P_1(1,0,0)$	$P_2(0,1,0)$	$P_3(0,0,1)$	$P_4(1,0,\beta)$	$P_5(1,\beta^5,\beta)$	$P_6(1,\beta^5,\beta^4)$
$P_7(1,\beta^2,1)$	$P_8(1,1,2)$	$P_9(1,\beta^6,\beta^3)$	$P_{10}(1,\beta^3,2)$	$P_{11}(1,\beta^6,2)$	$P_{12}(1,\beta^6,0)$
$P_{13}(0,1,\beta^6)$	$P_{14}(1,0,1)$	$P_{15}(1,1,\beta)$	$P_{16}(1,\beta^5,0)$	$P_{17}(0,1,\beta^5)$	$P_{18}(1,0,\beta^5)$
$P_{19}(1,\beta,\beta)$	$P_{20}(1,\beta^5,\beta^2)$	$P_{21}(1,\beta^4,2)$	$P_{22}(1,\beta^6,\beta^4)$	$P_{23}(1,\beta^2,\beta^5)$	$P_{24}(1,\beta,\beta^6)$
$P_{25}(1,2,2)$	$P_{26}(1,\beta^6,\beta^2)$	$P_{27}(1,\beta^4,\beta^4)$	$P_{28}(1,\beta^2,\beta^2)$	$P_{29}(1,\beta^4,\beta^2)$	$P_{30}(1,\beta^4,\beta^5)$
$P_{31}(1,\beta,\beta^3)$	$P_{32}(1,\beta^3,0)$	$P_{33}(0,1,\beta^3)$	$P_{34}(1,0,\beta^4)$	$P_{35}(1,\beta^2,\beta)$	$P_{36}(1,\beta^5,1)$
$P_{37}(1,1,0)$	$P_{38}(0,1,1)$	$P_{39}(1,0,4)$	$P_{40}(1,\beta^4,\beta)$	$P_{41}(1,\beta^5,2)$	$P_{42}(1,\beta^6,\beta^6)$
$P_{43}(1,2,\beta^2)$	$P_{44}(1,\beta^4,0)$	$P_{45}(0,1,\beta^4)$	$P_{46}(1,0,2)$	$P_{47}(1,\beta^6,\beta)$	$P_{48}(1,\beta^5,\beta^6)$
$P_{49}(1,2,\beta^3)$	$P_{50}(1,\beta^3,\beta^6)$	$P_{51}(1,2,\beta^6)$	$P_{52}(1,2,\beta^5)$	$P_{53}(1,\beta,2)$	$P_{54}(1,\beta^6,1)$
$P_{55}(1,1,\beta^3)$	$P_{56}(1,\beta^3,\beta^4)$	$P_{57}(1,\beta^2,\beta^3)$	$P_{58}(1,\beta^3,\beta^3)$	$P_{59}(1,\beta^3,\beta^2)$	$P_{60}(1,\beta^4,1)$
$P_{61}(1,1,\beta^6)$	$P_{62}(1,2,1)$	$P_{63}(1,1,1)$	$P_{64}(1,1,\beta^2)$	$P_{65}(1,\beta^4,\beta^6)$	$P_{66}(1,2,0)$
$P_{67}(0,1,2)$	$P_{68}(1,0,\beta^3)$	$P_{69}(1,\beta^3,\beta)$	$P_{70}(1,\beta^5,\beta^5)$	$P_{71}(1,\beta,\beta^2)$	$P_{72}(1,\beta^4,\beta^3)$
$P_{73}(1,\beta^3,1)$	$P_{74}(1,1,\beta^4)$	$P_{75}(1,\beta^2,2)$	$P_{76}(1,\beta^6,\beta^5)$	$P_{77}(1,\beta,1)$	$P_{78}(1,1,\beta^5)$
$P_{79}(1,\beta,\beta^5)$	$P_{80}(1,\beta,\beta^4)$	$P_{81}(1,\beta^2,\beta^6)$	$P_{82}(1,2,\beta^4)$	$P_{83}(1,\beta^2,\beta^4)$	$P_{84}(1,\beta^2,0)$
$P_{85}(0,1,\beta^2)$	$P_{86}(1,0,\beta^6)$	$P_{87}(1,2,\beta)$	$P_{88}(1,\beta^5,\beta^3)$	$P_{89}(1,\beta^3,\beta^57)$	$P_{90}(1,\beta,0)$
$P_{91}(0,1,\beta)$					

TABLE A.8: Lines of PG$(2, 9)$

L_1	P_1	P_2	P_{12}	P_{16}	P_{32}	P_{37}	P_{44}	P_{66}	P_{84}	P_{90}
L_2	P_2	P_3	P_{13}	P_{17}	P_{33}	P_{38}	P_{45}	P_{67}	P_{85}	P_{91}
$\vdots$										
L_{91}	P_{91}	P_1	P_{11}	P_{15}	P_{31}	P_{36}	P_{43}	P_{65}	P_{83}	P_{89}

A.5 Projective Plane of Order Eleven

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Table A.9 and Table A.10 show the points and lines of PG(2, 11).

TABLE A.9: Points of PG(2, 11)

$P_1(1,0,0)$	$P_2(0,1,0)$	$P_3(0,0,1)$	$P_4(1,4,4)$	$P_5(1,5,8)$	$P_6(1,10,1)$
$P_7(1,8,0)$	$P_8(0,1,8)$	$P_9(1,4,10)$	$P_{10}(1,0,10)$	$P_{11}(1,0,4)$	$P_{12}(1,5,4)$
$P_{13}(1,5,9)$	$P_{14}(1,2,5)$	$P_{15}(1,7,10)$	$P_{16}(1,0,9)$	$P_{17}(1,2,4)$	$P_{18}(1,5,6)$
$P_{19}(1,1,0)$	$P_{20}(0,1,1)$	$P_{21}(1,4,8)$	$P_{22}(1,10,6)$	$P_{23}(1,1,7)$	$P_{24}(1,3,3)$
$P_{25}(1,9,8)$	$P_{26}(1,10,3)$	$P_{27}(1,9,10)$	$P_{28}(1,0,1)$	$P_{29}(1,8,4)$	$P_{30}(1,5,1)$
$P_{31}(1,8,2)$	$P_{32}(1,6,9)$	$P_{33}(1,2,3)$	$P_{34}(1,9,3)$	$P_{35}(1,9,5)$	$P_{36}(1,7,9)$
$P_{37}(1,2,1)$	$P_{38}(1,8,1)$	$P_{39}(1,8,3)$	$P_{40}(1,9,0)$	$P_{41}(0,1,9)$	$P_{42}(1,4,2)$
$P_{43}(1,6,1)$	$P_{44}(1,8,6)$	$P_{45}(1,1,2)$	$P_{46}(1,6,6)$	$P_{47}(1,1,8)$	$P_{48}(1,10,10)$
$P_{49}(1,0,8)$	$P_{50}(1,10,4)$	$P_{51}(1,5,3)$	$P_{52}(1,9,7)$	$P_{53}(1,3,6)$	$P_{54}(1,1,6)$
$P_{55}(1,1,1)$	$P_{56}(1,8,8)$	$P_{57}(1,10,8)$	$P_{58}(1,10,9)$	$P_{59}(1,2,6)$	$P_{60}(1,1,9)$
$P_{61}(1,2,2)$	$P_{62}(1,6,8)$	$P_{63}(1,10,7)$	$P_{64}(1,3,5)$	$P_{65}(1,7,2)$	$P_{66}(1,6,7)$
$P_{67}(1,3,9)$	$P_{68}(1,2,9)$	$P_{69}(1,2,0)$	$P_{70}(0,1,2)$	$P_{71}(1,4,6)$	$P_{72}(1,1,3)$
$P_{73}(1,9,9)$	$P_{74}(1,2,8)$	$P_{75}(1,10,5)$	$P_{76}(1,7,1)$	$P_{77}(1,8,10)$	$P_{78}(1,0,5)$
$P_{79}(1,7,4)$	$P_{80}(1,5,0)$	$P_{81}(0,1,5)$	$P_{82}(1,4,7)$	$P_{83}(1,3,0)$	$P_{84}(0,1,3)$
$P_{85}(1,4,9)$	$P_{86}(1,2,7)$	$P_{87}(1,3,2)$	$P_{88}(1,6,10)$	$P_{89}(1,0,2)$	$P_{90}(1,6,4)$
$P_{91}(1,5,10)$	$P_{92}(1,0,6)$	$P_{93}(1,1,4)$	$P_{94}(1,5,5)$	$P_{95}(1,7,8)$	$P_{96}(1,10,2)$
$P_{97}(1,6,2)$	$P_{98}(1,6,5)$	$P_{99}(1,7,0)$	$P_{100}(0,1,7)$	$P_{101}(1,4,3)$	$P_{102}(1,9,2)$
$P_{103}(1,6,0)$	$P_{104}(0,1,6)$	$P_{105}(1,4,1)$	$P_{106}(1,8,9)$	$P_{107}(1,2,10)$	$P_{108}(1,0,7)$
$P_{109}(1,3,4)$	$P_{110}(1,5,7)$	$P_{111}(1,3,10)$	$P_{112}(1,0,3)$	$P_{113}(1,9,4)$	$P_{114}(1,5,2)$
$P_{115}(1,6,3)$	$P_{116}(1,9,1)$	$P_{117}(1,8,7)$	$P_{118}(1,3,7)$	$P_{119}(1,3,1)$	$P_{120}(1,8,5)$
$P_{121}(1,7,6)$	$P_{122}(1,1,5)$	$P_{123}(1,7,7)$	$P_{124}(1,3,8)$	$P_{125}(1,10,0)$	$P_{126}(0,1,10)$
$P_{127}(1,4,0)$	$P_{128}(0,1,4)$	$P_{129}(1,4,5)$	$P_{130}(1,7,5)$	$P_{131}(1,7,3)$	$P_{132}(1,9,6)$
$P_{133}(1,1,10)$					

TABLE A.10: Lines of PG(2, 11)

L_1	P_1	P_2	P_9	P_{22}	P_{40}	P_{44}	P_{49}	P_{55}	P_{74}	P_{106}	P_{118}	P_{132}
L_2	P_2	P_3	P_{10}	P_{23}	P_{41}	P_{45}	P_{50}	P_{56}	P_{75}	P_{107}	P_{119}	P_{133}
$\vdots$												
L_{133}	P_{133}	P_1	P_8	P_{21}	P_{39}	P_{43}	P_{48}	P_{54}	P_{73}	P_{105}	P_{117}	P_{131}

A.6 Projective Plane of Order Thirteen

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Table A.11, Table A.12 and Table A.13 show the points and lines of PG(2, 13).

TABLE A.11: $P_1, \ldots P_{84}$ of PG(2, 13)

$P_1(1,0,0)$	$P_2(0,1,0)$	$P_3(0,0,1)$	$P_4(1,0,7)$	$P_5(1,1,7)$	$P_6(1,1,8)$
$P_7(1,9,3)$	$P_8(1,11,2)$	$P_9(1,10,0)$	$P_{10}(0,1,10)$	$P_{11}(1,0,9)$	$P_{12}(1,8,7)$
$P_{13}(1,1,2)$	$P_{14}(1,10,4)$	$P_{15}(1,5,5)$	$P_{16}(1,4,1)$	$P_{17}(1,7,9)$	$P_{18}(1,8,11)$
$P_{19}(1,3,5)$	$P_{20}(1,4,6)$	$P_{21}(1,12,3)$	$P_{22}(1,11,9)$	$P_{23}(1,8,4)$	$P_{24}(1,5,8)$
$P_{25}(1,9,0)$	$P_{26}(0,1,9)$	$P_{27}(1,0,2)$	$P_{28}(1,10,7)$	$P_{29}(1,1,4)$	$P_{30}(1,5,12)$
$P_{31}(1,6,11)$	$P_{32}(1,3,12)$	$P_{33}(1,6,12)$	$P_{34}(1,6,4)$	$P_{35}(1,5,11)$	$P_{36}(1,3,9)$
$P_{37}(1,8,5)$	$P_{38}(1,4,0)$	$P_{39}(0,1,4)$	$P_{40}(1,0,12)$	$P_{41}(1,6,7)$	$P_{42}(1,1,0)$
$P_{43}(0,1,1)$	$P_{44}(1,0,1)$	$P_{45}(1,7,7)$	$P_{46}(1,1,1)$	$P_{47}(1,7,1)$	$P_{48}(1,7,4)$
$P_{49}(1,5,3)$	$P_{50}(1,11,10)$	$P_{51}(1,2,3)$	$P_{52}(1,11,3)$	$P_{53}(1,11,11)$	$P_{54}(1,3,1)$
$P_{55}(1,7,2)$	$P_{56}(1,10,12)$	$P_{57}(1,6,2)$	$P_{58}(1,10,2)$	$P_{59}(1,10,3)$	$P_{60}(1,11,0)$
$P_{61}(0,1,11)$	$P_{62}(1,0,10)$	$P_{63}(1,2,7)$	$P_{64}(1,1,9)$	$P_{65}(1,8,2)$	$P_{66}(1,10,9)$
$P_{67}(1,8,9)$	$P_{68}(1,8,6)$	$P_{69}(1,12,12)$	$P_{70}(1,6,1)$	$P_{71}(1,7,10)$	$P_{72}(1,2,8)$
$P_{73}(1,9,12)$	$P_{74}(1,6,9)$	$P_{75}(1,8,3)$	$P_{76}(1,11,4)$	$P_{77}(1,5,10)$	$P_{78}(1,2,4)$
$P_{79}(1,5,4)$	$P_{80}(1,5,6)$	$P_{81}(1,12,2)$	$P_{82}(1,10,10)$	$P_{83}(1,2,1)$	$P_{84}(1,7,8)$

TABLE A.12: $P_{85}, \dots P_{183}$ of $\mathrm{PG}(2,13)$

$P_{85}(1,9,5)$	$P_{86}(1,4,4)$	$P_{87}(1,5,1)$	$P_{88}(1,7,3)$	$P_{89}(1,11,6)$	$P_{90}(1,12,9)$
$P_{91}(1,8,12)$	$P_{92}(1,6,3)$	$P_{93}(1,11,8)$	$P_{94}(1,9,2)$	$P_{95}(1,10,6)$	$P_{96}(1,12,10)$
$P_{97}(1,2,5)$	$P_{98}(1,4,2)$	$P_{99}(1,10,8)$	$P_{100}(1,9,6)$	$P_{101}(1,12,11)$	$P_{102}(1,3,4)$
$P_{103}(1,5,9)$	$P_{104}(1,8,8)$	$P_{105}(1,9,1)$	$P_{106}(1,7,5)$	$P_{107}(1,4,9)$	$P_{108}(1,8,0)$
$P_{109}(0,1,8)$	$P_{110}(1,0,3)$	$P_{111}(1,11,7)$	$P_{112}(1,1,5)$	$P_{113}(1,4,11)$	$P_{114}(1,3,6)$
$P_{115}(1,12,4)$	$P_{116}(1,5,2)$	$P_{117}(1,10,5)$	$P_{118}(1,4,8)$	$P_{119}(1,9,4)$	$P_{120}(1,5,0)$
$P_{121}(0,1,5)$	$P_{122}(1,0,11)$	$P_{123}(1,3,7)$	$P_{124}(1,1,10)$	$P_{125}(1,2,9)$	$P_{126}(1,8,10)$
$P_{127}(1,2,10)$	$P_{128}(1,2,11)$	$P_{129}(1,3,0)$	$P_{130}(0,1,3)$	$P_{131}(1,0,5)$	$P_{132}(1,4,7)$
$P_{133}(1,1,11)$	$P_{134}(1,3,10)$	$P_{135}(1,2,0)$	$P_{136}(0,1,2)$	$P_{137}(1,0,4)$	$P_{138}(1,5,7)$
$P_{139}(1,1,12)$	$P_{140}(1,6,0)$	$P_{141}(0,1,6)$	$P_{142}(1,0,6)$	$P_{143}(1,12,7)$	$P_{144}(1,1,6)$
$P_{145}(1,12,6)$	$P_{146}(1,12,8)$	$P_{147}(1,9,11)$	$P_{148}(1,3,8)$	$P_{149}(1,9,8)$	$P_{150}(1,9,10)$
$P_{151}(1,2,12)$	$P_{152}(1,6,6)$	$P_{153}(1,12,1)$	$P_{154}(1,7,0)$	$P_{155}(0,1,7)$	$P_{156}(1,0,8)$
$P_{157}(1,9,7)$	$P_{158}(1,1,3)$	$P_{159}(1,11,5)$	$P_{160}(1,4,12)$	$P_{161}(1,6,5)$	$P_{162}(1,4,5)$
$P_{163}(1,4,10)$	$P_{164}(1,2,2)$	$P_{165}(1,10,1)$	$P_{166}(1,7,12)$	$P_{167}(1,6,10)$	$P_{168}(1,2,6)$
$P_{169}(1,12,5)$	$P_{170}(1,4,3)$	$P_{171}(1,11,12)$	$P_{172}(1,6,8)$	$P_{173}(1,9,9)$	$P_{174}(1,8,1)$
$P_{175}(1,7,11)$	$P_{176}(1,3,2)$	$P_{177}(1,10,11)$	$P_{178}(1,3,11)$	$P_{179}(1,3,3)$	$P_{180}(1,11,1)$
$P_{181}(1,7,6)$	$P_{182}(1,12,0)$	$P_{183}(0,1,12)$			

TABLE A.13: Lines of $\mathrm{PG}(2,13)$

L_1	P_1	P_2	P_9	P_{25}	P_{38}	P_{42}	P_{60}	P_{108}	P_{120}	P_{129}	P_{135}	P_{140}	P_{154}	P_{182}
L_2	P_2	P_3	P_{10}	P_{26}	P_{39}	P_{43}	P_{61}	P_{109}	P_{121}	P_{130}	P_{136}	P_{141}	P_{155}	P_{183}
$\vdots$														
L_{183}	P_{183}	P_1	P_8	P_{24}	P_{37}	P_{41}	P_{59}	P_{107}	P_{119}	P_{128}	P_{134}	P_{139}	P_{153}	P_{181}

A.7 Projective Plane of Order Sixteen

$\mathbb{F}_{16} = \{0, 1, \gamma, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^6, \gamma^7, \gamma^8, \gamma^9, \gamma^{10}, \gamma^{11}, \gamma^{12}, \gamma^{13}, \gamma^{14} : \gamma^4 + \gamma + 1 = 2 = 0\}$

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & 0 & \gamma \end{bmatrix}.$$

Table A.14, ..., Table A.17 show the points and lines of PG(2, 16).

TABLE A.14: $P_1, \ldots P_{85}$ of PG(2, 16)

$P_1(1,0,0)$	$P_2(0,1,0)$	$P_3(0,0,1)$	$P_4(1,0,1)$	$P_5(1,\gamma^{14},1)$
$P_6(1,\gamma^{14},\gamma^6)$	$P_7(1,\gamma^8,\gamma^9)$	$P_8(1,\gamma^5,\gamma^6)$	$P_9(1,\gamma^8,\gamma^6)$	$P_{10}(1,\gamma^8,\gamma^4)$
$P_{11}(1,\gamma^{10},\gamma^{14})$	$P_{12}(1,1,\gamma^5)$	$P_{13}(1,\gamma^9,\gamma^7)$	$P_{14}(1,\gamma^7,\gamma^4)$	$P_{15}(1,\gamma^{10},\gamma^8)$
$P_{16}(1,\gamma^6,\gamma^4)$	$P_{17}(1,\gamma^{10},\gamma^4)$	$P_{18}(1,\gamma^{10},\gamma^{10})$	$P_{19}(1,\gamma^4,\gamma^3)$	$P_{20}(1,\gamma^{11},0)$
$P_{21}(0,1,\gamma^{11})$	$P_{22}(1,0,\gamma^{14})$	$P_{23}(1,1,1)$	$P_{24}(1,\gamma^{14},\gamma^3)$	$P_{25}(1,\gamma^{11},\gamma^5)$
$P_{26}(1,\gamma^9,\gamma^{10})$	$P_{27}(1,\gamma^4,\gamma^6)$	$P_{28}(1,\gamma^8,\gamma^{11})$	$P_{29}(1,\gamma^3,\gamma^{12})$	$P_{30}(1,\gamma^2,\gamma^{10})$
$P_{31}(1,\gamma^4,\gamma^{13})$	$P_{32}(1,\gamma,\alpha^{10})$	$P_{33}(1,\gamma^4,\gamma^{10})$	$P_{34}(1,\gamma^4,\gamma^2)$	$P_{35}(1,\gamma^{12},\gamma^4)$
$P_{36}(1,\gamma^{10},\gamma^9)$	$P_{37}(1,\gamma^5,0)$	$P_{38}(0,1,\gamma^5)$	$P_{39}(1,0,\gamma^7)$	$P_{40}(1,\gamma^7,1)$
$P_{41}(1,\gamma^{14},\gamma^{13})$	$P_{42}(1,\gamma,0)$	$P_{43}(0,1,\gamma)$	$P_{44}(1,0,\gamma^6)$	$P_{45}(1,\gamma^8,1)$
$P_{46}(1,\gamma^{14},\gamma^9)$	$P_{47}(1,\gamma^5,\gamma)$	$P_{48}(1,\gamma^{13},\gamma^{14})$	$P_{49}(1,1,\gamma^6)$	$P_{50}(1,\gamma^8,\gamma^2)$
$P_{51}(1,\gamma^{12},\gamma^{10})$	$P_{52}(1,\gamma^4,\gamma^4)$	$P_{53}(1,\gamma^{10},\gamma^3)$	$P_{54}(1,\gamma^{11},\gamma^{13})$	$P_{55}(1,\gamma,\gamma^{11})$
$P_{56}(1,\gamma^3,\gamma)$	$P_{57}(1,\gamma^{13},\gamma^4)$	$P_{58}(1,\gamma^{10},\gamma^2)$	$P_{59}(1,\gamma^{12},\gamma^9)$	$P_{60}(1,\gamma^5,\gamma^8)$
$P_{61}(1,\gamma^6,\gamma^{12})$	$P_{62}(1,\gamma^2,\gamma^2)$	$P_{63}(1,\gamma^{12},\gamma^3)$	$P_{64}(1,\gamma^{11},\gamma^2)$	$P_{65}(1,\gamma^{12},\gamma^2)$
$P_{66}(1,\gamma^{12},\gamma^7)$	$P_{67}(1,\gamma^7,\gamma)$	$P_{68}(1,\gamma^{13},\gamma^{10})$	$P_{69}(1,\gamma^4,\gamma^8)$	$P_{70}(1,\gamma^6,\gamma^5)$
$P_{71}(1,\gamma^9,0)$	$P_{72}(0,1,\gamma^9)$	$P_{73}(1,0,\gamma^{10})$	$P_{74}(1,\gamma^4,1)$	$P_{75}(1,\gamma^{14},\gamma^{14})$
$P_{76}(1,1,\gamma^3)$	$P_{77}(1,\gamma^{11},\gamma^{12})$	$P_{78}(1,\gamma^2,\gamma^6)$	$P_{79}(1,\gamma^8,\gamma^5)$	$P_{80}(1,\gamma^9,\gamma^8)$
$P_{81}(1,\gamma^6,0)$	$P_{82}(0,1,\gamma^6)$	$P_{83}(1,0,\gamma^2)$	$P_{84}(1,\gamma^{12},1)$	$P_{85}(1,\gamma^{14},\gamma^{12})$

Table A.15: $P_{86}, \dots P_{200}$ of PG$(2,16)$

$P_{86}(1,\gamma^2,\gamma^4)$	$P_{87}(1,\gamma^{10},\gamma^{11})$	$P_{88}(1,\gamma^3,\gamma^6)$	$P_{89}(1,\gamma^8,\gamma^{12})$	$P_{90}(1,\gamma^2,\gamma^5)$
$P_{91}(1,\gamma^9,\gamma^{12})$	$P_{92}(1,\gamma^2,\gamma^{12})$	$P_{93}(1,\gamma^2,\gamma)$	$P_{94}(1,\gamma^{13},0)$	$P_{95}(0,1,\gamma^{13})$
$P_{96}(1,0,\gamma^4)$	$P_{97}(1,\gamma^{10},1)$	$P_{98}(1,\gamma^{14},8)$	$P_{99}(1,\gamma^7,\gamma^{13})$	$P_{100}(1,\gamma,\gamma^2)$
$P_{101}(1,\gamma^{12},\gamma^6)$	$P_{102}(1,\gamma^8,\gamma^{10})$	$P_{103}(1,\gamma^4,\gamma^{11})$	$P_{104}(1,\gamma^3,\gamma^9)$	$P_{105}(1,\gamma^5,\gamma^2)$
$P_{106}(1,\gamma^{12},\gamma^8)$	$P_{107}(1,\gamma^6,\gamma^{14})$	$P_{108}(1,1,\gamma^{13})$	$P_{109}(1,\gamma,\gamma^4)$	$P_{110}(1,\gamma^{10},\gamma^{12})$
$P_{111}(1,\gamma^2,\gamma^{11})$	$P_{112}(1,\gamma^3,\gamma^{10})$	$P_{113}(1,\gamma^4,\gamma^9)$	$P_{114}(1,\gamma^5,\gamma^7)$	$P_{115}(1,\gamma^7,\gamma^{11})$
$P_{116}(1,\gamma^3,\gamma^5)$	$P_{117}(1,\gamma^9,\gamma^{11})$	$P_{118}(1,\gamma^3,\gamma^{11})$	$P_{119}(1,\gamma^3,\gamma^{13})$	$P_{120}(1,2,2)$
$P_{121}(1,\gamma^{13},\gamma^3)$	$P_{122}(1,\gamma^{11},\gamma^7)$	$P_{123}(1,\gamma^7,\gamma^{14})$	$P_{124}(1,1,\gamma^9)$	$P_{125}(1,\gamma^5,\gamma^{10})$
$P_{126}(1,\gamma^4,\gamma^7)$	$P_{127}(1,\gamma^7,\gamma^{12})$	$P_{128}(1,\gamma^2,\gamma^7)$	$P_{129}(1,\gamma^7,\gamma^7)$	$P_{130}(1,\gamma^7,\gamma^3)$
$P_{131}(1,\gamma^{11},\gamma^{14})$	$P_{132}(1,1,\gamma^{12})$	$P_{133}(1,\gamma^2,\gamma^8)$	$P_{134}(1,\gamma^6,\gamma^2)$	$P_{135}(1,\gamma^{12},\gamma^{14})$
$P_{136}(1,1,\gamma^{11})$	$P_{137}(1,\gamma^3,\gamma^{14})$	$P_{138}(1,1,\gamma^{14})$	$P_{139}(1,1,0)$	$P_{140}(0,1,1)$
$P_{141}(1,0,\gamma^3)$	$P_{142}(1,\gamma^{11},1)$	$P_{143}(1,\gamma^{14},\gamma^5)$	$P_{144}(1,\gamma^9,\gamma^2)$	$P_{145}(1,\gamma^{12},\gamma^{13})$
$P_{146}(1,\gamma,\gamma^6)$	$P_{147}(1,\gamma^8,\gamma^7)$	$P_{148}(1,\gamma^7,0)$	$P_{149}(0,1,\gamma^7)$	$P_{150}(1,0,\gamma^9)$
$P_{151}(1,\gamma^5,1)$	$P_{152}(1,\gamma^{14},\gamma)$	$P_{153}(1,\gamma^{13},\gamma^{11})$	$P_{154}(1,\gamma^3,\gamma^4)$	$P_{155}(1,\gamma^{10},\gamma^6)$
$P_{156}(1,\gamma^8,\gamma^{14})$	$P_{157}(1,1,\gamma^2)$	$P_{158}(1,\gamma^{12},\gamma^{11})$	$P_{159}(1,\gamma^3,0)$	$P_{160}(0,1,\gamma^3)$
$P_{161}(1,0,\gamma^{12})$	$P_{162}(1,\gamma^2,1)$	$P_{163}(1,\gamma^{14},\gamma^4)$	$P_{164}(1,\gamma^{10},\gamma^7)$	$P_{165}(1,\gamma^7,\gamma^8)$
$P_{166}(1,\gamma^6,\gamma^6)$	$P_{167}(1,\gamma^8,\gamma^3)$	$P_{168}(1,\gamma^{11},\gamma)$	$P_{169}(1,\gamma^{13},\gamma^7)$	$P_{170}(1,\gamma^7,\gamma^{10})$
$P_{171}(1,\gamma^4,\gamma^{12})$	$P_{172}(1,\gamma^2,\gamma^{13})$	$P_{173}(1,\gamma,\gamma^{14})$	$P_{174}(1,1,\gamma^4)$	$P_{175}(1,\gamma^{10},\gamma^5)$
$P_{176}(1,\gamma^9,\gamma)$	$P_{177}(1,\gamma^{13},\gamma^9)$	$P_{178}(1,\gamma^5,\gamma^{14})$	$P_{179}(1,1,\gamma^{10})$	$P_{180}(1,\gamma^4,\gamma)$
$P_{181}(1,\gamma^{13},\gamma^8)$	$P_{182}(1,\gamma^6,\gamma)$	$P_{183}(1,\gamma^{13},\gamma)$	$P_{184}(1,\gamma^{13},\gamma^{12})$	$P_{185}(1,\gamma^2,0)$
$P_{186}(0,1,\gamma^2)$	$P_{187}(1,0,\gamma^{11})$	$P_{188}(1,\gamma^3,1)$	$P_{189}(1,\gamma^{14},\gamma^8)$	$P_{190}(1,\gamma^6,\gamma^{10})$
$P_{191}(1,\gamma^4,\gamma^5)$	$P_{192}(1,\gamma^9,\gamma^6)$	$P_{193}(1,\gamma^8,\gamma^8)$	$P_{194}(1,\gamma^6,\gamma^3)$	$P_{195}(1,\gamma^{11},\gamma^8)$
$P_{196}(1,\gamma^6,\gamma^8)$	$P_{197}(1,\gamma^6,\gamma^{11})$	$P_{198}(1,\gamma^3,\gamma^7)$	$P_{199}(1,\gamma^7,\gamma^5)$	$P_{200}(1,\gamma^9,\gamma^4)$

Table A.16: $P_{201}, \ldots P_{273}$ of PG$(2, 16)$

$P_{201}(1, \gamma^{10}, \gamma)$	$P_{202}(1, \gamma^{13}, \gamma^2)$	$P_{203}(1, \gamma^{12}, \gamma^5)$	$P_{204}(1, \gamma^9, \gamma^{13})$	$P_{205}(1, \gamma, \gamma^5)$
$P_{206}(1, \gamma^9, \gamma^5)$	$P_{207}(1, \gamma^9, \gamma^{14})$	$P_{208}(1, 1, \gamma^7)$	$P_{209}(1, \gamma^7, \gamma^9)$	$P_{210}(1, \gamma^5, \gamma^{11})$
$P_{211}(1, \gamma^3, \gamma^2)$	$P_{212}(1, \gamma^{12}, 0)$	$P_{213}(0, 1, \gamma^{12})$	$P_{214}(1, 0, \gamma^8)$	$P_{215}(1, \gamma^6, 1)$
$P_{216}(1, \gamma^{14}, \gamma^{10})$	$P_{217}(1, \gamma^4, \gamma^{14})$	$P_{218}(1, 1, \gamma)$	$P_{219}(1, \gamma^{13}, \gamma^6)$	$P_{220}(1, \gamma^8, \gamma^{13})$
$P_{221}(1, \gamma, \gamma^7)$	$P_{222}(1, \gamma^7, \gamma^2)$	$P_{223}(1, \gamma^{12}, \gamma)$	$P_{224}(1, \gamma^{13}, \gamma^5)$	$P_{225}(1, \gamma^9, \gamma^9)$
$P_{226}(1, \gamma^5, \gamma^3)$	$P_{227}(1, \gamma^{11}, \gamma^4)$	$P_{228}(1, \gamma^{10}, \gamma^{13})$	$P_{229}(1, \gamma, \gamma^{12})$	$P_{230}(1, \gamma, \gamma^{14})$
$P_{231}(1, 1, \gamma^8)$	$P_{232}(1, \gamma^6, \gamma^{13})$	$P_{233}(1, \gamma, \gamma^{11})$	$P_{234}(1, \gamma^5, \gamma^{13})$	$P_{235}(1, \gamma, \gamma^{13})$
$P_{236}(1, \gamma, \gamma^8)$	$P_{237}(1, \gamma^6, \gamma^9)$	$P_{238}(1, \gamma^5, \gamma^{12})$	$P_{239}(1, \gamma^2, \gamma^9)$	$P_{240}(1, \gamma^5, \gamma^9)$
$P_{241}(1, \gamma^5, \gamma^5)$	$P_{242}(1, \gamma^9, \gamma^3)$	$P_{243}(1, \gamma^{11}, \gamma^{10})$	$P_{244}(1, \gamma^4, 0)$	$P_{245}(0, 1, \gamma^4)$
$P_{246}(1, 0, \gamma^5)$	$P_{247}(1, \gamma^9, 1)$	$P_{248}(1, \gamma^{14}, \gamma^2)$	$P_{249}(1, \gamma^{12}, \gamma^{12})$	$P_{250}(1, \gamma, \gamma^3)$
$P_{251}(1, \gamma^{11}, \gamma^6)$	$P_{252}(1, \gamma^8, \gamma)$	$P_{253}(1, \gamma^{13}, \gamma^{13})$	$P_{254}(1, \gamma, \gamma^3)$	$P_{255}(1, \gamma^{11}, \gamma^{11})$
$P_{256}(1, \gamma^3, \gamma^3)$	$P_{257}(1, \gamma^{11}, \gamma^3)$	$P_{258}(1, \gamma^{11}, \gamma^9)$	$P_{259}(1, \gamma^5, \gamma^4)$	$P_{260}(1, \gamma^{10}, 0)$
$P_{261}(0, 1, \gamma^{10})$	$P_{262}(1, 0, \gamma)$	$P_{263}(1, \gamma^{13}, 1)$	$P_{264}(1, \gamma^{14}, \gamma^{11})$	$P_{265}(1, \gamma^3, \gamma^8)$
$P_{266}(1, \gamma^6, \gamma^7)$	$P_{267}(1, \gamma^7, \gamma^6)$	$P_{268}(1, \gamma^8, 0)$	$P_{269}(0, 1, \gamma^8)$	$P_{270}(1, 0, \gamma^{13})$
$P_{271}(1, \gamma, 1)$	$P_{272}(1, \gamma^{14}, 0)$	$P_{273}(0, 1, \gamma^{14})$		

Table A.17: Lines of PG$(2, 16)$

L_1	P_1	P_2	P_{20}	P_{37}	P_{42}	P_{71}	P_{81}	P_{94}	P_{139}	P_{148}	P_{159}	P_{185}	P_{212}	P_{244}
	P_{260}	P_{268}	P_{272}											
L_2	P_2	P_3	P_{21}	P_{38}	P_{43}	P_{72}	P_{82}	P_{95}	P_{140}	P_{149}	P_{160}	P_{186}	P_{213}	P_{245}
	P_{261}	P_{269}	P_{273}											
$\vdots$														
L_{307}	P_{273}	P_1	P_{19}	P_{36}	P_{41}	P_{70}	P_{80}	P_{93}	P_{138}	P_{147}	P_{158}	P_{184}	P_{211}	P_{243}
	P_{259}	P_{267}	P_{271}											

A.8 Projective Plane of Order Seventeen

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 8 \end{bmatrix}.$$

Table A.18, ..., Table A.21 show the points and lines of PG(2, 17).

TABLE A.18: $P_1, \ldots P_{114}$ of PG(2, 17)

$P_1(1,0,0)$	$P_2(0,1,0)$	$P_3(0,0,1)$	$P_4(1,0,8)$	$P_5(1,15,8)$	$P_6(1,15,12)$
$P_7(1,10,5)$	$P_8(1,7,10)$	$P_9(1,12,7)$	$P_{10}(1,5,0)$	$P_{11}(0,1,5)$	$P_{12}(1,0,15)$
$P_{13}(1,8,8)$	$P_{14}(1,15,9)$	$P_{15}(1,2,4)$	$P_{16}(1,13,0)$	$P_{17}(0,1,13)$	$P_{18}(1,0,12)$
$P_{19}(1,10,8)$	$P_{20}(1,15,5)$	$P_{21}(1,7,11)$	$P_{22}(1,14,4)$	$P_{23}(1,13,3)$	$P_{24}(1,6,1)$
$P_{25}(1,1,14)$	$P_{26}(1,11,2)$	$P_{27}(1,9,5)$	$P_{28}(1,7,3)$	$P_{29}(1,6,16)$	$P_{30}(1,16,2)$
$P_{31}(1,9,16)$	$P_{32}(1,16,16)$	$P_{33}(1,16,9)$	$P_{34}(1,2,6)$	$P_{35}(1,3,14)$	$P_{36}(1,11,7)$
$P_{37}(1,5,12)$	$P_{38}(1,10,7)$	$P_{39}(1,5,7)$	$P_{40}(1,5,16)$	$P_{41}(1,16,3)$	$P_{42}(1,6,2)$
$P_{43}(1,9,11)$	$P_{44}(1,14,15)$	$P_{45}(1,8,1)$	$P_{46}(1,1,16)$	$P_{47}(1,16,7)$	$P_{48}(1,5,3)$
$P_{49}(1,6,4)$	$P_{50}(1,13,1)$	$P_{51}(1,1,4)$	$P_{52}(1,13,4)$	$P_{53}(1,13,7)$	$P_{54}(1,5,5)$
$P_{55}(1,7,9)$	$P_{56}(1,2,5)$	$P_{57}(1,7,5)$	$P_{58}(1,7,6)$	$P_{59}(1,3,12)$	$P_{60}(1,10,4)$
$P_{61}(1,13,2)$	$P_{62}(1,9,6)$	$P_{63}(1,3,1)$	$P_{64}(1,1,11)$	$P_{65}(1,14,5)$	$P_{66}(1,7,4)$
$P_{67}(1,13,14)$	$P_{68}(1,11,15)$	$P_{69}(1,8,11)$	$P_{70}(1,14,1)$	$P_{71}(1,1,5)$	$P_{72}(1,7,15)$
$P_{73}(1,8,13)$	$P_{74}(1,4,6)$	$P_{75}(1,3,3)$	$P_{76}(1,6,9)$	$P_{77}(1,2,3)$	$P_{78}(1,6,3)$
$P_{79}(1,6,10)$	$P_{80}(1,12,12)$	$P_{81}(1,10,9)$	$P_{82}(1,2,11)$	$P_{83}(1,14,2)$	$P_{84}(1,9,15)$
$P_{85}(1,8,12)$	$P_{86}(1,10,3)$	$P_{87}(1,6,0)$	$P_{88}(0,1,6)$	$P_{89}(1,0,11)$	$P_{90}(1,14,8)$
$P_{91}(1,15,14)$	$P_{92}(1,11,3)$	$P_{93}(1,6,6)$	$P_{94}(1,3,9)$	$P_{95}(1,2,14)$	$P_{96}(1,11,13)$
$P_{97}(1,4,1)$	$P_{98}(1,1,12)$	$P_{99}(1,10,1)$	$P_{100}(1,1,1)$	$P_{101}(1,1,9)$	$P_{102}(1,2,10)$
$P_{103}(1,12,15)$	$P_{104}(1,8,2)$	$P_{105}(1,9,12)$	$P_{106}(1,10,13)$	$P_{107}(1,4,14)$	$P_{108}(1,11,1)$
$P_{109}(1,1,2)$	$P_{110}(1,9,0)$	$P_{111}(0,1,9)$	$P_{112}(1,0,10)$	$P_{113}(1,12,8)$	$P_{114}(1,15,1)$

Table A.19: $P_{115}, \dots P_{252}$ of PG(2, 17)

$P_{115}(1,1,6)$	$P_{116}(1,3,11)$	$P_{117}(1,14,16)$	$P_{118}(1,16,11)$	$P_{119}(1,14,11)$	$P_{120}(1,14,0)$
$P_{121}(0,1,14)$	$P_{122}(1,0,2)$	$P_{123}(1,9,8)$	$P_{124}(1,15,7)$	$P_{125}(1,5,15)$	$P_{126}(1,8,14)$
$P_{127}(1,11,11)$	$P_{128}(1,14,9)$	$P_{129}(1,2,2)$	$P_{130}(1,9,9)$	$P_{131}(1,2,9)$	$P_{132}(1,2,12)$
$P_{133}(1,10,11)$	$P_{134}(1,14,12)$	$P_{135}(1,10,12)$	$P_{136}(1,10,6)$	$P_{137}(1,3,4)$	$P_{138}(1,13,13)$
$P_{139}(1,4,9)$	$P_{140}(1,2,16)$	$P_{141}(1,16,6)$	$P_{142}(1,3,5)$	$P_{143}(1,7,12)$	$P_{144}(1,10,10)$
$P_{145}(1,12,9)$	$P_{146}(1,2,15)$	$P_{147}(1,8,7)$	$P_{148}(1,5,14)$	$P_{149}(1,11,12)$	$P_{150}(1,10,16)$
$P_{151}(1,16,15)$	$P_{152}(1,8,0)$	$P_{153}(0,1,8)$	$P_{154}(1,0,6)$	$P_{155}(1,3,8)$	$P_{156}(1,15,2)$
$P_{157}(1,9,7)$	$P_{158}(1,5,2)$	$P_{159}(1,9,2)$	$P_{160}(1,9,4)$	$P_{161}(1,13,6)$	$P_{162}(1,3,13)$
$P_{163}(1,4,3)$	$P_{164}(1,6,15)$	$P_{165}(1,8,5)$	$P_{166}(1,7,13)$	$P_{167}(1,4,2)$	$P_{168}(1,9,10)$
$P_{169}(1,12,14)$	$P_{170}(1,11,4)$	$P_{171}(1,13,15)$	$P_{172}(1,8,10)$	$P_{173}(1,12,2)$	$P_{174}(1,9,14)$
$P_{175}(1,11,5)$	$P_{176}(1,7,0)$	$P_{177}(0,1,7)$	$P_{178}(1,0,13)$	$P_{179}(1,4,8)$	$P_{180}(1,15,0)$
$P_{181}(0,1,15)$	$P_{182}(1,0,16)$	$P_{183}(1,16,8)$	$P_{184}(1,15,10)$	$P_{185}(1,12,1)$	$P_{186}(1,1,3)$
$P_{187}(1,6,14)$	$P_{188}(1,11,6)$	$P_{189}(1,3,7)$	$P_{190}(1,5,6)$	$P_{191}(1,3,6)$	$P_{192}(1,3,0)$
$P_{193}(0,1,3)$	$P_{194}(1,0,14)$	$P_{195}(1,11,8)$	$P_{196}(1,15,3)$	$P_{197}(1,6,13)$	$P_{198}(1,4,15)$
$P_{199}(1,8,6)$	$P_{200}(1,3,15)$	$P_{201}(1,8,15)$	$P_{202}(1,8,4)$	$P_{203}(1,13,10)$	$P_{204}(1,12,11)$
$P_{205}(1,14,6)$	$P_{206}(1,3,16)$	$P_{207}(1,16,5)$	$P_{208}(1,7,1)$	$P_{209}(1,1,15)$	$P_{210}(1,8,16)$
$P_{211}(1,16,0)$	$P_{212}(0,1,16)$	$P_{213}(1,0,7)$	$P_{214}(1,5,8)$	$P_{215}(1,15,15)$	$P_{216}(1,8,9)$
$P_{217}(1,2,7)$	$P_{218}(1,5,1)$	$P_{219}(1,1,13)$	$P_{220}(1,4,12)$	$P_{221}(1,10,14)$	$P_{222}(1,11,16)$
$P_{223}(1,16,14)$	$P_{224}(1,11,14)$	$P_{225}(1,11,10)$	$P_{226}(1,12,4)$	$P_{227}(1,13,11)$	$P_{228}(1,14,3)$
$P_{229}(1,6,7)$	$P_{230}(1,5,4)$	$P_{231}(1,13,5)$	$P_{232}(1,7,14)$	$P_{233}(1,11,0)$	$P_{234}(0,1,11)$
$P_{235}(1,0,5)$	$P_{236}(1,7,8)$	$P_{237}(1,15,11)$	$P_{238}(1,14,14)$	$P_{239}(1,11,9)$	$P_{240}(1,2,13)$
$P_{241}(1,4,16)$	$P_{242}(1,16,4)$	$P_{243}(1,13,12)$	$P_{244}(1,10,2)$	$P_{245}(1,9,13)$	$P_{246}(1,4,10)$
$P_{247}(1,12,5)$	$P_{248}(1,7,7)$	$P_{249}(1,5,9)$	$P_{250}(1,2,1)$	$P_{251}(1,1,10)$	$P_{252}(1,12,3)$

TABLE A.20: $P_{253}, \ldots P_{307}$ of $\mathrm{PG}(2,17)$

$P_{253}(1,6,12)$	$P_{254}(1,10,0)$	$P_{255}(0,1,10)$	$P_{256}(1,0,3)$	$P_{257}(1,6,8)$	$P_{258}(1,15,13)$
$P_{259}(1,4,0)$	$P_{260}(0,1,4)$	$P_{261}(1,0,4)$	$P_{262}(1,13,8)$	$P_{263}(1,15,16)$	$P_{264}(1,16,10)$
$P_{265}(1,12,13)$	$P_{266}(1,4,5)$	$P_{267}(1,7,2)$	$P_{268}(1,9,3)$	$P_{269}(1,6,11)$	$P_{270}(1,14,7)$
$P_{271}(1,5,10)$	$P_{272}(1,12,0)$	$P_{273}(0,1,12)$	$P_{274}(1,0,1)$	$P_{275}(1,1,8)$	$P_{276}(1,15,6)$
$P_{277}(1,3,2)$	$P_{278}(1,9,1)$	$P_{279}(1,1,0)$	$P_{280}(0,1,1)$	$P_{281}(1,0,9)$	$P_{282}(1,2,8)$
$P_{283}(1,15,4)$	$P_{284}(1,13,16)$	$P_{285}(1,16,12)$	$P_{286}(1,10,15)$	$P_{287}(1,8,3)$	$P_{288}(1,6,5)$
$P_{289}(1,7,16)$	$P_{290}(1,16,1)$	$P_{291}(1,1,7)$	$P_{292}(1,5,13)$	$P_{293}(1,4,11)$	$P_{294}(1,14,13)$
$P_{295}(1,4,13)$	$P_{296}(1,4,7)$	$P_{297}(1,5,11)$	$P_{298}(1,14,10)$	$P_{299}(1,12,6)$	$P_{300}(1,3,10)$
$P_{301}(1,12,10)$	$P_{302}(1,12,16)$	$P_{303}(1,16,13)$	$P_{304}(1,4,4)$	$P_{305}(1,13,9)$	$P_{306}(1,2,0)$
$P_{307}(0,1,2)$					

TABLE A.21: Lines of $\mathrm{PG}(2,17)$

L_1	P_1	P_2	P_{10}	P_{16}	P_{87}	P_{110}	P_{120}	P_{152}	P_{176}	P_{180}	P_{192}	P_{211}	P_{233}	P_{254}
	P_{259}	P_{272}	P_{279}	P_{306}										
$\vdots$														
L_{307}	P_{307}	P_1	P_9	P_{15}	P_{86}	P_{109}	P_{119}	P_{151}	P_{175}	P_{179}	P_{191}	P_{210}	P_{232}	P_{253}
	P_{258}	P_{271}	P_{278}	P_{305}										

Appendix B

GAP Program

B.1 GAP Program

$p_s := [];;\ r := [];;\ x := [];;\ \ell_1 := [];;\ \ell := [];;\ mm := [];;$

$oo := []\ ;;\ yy := []\ ;;\ nn := []\ ;;\ a_c := [];;$

$q := 13;$

$qq := 183;$

$n := 3;$

$u_1 := [1, 0, 0];;$

$c_m := [[0, 1, 0], [0, 0, 1], [2, 0, 1]];;$

Find the Points Of PG(2,13)

---

tic $:= 0; tt := 0;$

for i in $[0..qq - 1]$ do

$p := u_1 * c_m^i$ mod q;

if $\quad p[1] <> 0$ mod q then $z := p * p[1]^{-1}$ mod q;

elif $p[2] <> 0$ mod q then $z := p * p[2]^{-1}$ mod q;

elif $p[3] <> 0$ mod q then $z := p * p[3]^{-1}$ mod q;

fi;

$p_s[i+1] := z$;

Add (p_s, z);

if $p[3] = 0$ then $zz := z$;

$tt := tt + 1$;

$\ell_1[tt] := i$;

fi;

od;

$c := []$; $n_u := []$; $a_d := []$; $a_r := []$;

$a_c := [[1, 2, 3, 46, 7], [1, 2, 3, 46, 8], [1, 2, 3, 46, 18]]$;

for k in $[1 \cdots qq]$ do;

$n_u[k+1] := k$;

od;

---

for i in $[1 \cdots \text{Size}(a_c)]$ do

$w_q :=$ Difference $(n_u, a_c[i])$;

$c := a_c[i]$;

for j in w_q do

od; od;

Find type of Secants

---

for i in a_d do

$c_h := []$;;

for j in $[1 \cdots qq]$ do

Add$(c_h, [j])$;

fi;

od;

if Size$(c_h) = 0$ then

Add(a_r, i);

fi;

od;

Print(Size(a_r), " n");

$q_1 := 6$;

$s :=$ List ([$1 \cdots$ Size (a_r)], $i- >$ List($a_r[i], j- > p_s[j]$));

$w :=$ Size(s);;

for h in $[1 \cdots w-1]$ do

for j in $[h+1 \cdots w]$ do

$v :=$ [];;

Find the Permutations Of Arcs

$n := [s[j][i_1], s[j][i_2], s[j][i_3], s[j][i_4]]$;;

Add(v, n);

fi;

Find the Projectivity

---

for i in $[1 \cdots Length(v)]$ do

$t := v[i]$;

$a_1 := t[4][1] * (t[2][2] * t[3][3] - t[3][2] * t[2][3])$;

$a_2 := t[2][1] * (t[4][2] * t[3][3] - t[3][2] * t[4][3])$;

$a_3 := t[3][1] * (t[4][2] * t[2][3] - t[2][2] * t[4][3])$;

$b_1 := t[1][1] * (t[4][2] * t[3][3] - t[3][2] * t[4][3])$;

$b_2 := t[4][1] * (t[1][2] * t[3][3] - t[3][2] * t[1][3])$;

$b_3 := t[3][1] * (t[1][2] * t[4][3] - t[4][2] * t[1][3])$;

$c_1 := t[1][1] * (t[2][2] * t[4][3] - t[4][2] * t[2][3])$;

$c_2 := t[2][1] * (t[1][2] * t[4][3] - t[4][2] * t[1][3])$;

$c_3 := t[4][1] * (t[1][2] * t[2][3] - t[2][2] * t[1][3])$;

$a := (a_1 - a_2 + a_3)\text{mod } q;$

$b := (b_1 - b_2 + b_3)\text{mod } q;$

$c := (c_1 - c_2 + c_3)\text{mod } q;$

$T := [[a * t[1][1]\text{mod } q, a * t[1][2]\text{mod } q, a * t[1][3]\text{mod } q],$

$[b * t[2][1]\text{mod } q, b * t[2][2]\text{mod } q, b * t[2][3]\text{mod } q],$

$[c * t[3][1]\text{mod } q, c * t[3][2]\text{mod } q, c * t[3][3]\text{mod } q]];$

Find the Equivalents Arcs

---

for k in$[1 \cdots q_1]$ do

$r[k] := (s[h][k] * T)\text{mod } q;$

od;

for k in$[1 \cdots q_1]$ do

if $r[k][1] <> 0$ mod $q$then $x[k] := r[k] * r[k][1]^{-1}\text{mod } q;$

elif $r[k][2] <> 0$ mod $q$then $x[k] := r[k] * r[k][2]^{-1}\text{mod } q;$

elif $r[k][3] <> 0$ mod $q$then $x[k] := r[k] * r[k][3]^{-1}\text{mod } q;$

fi;

od;

if $\text{Set}(x) = \text{Set}(s[j])$ mod q then

for m_i in$[1 \cdots q_1]$ do

od;

if pos = fail then

Add (oo, j);

fi; fi;

od; od; od;

Find the Distinct Arcs

$r_y := []$;;$r_o := 0;$

for i in $[1 \cdots w]$ do

$yy[i] := i;$

od;

for i in xx do

$r_o := r_o + 1;$

$r_y[r_o] := a_r[i]$;

od;

Print("The ", Size(r_y), " distinct arcs-are: n", r_y," n");

Appendix C

Fortran Program

C.1 Fortran Program

C.1.1 Algorithm Two

program Main implicit none integer $G1, G2, G3, ci, cj, pi, pj, Li, Lj, adi, adj, ti, tj, tk, tr$

integer $sor, sor1, sor2, ui, uj, uk, xi, xj, O1, O2, i1, i2, i3, i4, uuii, uujj$

integer $Det, Det1, Det2, Det3, M1, LO, N1, di, dj, dk, ni, GH, complete, uukk$

integer $start_loop, tip, tin, comp, major, minor, dist, ii, jj, mm, oo, HH$

integer $loop1, loop2, clas, C, GF, B1, xr, i, j, IV, Lk, LL, s, ads, ts, tt$

integer, dimension (3) :: $P1, AB$

integer, dimension $(3, 3)$ :: $CM, A, A1, A2, A3, projective$

integer, dimension (4) :: INV

integer, dimension (6) :: L

integer, dimension $(31, 6)$ :: $LINE$

integer, dimension $(31, 3)$ :: $POINT, poin$

integer, dimension (4) :: $PERMUTATION$

integer, dimension (31) :: Td

integer, dimension (5) :: TRR, TYP, SEC

integer, allocatable :: $AD(:, :)$

integer, allocatable :: $ADD(:, :)$

integer, allocatable :: $ADD1(:,:)$

integer, allocatable :: $ADDEDS(:,:)$

integer, allocatable :: $ADDED(:,:)$

integer, allocatable :: $NEW(:,:)$

integer, allocatable :: $FATEN(:,:)$

integer, allocatable :: $TYPES(:)$

integer, allocatable :: $Distinct_Types(:,:)$

integer, allocatable :: $MULT(:)$

integer, allocatable :: $DISTINCT(:,:)$

open (unit=1 , file='COMPLETE-ARCS.fort')

open (unit=2 , file='NEW.fort')

open (unit=3 , file='Report.fort')

$IV = 0$

do $i = 1, q - 1$

do $j = 1, q - 1$

if (mod $(i * j, q).eq.1)$ then

$IV = IV + 1$

$INV(IV) = j$

end if

end do

end do

$P1 = (/1, 0, 0/)$

poin= 0

poin $(1, :) = P1$

POINT $(1, :) = poin(1, :)$

do $pi = 2, qq$

do $pj = 1, 3$

$poin(pi, :) =$ mod $(poin(pi, :) + P1(pj) * CM(pj, :), q)$

end do

if $(poin(pi, 1).ne.0)$ then

$poin =$ mod $(INV(poin(pi, 1)) * poin, q)$

end if

$P1 = poin(pi, :); POINT(pi, :) = poin(pi, :)$

end

$LL = 0$

do $Lk = 1, qq$

if $(POINT(Lk, 3).eq.0)$ then

$L(LL) = Lk$

end if

LINE $(1, :) = L$

end do

do $Li = 2, qq$

LINE $(Li, :) = L + 1$

do $Lj = 1, q + 1$

if $(LINE(Li, Lj).eq.qq + 1)$ then

$LINE(Li, Lj) = 1$

end if

end do

$L = LINE(Li, :)$

end do

$G1 = 2; G2 = 6$

allocate$(AD(G1, G2))$

$AD(1, 1) = 1; AD(1, 2) = 2; AD(1, 3) = 3; AD(1, 4) = 5; AD(1, 5) = 15; AD(1, 6) = 21$

$AD(2, 1) = 1; AD(2, 2) = 2; AD(2, 3) = 3; AD(2, 4) = 5; AD(2, 5) = 15; AD(2, 6) = 25$

do $start_loop = 1, 10$

allocate$(ADD1(G3, G2 + 6))$

allocate$(ADD(G3, G2 + 1))$

$tin = 0; complete = 0$

do $adi = 1, G1; comp = 0; HH = 0$

do $adj = 1, 31$

if$(all(AD(adi, :).ne.adj))$then;

$HH = HH + 1$

$ADD(HH,:) = (/AD(adi,:), adj/)$

end if

end do

do $ads = 1, HH; dott = 0, 5; TRR(tt) = 0$; end do

do $ti = 1, 31; tr = 0$

do $tj = 1, G2 + 1$

do $tk = 1, 6$

end do

end do

$Td(ti) = tr$

end do

do $ts = 1, 31$

do $tt = 0, 5$

end do

end do

$s = 0$

do $i = 4, 0, -1$

$TYP(i) = TRR(i)$

$s = s + TYP(i)$

end do

if($s.eq.31$) then;

do $sor1 = 1, G2$

do $sor2 = sor1 + 1, G2 + 1$

if($ADD(ads, sor1).gt.ADD(ads, sor2)$) then

$sor = ADD(ads, sor2)$

$ADD(ads, sor2) = ADD(ads, sor1)$

$ADD(ads, sor1) = sor$

end if

end do

end do

$comp = comp + 1$

$tin = tin + 1$

$ADD1(tin, :) = (/ADD(ads, :), TYP/)$

end if

end do

if($comp.eq.0$) then

$complete = complete + 1$

write$(1, *)AD(adi, :),'$ Is Complete$(', G2,' ,' , 4,') - arc'$

end if

end do

write $(1, *)$'———————————————————'

write $(1, *)'$Total number of complete$(', G2,' ,' , 4,')is :'$, complete

write $(1, *)$'———————————————————'

write $(1, *)'$COMPLETE$(', G2,' ,' , 4,') -$ arcs is$','$ [$'$, complete,$'$]$'$

Deallocate (AD)

Deallocate (ADD)

allocate $(ADDED(tin, G2 + 1))$

$dist = 0$

do $uuii = 1, tin$

$uukk = 0$

do $uujj = uuii + 1, tin$

$uukk = uukk + 1$

end if

end do

if ($uukk.eq.0$) then

$dist = dist + 1$

$ADDED(dist, :) = ADD1(uuii, 1 : G2 + 1)$

write $(2, *)ADDED(dist, :)$

end if

end do

write$(3, *)dist,' (', G2 + 1,' ,' , 4,') -$ arcs$'$

deallocate $(ADD1)$

$G1 = dist; G2 = G2 + 1$

allocate$(AD(G1, G2))$

do $i = 1, dist$

$AD(i, :) = ADDED(i, :)$

end do

Deallocate $(ADDED)$

end do

close (1)

close (3)

close (4)

close (6)

stop

end

C.1.2 Algorithm Five

program Main

implicit none

integer, parameter :: $q = 13, qq = q * q + q + 1, N = 2, NUM = q - 3$

integer $G1, G2, G3, ci, cj, pi, pj, Li, Lj, adi, adj, ti, tj, tk, tr, ts, tt$

integer $sor, sor1, sor2, ui, uj, uk, xi, xj, O1, O2, i1, i2, i3, i4, uuii, uujj$

integer $Det, Det1, Det2, Det3, M1, LO, N1, di, dj, dk, ni, GH, complete, uukk$

integer $start - loop, tip, tin, comp, major, minor, dist, ii, jj, mm, oo, HH$

integer $loop1, loop2, clas, C, GF, B1, xr, i, j, IV, Lk, LL, s, ff$

integer, dimension (3) :: $P1, AB$

integer, dimension $(3, 3)$ :: $CM, A, A1, A2, A3, projective$

integer, dimension $(q - 1)$ :: INV

integer, dimension $(q + 1)$ :: L

integer, dimension $(qq, q + 1)$ :: $LINE$

integer, dimension $(qq, 3)$ :: $POINT, poin$

integer, dimension (4) :: $PERMUTATION$

integer, dimension (qq) :: Td

integer, dimension $(N + 1)$ :: TRR, TYP, SEC

integer, allocatable :: $AD(:, :)$

integer, allocatable :: $ADD(:)$

integer, allocatable :: $ADD1(:, :)$

integer, allocatable :: $ADDEDS(:, :)$

integer, allocatable :: $ADDED(:, :)$

integer, allocatable :: $NEW(:, :)$

integer, allocatable :: $FATEN(:, :)$

integer, allocatable :: $TYPES(:)$

integer, allocatable :: $Distinct - Types(:, :)$

integer, allocatable :: $MULT(:)$

integer, allocatable :: $DISTINCT(:, :)$

open (unit=0 , file='ARCS.fort')

open (unit=1 , file='COMPLETE-ARCS.fort') open (unit=2 , file='TYPES.fort')

open (unit=3 , file='EQUIVALENT.fort')

open (unit=4 , file='RESULTS.fort')

open (unit=5 , file='INFORMATION.fort')

open (unit=8 , file='MAXIMUM.fort')

$IV = 0$

do $i = 1, q-1$

do $j = 1, q-1$

if (mod $(i*j, q).eq.1)$ then

$IV = IV + 1$

$INV(IV) = j$

end if

end do

end do

$P1 = (/1, 0, 0/)$

poin= 0

poin $(1, :) = P1$

POINT $(1, :) = poin(1, :)$

do $pi = 2, qq$

do $pj = 1, 3$

$poin(pi, :) =$ mod $(poin(pi, :) + P1(pj) * CM(pj, :), q)$

end do

if $(poin(pi, 1).ne.0)$ then

$poin =$ mod $(INV(poin(pi, 1)) * poin, q)$

end if

$P1 = poin(pi, :); POINT(pi, :) = poin(pi, :)$

end

$LL = 0$

do $Lk = 1, qq$

if $(POINT(Lk, 3).eq.0)$ then

$L(LL) = Lk$

end if

LINE $(1, :) = L$

end do

do $Li = 2, qq$

LINE $(Li, :) = L + 1$

do $Lj = 1, q + 1$

$LINE(Li, Lj) = 1$

end if

end do

$L = LINE(Li, :)$

end do

$G1 = 1$

$G2 = 4$

allocate $(AD(G1, G2))$

do $1ci = 1, G1$

1 read $(0, *)AD(ci, 1), AD(ci, 2), AD(ci, 3), AD(ci, 4)$

DO $start - loop = 1, NUM$

allocate $(ADD(G2 + 1))$

allocate $(ADD1(G3, G2 + 2 + N))$

$tin = 0$

$complete = 0$

$HH = 1$

do $adi = 1, G1$

comp= 0

do $adj = 1, qq$

$B1 = 0$

if(all$(AD(adi, :).ne.adj)$) then

$ADD(:) = (/AD(adi, :), adj/)$

do $tt = 0, N + 1$

$TRR(tt) = 0$

end do

do $ti = 1, qq$

$tr = 0$

do $tj = 1, G2 + 1$

do $tk = 1, q + 1$

if $(ADD(tj).eq.LINE(ti, tk))$ then

$tr = tr + 1$

end if

end do

end do

$Td(ti) = tr$

end do

do $ts = 1, qq$

do $tt = 0, N + 1$

if $(Td(ts).eq.tt)$ then

$TRR(tt) = TRR(tt) + 1$

end if

end do

end do

$ff = 0$

$s = 0$

do $i = N, 0, -1$

$ff = ff + 1$

$TYP(ff) = TRR(i)$

$s = s + TYP(ff)$

end do

if $(s.eq.qq)$ then

do $sor1 = 1, G2$

do $sor2 = sor1 + 1, G2 + 1$

if $(ADD(sor1).gt.ADD(sor2))$ then

$sor = ADD(sor2)$

ADD $(sor2) = ADD(sor1)$

ADD $(sor1) = sor$

end if

end do

end do

$comp = comp + 1$

do $xr = 1, tin$

if $(all(ADD1(xr, 1 : G2 + 1).eq.ADD(:)))$ then

$B1 = 1$

end if

end do

if $(B1.eq.0)$ then

$tin = tin + 1$

$ADD1(tin, :) = (/ADD(:), TYP/)$

end if

end if

end if

end do

if $(comp.eq.0)$ then

$complete = complete + 1$

write $(1, *)AD(adi, :),'$ Is Complete $(', G2,' ,' , N,') - arc'$

end if

end do

write $(1, *)$ '———————————————————'

write $(1, *)'$ Total number of complete $(', G2,' ,' , N,')is :'$, complete

write $(1, *)$ '———————————————————'

write $(1, *)$ ' COMPLETE $(', G2,' ,' , N,')$-arcs is $',' [', complete,']'$

Deallocate (AD)

Deallocate (ADD)

allocate $(Distinct - Types(tin, N + 1))$

$dist = 0$

do $uuii = 1, tin$

$uukk = 0$

do $uujj = uuii + 1, tin$

if $(all(ADD1(uujj, G2 + 2 : G2 + 2 + N).eq.ADD1(uuii, G2 + 2 : G2 + 2 + N)))$then

$uukk = uukk + 1$

end if

end do

if $(uukk.eq.0)$ then

$dist = dist + 1$

$Distinct - Types(dist, :) = ADD1(uuii, G2 + 2 : G2 + 2 + N)$

end if

end do

allocate $(TYPES(dist))$

allocate $(ADDED(tin, G2 + 1))$

$oo = 0$

do $ii = 1, dist$

$mm = 0$

do $jj = 1, tin$

if $(all(ADD1(jj, G2 + 2 : G2 + 2 + N).eq.Distinct - Types(ii, :)))$ then

$oo = oo + 1$

$ADDED(oo, :) = ADD1(jj, 1 : G2 + 1)$

$mm = mm + 1$

end if

end do

$TYPES(ii) = mm$

write $(2, *)TYPES(ii),'(', G2 + 1,',', N,') - arcoftype', Distinct - Types(ii, :)$

end do

if $(tin.eq.1)$ then

$N1 = 1$

goto 44

end if

deallocate $(ADD1)$

deallocate $(Distinct - Types)$

allocate $(NEW(tin, G2 + 1))$

allocate $(FATEN(G2 + 1, 3))$

allocate $(MULT(G2 + 1))$

$loop1 = 1$

$loop2 = TYPES(1)$

$GH = 0$

do $clas = 1, dist$

$GF = 0$

$C = 1 + loop2 - loop1$

Do $major = loop1, loop2$

$O1 = 0$

do $xi = 1, GH$

if $(all(NEW(xi, :).eq.ADDED(major, :)))$ then

$O1 = O1 + 1$

end if

end do

if $(O1.eq.0)$ then

Do $minor = major + 1, loop2$

$O2 = 0$

do $xj = 1, GH$

if $(all(NEW(xj, :).eq.ADDED(minor, :)))$ then

$O2 = O2 + 1$

end if

end do

if $(O2.eq.0)$ then

Do $i1 = 1, G2 + 1$

Do $i2 = 1, G2 + 1$

Do $i3 = 1, G2 + 1$

Do $i4 = 1, G2 + 1$

if ($i1$.ne.$i2$.And.$i1$.ne.$i3$.And.$i1$.ne.$i4$.And.$i2$.ne.$i3$.And.$i2$.ne.$i4$.And.$i3$.ne.$i4$) then

$PERMUTATION(:) = (/ADDED(minor, i1), ADDED(minor, i2),$

$ADDED(minor, i3), ADDED(minor, i4)/)$

$A(:, 1) = POINT(PERMUTATION(1), :)$

$A(:, 2) = POINT(PERMUTATION(2), :)$

$A(:, 3) = POINT(PERMUTATION(3), :)$

$AB = POINT(PERMUTATION(4), :)$

$A1 = A$

$A2 = A$

$A3 = A$

$A1(1, 1) = AB(1)$

$A1(2, 1) = AB(2)$

$A1(3, 1) = AB(3)$

$A2(1, 2) = AB(1)$

$A2(2, 2) = AB(2)$

$A2(3, 2) = AB(3)$

$A3(1, 3) = AB(1)$

$A3(2, 3) = AB(2)$

$A3(3, 3) = AB(3)$

$Det = mod(mod((A(1, 1) * A(2, 2) * A(3, 3) + A(1, 2) * A(2, 3) * A(3, 1)+$

$A(1, 3) * A(2, 1) * A(3, 2) - A(1, 3) * A(2, 2) * A(3, 1) - A(1, 1) * A(2, 3) * A(3, 2)-$

$A(1, 2) * A(2, 1) * A(3, 3)), q) + q, q)$

if $(Det.ne.0)$ then

$Det1 = \mathrm{mod}\ (\mathrm{mod}\ ((A1(1, 1) * A1(2, 2) * A1(3, 3)+$

$A1(1, 2) * A1(2, 3) * A1(3, 1) + A1(1, 3) * A1(2, 1) * A1(3, 2)-$

$A1(1, 3) * A1(2, 2) * A1(3, 1) - A1(1, 1) * A1(2, 3) * A1(3, 2)-$

$A1(1, 2) * A1(2, 1) * A1(3, 3)), q) + q, q)$

if $(Det1.ne.0)$ then

$Det2 = \mathrm{mod}\ (\mathrm{mod}((A2(1, 1) * A2(2, 2) * A2(3, 3)+$

$A2(1, 2) * A2(2, 3) * A2(3, 1) + A2(1, 3) * A2(2, 1) * A2(3, 2)-$

$A2(1, 3) * A2(2, 2) * A2(3, 1) - A2(1, 1) * A2(2, 3) * A2(3, 2)-$

$A2(1, 2) * A2(2, 1) * A2(3, 3)), q) + q, q)$

if $(Det2.ne.0)$ then

$Det3$ =mod (mod $((A3(1,1)*A3(2,2)*A3(3,3)+$

$A3(1,2)*A3(2,3)*A3(3,1)+A3(1,3)*A3(2,1)*A3(3,2)-$

$A3(1,3)*A3(2,2)*A3(3,1)-A3(1,1)*A3(2,3)*A3(3,2)-$

$A3(1,2)*A3(2,1)*A3(3,3)),q)+q,q)$

if $(Det3.ne.0)$ then

$PROJECTIVE(1,:)=mod(Det1*A(:,1),q)$

$PROJECTIVE(2,:)=mod(Det2*A(:,2),q)$

$PROJECTIVE(3,:)=mod(Det3*A(:,3),q)$

do $M1=1,G2+1$

$FATEN(M1,1)$ =mod $(POINT(ADDED(major,M1),1)*PROJECTIVE(1,1)+$

$POINT(ADDED(major,M1),2)*PROJECTIVE(2,1)+$

$POINT(ADDED(major,M1),3)*PROJECTIVE(3,1),q)$

$FATEN(M1,2)=mod(POINT(ADDED(major,M1),1)*PROJECTIVE(1,2)+$

$POINT(ADDED(major,M1),2)*PROJECTIVE(2,2)+$

$POINT(ADDED(major,M1),3)*PROJECTIVE(3,2),q)$

$FATEN(M1,3)=mod(POINT(ADDED(major,M1),1)*PROJECTIVE(1,3)+$

$POINT(ADDED(major,M1),2)*PROJECTIVE(2,3)+$

$POINT(ADDED(major,M1),3)*PROJECTIVE(3,3),q)$

if $(FATEN(M1,1).ne.0)$ then

$FATEN$ =mod $(INV(FATEN(M1,1))*FATEN,q)$

elseif $(FATEN(M1,2).ne.0)$ then

$FATEN$ =mod $(INV(FATEN(M1,2))*FATEN,q)$

elseif $(FATEN(M1,3).ne.0)$ then

$FATEN$ =mod $(INV(FATEN(M1,3))*FATEN,q)$

end if

do$LO=1,qq$

$MULT(M1)=LO$

exit

end if

end do

end do

do $sor1 = 1, G2$

do $sor2 = sor1 + 1, G2 + 1$

if $(MULT(sor1)$.gt.$MULT(sor2))$then

$sor = MULT(sor2)$

$MULT(sor2) = MULT(sor1)$

$MULT(sor1) = sor$

end if

end do

end do

if $(all(MULT(:)$.eq.$ADDED(minor, :)))$then

$GH = GH + 1$

$NEW(GH, :) = MULT(:)$

end if

end if

end if

end if

end if

end if

33 end do

end do

end do

end do

end if

99 end do

end if

$GF = GF + 1$

print*, $GF, ' of ', C$

100 end do

end do

allocate $(DISTINCT(tin, G2 + 1))$

$N1 = 0$

do $di = 1, tin$

$dk = 0$

do $dj = 1, GH$

if $(all(NEW(dj, :).\text{eq}.ADDED(di, :)))$ then

$dk = dk + 1$

end if

end do

if $(dk.eq.0)$ then

$N1 = N1 + 1$

DISTINCT $(N1, :) = ADDED(di, :)$

write $(4, *)DISTINCT(N1, :)$

end if

end do

write $(4, *)N1$

write $(1, *)$ 'Total number of complete $(', G2,' ,' , N,')$ is:$', complete$

write $(3, *)$ 'number of eq.$(', G2 + 1,' ,' , N,')$-arcs is $', GH,' oftotal', tin$

write $(5, *)$ 'Total $(', G2 + 1,' ,' , N,')$-arcs is$',' [', tin,']'$

write $(5, *)$ 'Equivalent$(', G2 + 1,' ,' , N,')$-arcs is$',' [', GH,']'$

write $(5, *)$ 'Distinc $(', G2 + 1,' ,' , N,')$-arcs is$',' [', N1,']'$

$G1 = N1$

$G2 = G2 + 1$

allocate $(AD(G1, G2))$

do$ni = 1, G1$

$AD(ni, :) = DISTINCT(ni, :)$

end do

Deallocate (NEW)

Deallocate $(DISTINCT)$

Deallocate $(ADDED)$

Deallocate $(FATEN)$

Deallocate $(MULT)$

Deallocate $(TYPES)$

END Do

goto 55

44 write $(8,*)$ 'MAXIMUM $(', G2+1,' ,' , N,')$-arcs is $', ADDED(1,:)$

write $(5,*)$ 'MAXIMUM $(', G2+1,' ,' , N,')$-arcs is $',' , 1$

write $(1,*)$ 'Total number of complete $(', G2+1,' ,' , N,')$ is:$', 1$

write $(1,*)ADDED(1,:),'$ Is Complete $(', G2+1,' ,' , N,')$-arc'

write $(4,*)ADDED(1,:)$

$write(4,*)1$

55 stop

end

Bibliography

[1] T. L. Alderson and A. A. Bruen, Maximal AMDS code, *Appl. Algebra Engrg. Comm. Comput.* **19** (2008), 87–98.

[2] A. Ali, *Classification of arcs in the Galois plane of order thirteen.* PhD thesis, University of Sussex, 1993.

[3] S. Ball, *On sets of points in finite planes.* PhD thesis, University of Sussex, 1994.

[4] S. Ball, Multiple blocking sets and arcs in finite planes, *J. London Math. Soc.* **54** (1996), 581–593.

[5] S. Ball, On the size of a triple blocking set in $\mathrm{PG}(2, q)$, *European J. Combin.* **17** (1996), 427–435.

[6] S. Ball, A. Blokhuis, On the size of a double blocking set in $\mathrm{PG}(2, q)$, *Finite Fields Appl.* **2** (1996), 125–137.

[7] S. Ball, Three-dimensional linear codes, Online table, (2014), https://mat-web.upc.edu/people/simeon.michael.ball/codebounds.html.

[8] D. Bartoli, S. Marcugini and F. Pambianco, The non-existence of some NMDS codes and the extremal size of complete $(n, 3)$-arcs in $\mathrm{PG}(2, 16)$, *Des. Codes Cryptogr.* **72** (2014), 129–134.

[9] J. Bierbrauer, (k, n)*-arcs of maximal size in the plane of order eight.* PhD thesis, Unpublished manuscript, 1988.

[10] A. Blokhuis, On multiple nuclei and a conjecture of Lurelli and Sce, *Bull. Belg. Math. Soc. Simon Steven* **3** (1994), 349–353.

[11] A. Blokhuis, L. Storme and T. Szőnyi, Multiple blocking sets and Baer-subplanes, *unpublished manuscipt* (1995), 351–356.

[12] A. E. Brouwer and H. A. Wilbrink, Blocking set in translation planes, *J. Geom.* **19** (1982), 200-200.

[13] A. A. Bruen, Baer subplanes and blocking sets, *Bull. Amer. Math. Soc.* **76** (1970), 342–344.

[14] A. A. Bruen, Blocking sets in finite projective planes, *SIAM J. Appl.Math.* **21** (1971), 380–392.

[15] A. A. Bruen and J. A. Thas, Blocking sets, *Geom. Dedicata* **6** (1977), 193–203.

[16] A. A. Bruen and R. Silverman, Arcs and blocking sets II, *European J.Combin.* **8** (1987), 351–356.

[17] A. A. Bruen, Polynomial multiplicities over finite and intersection sets, *J. Combin. Theory Ser.* **A 60** (1992), 19–33.

[18] P. Cameron, *Introduction to algebra*, Oxford University Press, New York, 1998, second edition.

[19] R. Casse, *Projective geometries over finite fields*, Oxford University Press, New York, 2006, second edition.

[20] R. Chandra, L. Dagum, D. Kohr, J. McDonald and R. Menon, *Parallel Programming in OpenMP*, Morgan Kaufmann, 2001.

[21] J. M. Chao and H. Kaneta, Classical arcs in $\mathrm{PG}(r, q)$ for $11 \leq q \leq 19$, *Discrete Math.* **174** (1997), 87–94.

[22] J. M. Chao and H. Kaneta, A complete 24-arc in $\mathrm{PG}(2, 29)$ with automorphism group $\mathrm{PSL}(2, 7)$, *Rend. Math. Appl.* **16** (1996), 537–544.

[23] J. M. Chao and H. Kaneta, Classical arcs in $\mathrm{PG}(r, q)$ for $23 \leq q \leq 29$, *Discrete Math.* **226** (2001), 377–385.

[24] C. Churchman, R. Ackoff and E. Arnoff, *Introduction to operations research*, New York, John Wiley and Sons, Inc., 1957, first edition.

[25] J. Ciechanowicz, *The non-existence of* $(37,6)$*-arcs in* $\mathrm{PG}(2,7)$ *and some other* (k,n)*-arcs in Galois planes.* PhD thesis, University of London, 1980.

[26] K. Coolsaet and H. Sticker, A full classification of the complete k-arcs in $\mathrm{PG}(2,23)$ and $\mathrm{PG}(2,25)$, *J. Combin. Des.* **17** (2009), 459–477.

[27] K. Coolsaet and H. Sticker, The complete k-arcs of $\mathrm{PG}(2,27)$ and $\mathrm{PG}(2,29)$, *J. Combin. Des.* **19** (2011), 111–130.

[28] K. Coolsaet and H. Sticker, The complete $(k,3)$-arcs of $\mathrm{PG}(2,q)$, $q \leq 13$, *J. Combin. Des.* **20** (2012), 89–111.

[29] K. Coolsaet, The complete arcs of $\mathrm{PG}(2,31)$, *J. Combin. Des.* **23** (2015), 522–533.

[30] R. Daskalov, On the existence and the non-existence of some (k,r)-arcs in $\mathrm{PG}(2,17)$, *Ninth International Workshop on Algebraic Combinatorial Coding Theory* (19-25 June, 2004), 95–100.

[31] S. Dodurekov and I. Landger, Near-MDS code, *J. Geom.* **54** (1995), 30–43.

[32] B. Fleischmann, R. Lasch, U. Derigs, W. Domschke and U. Rieder, *Operations research proceedings 2000*, Springer-Verlag Berlin Heidelberg, Germany, 2000, eight edition.

[33] W. Gropp, E. Lusk and A. Skjellum, *Portable parallel programming with the message-passing interface*, The MIT press, 1994.

[34] R. Hill, *A first course in coding theory*, Clarendon Press, Oxford, 1986, first edition.

[35] R. Hill and J. R. M. Mason, On (k,n)-arcs and the falsity of the Lunelli-Sce conjecture, *Finite Geometries and Designs (Proc. Conf., Chelwood Galo)*, 49, *Cambridge University* (1980), 153–168.

[36] J. W. P. Hirschfeld, *Projective geometries over finite fields*, Clarendon Press, Oxford, 1979, first edition.

[37] J. W. P. Hirschfeld, *Projective geometries over finite fields*, Oxford University Press, 1998, second edition.

[38] J. W. P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces, *J. Statist. Plann. Inference* **72** (1998), 355–380.

[39] J. W. P. Hirschfeld and E. V. D. Pichanick, Bounds for arcs of arbitrary degree in finite Desargusian planes, *J. Combin. Des.* **24** (2016), 184–196.

[40] J. Humphreys, *A Course in Group Theory*, Oxford University Press, New York, 1996, second edition.

[41] G. Kéri, Types of superregular matrices and the number of n-arcs and complete n-arcs in $\mathrm{PG}(r, q)$, *J. Combin. Des.* **14** (2006), 363–390.

[42] L. Lunelli and M. Sce, Considerazione arithmetiche e risultati sperimentali sui $\{K; n\}_q$-archi, *Ist. Lombardo Accad. Sci. Lett. Rend.* **A 98** (1964), 3–52.

[43] S. Marcugini, A. Milani and F. Pambianco, A computer based classification of the $(n, 3)$-arcs in $\mathrm{PG}(2, 8)$, *Rapporto Tecnico* (1998).

[44] S. Marcugini, A. Milani and F. Pambianco, Maximal $(n, 3)$-arcs in $\mathrm{PG}(2, 11)$, *Discrete Math.* **208** (1999), 421–426.

[45] S. Marcugini, A. Milani and F. Pambianco, Classification of the $[n, 3, n-3]_q$ NMDS codes over GF(7), GF(8) and GF(9), *Ars Combin.* **61** (2001), 263–269.

[46] S. Marcugini, A. Milani and F. Pambianco, Classification of the $(n, 3)$-arcs in $\mathrm{PG}(2, 7)$, *J. Geom.* **80** (2004), 179–184.

[47] S. Marcugini, A. Milani and F. Pambianco, Maximal $(n, 3)$-arcs in $\mathrm{PG}(2, 13)$, *Discrete Math.* **294** (2005), 139–145.

[48] S. Marcugini, A. Milani and F. Pambianco, Minimal complete arcs in $\mathrm{PG}(2, q)$, $q \leq 32$, *J. Combin. Math. Combin. Comput.* (2010), 217–222.

[49] J. R. M. Mason, *On (k, n)-arcs and (k, n)-caps in finite projective geometries.* PhD thesis, University of Salford, 1981.

[50] J. R. M. Monios and R. Bergqvist, *Inter modal freight transport and logistics*, CRC Press, 2017.

[51] J. W. Di Paola, On minimum blocking coalitions in small projective plane games, *SIAM J. Appl.Math.* **17** (1969), 378–392.

[52] J. Rajgopal, Principles and applications of operations research, *Maynard's Industrial Engineering Handbook* **1** (2004), 11–27.

[53] M. Richardson, On finite projective games, *Proc. Amer. Math. Soc.* **17** (1956), 458–465.

[54] A. Sadeh, *The classification of k-arcs and cubic surfaces with Twenty Seven Lines over the Field of Eleven Elements.* PhD thesis, University of Sussex, 1984.

[55] J. E. Savage, *Models of Computation*, Addison Wesley, 1998.

[56] B. Segre, Le geometrie di Galois, *Ann. Math. Pura Appl.* **48** (1959), 1–96.

[57] J. Singer, A theorem in finite projective geometry and some applications to number theory, *Trans. Amer. Math. Soc.* **43** (1938), 377–385.

[58] H. Taha, *Operations Research, an Introduction*, Pearson International, 2007, eighth edition.

[59] www.rcs.manchester.ac.uk/services/computational/parallel.

[60] A. Yasin, *Cubic Arcs in the Projective Plane of Order Eight.* PhD thesis, University of Sussex, 1986.