UNIVERSITY OF SUSSEX

THE CUBIC SURFACES WITH TWENTY-SEVEN LINES OVER FINITE FIELDS

Fatma Karaoglu

Thesis submitted for the Degree of Doctor of Philosophy in Mathematics

August 2018

Declaration

I hereby declare that this thesis has not been and will not be, submitted in whole or in part to another University for the award of any other degree.

I also declare that this thesis was composed by myself and that the work contained therein is my own, except where stated otherwise, such as citations.

Signature:

(Fatma Karaoglu)

Abstract

In this thesis, we classify the cubic surfaces with twenty-seven lines in three dimensional projective space over small finite fields. We use the Clebsch map to construct cubic surfaces with twenty-seven lines in $\mathrm{PG}(3,q)$ from 6-arcs not on a conic in $\mathrm{PG}(2,q)$. We introduce computational and geometrical procedures for the classification of cubic surfaces over the finite field $\mathbf{F}_q$. The performance of the algorithms is illustrated by the example of cubic surfaces over $\mathbf{F}_{13}$, $\mathbf{F}_{17}$ and $\mathbf{F}_{19}$.

Acknowledgements

I would like to express my thanks to Prof. James Hirschfeld by for giving me continuous guidance throughout this thesis and for all his advice and support. I am also appreciate for his remarks and suggestions in regard to typesetting problems in LaTeX. I am also deeply grateful to Prof. Anton Betten for the combined work contained in this thesis. I also thank him for his huge help with programming and his careful attention, detailed feedback, and patience on reading my multiple drafts. Without his advice and motivation, this work would not have been completed. Thank you James and Anton for introducing Finite Geometry to me. I hope this work proves to be worthy of your time and effort.

I thank all of my friends. They have been greatly tolerant and supportive. Finally, I would like to thank my mom and sister for their continuous support, encouragement and love.

Contents

Chapter 1

Introduction

This thesis combines some parts of two papers [9] and [10] and expands the work published in [9].

Classifying cubic surfaces over the complex numbers was one of the highlights of ninteenth century mathematics. In 1849, Cayley and Salmon showed that a general cubic surface over the complex field contains exactly 27 lines, [12]. However Cayley said *"there is great difficulty in conceiving the complete figure formed by the twenty-seven lines, indeed this can hardly be accomplished till a more perfect notation is discovered".* He observed that through each line of $\mathcal{F}$, there are 5 planes meeting it in two other lines and these planes are called *tritangent planes.* Further he showed that the equation of $\mathcal{F}$ can be written as $LMN + PQR = 0$ where L, M, N, P, Q, R are certain linear forms. These forms are associated to objects called trihedra, more specifically to a *Steiner trihedral pair* [30]. Finally in 1858 Schläfli answered Cayley's request by proving the double-six theorem [27]. The double-six as introduced by Schläfli is as follows: a double-six in $\mathrm{PG}(3, K)$ is the set of 12 lines

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix}$$

such that each line only meets those five lines which are not in the same row or column. The main property of a double-six is that it determines a unique cubic surface with 27 lines.

Let us now look at the case where $K = \mathbf{F}_q$ is a finite field of q elements. In 1915, Dickson [15] showed that a non-singular cubic surface $\mathcal{F}$ over $\mathbf{F}_2$ can only have $15, 9, 5, 3, 2, 1$, or

0 lines. In [28], Segre showed that a non-singular cubic surface $\mathcal{F}$ over $\mathbf{F}_q$ with q odd has $27, 15, 9, 7, 5, 3, 2, 1$, or 0 lines. Dickson [16] also classified all the projectively distinct non-degenerate cubic surfaces over $\mathbf{F}_2$.

The main related problems are stated in [19].

(i) When does a double-six exist over $\mathbf{F}_q$?

(ii) What are the particular properties of cubic surfaces over $\mathbf{F}_q$? For example, the number of Eckardt points on a line of the cubic surface, arithmetical properties of cubic surface and configuration of Eckardt points.

(iii) Classify the cubic surfaces with twenty-seven lines over $\mathbf{F}_q$.

Problem (i) has been settled, problem (ii) has seen much progress in [21]. This thesis is about problem (iii).

In [19] and [21], the smallest cases, namely, $\mathbf{F}_4, \mathbf{F}_7, \mathbf{F}_8, \mathbf{F}_9$, are resolved. Sadeh [26] classified the cubic surfaces with 27 lines in $\mathrm{PG}(3, 11)$.

Our work combines the classical theory around Schläfli's double-six and the birational Clebsch map with algebraic combinatorics around group actions on relations.

The classification problem for cubic surfaces is the problem of determining the orbits of $G = \mathrm{PGL}(4, q)$ on cubic surfaces. Two cubic surfaces are equivalent if there is a projectivity which maps one to the another. Thus, we consider the action of the group $G = \mathrm{PGL}(4, q)$ on $\mathrm{PG}(3, q)$ and hence on cubic surfaces.

In [9], an algorithm to classify cubic surfaces with 27 lines is presented that is based on the notion of lifting a planar 6-arc. This algorithm has certain limitations which make it unviable for larger values of q. In this work, we present a refined algorithm which can be used to classify cubic surfaces with 27 lines over any finite field. Here, the results of the classification of cubic surfaces with 27 lines over $\mathbf{F}_{13}, \mathbf{F}_{17}, \mathbf{F}_{19}$ is presented.

The structure of the thesis is as follows.

Chapter 2 introduces some basic concepts and definitions on projective geometry over finite fields and some results from group theory.

Chapter 3 explores properties of cubic surfaces and its associated structures; for example, double-six, trihedral pair and others.

The purpose of Chapter 4 is to construct cubic surfaces from 6-arcs in the plane. To do so, we explore the Clebsch map from a surface to a plane and study its properties. After this, an algorithm is described to create a cubic surface from a 6-arc in a plane.

In Chapter 5, the problem of classifying cubic surfaces is addressed. The approach taken in this chapter relies on the classification of the related non-conical 6-arcs in the projective plane over the same field. It utilizes the Clebsch map from Chapter 4 to create surfaces from arcs. For the classification, the theory developed earlier is used. A related but different approach to classifying cubic surfaces is described in [9]. In Section 5.1, the general theory of classification is discussed. The second section of Chapter 5 addresses the classification of cubic surfaces with 27 lines in $\mathrm{PG}(3, q)$ for small q. In Section 5.2.2, the theory developed in Section 5.1 is applied to the problem of classification of double-triplets in $\mathrm{PG}(3, q)$. Finally, in Section 5.3, we conjecture a mass formula for the total number of cubic surfaces with 27 lines (not up to isomorphism) as a function of q.

Chapter 2

Projective Geometry over Finite Fields

2.1 Finite fields

2.1.1 Definitions

Definition 2.1.1. A *field* is an ordered triple $(K, +, \times)$ such that K is a nonempty set and $+$, $\times$ are two binary operations on K satisfying the following axioms:

(i) *closure* of $+$ and $\times$: $a + b \in K$ and $a \times b \in K$ for all $a, b \in K$;

(ii) *associativity* of $+$ and $\times$: $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$ for all $a, b, c \in K$;

(iii) *commutativity* of $+$ and $\times$: $a + b = b + a$ and $a \times b = b \times a$ for all $a, b \in K$;

(iv) *identity elements* of two operations: $a + 0 = a$ such that $0 \in K$ and $b \times 1 = b$ such that $1 \in K$ and $0 \neq 1$ for all $a \in K$;

(v) *inverses* of two operations: there exist $-a \in K$ such that $a + (-a) = 0$ and $a^{-1} \in K$ such that $a \times a^{-1} = 1$ for all $0 \neq a \in K$;

(vi) *distributivity* of $\times$ over $+$: $a \times (b + c) = (a \times b) + (a \times c)$.

Definition 2.1.2. A *finite field* is a field with only a finite number of elements.

Definition 2.1.3. The *characteristic* of a finite field K is the smallest positive integer p such that $p \times 1 = \underbrace{1 + 1 + \cdots + 1}_{p \text{ times}} = 0$.

It is known due to work of Galois that finite fields exist for every prime power q and due to work of E.H. Moore that there is a unique finite field up to isomorphism for every such order q.

Let p be a prime and q be a prime power such that $q = p^h$ for some $h \in \mathbb{Z}$. There exists a field of order q and it is unique. Any field of q elements is isomorphic to $\mathbf{F}_q$, which can also be denoted by $GF(q)$. In this thesis, we use $\mathbf{F}_q$.

The additive structure of $\mathbf{F}_q$ where $q = p^h$, is given by

$$\mathbf{F}_q \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p. \ (h \text{ factors})$$

The multiplicative structure of $\mathbf{F}_q$, $q = p^h$, is given by

$$\mathbf{F}_q^* = \mathbf{F}_q \setminus \{0\} \cong \mathbb{Z}_{q-1}.$$

A generator of the multiplicative group of the finite field with order q is a *primitive element* of $\mathbf{F}_q^*$. The characteristic of $\mathbf{F}_q$ is p.

For instance, $\mathbf{F}_{13}$ consists of the residue classes of the integers modulo 13, $\mathbb{Z}/13\mathbb{Z}$, under the natural addition and multiplication, and is a finite field of 13 elements.

$$\mathbf{F}_{13} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \mid 13 = 0\}$$

A primitive element for $\mathbf{F}_{13}$ is 2, for example.

2.1.2 Automorphisms

Definition 2.1.4. Let $\mathbf{F}_q$ be a finite field with order $q = p^h$ for some $h \in \mathbb{N}$. An *automorphism* σ of $\mathbf{F}_q$ is a permutation of $\mathbf{F}_q$ such that

$$(x + y)\sigma = x\sigma + y\sigma, \quad (xy)\sigma = (x\sigma)(y\sigma) \text{ for all } x, y \in \mathbf{F}_q.$$

The *Frobenius automorphism* ϕ is the automorphism of $\mathbf{F}_q$ defined by $x\phi = x^p$ for

$x \in \mathbf{F}_q$.

The group $Aut(\mathbf{F}_q)$ of automorphisms of $\mathbf{F}_q$ is isomorphic to $\mathbb{Z}_h$. It is generated by the Frobenius automorphism ϕ,

$$Aut(\mathbf{F}_q) = \{\mathrm{I}, \phi, \phi^2, \ldots, \phi^{h-1}\}.$$

2.2 Projective space over a field

Definition 2.2.1. Let V be an $(n+1)$-dimensional vector space over a field K with zero element 0; it is denoted by $V(n+1, K)$. The n-dimensional projective space, denoted as $\mathrm{PG}(n, K)$, is the quotient of $V \backslash \{0\}$ by the equivalence relation

$$x \sim y \quad \iff \quad x = \lambda y \quad \text{for some} \quad \lambda \in K^* = K \backslash \{0\}$$

with the zero deleted. When $K = \mathbf{F}_q$, it is denoted by $\mathrm{PG}(n, q)$.

The elements of $\mathrm{PG}(n, q)$ are called points; the equivalence class of a vector X is the point $\mathbf{P}(X)$. Here, X is a vector representing $\mathbf{P}(X)$. The points $\mathbf{P}(X_1), \ldots, \mathbf{P}(X_k)$ are linearly independent if a set of vectors $X_1, \ldots, X_k$ representing them is linearly independent.

An r-dimensional subspace of $\mathrm{PG}(n, q)$ is an $(r+1)$-dimensional subspace of $V(n+1, K)$; it is denoted by Π_r. For example, *a point, a line,a plane and a solid* of the projective space are the vector subspaces of V of dimension $1, 2, 3$, and 4.

Let $\mathbf{P}(X)$ and $\mathbf{P}(Y)$ be points in $\mathrm{PG}(n, q)$ where $X = (x_0, \ldots, x_n)$ and $Y = (y_0, \ldots, y_n)$ with $x_i, y_i \in \mathbf{F}_q$. The line l which passes through $\mathbf{P}(X)$ and $\mathbf{P}(Y)$ is denoted by

$$l = \mathbf{L} \begin{bmatrix} X \\ Y \end{bmatrix}.$$

If $\dim(V) = 2$, then $\mathrm{PG}(1, q)$ is called the *projective line.*

Let $\mathbf{P}(X)$, $\mathbf{P}(Y)$ and $\mathbf{P}(Z)$ be points in $\mathrm{PG}(n, q)$ which are not collinear. The plane defined by $\mathbf{P}(X)$, $\mathbf{P}(Y)$ and $\mathbf{P}(Z)$ is denoted by

$$\boldsymbol{\pi}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

If $\dim(V) = 3$, then $\mathrm{PG}(2, q)$ is called the *projective plane*.

The projective plane $\mathrm{PG}(2, q)$ over $\mathbf{F}_q$ contains q^2+q+1 points and q^2+q+1 lines, [24]. There are $q+1$ points on a line and $q+1$ lines through a point. A point $\mathbf{P}(Y) = \mathbf{P}(y_0, y_1, y_2)$ is in $\mathrm{PG}(2, q)$ where y_0, y_1, y_2 are in $\mathbf{F}_q$ and not all zero.

The space $\mathrm{PG}(3, q)$ contains $q^3 + q^2 + q + 1$ points and equally many planes, as well as $(q^2 + q + 1)(q^2 + 1)$ lines. There are $q^2 + q + 1$ lines through every point and $q + 1$ planes through a line.

A point $\mathbf{P}(X) = \mathbf{P}(x_0, x_1, x_2, x_3)$ is in $\mathrm{PG}(3, q)$ where $x_0, x_1, x_2, x_3 \in \mathbf{F}_q$ are not all zero. Two distinct planes always intersect in a line. Three distinct planes in $\mathrm{PG}(3, q)$ intersect in either a line or a point.

Definition 2.2.2. A hyperplane is a subspace of $\mathrm{PG}(n, q)$ of codimension 1.

In $\mathrm{PG}(2, q)$, the hyperplanes are lines. Hyperplanes in $\mathrm{PG}(3, q)$ are planes.

Definition 2.2.3. A k-arc in $\mathrm{PG}(2, q)$ is a set of k points, no three of which are collinear.

Theorem 2.2.4 ([11])**.** *The largest arc in a projective plane of odd order q is a $(q+1)$-arc which is called an* oval*. The largest arc in a projective plane of even order q is a $(q+2)$-arc which is called a* hyperoval*.*

Definition 2.2.5. *A conic C* over $\mathbf{F}_q$ is the zero set of a homogeneous quadratic equation in 3 variables x_0, x_1, x_2 over a field $\mathbf{F}_q$. The equation of a conic can be written as

$$a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_0x_1 + a_4x_0x_2 + a_5x_1x_2 = 0$$

for constants $a_i \in \mathbf{F}_q$, $i = 0, \ldots, 5$.

Any 5-arc in $\mathrm{PG}(2, q)$ determines a unique conic passing through it.

Theorem 2.2.6 ([29])**.** *(Segre's theorem) Every $(q + 1)$-arc in* $\mathrm{PG}(2, q)$ *is a conic for q odd.*

A $(q+2)$-arc only exists when q is even. An example is a conic plus its nucleus which is the intersection of all its tangents.

Theorem 2.2.7 ([22])**.** *Let*

$$S = \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(a,b,1), \mathbf{P}(c,d,1)\}$$

be a set of 6 *points in* $\mathrm{PG}(2,q)$ *for* $a,b,c,d \in \mathbf{F}_q$. *Then* S *is a* $6-$*arc not on a conic if the following hold:*

(1) $a,b,c,d \neq 0,1$ *and* $a-c \neq 0$, $a-b \neq 0$, $b-d \neq 0$, $c-d \neq 0$

(2) $ad - bc \neq 0$,

(3) $(a-1)(d-1) - (b-1)(c-1) \neq 0$,

(4) $ad(b-1)(c-1) - bc(a-1)(d-1) \neq 0$.

Theorem 2.2.8 ([21])**.** *The number of* 6*-arcs not on a conic in* $\mathrm{PG}(2,q)$ *is*

$$\frac{q^3(q+1)(q-1)^2(q-2)(q-3)(q-5)^2(q^2+q+1)}{6\,!}.$$

Definition 2.2.9. Let $x_0 : x_1 : \ldots : x_n$ be homogeneous coordinates for $\mathrm{PG}(n,q)$. Let F be a homogeneous equation of degree r in the $n+1$ variables over $\mathbf{F}_q$.

(i) In $\mathrm{PG}(n,q)$, the *variety* $V(F)$ is the set of points $\mathbf{P}(x_0, x_1, \ldots, x_n)$ such that

$$F(x_0, x_1, \ldots, x_n) = 0$$

for the homogeneous polynomial F.

(ii) The variety $\mathcal{F} = V(F)$ is a *primal* or *hypersurface.*

(iii) When $r = 1$, then $\mathcal{F}$ is a *hyperplane* of $\mathrm{PG}(n,q)$ as defined in Definition 2.2.2.

(iv) A hypersurface in $\mathrm{PG}(2,q)$ is a *plane curve*, a hypersurface in $\mathrm{PG}(3,q)$ is a *surface.*

A hypersurface $\mathcal{F}$ is irreducible if F is irreducible over $\mathbf{F}_q$. The degree of the hypersurface $\mathcal{F}$ is the degree of F.

Definition 2.2.10. Let $P = \mathbf{P}(A)$ be a point on the hypersurface $\mathcal{F}$ where $A = (a_0, \dots, a_n)$. Then P is a *singular* point if

$$\frac{\partial F}{\partial x_0}(A) = \cdots = \frac{\partial F}{\partial x_n}(A) = 0.$$

Definition 2.2.11. The hypersurface $\mathcal{F}$ is *non-singular* if $\mathcal{F}$ does not have any singular point.

Definition 2.2.12. Let $\mathcal{F} = V(F)$ be the plane curve defined over $\mathbf{F}_q$ where

$$F = f_0 x_0^n + f_1 x_0^{n-1} + f_2 x_0^{n-2} + \cdots + f_n,$$

is in variables x_0, x_1, x_2 and f_i is a form in x_1, x_2 of degree i. If

$$f_0 = 0, \quad f_1 = 0, \quad f_2 \neq 0,$$

then $\mathcal{F}$ has a point of multiplicity 2 at $\mathbf{P}(1, 0, 0)$. This point is called a *double point*.

2.2.1 Fundamental theorem of projective geometry

The fundamental theorem of projective geometry determines the automorphism group of $\mathrm{PG}(n, q)$. Because an automorphism preserves collinearity, and hence incidence, it is also known as a collineation.

Definition 2.2.13. A *collineation* $\mathfrak{C} : \mathrm{PG}(n, q) \to \mathrm{PG}(n, q)$ is a bijection between subspaces which preserves incidence; that is, if $\Pi_r \subset \Pi_s$, then $\Pi_r\mathfrak{C} \subset \Pi_s\mathfrak{C}$. Thus a collineation of a projective geometry is a permutation of the points which maps subspaces onto subspaces.

Definition 2.2.14. A *correlation* is a bijection between subspaces of $\mathrm{PG}(n, q)$ which reverses incidence. It maps points to hyperplanes and vice-versa.

Definition 2.2.15. Let V be an $(n + 1)$-dimensional vector space over a field $\mathbf{F}_q$. A *semi-linear transformation* $\mathfrak{S} : V \to V$ is a transformation such that there exists a field automorphism $\sigma \in Aut(\mathbf{F}_q)$ satisfying the following.

(i) $(u + v)\mathfrak{S} = u\mathfrak{S} + v\mathfrak{S}$,

(ii) $(cu)\mathfrak{S} = (c\sigma)u\mathfrak{S}$ for all $u, v \in V$ and $c \in \mathbf{F}_q$.

Definition 2.2.16. Let V be an $(n+1)$-dimensional vector space over a field $\mathbf{F}_q$. With respect to a fixed basis of V, an automorphism σ of $\mathbf{F}_q$ can be extended to an *automorphic collineation* Φ_σ of $\mathrm{PG}(n, q)$; this is given by

$$\Phi_\sigma : \mathbf{P}(x_0, \ldots, x_n) \mapsto \mathbf{P}(x_0\sigma, \ldots, x_n\sigma).$$

It can be seen that Φ_σ is a semi-linear map.

Let

$$\Phi = \{\Phi_\sigma \mid \sigma \in Aut(\mathbf{F}_q) \text{ be the group of automorphic collineations of } V\}.$$

Definition 2.2.17. Let V be an $(n+1)$-dimensional vector space over a field $\mathbf{F}_q$ and $\mathrm{PG}(n, q)$ be the associated projective space.

(i) The *general linear group* $\mathrm{GL}(n+1, q)$ is the group of bijective linear transformations of $V(n+1, q)$ under composition.

(ii) The group $\Gamma L(n+1, q) = \mathrm{GL}(n+1, q) \rtimes \Phi$ is the semidirect product of $\mathrm{GL}(n+1, q)$ and Φ.

Definition 2.2.18. Any element $(T, \Phi_\sigma) \in \Gamma L(n+1, q)$ induces a collineation of $\mathrm{PG}(n, q)$. Namely it sends $\mathbf{P}(X)$ to $\mathbf{P}(X')$ where

$$X' = XT\Phi_\sigma.$$

(i) This resulting group is called $P\Gamma L(n+1, q)$.

(ii) Under this map, the image of $\mathrm{GL}(n+1, q)$ is $\mathrm{PGL}(n+1, q)$, the projectivity group of $\mathrm{PG}(n, q)$.

If there exists a projectivity which sends the object S to the object S', then S and S' are *projectively equivalent*; otherwise they are *projectively distinct*.

Any invertible semi-linear transformation of $V = V(n+1, q)$ will induce a collineation of $\mathrm{PG}(n, q)$. The converse is true also, as we will see next.

Lemma 2.2.19. *Let Z be the subgroup of $\Gamma L(n+1,q)$ and Z_0 be the subgroup of* $\mathrm{GL}(n+1,q)$ *such that* $Z = \{\mathfrak{s} \in \Gamma L(n+1,q) \mid \mathfrak{s} : v \mapsto tv, t \in \mathbf{F}_q, t \neq 0\}$ *and* $Z_0 = \{\mathfrak{a} \in \mathrm{GL}(n+1,q) \mid \forall \mathfrak{b} \in \mathrm{GL}(n+1,q), \mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}\}$. *Then*

$$P\Gamma L(n+1,q) = \Gamma L(n+1,q)/Z, \qquad \mathrm{PGL}(n+1,q) = \mathrm{GL}(n+1,q)/Z_0.$$

The order of the group of projectivities of $\mathrm{PG}(2,q)$ is

$$|\mathrm{PGL}(3,q)| = \frac{(q^3-1)(q^3-q)(q^3-q^2)}{q-1} = q^3(q^3-1)(q^2-1)$$

and the order of the group of projectivities of $\mathrm{PG}(3,q)$ is

$$|\mathrm{PGL}(4,q)| = \frac{(q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3)}{q-1} = q^6(q^4-1)(q^3-1)(q^2-1).$$

Theorem 2.2.20 ([4])**.** *If the rank of $V = V(n+1,q)$ is at least 3, then the automorphism group of* $\mathrm{PG}(n,q)$ *is* $\mathrm{P\Gamma L}(n+1,q)$.

For q prime, $\mathrm{P\Gamma L}(n+1,q)$ is equal to $\mathrm{PGL}(n+1,q)$.

2.2.2 The principle of duality

For any space S, there is a dual space S^*, whose points, lines, $\ldots$, and hyperplanes are respectively the hyperplanes, $\ldots$, lines, and points of S. If a theorem in S stated in terms of points, lines, $\ldots$, hyperplanes and their incidence, the same theorem is true in S^* and gives a dual theorem by substituting hyperplane for point and point for hyperplane. Thus join and meet are dual. Hence the dual of an r-space in $\mathrm{PG}(n,q)$ is an $(n-r-1)$-space.

In particular, in $\mathrm{PG}(2,q)$ the dual of a point is a line, whereas the dual of a line is a point; in $\mathrm{PG}(3,q)$ the dual of a point is a plane and vice versa, and the dual of a line is a line.

2.2.3 Quadrics in finite projective space

Definition 2.2.21. A *quadric* in $\mathrm{PG}(n,q)$ is a set of projective points

$$\mathscr{Q} := \{\mathbf{P}(X) \mid Q(X) = 0\}$$

where Q is a quadratic form, that is, Q is a homogeneous polynomial of degree 2 in $n+1$ variables. If the number of variables involved in the equation cannot be reduced under projective equivalence, then the corresponding quadric is said to be *non-degenerate*.

Definition 2.2.22. Let

$$\mathscr{B}(X,Y) = Q(X+Y) - Q(X) - Q(Y)$$

be the *bilinear form* associated to the quadratic form Q.

Definition 2.2.23. For a subspace U, define

$$U^{\perp} = \{v \in V \mid \mathscr{B}(u,v) = 0 \text{ for all } u \in U\}$$

the perpendicular subspace of U with respect to $\mathscr{B}$.

Theorem 2.2.24 ([24]). *In* $\mathrm{PG}(n,q)$,

(1) *the number of projectively distinct non-singular quadrics is one for n even; it is called parabolic, and denoted as $\mathscr{P}_n$.*

(2) *the number of projectively distinct non-singular quadrics is two for n odd, and they are called elliptic and hyperbolic, denoted by $\mathscr{E}_n$ and $\mathscr{H}_n$.*

The canonical forms for these quadrics are

$$\begin{aligned}
\mathscr{P}_n &= V(x_0^2 + x_1x_2 + x_3x_4 \cdots + x_{n-1}x_n),\\
\mathscr{E}_n &= V(f(x_0,x_1) + x_2x_3 + x_4x_5 \cdots + x_{n-1}x_n),\\
\mathscr{H}_n &= V(x_0x_1 + x_2x_3 + \cdots + x_{n-1}x_n)
\end{aligned}$$

where f is an irreducible binary quadratic form, that is, a quadratic homogeneous polynomial in two variables which cannot be factored over $\mathbf{F}_q$.

Theorem 2.2.25 ([24]). *The number of $\mathbf{F}_q$ points of the non-singular quadrics in* $\mathrm{PG}(n,q)$ *are as follows:*

(1) $(q^{n+1}-1)/(q-1)$ *on* $\mathscr{P}_n$,

(2) $(q^{\frac{n+1}{2}}+1)(q^{\frac{n+1}{2}-1}-1)/(q-1)$ *on* $\mathscr{E}_n$,

(3) $(q^{\frac{n+1}{2}-1}+1)(q^{\frac{n+1}{2}}-1)/(q-1)$ *on* $\mathscr{H}_n$.

Definition 2.2.26. The hyperbolic quadric $\mathscr{H}_3$ defined by the equation

$$x_0x_3 - x_1x_2 = 0$$

contains the $q+1$ lines

$$\left\{\mathbf{L}\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \end{bmatrix} \mid a \in \mathbf{F}_q\right\} \cup \left\{\mathbf{L}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\}.$$

This set of lines is called a *ruling* of $\mathscr{H}_3$. A ruling $\mathbf{R}$ is also called a *regulus.* The same quadric also contains the lines

$$\left\{\mathbf{L}\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & a \end{bmatrix} \mid a \in \mathbf{F}_q\right\} \cup \left\{\mathbf{L}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\},$$

which form a second ruling and is called the *opposite regulus*, denoted as $\mathbf{R}^{opp}$. Each point of $\mathscr{H}_3$ lies on exactly one line from each ruling. The lines of $\mathscr{H}_3$ are also known as the *generators* of $\mathscr{H}_3$.

On $\mathscr{H}_3$, there are $(q+1)^2$ points and $2(q+1)$ lines.

2.2.4 Polarities

Definition 2.2.27. Let S be a space $\mathrm{PG}(n,q)$ and S^* its dual space. The map $\mathfrak{p}$ is a *polarity* from S to S^* if the following holds.

(i) $\mathfrak{p}$ is a correlation,

(ii) $\mathfrak{p}^2 = I$, where I is an identity map.

$\mathfrak{p}$ maps points to hyperplanes and maps hyperplanes to points also.

A polarity is a correlation of order two. Here, only two types of polarities will be introduced. A more detailed account of the polarities can be found in ([24] Section 2.1.5).

Definition 2.2.28. Let $\mathfrak{p}$ be a polarity of $\mathrm{PG}(n,q)$ and assume the characteristic of $\mathbf{F}_q$ is odd . Suppose that $\mathfrak{p}$ is a projectivity of S to S^* given by the matrix T.

(i) If n is odd and $T^\top = -T$, then the polarity $\mathfrak{p}$ is called a *standard (null) polarity* on $\mathrm{PG}(n, q)$,

(ii) If $T^\top = T$, then the polarity $\mathfrak{p}$ is called an *orthogonal polarity* or *ordinary polarity with respect to a quadric.*

Here, $T^\top$ denotes the transpose of T.

2.2.5 Stereographic projection

Some problems in $\mathrm{PG}(3, q)$ can be more easily investigated in $\mathrm{PG}(2, q)$. For this reason, we consider the stereographic projection. Here we only state the projection from a point of a hyperbolic quadric. For other cases, see [23].

Definition 2.2.29. Let $\mathscr{H}_3$ be a nondegenerate hyperbolic quadric in $\mathrm{PG}(3, q)$. Let P be a point of $\mathscr{H}_3$ and ℓ_1, ℓ_2 be the generators through P and let π be a plane not containing P. Also, assume that π does not contain ℓ_1 and ℓ_2. The stereographic projection

$$\zeta\colon \mathscr{H}_3 \setminus \{P\} \to \pi$$

is given by $A\zeta = AP \cap \pi$ where $A \in \mathscr{H}_3 \setminus \{P\}$.

Remark 2.2.30. Let $\mathscr{H}_3$ be a nondegenerate hyperbolic quadric in $\mathrm{PG}(3, q)$. Let $P \in \mathscr{H}_3$ and let π be a plane not through ℓ_1 and ℓ_2, the generators through P. The stereographic projection ζ has the following properties:

(i) Every point of ℓ_1 other than P maps to Q_1 and every point of ℓ_2 other than P maps to Q_2 where $Q_1 = \ell_1 \cap \pi$ and $Q_2 = \ell_2 \cap \pi$.

(ii) Let v be the plane spanned by ℓ_1 and ℓ_2 and let $\ell_0 = Q_1Q_2 = \pi \cap v$. There are q^2 points on $\mathscr{H}_3$ not on ℓ_1 or ℓ_2. These points map to the points of $\pi \setminus \ell_0$.

(iii) Every line of π through Q_1 other than ℓ_0 is an image of a generator other than ℓ_2 meeting ℓ_1. Similarly, q generators of $\mathscr{H}_3$ meeting ℓ_2 other than ℓ_1 map to the q lines of π through Q_2 other than ℓ_0.

(iv) Recall that there are q^2+q+1 planes through P and $2q+1$ of them meet $\mathscr{H}_3$ in two lines. Therefore, there are q^2-q planes through P meeting $\mathscr{H}_3$ in a conic. These q^2-q conics map to the q^2-q lines in π containing neither Q_1 nor Q_2.

(v) There are q^3-q^2 planes not through P meeting $\mathscr{H}_3$ in a conic. These conics map bijectively to the q^3-q^2 conics in π through Q_1 and Q_2.

2.3 The Klein correspondence

Let $\mathbf{P}(X)$ and $\mathbf{P}(Y)$ be points in $\mathrm{PG}(3,q)$ where $X=(x_0,x_1,x_2,x_3)$ and $Y=(y_0,y_1,y_2,y_3)$ with $x_i, y_i \in \mathbf{F}_q$. The line l which passes through $\mathbf{P}(X)$ and $\mathbf{P}(Y)$ is denoted by

$$l = \mathbf{L}\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix}.$$

Let $\mathscr{L}$ be the set of lines in $\mathrm{PG}(3,q)$ and $\mathscr{H}_5$ be the Klein quadric in $\mathrm{PG}(5,q)$. Up to equivalence, $\mathscr{H}_5$ is the set of points in $\mathrm{PG}(5,q)$ whose coordinates $(x_0,\ldots,x_5)$ satisfy the equation

$$\mathfrak{Q}(X) = x_0x_1 + x_2x_3 + x_4x_5 = 0.$$

The lines in $\mathrm{PG}(3,q)$ are in bijection to the points on $\mathscr{H}_5$.

Definition 2.3.1. The mapping $\kappa : \mathscr{L} \to \mathscr{H}_5$ is given by

$$\mathbf{L}\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} \mapsto \mathbf{P}(l_{01}, l_{23}, -l_{02}, l_{13}, l_{03}, l_{12})$$

where $l_{ij} = x_iy_j - x_jy_i$ are the *Plücker coordinates.* This mapping is called the *Klein correspondence.*

Lemma 2.3.2. *Let $\mathbf{P}(X) = \mathbf{P}(x_0,x_1,x_2,x_3,x_4,x_5)$ and $\mathbf{P}(Y) = \mathbf{P}(y_0,y_1,y_2,y_3,y_4,y_5)$ be two points on $\mathscr{H}_5$. The line l through $\mathbf{P}(X)$ and $\mathbf{P}(Y)$ is completely contained in $\mathscr{H}_5$ if*

$$\mathscr{B}(X,Y) = x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2 + x_4y_5 + x_5y_4 = 0.$$

Here, $\mathscr{B}(X,Y)$ is the bilinear form associated to $\mathfrak{Q}(X)$.

A pencil of lines in $\mathrm{PG}(3,q)$ corresponds a line on $\mathscr{H}_5$. There are $(q^2+q+1)(q^2+1)$ points and $q^5+2q^4+3q^3+3q^2+2q+1$ lines on $\mathscr{H}_5$, ([8]).

Lemma 2.3.3. *Two lines ℓ_1 and ℓ_2 of $\mathrm{PG}(3,q)$ intersect if and only if the corresponding points P_1 and P_2 on $\mathscr{H}_5$ are collinear (in a line of $\mathscr{H}_5$) where*

$$P_1 = \kappa(\ell_1) = \mathbf{P}(l_{01}, l_{23}, -l_{02}, l_{13}, l_{03}, l_{12}), \text{ and } P_2 = \kappa(\ell_2) = \mathbf{P}(l'_{01}, l'_{23}, -l'_{02}, l'_{13}, l'_{03}, l'_{12}).$$

By Lemma 2.3.2, this occurs if and only if

$$\mathscr{B}(\kappa(\ell_1), \kappa(\ell_2)) = l_{01}l'_{23} + l_{23}l'_{01} - l_{02}l'_{13} - l_{13}l'_{02} + l_{03}l'_{12} + l_{12}l'_{03} = 0.$$

This means that two lines ℓ_1, ℓ_2 are skew if and only if $\mathscr{B}(\kappa(\ell_1), \kappa(\ell_2)) \neq 0$.

Example 2.3.4. In $\mathrm{PG}(3,7)$, let a_1, a_2 and b_1 be three lines as follows:

$$a_1 = \mathbf{L}\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}, a_2 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix}, b_1 = \mathbf{L}\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix},$$

Using Lemma 2.3.3, it can be seen that b_1 is skew to a_1 and intersects a_2 since

$$\kappa(a_1) = \mathbf{P}(1,1,0,0,3,2), \kappa(a_2) = \mathbf{P}(1,5,4,4,0,0), \text{ and } \kappa(b_1) = \mathbf{P}(0,0,6,1,5,3)$$

and $\mathscr{B}(\kappa(b_1), \kappa(a_1)) = 5 \neq 0$ and $\mathscr{B}(\kappa(b_1), \kappa(a_2)) = 0$.

2.4 Some results from group theory

Definition 2.4.1. Let $(G, *)$ be a group and S be a non-empty set. Let $g_1, g_2 \in G$ and $x \in S$. A *(right) action* of G on S is a function

$$\diamond : S \times G \to S$$

such that

(i) $x \diamond (g_1 * g_2) = (x \diamond g_1) \diamond g_2$, and

(ii) $x \diamond e = x$, where e is the identity element of G.

If there is an action of G on S, then S is called a G-set.

Example 2.4.2. Here are some examples of actions on the set of $\mathrm{PG}(n, q)$ under the group $\mathrm{PGL}(n+1, q)$.

(a) Let $G = \mathrm{PGL}(n+1, q)$ and $S = \mathrm{PG}(n, q)$. In this example, consider the right action of G on S given by vector matrix multiplication. That is, for $s \in \mathrm{PG}(n, q)$ considered as $1 \times (n+1)$ matrix and $g \in \mathrm{PGL}(n+1, q)$ considered as $(n+1) \times (n+1)$ matrix, the action is $(s, g) \mapsto s \cdot g$. We call this action a *basic action.*

(b) Let $G = \mathrm{PGL}(n+1, q)$ and $\mathcal{A}$ be the set of k-subsets of $\mathrm{PG}(n, q)$. For $A \in \mathcal{A}$ such that $A = \{P_1, P_2, \ldots, P_k\}$ and $g \in \mathrm{PGL}(n+1, q)$ consider the action $(A, g) \mapsto A \cdot_{\mathcal{A}} g$ where $A \cdot_{\mathcal{A}} g = \{P_1 \cdot g, P_2 \cdot g, \ldots, P_k \cdot g\}$. We call this action $\cdot_{\mathcal{A}}$ an *induced action on subsets of size k.*

(c) Let $G = \mathrm{PGL}(n+1, q)$, $\mathcal{A}$ be the set of k-subsets of $\mathrm{PG}(n, q)$ and $\mathcal{B}$ be the set of l-subsets of $\mathrm{PG}(n, q)$. Assume that G acts on both sets $\mathcal{A}$, $\mathcal{B}$ with actions $\cdot_{\mathcal{A}}$, $\cdot_{\mathcal{B}}$. Therefore, G also acts on the set $\mathcal{R} = \{(A, B) \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$. The action $\cdot_{\mathcal{R}}$ of G on $\mathcal{R}$ is $((A, B), g) \mapsto (A, B) \cdot_{\mathcal{R}} g$ where $(A, B) \cdot_{\mathcal{R}} = (A \cdot_{\mathcal{A}} g, B \cdot_{\mathcal{B}} g)$. We call this action an *induced action on pairs.*

Theorem 2.4.3 ([24])**.** *Let S be a G-set and an action $(s, g) \mapsto sg$ for $g \in G$ and $s \in S$. If the relation $\sim_G$ on S for all $x, y \in S$ satisfy*

$$x \sim_G y \textit{ if and only if } xg = y \textit{ for some } g \in G,$$

then $\sim_G$ is an equivalence relation on S.

Definition 2.4.4. Let G be a group acting on a set S with an action $(s, g) \mapsto sg$ for $g \in G$ and $s \in S$. Let x be an element of S.

(i) The orbit of an element x of S is the set of elements in S which are the images of x under elements of G. It is denoted by $\mathrm{Orb}_G(x)$:

$$\mathrm{Orb}_G(x) = \{y \in S \mid \exists g \in G \text{ such that } xg = y\}.$$

(ii) The stabiliser of an element x of S is the set of elements of G which move x to itself. It is denoted by $\mathrm{Stab}_G(x)$.

$$\mathrm{Stab}_G(x) = \{g \in G \mid xg = x\}.$$

It is a subgroup of G.

(iii) The group G is said to act transitively on S if $x \sim_G y$ for all $x, y \in S$. This means that there is only one orbit.

Theorem 2.4.5 ([24])**.** *(Conjugate-Stabiliser Theorem) Consider the group G acting on the set S and let x be in S and $g \in G$. Then,*

$$\mathrm{Stab}_G(xg) = g^{-1}\mathrm{Stab}_G(x)g.$$

Theorem 2.4.6 ([24])**.** *(Orbit-Stabiliser Theorem) Consider the finite group G acting on the finite set S and let x be in S. Then,*

$$|G| = |\mathrm{Stab}_G(x)| \cdot |\mathrm{Orb}_G(x)|.$$

Chapter 3

Double-sixes and Cubic Surfaces

The aim of this thesis is to classify the cubic surfaces with 27 lines in $\mathrm{PG}(3,q)$. Before the classification, all possible cubic surfaces need to be constructed. The structure of the cubic surface has to be considered. For this purpose, a structure known as a double-six is important.

3.1 Double-six

Definition 3.1.1. A *double-six* in $\mathrm{PG}(3,K)$ is the set of 12 lines

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{array} \tag{3.1}$$

such that a_i intersects b_j and is skew to a_j and b_i, whereas b_i intersects a_j and is skew to b_j and a_i for $i \neq j$.

Theorem 3.1.2 (Schläfli [27])**.** *Given five skew lines a_1, a_2, a_3, a_4, a_5 with a single transversal b_6 such that each set of four a_i omitting a_j has a unique further transversal b_j, then the five lines b_1, b_2, b_3, b_4, b_5 also have a transversal a_6. These twelve lines form a double-six.*

A double-six in $\mathrm{PG}(3,q)$ defines a cubic surface with 27 lines in $\mathrm{PG}(3,q)$. Here is an example of a double-six.

Example 3.1.3. Let us consider the twelve lines in $\mathrm{PG}(3,13)$ and their corresponding points on $\mathscr{H}_5$ such that

$$a_1 = \mathbf{L}\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad a_2 = \mathbf{L}\begin{bmatrix} 1 & 11 & 0 & 0 \\ 0 & 0 & 1 & 12 \end{bmatrix}, \quad a_3 = \mathbf{L}\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 0 & 12 \end{bmatrix},$$

$$a_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad a_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 11 & 0 \end{bmatrix}, \quad a_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix},$$

$$b_1 = \mathbf{L}\begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 12 \end{bmatrix}, \quad b_2 = \mathbf{L}\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad b_3 = \mathbf{L}\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$b_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & 11 & 0 \\ 0 & 1 & 0 & 12 \end{bmatrix}, \quad b_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 7 & 0 \end{bmatrix}, \quad b_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 6 & 0 \end{bmatrix},$$

$$\begin{aligned}
A_1 &= \kappa(a_1) = \mathbf{P}(0,0,12,2,1,2), & A_2 &= \kappa(a_2) = \mathbf{P}(0,0,12,2,12,11),\\
A_3 &= \kappa(a_3) = \mathbf{P}(1,7,0,0,12,7), & A_4 &= \kappa(a_4) = \mathbf{P}(1,7,0,0,1,6),\\
A_5 &= \kappa(a_5) = \mathbf{P}(1,11,2,1,0,0), & A_6 &= \kappa(a_6) = \mathbf{P}(1,11,11,12,0,0),\\
B_1 &= \kappa(b_1) = \mathbf{P}(0,0,12,7,12,6), & B_2 &= \kappa(b_2) = \mathbf{P}(0,0,12,7,1,7),\\
B_3 &= \kappa(b_3) = \mathbf{P}(1,2,0,0,1,11), & B_4 &= \kappa(b_4) = \mathbf{P}(1,2,0,0,12,2),\\
B_5 &= \kappa(b_5) = \mathbf{P}(1,6,6,12,0,0), & B_6 &= \kappa(b_6) = \mathbf{P}(1,6,7,1,0,0).
\end{aligned}$$

From Lemma 2.3.3, it can be seen that the line a_1 intersects b_2 and is skew to a_2 since $\mathscr{B}(A_1, B_2) = 0$ and $\mathscr{B}(A_1, A_2) = 5 \neq 0$. It can be seen that

$$a_1, \ldots, a_6, b_1, \ldots, b_6$$

form a double-six. This is the example from [10] with $a = 2$ and $b = 1$ in $\mathrm{PG}(3, 13)$. It is shown in Table 3.1.

Theorem 3.1.4 ([10])**.** *The cubic surface in* $\mathrm{PG}(3, q)$ *given by the equation*

$$x_3^3 - b^2(x_0^2 + x_1^2 + x_2^2)x_3 + \frac{b^3}{a}(a^2+1)x_0x_1x_2 = 0$$

where $a, b \in \mathbf{F}_q$ *with* $a \notin \{0, \pm 1\}$*,* $a^2 \neq \pm 1$ *and* $b \neq 0$ *has twenty-seven lines and the*

$$a_1 = \mathbf{L}\begin{bmatrix}1 & 2 & 0 & 0\\0 & 0 & 1 & 1\end{bmatrix} \qquad b_1 = \mathbf{L}\begin{bmatrix}1 & 6 & 0 & 0\\0 & 0 & 1 & 12\end{bmatrix}$$

$$a_2 = \mathbf{L}\begin{bmatrix}1 & 11 & 0 & 0\\0 & 0 & 1 & 12\end{bmatrix} \qquad b_2 = \mathbf{L}\begin{bmatrix}1 & 7 & 0 & 0\\0 & 0 & 1 & 1\end{bmatrix}$$

$$a_3 = \mathbf{L}\begin{bmatrix}1 & 0 & 6 & 0\\0 & 1 & 0 & 12\end{bmatrix} \qquad b_3 = \mathbf{L}\begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 0 & 1\end{bmatrix}$$

$$a_4 = \mathbf{L}\begin{bmatrix}1 & 0 & 7 & 0\\0 & 1 & 0 & 1\end{bmatrix} \qquad b_4 = \mathbf{L}\begin{bmatrix}1 & 0 & 11 & 0\\0 & 1 & 0 & 12\end{bmatrix}$$

$$a_5 = \mathbf{L}\begin{bmatrix}1 & 0 & 0 & 12\\0 & 1 & 11 & 0\end{bmatrix} \qquad b_5 = \mathbf{L}\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 7 & 0\end{bmatrix}$$

$$a_6 = \mathbf{L}\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 2 & 0\end{bmatrix} \qquad b_6 = \mathbf{L}\begin{bmatrix}1 & 0 & 0 & 12\\0 & 1 & 6 & 0\end{bmatrix}$$

Table 3.1: The double-six of Theorem 3.1.4 for $q = 13, a = 2, b = 1$

double-six in Table 3.2 is associated with it.

3.2 Cubic surfaces with twenty-seven lines in $\mathrm{PG}(3, q)$

Definition 3.2.1. *A cubic surface* $\mathcal{F}$ over $\mathbf{F}_q$ is the zero set of a homogeneous cubic equation in 4 variables over a field $\mathbf{F}_q$.

Let $m_1, \ldots, m_{20}$ be the 20 monomials of degree 3 in the four variables x_0, x_1, x_2 and x_3. The equation of the cubic surface can be written as

$$\sum_{j=1}^{20} \alpha_j m_j(x_0, x_1, x_2, x_3) = 0$$

for constants $\alpha_j \in \mathbf{F}_q$, $j = 1, \ldots, 20$. For instance;

$$\begin{aligned}\mathcal{F} &= V\left(x_0^3 + x_1^3 + x_2^3 + x_3^3\right)\\ &= \{\mathbf{P}(x_0, x_1, x_2, x_3) \in \mathrm{PG}(3, q) \mid x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\}.\end{aligned}$$

Therefore, to determine a cubic surface, 19 independent conditions are required since there are 20 monomials of degree 3 in four variables. The number of points on a cubic surface with 27 lines will be determined in Chapter 4 (Corollary 4.2.4).

Cubic surfaces and double-sixes are closely connected since a double-six determines a unique cubic surface.

$$a_1 = \mathbf{L}\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{bmatrix} \qquad b_1 = \mathbf{L}\begin{bmatrix} 1 & -\frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 & -b \end{bmatrix}$$

$$a_2 = \mathbf{L}\begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 0 & 1 & -b \end{bmatrix} \qquad b_2 = \mathbf{L}\begin{bmatrix} 1 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 & b \end{bmatrix}$$

$$a_3 = \mathbf{L}\begin{bmatrix} 1 & 0 & -\frac{1}{a} & 0 \\ 0 & 1 & 0 & -b \end{bmatrix} \qquad b_3 = \mathbf{L}\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \end{bmatrix}$$

$$a_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & \frac{1}{a} & 0 \\ 0 & 1 & 0 & b \end{bmatrix} \qquad b_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & -b \end{bmatrix}$$

$$a_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & -a & 0 \end{bmatrix} \qquad b_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & \frac{1}{a} & 0 \end{bmatrix}$$

$$a_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & a & 0 \end{bmatrix} \qquad b_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & -\frac{1}{a} & 0 \end{bmatrix}$$

Table 3.2: The double-six from Theorem 3.1.4

Theorem 3.2.2 ([23]). *A double-six lies on a unique cubic surface $\mathcal{F}$ with* 15 *further lines c_{ij} given by the intersection of $[a_i, b_j]$ and $[a_j, b_i]$. Here $[a_i, b_j]$ denotes the plane spanned by a_i and b_j.*

A cubic surface which admits a double-six has exactly twenty-seven lines in $\mathrm{PG}(3, q)$. From now on, $\mathcal{F}$ always denotes a cubic surface with 27 lines.

Assume that

$$\mathcal{F} = V\Big(\sum_{j=1}^{20} \alpha_j m_j(x_0, x_1, x_2, x_3)\Big). \tag{3.2}$$

The condition that every point on any of the 12 lines of the double-six lies on this surface can be expressed by requiring that (3.2) holds true for all those points. In order to find the cubic equation in four variables, it suffices to know 19 points on it which impose 19 linearly independent conditions. For instance, the set of points lying on a five-plus-one associated with a double-six have this property. Pick 4 points from the transversal and then 3 further points from each of the five lines but not on the transversal. These $4 + 5 \times 3 = 19$ points impose 19 conditions. Those 19 points are shown in Figure 3.1.

Example 3.2.3. Let $\mathscr{D}$ be the double-six of Example 3.1.3. Let $\mathcal{F}$ be the cubic surface which arises from $\mathscr{D}$ in $\mathrm{PG}(3, 13)$. Let us find the equation of the cubic surface $\mathcal{F}$. After substituting 19 points from Figure 3.1 into the equation (3.2), the equation of the cubic

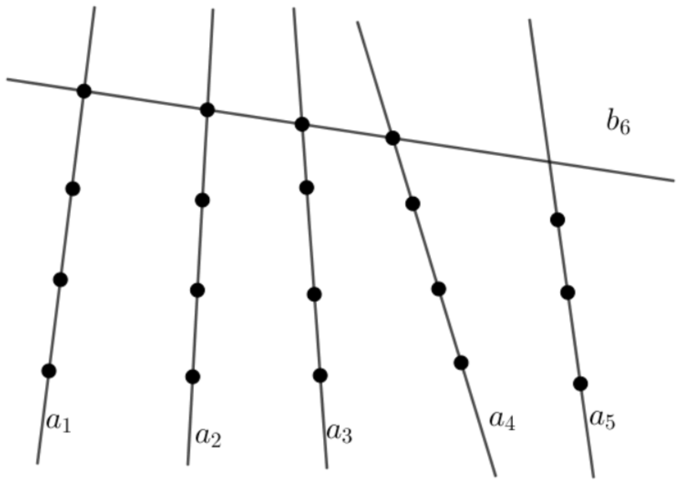


Figure 3.1: 19 points to substitute into (3.2)

surface which is associated with the double-six from Example 3.1.3 is

$$\mathcal{F} = V\Big(12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3\Big).$$

The points on $\mathcal{F}$ can be seen in the Appendix A.1.1.

Example 3.2.4. Let $\mathscr{D}$ be the double-six from Example 3.1.3. Let $\mathcal{F}$ be the cubic surface which arises from $\mathscr{D}$. Theorem 3.2.2 gives the following additional 15 lines of $\mathcal{F}$.

$$c_{12} = [a_1, b_2] \cap [a_2, b_1] = \boldsymbol{\pi} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 7 & 0 & 0 \end{bmatrix} \cap \boldsymbol{\pi} \begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 12 \\ 1 & 11 & 0 & 0 \end{bmatrix}$$

The planes $[a_1, b_2] = V(12x_2 + x_3)$ and $[a_2, b_1] = V(x_2 + x_3)$. Then the intersection of these two planes is

$$c_{12} = \mathbf{L} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Similarly, one finds fifteen further lines of the cubic surface $\mathcal{F}$ as in Table 3.3. So, the cubic

$$c_{12} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, c_{13} = \mathbf{L}\begin{bmatrix} 0 & 1 & 12 & 0 \\ 6 & 11 & 0 & 12 \end{bmatrix}, c_{14} = \mathbf{L}\begin{bmatrix} 0 & 12 & 12 & 0 \\ 7 & 2 & 0 & 12 \end{bmatrix},$$

$$c_{15} = \mathbf{L}\begin{bmatrix} 1 & 0 & 12 & 0 \\ 2 & 6 & 0 & 12 \end{bmatrix}, c_{16} = \mathbf{L}\begin{bmatrix} 12 & 0 & 12 & 0 \\ 11 & 7 & 0 & 12 \end{bmatrix}, c_{23} = \mathbf{L}\begin{bmatrix} 0 & 12 & 12 & 0 \\ 7 & 11 & 0 & 12 \end{bmatrix},$$

$$c_{24} = \mathbf{L}\begin{bmatrix} 0 & 1 & 12 & 0 \\ 6 & 2 & 0 & 12 \end{bmatrix}, c_{25} = \mathbf{L}\begin{bmatrix} 12 & 0 & 12 & 0 \\ 2 & 7 & 0 & 12 \end{bmatrix}, c_{26} = \mathbf{L}\begin{bmatrix} 1 & 0 & 12 & 0 \\ 11 & 6 & 0 & 12 \end{bmatrix},$$

$$c_{34} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad c_{35} = \mathbf{L}\begin{bmatrix} 12 & 0 & 10 & 6 \\ 0 & 12 & 3 & 7 \end{bmatrix}, c_{36} = \mathbf{L}\begin{bmatrix} 12 & 0 & 10 & 7 \\ 0 & 12 & 10 & 7 \end{bmatrix},$$

$$c_{45} = \mathbf{L}\begin{bmatrix} 12 & 0 & 3 & 6 \\ 0 & 12 & 3 & 6 \end{bmatrix}, c_{46} = \mathbf{L}\begin{bmatrix} 12 & 0 & 3 & 7 \\ 0 & 12 & 10 & 6 \end{bmatrix}, c_{56} = \mathbf{L}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Table 3.3: The fifteen further lines

surface $\mathcal{F}$ has the 27 lines in $\mathrm{PG}(3,q)$

$$\{a_i \mid i = 1, \ldots, 6\} \text{ (6 lines)}$$

$$\{b_i \mid i = 1, \ldots, 6\} \text{ (6 lines)}$$

$$\{c_{ij} \mid 1 \leq i < j \leq 6\} \text{ (15 lines)}.$$

In this thesis, only the cubic surfaces which arise from a double-six in $\mathrm{PG}(3,q)$ are considered.

The following result establishes a relation between cubic surfaces and planar 6-arcs which do not lie on a conic. Because this relation is very important for our approach to classifying cubic surfaces, we include a proof of part of it. This proof is an expanded version of the proof in [23].

Lemma 3.2.5 ([23]). *In* $\mathrm{PG}(3,q)$,

(1) *Let* a_1, a_2, a_3, a_4, a_5 *be skew lines with a transversal* b. *Each set of four* a_i *has a unique, distinct second transversal if and only if each set of five of six lines is linearly independent.*

(2) *The configuration above exists if and only if a* 6*-arc not on a conic exists in* PG$(2, q)$. *This occurs if and only if* $q \neq 2, 3,$ *or* 5.

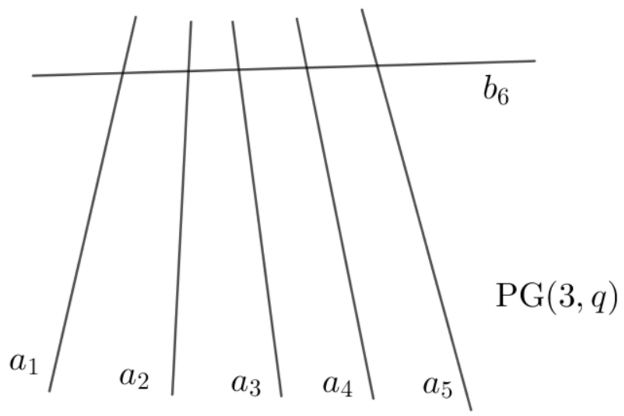
Proof (1) A set of lines in PG$(3, q)$ is linearly dependent if their corresponding coordinate vectors in $\mathscr{H}_5$ are linearly dependent. For more details on linear dependence of lines, see [23] Section 15.2 and 15.4. The following statements hold and the result follows.

(i) Five skew lines are linearly independent if and only if they only have one transversal.

(ii) Four skew lines with a transversal have second transversal if and only if the five lines are linearly independent.

(iii) Four skew lines are linearly independent if and only if these four have exactly two transversals.

(2) Let a_1, a_2, a_3, a_4, a_5 be five skew lines with a common transversal b_6 and let each set of four a_i has a unique, distinct second transversal. From (1), each set of five of the six lines is linearly independent.



Let $A_1, A_2, A_3, A_4, A_5, B_6$ be their corresponding point under the Klein correspondence, $A_i = \kappa(a_i)$ for $i = 1, \ldots, 5$ and $B_6 = \kappa(b_6)$ in $\mathscr{H}_5$. The A_i all lie in the perpendicular subspace of B_6, which is a cone $B_6\mathscr{H}_3$ with vertex B_6 and base $\mathscr{H}_3$ in a solid Π_3. In the first step, project $A_1, \ldots, A_5$ from B_6 to points $A'_1, \ldots, A'_5$ of $\mathscr{H}_3$ embedded in the Klein quadric (Figure 3.2).

$$A'_i = B_6 A_i \cap \Pi_3 \text{ where } i = 1, \ldots, 5.$$

These five points have the following properties.

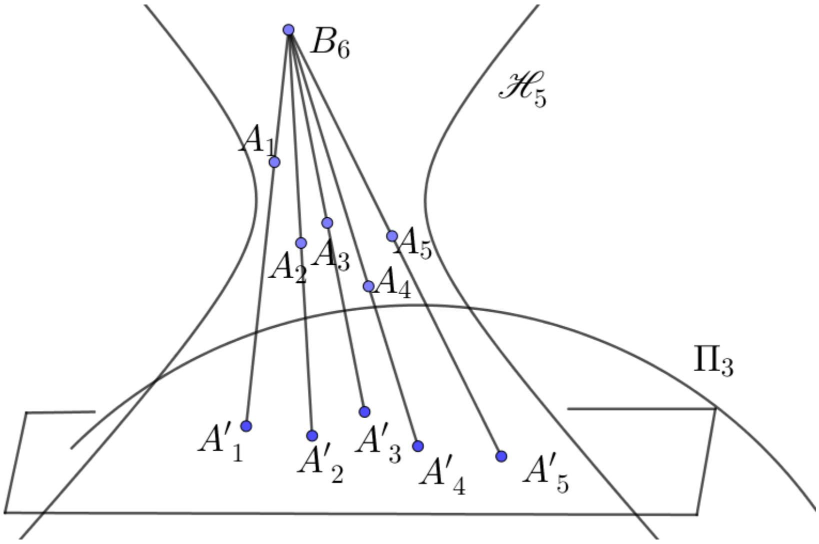


Figure 3.2: Projection from B_6 onto $\mathscr{H}_3$

(i) Since the a_i are skew, the join of two A_i is not on $\mathscr{H}_5$ by Lemma 2.3.3, and so no two A'_i lie on the same generator of $\mathscr{H}_3$.

(ii) If four A'_i were coplanar, then B_6 and the corresponding four A_i would lie in a solid, whence either b_6 is dependent on the four a_i or the four a_i are dependent. Both possibilities are contrary to the hypothesis in (1). So, no four of the five A'_i lie in a plane.

Consider the stereographic projection explained in Section 2.2.5. Let π be a plane not containing A'_1. Project $A'_2, \ldots, A'_5$ from A'_1 to points $A''_2, \ldots, A''_5$ of the plane π, Figure 3.3.

$$A''_i = A'_1 A'_i \cap \pi \text{ where } i = 2, \ldots, 5.$$

There are two generators ℓ_1 and ℓ_2 of $\mathscr{H}_3$ through the point A'_1. Let

$$P = \ell_1 \cap \pi \text{ and } Q = \ell_2 \cap \pi.$$

We claim that

$$\{P, Q, A''_2, A''_3, A''_4, A''_5\}$$

is a 6-arc not on a conic. In order to prove this, the following has to be shown.

(a) $\{P, Q, A''_2, A''_3, A''_4, A''_5\}$ is an arc, that is no three of its points are collinear.

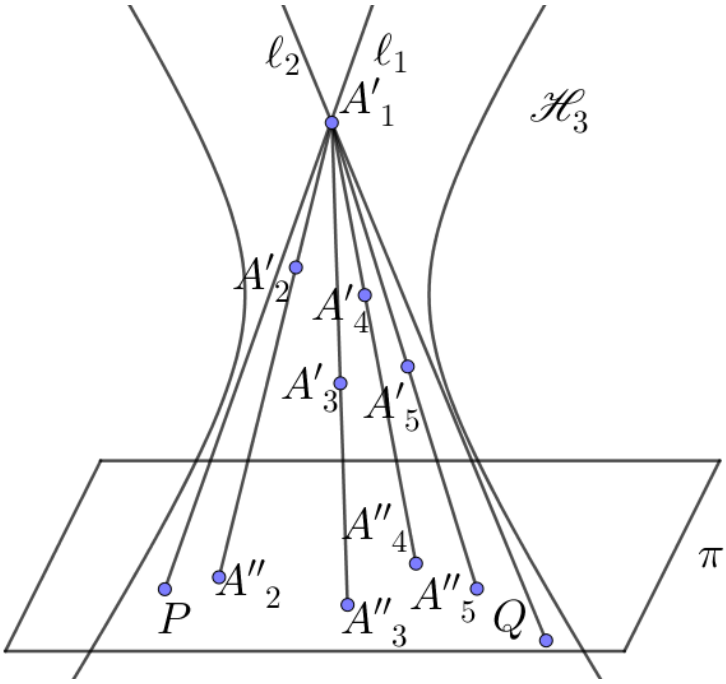


Figure 3.3: Projection from A'_1 to the plane π

The lines in π;

1- through P and Q,

There is no more point on this line which would be the image of any of the A_i under the projection by Remark 2.2.30 (i) and (ii).

2- through P,

If P, A''_i and A''_j were collinear, this line would be the image of the line in the opposite regulus of ℓ_1 where $i, j = \{2, 3, 4, 5\}$ by Remark 2.2.30 (iii). This is the contradiction to the property (2)- (i).

The case of lines through Q would be similar.

3- not through P and Q,

Let π_1 be the hyperplane passing through A'_1 in $\mathrm{PG}(3, q)$. If A''_i, A''_j and A''_k were collinear, such a line would be an image of a conic defined by a plane π_1 and so they would correspond to the points A'_i, A'_j and A'_k where $i, j, k \in \{2, 3, 4, 5\}$ by Remark 2.2.30 (iv). Together with A'_1, four A'_r would be on a plane, contradiction with the property (2)- (ii) where $r \in \{1, 2, 3, 4, 5\}$.

(b) $\{P, Q, A''_2, A''_3, A''_4, A''_5\}$ is not on a conic.

Let π_2 be the hyperplane which does not pass through A'_1 in $\mathrm{PG}(3,q)$. If

$$\{P, Q, A''_2, A''_3, A''_4, A''_5\}$$

was on a conic in the plane, then this conic would arise from intersections of $\mathscr{H}_3$ by hyperplane π_2 by Remark 2.2.30 (v). So, this means that the 4 points $A'_2, \ldots, A'_5$ lie on the conic which is the intersection of the hyperplane π_2 with $\mathscr{H}_3$. This contradicts with the property (2)- (ii).

Now, we show that a 6-arc not on a conic exists in $\mathrm{PG}(2,q)$ and hence a double-six and a cubic surface with 27 lines in $\mathrm{PG}(3,q)$ when $q \neq 2,3,5$.

When $q \leq 3$, there are no 5 arcs. For $q = 2$, the hyperoval is the largest arc with 4 points. For $q = 3$, the conic is the largest arc with 4 points. For $q = 4$, the hyperoval is a 6-arc consisting of the points of a conic together with its nucleus. This guarantees that there is a 6-arc not on a conic in $\mathrm{PG}(2,4)$; therefore a double-six exists. For $q = 5$, an oval is 6-arc but it is on a conic by Segre's Theorem; so a double-six does not exist. In $\mathrm{PG}(2,q)$, the sides of a quadrangle contain $6q-5$ points. A pentastigm is a 5-arc, together with the 10 lines that are joins of pairs of the points. The points and lines are called *vertices* and *sides* of the pentastigm. The sides of a pentastigm contain $10q-20$ points. For $q \geq 7$, a 5-arc exists since $10q - 20 < q^2 + q + 1$; then, the number of points on no chord of the 5-arc and also not on the conic through the 5-arc is

$$(q^2 + q + 1) - [(10q - 20) + (q - 4)] = (q - 5)^2 > 0.$$

This shows that a 6-arc not on a conic exists for all $q \geq 7$. Hence double-sixes and cubic surfaces with 27 lines exist for all $q \geq 7$. Consequently, a cubic surface exists over all finite fields $\mathbf{F}_q$ except when $q = 2,3,5$. □

Theorem 3.2.6 ([21]). *A necessary and sufficient condition for the existence of a double-six, and hence of a cubic surface with* 27 *lines, is the existence in a plane over the same field of a* 6*-arc not on a conic. This occurs when* $q \neq 2,3,5$.

Example 3.2.7. In $\mathrm{PG}(3,13)$, consider the five skew lines b_1, b_2, b_3, b_4, b_5 with the common transversal a_6.

$$b_1 = \mathbf{L}\begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 12 \end{bmatrix}, b_2 = \mathbf{L}\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, b_3 = \mathbf{L}\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$b_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & 11 & 0 \\ 0 & 1 & 0 & 12 \end{bmatrix}, b_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 7 & 0 \end{bmatrix}, a_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

In $\mathscr{H}_5 = V(x_0x_1 + x_2x_3 + x_4x_5)$,

$$B_1 = \kappa(b_1) = \mathbf{P}(0, 0, 11, 1, 2, 1),$$
$$B_2 = \kappa(b_2) = \mathbf{P}(0, 0, 2, 12, 2, 1),$$
$$B_3 = \kappa(b_3) = \mathbf{P}(6, 12, 0, 0, 6, 1),$$
$$B_4 = \kappa(b_4) = \mathbf{P}(7, 1, 0, 0, 6, 1),$$
$$B_5 = \kappa(b_5) = \mathbf{P}(1, 6, 7, 1, 0, 0),$$
$$A_6 = \kappa(a_6) = \mathbf{P}(1, 11, 2, 1, 0, 0).$$

The B_i's all lie in the perpendicular subspace of $A_6 = \kappa(a_6)$, which is a cone $A_6\mathscr{H}_3$ with vertex A_6 and base $\mathscr{H}_3$ in a solid $\Pi_3 = V(x_0, 6x_1 + 6x_2 - x_3)$ and $\mathscr{H}_3 = \mathscr{Q} \cap \Pi_3$. Here, we identify the factor space $A_6^{\perp}/\langle A_6\rangle$ with $V(x_0, 6x_1 + 6x_2 - x_3)$ and then $\mathscr{H}_3 = \mathscr{Q} \cap V(x_0, 6x_1 + 6x_2 - x_3)$.

For $i = 1, \ldots, 5$, let B_i' be the projection of B_i from A_6 onto $\mathscr{H}_3$. So

$$B_1' = \mathbf{P}(0, 0, 11, 1, 2, 1), \quad B_2' = \mathbf{P}(0, 0, 2, 12, 2, 1), \quad B_3' = \mathbf{P}(0, 11, 1, 7, 6, 1),$$
$$B_4' = \mathbf{P}(0, 2, 12, 6, 6, 1), \quad B_5' = \mathbf{P}(0, 8, 5, 0, 0, 0).$$

Let $\mathrm{PG}(3, q) = \Pi_3 = V(x_0, 6x_1 + 6x_2 - x_3)$ be defined on the four-dimensional vector space $V(4, q)$ with basis e_1, e_2, e_3, e_4 where

$$e_1 = \mathbf{P}(0, 1, 0, 6, 0, 0), e_2 = \mathbf{P}(0, 0, 1, 6, 0, 0), e_3 = \mathbf{P}(0, 0, 0, 0, 1, 0), e_4 = \mathbf{P}(0, 0, 0, 0, 0, 1).$$

Let $y_0 : y_1 : y_2 : y_3$ be homogeneous or local coordinates for the subspace with respect to this basis. The points of the hyperbolic quadric $\mathscr{H}_3$ in $\mathrm{PG}(3, q)$ are represented by local coordinates as $P = \mathbf{P}(y_0, y_1, y_2, y_3)$. With respect to local coordinates in Π_3, the points $B'_1, \ldots, B'_5$ are as follows:

$$B''_1 = \mathbf{P}(0, 11, 2, 1), \qquad B''_2 = \mathbf{P}(0, 2, 2, 1), \qquad B''_3 = \mathbf{P}(11, 1, 6, 1),$$
$$B''_4 = \mathbf{P}(2, 12, 6, 1), \qquad B''_5 = \mathbf{P}(8, 5, 0, 0).$$

Pick a vertex B''_5 and project $B''_1, \ldots, B''_4$ from B''_5 to points $R_1, \ldots, R_4$ of a plane $v = V(y_0)$ in Π_3 not containing B''_5. The generators of $\mathscr{H}_3$ through B''_5 are the two lines

$$L_1 = \mathbf{L}\begin{bmatrix} 8 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } L_2 = \mathbf{L}\begin{bmatrix} 8 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $R_5 = L_1 \cap v$ and $R_6 = L_2 \cap v$. Then,

$$R_1 = \mathbf{P}(0, 11, 2, 1), \qquad R_2 = \mathbf{P}(0, 2, 2, 1), \qquad R_3 = \mathbf{P}(0, 12, 6, 1),$$
$$R_4 = \mathbf{P}(0, 1, 6, 1), \qquad R_5 = \mathbf{P}(0, 0, 1, 0), \qquad R_6 = \mathbf{P}(0, 0, 0, 1).$$

Observe that

$$\{R_1, R_2, R_3, R_4, R_5, R_6\}$$

is a 6-arc not on a conic in a plane v. It is often convenient to choose a basis for v and express the arc in local coordinates with respect to this basis.

Let c_1, c_2, c_3, be the standard basis for $\mathrm{PG}(2, q)$ where

$$c_1 = \mathbf{P}(0, 1, 0, 0), c_2 = \mathbf{P}(0, 0, 1, 0), c_3 = \mathbf{P}(0, 0, 0, 1).$$

Let $z_0 : z_1 : z_2$ be local coordinates for this subspace. In $\mathrm{PG}(2, 13)$, the same set of points can be written in local coordinates as:

$$Q_1 = \mathbf{P}(11, 2, 1), \qquad Q_2 = \mathbf{P}(2, 2, 1), \qquad Q_3 = \mathbf{P}(12, 6, 1),$$
$$Q_4 = \mathbf{P}(1, 6, 1), \qquad Q_5 = \mathbf{P}(0, 1, 0), \qquad Q_6 = \mathbf{P}(0, 0, 1).$$

Observe that $S = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$ is a 6-arc not on a conic associated with the cubic surface $\mathcal{F}$ in $\mathrm{PG}(3, 13)$.

In order to classify cubic surfaces in $\mathrm{PG}(3,q)$, the action of the group $G = \mathrm{PGL}(4,q)$ on $\mathrm{PG}(3,q)$ and hence on cubic surfaces is considered. Testing whether two surfaces are in the same orbit is a difficult problem. Here is an instance of this problem:

Example 3.2.8. In $\mathrm{PG}(3,13)$, the two cubic surfaces

$$\begin{aligned}\mathcal{F} &= V\left(x_0^3 + x_1^3 + x_2^3 + x_3^3\right),\\ \mathcal{F}' &= V\left(12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3\right)\end{aligned}$$

are projectively equivalent. A projectivity $\Upsilon \in \mathrm{PGL}(4,13)$ taking $\mathcal{F}$ to $\mathcal{F}'$ is defined by the matrix

$$\begin{bmatrix} 4 & 1 & 3 & 4\\ 4 & 1 & 10 & 9\\ 4 & 12 & 3 & 9\\ 2 & 6 & 5 & 2 \end{bmatrix}.$$

The equation of $\mathcal{F}'$ with the variables x_0, x_1, x_2, x_3 renamed x_0', x_1', x_2', x_3' is obtained by substituting

$$\begin{aligned} x_0 =& 4x_0' + 4x_1' + 4x_2' + 2x_3'\\ x_1 =& x_0' + x_1' + 12x_2' + 6x_3'\\ x_2 =& 3x_0' + 10x_1' + 3x_2' + 5x_3'\\ x_3 =& 4x_0' + 9x_1' + 9x_2' + 2x_3' \end{aligned}$$

into the equation of $\mathcal{F}$.

Finding an isomorphism Υ as in the previous example is a difficult problem. We refer to [10].

3.3 The structure of a cubic surface

In this section, more details about the structure of cubic surfaces with twenty-seven lines will be given.

3.3.1 Double-sixes

Let $\mathcal{F}$ be a cubic surface with 27 lines over a field $\mathbf{F}_q$. Use Schläfli's notation for the lines as a_i, b_i, c_{ij} for $i \neq j$ and $i, j = 1, 2, \ldots, 6$ where $c_{ij} = [a_i, b_j] \cap [a_j, b_i]$. Each line of $\mathcal{F}$ meets 10 others. Namely,

$$\begin{array}{rll} a_i & \text{meets} & b_j,\ c_{ik},\ \text{where } i \neq j, k; \\ b_i & \text{meets} & a_j,\ c_{ik},\ \text{where } i \neq j, k; \\ c_{ij} & \text{meets} & a_i, a_j, b_i, b_j, c_{kl},\ \text{where } k, l \neq i, j. \end{array}$$

There is a 36 to 1 relation between double-sixes and the cubic surfaces. A double-six defines exactly one cubic surface with 27 lines and a cubic surface with 27 lines has exactly 36 double-sixes.

The 36 double-sixes associated to $\mathcal{F}$ are

$$\begin{array}{lccccccc} \mathscr{D}: & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \\ & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & (1 \text{ type}) \\ \mathscr{D}_{ij}: & a_i & b_i & c_{jk} & c_{jl} & c_{jm} & c_{jn} & \\ & a_j & b_j & c_{ik} & c_{il} & c_{im} & c_{in} & (15 \text{ types}) \\ \mathscr{D}_{ijk}: & a_i & a_j & a_k & c_{mn} & c_{ln} & c_{lm} & \\ & c_{jk} & c_{ik} & c_{ij} & b_l & b_m & b_n & (20 \text{ types}) \end{array}$$

3.3.2 Tritangent planes

Consider $\mathcal{F}$ with 27 lines of the form a_i, b_i, c_{ij}, where $i, j \in \{1, 2, 3, 4, 5, 6\}$, and with the twelve lines a_i, b_i in the form of a double-six (3.1), and the fifteen lines $c_{ij} = [a_i, b_j] \cap [a_j, b_i]$, for all $i \neq j$.

Lemma 3.3.1 ([23]). *Let $\mathcal{F}$ be a non-singular cubic surface, let P be a point of the surface and let $\pi_P(\mathcal{F})$ denote the tangent plane at P.*

(1) *If P is on no line of $\mathcal{F}$, then $\pi_P(\mathcal{F})$ meets $\mathcal{F}$ in an irreducible cubic with a double point at P.*

(2) *If P is on exactly one line of $\mathcal{F}$, then $\pi_P(\mathcal{F})$ meets $\mathcal{F}$ in the line plus a conic through*

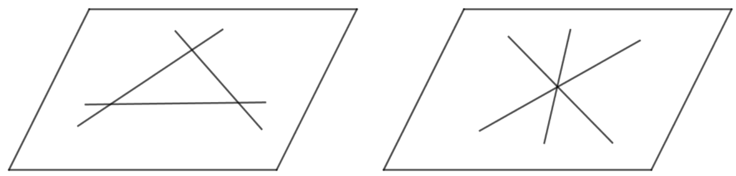


Figure 3.4: (iii) and (iv) of Lemma 3.3.1

P.

(3) *If P is on exactly two lines of $\mathcal{F}$, then $\pi_P(\mathcal{F})$ meets $\mathcal{F}$ these two lines plus another line forming a triangle.*

(4) *If P is on exactly three lines of $\mathcal{F}$, then $\pi_P(\mathcal{F})$ meets $\mathcal{F}$ in these three concurrent lines.*

Definition 3.3.2. A *tritangent plane* is a plane which intersects the cubic surface in three lines. In cases (3) and (4) of Lemma 3.3.1, $\pi_P(\mathcal{F})$ is a tritangent plane.

Theorem 3.3.3. *Through each line of $\mathcal{F}$ there are 5 tritangent planes. In fact,*

$$\begin{array}{lll} a_i & \textit{lies on} & [a_i, b_j, c_{ij}], [a_i, b_k, c_{ik}], [a_i, b_l, c_{il}], [a_i, b_m, c_{im}], [a_i, b_n, c_{in}] \\ b_i & \textit{lies on} & [a_j, b_i, c_{ij}], [a_k, b_i, c_{ik}], [a_l, b_i, c_{il}], [a_m, b_i, c_{im}], [a_n, b_i, c_{in}] \\ c_{ij} & \textit{lies on} & [c_{ij}, a_i, b_j], [c_{ij}, a_j, b_i], [c_{ij}, c_{kl}, c_{mn}], [c_{ij}, c_{km}, c_{ln}], [c_{ij}, c_{kn}, c_{lm}]. \end{array}$$

where i, j, k, l, m, n is any permutation of $1, 2, 3, 4, 5, 6$.

The existence of two lines of $\mathcal{F}$ on a tritangent plane gives the third line of $\mathcal{F}$ from Lemma 3.3.1 (iii). Since each line ℓ of $\mathcal{F}$ meets 10 other lines, double counting the possible tritangent planes though ℓ, implies that there are 5 tritangent planes through a line of $\mathcal{F}$. Therefore, there are $45 = 27 \times 5/3$ tritangent planes altogether, namely,

$$\begin{array}{lll} 30 & \times & [a_i, b_j, c_{ij}] = \pi_{ij}, \\ 15 & \times & [c_{ij}, c_{kl}, c_{mn}] = \pi_{ij,kl,mn}. \end{array}$$

Definition 3.3.4. An *incidence structure* $\mathscr{S}$ is a triple (A, B, I), where A and B are non-empty disjoint sets of objects called points and lines, respectively, and I is a symmetric incidence relation between points and lines.

Definition 3.3.5 ([25])**.** Let s and t be two positive integers. A *generalized quadrangle* $GQ(t,s)$ is an incidence structure $\mathscr{S} = (A, B, I)$ for which the incidence relation between points and lines satisfying the following axioms:

(i) Each point is incident with $1+t$ lines for $t > 1$ and two distinct points are incident with at most one line.

(ii) Each line is incident with $1+s$ points for $s > 1$ and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in A \times B$ for which x incident with M and y with L.

Under the inclusion relation, the lines of the cubic surface with 27 lines and tritangent planes form a generalized quadrangle of type $GQ(4,2)$ considering A as the lines of the surface and B as the tritangent planes.

Example 3.3.6. Let $\mathcal{F}$ be a cubic surface with twenty seven lines $a_i, b_i, c_{ij},\ i,j = 1,2,\ldots,6$ as in the Example 3.2.3 in $\mathrm{PG}(3,13)$.

$$\mathcal{F} = V\big(12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3\big).$$

Let $P_1 = \mathbf{P}(1,1,0,0), P_2 = \mathbf{P}(0,0,1,1)$ be two points of the surface $\mathcal{F}$.

The point P_1 is on exactly three lines of $\mathcal{F}$, namely c_{12}, c_{35}, c_{46}. Let $\pi_{P_1}(\mathcal{F})$ meet $\mathcal{F}$ in three lines such as

$$c_{12} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, c_{35} = \mathbf{L}\begin{bmatrix} 12 & 0 & 10 & 6 \\ 0 & 12 & 3 & 7 \end{bmatrix}, c_{46} = \mathbf{L}\begin{bmatrix} 12 & 0 & 3 & 7 \\ 0 & 12 & 10 & 6 \end{bmatrix}$$

from Example 3.2.4. Therefore,

$$\pi_{P_1}(\mathcal{F}) = [c_{12}, c_{35}, c_{46}] = \pi_{12,35,46} = V(10x_2 + 5x_3)$$

is a tritangent plane of $\mathcal{F}$.

The point P_2 is on exactly two lines of $\mathcal{F}$, namely a_1, b_2. The plane $\pi_{P_2}(\mathcal{F})$ meets $\mathcal{F}$

$$
\begin{array}{ll}
\pi_{12} = V(12x_2 + x_3) & \pi_{63} = V(12x_0 + 12x_1 + 7x_2 + x_3) \\
\pi_{21} = V(x_2 + x_3) & \pi_{45} = V(12x_0 + 12x_1 + 2x_2 + x_3) \\
\pi_{13} = V(2x_0 + 12x_1 + 12x_2 + x_3) & \pi_{54} = V(x_0 + x_1 + 7x_2 + x_3) \\
\pi_{31} = V(7x_0 + x_1 + x_2 + x_3) & \pi_{46} = V(x_0 + 12x_1 + 11x_2 + x_3) \\
\pi_{14} = V(11x_0 + x_1 + 12x_2 + x_3) & \pi_{64} = V(12x_0 + x_1 + 6x_2 + x_3) \\
\pi_{41} = V(6x_0 + 12x_1 + x_2 + x_3) & \pi_{56} = V(x_0 + x_3) \\
\pi_{15} = V(12x_0 + 7x_1 + 12x_2 + x_3) & \pi_{65} = V(12x_0 + x_3) \\
\pi_{51} = V(x_0 + 2x_1 + x_2 + x_3) & \pi_{12,34,56} = V(x_3) \\
\pi_{16} = V(x_0 + 6x_1 + 12x_2 + x_3) & \pi_{12,35,46} = V(2x_2 + x_3) \\
\pi_{61} = V(12x_0 + 11x_1 + x_2 + x_3) & \pi_{12,36,45} = V(11x_2 + x_3) \\
\pi_{23} = V(11x_0 + 12x_1 + x_2 + x_3) & \pi_{13,24,56} = V(11x_0 + x_3) \\
\pi_{32} = V(6x_0 + x_1 + 12x_2 + x_3) & \pi_{13,25,46} = V(5x_0 + 8x_1 + 8x_2 + x_3) \\
\pi_{24} = V(2x_0 + x_1 + x_2 + x_3) & \pi_{13,26,45} = V(10x_0 + 10x_1 + 10x_2 + x_3) \\
\pi_{42} = V(7x_0 + 12x_1 + 12x_2 + x_3) & \pi_{14,23,56} = V(2x_0 + x_3) \\
\pi_{25} = V(12x_0 + 6x_1 + x_2 + x_3) & \pi_{14,25,36} = V(3x_0 + 3x_1 + 10x_2 + x_3) \\
\pi_{52} = V(x_0 + 11x_1 + 12x_2 + x_3) & \pi_{14,26,35} = V(8x_0 + 5x_1 + 8x_2 + x_3) \\
\pi_{26} = V(x_0 + 7x_1 + x_2 + x_3) & \pi_{15,23,46} = V(3x_0 + 10x_1 + 3x_2 + x_3) \\
\pi_{62} = V(12x_0 + 2x_1 + 12x_2 + x_3) & \pi_{15,24,36} = V(5x_0 + 5x_1 + 5x_2 + x_3) \\
\pi_{34} = V(x_1 + x_3) & \pi_{15,26,34} = V(11x_1 + x_3) \\
\pi_{43} = V(12x_1 + x_3) & \pi_{16,23,45} = V(8x_0 + 8x_1 + 5x_2 + x_3) \\
\pi_{35} = V(12x_0 + x_1 + 11x_2 + x_3) & \pi_{16,24,35} = V(10x_0 + 3x_1 + 3x_2 + x_3) \\
\pi_{53} = V(x_0 + 12x_1 + 6x_2 + x_3) & \pi_{16,25,34} = V(2x_1 + x_3) \\
\pi_{36} = V(x_0 + x_1 + 2x_2 + x_3) &
\end{array}
$$

Table 3.4: The tritangent planes

in a_1, b_2 and c_{12} which forms a triangle. Recall from Example 3.2.4 that

$$a_1 = \mathbf{L}\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, b_2 = \mathbf{L}\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, c_{12} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$\pi_{P_2}(\mathcal{F}) = [a_1, b_2, c_{12}] = \pi_{12} = V(12x_2 + x_3)$$

is another tritangent plane of $\mathcal{F}$.

The 45 tritangent planes of $\mathcal{F}$ are listed in Table 3.4.

3.3.3 Eckardt points

The tritangent planes can occur in two types: either the three lines intersect in a point or they form a triangle.

Definition 3.3.7. A point is called an *Eckardt* point if it lies on exactly three lines of the cubic surface $\mathcal{F}$, [17].

In case (iv) of Lemma 3.3.1, P is an Eckardt point.

Corollary 3.3.8. *Every Eckardt point arises as the point of concurrency of the three lines in a tritangent plane.*

Since there are 45 tritangent planes there are at most 45 Eckardt points. With respect to the labeling of lines by a double-six, there are two types of Eckardt points:

$$\begin{aligned} E_{ij} &= a_i \cap b_j \cap c_{ij} \\ E_{ij,kl,mn} &= c_{ij} \cap c_{kl} \cap c_{mn} \end{aligned}$$

where i, j, k, l, m, n is a permutation of $\{1, 2, 3, 4, 5, 6\}$.

There are at most 30 of the first type and 15 of the second type.

Example 3.3.9. Consider the cubic surface of Example 3.2.3. The number of Eckardt points of this cubic surface $\mathcal{F}$ is 18. They can be seen with their coordinates in Table 3.5.

$$\begin{array}{lclclcl}
E_{12,35,46} &=& c_{12} \cap c_{35} \cap c_{46} = P(1,1,0,0), & E_{36} &=& a_3 \cap b_6 \cap c_{36} = P(12,12,7,1), \\
E_{16,25,34} &=& c_{16} \cap c_{25} \cap c_{34} = P(1,0,1,0), & E_{45} &=& a_4 \cap b_5 \cap c_{45} = P(1,1,7,1), \\
E_{14,23,56} &=& c_{14} \cap c_{23} \cap c_{56} = P(0,1,1,0), & E_{35} &=& a_3 \cap b_5 \cap c_{35} = P(1,12,6,1), \\
E_{12,36,45} &=& c_{12} \cap c_{36} \cap c_{45} = P(1,12,0,0), & E_{14} &=& a_1 \cap b_4 \cap c_{14} = P(6,12,1,1), \\
E_{13,24,56} &=& c_{13} \cap c_{24} \cap c_{56} = P(0,1,12,0), & E_{46} &=& a_4 \cap b_6 \cap c_{46} = P(12,1,6,1), \\
E_{15,26,34} &=& c_{15} \cap c_{26} \cap c_{34} = P(1,0,12,0), & E_{23} &=& a_2 \cap b_3 \cap c_{23} = P(6,1,12,1), \\
E_{13} &=& a_1 \cap b_3 \cap c_{13} = P(7,1,1,1), & E_{24} &=& a_2 \cap b_4 \cap c_{24} = P(7,12,12,1), \\
E_{61} &=& a_6 \cap b_1 \cap c_{16} = P(1,6,12,1), & E_{52} &=& a_5 \cap b_2 \cap c_{25} = P(12,6,1,1), \\
E_{51} &=& a_5 \cap b_1 \cap c_{15} = P(12,7,12,1), & E_{62} &=& a_6 \cap b_2 \cap c_{26} = P(1,7,1,1).
\end{array}$$

Table 3.5: The Eckardt points of $\mathcal{F}$

Lemma 3.3.10 ([23])**.** *Any two Eckardt points not on the same line of $\mathcal{F}$ are collinear with a third Eckardt point.*

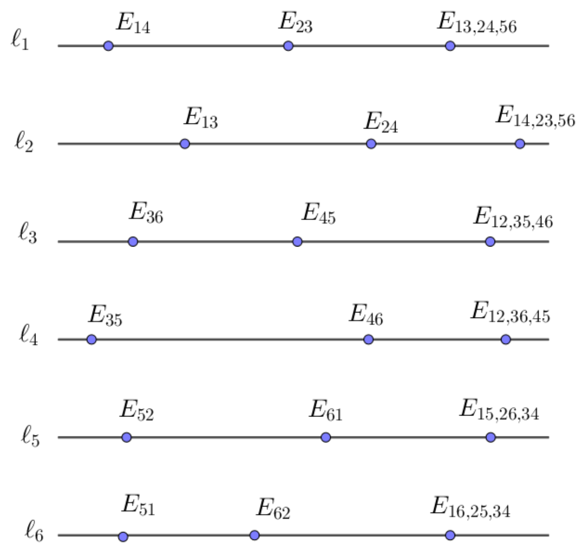


Figure 3.5: Some collinearities of Eckardt points

The possible collinearities are as follows:

40	E_{ij}	E_{jk}	E_{ki}
180	E_{ij}	E_{kl}	$E_{il,jk,rs}$
20	$E_{ij,kl,mn}$	$E_{il,kn,jm}$	$E_{in,jk,ml}$

Example 3.3.11. Considering Example 3.3.9, six of the possible collinearities between 18 Eckardt points presented in Table 3.5 are realised in the case of in Figure 3.5. The lines ℓ_i are lines of $\mathrm{PG}(3, q)$ not on $\mathcal{F}$.

The number of Eckardt points is an isomorphism invariant of a cubic surface. In some cases, cubic surfaces are characterised by their number of Eckardt points. If two cubic surfaces are projectively equivalent they must have the same number of Eckardt points. The converse is not true. It is possible to have projectively distinct cubic surfaces with the same number of Eckardt points. It follows from [10] that the smallest case for which this happens is when $q = 17$. Our work will show that in $\mathrm{PG}(3, 17)$ there are 2 projectively distinct cubic surfaces with 27 lines with exactly 4 Eckardt points and 3 projectively distinct cubic surfaces with 27 lines with exactly 6 Eckardt points.

3.3.4 The known families of cubic surfaces

The following three families of cubic surfaces with 27 lines over $\mathbf{F}_q$ are known. Interestingly, they are characterised by the number of Eckardt points.

Theorem 3.3.12 ([23])**.** (1) *The* diagonal *surface $\mathcal{D}$ is projectively equivalent to one with equation*

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 - (x_0 + x_1 + x_2 + x_3)^3 = 0.$$

If the surface has exactly 10 *Eckardt points, then it is the diagonal surface. This surface is also known as the Clebsch surface.*

(2) *There is a cubic surface with* 10 *Eckardt points if and only if $q \equiv \pm 1 (mod\ 5)$; that is, if and only if $x^2 - x - 1$ has two roots in $\mathbf{F}_q \setminus \{0\}$. Any two such surfaces are projectively equivalent.*

(3) *Let $\mathcal{F}_{10}$ be the cubic surface with* 10 *Eckardt points. The group $G(\mathcal{F}_{10})$ of projectivities of the diagonal surface $\mathcal{F}_{10}$ containing exactly* 10 *Eckardt points is isomorphic to* Sym_5 *of order* 120*. Here,* Sym_n *is the symmetric group of degree n.*

Theorem 3.3.13 ([23])**.** (1) *A surface which is projectively equivalent to one with equation*

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

is an equianharmonic *surface, and denoted by $\mathcal{E}$. If the surface has exactly* 18 *Eckardt points, then it is the equianharmonic surface. This surface is also known as the Fermat surface.*

(2) *There is a cubic surface with* 18 *Eckardt points if and only if $q \equiv 1 (mod\ 3)$; that is, if and only if $x^2 + x + 1$ has two roots in $\mathbf{F}_q \setminus \{0\}$. Any two such surfaces are projectively equivalent.*

(3) *Let $\mathcal{F}_{18}$ be the cubic surface with* 18 *Eckardt points. The group $G(\mathcal{F}_{18})$ of projectivities of an equianharmonic surface $\mathcal{F}_{18}$ is isomorphic to a split extension of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by* Sym_4 *and has order* 648*.*

Theorem 3.3.14 ([19])**.** *In* $\mathrm{PG}(3, q)$*, a cubic surface with* 45 *Eckardt points is both* equianharmonic and diagonal*, as well as projectively unique. Such a surface exists if and only if*

$q = 4^m$. The cubic surface with 45 Eckardt points has a projectivity group of order 25920. This group is the simple group associated with the Weyl group of type E_6, [14]. Other names for this group are

$$\mathrm{PSU}_4(2),\ \mathrm{PS}\Omega_6^-(2),\ \mathrm{PGSp}_4(3)\ \textit{or}\ \mathrm{PS}\Omega_5(3).$$

3.3.5 Configurations of Eckardt points

Lemma 3.3.15. *The number of Eckardt points on a line of a cubic surface is at most two for fields of odd characteristic, and it is at most five for fields of even characteristic.*

The following result is obtained in [21] by combinatorial methods using Lemma 3.3.10 and Lemma 3.3.15.

Theorem 3.3.16 ([21])**.** *Let $\mathcal{F}$ be a cubic surface with 27 lines over $\mathbf{F}_q$. Let e_3 be the number of Eckardt points of $\mathcal{F}$. Then*

(1) $e_3 \in \{0, 1, 2, 3, 4, 6, 9, 10, 18\}$ *when q is odd;*

(2) $e_3 \in \{0, 1, 3, 5, 9, 13, 45\}$ *when q is even.*

More information about these cases is given in the following list.

(a) $e_3 = 0$

(b) $e_3 = 1$

(c) $e_3 = 2$:

The two Eckardt points lie on a line of the cubic surface $\mathcal{F}$. For example, E_{12}, E_{13} both lie on the line a_1 of the cubic surface $\mathcal{F}$.

(d) $e_3 = 3$:

The three Eckardt points lie on a line off the surface. For example, E_{12}, E_{23}, E_{31}, are collinear by Lemma 3.3.10. See Figure 3.6.

(e) $e_3 = 4$:

The four Eckardt points are coplanar. For example, E_{12}, E_{23}, E_{31} are collinear. E_{21} is joined with each of E_{12}, E_{23}, E_{31} (cf. Fig. 3.7).

Figure 3.6: 3 Eckardt points

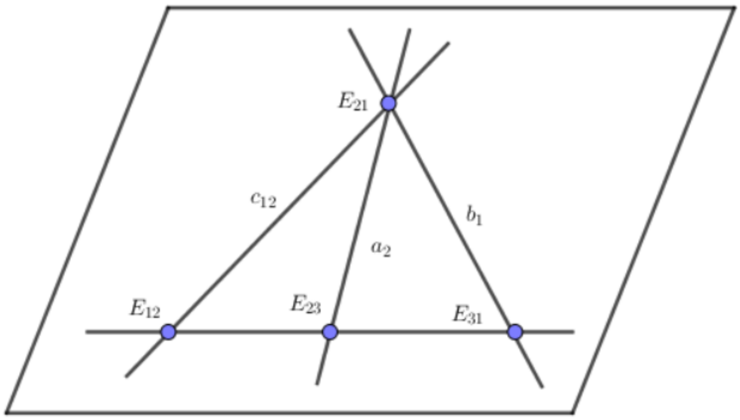


Figure 3.7: 4 Eckardt points

(f) $e_3 = 5$:

The five Eckardt points lie on a line on the cubic surface $\mathcal{F}$. For example,

$$E_{12}, E_{13}, E_{14}, E_{15}, E_{16}$$

lie on the line a_1 of $\mathcal{F}$. By Lemma 3.3.15, such a configuration only exists for q even (cf. Fig. 3.8).

(g) $e_3 = 6$:

The six Eckardt points appear as the vertices of a quadrilateral. For example,

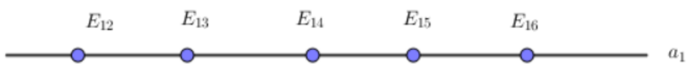


Figure 3.8: 5 Eckardt points

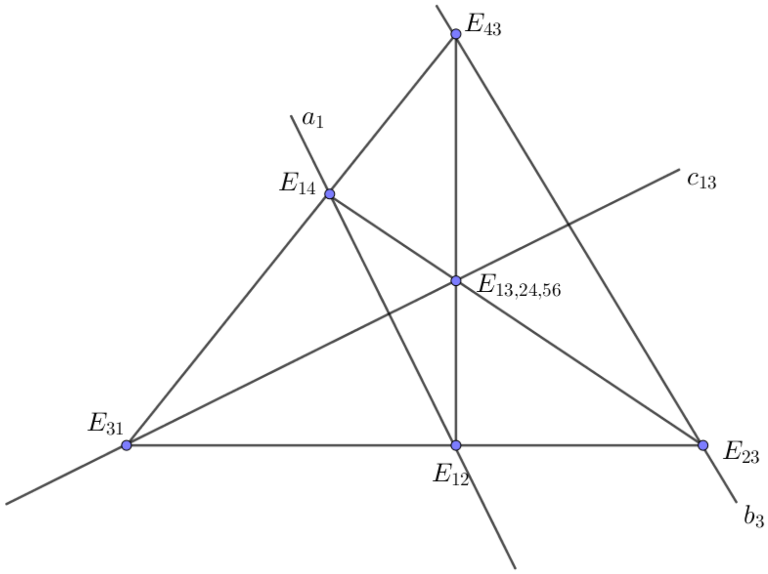


Figure 3.9: 6 Eckardt points

$E_{43}, E_{14}, E_{31}, E_{12}, E_{23}, E_{13,24,56}$. The three diagonal lines are the lines of a tritangent plane (cf. Fig. 3.9).

(h) $e_3 = 9$:

The nine Eckardt points are the inflexions of a plane cubic curve lying by threes on twelve lines as in rows, columns and diagonals of the array

$$\begin{array}{ccc} E_{12} & E_{45} & E_{15,24,36} \\ E_{23} & E_{56} & E_{14,26,35} \\ E_{31} & E_{64} & E_{16,25,34} \end{array}$$

In this case, each of the 27 lines of the cubic surface only has one Eckardt point (cf. Fig. 3.10).

Theorem 3.3.17. *If the cubic surface $\mathcal{F}$ has 9 Eckardt points, then $x^2 + x + 1$ has at least one root in $\mathbf{F}_q \setminus \{0\}$; that is, $q \not\equiv -1(mod\ 3)$.*

(i) $e_3 = 10$:

The ten Eckardt points appear as the vertices of a pentahedron and lie two on each

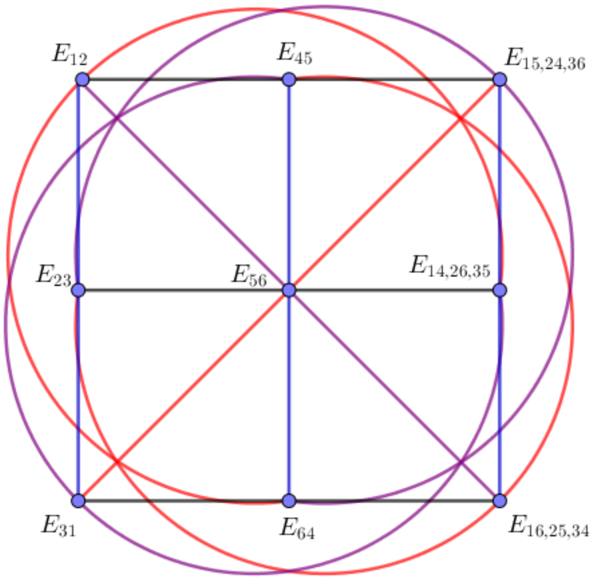


Figure 3.10: 9 Eckardt points

of 15 lines of the cubic surface. For example,

$$E_{12,36,45}, \quad E_{13,24,56}, \quad E_{13,26,45}, \quad E_{15,26,34}, \quad E_{14,23,56},$$
$$E_{12,35,46}, \quad E_{16,25,34}, \quad E_{15,23,46}, \quad E_{16,24,35}, \quad E_{14,25,36}$$

(cf. Fig. 3.11).

As stated in Theorem 3.3.12, the cubic surface with twenty-seven lines and 10 Eckardt points in $\mathrm{PG}(3,q)$ is projectively unique.

(j) $e_3 = 13$:

The thirteen Eckardt points lie on three coplanar lines of the cubic surface $\mathcal{F}$ concurrent at an Eckardt point. For example,

$$E_{12}, E_{13}, E_{14}, E_{15}, E_{16}, E_{32}, E_{42}, E_{52}, E_{62}, E_{12,34,56}, E_{21}, E_{12,35,46}, E_{13,26,45}$$

(cf. Fig. 3.12).

(k) $e_3 = 18$:

Each of the 27 line of the surface contains two Eckardt points. The cubic surface with

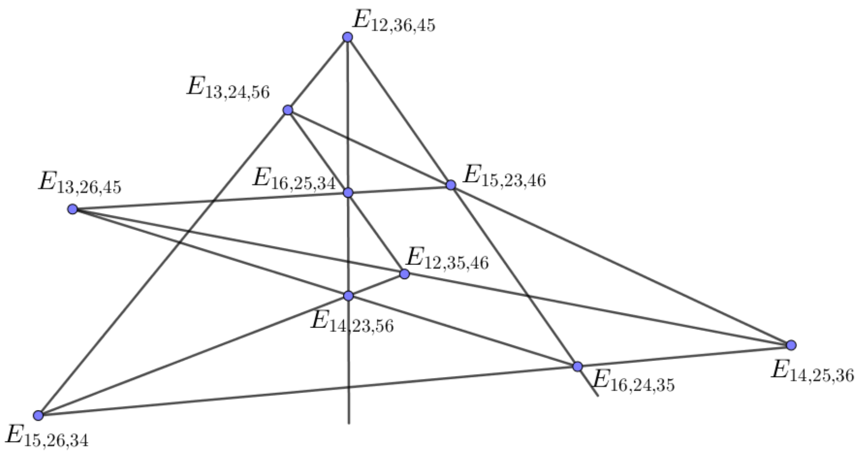


Figure 3.11: 10 Eckardt points

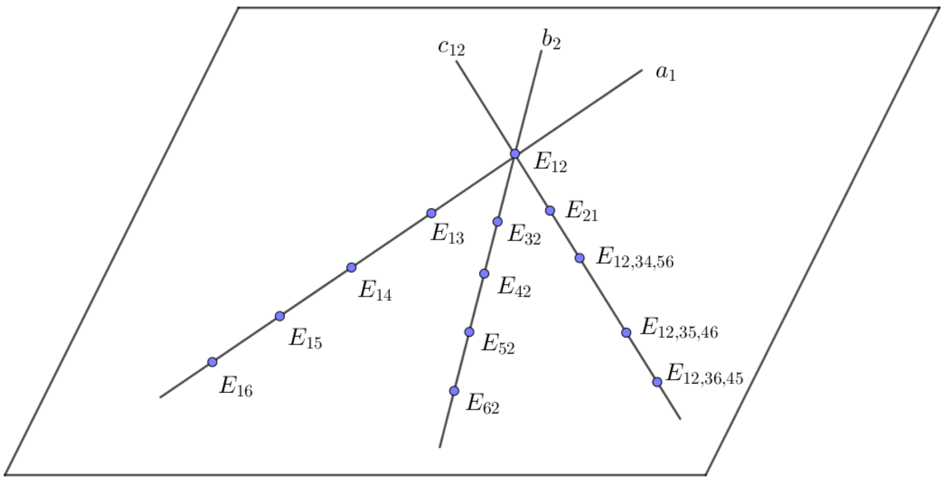


Figure 3.12: 13 Eckardt points

twenty-seven lines and 18 Eckardt points in $\mathrm{PG}(3,q)$ is projectively unique as stated Theorem 3.3.13.

(l) $e_3 = 45$:

Every tritangent plane of the cubic surface $\mathcal{F}$ intersects $\mathcal{F}$ in three concurrent lines of $\mathcal{F}$. These 45 Eckardt points are the all possible Eckardt points, each of the 27 lines has five of the Eckardt points. As stated in Theorem 3.3.14, the cubic surface with twenty-seven lines and 45 Eckardt points in $\mathrm{PG}(3,q)$ is projectively unique.

3.3.6 Steiner trihedral pairs

Definition 3.3.18. Let $\mathcal{F}$ be a cubic surface in $\mathrm{PG}(3,q)$. Two sets of three tritangent planes which pairwise intersect in 9 distinct lines of $\mathcal{F}$ form a *Steiner trihedral pair*, [30]. Each of the two sets of three planes is called a *trihedron* (plural trihedra).

In other words, a trihedral pair is two sets of three tritangent planes each which intersect transversally in 9 lines of the surface (cf. Fig. 3.13). A more general definition of a trihedral pair will be given in Definition 5.2.2 and it will be called a *double-triplet*. The difference is replacing tritangent planes by arbitrary planes which pairwise intersect in nine distinct lines of $\mathrm{PG}(3,q)$.

Theorem 3.3.19 ([21])**.** *If two tritangent planes $V(F_1)$ and $V(F_2)$ have no line of a cubic surface $\mathcal{F}$ in common, then they are uniquely associated with a third tritangent plane $V(F_3)$ in such a way that the 9 lines of $\mathcal{F}$ contained in these* 3 *planes belong to another set of* 3 *associated tritangent planes, $V(G_1), V(G_2), V(G_3)$: the two trihedra form a* trihedral pair.

For example,

$$\begin{array}{lcccc} S_{12,34}: & a_1 & b_4 & c_{14} & F_1 \\ & b_3 & a_2 & c_{23} & F_2 \\ & c_{13} & c_{24} & c_{56} & F_3 \\ & G_1 & G_2 & G_3 & \end{array}$$

Here, the rows represent for the planes of type F_i and the columns represent the planes of type G_i. This particular trihedral pair is denoted as $S_{12,34}$.

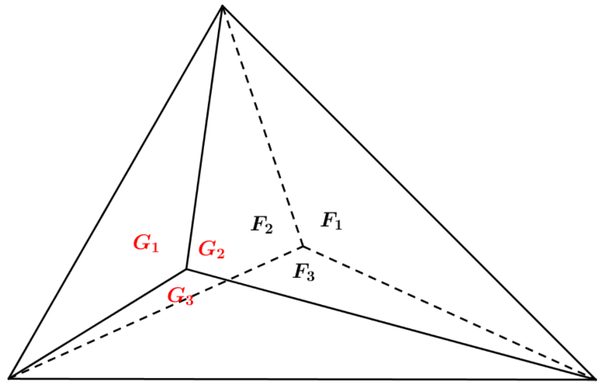


Figure 3.13: A trihedral pair

The 45 tritangent planes form $120 = 45 \times 32/12$ trihedral pairs which come in three types:

20	like	S_{123}	c_{23}	b_3	a_2
			a_3	c_{13}	b_1
			b_2	a_1	c_{12}
90	like	$S_{13,24}$	a_1	b_2	c_{12}
			b_4	a_3	c_{34}
			c_{14}	c_{23}	c_{56}
10	like	$S_{123,456}$	c_{14}	c_{25}	c_{36}
			c_{26}	c_{34}	c_{15}
			c_{35}	c_{16}	c_{24}

We utilize trihedral pairs to create cubic surfaces with 27 lines. The cubic surface with 27 lines can be created by any of the 120 trihedral pairs.

Theorem 3.3.20 ([23])**.** *Let the planes of a trihedral pair belonging to a cubic surface $\mathcal{F}$ be given by $V(F_1), V(F_2), V(F_3)$ and $V(G_1), V(G_2), V(G_3)$ for suitable linear forms F_1, F_2, F_3, G_1, G_2 and G_3. Then the cubic surface can be written as*

$$\mathcal{F} = V(F_1F_2F_3 + \lambda G_1G_2G_3)$$

for some λ in $\mathbf{F}_q \backslash \{0\}$.

A distinguished trihedral pair determines $q+1$ cubic surfaces. Not all of these surfaces have 27 lines though.

Example 3.3.21. Let $\mathcal{F}$ be the cubic surfaces in PG$(3,13)$ with twenty-seven lines associated with the double-six from Example 3.3.1, with fifteen further lines as in Example 3.2.4. From Example 3.2.3, we know that the equation of $\mathcal{F}$ is

$$12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3 = 0.$$

Let S_{123} be a trihedral pair. The tritangent planes associated with S_{123} are as follows:

$$\begin{aligned}
F_1 &= 11x_0 + 12x_1 + x_2 + x_3,\\
F_2 &= 7x_0 + x_1 + x_2 + x_3,\\
F_3 &= 12x_2 + x_3,\\
G_1 &= 6x_0 + x_1 + 12x_2 + x_3,\\
G_2 &= 2x_0 + 12x_1 + 12x_2 + x_3,\\
G_3 &= x_2 + x_3,
\end{aligned}$$

with $\pi_{23} = V(F_1), \pi_{31} = V(F_2), \pi_{12} = V(F_3), \pi_{32} = V(G_1), \pi_{13} = V(G_2), \pi_{21} = V(G_3)$.

From Theorem 3.3.20, it follows that

$$\begin{aligned}
\mathcal{F} &= V((11x_0 + 12x_1 + x_2 + x_3)(7x_0 + x_1 + x_2 + x_3)(12x_2 + x_3)\\
&\quad +\lambda(6x_0 + x_1 + 12x_2 + x_3)(2x_0 + 12x_1 + 12x_2 + x_3)(x_2 + x_3))
\end{aligned}$$

for some scalar $\lambda \in \mathbf{F}_q\backslash\{0\}$. In order to determine λ, one independent condition is required. For this, we use the fact that the point $\mathbf{P}(1,0,7,0)$ is on $\mathcal{F}$ but not on any of the planes of the chosen trihedral pair. Substituting $\mathbf{P}(1,0,7,0)$ into the equation yields $\lambda = 1$. Thus,

$$\begin{aligned}
\mathcal{F} &= V((11x_0 + 12x_1 + x_2 + x_3)(7x_0 + x_1 + x_2 + x_3)(12x_2 + x_3)\\
&\quad +(6x_0 + x_1 + 12x_2 + x_3)(2x_0 + 12x_1 + 12x_2 + x_3)(x_2 + x_3))\\
&= V(12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3).
\end{aligned}$$

This shows that the equation of the surface can be written in the form

$$LMN + PQR = 0,$$

where L, M, N, P, Q, R are linear forms: Simply put

$$L = F_1, M = F_2, N = F_3, P = \lambda G_1, Q = G_2, R = G_3.$$

Recall that three planes in 3-dimensional projective space over finite fields either intersect in a point or in a line.

Definition 3.3.22. A trihedral pair is *special* if the three planes of one trihedron meet in a line (cf. Fig. 3.14).

Example 3.3.23. The cubic surface $\mathcal{F}$ from Example 3.3.21

$$\mathcal{F} = V\big(12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3\big)$$

has 120 trihedral pairs, 36 of which are special. The special trihedral pairs are as follows:

$$\begin{array}{lllllllll}
S_{135}, & S_{136}, & S_{145}, & S_{146}, & S_{235}, & S_{236}, & S_{245}, & S_{246}, & S_{12,34}, \\
S_{15,23}, & S_{26,13}, & S_{25,13}, & S_{15,24}, & S_{25,14}, & S_{16,24}, & S_{56,12}, & S_{26,14}, & S_{13,45}, \\
S_{14,35}, & S_{13,46}, & S_{35,16}, & S_{36,15}, & S_{45,16}, & S_{46,15}, & S_{23,46}, & S_{23,45}, & S_{24,35}, \\
S_{45,26}, & S_{46,25}, & S_{35,26}, & S_{36,25}, & S_{34,56}, & S_{146,235}, & S_{145,236}, & S_{136,245}, & S_{135,246}.
\end{array}$$

Recall the collinearities between Eckardt points of $\mathcal{F}$ from Example 3.3.11. One of the special trihedral pairs is $S_{12,34}$,

$$\begin{array}{cccc}
a_1 & b_4 & c_{14} & F_1 \\
b_3 & a_2 & c_{23} & F_2 \\
c_{13} & c_{24} & c_{56} & F_3 \\
G_1 & G_2 & G_3 &
\end{array}$$

The planes $V(G_1), V(G_2), V(G_3)$ meet in a line which contains $E_{14}, E_{23}, E_{13,24,56}$. Also, the planes $V(F_1), V(F_2), V(F_3)$ meet in a line containing $E_{13}, E_{24}, E_{14,23,56}$. The trihedron represented by the columns of the array $\{V(G_1), V(G_2), V(G_3)\}$ is shown in Figure 3.14.

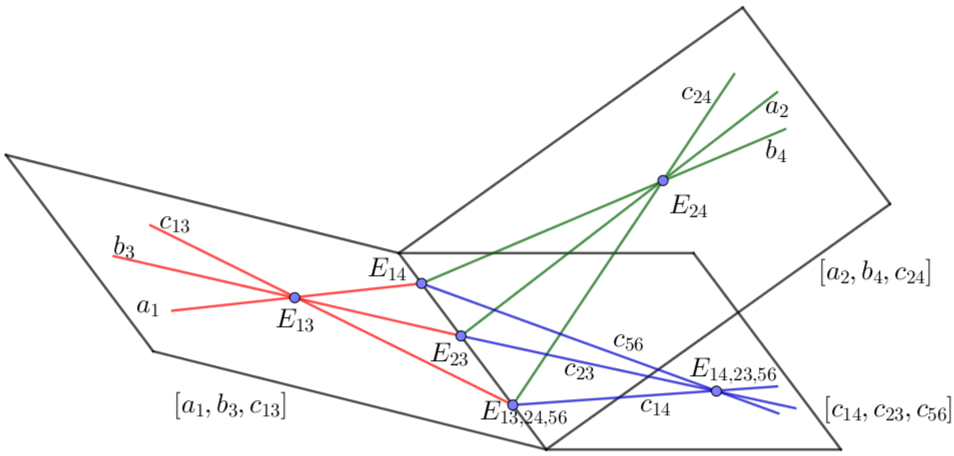


Figure 3.14: Special Trihedral pair

3.3.7 Triads

Definition 3.3.24. A *triad* is a set of three trihedral pairs which partitions the 27 lines into three sets of nine lines each.

The 120 trihedral pairs form 40 triads, namely

$$\begin{array}{rll} 10 & \text{of the kind} & S_{ijk}, S_{lmn}, S_{ijk,lmn} \\ 30 & \text{of the kind} & S_{ij,kl}, S_{ij,mn}, S_{kl,mn} \end{array}$$

where i, j, k, l, m, n is a permutation of $\{1, 2, 3, 4, 5, 6\}$.

3.3.8 Tactical configurations

Definition 3.3.25. An incidence structure $\mathscr{S} = (A, B, I)$, where the elements of A are called points and the elements of B are called lines, is a *tactical configuration* if

(i) every line is incident with k points;

(ii) every point is incident with r lines;

(iii) any two points are incident with at most one line.

Let $|A| = v$, $|B| = b$. Then

$$bk = rv.$$

Therefore, $\mathscr{S}$ is also called a (v_r, b_k) configuration.

Theorem 3.3.26. *Among the lines, tritangent planes, double-sixes, trihedral pairs, and triads, there are the following tactical configurations:*

(a) *lines, tritangent planes:* $(27_5, 45_3)$,

A cubic surface has exactly 27 *lines and* 45 *tritangent planes. Every tritangent plane contains three lines of the cubic surface and every line lies on five tritangent planes.*

(b) *lines, double-sixes:* $(27_{16}, 36_{12})$,

A cubic surface has exactly 27 *lines and* 36 *double-sixes. Every double-six has twelve lines of the cubic surface and each line is contained in sixteen double-sixes.*

(c) *lines, trihedral pairs:* $(27_{40}, 120_9)$,

A cubic surface has exactly 27 *lines and* 120 *trihedral pairs. Every trihedral pair contains nine lines of the cubic surface and each line is included in forty trihedral pairs.*

(d) *lines, triads:* $(27_{40}, 40_{27})$,

A cubic surface has exactly 27 *lines and* 40 *triads. Every triad has twenty-seven lines of the cubic surface and every line appears in every possible triad.*

(e) *tritangent planes, double-sixes:* $(45_{24}, 36_{30})$,

A cubic surface has exactly 45 *tritangent planes and* 36 *double-sixes. Every tritangent plane has twenty-four double-sixes of the cubic surface and from each double-six thirty tritangent planes can be created.*

(f) *tritangent planes, trihedral pairs:* $(45_{16}, 120_6)$,

A cubic surface has exactly 45 *tritangent planes and* 120 *trihedral pairs. Each tritangent planes is contained in sixteen trihedral pairs and from a trihedral pair six tritangent planes are obtained.*

(g) *tritangent planes, triads:* $(45_{16}, 40_{18})$,

A cubic surface has exactly 45 *tritangent planes and* 40 *triads. Each tritangent plane is contained in sixteen triads and a triad includes eighteen tritangent planes.*

(h) *double-sixes, trihedral pairs:* $(36_{20}, 120_6)$,

A cubic surface has exactly 36 *double-sixes and* 120 *trihedral pairs. A trihedral pair is associated with* 6 *double-sixes and a double-six is associated with* 20 *trihedral pairs.*

(i) *double-sixes, triads:* $(36_{10}, 40_9)$,

A cubic surface has exactly 36 *double-sixes and* 40 *triads. A triad is associated with* 9 *double-sixes and a double-six is associated with* 10 *triads.*

(j) *trihedral pairs, triads:* $(120_1, 40_3)$.

A cubic surface has exactly 120 *trihedral pairs and* 40 *triads pairs. Every trihedral pair belongs to one triad and a triad consists of three trihedral pairs.*

Chapter 4

The Representation of the Cubic Surface on a Plane

In Lemma 3.2.5, a connection between cubic surfaces with 27 lines and planar 6-arcs not on a conic was established. In this chapter, two further ways of associating planar 6-arcs and cubic surfaces with 27 lines are discussed. For this purpose, we will utilize a special kind of mapping, known as a birational map. Because of [13], we will often refer to these maps as Clebsch map (Fig. 4.1).

Recall that any 5-arc in a plane determines a conic but we assume that the 6^{th} point of the arc does not lie on this conic.

We will give an example for the Clebsch map in each model. We will continue to use the cubic surface $\mathcal{F}$ from Example 3.2.3.

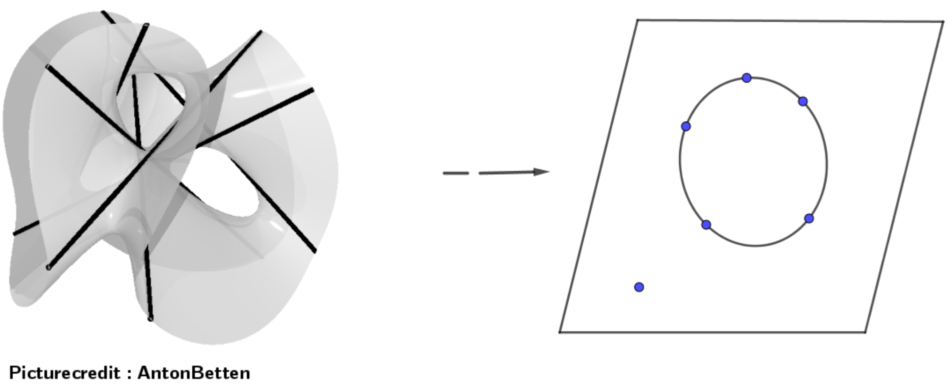


Figure 4.1: The Clebsch map from a surface to a plane

4.1 Clebsch map, model 1

Definition 4.1.1. Let N be a subspace of $\mathrm{PG}(n,q)$ and let M be a subspace of $\mathrm{PG}(m,q)$. Let $\Phi : M \to N$ be a map. Define the points of N as $\mathbf{P}(y_0, y_1, \ldots, y_n)$ and the points of M as $\mathbf{P}(x_0, x_1, \ldots, x_m)$. Let

$$(\mathbf{P}(x_0, x_1, \ldots, x_m))\Phi = \mathbf{P}(\mathfrak{r}_\mathfrak{o}(x_0, x_1, \ldots, x_m), \ldots, \mathfrak{r}_\mathfrak{n}(x_0, x_1, \ldots, x_m))$$

where $\mathfrak{r}_\mathfrak{i}$ are rational functions in $x_0, x_1, \ldots, x_m$ and

$$y_i = \mathfrak{r}_\mathfrak{i}(x_0, x_1, \ldots, x_m), \quad i \in \{0, 1, \ldots, n\}.$$

Then Φ is a *rational map.*

Rational maps do not need to be defined on all of M.

Definition 4.1.2. A rational map $\Phi : M \to N$ is birational if there exists a rational map $\mu : N \to M$ such that $\Phi\mu$ is the identity.

Theorem 4.1.3 (Clebsch [13])**.** *A general cubic surface is the image of a birational map from a projective plane given by the linear system of cubics through* 6 *points.*

A cubic surface with twenty-seven lines in $\mathrm{PG}(3,q)$ can be mapped onto the plane in the following way:

Recall from Theorem 3.3.20 that with each trihedral pair, an expression of the equation of the cubic surface with twenty-seven lines can be found as $V(F_1F_2F_3 + G_1G_2G_3)$, where $F_1, F_2, F_3, G_1, G_2, G_3$ are linear forms.

Let $\mathcal{F}$ be a cubic surface given by $V(F_1F_2F_3 + G_1G_2G_3)$, for some linear forms F_i and G_j. Using variables x_0, x_1, x_2, x_3, the forms can be written as

$$\begin{aligned}
F_1 &= a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + a_{03}x_3,\\
F_2 &= a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3,\\
F_3 &= a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + a_{23}x_3,\\
G_1 &= b_{00}x_0 + b_{01}x_1 + b_{02}x_2 + b_{03}x_3,\\
G_2 &= b_{10}x_0 + b_{11}x_1 + b_{12}x_2 + b_{13}x_3,\\
G_3 &= b_{20}x_0 + b_{21}x_1 + b_{22}x_2 + b_{23}x_3, \qquad (4.1)
\end{aligned}$$

where a_{ij} and b_{ij} are in $\mathbf{F}_q$.

Assume that $\mathbf{P}(X) = \mathbf{P}(x_0, x_1, x_2, x_3)$ is a point on the surface but not on any of the planes $F_i = 0$ and the $G_i = 0$. A point $\mathbf{P}(X) = \mathbf{P}(x_0, x_1, x_2, x_3)$ is on $\mathcal{F}$ if and only if

$$\begin{vmatrix} 0 & F_1 & G_3 \\ G_1 & 0 & F_2 \\ F_3 & G_2 & 0 \end{vmatrix} = 0,$$

that is, there exists $\mathbf{P}(Y) = \mathbf{P}(y_0, y_1, y_2)$ such that the following system has a non-trivial unique solution:

$$\begin{aligned} y_1 G_1(X) + y_2 F_3(X) &= 0 \\ y_0 F_1(X) + y_2 G_2(X) &= 0 \\ y_0 G_3(X) + y_1 F_2(X) &= 0. \end{aligned} \tag{4.2}$$

Let Φ be a map

$$\Phi \colon \mathcal{F} \to \mathrm{PG}(2, q)$$

which takes $\mathbf{P}(x_0, x_1, x_2, x_3)$ to $\mathbf{P}(y_0, y_1, y_2)$, where

$$\frac{y_1}{y_2} = -\frac{F_3(X)}{G_1(X)}, \frac{y_0}{y_2} = -\frac{G_2(X)}{F_1(X)}, \frac{y_0}{y_1} = -\frac{F_3(X)}{G_1(X)}.$$

Therefore,

$$\begin{aligned} y_0 &= F_2(X)F_3(X)G_2(X), \\ y_1 &= -F_3(X)G_2(X)G_3(X), \\ y_2 &= -F_1(X)F_2(X)F_3(X). \end{aligned} \tag{4.3}$$

Under the assumption $y_2 = 1$, this leads to

$$y_0 = \frac{F_2(X)F_3(X)}{G_1(X)G_3(X)}, y_1 = -\frac{F_3(X)}{G_1(X)}, y_2 = -\frac{F_1(X)F_2(X)F_3(X)}{G_1(X)G_2(X)G_3(X)} = 1.$$

In fact, we find rational functions $\mathfrak{r}_i$

$$
\begin{aligned}
y_0 &= \mathfrak{r}_0(x_0, x_1, x_2, x_3) = \frac{\displaystyle\sum_{i,j=0}^{3} x_i x_j (a_{1i} a_{2j})}{\displaystyle\sum_{i,j=0}^{3} x_i x_j (b_{0i} b_{1j})}, \\
y_1 &= \mathfrak{r}_1(x_0, x_1, x_2, x_3) = -\frac{\displaystyle\sum_{i=0}^{3} x_i a_{2i}}{\displaystyle\sum_{i=0}^{3} x_i b_{0i}}, \\
y_2 &= \mathfrak{r}_2(x_0, x_1, x_2, x_3) = 1.
\end{aligned}
\tag{4.4}
$$

Therefore, Φ is a rational map.

Conversely, we can write the map which expresses the x_i in terms of the y_j.

A point $\mathbf{P}(X)$ on the cubic surface $\mathcal{F}$ maps to a point $\mathbf{P}(Y)$ in $\mathrm{PG}(2, q)$. Rearranging the system of equations (4.2) together with (4.1) we get:

$$
\begin{aligned}
x_0(b_{00}y_1 + a_{20}y_2) + x_1(b_{01}y_1 + a_{21}y_2) + x_2(b_{02}y_1 + a_{22}y_2) + x_3(b_{03}y_1 + a_{23}y_2) &= 0 \\
x_0(a_{00}y_0 + b_{10}y_2) + x_1(a_{01}y_0 + b_{11}y_2) + x_2(a_{02}y_0 + b_{12}y_2) + x_3(a_{03}y_0 + b_{13}y_2) &= 0 \\
x_0(b_{20}y_0 + a_{10}y_1) + x_1(b_{21}y_0 + a_{11}y_1) + x_2(b_{22}y_0 + a_{12}y_1) + x_3(b_{23}y_0 + a_{13}y_1) &= 0
\end{aligned}
$$

Consider the equation

$$
\begin{pmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \end{pmatrix}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

where

$$
\begin{array}{lllllllll}
P_{00} &=& b_{00}y_1 + a_{20}y_2 & P_{01} &=& b_{01}y_1 + a_{21}y_2 & P_{02} &=& b_{02}y_1 + a_{22}y_2 \\
P_{10} &=& a_{00}y_0 + b_{10}y_2 & P_{11} &=& a_{01}y_0 + b_{11}y_2 & P_{12} &=& a_{02}y_0 + b_{12}y_2 \\
P_{20} &=& b_{20}y_0 + a_{10}y_1 & P_{21} &=& b_{21}y_0 + a_{11}y_1 & P_{22} &=& b_{22}y_0 + a_{12}y_1 \\
P_{03} &=& b_{03}y_1 + a_{23}y_2 & P_{13} &=& a_{03}y_0 + b_{13}y_2 & P_{02} &=& b_{02}y_1 + a_{22}y_2.
\end{array}
$$

Let

$$M = \begin{pmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{pmatrix}.$$

Solving the system for the x_i gives the following:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \mathrm{Adj}(M) \cdot \begin{pmatrix} P_{03} \\ P_{13} \\ P_{23} \end{pmatrix} \text{ and } x_3 = -\det(M)$$

where

$$\mathrm{Adj}(M) = \begin{pmatrix} \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} & -\begin{vmatrix} P_{01} & P_{02} \\ P_{21} & P_{22} \end{vmatrix} & \begin{vmatrix} P_{01} & P_{02} \\ P_{11} & P_{12} \end{vmatrix} \\ -\begin{vmatrix} P_{10} & P_{12} \\ P_{20} & P_{22} \end{vmatrix} & \begin{vmatrix} P_{00} & P_{02} \\ P_{20} & P_{22} \end{vmatrix} & -\begin{vmatrix} P_{00} & P_{02} \\ P_{10} & P_{12} \end{vmatrix} \\ \begin{vmatrix} P_{10} & P_{11} \\ P_{20} & P_{21} \end{vmatrix} & -\begin{vmatrix} P_{00} & P_{01} \\ P_{20} & P_{21} \end{vmatrix} & \begin{vmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{vmatrix} \end{pmatrix}$$

is the adjoint of M. This gives

$$x_0 : x_1 : x_2 : x_3 = h_0(y_0, y_1, y_2) : h_1(y_0, y_1, y_2) : h_2(y_0, y_1, y_2) : h_3(y_0, y_1, y_2),$$

where each h_i determines a cubic curve $V(h_i)$ in the plane. Explicitly, the cubic expressions

$x_i = h_i(y_0, y_1, y_2)$ are

$$
\begin{aligned}
x_0 &= h_0(y_0, y_1, y_2) \\
&= y_0^2 y_1(-a_{01}b_{02}b_{23} + a_{01}b_{03}b_{22} + a_{02}b_{01}b_{23} - a_{02}b_{03}b_{21} - a_{03}b_{01}b_{22} + a_{03}b_{02}b_{21}) \\
&+ y_0^2 y_2(-a_{01}a_{22}b_{23} + a_{01}a_{23}b_{22} + a_{02}a_{21}b_{23} - a_{02}a_{23}b_{21} - a_{03}a_{21}b_{22} + a_{03}a_{22}b_{21}) \\
&+ y_0 y_1^2(a_{01}a_{12}b_{03} - a_{01}a_{13}b_{02} - a_{02}a_{11}b_{03} + a_{02}a_{13}b_{01} + a_{03}a_{11}b_{02} - a_{03}a_{12}b_{01}) \\
&+ y_1^2 y_2(a_{11}b_{02}b_{13} - a_{11}b_{03}b_{12} - a_{12}b_{01}b_{13} + a_{12}b_{03}b_{11} + a_{13}b_{01}b_{12} - a_{13}b_{02}b_{11}) \\
&+ y_0 y_2^2(a_{21}b_{12}b_{23} - a_{21}b_{13}b_{22} - a_{22}b_{11}b_{23} + a_{22}b_{13}b_{21} + a_{23}b_{11}b_{22} - a_{23}b_{12}b_{21}) \\
&+ y_1 y_2^2(a_{11}a_{22}b_{13} - a_{11}a_{23}b_{12} - a_{12}a_{21}b_{13} + a_{12}a_{23}b_{11} + a_{13}a_{21}b_{12} - a_{13}a_{22}b_{11}) \\
&+ y_0 y_1 y_2(a_{01}a_{12}a_{23} - a_{01}a_{13}a_{22} - a_{02}a_{11}a_{23} + a_{02}a_{13}a_{21} + a_{03}a_{11}a_{22} - a_{03}a_{12}a_{21} \\
&+ b_{01}b_{12}b_{23} - b_{01}b_{13}b_{22} - b_{02}b_{11}b_{23} + b_{02}b_{13}b_{21} + b_{03}b_{11}b_{22} - b_{03}b_{12}b_{21})
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
x_1 &= h_1(y_0, y_1, y_2) \\
&= y_0^2 y_1(a_{00}b_{02}b_{23} - a_{00}b_{03}b_{22} - a_{02}b_{00}b_{23} + a_{02}b_{03}b_{20} + a_{03}b_{00}b_{22} - a_{03}b_{02}b_{20}) \\
&+ y_0^2 y_2(a_{00}a_{22}b_{23} - a_{00}a_{23}b_{22} - a_{02}a_{20}b_{23} + a_{02}a_{23}b_{20} + a_{03}a_{20}b_{22} - a_{03}a_{22}b_{20}) \\
&+ y_0 y_1^2(-a_{00}a_{12}b_{03} + a_{00}a_{13}b_{02} + a_{02}a_{10}b_{03} - a_{02}a_{13}b_{00} - a_{03}a_{10}b_{02} + a_{03}a_{12}b_{00}) \\
&+ y_1^2 y_2(-a_{10}b_{02}b_{13} + a_{10}b_{03}b_{12} + a_{12}b_{00}b_{13} - a_{12}b_{03}b_{10} - a_{13}b_{00}b_{12} + a_{13}b_{02}b_{10}) \\
&+ y_0 y_2^2(-a_{20}b_{12}b_{23} + a_{20}b_{13}b_{22} + a_{22}b_{10}b_{23} - a_{22}b_{13}b_{20} - a_{23}b_{10}b_{22} + a_{23}b_{12}b_{20}) \\
&+ y_1 y_2^2(-a_{10}a_{22}b_{13} + a_{10}a_{23}b_{12} + a_{12}a_{20}b_{13} - a_{12}a_{23}b_{10} - a_{13}a_{20}b_{12} + a_{13}a_{22}b_{10}) \\
&+ y_0 y_1 y_2(-a_{00}a_{12}a_{23} + a_{00}a_{13}a_{22} + a_{02}a_{10}a_{23} - a_{02}a_{13}a_{20} - a_{03}a_{10}a_{22} + a_{03}a_{12}a_{20} \\
&- b_{00}b_{12}b_{23} + b_{00}b_{13}b_{22} + b_{02}b_{10}b_{23} - b_{02}b_{13}b_{20} - b_{03}b_{10}b_{22} + b_{03}b_{12}b_{20})
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
x_2 &= h_2(y_0, y_1, y_2) \\
&= y_0^2 y_1(-a_{00}b_{01}b_{23} + a_{00}b_{03}b_{21} + a_{01}b_{00}b_{23} - a_{01}b_{03}b_{20} - a_{03}b_{00}b_{21} + a_{03}b_{01}b_{20}) \\
&+ y_0^2 y_2(-a_{00}a_{21}b_{23} + a_{00}a_{23}b_{21} + a_{01}a_{20}b_{23} - a_{01}a_{23}b_{20} - a_{03}a_{20}b_{21} + a_{03}a_{21}b_{20}) \\
&+ y_0 y_1^2(a_{00}a_{11}b_{03} - a_{00}a_{13}b_{01} - a_{01}a_{10}b_{03} + a_{01}a_{13}b_{00} + a_{03}a_{10}b_{01} - a_{03}a_{11}b_{00}) \\
&+ y_1^2 y_2(a_{10}b_{01}b_{13} - a_{10}b_{03}b_{11} - a_{11}b_{00}b_{13} + a_{11}b_{03}b_{10} + a_{13}b_{00}b_{11} - a_{13}b_{01}b_{10}) \\
&+ y_0 y_2^2(a_{20}b_{11}b_{23} - a_{20}b_{13}b_{21} - a_{21}b_{10}b_{23} + a_{21}b_{13}b_{20} + a_{23}b_{10}b_{21} - a_{23}b_{11}b_{20}) \\
&+ y_1 y_2^2(a_{10}a_{21}b_{13} - a_{10}a_{23}b_{11} - a_{11}a_{20}b_{13} + a_{11}a_{23}b_{10} + a_{13}a_{20}b_{11} - a_{13}a_{21}b_{10}) \\
&+ y_0 y_1 y_2(a_{00}a_{11}a_{23} - a_{00}a_{13}a_{21} - a_{01}a_{10}a_{23} + a_{01}a_{13}a_{20} + a_{03}a_{10}a_{21} - a_{03}a_{11}a_{20} \\
&+ b_{00}b_{11}b_{23} - b_{00}b_{13}b_{21} - b_{01}b_{10}b_{23} + b_{01}b_{13}b_{20} + b_{03}b_{10}b_{21} - b_{03}b_{11}b_{20})
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
x_3 &= h_3(y_0, y_1, y_2) \\
&= y_0^2 y_1(a_{00}b_{01}b_{22} - a_{00}b_{02}b_{21} - a_{01}b_{00}b_{22} + a_{01}b_{02}b_{20} + a_{02}b_{00}b_{21} - a_{02}b_{01}b_{20}) \\
&+ y_0^2 y_2(a_{00}a_{21}b_{22} - a_{00}a_{22}b_{21} - a_{01}a_{20}b_{22} + a_{01}a_{22}b_{20} + a_{02}a_{20}b_{21} - a_{02}a_{21}b_{20}) \\
&+ y_0 y_1^2(-a_{00}a_{11}b_{02} + a_{00}a_{12}b_{01} + a_{01}a_{10}b_{02} - a_{01}a_{12}b_{00} - a_{02}a_{10}b_{01} + a_{02}a_{11}b_{00}) \\
&+ y_1^2 y_2(-a_{10}b_{01}b_{12} + a_{10}b_{02}b_{11} + a_{11}b_{00}b_{12} - a_{11}b_{02}b_{10} - a_{12}b_{00}b_{11} + a_{12}b_{01}b_{10}) \\
&+ y_0 y_2^2(-a_{20}b_{11}b_{22} + a_{20}b_{12}b_{21} + a_{21}b_{10}b_{22} - a_{21}b_{12}b_{20} - a_{22}b_{10}b_{21} + a_{22}b_{11}b_{20}) \\
&+ y_1 y_2^2(-a_{10}a_{21}b_{12} + a_{10}a_{22}b_{11} + a_{11}a_{20}b_{12} - a_{11}a_{22}b_{10} - a_{12}a_{20}b_{11} + a_{12}a_{21}b_{10}) \\
&+ y_0 y_1 y_2(-a_{00}a_{11}a_{22} + a_{00}a_{12}a_{21} + a_{01}a_{10}a_{22} - a_{01}a_{12}a_{20} - a_{02}a_{10}a_{21} + a_{02}a_{11}a_{20} \\
&- b_{00}b_{11}b_{22} + b_{00}b_{12}b_{21} + b_{01}b_{10}b_{22} - b_{01}b_{12}b_{20} - b_{02}b_{10}b_{21} + b_{02}b_{11}b_{20})
\end{aligned} \tag{4.8}
$$

There is a map Φ^{-1} such that

$$\Phi^{-1} \colon \mathrm{PG}(2, q) \to \mathcal{F}$$

which maps $\mathbf{P}(Y) = (y_0, y_1, y_2)$ to $\mathbf{P}(X) = (x_0, x_1, x_2, x_3)$ where

$$x_0 : x_1 : x_2 : x_3 = h_0(y_0, y_1, y_2) : h_1(y_0, y_1, y_2) : h_2(y_0, y_1, y_2) : h_3(y_0, y_1, y_2).$$

Since Φ^{-1} is also a rational map and $\Phi\Phi^{-1} = I$, Φ is a birational map in terms of the input coefficients of F_i and G_i. This map called *Clebsch map*. Note that for the cubic surface

$\mathcal{F}$, 72 different systems (4.2) can be written: There are 3! ways to permute the F_i, there are 3! ways to permute the G_i, and there are two ways to either exchange all F_i with the corresponding G_i or to leave them as they are.

The four plane cubic curves h_0, h_1, h_2, h_3 are called *Clebsch cubics*. The system (4.2) is called *Clebsch system*. The zero set $\mathcal{S} = V(h_0, h_1, h_2, h_3)$ of these four Clebsch cubics is a 6-arc S not on a conic in $\mathrm{PG}(2, q)$. The Clebsch map Φ,

$$\Phi\colon \mathcal{F} \to \mathrm{PG}(2, q)$$

given by $\Phi(\mathbf{P}(X)) = \mathbf{P}(Y)$ maps plane sections of $\mathcal{F}$ to cubic curves through $\mathcal{S}$.

Example 4.1.4. Let

$$\mathcal{F} = V\left(12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3\right)$$

be a cubic surface with twenty-seven lines in $\mathrm{PG}(3, 13)$. The equation of $\mathcal{F}$ can be written as $V(F_1F_2F_3+G_1G_2G_3)$, where $F_1, F_2, F_3, G_1, G_2, G_3$ are linear forms in the four coordinates. From Example 3.3.6 and Example 3.3.21, we can write the equation of $\mathcal{F}$ as:

$$F_1F_2F_3 + G_1G_2G_3 = 0$$

where

$$\begin{aligned} F_1 &= 11x_0 + 12x_1 + x_2 + x_3, \\ F_2 &= 7x_0 + x_1 + x_2 + x_3, \\ F_3 &= 12x_2 + x_3, \\ G_1 &= 6x_0 + x_1 + 12x_2 + x_3, \\ G_2 &= 2x_0 + 12x_1 + 12x_2 + x_3, \\ G_3 &= x_2 + x_3, \end{aligned} \tag{4.9}$$

and $\pi_{23} = V(F_1)$, $\pi_{31} = V(F_2)$, $\pi_{12} = V(F_3)$, $\pi_{32} = V(G_1)$, $\pi_{13} = V(G_2)$, $\pi_{21} = V(G_3)$.

Let $\mathbf{P}(X) = \mathbf{P}(x_0, x_1, x_2, x_3)$ be a point on $\mathcal{F}$ not on the planes $V(F_i)$ and $V(G_i)$.

Consider the Clebsch system,

$$\begin{aligned} y_1 G_1(X) &+ y_2 F_3(X) &= 0, \\ y_0 F_1(X) &+ y_2 G_2(X) &= 0, \\ y_0 G_3(X) &+ y_1 F_2(X) &= 0. \end{aligned}$$

and the related Clebsch map Φ:

$$\Phi \;:\; \mathcal{F} \;\rightarrow\; \mathrm{PG}(2,q)$$

Let the image of $\mathbf{P}(X)$ under the Clebsch map Φ be $\mathbf{P}(Y) = P(y_0, y_1, y_2)$. From (4.9) and (4.3), the coordinates of $\mathbf{P}(Y)$ can be found:

$$\begin{aligned} y_0 &= 12x_0^2x_2 + x_0^2x_3 + 5x_0x_1x_2 + 8x_0x_1x_3 + 5x_0x_2^2 + 12x_0x_2x_3 + 9x_0x_3^2 + x_1^2x_2 + 12x_1^2x_3 \\ &\quad +2x_1x_2^2 + 11x_1x_2x_3 + x_2^3 + 12x_2^2x_3 + 12x_2x_3^2 + x_3^3, \\ y_1 &= 2x_0x_2^2 + 11x_0x_3^2 + 12x_1x_2^2 + x_1x_3^2 + 12x_2^3 + x_2^2x_3 + x_2x_3^2 + 12x_3^3, \\ y_2 &= 12x_0^2x_2 + x_0^2x_3 + 4x_0x_1x_2 + 9x_0x_1x_3 + 5x_0x_2^2 + 8x_0x_3^2 + 12x_1^2x_2 + x_1^2x_3 + x_2^3 + x_2^2x_3 \\ &\quad +12x_2x_3^2 + 12x_3^3. \end{aligned}$$

Let $\mathbf{P}(Y) = P(y_0, y_1, y_2)$ be a point on the plane. Considering (4.5), (4.6), (4.7), (4.8), and the coefficients of the plane equations F_i, G_i as in (4.9), $\mathbf{P}(Y)$ is mapped to $\mathbf{P}(X) = P(x_0, x_1, x_2, x_3)$ on the surface $\mathcal{F}$, where

$$\begin{aligned} x_0 &= 11y_0^2y_1 + 11y_0^2y_2 + 9y_0y_1^2 + 9y_1^2y_2 + 11y_0y_2^2 + 11y_1y_2^2 + 5y_0y_1y_2 \\ x_1 &= 4y_0^2y_1 + 4y_0^2y_2 + 5y_0y_1^2 + 8y_1^2y_2 + 9y_0y_2^2 + 9y_1y_2^2 \\ x_2 &= 9y_0^2y_1 + 2y_0y_1^2 + 2y_1^2y_2 + 9y_1y_2^2 + 10y_0y_1y_2 \\ x_3 &= 4y_0^2y_1 + 8y_0y_1^2 + 5y_1^2y_2 + 9y_1y_2^2. \end{aligned}$$

All points of a_1 map to the same point under Φ. Let us now associate an arc to the surface. Recall that

$$a_1 = \mathbf{L}\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consider a point of the form $\mathbf{P}(X_1) = \mathbf{P}(1, 2, \mu, \mu)$ where $\mu \in \mathbf{F}_q$. Substituting the point

$\mathbf{P}(X_1)$ into the Clebsch system, we get

$$\begin{array}{lcrcl} 8y_1 & + & 0 & = & 0, \\ (9+2\mu)y_0 & + & 0 & = & 0, \\ 2\mu y_0 & + & (9+2\mu)y_1 & = & 0. \end{array}$$

$y_0 = 0, y_1 = 0$ and $y_2 = 1$.

Also, the point $\mathbf{P}(0,0,1,1)$ on a_1 is mapped to $\mathbf{P}(0,0,1)$ under Φ. Therefore, every point on a_1 maps to a single point $\mathbf{P}(Y_1) = \mathbf{P}(0,0,1)$ in $\mathrm{PG}(2,13)$.

$$\Phi(a_1) = \mathbf{P}(Y_1) = \mathbf{P}(0,0,1).$$

Likewise, all points of a_2 map to one and the same point. In fact, for every $i = 1,\dots,6$, all points of a_i map to one and the same point under Φ:

$$\begin{aligned} \Phi(a_2) &= \mathbf{P}(Y_2) = \mathbf{P}(1,0,0), \\ \Phi(a_3) &= \mathbf{P}(Y_3) = \mathbf{P}(0,1,0), \\ \Phi(a_4) &= \mathbf{P}(Y_4) = \mathbf{P}(1,6,1), \\ \Phi(a_5) &= \mathbf{P}(Y_5) = \mathbf{P}(9,8,1), \\ \Phi(a_6) &= \mathbf{P}(Y_6) = \mathbf{P}(3,11,1). \end{aligned}$$

Observe that the set of points

$$S = \{\mathbf{P}(Y_1), \mathbf{P}(Y_2), \mathbf{P}(Y_3), \mathbf{P}(Y_4), \mathbf{P}(Y_5), \mathbf{P}(Y_6)\}$$

form a 6-arc not on a conic in $\mathrm{PG}(2,13)$. This is the 6-arc that is associated to $\mathcal{F}$. Other than the points on the six lines $a_1,\dots,a_6$, the map Φ induces a bijection into the points of the plane outside of S. The full map can be seen in Appendix A.1.2.

4.2 Clebsch map, model 2

In this section, a different but equivalent model for the Clebsch map will be given. This map can be derived as follows.

Let $\mathcal{F}$ be a cubic surface with 27 lines and let ℓ_1 and ℓ_2 be two skew lines of $\mathcal{F}$. There are 5 lines of $\mathcal{F}$ which are transversal to ℓ_1 and ℓ_2. Let ℓ_3 be one of the transversal lines.

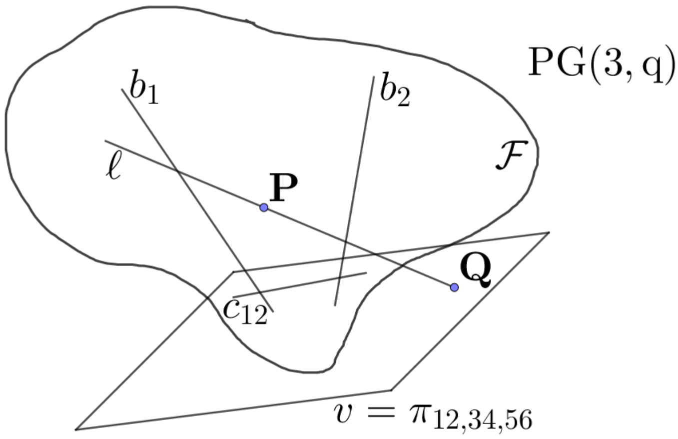


Figure 4.2: The Clebsch map from a surface to a plane, Model 2

There are 3 tritangent planes through ℓ_3 but not through either ℓ_1 or ℓ_2. Let v be one of these planes. Then through a point $P = \mathbf{P}(X)$ of $\mathcal{F}$ which is neither on ℓ_1 nor on ℓ_2, there exists a unique line ℓ meeting v in a unique point $Q = \mathbf{P}(Y)$ and the line ℓ meets ℓ_1 and ℓ_2 as well. Let Q be the image of P, so

$$\begin{aligned}\Phi\colon \mathcal{F} &\to v,\\ P &\mapsto Q.\end{aligned}$$

Conversely, through Q there is a unique line meeting ℓ_1 and ℓ_2, which has P as its third point of contact with $\mathcal{F}$ since a cubic surface meets a general line in three points.

Explicitly, let π_1 be the plane containing the line ℓ_1 and the point P and π_2 be the plane containing the line ℓ_2 and the point P. Two planes in $\mathrm{PG}(3, q)$ intersect in a line. Let $\ell = \pi_1 \cap \pi_2$ be the line where π_1 and π_2 meet. A line either lies in a plane or meets the plane in a unique point in $\mathrm{PG}(3, q)$. Here, the line ℓ cannot lie on the plane v. Therefore ℓ intersects the plane v in a unique point $Q = \Phi(P)$.

For instance, let $\mathcal{F}$ be a cubic surface with 27 lines and pick b_1 and b_2 to be two skew lines of $\mathcal{F}$. Let $v = \pi_{12,34,56}$ be the tritangent plane through the lines c_{12}, c_{34}, c_{56} of $\mathcal{F}$. Then through a point $P = \mathbf{P}(X)$ of $\mathcal{F}$ neither on b_1 nor on b_2, there exists a unique line ℓ meeting $v = \pi_{12,34,56}$ in a unique point $Q = \mathbf{P}(Y) = \Phi(P)$ (cf. Fig. 4.2).

Conversely, through Q there is a unique line intersecting b_1 and b_2. Besides those two points, the line meets the cubic surface in one third point $P = \Phi^{-1}(Q)$.

Example 4.2.1. Let $\mathcal{F}$ be a cubic surface with 27 lines as in Example 3.2.3 in $\mathrm{PG}(3, 13)$

with equation

$$12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3 = 0.$$

Recall that

$$b_1 = \mathbf{L}\begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 12 \end{bmatrix}, b_2 = \mathbf{L}\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and $\pi_{12,34,56} = V(x_3)$.

Now, express the x_i in terms of the y_i and the y_i in terms of the x_i.

Let $\mathbf{P}(X) = \mathbf{P}(x_0, x_1, x_2, x_3)$ be a point on $\mathcal{F}$ not on b_1 or b_2. Let ℓ be the unique line through $\mathbf{P}(X)$ which is the transversal of b_1, b_2. Then $\ell = \pi_1 \cap \pi_2$, where $\pi_i = \langle b_i, \mathbf{P}(X)\rangle$. In order to find this line, do the following.

(1) Let π_1 be the plane spanned by the line b_1 and the point $\mathbf{P}(X)$,

$$\pi_1 = \boldsymbol{\pi}\begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 12 \\ x_0 & x_1 & x_2 & x_3 \end{bmatrix}.$$

Find the dual of π_1, $\pi_1^{\perp} = \mathbf{P}(7x_2 + 7x_3, x_2 + x_3, 6x_0 + 12x_1, 6x_0 + 12x_1)$.

(2) Let π_2 be the plane spanned by the line b_2 and the point $\mathbf{P}(X)$,

$$\pi_2 = \boldsymbol{\pi}\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \end{bmatrix}.$$

Find the dual of π_2, $\pi_2^{\perp} = \mathbf{P}(7x_2 + 6x_3, 12x_2 + x_3, 6x_0 + x_1, 7x_0 + 12x_1)$.

(3) Paste the solutions together in a matrix

$$L = \mathbf{L}\begin{bmatrix} 7x_2 + 7x_3 & x_2 + x_3 & 6x_0 + 12x_1 & 6x_0 + 12x_1 \\ 7x_2 + 6x_3 & 12x_2 + x_3 & 6x_0 + x_1 & 7x_0 + 12x_1 \end{bmatrix}.$$

The line through $\mathbf{P}(X)$ and transversal to b_1 and b_2 is defined by the perpendicular subspace of L, which is

$$\ell = L^{\perp} = \mathbf{L}\begin{bmatrix} x_0x_3 + 11x_1x_2 & 6x_0x_2 + x_1x_3 & 0 & 12x_2^2 + x_3^2 \\ 12x_0x_2 + 2x_1x_3 & 7x_0x_3 + 12x_1x_2 & 12x_2^2 + x_3^2 & 0 \end{bmatrix}$$

Let $\mathbf{P}(Y)$ be the image of $\mathbf{P}(X)$ under the Clebsch map, Φ. Then,

$$\mathbf{P}(Y) = \mathbf{P}(y_0, y_1, y_2, 0) = \ell \cap \pi_{12,34,56} = \ell \cap V(x_3)$$

so,

$$\begin{aligned} y_0 &= 12x_0x_2 + 2x_1x_3, \\ y_1 &= 7x_0x_3 + 12x_1x_2, \\ y_2 &= 12x_2^2 + x_3^2. \end{aligned}$$

Conversely, assume $\mathbf{P}(Y) = \mathbf{P}(y_0, y_1, y_2, 0)$ is a point on the plane $\pi_{12,34,56}$. To find the inverse image $\mathbf{P}(X)$, let ℓ_1 be the unique line through $\mathbf{P}(Y)$ which is transversal to b_1 and b_2. In order to find this line, do the following.

(1) Let π_3 be the plane spanned by the line b_1 and the point $\mathbf{P}(Y)$,

$$\pi_3 = \boldsymbol{\pi}\begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 12 \\ y_0 & y_1 & y_2 & 0 \end{bmatrix}.$$

The dual of π_3 is $\pi_3^{\perp} = \mathbf{P}(7y_2, y_2, 12y_1 + 6y_0, 12y_1 + 6y_0)$.

(2) Let π_4 be the plane spanned by the line b_2 and the point $\mathbf{P}(Y)$,

$$\pi_4 = \boldsymbol{\pi}\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ y_0 & y_1 & y_2 & 0 \end{bmatrix}.$$

The dual of π_4 is $\pi_4^{\perp} = \mathbf{P}(7y_2, 12y_2, y_1 + 6y_0, 12y_1 + 7y_0)$.

(3) Paste the solutions together in a matrix

$$L_1 = \mathbf{L}\begin{bmatrix} 7y_2 & y_2 & 12y_1 + 6y_0 & 12y_1 + 6y_0 \\ 7y_2 & 12y_2 & y_1 + 6y_0 & 12y_1 + 7y_0 \end{bmatrix}.$$

The line through $\mathbf{P}(Y)$ and transversal to b_1 and b_2 is defined as

$$\ell_1 = L_1^{\perp} = \mathbf{L}\begin{bmatrix} 2y_1 & 7y_0 & 0 & y_2 \\ y_0 & y_1 & y_2 & 0 \end{bmatrix}.$$

The line ℓ_1 meets $\mathcal{F}$ in three points. One point is on b_1, one point is on b_2, and one point is the inverse image of $\mathbf{P}(Y)$ under Φ.

The point on b_1 is

$$\begin{aligned} P_1 = \ell_1 \cap b_1 &= \mathbf{P}(2y_1 + 12y_0, 7y_0 + 12y_1, 12y_2, y_2) \\ &= (2y_1 + 12y_0)\mathbf{P}(1, 6, 0, 0) + 12y_2\mathbf{P}(0, 0, 1, 12). \end{aligned}$$

The point on b_2 is

$$\begin{aligned} P_2 = \ell_1 \cap b_2 &= \mathbf{P}(2y_1 + y_0, 7y_0 + y_1, y_2, y_2) \\ &= (2y_1 + y_0)\mathbf{P}(1, 7, 0, 0) + y_2\mathbf{P}(0, 0, 1, 1)) \end{aligned}$$

Introducing a parameter t for the points on ℓ_1 as

$$\mathbf{P}(2y_1, 7y_0, 0, y_2) + t\mathbf{P}(y_0, y_1, y_2, 0) = \mathbf{P}(2y_1 + ty_0, 7y_0 + ty_1, ty_2, y_2),$$

we see that P_1 corresponds to $t = t_1 = 12$ and P_2 corresponds to $t = t_2 = 1$.

Restricting the equation of $\mathcal{F}$ to ℓ_1 means substituting

$$x_0 = 2y_1 + ty_0$$

$$x_1 = 7y_0 + ty_1$$

$$x_2 = ty_2$$

$$x_3 = y_2$$

into

$$12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3 = 0,$$

which yields

$$\begin{aligned}
0 &= 12(2y_1 + ty_0)^2y_2 + 12(7y_0 + ty_1)^2y_2 + 12(ty_2)^2y_2 + 9(2y_1 + ty_0)(7y_0 + ty_1)(ty_2) + y_2^3 \\
0 &= (9y_0y_1y_2)t^3 + (10y_0^2y_2 + 4y_1^2y_2 + 12y_2^3)t^2 + 4y_0y_1y_2t + 3y_0^2y_2 + 9y_1^2y_2 + y_2^3
\end{aligned}$$

Dividing by the leading coefficient $9y_0y_1y_2$ yields the cubic

$$\begin{aligned}
0 &= t^3 + \frac{10y_0^2y_2 + 4y_1^2y_2 + 12y_2^3}{9y_0y_1y_2}t^2 + \frac{4}{9}t + \frac{3y_0^2y_2 + 9y_1^2y_2 + y_2^3}{9y_0y_1y_2} \\
&= (t - t_1)(t - t_2)(t - t_3)
\end{aligned}$$

where t_3 is the parameter for the third point of intersection of ℓ_1 with $\mathcal{F}$. Comparing coefficients of t^2 yields

$$\frac{10y_0^2y_2 + 4y_1^2y_2 + 12y_2^3}{9y_0y_1y_2} = -(t_1 + t_2 + t_3).$$

Substituting $t_1 = -1$ and $t_2 = 1$ yields

$$t_3 = -\frac{10y_0^2y_2 + 4y_1^2y_2 + 12y_2^3}{9y_0y_1y_2}.$$

Therefore, $P_3 = \mathbf{P}(X) = \Phi^{-1}(Y) = \mathbf{P}(2y_1 + t_3y_0, 7y_0 + t_3y_1, t_3y_2, y_2)$

$$\begin{aligned}
x_0 &= 3y_0y_1^2y_2 + 9y_0^3y_2 + 3y_0y_2^3, \\
x_1 &= 3y_0^2y_1y_2 + y_1^3y_2 + 3y_1y_2^3, \\
x_2 &= 9y_0^2y_2^2 + y_1^2y_2^2 + 3y_2^4, \\
x_3 &= y_0y_1y_2^2.
\end{aligned}$$

Therefore, there is a mapping

$$\Phi^{-1} \colon \mathrm{PG}(2, q) \to \mathcal{F}$$

given by $\Phi^{-1}(\mathbf{P}(Y)) = \mathbf{P}(X)$. The whole map is listed in Appendix A.1.3.

Lemma 4.2.2. *Let $\mathcal{F}$ be a cubic surface with twenty-seven lines with the Schläfli labeling of* *the lines a_i, b_i, c_{ij}. Let $\Phi_{\mathscr{D}^1}$ be the Clebsch map where P maps to Q such that there exists a line ℓ through P which is the transversal to b_1 and b_2 and meets the plane $v = [c_{12}, c_{34}, c_{56}]$ at Q. Then, each line of the half double-six $\mathscr{D}^1$ of $\mathcal{F}$, $a_1, a_2, a_3, a_4, a_5, a_6$, maps to a single point in v, $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ under $\Phi_{\mathscr{D}^1}$.*

Proof Let P_3 be an arbitrary point on a_3. The line a_3 is the unique line through P_3 and transversal to b_1 and b_2. Since a_3 does not lie on the plane v, a_3 meets v in a unique point, say Q_3, which is the image of P_3 under $\Phi_{\mathscr{D}^1}$. Since P_3 is arbitrary on a_3, all points on a_3 map to Q_3.

Therefore,

$$\Phi_{\mathscr{D}^1}(P_3) = Q_3 = a_3 \cap v \quad \Rightarrow \quad \Phi_{\mathscr{D}^1}(a_3) = Q_3.$$

In the same way, it can be seen that

$$\begin{aligned} \Phi_{\mathscr{D}^1}(a_4) &= Q_4 = a_4 \cap v, \\ \Phi_{\mathscr{D}^1}(a_5) &= Q_5 = a_5 \cap v, \\ \Phi_{\mathscr{D}^1}(a_6) &= Q_6 = a_6 \cap v. \end{aligned}$$

The situation for $\Phi_{\mathscr{D}^1}(a_1)$ and $\Phi_{\mathscr{D}^1}(a_2)$ is a bit different. We claim that

$$\begin{aligned} \Phi_{\mathscr{D}^1}(a_1) &= Q_1 = b_1 \cap c_{12} \\ \Phi_{\mathscr{D}^1}(a_2) &= Q_2 = b_2 \cap c_{12}. \end{aligned}$$

Consider an arbitrary point P_1 on a_1. Let ℓ be the line P_1Q_1 where $Q_1 = b_1 \cap c_{12}$. Since

$$c_{12} = [a_1, b_2] \cap [a_2, b_1]$$

we find that

$$Q_1 \in c_{12} \subseteq [a_1, b_2]$$

and hence a_1, b_2 and P_1Q_1 are three lines in the plane $[a_1, b_2]$. Therefore, P_1Q_1 and b_2 intersect. Thus P_1Q_1 is the unique transversal of b_1 and b_2 through P_1. Since

$$P_1Q_1 \cap v = Q_1,$$

we see that

$$\Phi_{\mathscr{D}^1}(P_1) = Q_1.$$

Since P_1 was arbitrary on a_1 it follows that $\Phi_{\mathscr{D}^1}(a_1) = Q_1$.

In the same way it follows that

$$\Phi_{\mathscr{D}^1}(a_2) = Q_2 = b_2 \cap c_{12}.$$

$\square$

Example 4.2.3. Let Φ be the Clebsch map defined as in Example 4.2.1. In the notation introduced in Lemma 4.2.2, $\Phi = \Phi_{\mathscr{D}^1}$. Recall that $\mathcal{F}$ is the cubic surface with twenty seven lines in $\mathrm{PG}(3, q)$ given by

$$12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3 = 0.$$

The mapping $\Phi_{\mathscr{D}^1}$ was defined by picking the two skew lines b_1 and b_2 and by picking the plane $\pi_{12,34,56} = V(x_3)$. Here,

$$b_1 = \mathbf{L}\begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 12 \end{bmatrix}, b_2 = \mathbf{L}\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, c_{12} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

Recall that

$$a_3 = \mathbf{L}\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 0 & 12 \end{bmatrix}, a_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, a_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 11 & 0 \end{bmatrix}, a_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

Then

$$\Phi_{\mathscr{D}^1}(a_i) = a_i \cap v = R_i \text{ for } i = 3, 4, 5, 6,$$

where $R_3 = \mathbf{P}(1, 0, 6), R_4 = \mathbf{P}(1, 0, 7), R_5 = \mathbf{P}(0, 1, 11)$, and $R_6 = \mathbf{P}(0, 1, 2)$. From the proof of Lemma 4.2.2, $R_i = b_i \cap v$ for $i = 1, 2$, so

$$R_1 = \mathbf{P}(1, 6, 0) = b_1 \cap v \text{ and } R_2 = \mathbf{P}(1, 7, 0) = b_2 \cap v.$$

This shows that

$$S_2 = \{R_1, R_2, R_3, R_4, R_5, R_6\}$$

is a 6 arc not on a conic in $\mathrm{PG}(2, 13)$ associated with the cubic surface

$$\mathcal{F} = V(12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3)$$

in $\mathrm{PG}(3, 13)$.

The points on the surface other than the points on $a_1, a_2, \ldots, a_6$ are mapped to distinct points in the plane but not in S.

Corollary 4.2.4. *In* $\mathrm{PG}(3, q)$, *a cubic surface with* 27 *lines has* $q^2 + 7q + 1$ *points.*

Proof From above, there are $q^2 + q + 1 - 6 + 6(q + 1) = q^2 + 7q + 1$ points on $\mathcal{F}$. $\square$

Note 4.2.5. In $\mathrm{PG}(3, q)$, the number of points on the 27 lines of a cubic surface $\mathcal{F}$ is $27(q - 4) + e_3$ where e_3 is the total number of Eckardt points of $\mathcal{F}$.

The Clebsch map collapses six lines of the surface $\mathcal{F}$ to points in the plane forming a non-conical arc. The six lines that are collapsed from one half of a double-six. Since there are 36 double-sixes associated with $\mathcal{F}$, we will need to consider 72 Clebsch maps, one for each set of six disjoint lines on $\mathcal{F}$. Our notation for these Clebsch maps is as follows:

$$\Phi_{\mathscr{D}^1}, \Phi_{\mathscr{D}^2},\ 15\ \Phi_{\mathscr{D}^1_{ij}}, 15\ \Phi_{\mathscr{D}^2_{ij}},\ 20\ \Phi_{\mathscr{D}^1_{ijk}},\ 20\ \Phi_{\mathscr{D}^2_{ijk}}.$$

Here,

$\mathscr{D}^1$	a_1	a_2	a_3	a_4	a_5	a_6
$\mathscr{D}^2$	b_1	b_2	b_3	b_4	b_5	b_6
$\mathscr{D}^1_{ij}$	a_i	b_i	c_{jk}	c_{jl}	c_{jm}	c_{jn}
$\mathscr{D}^2_{ij}$	a_j	b_j	c_{jk}	c_{jl}	c_{jm}	c_{jn}
$\mathscr{D}^1_{ijk}$	a_i	a_j	a_k	c_{mn}	c_{ln}	c_{lm}
$\mathscr{D}^2_{ijk}$	c_{jk}	c_{ik}	c_{ij}	b_l	b_m	b_n

In Appendix B, a possible arrangement for 72 Clebsch maps of the cubic surface $\mathcal{F}$ with 27 lines in $\mathrm{PG}(3, q)$ can be seen. For each half double-six, one Clebsch map is listed which sends these 6 lines to the 6 points of a non-conical arc.

Lemma 4.2.6. *Let $S = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$ be a 6 arc not on a conic in $\mathrm{PG}(2, q)$ and let $\mathcal{F}$ be the related cubic surface with 27 lines in $\mathrm{PG}(3, q)$ to the 6-arc S. Then there exists a half double-six on $\mathcal{F}$, say $\mathscr{D}^1$, such that if A is the points on the lines a_i, then the restriction of $\Phi_{\mathscr{D}^1}$ is a bijection*

$$\Phi^A_{\mathscr{D}^1} : \mathcal{F} \setminus A \to \mathrm{PG}(2, q) \setminus S.$$

Proof Let $Q \in \mathrm{PG}(2, q) \setminus S$. Through Q there is a unique line meeting b_1 and b_2, which has P as its third point of contact with $\mathcal{F}$ since a cubic surface (degree three in $\mathrm{PG}(3, q)$) meets a line (off the cubic surface) in three points, $\Phi^A_{\mathscr{D}^1}(P) = Q$. It means that there exists a point P on $\mathcal{F} \setminus A$ as the preimage of Q under $\Phi^A_{\mathscr{D}^1}$. Therefore, since Q is an arbitrary point in $\mathrm{PG}(2, q) \setminus S$, $\Phi^A_{\mathscr{D}^1}$ is onto. Since P is unique, this also shows that $\Phi^A_{\mathscr{D}^1}$ is one-to-one.

□

4.3 The arc lifting algorithm

In this section, the construction of the cubic surface with 27 lines over $\mathbf{F}_q$ from a 6-arc not on a conic in $\mathrm{PG}(2, q)$ will be described.

Let $S = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$ be a 6-arc not on a conic in $\mathrm{PG}(2, q)$. Let Q_iQ_j be the line through Q_i and Q_j, and let C_j be the conic through the five points of $S \setminus \{Q_j\}$. Let $\mathcal{F}$ be any of the associated cubic surfaces with twenty-seven lines in $\mathrm{PG}(3, q)$. Consider Clebsch's mapping $\Phi_{\mathscr{D}^1}$ of $\mathcal{F}$ to the plane:

$$\Phi_{\mathscr{D}^1} : \quad \mathcal{F} \to \mathrm{PG}(2, q).$$

The $q+1$ points of the line a_i of $\mathcal{F}$ in $\mathrm{PG}(3, q)$ are all mapped to same point $Q_i \in \mathrm{PG}(2, q)$. This is true for $i = 1, \ldots, 6$. So there is a set of 6 points forming a 6-arc not on a conic under $\Phi_{\mathscr{D}^1}$. Each of the remaining points of $\mathcal{F}$ is mapped to a separate point of the plane. The points of the line b_i map to the points of the conic C_i, and the points of the line c_{ij}

map to the points of the line Q_iQ_j:

$$\begin{aligned}\Phi_{\mathscr{D}^1}(a_i) &= Q_i,\\ \Phi_{\mathscr{D}^1}(b_i) &= C_i,\\ \Phi_{\mathscr{D}^1}(c_{ij}) &= Q_iQ_j.\end{aligned}$$

Lemma 4.3.1 ([23]). *Consider the Clebsch map $\Phi_{\mathscr{D}^1}$. The points of b_i are mapped bijectively to the points of the conic C_i, where C_i is the conic through the 5 points of S other than Q_i, $\Phi_{\mathscr{D}^1}(b_i) = C_i$. The points of c_{ij} are mapped bijectively to the points of the bisecant Q_iQ_j, so $\Phi_{\mathscr{D}^1}(c_{ij}) = Q_iQ_j$.*

Let $V(F_1)$ be the plane section of the corresponding cubic surface $\mathcal{F}$ consisting of c_{ij}, c_{kl}, c_{mn} and let $V(f_1)$ be the cubic curve through S in $\mathrm{PG}(2,q)$ which is made up of the three lines Q_iQ_j, Q_kQ_l, and Q_mQ_n. Then

$$\Phi_{\mathscr{D}^1}(V(F_1)) = V(f_1).$$

Since $E_{ij,kl,mn} = c_{ij} \cap c_{kl} \cap c_{mn}$, the image of such an Eckardt point is the intersection of $Q_iQ_j \cap Q_kQ_l \cap Q_mQ_n$ of three bisecants of S. Such a point is called *Brianchon point.*

Let $V(F_2)$ be the tritangent plane consisting of a_i, b_j, c_{ij}, and let

$$\Phi_{\mathscr{D}^1}(V(F_2)) = V(f_2).$$

Then $V(f_2)$ is the cubic curve through S which consists of the conic C_j and the line Q_iQ_j. If the tritangent plane through a_i, b_j, and c_{ij} contains an Eckardt point E_{ij} then the tangent line Q_iQ_j touches C_j in P_i.

In particular, we note that the existence of Eckardt points can be predicted from the properties of the arc S.

The *arc lifting* algorithm is used for the construction of the cubic surface with twenty-seven lines in $\mathrm{PG}(3,q)$ arising from a 6-arc not on a conic in $\mathrm{PG}(2,q)$.

The input of the algorithm is

$$S = \text{6-arc not on a conic in } \mathrm{PG}(2,q)$$

and the output of the algorithm is

$$\mathcal{F} \;=\; \text{a cubic surface with twenty-seven lines in PG}(3,q) \text{ associated to } S.$$

The algorithm proceeds in four steps.

At the first step, for a given 6-arc S not on a conic, the plane cubic curves through S are found. In the second step, the Clebsch map is used to obtain the equations of the tritangent planes corresponding to these cubics. Only those cubics are found which correspond to the images of tritangent planes under Φ. In the third step, the possible arrangements of 120 trihedral pairs on this surface are found. This yields eighteen conditions to determine the cubic surface. In the last step, an extra point in the plane is considered to determine a cubic surface $\mathcal{F}$ together with a distinguished double-six which has the property that it has all previously constructed tritangent planes. Moreover, via the Clebsch map, the constructed surface is associated to the non-conical 6-arc S that we started with. Here is a somewhat more detailed description of the algorithm.

Step 1: Determine the linear system of cubic curves through $S = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$ in PG$(2,q)$.

(1) Calculate the 6 conics C_j through $S \setminus \{Q_j\}$, where $j = 1, \ldots, 6$.

(2) Calculate the 15 bisecants Q_iQ_j through each pair of points Q_i and Q_j, where $i, j = 1, \ldots, 6$, and $i \neq j$.

(3) Calculate the 30 cubic curves through S of the form $C_j \cdot Q_iQ_j$, denoted by W_{ij}, where $i, j = 1, \ldots, 6$ and $i \neq j$.

(4) Calculate the 15 cubic curves through S of the form $Q_iQ_j \cdot Q_kQ_l \cdot Q_mQ_n$, denoted by $W_{ij,kl,mn}$, where i, j, k, l, m, n is a permutation of $\{1, \ldots, 6\}$.

Step 2: Find the tritangent planes and the 27 lines of $\mathcal{F}$ in PG$(3,q)$.

(1) Fix 4 linearly independent cubic curves through S as base curves, say

$$\begin{aligned} V(f_1) &= V(C_3 \cdot Q_2Q_3) = W_{23}, \quad V(f_2) = V(C_1 \cdot Q_1Q_3) = W_{31}, \\ V(g_1) &= V(C_2 \cdot Q_2Q_3) = W_{32}, \quad V(g_2) = V(C_3 \cdot Q_1Q_3) = W_{13}. \end{aligned}$$

The tritangent plane $[a_i b_j c_{ij}]$ is the inverse image of the cubic curve

$$W_{ij} = V(C_j \cdot Q_i Q_j).$$

The tritangent plane $[c_{ij} c_{kl} c_{mn}]$ is the inverse image of the cubic curve

$$W_{ij,kl,mn} = V(Q_i Q_j \cdot Q_k Q_l \cdot Q_m Q_n).$$

(2) Associate the 4 chosen tritangent planes with the coordinate hyperplanes in the following way:

$$V(F_1) \to V(f_1) \text{ with } V(F_1) = [a_2 b_3 c_{23}] = \pi_{23}, \quad V(F_1) = V(x_0);$$
$$V(F_2) \to V(f_2) \text{ with } V(F_2) = [a_3 b_1 c_{13}] = \pi_{31}, \quad V(F_2) = V(x_1);$$
$$V(G_1) \to V(g_1) \text{ with } V(G_1) = [a_3 b_2 c_{23}] = \pi_{32}, \quad V(G_1) = V(x_2);$$
$$V(G_2) \to V(g_2) \text{ with } V(G_2) = [a_1 b_3 c_{13}] = \pi_{13}, \quad V(G_2) = V(x_3).$$

The 4 chosen tritangent planes will be referred to as the base tritangent planes.

The defining equation of any cubic curve passing through the 6 points $Q_1, \ldots, Q_6$ can be written as a linear combination of the equations f_1, f_2, g_1, g_2 of 4 base cubic curves. Moreover, the defining equation of every tritangent plane of $\mathcal{F}$ can be written as a linear combination of the equations F_1, F_2, G_1, G_2 of the 4 base tritangent planes. Applying a linear substitution to the base planes means applying a linear transformation to the coefficients of the equation of the tritangent planes.

Moreover, the coefficients in a linear combination expressing a given cubic curve in terms of the 4 base cubics are the same as the coefficients in a linear combinations expressing the corresponding tritangent plane in terms of the corresponding 4 base tritangent plane, that is if

$$h = \sum_{i=0}^{1} a_i f_i + \sum_{i=0}^{1} b_i g_i$$

and h corresponds to the tritangent plane H and the f_i correspond to the F_i and the

g_i corresponds to the G_i then

$$H = \sum_{i=0}^{1} a_i F_i + \sum_{i=0}^{1} b_i G_i.$$

(3) Calculate the remaining 41 tritangent planes as linear combinations of x_0, x_1, x_2, x_3.

(4) Using

$$\begin{aligned} a_i &= \pi_{ij} \cap \pi_{ik} = [a_i, b_j, c_{ij}] \cap [a_i, b_k, c_{ik}], \\ b_i &= \pi_{ji} \cap \pi_{ki} = [a_j, b_i, c_{ij}] \cap [a_k, b_i, c_{ik}], \\ c_{ij} &= \pi_{ij} \cap \pi_{ji} = [a_i, b_j, c_{ij}] \cap [a_j, b_i, c_{ij}], \end{aligned}$$

all 27 lines are found.

(5) Each Steiner trihedral pair consists of two set of three tritangent planes. Since the 45 tritangent planes are known, all 120 arrangements for the Steiner trihedral pairs can be found. Recall that

$$\begin{aligned} S_{ijk} &\text{ has planes } \{\pi_{ij}, \pi_{jk}, \pi_{ki}\} \text{ and } \{\pi_{ji}, \pi_{kj}, \pi_{ik}\}, \\ S_{ij,kl} &\text{ has planes } \{\pi_{ij}, \pi_{kl}, \pi_{il,jk,mn}\} \text{ and } \{\pi_{il}, \pi_{kj}, \pi_{ij,kl,mn}\}, \\ S_{ijk,lmn} &\text{ has planes } \{\pi_{il,jm,kn}, \pi_{im,kl,jn}, \pi_{in,jl,km}\} \text{ and } \{\pi_{il,jn,km}, \pi_{in,jm,kl}, \pi_{im,jl,kn}\}. \end{aligned}$$

Step 3: Find the trihedral pairs of $\mathcal{F}$ in $\mathrm{PG}(3, q)$.

Pick a trihedral pair related to 4 base tritangent planes $V(F_1), V(F_2), V(G_1), V(G_2)$ and 4 base cubic curves $V(f_1), V(f_2), V(g_1), V(g_2)$, say S_{123}:

$$\begin{array}{cccc} c_{23} & b_3 & a_2 & F_1 \\ a_3 & c_{13} & b_1 & F_2 \\ b_2 & a_1 & c_{12} & F_3 \\ G_1 & G_2 & G_3 & \end{array}$$

In step two, 45 tritangent planes are already calculated as the linear combination of x_0, x_1, x_2, x_3. Therefore, F_i and G_i are linear forms in the four coordinates. From Theorem

3.3.20, the cubic surface with trihedral pair S_{123} can be written in the form

$$\mathcal{F} = V(F_1F_2F_3 + \lambda G_1G_2G_3). \tag{4.10}$$

Step 4: Find the equation of the cubic surface $\mathcal{F}$ in $\mathrm{PG}(3,q)$ with twenty-seven lines.

The parameter λ can be found in the following way.

The Clebsch map is onto; that is, every point in $\mathrm{PG}(2,q)$ is an image of a point (or points) on the cubic surface $\mathcal{F}$ as follows:

$$x_0 : x_1 : x_2 : x_3 = f_1(y_0,y_1,y_2) : f_2(y_0,y_1,y_2) : g_1(y_0,y_1,y_2) : g_2(y_0,y_1,y_2),$$

where $V(f_1), V(f_2), V(g_1), V(g_2)$ are base cubic curves in the plane. Picking a point in the plane $\mathbf{P}(y_0,y_1,y_2)$ not on the base cubic curves, a point $\mathbf{P}(x_0,x_1,x_2,x_3)$ on $\mathcal{F}$ is found by evaluating the base cubic curves f_1, f_2, g_1, g_2 at the point $\mathbf{P}(y_0,y_1,y_2)$. The parameter λ is found by evaluating $\mathbf{P}(x_0,x_1,x_2,x_3)$ in (4.10).

This algorithm creates a surface from a 6-arc not on a conic in the plane.

Example 4.3.2. In $\mathrm{PG}(2,13)$, consider the 6-arc not on a conic $S = \{Q_1,Q_2,Q_3,Q_4,Q_5,Q_6\}$ where

$$\begin{array}{lll} Q_1 = \mathbf{P}(2,1,0), & Q_2 = \mathbf{P}(11,1,0), & Q_3 = \mathbf{P}(11,0,1), \\ Q_4 = \mathbf{P}(2,0,1), & Q_5 = \mathbf{P}(0,6,1), & Q_6 = \mathbf{P}(0,7,1). \end{array}$$

Recall that this is the 6-arc not on a conic in Example 4.2.3.

The related cubic surface with twenty seven lines $\mathcal{F}$ in $\mathrm{PG}(3,13)$ can be determined in 4 step as follows:

Step 1:

In order to find the linear system of cubic curves through S, we need to consider the bisecants Q_iQ_j of the 6-arc S and the conics C_i through $S \setminus \{Q_i\}$. The 15 bisecants of S are listed in Table 4.1 and the 6 conics is listed in Table 4.2. The resulting the 45 cubic curves through S are given in Table 4.3 and in Table 4.4.

$Q_1Q_2 = V(y_2)$	$Q_2Q_6 = V(12y_0 + 11y_1 + y_2)$
$Q_1Q_3 = V(7y_0 + 12y_1 + y_2)$	$Q_3Q_4 = V(y_1)$
$Q_1Q_4 = V(6y_0 + y_1 + y_2)$	$Q_3Q_5 = V(7y_0 + 2y_1 + y_2)$
$Q_1Q_5 = V(12y_0 + 2y_1 + y_2)$	$Q_3Q_6 = V(7y_0 + 11y_1 + y_2)$
$Q_1Q_6 = V(y_0 + 11y_1 + y_2)$	$Q_4Q_5 = V(6y_0 + 2y_1 + y_2)$
$Q_2Q_3 = V(7y_0 + y_1 + y_2)$	$Q_4Q_6 = V(6y_0 + 11y_1 + y_2)$
$Q_2Q_4 = V(6y_0 + 12y_1 + y_2)$	$Q_5Q_6 = V(y_0)$
$Q_2Q_5 = V(y_0 + 2y_1 + y_2)$	

Table 4.1: The bisecants of S

$$
\begin{aligned}
C_1 &= V(y_0^2 + 3y_1^2 + 9y_2^2 + 10y_0y_1) \\
C_2 &= V(y_0^2 + 3y_1^2 + 9y_2^2 + 3y_0y_1) \\
C_3 &= V(y_0^2 + 9y_1^2 + y_2^2 + 4y_0y_2) \\
C_4 &= V(y_0^2 + 9y_1^2 + y_2^2 + 9y_0y_2) \\
C_5 &= V(y_0^2 + 9y_1^2 + 9y_2^2 + 10y_1y_2) \\
C_6 &= V(y_0^2 + 9y_1^2 + 9y_2^2 + 3y_1y_2)
\end{aligned}
$$

Table 4.2: The conics C_i through $S \setminus \{Q_i\}$

$$
\begin{aligned}
&W_{12} = V(9y_2^3 + y_0^2y_2 + 3y_1^2y_2 + 3y_0y_1y_2)\\
&W_{21} = V(9y_2^3 + y_0^2y_2 + 3y_1^2y_2 + 10y_0y_1y_2)\\
&W_{13} = V(y_0^3 + 8y_1^3 + 2y_2^3 + 11y_0^2y_1 + 6y_0^2y_2 + 9y_0y_1^2 + 5y_1^2y_2 + 9y_0y_2^2 + 11y_1y_2^2 + 5y_0y_1y_2)\\
&W_{31} = V(y_0^3 + 7y_1^3 + 5y_2^3 + 8y_0^2y_1 + 2y_0^2y_2 + 9y_0y_1^2 + 6y_1^2y_2 + 9y_0y_2^2 + 8y_1y_2^2 + 7y_0y_1y_2)\\
&W_{14} = V(y_0^3 + 8y_1^3 + 11y_2^3 + 11y_0^2y_1 + 7y_0^2y_2 + 9y_0y_1^2 + 8y_1^2y_2 + 9y_0y_2^2 + 11y_1y_2^2 + 8y_0y_1y_2)\\
&W_{41} = V(y_0^3 + 7y_1^3 + 8y_2^3 + 8y_0^2y_1 + 11y_0^2y_2 + 9y_0y_1^2 + 7y_1^2y_2 + 9y_0y_2^2 + 8y_1y_2^2 + 6y_0y_1y_2)\\
&W_{15} = V(y_0^3 + 8y_1^3 + 4y_2^3 + 11y_0^2y_1 + 12y_0^2y_2 + 9y_0y_1^2 + 10y_1^2y_2 + 9y_0y_2^2 + 11y_1y_2^2 + 10y_0y_1y_2)\\
&W_{51} = V(y_0^3 + 7y_1^3 + 4y_2^3 + 8y_0^2y_1 + 12y_0^2y_2 + 9y_0y_1^2 + 10y_1^2y_2 + 9y_0y_2^2 + 8y_1y_2^2 + 3y_0y_1y_2)\\
&W_{16} = V(y_0^3 + 8y_1^3 + 9y_2^3 + 11y_0^2y_1 + y_0^2y_2 + 9y_0y_1^2 + 3y_1^2y_2 + 9y_0y_2^2 + 11y_1y_2^2 + 3y_0y_1y_2)\\
&W_{61} = V(y_0^3 + 7y_1^3 + 9y_2^3 + 8y_0^2y_1 + y_0^2y_2 + 9y_0y_1^2 + 3y_1^2y_2 + 9y_0y_2^2 + 8y_1y_2^2 + 10y_0y_1y_2)\\
&W_{23} = V(y_0^3 + 5y_1^3 + 2y_2^3 + 2y_0^2y_1 + 6y_0^2y_2 + 9y_0y_1^2 + 5y_1^2y_2 + 9y_0y_2^2 + 2y_1y_2^2 + 8y_0y_1y_2)\\
&W_{32} = V(y_0^3 + 6y_1^3 + 5y_2^3 + 5y_0^2y_1 + 2y_0^2y_2 + 9y_0y_1^2 + 6y_1^2y_2 + 9y_0y_2^2 + 5y_1y_2^2 + 6y_0y_1y_2)\\
&W_{24} = V(y_0^3 + 5y_1^3 + 11y_2^3 + 2y_0^2y_1 + 7y_0^2y_2 + 9y_0y_1^2 + 8y_1^2y_2 + 9y_0y_2^2 + 2y_1y_2^2 + 5y_0y_1y_2)\\
&W_{42} = V(y_0^3 + 6y_1^3 + 8y_2^3 + 5y_0^2y_1 + 11y_0^2y_2 + 9y_0y_1^2 + 7y_1^2y_2 + 9y_0y_2^2 + 5y_1y_2^2 + 7y_0y_1y_2)\\
&W_{25} = V(y_0^3 + 5y_1^3 + 9y_2^3 + 2y_0^2y_1 + y_0^2y_2 + 9y_0y_1^2 + 3y_1^2y_2 + 9y_0y_2^2 + 2y_1y_2^2 + 10y_0y_1y_2)\\
&W_{52} = V(y_0^3 + 6y_1^3 + 9y_2^3 + 5y_0^2y_1 + y_0^2y_2 + 9y_0y_1^2 + 3y_1^2y_2 + 9y_0y_2^2 + 5y_1y_2^2 + 3y_0y_1y_2)\\
&W_{26} = V(y_0^3 + 5y_1^3 + 4y_2^3 + 2y_0^2y_1 + 12y_0^2y_2 + 9y_0y_1^2 + 10y_1^2y_2 + 9y_0y_2^2 + 2y_1y_2^2 + 3y_0y_1y_2)\\
&W_{62} = V(y_0^3 + 6y_1^3 + 4y_2^3 + 5y_0^2y_1 + 12y_0^2y_2 + 9y_0y_1^2 + 10y_1^2y_2 + 9y_0y_2^2 + 5y_1y_2^2 + 10y_0y_1y_2)\\
&W_{34} = V(9y_1^3 + y_0^2y_1 + y_1y_2^2 + 9y_0y_1y_2)\\
&W_{43} = V(9y_1^3 + y_0^2y_1 + y_1y_2^2 + 4y_0y_1y_2)\\
&W_{35} = V(y_0^3 + 10y_1^3 + 5y_2^3 + 4y_0^2y_1 + 2y_0^2y_2 + 9y_0y_1^2 + 6y_1^2y_2 + 9y_0y_2^2 + 4y_1y_2^2 + 10y_0y_1y_2)\\
&W_{53} = V(y_0^3 + 10y_1^3 + 2y_2^3 + 4y_0^2y_1 + 6y_0^2y_2 + 9y_0y_1^2 + 5y_1^2y_2 + 9y_0y_2^2 + 4y_1y_2^2 + 3y_0y_1y_2)\\
&W_{36} = V(y_0^3 + 3y_1^3 + 5y_2^3 + 9y_0^2y_1 + 2y_0^2y_2 + 9y_0y_1^2 + 6y_1^2y_2 + 9y_0y_2^2 + 9y_1y_2^2 + 3y_0y_1y_2)\\
&W_{63} = V(y_0^3 + 3y_1^3 + 2y_2^3 + 9y_0^2y_1 + 6y_0^2y_2 + 9y_0y_1^2 + 5y_1^2y_2 + 9y_0y_2^2 + 9y_1y_2^2 + 10y_0y_1y_2)\\
&W_{45} = V(y_0^3 + 3y_1^3 + 8y_2^3 + 9y_0^2y_1 + 11y_0^2y_2 + 9y_0y_1^2 + 7y_1^2y_2 + 9y_0y_2^2 + 9y_1y_2^2 + 10y_0y_1y_2)\\
&W_{54} = V(y_0^3 + 3y_1^3 + 11y_2^3 + 9y_0^2y_1 + 7y_0^2y_2 + 9y_0y_1^2 + 8y_1^2y_2 + 9y_0y_2^2 + 9y_1y_2^2 + 3y_0y_1y_2)\\
&W_{46} = V(y_0^3 + 10y_1^3 + 8y_2^3 + 4y_0^2y_1 + 11y_0^2y_2 + 9y_0y_1^2 + 7y_1^2y_2 + 9y_0y_2^2 + 4y_1y_2^2 + 3y_0y_1y_2)\\
&W_{64} = V(y_0^3 + 10y_1^3 + 11y_2^3 + 4y_0^2y_1 + 7y_0^2y_2 + 9y_0y_1^2 + 8y_1^2y_2 + 9y_0y_2^2 + 4y_1y_2^2 + 10y_0y_1y_2)\\
&W_{56} = V(y_0^3 + 9y_0y_1^2 + 9y_0y_2^2 + 3y_0y_1y_2)\\
&W_{65} = V(y_0^3 + 9y_0y_1^2 + 9y_0y_2^2 + 10y_0y_1y_2)
\end{aligned}
$$

Table 4.3: The 30 cubic curves of the form $W_{ij} = C_j \cdot Q_iQ_j = \Phi(\pi_{ij})$

$$W_{12,34,56} = V(y_0y_1y_2)$$
$$W_{12,35,46} = V(9y_2^3 + y_0^2y_2 + 3y_1^2y_2 + 8y_0y_1y_2)$$
$$W_{12,36,45} = V(9y_2^3 + y_0^2y_2 + 3y_1^2y_2 + 5y_0y_1y_2)$$
$$W_{13,24,56} = V(y_0^3 + 9y_0y_1^2 + 9y_0y_2^2 + 8y_0y_1y_2)$$
$$W_{13,25,46} = V(y_0^3 + 10y_1^3 + 9y_2^3 + 4y_0^2y_1 + y_0^2y_2 + 9y_0y_1^2 + 3y_1^2y_2 + 9y_0y_2^2 + 4y_1y_2^2 + y_0y_1y_2)$$
$$W_{13,26,45} = V(y_0^3 + 3y_1^3 + 4y_2^3 + 9y_0^2y_1 + 12y_0^2y_2 + 9y_0y_1^2 + 10y_1^2y_2 + 9y_0y_2^2 + 9y_1y_2^2 + 2y_0y_1y_2)$$
$$W_{14,23,56} = V(y_0^3 + 9y_0y_1^2 + 9y_0y_2^2 + 5y_0y_1y_2)$$
$$W_{14,25,36} = V(y_0^3 + 3y_1^3 + 9y_2^3 + 9y_0^2y_1 + y_0^2y_2 + 9y_0y_1^2 + 3y_1^2y_2 + 9y_0y_2^2 + 9y_1y_2^2 + 11y_0y_1y_2)$$
$$W_{14,26,35} = V(y_0^3 + 10y_1^3 + 4y_2^3 + 4y_0^2y_1 + 12y_0^2y_2 + 9y_0y_1^2 + 10y_1^2y_2 + 9y_0y_2^2 + 4y_1y_2^2 + 12y_0y_1y_2)$$
$$W_{15,23,46} = V(y_0^3 + 10y_1^3 + 4y_2^3 + 4y_0^2y_1 + 12y_0^2y_2 + 9y_0y_1^2 + 10y_1^2y_2 + 9y_0y_2^2 + 4y_1y_2^2 + 11y_0y_1y_2)$$
$$W_{15,24,36} = V(y_0^3 + 3y_1^3 + 4y_2^3 + 9y_0^2y_1 + 12y_0^2y_2 + 9y_0y_1^2 + 10y_1^2y_2 + 9y_0y_2^2 + 9y_1y_2^2 + y_0y_1y_2)$$
$$W_{15,26,34} = V(9y_1^3 + y_0^2y_1 + y_1y_2^2 + 11y_0y_1y_2)$$
$$W_{16,23,45} = V(y_0^3 + 3y_1^3 + 9y_2^3 + 9y_0^2y_1 + y_0^2y_2 + 9y_0y_1^2 + 3y_1^2y_2 + 9y_0y_2^2 + 9y_1y_2^2 + 12y_0y_1y_2)$$
$$W_{16,24,35} = V(y_0^3 + 10y_1^3 + 9y_2^3 + 4y_0^2y_1 + y_0^2y_2 + 9y_0y_1^2 + 3y_1^2y_2 + 9y_0y_2^2 + 4y_1y_2^2 + 2y_0y_1y_2)$$
$$W_{16,25,34} = V(9y_1^3 + y_0^2y_1 + y_1y_2^2 + 2y_0y_1y_2)$$

Table 4.4: The 15 cubic curves of the form $W_{ij,kl,mn} = Q_iQ_j \cdot Q_kQ_l \cdot Q_mQ_n = \Phi(\pi_{ij,kl,mn})$

Step 2:

Choose four base cubic curves $W_{23}, W_{31}, W_{32}, W_{13}$ through S:

$$W_{23} = V(y_0^3 + 5y_1^3 + 2y_2^3 + 2y_0^2y_1 + 6y_0^2y_2 + 9y_0y_1^2 + 5y_1^2y_2 + 9y_0y_2^2 + 2y_1y_2^2 + 8y_0y_1y_2)$$
$$W_{31} = V(y_0^3 + 7y_1^3 + 5y_2^3 + 8y_0^2y_1 + 2y_0^2y_2 + 9y_0y_1^2 + 6y_1^2y_2 + 9y_0y_2^2 + 8y_1y_2^2 + 7y_0y_1y_2)$$
$$W_{32} = V(y_0^3 + 6y_1^3 + 5y_2^3 + 5y_0^2y_1 + 2y_0^2y_2 + 9y_0y_1^2 + 6y_1^2y_2 + 9y_0y_2^2 + 5y_1y_2^2 + 6y_0y_1y_2)$$
$$W_{13} = V(y_0^3 + 8y_1^3 + 2y_2^3 + 11y_0^2y_1 + 6y_0^2y_2 + 9y_0y_1^2 + 5y_1^2y_2 + 9y_0y_2^2 + 11y_1y_2^2 + 5y_0y_1y_2)$$

The corresponding tritangent planes, called base tritangent planes, are chosen as

$$\pi_{23} = V(x_0), \quad \pi_{31} = V(x_1), \quad \pi_{32} = V(x_2), \quad \pi_{13} = V(x_3).$$

Using the linear system of cubic curves, every tritangent plane can be written as a linear combination of x_0, x_1, x_2, x_3. For instance, the plane cubic curve W_{12} is a linear combination of the 4 base cubic curves:

$$\begin{aligned}
W_{12} &= V(9y_2^3 + y_0^2y_2 + 3y_1^2y_2 + 3y_0y_1y_2) \\
&= aW_{23} + bW_{31} + cW_{32} + dW_{13} \\
&= V\big((a+b+c+d)y_0^3 + (5a+7b+6c+8d)y_1^3 + (2a+5b+5c+2d)y_2^3 \\
&\quad + (2a+8b+5c+11d)y_0^2y_1 + (6a+2b+2c+6d)y_0^2y_2 + (9a+9b+9c+9d)y_0y_1^2 \\
&\quad + (5a+6b+6c+5d)y_1^2y_2 + (9a+9b+9c+9d)y_0y_2^2 + (2a+8b+5c+11d)y_1y_2^2 \\
&\quad + (8a+7b+6c+5d)y_0y_1y_2\big)
\end{aligned}$$

Hence $a = 7, b = 5, c = 0, d = 1$.

Therefore, the corresponding tritangent plane π_{12} can be written as

$$\begin{aligned}
\pi_{12} &= 7\pi_{23} + 5\pi_{31} + \pi_{13} \\
&= V(7x_0 + 5x_1 + x_3).
\end{aligned}$$

The resulting equations of all tritangent planes are given in Table 4.5. The 27 lines are listed in Table 4.6. Also, all 120 Steiner trihedral pairs can be observed by considering the possible arrangements of tritangent planes.

$$
\begin{aligned}
\pi_{12} &= V(7x_0 + 5x_1 + x_3) \\
\pi_{21} &= V(2x_0 + 10x_2 + x_3) \\
\pi_{13} &= V(x_3) \\
\pi_{31} &= V(x_1) \\
\pi_{14} &= V(x_0 + 10x_1 + x_3) \\
\pi_{41} &= V(2x_0 + 10x_1 + 10x_2 + x_3) \\
\pi_{15} &= V(10x_0 + 9x_1 + x_3) \\
\pi_{51} &= V(2x_0 + 9x_1 + 10x_2 + x_3) \\
\pi_{16} &= V(4x_0 + x_1 + x_3) \\
\pi_{61} &= V(2x_0 + x_1 + 10x_2 + x_3) \\
\pi_{23} &= V(x_0) \\
\pi_{32} &= V(x_2) \\
\pi_{24} &= V(x_0 + 10x_2 + x_3) \\
\pi_{42} &= V(7x_0 + 5x_1 + 5x_2 + x_3) \\
\pi_{25} &= V(10x_0 + 10x_2 + x_3) \\
\pi_{52} &= V(7x_0 + 5x_1 + 7x_2 + x_3) \\
\pi_{26} &= V(4x_0 + 10x_2 + x_3) \\
\pi_{62} &= V(7x_0 + 5x_1 + 11x_2 + x_3) \\
\pi_{34} &= V(12x_1 + x_2) \\
\pi_{43} &= V(12x_0 + x_3) \\
\pi_{35} &= V(3x_1 + x_2) \\
\pi_{53} &= V(10x_0 + x_3)
\end{aligned}
$$

$$
\begin{aligned}
\pi_{36} &= V(9x_1 + x_2) \\
\pi_{63} &= V(4x_0 + x_3) \\
\pi_{45} &= V(10x_0 + 2x_1 + 2x_2 + x_3) \\
\pi_{54} &= V(x_0 + 2x_1 + 8x_2 + x_3) \\
\pi_{46} &= V(4x_0 + 8x_1 + 8x_2 + x_3) \\
\pi_{64} &= V(x_0 + 8x_1 + 2x_2 + x_3) \\
\pi_{56} &= V(4x_0 + 10x_1 + x_2 + x_3) \\
\pi_{65} &= V(10x_0 + 10x_1 + 9x_2 + x_3) \\
\pi_{12,34,56} &= V(12x_0 + 10x_1 + 3x_2 + x_3) \\
\pi_{12,35,46} &= V(10x_0 + 8x_1 + 7x_2 + x_3) \\
\pi_{12,36,45} &= V(4x_0 + 2x_1 + 6x_2 + x_3) \\
\pi_{13,24,56} &= V(10x_1 + x_3) \\
\pi_{13,25,46} &= V(8x_1 + x_3) \\
\pi_{13,26,45} &= V(2x_1 + x_3) \\
\pi_{14,23,56} &= V(4x_0 + x_2) \\
\pi_{14,25,36} &= V(4x_0 + 10x_1 + 4x_2 + x_3) \\
\pi_{14,26,35} &= V(10x_0 + 10x_1 + 12x_2 + x_3) \\
\pi_{15,23,46} &= V(7x_0 + x_2) \\
\pi_{15,24,36} &= V(4x_0 + 9x_1 + x_2 + x_3) \\
\pi_{15,26,34} &= V(12x_0 + 9x_1 + 4x_2 + x_3) \\
\pi_{16,23,45} &= V(5x_0 + x_2) \\
\pi_{16,24,35} &= V(10x_0 + x_1 + 9x_2 + x_3) \\
\pi_{16,25,34} &= V(12x_0 + x_1 + 12x_2 + x_3)
\end{aligned}
$$

Table 4.5: The tritangent planes of the associated cubic surface with S

$$a_1 = \mathbf{L}\begin{bmatrix} 1 & 9 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad a_2 = \mathbf{L}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad a_3 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$a_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 12 & 0 \end{bmatrix} \quad a_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 6 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix} \quad a_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 8 & 9 \\ 0 & 1 & 9 & 0 \end{bmatrix}$$

$$b_1 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad b_2 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 8 \end{bmatrix} \quad b_3 = \mathbf{L}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$b_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad b_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 10 & 4 \end{bmatrix} \quad b_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 4 & 12 \end{bmatrix}$$

$$c_{12} = \mathbf{L}\begin{bmatrix} 1 & 0 & 7 & 6 \\ 0 & 1 & 7 & 8 \end{bmatrix} \quad c_{13} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad c_{14} = \mathbf{L}\begin{bmatrix} 1 & 0 & 9 & 12 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

$$c_{15} = \mathbf{L}\begin{bmatrix} 1 & 0 & 6 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix} \quad c_{16} = \mathbf{L}\begin{bmatrix} 1 & 0 & 8 & 9 \\ 0 & 1 & 0 & 12 \end{bmatrix} \quad c_{23} = \mathbf{L}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c_{24} = \mathbf{L}\begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad c_{25} = \mathbf{L}\begin{bmatrix} 1 & 0 & 12 & 0 \\ 0 & 1 & 6 & 5 \end{bmatrix} \quad c_{26} = \mathbf{L}\begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & 8 & 11 \end{bmatrix}$$

$$c_{34} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad c_{35} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 10 & 0 \end{bmatrix} \quad c_{36} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

$$c_{45} = \mathbf{L}\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 0 & 11 \end{bmatrix} \quad c_{46} = \mathbf{L}\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 0 & 5 \end{bmatrix} \quad c_{56} = \mathbf{L}\begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Table 4.6: The 27 lines of the associated cubic surface with S

Step 3:

The chosen abstract trihedral pair is S_{123} :

$$\begin{array}{ccc|c} c_{23} & b_3 & a_2 & \pi_{23} \\ a_3 & c_{13} & b_1 & \pi_{31} \\ b_2 & a_1 & c_{12} & \pi_{12} \\ \hline \pi_{32} & \pi_{13} & \pi_{21} & \end{array}$$

From Table 4.5, the six plane equations are:

$$\pi_{23} = V(x_0), \qquad \pi_{31} = V(x_1), \qquad \pi_{12} = V(7x_0 + 5x_1 + x_3),$$

$$\pi_{32} = V(x_2), \qquad \pi_{13} = V(x_3), \qquad \pi_{21} = V(2x_0 + 10x_2 + x_3).$$

The equation of the corresponding cubic surface is

$$\mathcal{F}_{a,b} = V\big(ax_0x_1(7x_0 + 5x_1 + x_3) + bx_2x_3(2x_0 + 10x_2 + x_3)\big)$$

$$
\begin{aligned}
\mathcal{F}_{1,0} &= V(7x_0^2x_1 + 5x_0x_1^2 + x_0x_1x_3)\\
\mathcal{F}_{0,1} &= V(5x_2^2x_3 + 7x_2x_3^2 + x_0x_2x_3)\\
\mathcal{F}_{1,1} &= V(10x_0^2x_1 + 9x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 7x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,2} &= V(5x_0^2x_1 + 11x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 10x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,3} &= V(12x_0^2x_1 + 3x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 11x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,4} &= V(9x_0^2x_1 + 12x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 5x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,5} &= V(2x_0^2x_1 + 7x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 4x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,6} &= V(6x_0^2x_1 + 8x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 12x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,7} &= V(7x_0^2x_1 + 5x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,8} &= V(11x_0^2x_1 + 6x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 9x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,9} &= V(4x_0^2x_1 + x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 8x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,10} &= V(x_0^2x_1 + 10x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 2x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,11} &= V(8x_0^2x_1 + 2x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 3x_0x_1x_3 + x_0x_2x_3)\\
\mathcal{F}_{1,12} &= V(3x_0^2x_1 + 4x_0x_1^2 + 5x_2^2x_3 + 7x_2x_3^2 + 6x_0x_1x_3 + x_0x_2x_3)
\end{aligned}
$$

Table 4.7: The $q+1$ cubic surfaces arising from S_{123}

where a, b are homogeneous coordinates for $\mathrm{PG}(1,q)$. The $q+1$ equations on the line are listed in Table 4.7.

Step 4:

The point $\mathbf{P}(1,1,1)$ in $\mathrm{PG}(2,q)$ is not on any of the base cubic curves $W_{23}, W_{31}, W_{32}, W_{13}$. By evaluating the base cubic curves at this point, a point $\mathbf{P}(1,12,12,1)$ on $\mathcal{F}$ is found. Therefore $\lambda = 6$ and the equation of the surface is:

$$7x_0^2x_1 + 5x_0x_1^2 + 8x_2^2x_3 + 6x_2x_3^2 + x_0x_1x_3 + 12x_0x_2x_3 = 0.$$

This is $\mathcal{F}_{1,6}$ from Table 4.7. Let us denote this cubic surface as $\mathcal{F}_S$. It is isomorphic to the cubic surface $\mathcal{F}$ in Example 4.2.3:

$$
\begin{aligned}
\mathcal{F} &= V\big(12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3\big),\\
\mathcal{F}_S &= V\big(7x_0^2x_1 + 5x_0x_1^2 + 8x_2^2x_3 + 6x_2x_3^2 + x_0x_1x_3 + 12x_0x_2x_3\big).
\end{aligned}
$$

The projectivity associated to

$$R = \begin{bmatrix} 8 & 10 & 0 & 0 \\ 8 & 5 & 4 & 5 \\ 12 & 12 & 0 & 0 \\ 4 & 5 & 4 & 9 \end{bmatrix}$$

takes $\mathcal{F}$ to $\mathcal{F}_S$. We get the equation of $\mathcal{F}_S$ in terms of x_0', x_1', x_2', x_3' by substituting

$$x_0 = 8x_0' + 8x_1' + 12x_2' + 4x_3'$$
$$x_1 = 10x_0' + 5x_1' + 12x_2' + 5x_3'$$
$$x_2 = 4x_1' + 4x_3'$$
$$x_3 = 5x_1' + 9x_3'$$

in the equation of $\mathcal{F}$.

Chapter 5

The Classification Problem for Cubic Surfaces with 27 Lines over a Finite Field

5.1 General theory of classification

Let G be a group and S be a set on which G acts. We assume that both G and S are finite. Classifying S under G means determining a list $s_1, s_2, \ldots, s_h$ of elements in S such that each orbit of G on S is represented by exactly one element in the list. The set $\mathrm{T} = \{s_1, s_2, \ldots, s_h\}$ is a *transversal* of the G-orbits on S. The elements s_i are the *orbit representatives.*

The condition that T intersects each G-orbit in exactly one element means that we have a partition of S into pairwise disjoint orbits:

$$S = \bigcup_{s \in \mathrm{T}} \mathrm{Orb}_G(s).$$

The particular choice of s in its orbit is irrelevant. The problem of determining a transversal of the G-orbits on S is the classification problem (for S under G).

Several related problems are of interest. First, it is desirable to compute stabiliser groups for each of the elements $s_i \in \mathrm{T}$. Secondly, for an element x in S, we would like to determine the G-orbit containing x. This could mean finding the orbit representative

$s_i \in \mathrm{T}$ with $s_i \sim_G x$. Finally, the isomorphism problem is of interest. Given two elements $x, y \in S$, we would like to know whether $x \sim_G y$ and if so, we would like to find an element $g \in G$ with $xg = y$.

The problem at hand is of course that of classifying the orbits of G on S where

$$G = \mathrm{PGL}(4, q),$$

$$S = \{\text{cubic surfaces in } \mathrm{PG}(3, q) \text{ with exactly 27 lines}\}.$$

Because the order of the group G and the size of the set S grow rapidly as functions of q, efficient algorithms for solving the classification problem are required. One of the main ideas is to consider a set of smaller and related objects which are easier to classify. The classification of cubic surfaces is facilitated by reducing to the smaller objects, classifying the smaller objects and then lifting the classification of smaller objects to the classification of objects. At times, the smaller objects are in turn reduced to even smaller objects, which are even easier to classify. In order to facilitate such a reduction, the following lemma from [6] can be used. Before we state the result, some notation has to be introduced.

Let G be the group which acts on the finite sets $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{R}$ be a relation between $\mathcal{A}$ and $\mathcal{B}$.

$$\mathcal{R} \subseteq \{(a, b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Also, G acts on $\mathcal{R}$. $\mathcal{R}$ is G-invariant, so

$$(a, b) \in \mathcal{R} \Rightarrow (ag, bg) \in \mathcal{R} \text{ for all } g \in G.$$

A pair $(a, b) \in \mathcal{R}$ is called a flag. For $a \in \mathcal{A}$,

$$\mathrm{Up}(a) = \{(a, b) \mid b \in \mathcal{B}, (a, b) \in \mathcal{R}\}$$

and for $b \in \mathcal{B}$,

$$\mathrm{Down}(b) = \{(a, b) \mid a \in \mathcal{A}, (a, b) \in \mathcal{R}\}.$$

Let $P_1, \ldots, P_m$ be the representatives for the orbits of G on $\mathcal{A}$; so

$$\mathcal{A} = \bigcup_{i=1,\ldots,m} \mathrm{Orb}_G(P_i).$$

Let $Q_1, \ldots, Q_n$ be the representatives for orbits of G on $\mathcal{B}$; so

$$\mathcal{B} = \bigcup_{j=1,\ldots,n} \mathrm{Orb}_G(Q_j).$$

Let $\mathcal{T}_{i,r}$ be the orbits of $\mathrm{Stab}_G(P_i)$ on $\mathrm{Up}(P_i)$ and let $t_{i,r} = (P_i, b_{ir})$ be the orbit representatives of $\mathcal{T}_{i,r}$; so

$$\mathrm{Up}(P_i) = \bigcup_{r=1,\ldots,r_i} \mathcal{T}_{i,r},$$

where $r = 1, \ldots, r_i$ and $\mathcal{T}_{i,r} = \mathrm{Orb}_{\mathrm{Stab}_G(P_i)}(t_{i,r})$.

Let $\mathcal{S}_{j,l}$ be the orbits of $\mathrm{Stab}_G(Q_j)$ on $\mathrm{Down}(Q_j)$ and let $s_{j,l}$ be the orbit representatives of $\mathcal{S}_{j,l}$; so

$$\mathrm{Down}(Q_j) = \bigcup_{l=1,\ldots,l_j} \mathcal{S}_{j,l}$$

where $l = 1, \ldots, l_j$ and $\mathcal{S}_{j,l} = \mathrm{Orb}_{\mathrm{Stab}_G(Q_j)}(s_{j,l})$.

Lemma 5.1.1 ([6])**.** (1) *There is a canonical bijection between the set of orbits*

$$\{\mathcal{T}_{i,r} \mid i = 1, \ldots, m, \quad r = 1, \ldots, r_i\}$$

and the set of orbits

$$\{\mathcal{S}_{j,l} \mid j = 1, \ldots, n, \quad l = 1, \ldots, l_j\}.$$

(2) $\displaystyle\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} l_j.$

(3) *If $\mathcal{T}_{i,r}$ and $\mathcal{S}_{j,l}$ are corresponding orbits under the canonical bijection from* 1*, then*

$$|\mathrm{Orb}_G(P_i)| \cdot |\mathcal{T}_{i,r}| = |\mathrm{Orb}_G(Q_j)| \cdot |\mathcal{S}_{j,l}|.$$

Proof Let $\mathcal{R}_1, \ldots, \mathcal{R}_\delta$ be the orbits of G on the set $\mathcal{R}$. We claim that for each $\mathcal{R}_k$, $k = 1, \ldots, \delta$, there exists exactly one $\mathcal{T}_{i,r}$ such that $\mathcal{T}_{i,r} \subset \mathcal{R}_k$. For $i = 1, 2$ let Π_i denote the projection onto the i^{th} coordinate.

If $\mathcal{T}_{i,r} \subset \mathcal{R}_k$ and $\mathcal{T}_{u,v} \subset \mathcal{R}_k$ (with $(i,r) \neq (u,v)$) then $t_{i,r} \in \mathcal{R}_k$ and $t_{u,v} \in \mathcal{R}_k$, hence there exists an element $g \in G$ with $t_{i,r}g = t_{u,v}$ and hence

$$P_i g = \Pi_1(t_{i,r}g) = \Pi_1(t_{u,v}) = P_u,$$

and so $P_i \sim_G P_u$, which means that $i = u$ and hence $g \in \mathrm{Stab}_G(P_i)$. Therefore, $r \neq v$. But

$$b_{ir}g = \Pi_2(t_{i,r}g) = \Pi_2(t_{u,v}) = b_{iv}$$

which means that

$$b_{ir} \sim_{\mathrm{Stab}_G(P_i)} b_{iv},$$

which is impossible because $r \neq v$.

Likewise, one shows that for each $\mathcal{R}_k$, $k = 1, \ldots, \delta$, there exists exactly one $\mathcal{S}_{j,l}$ such that $\mathcal{S}_{j,l} \subset \mathcal{R}_k$. Thus we have bijections

$$(i,r) \longleftrightarrow k \longleftrightarrow (j,l).$$

Therefore, if (i,r) and (j,l) correspond in this way, then there exists a group element $g \in G$ with

$$t_{i,r}g = s_{j,l}.$$

□

5.2 Application of the theory to the classification of cubic surfaces over a finite field

The most convenient structures associated with cubic surfaces with 27 lines are the following:

(a) a double-six,

(b) a 6-arc in a plane but not on a conic.

Theorem 3.2.2 shows that a double-six determines a unique cubic surface with 27 lines.

Theorem 3.2.5 and 3.1.2 show that a 6-arc not on a conic determines a cubic surface with 27 lines.

In [10], the double-sixes are used as substructure. In [9], a GAP program [18] is used to compute the projectivities between two sets of 6 points in $\mathrm{PG}(3, q)$. This is helpful to compute the stabiliser of a chosen special trihedral pair and the stabiliser of the associated cubic surface with 27 lines. Special trihedral pairs exist for all $q \leq 13$, so this method will work for all fields of order $q \leq 13$. However, for larger q, there are cubic surfaces with less than 3 Eckardt points, and so they do not have any special trihedral pair. For those trihedral pairs, the GAP program would compute projectivities which do not preserve the partition of the 6 planes into two sets of size three, and hence be unable to compute the stabiliser of the non-special trihedral pair. For example, there is a cubic surface with twenty-seven lines with only one Eckardt point in $\mathrm{PG}(3, 17)$, [10]. To fix this problem, we decided to use Lemma 5.1.1 to classify the trihedral pairs, using an implementation in Orbiter [7]. This generalizes the algorithm described in [9] and eliminates the need to consider special trihedral pairs. Summarizing, we use Orbiter to perform the following classification tasks for us:

(i) The classification of 6-arcs not on a conic in $\mathrm{PG}(2, q)$,

(ii) The classification of double-triplets in $\mathrm{PG}(3, q)$. For the definition of double-triplet, see Section 5.2.2.

A summary of the projectively distinct cubic surfaces with 27 lines in $\mathrm{PG}(3, q)$, $q \leq 11$, are presented in Table 5.1. In Table 5.1, $G(\mathcal{F}_q^i)$ is the group of projectivities which fixes the cubic surface $\mathcal{F}_q^i$ and e_3 is the total number of Eckardt points of $\mathcal{F}_q^i$.

5.2.1 The classification of 6-arcs not on a conic in $\mathrm{PG}(2, q)$

The projectively distinct 6-arcs not on a conic over the field for $q = 13, 17, 19$ are given in Table 5.2, Table 5.3 and Table 5.4. These results are from Ali [1], Al-Seraji [2] and Al-Zangana [3], and they have been verified using Orbiter, [7].

q	$\mathcal{F}_q^i$	e_3	$\lvert G(\mathcal{F}_q^i)\rvert$	Common Name	Classified by
4	$\mathcal{F}_4$	45	25 920		Hirschfeld
7	$\mathcal{F}_7$	18	648	Equianharmonic	Hirschfeld
8	$\mathcal{F}_8$	13	192		Hirschfeld
9	$\mathcal{F}_9^0$	10	120	Diagonal	Hirschfeld
9	$\mathcal{F}_9^1$	9	216		Hirschfeld
11	$\mathcal{F}_{11}^0$	6	24		Sadeh
11	$\mathcal{F}_{11}^1$	10	120	Diagonal	Sadeh

Table 5.1: Cubic Surfaces for $q \leq 11$

$s_0 = A_1 \cup \{\mathbf{P}(2,3,1)\}$	$s_7 = A_1 \cup \{\mathbf{P}(2,4,1)\}$	$s_{14} = A_1 \cup \{\mathbf{P}(5,6,1)\}$
$s_1 = A_1 \cup \{\mathbf{P}(4,3,1)\}$	$s_8 = A_1 \cup \{\mathbf{P}(9,4,1)\}$	$s_{15} = A_1 \cup \{\mathbf{P}(7,6,1)\}$
$s_2 = A_1 \cup \{\mathbf{P}(6,3,1)\}$	$s_9 = A_1 \cup \{\mathbf{P}(10,4,1)\}$	$s_{16} = A_1 \cup \{\mathbf{P}(8,7,1)\}$
$s_3 = A_1 \cup \{\mathbf{P}(7,3,1)\}$	$s_{10} = A_1 \cup \{\mathbf{P}(4,5,1)\}$	$s_{17} = A_1 \cup \{\mathbf{P}(12,7,1)\}$
$s_4 = A_1 \cup \{\mathbf{P}(8,3,1)\}$	$s_{11} = A_1 \cup \{\mathbf{P}(6,5,1)\}$	$s_{18} = A_2 \cup \{\mathbf{P}(2,6,1)\}$
$s_5 = A_1 \cup \{\mathbf{P}(10,3,1)\}$	$s_{12} = A_1 \cup \{\mathbf{P}(8,5,1)\}$	$s_{19} = A_3 \cup \{\mathbf{P}(5,4,1)\}$
$s_6 = A_1 \cup \{\mathbf{P}(12,3,1)\}$	$s_{13} = A_1 \cup \{\mathbf{P}(12,5,1)\}$	$s_{20} = A_3 \cup \{\mathbf{P}(6,4,1)\}$

$$\begin{aligned}
A_1 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(3,2,1)\},\\
A_2 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(6,2,1)\},\\
A_3 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(4,3,1)\}.
\end{aligned}$$

Table 5.2: The 6-arcs not on a conic $\mathcal{S}$ in $\mathrm{PG}(2,13)$

$s_0 = A_1 \cup \{\mathbf{P}(2,3,1)\}$	$s_{22} = A_1 \cup \{\mathbf{P}(14,5,1)\}$	$s_{43} = A_1 \cup \{\mathbf{P}(2,11,1)\}$
$s_1 = A_1 \cup \{\mathbf{P}(4,3,1)\}$	$s_{23} = A_1 \cup \{\mathbf{P}(4,6,1)\}$	$s_{44} = A_1 \cup \{\mathbf{P}(12,11,1)\}$
$s_2 = A_1 \cup \{\mathbf{P}(6,3,1)\}$	$s_{24} = A_1 \cup \{\mathbf{P}(5,6,1)\}$	$s_{45} = A_1 \cup \{\mathbf{P}(15,11,1)\}$
$s_3 = A_1 \cup \{\mathbf{P}(7,3,1)\}$	$s_{25} = A_1 \cup \{\mathbf{P}(7,6,1)\}$	$s_{46} = A_1 \cup \{\mathbf{P}(5,12,1)\}$
$s_4 = A_1 \cup \{\mathbf{P}(8,3,1)\}$	$s_{26} = A_1 \cup \{\mathbf{P}(10,6,1)\}$	$s_{47} = A_1 \cup \{\mathbf{P}(13,12,1)\}$
$s_5 = A_1 \cup \{\mathbf{P}(10,3,1)\}$	$s_{27} = A_1 \cup \{\mathbf{P}(12,6,1)\}$	$s_{48} = A_1 \cup \{\mathbf{P}(10,13,1)\}$
$s_6 = A_1 \cup \{\mathbf{P}(11,3,1)\}$	$s_{28} = A_1 \cup \{\mathbf{P}(13,6,1)\}$	$s_{49} = A_1 \cup \{\mathbf{P}(15,14,1)\}$
$s_7 = A_1 \cup \{\mathbf{P}(12,3,1)\}$	$s_{29} = A_1 \cup \{\mathbf{P}(14,6,1)\}$	$s_{50} = A_2 \cup \{\mathbf{P}(2,5,1)\}$
$s_8 = A_1 \cup \{\mathbf{P}(14,3,1)\}$	$s_{30} = A_1 \cup \{\mathbf{P}(15,6,1)\}$	$s_{51} = A_2 \cup \{\mathbf{P}(6,5,1)\}$
$s_9 = A_1 \cup \{\mathbf{P}(15,3,1)\}$	$s_{31} = A_1 \cup \{\mathbf{P}(16,6,1)\}$	$s_{52} = A_2 \cup \{\mathbf{P}(7,5,1)\}$
$s_{10} = A_1 \cup \{\mathbf{P}(16,3,1)\}$	$s_{32} = A_1 \cup \{\mathbf{P}(5,7,1)\}$	$s_{53} = A_2 \cup \{\mathbf{P}(8,5,1)\}$
$s_{11} = A_1 \cup \{\mathbf{P}(2,4,1)\}$	$s_{33} = A_1 \cup \{\mathbf{P}(8,7,1)\}$	$s_{54} = A_2 \cup \{\mathbf{P}(13,6,1)\}$
$s_{12} = A_1 \cup \{\mathbf{P}(8,4,1)\}$	$s_{34} = A_1 \cup \{\mathbf{P}(2,8,1)\}$	$s_{55} = A_2 \cup \{\mathbf{P}(7,14,1)\}$
$s_{13} = A_1 \cup \{\mathbf{P}(9,4,1)\}$	$s_{35} = A_1 \cup \{\mathbf{P}(4,8,1)\}$	$s_{56} = A_3 \cup \{\mathbf{P}(5,3,1)\}$
$s_{14} = A_1 \cup \{\mathbf{P}(11,4,1)\}$	$s_{36} = A_1 \cup \{\mathbf{P}(7,8,1)\}$	$s_{57} = A_3 \cup \{\mathbf{P}(2,6,1)\}$
$s_{15} = A_1 \cup \{\mathbf{P}(12,4,1)\}$	$s_{37} = A_1 \cup \{\mathbf{P}(9,8,1)\}$	$s_{58} = A_3 \cup \{\mathbf{P}(10,6,1)\}$
$s_{16} = A_1 \cup \{\mathbf{P}(13,4,1)\}$	$s_{38} = A_1 \cup \{\mathbf{P}(10,8,1)\}$	$s_{59} = A_3 \cup \{\mathbf{P}(13,6,1)\}$
$s_{17} = A_1 \cup \{\mathbf{P}(4,5,1)\}$	$s_{39} = A_1 \cup \{\mathbf{P}(2,9,1)\}$	$s_{60} = A_3 \cup \{\mathbf{P}(14,6,1)\}$
$s_{18} = A_1 \cup \{\mathbf{P}(6,5,1)\}$	$s_{40} = A_1 \cup \{\mathbf{P}(7,9,1)\}$	$s_{61} = A_3 \cup \{\mathbf{P}(2,11,1)\}$
$s_{19} = A_1 \cup \{\mathbf{P}(8,5,1)\}$	$s_{41} = A_1 \cup \{\mathbf{P}(10,9,1)\}$	$s_{62} = A_3 \cup \{\mathbf{P}(3,11,1)\}$
$s_{20} = A_1 \cup \{\mathbf{P}(10,5,1)\}$	$s_{42} = A_1 \cup \{\mathbf{P}(14,10,1)\}$	$s_{63} = A_4 \cup \{\mathbf{P}(15,7,1)\}$
$s_{21} = A_1 \cup \{\mathbf{P}(11,5,1)\}$		

$$\begin{aligned}
A_1 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(3,2,1)\},\\
A_2 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(5,2,1)\},\\
A_3 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(6,2,1)\},\\
A_4 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(7,3,1)\}
\end{aligned}$$

Table 5.3: The 6-arcs not on a conic $\mathcal{S}$ in $\mathrm{PG}(2,17)$

$s_0 = A_1 \cup \{\mathbf{P}(2,3,1)\}$	$s_{35} = A_1 \cup \{\mathbf{P}(15,6,1)\}$	$s_{70} = A_1 \cup \{\mathbf{P}(17,14,1)\}$
$s_1 = A_1 \cup \{\mathbf{P}(4,3,1)\}$	$s_{36} = A_1 \cup \{\mathbf{P}(16,6,1)\}$	$s_{71} = A_1 \cup \{\mathbf{P}(2,15,1)\}$
$s_2 = A_1 \cup \{\mathbf{P}(6,3,1)\}$	$s_{37} = A_1 \cup \{\mathbf{P}(17,6,1)\}$	$s_{72} = A_1 \cup \{\mathbf{P}(17,16,1)\}$
$s_3 = A_1 \cup \{\mathbf{P}(7,3,1)\}$	$s_{38} = A_1 \cup \{\mathbf{P}(2,7,1)\}$	$s_{73} = A_2 \cup \{\mathbf{P}(14,3,1)\}$
$s_4 = A_1 \cup \{\mathbf{P}(8,3,1)\}$	$s_{39} = A_1 \cup \{\mathbf{P}(4,7,1)\}$	$s_{74} = A_2 \cup \{\mathbf{P}(11,4,1)\}$
$s_5 = A_1 \cup \{\mathbf{P}(10,3,1)\}$	$s_{40} = A_1 \cup \{\mathbf{P}(5,7,1)\}$	$s_{75} = A_2 \cup \{\mathbf{P}(2,5,1)\}$
$s_6 = A_1 \cup \{\mathbf{P}(11,3,1)\}$	$s_{41} = A_1 \cup \{\mathbf{P}(8,7,1)\}$	$s_{76} = A_2 \cup \{\mathbf{P}(6,5,1)\}$
$s_7 = A_1 \cup \{\mathbf{P}(12,3,1)\}$	$s_{42} = A_1 \cup \{\mathbf{P}(11,7,1)\}$	$s_{77} = A_2 \cup \{\mathbf{P}(11,5,1)\}$
$s_8 = A_1 \cup \{\mathbf{P}(13,3,1)\}$	$s_{43} = A_1 \cup \{\mathbf{P}(14,7,1)\}$	$s_{78} = A_2 \cup \{\mathbf{P}(14,5,1)\}$
$s_9 = A_1 \cup \{\mathbf{P}(15,3,1)\}$	$s_{44} = A_1 \cup \{\mathbf{P}(15,7,1)\}$	$s_{79} = A_2 \cup \{\mathbf{P}(16,5,1)\}$
$s_{10} = A_1 \cup \{\mathbf{P}(16,3,1)\}$	$s_{45} = A_1 \cup \{\mathbf{P}(17,7,1)\}$	$s_{80} = A_2 \cup \{\mathbf{P}(7,6,1)\}$
$s_{11} = A_1 \cup \{\mathbf{P}(17,3,1)\}$	$s_{46} = A_1 \cup \{\mathbf{P}(7,8,1)\}$	$s_{81} = A_2 \cup \{\mathbf{P}(12,6,1)\}$
$s_{12} = A_1 \cup \{\mathbf{P}(18,3,1)\}$	$s_{47} = A_1 \cup \{\mathbf{P}(9,8,1)\}$	$s_{82} = A_2 \cup \{\mathbf{P}(18,7,1)\}$
$s_{13} = A_1 \cup \{\mathbf{P}(2,4,1)\}$	$s_{48} = A_1 \cup \{\mathbf{P}(10,8,1)\}$	$s_{83} = A_2 \cup \{\mathbf{P}(2,8,1)\}$
$s_{14} = A_1 \cup \{\mathbf{P}(8,4,1)\}$	$s_{49} = A_1 \cup \{\mathbf{P}(11,8,1)\}$	$s_{84} = A_2 \cup \{\mathbf{P}(4,8,1)\}$
$s_{15} = A_1 \cup \{\mathbf{P}(9,4,1)\}$	$s_{50} = A_1 \cup \{\mathbf{P}(14,8,1)\}$	$s_{85} = A_2 \cup \{\mathbf{P}(12,8,1)\}$
$s_{16} = A_1 \cup \{\mathbf{P}(10,4,1)\}$	$s_{51} = A_1 \cup \{\mathbf{P}(16,8,1)\}$	$s_{86} = A_2 \cup \{\mathbf{P}(13,8,1)\}$
$s_{17} = A_1 \cup \{\mathbf{P}(12,4,1)\}$	$s_{52} = A_1 \cup \{\mathbf{P}(18,8,1)\}$	$s_{87} = A_2 \cup \{\mathbf{P}(16,8,1)\}$
$s_{18} = A_1 \cup \{\mathbf{P}(13,4,1)\}$	$s_{53} = A_1 \cup \{\mathbf{P}(7,9,1)\}$	$s_{88} = A_2 \cup \{\mathbf{P}(17,8,1)\}$
$s_{19} = A_1 \cup \{\mathbf{P}(16,4,1)\}$	$s_{54} = A_1 \cup \{\mathbf{P}(8,9,1)\}$	$s_{89} = A_2 \cup \{\mathbf{P}(18,8,1)\}$
$s_{20} = A_1 \cup \{\mathbf{P}(2,5,1)\}$	$s_{55} = A_1 \cup \{\mathbf{P}(10,9,1)\}$	$s_{90} = A_2 \cup \{\mathbf{P}(2,9,1)\}$
$s_{21} = A_1 \cup \{\mathbf{P}(6,5,1)\}$	$s_{56} = A_1 \cup \{\mathbf{P}(12,9,1)\}$	$s_{91} = A_2 \cup \{\mathbf{P}(7,9,1)\}$
$s_{22} = A_1 \cup \{\mathbf{P}(8,5,1)\}$	$s_{57} = A_1 \cup \{\mathbf{P}(14,9,1)\}$	$s_{92} = A_2 \cup \{\mathbf{P}(8,13,1)\}$
$s_{23} = A_1 \cup \{\mathbf{P}(10,5,1)\}$	$s_{58} = A_1 \cup \{\mathbf{P}(15,9,1)\}$	$s_{93} = A_2 \cup \{\mathbf{P}(6,15,1)\}$
$s_{24} = A_1 \cup \{\mathbf{P}(11,5,1)\}$	$s_{59} = A_1 \cup \{\mathbf{P}(16,9,1)\}$	$s_{94} = A_2 \cup \{\mathbf{P}(13,15,1)\}$
$s_{25} = A_1 \cup \{\mathbf{P}(12,5,1)\}$	$s_{60} = A_1 \cup \{\mathbf{P}(6,10,1)\}$	$s_{95} = A_2 \cup \{\mathbf{P}(17,15,1)\}$
$s_{26} = A_1 \cup \{\mathbf{P}(13,5,1)\}$	$s_{61} = A_1 \cup \{\mathbf{P}(7,10,1)\}$	$s_{96} = A_3 \cup \{\mathbf{P}(5,4,1)\}$
$s_{27} = A_1 \cup \{\mathbf{P}(14,5,1)\}$	$s_{62} = A_1 \cup \{\mathbf{P}(9,10,1)\}$	$s_{97} = A_3 \cup \{\mathbf{P}(9,8,1)\}$
$s_{28} = A_1 \cup \{\mathbf{P}(15,5,1)\}$	$s_{63} = A_1 \cup \{\mathbf{P}(11,10,1)\}$	$s_{98} = A_3 \cup \{\mathbf{P}(16,9,1)\}$
$s_{29} = A_1 \cup \{\mathbf{P}(16,5,1)\}$	$s_{64} = A_1 \cup \{\mathbf{P}(18,10,1)\}$	$s_{99} = A_3 \cup \{\mathbf{P}(14,13,1)\}$
$s_{30} = A_1 \cup \{\mathbf{P}(2,6,1)\}$	$s_{65} = A_1 \cup \{\mathbf{P}(6,11,1)\}$	$s_{100} = A_3 \cup \{\mathbf{P}(16,15,1)\}$
$s_{31} = A_1 \cup \{\mathbf{P}(4,6,1)\}$	$s_{66} = A_1 \cup \{\mathbf{P}(6,12,1)\}$	$s_{101} = A_4 \cup \{\mathbf{P}(7,8,1)\}$
$s_{32} = A_1 \cup \{\mathbf{P}(5,6,1)\}$	$s_{67} = A_1 \cup \{\mathbf{P}(13,12,1)\}$	$s_{102} = A_4 \cup \{\mathbf{P}(12,17,1)\}$
$s_{33} = A_1 \cup \{\mathbf{P}(7,6,1)\}$	$s_{68} = A_1 \cup \{\mathbf{P}(14,12,1)\}$	$s_{103} = A_5 \cup \{\mathbf{P}(6,5,1)\}$
$s_{34} = A_1 \cup \{\mathbf{P}(8,6,1)\}$	$s_{69} = A_1 \cup \{\mathbf{P}(14,13,1)\}$	

$$
\begin{aligned}
A_1 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(3,2,1)\}, \\
A_2 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(5,2,1)\}, \\
A_3 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(4,3,1)\}, \\
A_4 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(8,3,1)\}, \\
A_5 &= \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(5,4,1)\}
\end{aligned}
$$

Table 5.4: The 6-arcs not on a conic $\mathcal{S}$ in $\mathrm{PG}(2,19)$

5.2.2 The classification of double-triplets in $\mathrm{PG}(3, q)$

Definition 5.2.1. Three planes in $\mathrm{PG}(3, q)$ are called a *triplet*.

Definition 5.2.2. A *double-triplet* is an unordered pair of triplets intersecting in nine distinct lines in $\mathrm{PG}(3, q)$.

Remark 5.2.3. Let $\mathcal{F}$ be a cubic surface with 27 lines. Any Steiner trihedral pair of $\mathcal{F}$ is a double-triplet.

The classification of double-triplets in $\mathrm{PG}(3, q)$ under the group $G = \mathrm{PGL}(4, q)$ is based on the classification of triplets. So, first the classification of triplets under the group $\mathrm{PGL}(4, q)$ will be given. Applying Lemma 5.1.1 yields the classification of double-triplets.

Dually, a triplet is a 3-subset of $\mathrm{PG}(3, q)$.

Theorem 5.2.4. *There are two projectively distinct* 3*-subsets of* $\mathrm{PG}(3, q)$*. Three points either are collinear or not. Dually, there are two triplets in* $\mathrm{PG}(3, q)$ *up to equivalence. Three planes intersect in a line or in a point.*

Some more information about these two orbits will be given next.

Let $\mathscr{P}$ be the set of points of $\mathrm{PG}(3, q)$ and let $\mathscr{P}^3$ be the set of 3-subsets of points. Let $\mathscr{U}$ be the set of 3-subsets of points which are collinear and let $\mathbb{U} = \{U_1, U_2, U_3\}$ be the orbit representative of $\mathscr{U}$ where

$$U_1 = \mathbf{P}(1, 0, 0, 0), U_2 = \mathbf{P}(0, 1, 0, 0) \text{ and } U_3 = \mathbf{P}(1, 1, 0, 0).$$

U_1, U_2, U_3 are collinear in the line

$$l = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The stabiliser of $\mathbb{U}$ is the group

$$\mathrm{Stab}_G(\mathbb{U}) = \left\{ \begin{bmatrix} C & 0 \\ B & A \end{bmatrix}, \text{ where } C \in \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \rangle, A \in \mathrm{GL}(2, q), \right.$$
$$\left. B \in \mathbf{F}_q^{2\times 2} \right\}$$

of order

$$|\mathrm{Stab}_G(\mathbb{U})| = 3!(q-1)(q+1)q(q-1)q^4.$$

Let U_1', U_2', U_3' be the points such that $U_3' = \lambda U_1' + \mu U_2'$ where $\lambda \neq 0$ and $\mu \neq 0$. There exists a transformation N in G such that $\mathbb{U}N = \{U_1', U_2', U_3'\}$. Namely,

$$N = \begin{bmatrix} \lambda U_1' \\ \mu U_2' \\ q_3 \\ q_4 \end{bmatrix}, \tag{5.1}$$

where q_4, q_3 are chosen to make N invertible.

The orbit of $\mathbb{U}$ is

$$\mathrm{Orb}_G(\mathbb{U}) = \{\{U_1', U_2', U_3'\} \in \mathscr{P}^3 \mid rk(\langle U_1', U_2', U_3'\rangle) = 2\}.$$

Let $\mathscr{V}$ be the set of 3-subsets which form a triangle and $\mathbb{V} = \{V_1, V_2, V_3\}$ be the orbit representative of $\mathscr{V}$ where

$$V_1 = \mathbf{P}(1,0,0,0), V_2 = \mathbf{P}(0,1,0,0) \text{ and } V_3 = \mathbf{P}(0,0,1,0).$$

The stabiliser of $\mathbb{V}$ is the group

$$\mathrm{Stab}_G(\mathbb{V}) = \left\{ \left[\begin{array}{ccc|c} & & & 0 \\ & N & & 0 \\ & & & 0 \\ \hline b_1 & b_2 & b_3 & a \end{array}\right], \text{ where } M = P \begin{bmatrix} \lambda \\ \mu \\ \varphi \end{bmatrix} \text{ with } P \text{ any } 3\times 3 \text{ permutation} \right.$$

$$\left. \text{matrix and } \lambda, \mu, \varphi, a \in \mathbf{F}_q \setminus \{0\} \text{ and } b_1, b_2, b_3 \in \mathbf{F}_q \right\}$$

of order

$$|\mathrm{Stab}_G(\mathbb{V})| = 3!(q-1)^3 q^3.$$

Let V_1', V_2', V_3' be any three non-collinear points. There exists a transformation N in G such that $\mathbb{V}N = \{V_1', V_2', V_3'\}$. Namely,

$$N = \begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ q_4 \end{bmatrix}, \tag{5.2}$$

where q_4 is linearly independent from V_1', V_2', V_3'.

The orbit of $\mathbb{V}$ is

$$\mathrm{Orb}_G(\mathbb{V}) = \{\{V_1', V_2', V_3'\} \in \mathscr{P}^3 \mid rk(\langle V_1', V_2', V_3'\rangle) = 3\}.$$

Lemma 5.2.5. *The number of collinear* 3*-subsets of points in* $\mathrm{PG}(3,q)$ *is*

$$|\mathscr{U}| = \binom{q+1}{3}(q^2+q+1)(q^2+1).$$

The number of non-collinear 3*-subsets of points in* $\mathrm{PG}(3,q)$ *is*

$$|\mathscr{V}| = \binom{q^3+q^2+q+1}{3} - \binom{q+1}{3}(q^2+q+1)(q^2+1)$$

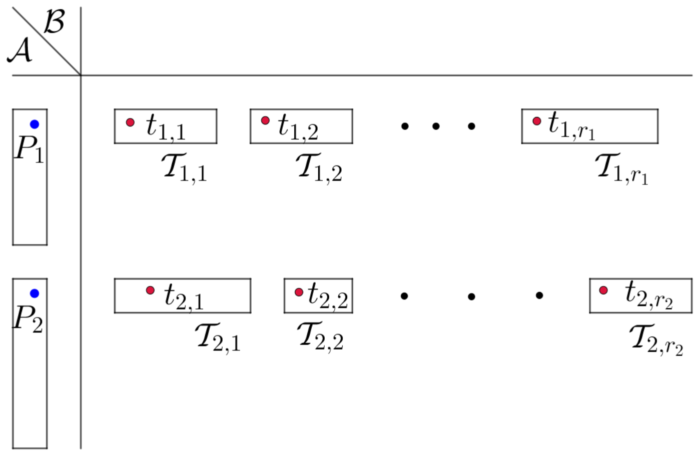
Remark 5.2.6. Let g be the order of the group $\mathrm{PGL}(4,q)$. Using Theorem 2.4.6, it follows that

$$\binom{q^3+q^2+q+1}{3} = g \cdot \Big(\frac{1}{3!(q+1)(q-1)^2q^5} + \frac{1}{3!(q-1)^3q^3}\Big).$$

Under the standard polarity, the previous results about 3-subsets of points become statements about 3-subsets of planes, or triplets. The next result shows that the double-triplets can be classified.

Theorem 5.2.7. *There is an algorithm which classifies the double-triplets in* $\mathrm{PG}(3,q)$.

Proof We rely on Lemma 5.1.1. Let $\mathcal{A}$ be the set of triplets and let $\mathcal{B}$ be the set of double-triplets in $\mathrm{PG}(3,q)$. Let $G = \mathrm{PGL}(4,q)$. The classification of triplets under the group G can be seen in Theorem 5.2.4. The information about the stabilisers is contained in the remarks after the theorem.



The relation $\mathcal{R}$ is given by

$$\mathcal{R} = \{(A, B) \in \mathcal{A} \times \mathcal{B} \mid B = \{A_1, A_2\} \subset \mathcal{A}, \text{ either } A = A_1 \text{ or } A = A_2\}.$$

The relation $\mathcal{R}$ is G-invariant.

The double-triplets Q_r associated to P_i can be obtained from the set

$$\text{Up}(P_i) = \{(P_i, Q_r) \in \mathcal{A} \times \mathcal{B} \mid (P_i, Q_r) \in \mathcal{R} \text{ for } Q_r = \{P_i, A\}\}.$$

Let $\mathcal{T}_{i,r}$, $r = 1, \ldots, r_i$ be the orbits of $\text{Stab}_G(P_i)$ on $\text{Up}(P_i)$. On the other hand, for a given double-triplet $T = \{A_1, A_2\} \subset \mathcal{A}$, there are exactly the 2 triplets A_1, A_2 which are related to T. Hence the down-set of T is the set of pairs

$$\text{Down}(T) = \{(A_1, T), (A_2, T)\}.$$

Let $Q_1, \ldots, Q_n$ be a set of orbit representatives for the action of G on $\mathcal{B}$. For $k = 1, \ldots, n$, the set $\text{Down}(Q_k)$ is partitioned into orbits of the group $\text{Stab}_G(Q_k)$. These are the $\mathcal{S}_{j,l}$ in the Lemma. The orbits $\mathcal{S}_{j,l}$ are paired with certain orbits of the form $\mathcal{T}_{i,r}$.

Let $P_1 = \mathbb{U}$, and $P_2 = \mathbb{V}$ be a transversal of the triplets in $\text{PG}(3, q)$, as in Theorem 5.2.4. Let $\text{Stab}_G(P_i) = G_i$ for $i = 1, 2$ be the associated stabiliser subgroups. Assume that the orbits of G_i on $\text{Up}(P_i)$ is known as $\mathcal{T}_{i,r}$ with orbit representatives $t_{i,r} = (P_i, b_{ir})$ where $b_{ir} \in \mathcal{B}$ for $i = 1, 2$ and $r = 1, 2, \ldots, r_i$. Assume that the associated stabilisers $\text{Stab}_{G_i}(t_{i,r}) = \text{Stab}_G(t_{i,r}) = G_{i,r}$ are known also.

The following algorithm is an adaptation of an algorithm from [6]. As input, we have a transversal $P_1 = \mathbb{U}, P_2 = \mathbb{V}$ of G-orbits on $\mathcal{A}$ with stabilisers $G_i = \mathrm{Stab}_G(P_i)$. We also have orbits $\mathcal{T}_{i,r}$, $r = 1, \ldots, r_i$ of $\mathrm{Stab}_G(P_i)$ on $\mathrm{Up}(P_i)$ for $i = 1$ and $i = 2$. In addition, we have chosen orbit representatives $t_{i,r} = (P_i, b_{ir}) \in \mathcal{T}_{i,r}$ and we know the associated stabiliser subgroups

$$G_{i,r} = \mathrm{Stab}_{G_i}(t_{i,r})$$

for $r = 1, \ldots, r_i$ and $i = 1, 2$.

The algorithm will output a list $Q_1, \ldots, Q_n$ of G-orbit representatives for the action of G on $\mathcal{B}$, with corresponding stabilisers

$$\mathrm{Stab}_G(Q_k).$$

(1) Initialize by marking all orbits $\mathcal{T}_{i,r}$ as unprocessed.

(2) Consider the first/ next unprocessed orbit $\mathcal{T}_{i,r}$.

(3) Mark $\mathcal{T}_{i,r}$ as processed.

(4) Define a new isomorphism type of double-triplets represented by $Q_k := \Pi_2(t_{i,r}) = b_{ir}$. Record $\mathrm{Stab}_{G_i}(t_{i,r})$ as subgroup of $\mathrm{Stab}_G(Q_k)$.

(5) Suppose that $Q_k = \{A_1, A_2\} \subset \mathcal{A}$, where A_1 and A_2 are triplets. Thus

$$\mathrm{Down}(Q_k) = \{(A_1, Q_k), (A_2, Q_k)\}.$$

Loop over the two elements in $\mathrm{Down}(Q_k)$.

(6) Let H be the first/next unprocessed in $\mathrm{Down}(Q_k)$. Let $A = \Pi_1(H)$.

(7) Determine whether $rk(A) = 2$ or 3. If $rk(A) = 2$, find a matrix N as in (5.1) such that $AN = P_1$. If $rk(A) = 3$, find a matrix N as in (5.2) such that $AN = P_2$.

(8) Let α be the projectivity associated to N. Thus, $A\alpha = P_j$.

(9) Determine the index $s \leq r_j$ such that

$$H\alpha \in \mathcal{T}_{j,s}.$$

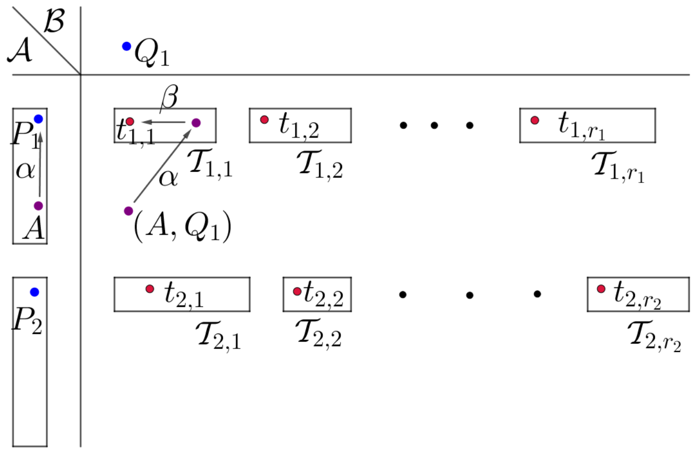


Figure 5.1: $\alpha\beta$ is the generator for $\mathrm{Stab}_G(T)$

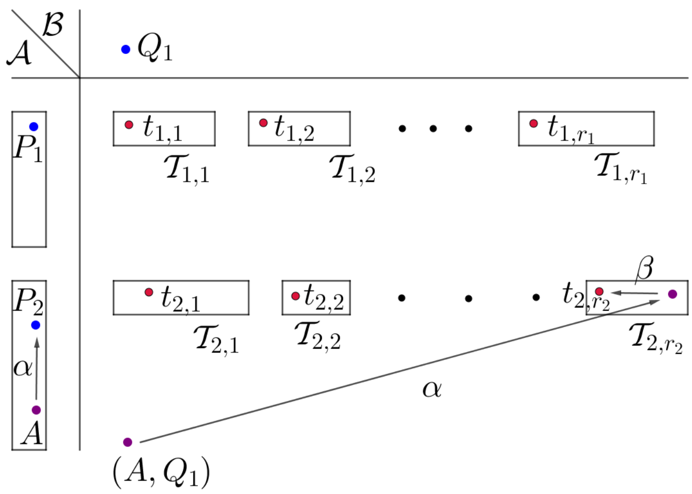


Figure 5.2: Remove $\mathcal{T}_{2,r_2}$

(10) Let $B = \Pi_2(H)$. Determine a matrix $M \in G_j = \mathrm{Stab}_G(P_j)$ such that

$$B\alpha M = t_{j,s}.$$

Let β be the projectivity associated to M.

(11) If $j = i$ and $s = r$, record $\alpha\beta$ as a generator for $\mathrm{Stab}_G(Q_k)$. An example can be seen in Figure 5.1.

(12) Otherwise, mark $\mathcal{T}_{j,s}$ as processed and define the group element $\Delta_{j,s} = (\alpha\beta)^{-1}$. An example can be seen in Figure 5.2.

(13) Continue with the next unprocessed element $H \in \text{Down}(Q_k)$ in Step 6 until done.

(14) Continue with the next unprocessed orbit $\mathcal{T}_{i,r}$ in Step 2 until done.

(15) The collection $Q_1, \ldots, Q_n$ of elements discovered in Step 4 is the desired transversal of G-orbits on $\mathcal{B}$, that is, the isomorphism types of double-triplets. The associated stabiliser groups are the groups $\text{Stab}_G(Q_k)$ initialized in Step 4 and possibly extended in Step 11.

□

5.2.3 The classification algorithm for the cubic surfaces with 27 lines

This algorithm is used for the classification of cubic surfaces with twenty-seven lines in $\text{PG}(3, q)$. The inputs of the algorithm are *projectively distinct* 6*-arcs* S *not on a conic in* $\text{PG}(2, q)$ and *the classification of double-triplets under the group* $\text{PGL}(4, q)$ and the output of the algorithm is *projectively distinct cubic surfaces* $\mathcal{F}$ *with twenty-seven lines in* $\text{PG}(3, q)$ *with a distinguished double-six.*

Assume that we have the classification of 6-arcs not on a conic in $\text{PG}(2, q)$ under the action of $\text{PGL}(3, q)$. Let

$$\mathcal{S} = \{s_1, s_2, \ldots, s_a\}$$

be a transversal for these orbits, where a is the number of orbits. Assume that the classification of double-triplets in $\text{PG}(3, q)$ is known under the group $G = \text{PGL}(4, q)$.

Let $j = 1$*. While* $\mathcal{S}$ *is nonempty, do the following:*

Step 1: Let s_i be the next element in $\mathcal{S}$. Perform the arc lifting algorithm from Section 4.3 for s_i. This yields a surface $\mathcal{F}_{s_i}$ with a distinguished double-six $\mathscr{D}$. Say $Q_1 = \mathcal{F}_{s_i}$.

The arc lifting algorithm uses S_{123} to assign the basis vectors of $\text{PG}(3, q)$ to

$$\pi_{23}, \pi_{31}, \pi_{32}, \pi_{13}.$$

These basis vectors form four of the six planes in the trihedral pair.

$$S_{123} = (\pi_{23}, \pi_{31}, \pi_{12}; \pi_{32}, \pi_{13}, \pi_{21}).$$

Let $F_1, F_2, F_3, G_1, G_2, G_3$ be the equations of the six planes, so that

$$\begin{aligned}
\pi_{23} &= V(F_1), \quad \pi_{31} = V(F_2), \quad \pi_{12} = V(F_3), \\
\pi_{32} &= V(G_1), \quad \pi_{13} = V(G_2), \quad \pi_{21} = V(G_3).
\end{aligned}$$

Let

$$E = \{\mathcal{F}_1, \dots, \mathcal{F}_{q+1}\}$$

be the cubic surfaces associated with

$$V(aF_1F_2F_3 + bG_1G_2G_3),$$

where $a : b$ are homogeneous coordinates of $\mathrm{PG}(1, q)$. The arc lifting algorithm also ensures that $Q_1 \in E$. Without loss of generality, we may assume that $Q_1 = \mathcal{F}_1$. Every trihedral pair is a double-triplet. From the classification of double-triplets, we know the group

$$G_T = G_{\{F_1,F_2,F_3;G_1,G_2,G_3\}},$$

which is the stabiliser of the double-triplet S_{123}.

Step 2: Compute the orbits of G_T on E using the dual action.

By considering the orbit of $\mathcal{F}_1$ under G_T, we can compute the subgroup

$$H = \mathrm{Stab}_{G_T}(\mathcal{F}_1)$$

of G_T. Let $D = \{d_1, \dots, d_r\}$ be a set of coset representatives of $\mathrm{Stab}_{G_T}(\mathcal{F}_1)$ in G_T, so

$$G_T = \bigcup_{d \in D} Hd.$$

The elements of D correspond one-to-one to the elements of the orbit

$$\mathrm{Orb}_{G_T}(\mathcal{F}_1).$$

For the sake of notational simplicity, let

$$\mathrm{Orb}_{G_T}(\mathcal{F}_1) = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r\},$$

and assume that

$$\mathcal{F}_1 d_h^\top = \mathcal{F}_h \text{ for } h = 1, \ldots, r.$$

Here, we use $g^\top$ to denote the action of g on the dual space. This is necessary because the classification of double-triplets was performed using dual coordinates.

Step 3: From the arc lifting, the surface $\mathcal{F}_1$ is created together with a distinguished double-six $\mathscr{D}$. Using this double-six, the 120 Steiner trihedral pairs on $\mathcal{F}_1$ are known. Let

$$T_1, \ldots, T_{120}$$

be all Steiner trihedral pairs on $\mathcal{F}_1$, with $T_1 = S_{123}$. Here, S_{ijk} are from T_1 to T_{20}, $S_{ij,kl}$ are from T_{21} to T_{110} and $S_{ijk,lmn}$ are from T_{111} to T_{120}. Each Steiner trihedral pair is considered as a double-triplet. We perform the following algorithm.

Let A be the empty set. For $l = 1, \ldots, 120$, test if the double-triplet T_l can be mapped to the double-triplet T_1. If so, let c_l be a group element which maps T_l to T_1.

Perform a loop over all $d_h \in D$ and see if

$$\mathcal{F}_1 (c_l d_h)^\top = \mathcal{F}_1.$$

If so, add $c_l d_h$ to A.

Step 4: The stabiliser of $\mathcal{F}_1$ is the group

$$G(\mathcal{F}_1) = \bigcup_{a \in A} H^\top a^\top.$$

Step 5: Let $\mathfrak{H}$ be the set of 72 half double-sixes on $\mathcal{F}_1$.

Compute a transversal $\mathcal{H}_1, \ldots, \mathcal{H}_x$ of the orbits of $G(\mathcal{F}_1)$ on $\mathfrak{H}$.

Step 6: For $k = 1, \ldots, x$, construct the Clebsch map $\Phi_{\mathcal{H}_k}$. Let s be the 6-arc associated to $\Phi_{\mathcal{H}_k}$.

Perform isomorphism testing to determine the representative $s_{i'}$ in the classification of 6-arcs not on a conic which is isomorphic to s. Remove $s_{i'}$ from $\mathcal{S}$.

Step 7: Increment j and continue with step 1 until $\mathcal{S}$ is empty.

Let $n = j$. The elements $Q_1, Q_2, \ldots, Q_n$ form a transversal of the G-orbits on the set of cubic surfaces with 27 lines in $\mathrm{PG}(3, q)$*.*

We illustrate how to find the automorphism group of a cubic surface with twenty-seven lines over $\mathbf{F}_q$.

Example 5.2.8. Let

$$S = \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1), \mathbf{P}(3,2,1), \mathbf{P}(7,6,1)\}$$

be a 6-arc not on a conic in $\mathrm{PG}(2, 13)$.

Step 1: The arc lifting algorithm from Section 4.3 yields the cubic surface

$$\mathcal{F}_S = V\big(x_0^2x_1 + 5x_0x_1^2 + 12x_2^2x_3 + 8x_2x_3^2 + 4x_0x_1x_2 + 6x_0x_2x_3\big).$$

together with a distinguished double-six $\mathscr{D}$. Let $T_1, \ldots, T_{120}$ be the Steiner trihedral pairs with respect to $\mathscr{D}$. Considering each Steiner trihedral pair as a double-triplet, we can determine its isomorphism type. This information is displayed in Table 5.5. In these tables, the double-triplet is shown using dual coordinates, using labelling of points shown in Appendix C. Also, N refers to the isomorphism type in the classification of double-triplets in Appendix C.

We find that the Steiner trihedral pair $S_{123} = T_1$ is the double-triplet

$$\{0, 1, 169; 2, 3, 1539\},$$

where the planes are given using dual coordinates as

$$\begin{aligned}
\pi_0 : &\quad x_0 = 0, \\
\pi_1 : &\quad x_1 = 0, \\
\pi_{169} : &\quad 10x_0 + 11x_1 + x_2, \\
\pi_2 : &\quad x_2 = 0, \\
\pi_3 : &\quad x_3 = 0, \\
\pi_{1539} : &\quad 4x_0 + 8x_2 + x_3 = 0.
\end{aligned}$$

This double-triplet is isomorphic to the double-triplet number 55 from Appendix C, namely

$$\{0, 1, 2; 3, 4, 41\},$$

with equations

$$\begin{aligned}
\pi_0 : &\quad x_0 = 0, \\
\pi_1 : &\quad x_1 = 0, \\
\pi_2 : &\quad x_2 = 0, \\
\pi_3 : &\quad x_3 = 0, \\
\pi_4 : &\quad x_3 = 0, \\
\pi_{41} : &\quad 12x + 0 + x_1 + x_2.
\end{aligned}$$

The stabiliser of this double-triplet is a group of order 8, generated by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 12 & 12 & 12 & 12 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 12 & 12 & 0 \\ 0 & 0 & 0 & 1 \\ 12 & 12 & 12 & 12 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

After conjugating by a projectivity which takes

$$\{0, 1, 169; 2, 3, 1539\} \text{ to } \{0, 1, 2; 3, 4, 41\},$$

the stabiliser of S_{123} is

$$G_T = \mathrm{Stab}_G(S_{123}) = \langle \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 12 \end{bmatrix} \rangle.$$

Step 2: The group G_T acts on the set E of cubic surfaces associated with T and shown in Table 5.6. We know from the arc lifting that $\mathcal{F}_S = \mathcal{F}_{1,2}$. The orbit of $\mathcal{F}_S$ under the dual action of G_T has size two:

$$\mathrm{Orb}_{G_T}(\mathcal{F}_S) = \{\mathcal{F}_{1,2}, \mathcal{F}_{1,10}\}.$$

The associated system of coset representatives is

$$D = \{d_1, d_2\},$$

where

$$d_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, d_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{bmatrix}.$$

We can check that

$$\mathcal{F}_{1,2} d_2^\top = \mathcal{F}_{1,10}.$$

The stabiliser of $\mathcal{F}_S = \mathcal{F}_{1,2}$ is the following group of order 4.

$$H = \mathrm{Stab}_{G_T}(\mathcal{F}_S) = \langle \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 12 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rangle.$$

Moreover,

$$G_T = \bigcup_{i=1}^{2} H d_i.$$

	double-triplet	N
S_{123}	$\{0, 1, 169; 2, 3, 1539\}$	55
S_{124}	$\{1540, 1695, 169; 132, 1527, 1539\}$	55
S_{125}	$\{1544, 1617, 169; 74, 1960, 1539\}$	106
S_{126}	$\{1538, 1565, 169; 148, 2240, 1539\}$	106
S_{134}	$\{29, 1695, 3; 196, 1527, 1\}$	55
S_{135}	$\{55, 1617, 3; 192, 1960, 1\}$	106
S_{136}	$\{107, 1565, 3; 185, 2240, 1\}$	106
S_{145}	$\{842, 1617, 1527; 1345, 1960, 1695\}$	51
S_{146}	$\{1798, 1565, 1527; 617, 2240, 1695\}$	50
S_{156}	$\{1889, 1565, 1960; 881, 2240, 1617\}$	115
S_{234}	$\{29, 132, 0; 196, 1540, 2\}$	55
S_{235}	$\{55, 74, 0; 192, 1544, 2\}$	50
S_{236}	$\{107, 148, 0; 185, 1538, 2\}$	51
S_{245}	$\{842, 74, 1540; 1345, 1544, 132\}$	106
S_{246}	$\{1798, 148, 1540; 617, 1538, 132\}$	106
S_{256}	$\{1889, 148, 1544; 881, 1538, 74\}$	115
S_{345}	$\{842, 192, 29; 1345, 55, 196\}$	106
S_{346}	$\{1798, 185, 29; 617, 107, 196\}$	106
S_{356}	$\{1889, 185, 55; 881, 107, 192\}$	115
S_{456}	$\{1889, 617, 842; 881, 1798, 1345\}$	115
$S_{12,34}$	$\{1527, 0, 197; 3, 1540, 28\}$	6
$S_{13,24}$	$\{1527, 2, 377; 169, 29, 28\}$	6
$S_{14,23}$	$\{3, 132, 377; 169, 196, 197\}$	55
$S_{23,14}$	$\{1540, 1, 377; 1539, 29, 197\}$	55
$S_{24,13}$	$\{0, 1695, 377; 1539, 196, 28\}$	6
$S_{34,12}$	$\{2, 1695, 197; 1, 132, 28\}$	6
$S_{12,35}$	$\{1960, 0, 275; 3, 1544, 25\}$	73
$S_{13,25}$	$\{1960, 2, 2141; 169, 55, 25\}$	61
$S_{15,23}$	$\{3, 74, 2141; 169, 192, 275\}$	106
$S_{23,15}$	$\{1544, 1, 2141; 1539, 55, 275\}$	106
$S_{25,13}$	$\{0, 1617, 2141; 1539, 192, 25\}$	73
$S_{35,12}$	$\{2, 1617, 275; 1, 74, 25\}$	61
$S_{12,36}$	$\{2240, 0, 327; 3, 1538, 19\}$	56
$S_{13,26}$	$\{2240, 2, 1848; 169, 107, 19\}$	71
$S_{16,23}$	$\{3, 148, 1848; 169, 185, 327\}$	106
$S_{23,16}$	$\{1538, 1, 1848; 1539, 107, 327\}$	106
$S_{26,13}$	$\{0, 1565, 1848; 1539, 185, 19\}$	56
$S_{36,12}$	$\{2, 1565, 327; 1, 148, 19\}$	71
$S_{12,45}$	$\{1960, 1540, 2199; 1527, 1544, 2290\}$	71
$S_{14,25}$	$\{1960, 132, 1848; 169, 842, 2290\}$	106
$S_{15,24}$	$\{1527, 74, 1848; 169, 1345, 2199\}$	71
$S_{24,15}$	$\{1544, 1695, 1848; 1539, 842, 2199\}$	56
$S_{25,14}$	$\{1540, 1617, 1848; 1539, 1345, 2290\}$	106

Table 5.5: The isomorphism classes of Steiner trihedral pairs

	double-triplet	N
$S_{45,12}$	$\{132, 1617, 2199; 1695, 74, 2290\}$	56
$S_{12,46}$	$\{2240, 1540, 1023; 1527, 1538, 1062\}$	61
$S_{14,26}$	$\{2240, 132, 2141; 169, 1798, 1062\}$	106
$S_{16,24}$	$\{1527, 148, 2141; 169, 617, 1023\}$	61
$S_{24,16}$	$\{1538, 1695, 2141; 1539, 1798, 1023\}$	73
$S_{26,14}$	$\{1540, 1565, 2141; 1539, 617, 1062\}$	106
$S_{46,12}$	$\{132, 1565, 1023; 1695, 148, 1062\}$	73
$S_{12,56}$	$\{2240, 1544, 1287; 1960, 1538, 559\}$	126
$S_{15,26}$	$\{2240, 74, 377; 169, 1889, 559\}$	71
$S_{16,25}$	$\{1960, 148, 377; 169, 881, 1287\}$	61
$S_{25,16}$	$\{1538, 1617, 377; 1539, 1889, 1287\}$	73
$S_{26,15}$	$\{1544, 1565, 377; 1539, 881, 559\}$	56
$S_{56,12}$	$\{74, 1565, 1287; 1617, 148, 559\}$	126
$S_{13,45}$	$\{1960, 29, 1023; 1527, 55, 1287\}$	56
$S_{14,35}$	$\{1960, 196, 327; 3, 842, 1287\}$	106
$S_{15,34}$	$\{1527, 192, 327; 3, 1345, 1023\}$	56
$S_{34,15}$	$\{55, 1695, 327; 1, 842, 1023\}$	71
$S_{35,14}$	$\{29, 1617, 327; 1, 1345, 1287\}$	106
$S_{45,13}$	$\{196, 1617, 1023; 1695, 192, 1287\}$	71
$S_{13,46}$	$\{2240, 29, 2199; 1527, 107, 559\}$	73
$S_{14,36}$	$\{2240, 196, 275; 3, 1798, 559\}$	106
$S_{16,34}$	$\{1527, 185, 275; 3, 617, 2199\}$	73
$S_{34,16}$	$\{107, 1695, 275; 1, 1798, 2199\}$	61
$S_{36,14}$	$\{29, 1565, 275; 1, 617, 559\}$	106
$S_{46,13}$	$\{196, 1565, 2199; 1695, 185, 559\}$	61
$S_{13,56}$	$\{2240, 55, 2290; 1960, 107, 1062\}$	126
$S_{15,36}$	$\{2240, 192, 197; 3, 1889, 1062\}$	56
$S_{16,35}$	$\{1960, 185, 197; 3, 881, 2290\}$	73
$S_{35,16}$	$\{107, 1617, 197; 1, 1889, 2290\}$	61
$S_{36,15}$	$\{55, 1565, 197; 1, 881, 1062\}$	71
$S_{56,13}$	$\{192, 1565, 2290; 1617, 185, 1062\}$	126
$S_{14,56}$	$\{2240, 842, 25; 1960, 1798, 19\}$	115
$S_{15,46}$	$\{2240, 1345, 28; 1527, 1889, 19\}$	66
$S_{16,45}$	$\{1960, 617, 28; 1527, 881, 25\}$	66
$S_{45,16}$	$\{1798, 1617, 28; 1695, 1889, 25\}$	66
$S_{46,15}$	$\{842, 1565, 28; 1695, 881, 19\}$	66
$S_{56,14}$	$\{1345, 1565, 25; 1617, 617, 19\}$	115
$S_{23,45}$	$\{1544, 29, 1062; 1540, 55, 559\}$	106
$S_{24,35}$	$\{1544, 196, 19; 0, 842, 559\}$	61
$S_{25,34}$	$\{1540, 192, 19; 0, 1345, 1062\}$	61
$S_{34,25}$	$\{55, 132, 19; 2, 842, 1062\}$	73
$S_{35,24}$	$\{29, 74, 19; 2, 1345, 559\}$	73
$S_{45,23}$	$\{196, 74, 1062; 132, 192, 559\}$	106

Table 5.5 (continued)

	double-triplet	N
$S_{23,46}$	$\{1538, 29, 2290; 1540, 107, 1287\}$	106
$S_{24,36}$	$\{1538, 196, 25; 0, 1798, 1287\}$	71
$S_{26,34}$	$\{1540, 185, 25; 0, 617, 2290\}$	71
$S_{34,26}$	$\{107, 132, 25; 2, 1798, 2290\}$	56
$S_{36,24}$	$\{29, 148, 25; 2, 617, 1287\}$	56
$S_{46,23}$	$\{196, 148, 2290; 132, 185, 1287\}$	106
$S_{23,56}$	$\{1538, 55, 2199; 1544, 107, 1023\}$	115
$S_{25,36}$	$\{1538, 192, 28; 0, 1889, 1023\}$	66
$S_{26,35}$	$\{1544, 185, 28; 0, 881, 2199\}$	66
$S_{35,26}$	$\{107, 74, 28; 2, 1889, 2199\}$	66
$S_{36,25}$	$\{55, 148, 28; 2, 881, 1023\}$	66
$S_{56,23}$	$\{192, 148, 2199; 74, 185, 1023\}$	115
$S_{24,56}$	$\{1538, 842, 275; 1544, 1798, 327\}$	126
$S_{25,46}$	$\{1538, 1345, 197; 1540, 1889, 327\}$	61
$S_{26,45}$	$\{1544, 617, 197; 1540, 881, 275\}$	71
$S_{45,26}$	$\{1798, 74, 197; 132, 1889, 275\}$	56
$S_{46,25}$	$\{842, 148, 197; 132, 881, 327\}$	73
$S_{56,24}$	$\{1345, 148, 275; 74, 617, 327\}$	126
$S_{34,56}$	$\{107, 842, 2141; 55, 1798, 1848\}$	126
$S_{35,46}$	$\{107, 1345, 377; 29, 1889, 1848\}$	73
$S_{36,45}$	$\{55, 617, 377; 29, 881, 2141\}$	56
$S_{45,36}$	$\{1798, 192, 377; 196, 1889, 2141\}$	71
$S_{46,35}$	$\{842, 185, 377; 196, 881, 1848\}$	61
$S_{56,34}$	$\{1345, 185, 2141; 192, 617, 1848\}$	126
$S_{123,456}$	$\{2199, 1287, 1062; 1023, 559, 2290\}$	115
$S_{124,356}$	$\{275, 1287, 19; 327, 559, 25\}$	115
$S_{125,346}$	$\{197, 1023, 19; 327, 1062, 28\}$	66
$S_{126,345}$	$\{197, 2199, 25; 275, 2290, 28\}$	66
$S_{134,256}$	$\{2141, 2290, 19; 1848, 1062, 25\}$	115
$S_{135,246}$	$\{377, 2199, 19; 1848, 559, 28\}$	66
$S_{136,245}$	$\{377, 1023, 25; 2141, 1287, 28\}$	66
$S_{145,236}$	$\{377, 275, 1062; 2141, 559, 197\}$	51
$S_{146,235}$	$\{377, 327, 2290; 1848, 1287, 197\}$	50
$S_{156,234}$	$\{2141, 327, 2199; 1848, 1023, 275\}$	115

Table 5.5 (continued)

$$
\begin{aligned}
\mathcal{F}_{1,0} &= V(10x_0^2x_1 + 11x_0x_1^2 + x_0x_1x_2)\\
\mathcal{F}_{0,1} &= V(2x_2^2x_3 + 10x_2x_3^2 + x_0x_2x_3)\\
\mathcal{F}_{1,1} &= V(9x_0^2x_1 + 6x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 10x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,2} &= V(11x_0^2x_1 + 3x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 5x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,3} &= V(3x_0^2x_1 + 2x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 12x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,4} &= V(12x_0^2x_1 + 8x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 9x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,5} &= V(7x_0^2x_1 + 9x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 2x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,6} &= V(8x_0^2x_1 + x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 6x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,7} &= V(5x_0^2x_1 + 12x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 7x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,8} &= V(6x_0^2x_1 + 4x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 11x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,9} &= V(x_0^2x_1 + 5x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 4x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,10} &= V(10x_0^2x_1 + 11x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,11} &= V(2x_0^2x_1 + 10x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 8x_0x_1x_2 + x_0x_2x_3)\\
\mathcal{F}_{1,12} &= V(4x_0^2x_1 + 7x_0x_1^2 + 2x_2^2x_3 + 10x_2x_3^2 + 3x_0x_1x_2 + x_0x_2x_3)
\end{aligned}
$$

Table 5.6: The elements of the set E

Step 3: Appealing to Table 5.5 again, we find the set of double-triplets which can be mapped to S_{123}:

$$\mathscr{T} = \{S_{123}, S_{124}, S_{134}, S_{234}, S_{14,23}, S_{23,14}\}.$$

These are simply the double-triplets of isomorphism type 55. In addition, we find that

$$
\begin{aligned}
S_{123}c_1 &= S_{123}, & S_{234}c_4 &= S_{123},\\
S_{124}c_2 &= S_{123}, & S_{14,23}c_5 &= S_{123},\\
S_{134}c_3 &= S_{123}, & S_{23,14}c_6 &= S_{123}.
\end{aligned}
$$

By testing the elements $d \in D$, we find that

$$\mathcal{F}_S(c_1 d_1)^\top = \mathcal{F}_S,$$
$$\mathcal{F}_S(c_2 d_1)^\top = \mathcal{F}_S,$$
$$\mathcal{F}_S(c_3 d_2)^\top = \mathcal{F}_S,$$
$$\mathcal{F}_S(c_4 d_2)^\top = \mathcal{F}_S,$$
$$\mathcal{F}_S(c_5 d_2)^\top = \mathcal{F}_S,$$
$$\mathcal{F}_S(c_6 d_1)^\top = \mathcal{F}_S,$$

which leads us to define the set

$$\begin{aligned} A &= \{c_1d_1, c_2d_1, c_3d_2, c_4d_2, c_5d_2, c_6d_1\} \\ &= \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 11 & 11 \\ 3 & 0 & 3 & 3 \\ 2 & 11 & 5 & 2 \\ 10 & 3 & 10 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 6 & 0 \\ 5 & 8 & 5 & 0 \\ 4 & 0 & 12 & 0 \\ 8 & 0 & 8 & 8 \end{bmatrix} \right. \\ &\qquad \left. \begin{bmatrix} 0 & 1 & 5 & 12 \\ 0 & 0 & 0 & 8 \\ 5 & 12 & 0 & 1 \\ 0 & 8 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & 12 \\ 8 & 5 & 8 & 8 \\ 0 & 4 & 8 & 1 \\ 8 & 0 & 8 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 0 & 0 \\ 12 & 0 & 12 & 12 \\ 12 & 8 & 0 & 12 \\ 12 & 1 & 12 & 0 \end{bmatrix} \right\}. \end{aligned}$$

Step 4: The stabiliser of $\mathcal{F}_S$ is the group

$$G(\mathcal{F}_S) = \bigcup_{a \in A} H^\top a^\top$$

of order

$$|G(\mathcal{F}_S)| = |H| \cdot |A| = 4 \cdot 6 = 24,$$

generated by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 12 & 1 & 12 & 12 \\ 12 & 0 & 0 & 12 \\ 12 & 0 & 12 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 1 & 5 & 12 \\ 8 & 5 & 8 & 8 \\ 0 & 4 & 8 & 1 \\ 8 & 0 & 8 & 0 \end{bmatrix}.$$

Our next example is the equianharmonic surface. This continues Example 4.3.2.

Example 5.2.9. The cubic surface $\mathcal{F}_S$ in $\mathrm{PG}(3,13)$ was created from the 6-arc S not on a conic in $\mathrm{PG}(2,13)$. In Step 3 of the Example 4.3.2, the trihedral pair S_{123} was expressed as two sets of three tritangent planes. The first trihedron consists of the planes

$$\pi_{23} = V(x_0), \pi_{31} = V(x_1), \text{ and } \pi_{12} = V(7x_0 + 5x_1 + x_3).$$

The second trihedron is

$$\pi_{32} = V(x_2), \pi_{13} = V(x_3), \text{ and } \pi_{21} = V(2x_0 + 10x_2 + x_3).$$

Therefore, S_{123} is a double-triplet. As is explained in the proof of the Theorem 5.2.7, the orbits of G on the set of double-triplets in $\mathrm{PG}(3,13)$ can be found with the stabiliser of each double-triplets under the group G. It can be seen that the trihedral pair S_{123} is isomorphic to double-triplet no 47 in the classification displayed in Appendix C. Therefore, the stabiliser $G_T = \mathrm{Stab}_G(S_{123})$ of S_{123} is the group of order 8 with the generators

$$\begin{bmatrix} 0 & 1 & 0 & 8 \\ 0 & 0 & 8 & 0 \\ 0 & 5 & 0 & 0 \\ 8 & 7 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 9 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix}.$$

There are 8 orbits of G_T on the $q+1$ surfaces on the line which are given in the step 3 of the Example 4.3.2. We focus on the equation in row 12 which was the equation of $\mathcal{F}_S$. Moreover, the orbit of the associated cubic surface $\mathcal{F}_S$ with the arc S has length 1. So, the subgroup which stabilizes the equation of $\mathcal{F}_S$ has 1 coset in G_T which is the identity.

Regarding the notation in the classification algorithm, the set $D = \{I\}$ and the stabiliser of the equation $\mathcal{F}_S$ under G_T is denoted by

$$H = \mathrm{Stab}_{G_T}(\mathcal{F}_S)$$

and it is the group of order 8 with the generators

$$\begin{bmatrix} 0 & 1 & 0 & 8 \\ 0 & 0 & 8 & 0 \\ 0 & 5 & 0 & 0 \\ 8 & 7 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 9 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix}.$$

The cubic surface $\mathcal{F}_S$ has 120 Steiner trihedral pairs. In Example 4.3.2 those 120 trihedral pairs are expressed by double-triplets. From the classification of double-triplets in $\mathrm{PG}(3, 13)$ it can be seen that there are 81 Steiner trihedral pairs of $\mathcal{F}_S$ in the orbit of S_{123} under the group $\mathrm{PGL}(4, 13)$. Regarding the notation in classification algorithm $C = \{c_1, \ldots, c_{81}\}$. Moreover, $A = \{c_1, \ldots, c_{81}\}$. The automorphism group of the cubic surface $\mathcal{F}_S$,

$$\mathrm{Stab}_G(\mathcal{F}_S) = \bigcup_{\delta \in A} G_T\delta,$$

is a group of order 648. Generators are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 12 & 0 & 10 \\ 10 & 0 & 12 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 12 & 3 & 2 \\ 2 & 0 & 0 & 6 \\ 10 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 5 & 0 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 12 & 0 \\ 2 & 9 & 12 & 6 \\ 5 & 1 & 4 & 5 \\ 0 & 10 & 9 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 4 & 3 & 5 \\ 11 & 0 & 0 & 7 \\ 5 & 0 & 0 & 6 \\ 7 & 9 & 12 & 8 \end{bmatrix}.$$

Here is the main results of this thesis.

Theorem 5.2.10. (1) *The number of isomorphism classes of cubic surfaces with 27 lines* *over the field for* $q = 13, 17, 19$ *and the classification of these surfaces by the number of Eckardt points and the automorphism group orders of these surfaces are as listed in Tables 5.7 and 5.8. Here, the automorphism group means the projective stabiliser of a surface.*

(2) *In Table 5.7,* a *is the number of* $\mathrm{PGL}(3, q)$*-orbits on the set of* 6*-arcs not on a conic in* $\mathrm{PG}(2, q)$*,* t *is the number of* $\mathrm{PGL}(4, q)$*-orbits on the set of triplets,* t_1 *is the number of orbits of the stabiliser of a triplet type one* P_1 *on* $\mathrm{Up}(P_1)$*,* t_2 *is the number of orbits of the stabiliser of a triplet type two* P_2 *on* $\mathrm{Up}(P_2)$*,* d *is the number of* $\mathrm{PGL}(4, q)$*-orbits on the set of double-triplets, and* c *is the number of* $\mathrm{PGL}(4, q)$*-orbits on the set of cubic surfaces with* 27 *lines in* $\mathrm{PG}(3, q)$*.*

(3) *In Table 5.8,* $G(\mathcal{F}_q^i)$ *is the group of projectivities which fixes the cubic surface* $\mathcal{F}_q^i$*,* s *is the number of projectively distinct* 6*-arcs not on a conic associated with* $\mathcal{F}_q^i$*,* e_3 *is the total number of Eckardt points of* $\mathcal{F}_q^i$ *and* n_i *is the number of orbits of the stabiliser of the cubic surface on half double-sixes of* $\mathcal{F}_q^i$*.*

(4) *There is a canonical one-to-one correspondence between the isomorphism types of* 6*-arcs not on a conic in* $\mathrm{PG}(2, q)$ *and the orbits of the stabiliser of the cubic surface* $\mathcal{F}$ *acting on the set of half double-sixes, where* $\mathcal{F}$ *ranges over a system of representatives of the isomorphisms types of cubic surfaces with* 27 *lines in* $\mathrm{PG}(3, q)$*,*

$$\sum_{i=1}^{c} n_i = a.$$

	$q = 13$	$q = 17$	$q = 19$
a	21	64	104
t	2	2	2
t_1	48	98	145
t_2	449	1361	2194
d	282	784	1237
c	4	7	10

Table 5.7: Summary for classification results of $q = 13, 17, 19$

q	$\mathcal{F}_q^i$	s	e_3	$\lvert G(\mathcal{F}_q^i)\rvert$	n_i	Type
13	$\mathcal{F}_{13}^0$	10	4	12	10	
13	$\mathcal{F}_{13}^1$	7	6	24	7	
13	$\mathcal{F}_{13}^2$	2	9	108	2	
13	$\mathcal{F}_{13}^3$	2	18	648	2	$\mathcal{E}$
17	$\mathcal{F}_{17}^0$	9	1	8	9	
17	$\mathcal{F}_{17}^1$	14	3	6	14	
17	$\mathcal{F}_{17}^2$	10	4	12	10	
17	$\mathcal{F}_{17}^3$	10	4	12	10	
17	$\mathcal{F}_{17}^4$	7	6	24	7	
17	$\mathcal{F}_{17}^5$	7	6	24	7	
17	$\mathcal{F}_{17}^6$	7	6	24	7	
19	$\mathcal{F}_{19}^0$	21	2	4	21	
19	$\mathcal{F}_{19}^1$	21	2	4	21	
19	$\mathcal{F}_{19}^2$	14	3	6	14	
19	$\mathcal{F}_{19}^3$	14	3	6	14	
19	$\mathcal{F}_{19}^4$	10	4	12	10	
19	$\mathcal{F}_{19}^5$	7	6	24	7	
19	$\mathcal{F}_{19}^6$	7	6	24	7	
19	$\mathcal{F}_{19}^7$	4	9	54	4	
19	$\mathcal{F}_{19}^8$	4	10	120	4	$\mathcal{D}$
19	$\mathcal{F}_{19}^9$	2	18	648	2	$\mathcal{E}$

Table 5.8: Cubic Surfaces for $q = 13, 17, 19$

Proof The algorithms from Section 4.3 and Section 5.2.3 have been used to prove the theorem. Here are the computing times using Orbiter:

q	timing
13	25 sec
17	1 min 22 sec
19	2 min 42 sec

In Appendix D, the Orbiter makefile is given in order to get much detailed classification results. $\square$

5.3 Counting formula for the cubic surface in $\mathrm{PG}(3, q)$

Conjecture 5.3.1. *The number of cubic surfaces with twenty-seven lines in* $\mathrm{PG}(3, q)$ *is*

$$\frac{q^6(q^2-1)(q^3-1)(q^4-1)(q-2)(q-3)(q-5)^2}{51840}.$$

Some numerical data associated to the classification is shown in Table 5.9 for the fields of order $q = 13, 17, 19$.

Let g be the order of $\mathrm{PGL}(4, q)$. Let $G(\mathcal{F})$ be the group of projectivities of the cubic surface $\mathcal{F}$. It follows from Theorem 2.4.6 that

$$C_q := \sum_{\text{iso type } \mathcal{F}} \frac{g}{|G(\mathcal{F})|}.$$

For $q = 13$, the distribution of the automorphism group orders of the projectively distinct cubic surface with 27 lines is $12, 24, 108, 648$ as shown in Table 5.8. Therefore, the number of cubic surfaces with twenty-seven lines in $\mathrm{PG}(3, 13)$ is

$$\begin{aligned}
C_{13} &= |\mathrm{PGL}(4,13)| \cdot \left(\frac{1}{12} + \frac{1}{24} + \frac{1}{108} + \frac{1}{648}\right) \\
&= 50858076935877120 \cdot \left(\frac{1}{12} + \frac{1}{24} + \frac{1}{108} + \frac{1}{648}\right) \\
&= 6906652423390720 \\
&= \frac{13^6(13^2-1)(13^3-1)(13^4-1)(13-2)(13-3)(13-5)^2}{51840}
\end{aligned}$$

For $q = 17$, the distribution of the automorphism group orders of the projectively

	$q = 13$	$q = 17$	$q = 19$
p_2	183	307	381
l_2	183	307	381
m_2	14	18	20
n_2	14	18	20
A_q	7925229312	291908606976	1253872170432
g_3	810534816	6950204928	16934047920
p_3	2380	5220	7240
P_3	2380	5220	7240
l_3	31110	89030	137922
m_3	14	18	20
n_3	183	307	381
g_4	50858076935877120	2851903720876769280	15136750711925049600
C_q	6906652423390720	1663610503844782080	15566559682757489280

$$
\begin{aligned}
p_2 &:= \text{the total number of points in PG}(2,q),\\
l_2 &:= \text{the total number of lines in PG}(2,q),\\
m_2 &:= \text{the total number of points on a line in PG}(2,q),\\
n_2 &:= \text{the total number of lines passing through a point of PG}(2,q),\\
A_q &:= \text{the total number of 6-arcs not on a conic in PG}(2,q),\\
g_3 &:= |\mathrm{PGL}(3,q)|,\\
p_3 &:= \text{the total number of points in PG}(3,q),\\
P_3 &:= \text{the total number of planes in PG}(3,q)\ ,\\
l_3 &:= \text{the total number of lines in PG}(3,q),\\
m_3 &:= \text{the total number of planes through a line in PG}(3,q),\\
n_3 &:= \text{the total number of lines passing through a point of PG}(3,q),\\
g_4 &:= |\mathrm{PGL}(4,q)|,\\
C_q &:= \text{the total number of cubic surfaces with 27 lines in PG}(3,q).
\end{aligned}
$$

Table 5.9: Counting properties for $q = 13, 17, 19$

distinct cubic surfaces is $8, 6, 12^2, 24^3$. Therefore, the number of cubic surfaces with twenty-seven lines in PG(3, 17) is

$$\begin{aligned}
C_{17} &= |\mathrm{PGL}(4,17)| \cdot \Big(\frac{1}{8} + \frac{1}{6} + \frac{2}{12} + \frac{3}{24}\Big) \\
&= 2851903720876769280 \cdot \Big(\frac{1}{8} + \frac{1}{6} + \frac{2}{12} + \frac{3}{24}\Big) \\
&= 1663610503844782080 \\
&= \frac{17^6(17^2-1)(17^3-1)(17^4-1)(17-2)(17-3)(17-5)^2}{51840}
\end{aligned}$$

For $q = 19$, the distribution of the automorphism group orders of the projectively distinct cubic surface with 27 lines is $4^2, 6^2, 12, 24^2, 54, 120, 648$ as shown in Table 5.8. Therefore, the number of cubic surfaces with twenty-seven lines in PG(3, 19) is

$$\begin{aligned}
C_{19} &= |\mathrm{PGL}(4,19)| \cdot \Big(\frac{2}{4} + \frac{2}{6} + \frac{1}{12} + \frac{2}{24} + \frac{1}{54} + \frac{1}{120} + \frac{1}{648}\Big) \\
&= 151367507119250496000 \cdot \Big(\frac{2}{4} + \frac{2}{6} + \frac{1}{12} + \frac{2}{24} + \frac{1}{54} + \frac{1}{120} + \frac{1}{648}\Big) \\
&= 155665596827574892800 \\
&= \frac{19^6(19^2-1)(19^3-1)(19^4-1)(19-2)(19-3)(19-5)^2}{51840}
\end{aligned}$$

Appendices

Appendix A

An example of a cubic surface with 27 lines in $\mathrm{PG}(3, 13)$

A.1 Cubic surface $\mathcal{F}$

$\mathcal{F} = V(12x_0^2x_3 + 12x_1^2x_3 + 12x_2^2x_3 + 9x_0x_1x_2 + x_3^3)$

A.1.1 Points on $\mathcal{F}$

$P'_0 = (1, 0, 0, 0)$	$P'_{12} = (10, 1, 0, 0)$	$P'_{24} = (10, 0, 1, 0)$	$P'_{36} = (0, 10, 1, 0)$
$P'_1 = (0, 1, 0, 0)$	$P'_{13} = (11, 1, 0, 0)$	$P'_{25} = (11, 0, 1, 0)$	$P'_{37} = (0, 11, 1, 0)$
$P'_2 = (0, 0, 1, 0)$	$P'_{14} = (12, 1, 0, 0)$	$P'_{26} = (12, 0, 1, 0)$	$P'_{38} = (0, 12, 1, 0)$
$P'_3 = (1, 1, 0, 0)$	$P'_{15} = (1, 0, 1, 0)$	$P'_{27} = (0, 1, 1, 0)$	$P'_{39} = (1, 0, 0, 1)$
$P'_4 = (2, 1, 0, 0)$	$P'_{16} = (2, 0, 1, 0)$	$P'_{28} = (0, 2, 1, 0)$	$P'_{40} = (12, 0, 0, 1)$
$P'_5 = (3, 1, 0, 0)$	$P'_{17} = (3, 0, 1, 0)$	$P'_{29} = (0, 3, 1, 0)$	$P'_{41} = (0, 1, 0, 1)$
$P'_6 = (4, 1, 0, 0)$	$P'_{18} = (4, 0, 1, 0)$	$P'_{30} = (0, 4, 1, 0)$	$P'_{42} = (6, 2, 0, 1)$
$P'_7 = (5, 1, 0, 0)$	$P'_{19} = (5, 0, 1, 0)$	$P'_{31} = (0, 5, 1, 0)$	$P'_{43} = (7, 2, 0, 1)$
$P'_8 = (6, 1, 0, 0)$	$P'_{20} = (6, 0, 1, 0)$	$P'_{32} = (0, 6, 1, 0)$	$P'_{44} = (2, 6, 0, 1)$
$P'_9 = (7, 1, 0, 0)$	$P'_{21} = (7, 0, 1, 0)$	$P'_{33} = (0, 7, 1, 0)$	$P'_{45} = (11, 6, 0, 1)$
$P'_{10} = (8, 1, 0, 0)$	$P'_{22} = (8, 0, 1, 0)$	$P'_{34} = (0, 8, 1, 0)$	$P'_{46} = (2, 7, 0, 1)$
$P'_{11} = (9, 1, 0, 0)$	$P'_{23} = (9, 0, 1, 0)$	$P'_{35} = (0, 9, 1, 0)$	$P'_{47} = (11, 7, 0, 1)$

$P'_{48} = (6, 11, 0, 1)$
$P'_{49} = (7, 11, 0, 1)$
$P'_{50} = (0, 12, 0, 1)$
$P'_{51} = (0, 0, 1, 1)$
$P'_{52} = (2, 1, 1, 1)$
$P'_{53} = (7, 1, 1, 1)$
$P'_{54} = (1, 2, 1, 1)$
$P'_{55} = (4, 2, 1, 1)$
$P'_{56} = (6, 3, 1, 1)$
$P'_{57} = (8, 3, 1, 1)$
$P'_{58} = (2, 4, 1, 1)$
$P'_{59} = (8, 4, 1, 1)$
$P'_{60} = (9, 5, 1, 1)$
$P'_{61} = (10, 5, 1, 1)$
$P'_{62} = (3, 6, 1, 1)$
$P'_{63} = (12, 6, 1, 1)$
$P'_{64} = (1, 7, 1, 1)$
$P'_{65} = (10, 7, 1, 1)$
$P'_{66} = (3, 8, 1, 1)$
$P'_{67} = (4, 8, 1, 1)$
$P'_{68} = (5, 9, 1, 1)$
$P'_{69} = (11, 9, 1, 1)$
$P'_{70} = (5, 10, 1, 1)$
$P'_{71} = (7, 10, 1, 1)$
$P'_{72} = (9, 11, 1, 1)$
$P'_{73} = (12, 11, 1, 1)$
$P'_{74} = (6, 12, 1, 1)$
$P'_{75} = (11, 12, 1, 1)$
$P'_{76} = (6, 0, 2, 1)$
$P'_{77} = (7, 0, 2, 1)$
$P'_{78} = (1, 1, 2, 1)$
$P'_{79} = (4, 1, 2, 1)$
$P'_{80} = (1, 4, 2, 1)$
$P'_{81} = (6, 4, 2, 1)$
$P'_{82} = (0, 6, 2, 1)$
$P'_{83} = (4, 6, 2, 1)$
$P'_{84} = (0, 7, 2, 1)$
$P'_{85} = (9, 7, 2, 1)$
$P'_{86} = (7, 9, 2, 1)$
$P'_{87} = (12, 9, 2, 1)$
$P'_{88} = (9, 12, 2, 1)$
$P'_{89} = (12, 12, 2, 1)$
$P'_{90} = (6, 1, 3, 1)$
$P'_{91} = (8, 1, 3, 1)$
$P'_{92} = (6, 5, 3, 1)$
$P'_{93} = (12, 5, 3, 1)$
$P'_{94} = (1, 6, 3, 1)$
$P'_{95} = (5, 6, 3, 1)$
$P'_{96} = (8, 7, 3, 1)$
$P'_{97} = (12, 7, 3, 1)$
$P'_{98} = (1, 8, 3, 1)$
$P'_{99} = (7, 8, 3, 1)$
$P'_{100} = (5, 12, 3, 1)$
$P'_{101} = (7, 12, 3, 1)$
$P'_{102} = (2, 1, 4, 1)$
$P'_{103} = (8, 1, 4, 1)$
$P'_{104} = (1, 2, 4, 1)$
$P'_{105} = (6, 2, 4, 1)$
$P'_{106} = (12, 5, 4, 1)$
$P'_{107} = (2, 6, 4, 1)$
$P'_{108} = (6, 6, 4, 1)$
$P'_{109} = (7, 7, 4, 1)$
$P'_{110} = (11, 7, 4, 1)$
$P'_{111} = (1, 8, 4, 1)$
$P'_{112} = (7, 11, 4, 1)$
$P'_{113} = (12, 11, 4, 1)$
$P'_{114} = (5, 12, 4, 1)$
$P'_{115} = (11, 12, 4, 1)$
$P'_{116} = (9, 1, 5, 1)$
$P'_{117} = (10, 1, 5, 1)$
$P'_{118} = (6, 3, 5, 1)$
$P'_{119} = (12, 3, 5, 1)$
$P'_{120} = (12, 4, 5, 1)$
$P'_{121} = (3, 6, 5, 1)$
$P'_{122} = (7, 6, 5, 1)$
$P'_{123} = (6, 7, 5, 1)$
$P'_{124} = (10, 7, 5, 1)$
$P'_{125} = (1, 9, 5, 1)$
$P'_{126} = (1, 10, 5, 1)$
$P'_{127} = (7, 10, 5, 1)$
$P'_{128} = (3, 12, 5, 1)$
$P'_{129} = (4, 12, 5, 1)$
$P'_{130} = (2, 0, 6, 1)$
$P'_{131} = (11, 0, 6, 1)$
$P'_{132} = (3, 1, 6, 1)$
$P'_{133} = (12, 1, 6, 1)$
$P'_{134} = (0, 2, 6, 1)$
$P'_{135} = (4, 2, 6, 1)$
$P'_{136} = (1, 3, 6, 1)$
$P'_{137} = (5, 3, 6, 1)$
$P'_{138} = (2, 4, 6, 1)$
$P'_{139} = (6, 4, 6, 1)$
$P'_{140} = (3, 5, 6, 1)$
$P'_{141} = (7, 5, 6, 1)$
$P'_{142} = (4, 6, 6, 1)$
$P'_{143} = (8, 6, 6, 1)$
$P'_{144} = (5, 7, 6, 1)$
$P'_{145} = (9, 7, 6, 1)$
$P'_{146} = (6, 8, 6, 1)$
$P'_{147} = (10, 8, 6, 1)$
$P'_{148} = (7, 9, 6, 1)$
$P'_{149} = (11, 9, 6, 1)$
$P'_{150} = (8, 10, 6, 1)$
$P'_{151} = (12, 10, 6, 1)$
$P'_{152} = (0, 11, 6, 1)$
$P'_{153} = (9, 11, 6, 1)$
$P'_{154} = (1, 12, 6, 1)$
$P'_{155} = (10, 12, 6, 1)$
$P'_{156} = (2, 0, 7, 1)$
$P'_{157} = (11, 0, 7, 1)$
$P'_{158} = (1, 1, 7, 1)$
$P'_{159} = (10, 1, 7, 1)$
$P'_{160} = (0, 2, 7, 1)$
$P'_{161} = (9, 2, 7, 1)$
$P'_{162} = (8, 3, 7, 1)$
$P'_{163} = (12, 3, 7, 1)$
$P'_{164} = (7, 4, 7, 1)$
$P'_{165} = (11, 4, 7, 1)$
$P'_{166} = (6, 5, 7, 1)$
$P'_{167} = (10, 5, 7, 1)$
$P'_{168} = (5, 6, 7, 1)$
$P'_{169} = (9, 6, 7, 1)$
$P'_{170} = (4, 7, 7, 1)$
$P'_{171} = (8, 7, 7, 1)$

$P'_{172} = (3, 8, 7, 1)$
$P'_{173} = (7, 8, 7, 1)$
$P'_{174} = (2, 9, 7, 1)$
$P'_{175} = (6, 9, 7, 1)$
$P'_{176} = (1, 10, 7, 1)$
$P'_{177} = (5, 10, 7, 1)$
$P'_{178} = (0, 11, 7, 1)$
$P'_{179} = (4, 11, 7, 1)$
$P'_{180} = (3, 12, 7, 1)$
$P'_{181} = (12, 12, 7, 1)$
$P'_{182} = (3, 1, 8, 1)$
$P'_{183} = (4, 1, 8, 1)$
$P'_{184} = (1, 3, 8, 1)$
$P'_{185} = (7, 3, 8, 1)$
$P'_{186} = (1, 4, 8, 1)$
$P'_{187} = (6, 6, 8, 1)$
$P'_{188} = (10, 6, 8, 1)$
$P'_{189} = (3, 7, 8, 1)$
$P'_{190} = (7, 7, 8, 1)$
$P'_{191} = (12, 9, 8, 1)$
$P'_{192} = (6, 10, 8, 1)$
$P'_{193} = (12, 10, 8, 1)$
$P'_{194} = (9, 12, 8, 1)$
$P'_{195} = (10, 12, 8, 1)$
$P'_{196} = (5, 1, 9, 1)$
$P'_{197} = (11, 1, 9, 1)$
$P'_{198} = (7, 2, 9, 1)$
$P'_{199} = (12, 2, 9, 1)$
$P'_{200} = (1, 5, 9, 1)$
$P'_{201} = (7, 6, 9, 1)$
$P'_{202} = (11, 6, 9, 1)$
$P'_{203} = (2, 7, 9, 1)$
$P'_{204} = (6, 7, 9, 1)$
$P'_{205} = (12, 8, 9, 1)$
$P'_{206} = (1, 11, 9, 1)$
$P'_{207} = (6, 11, 9, 1)$
$P'_{208} = (2, 12, 9, 1)$
$P'_{209} = (8, 12, 9, 1)$
$P'_{210} = (5, 1, 10, 1)$
$P'_{211} = (7, 1, 10, 1)$
$P'_{212} = (1, 5, 10, 1)$
$P'_{213} = (7, 5, 10, 1)$
$P'_{214} = (8, 6, 10, 1)$
$P'_{215} = (12, 6, 10, 1)$
$P'_{216} = (1, 7, 10, 1)$
$P'_{217} = (5, 7, 10, 1)$
$P'_{218} = (6, 8, 10, 1)$
$P'_{219} = (12, 8, 10, 1)$
$P'_{220} = (6, 12, 10, 1)$
$P'_{221} = (8, 12, 10, 1)$
$P'_{222} = (6, 0, 11, 1)$
$P'_{223} = (7, 0, 11, 1)$
$P'_{224} = (9, 1, 11, 1)$
$P'_{225} = (12, 1, 11, 1)$
$P'_{226} = (7, 4, 11, 1)$
$P'_{227} = (12, 4, 11, 1)$
$P'_{228} = (0, 6, 11, 1)$
$P'_{229} = (9, 6, 11, 1)$
$P'_{230} = (0, 7, 11, 1)$
$P'_{231} = (4, 7, 11, 1)$
$P'_{232} = (1, 9, 11, 1)$
$P'_{233} = (6, 9, 11, 1)$
$P'_{234} = (1, 12, 11, 1)$
$P'_{235} = (4, 12, 11, 1)$
$P'_{236} = (0, 0, 12, 1)$
$P'_{237} = (6, 1, 12, 1)$
$P'_{238} = (11, 1, 12, 1)$
$P'_{239} = (9, 2, 12, 1)$
$P'_{240} = (12, 2, 12, 1)$
$P'_{241} = (5, 3, 12, 1)$
$P'_{242} = (7, 3, 12, 1)$
$P'_{243} = (5, 4, 12, 1)$
$P'_{244} = (11, 4, 12, 1)$
$P'_{245} = (3, 5, 12, 1)$
$P'_{246} = (4, 5, 12, 1)$
$P'_{247} = (1, 6, 12, 1)$
$P'_{248} = (10, 6, 12, 1)$
$P'_{249} = (3, 7, 12, 1)$
$P'_{250} = (12, 7, 12, 1)$
$P'_{251} = (9, 8, 12, 1)$
$P'_{252} = (10, 8, 12, 1)$
$P'_{253} = (2, 9, 12, 1)$
$P'_{254} = (8, 9, 12, 1)$
$P'_{255} = (6, 10, 12, 1)$
$P'_{256} = (8, 10, 12, 1)$
$P'_{257} = (1, 11, 12, 1)$
$P'_{258} = (4, 11, 12, 1)$
$P'_{259} = (2, 12, 12, 1)$
$P'_{260} = (7, 12, 12, 1)$

A.1.2 Model 1: Clebsch map of $\mathcal{F}$ for $\mathscr{D}^1$

i	P_i'	lines	$\Phi_{\mathscr{D}^1}(P_i')$
0	$(1,0,0,0)$	c_{12}, c_{34}	undef
1	$(0,1,0,0)$	c_{12}, c_{56}	undef
2	$(0,0,1,0)$	c_{34}, c_{56}	$(1,12,1)$
3	$(1,1,0,0)$	c_{12}, c_{35}, c_{46}	undef
4	$(2,1,0,0)$	b_2, c_{12}	undef
5	$(3,1,0,0)$	c_{12}	undef
6	$(4,1,0,0)$	c_{12}	undef
7	$(5,1,0,0)$	c_{12}	undef
8	$(6,1,0,0)$	a_2, c_{12}	$(1,0,0)$
9	$(7,1,0,0)$	a_1, c_{12}	$(0,0,1)$
10	$(8,1,0,0)$	c_{12}	undef
11	$(9,1,0,0)$	c_{12}	undef
12	$(10,1,0,0)$	c_{12}	undef
13	$(11,1,0,0)$	b_1, c_{12}	undef
14	$(12,1,0,0)$	c_{12}, c_{36}, c_{45}	undef
15	$(1,0,1,0)$	c_{16}, c_{25}, c_{34}	$(1,8,1)$
16	$(2,0,1,0)$	a_4, c_{34}	$(1,6,1)$
17	$(3,0,1,0)$	c_{34}	$(1,10,1)$
18	$(4,0,1,0)$	c_{34}	$(1,4,1)$
19	$(5,0,1,0)$	c_{34}	$(1,9,1)$
20	$(6,0,1,0)$	b_4, c_{34}	$(1,3,1)$
21	$(7,0,1,0)$	b_3, c_{34}	undef
22	$(8,0,1,0)$	c_{34}	$(1,5,1)$
23	$(9,0,1,0)$	c_{34}	$(1,1,1)$
24	$(10,0,1,0)$	c_{34}	$(1,2,1)$
25	$(11,0,1,0)$	a_3, c_{34}	$(0,1,0)$
26	$(12,0,1,0)$	c_{15}, c_{26}, c_{34}	$(1,11,1)$

i	P_i'	lines	$\Phi_{\mathscr{D}^1}(P_i')$
27	$(0,1,1,0)$	c_{14}, c_{23}, c_{56}	$(11,1,0)$
28	$(0,2,1,0)$	b_5, c_{56}	$(10,1,1)$
29	$(0,3,1,0)$	c_{56}	$(11,7,1)$
30	$(0,4,1,0)$	c_{56}	$(7,9,1)$
31	$(0,5,1,0)$	c_{56}	$(5,10,1)$
32	$(0,6,1,0)$	a_5, c_{56}	$(9,8,1)$
33	$(0,7,1,0)$	a_6, c_{56}	$(3,11,1)$
34	$(0,8,1,0)$	c_{56}	$(8,2,1)$
35	$(0,9,1,0)$	c_{56}	$(2,5,1)$
36	$(0,10,1,0)$	c_{56}	$(6,3,1)$
37	$(0,11,1,0)$	b_6, c_{56}	$(4,4,1)$
38	$(0,12,1,0)$	c_{13}, c_{24}, c_{56}	undef
39	$(1,0,0,1)$	a_6, b_5	$(3,11,1)$
40	$(12,0,0,1)$	a_5, b_6	$(9,8,1)$
41	$(0,1,0,1)$	a_4, b_3	$(1,6,1)$
42	$(6,2,0,1)$	c_{23}	$(7,1,0)$
43	$(7,2,0,1)$	c_{13}	undef
44	$(2,6,0,1)$	c_{16}	$(10,2,1)$
45	$(11,6,0,1)$	c_{25}	$(4,8,1)$
46	$(2,7,0,1)$	c_{26}	$(5,11,1)$
47	$(11,7,0,1)$	c_{15}	$(8,10,1)$
48	$(6,11,0,1)$	c_{14}	$(6,10,1)$
49	$(7,11,0,1)$	c_{24}	$(11,6,1)$
50	$(0,12,0,1)$	a_3, b_4	$(0,1,0)$
51	$(0,0,1,1)$	a_1, b_2	$(0,0,1)$
52	$(2,1,1,1)$	a_4, b_2	$(1,6,1)$
53	$(7,1,1,1)$	a_1, b_3, c_{13}	$(0,0,1)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$
54	$(1,2,1,1)$	a_1, b_5	$(0,0,1)$
55	$(4,2,1,1)$	b_2	undef
56	$(6,3,1,1)$	b_2, c_{23}	undef
57	$(8,3,1,1)$	a_1	$(0,0,1)$
58	$(2,4,1,1)$	a_1	$(0,0,1)$
59	$(8,4,1,1)$	b_2	undef
60	$(9,5,1,1)$	a_1	$(0,0,1)$
61	$(10,5,1,1)$	b_2	undef
62	$(3,6,1,1)$	a_1, c_{16}	$(0,0,1)$
63	$(12,6,1,1)$	a_5, b_2, c_{25}	$(9,8,1)$
64	$(1,7,1,1)$	a_6, b_2, c_{26}	$(3,11,1)$
65	$(10,7,1,1)$	a_1, c_{15}	$(0,0,1)$
66	$(3,8,1,1)$	b_2	undef
67	$(4,8,1,1)$	a_1	$(0,0,1)$
68	$(5,9,1,1)$	b_2	undef
69	$(11,9,1,1)$	a_1	$(0,0,1)$
70	$(5,10,1,1)$	a_1	$(0,0,1)$
71	$(7,10,1,1)$	b_2, c_{24}	undef
72	$(9,11,1,1)$	b_2	undef
73	$(12,11,1,1)$	a_1, b_6	$(0,0,1)$
74	$(6,12,1,1)$	a_1, b_4, c_{14}	$(0,0,1)$
75	$(11,12,1,1)$	a_3, b_2	$(0,1,0)$
76	$(6,0,2,1)$	c_{14}	$(7,3,1)$
77	$(7,0,2,1)$	c_{13}	undef
78	$(1,1,2,1)$	a_6, b_3	$(3,11,1)$
79	$(4,1,2,1)$	a_4	$(1,6,1)$
80	$(1,4,2,1)$	b_5	$(12,3,1)$
81	$(6,4,2,1)$	c_{23}	$(1,1,0)$
82	$(0,6,2,1)$	c_{25}	$(2,8,1)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$
83	$(4,6,2,1)$	c_{16}	$(6,9,1)$
84	$(0,7,2,1)$	c_{26}	$(11,11,1)$
85	$(9,7,2,1)$	c_{15}	$(4,5,1)$
86	$(7,9,2,1)$	c_{24}	$(8,6,1)$
87	$(12,9,2,1)$	b_6	$(10,7,1)$
88	$(9,12,2,1)$	a_3	$(0,1,0)$
89	$(12,12,2,1)$	a_5, b_4	$(9,8,1)$
90	$(6,1,3,1)$	a_4, c_{14}	$(1,6,1)$
91	$(8,1,3,1)$	b_3	undef
92	$(6,5,3,1)$	c_{23}	$(10,1,0)$
93	$(12,5,3,1)$	a_5	$(9,8,1)$
94	$(1,6,3,1)$	b_5, c_{25}	$(5,8,1)$
95	$(5,6,3,1)$	c_{16}	$(11,10,1)$
96	$(8,7,3,1)$	c_{15}	$(12,2,1)$
97	$(12,7,3,1)$	b_6, c_{26}	$(2,11,1)$
98	$(1,8,3,1)$	a_6	$(3,11,1)$
99	$(7,8,3,1)$	c_{24}	$(6,6,1)$
100	$(5,12,3,1)$	b_4	$(7,2,1)$
101	$(7,12,3,1)$	a_3, c_{13}	$(0,1,0)$
102	$(2,1,4,1)$	b_3	undef
103	$(8,1,4,1)$	a_4	$(1,6,1)$
104	$(1,2,4,1)$	a_6	$(3,11,1)$
105	$(6,2,4,1)$	c_{14}	$(8,9,1)$
106	$(12,5,4,1)$	b_6	$(5,9,1)$
107	$(2,6,4,1)$	c_{25}	$(12,8,1)$
108	$(6,6,4,1)$	c_{16}, c_{23}	$(5,1,0)$
109	$(7,7,4,1)$	c_{15}, c_{24}	$(10,6,1)$
110	$(11,7,4,1)$	c_{26}	$(7,11,1)$
111	$(1,8,4,1)$	b_5	$(6,5,1)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$
112	$(7,11,4,1)$	c_{13}	undef
113	$(12,11,4,1)$	a_5	$(9,8,1)$
114	$(5,12,4,1)$	a_3	$(0,1,0)$
115	$(11,12,4,1)$	b_4	$(11,12,1)$
116	$(9,1,5,1)$	b_3	undef
117	$(10,1,5,1)$	a_4	$(1,6,1)$
118	$(6,3,5,1)$	c_{14}	$(2,12,1)$
119	$(12,3,5,1)$	b_6	$(7,5,1)$
120	$(12,4,5,1)$	a_5	$(9,8,1)$
121	$(3,6,5,1)$	c_{25}	$(8,8,1)$
122	$(7,6,5,1)$	c_{16},c_{24}	$(4,6,1)$
123	$(6,7,5,1)$	c_{15},c_{23}	$(6,1,0)$
124	$(10,7,5,1)$	c_{26}	$(6,11,1)$
125	$(1,9,5,1)$	a_6	$(3,11,1)$
126	$(1,10,5,1)$	b_5	$(11,9,1)$
127	$(7,10,5,1)$	c_{13}	undef
128	$(3,12,5,1)$	a_3	$(0,1,0)$
129	$(4,12,5,1)$	b_4	$(5,5,1)$
130	$(2,0,6,1)$	c_{35}	$(9,10,1)$
131	$(11,0,6,1)$	c_{46}	$(2,2,1)$
132	$(3,1,6,1)$	b_3,c_{35}	undef
133	$(12,1,6,1)$	a_4,b_6,c_{46}	$(1,6,1)$
134	$(0,2,6,1)$	c_{46}	$(4,7,1)$
135	$(4,2,6,1)$	c_{35}	$(9,12,1)$
136	$(1,3,6,1)$	a_6,c_{46}	$(3,11,1)$
137	$(5,3,6,1)$	c_{35}	$(9,9,1)$
138	$(2,4,6,1)$	c_{46}	$(8,4,1)$
139	$(6,4,6,1)$	c_{14},c_{35}	$(9,2,1)$
140	$(3,5,6,1)$	c_{46}	$(12,1,1)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$
141	$(7,5,6,1)$	c_{24},c_{35}	$(9,6,1)$
142	$(4,6,6,1)$	c_{25},c_{46}	$(7,8,1)$
143	$(8,6,6,1)$	c_{16},c_{35}	$(9,7,1)$
144	$(5,7,6,1)$	c_{15},c_{46}	$(5,3,1)$
145	$(9,7,6,1)$	c_{26},c_{35}	$(9,11,1)$
146	$(6,8,6,1)$	c_{23},c_{46}	$(3,1,0)$
147	$(10,8,6,1)$	c_{35}	$(9,4,1)$
148	$(7,9,6,1)$	c_{13},c_{46}	undef
149	$(11,9,6,1)$	c_{35}	$(9,1,1)$
150	$(8,10,6,1)$	c_{46}	$(11,5,1)$
151	$(12,10,6,1)$	a_5,c_{35}	$(9,8,1)$
152	$(0,11,6,1)$	c_{35}	$(9,3,1)$
153	$(9,11,6,1)$	c_{46}	$(6,12,1)$
154	$(1,12,6,1)$	a_3,b_5,c_{35}	$(0,1,0)$
155	$(10,12,6,1)$	b_4,c_{46}	$(10,9,1)$
156	$(2,0,7,1)$	c_{45}	$(7,1,1)$
157	$(11,0,7,1)$	c_{36}	$(3,4,1)$
158	$(1,1,7,1)$	a_4,b_5,c_{45}	$(1,6,1)$
159	$(10,1,7,1)$	b_3,c_{36}	undef
160	$(0,2,7,1)$	c_{45}	$(10,5,1)$
161	$(9,2,7,1)$	c_{36}	$(3,10,1)$
162	$(8,3,7,1)$	c_{36}	$(3,1,1)$
163	$(12,3,7,1)$	a_5,c_{45}	$(9,8,1)$
164	$(7,4,7,1)$	c_{24},c_{36}	$(3,6,1)$
165	$(11,4,7,1)$	c_{45}	$(5,7,1)$
166	$(6,5,7,1)$	c_{14},c_{36}	$(3,5,1)$
167	$(10,5,7,1)$	c_{45}	$(12,12,1)$
168	$(5,6,7,1)$	c_{25},c_{36}	$(3,8,1)$
169	$(9,6,7,1)$	c_{16},c_{45}	$(2,3,1)$

i	P_i'	lines	$\Phi_{\mathscr{D}^1}(P_i')$
170	$(4,7,7,1)$	c_{15}, c_{36}	$(3,7,1)$
171	$(8,7,7,1)$	c_{26}, c_{45}	$(8,11,1)$
172	$(3,8,7,1)$	c_{36}	$(3,12,1)$
173	$(7,8,7,1)$	c_{13}, c_{45}	undef
174	$(2,9,7,1)$	c_{36}	$(3,3,1)$
175	$(6,9,7,1)$	c_{23}, c_{45}	$(4,1,0)$
176	$(1,10,7,1)$	a_6, c_{36}	$(3,11,1)$
177	$(5,10,7,1)$	c_{45}	$(6,4,1)$
178	$(0,11,7,1)$	c_{36}	$(3,9,1)$
179	$(4,11,7,1)$	c_{45}	$(11,2,1)$
180	$(3,12,7,1)$	b_4, c_{45}	$(4,10,1)$
181	$(12,12,7,1)$	a_3, b_6, c_{36}	$(0,1,0)$
182	$(3,1,8,1)$	a_4	$(1,6,1)$
183	$(4,1,8,1)$	b_3	undef
184	$(1,3,8,1)$	b_5	$(2,10,1)$
185	$(7,3,8,1)$	c_{24}	$(7,6,1)$
186	$(1,4,8,1)$	a_6	$(3,11,1)$
187	$(6,6,8,1)$	c_{14}, c_{25}	$(10,8,1)$
188	$(10,6,8,1)$	c_{16}	$(5,1,1)$
189	$(3,7,8,1)$	c_{15}	$(11,4,1)$
190	$(7,7,8,1)$	c_{13}, c_{26}	undef
191	$(12,9,8,1)$	a_5	$(9,8,1)$
192	$(6,10,8,1)$	c_{23}	$(12,1,0)$
193	$(12,10,8,1)$	b_6	$(6,2,1)$
194	$(9,12,8,1)$	b_4	$(8,1,1)$
195	$(10,12,8,1)$	a_3	$(0,1,0)$
196	$(5,1,9,1)$	a_4	$(1,6,1)$
197	$(11,1,9,1)$	b_3	undef
198	$(7,2,9,1)$	c_{24}	$(5,6,1)$

i	P_i'	lines	$\Phi_{\mathscr{D}^1}(P_i')$
199	$(12,2,9,1)$	a_5	$(9,8,1)$
200	$(1,5,9,1)$	b_5	$(8,7,1)$
201	$(7,6,9,1)$	c_{13}, c_{25}	undef
202	$(11,6,9,1)$	c_{16}	$(12,5,1)$
203	$(2,7,9,1)$	c_{15}	$(2,9,1)$
204	$(6,7,9,1)$	c_{14}, c_{26}	$(4,11,1)$
205	$(12,8,9,1)$	b_6	$(11,3,1)$
206	$(1,11,9,1)$	a_6	$(3,11,1)$
207	$(6,11,9,1)$	c_{23}	$(8,1,0)$
208	$(2,12,9,1)$	b_4	$(6,7,1)$
209	$(8,12,9,1)$	a_3	$(0,1,0)$
210	$(5,1,10,1)$	b_3	undef
211	$(7,1,10,1)$	a_4, c_{24}	$(1,6,1)$
212	$(1,5,10,1)$	a_6	$(3,11,1)$
213	$(7,5,10,1)$	c_{13}	undef
214	$(8,6,10,1)$	c_{25}	$(6,8,1)$
215	$(12,6,10,1)$	b_6, c_{16}	$(8,12,1)$
216	$(1,7,10,1)$	b_5, c_{15}	$(7,12,1)$
217	$(5,7,10,1)$	c_{26}	$(12,11,1)$
218	$(6,8,10,1)$	c_{14}	$(11,1,1)$
219	$(12,8,10,1)$	a_5	$(9,8,1)$
220	$(6,12,10,1)$	a_3, c_{23}	$(0,1,0)$
221	$(8,12,10,1)$	b_4	$(2,4,1)$
222	$(6,0,11,1)$	c_{23}	$(2,1,0)$
223	$(7,0,11,1)$	c_{24}	$(2,6,1)$
224	$(9,1,11,1)$	a_4	$(1,6,1)$
225	$(12,1,11,1)$	a_5, b_3	$(9,8,1)$
226	$(7,4,11,1)$	c_{13}	undef
227	$(12,4,11,1)$	b_6	$(12,10,1)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$
228	$(0,6,11,1)$	c_{16}	$(7,4,1)$
229	$(9,6,11,1)$	c_{25}	$(11,8,1)$
230	$(0,7,11,1)$	c_{15}	$(6,1,1)$
231	$(4,7,11,1)$	c_{26}	$(10,11,1)$
232	$(1,9,11,1)$	b_5	$(4,2,1)$
233	$(6,9,11,1)$	c_{14}	$(5,4,1)$
234	$(1,12,11,1)$	a_6, b_4	$(3,11,1)$
235	$(4,12,11,1)$	a_3	$(0,1,0)$
236	$(0,0,12,1)$	a_2, b_1	$(1,0,0)$
237	$(6,1,12,1)$	a_2, b_3, c_{23}	$(1,0,0)$
238	$(11,1,12,1)$	a_4, b_1	$(1,6,1)$
239	$(9,2,12,1)$	b_1	undef
240	$(12,2,12,1)$	a_2, b_6	$(1,0,0)$
241	$(5,3,12,1)$	a_2	$(1,0,0)$
242	$(7,3,12,1)$	b_1, c_{13}	undef
243	$(5,4,12,1)$	b_1	undef
244	$(11,4,12,1)$	a_2	$(1,0,0)$
245	$(3,5,12,1)$	b_1	undef
246	$(4,5,12,1)$	a_2	$(1,0,0)$
247	$(1,6,12,1)$	a_6, b_1, c_{16}	$(3,11,1)$
248	$(10,6,12,1)$	a_2, c_{25}	$(1,0,0)$
249	$(3,7,12,1)$	a_2, c_{26}	$(1,0,0)$
250	$(12,7,12,1)$	a_5, b_1, c_{15}	$(9,8,1)$
251	$(9,8,12,1)$	a_2	$(1,0,0)$
252	$(10,8,12,1)$	b_1	undef
253	$(2,9,12,1)$	a_2	$(1,0,0)$
254	$(8,9,12,1)$	b_1	undef
255	$(6,10,12,1)$	b_1, c_{14}	undef
256	$(8,10,12,1)$	a_2	$(1,0,0)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$
257	$(1,11,12,1)$	a_2, b_5	$(1,0,0)$
258	$(4,11,12,1)$	b_1	undef
259	$(2,12,12,1)$	a_3, b_1	$(0,1,0)$
260	$(7,12,12,1)$	a_2, b_4, c_{24}	$(1,0,0)$

A.1.3 Model 2: Clebsch map of $\mathcal{F}$ for $\mathscr{D}^1$

i	P_i'	lines	$\Phi_{\mathscr{D}^1}(P_i')$	$\Phi_{\mathscr{D}^1}(P_i')$
0	$(1,0,0,0)$	c_{12}, c_{34}	undef	undef
1	$(0,1,0,0)$	c_{12}, c_{56}	undef	undef
2	$(0,0,1,0)$	c_{34}, c_{56}	$(0,0,1,0)$	$(0,0,1)$
3	$(1,1,0,0)$	c_{12}, c_{35}, c_{46}	undef	undef
4	$(2,1,0,0)$	b_2, c_{12}	undef	undef
5	$(3,1,0,0)$	c_{12}	undef	undef
6	$(4,1,0,0)$	c_{12}	undef	undef
7	$(5,1,0,0)$	c_{12}	undef	undef
8	$(6,1,0,0)$	a_2, c_{12}	undef	undef
9	$(7,1,0,0)$	a_1, c_{12}	undef	undef
10	$(8,1,0,0)$	c_{12}	undef	undef
11	$(9,1,0,0)$	c_{12}	undef	undef
12	$(10,1,0,0)$	c_{12}	undef	undef
13	$(11,1,0,0)$	b_1, c_{12}	undef	undef
14	$(12,1,0,0)$	c_{12}, c_{36}, c_{45}	undef	undef
15	$(1,0,1,0)$	c_{16}, c_{25}, c_{34}	$(1,0,1,0)$	$(1,0,1)$
16	$(2,0,1,0)$	a_4, c_{34}	$(2,0,1,0)$	$(2,0,1)$
17	$(3,0,1,0)$	c_{34}	$(3,0,1,0)$	$(3,0,1)$
18	$(4,0,1,0)$	c_{34}	$(4,0,1,0)$	$(4,0,1)$
19	$(5,0,1,0)$	c_{34}	$(5,0,1,0)$	$(5,0,1)$
20	$(6,0,1,0)$	b_4, c_{34}	$(6,0,1,0)$	$(6,0,1)$
21	$(7,0,1,0)$	b_3, c_{34}	$(7,0,1,0)$	$(7,0,1)$
22	$(8,0,1,0)$	c_{34}	$(8,0,1,0)$	$(8,0,1)$
23	$(9,0,1,0)$	c_{34}	$(9,0,1,0)$	$(9,0,1)$
24	$(10,0,1,0)$	c_{34}	$(10,0,1,0)$	$(10,0,1)$
25	$(11,0,1,0)$	a_3, c_{34}	$(11,0,1,0)$	$(11,0,1)$
26	$(12,0,1,0)$	c_{15}, c_{26}, c_{34}	$(12,0,1,0)$	$(12,0,1)$
27	$(0,1,1,0)$	c_{14}, c_{23}, c_{56}	$(0,1,1,0)$	$(0,1,1)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$	$\Phi_{\mathscr{D}^1}(P'_i)$
28	$(0,2,1,0)$	b_5, c_{56}	$(0,2,1,0)$	$(0,2,1)$
29	$(0,3,1,0)$	c_{56}	$(0,3,1,0)$	$(0,3,1)$
30	$(0,4,1,0)$	c_{56}	$(0,4,1,0)$	$(0,4,1)$
31	$(0,5,1,0)$	c_{56}	$(0,5,1,0)$	$(0,5,1)$
32	$(0,6,1,0)$	a_5, c_{56}	$(0,6,1,0)$	$(0,6,1)$
33	$(0,7,1,0)$	a_6, c_{56}	$(0,7,1,0)$	$(0,7,1)$
34	$(0,8,1,0)$	c_{56}	$(0,8,1,0)$	$(0,8,1)$
35	$(0,9,1,0)$	c_{56}	$(0,9,1,0)$	$(0,9,1)$
36	$(0,10,1,0)$	c_{56}	$(0,10,1,0)$	$(0,10,1)$
37	$(0,11,1,0)$	b_6, c_{56}	$(0,11,1,0)$	$(0,11,1)$
38	$(0,12,1,0)$	c_{13}, c_{24}, c_{56}	$(0,12,1,0)$	$(0,12,1)$
39	$(1,0,0,1)$	a_6, b_5	$(0,7,1,0)$	$(0,7,1)$
40	$(12,0,0,1)$	a_5, b_6	$(0,6,1,0)$	$(0,6,1)$
41	$(0,1,0,1)$	a_4, b_3	$(2,0,1,0)$	$(2,0,1)$
42	$(6,2,0,1)$	c_{23}	$(4,3,1,0)$	$(4,3,1)$
43	$(7,2,0,1)$	c_{13}	$(4,10,1,0)$	$(4,10,1)$
44	$(2,6,0,1)$	c_{16}	$(12,1,1,0)$	$(12,1,1)$
45	$(11,6,0,1)$	c_{25}	$(12,12,1,0)$	$(12,12,1)$
46	$(2,7,0,1)$	c_{26}	$(1,1,1,0)$	$(1,1,1)$
47	$(11,7,0,1)$	c_{15}	$(1,12,1,0)$	$(1,12,1)$
48	$(6,11,0,1)$	c_{14}	$(9,3,1,0)$	$(9,3,1)$
49	$(7,11,0,1)$	c_{24}	$(9,10,1,0)$	$(9,10,1)$
50	$(0,12,0,1)$	a_3, b_4	$(11,0,1,0)$	$(11,0,1)$
51	$(0,0,1,1)$	a_1, b_2	undef	undef
52	$(2,1,1,1)$	a_4, b_2	undef	undef
53	$(7,1,1,1)$	a_1, b_3, c_{13}	$(11,1,0,0)$	$(11,1,0)$
54	$(1,2,1,1)$	a_1, b_5	$(11,1,0,0)$	$(11,1,0)$
55	$(4,2,1,1)$	b_2	undef	undef
56	$(6,3,1,1)$	b_2, c_{23}	undef	undef
57	$(8,3,1,1)$	a_1	$(11,1,0,0)$	$(11,1,0)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$	$\Phi_{\mathscr{D}^1}(P'_i)$
58	$(2,4,1,1)$	a_1	$(11,1,0,0)$	$(11,1,0)$
59	$(8,4,1,1)$	b_2	undef	undef
60	$(9,5,1,1)$	a_1	$(11,1,0,0)$	$(11,1,0)$
61	$(10,5,1,1)$	b_2	undef	undef
62	$(3,6,1,1)$	a_1, c_{16}	$(11,1,0,0)$	$(11,1,0)$
63	$(12,6,1,1)$	a_5, b_2, c_{25}	undef	undef
64	$(1,7,1,1)$	a_6, b_2, c_{26}	undef	undef
65	$(10,7,1,1)$	a_1, c_{15}	$(11,1,0,0)$	$(11,1,0)$
66	$(3,8,1,1)$	b_2	undef	undef
67	$(4,8,1,1)$	a_1	$(11,1,0,0)$	$(11,1,0)$
68	$(5,9,1,1)$	b_2	undef	undef
69	$(11,9,1,1)$	a_1	$(11,1,0,0)$	$(11,1,0)$
70	$(5,10,1,1)$	a_1	$(11,1,0,0)$	$(11,1,0)$
71	$(7,10,1,1)$	b_2, c_{24}	undef	undef
72	$(9,11,1,1)$	b_2	undef	undef
73	$(12,11,1,1)$	a_1, b_6	$(11,1,0,0)$	$(11,1,0)$
74	$(6,12,1,1)$	a_1, b_4, c_{14}	$(11,1,0,0)$	$(11,1,0)$
75	$(11,12,1,1)$	a_3, b_2	undef	undef
76	$(6,0,2,1)$	c_{14}	$(4,12,1,0)$	$(4,12,1)$
77	$(7,0,2,1)$	c_{13}	$(9,1,1,0)$	$(9,1,1)$
78	$(1,1,2,1)$	a_6, b_3	$(0,7,1,0)$	$(0,7,1)$
79	$(4,1,2,1)$	a_4	$(2,0,1,0)$	$(2,0,1)$
80	$(1,4,2,1)$	b_5	$(11,9,1,0)$	$(11,9,1)$
81	$(6,4,2,1)$	c_{23}	$(10,6,1,0)$	$(10,6,1)$
82	$(0,6,2,1)$	c_{25}	$(9,4,1,0)$	$(9,4,1)$
83	$(4,6,2,1)$	c_{16}	$(3,12,1,0)$	$(3,12,1)$
84	$(0,7,2,1)$	c_{26}	$(4,9,1,0)$	$(4,9,1)$
85	$(9,7,2,1)$	c_{15}	$(10,1,1,0)$	$(10,1,1)$
86	$(7,9,2,1)$	c_{24}	$(3,7,1,0)$	$(3,7,1)$
87	$(12,9,2,1)$	b_6	$(2,4,1,0)$	$(2,4,1)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$	$\Phi_{\mathscr{D}^1}(P'_i)$
88	$(9,12,2,1)$	a_3	$(11,0,1,0)$	$(11,0,1)$
89	$(12,12,2,1)$	a_5, b_4	$(0,6,1,0)$	$(0,6,1)$
90	$(6,1,3,1)$	a_4, c_{14}	$(2,0,1,0)$	$(2,0,1)$
91	$(8,1,3,1)$	b_3	$(6,8,1,0)$	$(6,8,1)$
92	$(6,5,3,1)$	c_{23}	$(1,8,1,0)$	$(1,8,1)$
93	$(12,5,3,1)$	a_5	$(0,6,1,0)$	$(0,6,1)$
94	$(1,6,3,1)$	b_5, c_{25}	$(7,3,1,0)$	$(7,3,1)$
95	$(5,6,3,1)$	c_{16}	$(2,6,1,0)$	$(2,6,1)$
96	$(8,7,3,1)$	c_{15}	$(11,7,1,0)$	$(11,7,1)$
97	$(12,7,3,1)$	b_6, c_{26}	$(6,10,1,0)$	$(6,10,1)$
98	$(1,8,3,1)$	a_6	$(0,7,1,0)$	$(0,7,1)$
99	$(7,8,3,1)$	c_{24}	$(12,5,1,0)$	$(12,5,1)$
100	$(5,12,3,1)$	b_4	$(7,5,1,0)$	$(7,5,1)$
101	$(7,12,3,1)$	a_3, c_{13}	$(11,0,1,0)$	$(11,0,1)$
102	$(2,1,4,1)$	b_3	$(3,8,1,0)$	$(3,8,1)$
103	$(8,1,4,1)$	a_4	$(2,0,1,0)$	$(2,0,1)$
104	$(1,2,4,1)$	a_6	$(0,7,1,0)$	$(0,7,1)$
105	$(6,2,4,1)$	c_{14}	$(10,9,1,0)$	$(10,9,1)$
106	$(12,5,4,1)$	b_6	$(6,7,1,0)$	$(6,7,1)$
107	$(2,6,4,1)$	c_{25}	$(11,5,1,0)$	$(11,5,1)$
108	$(6,6,4,1)$	c_{16}, c_{23}	$(6,4,1,0)$	$(6,4,1)$
109	$(7,7,4,1)$	c_{15}, c_{24}	$(7,9,1,0)$	$(7,9,1)$
110	$(11,7,4,1)$	c_{26}	$(2,8,1,0)$	$(2,8,1)$
111	$(1,8,4,1)$	b_5	$(7,6,1,0)$	$(7,6,1)$
112	$(7,11,4,1)$	c_{13}	$(3,4,1,0)$	$(3,4,1)$
113	$(12,11,4,1)$	a_5	$(0,6,1,0)$	$(0,6,1)$
114	$(5,12,4,1)$	a_3	$(11,0,1,0)$	$(11,0,1)$
115	$(11,12,4,1)$	b_4	$(10,5,1,0)$	$(10,5,1)$
116	$(9,1,5,1)$	b_3	$(11,3,1,0)$	$(11,3,1)$
117	$(10,1,5,1)$	a_4	$(2,0,1,0)$	$(2,0,1)$

i	P_i'	lines	$\Phi_{\mathscr{D}^1}(P_i')$	$\Phi_{\mathscr{D}^1}(P_i')$
118	$(6,3,5,1)$	c_{14}	$(1,7,1,0)$	$(1,7,1)$
119	$(12,3,5,1)$	b_6	$(12,2,1,0)$	$(12,2,1)$
120	$(12,4,5,1)$	a_5	$(0,6,1,0)$	$(0,6,1)$
121	$(3,6,5,1)$	c_{25}	$(5,2,1,0)$	$(5,2,1)$
122	$(7,6,5,1)$	c_{16}, c_{24}	$(8,3,1,0)$	$(8,3,1)$
123	$(6,7,5,1)$	c_{15}, c_{23}	$(5,10,1,0)$	$(5,10,1)$
124	$(10,7,5,1)$	c_{26}	$(8,11,1,0)$	$(8,11,1)$
125	$(1,9,5,1)$	a_6	$(0,7,1,0)$	$(0,7,1)$
126	$(1,10,5,1)$	b_5	$(1,11,1,0)$	$(1,11,1)$
127	$(7,10,5,1)$	c_{13}	$(12,6,1,0)$	$(12,6,1)$
128	$(3,12,5,1)$	a_3	$(11,0,1,0)$	$(11,0,1)$
129	$(4,12,5,1)$	b_4	$(2,10,1,0)$	$(2,10,1)$
130	$(2,0,6,1)$	c_{35}	$(10,10,1,0)$	$(10,10,1)$
131	$(11,0,6,1)$	c_{46}	$(3,3,1,0)$	$(3,3,1)$
132	$(3,1,6,1)$	b_3, c_{35}	$(9,7,1,0)$	$(9,7,1)$
133	$(12,1,6,1)$	a_4, b_6, c_{46}	$(2,0,1,0)$	$(2,0,1)$
134	$(0,2,6,1)$	c_{46}	$(1,10,1,0)$	$(1,10,1)$
135	$(4,2,6,1)$	c_{35}	$(8,4,1,0)$	$(8,4,1)$
136	$(1,3,6,1)$	a_6, c_{46}	$(0,7,1,0)$	$(0,7,1)$
137	$(5,3,6,1)$	c_{35}	$(7,1,1,0)$	$(7,1,1)$
138	$(2,4,6,1)$	c_{46}	$(12,4,1,0)$	$(12,4,1)$
139	$(6,4,6,1)$	c_{14}, c_{35}	$(6,11,1,0)$	$(6,11,1)$
140	$(3,5,6,1)$	c_{46}	$(11,1,1,0)$	$(11,1,1)$
141	$(7,5,6,1)$	c_{24}, c_{35}	$(5,8,1,0)$	$(5,8,1)$
142	$(4,6,6,1)$	c_{25}, c_{46}	$(10,11,1,0)$	$(10,11,1)$
143	$(8,6,6,1)$	c_{16}, c_{35}	$(4,5,1,0)$	$(4,5,1)$
144	$(5,7,6,1)$	c_{15}, c_{46}	$(9,8,1,0)$	$(9,8,1)$
145	$(9,7,6,1)$	c_{26}, c_{35}	$(3,2,1,0)$	$(3,2,1)$
146	$(6,8,6,1)$	c_{23}, c_{46}	$(8,5,1,0)$	$(8,5,1)$
147	$(10,8,6,1)$	c_{35}	$(2,12,1,0)$	$(2,12,1)$

i	P_i'	lines	$\Phi_{\mathscr{D}^1}(P_i')$	$\Phi_{\mathscr{D}^1}(P_i')$
148	$(7,9,6,1)$	c_{13}, c_{46}	$(7,2,1,0)$	$(7,2,1)$
149	$(11,9,6,1)$	c_{35}	$(1,9,1,0)$	$(1,9,1)$
150	$(8,10,6,1)$	c_{46}	$(6,12,1,0)$	$(6,12,1)$
151	$(12,10,6,1)$	a_5, c_{35}	$(0,6,1,0)$	$(0,6,1)$
152	$(0,11,6,1)$	c_{35}	$(12,3,1,0)$	$(12,3,1)$
153	$(9,11,6,1)$	c_{46}	$(5,9,1,0)$	$(5,9,1)$
154	$(1,12,6,1)$	a_3, b_5, c_{35}	$(11,0,1,0)$	$(11,0,1)$
155	$(10,12,6,1)$	b_4, c_{46}	$(4,6,1,0)$	$(4,6,1)$
156	$(2,0,7,1)$	c_{45}	$(3,10,1,0)$	$(3,10,1)$
157	$(11,0,7,1)$	c_{36}	$(10,3,1,0)$	$(10,3,1)$
158	$(1,1,7,1)$	a_4, b_5, c_{45}	$(2,0,1,0)$	$(2,0,1)$
159	$(10,1,7,1)$	b_3, c_{36}	$(9,6,1,0)$	$(9,6,1)$
160	$(0,2,7,1)$	c_{45}	$(1,3,1,0)$	$(1,3,1)$
161	$(9,2,7,1)$	c_{36}	$(8,9,1,0)$	$(8,9,1)$
162	$(8,3,7,1)$	c_{36}	$(7,12,1,0)$	$(7,12,1)$
163	$(12,3,7,1)$	a_5, c_{45}	$(0,6,1,0)$	$(0,6,1)$
164	$(7,4,7,1)$	c_{24}, c_{36}	$(6,2,1,0)$	$(6,2,1)$
165	$(11,4,7,1)$	c_{45}	$(12,9,1,0)$	$(12,9,1)$
166	$(6,5,7,1)$	c_{14}, c_{36}	$(5,5,1,0)$	$(5,5,1)$
167	$(10,5,7,1)$	c_{45}	$(11,12,1,0)$	$(11,12,1)$
168	$(5,6,7,1)$	c_{25}, c_{36}	$(4,8,1,0)$	$(4,8,1)$
169	$(9,6,7,1)$	c_{16}, c_{45}	$(10,2,1,0)$	$(10,2,1)$
170	$(4,7,7,1)$	c_{15}, c_{36}	$(3,11,1,0)$	$(3,11,1)$
171	$(8,7,7,1)$	c_{26}, c_{45}	$(9,5,1,0)$	$(9,5,1)$
172	$(3,8,7,1)$	c_{36}	$(2,1,1,0)$	$(2,1,1)$
173	$(7,8,7,1)$	c_{13}, c_{45}	$(8,8,1,0)$	$(8,8,1)$
174	$(2,9,7,1)$	c_{36}	$(1,4,1,0)$	$(1,4,1)$
175	$(6,9,7,1)$	c_{23}, c_{45}	$(7,11,1,0)$	$(7,11,1)$
176	$(1,10,7,1)$	a_6, c_{36}	$(0,7,1,0)$	$(0,7,1)$
177	$(5,10,7,1)$	c_{45}	$(6,1,1,0)$	$(6,1,1)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$	$\Phi_{\mathscr{D}^1}(P'_i)$
178	$(0,11,7,1)$	c_{36}	$(12,10,1,0)$	$(12,10,1)$
179	$(4,11,7,1)$	c_{45}	$(5,4,1,0)$	$(5,4,1)$
180	$(3,12,7,1)$	b_4, c_{45}	$(4,7,1,0)$	$(4,7,1)$
181	$(12,12,7,1)$	a_3, b_6, c_{36}	$(11,0,1,0)$	$(11,0,1)$
182	$(3,1,8,1)$	a_4	$(2,0,1,0)$	$(2,0,1)$
183	$(4,1,8,1)$	b_3	$(11,10,1,0)$	$(11,10,1)$
184	$(1,3,8,1)$	b_5	$(12,11,1,0)$	$(12,11,1)$
185	$(7,3,8,1)$	c_{24}	$(1,6,1,0)$	$(1,6,1)$
186	$(1,4,8,1)$	a_6	$(0,7,1,0)$	$(0,7,1)$
187	$(6,6,8,1)$	c_{14}, c_{25}	$(8,10,1,0)$	$(8,10,1)$
188	$(10,6,8,1)$	c_{16}	$(5,11,1,0)$	$(5,11,1)$
189	$(3,7,8,1)$	c_{15}	$(8,2,1,0)$	$(8,2,1)$
190	$(7,7,8,1)$	c_{13}, c_{26}	$(5,3,1,0)$	$(5,3,1)$
191	$(12,9,8,1)$	a_5	$(0,6,1,0)$	$(0,6,1)$
192	$(6,10,8,1)$	c_{23}	$(12,7,1,0)$	$(12,7,1)$
193	$(12,10,8,1)$	b_6	$(1,2,1,0)$	$(1,2,1)$
194	$(9,12,8,1)$	b_4	$(2,3,1,0)$	$(2,3,1)$
195	$(10,12,8,1)$	a_3	$(11,0,1,0)$	$(11,0,1)$
196	$(5,1,9,1)$	a_4	$(2,0,1,0)$	$(2,0,1)$
197	$(11,1,9,1)$	b_3	$(3,5,1,0)$	$(3,5,1)$
198	$(7,2,9,1)$	c_{24}	$(10,4,1,0)$	$(10,4,1)$
199	$(12,2,9,1)$	a_5	$(0,6,1,0)$	$(0,6,1)$
200	$(1,5,9,1)$	b_5	$(6,6,1,0)$	$(6,6,1)$
201	$(7,6,9,1)$	c_{13}, c_{25}	$(6,9,1,0)$	$(6,9,1)$
202	$(11,6,9,1)$	c_{16}	$(11,8,1,0)$	$(11,8,1)$
203	$(2,7,9,1)$	c_{15}	$(2,5,1,0)$	$(2,5,1)$
204	$(6,7,9,1)$	c_{14}, c_{26}	$(7,4,1,0)$	$(7,4,1)$
205	$(12,8,9,1)$	b_6	$(7,7,1,0)$	$(7,7,1)$
206	$(1,11,9,1)$	a_6	$(0,7,1,0)$	$(0,7,1)$
207	$(6,11,9,1)$	c_{23}	$(3,9,1,0)$	$(3,9,1)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$	$\Phi_{\mathscr{D}^1}(P'_i)$
208	$(2,12,9,1)$	b_4	$(10,8,1,0)$	$(10,8,1)$
209	$(8,12,9,1)$	a_3	$(11,0,1,0)$	$(11,0,1)$
210	$(5,1,10,1)$	b_3	$(6,5,1,0)$	$(6,5,1)$
211	$(7,1,10,1)$	a_4, c_{24}	$(2,0,1,0)$	$(2,0,1)$
212	$(1,5,10,1)$	a_6	$(0,7,1,0)$	$(0,7,1)$
213	$(7,5,10,1)$	c_{13}	$(1,5,1,0)$	$(1,5,1)$
214	$(8,6,10,1)$	c_{25}	$(2,7,1,0)$	$(2,7,1)$
215	$(12,6,10,1)$	b_6, c_{16}	$(7,10,1,0)$	$(7,10,1)$
216	$(1,7,10,1)$	b_5, c_{15}	$(6,3,1,0)$	$(6,3,1)$
217	$(5,7,10,1)$	c_{26}	$(11,6,1,0)$	$(11,6,1)$
218	$(6,8,10,1)$	c_{14}	$(12,8,1,0)$	$(12,8,1)$
219	$(12,8,10,1)$	a_5	$(0,6,1,0)$	$(0,6,1)$
220	$(6,12,10,1)$	a_3, c_{23}	$(11,0,1,0)$	$(11,0,1)$
221	$(8,12,10,1)$	b_4	$(7,8,1,0)$	$(7,8,1)$
222	$(6,0,11,1)$	c_{23}	$(9,12,1,0)$	$(9,12,1)$
223	$(7,0,11,1)$	c_{24}	$(4,1,1,0)$	$(4,1,1)$
224	$(9,1,11,1)$	a_4	$(2,0,1,0)$	$(2,0,1)$
225	$(12,1,11,1)$	a_5, b_3	$(0,6,1,0)$	$(0,6,1)$
226	$(7,4,11,1)$	c_{13}	$(10,7,1,0)$	$(10,7,1)$
227	$(12,4,11,1)$	b_6	$(11,4,1,0)$	$(11,4,1)$
228	$(0,6,11,1)$	c_{16}	$(9,9,1,0)$	$(9,9,1)$
229	$(9,6,11,1)$	c_{25}	$(3,1,1,0)$	$(3,1,1)$
230	$(0,7,11,1)$	c_{15}	$(4,4,1,0)$	$(4,4,1)$
231	$(4,7,11,1)$	c_{26}	$(10,12,1,0)$	$(10,12,1)$
232	$(1,9,11,1)$	b_5	$(2,9,1,0)$	$(2,9,1)$
233	$(6,9,11,1)$	c_{14}	$(3,6,1,0)$	$(3,6,1)$
234	$(1,12,11,1)$	a_6, b_4	$(0,7,1,0)$	$(0,7,1)$
235	$(4,12,11,1)$	a_3	$(11,0,1,0)$	$(11,0,1)$
236	$(0,0,12,1)$	a_2, b_1	undef	undef
237	$(6,1,12,1)$	a_2, b_3, c_{23}	$(2,1,0,0)$	$(2,1,0)$

i	P'_i	lines	$\Phi_{\mathscr{D}^1}(P'_i)$	$\Phi_{\mathscr{D}^1}(P'_i)$
238	$(11, 1, 12, 1)$	a_4, b_1	undef	undef
239	$(9, 2, 12, 1)$	b_1	undef	undef
240	$(12, 2, 12, 1)$	a_2, b_6	$(2, 1, 0, 0)$	$(2, 1, 0)$
241	$(5, 3, 12, 1)$	a_2	$(2, 1, 0, 0)$	$(2, 1, 0)$
242	$(7, 3, 12, 1)$	b_1, c_{13}	undef	undef
243	$(5, 4, 12, 1)$	b_1	undef	undef
244	$(11, 4, 12, 1)$	a_2	$(2, 1, 0, 0)$	$(2, 1, 0)$
245	$(3, 5, 12, 1)$	b_1	undef	undef
246	$(4, 5, 12, 1)$	a_2	$(2, 1, 0, 0)$	$(2, 1, 0)$
247	$(1, 6, 12, 1)$	a_6, b_1, c_{16}	undef	undef
248	$(10, 6, 12, 1)$	a_2, c_{25}	$(2, 1, 0, 0)$	$(2, 1, 0)$
249	$(3, 7, 12, 1)$	a_2, c_{26}	$(2, 1, 0, 0)$	$(2, 1, 0)$
250	$(12, 7, 12, 1)$	a_5, b_1, c_{15}	undef	undef
251	$(9, 8, 12, 1)$	a_2	$(2, 1, 0, 0)$	$(2, 1, 0)$
252	$(10, 8, 12, 1)$	b_1	undef	undef
253	$(2, 9, 12, 1)$	a_2	$(2, 1, 0, 0)$	$(2, 1, 0)$
254	$(8, 9, 12, 1)$	b_1	undef	undef
255	$(6, 10, 12, 1)$	b_1, c_{14}	undef	undef
256	$(8, 10, 12, 1)$	a_2	$(2, 1, 0, 0)$	$(2, 1, 0)$
257	$(1, 11, 12, 1)$	a_2, b_5	$(2, 1, 0, 0)$	$(2, 1, 0)$
258	$(4, 11, 12, 1)$	b_1	undef	undef
259	$(2, 12, 12, 1)$	a_3, b_1	undef	undef
260	$(7, 12, 12, 1)$	a_2, b_4, c_{24}	$(2, 1, 0, 0)$	$(2, 1, 0)$

Appendix B

72 Clebsch maps of $\mathcal{F}$ in Model 2

Let ℓ_1 and ℓ_2 be two skew lines of the cubic surface $\mathcal{F}$. Φ maps from $\mathcal{F}$ to v.

	ℓ_1	ℓ_2	v
$\Phi_{\mathscr{D}^1}$	b_1	b_2	$\pi_{12,34,56}$
$\Phi_{\mathscr{D}^2}$	a_1	a_2	$\pi_{12,34,56}$
$\Phi_{\mathscr{D}^1_{12}}$	a_2	b_2	$\pi_{12,34,56}$
$\Phi_{\mathscr{D}^2_{12}}$	a_1	b_1	$\pi_{12,34,56}$
$\Phi_{\mathscr{D}^1_{13}}$	a_3	b_3	$\pi_{13,24,56}$
$\Phi_{\mathscr{D}^2_{13}}$	a_1	b_1	$\pi_{13,24,56}$
$\Phi_{\mathscr{D}^1_{14}}$	a_4	b_4	$\pi_{14,23,56}$
$\Phi_{\mathscr{D}^2_{14}}$	a_1	b_1	$\pi_{14,23,56}$
$\Phi_{\mathscr{D}^1_{15}}$	a_5	b_5	$\pi_{15,23,46}$
$\Phi_{\mathscr{D}^2_{15}}$	a_1	b_1	$\pi_{15,23,46}$
$\Phi_{\mathscr{D}^1_{16}}$	a_6	b_6	$\pi_{16,23,45}$
$\Phi_{\mathscr{D}^2_{16}}$	a_1	b_1	$\pi_{16,23,45}$
$\Phi_{\mathscr{D}^1_{23}}$	a_3	b_3	$\pi_{14,23,56}$
$\Phi_{\mathscr{D}^2_{23}}$	a_2	b_2	$\pi_{14,23,56}$
$\Phi_{\mathscr{D}^1_{24}}$	a_4	b_4	$\pi_{13,24,56}$
$\Phi_{\mathscr{D}^2_{24}}$	a_2	b_2	$\pi_{13,24,56}$
$\Phi_{\mathscr{D}^1_{25}}$	a_5	b_5	$\pi_{13,25,46}$
$\Phi_{\mathscr{D}^2_{25}}$	a_2	b_2	$\pi_{13,25,46}$
$\Phi_{\mathscr{D}^1_{26}}$	a_6	b_6	$\pi_{13,26,45}$
$\Phi_{\mathscr{D}^2_{26}}$	a_2	b_2	$\pi_{13,26,45}$
$\Phi_{\mathscr{D}^1_{34}}$	a_4	b_4	$\pi_{12,34,56}$
$\Phi_{\mathscr{D}^2_{34}}$	a_3	b_3	$\pi_{12,34,56}$
$\Phi_{\mathscr{D}^1_{35}}$	a_5	b_5	$\pi_{12,35,46}$
$\Phi_{\mathscr{D}^2_{35}}$	a_3	b_3	$\pi_{12,35,46}$

	ℓ_1	ℓ_2	v
$\Phi_{\mathscr{D}^1_{36}}$	a_6	b_6	$\pi_{12,36,45}$
$\Phi_{\mathscr{D}^2_{36}}$	a_3	b_3	$\pi_{12,36,45}$
$\Phi_{\mathscr{D}^1_{45}}$	a_5	b_5	$\pi_{12,36,45}$
$\Phi_{\mathscr{D}^2_{45}}$	a_4	b_4	$\pi_{12,36,45}$
$\Phi_{\mathscr{D}^1_{46}}$	a_6	b_6	$\pi_{12,35,46}$
$\Phi_{\mathscr{D}^2_{46}}$	a_4	b_4	$\pi_{12,35,46}$
$\Phi_{\mathscr{D}^1_{56}}$	a_6	b_6	$\pi_{12,34,56}$
$\Phi_{\mathscr{D}^2_{56}}$	a_5	b_5	$\pi_{12,34,56}$
$\Phi_{\mathscr{D}^1_{123}}$	b_4	b_5	π_{61}
$\Phi_{\mathscr{D}^2_{123}}$	a_1	a_2	π_{43}
$\Phi_{\mathscr{D}^1_{124}}$	b_3	b_5	π_{61}
$\Phi_{\mathscr{D}^2_{124}}$	a_1	a_2	π_{34}
$\Phi_{\mathscr{D}^1_{125}}$	b_3	b_4	π_{61}
$\Phi_{\mathscr{D}^2_{125}}$	a_1	a_2	π_{35}
$\Phi_{\mathscr{D}^1_{126}}$	b_3	b_4	π_{51}
$\Phi_{\mathscr{D}^2_{126}}$	a_1	a_2	π_{36}
$\Phi_{\mathscr{D}^1_{134}}$	b_2	b_5	π_{61}
$\Phi_{\mathscr{D}^2_{134}}$	a_1	a_3	π_{24}
$\Phi_{\mathscr{D}^1_{135}}$	b_2	b_4	π_{61}
$\Phi_{\mathscr{D}^2_{135}}$	a_1	a_3	π_{25}
$\Phi_{\mathscr{D}^1_{136}}$	b_2	b_4	π_{51}
$\Phi_{\mathscr{D}^2_{136}}$	a_1	a_3	π_{26}
$\Phi_{\mathscr{D}^1_{145}}$	b_2	b_3	π_{61}
$\Phi_{\mathscr{D}^2_{145}}$	a_1	a_4	π_{25}

	ℓ_1	ℓ_2	v
$\Phi_{\mathscr{D}^1_{146}}$	b_2	b_3	π_{51}
$\Phi_{\mathscr{D}^2_{146}}$	a_1	a_4	π_{26}
$\Phi_{\mathscr{D}^1_{156}}$	b_2	b_3	π_{41}
$\Phi_{\mathscr{D}^2_{156}}$	a_1	a_5	π_{26}
$\Phi_{\mathscr{D}^1_{234}}$	b_1	b_5	π_{62}
$\Phi_{\mathscr{D}^2_{234}}$	a_2	a_3	π_{14}
$\Phi_{\mathscr{D}^1_{235}}$	b_1	b_4	π_{62}
$\Phi_{\mathscr{D}^2_{235}}$	a_2	a_3	π_{15}
$\Phi_{\mathscr{D}^1_{236}}$	b_1	b_4	π_{52}
$\Phi_{\mathscr{D}^2_{236}}$	a_2	a_3	π_{16}
$\Phi_{\mathscr{D}^1_{245}}$	b_1	b_3	π_{62}
$\Phi_{\mathscr{D}^2_{245}}$	a_2	a_4	π_{15}
$\Phi_{\mathscr{D}^1_{246}}$	b_1	b_3	π_{52}
$\Phi_{\mathscr{D}^2_{246}}$	a_2	a_4	π_{16}
$\Phi_{\mathscr{D}^1_{256}}$	b_1	b_3	π_{42}
$\Phi_{\mathscr{D}^2_{256}}$	a_2	a_5	π_{16}
$\Phi_{\mathscr{D}^1_{345}}$	b_1	b_2	π_{63}
$\Phi_{\mathscr{D}^2_{345}}$	a_3	a_4	π_{15}
$\Phi_{\mathscr{D}^1_{346}}$	b_1	b_2	π_{53}
$\Phi_{\mathscr{D}^2_{346}}$	a_3	a_4	π_{16}
$\Phi_{\mathscr{D}^1_{356}}$	b_1	b_2	π_{43}
$\Phi_{\mathscr{D}^2_{356}}$	a_3	a_5	π_{16}
$\Phi_{\mathscr{D}^1_{456}}$	b_1	b_2	π_{34}
$\Phi_{\mathscr{D}^2_{456}}$	a_4	a_5	π_{16}

Appendix C

The points of $\mathrm{PG}(3,13)$ and the classification of double-triplets in $\mathrm{PG}(3,13)$

There are 2380 points in $\mathrm{PG}(3,13)$. Let P_n denote the point of $\mathrm{PG}(3,13)$ where $n = 0,\ldots,2379$.

The order of the group $\mathrm{PGL}(4,13)$ is 50858076935877120.

There are 282 orbits of $\mathrm{PGL}(4,13)$ on the set of double-triplets in $\mathrm{PG}(3,13)$. Let n denote the dual plane of the point P_n of $\mathrm{PG}(3,13)$ and let o be the length of the orbit. Here,

$$\{0,1,5;2,3,4\}_{12}$$

means that the double-triplet has two triplets $\{0,1,5\}$ and $\{2,3,4\}$. The order of the stabiliser of this double-triplet under the group $\mathrm{PGL}(4,13)$ is 12.

The overall number of double-triplets in $\mathrm{PG}(3,13)$ is 2287884218987246880.

The points of $\mathrm{PG}(3,13)$ and the orbits of $\mathrm{PGL}(4,13)$ on the set of double-triplets are in the following.

$P_0 = (1,0,0,0)$
$P_1 = (0,1,0,0)$
$P_2 = (0,0,1,0)$
$P_3 = (0,0,0,1)$
$P_4 = (1,1,1,1)$
$P_5 = (1,1,0,0)$
$P_6 = (2,1,0,0)$
$P_7 = (3,1,0,0)$
$P_8 = (4,1,0,0)$
$P_9 = (5,1,0,0)$
$P_{10} = (6,1,0,0)$
$P_{11} = (7,1,0,0)$
$P_{12} = (8,1,0,0)$
$P_{13} = (9,1,0,0)$
$P_{14} = (10,1,0,0)$
$P_{15} = (11,1,0,0)$
$P_{16} = (12,1,0,0)$
$P_{17} = (1,0,1,0)$
$P_{18} = (2,0,1,0)$
$P_{19} = (3,0,1,0)$
$P_{20} = (4,0,1,0)$
$P_{21} = (5,0,1,0)$
$P_{22} = (6,0,1,0)$
$P_{23} = (7,0,1,0)$
$P_{24} = (8,0,1,0)$
$P_{25} = (9,0,1,0)$
$P_{26} = (10,0,1,0)$
$P_{27} = (11,0,1,0)$
$P_{28} = (12,0,1,0)$
$P_{29} = (0,1,1,0)$
$P_{30} = (1,1,1,0)$
$P_{31} = (2,1,1,0)$
$P_{32} = (3,1,1,0)$
$P_{33} = (4,1,1,0)$
$P_{34} = (5,1,1,0)$
$P_{35} = (6,1,1,0)$
$P_{36} = (7,1,1,0)$
$P_{37} = (8,1,1,0)$
$P_{38} = (9,1,1,0)$
$P_{39} = (10,1,1,0)$
$P_{40} = (11,1,1,0)$
$P_{41} = (12,1,1,0)$
$P_{42} = (0,2,1,0)$
$P_{43} = (1,2,1,0)$
$P_{44} = (2,2,1,0)$
$P_{45} = (3,2,1,0)$
$P_{46} = (4,2,1,0)$
$P_{47} = (5,2,1,0)$
$P_{48} = (6,2,1,0)$
$P_{49} = (7,2,1,0)$
$P_{50} = (8,2,1,0)$
$P_{51} = (9,2,1,0)$
$P_{52} = (10,2,1,0)$
$P_{53} = (11,2,1,0)$
$P_{54} = (12,2,1,0)$
$P_{55} = (0,3,1,0)$
$P_{56} = (1,3,1,0)$
$P_{57} = (2,3,1,0)$
$P_{58} = (3,3,1,0)$
$P_{59} = (4,3,1,0)$
$P_{60} = (5,3,1,0)$
$P_{61} = (6,3,1,0)$
$P_{62} = (7,3,1,0)$
$P_{63} = (8,3,1,0)$
$P_{64} = (9,3,1,0)$
$P_{65} = (10,3,1,0)$
$P_{66} = (11,3,1,0)$
$P_{67} = (12,3,1,0)$
$P_{68} = (0,4,1,0)$
$P_{69} = (1,4,1,0)$
$P_{70} = (2,4,1,0)$
$P_{71} = (3,4,1,0)$
$P_{72} = (4,4,1,0)$
$P_{73} = (5,4,1,0)$
$P_{74} = (6,4,1,0)$
$P_{75} = (7,4,1,0)$
$P_{76} = (8,4,1,0)$
$P_{77} = (9,4,1,0)$
$P_{78} = (10,4,1,0)$
$P_{79} = (11,4,1,0)$
$P_{80} = (12,4,1,0)$
$P_{81} = (0,5,1,0)$
$P_{82} = (1,5,1,0)$
$P_{83} = (2,5,1,0)$
$P_{84} = (3,5,1,0)$
$P_{85} = (4,5,1,0)$
$P_{86} = (5,5,1,0)$
$P_{87} = (6,5,1,0)$
$P_{88} = (7,5,1,0)$
$P_{89} = (8,5,1,0)$
$P_{90} = (9,5,1,0)$
$P_{91} = (10,5,1,0)$
$P_{92} = (11,5,1,0)$
$P_{93} = (12,5,1,0)$
$P_{94} = (0,6,1,0)$
$P_{95} = (1,6,1,0)$
$P_{96} = (2,6,1,0)$
$P_{97} = (3,6,1,0)$
$P_{98} = (4,6,1,0)$
$P_{99} = (5,6,1,0)$
$P_{100} = (6,6,1,0)$
$P_{101} = (7,6,1,0)$
$P_{102} = (8,6,1,0)$
$P_{103} = (9,6,1,0)$
$P_{104} = (10,6,1,0)$
$P_{105} = (11,6,1,0)$
$P_{106} = (12,6,1,0)$
$P_{107} = (0,7,1,0)$
$P_{108} = (1,7,1,0)$
$P_{109} = (2,7,1,0)$
$P_{110} = (3,7,1,0)$
$P_{111} = (4,7,1,0)$
$P_{112} = (5,7,1,0)$
$P_{113} = (6,7,1,0)$
$P_{114} = (7,7,1,0)$
$P_{115} = (8,7,1,0)$
$P_{116} = (9,7,1,0)$
$P_{117} = (10,7,1,0)$
$P_{118} = (11,7,1,0)$
$P_{119} = (12,7,1,0)$
$P_{120} = (0,8,1,0)$
$P_{121} = (1,8,1,0)$
$P_{122} = (2,8,1,0)$
$P_{123} = (3,8,1,0)$
$P_{124} = (4,8,1,0)$
$P_{125} = (5,8,1,0)$
$P_{126} = (6,8,1,0)$
$P_{127} = (7,8,1,0)$
$P_{128} = (8,8,1,0)$
$P_{129} = (9,8,1,0)$
$P_{130} = (10,8,1,0)$
$P_{131} = (11,8,1,0)$
$P_{132} = (12,8,1,0)$
$P_{133} = (0,9,1,0)$
$P_{134} = (1,9,1,0)$
$P_{135} = (2,9,1,0)$
$P_{136} = (3,9,1,0)$
$P_{137} = (4,9,1,0)$
$P_{138} = (5,9,1,0)$
$P_{139} = (6,9,1,0)$
$P_{140} = (7,9,1,0)$
$P_{141} = (8,9,1,0)$
$P_{142} = (9,9,1,0)$
$P_{143} = (10,9,1,0)$
$P_{144} = (11,9,1,0)$
$P_{145} = (12,9,1,0)$

$P_{146} = (0, 10, 1, 0)$
$P_{147} = (1, 10, 1, 0)$
$P_{148} = (2, 10, 1, 0)$
$P_{149} = (3, 10, 1, 0)$
$P_{150} = (4, 10, 1, 0)$
$P_{151} = (5, 10, 1, 0)$
$P_{152} = (6, 10, 1, 0)$
$P_{153} = (7, 10, 1, 0)$
$P_{154} = (8, 10, 1, 0)$
$P_{155} = (9, 10, 1, 0)$
$P_{156} = (10, 10, 1, 0)$
$P_{157} = (11, 10, 1, 0)$
$P_{158} = (12, 10, 1, 0)$
$P_{159} = (0, 11, 1, 0)$
$P_{160} = (1, 11, 1, 0)$
$P_{161} = (2, 11, 1, 0)$
$P_{162} = (3, 11, 1, 0)$
$P_{163} = (4, 11, 1, 0)$
$P_{164} = (5, 11, 1, 0)$
$P_{165} = (6, 11, 1, 0)$
$P_{166} = (7, 11, 1, 0)$
$P_{167} = (8, 11, 1, 0)$
$P_{168} = (9, 11, 1, 0)$
$P_{169} = (10, 11, 1, 0)$
$P_{170} = (11, 11, 1, 0)$
$P_{171} = (12, 11, 1, 0)$
$P_{172} = (0, 12, 1, 0)$
$P_{173} = (1, 12, 1, 0)$
$P_{174} = (2, 12, 1, 0)$
$P_{175} = (3, 12, 1, 0)$
$P_{176} = (4, 12, 1, 0)$
$P_{177} = (5, 12, 1, 0)$
$P_{178} = (6, 12, 1, 0)$
$P_{179} = (7, 12, 1, 0)$
$P_{180} = (8, 12, 1, 0)$
$P_{181} = (9, 12, 1, 0)$
$P_{182} = (10, 12, 1, 0)$
$P_{183} = (11, 12, 1, 0)$
$P_{184} = (12, 12, 1, 0)$
$P_{185} = (1, 0, 0, 1)$
$P_{186} = (2, 0, 0, 1)$
$P_{187} = (3, 0, 0, 1)$
$P_{188} = (4, 0, 0, 1)$
$P_{189} = (5, 0, 0, 1)$
$P_{190} = (6, 0, 0, 1)$
$P_{191} = (7, 0, 0, 1)$
$P_{192} = (8, 0, 0, 1)$
$P_{193} = (9, 0, 0, 1)$
$P_{194} = (10, 0, 0, 1)$
$P_{195} = (11, 0, 0, 1)$
$P_{196} = (12, 0, 0, 1)$
$P_{197} = (0, 1, 0, 1)$
$P_{198} = (1, 1, 0, 1)$
$P_{199} = (2, 1, 0, 1)$
$P_{200} = (3, 1, 0, 1)$
$P_{201} = (4, 1, 0, 1)$
$P_{202} = (5, 1, 0, 1)$
$P_{203} = (6, 1, 0, 1)$
$P_{204} = (7, 1, 0, 1)$
$P_{205} = (8, 1, 0, 1)$
$P_{206} = (9, 1, 0, 1)$
$P_{207} = (10, 1, 0, 1)$
$P_{208} = (11, 1, 0, 1)$
$P_{209} = (12, 1, 0, 1)$
$P_{210} = (0, 2, 0, 1)$
$P_{211} = (1, 2, 0, 1)$
$P_{212} = (2, 2, 0, 1)$
$P_{213} = (3, 2, 0, 1)$
$P_{214} = (4, 2, 0, 1)$
$P_{215} = (5, 2, 0, 1)$
$P_{216} = (6, 2, 0, 1)$
$P_{217} = (7, 2, 0, 1)$
$P_{218} = (8, 2, 0, 1)$
$P_{219} = (9, 2, 0, 1)$
$P_{220} = (10, 2, 0, 1)$
$P_{221} = (11, 2, 0, 1)$
$P_{222} = (12, 2, 0, 1)$
$P_{223} = (0, 3, 0, 1)$
$P_{224} = (1, 3, 0, 1)$
$P_{225} = (2, 3, 0, 1)$
$P_{226} = (3, 3, 0, 1)$
$P_{227} = (4, 3, 0, 1)$
$P_{228} = (5, 3, 0, 1)$
$P_{229} = (6, 3, 0, 1)$
$P_{230} = (7, 3, 0, 1)$
$P_{231} = (8, 3, 0, 1)$
$P_{232} = (9, 3, 0, 1)$
$P_{233} = (10, 3, 0, 1)$
$P_{234} = (11, 3, 0, 1)$
$P_{235} = (12, 3, 0, 1)$
$P_{236} = (0, 4, 0, 1)$
$P_{237} = (1, 4, 0, 1)$
$P_{238} = (2, 4, 0, 1)$
$P_{239} = (3, 4, 0, 1)$
$P_{240} = (4, 4, 0, 1)$
$P_{241} = (5, 4, 0, 1)$
$P_{242} = (6, 4, 0, 1)$
$P_{243} = (7, 4, 0, 1)$
$P_{244} = (8, 4, 0, 1)$
$P_{245} = (9, 4, 0, 1)$
$P_{246} = (10, 4, 0, 1)$
$P_{247} = (11, 4, 0, 1)$
$P_{248} = (12, 4, 0, 1)$
$P_{249} = (0, 5, 0, 1)$
$P_{250} = (1, 5, 0, 1)$
$P_{251} = (2, 5, 0, 1)$
$P_{252} = (3, 5, 0, 1)$
$P_{253} = (4, 5, 0, 1)$
$P_{254} = (5, 5, 0, 1)$
$P_{255} = (6, 5, 0, 1)$
$P_{256} = (7, 5, 0, 1)$
$P_{257} = (8, 5, 0, 1)$
$P_{258} = (9, 5, 0, 1)$
$P_{259} = (10, 5, 0, 1)$
$P_{260} = (11, 5, 0, 1)$
$P_{261} = (12, 5, 0, 1)$
$P_{262} = (0, 6, 0, 1)$
$P_{263} = (1, 6, 0, 1)$
$P_{264} = (2, 6, 0, 1)$
$P_{265} = (3, 6, 0, 1)$
$P_{266} = (4, 6, 0, 1)$
$P_{267} = (5, 6, 0, 1)$
$P_{268} = (6, 6, 0, 1)$
$P_{269} = (7, 6, 0, 1)$
$P_{270} = (8, 6, 0, 1)$
$P_{271} = (9, 6, 0, 1)$
$P_{272} = (10, 6, 0, 1)$
$P_{273} = (11, 6, 0, 1)$
$P_{274} = (12, 6, 0, 1)$
$P_{275} = (0, 7, 0, 1)$
$P_{276} = (1, 7, 0, 1)$
$P_{277} = (2, 7, 0, 1)$
$P_{278} = (3, 7, 0, 1)$
$P_{279} = (4, 7, 0, 1)$
$P_{280} = (5, 7, 0, 1)$
$P_{281} = (6, 7, 0, 1)$
$P_{282} = (7, 7, 0, 1)$
$P_{283} = (8, 7, 0, 1)$
$P_{284} = (9, 7, 0, 1)$
$P_{285} = (10, 7, 0, 1)$
$P_{286} = (11, 7, 0, 1)$
$P_{287} = (12, 7, 0, 1)$
$P_{288} = (0, 8, 0, 1)$
$P_{289} = (1, 8, 0, 1)$
$P_{290} = (2, 8, 0, 1)$
$P_{291} = (3, 8, 0, 1)$

$P_{292} = (4, 8, 0, 1)$
$P_{293} = (5, 8, 0, 1)$
$P_{294} = (6, 8, 0, 1)$
$P_{295} = (7, 8, 0, 1)$
$P_{296} = (8, 8, 0, 1)$
$P_{297} = (9, 8, 0, 1)$
$P_{298} = (10, 8, 0, 1)$
$P_{299} = (11, 8, 0, 1)$
$P_{300} = (12, 8, 0, 1)$
$P_{301} = (0, 9, 0, 1)$
$P_{302} = (1, 9, 0, 1)$
$P_{303} = (2, 9, 0, 1)$
$P_{304} = (3, 9, 0, 1)$
$P_{305} = (4, 9, 0, 1)$
$P_{306} = (5, 9, 0, 1)$
$P_{307} = (6, 9, 0, 1)$
$P_{308} = (7, 9, 0, 1)$
$P_{309} = (8, 9, 0, 1)$
$P_{310} = (9, 9, 0, 1)$
$P_{311} = (10, 9, 0, 1)$
$P_{312} = (11, 9, 0, 1)$
$P_{313} = (12, 9, 0, 1)$
$P_{314} = (0, 10, 0, 1)$
$P_{315} = (1, 10, 0, 1)$
$P_{316} = (2, 10, 0, 1)$
$P_{317} = (3, 10, 0, 1)$
$P_{318} = (4, 10, 0, 1)$
$P_{319} = (5, 10, 0, 1)$
$P_{320} = (6, 10, 0, 1)$
$P_{321} = (7, 10, 0, 1)$
$P_{322} = (8, 10, 0, 1)$
$P_{323} = (9, 10, 0, 1)$
$P_{324} = (10, 10, 0, 1)$
$P_{325} = (11, 10, 0, 1)$
$P_{326} = (12, 10, 0, 1)$
$P_{327} = (0, 11, 0, 1)$
$P_{328} = (1, 11, 0, 1)$
$P_{329} = (2, 11, 0, 1)$
$P_{330} = (3, 11, 0, 1)$
$P_{331} = (4, 11, 0, 1)$
$P_{332} = (5, 11, 0, 1)$
$P_{333} = (6, 11, 0, 1)$
$P_{334} = (7, 11, 0, 1)$
$P_{335} = (8, 11, 0, 1)$
$P_{336} = (9, 11, 0, 1)$
$P_{337} = (10, 11, 0, 1)$
$P_{338} = (11, 11, 0, 1)$
$P_{339} = (12, 11, 0, 1)$
$P_{340} = (0, 12, 0, 1)$
$P_{341} = (1, 12, 0, 1)$
$P_{342} = (2, 12, 0, 1)$
$P_{343} = (3, 12, 0, 1)$
$P_{344} = (4, 12, 0, 1)$
$P_{345} = (5, 12, 0, 1)$
$P_{346} = (6, 12, 0, 1)$
$P_{347} = (7, 12, 0, 1)$
$P_{348} = (8, 12, 0, 1)$
$P_{349} = (9, 12, 0, 1)$
$P_{350} = (10, 12, 0, 1)$
$P_{351} = (11, 12, 0, 1)$
$P_{352} = (12, 12, 0, 1)$
$P_{353} = (0, 0, 1, 1)$
$P_{354} = (1, 0, 1, 1)$
$P_{355} = (2, 0, 1, 1)$
$P_{356} = (3, 0, 1, 1)$
$P_{357} = (4, 0, 1, 1)$
$P_{358} = (5, 0, 1, 1)$
$P_{359} = (6, 0, 1, 1)$
$P_{360} = (7, 0, 1, 1)$
$P_{361} = (8, 0, 1, 1)$
$P_{362} = (9, 0, 1, 1)$
$P_{363} = (10, 0, 1, 1)$
$P_{364} = (11, 0, 1, 1)$
$P_{365} = (12, 0, 1, 1)$
$P_{366} = (0, 1, 1, 1)$
$P_{367} = (2, 1, 1, 1)$
$P_{368} = (3, 1, 1, 1)$
$P_{369} = (4, 1, 1, 1)$
$P_{370} = (5, 1, 1, 1)$
$P_{371} = (6, 1, 1, 1)$
$P_{372} = (7, 1, 1, 1)$
$P_{373} = (8, 1, 1, 1)$
$P_{374} = (9, 1, 1, 1)$
$P_{375} = (10, 1, 1, 1)$
$P_{376} = (11, 1, 1, 1)$
$P_{377} = (12, 1, 1, 1)$
$P_{378} = (0, 2, 1, 1)$
$P_{379} = (1, 2, 1, 1)$
$P_{380} = (2, 2, 1, 1)$
$P_{381} = (3, 2, 1, 1)$
$P_{382} = (4, 2, 1, 1)$
$P_{383} = (5, 2, 1, 1)$
$P_{384} = (6, 2, 1, 1)$
$P_{385} = (7, 2, 1, 1)$
$P_{386} = (8, 2, 1, 1)$
$P_{387} = (9, 2, 1, 1)$
$P_{388} = (10, 2, 1, 1)$
$P_{389} = (11, 2, 1, 1)$
$P_{390} = (12, 2, 1, 1)$
$P_{391} = (0, 3, 1, 1)$
$P_{392} = (1, 3, 1, 1)$
$P_{393} = (2, 3, 1, 1)$
$P_{394} = (3, 3, 1, 1)$
$P_{395} = (4, 3, 1, 1)$
$P_{396} = (5, 3, 1, 1)$
$P_{397} = (6, 3, 1, 1)$
$P_{398} = (7, 3, 1, 1)$
$P_{399} = (8, 3, 1, 1)$
$P_{400} = (9, 3, 1, 1)$
$P_{401} = (10, 3, 1, 1)$
$P_{402} = (11, 3, 1, 1)$
$P_{403} = (12, 3, 1, 1)$
$P_{404} = (0, 4, 1, 1)$
$P_{405} = (1, 4, 1, 1)$
$P_{406} = (2, 4, 1, 1)$
$P_{407} = (3, 4, 1, 1)$
$P_{408} = (4, 4, 1, 1)$
$P_{409} = (5, 4, 1, 1)$
$P_{410} = (6, 4, 1, 1)$
$P_{411} = (7, 4, 1, 1)$
$P_{412} = (8, 4, 1, 1)$
$P_{413} = (9, 4, 1, 1)$
$P_{414} = (10, 4, 1, 1)$
$P_{415} = (11, 4, 1, 1)$
$P_{416} = (12, 4, 1, 1)$
$P_{417} = (0, 5, 1, 1)$
$P_{418} = (1, 5, 1, 1)$
$P_{419} = (2, 5, 1, 1)$
$P_{420} = (3, 5, 1, 1)$
$P_{421} = (4, 5, 1, 1)$
$P_{422} = (5, 5, 1, 1)$
$P_{423} = (6, 5, 1, 1)$
$P_{424} = (7, 5, 1, 1)$
$P_{425} = (8, 5, 1, 1)$
$P_{426} = (9, 5, 1, 1)$
$P_{427} = (10, 5, 1, 1)$
$P_{428} = (11, 5, 1, 1)$
$P_{429} = (12, 5, 1, 1)$
$P_{430} = (0, 6, 1, 1)$
$P_{431} = (1, 6, 1, 1)$
$P_{432} = (2, 6, 1, 1)$
$P_{433} = (3, 6, 1, 1)$
$P_{434} = (4, 6, 1, 1)$
$P_{435} = (5, 6, 1, 1)$
$P_{436} = (6, 6, 1, 1)$
$P_{437} = (7, 6, 1, 1)$

$P_{438} = (8, 6, 1, 1)$
$P_{439} = (9, 6, 1, 1)$
$P_{440} = (10, 6, 1, 1)$
$P_{441} = (11, 6, 1, 1)$
$P_{442} = (12, 6, 1, 1)$
$P_{443} = (0, 7, 1, 1)$
$P_{444} = (1, 7, 1, 1)$
$P_{445} = (2, 7, 1, 1)$
$P_{446} = (3, 7, 1, 1)$
$P_{447} = (4, 7, 1, 1)$
$P_{448} = (5, 7, 1, 1)$
$P_{449} = (6, 7, 1, 1)$
$P_{450} = (7, 7, 1, 1)$
$P_{451} = (8, 7, 1, 1)$
$P_{452} = (9, 7, 1, 1)$
$P_{453} = (10, 7, 1, 1)$
$P_{454} = (11, 7, 1, 1)$
$P_{455} = (12, 7, 1, 1)$
$P_{456} = (0, 8, 1, 1)$
$P_{457} = (1, 8, 1, 1)$
$P_{458} = (2, 8, 1, 1)$
$P_{459} = (3, 8, 1, 1)$
$P_{460} = (4, 8, 1, 1)$
$P_{461} = (5, 8, 1, 1)$
$P_{462} = (6, 8, 1, 1)$
$P_{463} = (7, 8, 1, 1)$
$P_{464} = (8, 8, 1, 1)$
$P_{465} = (9, 8, 1, 1)$
$P_{466} = (10, 8, 1, 1)$
$P_{467} = (11, 8, 1, 1)$
$P_{468} = (12, 8, 1, 1)$
$P_{469} = (0, 9, 1, 1)$
$P_{470} = (1, 9, 1, 1)$
$P_{471} = (2, 9, 1, 1)$
$P_{472} = (3, 9, 1, 1)$
$P_{473} = (4, 9, 1, 1)$
$P_{474} = (5, 9, 1, 1)$
$P_{475} = (6, 9, 1, 1)$
$P_{476} = (7, 9, 1, 1)$
$P_{477} = (8, 9, 1, 1)$
$P_{478} = (9, 9, 1, 1)$
$P_{479} = (10, 9, 1, 1)$
$P_{480} = (11, 9, 1, 1)$
$P_{481} = (12, 9, 1, 1)$
$P_{482} = (0, 10, 1, 1)$
$P_{483} = (1, 10, 1, 1)$
$P_{484} = (2, 10, 1, 1)$
$P_{485} = (3, 10, 1, 1)$
$P_{486} = (4, 10, 1, 1)$
$P_{487} = (5, 10, 1, 1)$
$P_{488} = (6, 10, 1, 1)$
$P_{489} = (7, 10, 1, 1)$
$P_{490} = (8, 10, 1, 1)$
$P_{491} = (9, 10, 1, 1)$
$P_{492} = (10, 10, 1, 1)$
$P_{493} = (11, 10, 1, 1)$
$P_{494} = (12, 10, 1, 1)$
$P_{495} = (0, 11, 1, 1)$
$P_{496} = (1, 11, 1, 1)$
$P_{497} = (2, 11, 1, 1)$
$P_{498} = (3, 11, 1, 1)$
$P_{499} = (4, 11, 1, 1)$
$P_{500} = (5, 11, 1, 1)$
$P_{501} = (6, 11, 1, 1)$
$P_{502} = (7, 11, 1, 1)$
$P_{503} = (8, 11, 1, 1)$
$P_{504} = (9, 11, 1, 1)$
$P_{505} = (10, 11, 1, 1)$
$P_{506} = (11, 11, 1, 1)$
$P_{507} = (12, 11, 1, 1)$
$P_{508} = (0, 12, 1, 1)$
$P_{509} = (1, 12, 1, 1)$
$P_{510} = (2, 12, 1, 1)$
$P_{511} = (3, 12, 1, 1)$
$P_{512} = (4, 12, 1, 1)$
$P_{513} = (5, 12, 1, 1)$
$P_{514} = (6, 12, 1, 1)$
$P_{515} = (7, 12, 1, 1)$
$P_{516} = (8, 12, 1, 1)$
$P_{517} = (9, 12, 1, 1)$
$P_{518} = (10, 12, 1, 1)$
$P_{519} = (11, 12, 1, 1)$
$P_{520} = (12, 12, 1, 1)$
$P_{521} = (0, 0, 2, 1)$
$P_{522} = (1, 0, 2, 1)$
$P_{523} = (2, 0, 2, 1)$
$P_{524} = (3, 0, 2, 1)$
$P_{525} = (4, 0, 2, 1)$
$P_{526} = (5, 0, 2, 1)$
$P_{527} = (6, 0, 2, 1)$
$P_{528} = (7, 0, 2, 1)$
$P_{529} = (8, 0, 2, 1)$
$P_{530} = (9, 0, 2, 1)$
$P_{531} = (10, 0, 2, 1)$
$P_{532} = (11, 0, 2, 1)$
$P_{533} = (12, 0, 2, 1)$
$P_{534} = (0, 1, 2, 1)$
$P_{535} = (1, 1, 2, 1)$
$P_{536} = (2, 1, 2, 1)$
$P_{537} = (3, 1, 2, 1)$
$P_{538} = (4, 1, 2, 1)$
$P_{539} = (5, 1, 2, 1)$
$P_{540} = (6, 1, 2, 1)$
$P_{541} = (7, 1, 2, 1)$
$P_{542} = (8, 1, 2, 1)$
$P_{543} = (9, 1, 2, 1)$
$P_{544} = (10, 1, 2, 1)$
$P_{545} = (11, 1, 2, 1)$
$P_{546} = (12, 1, 2, 1)$
$P_{547} = (0, 2, 2, 1)$
$P_{548} = (1, 2, 2, 1)$
$P_{549} = (2, 2, 2, 1)$
$P_{550} = (3, 2, 2, 1)$
$P_{551} = (4, 2, 2, 1)$
$P_{552} = (5, 2, 2, 1)$
$P_{553} = (6, 2, 2, 1)$
$P_{554} = (7, 2, 2, 1)$
$P_{555} = (8, 2, 2, 1)$
$P_{556} = (9, 2, 2, 1)$
$P_{557} = (10, 2, 2, 1)$
$P_{558} = (11, 2, 2, 1)$
$P_{559} = (12, 2, 2, 1)$
$P_{560} = (0, 3, 2, 1)$
$P_{561} = (1, 3, 2, 1)$
$P_{562} = (2, 3, 2, 1)$
$P_{563} = (3, 3, 2, 1)$
$P_{564} = (4, 3, 2, 1)$
$P_{565} = (5, 3, 2, 1)$
$P_{566} = (6, 3, 2, 1)$
$P_{567} = (7, 3, 2, 1)$
$P_{568} = (8, 3, 2, 1)$
$P_{569} = (9, 3, 2, 1)$
$P_{570} = (10, 3, 2, 1)$
$P_{571} = (11, 3, 2, 1)$
$P_{572} = (12, 3, 2, 1)$
$P_{573} = (0, 4, 2, 1)$
$P_{574} = (1, 4, 2, 1)$
$P_{575} = (2, 4, 2, 1)$
$P_{576} = (3, 4, 2, 1)$
$P_{577} = (4, 4, 2, 1)$
$P_{578} = (5, 4, 2, 1)$
$P_{579} = (6, 4, 2, 1)$
$P_{580} = (7, 4, 2, 1)$
$P_{581} = (8, 4, 2, 1)$
$P_{582} = (9, 4, 2, 1)$
$P_{583} = (10, 4, 2, 1)$

$P_{584} = (11, 4, 2, 1)$
$P_{585} = (12, 4, 2, 1)$
$P_{586} = (0, 5, 2, 1)$
$P_{587} = (1, 5, 2, 1)$
$P_{588} = (2, 5, 2, 1)$
$P_{589} = (3, 5, 2, 1)$
$P_{590} = (4, 5, 2, 1)$
$P_{591} = (5, 5, 2, 1)$
$P_{592} = (6, 5, 2, 1)$
$P_{593} = (7, 5, 2, 1)$
$P_{594} = (8, 5, 2, 1)$
$P_{595} = (9, 5, 2, 1)$
$P_{596} = (10, 5, 2, 1)$
$P_{597} = (11, 5, 2, 1)$
$P_{598} = (12, 5, 2, 1)$
$P_{599} = (0, 6, 2, 1)$
$P_{600} = (1, 6, 2, 1)$
$P_{601} = (2, 6, 2, 1)$
$P_{602} = (3, 6, 2, 1)$
$P_{603} = (4, 6, 2, 1)$
$P_{604} = (5, 6, 2, 1)$
$P_{605} = (6, 6, 2, 1)$
$P_{606} = (7, 6, 2, 1)$
$P_{607} = (8, 6, 2, 1)$
$P_{608} = (9, 6, 2, 1)$
$P_{609} = (10, 6, 2, 1)$
$P_{610} = (11, 6, 2, 1)$
$P_{611} = (12, 6, 2, 1)$
$P_{612} = (0, 7, 2, 1)$
$P_{613} = (1, 7, 2, 1)$
$P_{614} = (2, 7, 2, 1)$
$P_{615} = (3, 7, 2, 1)$
$P_{616} = (4, 7, 2, 1)$
$P_{617} = (5, 7, 2, 1)$
$P_{618} = (6, 7, 2, 1)$
$P_{619} = (7, 7, 2, 1)$
$P_{620} = (8, 7, 2, 1)$
$P_{621} = (9, 7, 2, 1)$
$P_{622} = (10, 7, 2, 1)$
$P_{623} = (11, 7, 2, 1)$
$P_{624} = (12, 7, 2, 1)$
$P_{625} = (0, 8, 2, 1)$
$P_{626} = (1, 8, 2, 1)$
$P_{627} = (2, 8, 2, 1)$
$P_{628} = (3, 8, 2, 1)$
$P_{629} = (4, 8, 2, 1)$
$P_{630} = (5, 8, 2, 1)$
$P_{631} = (6, 8, 2, 1)$
$P_{632} = (7, 8, 2, 1)$
$P_{633} = (8, 8, 2, 1)$
$P_{634} = (9, 8, 2, 1)$
$P_{635} = (10, 8, 2, 1)$
$P_{636} = (11, 8, 2, 1)$
$P_{637} = (12, 8, 2, 1)$
$P_{638} = (0, 9, 2, 1)$
$P_{639} = (1, 9, 2, 1)$
$P_{640} = (2, 9, 2, 1)$
$P_{641} = (3, 9, 2, 1)$
$P_{642} = (4, 9, 2, 1)$
$P_{643} = (5, 9, 2, 1)$
$P_{644} = (6, 9, 2, 1)$
$P_{645} = (7, 9, 2, 1)$
$P_{646} = (8, 9, 2, 1)$
$P_{647} = (9, 9, 2, 1)$
$P_{648} = (10, 9, 2, 1)$
$P_{649} = (11, 9, 2, 1)$
$P_{650} = (12, 9, 2, 1)$
$P_{651} = (0, 10, 2, 1)$
$P_{652} = (1, 10, 2, 1)$
$P_{653} = (2, 10, 2, 1)$
$P_{654} = (3, 10, 2, 1)$
$P_{655} = (4, 10, 2, 1)$
$P_{656} = (5, 10, 2, 1)$
$P_{657} = (6, 10, 2, 1)$
$P_{658} = (7, 10, 2, 1)$
$P_{659} = (8, 10, 2, 1)$
$P_{660} = (9, 10, 2, 1)$
$P_{661} = (10, 10, 2, 1)$
$P_{662} = (11, 10, 2, 1)$
$P_{663} = (12, 10, 2, 1)$
$P_{664} = (0, 11, 2, 1)$
$P_{665} = (1, 11, 2, 1)$
$P_{666} = (2, 11, 2, 1)$
$P_{667} = (3, 11, 2, 1)$
$P_{668} = (4, 11, 2, 1)$
$P_{669} = (5, 11, 2, 1)$
$P_{670} = (6, 11, 2, 1)$
$P_{671} = (7, 11, 2, 1)$
$P_{672} = (8, 11, 2, 1)$
$P_{673} = (9, 11, 2, 1)$
$P_{674} = (10, 11, 2, 1)$
$P_{675} = (11, 11, 2, 1)$
$P_{676} = (12, 11, 2, 1)$
$P_{677} = (0, 12, 2, 1)$
$P_{678} = (1, 12, 2, 1)$
$P_{679} = (2, 12, 2, 1)$
$P_{680} = (3, 12, 2, 1)$
$P_{681} = (4, 12, 2, 1)$
$P_{682} = (5, 12, 2, 1)$
$P_{683} = (6, 12, 2, 1)$
$P_{684} = (7, 12, 2, 1)$
$P_{685} = (8, 12, 2, 1)$
$P_{686} = (9, 12, 2, 1)$
$P_{687} = (10, 12, 2, 1)$
$P_{688} = (11, 12, 2, 1)$
$P_{689} = (12, 12, 2, 1)$
$P_{690} = (0, 0, 3, 1)$
$P_{691} = (1, 0, 3, 1)$
$P_{692} = (2, 0, 3, 1)$
$P_{693} = (3, 0, 3, 1)$
$P_{694} = (4, 0, 3, 1)$
$P_{695} = (5, 0, 3, 1)$
$P_{696} = (6, 0, 3, 1)$
$P_{697} = (7, 0, 3, 1)$
$P_{698} = (8, 0, 3, 1)$
$P_{699} = (9, 0, 3, 1)$
$P_{700} = (10, 0, 3, 1)$
$P_{701} = (11, 0, 3, 1)$
$P_{702} = (12, 0, 3, 1)$
$P_{703} = (0, 1, 3, 1)$
$P_{704} = (1, 1, 3, 1)$
$P_{705} = (2, 1, 3, 1)$
$P_{706} = (3, 1, 3, 1)$
$P_{707} = (4, 1, 3, 1)$
$P_{708} = (5, 1, 3, 1)$
$P_{709} = (6, 1, 3, 1)$
$P_{710} = (7, 1, 3, 1)$
$P_{711} = (8, 1, 3, 1)$
$P_{712} = (9, 1, 3, 1)$
$P_{713} = (10, 1, 3, 1)$
$P_{714} = (11, 1, 3, 1)$
$P_{715} = (12, 1, 3, 1)$
$P_{716} = (0, 2, 3, 1)$
$P_{717} = (1, 2, 3, 1)$
$P_{718} = (2, 2, 3, 1)$
$P_{719} = (3, 2, 3, 1)$
$P_{720} = (4, 2, 3, 1)$
$P_{721} = (5, 2, 3, 1)$
$P_{722} = (6, 2, 3, 1)$
$P_{723} = (7, 2, 3, 1)$
$P_{724} = (8, 2, 3, 1)$
$P_{725} = (9, 2, 3, 1)$
$P_{726} = (10, 2, 3, 1)$
$P_{727} = (11, 2, 3, 1)$
$P_{728} = (12, 2, 3, 1)$
$P_{729} = (0, 3, 3, 1)$

$P_{730} = (1, 3, 3, 1)$
$P_{731} = (2, 3, 3, 1)$
$P_{732} = (3, 3, 3, 1)$
$P_{733} = (4, 3, 3, 1)$
$P_{734} = (5, 3, 3, 1)$
$P_{735} = (6, 3, 3, 1)$
$P_{736} = (7, 3, 3, 1)$
$P_{737} = (8, 3, 3, 1)$
$P_{738} = (9, 3, 3, 1)$
$P_{739} = (10, 3, 3, 1)$
$P_{740} = (11, 3, 3, 1)$
$P_{741} = (12, 3, 3, 1)$
$P_{742} = (0, 4, 3, 1)$
$P_{743} = (1, 4, 3, 1)$
$P_{744} = (2, 4, 3, 1)$
$P_{745} = (3, 4, 3, 1)$
$P_{746} = (4, 4, 3, 1)$
$P_{747} = (5, 4, 3, 1)$
$P_{748} = (6, 4, 3, 1)$
$P_{749} = (7, 4, 3, 1)$
$P_{750} = (8, 4, 3, 1)$
$P_{751} = (9, 4, 3, 1)$
$P_{752} = (10, 4, 3, 1)$
$P_{753} = (11, 4, 3, 1)$
$P_{754} = (12, 4, 3, 1)$
$P_{755} = (0, 5, 3, 1)$
$P_{756} = (1, 5, 3, 1)$
$P_{757} = (2, 5, 3, 1)$
$P_{758} = (3, 5, 3, 1)$
$P_{759} = (4, 5, 3, 1)$
$P_{760} = (5, 5, 3, 1)$
$P_{761} = (6, 5, 3, 1)$
$P_{762} = (7, 5, 3, 1)$
$P_{763} = (8, 5, 3, 1)$
$P_{764} = (9, 5, 3, 1)$
$P_{765} = (10, 5, 3, 1)$
$P_{766} = (11, 5, 3, 1)$
$P_{767} = (12, 5, 3, 1)$
$P_{768} = (0, 6, 3, 1)$
$P_{769} = (1, 6, 3, 1)$
$P_{770} = (2, 6, 3, 1)$
$P_{771} = (3, 6, 3, 1)$
$P_{772} = (4, 6, 3, 1)$
$P_{773} = (5, 6, 3, 1)$
$P_{774} = (6, 6, 3, 1)$
$P_{775} = (7, 6, 3, 1)$
$P_{776} = (8, 6, 3, 1)$
$P_{777} = (9, 6, 3, 1)$
$P_{778} = (10, 6, 3, 1)$
$P_{779} = (11, 6, 3, 1)$
$P_{780} = (12, 6, 3, 1)$
$P_{781} = (0, 7, 3, 1)$
$P_{782} = (1, 7, 3, 1)$
$P_{783} = (2, 7, 3, 1)$
$P_{784} = (3, 7, 3, 1)$
$P_{785} = (4, 7, 3, 1)$
$P_{786} = (5, 7, 3, 1)$
$P_{787} = (6, 7, 3, 1)$
$P_{788} = (7, 7, 3, 1)$
$P_{789} = (8, 7, 3, 1)$
$P_{790} = (9, 7, 3, 1)$
$P_{791} = (10, 7, 3, 1)$
$P_{792} = (11, 7, 3, 1)$
$P_{793} = (12, 7, 3, 1)$
$P_{794} = (0, 8, 3, 1)$
$P_{795} = (1, 8, 3, 1)$
$P_{796} = (2, 8, 3, 1)$
$P_{797} = (3, 8, 3, 1)$
$P_{798} = (4, 8, 3, 1)$
$P_{799} = (5, 8, 3, 1)$
$P_{800} = (6, 8, 3, 1)$
$P_{801} = (7, 8, 3, 1)$
$P_{802} = (8, 8, 3, 1)$
$P_{803} = (9, 8, 3, 1)$
$P_{804} = (10, 8, 3, 1)$
$P_{805} = (11, 8, 3, 1)$
$P_{806} = (12, 8, 3, 1)$
$P_{807} = (0, 9, 3, 1)$
$P_{808} = (1, 9, 3, 1)$
$P_{809} = (2, 9, 3, 1)$
$P_{810} = (3, 9, 3, 1)$
$P_{811} = (4, 9, 3, 1)$
$P_{812} = (5, 9, 3, 1)$
$P_{813} = (6, 9, 3, 1)$
$P_{814} = (7, 9, 3, 1)$
$P_{815} = (8, 9, 3, 1)$
$P_{816} = (9, 9, 3, 1)$
$P_{817} = (10, 9, 3, 1)$
$P_{818} = (11, 9, 3, 1)$
$P_{819} = (12, 9, 3, 1)$
$P_{820} = (0, 10, 3, 1)$
$P_{821} = (1, 10, 3, 1)$
$P_{822} = (2, 10, 3, 1)$
$P_{823} = (3, 10, 3, 1)$
$P_{824} = (4, 10, 3, 1)$
$P_{825} = (5, 10, 3, 1)$
$P_{826} = (6, 10, 3, 1)$
$P_{827} = (7, 10, 3, 1)$
$P_{828} = (8, 10, 3, 1)$
$P_{829} = (9, 10, 3, 1)$
$P_{830} = (10, 10, 3, 1)$
$P_{831} = (11, 10, 3, 1)$
$P_{832} = (12, 10, 3, 1)$
$P_{833} = (0, 11, 3, 1)$
$P_{834} = (1, 11, 3, 1)$
$P_{835} = (2, 11, 3, 1)$
$P_{836} = (3, 11, 3, 1)$
$P_{837} = (4, 11, 3, 1)$
$P_{838} = (5, 11, 3, 1)$
$P_{839} = (6, 11, 3, 1)$
$P_{840} = (7, 11, 3, 1)$
$P_{841} = (8, 11, 3, 1)$
$P_{842} = (9, 11, 3, 1)$
$P_{843} = (10, 11, 3, 1)$
$P_{844} = (11, 11, 3, 1)$
$P_{845} = (12, 11, 3, 1)$
$P_{846} = (0, 12, 3, 1)$
$P_{847} = (1, 12, 3, 1)$
$P_{848} = (2, 12, 3, 1)$
$P_{849} = (3, 12, 3, 1)$
$P_{850} = (4, 12, 3, 1)$
$P_{851} = (5, 12, 3, 1)$
$P_{852} = (6, 12, 3, 1)$
$P_{853} = (7, 12, 3, 1)$
$P_{854} = (8, 12, 3, 1)$
$P_{855} = (9, 12, 3, 1)$
$P_{856} = (10, 12, 3, 1)$
$P_{857} = (11, 12, 3, 1)$
$P_{858} = (12, 12, 3, 1)$
$P_{859} = (0, 0, 4, 1)$
$P_{860} = (1, 0, 4, 1)$
$P_{861} = (2, 0, 4, 1)$
$P_{862} = (3, 0, 4, 1)$
$P_{863} = (4, 0, 4, 1)$
$P_{864} = (5, 0, 4, 1)$
$P_{865} = (6, 0, 4, 1)$
$P_{866} = (7, 0, 4, 1)$
$P_{867} = (8, 0, 4, 1)$
$P_{868} = (9, 0, 4, 1)$
$P_{869} = (10, 0, 4, 1)$
$P_{870} = (11, 0, 4, 1)$
$P_{871} = (12, 0, 4, 1)$
$P_{872} = (0, 1, 4, 1)$
$P_{873} = (1, 1, 4, 1)$
$P_{874} = (2, 1, 4, 1)$
$P_{875} = (3, 1, 4, 1)$

$P_{876} = (4, 1, 4, 1)$
$P_{877} = (5, 1, 4, 1)$
$P_{878} = (6, 1, 4, 1)$
$P_{879} = (7, 1, 4, 1)$
$P_{880} = (8, 1, 4, 1)$
$P_{881} = (9, 1, 4, 1)$
$P_{882} = (10, 1, 4, 1)$
$P_{883} = (11, 1, 4, 1)$
$P_{884} = (12, 1, 4, 1)$
$P_{885} = (0, 2, 4, 1)$
$P_{886} = (1, 2, 4, 1)$
$P_{887} = (2, 2, 4, 1)$
$P_{888} = (3, 2, 4, 1)$
$P_{889} = (4, 2, 4, 1)$
$P_{890} = (5, 2, 4, 1)$
$P_{891} = (6, 2, 4, 1)$
$P_{892} = (7, 2, 4, 1)$
$P_{893} = (8, 2, 4, 1)$
$P_{894} = (9, 2, 4, 1)$
$P_{895} = (10, 2, 4, 1)$
$P_{896} = (11, 2, 4, 1)$
$P_{897} = (12, 2, 4, 1)$
$P_{898} = (0, 3, 4, 1)$
$P_{899} = (1, 3, 4, 1)$
$P_{900} = (2, 3, 4, 1)$
$P_{901} = (3, 3, 4, 1)$
$P_{902} = (4, 3, 4, 1)$
$P_{903} = (5, 3, 4, 1)$
$P_{904} = (6, 3, 4, 1)$
$P_{905} = (7, 3, 4, 1)$
$P_{906} = (8, 3, 4, 1)$
$P_{907} = (9, 3, 4, 1)$
$P_{908} = (10, 3, 4, 1)$
$P_{909} = (11, 3, 4, 1)$
$P_{910} = (12, 3, 4, 1)$
$P_{911} = (0, 4, 4, 1)$
$P_{912} = (1, 4, 4, 1)$
$P_{913} = (2, 4, 4, 1)$
$P_{914} = (3, 4, 4, 1)$
$P_{915} = (4, 4, 4, 1)$
$P_{916} = (5, 4, 4, 1)$
$P_{917} = (6, 4, 4, 1)$
$P_{918} = (7, 4, 4, 1)$
$P_{919} = (8, 4, 4, 1)$
$P_{920} = (9, 4, 4, 1)$
$P_{921} = (10, 4, 4, 1)$
$P_{922} = (11, 4, 4, 1)$
$P_{923} = (12, 4, 4, 1)$
$P_{924} = (0, 5, 4, 1)$
$P_{925} = (1, 5, 4, 1)$
$P_{926} = (2, 5, 4, 1)$
$P_{927} = (3, 5, 4, 1)$
$P_{928} = (4, 5, 4, 1)$
$P_{929} = (5, 5, 4, 1)$
$P_{930} = (6, 5, 4, 1)$
$P_{931} = (7, 5, 4, 1)$
$P_{932} = (8, 5, 4, 1)$
$P_{933} = (9, 5, 4, 1)$
$P_{934} = (10, 5, 4, 1)$
$P_{935} = (11, 5, 4, 1)$
$P_{936} = (12, 5, 4, 1)$
$P_{937} = (0, 6, 4, 1)$
$P_{938} = (1, 6, 4, 1)$
$P_{939} = (2, 6, 4, 1)$
$P_{940} = (3, 6, 4, 1)$
$P_{941} = (4, 6, 4, 1)$
$P_{942} = (5, 6, 4, 1)$
$P_{943} = (6, 6, 4, 1)$
$P_{944} = (7, 6, 4, 1)$
$P_{945} = (8, 6, 4, 1)$
$P_{946} = (9, 6, 4, 1)$
$P_{947} = (10, 6, 4, 1)$
$P_{948} = (11, 6, 4, 1)$
$P_{949} = (12, 6, 4, 1)$
$P_{950} = (0, 7, 4, 1)$
$P_{951} = (1, 7, 4, 1)$
$P_{952} = (2, 7, 4, 1)$
$P_{953} = (3, 7, 4, 1)$
$P_{954} = (4, 7, 4, 1)$
$P_{955} = (5, 7, 4, 1)$
$P_{956} = (6, 7, 4, 1)$
$P_{957} = (7, 7, 4, 1)$
$P_{958} = (8, 7, 4, 1)$
$P_{959} = (9, 7, 4, 1)$
$P_{960} = (10, 7, 4, 1)$
$P_{961} = (11, 7, 4, 1)$
$P_{962} = (12, 7, 4, 1)$
$P_{963} = (0, 8, 4, 1)$
$P_{964} = (1, 8, 4, 1)$
$P_{965} = (2, 8, 4, 1)$
$P_{966} = (3, 8, 4, 1)$
$P_{967} = (4, 8, 4, 1)$
$P_{968} = (5, 8, 4, 1)$
$P_{969} = (6, 8, 4, 1)$
$P_{970} = (7, 8, 4, 1)$
$P_{971} = (8, 8, 4, 1)$
$P_{972} = (9, 8, 4, 1)$
$P_{973} = (10, 8, 4, 1)$
$P_{974} = (11, 8, 4, 1)$
$P_{975} = (12, 8, 4, 1)$
$P_{976} = (0, 9, 4, 1)$
$P_{977} = (1, 9, 4, 1)$
$P_{978} = (2, 9, 4, 1)$
$P_{979} = (3, 9, 4, 1)$
$P_{980} = (4, 9, 4, 1)$
$P_{981} = (5, 9, 4, 1)$
$P_{982} = (6, 9, 4, 1)$
$P_{983} = (7, 9, 4, 1)$
$P_{984} = (8, 9, 4, 1)$
$P_{985} = (9, 9, 4, 1)$
$P_{986} = (10, 9, 4, 1)$
$P_{987} = (11, 9, 4, 1)$
$P_{988} = (12, 9, 4, 1)$
$P_{989} = (0, 10, 4, 1)$
$P_{990} = (1, 10, 4, 1)$
$P_{991} = (2, 10, 4, 1)$
$P_{992} = (3, 10, 4, 1)$
$P_{993} = (4, 10, 4, 1)$
$P_{994} = (5, 10, 4, 1)$
$P_{995} = (6, 10, 4, 1)$
$P_{996} = (7, 10, 4, 1)$
$P_{997} = (8, 10, 4, 1)$
$P_{998} = (9, 10, 4, 1)$
$P_{999} = (10, 10, 4, 1)$
$P_{1000} = (11, 10, 4, 1)$
$P_{1001} = (12, 10, 4, 1)$
$P_{1002} = (0, 11, 4, 1)$
$P_{1003} = (1, 11, 4, 1)$
$P_{1004} = (2, 11, 4, 1)$
$P_{1005} = (3, 11, 4, 1)$
$P_{1006} = (4, 11, 4, 1)$
$P_{1007} = (5, 11, 4, 1)$
$P_{1008} = (6, 11, 4, 1)$
$P_{1009} = (7, 11, 4, 1)$
$P_{1010} = (8, 11, 4, 1)$
$P_{1011} = (9, 11, 4, 1)$
$P_{1012} = (10, 11, 4, 1)$
$P_{1013} = (11, 11, 4, 1)$
$P_{1014} = (12, 11, 4, 1)$
$P_{1015} = (0, 12, 4, 1)$
$P_{1016} = (1, 12, 4, 1)$
$P_{1017} = (2, 12, 4, 1)$
$P_{1018} = (3, 12, 4, 1)$
$P_{1019} = (4, 12, 4, 1)$
$P_{1020} = (5, 12, 4, 1)$
$P_{1021} = (6, 12, 4, 1)$

$P_{1022} = (7, 12, 4, 1)$
$P_{1023} = (8, 12, 4, 1)$
$P_{1024} = (9, 12, 4, 1)$
$P_{1025} = (10, 12, 4, 1)$
$P_{1026} = (11, 12, 4, 1)$
$P_{1027} = (12, 12, 4, 1)$
$P_{1028} = (0, 0, 5, 1)$
$P_{1029} = (1, 0, 5, 1)$
$P_{1030} = (2, 0, 5, 1)$
$P_{1031} = (3, 0, 5, 1)$
$P_{1032} = (4, 0, 5, 1)$
$P_{1033} = (5, 0, 5, 1)$
$P_{1034} = (6, 0, 5, 1)$
$P_{1035} = (7, 0, 5, 1)$
$P_{1036} = (8, 0, 5, 1)$
$P_{1037} = (9, 0, 5, 1)$
$P_{1038} = (10, 0, 5, 1)$
$P_{1039} = (11, 0, 5, 1)$
$P_{1040} = (12, 0, 5, 1)$
$P_{1041} = (0, 1, 5, 1)$
$P_{1042} = (1, 1, 5, 1)$
$P_{1043} = (2, 1, 5, 1)$
$P_{1044} = (3, 1, 5, 1)$
$P_{1045} = (4, 1, 5, 1)$
$P_{1046} = (5, 1, 5, 1)$
$P_{1047} = (6, 1, 5, 1)$
$P_{1048} = (7, 1, 5, 1)$
$P_{1049} = (8, 1, 5, 1)$
$P_{1050} = (9, 1, 5, 1)$
$P_{1051} = (10, 1, 5, 1)$
$P_{1052} = (11, 1, 5, 1)$
$P_{1053} = (12, 1, 5, 1)$
$P_{1054} = (0, 2, 5, 1)$
$P_{1055} = (1, 2, 5, 1)$
$P_{1056} = (2, 2, 5, 1)$
$P_{1057} = (3, 2, 5, 1)$
$P_{1058} = (4, 2, 5, 1)$
$P_{1059} = (5, 2, 5, 1)$
$P_{1060} = (6, 2, 5, 1)$
$P_{1061} = (7, 2, 5, 1)$
$P_{1062} = (8, 2, 5, 1)$
$P_{1063} = (9, 2, 5, 1)$
$P_{1064} = (10, 2, 5, 1)$
$P_{1065} = (11, 2, 5, 1)$
$P_{1066} = (12, 2, 5, 1)$
$P_{1067} = (0, 3, 5, 1)$
$P_{1068} = (1, 3, 5, 1)$
$P_{1069} = (2, 3, 5, 1)$
$P_{1070} = (3, 3, 5, 1)$
$P_{1071} = (4, 3, 5, 1)$
$P_{1072} = (5, 3, 5, 1)$
$P_{1073} = (6, 3, 5, 1)$
$P_{1074} = (7, 3, 5, 1)$
$P_{1075} = (8, 3, 5, 1)$
$P_{1076} = (9, 3, 5, 1)$
$P_{1077} = (10, 3, 5, 1)$
$P_{1078} = (11, 3, 5, 1)$
$P_{1079} = (12, 3, 5, 1)$
$P_{1080} = (0, 4, 5, 1)$
$P_{1081} = (1, 4, 5, 1)$
$P_{1082} = (2, 4, 5, 1)$
$P_{1083} = (3, 4, 5, 1)$
$P_{1084} = (4, 4, 5, 1)$
$P_{1085} = (5, 4, 5, 1)$
$P_{1086} = (6, 4, 5, 1)$
$P_{1087} = (7, 4, 5, 1)$
$P_{1088} = (8, 4, 5, 1)$
$P_{1089} = (9, 4, 5, 1)$
$P_{1090} = (10, 4, 5, 1)$
$P_{1091} = (11, 4, 5, 1)$
$P_{1092} = (12, 4, 5, 1)$
$P_{1093} = (0, 5, 5, 1)$
$P_{1094} = (1, 5, 5, 1)$
$P_{1095} = (2, 5, 5, 1)$
$P_{1096} = (3, 5, 5, 1)$
$P_{1097} = (4, 5, 5, 1)$
$P_{1098} = (5, 5, 5, 1)$
$P_{1099} = (6, 5, 5, 1)$
$P_{1100} = (7, 5, 5, 1)$
$P_{1101} = (8, 5, 5, 1)$
$P_{1102} = (9, 5, 5, 1)$
$P_{1103} = (10, 5, 5, 1)$
$P_{1104} = (11, 5, 5, 1)$
$P_{1105} = (12, 5, 5, 1)$
$P_{1106} = (0, 6, 5, 1)$
$P_{1107} = (1, 6, 5, 1)$
$P_{1108} = (2, 6, 5, 1)$
$P_{1109} = (3, 6, 5, 1)$
$P_{1110} = (4, 6, 5, 1)$
$P_{1111} = (5, 6, 5, 1)$
$P_{1112} = (6, 6, 5, 1)$
$P_{1113} = (7, 6, 5, 1)$
$P_{1114} = (8, 6, 5, 1)$
$P_{1115} = (9, 6, 5, 1)$
$P_{1116} = (10, 6, 5, 1)$
$P_{1117} = (11, 6, 5, 1)$
$P_{1118} = (12, 6, 5, 1)$
$P_{1119} = (0, 7, 5, 1)$
$P_{1120} = (1, 7, 5, 1)$
$P_{1121} = (2, 7, 5, 1)$
$P_{1122} = (3, 7, 5, 1)$
$P_{1123} = (4, 7, 5, 1)$
$P_{1124} = (5, 7, 5, 1)$
$P_{1125} = (6, 7, 5, 1)$
$P_{1126} = (7, 7, 5, 1)$
$P_{1127} = (8, 7, 5, 1)$
$P_{1128} = (9, 7, 5, 1)$
$P_{1129} = (10, 7, 5, 1)$
$P_{1130} = (11, 7, 5, 1)$
$P_{1131} = (12, 7, 5, 1)$
$P_{1132} = (0, 8, 5, 1)$
$P_{1133} = (1, 8, 5, 1)$
$P_{1134} = (2, 8, 5, 1)$
$P_{1135} = (3, 8, 5, 1)$
$P_{1136} = (4, 8, 5, 1)$
$P_{1137} = (5, 8, 5, 1)$
$P_{1138} = (6, 8, 5, 1)$
$P_{1139} = (7, 8, 5, 1)$
$P_{1140} = (8, 8, 5, 1)$
$P_{1141} = (9, 8, 5, 1)$
$P_{1142} = (10, 8, 5, 1)$
$P_{1143} = (11, 8, 5, 1)$
$P_{1144} = (12, 8, 5, 1)$
$P_{1145} = (0, 9, 5, 1)$
$P_{1146} = (1, 9, 5, 1)$
$P_{1147} = (2, 9, 5, 1)$
$P_{1148} = (3, 9, 5, 1)$
$P_{1149} = (4, 9, 5, 1)$
$P_{1150} = (5, 9, 5, 1)$
$P_{1151} = (6, 9, 5, 1)$
$P_{1152} = (7, 9, 5, 1)$
$P_{1153} = (8, 9, 5, 1)$
$P_{1154} = (9, 9, 5, 1)$
$P_{1155} = (10, 9, 5, 1)$
$P_{1156} = (11, 9, 5, 1)$
$P_{1157} = (12, 9, 5, 1)$
$P_{1158} = (0, 10, 5, 1)$
$P_{1159} = (1, 10, 5, 1)$
$P_{1160} = (2, 10, 5, 1)$
$P_{1161} = (3, 10, 5, 1)$
$P_{1162} = (4, 10, 5, 1)$
$P_{1163} = (5, 10, 5, 1)$
$P_{1164} = (6, 10, 5, 1)$
$P_{1165} = (7, 10, 5, 1)$
$P_{1166} = (8, 10, 5, 1)$
$P_{1167} = (9, 10, 5, 1)$

$P_{1168} = (10, 10, 5, 1)$
$P_{1169} = (11, 10, 5, 1)$
$P_{1170} = (12, 10, 5, 1)$
$P_{1171} = (0, 11, 5, 1)$
$P_{1172} = (1, 11, 5, 1)$
$P_{1173} = (2, 11, 5, 1)$
$P_{1174} = (3, 11, 5, 1)$
$P_{1175} = (4, 11, 5, 1)$
$P_{1176} = (5, 11, 5, 1)$
$P_{1177} = (6, 11, 5, 1)$
$P_{1178} = (7, 11, 5, 1)$
$P_{1179} = (8, 11, 5, 1)$
$P_{1180} = (9, 11, 5, 1)$
$P_{1181} = (10, 11, 5, 1)$
$P_{1182} = (11, 11, 5, 1)$
$P_{1183} = (12, 11, 5, 1)$
$P_{1184} = (0, 12, 5, 1)$
$P_{1185} = (1, 12, 5, 1)$
$P_{1186} = (2, 12, 5, 1)$
$P_{1187} = (3, 12, 5, 1)$
$P_{1188} = (4, 12, 5, 1)$
$P_{1189} = (5, 12, 5, 1)$
$P_{1190} = (6, 12, 5, 1)$
$P_{1191} = (7, 12, 5, 1)$
$P_{1192} = (8, 12, 5, 1)$
$P_{1193} = (9, 12, 5, 1)$
$P_{1194} = (10, 12, 5, 1)$
$P_{1195} = (11, 12, 5, 1)$
$P_{1196} = (12, 12, 5, 1)$
$P_{1197} = (0, 0, 6, 1)$
$P_{1198} = (1, 0, 6, 1)$
$P_{1199} = (2, 0, 6, 1)$
$P_{1200} = (3, 0, 6, 1)$
$P_{1201} = (4, 0, 6, 1)$
$P_{1202} = (5, 0, 6, 1)$
$P_{1203} = (6, 0, 6, 1)$
$P_{1204} = (7, 0, 6, 1)$
$P_{1205} = (8, 0, 6, 1)$
$P_{1206} = (9, 0, 6, 1)$
$P_{1207} = (10, 0, 6, 1)$
$P_{1208} = (11, 0, 6, 1)$
$P_{1209} = (12, 0, 6, 1)$
$P_{1210} = (0, 1, 6, 1)$
$P_{1211} = (1, 1, 6, 1)$
$P_{1212} = (2, 1, 6, 1)$
$P_{1213} = (3, 1, 6, 1)$
$P_{1214} = (4, 1, 6, 1)$
$P_{1215} = (5, 1, 6, 1)$
$P_{1216} = (6, 1, 6, 1)$
$P_{1217} = (7, 1, 6, 1)$
$P_{1218} = (8, 1, 6, 1)$
$P_{1219} = (9, 1, 6, 1)$
$P_{1220} = (10, 1, 6, 1)$
$P_{1221} = (11, 1, 6, 1)$
$P_{1222} = (12, 1, 6, 1)$
$P_{1223} = (0, 2, 6, 1)$
$P_{1224} = (1, 2, 6, 1)$
$P_{1225} = (2, 2, 6, 1)$
$P_{1226} = (3, 2, 6, 1)$
$P_{1227} = (4, 2, 6, 1)$
$P_{1228} = (5, 2, 6, 1)$
$P_{1229} = (6, 2, 6, 1)$
$P_{1230} = (7, 2, 6, 1)$
$P_{1231} = (8, 2, 6, 1)$
$P_{1232} = (9, 2, 6, 1)$
$P_{1233} = (10, 2, 6, 1)$
$P_{1234} = (11, 2, 6, 1)$
$P_{1235} = (12, 2, 6, 1)$
$P_{1236} = (0, 3, 6, 1)$
$P_{1237} = (1, 3, 6, 1)$
$P_{1238} = (2, 3, 6, 1)$
$P_{1239} = (3, 3, 6, 1)$
$P_{1240} = (4, 3, 6, 1)$
$P_{1241} = (5, 3, 6, 1)$
$P_{1242} = (6, 3, 6, 1)$
$P_{1243} = (7, 3, 6, 1)$
$P_{1244} = (8, 3, 6, 1)$
$P_{1245} = (9, 3, 6, 1)$
$P_{1246} = (10, 3, 6, 1)$
$P_{1247} = (11, 3, 6, 1)$
$P_{1248} = (12, 3, 6, 1)$
$P_{1249} = (0, 4, 6, 1)$
$P_{1250} = (1, 4, 6, 1)$
$P_{1251} = (2, 4, 6, 1)$
$P_{1252} = (3, 4, 6, 1)$
$P_{1253} = (4, 4, 6, 1)$
$P_{1254} = (5, 4, 6, 1)$
$P_{1255} = (6, 4, 6, 1)$
$P_{1256} = (7, 4, 6, 1)$
$P_{1257} = (8, 4, 6, 1)$
$P_{1258} = (9, 4, 6, 1)$
$P_{1259} = (10, 4, 6, 1)$
$P_{1260} = (11, 4, 6, 1)$
$P_{1261} = (12, 4, 6, 1)$
$P_{1262} = (0, 5, 6, 1)$
$P_{1263} = (1, 5, 6, 1)$
$P_{1264} = (2, 5, 6, 1)$
$P_{1265} = (3, 5, 6, 1)$
$P_{1266} = (4, 5, 6, 1)$
$P_{1267} = (5, 5, 6, 1)$
$P_{1268} = (6, 5, 6, 1)$
$P_{1269} = (7, 5, 6, 1)$
$P_{1270} = (8, 5, 6, 1)$
$P_{1271} = (9, 5, 6, 1)$
$P_{1272} = (10, 5, 6, 1)$
$P_{1273} = (11, 5, 6, 1)$
$P_{1274} = (12, 5, 6, 1)$
$P_{1275} = (0, 6, 6, 1)$
$P_{1276} = (1, 6, 6, 1)$
$P_{1277} = (2, 6, 6, 1)$
$P_{1278} = (3, 6, 6, 1)$
$P_{1279} = (4, 6, 6, 1)$
$P_{1280} = (5, 6, 6, 1)$
$P_{1281} = (6, 6, 6, 1)$
$P_{1282} = (7, 6, 6, 1)$
$P_{1283} = (8, 6, 6, 1)$
$P_{1284} = (9, 6, 6, 1)$
$P_{1285} = (10, 6, 6, 1)$
$P_{1286} = (11, 6, 6, 1)$
$P_{1287} = (12, 6, 6, 1)$
$P_{1288} = (0, 7, 6, 1)$
$P_{1289} = (1, 7, 6, 1)$
$P_{1290} = (2, 7, 6, 1)$
$P_{1291} = (3, 7, 6, 1)$
$P_{1292} = (4, 7, 6, 1)$
$P_{1293} = (5, 7, 6, 1)$
$P_{1294} = (6, 7, 6, 1)$
$P_{1295} = (7, 7, 6, 1)$
$P_{1296} = (8, 7, 6, 1)$
$P_{1297} = (9, 7, 6, 1)$
$P_{1298} = (10, 7, 6, 1)$
$P_{1299} = (11, 7, 6, 1)$
$P_{1300} = (12, 7, 6, 1)$
$P_{1301} = (0, 8, 6, 1)$
$P_{1302} = (1, 8, 6, 1)$
$P_{1303} = (2, 8, 6, 1)$
$P_{1304} = (3, 8, 6, 1)$
$P_{1305} = (4, 8, 6, 1)$
$P_{1306} = (5, 8, 6, 1)$
$P_{1307} = (6, 8, 6, 1)$
$P_{1308} = (7, 8, 6, 1)$
$P_{1309} = (8, 8, 6, 1)$
$P_{1310} = (9, 8, 6, 1)$
$P_{1311} = (10, 8, 6, 1)$
$P_{1312} = (11, 8, 6, 1)$
$P_{1313} = (12, 8, 6, 1)$

$P_{1314} = (0, 9, 6, 1)$
$P_{1315} = (1, 9, 6, 1)$
$P_{1316} = (2, 9, 6, 1)$
$P_{1317} = (3, 9, 6, 1)$
$P_{1318} = (4, 9, 6, 1)$
$P_{1319} = (5, 9, 6, 1)$
$P_{1320} = (6, 9, 6, 1)$
$P_{1321} = (7, 9, 6, 1)$
$P_{1322} = (8, 9, 6, 1)$
$P_{1323} = (9, 9, 6, 1)$
$P_{1324} = (10, 9, 6, 1)$
$P_{1325} = (11, 9, 6, 1)$
$P_{1326} = (12, 9, 6, 1)$
$P_{1327} = (0, 10, 6, 1)$
$P_{1328} = (1, 10, 6, 1)$
$P_{1329} = (2, 10, 6, 1)$
$P_{1330} = (3, 10, 6, 1)$
$P_{1331} = (4, 10, 6, 1)$
$P_{1332} = (5, 10, 6, 1)$
$P_{1333} = (6, 10, 6, 1)$
$P_{1334} = (7, 10, 6, 1)$
$P_{1335} = (8, 10, 6, 1)$
$P_{1336} = (9, 10, 6, 1)$
$P_{1337} = (10, 10, 6, 1)$
$P_{1338} = (11, 10, 6, 1)$
$P_{1339} = (12, 10, 6, 1)$
$P_{1340} = (0, 11, 6, 1)$
$P_{1341} = (1, 11, 6, 1)$
$P_{1342} = (2, 11, 6, 1)$
$P_{1343} = (3, 11, 6, 1)$
$P_{1344} = (4, 11, 6, 1)$
$P_{1345} = (5, 11, 6, 1)$
$P_{1346} = (6, 11, 6, 1)$
$P_{1347} = (7, 11, 6, 1)$
$P_{1348} = (8, 11, 6, 1)$
$P_{1349} = (9, 11, 6, 1)$
$P_{1350} = (10, 11, 6, 1)$
$P_{1351} = (11, 11, 6, 1)$
$P_{1352} = (12, 11, 6, 1)$
$P_{1353} = (0, 12, 6, 1)$
$P_{1354} = (1, 12, 6, 1)$
$P_{1355} = (2, 12, 6, 1)$
$P_{1356} = (3, 12, 6, 1)$
$P_{1357} = (4, 12, 6, 1)$
$P_{1358} = (5, 12, 6, 1)$
$P_{1359} = (6, 12, 6, 1)$
$P_{1360} = (7, 12, 6, 1)$
$P_{1361} = (8, 12, 6, 1)$
$P_{1362} = (9, 12, 6, 1)$
$P_{1363} = (10, 12, 6, 1)$
$P_{1364} = (11, 12, 6, 1)$
$P_{1365} = (12, 12, 6, 1)$
$P_{1366} = (0, 0, 7, 1)$
$P_{1367} = (1, 0, 7, 1)$
$P_{1368} = (2, 0, 7, 1)$
$P_{1369} = (3, 0, 7, 1)$
$P_{1370} = (4, 0, 7, 1)$
$P_{1371} = (5, 0, 7, 1)$
$P_{1372} = (6, 0, 7, 1)$
$P_{1373} = (7, 0, 7, 1)$
$P_{1374} = (8, 0, 7, 1)$
$P_{1375} = (9, 0, 7, 1)$
$P_{1376} = (10, 0, 7, 1)$
$P_{1377} = (11, 0, 7, 1)$
$P_{1378} = (12, 0, 7, 1)$
$P_{1379} = (0, 1, 7, 1)$
$P_{1380} = (1, 1, 7, 1)$
$P_{1381} = (2, 1, 7, 1)$
$P_{1382} = (3, 1, 7, 1)$
$P_{1383} = (4, 1, 7, 1)$
$P_{1384} = (5, 1, 7, 1)$
$P_{1385} = (6, 1, 7, 1)$
$P_{1386} = (7, 1, 7, 1)$
$P_{1387} = (8, 1, 7, 1)$
$P_{1388} = (9, 1, 7, 1)$
$P_{1389} = (10, 1, 7, 1)$
$P_{1390} = (11, 1, 7, 1)$
$P_{1391} = (12, 1, 7, 1)$
$P_{1392} = (0, 2, 7, 1)$
$P_{1393} = (1, 2, 7, 1)$
$P_{1394} = (2, 2, 7, 1)$
$P_{1395} = (3, 2, 7, 1)$
$P_{1396} = (4, 2, 7, 1)$
$P_{1397} = (5, 2, 7, 1)$
$P_{1398} = (6, 2, 7, 1)$
$P_{1399} = (7, 2, 7, 1)$
$P_{1400} = (8, 2, 7, 1)$
$P_{1401} = (9, 2, 7, 1)$
$P_{1402} = (10, 2, 7, 1)$
$P_{1403} = (11, 2, 7, 1)$
$P_{1404} = (12, 2, 7, 1)$
$P_{1405} = (0, 3, 7, 1)$
$P_{1406} = (1, 3, 7, 1)$
$P_{1407} = (2, 3, 7, 1)$
$P_{1408} = (3, 3, 7, 1)$
$P_{1409} = (4, 3, 7, 1)$
$P_{1410} = (5, 3, 7, 1)$
$P_{1411} = (6, 3, 7, 1)$
$P_{1412} = (7, 3, 7, 1)$
$P_{1413} = (8, 3, 7, 1)$
$P_{1414} = (9, 3, 7, 1)$
$P_{1415} = (10, 3, 7, 1)$
$P_{1416} = (11, 3, 7, 1)$
$P_{1417} = (12, 3, 7, 1)$
$P_{1418} = (0, 4, 7, 1)$
$P_{1419} = (1, 4, 7, 1)$
$P_{1420} = (2, 4, 7, 1)$
$P_{1421} = (3, 4, 7, 1)$
$P_{1422} = (4, 4, 7, 1)$
$P_{1423} = (5, 4, 7, 1)$
$P_{1424} = (6, 4, 7, 1)$
$P_{1425} = (7, 4, 7, 1)$
$P_{1426} = (8, 4, 7, 1)$
$P_{1427} = (9, 4, 7, 1)$
$P_{1428} = (10, 4, 7, 1)$
$P_{1429} = (11, 4, 7, 1)$
$P_{1430} = (12, 4, 7, 1)$
$P_{1431} = (0, 5, 7, 1)$
$P_{1432} = (1, 5, 7, 1)$
$P_{1433} = (2, 5, 7, 1)$
$P_{1434} = (3, 5, 7, 1)$
$P_{1435} = (4, 5, 7, 1)$
$P_{1436} = (5, 5, 7, 1)$
$P_{1437} = (6, 5, 7, 1)$
$P_{1438} = (7, 5, 7, 1)$
$P_{1439} = (8, 5, 7, 1)$
$P_{1440} = (9, 5, 7, 1)$
$P_{1441} = (10, 5, 7, 1)$
$P_{1442} = (11, 5, 7, 1)$
$P_{1443} = (12, 5, 7, 1)$
$P_{1444} = (0, 6, 7, 1)$
$P_{1445} = (1, 6, 7, 1)$
$P_{1446} = (2, 6, 7, 1)$
$P_{1447} = (3, 6, 7, 1)$
$P_{1448} = (4, 6, 7, 1)$
$P_{1449} = (5, 6, 7, 1)$
$P_{1450} = (6, 6, 7, 1)$
$P_{1451} = (7, 6, 7, 1)$
$P_{1452} = (8, 6, 7, 1)$
$P_{1453} = (9, 6, 7, 1)$
$P_{1454} = (10, 6, 7, 1)$
$P_{1455} = (11, 6, 7, 1)$
$P_{1456} = (12, 6, 7, 1)$
$P_{1457} = (0, 7, 7, 1)$
$P_{1458} = (1, 7, 7, 1)$
$P_{1459} = (2, 7, 7, 1)$

$P_{1460} = (3, 7, 7, 1)$
$P_{1461} = (4, 7, 7, 1)$
$P_{1462} = (5, 7, 7, 1)$
$P_{1463} = (6, 7, 7, 1)$
$P_{1464} = (7, 7, 7, 1)$
$P_{1465} = (8, 7, 7, 1)$
$P_{1466} = (9, 7, 7, 1)$
$P_{1467} = (10, 7, 7, 1)$
$P_{1468} = (11, 7, 7, 1)$
$P_{1469} = (12, 7, 7, 1)$
$P_{1470} = (0, 8, 7, 1)$
$P_{1471} = (1, 8, 7, 1)$
$P_{1472} = (2, 8, 7, 1)$
$P_{1473} = (3, 8, 7, 1)$
$P_{1474} = (4, 8, 7, 1)$
$P_{1475} = (5, 8, 7, 1)$
$P_{1476} = (6, 8, 7, 1)$
$P_{1477} = (7, 8, 7, 1)$
$P_{1478} = (8, 8, 7, 1)$
$P_{1479} = (9, 8, 7, 1)$
$P_{1480} = (10, 8, 7, 1)$
$P_{1481} = (11, 8, 7, 1)$
$P_{1482} = (12, 8, 7, 1)$
$P_{1483} = (0, 9, 7, 1)$
$P_{1484} = (1, 9, 7, 1)$
$P_{1485} = (2, 9, 7, 1)$
$P_{1486} = (3, 9, 7, 1)$
$P_{1487} = (4, 9, 7, 1)$
$P_{1488} = (5, 9, 7, 1)$
$P_{1489} = (6, 9, 7, 1)$
$P_{1490} = (7, 9, 7, 1)$
$P_{1491} = (8, 9, 7, 1)$
$P_{1492} = (9, 9, 7, 1)$
$P_{1493} = (10, 9, 7, 1)$
$P_{1494} = (11, 9, 7, 1)$
$P_{1495} = (12, 9, 7, 1)$
$P_{1496} = (0, 10, 7, 1)$
$P_{1497} = (1, 10, 7, 1)$
$P_{1498} = (2, 10, 7, 1)$
$P_{1499} = (3, 10, 7, 1)$
$P_{1500} = (4, 10, 7, 1)$
$P_{1501} = (5, 10, 7, 1)$
$P_{1502} = (6, 10, 7, 1)$
$P_{1503} = (7, 10, 7, 1)$
$P_{1504} = (8, 10, 7, 1)$
$P_{1505} = (9, 10, 7, 1)$
$P_{1506} = (10, 10, 7, 1)$
$P_{1507} = (11, 10, 7, 1)$
$P_{1508} = (12, 10, 7, 1)$
$P_{1509} = (0, 11, 7, 1)$
$P_{1510} = (1, 11, 7, 1)$
$P_{1511} = (2, 11, 7, 1)$
$P_{1512} = (3, 11, 7, 1)$
$P_{1513} = (4, 11, 7, 1)$
$P_{1514} = (5, 11, 7, 1)$
$P_{1515} = (6, 11, 7, 1)$
$P_{1516} = (7, 11, 7, 1)$
$P_{1517} = (8, 11, 7, 1)$
$P_{1518} = (9, 11, 7, 1)$
$P_{1519} = (10, 11, 7, 1)$
$P_{1520} = (11, 11, 7, 1)$
$P_{1521} = (12, 11, 7, 1)$
$P_{1522} = (0, 12, 7, 1)$
$P_{1523} = (1, 12, 7, 1)$
$P_{1524} = (2, 12, 7, 1)$
$P_{1525} = (3, 12, 7, 1)$
$P_{1526} = (4, 12, 7, 1)$
$P_{1527} = (5, 12, 7, 1)$
$P_{1528} = (6, 12, 7, 1)$
$P_{1529} = (7, 12, 7, 1)$
$P_{1530} = (8, 12, 7, 1)$
$P_{1531} = (9, 12, 7, 1)$
$P_{1532} = (10, 12, 7, 1)$
$P_{1533} = (11, 12, 7, 1)$
$P_{1534} = (12, 12, 7, 1)$
$P_{1535} = (0, 0, 8, 1)$
$P_{1536} = (1, 0, 8, 1)$
$P_{1537} = (2, 0, 8, 1)$
$P_{1538} = (3, 0, 8, 1)$
$P_{1539} = (4, 0, 8, 1)$
$P_{1540} = (5, 0, 8, 1)$
$P_{1541} = (6, 0, 8, 1)$
$P_{1542} = (7, 0, 8, 1)$
$P_{1543} = (8, 0, 8, 1)$
$P_{1544} = (9, 0, 8, 1)$
$P_{1545} = (10, 0, 8, 1)$
$P_{1546} = (11, 0, 8, 1)$
$P_{1547} = (12, 0, 8, 1)$
$P_{1548} = (0, 1, 8, 1)$
$P_{1549} = (1, 1, 8, 1)$
$P_{1550} = (2, 1, 8, 1)$
$P_{1551} = (3, 1, 8, 1)$
$P_{1552} = (4, 1, 8, 1)$
$P_{1553} = (5, 1, 8, 1)$
$P_{1554} = (6, 1, 8, 1)$
$P_{1555} = (7, 1, 8, 1)$
$P_{1556} = (8, 1, 8, 1)$
$P_{1557} = (9, 1, 8, 1)$
$P_{1558} = (10, 1, 8, 1)$
$P_{1559} = (11, 1, 8, 1)$
$P_{1560} = (12, 1, 8, 1)$
$P_{1561} = (0, 2, 8, 1)$
$P_{1562} = (1, 2, 8, 1)$
$P_{1563} = (2, 2, 8, 1)$
$P_{1564} = (3, 2, 8, 1)$
$P_{1565} = (4, 2, 8, 1)$
$P_{1566} = (5, 2, 8, 1)$
$P_{1567} = (6, 2, 8, 1)$
$P_{1568} = (7, 2, 8, 1)$
$P_{1569} = (8, 2, 8, 1)$
$P_{1570} = (9, 2, 8, 1)$
$P_{1571} = (10, 2, 8, 1)$
$P_{1572} = (11, 2, 8, 1)$
$P_{1573} = (12, 2, 8, 1)$
$P_{1574} = (0, 3, 8, 1)$
$P_{1575} = (1, 3, 8, 1)$
$P_{1576} = (2, 3, 8, 1)$
$P_{1577} = (3, 3, 8, 1)$
$P_{1578} = (4, 3, 8, 1)$
$P_{1579} = (5, 3, 8, 1)$
$P_{1580} = (6, 3, 8, 1)$
$P_{1581} = (7, 3, 8, 1)$
$P_{1582} = (8, 3, 8, 1)$
$P_{1583} = (9, 3, 8, 1)$
$P_{1584} = (10, 3, 8, 1)$
$P_{1585} = (11, 3, 8, 1)$
$P_{1586} = (12, 3, 8, 1)$
$P_{1587} = (0, 4, 8, 1)$
$P_{1588} = (1, 4, 8, 1)$
$P_{1589} = (2, 4, 8, 1)$
$P_{1590} = (3, 4, 8, 1)$
$P_{1591} = (4, 4, 8, 1)$
$P_{1592} = (5, 4, 8, 1)$
$P_{1593} = (6, 4, 8, 1)$
$P_{1594} = (7, 4, 8, 1)$
$P_{1595} = (8, 4, 8, 1)$
$P_{1596} = (9, 4, 8, 1)$
$P_{1597} = (10, 4, 8, 1)$
$P_{1598} = (11, 4, 8, 1)$
$P_{1599} = (12, 4, 8, 1)$
$P_{1600} = (0, 5, 8, 1)$
$P_{1601} = (1, 5, 8, 1)$
$P_{1602} = (2, 5, 8, 1)$
$P_{1603} = (3, 5, 8, 1)$
$P_{1604} = (4, 5, 8, 1)$
$P_{1605} = (5, 5, 8, 1)$

$P_{1606} = (6, 5, 8, 1)$
$P_{1607} = (7, 5, 8, 1)$
$P_{1608} = (8, 5, 8, 1)$
$P_{1609} = (9, 5, 8, 1)$
$P_{1610} = (10, 5, 8, 1)$
$P_{1611} = (11, 5, 8, 1)$
$P_{1612} = (12, 5, 8, 1)$
$P_{1613} = (0, 6, 8, 1)$
$P_{1614} = (1, 6, 8, 1)$
$P_{1615} = (2, 6, 8, 1)$
$P_{1616} = (3, 6, 8, 1)$
$P_{1617} = (4, 6, 8, 1)$
$P_{1618} = (5, 6, 8, 1)$
$P_{1619} = (6, 6, 8, 1)$
$P_{1620} = (7, 6, 8, 1)$
$P_{1621} = (8, 6, 8, 1)$
$P_{1622} = (9, 6, 8, 1)$
$P_{1623} = (10, 6, 8, 1)$
$P_{1624} = (11, 6, 8, 1)$
$P_{1625} = (12, 6, 8, 1)$
$P_{1626} = (0, 7, 8, 1)$
$P_{1627} = (1, 7, 8, 1)$
$P_{1628} = (2, 7, 8, 1)$
$P_{1629} = (3, 7, 8, 1)$
$P_{1630} = (4, 7, 8, 1)$
$P_{1631} = (5, 7, 8, 1)$
$P_{1632} = (6, 7, 8, 1)$
$P_{1633} = (7, 7, 8, 1)$
$P_{1634} = (8, 7, 8, 1)$
$P_{1635} = (9, 7, 8, 1)$
$P_{1636} = (10, 7, 8, 1)$
$P_{1637} = (11, 7, 8, 1)$
$P_{1638} = (12, 7, 8, 1)$
$P_{1639} = (0, 8, 8, 1)$
$P_{1640} = (1, 8, 8, 1)$
$P_{1641} = (2, 8, 8, 1)$
$P_{1642} = (3, 8, 8, 1)$
$P_{1643} = (4, 8, 8, 1)$
$P_{1644} = (5, 8, 8, 1)$
$P_{1645} = (6, 8, 8, 1)$
$P_{1646} = (7, 8, 8, 1)$
$P_{1647} = (8, 8, 8, 1)$
$P_{1648} = (9, 8, 8, 1)$
$P_{1649} = (10, 8, 8, 1)$
$P_{1650} = (11, 8, 8, 1)$
$P_{1651} = (12, 8, 8, 1)$
$P_{1652} = (0, 9, 8, 1)$
$P_{1653} = (1, 9, 8, 1)$
$P_{1654} = (2, 9, 8, 1)$
$P_{1655} = (3, 9, 8, 1)$
$P_{1656} = (4, 9, 8, 1)$
$P_{1657} = (5, 9, 8, 1)$
$P_{1658} = (6, 9, 8, 1)$
$P_{1659} = (7, 9, 8, 1)$
$P_{1660} = (8, 9, 8, 1)$
$P_{1661} = (9, 9, 8, 1)$
$P_{1662} = (10, 9, 8, 1)$
$P_{1663} = (11, 9, 8, 1)$
$P_{1664} = (12, 9, 8, 1)$
$P_{1665} = (0, 10, 8, 1)$
$P_{1666} = (1, 10, 8, 1)$
$P_{1667} = (2, 10, 8, 1)$
$P_{1668} = (3, 10, 8, 1)$
$P_{1669} = (4, 10, 8, 1)$
$P_{1670} = (5, 10, 8, 1)$
$P_{1671} = (6, 10, 8, 1)$
$P_{1672} = (7, 10, 8, 1)$
$P_{1673} = (8, 10, 8, 1)$
$P_{1674} = (9, 10, 8, 1)$
$P_{1675} = (10, 10, 8, 1)$
$P_{1676} = (11, 10, 8, 1)$
$P_{1677} = (12, 10, 8, 1)$
$P_{1678} = (0, 11, 8, 1)$
$P_{1679} = (1, 11, 8, 1)$
$P_{1680} = (2, 11, 8, 1)$
$P_{1681} = (3, 11, 8, 1)$
$P_{1682} = (4, 11, 8, 1)$
$P_{1683} = (5, 11, 8, 1)$
$P_{1684} = (6, 11, 8, 1)$
$P_{1685} = (7, 11, 8, 1)$
$P_{1686} = (8, 11, 8, 1)$
$P_{1687} = (9, 11, 8, 1)$
$P_{1688} = (10, 11, 8, 1)$
$P_{1689} = (11, 11, 8, 1)$
$P_{1690} = (12, 11, 8, 1)$
$P_{1691} = (0, 12, 8, 1)$
$P_{1692} = (1, 12, 8, 1)$
$P_{1693} = (2, 12, 8, 1)$
$P_{1694} = (3, 12, 8, 1)$
$P_{1695} = (4, 12, 8, 1)$
$P_{1696} = (5, 12, 8, 1)$
$P_{1697} = (6, 12, 8, 1)$
$P_{1698} = (7, 12, 8, 1)$
$P_{1699} = (8, 12, 8, 1)$
$P_{1700} = (9, 12, 8, 1)$
$P_{1701} = (10, 12, 8, 1)$
$P_{1702} = (11, 12, 8, 1)$
$P_{1703} = (12, 12, 8, 1)$
$P_{1704} = (0, 0, 9, 1)$
$P_{1705} = (1, 0, 9, 1)$
$P_{1706} = (2, 0, 9, 1)$
$P_{1707} = (3, 0, 9, 1)$
$P_{1708} = (4, 0, 9, 1)$
$P_{1709} = (5, 0, 9, 1)$
$P_{1710} = (6, 0, 9, 1)$
$P_{1711} = (7, 0, 9, 1)$
$P_{1712} = (8, 0, 9, 1)$
$P_{1713} = (9, 0, 9, 1)$
$P_{1714} = (10, 0, 9, 1)$
$P_{1715} = (11, 0, 9, 1)$
$P_{1716} = (12, 0, 9, 1)$
$P_{1717} = (0, 1, 9, 1)$
$P_{1718} = (1, 1, 9, 1)$
$P_{1719} = (2, 1, 9, 1)$
$P_{1720} = (3, 1, 9, 1)$
$P_{1721} = (4, 1, 9, 1)$
$P_{1722} = (5, 1, 9, 1)$
$P_{1723} = (6, 1, 9, 1)$
$P_{1724} = (7, 1, 9, 1)$
$P_{1725} = (8, 1, 9, 1)$
$P_{1726} = (9, 1, 9, 1)$
$P_{1727} = (10, 1, 9, 1)$
$P_{1728} = (11, 1, 9, 1)$
$P_{1729} = (12, 1, 9, 1)$
$P_{1730} = (0, 2, 9, 1)$
$P_{1731} = (1, 2, 9, 1)$
$P_{1732} = (2, 2, 9, 1)$
$P_{1733} = (3, 2, 9, 1)$
$P_{1734} = (4, 2, 9, 1)$
$P_{1735} = (5, 2, 9, 1)$
$P_{1736} = (6, 2, 9, 1)$
$P_{1737} = (7, 2, 9, 1)$
$P_{1738} = (8, 2, 9, 1)$
$P_{1739} = (9, 2, 9, 1)$
$P_{1740} = (10, 2, 9, 1)$
$P_{1741} = (11, 2, 9, 1)$
$P_{1742} = (12, 2, 9, 1)$
$P_{1743} = (0, 3, 9, 1)$
$P_{1744} = (1, 3, 9, 1)$
$P_{1745} = (2, 3, 9, 1)$
$P_{1746} = (3, 3, 9, 1)$
$P_{1747} = (4, 3, 9, 1)$
$P_{1748} = (5, 3, 9, 1)$
$P_{1749} = (6, 3, 9, 1)$
$P_{1750} = (7, 3, 9, 1)$
$P_{1751} = (8, 3, 9, 1)$

$P_{1752} = (9, 3, 9, 1)$
$P_{1753} = (10, 3, 9, 1)$
$P_{1754} = (11, 3, 9, 1)$
$P_{1755} = (12, 3, 9, 1)$
$P_{1756} = (0, 4, 9, 1)$
$P_{1757} = (1, 4, 9, 1)$
$P_{1758} = (2, 4, 9, 1)$
$P_{1759} = (3, 4, 9, 1)$
$P_{1760} = (4, 4, 9, 1)$
$P_{1761} = (5, 4, 9, 1)$
$P_{1762} = (6, 4, 9, 1)$
$P_{1763} = (7, 4, 9, 1)$
$P_{1764} = (8, 4, 9, 1)$
$P_{1765} = (9, 4, 9, 1)$
$P_{1766} = (10, 4, 9, 1)$
$P_{1767} = (11, 4, 9, 1)$
$P_{1768} = (12, 4, 9, 1)$
$P_{1769} = (0, 5, 9, 1)$
$P_{1770} = (1, 5, 9, 1)$
$P_{1771} = (2, 5, 9, 1)$
$P_{1772} = (3, 5, 9, 1)$
$P_{1773} = (4, 5, 9, 1)$
$P_{1774} = (5, 5, 9, 1)$
$P_{1775} = (6, 5, 9, 1)$
$P_{1776} = (7, 5, 9, 1)$
$P_{1777} = (8, 5, 9, 1)$
$P_{1778} = (9, 5, 9, 1)$
$P_{1779} = (10, 5, 9, 1)$
$P_{1780} = (11, 5, 9, 1)$
$P_{1781} = (12, 5, 9, 1)$
$P_{1782} = (0, 6, 9, 1)$
$P_{1783} = (1, 6, 9, 1)$
$P_{1784} = (2, 6, 9, 1)$
$P_{1785} = (3, 6, 9, 1)$
$P_{1786} = (4, 6, 9, 1)$
$P_{1787} = (5, 6, 9, 1)$
$P_{1788} = (6, 6, 9, 1)$
$P_{1789} = (7, 6, 9, 1)$
$P_{1790} = (8, 6, 9, 1)$
$P_{1791} = (9, 6, 9, 1)$
$P_{1792} = (10, 6, 9, 1)$
$P_{1793} = (11, 6, 9, 1)$
$P_{1794} = (12, 6, 9, 1)$
$P_{1795} = (0, 7, 9, 1)$
$P_{1796} = (1, 7, 9, 1)$
$P_{1797} = (2, 7, 9, 1)$
$P_{1798} = (3, 7, 9, 1)$
$P_{1799} = (4, 7, 9, 1)$
$P_{1800} = (5, 7, 9, 1)$
$P_{1801} = (6, 7, 9, 1)$
$P_{1802} = (7, 7, 9, 1)$
$P_{1803} = (8, 7, 9, 1)$
$P_{1804} = (9, 7, 9, 1)$
$P_{1805} = (10, 7, 9, 1)$
$P_{1806} = (11, 7, 9, 1)$
$P_{1807} = (12, 7, 9, 1)$
$P_{1808} = (0, 8, 9, 1)$
$P_{1809} = (1, 8, 9, 1)$
$P_{1810} = (2, 8, 9, 1)$
$P_{1811} = (3, 8, 9, 1)$
$P_{1812} = (4, 8, 9, 1)$
$P_{1813} = (5, 8, 9, 1)$
$P_{1814} = (6, 8, 9, 1)$
$P_{1815} = (7, 8, 9, 1)$
$P_{1816} = (8, 8, 9, 1)$
$P_{1817} = (9, 8, 9, 1)$
$P_{1818} = (10, 8, 9, 1)$
$P_{1819} = (11, 8, 9, 1)$
$P_{1820} = (12, 8, 9, 1)$
$P_{1821} = (0, 9, 9, 1)$
$P_{1822} = (1, 9, 9, 1)$
$P_{1823} = (2, 9, 9, 1)$
$P_{1824} = (3, 9, 9, 1)$
$P_{1825} = (4, 9, 9, 1)$
$P_{1826} = (5, 9, 9, 1)$
$P_{1827} = (6, 9, 9, 1)$
$P_{1828} = (7, 9, 9, 1)$
$P_{1829} = (8, 9, 9, 1)$
$P_{1830} = (9, 9, 9, 1)$
$P_{1831} = (10, 9, 9, 1)$
$P_{1832} = (11, 9, 9, 1)$
$P_{1833} = (12, 9, 9, 1)$
$P_{1834} = (0, 10, 9, 1)$
$P_{1835} = (1, 10, 9, 1)$
$P_{1836} = (2, 10, 9, 1)$
$P_{1837} = (3, 10, 9, 1)$
$P_{1838} = (4, 10, 9, 1)$
$P_{1839} = (5, 10, 9, 1)$
$P_{1840} = (6, 10, 9, 1)$
$P_{1841} = (7, 10, 9, 1)$
$P_{1842} = (8, 10, 9, 1)$
$P_{1843} = (9, 10, 9, 1)$
$P_{1844} = (10, 10, 9, 1)$
$P_{1845} = (11, 10, 9, 1)$
$P_{1846} = (12, 10, 9, 1)$
$P_{1847} = (0, 11, 9, 1)$
$P_{1848} = (1, 11, 9, 1)$
$P_{1849} = (2, 11, 9, 1)$
$P_{1850} = (3, 11, 9, 1)$
$P_{1851} = (4, 11, 9, 1)$
$P_{1852} = (5, 11, 9, 1)$
$P_{1853} = (6, 11, 9, 1)$
$P_{1854} = (7, 11, 9, 1)$
$P_{1855} = (8, 11, 9, 1)$
$P_{1856} = (9, 11, 9, 1)$
$P_{1857} = (10, 11, 9, 1)$
$P_{1858} = (11, 11, 9, 1)$
$P_{1859} = (12, 11, 9, 1)$
$P_{1860} = (0, 12, 9, 1)$
$P_{1861} = (1, 12, 9, 1)$
$P_{1862} = (2, 12, 9, 1)$
$P_{1863} = (3, 12, 9, 1)$
$P_{1864} = (4, 12, 9, 1)$
$P_{1865} = (5, 12, 9, 1)$
$P_{1866} = (6, 12, 9, 1)$
$P_{1867} = (7, 12, 9, 1)$
$P_{1868} = (8, 12, 9, 1)$
$P_{1869} = (9, 12, 9, 1)$
$P_{1870} = (10, 12, 9, 1)$
$P_{1871} = (11, 12, 9, 1)$
$P_{1872} = (12, 12, 9, 1)$
$P_{1873} = (0, 0, 10, 1)$
$P_{1874} = (1, 0, 10, 1)$
$P_{1875} = (2, 0, 10, 1)$
$P_{1876} = (3, 0, 10, 1)$
$P_{1877} = (4, 0, 10, 1)$
$P_{1878} = (5, 0, 10, 1)$
$P_{1879} = (6, 0, 10, 1)$
$P_{1880} = (7, 0, 10, 1)$
$P_{1881} = (8, 0, 10, 1)$
$P_{1882} = (9, 0, 10, 1)$
$P_{1883} = (10, 0, 10, 1)$
$P_{1884} = (11, 0, 10, 1)$
$P_{1885} = (12, 0, 10, 1)$
$P_{1886} = (0, 1, 10, 1)$
$P_{1887} = (1, 1, 10, 1)$
$P_{1888} = (2, 1, 10, 1)$
$P_{1889} = (3, 1, 10, 1)$
$P_{1890} = (4, 1, 10, 1)$
$P_{1891} = (5, 1, 10, 1)$
$P_{1892} = (6, 1, 10, 1)$
$P_{1893} = (7, 1, 10, 1)$
$P_{1894} = (8, 1, 10, 1)$
$P_{1895} = (9, 1, 10, 1)$
$P_{1896} = (10, 1, 10, 1)$
$P_{1897} = (11, 1, 10, 1)$

$P_{1898} = (12, 1, 10, 1)$
$P_{1899} = (0, 2, 10, 1)$
$P_{1900} = (1, 2, 10, 1)$
$P_{1901} = (2, 2, 10, 1)$
$P_{1902} = (3, 2, 10, 1)$
$P_{1903} = (4, 2, 10, 1)$
$P_{1904} = (5, 2, 10, 1)$
$P_{1905} = (6, 2, 10, 1)$
$P_{1906} = (7, 2, 10, 1)$
$P_{1907} = (8, 2, 10, 1)$
$P_{1908} = (9, 2, 10, 1)$
$P_{1909} = (10, 2, 10, 1)$
$P_{1910} = (11, 2, 10, 1)$
$P_{1911} = (12, 2, 10, 1)$
$P_{1912} = (0, 3, 10, 1)$
$P_{1913} = (1, 3, 10, 1)$
$P_{1914} = (2, 3, 10, 1)$
$P_{1915} = (3, 3, 10, 1)$
$P_{1916} = (4, 3, 10, 1)$
$P_{1917} = (5, 3, 10, 1)$
$P_{1918} = (6, 3, 10, 1)$
$P_{1919} = (7, 3, 10, 1)$
$P_{1920} = (8, 3, 10, 1)$
$P_{1921} = (9, 3, 10, 1)$
$P_{1922} = (10, 3, 10, 1)$
$P_{1923} = (11, 3, 10, 1)$
$P_{1924} = (12, 3, 10, 1)$
$P_{1925} = (0, 4, 10, 1)$
$P_{1926} = (1, 4, 10, 1)$
$P_{1927} = (2, 4, 10, 1)$
$P_{1928} = (3, 4, 10, 1)$
$P_{1929} = (4, 4, 10, 1)$
$P_{1930} = (5, 4, 10, 1)$
$P_{1931} = (6, 4, 10, 1)$
$P_{1932} = (7, 4, 10, 1)$
$P_{1933} = (8, 4, 10, 1)$
$P_{1934} = (9, 4, 10, 1)$
$P_{1935} = (10, 4, 10, 1)$
$P_{1936} = (11, 4, 10, 1)$
$P_{1937} = (12, 4, 10, 1)$
$P_{1938} = (0, 5, 10, 1)$
$P_{1939} = (1, 5, 10, 1)$
$P_{1940} = (2, 5, 10, 1)$
$P_{1941} = (3, 5, 10, 1)$
$P_{1942} = (4, 5, 10, 1)$
$P_{1943} = (5, 5, 10, 1)$
$P_{1944} = (6, 5, 10, 1)$
$P_{1945} = (7, 5, 10, 1)$
$P_{1946} = (8, 5, 10, 1)$
$P_{1947} = (9, 5, 10, 1)$
$P_{1948} = (10, 5, 10, 1)$
$P_{1949} = (11, 5, 10, 1)$
$P_{1950} = (12, 5, 10, 1)$
$P_{1951} = (0, 6, 10, 1)$
$P_{1952} = (1, 6, 10, 1)$
$P_{1953} = (2, 6, 10, 1)$
$P_{1954} = (3, 6, 10, 1)$
$P_{1955} = (4, 6, 10, 1)$
$P_{1956} = (5, 6, 10, 1)$
$P_{1957} = (6, 6, 10, 1)$
$P_{1958} = (7, 6, 10, 1)$
$P_{1959} = (8, 6, 10, 1)$
$P_{1960} = (9, 6, 10, 1)$
$P_{1961} = (10, 6, 10, 1)$
$P_{1962} = (11, 6, 10, 1)$
$P_{1963} = (12, 6, 10, 1)$
$P_{1964} = (0, 7, 10, 1)$
$P_{1965} = (1, 7, 10, 1)$
$P_{1966} = (2, 7, 10, 1)$
$P_{1967} = (3, 7, 10, 1)$
$P_{1968} = (4, 7, 10, 1)$
$P_{1969} = (5, 7, 10, 1)$
$P_{1970} = (6, 7, 10, 1)$
$P_{1971} = (7, 7, 10, 1)$
$P_{1972} = (8, 7, 10, 1)$
$P_{1973} = (9, 7, 10, 1)$
$P_{1974} = (10, 7, 10, 1)$
$P_{1975} = (11, 7, 10, 1)$
$P_{1976} = (12, 7, 10, 1)$
$P_{1977} = (0, 8, 10, 1)$
$P_{1978} = (1, 8, 10, 1)$
$P_{1979} = (2, 8, 10, 1)$
$P_{1980} = (3, 8, 10, 1)$
$P_{1981} = (4, 8, 10, 1)$
$P_{1982} = (5, 8, 10, 1)$
$P_{1983} = (6, 8, 10, 1)$
$P_{1984} = (7, 8, 10, 1)$
$P_{1985} = (8, 8, 10, 1)$
$P_{1986} = (9, 8, 10, 1)$
$P_{1987} = (10, 8, 10, 1)$
$P_{1988} = (11, 8, 10, 1)$
$P_{1989} = (12, 8, 10, 1)$
$P_{1990} = (0, 9, 10, 1)$
$P_{1991} = (1, 9, 10, 1)$
$P_{1992} = (2, 9, 10, 1)$
$P_{1993} = (3, 9, 10, 1)$
$P_{1994} = (4, 9, 10, 1)$
$P_{1995} = (5, 9, 10, 1)$
$P_{1996} = (6, 9, 10, 1)$
$P_{1997} = (7, 9, 10, 1)$
$P_{1998} = (8, 9, 10, 1)$
$P_{1999} = (9, 9, 10, 1)$
$P_{2000} = (10, 9, 10, 1)$
$P_{2001} = (11, 9, 10, 1)$
$P_{2002} = (12, 9, 10, 1)$
$P_{2003} = (0, 10, 10, 1)$
$P_{2004} = (1, 10, 10, 1)$
$P_{2005} = (2, 10, 10, 1)$
$P_{2006} = (3, 10, 10, 1)$
$P_{2007} = (4, 10, 10, 1)$
$P_{2008} = (5, 10, 10, 1)$
$P_{2009} = (6, 10, 10, 1)$
$P_{2010} = (7, 10, 10, 1)$
$P_{2011} = (8, 10, 10, 1)$
$P_{2012} = (9, 10, 10, 1)$
$P_{2013} = (10, 10, 10, 1)$
$P_{2014} = (11, 10, 10, 1)$
$P_{2015} = (12, 10, 10, 1)$
$P_{2016} = (0, 11, 10, 1)$
$P_{2017} = (1, 11, 10, 1)$
$P_{2018} = (2, 11, 10, 1)$
$P_{2019} = (3, 11, 10, 1)$
$P_{2020} = (4, 11, 10, 1)$
$P_{2021} = (5, 11, 10, 1)$
$P_{2022} = (6, 11, 10, 1)$
$P_{2023} = (7, 11, 10, 1)$
$P_{2024} = (8, 11, 10, 1)$
$P_{2025} = (9, 11, 10, 1)$
$P_{2026} = (10, 11, 10, 1)$
$P_{2027} = (11, 11, 10, 1)$
$P_{2028} = (12, 11, 10, 1)$
$P_{2029} = (0, 12, 10, 1)$
$P_{2030} = (1, 12, 10, 1)$
$P_{2031} = (2, 12, 10, 1)$
$P_{2032} = (3, 12, 10, 1)$
$P_{2033} = (4, 12, 10, 1)$
$P_{2034} = (5, 12, 10, 1)$
$P_{2035} = (6, 12, 10, 1)$
$P_{2036} = (7, 12, 10, 1)$
$P_{2037} = (8, 12, 10, 1)$
$P_{2038} = (9, 12, 10, 1)$
$P_{2039} = (10, 12, 10, 1)$
$P_{2040} = (11, 12, 10, 1)$
$P_{2041} = (12, 12, 10, 1)$
$P_{2042} = (0, 0, 11, 1)$
$P_{2043} = (1, 0, 11, 1)$

$P_{2044} = (2, 0, 11, 1)$
$P_{2045} = (3, 0, 11, 1)$
$P_{2046} = (4, 0, 11, 1)$
$P_{2047} = (5, 0, 11, 1)$
$P_{2048} = (6, 0, 11, 1)$
$P_{2049} = (7, 0, 11, 1)$
$P_{2050} = (8, 0, 11, 1)$
$P_{2051} = (9, 0, 11, 1)$
$P_{2052} = (10, 0, 11, 1)$
$P_{2053} = (11, 0, 11, 1)$
$P_{2054} = (12, 0, 11, 1)$
$P_{2055} = (0, 1, 11, 1)$
$P_{2056} = (1, 1, 11, 1)$
$P_{2057} = (2, 1, 11, 1)$
$P_{2058} = (3, 1, 11, 1)$
$P_{2059} = (4, 1, 11, 1)$
$P_{2060} = (5, 1, 11, 1)$
$P_{2061} = (6, 1, 11, 1)$
$P_{2062} = (7, 1, 11, 1)$
$P_{2063} = (8, 1, 11, 1)$
$P_{2064} = (9, 1, 11, 1)$
$P_{2065} = (10, 1, 11, 1)$
$P_{2066} = (11, 1, 11, 1)$
$P_{2067} = (12, 1, 11, 1)$
$P_{2068} = (0, 2, 11, 1)$
$P_{2069} = (1, 2, 11, 1)$
$P_{2070} = (2, 2, 11, 1)$
$P_{2071} = (3, 2, 11, 1)$
$P_{2072} = (4, 2, 11, 1)$
$P_{2073} = (5, 2, 11, 1)$
$P_{2074} = (6, 2, 11, 1)$
$P_{2075} = (7, 2, 11, 1)$
$P_{2076} = (8, 2, 11, 1)$
$P_{2077} = (9, 2, 11, 1)$
$P_{2078} = (10, 2, 11, 1)$
$P_{2079} = (11, 2, 11, 1)$
$P_{2080} = (12, 2, 11, 1)$
$P_{2081} = (0, 3, 11, 1)$
$P_{2082} = (1, 3, 11, 1)$
$P_{2083} = (2, 3, 11, 1)$
$P_{2084} = (3, 3, 11, 1)$
$P_{2085} = (4, 3, 11, 1)$
$P_{2086} = (5, 3, 11, 1)$
$P_{2087} = (6, 3, 11, 1)$
$P_{2088} = (7, 3, 11, 1)$
$P_{2089} = (8, 3, 11, 1)$
$P_{2090} = (9, 3, 11, 1)$
$P_{2091} = (10, 3, 11, 1)$
$P_{2092} = (11, 3, 11, 1)$
$P_{2093} = (12, 3, 11, 1)$
$P_{2094} = (0, 4, 11, 1)$
$P_{2095} = (1, 4, 11, 1)$
$P_{2096} = (2, 4, 11, 1)$
$P_{2097} = (3, 4, 11, 1)$
$P_{2098} = (4, 4, 11, 1)$
$P_{2099} = (5, 4, 11, 1)$
$P_{2100} = (6, 4, 11, 1)$
$P_{2101} = (7, 4, 11, 1)$
$P_{2102} = (8, 4, 11, 1)$
$P_{2103} = (9, 4, 11, 1)$
$P_{2104} = (10, 4, 11, 1)$
$P_{2105} = (11, 4, 11, 1)$
$P_{2106} = (12, 4, 11, 1)$
$P_{2107} = (0, 5, 11, 1)$
$P_{2108} = (1, 5, 11, 1)$
$P_{2109} = (2, 5, 11, 1)$
$P_{2110} = (3, 5, 11, 1)$
$P_{2111} = (4, 5, 11, 1)$
$P_{2112} = (5, 5, 11, 1)$
$P_{2113} = (6, 5, 11, 1)$
$P_{2114} = (7, 5, 11, 1)$
$P_{2115} = (8, 5, 11, 1)$
$P_{2116} = (9, 5, 11, 1)$
$P_{2117} = (10, 5, 11, 1)$
$P_{2118} = (11, 5, 11, 1)$
$P_{2119} = (12, 5, 11, 1)$
$P_{2120} = (0, 6, 11, 1)$
$P_{2121} = (1, 6, 11, 1)$
$P_{2122} = (2, 6, 11, 1)$
$P_{2123} = (3, 6, 11, 1)$
$P_{2124} = (4, 6, 11, 1)$
$P_{2125} = (5, 6, 11, 1)$
$P_{2126} = (6, 6, 11, 1)$
$P_{2127} = (7, 6, 11, 1)$
$P_{2128} = (8, 6, 11, 1)$
$P_{2129} = (9, 6, 11, 1)$
$P_{2130} = (10, 6, 11, 1)$
$P_{2131} = (11, 6, 11, 1)$
$P_{2132} = (12, 6, 11, 1)$
$P_{2133} = (0, 7, 11, 1)$
$P_{2134} = (1, 7, 11, 1)$
$P_{2135} = (2, 7, 11, 1)$
$P_{2136} = (3, 7, 11, 1)$
$P_{2137} = (4, 7, 11, 1)$
$P_{2138} = (5, 7, 11, 1)$
$P_{2139} = (6, 7, 11, 1)$
$P_{2140} = (7, 7, 11, 1)$
$P_{2141} = (8, 7, 11, 1)$
$P_{2142} = (9, 7, 11, 1)$
$P_{2143} = (10, 7, 11, 1)$
$P_{2144} = (11, 7, 11, 1)$
$P_{2145} = (12, 7, 11, 1)$
$P_{2146} = (0, 8, 11, 1)$
$P_{2147} = (1, 8, 11, 1)$
$P_{2148} = (2, 8, 11, 1)$
$P_{2149} = (3, 8, 11, 1)$
$P_{2150} = (4, 8, 11, 1)$
$P_{2151} = (5, 8, 11, 1)$
$P_{2152} = (6, 8, 11, 1)$
$P_{2153} = (7, 8, 11, 1)$
$P_{2154} = (8, 8, 11, 1)$
$P_{2155} = (9, 8, 11, 1)$
$P_{2156} = (10, 8, 11, 1)$
$P_{2157} = (11, 8, 11, 1)$
$P_{2158} = (12, 8, 11, 1)$
$P_{2159} = (0, 9, 11, 1)$
$P_{2160} = (1, 9, 11, 1)$
$P_{2161} = (2, 9, 11, 1)$
$P_{2162} = (3, 9, 11, 1)$
$P_{2163} = (4, 9, 11, 1)$
$P_{2164} = (5, 9, 11, 1)$
$P_{2165} = (6, 9, 11, 1)$
$P_{2166} = (7, 9, 11, 1)$
$P_{2167} = (8, 9, 11, 1)$
$P_{2168} = (9, 9, 11, 1)$
$P_{2169} = (10, 9, 11, 1)$
$P_{2170} = (11, 9, 11, 1)$
$P_{2171} = (12, 9, 11, 1)$
$P_{2172} = (0, 10, 11, 1)$
$P_{2173} = (1, 10, 11, 1)$
$P_{2174} = (2, 10, 11, 1)$
$P_{2175} = (3, 10, 11, 1)$
$P_{2176} = (4, 10, 11, 1)$
$P_{2177} = (5, 10, 11, 1)$
$P_{2178} = (6, 10, 11, 1)$
$P_{2179} = (7, 10, 11, 1)$
$P_{2180} = (8, 10, 11, 1)$
$P_{2181} = (9, 10, 11, 1)$
$P_{2182} = (10, 10, 11, 1)$
$P_{2183} = (11, 10, 11, 1)$
$P_{2184} = (12, 10, 11, 1)$
$P_{2185} = (0, 11, 11, 1)$
$P_{2186} = (1, 11, 11, 1)$
$P_{2187} = (2, 11, 11, 1)$
$P_{2188} = (3, 11, 11, 1)$
$P_{2189} = (4, 11, 11, 1)$

$P_{2190} = (5, 11, 11, 1)$
$P_{2191} = (6, 11, 11, 1)$
$P_{2192} = (7, 11, 11, 1)$
$P_{2193} = (8, 11, 11, 1)$
$P_{2194} = (9, 11, 11, 1)$
$P_{2195} = (10, 11, 11, 1)$
$P_{2196} = (11, 11, 11, 1)$
$P_{2197} = (12, 11, 11, 1)$
$P_{2198} = (0, 12, 11, 1)$
$P_{2199} = (1, 12, 11, 1)$
$P_{2200} = (2, 12, 11, 1)$
$P_{2201} = (3, 12, 11, 1)$
$P_{2202} = (4, 12, 11, 1)$
$P_{2203} = (5, 12, 11, 1)$
$P_{2204} = (6, 12, 11, 1)$
$P_{2205} = (7, 12, 11, 1)$
$P_{2206} = (8, 12, 11, 1)$
$P_{2207} = (9, 12, 11, 1)$
$P_{2208} = (10, 12, 11, 1)$
$P_{2209} = (11, 12, 11, 1)$
$P_{2210} = (12, 12, 11, 1)$
$P_{2211} = (0, 0, 12, 1)$
$P_{2212} = (1, 0, 12, 1)$
$P_{2213} = (2, 0, 12, 1)$
$P_{2214} = (3, 0, 12, 1)$
$P_{2215} = (4, 0, 12, 1)$
$P_{2216} = (5, 0, 12, 1)$
$P_{2217} = (6, 0, 12, 1)$
$P_{2218} = (7, 0, 12, 1)$
$P_{2219} = (8, 0, 12, 1)$
$P_{2220} = (9, 0, 12, 1)$
$P_{2221} = (10, 0, 12, 1)$
$P_{2222} = (11, 0, 12, 1)$
$P_{2223} = (12, 0, 12, 1)$
$P_{2224} = (0, 1, 12, 1)$
$P_{2225} = (1, 1, 12, 1)$
$P_{2226} = (2, 1, 12, 1)$
$P_{2227} = (3, 1, 12, 1)$
$P_{2228} = (4, 1, 12, 1)$
$P_{2229} = (5, 1, 12, 1)$
$P_{2230} = (6, 1, 12, 1)$
$P_{2231} = (7, 1, 12, 1)$
$P_{2232} = (8, 1, 12, 1)$
$P_{2233} = (9, 1, 12, 1)$
$P_{2234} = (10, 1, 12, 1)$
$P_{2235} = (11, 1, 12, 1)$
$P_{2236} = (12, 1, 12, 1)$
$P_{2237} = (0, 2, 12, 1)$
$P_{2238} = (1, 2, 12, 1)$
$P_{2239} = (2, 2, 12, 1)$
$P_{2240} = (3, 2, 12, 1)$
$P_{2241} = (4, 2, 12, 1)$
$P_{2242} = (5, 2, 12, 1)$
$P_{2243} = (6, 2, 12, 1)$
$P_{2244} = (7, 2, 12, 1)$
$P_{2245} = (8, 2, 12, 1)$
$P_{2246} = (9, 2, 12, 1)$
$P_{2247} = (10, 2, 12, 1)$
$P_{2248} = (11, 2, 12, 1)$
$P_{2249} = (12, 2, 12, 1)$
$P_{2250} = (0, 3, 12, 1)$
$P_{2251} = (1, 3, 12, 1)$
$P_{2252} = (2, 3, 12, 1)$
$P_{2253} = (3, 3, 12, 1)$
$P_{2254} = (4, 3, 12, 1)$
$P_{2255} = (5, 3, 12, 1)$
$P_{2256} = (6, 3, 12, 1)$
$P_{2257} = (7, 3, 12, 1)$
$P_{2258} = (8, 3, 12, 1)$
$P_{2259} = (9, 3, 12, 1)$
$P_{2260} = (10, 3, 12, 1)$
$P_{2261} = (11, 3, 12, 1)$
$P_{2262} = (12, 3, 12, 1)$
$P_{2263} = (0, 4, 12, 1)$
$P_{2264} = (1, 4, 12, 1)$
$P_{2265} = (2, 4, 12, 1)$
$P_{2266} = (3, 4, 12, 1)$
$P_{2267} = (4, 4, 12, 1)$
$P_{2268} = (5, 4, 12, 1)$
$P_{2269} = (6, 4, 12, 1)$
$P_{2270} = (7, 4, 12, 1)$
$P_{2271} = (8, 4, 12, 1)$
$P_{2272} = (9, 4, 12, 1)$
$P_{2273} = (10, 4, 12, 1)$
$P_{2274} = (11, 4, 12, 1)$
$P_{2275} = (12, 4, 12, 1)$
$P_{2276} = (0, 5, 12, 1)$
$P_{2277} = (1, 5, 12, 1)$
$P_{2278} = (2, 5, 12, 1)$
$P_{2279} = (3, 5, 12, 1)$
$P_{2280} = (4, 5, 12, 1)$
$P_{2281} = (5, 5, 12, 1)$
$P_{2282} = (6, 5, 12, 1)$
$P_{2283} = (7, 5, 12, 1)$
$P_{2284} = (8, 5, 12, 1)$
$P_{2285} = (9, 5, 12, 1)$
$P_{2286} = (10, 5, 12, 1)$
$P_{2287} = (11, 5, 12, 1)$
$P_{2288} = (12, 5, 12, 1)$
$P_{2289} = (0, 6, 12, 1)$
$P_{2290} = (1, 6, 12, 1)$
$P_{2291} = (2, 6, 12, 1)$
$P_{2292} = (3, 6, 12, 1)$
$P_{2293} = (4, 6, 12, 1)$
$P_{2294} = (5, 6, 12, 1)$
$P_{2295} = (6, 6, 12, 1)$
$P_{2296} = (7, 6, 12, 1)$
$P_{2297} = (8, 6, 12, 1)$
$P_{2298} = (9, 6, 12, 1)$
$P_{2299} = (10, 6, 12, 1)$
$P_{2300} = (11, 6, 12, 1)$
$P_{2301} = (12, 6, 12, 1)$
$P_{2302} = (0, 7, 12, 1)$
$P_{2303} = (1, 7, 12, 1)$
$P_{2304} = (2, 7, 12, 1)$
$P_{2305} = (3, 7, 12, 1)$
$P_{2306} = (4, 7, 12, 1)$
$P_{2307} = (5, 7, 12, 1)$
$P_{2308} = (6, 7, 12, 1)$
$P_{2309} = (7, 7, 12, 1)$
$P_{2310} = (8, 7, 12, 1)$
$P_{2311} = (9, 7, 12, 1)$
$P_{2312} = (10, 7, 12, 1)$
$P_{2313} = (11, 7, 12, 1)$
$P_{2314} = (12, 7, 12, 1)$
$P_{2315} = (0, 8, 12, 1)$
$P_{2316} = (1, 8, 12, 1)$
$P_{2317} = (2, 8, 12, 1)$
$P_{2318} = (3, 8, 12, 1)$
$P_{2319} = (4, 8, 12, 1)$
$P_{2320} = (5, 8, 12, 1)$
$P_{2321} = (6, 8, 12, 1)$
$P_{2322} = (7, 8, 12, 1)$
$P_{2323} = (8, 8, 12, 1)$
$P_{2324} = (9, 8, 12, 1)$
$P_{2325} = (10, 8, 12, 1)$
$P_{2326} = (11, 8, 12, 1)$
$P_{2327} = (12, 8, 12, 1)$
$P_{2328} = (0, 9, 12, 1)$
$P_{2329} = (1, 9, 12, 1)$
$P_{2330} = (2, 9, 12, 1)$
$P_{2331} = (3, 9, 12, 1)$
$P_{2332} = (4, 9, 12, 1)$
$P_{2333} = (5, 9, 12, 1)$
$P_{2334} = (6, 9, 12, 1)$
$P_{2335} = (7, 9, 12, 1)$

$P_{2336} = (8, 9, 12, 1)$
$P_{2337} = (9, 9, 12, 1)$
$P_{2338} = (10, 9, 12, 1)$
$P_{2339} = (11, 9, 12, 1)$
$P_{2340} = (12, 9, 12, 1)$
$P_{2341} = (0, 10, 12, 1)$
$P_{2342} = (1, 10, 12, 1)$
$P_{2343} = (2, 10, 12, 1)$
$P_{2344} = (3, 10, 12, 1)$
$P_{2345} = (4, 10, 12, 1)$
$P_{2346} = (5, 10, 12, 1)$
$P_{2347} = (6, 10, 12, 1)$
$P_{2348} = (7, 10, 12, 1)$
$P_{2349} = (8, 10, 12, 1)$
$P_{2350} = (9, 10, 12, 1)$
$P_{2351} = (10, 10, 12, 1)$
$P_{2352} = (11, 10, 12, 1)$
$P_{2353} = (12, 10, 12, 1)$
$P_{2354} = (0, 11, 12, 1)$
$P_{2355} = (1, 11, 12, 1)$
$P_{2356} = (2, 11, 12, 1)$
$P_{2357} = (3, 11, 12, 1)$
$P_{2358} = (4, 11, 12, 1)$
$P_{2359} = (5, 11, 12, 1)$
$P_{2360} = (6, 11, 12, 1)$
$P_{2361} = (7, 11, 12, 1)$
$P_{2362} = (8, 11, 12, 1)$
$P_{2363} = (9, 11, 12, 1)$
$P_{2364} = (10, 11, 12, 1)$
$P_{2365} = (11, 11, 12, 1)$
$P_{2366} = (12, 11, 12, 1)$
$P_{2367} = (0, 12, 12, 1)$
$P_{2368} = (1, 12, 12, 1)$
$P_{2369} = (2, 12, 12, 1)$
$P_{2370} = (3, 12, 12, 1)$
$P_{2371} = (4, 12, 12, 1)$
$P_{2372} = (5, 12, 12, 1)$
$P_{2373} = (6, 12, 12, 1)$
$P_{2374} = (7, 12, 12, 1)$
$P_{2375} = (8, 12, 12, 1)$
$P_{2376} = (9, 12, 12, 1)$
$P_{2377} = (10, 12, 12, 1)$
$P_{2378} = (11, 12, 12, 1)$
$P_{2379} = (12, 12, 12, 1)$

$0 : \{0, 1, 5; 2, 3, 4\}_{12}$
$1 : \{0, 1, 5; 2, 3, 31\}_{48}$
$2 : \{0, 1, 5; 2, 3, 32\}_{24}$
$3 : \{0, 1, 5; 2, 3, 33\}_{72}$
$4 : \{0, 1, 5; 2, 3, 353\}_{864}$
$5 : \{0, 1, 5; 2, 3, 367\}_{12}$
$6 : \{0, 1, 5; 2, 3, 368\}_{6}$
$7 : \{0, 1, 5; 2, 3, 369\}_{18}$
$8 : \{0, 1, 5; 2, 31, 43\}_{158184}$
$9 : \{0, 1, 5; 2, 31, 46\}_{210912}$
$10 : \{0, 1, 5; 2, 31, 47\}_{26364}$
$11 : \{0, 1, 5; 2, 31, 48\}_{26364}$
$12 : \{0, 1, 5; 2, 31, 49\}_{26364}$
$13 : \{0, 1, 5; 2, 31, 50\}_{26364}$
$14 : \{0, 1, 5; 2, 31, 51\}_{26364}$
$15 : \{0, 1, 5; 2, 31, 52\}_{26364}$
$16 : \{0, 1, 5; 2, 31, 54\}_{26364}$
$17 : \{0, 1, 5; 2, 31, 56\}_{52728}$
$18 : \{0, 1, 5; 2, 31, 61\}_{52728}$
$19 : \{0, 1, 5; 2, 31, 62\}_{26364}$
$20 : \{0, 1, 5; 2, 31, 63\}_{26364}$
$21 : \{0, 1, 5; 2, 31, 64\}_{26364}$
$22 : \{0, 1, 5; 2, 31, 65\}_{26364}$
$23 : \{0, 1, 5; 2, 31, 66\}_{26364}$
$24 : \{0, 1, 5; 2, 31, 67\}_{26364}$
$25 : \{0, 1, 5; 2, 31, 69\}_{52728}$
$26 : \{0, 1, 5; 2, 31, 76\}_{158184}$
$27 : \{0, 1, 5; 2, 31, 77\}_{26364}$
$28 : \{0, 1, 5; 2, 31, 79\}_{26364}$
$29 : \{0, 1, 5; 2, 31, 82\}_{52728}$
$30 : \{0, 1, 5; 2, 31, 93\}_{26364}$
$31 : \{0, 1, 5; 2, 31, 95\}_{52728}$
$32 : \{0, 1, 5; 2, 32, 49\}_{26364}$
$33 : \{0, 1, 5; 2, 32, 50\}_{26364}$
$34 : \{0, 1, 5; 2, 32, 51\}_{26364}$
$35 : \{0, 1, 5; 2, 32, 54\}_{26364}$
$36 : \{0, 1, 5; 2, 32, 57\}_{79092}$
$37 : \{0, 1, 5; 2, 32, 64\}_{52728}$
$38 : \{0, 1, 5; 2, 32, 66\}_{26364}$
$39 : \{0, 1, 5; 2, 32, 67\}_{26364}$
$40 : \{0, 1, 5; 2, 32, 75\}_{26364}$
$41 : \{0, 1, 5; 2, 32, 80\}_{79092}$
$42 : \{0, 1, 5; 2, 32, 82\}_{26364}$
$43 : \{0, 1, 5; 2, 32, 87\}_{26364}$
$44 : \{0, 1, 5; 2, 33, 71\}_{474552}$
$45 : \{0, 1, 2; 3, 4, 31\}_{8}$
$46 : \{0, 1, 2; 3, 4, 32\}_{8}$
$47 : \{0, 1, 2; 3, 4, 33\}_{8}$
$48 : \{0, 1, 2; 3, 4, 34\}_{8}$
$49 : \{0, 1, 2; 3, 4, 35\}_{8}$
$50 : \{0, 1, 2; 3, 4, 36\}_{8}$
$51 : \{0, 1, 2; 3, 4, 37\}_{8}$
$52 : \{0, 1, 2; 3, 4, 38\}_{8}$
$53 : \{0, 1, 2; 3, 4, 39\}_{8}$
$54 : \{0, 1, 2; 3, 4, 40\}_{8}$
$55 : \{0, 1, 2; 3, 4, 41\}_{8}$
$56 : \{0, 1, 2; 3, 4, 45\}_{2}$
$57 : \{0, 1, 2; 3, 4, 46\}_{2}$
$58 : \{0, 1, 2; 3, 4, 47\}_{2}$
$59 : \{0, 1, 2; 3, 4, 48\}_{2}$
$60 : \{0, 1, 2; 3, 4, 50\}_{2}$

$61 : \{0, 1, 2; 3, 4, 51\}_2$
$62 : \{0, 1, 2; 3, 4, 52\}_2$
$63 : \{0, 1, 2; 3, 4, 53\}_2$
$64 : \{0, 1, 2; 3, 4, 54\}_2$
$65 : \{0, 1, 2; 3, 4, 59\}_2$
$66 : \{0, 1, 2; 3, 4, 60\}_2$
$67 : \{0, 1, 2; 3, 4, 64\}_6$
$68 : \{0, 1, 2; 3, 4, 65\}_2$
$69 : \{0, 1, 2; 3, 4, 66\}_2$
$70 : \{0, 1, 2; 3, 4, 67\}_2$
$71 : \{0, 1, 2; 3, 4, 73\}_2$
$72 : \{0, 1, 2; 3, 4, 74\}_2$
$73 : \{0, 1, 2; 3, 4, 79\}_2$
$74 : \{0, 1, 2; 3, 4, 89\}_2$
$75 : \{0, 1, 2; 3, 4, 199\}_2$
$76 : \{0, 1, 2; 3, 4, 200\}_2$
$77 : \{0, 1, 2; 3, 4, 201\}_2$
$78 : \{0, 1, 2; 3, 4, 202\}_2$
$79 : \{0, 1, 2; 3, 4, 203\}_2$
$80 : \{0, 1, 2; 3, 4, 204\}_4$
$81 : \{0, 1, 2; 3, 4, 212\}_4$
$82 : \{0, 1, 2; 3, 4, 213\}_1$
$83 : \{0, 1, 2; 3, 4, 214\}_1$
$84 : \{0, 1, 2; 3, 4, 215\}_2$
$85 : \{0, 1, 2; 3, 4, 216\}_1$
$86 : \{0, 1, 2; 3, 4, 217\}_1$
$87 : \{0, 1, 2; 3, 4, 226\}_2$
$88 : \{0, 1, 2; 3, 4, 227\}_2$
$89 : \{0, 1, 2; 3, 4, 228\}_2$
$90 : \{0, 1, 2; 3, 4, 229\}_1$
$91 : \{0, 1, 2; 3, 4, 231\}_2$
$92 : \{0, 1, 2; 3, 4, 232\}_1$
$93 : \{0, 1, 2; 3, 4, 235\}_1$
$94 : \{0, 1, 2; 3, 4, 240\}_2$
$95 : \{0, 1, 2; 3, 4, 242\}_2$
$96 : \{0, 1, 2; 3, 4, 243\}_1$
$97 : \{0, 1, 2; 3, 4, 246\}_2$
$98 : \{0, 1, 2; 3, 4, 248\}_1$
$99 : \{0, 1, 2; 3, 4, 254\}_2$
$100 : \{0, 1, 2; 3, 4, 255\}_1$
$101 : \{0, 1, 2; 3, 4, 268\}_2$
$102 : \{0, 1, 2; 3, 4, 271\}_4$
$103 : \{0, 1, 2; 3, 4, 274\}_2$
$104 : \{0, 1, 2; 3, 4, 282\}_2$
$105 : \{0, 1, 2; 3, 4, 564\}_2$
$106 : \{0, 1, 2; 3, 4, 565\}_1$
$107 : \{0, 1, 2; 3, 4, 566\}_1$
$108 : \{0, 1, 2; 3, 4, 567\}_1$
$109 : \{0, 1, 2; 3, 4, 568\}_4$
$110 : \{0, 1, 2; 3, 4, 569\}_2$
$111 : \{0, 1, 2; 3, 4, 570\}_1$
$112 : \{0, 1, 2; 3, 4, 571\}_2$
$113 : \{0, 1, 2; 3, 4, 572\}_1$
$114 : \{0, 1, 2; 3, 4, 578\}_1$
$115 : \{0, 1, 2; 3, 4, 579\}_2$
$116 : \{0, 1, 2; 3, 4, 582\}_2$
$117 : \{0, 1, 2; 3, 4, 583\}_4$
$118 : \{0, 1, 2; 3, 4, 584\}_2$
$119 : \{0, 1, 2; 3, 4, 592\}_1$
$120 : \{0, 1, 2; 3, 4, 597\}_4$
$121 : \{0, 1, 2; 3, 4, 598\}_2$
$122 : \{0, 1, 2; 3, 4, 608\}_4$
$123 : \{0, 1, 2; 3, 4, 624\}_{12}$
$124 : \{0, 1, 2; 3, 4, 750\}_2$
$125 : \{0, 1, 2; 3, 4, 753\}_2$
$126 : \{0, 1, 2; 3, 4, 761\}_6$
$127 : \{0, 1, 2; 3, 4, 764\}_2$
$128 : \{0, 1, 2; 3, 4, 765\}_2$
$129 : \{0, 1, 2; 3, 30, 45\}_{12}$
$130 : \{0, 1, 2; 3, 30, 46\}_{24}$
$131 : \{0, 1, 2; 3, 30, 48\}_{12}$
$132 : \{0, 1, 2; 3, 30, 50\}_{12}$
$133 : \{0, 1, 2; 3, 30, 53\}_{12}$
$134 : \{0, 1, 2; 3, 30, 59\}_{24}$
$135 : \{0, 1, 2; 3, 30, 60\}_{12}$
$136 : \{0, 1, 2; 3, 30, 64\}_{72}$
$137 : \{0, 1, 2; 3, 30, 65\}_{12}$
$138 : \{0, 1, 2; 3, 30, 73\}_{12}$
$139 : \{0, 1, 2; 3, 30, 79\}_{24}$
$140 : \{0, 1, 2; 3, 30, 89\}_{24}$
$141 : \{0, 1, 2; 3, 30, 198\}_8$
$142 : \{0, 1, 2; 3, 198, 354\}_{12}$
$143 : \{0, 1, 2; 30, 45, 57\}_{52728}$
$144 : \{0, 1, 2; 30, 45, 59\}_{26364}$
$145 : \{0, 1, 2; 30, 45, 61\}_{26364}$
$146 : \{0, 1, 2; 30, 45, 62\}_{26364}$
$147 : \{0, 1, 2; 30, 45, 63\}_{79092}$
$148 : \{0, 1, 2; 30, 45, 64\}_{105456}$
$149 : \{0, 1, 2; 30, 45, 65\}_{26364}$
$150 : \{0, 1, 2; 30, 45, 67\}_{26364}$
$151 : \{0, 1, 2; 30, 45, 70\}_{52728}$
$152 : \{0, 1, 2; 30, 45, 73\}_{26364}$
$153 : \{0, 1, 2; 30, 45, 76\}_{26364}$
$154 : \{0, 1, 2; 30, 45, 77\}_{26364}$
$155 : \{0, 1, 2; 30, 45, 78\}_{26364}$
$156 : \{0, 1, 2; 30, 45, 79\}_{26364}$
$157 : \{0, 1, 2; 30, 45, 80\}_{26364}$

158 : $\{0, 1, 2; 30, 45, 85\}_{26364}$
159 : $\{0, 1, 2; 30, 45, 87\}_{26364}$
160 : $\{0, 1, 2; 30, 45, 88\}_{26364}$
161 : $\{0, 1, 2; 30, 45, 89\}_{52728}$
162 : $\{0, 1, 2; 30, 45, 91\}_{26364}$
163 : $\{0, 1, 2; 30, 45, 92\}_{52728}$
164 : $\{0, 1, 2; 30, 45, 93\}_{26364}$
165 : $\{0, 1, 2; 30, 45, 98\}_{26364}$
166 : $\{0, 1, 2; 30, 45, 99\}_{26364}$
167 : $\{0, 1, 2; 30, 45, 101\}_{26364}$
168 : $\{0, 1, 2; 30, 45, 102\}_{26364}$
169 : $\{0, 1, 2; 30, 45, 104\}_{26364}$
170 : $\{0, 1, 2; 30, 45, 106\}_{26364}$
171 : $\{0, 1, 2; 30, 45, 109\}_{26364}$
172 : $\{0, 1, 2; 30, 45, 112\}_{26364}$
173 : $\{0, 1, 2; 30, 45, 113\}_{26364}$
174 : $\{0, 1, 2; 30, 45, 115\}_{52728}$
175 : $\{0, 1, 2; 30, 45, 118\}_{26364}$
176 : $\{0, 1, 2; 30, 45, 119\}_{26364}$
177 : $\{0, 1, 2; 30, 45, 124\}_{26364}$
178 : $\{0, 1, 2; 30, 45, 125\}_{26364}$
179 : $\{0, 1, 2; 30, 45, 126\}_{26364}$
180 : $\{0, 1, 2; 30, 45, 127\}_{26364}$
181 : $\{0, 1, 2; 30, 45, 129\}_{26364}$
182 : $\{0, 1, 2; 30, 45, 130\}_{26364}$
183 : $\{0, 1, 2; 30, 45, 131\}_{26364}$
184 : $\{0, 1, 2; 30, 45, 135\}_{26364}$
185 : $\{0, 1, 2; 30, 45, 139\}_{26364}$
186 : $\{0, 1, 2; 30, 45, 141\}_{26364}$
187 : $\{0, 1, 2; 30, 45, 143\}_{26364}$
188 : $\{0, 1, 2; 30, 45, 145\}_{26364}$
189 : $\{0, 1, 2; 30, 45, 151\}_{26364}$
190 : $\{0, 1, 2; 30, 45, 153\}_{26364}$
191 : $\{0, 1, 2; 30, 45, 154\}_{52728}$
192 : $\{0, 1, 2; 30, 45, 155\}_{26364}$
193 : $\{0, 1, 2; 30, 45, 157\}_{26364}$
194 : $\{0, 1, 2; 30, 45, 158\}_{26364}$
195 : $\{0, 1, 2; 30, 45, 163\}_{26364}$
196 : $\{0, 1, 2; 30, 45, 164\}_{26364}$
197 : $\{0, 1, 2; 30, 45, 165\}_{26364}$
198 : $\{0, 1, 2; 30, 45, 166\}_{26364}$
199 : $\{0, 1, 2; 30, 45, 168\}_{26364}$
200 : $\{0, 1, 2; 30, 45, 171\}_{52728}$
201 : $\{0, 1, 2; 30, 45, 174\}_{52728}$
202 : $\{0, 1, 2; 30, 45, 176\}_{26364}$
203 : $\{0, 1, 2; 30, 45, 179\}_{26364}$
204 : $\{0, 1, 2, 30, 45, 180\}_{26364}$
205 : $\{0, 1, 2; 30, 46, 65\}_{26364}$
206 : $\{0, 1, 2; 30, 46, 70\}_{158184}$
207 : $\{0, 1, 2; 30, 46, 71\}_{52728}$
208 : $\{0, 1, 2; 30, 46, 73\}_{52728}$
209 : $\{0, 1, 2; 30, 46, 74\}_{26364}$
210 : $\{0, 1, 2; 30, 46, 77\}_{26364}$
211 : $\{0, 1, 2; 30, 46, 79\}_{26364}$
212 : $\{0, 1, 2; 30, 46, 87\}_{26364}$
213 : $\{0, 1, 2; 30, 46, 101\}_{26364}$
214 : $\{0, 1, 2; 30, 46, 103\}_{26364}$
215 : $\{0, 1, 2; 30, 46, 104\}_{26364}$
216 : $\{0, 1, 2; 30, 46, 105\}_{26364}$
217 : $\{0, 1, 2; 30, 46, 109\}_{158184}$
218 : $\{0, 1, 2; 30, 46, 110\}_{52728}$
219 : $\{0, 1, 2; 30, 46, 112\}_{26364}$
220 : $\{0, 1, 2; 30, 46, 119\}_{26364}$
221 : $\{0, 1, 2; 30, 46, 122\}_{105456}$
222 : $\{0, 1, 2; 30, 46, 130\}_{26364}$
223 : $\{0, 1, 2; 30, 46, 132\}_{26364}$
224 : $\{0, 1, 2; 30, 46, 135\}_{105456}$
225 : $\{0, 1, 2; 30, 46, 141\}_{26364}$
226 : $\{0, 1, 2; 30, 46, 161\}_{52728}$
227 : $\{0, 1, 2; 30, 46, 174\}_{52728}$
228 : $\{0, 1, 2; 30, 48, 75\}_{26364}$
229 : $\{0, 1, 2; 30, 48, 76\}_{26364}$
230 : $\{0, 1, 2; 30, 48, 88\}_{26364}$
231 : $\{0, 1, 2; 30, 48, 96\}_{52728}$
232 : $\{0, 1, 2; 30, 48, 97\}_{158184}$
233 : $\{0, 1, 2; 30, 48, 98\}_{26364}$
234 : $\{0, 1, 2; 30, 48, 104\}_{52728}$
235 : $\{0, 1, 2; 30, 48, 105\}_{26364}$
236 : $\{0, 1, 2; 30, 48, 119\}_{26364}$
237 : $\{0, 1, 2; 30, 48, 122\}_{26364}$
238 : $\{0, 1, 2; 30, 48, 124\}_{26364}$
239 : $\{0, 1, 2; 30, 48, 125\}_{26364}$
240 : $\{0, 1, 2; 30, 48, 137\}_{26364}$
241 : $\{0, 1, 2; 30, 48, 143\}_{26364}$
242 : $\{0, 1, 2; 30, 48, 144\}_{52728}$
243 : $\{0, 1, 2; 30, 48, 151\}_{26364}$
244 : $\{0, 1, 2; 30, 48, 154\}_{26364}$
245 : $\{0, 1, 2; 30, 48, 155\}_{26364}$
246 : $\{0, 1, 2; 30, 48, 157\}_{26364}$
247 : $\{0, 1, 2; 30, 48, 161\}_{26364}$
248 : $\{0, 1, 2; 30, 48, 162\}_{26364}$
249 : $\{0, 1, 2; 30, 48, 167\}_{26364}$
250 : $\{0, 1, 2; 30, 48, 175\}_{26364}$
251 : $\{0, 1, 2; 30, 48, 179\}_{52728}$
252 : $\{0, 1, 2; 30, 48, 181\}_{26364}$
253 : $\{0, 1, 2; 30, 50, 60\}_{26364}$
254 : $\{0, 1, 2; 30, 50, 64\}_{26364}$

$255 : \{0, 1, 2; 30, 50, 87\}_{26364}$
$256 : \{0, 1, 2; 30, 50, 124\}_{79092}$
$257 : \{0, 1, 2; 30, 50, 125\}_{52728}$
$258 : \{0, 1, 2; 30, 50, 132\}_{52728}$
$259 : \{0, 1, 2; 30, 50, 136\}_{26364}$
$260 : \{0, 1, 2; 30, 50, 148\}_{79092}$
$261 : \{0, 1, 2; 30, 50, 152\}_{26364}$
$262 : \{0, 1, 2; 30, 50, 155\}_{52728}$
$263 : \{0, 1, 2; 30, 50, 163\}_{26364}$
$264 : \{0, 1, 2; 30, 50, 182\}_{26364}$
$265 : \{0, 1, 2; 30, 53, 162\}_{26364}$
$266 : \{0, 1, 2; 30, 53, 171\}_{79092}$
$267 : \{0, 1, 2; 30, 53, 177\}_{52728}$
$268 : \{0, 1, 2; 30, 53, 178\}_{52728}$
$269 : \{0, 1, 2; 30, 59, 71\}_{158184}$
$270 : \{0, 1, 2; 30, 59, 73\}_{105456}$
$271 : \{0, 1, 2; 30, 59, 74\}_{52728}$
$272 : \{0, 1, 2; 30, 59, 77\}_{105456}$
$273 : \{0, 1, 2; 30, 59, 79\}_{52728}$
$274 : \{0, 1, 2; 30, 60, 84\}_{105456}$
$275 : \{0, 1, 2; 30, 60, 87\}_{79092}$
$276 : \{0, 1, 2; 30, 60, 93\}_{105456}$
$277 : \{0, 1, 2; 30, 60, 175\}_{52728}$
$278 : \{0, 1, 2; 30, 64, 136\}_{949104}$
$279 : \{0, 1, 2; 30, 64, 145\}_{52728}$
$280 : \{0, 1, 2; 30, 73, 92\}_{79092}$
$281 : \{0, 1, 2; 30, 79, 165\}_{316368}$

Appendix D

The Orbiter makefile

```
MY_PATH=~/DEV.18
SRC=$(MY_PATH)/GITHUB/orbiter/ORBITER/SRC

PROJECTIVE_SPACE_PATH=$(SRC)/APPS/PROJECTIVE_SPACE
SURFACES_PATH=$(SRC)/APPS/SURFACES




#########################################################################
# F_2:
#########################################################################

cheat2:
	$(PROJECTIVE_SPACE_PATH)/cheat_sheet_PG.out -n 2 -q 2
	pdflatex PG_2_2.tex
	open PG_2_2.pdf


cheat32:
	$(PROJECTIVE_SPACE_PATH)/cheat_sheet_PG.out -n 3 -q 2
```

```
	pdflatex PG_3_2.tex
	open PG_3_2.pdf

classify2:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 2 -classify
	pdflatex arc_lifting_q2.tex
	open arc_lifting_q2.pdf





#########################################################################
# F_3:
#########################################################################

cheat3:
	$(PROJECTIVE_SPACE_PATH)/cheat_sheet_PG.out -n 3 -q 3 -surface
	pdflatex PG_3_3.tex
	open PG_3_3.pdf


cheat33:
	$(PROJECTIVE_SPACE_PATH)/cheat_sheet_PG.out -n 3 -q 3
	pdflatex PG_3_3.tex
	open PG_3_3.pdf


classify3:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 3 -classify
	pdflatex arc_lifting_q3.tex
```

```
	open arc_lifting_q3.pdf






#########################################################################
# F_4:
#########################################################################


classify4:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 4 -classify
	pdflatex arc_lifting_q4.tex
	open arc_lifting_q4.pdf


#########################################################################
# F_5:
#########################################################################


classify5:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 5 -classify
	pdflatex arc_lifting_q5.tex
	open arc_lifting_q5.pdf



#########################################################################
```

```
# F_7:
##########################################################################


cheat2_7:
	$(PROJECTIVE_SPACE_PATH)/cheat_sheet_PG.out -n 2 -q 7
	pdflatex PG_2_7.tex
	open PG_2_7.pdf



arc_lifting_7:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 7 \
		-arc "12,25,34,42,28,14"
	pdflatex single_arc_lifting_q7.tex
	open single_arc_lifting_q7.pdf


classify7:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 7 -classify
	pdflatex arc_lifting_q7.tex
	open arc_lifting_q7.pdf



##########################################################################
# F_8:
##########################################################################


classify8:
```

```
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 8 -classify
	pdflatex arc_lifting_q8.tex
	open arc_lifting_q8.pdf






cheat_8_0:
	$(SURFACE_PATH)/cheat_sheet_surface.out -v 2 -q 8 -no 0
	pdflatex Surface_q8_iso0.tex
	open Surface_q8_iso0.pdf


############################################################################
# F_9:
############################################################################

cheat9:
	$(PROJECTIVE_SPACE_PATH)/cheat_sheet_PG.out -n 2 -q 9
	pdflatex PG_2_9.tex
	open PG_2_9.pdf



classify9:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 9 -classify
	pdflatex arc_lifting_q9.tex
	open arc_lifting_q9.pdf

```

```
#########################################################################
# F_11:
#########################################################################


classify11:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 11 -classify
	#pdflatex arc_lifting_q11.tex
	#open arc_lifting_q11.pdf


#########################################################################
# F_13:
#########################################################################



cheat2_13:
	$(PROJECTIVE_SPACE_PATH)/cheat_sheet_PG.out -n 2 -q 13
	pdflatex PG_2_13.tex
	open PG_2_13.pdf



arc_lifting_13:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 13 \
		-arc "5,14,26,17,92,105"

```

```
classify13:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 13 -classify
	pdflatex arc_lifting_q13.tex
	open arc_lifting_q13.pdf






#########################################################################
# F_17:
#########################################################################


classify17:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 17 -classify
	#pdflatex arc_lifting_q17.tex
	#open arc_lifting_q17.pdf



#########################################################################
# F_19:
#########################################################################


classify19:
	$(SURFACES_PATH)/arc_lifting_main.out -v 2 -n 2 -q 19 -classify
```

```
#pdflatex arc_lifting_q19.tex
#open arc_lifting_q19.pdf
```

Bibliography

[1] A. H. Ali, The classification of arcs in Galois plane of order thirteen, Ph.D. thesis, University of Sussex, 1993.

[2] N. A. M. Al-Seraji, The Geometry of the Plane of Order Seventeen and its Application to Error-Correcting Codes, Ph.D. thesis, University of Sussex, 2010.

[3] E. B. Al-Zangana, The Geometry of The Plane of Order Nineteen and its Application to Error-Correcting Codes, Ph.D. thesis, University of Sussex, 2011.

[4] R. Baer, *Linear Algebra and Projective Geometry*, Academic Press, New York, 1952, 318 pp.

[5] H. F. Baker, *Principles of Geometry*, *III*, Cambridge University Press, Cambridge, 1921–1933, (Ungar, 1960–1968).

[6] A. Betten, The packings of $\mathrm{PG}(3,3)$, *Des. Codes Cryptogr.* **79** (2016), 583–595.

[7] A. Betten, Orbiter – A program to classify discrete objects; 2018, (`https://github.com/abetten/orbiter`)

[8] A. Betten, Finite orthogonal spaces with respect to a hyperbolic pair, preprint.

[9] A. Betten, J. W. P. Hirschfeld, F. Karaoglu, The classification of cubic surfaces with twenty-seven lines over the field of thirteen elements, *European Journal of Mathematics* **4** (2018), 37–50.

[10] A. Betten, F. Karaoglu, Cubic surfaces over small finite fields, submitted.

[11] R. C. Bose, Mathematical theory of the symmetrical factorial design, *Sankhyā* **8** (1947), 107–166.

[12] A. Cayley, On the triple tangent planes of surfaces of the third order, *The Cambridge and Dublin Mathematical Journal* **5** (1849), 118-132.

[13] A. Clebsch, Die Geometrie auf den Flächen dritter Ordnung, *J. Reine Angew. Math.* **65** (1866), 359–380.

[14] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, ATLAS *of Finite Groups*, Oxford University Press, Oxford, 1985, 252 pp.

[15] L. E. Dickson, The straight lines on modular cubic surfaces, *Proc. Nat. Acad. Sci. U.S.A.* **1** (1915), 248–253.

[16] L. E. Dickson, Projective classification of cubic surfaces modulo 2, *Ann. of Math.* **16** (1915), 139–157.

[17] F. E. Eckardt, Ueber diejenigen Flächen dritter Grades, auf denen sich drei gerade Linien in einem Punkte schneiden, *Math. Ann.* **10** (1876), 227-272.

[18] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.7;2017, `(http://www.gap-system.org)`

[19] J. W. P. Hirschfeld, The double-six of lines over $\mathrm{PG}(3,4)$, *J. Austral. Math. Soc.* **4** (1964), 83–89.

[20] J. W. P. Hirschfeld, The geometry of cubic surfaces, and Grace's extension of the double-six, over finite fields, Ph.D. thesis, University of Edinburgh, 1965.

[21] J. W. P. Hirschfeld, Classical configurations over finite fields: I. The double-six and the cubic surface with 27 lines, *Rend. Mat. e Appl.* **26** (1967), 115–152.

[22] J. W. P. Hirschfeld, Del Pezzo surfaces over finite fields, *Tensor* **37** (1982), 79–84.

[23] J. W. P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985, x+316 pp.

[24] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, second ed., Oxford University Press, Oxford, 1998, xiv+555 pp.

[25] S. E. Payne, J. A. Thas, *Finite Generalized Quadrangles*, Pitman, London, 1984, 312 pp.

[26] A. R. Sadeh, The classification of k-arcs and cubic surfaces with twenty-seven lines over the field of eleven elements, Ph.D. thesis, University of Sussex, 1984.

[27] L. Schläfli, An attempt to determine the twenty-seven lines upon a surface of the third order and to divide such surfaces into species in reference to the reality of the lines upon the surface, *Quart. J. Math.* **2** (1858), 55–110.

[28] B. Segre, *The Non-Singular Cubic Surfaces*, Oxford University Press, Oxford, 1942, 180 pp.

[29] B. Segre, Ovals in a finite projective plane, *Canad. J. Math.* **7** (1955), 414–416.

[30] J. Steiner, Über die Flächen dritten Grades, *J. Reine Angew. Math.* **53** (1857), 133-141.