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# Aspects of Non-locality in Gravity

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Submitted for the degree of Doctor of Philosophy

University of Sussex

October 2018

## Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree. It consists of three papers, presented in their published versions and which were done in collaboration with Xavier Calmet, Shahn Majid and Djuna Croon.

- Non-locality in Quantum Field Theory due to General Relativity, X. Calmet, D. Croon, C. Fritz, *Eur.Phys.J. C* 75 (2015) no.12, 605, arXiv:1505.04517 [hep-th] The original idea for the paper was proposed by Xavier Calmet with input from Djuna Croon and myself. Calculations were completed by all collaborators with myself contributing mainly to the last section. Writing was mostly done by Xavier Calmet
- Inflation on a non-commutative space-time, X. Calmet, C. Fritz, *Phys.Lett.B* 747 (2015) 406-409, arXiv:1506.04049 [hep-th] The original idea for the paper was proposed by Xavier Calmet. Calculations were completed by myself under his supervision. Writing of the paper was done by me under supervision and with modifications by Xavier Calmet.
- Noncommutative spherically symmetric spacetimes at semiclassical order, C. Fritz, S. Majid, *Class.Quant.Grav.* 34 (2017) no.13, 135013, arXiv:1611.04971 [gr-qc] The original concept for the paper came from both Shahn Majid and myself. I wrote the maple code used in this paper and was responsible for all the calculations from section 3 onwards. Calculations in earlier sections were performed by both Shahn Majid and myself. He wrote the introduction and most of sections 1 and 2 while I wrote the later sections with his input.

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ASPECTS OF NON-LOCALITY IN GRAVITY

Since the beginning of the 20th century, much time and effort has been invested in the search for a theory of quantum gravity. While this provided a myriad of possibilities, it has so far failed to find a definitive answer. Here we take an alternative approach: instead of constructing a theory of quantum gravity and examining its low energy limit, we start with the conventional theory and ask what are the first deviations induced by a possible quantization of gravity. It is proposed that in this limit quantum gravity, whatever the ultimate theory might be, manifests itself as non-locality.

In this thesis are explored two different approaches to effective theories. In the first, it is demonstrated how combining quantum field theory with general relativity naturally gives rise to non-locality. This is explored in the context of inflation, a natural place to look for high energy phenomena. By considering a simple scalar field theory, it is shown how non-locality results in higher dimensional operators and what the effects are on inflationary models.

The second approach looks at a theory which naturally incorporates a minimal scale. Non-commutative geometry parallels the phase space or deformation quantization approach of quantum mechanics. It supposes that at short scales, the structure of spacetime is algebraic rather than geometric. In the first instance, we follow the first section and look at cosmological implications by replacing normal scalar theory with its noncommutative counterpart. In the second, we take a step back and examine the implications of quantization on the differential geometry. The formalism is developed and applied to generic spherically symmetric spacetimes where it is shown that to first order in deformation, the quantization is unique.

## Acknowledgements

Above all, I would like to thank my Supervisor Xavier Calmet. His guidance, support and patience, even when I didn't make it easy, throughout my time at Sussex are what made this thesis possible.

I am also very grateful to my parents whose support was what made it possible in the first place to begin and finish a PhD.

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## Introduction

Understanding quantum gravity remains one of the holy grails of modern physics. However, despite many approaches being pioneered over the years there is, as of yet, no conclusive answer regarding the underlying theory. What can be said for certain, is that classical gravity must be recovered in some limit of the quantum theory. A simple and obvious statement, it is no less profound, representing a unifying feature that all approaches must have in common. But we can also turn it on its head by taking a step back from this limit and asking: what do we expect the first deviation from the classical theory to be? It is here that we find a simple, heuristic motivation which brings together the works presented here. Ranging from quantum field theory calculations to relatively formal mathematical proofs, they all describe effective theories. Unifying them is the idea that there exists in nature some sort of minimal length scale, showing itself as non-locality.

Both the motivating example shown below and the first section demonstrate that by simply including gravitational effects in conventional quantum theory, we see this phenomenon arise quite naturally. After that is illustrated a more constructive approach which incorporates the idea at a fundamental level. The argument whether or not the theory presented there is in fact a self contained description of quantum gravity, or just another step towards one, is rather more philosophical than physical. Exhaustively arguing for either is beyond the scope of this thesis, but while the former position is doubtlessly more bold, the latter is perhaps more in keeping with the overarching theme. More important is the principle allowing the construction of a (semi)quantum spacetime and how this manifests in the classical theory. So we look at the scales at which effects become apparent as well as the size of these scales.

Now consider the following thought experiment:

On the one hand we have quantum mechanics. Through it, a massive particle has a de Broglie wavelength inversely proportional to its total energy  $\lambda_C \sim 1/E$ . Suppose that

we wanted to probe some event or region of size  $l$ . This can be resolved by a particle with  $\lambda_C \sim l$ . It follows that the higher the energy, the smaller the resolution. Note that in order to fully localize a measurement, we would hypothetically need infinite energy. This is nothing more than a qualitative restatement of Heisenberg's uncertainty relation (since  $E \sim p$  for high momentum)

$$\Delta p \Delta x \geq \frac{\hbar}{2} \quad (1.0.1)$$

which limits the accuracy with which a particles position can be measured in relation to its momentum.

Now consider on the other hand gravity. Here we know that taking some mass (or equivalently, energy) and confining it within its Schwarzschild radius, the region containing it will form a black hole of size  $r_S = 2Gm$ . This forms the basis of the *hoop conjecture* proposed by Kip Thorne [1]. A black hole is an extended object whose size grows with increasing energy  $r_S \propto E$  after being formed. Juxtapose this with the previous paragraph where we measure an increasingly small region with increasing energy. Now suppose we take some particle and keep accelerating it, thereby probing an ever shrinking  $l$ . Sooner or later we reach a critical point,  $\lambda_C \sim l \sim r_S$ . What happens now? Well, rather than continued acceleration giving increased resolution, by the hoop conjecture our particle undergoes gravitational collapse and forms a black hole, increasing in size with energy, concealing whatever happens behind its event horizon. So it would appear that once we reach  $E \propto r_S$ , no matter how hard we try, no further improvements to measurement accuracy can be achieved. Even worse, as energy increases the black hole grows. This leads us to the idea of a minimum resolvable length

$$l \sim \min\left(\frac{1}{E}, E\right). \quad (1.0.2)$$

Now, this is a simple and naive illustration, but nonetheless illustrates the limiting effects of gravity when incorporated into quantum mechanics. Indeed we can posit the mass of these smallest of black holes as  $m = m_P$  giving physical meaning to the Plank scale. However, it is not (or not only) the value of this scale which interests us here. Rather we want to pay attention to the fact that though small, these black holes do nonetheless have a nonzero extent given by the Schwarzschild radius as opposed to the pointlike nature of particles. This extent is then the smallest possible measurable spacetime interval and simplest illustration of non-locality. That is not to say however, that black holes need

represent the smallest unit of spacetime and we can instead consider all and any possibilities that give some sort of tessellation. Most important is the idea of a scale at which the effects of quantum spacetime become important.

This thesis is laid out in a paper style fashion, based on three published works. The introductory section is intended to complement the introductions and materials in these papers as well as provide a thematic link. It is also intended to expand on the basics not included in the papers and put their content in context. A reader might wonder at the comparative lengths of the sections in this introduction, with much more space being dedicated to the last than the first. At heart, this is a work of physics and not mathematics. However, the approach of the last section is rooted in the mathematical abstraction of familiar physical principles and it would seem insincere to simply state the results without imparting some degree of understanding of how they came to be. Especially since the formalism is comparatively new and possibly unfamiliar. It therefore seems appropriate to spend more time introducing, hopefully to a sufficient degree, the technicalities behind the approach.

We look at the subject from two perspectives: the first is to explore how non-locality arises naturally in the context of quantum field theory. In a way not too dissimilar from the above example, it is demonstrated how coupling gravity to some low energy sector results in interaction separated by a non-zero spacetime interval. The second is a constructive approach, namely noncommutative geometry. It posits that spacetime is quantized and its underlying structure is algebraic rather than geometric. First is taken what might be called a “dynamic approach”, where the Physics on such a spacetime is explored in the context of an effective field theory. Here are introduced the basic ingredients which are used in that particular publication but also serve as a basis for the following section. There we take a closer look at geometry itself and extend the analysis to the differential geometry. This is used to abstract and define familiar objects like the metric and connection in algebraic terms. We then examine the relationship between the classical geometric and the semiclassical algebraic structure within the context of Poisson-Riemannian geometry.

## 1.1 Non-locality in General Relativity

We must however, practice caution in dealing with effective theories, specifically about the issue of perturbative unitarity. We consider the work in [2,3], where the authors examined the perturbative unitarity implications of 2 to 2  $s$ -channel graviton scattering as shown in 1.1. These works were concerned with the so called *self healing* of perturbative unitarity.

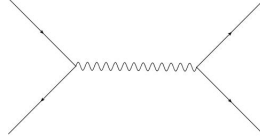


Figure 1.1: S-channel graviton exchange

For the  $S$ -matrix element

$$\langle p_1 p_2 | S | k_1 k_2 \rangle$$

which in an interacting theory is written as  $S = 1 + iT$ , we require  $S^\dagger S = S S^\dagger = 1$  which implies  $i(T^\dagger - T) = T^\dagger T$  giving rise to the optical theorem [4]

$$\text{Im}(\mathcal{M}) = 2E_{cm}p_{cm}\sigma$$

where  $\mathcal{M}$  is the scattering amplitude for a given process,  $\sigma$  the cross-section,  $E_{cm}$  and  $p_{cm}$  the centre of mass energy and momentum respectively. In a renormalizable theory this is required to be satisfied order by order. However, in an effective theory, we are expanding in powers of the energy which leads to the amplitude itself becoming energy dependent and giving rise to problems in respecting tree level unitarity. Violation is usually interpreted as the emergence of new Physics, since we suppose that there may be unknown effects at higher energies.

In [2] the authors considered the process 1.1, giving rise to the perturbative unitarity bound

$$|\text{Re}(\mathcal{M})| \geq \frac{1}{2} \tag{1.1.1}$$

and asking at what energy this is violated. The tree level scattering amplitude is given by

$$\mathcal{M}_{Tree} = -\frac{G_N N E_{cm}}{40} \tag{1.1.2}$$

where we have  $N = N_s + 3N_V + 12N_f$ , where  $N$  is the total number of degrees of freedom in the theory, so  $N_s$  for scalar,  $N_V$  for vector particles and  $N_f$  for Fermions. This is calculated for  $E_{cm}$  much greater than the masses of the particles involved. It was shown that perturbative unitarity violation takes place at energy

$$E_{cm} = \frac{20}{G_N N}. \quad (1.1.3)$$

In the standard model,  $E_{cm} \sim 6 \times 10^{18} GeV$  or about half of the Plank mass. This result indicates that new Physics occurs well below the Planck scale and we could posit that the ‘actual’ Planck scale ought to be around  $(G_N N)^{-1/2}$ . Now the question arises how this links up to earlier arguments given about a minimal length.

So we consider arguments given in [3] that perturbative unitarity can *heals itself*. They demonstrated that by iterating the one loop diagrams and resumming the multi loop cor-

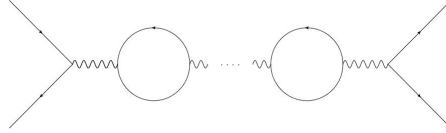


Figure 1.2: Loop correction to s-channel graviton exchange. Loops contain a sum over all elementary particles

rections one obtains

$$\mathcal{M} = \frac{\alpha E_{cm}}{1 - \frac{\alpha E_{cm}}{\pi} \ln(-\frac{E_{cm}}{\mu})} \quad (1.1.4)$$

where  $\alpha = \frac{NG_N}{120\pi}$  and  $\mu$  is the renormalization scale. The loops are calculated using dimensional regularization. This is referred to as the *dressed* amplitude and satisfies (1.1.1). The corresponding dressed propagator is what we will actually be interested in and reads

$$i\mathcal{D}^{\alpha\beta,\mu\nu}(q^2) = \frac{i(L^{\alpha\mu}L^{\beta\nu} + L^{\alpha\nu}L^{\beta\mu} - L^{\alpha\beta}L^{\mu\nu})}{2q^2 \left(1 - \alpha q^2 \ln(-\frac{q^2}{\mu^2})\right)}. \quad (1.1.5)$$

It has been observed [2] that the second term in the denominator gives rise to two additional complex poles leading to acausal effects. These become appreciable at energies approaching  $1/\sqrt{NG_N} = M_p/\sqrt{N}$ . It is also interesting to note the dependence of the Planck scale on the number of particles so that a large sector beyond the standard model, could render Planck sized objects detectable with current experiments. The absence of observed quantum gravity effects at the  $TeV$  scale at the LHC, puts an upper bound on

the number of particles which couple gravitationally to the standard model at  $N \sim 10^{33}$  [5].

Now, the propagator shown above was examined by the authors in [6] where it was shown to correspond to an effective action

$$\Gamma = \Gamma_L + \Gamma_{NL} \quad (1.1.6)$$

where the local, gravitational effective action is

$$\Gamma_L = \int d^4x \sqrt{g} \left( \frac{1}{\kappa^2} R + c_1(\mu) R^2 + c_2(\mu) R_{\mu\nu} R^{\mu\nu} + c_3(\mu) R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) \quad (1.1.7)$$

where  $c_n(\mu)$  are the renormalized coupling constants depending on the renormalization scale  $\mu$  and we keep only parts up to second order in curvature. The non-local part is then

$$\Gamma_{NL} = \int d^4x \sqrt{g} \left( \alpha R \ln \left( \frac{\square}{\mu^2} \right) R + \beta R_{\mu\nu} \ln \left( \frac{\square}{\mu^2} \right) R^{\mu\nu} + \gamma R_{\alpha\beta\mu\nu} \ln \left( \frac{\square}{\mu^2} \right) R^{\alpha\beta\mu\nu} \right). \quad (1.1.8)$$

In [7] the authors performed a comprehensive study of this type of action containing infinite order differential operators which appear widely when studying non-local phenomena. However, we will follow the analysis of [6] where terms like those above were referred to as *quasi-local*. The full non-locality is made explicit by resolving the  $\ln(\square/\mu^2)$  operator in the following way.

First are introduced the position space eigenstates, which are normalized covariantly

$$\langle x|y \rangle = \frac{\delta^{(4)}(x-y)}{(\sqrt{\sqrt{g(x)}\sqrt{g(y)}})} \quad (1.1.9)$$

where by  $x$  and  $y$  we denote two separate spacetime points. Now, taking as an example the Ricci term, it can be resolved as

$$S = \int d^4x \sqrt{g(x)} R(x) \int d^4y \sqrt{g(y)} \langle x| \ln \left( \frac{\square}{\mu^2} \right) |y \rangle R(y), \quad (1.1.10)$$

allowing us to see the non-locality more explicitly by writing the action in such a way that it depends on separate points. In fact we can go even further by defining the interpolating function as

$$\mathcal{L}(x, y) = \langle x| \ln \left( \frac{\square}{\mu^2} \right) |y \rangle \quad (1.1.11)$$

thereby rendering the above as

$$S = \int d^4x \int d^4y \sqrt{g(x)} R(x) \mathcal{L}(x, y) \sqrt{g(y)} R(y). \quad (1.1.12)$$

The same can be done for the remaining terms in (1.1.8). These considerations lead to the first paper in chapter 2. Here it is shown that non-local terms emerge as higher order effective operators in the matter sector of the Lagrangian.

### 1.1.1 Summary

This first paper essentially follows the procedure shown above with the first steps reversed. Starting out with the resummed propagator (1.1.5), the amplitude for gravitational scattering of two scalars is rewritten in a manner similar to (1.1.4). It is then shown that the same amplitude results from a non-local dimension 8 effective operator

$$\frac{2}{15} G_N^2 N (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \ln \left( -\frac{\square}{\mu^2} \right) (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2). \quad (1.1.13)$$

The  $\ln \left( -\frac{\square}{\mu^2} \right)$  term is resolved as a interpolating function  $\mathcal{L}(x-y)$ . The next section derives bounds on this operator from the cosmic microwave background using the standard form for the speed of sound in cosmology. From this is obtained

$$c_s^2 = 1 - \frac{2}{15\pi} H^2 \epsilon N G_N \quad (1.1.14)$$

where  $H$  is the Hubble parameter and  $\epsilon = \frac{1}{16\pi G_N} \frac{1}{V^2} \left( \frac{\partial V}{\partial \phi} \right)^2$  is the slow roll parameter. This indicates that one would expect a small amount of nongaussianity in the CMB.

## 1.2 Noncommutative Geometry

Expecting some sort of non-locality at very high energy scales is reasonable and we have seen how a very conservative approach gives rise to phenomena that can be interpreted as exactly that. Another approach, the one we will be concentrating on, is noncommutative geometry. Instead of considering fields propagating on some background spacetime, the aim is to quantize spacetime itself. The methodology is inspired by conventional quantum mechanics: Quantizing a classical phase space with coordinates  $x$  and  $p$ , means replacing these with their corresponding hermitian Hilbert space operators,  $\hat{x}$  and  $\hat{p}$ . A well known implication is the Heisenberg commutator  $[\hat{x}, \hat{p}] = i\hbar$  which results in the uncertainty relation (1.0.1). This, as discussed, limits the accuracy of any sort of measurement. In

phase space, we can see this as a sort of ‘smearing’ where points can no longer be localized with infinite precision but we have a resolution  $\sim \hbar$ . This can be thought of as a form of non-locality with Planck’s constant providing the scale.

One may well ask whether the notion of a point still makes sense, as it could be argued to be a fundamentally meaningless in this case. More generally even, whether geometry is at all suitable: If we mean to quantize gravity, doesn’t that mean we need to quantize geometry itself? To get around this, we think of quantities like functions not as geometric objects, but rather more abstractly as elements of some algebra which will ultimately allow us to apply a quantization procedure based on that of quantum mechanics.

More explicitly, it is based on a formalism developed by Herman Weyl [8], which made explicit the relationship between the algebra of Hilbert space operators and phase space coordinates. It assigned to each *phase space observable* or function,  $f$  say, its Weyl symbol  $\mathcal{W}[f]$  corresponding to the hermitian *quantum observable* associated to  $f$ . The composition of two such operators is given by the convolution of their arguments  $\mathcal{W}[f]\mathcal{W}[g] = \mathcal{W}[f \star g]$ . Later on, both Groenewold [9] and Moyal [10] independently developed the phase space formulation of quantum mechanics, whereby Hilbert space operators are replaced entirely by functions in phase space. It relied on invertibility of the Weyl mapping allowing, one two work with functions directly, in particular  $f \star g = \mathcal{W}^{-1}[\mathcal{W}[f]\mathcal{W}[g]]$  where  $\star$  is the famed noncommutative and associative Moyal product. Note that the functions themselves still take their classical values, only the product between them is changed. The Weyl mapping can be considered as an algebra homomorphism with respect to the Moyal product  $\mathcal{W} : (C^\infty(\mathcal{M}), \cdot) \rightarrow (C^\infty(\mathcal{M}), \star)$  for  $\mathcal{M} = \mathbb{R}^{2d}$ . We can thus view quantization as such a mapping between algebras. In particular, for any two functions  $f \star g = fg + \mathcal{O}(\hbar)$  we can write

$$[f, g]_\hbar = f \star g - g \star f = i\hbar\{f, g\} + \mathcal{O}(\hbar^2). \quad (1.2.1)$$

Since  $\{, \}$  is the Poisson bracket on  $\mathbb{R}^{2d}$ , in the 2-dimensional case this reduces to the Heisenberg uncertainty relation  $[x, p]_\hbar = i\hbar$ . Quantization therefore amounts to replacing ordinary multiplication with a noncommutative product. This is referred to as *deformation quantization* and forms the basis of everything that is to follow.

Noncommutative geometry can then be thought of as the generalization of this to arbitrary manifolds, where we seek to apply a similar quantization procedure to spacetime



rather than just phase space. The algebra of functions over a spacetime manifold  $C^\infty(\mathcal{M})$  generated by the coordinate functions  $x^i$  is replaced with the noncommutative  $C^*$ -algebra of Hilbert space operators with hermitian generators  $\hat{x}^i$ . We then impose on these some form of commutation relation, for example

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \quad (1.2.2)$$

where  $\theta^{ij}$  is an antisymmetric, rank 2 tensor. Analogous to quantum mechanics, this leads to the uncertainty relation

$$\Delta x^i \Delta x^j \geq \frac{1}{2} |\theta^{ij}|. \quad (1.2.3)$$

So we see the same sort of smearing phenomenon of Heisenberg's principle, but now in spacetime. By this method we introduce into geometry a scale controlled by coordinates failure to commute. It indicates that what we might call classical geometry is valid down to scales  $\sim \sqrt{|\theta|}$  after which it breaks down. Indeed, before the advent of renormalization in QFT, it was thought that this coarse graining could provide a natural cutoff to control the infinities arising there. For example Snyder developed a theory of discrete spacetime without breaking Lorentz invariance [11]. The scale itself could be identified with the Planck length, although there is no a priori reason to do so. In any case, this represents a theory which has the notion of a fundamental length scale 'built in' in contrast to the previous section where it was emergent.

### 1.2.1 Poisson Structure

Definitions of quantities used here and elsewhere are given in Appendix A. Poisson structures are of central importance in noncommutative geometry. So we first make some definitions that will be useful in the rest of this thesis, in particular the notion of a *Poisson algebra* by which we mean:

A vector space  $\mathcal{A}$  over some field  $k$  which, for  $f, g, h \in \mathcal{A}$ , has the following properties

- A commutative, associative product  $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$
- A Lie algebra structure  $\{, \} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  called the Poisson bracket which satisfies the compatibility condition

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

and the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0.$$

A *Poisson Manifold* is a manifold  $\mathcal{M}$  where the vector space of smooth functions  $C^\infty(\mathcal{M})$  forms a Poisson algebra. We also define the skew-symmetric *Poisson tensor* or *bivector* field by

$$\omega(df, dg) = \{f, g\}. \quad (1.2.4)$$

It will prove useful to express this locally, so suppose we coordinate  $\mathcal{M}$  with  $\{x^i\}$  for  $i \in [1, \dots, d]$ , also defining a basis  $\{\partial_i\} \in T(\mathcal{M})$ . Hence  $\omega = \omega^{ij}\partial_i \wedge \partial_j$  so the Poisson bracket can be written as

$$\{f, g\} = \omega^{ij}\partial_i f \partial_j g \quad (1.2.5)$$

from which it follows that  $\{x^i, x^j\} = \omega^{ij}$ , so obviously  $\omega^{ij} = -\omega^{ji}$ . Using this form, the Jacobi identity can be written as

$$\sum_{cyclic(i,j,k)} \omega^{im}\partial_m \omega^{jk} = 0. \quad (1.2.6)$$

A Hamiltonian vector field is defined by

$$\hat{f} \equiv \{f, \} \quad (1.2.7)$$

which, using the above, can be expressed locally by  $\hat{f} = \omega^{ij}(\partial_i f)\partial_j$  and hence

$$\hat{f}(g) = \{f, g\}. \quad (1.2.8)$$

In everything that follows, it will be implicitly assumed that we are working over a Poisson manifold  $(\mathcal{M}, \{, \})$ , the importance of which is due to the famous result by Kontsevich [12] where it was demonstrated that the formalism to be outlined below can always be applied in this case.

### 1.2.2 Star Products

This section is intended to give a very brief and condensed overview of deformation quantization as defined by Kontsevich [12]. It is not intended to be comprehensive, but rather explain some of the aspects relevant to the later sections. For this reason we will mainly

be interested in deformations at first order in the deformation parameter and how they relate to the Poisson bracket.

Let  $(\mathcal{A}, \{, \})$  be a Poisson algebra and take  $\mathcal{A} = C^\infty(\mathcal{M})$ . For  $k$  we usually take either the real or complex numbers. By a deformation is meant the extension of  $\mathcal{A}$  by a formal power series<sup>1</sup> in a parameter  $\lambda \in \mathbb{C}$  to  $\mathcal{A}[[\lambda]]$  over  $k[[\lambda]]$ . The deformed algebra  $\mathcal{A}[[\lambda]]$  is associative (but not necessarily commutative) and with  $\mathcal{A} = \mathcal{A}[[\lambda]]/\lambda\mathcal{A}[[\lambda]]$ . We take  $f, g, h \in \mathcal{A} \subset \mathcal{A}[[\lambda]]$ . Following [12], a star product is a  $k[[\lambda]]$ -linear, associative product  $\star : \mathcal{A}[[\lambda]] \times \mathcal{A}[[\lambda]] \rightarrow \mathcal{A}[[\lambda]]$  defined by

$$f \star g = \sum_{n=0}^{\infty} \lambda^n B_n(f, g) \quad (1.2.9)$$

where  $B_n : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is a bi-differential operator with  $B_n(1, f) = 0$  and  $B_0(f, g) = fg$  that can be written locally as

$$B_n(f, g) = \sum_{A, B} \alpha_n^{A, B} \partial_A f \partial_B g \quad (1.2.10)$$

for multi-indices  $A = (a_1, \dots, a_i)$  and  $B = (b_1, \dots, b_j)$  for any  $i, j \in \mathbb{N}$ . We ask that this be associative so that

$$f \star (g \star h) = (f \star g) \star h \quad (1.2.11)$$

which can also be expressed as a power series

$$\sum_i \sum_j \lambda^{i+j} B_i(f, B_j(g, h)) = \sum_i \sum_j \lambda^{i+j} B_i(B_j(f, g), h). \quad (1.2.12)$$

Now concentrate on the  $\mathcal{O}(\lambda)$  part. Decompose the first order bi-differential operator into symmetric and antisymmetric parts  $B_1 = B_1^- + B_1^+$  so  $B_1^-(f, g) = B_1(f, g) - B_1(g, f)$  and  $B_1^+(f, g) = B_1(f, g) + B_1(g, f)$ . From this we see that

$$[f, g]_\lambda \equiv f \star g - g \star f = \lambda B_1^-(f, g) + \mathcal{O}(\lambda^2) \quad (1.2.13)$$

which already indicates the relationship between  $B_1$  and the Poisson bracket. Additionally, we can check the condition for associativity at this order. Thus, by expanding (1.2.12)

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<sup>1</sup>By formal is meant that the series needn't be, in some sense, convergent. In physical applications, convergence is usually assumed.

and considering cyclic permutations we derive the relations

$$\begin{aligned} B_1^-(fg, h) &= fB_1^-(g, h) + B_1^-(f, h)g \\ B_1^-(f, B_1^-(g, h)) + B_1^-(h, B_1^-(f, g)) + B_1^-(g, B_1^-(h, f)) &= 0. \end{aligned} \quad (1.2.14)$$

which we recognize as the product and Leibniz rules from the previous section. We can hence identify the first order antisymmetric part of the bi-differential operator with the Poisson bracket

$$B_1^-(f, g) = \{f, g\}. \quad (1.2.15)$$

An important property of the star product arises from asking the question whether there exists a relation between  $\star$  and some other  $\star'$ . Define  $C : \mathcal{A}[[\lambda]] \rightarrow \mathcal{A}[[\lambda]]$  which is  $k[[\lambda]]$ -linear map referred to as a *gauge transformation*. This can be expanded as a power series in  $\lambda$  as  $C = 1 + \sum_n \lambda^n C_n$  where  $C_n : \mathcal{A} \rightarrow \mathcal{A}$ , analogously to  $B$ . It then acts as  $C(f) = f + \sum_n \lambda^n C_n(f)$  and, because of  $k[[\lambda]]$ -linearity, we can write

$$f \star' g = C^{-1}(C(f) \star C(g)). \quad (1.2.16)$$

At first order in deformation expanding gives the relationship

$$B_1'(f, g) = B_1(f, g) + fC_1(g) + C_1(f)g - C_1(fg) \quad (1.2.17)$$

i.e. the gauge transform to first order is symmetric in  $f$  and  $g$ . The antisymmetric part is not affected i.e.  $B_1'^- = B_1^-$  and we can always choose some  $C$  to eliminate  $B_1^+$ , so the first order deformed product can always be given in terms of the Poisson bracket. Indeed, Kontsevich showed that gauge equivalent star products are classified by Poisson structures [12]. Therefore it is always possible write

$$f \star g = fg + \frac{\lambda}{2}\{f, g\} + \mathcal{O}(\lambda^2) \quad (1.2.18)$$

modulo gauge transform. He also showed that this could be associatively extended to all orders as well as proving a procedure for doing so based on (1.2.9). While we could go into further details, for the rest of this thesis we will not require more than what is outlined here. The most important part is to note the general principle that there always exists a full, associative deformation quantization for any given Poisson manifold where the product can be expressed as above.

### 1.2.3 Constant Poisson Tensor

The previous sections establish two things. Firstly, quantization can be performed algebraically by replacing the pointwise product of functions. Secondly, such a product always exists and can be defined to all orders if there exists a Poisson tensor. Given this, we can apply deformation quantization to spacetime functions and take the first steps in testing physics with quantized gravity. The essence of this approach is to suppose that the underlying spacetime is quantized, with the quantum structure encoded in the deformed product. We then further suppose that this can be used to construct an effective field theory where the expansion gives rise to higher order operators. This in turn gives rise to the interpretation of the expansion parameter as the scale of quantum gravity, so it could in some way be related to the Planck mass. For example in (1.2.2), we could identify  $\sqrt{|\theta|} \sim M_p$ . Indeed, taking this particular example, we can now directly work with functions rather than Hilbert space operators. Thus, using (1.2.13)

$$[x^i, x^j]_\theta = \theta^{ij} \quad (1.2.19)$$

where  $\theta^{ij}$  is the Poisson tensor<sup>2</sup>. For the sake of keeping notation consistent with the literature, we say that  $|\theta| \sim \lambda$ . Interest in this particular algebra was inspired by Seiberg and Witten with the formulation of noncommutative geometry as a low energy effective theory of string theory [13]. It has been studied in relation to gauge theory [14], particle physics [15] and general relativity [16, 17] amongst other things. A comprehensive overview using this particular algebra and its relation to physics can be found in [18]. The product can be obtained from

$$f \star g = \cdot \circ e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j} (f \otimes g) = fg + \frac{1}{2}\theta^{ij}\partial_i f \partial_j g + \mathcal{O}(\theta^2). \quad (1.2.20)$$

This connects to a particular type of deformation known as a *twist deformation* which leads into the mathematically rich formalism of *quantum groups*, see [19] for a physically relevant example. However the details of this are not important to the present work so we forgo introducing the formalism here. Indeed we will later examine a general approach where the method of deformation is irrelevant up to first order beyond having a Poisson structure. Rather we will focus on a simple prescription, which is to exchange ordinary

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<sup>2</sup>This is just one of many possible algebras. Another example would be the Lie algebra type  $[x^i, x^j] = c^{ij}{}_k x^k$

pointwise multiplication with star products. So for example, if we were to take

$$S = \int d^4x \sqrt{g} (\partial^i \phi \partial_j \phi + \phi^4) \quad (1.2.21)$$

it would then become

$$S = \int d^4x \sqrt{g} \star (\partial^i \phi \star \partial_j \phi + \phi \star \phi \star \phi \star \phi). \quad (1.2.22)$$

Expanding the product using (1.2.20) then gives an effective field theory where the higher order operators are dictated by the deformation. This is an efficient way of investigating the effects of quantum spacetimes and is the topic of the second paper in chapter 3.

#### 1.2.4 Summary

This paper deals with the application of the above to cosmology in order to set bounds on the scale of  $\theta$ . Taking a general scalar action (9), the product is replaced (10) (equations in 3). At this point an interesting issue arises with the term  $\partial_\mu \phi \star \partial^\mu \phi$ . As pointed out, it is required that the upper index be lowered and the metric extracted to obtain a complete result. However, whereas scalars are not affected by deformation, the same isn't necessarily true of tensors. In [20], the authors show how the  $\star$ -product alters classical coordinate invariance. Maintaining the correct transformation properties means considering noncommutative tensors with certain conditions imposed. In the case of the noncommutative metric, which we denote  $G^{\mu\nu}$ , this amounts to

$$G^{\mu\nu}|_{\theta=0} = g^{\mu\nu} \quad G^{\mu\alpha} \star G_{\star\alpha\nu} = \delta^\mu_\nu \quad (1.2.23)$$

where  $G_{\star\alpha\nu}$  is called the  $\star$ -inverse of  $G^{\mu\alpha}$ <sup>3</sup>. The latter condition in particular, ensures that one can raise and lower indices consistently. The simplest choice is to take  $G^{\mu\alpha} = g^{\mu\alpha}$  and is indeed the one made in 3. This is consistent with the above conditions and much the simplest way of specifying the kinetic term. One could use this in conjunction with the above to find  $G_{\star\alpha\nu}$ , however this isn't necessary for the discussion presented here. As noted by the authors in [20], an alternative would be to start out with the vielbein  $E^\mu_a$  and write

$$G^{\mu\nu} = \frac{1}{2} (E^\mu_a \star E^\nu_b + E^\nu_a \star E^\mu_b) \eta^{ab}. \quad (1.2.24)$$

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<sup>3</sup>One could equally take  $G_{\mu\alpha}$  as a starting point and consider  $G^{\star\alpha\nu}$ . However, we want to find a noncommutative analogue of  $g^{\alpha\nu}$  for the kinetic term.

Although even here,  $E^\mu_a$  needn't be a classical vector field. The difficulty and ambiguity in specifying a metric may be somewhat disconcerting, however, the next section deals with it in detail. Indeed constructing a noncommutative (or quantum) metric is one the central aims and issues like the raising and lowering of indices turn out to result in interesting conditions. However, doing so is outside of the scope of this section and the attendant paper.

It is also remarked that the algebra (1.2.19) is only left invariant under certain coordinate transformations, which by considering  $x^\mu \rightarrow x^\mu + \xi^\mu(x)$ , turn out to be

$$\xi^\mu(x) = \theta^{\mu\nu} \partial_\nu f(x) \quad (1.2.25)$$

for some arbitrary function  $f(x)$ . The Jacobian of these transformations is equal to one, leaving the volume element invariant. This means we are considering unimodular gravity which has  $\det(g_{\mu\nu}) = 1$ . As argued in [16, 21], the condition (1.2.25) does not place any additional constraints on the metric. Furthermore, it is shown in [22] that the details of the analysis of small cosmological perturbations is not affected by having  $\det(g_{\mu\nu}) = 1$ .

The paper demonstrates that by expanding out the  $\star$ -product, the lowest order contributions are at  $\mathcal{O}(\theta^2)$ . It is then further argued that in the case of cosmology, where the inflaton field is homogeneous, all corrections vanish and the theory is reduced to the classical version. However, the following section points out that this isn't true of perturbations and gives a modified power spectrum with  $\mathcal{O}(\theta^2)$  corrections.

### 1.2.5 Note

It is pointed out here that the units of the result given in 3 are incorrect: It is stated that the bounds are  $\sqrt{\theta} \sim 19$  TeV, when in actual fact it ought to be  $\theta^{-1/2} \sim 19$  TeV.

## 1.3 Noncommutative Differential Geometry

The previous sections gave a grounding in noncommutative geometry, but could be argued not to tell the full story. Classically, gravity is described by geometry. The central conceit of noncommutative geometry is that in order to describe quantum gravity, one requires a quantum geometry. This then ought to, in the classical limit, reduce to classical geometry in what has been called the *quantum spacetime hypothesis* [23, 24]. Introducing the star

product in the way done above, gives an effective field theory where the resultant higher order operators encode the quantum structure of spacetime. However this falls somewhat short of a true quantum geometry, since at the level of the action it is just that, an effective field theory. That is to say, the underlying spacetime is still classical, all be it with additional dynamical terms. It is of course reasonable to expect that a theory of quantum gravity can be described in this way. However, rather than just looking at the dynamics, the formalism behind the previous section gives rise to much richer structures which are more suited to looking for quantum versions of Riemannian geometry. However, simply considering deformed functions is not sufficient for this purpose and we need to think about what exactly is meant when we talk about a *quantum Riemannian geometry*.

The introduction introduced the idea of a fundamental problem with the formulation of quantum field theory is its definition on a continuous background. It also argued that even at a naive level, spacetime may not be pointlike at short scales. An effective field theory avoids this problem by only being valid in a certain energy range, but is unsatisfactory at a fundamental level. One would desire to have the classical geometry emerge from the quantum theory in some limit. But how is that to work if we need to presuppose that geometry to define a quantum theory in the first place as with any field theory? Instead we can look to drop the assumption of a continuum. This is in fact, somewhat implied by the previous sections when we talk about a minimum length scale. Taking a step further, we might try to find a true quantum geometry. Noncommutative geometry provides a possible path by supposing that the underlying quantum structure of spacetime is algebraic rather than geometric. This framework includes the classical continuous case but also allows for a generalisation to noncommutative spaces

To be able to fully understand Riemannian geometry, beyond the algebra of functions, we also need to consider the exterior algebra of differential forms, which is necessary in order to talk about (quantum) tensors. The approach examined here originates from the idea of quantum groups mentioned briefly earlier. Early developments on bimodule connections in [25–28] form the basis of quantum Riemannian geometry [24, 29–31] some of which will be introduced here. Just as the algebra is controlled to first order by the Poisson bracket, so too is the differential algebra controlled by what will be introduced as the Poisson connection. The combination of an algebra and a choice of differential algebra constrains the possible metrics, implying the geometry to be in some sense intrinsic. As



will be shown, one can derive compatibility conditions between the classical geometric structure and the quantum structure at the semiclassical level. This is the basis for the recently developed formalism of Poisson-Riemannian geometry [32] on which the final paper of this thesis is based.

For our part, rather than constructing and examining the full quantum geometry, we are mainly interested in the quantization at the semiclassical level. This is for several reasons. As was shown in [30, 33] and will be demonstrated here, attempting to quantize the exterior algebra results in nonassociativity at second order in the deformation parameter. While noncommutativity is well understood and motivated within Physics, nonassociativity is another matter. At the very least it has the capacity to enormously complicate any calculations but also poses very formidable formal challenges. From a physical standpoint, if as before we consider the deformation as a ‘perturbative’ phenomenon in the deformation parameter, it corresponds to the scale at which quantum effects become important. So we will only be interested in terms at linear order hence avoiding any nonassociativity.

### 1.3.1 Tensors and Notation

This section is intended to give an overview of the notation and conventions of the next section which might be unfamiliar. We will use definitions given in appendix A for vector spaces and tensor products. These allow us to define the notion of a tensor as a multilinear map taking several vectors and dual vectors to an element in the corresponding field  $k$ . Thus, abstractly, a tensor  $T$  of type  $(p, q)$  is defined as the map

$$T : \bigotimes^p V \otimes^q V^* \rightarrow k \quad (1.3.1)$$

for a vector space  $V$  and its dual  $V^*$ . The notation above is a shorthand for

$$\bigotimes^p V = \underbrace{V \otimes V \otimes \cdots \otimes V}_{p \text{ times}} \quad (1.3.2)$$

so we simply take  $p$  copies of  $V$ .

It certainly seems very different to the usual treatment of tensors from previous sections and indeed, Physics in general. At the very least, one might expect to see a bunch of indices. However, writing tensors in that way doesn’t show the full picture, in particular coordinate independence. An element  $v \in V$  is written without explicit reference to any

basis, but we can always choose a set  $\{e^a\} \in V$  for which  $v = v_a e^a$ . The same can be done with the dual space: If  $\{e_a\} \in V^*$ , then  $\phi \in V^*$  can be written as  $\phi = \phi^a e_a$ . According to the above definitions, these are  $(1, 0)$  and  $(0, 1)$  tensors respectively. We see the split into basis and coefficients, which show the more familiar ‘indexed’ form. This applies also to higher ranked tensors. For example, a  $(2, 1)$  tensor  $T$  is decomposed as  $T = T^a_{bc} e_a \otimes e^b \otimes e^c$ .

In our case, we will exclusively work with the cotangent space basis  $\{dx^\mu\}$ , also called *one-forms*. The set containing these is denoted by  $\Omega^1(\mathcal{A})$  and also referred to as the *exterior* or *differential algebra*. We do this since if we were to take some manifold with coordinates (or function algebra with generators)  $x^\mu$ , the exterior algebra can be simply obtained by the action of the exterior derivative. That is, taking  $\{x^\mu\} \in \mathcal{A}$ , the associated differential algebra is obtained by  $d : \mathcal{A} \rightarrow \Omega^1(\mathcal{A})$ . So for example, the basis  $(t, x, y, z)$  in function space becomes  $(dt, dx, dy, dz)$  in  $\Omega^1(\mathcal{A})$ . From the discussion above, we can use this to express tensors: The metric tensor, denoted  $g$ , can be decomposed as  $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$ . In everything that follows, when we speak of tensors, we are referring to the object  $g \in \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A})$ . On the other hand, we think of  $g_{\mu\nu}$  as tensor coefficients which are in  $\mathcal{A}$ .

Distinguishing between the basis elements and tensor coefficients will be very important in what is to come: One needs to differentiate between elements of the function algebra (to which tensor coefficients are counted) and the exterior algebra (which contains the  $dx^\mu$ ’s). As will be shown, in the noncommutative case multiplying elements of  $\mathcal{A}$  with other elements of  $\mathcal{A}$  is structurally very different to multiplying elements of  $\mathcal{A}$  with those of  $\Omega^1(\mathcal{A})$ .

### 1.3.2 Preliminaries

In order to examine the quantum geometry, we must first define it. This is done by casting classical geometry in algebraic terms which allows us to generalize to cases where the coordinate algebra is no longer commutative, pursuing the same idea as the previous section. We generally follow [34] for the first part this section and take  $f, g, h \in \mathcal{A}$ ,  $\alpha \in \Omega^n$  and  $\beta \in \Omega^m$  unless stated otherwise. Take an algebra with some associative but not necessarily commutative product  $(\mathcal{A}, \cdot)$  and  $\Omega(\mathcal{A}) = \bigoplus^i \Omega^i(\mathcal{A})$ . We say, that we have a

*differential graded algebra*  $(\Omega^1, d)$  of  $\mathcal{A}$  if

- $\Omega^1(\mathcal{A})$  is an  $\mathcal{A}$ -bimodule
- $d : \mathcal{A} \rightarrow \Omega^1(\mathcal{A})$  obeying  $d(fg) = d(f)g + fdg$  and  $d^2 = 0$
- There is a product  $\wedge : \Omega^n(\mathcal{A}) \otimes \Omega^m(\mathcal{A}) \rightarrow \Omega^{n+m}(\mathcal{A})$  so that  $d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d(\beta)$
- $\Omega^1 = \text{span}\{fdg\} \ \forall f, g \in \mathcal{A}$

The first condition means that multiplication of elements in  $\Omega^1(\mathcal{A})$  by elements in  $\mathcal{A}$  can be done from both the left and the right. We also say that  $\Omega^1(\mathcal{A})$  is a *calculus* over  $\mathcal{A}$  with elements in  $\Omega^m(\mathcal{A})$  referred to as  $m$ -forms. An obvious example is  $\mathcal{A} = C^\infty(\mathcal{M})$  with classical pointwise multiplication and  $\Omega^1(\mathcal{A}) = T^*\mathcal{M}$  in which case  $d$  is the usual exterior derivative and  $T^*\mathcal{M}$  is spanned by the basis one-forms  $\{dx^\mu\}$ . However, this abstract construction is more general and includes cases where the product is not commutative so

$$\begin{aligned} fg &\neq gf \\ \alpha \wedge \beta &\neq -\beta \wedge \alpha. \end{aligned} \tag{1.3.3}$$

Taking two algebras  $\mathcal{A}$  and  $\mathcal{A}'$  with calculi  $\Omega(\mathcal{A})$  and  $\Omega(\mathcal{A}')$  one also has tensor product

$$\Omega(\mathcal{A} \otimes \mathcal{A}') = \Omega(\mathcal{A}) \otimes_{\mathcal{A}} \Omega(\mathcal{A}'). \tag{1.3.4}$$

The subscript is indicative of the property

$$\alpha \otimes_{\mathcal{A}} f\beta = \alpha f \otimes_{\mathcal{A}} \beta. \tag{1.3.5}$$

Now consider  $\wedge$  applied to one-forms  $\wedge : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^2(\mathcal{A})$ . It allows us to define a notion of something being ‘symmetric’. So, if for some  $a \in \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$  we have  $\wedge(a) = 0$ ,  $a$  will be referred to as being *quantum symmetric*. In the classical case this is obviously the wedge product with the usual skew-symmetry. Conversely, one can also consider the inverse lift map which takes 2-forms into antisymmetric 1-1-forms

$$i : \Omega^2(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{1.3.6}$$

so that  $\wedge \circ i = \text{id}$ . Next consider some arbitrary  $\mathcal{A}$ -bimodule  $E$ . By this we mean that  $E$  can be a tensor or wedge product of bimodules since they both preserve the bimodule

property, so for example  $E = \Omega^m(\mathcal{A})$ . It will prove useful to define the map

$$\sigma_E : E \otimes_{\mathcal{A}} E \rightarrow E \otimes_{\mathcal{A}} E \quad (1.3.7)$$

called the *generalized braiding* which has the function of flipping the tensor factors around the tensor product with the property  $\sigma_E^2 = \text{id}$ . Classically this is simply  $\sigma_E(\alpha \otimes_{\mathcal{A}} \beta) = \beta \otimes_{\mathcal{A}} \alpha$ , for example.

Next we extend the notion of a connection to the bimodule construction. A (left) connection is a map

$$\nabla_E : E \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} E \quad (1.3.8)$$

which obeys the (left) Leibniz rule  $\nabla_E(fe) = df \otimes_{\mathcal{A}} e + f\nabla_E e$  for  $e \in E$ . This is extended to a bimodule connection [25] using the braiding defined in (1.3.7) giving

$$\nabla_E(e f) = \nabla_E(e) f + \sigma_E(e \otimes_{\mathcal{A}} df). \quad (1.3.9)$$

So a bimodule connection on  $E$  is given by  $(\nabla_E, \sigma_E)$ . Further, if such a  $\sigma_E$  associated with a left connection exists, then it is uniquely determined by the formula

$$\sigma_E(e \otimes da) = da \otimes e + \nabla_E[e, a] + [a, \nabla_E e]. \quad (1.3.10)$$

This construction can be extended to tensor products of bimodules. Take  $(\nabla_E, \sigma_E)$  on  $E$  and  $(\nabla_F, \sigma_F)$  on  $F$ . Now the connection extends to tensor products as

$$\nabla_{E \otimes_{\mathcal{A}} F} = \nabla_E \otimes_{\mathcal{A}} \text{id}_F + (\sigma_E \otimes_{\mathcal{A}} \text{id}_F)(\text{id}_E \otimes_{\mathcal{A}} \nabla_F) \quad (1.3.11)$$

with the braiding

$$\sigma_{E \otimes_{\mathcal{A}} F} = (\sigma_E \otimes_{\mathcal{A}} \text{id}_F)(\text{id}_E \otimes_{\mathcal{A}} \sigma_F) \quad (1.3.12)$$

thus giving the bimodule connection  $(\nabla_{E \otimes_{\mathcal{A}} F}, \sigma_{E \otimes_{\mathcal{A}} F})$  on  $E \otimes_{\mathcal{A}} F$ .

What we have seen so far can easily be identified with classical differential geometry, but also includes far more general cases. It is at this point we will refer to Physics as guide to continue our construction. After all, the aim is to apply this formalism to the problem of quantum gravity and since the usual starting point is the metric, that is exactly where we turn our attention to next. But this does present some conceptual problems. In Rieman-

nian geometry, the metric measures distance between points through the line element. However, the whole purpose of moving to noncommutative geometry and its algebraic formulation in the first place is to get around the problem of the continuum assumption of differential geometry. That is, the idea that at its most abstract level, a manifold is a collection points. What then is the use of a metric? Well here is exactly where we take our cues from Physics; since one would hope to have the classical case emerge from the quantum one, it makes sense to start from the same place we would for any physics problem. Construction of the quantum geometry starts from the same point but rather than thinking of the metric in geometric terms, we instead concentrate its abstract properties. We will focus on the construction of [24] since this is what will be relevant to later sections. It seeks to maintain the role of the metric in contracting tensor indicies as is familiar in GR and is necessary for the present formulation of noncommutative Riemannian geometry.

So in this vein, the metric is taken to be an element

$$g \in \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}). \quad (1.3.13)$$

We can impose on it the requirement that it be symmetric in the sense  $\wedge(g) = 0$ . Now suppose that there exists an inverse giving rise to an inner product  $(, ) : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \mathcal{A}$  so that for a one-form

$$((, ) \otimes \text{id})(\alpha \otimes_{\mathcal{A}} g) = \alpha = (\text{id} \otimes (, ))(g \otimes_{\mathcal{A}} \alpha) \quad (1.3.14)$$

which is also a bimodule map so that  $f(, ) = (f(, ), )$  and  $(, )f = (, (, )f)$ . In the commutative case this is simply  $(dx^\mu, dx^\nu) = g^{\mu\nu}$ . Then, take  $g = g_1 \otimes_{\mathcal{A}} g_2$  and write

$$f\alpha = ((, ) \otimes \text{id})(f\alpha \otimes_{\mathcal{A}} g) = (f\alpha, g_1)g_2 = f(\alpha, g_1)g_2. \quad (1.3.15)$$

Using the bimodule property this can also be rendered as

$$f\alpha = (\text{id} \otimes (, ))(g \otimes_{\mathcal{A}} f\alpha) = (\text{id} \otimes (, ))(gf \otimes_{\mathcal{A}} \alpha) = g_1(g_2 f, \alpha). \quad (1.3.16)$$

Since the rightmost expressions must be equal, we have  $[f, g] = 0$  for all  $f \in \mathcal{A}$ , i.e. the metric commutes with functions. It is said to be central in the algebra.

Now, putting everything together, it is natural to ask about the existence of a connection

associated with the metric. This can be analogous to the Levi-Civita connection which is torsion free and metric compatible. The torsion of some  $\nabla$  is the map  $T : \Omega^1 \mathcal{A} \rightarrow \Omega^2 \mathcal{A}$  and defined as

$$T_\nabla = d - \wedge \nabla \quad (1.3.17)$$

so torsion free means having  $\wedge \nabla = d$ . We can then define metric compatibility by the vanishing of

$$\nabla g \equiv (\nabla \otimes \text{id})g + (\sigma \otimes \text{id})(\text{id} \otimes \nabla)g \in \Omega^1 \otimes \Omega^1 \otimes \Omega^1 \quad (1.3.18)$$

for some bimodule connection. A connection that is both torsion free and metric compatible will be referred to as a *quantum Levi-Civita connection*.

### 1.3.3 Deformation

We now focus on implementing the previous section in a deformation setting following [35]. So take  $\mathcal{A}$  to be  $C^\infty(\mathcal{M})[[\lambda]]$  with deformed product  $\bullet$  and deformation parameter  $\lambda$ . Likewise for the exterior algebra. Functions and one-forms take their classical values and  $f \bullet g = fg + \mathcal{O}(\lambda)$ . For reasons that were mentioned earlier, we are only interested in the semi-classical data, that is terms at  $\mathcal{O}(\lambda)$  and will generally neglect anything at higher order. Denote the commutator by

$$[f, g]_\lambda = f \bullet g - g \bullet f = \lambda \{f, g\} + \mathcal{O}(\lambda^2) \quad (1.3.19)$$

where  $\{, \}$  is the Poisson bracket. A note on notation: In contrast to the previous section, deformed or quantum quantities will carry the subscript  $\lambda$  to distinguish them from their classical counterparts. For example, we use  $d$  for the classical exterior derivative and use  $d_\lambda$  for its deformation related by  $d_\lambda f = df + \mathcal{O}(\lambda)$ . We also have the deformed exterior product  $\wedge_\lambda$  which is similarly related by  $\alpha \wedge_\lambda \beta = \alpha \wedge \beta + \mathcal{O}(\lambda)$  and assumed to be associative at  $\mathcal{O}(\lambda)$ .  $d_\lambda$  obeys the graded Leibniz rule

$$d_\lambda(\alpha \wedge_\lambda \beta) = d_\lambda(\alpha) \wedge_\lambda \beta + (-1)^{|\alpha|} \alpha \wedge_\lambda d_\lambda(\beta). \quad (1.3.20)$$

Since  $\Omega^1(\mathcal{A})$  is a bimodule over  $\mathcal{A}$ , we also have

$$[f, \alpha]_\lambda = \lambda \gamma(f, \alpha) + \mathcal{O}(\lambda^2). \quad (1.3.21)$$

The map  $\gamma$  is said to be Poisson compatible if

$$d\{f, g\} = \gamma(f, dg) + \gamma(g, df). \quad (1.3.22)$$

Further, analogous to the Leibniz rule for the Poisson bracket  $\{f, gh\} = g\{f, h\} + \{f, g\}h$ , it satisfies

$$\gamma(f, \alpha g) = \gamma(f, \alpha)g + \alpha\{f, g\} \quad (1.3.23)$$

and thus is a derivation on  $\mathcal{A}$ . The meaning of  $\gamma$  perhaps becomes more clear when cast in the form

$$\gamma(f, \alpha) = \nabla_{\hat{f}}\alpha \quad (1.3.24)$$

and we see that it can be thought of as a connection along the Hamiltonian vector field  $\hat{f} = \{f, \}$  which we call a *Poisson connection*. Now it should be noted that  $\nabla_{\hat{f}}$  is only partially defined in that it lies *only* along  $\hat{f}$  and is therefore referred to as a *preconnection*. However, in the application of this in a later chapter we will actually take a Riemannian manifold with a metric as starting point and find a Poisson connection compatible with the available structure, in which case we can safely think of  $\nabla_{\hat{f}}$  as a (but not usually Levi-Civita) covariant derivative. It will be useful to bear in mind the curvature which is defined in the usual way

$$R(\hat{f}, \hat{g}) = \nabla_{\hat{f}}\nabla_{\hat{g}} - \nabla_{\hat{g}}\nabla_{\hat{f}} - \nabla_{\{f, g\}}. \quad (1.3.25)$$

Returning to the algebra (1.3.19), this clearly follows the Kontsevich construction in section 1.2.2 and can be extended associatively to all orders. Thus it satisfies the Jacobi identity

$$[f, [g, h]_{\lambda}]_{\lambda} + [h, [f, g]_{\lambda}]_{\lambda} + [g, [h, f]_{\lambda}]_{\lambda} = 0. \quad (1.3.26)$$

Now, extend this to the exterior algebra and check for associativity by defining the *super-Jacobiator* as

$$J(f, g, h) = [f, [g, dh]] + [dh, [f, g]] + [g, [dh, f]] + \mathcal{O}(\lambda^3). \quad (1.3.27)$$

Expanding the right hand side and using (1.3.25) gives

$$\begin{aligned}
J(f, g, h) &= \lambda[f, \nabla_{\hat{g}} dh] - \lambda[g, \nabla_{\hat{h}} dh] - \lambda[\{f, g\}, dh] + \mathcal{O}(\lambda^3) \\
&= \lambda^2 \nabla_{\hat{f}} \nabla_{\hat{g}} dh - \lambda^2 \nabla_{\hat{g}} \nabla_{\hat{f}} dh - \lambda^2 \nabla_{\{f, g\}} dh + \mathcal{O}(\lambda^3) \\
&= \lambda^2 R(\hat{f}, \hat{g})(dh) + \mathcal{O}(\lambda^3)
\end{aligned}$$

truncating at  $\mathcal{O}(\lambda^2)$ . Now what does this mean? If the product between the algebra and exterior algebra were associative, we would expect  $J$  to vanish, conversely requiring a flat Poisson connection. Hence associativity (or its failure) between one forms and functions is given by the curvature of the Poisson connection. This is interesting since it links properties of the algebra to those of the geometry and lends them some physical intuition. It demonstrates that a non-flat connection will generally result in a nonassociative differential algebra. Although  $\nabla_{\hat{f}}$  here needn't be the Levi-Civita connection and thus not necessarily have anything to say about spacetime curvature, they could nonetheless be related. It lends itself to the generic statement that a curved spacetime may give rise to a nonassociative quantization at second order in deformation.

This is quite disconcerting since we would usually want to avoid having to work with nonassociative algebras. But we must then contend with the very rigid conditions imposed by requiring a flat connection. Indeed, if the aim is to find some form of quantum spacetime in which the semiclassical data is related to the Levi-Civita connection, this restriction may even too stringent. A similar conclusion was reached in [33] using the alternative notion of a contravariant connection.

However, the construction given here results in non-associativity only at  $\mathcal{O}(\lambda^2)$ . Since the scale of quantum gravity  $\mathcal{O}(\lambda)$  is supposed to be around the Planck scale, and this would be extremely difficult to detect, it seems reasonable to neglect contributions at this order. So at least from a physical standpoint, we are justified to confine ourselves to the semi-classical regime.

### 1.3.4 Semiquantization Functor

It is interesting to consider the implication of the previous section, which shows how classical structures affects quantization. Or perhaps it should rather be thought of as quantizability restricting classical structure. In this view, the constraints amount the ex-



istence of certain physical ‘fields’ as displayed in (1.3.19) and (1.3.21). At the level of the algebra, the quantum structure is controlled by a Poisson tensor, whereas the differential algebra is controlled by a Poisson connection. The latter in particular, allows us to make a direct connection to Physics. Even when the Poisson connection isn’t itself Levi-Civita, if the two can be related the presence of curvature strongly implies a non-trivial form of (1.3.21). So the choice exterior algebra puts constraints on the curvature.

An application is presented in [32], which gives a categorical construction for obtaining contributions at (and only at)  $\mathcal{O}(\lambda)$ . This is referred to as *semiquantization* to distinguish it from full quantization. On the other hand, we know that for the algebra the semiclassical data is canonically given by the Poisson bracket which can be extended to all orders by Kontsevich’s deformation procedure. One could perhaps imagine that knowing the theory to first order, it would then be possible to extend this to higher orders by identifying some appropriate quantization scheme, something in regards to which this theory remains essentially agnostic.

What follows is a brief outline and summary of the main results in [32]. Using the language of categories, as illustrated in appendix B, the aim is to construct a functor  $Q$  which has the effect of mapping the classical data which consists of vector bundles and bundle maps over some manifold  $\mathcal{M}$ , to (semi)quantum data which is a first order deformation. The data we consider is  $\mathcal{E}$ , the monoidal category of bimodules with  $\otimes_{\mathcal{A}}$  and  $\mathcal{D}$ , the monoidal category of pairs  $(E, \nabla_E)$  and morphisms consisting of bimodule maps intertwining the connections.

## Classical Data

On the classical side, we take the  $\mathcal{A} = C^\infty(\mathcal{M})$  with ordinary multiplication and can think of sections of a vector bundle as a  $C^\infty(\mathcal{M})$ -bimodule  $E$ . This is just a special case of section 1.3.2 with a commutative product. We denote the corresponding categories by

Name	Objects	Morphisms
$\mathcal{E}_0$	Vector bundles	Bundle maps
$\tilde{\mathcal{D}}_0$	Vector bundles and connection	Bundle maps
$\mathcal{D}_0$	Vector bundles and connection	Bundle maps intertwining the connections

Taking two vector bundles with connections  $(E, \nabla_E)$  and  $(V, \nabla_V)$ , a bimodule (or in this case a bundle) map  $T$  is a morphism between  $E$  and  $V$  so that

$$\begin{array}{ccc} E & \xrightarrow{T} & V \\ \pi_E \searrow & & \swarrow \pi_V \\ & \mathcal{M} & \end{array} \quad (1.3.28)$$

To be a morphism in  $\mathcal{D}_0$  it also intertwines the connections in that  $\nabla_V(T(e)) = (\text{id} \otimes T)\nabla_E(e)$ . The Poisson connection can be written as

$$\nabla_{\hat{f}}\alpha = \omega^{\mu\nu}\partial_\mu f \nabla_\nu \alpha \quad (1.3.29)$$

with  $\nabla_\mu$  acting as a covariant derivative which can be expressed in terms of Christoffel symbols

$$\nabla_\mu(\alpha_\nu dx^\nu) = (\alpha_{\nu,\mu} - \alpha_\gamma \Gamma_{\mu\nu}^\gamma) dx^\nu. \quad (1.3.30)$$

Now, Poisson compatibility in (1.3.22) can be written

$$\omega^{\mu\nu}_{,\gamma} + \omega^{\mu\kappa}\Gamma_{\kappa\gamma}^\nu + \omega^{\kappa\nu}\Gamma_{\kappa\gamma}^\mu = 0. \quad (1.3.31)$$

Note the order of the lower indices. We also have (1.2.6) and  $\omega$  is generally assumed to satisfy this condition since it in principle allows for the theory to be extended to higher orders at the level of the algebra. However, in general this isn't necessary for the analysis given here and  $\omega$  could denote any bivector satisfying (1.3.31). We also add the further requirement

$$\nabla(g) = 0 \quad (1.3.32)$$

which ensures that the metric will be central in the algebra, in the sense demonstrated toward the end of section 1.3.2. A connection satisfying both conditions will sometimes be referred to as a *quantizing connection*.

## Quantum Data

For the quantum algebra we take the product (1.3.19). Because of the possibility of nonassociativity at  $\mathcal{O}(\lambda^2)$  and above, we work strictly at  $\mathcal{O}(\lambda)$  i.e. we consider only

$\mathcal{A}_1 = \mathcal{A}[[\lambda]]/\lambda^2\mathcal{A}[[\lambda]]$  with  $\mathcal{A}$  as above. So, for  $e \in E$  we define the bimodule product as

$$\begin{aligned} f \bullet e &= fe + \frac{\lambda}{2} \omega^{\mu\nu} \partial_\mu f \nabla_\nu e \\ e \bullet f &= ef - \frac{\lambda}{2} \omega^{\mu\nu} \partial_\mu f \nabla_\nu e \end{aligned} \tag{1.3.33}$$

To simplify notation, we will generally take  $\nabla_\mu \equiv (\nabla_E)_\mu$  in cases where the meaning is implicitly clear. This is by no means the only way of defining a product between  $\mathcal{A}_1$  and  $E$ , but is perhaps the most natural one and can easily be verified to reproduce the commutator (1.3.21). Now for the quantum data we have

Name	Objects	Morphisms
$\tilde{\mathcal{E}}_1$	Bimodules	Left module maps
$\mathcal{E}_1$	Bimodules	Bimodule maps
$\mathcal{D}_1$	Bimodules with connection	Bimodule maps intertwining the connections

Note that bimodule *map* means the same as bimodule in general i.e. it can be multiplied by elements of  $\mathcal{A}_1$  on both sides.

## Functor

As indicated above, the underlying vector spaces are undeformed, so the functor  $Q$  acts as the identity on elements i.e.  $Q(e) = e$  affecting only the product  $Q(fe) = f \bullet e$ . We can view  $Q(E)$  simply as  $E$  with ordinary multiplication replaced by  $\bullet$  and since elements are undeformed,  $Q(e) = e \in Q(E)$ . Next we are interested in the action of  $Q$  on morphisms. Now, the previous discussion has hopefully demonstrated is the important role the connection plays in the interaction between quantum and classical data. For this reason we restrict ourselves to  $\tilde{\mathcal{D}}_0$  for the latter and distinguish connections as a special class of morphisms. However, there is no reason  $Q$  ought to respect this division, so in the first instance we consider  $Q : \tilde{\mathcal{D}}_0 \rightarrow \tilde{\mathcal{E}}_1$  and take its action on bundle maps. So now  $T : E \rightarrow V$  becomes

$$Q(T) : Q(E) \rightarrow Q(V). \tag{1.3.34}$$

This can be expanded as  $Q(T) = T_0 + \lambda T_1$  where  $T_0 = T$ . For left module maps require  $Q(T)(f \bullet e) = f \bullet Q(T)(e)$ . By expanding and taking the product according to (1.3.33), we eventually find

$$T_1 = \omega^{\mu\nu} (\nabla_V)_\mu \circ \nabla_\nu (T_0) \tag{1.3.35}$$

where  $\nabla_\nu(T_0) = (\nabla_V)_\nu \circ T_0 - T_0 \circ (\nabla_E)_\nu$ . Thus we have

$$Q(T) = T + \frac{\lambda}{2} \omega^{\mu\nu} (\nabla_V)_\mu \circ \nabla_\nu(T). \quad (1.3.36)$$

Note however, that this is not functorial: That would require the map to respect composition which the above does not, but rather

$$Q(T \circ S) = Q(T) \circ Q(S) + \frac{\lambda}{2} \omega^{\mu\nu} \nabla_\mu(T) \circ \nabla_\nu(S) \quad (1.3.37)$$

For some bundle map  $S$ . We can then restrict ourselves to bimodule maps i.e. consider  $Q : \tilde{\mathcal{D}}_0 \rightarrow \mathcal{E}_1$ . So in addition to the above, we further impose  $Q(T)(e \bullet f) = Q(T)(e) \bullet f$  which turns into the requirement  $T_1 = -\omega^{\mu\nu} (\nabla_V)_\mu \circ \nabla_\nu(T_0)$ . Comparing this to (1.3.35) see that  $T_1 = 0$ . Furthermore, this results in the condition  $\nabla_V \circ T_0 = T_0 \circ \nabla_E$  i.e. the bundle map must conserve (that is to say intertwine) the covariant derivative. So by restricting to bimodules we automatically get  $Q : \mathcal{D}_0 \rightarrow \mathcal{E}_1$  and since  $Q(T) = T$  along with the above conditions, this is now also functorial.

This is extended to tensor products over  $\mathcal{A}_1$ . In the classical case we denote the tensor product by  $\otimes_0 \equiv \otimes_{\mathcal{A}}$ . Independently of this, we also have a tensor product of bimodules  $\otimes_1 \equiv \otimes_{\mathcal{A}_1}$  satisfying (1.3.5), seemingly giving two different compositions i.e. we can take the classical composition of vector bundles and apply  $Q$  so  $Q(E \otimes_0 F)$  or apply  $Q$  and then compose the result  $Q(E) \otimes_1 Q(F)$ . However, these are related by a *natural transformation*, which in category theory is a map between functors. In our case, denoted by  $q$  it gives

$$\begin{array}{ccc} E \otimes_0 V & \xrightarrow{Q} & Q(E \otimes_0 V) \\ & \searrow Q \otimes_1 Q & \uparrow q_{E,F} \\ & & Q(E) \otimes_1 Q(V) \end{array}$$

As a result,  $Q$  is said to be monoidal i.e. it preserves the monoidal structure of the category. Explicitly, the natural transformation is

$$q_{E,V}(Q(e) \otimes_1 Q(v)) = Q(e \otimes_0 v) + \frac{\lambda}{2} Q(\omega^{\mu\nu} (\nabla_E)_\mu e \otimes_0 (\nabla_V)_\nu v). \quad (1.3.38)$$

Allowing us to identify  $\otimes_0$  and  $\otimes_1$ . Its inverse is simply

$$q_{E,V}^{-1}(Q(e \otimes_0 v)) = Q(e) \otimes_1 Q(v) - \frac{\lambda}{2} \omega^{\mu\nu} (\nabla_E)_\mu e \otimes_1 (\nabla_V)_\nu v. \quad (1.3.39)$$

This is compatible with the territoriality property (1.3.5) and the requirement that it be a bimodule map  $q_{E,F}(f \bullet e \otimes_1 v) = f \bullet q_{E,F}(e \otimes_1 v)$  and  $q_{E,F}(e \otimes_1 v \bullet f) = q_{E,F}(e \otimes_1 v) \bullet f$ .

### Quantum Connection

Next we examine the connection, which means extending the functor to  $Q : \mathcal{D}_0 \rightarrow \mathcal{D}_1$  by defining  $Q(\nabla_E) = \nabla_{Q(E)} : E_A \rightarrow \Omega(\mathcal{A}) \otimes_1 E_A$  so that  $Q(E, \nabla_E) = (Q(E), \nabla_{Q(E)})$ . First consider the connection as a left module map satisfying the left Leibniz rule, require  $\nabla_{Q(E)}(f \bullet e) = df \otimes_1 e + f \bullet \nabla_{Q(E)}(e)$  with respect to the deformed tensor product. This can be achieved by setting

$$\nabla_{Q(E)} \equiv q_{\Omega^1, E}^{-1} \nabla_E - \frac{\lambda}{2} \omega^{\mu\nu} dx^\gamma \otimes_1 [(\nabla_E)_\gamma, (\nabla_E)_\nu](\nabla_E)_\mu. \quad (1.3.40)$$

Now remember that that associated to any left connection is a braiding as explained in section 1.3.2 is then calculated using (1.3.10)

$$\sigma_{Q(E)}(e \otimes_1 \alpha) = \alpha \otimes_1 e + \lambda \omega^{\mu\nu} (\nabla_\nu \alpha \otimes_1 (\nabla_E)_\mu e + \alpha_\nu dx^\gamma \otimes_1 [(\nabla_E)_\gamma, (\nabla_E)_\mu] e). \quad (1.3.41)$$

Thus giving a bimodule connection. Combine this with the previous section where we saw that restricting  $Q$  to bimodule maps automatically meant going from  $\tilde{\mathcal{D}}_0$  to  $\mathcal{D}_0$  and  $Q$  functorial. We can use (1.3.40) to see

$$\nabla_E \circ T = T \circ \nabla_E \implies \nabla_{Q(E)} \circ Q(T) = Q(T) \circ \nabla_{Q(E)}. \quad (1.3.42)$$

Thus, we have that  $Q : \tilde{\mathcal{D}}_0 \rightarrow \tilde{\mathcal{E}}_1$  is a mapping while  $Q : \mathcal{D}_0 \rightarrow \mathcal{E}_1$  and  $Q : \mathcal{D}_0 \rightarrow \mathcal{D}_1$  are functorial. These are used as a basis throughout the rest of this section and the paper to which it pertains.

Before continuing however, we want to complete our treatment of connections. Since classically, the covariant derivative extends to tensor products of bundles and we want (1.3.11), we expect the quantization to respect this property i.e. the quantization of the classical connection over tensor products is the same as the bimodule connection over  $\otimes_1$

$$\begin{array}{ccc} Q(E) \otimes_1 Q(F) & \xrightarrow{q_{E,F}} & Q(E \otimes_0 F) \\ \downarrow \nabla_{Q(E) \otimes_1 Q(F)} & & \downarrow \nabla_{Q(E \otimes_0 F)} \\ \Omega^1(\mathcal{A}) \otimes_1 Q(E) \otimes_1 Q(F) & \xrightarrow{\text{id} \otimes q} & \Omega^1(\mathcal{A}) \otimes_1 Q(E \otimes_0 F) \end{array} \quad (1.3.43)$$

Though this isn't the complete picture since  $\nabla_{Q(E)}$  is only a quantization of the quantizing connection satisfying (1.3.31). So, in order to describe an arbitrary connection, we use the fact that any two can be related through a bundle map  $S : E \rightarrow \Omega^1(\mathcal{M}) \otimes_0 E$  as  $\nabla_S = \nabla_E + S$  which is quantized as a left module map. So applying  $Q$  gives

$$\nabla_{QS} \equiv \nabla_{Q(E)} + q_{\Omega^1, E}^{-1} Q(S) \quad (1.3.44)$$

and braiding

$$\sigma_{QS}(e \otimes_1 \alpha) = \sigma_{Q(E)}(e \otimes_1 \alpha) + \lambda \omega^{\mu\nu} \alpha_\mu \nabla_\nu(S)(e). \quad (1.3.45)$$

What this allows for is in cases where  $\nabla_E$  isn't the Levi-Civita connection, as will be the case later, we can nonetheless apply the same procedure where  $S$  will turn out to be the contorsion tensor. We note here an important observation, if classically we have  $\nabla_E(e) = 0$  for all  $e \in E$  then  $e$  is central in the quantized bimodule by applying (1.3.33) and (1.3.21). What's more

$$\nabla_E(e) = 0 \implies \nabla_{Q(E)}(e) = 0. \quad (1.3.46)$$

This is simply an application of (1.3.40). Furthermore, if  $S(e) = 0$  then

$$\nabla_S(e) = 0 \implies \nabla_{QS}(e) = 0. \quad (1.3.47)$$

We can see in this a precursor to the quantum metric compatibility.

## Exterior Algebra

The semiquantization functor acts on the exterior algebra as  $Q(\Omega^1(\mathcal{M})) = \Omega^1(\mathcal{A})$  so we want to extend this to  $\Omega^n(\mathcal{A}) = Q(\Omega^n(\mathcal{M}))$ . This achieved by means of the composition

$$Q(\Omega^n(\mathcal{M})) \otimes_1 Q(\Omega^m(\mathcal{M})) \xrightarrow{q} Q(\Omega^n(\mathcal{M}) \otimes_0 \Omega^m(\mathcal{M})) \xrightarrow{Q(\wedge)} Q(\Omega^{n+m}(\mathcal{M})). \quad (1.3.48)$$

The quantity  $Q(\wedge)$  results from an application of the functor  $Q : \mathcal{D}_0 \rightarrow \mathcal{E}_1$  defined earlier. Classically, the connection extends to forms of all degrees since it preserves the symmetry when acting on tensor products. So for example, when wanting to act it on  $\Omega^2(\mathcal{M})$ , we can do this directly or equivalently, act it on the tensor product  $\Omega^1(\mathcal{M}) \otimes \Omega^1(\mathcal{M})$  and then antisymmetrize the result. This amounts to  $\nabla \wedge = (\text{id} \otimes \wedge) \nabla$ , thus making  $\wedge$  a morphism in  $\mathcal{D}_0$ . Hence  $Q(\wedge) = \wedge$  so that we can define the functorial wedge product as  $\wedge_Q \equiv \wedge \circ q$ .

Explicitly this is

$$\alpha \wedge_Q \beta = \alpha \wedge \beta + \frac{\lambda}{2} \omega^{\mu\nu} \nabla_\mu \alpha \wedge \nabla_\nu \beta \quad (1.3.49)$$

which is associative up to  $\mathcal{O}(\lambda^2)$ . Now we also need the graded Leibniz rule for the exterior derivative to hold. Since from the above  $d(\alpha \wedge_Q \beta) \neq d\alpha \wedge_Q \beta + (-1)^{|\alpha|} \alpha \wedge_Q d\beta$  one possible choice would be to deform  $d$ , however this is only one of three possible options. The other two are either to consider some modified Leibniz rule or to add a further modification to  $\wedge_Q$ . All of these are equivalent and consistent since  $Q$  does not single out any particular choice, so we choose the latter. At least part of the rationale may be that it is the simplest from a calculational standpoint. For this we define the two form

$$H^{\alpha\beta} \equiv \frac{1}{2} \omega^{\alpha\gamma} (T^\beta_{\nu\mu;\gamma} - 2R^\beta_{\nu\mu\gamma}) dx^\mu \wedge dx^\nu \quad (1.3.50)$$

which measures the failure of the Leibniz rule at  $\mathcal{O}(\lambda)$

$$d(\alpha \wedge_Q \beta) - d\alpha \wedge_Q \beta - (-1)^{|\alpha|} \alpha \wedge_Q d\beta = -\lambda H^{\mu\nu} \wedge \iota_{\partial_\mu} \alpha \wedge \nabla_\nu \beta + \lambda H^{\mu\nu} \wedge \nabla_\mu \alpha \wedge \iota_{\partial_\nu} \beta. \quad (1.3.51)$$

Where  $\iota_\gamma : \Omega^n(\mathcal{M}) \rightarrow \Omega^{n-1}(\mathcal{M})$  is the interior product. Now we can define

$$\alpha \wedge_1 \beta = \alpha \wedge_Q \beta + \lambda (-1)^{|\alpha|+1} H^{\mu\nu} \wedge \iota_{\partial_\mu} \alpha \wedge \iota_{\partial_\nu} \beta \quad (1.3.52)$$

which allows us to keep the classical exterior derivative and Leibniz rule, provided  $H^{\mu\nu} = H^{\nu\mu}$  so  $d(\alpha \wedge_1 \beta) = d\alpha \wedge_1 \beta + (-1)^{|\alpha|} \alpha \wedge_1 d\beta$ .

### Quantum Torsion

Next we turn to the Torsion as defined in (1.3.17). First, for the quantizing connection, for a one form  $\alpha$  it comes out as

$$T_{\nabla_{Q(E)}}(\alpha) = \wedge_1 \nabla_{Q(E)}(\alpha) - d\alpha = T(\alpha) + \frac{\lambda}{4} \omega^{\gamma\kappa} T^\delta_{\nu\mu;\kappa} (\iota_{\partial_\delta} \nabla_\gamma \alpha) dx^\mu \wedge dx^\nu. \quad (1.3.53)$$

So if  $\nabla_E$  is torsion free, so is  $\nabla_{Q(E)}$ . This is in contrast to the case of some arbitrary connection  $\nabla_S$  in which case the quantum torsion comes out as

$$\begin{aligned} T_{\nabla_{QS}}(\alpha) &= T_{\nabla_Q}(\alpha) + \wedge_1 q^{-1} S(\alpha) + \frac{\lambda}{2} \wedge \omega^{ij} \nabla_i \circ \mathbb{W}_j(S)(\alpha) \\ &= T_{\nabla_S}(\alpha) + \frac{\lambda}{4} \omega^{\gamma\delta} (\alpha_{\rho;\gamma} (T^\rho_{\nu\mu;\delta} - 2S^\rho_{\nu\mu;\delta}) - 2S^\rho_{\nu\mu;\delta\gamma}) dx^\mu \wedge dx^\nu \\ &\quad + \lambda \alpha_p S^p_{\nu\mu} H^{\nu\mu}. \end{aligned}$$

Now let us suppose that  $\nabla_S$  is torsion free i.e.  $T_{\nabla_S} = 0$  and thus giving  $0 = T_{\nabla}(dx^\gamma) + \wedge S(dx^\gamma)$ . Then by taking  $S(dx^\gamma) = S^\gamma_{\mu\nu} dx^\mu \otimes dx^\nu$  we get

$$S^\gamma_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} T^\gamma_{\mu\nu} dx^\mu \wedge dx^\nu. \quad (1.3.54)$$

Then the above becomes

$$T_{\nabla_{QS}}(\alpha) = \frac{\lambda}{4} \omega^{\mu\nu} \alpha_\rho [\nabla_\mu, \nabla_\nu] T^\rho_{\delta\gamma} dx^\gamma \wedge dx^\delta + \lambda \alpha_p S^p_{\mu\nu} H^{\mu\nu}. \quad (1.3.55)$$

So while classically  $\nabla_S$  may be torsion free, the same needn't be true in the quantum case unlike  $\nabla_{Q(E)}$ . We can fix this by modifying the connection

$$\nabla_1 = \nabla_{QS} + \lambda K \quad (1.3.56)$$

where we have introduced the bundle map  $K : \Omega(\mathcal{M}) \rightarrow \Omega(\mathcal{M}) \otimes_0 \Omega(\mathcal{M})$  given by  $K(dx^a) = K^a_{mn} dx^m \otimes_0 dx^n$ . This is fixed so that

$$K^\rho_{\gamma\delta} dx^\gamma \wedge dx^\delta = -\frac{1}{4} \omega^{\mu\nu} [\nabla_\mu, \nabla_\nu] S^\rho_{\delta\gamma} dx^\gamma \wedge dx^\delta - S^\rho_{\mu\nu} H^{\mu\nu}. \quad (1.3.57)$$

Which cancels the additional in (1.3.55). Thus we have that  $T_{\nabla_1} = 0$  for any (classically) torsion free connection  $\nabla_S$ .

## Quantum Metric

This gives a good candidate for a quantum Levi-Civita connection, all we need now is metric compatibility. So next we turn to the quantum metric itself, treated as an element  $g_1 \in \Omega^1(\mathcal{A}) \otimes_1 \Omega^1(\mathcal{A})$ . Recall that it ought to have quantum symmetry i.e.  $\wedge_1(g_1) = 0$  and be central in the algebra if it is to have a well defined inverse. First, consider

$$g_Q \equiv q_{\Omega^1, \Omega^1}^{-1}(g) = g_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} \omega^{\gamma\delta} (g_{\mu\kappa, \gamma} - \Gamma_{k\gamma\mu}) \Gamma^\kappa_{\delta\nu} dx^\mu \otimes_1 dx^\nu \quad (1.3.58)$$

the functorial quantum metric. While this satisfies  $\wedge_Q(g_Q) = 0$ , we still need an  $\mathcal{O}(\lambda)$  modification to compensate for the one in  $\wedge_1$ . It is relatively straightforward to see that  $\wedge_1(g_Q) = \lambda g_{ij} H^{ij}$ , so we define

$$\mathcal{R} \equiv g_{\mu\nu} H^{\mu\nu}, \quad g_1 \equiv g_Q - \lambda q_{\Omega^1, \Omega^1}^{-1}(\mathcal{R}) \quad (1.3.59)$$



where  $\mathcal{R}$  is called the *Ricci two-form* and is

$$\mathcal{R} = \frac{1}{4} g_{\alpha\beta} \omega^{\alpha\gamma} (\nabla_\gamma T^\beta_{\mu\nu} - R^\beta_{\mu\nu\gamma} + R^\beta_{\nu\mu\gamma}) dx^\nu \wedge dx^\mu. \quad (1.3.60)$$

This now gives the required  $\wedge_1(g_1) = 0$ . Centrality of  $g_1$  up to  $\mathcal{O}(\lambda^2)$  follows trivially from the fact that the Poisson connection is chosen to be metric compatible and  $\mathcal{O}(\lambda)$  modifications do not contribute to the commutators at this order, thus  $[f, g_1] = \lambda \omega^{\mu\nu} \partial_\mu f \nabla_\nu g$  is guaranteed to be 0 an an application of (1.3.47).

### Quantum Levi-Civita Connection

Now turning back to the connection, we already have star preservation and torsion free, so now just need compatibility with the metric defined above. First consider the functorial connection (1.3.40) given explicitly for  $E = \Omega^1(\mathcal{M})$  as

$$\nabla_Q(dx^\iota) = - \left( \Gamma^\iota_{\mu\nu} + \frac{\lambda}{2} \omega^{\alpha\beta} (\Gamma^\iota_{\mu\kappa, \alpha} \Gamma^\kappa_{\beta\nu} - \Gamma^\iota_{\kappa\tau} \Gamma^\kappa_{\alpha\mu} \Gamma^\tau_{\beta\nu} - \Gamma^\iota_{\beta\kappa} R^\kappa_{\nu\mu\alpha}) \right) dx^\mu \otimes_1 dx^\nu \quad (1.3.61)$$

and also

$$\sigma_Q(dx^\alpha \otimes_1 dx^\beta) = dx^\beta \otimes_1 dx^\alpha + \lambda \left( \omega^{\mu\nu} \Gamma^\alpha_{\mu\gamma} \Gamma^\beta_{\nu\delta} - \omega^{\mu\beta} R^\alpha_{\gamma\delta\mu} \right) dx^\delta \otimes_1 dx^\gamma. \quad (1.3.62)$$

Now,  $\nabla_{\Omega^1 \otimes_0 \Omega^1}(g) = 0$  implies that  $\nabla_{Q(\Omega^1 \otimes_0 \Omega^1)}(g) = 0$  due to (1.3.46). Then (1.3.43) gives that  $\nabla_{Q(\Omega^1 \otimes_0 \Omega^1)}(g_Q) = 0$ . This could also be calculated explicitly using (1.3.58), (1.3.61) and (1.3.62) by

$$(\nabla_Q \otimes \text{id})g + (\sigma_Q \otimes \text{id})(\text{id} \otimes \nabla_Q)(g_Q) = 0. \quad (1.3.63)$$

We take  $\widehat{\nabla}$  as the Levi-Civita connection to distinguish it from some general  $\nabla_S$ . The quantization condition on the metric still requires  $\nabla(g) = 0$ , but this needn't be torsion free allowing us to write  $\widehat{\nabla} = \nabla + S$  where the bimodule map  $S$  is now the *contorsion tensor*. Then, using  $\nabla(dx^\alpha) = -\Gamma^\alpha_{\mu\nu} dx^\mu \otimes_0 dx^\nu$  and  $S(dx^\alpha) = S^\alpha_{\mu\nu} dx^\mu \otimes_0 dx^\nu$ , we have for the components

$$T^\gamma_{\mu\nu} = \Gamma^\gamma_{\mu\nu} - \Gamma^\gamma_{\nu\mu}, \quad S^\gamma_{\mu\nu} = \frac{1}{2} g^{\gamma\kappa} (T_{\kappa\mu\nu} + T_{\mu\kappa\nu} + T_{\nu\kappa\mu}) \quad (1.3.64)$$

and letting  $\widehat{\Gamma}$  denote the Levi-Civita connection

$$\Gamma^\gamma_{\mu\nu} = \widehat{\Gamma}^\gamma_{\mu\nu} + S^\gamma_{\mu\nu}. \quad (1.3.65)$$

However, in the quantum case, we look not only for  $\nabla_{QS}$ , but rather the full  $\nabla_1$  as defined by (1.3.56) so as to have zero quantum torsion. Then, taking  $\nabla_1(g_1)$ , it turns out that the symmetric part vanishes and we are left with an antisymmetric metric compatibility tensor

$$(\text{id} \otimes \wedge) q^2 \nabla_1(g_1) = -\lambda \widehat{\nabla} \mathcal{R} - \lambda \omega^{\gamma\delta} S_{\rho\delta\nu} (R^\rho_{\mu\kappa\gamma} - S^\rho_{\kappa\mu;\gamma}) dx^\kappa \otimes_0 dx^\mu \wedge dx^\nu. \quad (1.3.66)$$

What is more, this cannot be eliminated by some choice of  $K$  in (1.3.56) and has to vanish independently. So a fully metric compatible, torsion free connection exists if and only if

$$\widehat{\nabla} \mathcal{R} + \omega^{\gamma\delta} S_{\rho\delta\nu} (R^\rho_{\mu\kappa\gamma} - S^\rho_{\kappa\mu;\gamma}) dx^\kappa \otimes_0 dx^\mu \wedge dx^\nu = 0. \quad (1.3.67)$$

This could be thought of as a sort of *quantum Koszul formula* and we refer to a connection satisfying it as the *quantum Levi-Civita connection*.

### 1.3.5 Summary

In section 2 of 4, the formalism is further developed with regard to the inverse quantum metric  $(, )_1$ . In particular, the quantum Laplace operator is derived to  $\mathcal{O}(\lambda)$  which has the form

$$\square_1 f = (, )_1 \nabla_1 df = \square f + \frac{\lambda}{2} \omega^{\alpha\beta} (\text{Ric}^\gamma_\alpha - S^\gamma_{;\alpha}) (\widehat{\nabla}_\beta df)_\gamma \quad (1.3.68)$$

where  $\text{Ric}^\gamma_\alpha = g^{\gamma\nu} R^\beta_{\nu\beta\alpha}$  is the Ricci tensor. Additionally, expressions for the quantum Riemann tensor

$$\text{Riem}_1 = (d \otimes_1 \text{id} - (\wedge_1 \otimes_1 \text{id})(\text{id} \otimes_1 \nabla_1)) \nabla_1 \quad (1.3.69)$$

and quantum Ricci tensor

$$\text{Ricci}_1 = ((, )_1 \otimes_1 \text{id} \otimes_1 \text{id})(\text{id} \otimes_1 i_1 \otimes_1 \text{id})(\text{id} \otimes_1 \text{Riem}_1)(g_1) \quad (1.3.70)$$

are derived. The results are applied to known quantum spacetimes: Bicrossproduct [36] and 2D Bertotti-Robinson [23] and matched with known results.

Section 4 looks at the application of the above to the case of spherically symmetric metrics, in particular it seeks to solve (1.3.31), (1.3.32) and (1.3.67) for a metric of the form

$$g = a^2(r, t) dt \otimes dt + b^2(r, t) dr \otimes dr + c^2(r, t) (d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi) \quad (1.3.71)$$

along with the associated Levi-Civita connection and Poisson tensor

$$\omega^{23} = \frac{f(t, r)}{\sin \theta} = -\omega^{32}, \quad \omega^{01} = g(t, r) = -\omega^{10} \quad (1.3.72)$$

It is found that the quantum algebra is uniquely given by

$$[z^i, z^j] = \lambda \epsilon^{ij}{}_k z^k, \quad [z^i, dz^j] = \lambda z^j \epsilon^i{}_{mn} z^m dz^n \quad (1.3.73)$$

where we used the angular coordinates  $z^i$ ,  $i = 1, 2, 3$  which relate back to Cartesian coordinates by  $z^i = \frac{x^i}{r}$  and satisfy  $\sum_i z^{i2} = 1$ . It is observed that this corresponds to the algebra of a nonassociative fuzzy sphere as shown in [37] and which is examined in the context of Poisson-Riemannian geometry in section 2.5. Furthermore, the algebra also corresponds to that of the flat FLRW metric examined separately in section 3. We also find that

$$[t, x^\mu] = [r, x^\mu] = 0, \quad [x^\mu, dt] = [x^\mu, dr] = 0 \quad (1.3.74)$$

so that  $t, r, dt, dr$  are central at order  $\lambda$ . This gives the quantum metric on the right hand side of (1.3.59) as

$$g_1 = g_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda c^2}{2(z^3)^2} \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j) \quad (1.3.75)$$

and corresponding quantum Levi-Civita connection

$$\nabla_1(dt) = -\hat{\Gamma}^0{}_{\mu\nu} dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{2(z^3)^2} \frac{c \partial_t c}{a^2} \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j)$$

$$\nabla_1(dr) = -\hat{\Gamma}^1{}_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2(z^3)^2} \frac{c \partial_r c}{b^2} \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j) \quad (1.3.76)$$

$$\nabla_1(dz^a) = -\hat{\Gamma}^i{}_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} \left( \epsilon_{ijk} z^k z^a dz^i \otimes_1 dz^j - \frac{1}{(z^3)^2} \epsilon^a{}_{i3} dz^3 \otimes_1 dz^i \right)$$

The result is then applied to various metrics: FLRW, Schwarzschild and Bertotti-Robinson which simply correspond to different parameter choices in (1.3.71). As is pointed out in the conclusion, it is unclear at present, how to draw physical conclusions from this analysis. For example, while the zeroth order part of (1.3.75) looks like the classical metric, it is over the quantum tensor product. For one, it fails to satisfy (1.3.5) and so expressions like  $g_{\mu\nu} dx^\mu \otimes_1 dx^\nu$  or  $dx^\mu \otimes_1 g_{\mu\nu} dx^\nu$  aren't equivalent, which indicates one ought to at

least tread with caution when trying to work with the components as usual. The physical relation between the quantum and classical tensor products is suggested as an area for further research as it is unknown at present. This difficulty could be avoided by considering scalar quantities, such as the Laplace operator. However, as shown in the paper, this turns out to have a vanishing  $\mathcal{O}(\lambda)$  contribution.

## 1.4 Outline

This completes the introduction. What follows are the three papers described. The layout is such that each paper represents a chapter and they appear here in their published form.

# Non-locality in quantum field theory due to general relativity

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Received: 21 October 2015 / Accepted: 6 December 2015 / Published online: 19 December 2015  
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**Abstract** We show that general relativity coupled to a quantum field theory generically leads to non-local effects in the matter sector. These non-local effects can be described by non-local higher dimensional operators which remarkably have an approximate shift symmetry. When applied to inflationary models, our results imply that small non-Gaussianities are a generic feature of models based on general relativity coupled to matter fields. However, these effects are too small to be observable in the cosmic microwave background.

## 1 Introduction

A century after the introduction of general relativity by Einstein, finding a quantum mechanical description of general relativity remains one of the holy grails of theoretical physics and one of the few unresolved problems in modern physics. At this stage of our understanding of nature, it is not clear whether the quantization of general relativity is so difficult because of technical issues, essentially having to deal with a dimensionful coupling constant which is the Planck mass or whether general relativity or quantum mechanics need to be modified at very short distances. Given the current state of the art, it is important to investigate general relativity and quantum mechanics in the energy region where we expect them to work, i.e., below the Planck mass  $M_P = 1/\sqrt{G_N}$ . The concept of effective field theory provides a very powerful framework to investigate the quantization of general relativity in this energy regime, i.e. below the Planck mass. Effective field theory methods are powerful tools to deal with quantum gravity below the Planck mass [1–7].

An important question is to identify the energy scale at which the effective theory might break down. A well-

established criterion is that of perturbative unitarity. Treating general relativity as an effective field theory, several groups have investigated the gravitational scattering of fields studying whether perturbative unitarity could be violated below the Planck scale [7–13]. It was shown in [9] that in linearized general relativity with a Minkowski background perturbative unitarity is restored by resumming an infinite series of matter loops on a graviton line in the large  $N$  limit, where  $N = N_s + 3N_f + 12N_V$  ( $N_s$ ,  $N_f$  and  $N_V$  are, respectively, the number of real scalar fields, fermions and spin 1 fields in the model), while keeping  $NG_N$  small. This large  $N$  resummation leads to a resummed graviton propagator given by

$$iD^{\alpha\beta,\mu\nu}(q^2) = \frac{i(L^{\alpha\mu}L^{\beta\nu} + L^{\alpha\nu}L^{\beta\mu} - L^{\alpha\beta}L^{\mu\nu})}{2q^2 \left(1 - \frac{NG_N q^2}{120\pi} \log\left(-\frac{q^2}{\mu^2}\right)\right)} \quad (1)$$

with  $L^{\mu\nu}(q) = \eta^{\mu\nu} - q^\mu q^\nu / q^2$ ,  $N = N_s + 3N_f + 12N_V$ . Note that in the standard model of particle physics  $N = 283$ . It is thus a large number which justifies our calculation. This resummation is valid for energies  $E < 1/\sqrt{G_N}$ . In the case, graviton loops are suppressed by factors of  $N$  compared to matter loops and we do not need to worry about quantum gravity corrections. A similar calculation has been done by the authors of [8] who have pointed out that the denominator of this resummed propagator has a pair of complex poles which lead to acausal effects (see also [14, 15] for earlier work in the same direction and where essentially the same conclusion was reached). These acausal effects should become appreciable at energies near  $(G_N N)^{-1/2}$ . Thus, unitarity is restored but at the price of non-causality. We shall see that causality can be restored as well by replacing the log term by an interpolating non-local function. However, this procedure does not remove the poles which can be interpreted as black hole precursors [16] and correspond to the energy scale at which strong gravitational effects become important and thus the energy scale at which the effective field theory treatment of general relativity should break down. Note that this scale depends on the number of fields in the theory. Studying

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quantum effects in general relativity in the large  $N$  limit is not new and has been considered as well by e.g. Smolin [17] and Tomboulis [14, 15].

Thought experiments based on general relativity and quantum mechanics [18–21] lead to the conclusion that distances smaller than the Planck length are not observable. These results can be interpreted as a form of non-locality around the Planck scale. The results obtained in [16] are thus not very surprising. The position of the poles define the energy scale at which the effective theory should break down and the non-local effects correspond to the minimal length expected around the mass scale of the first quantum black holes which are extended objects of the size of the inverse of the Planck mass.

The consequences of these non-local effects for the FLRW metric have been investigated in [22]. The aim of the paper is to derive an effective field theory for a scalar field, such as the inflaton, coupled to general relativity. We will show that this gives rise to some non-local effects in the interactions of this scalar field. These results only assume linearized general relativity and quantum field theory and are as such non-speculative.

## 2 Effective theory and non-locality

The tree-level gravitational scattering of two scalars has been considered already [23]. The invariant amplitude is given by

$$A_{\text{tree}} = 16\pi G \left( m^4 \left( \frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right) + \frac{1}{2s} (2m^2 + t)(2m^2 + u) + \frac{1}{2t} (2m^2 + s)(2m^2 + u) + \frac{1}{2u} (2m^2 + s)(2m^2 + t) \right) \quad (2)$$

with  $s = -(p_1 + q_1)^2 = (p_2 + q_2)^2$ ,  $t = -(p_1 - p_2)^2 = (q_1 - q_2)^2$  and  $u = -(p_1 - q_2)^2 = (p_2 - q_1)^2$ . Note that we are using the signature  $(+, -, -, -)$ . It is straightforward to calculate the dressed amplitude using the resummed graviton propagator (1). Let us rewrite

$$iD^{\alpha\beta,\mu\nu}(q^2) = \frac{P^{\alpha\beta,\mu\nu}(q^2)}{1 + f(q^2)}, \quad (3)$$

where  $P^{\alpha\beta,\mu\nu}(q^2)$  is the usual graviton propagator and where  $f(q^2)$  is given by

$$f(q^2) = -\frac{NG_N q^2}{120\pi} \log \left( -\frac{q^2}{\mu^2} \right). \quad (4)$$

The dressed amplitude is then given by

$$A_{\text{dressed}} = 16\pi G \left( m^4 \left( \frac{1}{s(1+f(s))} + \frac{1}{t(1+f(t))} + \frac{1}{u(1+f(u))} \right) \right)$$

$$+ \frac{1}{2s(1+f(s))} (2m^2 + t)(2m^2 + u) + \frac{1}{2t(1+f(t))} (2m^2 + s)(2m^2 + u) + \frac{1}{2u(1+f(u))} (2m^2 + s)(2m^2 + t). \quad (5)$$

We emphasize that this calculation is done in linearized general relativity with a Minkowski background treating general relativity as an effective theory that is valid at energies smaller than the Planck scale.

We can now Taylor expand this amplitude around the massive pole of the dressed propagator and obtain

$$A_{\text{dressed}} = A_{\text{tree}} + A^{(1)} + \dots \quad (6)$$

with

$$A^{(1)} = \frac{2}{15} G_N^2 N \left( m^4 \left( \log \left( -\frac{stu}{\mu^6} \right) \right) + \log \left( -\frac{s}{\mu^2} \right) (2m^2 + t)(2m^2 + u) + \log \left( -\frac{t}{\mu^2} \right) (2m^2 + s)(2m^2 + u) + \log \left( -\frac{u}{\mu^2} \right) (2m^2 + s)(2m^2 + t) \right). \quad (7)$$

It is easy to see that  $A^{(1)}$  can be obtained from the following non-local dimension 8 effective operator  $O_8$ :

$$O_8 = \frac{2}{15} G_N^2 N (\partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi(x)^2) \log \left( -\frac{\square}{\mu^2} \right) \times (\partial_\nu \phi(x) \partial^\nu \phi(x) - m^2 \phi(x)^2), \quad (8)$$

where  $\square = g^{\mu\nu} \partial_\mu \partial_\nu$ .

As mentioned before, the resummed graviton propagator (1) has a pair of complex poles which lead to acausal effects. We are expanding the effective action around these poles. The situation is very similar to that observed in [22]. The effective operator cannot be used to generate causal effects in the equations of motion as the Feynman propagators involve both advanced and retarded solutions. This is appropriate for scattering amplitudes but not for the equations of motion. There is a well-established procedure to ensure that the effective operator leads to causal effect at the level of the equations of motion as well. We follow the procedure outlined in [22, 24] to generate a causal action (see also [25–28] for earlier works in that direction). This requires a reinterpretation of the log-term which can be interpreted as an interpolating non-local function of the type  $\mathcal{L}(x, y)$ . We consider the following action:

$$S = \int d^4x \sqrt{-g} \times \left( \frac{1}{16\pi G_N} R(x) - \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{m^2}{2} \phi^2(x) \right)$$

$$+ \frac{2}{15} G_N^2 N \left( (\partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi(x)^2) \log \left( -\frac{\square}{\mu^2} \right) \right. \\ \left. \times \left( \partial_\nu \phi(x) \partial^\nu \phi(x) - m^2 \phi(x)^2 \right) \right), \quad (9)$$

where the log term is interpreted as an interpolating function,

$$S = \int d^4x \sqrt{-g} \\ \times \left( \frac{1}{16\pi G_N} R - \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{m^2}{2} \phi^2 + \left( \frac{2}{15} G_N^2 N \right) \right. \\ \times \left( \left( \partial_\mu \phi(x) \partial^\mu \phi(x) + m^2 \phi(x)^2 \right) \int d^4y \sqrt{-g(y)} \right. \\ \left. \times \left\langle x \left| \log \left( -\frac{\square}{\mu^2} \right) \right| y \right\rangle (\partial_\nu \phi(y) \partial^\nu \phi(y) - m^2 \phi(y)^2) \right) \Bigg). \quad (10)$$

Let us define the interpolating function by

$$\mathcal{L}(x, y) = \left\langle x \left| \log \left( -\frac{\square}{\mu^2} \right) \right| y \right\rangle. \quad (11)$$

The specific form for  $\mathcal{L}(x, y)$  depends on the system to which we want to apply this effective theory. It is straightforward to find a specific representation for a flat Minkowski background. One can use the following approximation valid for small  $\epsilon$ :  $\log(x) \approx -1/\epsilon + x^\epsilon/\epsilon$ . One then finds [24]

$$- \left\langle x \left| \frac{1}{\epsilon} \right| y \right\rangle + \left\langle x \left| \frac{(\square/\mu^2)^\epsilon}{\epsilon} \right| y \right\rangle \\ = -\frac{1}{\epsilon} \delta(x-y) + \frac{1}{\epsilon} \frac{2\pi^2}{\mu^{2\epsilon}} \int d^4k k^{2+2\epsilon} \frac{1}{|x-y|} J_1(k|x-y|) \\ \sim -\frac{1}{\epsilon} \delta(x-y) - \frac{8\pi^2}{\mu^{2\epsilon}} \frac{1}{|x-y|^{4+2\epsilon}}, \quad (12)$$

where  $J_1$  is the Bessel function. We see that  $\mathcal{L}(x, y)$  is a function of  $x - y$ . For a purely time dependent problem in curved space-time (i.e. in cosmology),  $\mathcal{L}(x, y)$  takes the form

$$\mathcal{L}(t, t') \\ = -2 \lim_{\epsilon \rightarrow 0} \left( \frac{\Theta(t-t'-\epsilon)}{t-t'} + \delta(t-t')(\log(\mu\epsilon) + \gamma) \right), \quad (13)$$

which is the appropriate form for the in-in formalism [22].

Before applying these results to cosmology, let us emphasize an important point. The amplitude in Eq. (7) is calculated in the approximation described above (i.e. large  $N$  while keeping  $NG_N$  small) in Minkowski space-time. We then proceed to identify the effective higher dimensional operator which can reproduce Eq. (7). We derive that the operator given in Eq. (8) when taking a Minkowski metric reproduces the amplitude given in Eq. (7). This enables us to calculate the coefficient of  $O_8$  by matching it with our perturbative calculation. At this stage the coefficient of  $O_8$  is uniquely

determined and we can also use  $O_8$  in curved space-time and hence for inflation calculations as well.

In this section, we have shown that the non-locality induced in the resummed graviton propagator leads to non-locality in the self-interactions of a scalar field coupled to graviton. The same would be true of any spin state as well. Non-locality is an intrinsic feature of a quantum mechanical description of general relativity as emphasized in the introduction. Note that remarkably, the higher dimensional scalar field operator obtained by integrating out the poles (quantum black holes) in the graviton propagator, are invariant under approximative shift symmetry ( $\phi \rightarrow \phi + c$ , where  $c$  is a constant) in the limit of the mass of the scalar field going to zero. The breaking of this shift symmetry is proportional to the mass of the scalar field. If we apply this construction to an inflation scenario as we shall do below, this implies that any contribution to the flatness of the potential will be suppressed by powers of  $m_s/M_P$  where  $m_s$  is the inflaton mass, which is of the order of  $10^9$  GeV. Quantum gravitational effects arising from quantum black holes are thus small and cannot affect the flatness of the potential. A potential for the scalar field may lead to breaking of the shift symmetry; however, one of our main points is that such a symmetry breaking will not be generated by quantum effects in general relativity if not introduced explicitly in the model. We shall now consider non-local effects due to the dimension 8 operator introduced in this section. We stress that this operator is an intrinsic feature of general relativity and scalar fields coupled to gravity.

### 3 Bounds from cosmic microwave background

We can now study the implications of this non-local effect, which is purely obtained by considering quantum field theory coupled to general relativity. These effects will be imprinted on the CMB as a deviation in the speed of sound. Focussing on  $O_8$ , we consider the  $x$ -dependent Lagrangian for an inflaton,

$$L(x) = X + \frac{m^2}{2} \phi^2(x) + \frac{8}{15} G_N^2 N \left( X(x) + \frac{m^2}{2} \phi^2(x) \right) \\ \times \int d^4y \sqrt{-g(y)} \mathcal{L}(x, y) \left( X(y) + \frac{m^2}{2} \phi^2(y) \right), \quad (14)$$

where we have introduced the standard notation  $X(x) = -1/2 \partial_\mu \phi(x) \partial^\mu \phi(x)$  and  $X(y) = -1/2 \partial_\mu \phi(y) \partial^\mu \phi(y)$ . Remarkably, our calculation does not depend on the specific form of  $\mathcal{L}(x, y)$ ; we shall merely require that  $\int d^4y \sqrt{-g(y)} \mathcal{L}(x, y) \delta(x-y) = 1$ . The speed of sound can be calculated using the standard procedure [29]; since  $L(x)$  is a polynomial in  $X$ , keeping in mind that  $dX(y)/dX(x) = \delta(y-x)$ , we find



$$c_s^2 = \frac{L(x),_{X(x)}}{L(x),_{X(x)} + 2X(x)L(x),_{X(x)X(x)}} \approx 1 - \frac{32}{15}X(x)G_N^2N, \quad (15)$$

which remarkably does not depend on the specific representation chosen for  $\mathcal{L}(x, y)$  to leading order in the  $\sqrt{G_N}$  expansion. Restricting ourselves to a spatially homogeneous scalar field, we get

$$c_s \approx 1 - \frac{8}{15}\phi^2G_N^2N \approx 1 - \frac{2}{15\pi}H^2\epsilon G_NN, \quad (16)$$

where  $H$  is the Hubble parameter,  $\epsilon = 1/(16\pi G_N)1/V^2(\partial V(\phi)/\partial\phi)^2$  is the slow roll parameter (the slow roll condition is that  $\epsilon \ll 1$ ) and where we have used the approximation  $X \approx XL_{,X}$ . Quantum effects in general relativity thus lead to a speed of sound which is not exactly one but close to it. This is a generic feature of general relativity coupled to matter. Small non-Gaussianities are expected to appear in models of inflation based on general relativity and quantum field theory even in inflationary models with just one scalar field. However, these effects are too small to be observable since the speed of sound would typically be close to unity.

It is worth mentioning that because our action is nearly local (the non-locality is only apparent at short distances and in the interactions of the scalar field with itself), the Lagrangian given in Eq. (14) can be quantized the usual way (as done in [29]), as the kinetic term for the scalar field has its usual local appearance.

Finally, we emphasize that, while  $O_8$  leads to the leading contribution to deviations in the speed of sound, graviton loop corrections to the scalar propagator may be present; they will be imprinted differently in the cosmic microwave background [30].

#### 4 Conclusion

In this paper we have shown that general relativity coupled to scalar fields naturally leads to non-local effects. This non-locality can be associated with the existence of black hole precursors or quantum black holes [16]. We have shown that the amount of non-locality is determined by the number of matter fields in the theory, since it determines the location of the poles in the resummed graviton propagator. General relativity induces non-local effects in the scalar field sector. These effects can be described in terms of an effective higher non-local dimensional operator which remarkably has an approximate shift symmetry. When applied to inflationary models, we have shown that these non-local effects lead to a small non-Gaussianities in models of inflation involving a scalar field and general relativity.

**Acknowledgments** We would like to thank Veronica Sanz for helpful discussions. This work is supported in part by the Science and Technology Facilities Council (Grant No. ST/J000477/1).

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# Inflation on a non-commutative space–time



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## ARTICLE INFO

### Article history:

Received 24 April 2015

Received in revised form 9 June 2015

Accepted 11 June 2015

Available online 16 June 2015

Editor: M. Trodden

## ABSTRACT

We study inflation on a non-commutative space–time within the framework of enveloping algebra approach which allows for a consistent formulation of general relativity and of the standard model of particle physics. We show that within this framework, the effects of the non-commutativity of spacetime are very subtle. The dominant effect comes from contributions to the process of structure formation. We describe the bound relevant to this class of non-commutative theories and derive the tightest bound to date of the value of the non-commutative scale within this framework. Assuming that inflation took place, we get a model independent bound on the scale of space–time non-commutativity of the order of 19 TeV.

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## 1. Introduction

The idea of space–time non-commutativity dates back to the early days of quantum field theory when it was hoped that it may help to make sense of UV divergences which are intrinsic to this framework [1,2]. With the advent of renormalization and the proof that physically relevant Yang–Mills theories were renormalizable, non-commutative gauge theories lost much of their appeal. However, there was a renewal of interest for such theories when they reappeared as a certain limit in string theory [3,4]. In [4], it was shown that the end points of open strings ending on a  $Dp$ -brane with a Neveu–Schwarz two form flux  $B$  background do not commute. String theory has an additional symmetry transformation known as T-duality, which relates geometric structures in different topologies. It naturally gives rise to non-commutative geometry. Independently of string theory, quantum gravity is likely to involve the notion of a minimal length, see e.g. [5,6], which could imply a non-commutativity of space–time at short distances. This may help to alleviate the problem of the non-renormalizability of perturbative quantum gravity.

There are different approaches to non-commutative geometry, which can be divided in roughly two classes. The first approach is due to Alain Connes. It is based on the notion of the spectral triple and has its origin in mathematical physics. The second approach indeed goes back to Moyal and Groenewold [1,2] and emphasizes

that space–time itself might be non-commutativity at short distance. The non-commutativity of space–time leads to issues with space–time and gauge symmetries. There are two distinct ways to deal with these issues. One is to take gauge fields to be as usual Lie algebra valued and to restrict the gauge symmetries which can be considered (see e.g. [4]). The other one is to take gauge fields in the enveloping algebra which enables one to consider any gauge group with any representation for the matter fields [7–11]. In this article, we will consider the latter approach and derive the tightest bound to date on the non-commutative scale within this approach.

We shall focus here on the simplest model of space–time non-commutativity which has been extensively studied and will consider non-commuting coordinates with a canonical structure

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (1)$$

where  $\theta^{\mu\nu}$  is a constant tensor of mass dimension  $-2$ .

Our aim is to investigate effects of space–time non-commutativity in the early universe. We thus have to select a framework which enables us to formulate both field theories and general relativity on a non-commutative space–time. While there are different approaches to space–time non-commutativity, there is only one which leads to the well-known standard model of particle physics and general relativity in the low energy regime. We shall thus use the enveloping algebra approach [7–11] which enables one to formulate any gauge theory including arbitrary representations for the gauge and matter fields on a non-commutative space–time. This approach has led to a consistent formulation of the standard model of particle physics on such a space–time [12]. Treating General Relativity as a gauge theory, one can also formulate General Relativity on a non-commutative space–time [13–15]. It turns

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out that one needs to limit general coordinate transformations to those which are volume preserving diffeomorphisms. This leads to unimodular gravity which is known to be, at least classically, equivalent to general relativity. Following the enveloping algebra approach has several benefits. First of all, it makes use of real symmetries which imply a conserved charge via Noether's theorem. Such theories have an exact space-time symmetry [16,17] which corresponds to Lorentz invariance in the limit of  $\theta^{\mu\nu} \rightarrow 0$ . The implication of this symmetry is that all the bounds on space-time non-commutativity are weak [18], typically of the order of a TeV [19,20].

Using this framework, we will consider inflation and the cosmic microwave background on a non-commutative space-time. There are many attempts to study inflation in the context of a non-commutative space-time [21–29],<sup>1</sup> but as far as we know this is the first study of early universe physics using the enveloping algebra approach which allows to study in details the effects of the non-commutativity of space-time on the metric. As an example we will consider chaotic inflation [30] on a non-commutative space-time and show that the effects of non-commutativity vanish both for the scalar field and for the metric. This is a rather surprising and interesting result since one might have expected that a preferred direction in space-time could lead to large effects in the slow roll parameters since inflation could have exponentially increased the original asymmetry in space-time. We then consider the effects of space-time non-commutativity on the CMB which are this time non-vanishing. This is not surprising as non-commutative gauge theories are a special case of non-local theories which are known to affect the CMB. We derive the tightest bound to date on the scale of space-time non-commutativity within this framework.

## 2. Theoretical framework

We consider here the algebra  $\hat{\mathcal{A}}$  of non-commutative space-time coordinates  $\{\hat{x}^\mu\}$  satisfying the canonical relation

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (2)$$

where  $\theta \in \Omega^2(TM)$  is a constant tensor and can be locally expressed as  $\theta = \theta^{\mu\nu} \partial_\mu \otimes \partial_\nu$  with  $\theta^{\mu\nu} = -\theta^{\nu\mu}$ . As usual, we want to represent functions in  $\hat{\mathcal{A}}$  as elements in the space of linear complex functions  $\mathcal{F}$ . To do so we introduce the Moyal star product

$$\begin{aligned} (f_1 \cdot f_2)(\hat{x}) &= (f_1 \star f_2)(x) \\ &= \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1 \nu_1} \dots \theta^{\mu_n \nu_n} \partial_{\mu_1} \dots \partial_{\mu_n} f_1 \partial_{\nu_1} \dots \partial_{\nu_n} f_2. \end{aligned} \quad (3)$$

Before continuing on to the main discussion, it will be useful to note some useful properties of the star product. Firstly, under complex conjugation one has

$$(f_1 \star f_2)^* = f_2^* \star f_1^*. \quad (4)$$

Secondly, the trace property under integration implies that

$$\int d^4x (f_1 \star f_2)(x) = \int d^4x (f_1 \cdot f_2)(x) \quad (5)$$

and more generally, one also has the cyclicity property

<sup>1</sup> These previous studies have mainly focussed on a non-commutative inflaton without considering non-commutative effects in the gravity sector. They have obtained bounds of the order of 10 TeV.

$$\begin{aligned} &\int d^4x (f_1 \star \dots \star f_n)(x) \\ &= \int d^4x (f_1 \star \dots \star f_{m-1}) \cdot (f_m \star \dots \star f_n)(x) \\ &= \int d^4x (f_m \star \dots \star f_n) \cdot (f_1 \star \dots \star f_{m-1})(x). \end{aligned} \quad (6)$$

It is important to note, given that  $\theta$  is constant, that this theory violates general diffeomorphism invariance. However, as shown in [13] we may recover a reduced group of diffeomorphisms compatible with (2) parametrized by

$$\hat{x}'^\mu = \hat{x}^\mu + \hat{\xi}^\mu. \quad (7)$$

A subset of these transformations given by

$$\hat{\xi}^\mu = \theta^{\mu\nu} \partial_\nu \hat{f}(\hat{x}) \quad (8)$$

leaves  $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$  invariant. We shall thus only consider such transformations. Note that the Jacobian of these transformations is equal to one. The transformations which preserve the non-commutative algebra correspond to the reduced group of diffeomorphisms which are volume preserving. In other words, on a non-commutative space-time, we are forced to consider unimodular gravity. This is the main difference between our work and previous attempts at formulating inflation on a non-commutative space-time [22,25,26,31]. The approach to general relativity on a non-commutative space-time formulated in [13] relies on gauging a local  $SO(3, 1)$  (the tetrad approach). The local  $SO(3, 1)$  gauge symmetry is implemented using the enveloping algebra approach. This means that the gauge fields are assumed to be in the enveloping algebra instead of the usual Lie algebra. The local gauge invariance is enforced using the Seiberg–Witten maps order by order in  $\theta$  [13]. We now have all the tools needed to formulate a consistent scalar field action in a curved space-time on a non-commutative space-time.

## 3. Non-commutative scalar action

We consider inflation driven a single scalar field with a potential  $V(\phi^n)$  and denote for convenience  $\phi \equiv \phi(x)$ . In the commutative case, the action may be written

$$S = \int d^4x e \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \sum_n c_n \frac{\phi^n}{\Lambda^{(n-4)}} \right), \quad (9)$$

where  $e$  is the tetrad determinant,  $\Lambda$  is an energy scale and  $c_n$  are dimensionless Wilson coefficients of order unity. The choice of this frame follows from the derivation of non-commutative general relativity from the Seiberg–Witten map, as in [13,14], for which gravity is treated as a gauge theory. Another reason is that when mapping quantities on to a non-commutative space, it is very difficult to do so for a square root (which may not even exist in  $\mathcal{A}_\theta$ ) and  $e$  is used as an effective way to represent  $\sqrt{g}$ . Setting the tetrad determinant to one, the action for the non-commutative scalar field may be written

$$\begin{aligned} S &= \int d^4x \left( \frac{1}{2} G^{\mu\nu} \star \partial_\mu \phi \star \partial_\nu \phi \right. \\ &\quad \left. - \frac{1}{2} m^2 \phi \star \phi - \sum_n c_n \frac{\phi^{n\star}}{\Lambda^{(n-4)}} \right). \end{aligned} \quad (10)$$

One might be tempted to take  $\partial_\mu \phi \star \partial^\mu \phi$  and use (5) to eliminate the star product, as is done with, e.g., the mass term.

However, this is not possible here because of the space-time dependent metric. Let us add a quick comment on our conventions here: when defining a derivative operator in  $\mathcal{A}_\theta$ , one naturally has a map  $\partial_\mu^* : \mathcal{A}_\theta \rightarrow \mathcal{A}_\theta$  whose (left) action is defined to be  $\partial_\mu^* f \equiv \partial_\mu f$  with  $f \in \mathcal{A}_\theta$ . However, the same definition does not hold for  $\partial^{*\mu}$ , which, in general, is a power series in  $\theta$  and a higher order differential operator. It may be obtained from the relation  $\partial^{*\mu} = G^{\mu\nu} \star \partial_\nu$ . For the non-commutative metric, the condition  $G^{\mu\nu}|_{\theta=0} = g^{\mu\nu}$  applies. Furthermore, we are free to choose a frame where  $G^{\mu\nu} = g^{\mu\nu}$  but we must keep in mind that  $G^{\mu\nu} \star G_{\mu\alpha} \neq \delta^\mu_\alpha$ . We thus require a ‘star inverse’ to be defined such that  $G^{\mu\nu} \star G_{\star\mu\alpha} = \delta^\mu_\alpha$ , see for example [32]. It will, however, not be necessary for the analysis presented here. Indeed making this choice for the metric leads to a significant simplification. It is unnecessary to find an expansion of  $G^{\mu\nu}$  in terms of  $\theta$ .

We now expand out the star products in the action, mapping the non-commutative theory to a commutative space-time. Note that, as we have just explained, the Seiberg–Witten map for the metric is trivial. For the kinetic term we find

$$g^{\mu\nu} \star \partial_\mu \phi \star \partial_\nu \phi = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{8} g^{\mu\nu} \theta^{\alpha\beta} \theta^{\gamma\delta} \partial_\alpha \partial_\gamma \partial_\mu \phi \partial_\beta \partial_\delta \partial_\nu \phi + \mathcal{O}(\theta^4) \quad (11)$$

and for the potential we get

$$\phi^{n\star} = \phi^n - \frac{1}{8} \theta^{\mu\nu} \theta^{\alpha\beta} \sum_{m=1}^{n-2} \phi^{n-m-1} \partial_\nu \partial_\beta \phi^m \partial_\mu \partial_\alpha \phi + \mathcal{O}(\theta^3), \quad (12)$$

where (6) has been used. We see non-commutative corrections appear only at second order in  $\theta$  meaning that any effects are going to be strongly suppressed. It is worth noting the appearance of corrections in the kinetic term. This feature is absent in [25,26,31]. In a flat space-time, one may use the cyclicity of the star product to cancel corrections to quadratic terms such as these. However, since we are dealing with a curved space-time, we cannot do this here.

We now have all the tools to consider inflation on a non-commutative space-time using the enveloping algebra approach. Here is the set of assumptions we are making. Firstly, the inflaton field is taken to be homogenous i.e.  $\phi \equiv \phi(t)$ . Secondly, we assume that the same is true of the metric for a spatially flat, FLRW like cosmology: We know that there are second order in  $\theta$  corrections to Einstein’s equations as shown in [13], but these corrections vanish for a metric which is purely time-dependent. Feeding these assumptions into the above equations, one quickly sees that, owing to the antisymmetry of  $\theta^{\mu\nu}$ , all but the zeroth order terms vanish. We thus see that the inflation does not feel the non-commutativity of space-time. In particular the slow roll parameters are given by their usual commutative expression

$$\epsilon = \frac{M_P^2}{2} \left( \frac{1}{V_0} \frac{\partial V_0}{\partial \phi} \right)^2 \quad \eta = \frac{M_P^2}{V_0} \left| \frac{\partial^2 V_0}{\partial \phi^2} \right|. \quad (13)$$

This result is somewhat surprising; intuitively, one would expect that the presence of a preferred direction in space-time would result in anisotropic contributions to the metric at some order in  $\theta$ . However, the nature of the corrections is such that they vanish to all orders conserving the initial isotropy. It is interesting to consider this against cosmological paradigms, such as the flatness problem, which are generally amplified throughout time. As usual, space-time non-commutative effects are very elusive [18]!

#### 4. CMB corrections

While, within our framework, there are no effects of space-time non-commutativity on the slow roll parameters, we now show

that there are interesting observable effects on the CMB. A homogenous field may not have any corrections, but the same is not necessarily true of perturbations to that field.

$$\phi(t, \mathbf{x}) = \phi(t) + \delta\phi(t, \mathbf{x}). \quad (14)$$

While the overall evolution of the universe may be unaffected, space-time non-commutativity could have some influence on structure formation. We thus need to consider non-commutative corrections to inflaton perturbations. It is well known that general relativity and unimodular gravity, at least in the classical regime, are equivalent [33]. It is generally possible to find a subset of spacetime where we can write Einstein’s equations such that  $\det(g_{\mu\nu}) = 1$ . This implies that the predictions for inflation in unimodular gravity are the same as in the full general relativity framework on a classical space-time. This has been explicitly shown in [34]. As emphasized already, our approach to general relativity formulated on a non-commutative space-time forces us to consider unimodular gravity. However, the work in [34] implies that the details of the analysis of small perturbations do not depend on whether the underlying theory of gravity is general relativity or unimodular gravity. The analysis performed in [27–29] where statistical anisotropies of the CMB were studied without paying attention to non-commutative corrections to general relativity thus applies to the enveloping algebra. However, our framework enables us to justify the assumption that non-commutative corrections to metric can be neglected. Indeed, using the results presented in [13,14], it is straightforward to see that for a purely time-dependent metric such as the FLRW metric, the non-commutative corrections to the classical metric vanish to all orders in  $\theta$ .

The calculation of the  $n$ -point correlators takes place at the level of the equations of motion. This calculation will be unaltered for a unimodular metric and will thus apply to the enveloping algebra approach considered here. We can thus follow the technique developed in [27–29]. We first consider the contributions from non-commutativity to the power spectrum of the CMB which are obtained from calculating the two-point correlation function for scalar perturbations. For a co-moving (commutative) scalar field  $\tilde{\zeta}(\eta, \mathbf{k})$ , where the tilde indicates that this is a Fourier mode of  $\zeta(\eta, \mathbf{x})$ , the power spectrum in terms of the two-point function is

$$\langle 0 | \tilde{\zeta}^\dagger(\eta, \mathbf{k}) \tilde{\zeta}(\eta, \mathbf{k}') | 0 \rangle = (2\pi)^3 P_{\theta=0}(\eta, \mathbf{k}) \delta^3(\mathbf{k} - \mathbf{k}'). \quad (15)$$

We have defined the conformal time

$$d\eta = \frac{1}{a(t)} dt, \quad (16)$$

where  $a(t)$  is the cosmological scale factor. At the time horizon crossing  $\eta_0$ , this is given by

$$P_0(\eta, \mathbf{k}) = \frac{16\pi}{9E} \frac{H^2}{2k^3} \Big|_{a(\eta_0)H=k}. \quad (17)$$

It was found in [29], that this is modified by non-commutativity and that a revised expression for the power spectrum can be derived.

$$P_\theta(\eta, \mathbf{k}) = P_0(\eta, \mathbf{k}) \cosh(H\theta^{0a} k_a), \quad (18)$$

where  $H$  is the Hubble parameter. Expanding to leading order gives

$$P_\theta(\eta, \mathbf{k}) = P_0(\eta, \mathbf{k}) + \frac{H^2}{2!} P_0(\eta, \mathbf{k}) \theta^{0a} \theta^{0b} k_b k_a. \quad (19)$$

Again, we note the absence of first order contributions in the non-commutative parameter. Also interesting to note is that a more

general treatment of rotational invariance violation in [35] gives a similar result. We now see that non-commutativity indeed has an effect that may be measured by CMB experiments.

By doing so, bounds on the scale of space–time non-commutativity have been derived in [29] using WMAP, ACBAR and CBI and in [28] using PLANCK data. In [28] the authors found a bound of 19 TeV on the scale of spacetime non-commutativity. Since the same derivation goes through in our formalism as well, this leads to the tightest bound to date on the energy scale of spacetime non-commutativity within the framework of the enveloping algebra approach.

## 5. Conclusion

We have considered corrections induced by non-commutativity on a scalar inflaton field. Specifically, we consider the enveloping algebra approach to space–time non-commutativity with a constant non-commutative parameter. In this approach the reduced group of diffeomorphisms, chosen so as to leave (2) invariant, leads to unimodular gravity. By replacing conventional multiplication by the Moyal star product and expanding in terms of the non-commutative parameter  $\theta$  as well as using the Seiberg–Witten maps for the local  $SO(3, 1)$  gravitational theory, it was shown that no corrections enter the inflationary action. This leads us to the somewhat surprising realization that even in the presence of a preferred direction in space–time, it does not affect the overall evolution of the universe. Instead, one must examine primordial perturbations to the inflaton field which do experience the non-commutativity and look for their imprints in the CMB. Owing to the classical equivalence of general relativity and unimodular gravity, the analysis necessary for doing so is the same in both cases. We can derive the bound  $\sqrt{\theta}$  to  $\sim 19$  TeV within our approach to space–time non-commutativity. This is the tightest limit to date on the scale of space–time non-commutativity within the enveloping algebra approach to space–time non-commutativity.

## Acknowledgement

This work is supported in part by the Science and Technology Facilities Council (grant number ST/J000477/1).

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PAPER

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To cite this article: Christopher Fritz and Shahn Majid 2017 *Class. Quantum Grav.* **34** 135013

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# Noncommutative spherically symmetric spacetimes at semiclassical order

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Received 5 January 2017, revised 21 April 2017

Accepted for publication 12 May 2017

Published 14 June 2017



## Abstract

Working within the recent formalism of Poisson–Riemannian geometry, we completely solve the case of generic spherically symmetric metric and spherically symmetric Poisson-bracket to find a unique answer for the quantum differential calculus, quantum metric and quantum Levi-Civita connection at semiclassical order  $O(\lambda)$ . Here  $\lambda$  is the deformation parameter, plausibly the Planck scale. We find that  $r, t, dr, dt$  are all forced to be central, i.e. undeformed at order  $\lambda$ , while for each value of  $r, t$  we are forced to have a fuzzy sphere of radius  $r$  with a unique differential calculus which is necessarily nonassociative at order  $\lambda^2$ . We give the spherically symmetric quantisation of the FLRW cosmology in detail and also recover a previous analysis for the Schwarzschild black hole, now showing that the quantum Ricci tensor for the latter vanishes at order  $\lambda$ . The quantum Laplace–Beltrami operator for spherically symmetric models turns out to be undeformed at order  $\lambda$  while more generally in Poisson–Riemannian geometry we show that it deforms to

$$\square f + \frac{\lambda}{2} \omega^{\alpha\beta} (\text{Ric}^\gamma{}_\alpha - S^\gamma{}_{;\alpha}) (\widehat{\nabla}_\beta df)_\gamma + O(\lambda^2)$$

in terms of the classical Levi-Civita connection  $\widehat{\nabla}$ , the contorsion tensor  $S$ , the Poisson-bivector  $\omega$  and the Ricci curvature of the Poisson-connection that controls the quantum differential structure. The Majid–Ruegg spacetime  $[x, t] = \lambda x$  with its standard calculus and unique quantum metric provides an example with nontrivial correction to the Laplacian at order  $\lambda$ .

**Keywords:** noncommutative geometry, quantum gravity, Poisson geometry, semiclassical limit, quantum cosmology

(Some figures may appear in colour only in the online journal)

<sup>3</sup>This work is supported in part by the Science and Technology Facilities Council (grant number ST/J000477/1)

## 1. Introduction

In recent years it has come to be fairly widely accepted that quantum gravity effects could render spacetime better modelled as a ‘quantum’ geometry than a classical one, with coordinates  $x^\mu$  now generating a noncommutative coordinate algebra. We refer to [12, 14, 21, 24, 26] for some early works, as well as [34] from the 1940s although this did not propose a closed coordinate algebra as such. A further ingredient to such a *quantum spacetime hypothesis* was to include differential forms  $dx^\mu$  such as in [1, 31, 33], while in recent years one also has quantum metrics and quantum bimodule connections within a systematic framework of ‘noncommutative Riemannian geometry’ [9, 16, 28, 29]. The latter links to spectral triples or ‘Dirac operators’ in the general approach to noncommutative geometry of Connes [13] as well as to quantum group frame bundles in 2 + 1 quantum gravity [27]. It may also relate to other ideas for ‘quantum geometry’ from spin foams and loop quantum cosmology, see for example [4, 5, 18].

In the present paper we continue recent work [10] which explores the content of such noncommutative Riemannian geometry at the Poisson level of first order in a deformation or ‘quantisation’ parameter  $\lambda$ . This is obviously useful to understand issues at order  $\lambda$  before attempting the full theory, but it also turns out to be surprisingly rich with compatibility conditions between the Poisson bivector  $\omega^{\mu\nu}$  that controls the quantum spacetime relations  $[x^\mu, x^\nu] = \lambda\omega^{\mu\nu} + O(\lambda^2)$  and the classical Riemannian metric  $g_{\mu\nu}$  that we also want to quantise. This emergence of a well-defined order  $\lambda$  ‘Poisson–Riemannian geometry’ in [10] implies a specific paradigm of physics governing first order corrections and coming out of the quantum spacetime hypothesis in much the same way as classical mechanics emerges from quantum mechanics at first order in  $\hbar$ . In our case  $\lambda$  is plausibly the Planck scale so, although this is a Poisson-level theory, it includes quantum gravity effects and could be called ‘semi-quantum gravity’ [10] or ‘classical-quantum gravity’.

The key further ingredient in this theory is a type of connection  $\nabla$  which controls commutators such as

$$[x^\mu, dx^\nu] = -\lambda\omega^{\mu\rho}\Gamma_{\rho\sigma}^\nu dx^\sigma + O(\lambda^2)$$

where  $\Gamma$  are the Christoffel symbols of  $\nabla$ . It is only the combination  $\omega^{\mu\rho}\Gamma_{\rho\sigma}^\nu$  which we actually need here and which can be seen as the structure constants of a Lie–Rinehart or ‘contravariant’ connection known to be relevant to quantising vector bundles [6, 11, 17, 20, 22]. One can also think of these as covariant derivatives partially defined just along hamiltonian vector fields. In our case we follow [10] and suppose a full ordinary covariant derivative  $\nabla$  of which only the hamiltonian vector field directions are relevant to the commutation relations. This is physically reasonable given that covariant derivatives already arise extensively in General Relativity but does mean that our covariant derivatives have extra directions that do not play an immediate role for the quantisation (but which could couple to physical fields later on). The field equations for this connection  $\nabla$  are [10]:

- (1) Poisson compatibility  $\nabla_\gamma\omega^{\alpha\beta} + T^\alpha_{\delta\gamma}\omega^{\delta\beta} + T^\beta_{\delta\gamma}\omega^{\alpha\delta} = 0$  where  $T$  is the torsion of  $\nabla$ ;
- (2) Metric compatibility  $\nabla_\gamma g_{\mu\nu} = 0$ ;
- (3) A condition on the curvature and torsion of  $\nabla$  (see (2.14) in section 2).

It was shown in [10] that (1) allows for the entire classical exterior algebra to quantise uniquely at lowest order, now with a quantum wedge product  $\wedge_1$ ; (2) allows for the metric similarly to quantise to a quantum metric  $g_1$  and (3) for the classical Levi-Civita connection  $\hat{\nabla}$  to quantise to a quantum Levi-Civita  $\nabla_1$ . In fact the formulae for  $\nabla_1$  in [10] give a



unique ‘closest to quantum Levi-Civita’ connection at order  $\lambda$  even when (3) does not hold but in that case  $\nabla_{1g_1}$  has an order  $\lambda$  correction. Our first main goal of the present work is to describe these results more explicitly using tensor calculus methods as in classical General Relativity (starting with lemma 2.2) and also to extend them to cover the quantum Laplacian and quantum Ricci tensor in theorem 2.3 and section 2.2. This takes considerable work and occupies our ‘formalism’ section 2.

We believe that these Poisson–Riemannian field equations deserve further study as an extension of classical General Relativity. In this respect our second main goal is a full analysis of their content in the spherically symmetric case. This includes the example of the Schwarzschild black hole already covered in [10] but now taken further and also, which is new, the FLRW or big-bang cosmological model. In our class of quantisations we assume that both the metric and the Poisson tensor are spherically symmetric and find generically that  $t$  must be central. The radius variable  $r$  and the differentials  $dt, dr$  are then also central as an outcome of our analysis. This means that the only quantisation that can take place is on the spheres at each fixed  $r, t$  and we find that these are necessarily the ‘nonassociative fuzzy sphere’ quantisation of  $S^2$  and calculus at order  $\lambda$  obtained in [7] as a cochain twist and later in [10] within Poisson–Riemannian geometry. This result is both positive and negative. It is positive because our analysis says that this simple form of quantisation is unique under our assumptions at order  $\lambda$ , it is negative because it is hard to extract physical predictions in this model and we show in particular that more obvious sources such as corrections to the quantum scalar curvature and quantum Laplace–Beltrami operator vanish at order  $\lambda$ , in line with cochain twist as a kind of ‘gauge transformation’. We do still have changes to the form of the quantum metric (and quantum Ricci tensor) and more subtle effects such as nonassociativity of the differential calculus at order  $\lambda^2$ .

To explain this latter point in more detail, one can see [6, 10] that the Jacobi identity in the form  $0 = [x^\mu, [dx^\nu, x^\rho]] + \text{cyclic}$  at order  $\lambda^2$  amounts to vanishing of the curvature of  $\nabla$  after contraction with  $\omega$ . Thus, usual associative noncommutative geometry [13] where the quantum differential forms define a differential graded algebra corresponds essentially to  $\nabla$  a flat connection (this being precisely true in the symplectic case). In general, the existence of a flat connection respecting a Lie symmetry can have a topological obstruction (it is governed by the relevant Atiyah class) and this goes some way towards understanding why some noncommutative algebras [9, 28] admit few covariant noncommutative geometries. At the semiclassical level we can see this as fixing  $\omega$  and finding only very restricted solutions for  $\nabla, g$  in the presence of symmetry. Our new result in theorem 4.1 is a similar rigidity where we fix  $g$  and find no flat  $\nabla$  and  $\omega$  with rotational symmetry. We are not limited to flat  $\nabla$  in Poisson–Riemannian geometry as the nonassociativity shows up at order  $\lambda^2$  not order  $\lambda$  and indeed from a General Relativity point of view if assuming a flat connection is too restrictive then it is reasonable to accept that we need a curved one. It is also worth remembering that noncommutative geometry was only meant to be an effective description and  $\lambda$  is so small that  $\lambda^2$  is not relevant in practice away from singular situations that blow up its effective value. Therefore we have no real evidence that the world is in fact ‘flat’ in this respect. It is therefore one of the notable outcomes of our analysis that spherical symmetry generically requires such nonassociativity of differentials at order  $\lambda^2$ .

It is worth noting that a primary reason for wanting associative algebras is a practical one that these are much easier and more familiar to work with. In modern thinking, however, there is a class of *quasiassociative algebras*, shown in the 1990s to include the octonions, where the breakdown of associativity is nevertheless strictly controlled by a certain 3-cocycle ‘associator’. In formal terms the algebra is associative in a monoidal category, where a coherence

theorem of Mac Lane [23] says that one can work *as if* the algebra is strictly associative; one can put in brackets as needed for compositions to make sense (this involves inserting the 3-cocycle) and different ways to do this give the same final result. Such categories are familiar in topological quantum field theory and in quantum group theory, where they are induced by a ‘Drinfeld cochain twist’ [15] of an underlying symmetry. The quantum group  $\mathbb{C}_q[SU_2]$ , for example, has no bicovariant differential calculus quantising the classical one but does have a nonassociative one where the exterior algebra is a super co-quasi-quantum group [6]. Our nonassociative fuzzy spheres have similarly been conjectured in [7] to extend to all orders in  $\lambda$  as quasiassociative cochain twists for a certain action of the Lorentz group. This is recalled in section 2.5. In general there is a considerable amount of current interest in nonassociative twists in various contexts [3, 8, 30] including in relation to contravariant connections [2].

Finally on this topic, although not exactly the same as far as we know, there is a similarity here with quantum anomalies in physics where symmetries do not survive quantisation due to curvature obstructions. In that context it is sometime possible to cancel anomalies by introducing extra dimensions and in quantum group examples one can often do something similar (thus  $\mathbb{C}_q[SU_2]$  does have a bicovariant associative differential calculus [35], but it is 4-dimensional). It is not known if we can do the same for the nonassociative fuzzy sphere to make it associative or by implication for spherically symmetric mildly nonassociative spacetimes in our analysis, but if so it may link up with the associative noncommutative Schwarzschild black hole with a 5-dimensional differential calculus in [25]. This is outside our current scope since it leaves our Poisson–Riemannian deformation theory setting, but could provide an alternative extra dimensions ‘consequence’ of our analysis.

Other possible effects include the form of the quantum metric  $g_1$  and its inverse  $(\cdot)_1$ . Here a natural way to write its coefficients is as  $g_1 = dx^\mu \bullet \tilde{g}_{\mu\nu} \otimes_1 dx^\nu$  where  $\bullet$  is the quantum product, which is arranged so that  $\tilde{g}_{\mu\nu}$  is inverse to the equally natural matrix  $\tilde{g}^{\mu\nu} = (dx^\mu, dx^\nu)_1$ . Then we find in the general analysis that

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{\lambda}{2} h_{\mu\nu}$$

at order  $\lambda$  where  $h_{\mu\nu}$  is a certain antisymmetric tensor (or 2-form) built from the classical data in (2.12). The physical interpretation of this is not clear but *if* we suppose that the  $\tilde{g}_{\mu\nu}$  are the observed ‘effective metric’ then we see that this acquires an anti-symmetric or spin 1 component, making contact with other scenarios where non-symmetric metrics have been studied. On the other hand,  $h_{\mu\nu}$  is not tensorial i.e. transforms in a more complicated way if we change coordinates, albeit in such a way that when proper account is taken of the quantum tensor product  $\otimes_1$ , our constructions themselves are coordinate invariant. We look at this closely on one of the models in section 4.3. The same applies for the quantum Ricci tensor. Theorem 2.3 also shows that the quantum Laplacian  $\square_1 = (\cdot)_1 \nabla_1 d$  gets generically an order  $\lambda$  correction given by the Ricci curvature of  $\nabla$  and the covariant derivative of the contorsion tensor of  $\nabla$ . In both cases Poisson–Riemannian geometry leads in principle to calculable effects due to our standing assumption that quantum fields are identified with classical ones just with modified operations. The precise *physical* significance of these effects, however, is a much more involved question due to the necessity of working on a curved background, but frequency dependence of the speed of light and of gravitational redshift could both be expected features based on limited ad-hoc experience from other models [1, 25]. The difference is that Poisson–Riemannian geometry now offers the possibility of a systematic geometric treatment of such phenomena as an important direction for further work.

A third main goal of the paper is the detailed computation of some examples so as to explore some of the theory and issues above. Here the 2-dimensional Majid–Ruegg spacetime  $[x, t] = \lambda x$  is explored at the Poisson level in section 2.3 and has the merit that its full noncommutative geometry is known already by algebraic means [9] (our results are reconciled with that work in the appendix). The classical metric

$$g = dx^2 + bv^2; \quad v = xdt - tdx$$

describes either a strong gravitational source with a Ricci singularity at  $x = 0$  or an expanding universe, depending on the sign of the parameter  $b$ . We find now that there are order  $\lambda$  corrections in the quantum wave equation, with plane waves at first order now provided by Kummer  $M$  and  $U$  functions. One of the surprising outcomes of the paper is that such cases are relatively rare and for example in section 2.4 we find no order  $\lambda$  correction to the quantum Laplacian for the Bertotti–Robinson metric quantisation of [28] on the same coordinate algebra (this has the same  $\omega$  but a different  $\nabla, g$ ). The other model that we look at in particular detail, in section 3, is the rotationally invariant quantisation of the classical spatially-flat FLRW metric

$$g = -dt^2 + a^2(t) \sum_i (dx^i)^2.$$

We find that everything works in the sense that, as for the black hole, there do exist  $\omega, \nabla$  solving our field equations. Here  $\nabla$  pulls back to a unique contravariant connection, so there is a unique noncommutative geometry at order  $\lambda$ . We further find that  $h_{\mu\nu} = 0$  when computed in this section so that the quantum metric, and also the quantum Levi-Civita connection, look remarkably undeformed at order  $\lambda$ . This model is a warm up to the general analysis but because it is done in Cartesian and not the polar coordinates used in section 4, it provides a good illustration of the subtle issues concerning changes of coordinates as reconciled in section 4.3. The paper ends with some further discussion in section 5.

## 2. Formalism

Throughout this paper by ‘quantum’ we mean extended to noncommutative geometry to order  $\lambda$ . There is a physical assumption that quantities will extend further to all orders according to axioms yet to be determined but we do not consider the details of that yet (the idea is to proceed order by order strictly as necessary). This is for convenience and one could more precisely say ‘semiquantum’ as in [10]. We use  $\bullet$  for the deformed product and  $\nabla$  for the covariant derivative with respect to the Poisson compatible ‘quantising’ connection  $\nabla$ . This usually has torsion and should not be confused with the Levi-Civita connection.

### 2.1. Poisson–Riemannian geometry and the quantum Laplacian

We start with a short recap of results we need from [10] but in a more explicit tensorial form, along with some new general results in the same spirit. We let  $M$  be a smooth manifold with exterior algebra  $\Omega$ , equipped with a metric tensor  $g$  and torsion free metric compatible Levi-Civita connection  $\widehat{\nabla}$  on  $\Omega^1$ , with Christoffel symbols  $\widehat{\Gamma}$ . We let  $\nabla : \Omega^1 \rightarrow \Omega^1 \otimes \Omega^1$  be another connection on  $\Omega^1$  with similarly defined ‘Christoffel symbols’  $\Gamma$ , so that

$$\widehat{\nabla}_\beta dx^\alpha = -\widehat{\Gamma}^\alpha_{\beta\gamma} dx^\gamma, \quad \nabla_\beta dx^\alpha = -\Gamma^\alpha_{\beta\gamma} dx^\gamma$$

respectively for the two connections. The tensor product here is over  $C^\infty(M)$  and we view the covariant derivative abstractly as a map or in practice with its first output

against  $\partial/\partial x^\beta$  to define  $\nabla_\beta$ . Its action on the component tensor of a 1-form  $\eta = \eta_\alpha dx^\alpha$  is  $\nabla_\beta \eta_\alpha = (\nabla_\beta \eta)_\alpha = \partial_\beta \eta_\alpha - \eta_\gamma \Gamma^\gamma_{\beta\alpha}$ , which fixes its extension to other tensors. The torsion and curvature tensors of  $\nabla$  are

$$T^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}, \quad R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\delta\beta,\gamma} - \Gamma^\alpha_{\gamma\beta,\delta} + \Gamma^\kappa_{\delta\beta} \Gamma^\alpha_{\gamma\kappa} - \Gamma^\kappa_{\gamma\beta} \Gamma^\alpha_{\delta\kappa}.$$

in the conventions of [10]. In the presence of a metric we have a contorsion tensor  $S$  defined by

$$\Gamma^\alpha_{\beta\gamma} = \hat{\Gamma}^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma}$$

where metric compatibility  $\nabla g = 0$  is equivalent to the first of

$$S^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (T_{\delta\beta\gamma} + T_{\beta\delta\gamma} + T_{\gamma\delta\beta}), \quad T^\alpha_{\beta\gamma} = S^\alpha_{\beta\gamma} - S^\alpha_{\gamma\beta}$$

and also implies that the lowered  $R_{\alpha\beta\gamma\delta}$  is antisymmetric in the first pair of indices (as well as the second). Also note that the lowered index contorsion here obeys  $S_{\alpha\beta\gamma} = -S_{\gamma\beta\alpha}$ . In this setup, torsion becomes the relevant field which determines metric compatible  $\nabla$  via  $S$ . This set up is slightly more than we strictly needed for the noncommutative geometry itself as explained in the introduction.

We let  $\omega^{\alpha\beta}$  define the Poisson bracket  $\{f, h\} = \omega^{\alpha\beta} (\partial_\alpha f) \partial_\beta h$ . This is a bivector field and it is shown in [10] that  $\nabla$  in our full sense is Poisson compatible if and only if

$$\nabla_\gamma \omega^{\alpha\beta} + T^\alpha_{\delta\gamma} \omega^{\delta\beta} + T^\beta_{\delta\gamma} \omega^{\alpha\delta} = 0 \quad (2.1)$$

or equivalently in the Riemannian case that

$$\hat{\nabla}_\gamma \omega^{\alpha\beta} + S^\alpha_{\delta\gamma} \omega^{\delta\beta} + S^\beta_{\delta\gamma} \omega^{\alpha\delta} = 0. \quad (2.2)$$

We also want  $\omega$  to be a Poisson tensor even though this is not strictly needed at order  $\lambda$ ,

$$\sum_{\text{cyclic}(\alpha,\beta,\gamma)} \omega^{\alpha\mu} \omega^{\beta\gamma}{}_{,\mu} = 0 \quad (2.3)$$

which given Poisson-compatibility is equivalent [10] to

$$\sum_{\text{cyclic}(\alpha,\beta,\gamma)} \omega^{\alpha\mu} \omega^{\beta\nu} T^\gamma_{\mu\nu} = 0. \quad (2.4)$$

Given the Poisson tensor and (2.1) respectively we quantize the product of functions with each other and with 1-forms,

$$f \bullet h = fh + \frac{\lambda}{2} \{f, h\}, \quad f \bullet \eta = f\eta + \frac{\lambda}{2} \omega^{ab} f_{,a} \nabla_b \eta, \quad \eta \bullet f = \eta f - \frac{\lambda}{2} \omega^{ab} f_{,a} \nabla_b \eta$$

to order  $\lambda$ , so that

$$[x^\alpha, \eta] = \lambda \omega^{\alpha\beta} \nabla_\beta \eta \quad (2.5)$$

to order  $\lambda$  in the quantum algebra. It is shown in [10] that we also can quantize the wedge product of 1-forms and higher,

$$dx^\alpha \wedge_1 dx^\beta = dx^\alpha \wedge dx^\beta + \frac{\lambda}{2} \omega^{\gamma\delta} \nabla_\gamma dx^\alpha \wedge \nabla_\delta dx^\beta + \lambda H^{\alpha\beta} \quad (2.6)$$

to order  $\lambda$ . This gives anticommutation relations

$$\{dx^\alpha, dx^\beta\}_1 = \lambda \omega^{\gamma\delta} \Gamma^\alpha_{\gamma\mu} \Gamma^\beta_{\delta\nu} dx^\mu \wedge dx^\nu + 2\lambda H^{\alpha\beta}. \quad (2.7)$$

Here the extra ‘non-functorial’ term needed is given by a family of 2-forms

$$H^{\alpha\beta} = \frac{1}{4} \omega^{\alpha\gamma} (\nabla_\gamma T^\beta_{\nu\mu} - 2R^\beta_{\nu\mu\gamma}) dx^\mu \wedge dx^\nu.$$

The exterior derivative  $d$  is taken as undeformed on the underlying vector spaces. Note that because the products by functions is modified, the quantum tensor product  $\otimes_1$ , i.e. over the quantum algebra is not the usual tensor product. It is characterised by

$$\eta \otimes_1 f \bullet \zeta = \eta \bullet f \otimes_1 \zeta$$

for all functions  $f$  and any  $\eta, \zeta$ . If we denote by  $A$  the vector space  $C^\infty(M)$  with this modified product, which can always be taken to be associative, and if  $\Omega^1$  with  $\bullet$  is separately a left and right action of  $A$  (even if they do not associate) then  $\otimes_1$  is just the usual tensor product  $\otimes_A$  over  $A$ . Note that  $\Omega^1 \otimes_1 \Omega^1 \otimes_1 \Omega^1$  in the case of nonassociative differentials will still have ambiguities at order  $\lambda^2$ . The quantum and classical tensor products are in fact identified by a natural transformation  $q$  to order  $\lambda$ , as explained in [10]. The fact that everything works and is consistent at order  $\lambda$  is a nontrivial part of that work, where we work associatively modulo  $\lambda^2$ .

Next, although not strictly necessary, it is useful to optionally require that the  $\nabla_Q$  is geometrically well behaved in noncommutative geometry in that its associated quantum torsion commutes with the quantum product by functions (a ‘bimodule map’). This is a quantum tensoriality property in the same spirit as requiring centrality of the quantum metric. We say that such a  $\nabla$  is *regular*, amounting to

$$\omega^{\alpha\beta} \nabla_\beta T^\gamma_{\mu\nu} = 0. \quad (2.8)$$

With or without this simplifying assumption, there is a quantum metric  $g_1 \in \Omega^1 \otimes_1 \Omega^1$  to order  $\lambda$  given by [10]

$$g_1 = g_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} \omega^{\alpha\beta} \Gamma_{\mu\alpha\kappa} \Gamma^\kappa_{\beta\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} \mathcal{R}_{\mu\nu} dx^\mu \otimes_1 dx^\nu \quad (2.9)$$

and obeying  $\wedge_1(g_1) = 0$  as well as a ‘reality’ property  $\text{flip}(* \otimes *)g_1 = g_1$ . Note the quantum  $\Omega^1$  as a vector space is identified with the classical  $\Omega^1$  and the above formula specifies an element of  $\Omega^1 \otimes_1 \Omega^1$  by giving the classical 1-forms for each factor in each term. This should not be confused with  $\tilde{g}_{\mu\nu}$  which we will introduce later as coefficients with respect to the  $\bullet$  product. In our case  $x^{\mu*} = x^\mu$  since our classical manifold has real coordinates and also acts trivially on all classical (real) tensor components, while  $\lambda^* = -\lambda$ . The action of  $*$  on a  $\bullet$ -product reverse orders while on a  $\wedge_1$ -product it reverse order with sign according to the degrees. For the most part this  $*$ -operation takes care of itself given that our classical tensors are real, so we will not emphasise it. The first two terms in (2.9) are the functorial part  $g_Q$  and the last term is a correction. Here

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} g_{\alpha\beta} \omega^{\alpha\gamma} (\nabla_\gamma T^\beta_{\mu\nu} - R^\beta_{\mu\nu\gamma} + R^\beta_{\nu\mu\gamma}) \quad (2.10)$$

is antisymmetric and can be viewed as the *generalised Ricci 2-form*

$$\mathcal{R} = \frac{1}{2} \mathcal{R}_{\mu\nu} dx^\nu \wedge dx^\mu = g_{\mu\nu} H^{\mu\nu}$$

(note the sign and factor in our conventions for 2-form components). Next we let  $(\cdot)_1 : \Omega^1 \otimes_1 \Omega^1 \rightarrow A$  be the inverse metric as a bimodule map. We define  $A$ -valued coefficients  $g_{1\mu\nu}, \tilde{g}_{\mu\nu}$  by

$$g_1 = g_{1\mu\nu} dx^\mu \otimes_1 dx^\nu = dx^\mu \bullet \tilde{g}_{\mu\nu} \otimes_1 dx^\nu = dx^\mu \otimes_1 \tilde{g}_{\mu\nu} \bullet dx^\nu$$

so that

$$\tilde{g}_{\mu\nu} = g_{1\mu\nu} + \frac{\lambda}{2} \omega^{\alpha\beta} \Gamma^\gamma_{\alpha\mu} g_{\gamma\nu,\beta} = g_{\mu\nu} + \frac{\lambda}{2} h_{\mu\nu} \quad (2.11)$$

to order  $\lambda$ , where we also write  $h_{\mu\nu}$  for the leading order correction in  $\tilde{g}_{\mu\nu}$ . Here  $g_{1\mu\nu}$  is read off from (2.9) as the quantum metric coefficients when we choose to use the undeformed product and  $\tilde{g}_{\mu\nu}$  are the coefficients when we choose to reorder and use the deformed product as stated (we can also place the  $\tilde{g}_{\mu\nu}$  with the second factor since  $\otimes_1$  behaves well with respect to the  $\bullet$  product as we explained above). The two sets of coefficients are related by (2.11) but in different calculations one or the other may be easier to work depending on the context (the same remark will apply to all our other quantum tensors). From (2.9) and (2.11) we find

$$h_{\mu\nu} = \mathcal{R}_{\mu\nu} + \omega^{\alpha\beta} (\Gamma_{\mu\alpha\kappa} \Gamma^\kappa_{\beta\nu} + \Gamma^\gamma_{\alpha\mu} g_{\gamma\nu,\beta}) = -h_{\nu\mu} \quad (2.12)$$

where we use metric compatibility of  $\nabla$  in the form  $g_{\gamma\nu,\beta} = \Gamma_{\gamma\beta\nu} + \Gamma_{\nu\beta\gamma}$  to replace the second term to more easily verify antisymmetry. We let  $\tilde{g}^{\mu\nu}$  be the  $A$ -valued matrix inverse so that  $\tilde{g}_{\mu\nu} \bullet \tilde{g}^{\nu\gamma} = \delta_\mu^\gamma = \tilde{g}^{\gamma\nu} \bullet \tilde{g}_{\nu\mu}$  and define

$$(dx^\mu, dx^\nu)_1 = \tilde{g}^{\mu\nu} = g^{\mu\nu} - \frac{\lambda}{2} \tilde{h}^{\mu\nu} \quad (2.13)$$

which we extend by  $(f \bullet dx^\mu, dx^\nu \bullet \tilde{f})_1 = f \bullet (dx^\mu, dx^\nu)_1 \bullet \tilde{f}$  for any functions  $f, \tilde{f}$ . This gives us a bimodule map  $(\cdot)_1 : \Omega^1 \otimes_1 \Omega^1 \rightarrow A$  inverse to  $g_1$  in the usual sense of noncommutative geometry [9], namely

$$((\cdot)_1 \otimes \text{id})(\eta \otimes_1 g_1) = \eta = (\text{id} \otimes (\cdot)_1)(g_1 \otimes_1 \eta)$$

for all  $\eta \in \Omega^1$ , except that we only claim these facts to order  $\lambda$ . From the above,

$$\begin{aligned} \tilde{h}^{\mu\nu} &= g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta} + g^{\mu\alpha} \{g_{\alpha\beta}, g^{\beta\nu}\} = \mathcal{R}^{\mu\nu} + \omega^{\alpha\beta} (\Gamma^\mu_{\alpha\kappa} \Gamma^\kappa_{\beta\gamma} g^{\nu\gamma} + \Gamma^\mu_{\alpha\gamma} g^{\nu\gamma}_{,\beta}) \\ &= \mathcal{R}^{\mu\nu} - \omega^{\alpha\beta} g^{\eta\zeta} \Gamma^\mu_{\alpha\eta} \Gamma^\nu_{\beta\zeta} = -\tilde{h}^{\nu\mu} \end{aligned}$$

and  $\mathcal{R}$  has indices raised by  $g$ . As an application, in bimodule noncommutative geometry there is a quantum dimension [9] which we can now compute.

**Proposition 2.1.** *In the setting above, the ‘quantum dimension’ to order  $\lambda$  is*

$$\dim_1 := (\cdot)_1(g_1) = \dim(M) + \frac{\lambda}{2} \{g_{\mu\nu}, g^{\mu\nu}\}.$$

**Proof.** Given the above results, we have

$$\begin{aligned} \dim_1 &= (dx^\mu \bullet \tilde{g}_{\mu\nu}, dx^\nu)_1 = \tilde{g}_{\mu\nu} \bullet \tilde{g}^{\mu\nu} + ([dx^\mu, g_{\mu\nu}], dx^\nu) \\ &= \dim(M) + \frac{\lambda}{2} (h_{\mu\nu} - h_{\nu\mu}) g^{\mu\nu} + \lambda \omega^{\alpha\beta} g_{\mu\nu,\alpha} \Gamma^\mu_{\beta\gamma} g^{\gamma\nu} = \dim(M) - \lambda \omega^{\alpha\beta} g^{\mu\nu}_{,\alpha} \Gamma_{\nu\beta\mu} \end{aligned}$$

where the middle term vanishes as  $g^{\mu\nu}$  is symmetric and we transferred to the derivative to the inverse metric. We can now use metric compatibility in the form  $\Gamma_{\mu\beta\nu} + \Gamma_{\nu\beta\mu} = g_{\mu\nu,\beta}$  to obtain the answer.  $\square$

Finally, the theory in [10] says that there is a quantum torsion free quantum metric compatible (or *quantum Levi-Civita*) connection  $\nabla_1 : \Omega^1 \rightarrow \Omega^1 \otimes_1 \Omega^1$  to order  $\lambda$  if and only if

$$\widehat{\nabla}\mathcal{R} + \omega^{\alpha\beta} g_{\rho\sigma} S^\sigma{}_{\beta\nu} (R^\rho{}_{\mu\gamma\alpha} + \nabla_\alpha S^\rho{}_{\gamma\mu}) dx^\gamma \otimes dx^\mu \wedge dx^\nu = 0. \quad (2.14)$$

In fact the theory always gives a unique ‘best possible’  $\nabla_1$  at this order for which the symmetric part of  $\nabla_1 g_1$  vanishes. This leaves open that  $\nabla_1 g_1 = O(\lambda)$ , namely proportional to the left hand side of (2.14). The construction of  $\nabla_1$  takes the form

$$\nabla_1 = \nabla_Q + q^{-1}Q(S) + \lambda K$$

where the first two terms are functorial and the last term is a further correction. Translating the formulae in [10] into indices and combining, one has

**Lemma 2.2.** *Writing  $\nabla_1 dx^\ell = -\Gamma_1{}^\ell{}_{\mu\nu} dx^\mu \otimes_1 dx^\nu$ , the construction of [10] can be written explicitly as*

$$\Gamma_1{}^\ell{}_{\mu\nu} = \widehat{\Gamma}^\ell{}_{\mu\nu} + \frac{\lambda}{2} \omega^{\alpha\beta} \left( \widehat{\Gamma}^\ell{}_{\mu\kappa, \alpha} \Gamma^\kappa{}_{\beta\nu} - \widehat{\Gamma}^\ell{}_{\kappa\tau} \Gamma^\kappa{}_{\alpha\mu} \Gamma^\tau{}_{\beta\nu} + \widehat{\Gamma}^\ell{}_{\alpha\kappa} (R^\kappa{}_{\nu\mu\beta} + \nabla_\beta S^\kappa{}_{\mu\nu}) \right).$$

**Proof.** It is already stated in [10] that

$$\nabla_Q(dx^\ell) = - \left( \Gamma^\ell{}_{\mu\nu} + \frac{\lambda}{2} \omega^{\alpha\beta} (\Gamma^\ell{}_{\mu\kappa, \alpha} \Gamma^\kappa{}_{\beta\nu} - \Gamma^\ell{}_{\kappa\tau} \Gamma^\kappa{}_{\alpha\mu} \Gamma^\tau{}_{\beta\nu} - \Gamma^\ell{}_{\beta\kappa} R^\kappa{}_{\nu\mu\alpha}) \right) dx^\mu \otimes_1 dx^\nu$$

Next, we carefully we write the term  $\omega^{ij} \nabla_i \circ \nabla_j(S)$  in  $Q(S)$  in [10, lemma 3.2] as curvature plus an extra term involving  $\nabla S$  and  $\Gamma$ , to give

$$\begin{aligned} q^{-1}Q(S)(dx^\ell) &= (S^\ell{}_{\mu\nu} + \frac{\lambda}{2} \omega^{\alpha\beta} (S^\ell{}_{\mu\kappa, \alpha} \Gamma^\kappa{}_{\beta\nu} - S^\ell{}_{\kappa\tau} \Gamma^\kappa{}_{\alpha\mu} \Gamma^\tau{}_{\beta\nu}) + \frac{\lambda}{4} R_\omega(S)^\ell{}_{\mu\nu} \\ &\quad - \frac{\lambda}{2} \omega^{\alpha\beta} \Gamma^\ell{}_{\alpha\kappa} \nabla_\beta S^\kappa{}_{\mu\nu}) dx^\mu \otimes_1 dx^\nu \end{aligned}$$

where

$$R_\omega(S)^\ell{}_{\mu\nu} = \omega^{\alpha\beta} (R^\ell{}_{\kappa\alpha\beta} S^\kappa{}_{\mu\nu} - R^\kappa{}_{\mu\alpha\beta} S^\ell{}_{\kappa\nu} - R^\kappa{}_{\nu\alpha\beta} S^\ell{}_{\mu\kappa})$$

is the curvature of  $\nabla$  evaluated on the Poisson bivector and acting on the contorsion tensor  $S$ . Finally, we take  $K$  given explicitly in [10, corollary 5.9],

$$K(dx^\ell) = \left( \frac{1}{2} \omega^{\alpha\beta} (S^\ell{}_{\alpha\kappa} \nabla_\beta S^\kappa{}_{\mu\nu} - S^\ell{}_{\beta\kappa} R^\kappa{}_{\nu\mu\alpha}) - \frac{1}{4} R_\omega(S)^\ell{}_{\mu\nu} \right) dx^\mu \otimes_1 dx^\nu$$

and combine all the terms to give the compact formula stated.  $\square$

As a bimodule connection there is also a generalised braiding  $\sigma_1 : \Omega^1 \otimes_1 \Omega^1 \rightarrow \Omega^1 \otimes_1 \Omega^1$  that expresses the right-handed Liebniz rule for a bimodule left connection, namely

$$\sigma_1(dx^\alpha \otimes_1 dx^\beta) = \sigma_Q(dx^\alpha \otimes_1 dx^\beta) + \lambda \omega^{\beta\mu} (\nabla_\mu S)(dx^\alpha) \quad (2.15)$$

which comes out as

$$\sigma_1(dx^\alpha \otimes_1 dx^\beta) = dx^\beta \otimes_1 dx^\alpha + \lambda (\omega^{\mu\nu} \Gamma^\alpha{}_{\mu\gamma} \Gamma^\beta{}_{\nu\delta} - \omega^{\mu\beta} (R^\alpha{}_{\gamma\delta\mu} + S^\alpha{}_{\delta\gamma;\mu})) dx^\delta \otimes_1 dx^\gamma \quad (2.16)$$

The bimodule noncommutative geometry also has a natural definition of quantum Laplacian [9] and we can now compute this

**Theorem 2.3.** *In Poisson–Riemannian geometry the quantum Laplacian to order  $\lambda$  is*

$$\square_1 f := (\cdot)_1 \nabla_1 df = \square f + \frac{\lambda}{2} \omega^{\alpha\beta} (\text{Ric}^\gamma{}_\alpha - S^\gamma{}_{;\alpha}) (\widehat{\nabla}_\beta df)_\gamma$$

**Proof.** Here  $\text{Ric}^\gamma{}_\alpha = g^{\gamma\nu} R^\beta{}_{\nu\beta\alpha} = -R^\gamma{}_{\nu\beta\alpha} g^{\nu\beta}$  and  $(\widehat{\nabla}_\alpha df)_\gamma = f_{,\alpha\gamma} - \widehat{\Gamma}^\ell{}_{\alpha\gamma} f_{,\ell}$  as usual. Let us also note that  $d$  is not deformed but can look different, namely write  $df = (\tilde{\partial}_\alpha f) \bullet dx^\alpha$  so that

$$\tilde{\partial}_\mu = \partial_\mu + \frac{\lambda}{2} \omega^{\alpha\beta} \Gamma^\nu{}_{\beta\mu} \partial_\alpha \partial_\nu$$

and we similarly write  $\nabla_1 dx^\ell = -\tilde{\Gamma}_1{}^\ell{}_{\mu\nu} \bullet dx^\mu \otimes dx^\nu$  so that

$$\tilde{\Gamma}_1{}^\ell{}_{\mu\nu} = \Gamma_1{}^\ell{}_{\mu\nu} + \frac{\lambda}{2} \omega^{\alpha\beta} \widehat{\Gamma}^\ell{}_{\nu\kappa,\alpha} \Gamma^\kappa{}_{\beta\mu} = \widehat{\Gamma}^\ell{}_{\mu\nu} + \frac{\lambda}{2} \gamma^\ell{}_{\mu\nu},$$

say, using symmetry of the last two indices of  $\widehat{\Gamma}$ . Then by the bimodule and derivation properties at the quantum level, we deduce

$$\square_1 f = (\cdot)_1 (d\tilde{\partial}_\nu f \otimes dx^\nu + (\tilde{\partial}_\alpha f) \bullet \nabla_1 dx^\alpha) = \left( \tilde{\partial}_\mu \tilde{\partial}_\nu f - (\tilde{\partial}_\alpha f) \bullet \tilde{\Gamma}_1{}^\alpha{}_{\mu\nu} \right) \bullet \tilde{g}^{\mu\nu}$$

We then expand this out to obtain the classical  $\square f$  and five corrections times  $\lambda/2$  as follows:

(i) From the deformed product with  $\tilde{g}^{\mu\nu}$  we obtain

$$\{\partial_\mu \partial_\nu f - (\partial_\alpha f) \widehat{\Gamma}^\alpha{}_{\mu\nu}, g^{\mu\nu}\}$$

(ii) From the deformation in  $\tilde{g}^{\mu\nu}$  we obtain

$$-(\partial_\mu \partial_\nu f - (\partial_\alpha f) \widehat{\Gamma}^\alpha{}_{\mu\nu}) \tilde{h}^{\mu\nu} = 0$$

by the antisymmetry of  $\tilde{h}^{\mu\nu}$  compared to symmetry of  $\widehat{\Gamma}^\alpha{}_{\mu\nu}$  and of  $\partial_\mu \partial_\nu f$ . So there is no contribution from this aspect at order  $\lambda$ .

(iii) From the deformation in  $\tilde{\partial}_\mu \tilde{\partial}_\nu f$  we obtain

$$\begin{aligned} & \omega^{\alpha\beta} \Gamma^\gamma{}_{\beta\mu} g^{\mu\nu} \partial_\alpha \partial_\gamma \partial_\nu f + g^{\mu\nu} \partial_\mu (\omega^{\alpha\beta} \Gamma^\gamma{}_{\beta\nu} \partial_\alpha \partial_\gamma f) \\ &= 2\omega^{\alpha\beta} \Gamma^\gamma{}_{\beta\mu} g^{\mu\nu} \partial_\alpha \partial_\gamma \partial_\nu f + g^{\mu\nu} (\partial_\alpha \partial_\gamma f) \partial_\mu (\omega^{\alpha\beta} \Gamma^\gamma{}_{\beta\nu}) \end{aligned}$$

(iv) From the deformation in  $-\tilde{\partial}_\alpha f \bullet \widehat{\Gamma}^\alpha{}_{\mu\nu}$  we obtain

$$-\omega^{\alpha\beta} \Gamma^\gamma{}_{\beta\kappa} g^{\mu\nu} \widehat{\Gamma}^\kappa{}_{\mu\nu} \partial_\alpha \partial_\gamma f - \{\partial_\alpha f, \widehat{\Gamma}^\alpha{}_{\mu\nu}\} g^{\mu\nu}$$

(v) From the deformation in  $\tilde{\Gamma}^1$  and our above formulae for that, we obtain

$$-\gamma^\ell{}_{\mu\nu} g^{\mu\nu} \partial_\ell f = -(\partial_\ell f) \omega^{\alpha\beta} \left( 2\widehat{\Gamma}^\ell{}_{\mu\kappa,\alpha} \Gamma^\kappa{}_{\beta\nu} g^{\mu\nu} + \widehat{\Gamma}^\ell{}_{\beta\kappa} \text{Ric}^\kappa{}_\alpha + \widehat{\Gamma}^\ell{}_{\alpha\kappa} S^\kappa{}_{;\beta} \right)$$

where  $S^\kappa = S^\kappa{}_{\mu\nu} g^{\mu\nu}$  is the ‘contorsion vector field’ and  $;$  is with respect to  $\nabla$ .

Now, comparing, we see that the cubic derivatives of  $f$  in (i) and (iii) cancel using metric compatibility to write a derivative of the metric in terms of  $\Gamma$ . Similarly the 1-derivative term from (i) is  $-\partial_\ell f$  times



$$\begin{aligned}\{\widehat{\Gamma}^\iota_{\mu\nu}, g^{\mu\nu}\} &= \omega^{\alpha\beta} \widehat{\Gamma}^\iota_{\mu\kappa,\alpha} g^{\mu\nu}{}_{,\beta} g_{\eta\nu} g^{\kappa\eta} = -\omega^{\alpha\beta} \widehat{\Gamma}^\iota_{\mu\kappa,\alpha} g^{\mu\nu} (\Gamma_{\eta\beta\nu} + \Gamma_{\nu\beta\eta}) g^{\kappa\eta} \\ &= -2\omega^{\alpha\beta} \widehat{\Gamma}^\iota_{\mu\kappa,\alpha} g^{\mu\nu} \Gamma^\kappa_{\beta\nu}\end{aligned}$$

where we inserted  $g_{\eta\nu}$ , turned  $\partial_\beta$  onto this and used metric compatibility of  $\nabla$ . In the last step we used that  $\widehat{\Gamma}$  is torsion free so symmetric in the last two indices. The result exactly cancels with a term in (v) giving

$$\square_I f = \square f + \frac{\lambda}{2} (\partial_\iota f) \omega^{\alpha\beta} \widehat{\Gamma}^\iota_{\alpha\gamma} (\text{Ric}^\gamma_\beta - \mathcal{S}^\gamma{}_{;\beta}) + \mathcal{O}(\partial^2) f$$

where we have not yet analysed corrections with quadratic derivatives of  $f$ . Turning to these, the remainder of (i) and (iv) contribute

$$\begin{aligned}-\{\partial_\gamma f, \widehat{\Gamma}^\gamma\} - (\partial_\alpha \partial_\gamma f) \omega^{\alpha\beta} \Gamma^\gamma_{\beta\kappa} \widehat{\Gamma}^\kappa &= -\omega^{\alpha\beta} (\partial_\alpha \partial_\gamma f) \widehat{\Gamma}^\gamma{}_{;\beta} \\ &= \omega^{\alpha\beta} (\partial_\alpha \partial_\gamma f) \mathcal{S}^\gamma{}_{;\beta} - (\partial_\alpha \partial_\gamma f) g^{\mu\nu} \omega^{\alpha\beta} \Gamma^\gamma_{\mu\nu;\beta} \\ &= \omega^{\alpha\beta} (\partial_\alpha \partial_\gamma f) \mathcal{S}^\gamma{}_{;\beta} - (\partial_\alpha \partial_\gamma f) g^{\mu\nu} \omega^{\alpha\beta} (\Gamma^\gamma_{\mu\nu,\beta} + \Gamma^\gamma_{\beta\kappa} \Gamma^\kappa_{\mu\nu} - \Gamma^\gamma_{\kappa\nu} \Gamma^\kappa_{\beta\mu} - \Gamma^\gamma_{\mu\kappa} \Gamma^\kappa_{\beta\nu}).\end{aligned}$$

Meanwhile in (iii), we use poisson-compatibility in the direct form [10]

$$\omega^{\alpha\beta}{}_{,\mu} = \omega^{\beta\eta} \Gamma^\alpha_{\eta\mu} + \omega^{\eta\alpha} \Gamma^\beta_{\eta\mu}$$

to obtain

$$(\partial_\alpha \partial_\gamma f) g^{\mu\nu} (\omega^{\alpha\beta} \Gamma^\gamma_{\beta\nu,\mu} + \omega^{\beta\eta} \Gamma^\alpha_{\eta\mu} \Gamma^\gamma_{\beta\nu} - \omega^{\alpha\beta} \Gamma^\kappa_{\beta\nu} \Gamma^\gamma_{\kappa\mu})$$

using  $g^{\mu\nu}$  symmetric to massage the last term. The middle term vanishes as it is antisymmetric in  $\alpha, \gamma$  and the remaining two terms together with the above terms from  $\Gamma^\gamma_{\mu\nu;\beta}$  combine to give  $(\partial_\alpha \partial_\gamma f) g^{\mu\nu} \omega^{\alpha\beta} R^\gamma_{\nu\mu\beta}$ . This gives our 2-derivative corrections at order  $\lambda$  as

$$\frac{\lambda}{2} (\partial_\alpha \partial_\gamma f) \omega^{\alpha\beta} (\mathcal{S}^\gamma{}_{;\beta} - \text{Ric}^\gamma_\beta).$$

We then combine our results to the expression stated.  $\square$

## 2.2. Quantum Riemann and Ricci curvatures

The quantum Riemann curvature in noncommutative geometry is defined by

$$\text{Riem}_1 = (d \otimes_1 \text{id} - (\wedge_1 \otimes_1 \text{id})(\text{id} \otimes_1 \nabla_1)) \nabla_1 \quad (2.17)$$

and we start by obtaining an expression for it to semiclassical order in terms of tensors. It will be convenient to define components by

$$\text{Riem}_1(dx^\alpha) := -\frac{1}{2} \widehat{R}_1^\alpha{}_{\beta\mu\nu} dx^\mu \wedge dx^\nu \otimes_1 dx^\beta := -\frac{1}{2} \widetilde{R}_1^\alpha{}_{\beta\mu\nu} \bullet (dx^\mu \wedge dx^\nu) \otimes_1 dx^\beta$$

$$\widetilde{R}_1^\alpha{}_{\beta\mu\nu} = \widehat{R}_1^\alpha{}_{\beta\mu\nu} + \frac{\lambda}{2} \omega^{\delta\gamma} \left( \widehat{R}^\alpha{}_{\beta\eta\nu,\delta} \Gamma^\eta_{\gamma\mu} + \widehat{R}^\alpha{}_{\beta\mu\eta,\delta} \Gamma^\eta_{\gamma\nu} \right)$$

depending on how the coefficients enter. If we write  $\Gamma_1 = \widehat{\Gamma} + \frac{\lambda}{2} \widehat{\gamma}$  then

$$\begin{aligned}
\text{Riem}_1(\text{d}x^\alpha) &= (\text{d} \otimes_1 \text{id} - (\wedge_1 \otimes_1 \text{id})(\text{id} \otimes_1 \nabla_1)) \nabla_1(\text{d}x^\alpha) \\
&= -(\text{d} \otimes_1 \text{id} - (\wedge_1 \otimes_1 \text{id})(\text{id} \otimes_1 \nabla_1)) \Gamma_1^\alpha{}_{\mu\beta} \text{d}x^\mu \otimes_1 \text{d}x^\beta \\
&= -(\Gamma_1^\alpha{}_{\mu\beta,\nu} \text{d}x^\nu \wedge \text{d}x^\mu \otimes_1 \text{d}x^\beta + (\Gamma_1^\alpha{}_{\mu\gamma} \text{d}x^\mu) \wedge_1 (\Gamma_1^\gamma{}_{\nu\beta} \text{d}x^\nu) \otimes_1 \text{d}x^\beta) \\
&= -\left( \hat{\Gamma}^\alpha{}_{\nu\beta,\mu} \text{d}x^\mu \wedge \text{d}x^\nu \otimes_1 \text{d}x^\beta + (\hat{\Gamma}^\alpha{}_{\mu\gamma} \text{d}x^\mu) \wedge_1 (\hat{\Gamma}^\gamma{}_{\nu\beta} \text{d}x^\nu) \otimes_1 \text{d}x^\beta \right) \\
&\quad + \frac{\lambda}{2} \left( \hat{\gamma}^\alpha{}_{\mu\beta,\nu} + \hat{\Gamma}^\alpha{}_{\nu\gamma} \hat{\gamma}^\gamma{}_{\mu\beta} - \hat{\gamma}^\alpha{}_{\mu\gamma} \hat{\Gamma}^\gamma{}_{\nu\beta} \right) \text{d}x^\mu \wedge \text{d}x^\nu \otimes_1 \text{d}x^\beta \\
&= -\frac{1}{2} \hat{R}^\alpha{}_{\beta\mu\nu} \text{d}x^\mu \wedge \text{d}x^\nu \otimes_1 \text{d}x^\beta + \frac{\lambda}{2} \hat{\gamma}^\alpha{}_{\mu\beta;\nu} \text{d}x^\mu \wedge \text{d}x^\nu \otimes_1 \text{d}x^\beta \\
&\quad - \frac{\lambda}{2} \omega^{\eta\zeta} \nabla_\eta \left( \hat{\Gamma}^\alpha{}_{\mu\gamma} \text{d}x^\mu \right) \wedge \nabla_\zeta \left( \hat{\Gamma}^\gamma{}_{\nu\beta} \text{d}x^\nu \right) \otimes_1 \text{d}x^\beta \\
&\quad - \lambda \hat{\Gamma}^\alpha{}_{\mu\gamma} \hat{\Gamma}^\gamma{}_{\nu\beta} H^{\mu\nu} \otimes_1 \text{d}x^\beta
\end{aligned} \tag{2.18}$$

to  $O(\lambda^2)$ , where  $\hat{\cdot}$  is with respect to the Levi-Civita connection. The term  $\hat{\gamma}^\alpha{}_{\gamma\beta} \hat{\Gamma}^\gamma{}_{\nu\mu}$  does not contribute due to the antisymmetry of the wedge product. This implies for the components

$$\begin{aligned}
\hat{R}_1^\alpha{}_{\beta\mu\nu} &= \hat{R}^\alpha{}_{\beta\mu\nu} + \lambda \left( \frac{1}{2} (\hat{\gamma}^\alpha{}_{\nu\beta;\mu} - \hat{\gamma}^\alpha{}_{\mu\beta;\nu}) + \omega^{\eta\zeta} (\hat{\Gamma}^\alpha{}_{\mu\gamma,\eta} - \hat{\Gamma}^\alpha{}_{\kappa\gamma} \Gamma^\kappa{}_{\eta\mu}) (\hat{\Gamma}^\gamma{}_{\nu\beta,\zeta} - \hat{\Gamma}^\gamma{}_{\kappa\beta} \Gamma^\kappa{}_{\zeta\nu}) \right. \\
&\quad \left. + \frac{\omega^{\eta\kappa}}{2} \hat{\Gamma}^\alpha{}_{\eta\gamma} \hat{\Gamma}^\gamma{}_{\zeta\beta} (R^\zeta{}_{\mu\nu\kappa} - R^\zeta{}_{\nu\mu\kappa} - T^\zeta{}_{\mu\nu;\kappa}) \right)
\end{aligned}$$

where we inserted a previous formula for  $H$  in terms of the curvature and torsion of  $\nabla$ . One can similarly read off  $\hat{\gamma}$  from the quantum Levi-Civita connection in lemma 2.2.

Next, following [9], we consider the classical map  $i : \Omega^2 \rightarrow \Omega^1 \otimes \Omega^1$  that sends a 2-form to an antisymmetric 1-1 form in the obvious way.

**Proposition 2.4.** *The map  $i$  quantises to a bimodule map such that  $\wedge_1 i_1 = \text{id}$  to order  $\lambda$  by*

$$i_1(\text{d}x^\mu \wedge \text{d}x^\nu) = \frac{1}{2}(\text{d}x^\mu \otimes_1 \text{d}x^\nu - \text{d}x^\nu \otimes_1 \text{d}x^\mu) + \lambda I(\text{d}x^\mu \wedge \text{d}x^\nu)$$

for any tensor map  $I(\text{d}x^\mu \wedge \text{d}x^\nu) = I^{\mu\nu}{}_{\alpha\beta} \text{d}x^\alpha \otimes \text{d}x^\beta$  where the tensor  $I$  is antisymmetric in  $\mu, \nu$  and symmetric in  $\alpha, \beta$ . The functorial choice here is

$$I^{\mu\nu}{}_{\alpha\beta} = -\frac{1}{4} \omega^{\kappa\tau} (\Gamma^\mu{}_{\kappa\alpha} \Gamma^\nu{}_{\tau\beta} + \Gamma^\mu{}_{\tau\alpha} \Gamma^\nu{}_{\kappa\beta}).$$

**Proof.** The functorial construction in [10] gives  $i_Q : \Omega^2 \rightarrow \Omega^1 \otimes_1 \Omega^1$  necessarily obeying  $\wedge_Q i_Q = \text{id}$ . Here  $\nabla(i) = 0$  since the connection on  $\Omega^2$  is descended from the connection on  $\Omega^1 \otimes \Omega^1$  so that  $i_Q = q^{-1}i$  on identifying the vector spaces. This gives the expression stated for the canonical  $I$  and this also works for  $\wedge_1$  since this on  $\frac{1}{2}(\text{d}x^\mu \otimes_1 \text{d}x^\nu - \text{d}x^\nu \otimes_1 \text{d}x^\mu)$  differs from  $\wedge_Q$  by  $\frac{\lambda}{2}(H^{\mu\nu} - H^{\nu\mu}) = 0$ . Finally, if we change the canonical  $I$  to any other tensor with the same symmetries then its wedge is not changed and we preserve all required properties. Note that canonical choice can also be written as

$$i_1(\text{d}x^\mu \wedge_1 \text{d}x^\nu) = \frac{1}{2}(\text{d}x^\mu \otimes_1 \text{d}x^\nu - \text{d}x^\nu \otimes_1 \text{d}x^\mu) - \frac{\lambda}{2} \omega^{\alpha\beta} \Gamma^\mu{}_{\alpha\kappa} \Gamma^\nu{}_{\beta\tau} \text{d}x^\tau \otimes_1 \text{d}x^\kappa + \lambda i(H^{\mu\nu}) \tag{2.19}$$

when we allow for the relations of  $\wedge_1$ .  $\square$

Now we can follow [9] and use  $i_1$  to lift the first output of  $\text{Riem}_1$  and take a trace of this to compute the quantum Ricci tensor. To take the trace it is convenient, but not necessary, to use the quantum metric and its inverse, so

$$\text{Ricci}_1 = ((\cdot)_1 \otimes_1 \text{id} \otimes_1 \text{id})(\text{id} \otimes_1 i_1 \otimes_1 \text{id})(\text{id} \otimes_1 \text{Riem}_1)(g_1) \quad (2.20)$$

We now calculate  $\text{Ricci}_1$  from (2.20) taking first the ‘classical’ antisymmetric lift

$$i_1(\text{dx}^\mu \wedge \text{dx}^\nu) = \frac{1}{2}(\text{dx}^\mu \otimes_1 \text{dx}^\nu - \text{dx}^\nu \otimes_1 \text{dx}^\mu)$$

corresponding to  $I = 0$ . Then using the second form of the components of  $\text{Riem}_1$ ,

$$\begin{aligned} \text{Ricci}_1 &= ((\cdot)_1 \otimes_1 \text{id} \otimes_1 \text{id})(\text{id} \otimes_1 i_1 \otimes_1 \text{id})(\text{id} \otimes_1 \text{Riem}_1)(g_1) \\ &= -\frac{1}{2}((\cdot)_1 \otimes_1 \text{id} \otimes_1 \text{id})(\text{id} \otimes_1 i_1 \otimes_1 \text{id})(\text{dx}^\alpha \bullet \tilde{g}_{\alpha\beta} \otimes_1 \tilde{R}_1^\beta{}_{\gamma\mu\nu} \bullet (\text{dx}^\mu \wedge \text{dx}^\nu) \otimes_1 \text{dx}^\gamma) \\ &= -\frac{1}{2}((\cdot)_1 \otimes_1 \text{id} \otimes_1 \text{id})(\text{dx}^\alpha \otimes_1 \tilde{g}_{\alpha\beta} \bullet \tilde{R}_1^\beta{}_{\gamma\mu\nu} \bullet i_1(\text{dx}^\mu \wedge \text{dx}^\nu) \otimes_1 \text{dx}^\gamma) \\ &= -\frac{1}{2}(\text{dx}^\alpha, \tilde{g}_{\alpha\beta} \bullet \tilde{R}_1^\beta{}_{\gamma\mu\nu} \bullet \text{dx}^\mu)_1 \bullet \text{dx}^\nu \otimes_1 \text{dx}^\gamma \\ &= -\frac{1}{2}\tilde{g}^{\alpha\mu} \bullet \tilde{g}_{\alpha\beta} \bullet \tilde{R}_1^\beta{}_{\gamma\mu\nu} \bullet \text{dx}^\nu \otimes_1 \text{dx}^\gamma - \frac{1}{2}(\text{dx}^\alpha, [\hat{R}_{\alpha\gamma\mu\nu}, \text{dx}^\mu])\text{dx}^\nu \otimes_1 \text{dx}^\gamma. \end{aligned}$$

In the fourth line we used the fact that the Riemann tensor is already antisymmetric in  $\mu$  and  $\nu$ . Note that  $\tilde{g}^{\alpha\mu} \bullet \tilde{g}_{\alpha\beta} = (\tilde{g}^{\mu\alpha} + \lambda \tilde{h}^{\mu\alpha}) \bullet \tilde{g}_{\alpha\beta} = \delta^\mu_\beta + \lambda \tilde{h}^\mu{}_\beta$  to  $O(\lambda^2)$  where we lower an index using the classical metric. Meanwhile, putting in general  $I$  adds a term

$$-\frac{\lambda}{2}\hat{R}_{\alpha\gamma\mu\nu}((\cdot) \otimes \text{id} \otimes \text{id})(\text{dx}^\alpha \otimes_1 I(\text{dx}^\mu \wedge \text{dx}^\nu) \otimes_1 \text{dx}^\gamma)$$

and we therefore obtain

$$\begin{aligned} \text{Ricci}_1 &= -\frac{1}{2}\tilde{R}_1^\mu{}_{\gamma\mu\nu} \bullet \text{dx}^\nu \otimes_1 \text{dx}^\gamma - \frac{\lambda}{2}\left(\tilde{h}^{\mu\beta}\hat{R}_{\beta\gamma\mu\nu} - \omega^{\eta\zeta}g^{\alpha\beta}\hat{R}_{\alpha\gamma\mu\nu,\eta}\Gamma^\mu{}_{\zeta\beta}\right)\text{dx}^\nu \otimes_1 \text{dx}^\gamma \\ &\quad - \frac{\lambda}{2}\hat{R}^\alpha{}_{\gamma\eta\zeta}I^{\eta\zeta}{}_{\alpha\nu}\text{dx}^\nu \otimes_1 \text{dx}^\gamma \end{aligned} \quad (2.21)$$

The idea of [9] is then to use the freedom in  $I$  to arrange that

$$\wedge_1(\text{Ricci}_1) = 0, \quad \text{flip}(* \otimes *)\text{Ricci}_1 = \text{Ricci}_1$$

to order  $\lambda$  so that  $\text{Ricci}_1$  enjoys the same quantum symmetry and ‘reality’ properties (to order  $\lambda$ ) as  $g_1$ . (A further ‘reality’ condition on the map  $i_1$  in [9] just amounts in our case to the entries of the tensor  $I$  being real.) If we write components

$$\text{Ricci}_1 := -\frac{1}{2}\tilde{R}_{1\mu\nu} \bullet \text{dx}^\nu \otimes_1 \text{dx}^\mu = -\frac{1}{2}\text{dx}^\nu \bullet \tilde{\tilde{R}}_{1\nu\mu} \otimes_1 \text{dx}^\mu$$

then (2.21) is equivalent to

$$\tilde{\tilde{R}}_{1\mu\nu} = \tilde{R}_1^\alpha{}_{\mu\alpha\nu} + \lambda\left(\tilde{h}^{\alpha\beta}\hat{R}_{\beta\mu\alpha\nu} - \omega^{\eta\zeta}g^{\alpha\beta}\hat{R}_{\alpha\mu\kappa\nu,\eta}\Gamma^\kappa{}_{\zeta\beta} + \hat{R}^\alpha{}_{\mu\eta\zeta}I^{\eta\zeta}{}_{\alpha\nu}\right) \quad (2.22)$$

and

$$\widetilde{\widetilde{R}}_{1\nu\mu} = \widetilde{R}_{1\mu\nu} - \lambda\omega^{\alpha\beta}\widehat{R}_{\mu\delta,\alpha}\Gamma^{\delta}_{\beta\nu}$$

respectively, where  $\widehat{R}_{\mu\nu}$  is the classical Ricci tensor. This second version is useful for the quantum reality condition, which says that if we write  $\widetilde{\widetilde{R}}_{1\mu\nu} = \widehat{R}_{\mu\nu} + \lambda\rho_{\mu\nu}$  then the quantum correction  $\rho_{\mu\nu}$  is required to be antisymmetric. Remember that this will have contributions from  $\widetilde{R}_1$  as well as the terms directly visible in (2.22).

We then define the quantum Ricci scalar as

$$S_1 = (\cdot)_1 \text{Ricci}_1 = -\frac{1}{2}\widetilde{R}_{1\mu\nu} \bullet \tilde{g}^{\nu\mu} \quad (2.23)$$

which does not depend on the lifting tensor  $I$  due to the antisymmetry of the first two indices of  $\widehat{R}$ . There does not appear to be a completely canonical choice of Ricci in noncommutative geometry as it depends on the choice of lifting for which we have not done a general analysis, but this constructive approach allows us to begin to explore it. The reader should note that the natural conventions in our context reduce in the classical limit to  $-\frac{1}{2}$  of the usual Riemann and Ricci curvatures, which we have handled by putting this factor into the definition of the tensor components so that these all have limits that match standard conventions.

### 2.3. Laplacian in the bicrossproduct model

We apply the above formalism to the bicrossproduct model quantum spacetime [26]. Much of the quantum geometry (but not the Laplacian) was already solved to all orders by algebraic methods in [9] and the appendix carefully checks that our new tensor calculus formulae agree with that to order  $\lambda$  (this is not easy and provides a critical check).

The 2D version here has coordinates  $t, r$  with  $r$  invertible and Poisson bracket  $\{r, t\} = r$  or  $\omega^{10} = r$  in the coordinate basis. The work [9] used  $r$  rather than  $x$  as this is also the radial geometry of a higher-dimensional model. The Poisson-compatible ‘quantising’ connection is given by  $\Gamma^0_{01} = -r^{-1}$ ,  $\Gamma^0_{10} = r^{-1}$  or in abstract terms  $\nabla dr = 0$  and  $\nabla dt = r^{-1}(dt \otimes dr - dr \otimes dt)$ . Letting  $v = rdt - tdr$ , we have  $\nabla dr = \nabla v = 0$  so a pair of central 1-forms  $v, dr$  at least at first order. This model has trivial curvature of  $\nabla$  (and is indeed associative) but in other respects is a good test of our formulae with nontrivial torsion and contorsion and curvature of the Levi-Civita connection.

Next we take classical metric  $g = dr \otimes dr + bv \otimes v$  where  $b$  is a nonzero real parameter. It clearly has inverse  $(dr, dr) = 1$ ,  $(v, v) = b^{-1}$ ,  $(dr, v) = (v, dr) = 0$  and is the unique form of classical metric for which  $\nabla g = 0$  for the above Poisson-compatible connection. This was shown in [9] where it was also shown that the classical Riemannian geometry is that of either a strongly gravitating particle or an expanding universe according to the sign of  $b$ . The Levi-Civita connection for  $g$  is

$$\widehat{\nabla}v = -\frac{2v}{r} \otimes dr, \quad \widehat{\nabla}dr = \frac{2bv}{r} \otimes v.$$

In tensor terms, now in the coordinate basis  $x^0 = t$  and  $x^1 = r$ , the metric tensor and Levi-Civita connection are see [9, appendix]

$$g_{\mu\nu} = \begin{pmatrix} br^2 & -brt \\ -brt & 1 + bt^2 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \frac{1 + bt^2}{br^2} & \frac{t}{r} \\ \frac{t}{r} & 1 \end{pmatrix}$$

$$\hat{\Gamma}_{\mu\nu}^0 = \begin{pmatrix} -2bt & r^{-1}(1+2bt^2) \\ r^{-1}(1+2bt^2) & -2r^{-2}t(1+bt^2) \end{pmatrix}, \quad \hat{\Gamma}_{\mu\nu}^1 = \begin{pmatrix} -2br & 2bt \\ 2bt & -2br^{-1}t^2 \end{pmatrix}$$

The contorsion tensor can be written [9]

$$S^\kappa_{\alpha\beta} = 2b\epsilon_{\alpha\mu}x^\mu\epsilon_{\beta\nu}g^{\kappa\nu}, \quad S^\mu = 2\frac{x^\mu}{r^2}$$

where  $\epsilon_{01} = 1$  is antisymmetric. Then formula (2.10) gives  $\mathcal{R}_{10} = br$  or  $\mathcal{R}_{\mu\nu} = -br\epsilon_{\mu\nu}$  and hence  $\mathcal{R} = \mathcal{R}_{10}dt \wedge dr = br \wedge dr$  as in [10, section 7.1]

We also write

$$df = f_{,r}dr + f_{,t}dt = (\partial_r f)dr + (\partial_v f)v; \quad \partial_v f = \frac{1}{r}f_{,t}, \quad \partial_r f = f_{,r} + \frac{t}{r}f_{,t}$$

Then

$$\begin{aligned} \square f &= (\cdot)\hat{\nabla}df = (\cdot)((\partial_r f)\hat{\nabla}dr + (\partial_v f)\hat{\nabla}v + d\partial_r f \otimes dr + d\partial_v f \otimes v) \\ &= \frac{2}{r}(\partial_r f) + \partial_r^2 f + b^{-1}\partial_v^2 f \end{aligned}$$

is the classical Laplacian for  $g$ . When  $b < 0$  the interpretation of the classical geometry is that of a strong gravitational source and curvature singularity at  $r = 0$ . Being conformally flat after a change of variables to  $r' = 1/r, t' = t/r$  the massless waves or zero eigenfunctions of the classical Laplacian are plane waves in  $t', r'$  space of the form  $e^{i\omega t'}e^{\pm \frac{\omega r'}{\sqrt{-b}}}$  or

$$\psi_\omega^\pm(t, r) = e^{i\frac{\omega t}{r}}e^{\pm i\frac{\omega}{r\sqrt{-b}}}$$

while the massive modes are harder to describe due to the conformal factor. One can similarly solve the expanding universe case where  $b > 0$  and the interpretation of the  $r, t$  variables is swapped. This completes the classical data.

Next, the quantum metric at semiclassical order from (2.9) is

$$\begin{aligned} g_1 &= g_{\mu\nu}dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2}\mathcal{R}_{01}(dt \otimes_1 dr - dr \otimes_1 dt) + \frac{\lambda}{2}\omega^{10}g_{\mu 0}\Gamma^0_{10}\Gamma^0_{01}dx^\mu \otimes_1 dr \\ &= g_{\mu\nu}dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2}btdr \otimes_1 dr + \frac{\lambda}{2}brdr \otimes_1 dt - \lambda brdt \otimes_1 dr \end{aligned}$$

Also, from the formula (2.12), we have

$$h_{01} = \mathcal{R}_{01} + g_{00}\omega^{10}\Gamma^0_{10}\Gamma^0_{01} + \omega^{10}\Gamma^0_{10}g_{01,0} = -3br$$

and similarly  $h_{10} = 3br$ , so that

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{\lambda}{2}\begin{pmatrix} 0 & -3br \\ 3br & 0 \end{pmatrix} = g_{\mu\nu} - \frac{\lambda 3br}{2}\epsilon_{\mu\nu}.$$

For the correction in  $\Gamma_1^1{}_{\mu\nu}$  for the quantum Levi-Civita connection in lemma 2.2 we have

$$\begin{aligned} &\frac{\lambda}{2}\omega^{\alpha\beta}\hat{\Gamma}^1_{\mu 0, \alpha}\Gamma^0_{\beta\nu} - \frac{\lambda}{2}\omega^{\alpha\beta}\hat{\Gamma}^1_{00}\Gamma^0_{\alpha\mu}\Gamma^0_{\beta\nu} - \frac{\lambda}{2}\omega^{01}S^1_{0\kappa}\nabla_1 S^\kappa_{\mu\nu} \\ &= -\frac{\lambda}{2}\hat{\Gamma}^1_{\mu 0, \nu} - \lambda b\epsilon_{\mu\nu} - \frac{\lambda}{2}r\hat{\Gamma}^1_{0\kappa}\nabla_1 S^\kappa_{\mu\nu} = 2\lambda b\begin{pmatrix} 0 & 1 \\ 0 & -r^{-1}t \end{pmatrix} \end{aligned}$$

and

$$\nabla_1 S^0_{\mu\nu} = \begin{pmatrix} -\frac{2bt}{r} & \frac{2(1+bt^2)}{r^2} \\ \frac{2bt^2}{r^2} & -\frac{2t(1+bt^2)}{r^3} \end{pmatrix}, \quad \nabla_1 S^1_{\mu\nu} = \begin{pmatrix} -2b & \frac{2bt}{r} \\ \frac{2bt}{r} & -\frac{2bt^2}{r^2} \end{pmatrix}$$

where  $\nabla_1$  in this context means with respect to  $r$ . Similarly, the correction to  $\Gamma_1^0{}_{\mu\nu}$  in lemma 2.2 is

$$\begin{aligned} & \frac{\lambda}{2} \omega^{\alpha\beta} \hat{\Gamma}^0_{\mu 0, \alpha} \Gamma^0_{\beta\nu} - \frac{\lambda}{2} \omega^{\alpha\beta} \hat{\Gamma}^0_{00} \Gamma^0_{\alpha\mu} \Gamma^0_{\beta\nu} - \frac{\lambda}{2} \omega^{01} S^0_{0\kappa} \nabla_1 S^\kappa_{\mu\nu} \\ &= -\frac{\lambda}{2} \hat{\Gamma}^0_{\mu 0, \nu} - \lambda b r^{-1} t \epsilon_{\mu\nu} - \frac{\lambda}{2} r \hat{\Gamma}^0_{0\kappa} \nabla_1 S^\kappa_{\mu\nu} = \lambda \begin{pmatrix} 2b & 0 \\ -\frac{2bt}{r} & \frac{1}{2r^2} \end{pmatrix} \end{aligned}$$

giving

$$\begin{aligned} \nabla_1 dr &= -\hat{\Gamma}^1_{\mu\nu} dx^\mu \otimes_1 dx^\nu - 2\lambda b(dt - r^{-1}tdr) \otimes_1 dr \\ \nabla_1 dt &= -\hat{\Gamma}^0_{\mu\nu} dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{2} r^{-2} dr \otimes_1 dr - 2\lambda b(dt - r^{-1}tdr) \otimes_1 dt. \end{aligned}$$

One can check that the condition (2.14) holds so that this is the quantum Levi-Civita connection at order  $\lambda$ .

The quantum Riemann tensor by direct computation (using Maple) from (2.18) comes out as

$$\text{Riem}_1(dx^\alpha) = -\frac{1}{2} \hat{R}^\alpha_{\beta\mu\nu} dx^\mu \wedge dx^\nu \otimes_1 dx^\beta + \frac{5b\lambda}{r} dt \wedge dr \otimes_1 dx^\alpha \quad (2.24)$$

For the quantum Ricci tensor we need a lift map  $i_1$  and we take

$$i_1(dt \wedge dr) = \frac{1}{2}(dt \otimes_1 dr - dr \otimes_1 dt) + \frac{7\lambda}{4r} g$$

where  $dt \wedge_1 dr = dt \wedge dr$  since  $\nabla dr = 0$  and only  $H^{00}$  is non-zero. The first term is the functorial term and the second term is  $\lambda I(dt \wedge dr)$ . Then (2.21) gives us  $\text{Ricci}_1 = g_1/r^2$  to order  $\lambda$  in agreement with the algebraic result in [9]. This means that the quantum Ricci scalar is undeformed.

Finally, the contracted contorsion tensor obeys

$$S^\mu{}_{;0} = 0, \quad S^\mu{}_{;1} = -2\frac{x^\mu}{r^3}$$

while the curvature of  $\nabla$  vanishes. Hence the Laplacian in theorem 2.3 is

$$\square_1 f = \square f + \frac{\lambda}{2} \omega^{10} (-S^\mu{}_{;1}) (\hat{\nabla}_0 df)_\mu = \square f + \frac{\lambda}{r^2} x^\mu (\hat{\nabla}_0 df)_\mu$$

which can be further expanded out using the values of  $\Gamma^0_{\mu\nu}$ . We see that there is an order  $\lambda$  correction. It is not so clear how to immediately read off physical predictions from this but one thing we can still do in the deformed case is make the conformal change of variables as classically and separate off  $\psi = e^{i\omega t'} f$ , to give an equation for null modes

$$\left( \frac{\partial^2}{\partial r'^2} + \lambda_P \omega \left( \frac{2}{r'} + \frac{\partial}{\partial r'} \right) - \frac{\omega^2}{b} \right) f = 0$$

where  $\lambda = i\lambda_P$ . This is solved by

$$f = r' e^{-\frac{1}{2}\omega r'} \left( \lambda_P + \frac{i\sqrt{b\lambda_P^2+4}}{\sqrt{-b}} \right) \left( AM\left(1 - \frac{i\sqrt{-b}\lambda_P}{\sqrt{b\lambda_P^2+4}}, 2, i\sqrt{b\lambda_P^2+4} \frac{\omega r'}{\sqrt{-b}}\right) \right. \\ \left. + BU\left(1 - \frac{i\sqrt{-b}\lambda_P}{\sqrt{b\lambda_P^2+4}}, 2, i\sqrt{b\lambda_P^2+4} \frac{\omega r'}{\sqrt{-b}}\right) \right)$$

for constants  $A$  and  $B$ .  $M(a, b, z)$  and  $U(a, b, z)$  denote the Kummer  $M$  and  $U$  functions (or hypergeometric  ${}_1F_1$ ,  $U$  respectively in Mathematica). In the limit  $\lambda_P \rightarrow 0$ , this becomes

$$f = \frac{i}{2} \frac{\sqrt{-b}A}{\omega} e^{i\frac{\omega r'}{\sqrt{-b}}} + \frac{i}{2} \frac{\sqrt{-b}(B-A)}{\omega} e^{-i\frac{\omega r'}{\sqrt{-b}}}$$

which means we recover our two independent solutions  $\psi_\omega^\pm$  as a check. Bearing in mind that our equations are only justified to order  $\lambda$ , we can equally well write

$$\psi_\omega(t, r) = \frac{e^{i\frac{\omega t}{r}}}{r} e^{-i\frac{\omega}{r\sqrt{-b}} \left(1 - \frac{i\sqrt{-b}\lambda_P}{2}\right)} \\ \left( AM\left(1 - \frac{i\sqrt{-b}\lambda_P}{2}, 2, i\frac{2\omega}{r\sqrt{-b}}\right) + BU\left(1 - \frac{i\sqrt{-b}\lambda_P}{2}, 2, i\frac{2\omega}{r\sqrt{-b}}\right) \right)$$

and proceed to analyse the behaviour for small  $\lambda_P$  in terms of integral formulae. Thus

$$M(1-a, 2, z) = \frac{1}{\Gamma(1-a)\Gamma(1+a)} \int_0^1 e^{zu} \left(\frac{1-u}{u}\right)^a du = \int_0^1 e^{zu} (1+a \ln(\frac{1-u}{u})) du + O(a^2)$$

which we evaluate for  $z = 2is$  and  $s$  real in terms of the function

$$\tau_M(s) = i \frac{M^{(1,0,0)}(1, 2, 2is)}{M(1, 2, 2is)} = \frac{\int_0^1 e^{2isu} \ln(\frac{1-u}{u}) du}{i \int_0^1 e^{2isu} du}$$

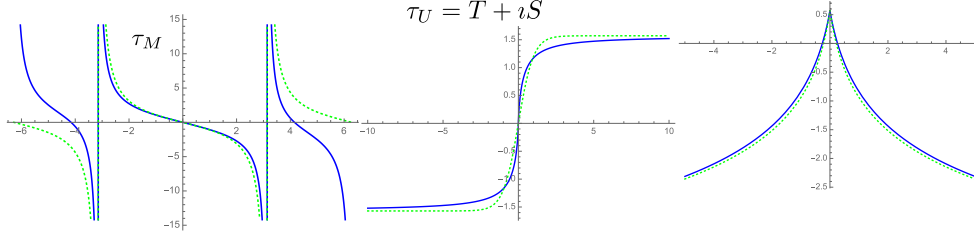
shown in figure 1. This function in the principal region (containing  $s = 0$ ) is qualitatively identical to the trig function  $-2 \tan(s/2)$  but blows up slightly more slowly as  $s \rightarrow \pm\pi$ . This gives us  $M(1-a, 2, 2is) = \frac{1}{2is}(e^{2is} - 1)(1 + a i \tau_M(s)) + O(a^2)$  and hence with  $a = i\frac{\sqrt{-b}\lambda_P}{2}$  and  $s = \frac{\omega}{r\sqrt{-b}}$ , we have up to normalisation

$$\psi_\omega^M(t, r) = e^{i\frac{\omega t}{r}} \sin\left(\frac{\omega}{r\sqrt{-b}}\right) e^{-\frac{\omega}{2r}\lambda_P \left(1 + \frac{i}{\omega}\sqrt{-b}\tau_M\left(\frac{\omega}{r\sqrt{-b}}\right)\right)} + O(\lambda_P^2), \quad |r| > \frac{|\omega|}{\pi\sqrt{-b}}$$

as one of our independent solutions. Notice that for  $\lambda_P \neq 0$  our solution blows up and our approximations break down as  $r$  approaches a certain minimum distance as shown to the classical Ricci singularity at  $r = 0$ , depending on the frequency. This is a geometric ‘horizon’ of some sort (with scale controlled by  $\sqrt{-b}$ ) but frequency dependent, and very different effect from the usual Planck scale bound  $|r| \gg |\omega|\lambda_P$  needed in any case for our general analysis. Meanwhile for large  $|r|$ , the effective  $\lambda_P$  is suppressed as  $\tau_M'(0) = -1$ .

For the other mode, the similar integral

$$U(1-a, 2, z) = \frac{1}{\Gamma(1-a)} \int_0^\infty e^{-zu} \left(\frac{1+u}{u}\right)^a du$$



**Figure 1.** Functions  $\tau_M$  and  $\tau_U$  related to differentials of Kummer  $M(1-a, 2, 2is)$  and  $U(1-a, 2, 2is)$  at  $a=0$  and similar to  $\tan$ ,  $\tanh$  and a shifted  $\ln$  (shown dashed for reference).

is not directly applicable as it is not valid on the imaginary axis but we can still proceed in a similar way for the other mode by defining

$$\tau_U(s) = i \frac{U^{(1,0,0)}(1, 2, 2is)}{U(1, 2, 2is)} = T(s) + iS(s)$$

where the real function  $T(s)$  resembles  $\frac{\pi}{2} \tanh(s)$  (but is vertical at the origin) and  $S(s)$  resembles  $-\ln(e^{-\gamma} + 2|s|)$  as also shown in figure 1, where  $\gamma \approx 0.577$  is the Euler constant. Then  $U(1-a, 2, 2is) = \frac{1}{2is}(1 + a\tau_U(s)) + O(a^2)$  giving up to normalisation

$$\psi_\omega^U(t, r) = e^{i\frac{\omega t}{r}} e^{-i\frac{\omega}{r\sqrt{-b}}} e^{-\frac{\omega}{2r}\lambda_P \left(1 + \frac{r}{\omega} \sqrt{-b}\tau_U\left(\frac{\omega}{r\sqrt{-b}}\right)\right)} + O(\lambda_P^2)$$

as a second solution. This still has our general Planck scale lower bound needed for the general analysis but no specific geometric bound at finite radius as  $\tau_U$  does not blow up and moreover has only a mild log divergence as  $s \rightarrow \infty$  or  $r \rightarrow 0$ . There is no particular suppression of  $\lambda_P$  as  $s \rightarrow 0$  or  $r \rightarrow \infty$  and indeed  $\tau_U$  tends to a constant nonzero imaginary value (the meaning of which is unclear as it can be absorbed in a normalisation).

Both of our solutions have been exhibited as deviations from the classical solutions and consequently they can reasonably be expected to lead to physical predictions, such as a change of the group velocity along the lines of [1] and of gravitational redshift along the lines of [25]. However, doing this in a convincing way in a GR setting requires rather more analysis and is beyond our scope here.

#### 2.4. Laplacian in the 2D Bertotti–Robinson model

By way of contrast we note that the bicrossproduct spacetime algebra has an alternative differential structure for which the full quantum geometry was also already solved, in [28]. We have the same Poisson bracket as above but this time the zero curvature ‘quantising’ connection  $\nabla dr = \frac{1}{r} dr \otimes dr$ ,  $\nabla dt = -\frac{\alpha}{r} dt$  or non-zero Christoffel symbols  $\Gamma^1_{11} = -r^{-1}$  and  $\Gamma^0_{10} = \alpha r^{-1}$  and the de Sitter metric in the form

$$g = ar^{-2} dr \otimes dr + br^{\alpha-1} (dr \otimes dt + dt \otimes dr) + cr^{2\alpha} dt \otimes dt$$

where only the nonzero combination  $\bar{\delta} = c\alpha^2/(b^2 - ac)$  of parameters is relevant up coordinate transformations. One can easily compute the classical Levi-Civita connection in these coordinates as

$$\hat{\Gamma}^0_{00} = -\frac{bca\alpha r^\alpha}{b^2 - ac}, \quad \hat{\Gamma}^0_{10} = -\frac{aca}{r(b^2 - ac)}, \quad \hat{\Gamma}^0_{11} = -\frac{ba\alpha r^{-\alpha-2}}{b^2 - ac}$$



$$\hat{\Gamma}^1_{00} = \frac{c^2 \alpha r^{1+2\alpha}}{b^2 - ac}, \quad \hat{\Gamma}^1_{10} = \frac{bc \alpha r^\alpha}{b^2 - ac}, \quad \hat{\Gamma}^1_{11} = \frac{-b^2(1 - \alpha) + ac}{r(b^2 - ac)}$$

Combing this with the ‘quantising’ connection yields the contorsion tensor

$$S^0_{00} = -S^1_{01} = -S^1_{10} = \frac{bc \alpha r^\alpha}{b^2 - ac}, \quad S^0_{10} = -S^1_{11} = S^0_{01} + \frac{\alpha}{r} = \frac{b^2 \alpha}{b^2 - ac}$$

$$S^0_{11} = \frac{ba \alpha r^{-\alpha-2}}{b^2 - ac}, \quad S^1_{00} = -\frac{c^2 \alpha r^{1+2\alpha}}{b^2 - ac}$$

From here we compute  $S^\mu = \frac{\alpha}{b^2 - ac}(br^{-\alpha}, -cr)$  for the  $t, r$  components giving  $\nabla_\nu S^\mu = 0$  so that in conjunction with flatness of  $\nabla$ , theorem 2.3 shows that there is no order  $\lambda$  correction to the Laplace operator.

We can also find the geometric quantum Laplacian to all orders directly from the full quantum geometry at least after a convenient but non-algebraic coordinate transformation in [28]. If we allow this then the model has generators  $R, T$  with the only non-zero commutation relations  $[T, R] = \lambda'$ ,  $[R, dR] = \lambda' \sqrt{\delta} dR$  where  $\lambda' = \lambda \sqrt{b^2 - ac}$  and the quantum metric and quantum Levi-Civita connection [28]

$$g_1 = dR \bullet e^{2T\sqrt{\delta}} \otimes_1 dR - dT \otimes_1 dT,$$

$$\nabla_1 dT = -\sqrt{\delta} e^{2T\sqrt{\delta}} \bullet dR \otimes_1 dR, \quad \nabla_1 dR = -\sqrt{\delta} (dR \otimes_1 dT + dT \otimes_1 dR),$$

which immediately gives us

$$(dR, dR)_1 = e^{-2T\sqrt{\delta}}, \quad (dT, dT)_1 = -1, \quad \square_1 T = (\cdot) \nabla_1 dT = -\sqrt{\delta}, \quad \square_1 R = 0.$$

Finally, for a general normal-ordered function  $f(T, R)$  with  $T$ 's to the left, we have

$$df = \frac{\partial f}{\partial T} \bullet dT + \partial^1 f \bullet dR; \quad \partial^1 f(R) = \frac{f(R) - f(R - \lambda' \sqrt{\delta})}{\lambda' \sqrt{\delta}}$$

due to the standard form of the commutation relations. With these ingredients and following exactly the same method as in the appendix, we have

$$\square_1 f = (\cdot)_1 \nabla_1 (df) = -\sqrt{\delta} \frac{\partial f}{\partial T} - \frac{\partial^2 f}{\partial T^2} + (\partial^1)^2 f \bullet e^{-2T\sqrt{\delta}} = \square f + O(\lambda^2)$$

when we expand  $\partial^1 = \frac{\partial}{\partial R} - \frac{\lambda' \sqrt{\delta}}{2} \frac{\partial^2}{\partial R^2} + O(\lambda^2)$  and write the bullet as classical plus Poisson bracket. This confirms what we found from theorem 2.3. We can also use identities from quantum mechanics applied to  $R, T$  in our case to further write

$$\square_1 f = -\sqrt{\delta} \frac{\partial f}{\partial T} - \frac{\partial^2 f}{\partial T^2} + e^{-2T\sqrt{\delta}} \bullet \Delta_1 f$$

where

$$\Delta_1 f(R) = \frac{f(R + 2\lambda' \sqrt{\delta}) - 2f(R - \lambda' \sqrt{\delta}) + f(R)}{(\lambda' \sqrt{\delta})^2}.$$

We see that the quantum Laplacian working in the quantum algebra with normal-ordered quantum wave functions has the classical form except that the derivative in the  $R$  direction is a finite difference one. It is also clear that we have eigenfunctions  $\psi(T, R) = e^{i\omega T} e^{ikR}$ . This is an identical situation to the standard Minkowski spacetime bicrossproduct model in [1] except that there time became a finite difference and there was no actual quantum geometry. Like there, one *could* claim that there is an order  $\lambda$  correction provided classical fields are identified with normal ordered ones, but from the point of view of Poisson–Riemannian geometry this is an artefact of such an assumption (the Poisson geometry being closer to Weyl ordering). We have focussed on the 2D case but the same conclusion holds for the Bertotti–Robinson quantum metric on  $S^{n-1} \times dS_2$  in [28] keeping the angular coordinates to the left along with  $T$ ; then only the double  $R$ -derivative deforms namely to  $\Delta_1$  on normal-ordered functions. [28] already obtained the quantum Ricci and scalar curvatures in the same form as classically (normal ordered in the former case).

### 2.5. Fuzzy nonassociative sphere revisited

The case of the sphere in Poisson–Riemannian geometry is covered in [10] mainly in very explicit cartesian coordinates where we broke the rotational symmetry. However, the results are fully rotationally invariant as is more evident if we work with  $z^i$ ,  $i = 1, 2, 3$  and the relation  $\sum_i z^{i2} = 1$ . We took  $\nabla = \widehat{\nabla}$  (the Levi-Civita connection) so  $S = 0$ , and  $\omega$  the inverse of the canonical volume 2-form on the unit sphere. Then the results of [10] give us a particular ‘fuzzy sphere’ differential calculus

$$[z^i, z^j]_\bullet = \lambda \epsilon^{ij}_k z^k, \quad [z^i, dz^j]_\bullet = \lambda z^j \epsilon^i_{mn} z^m dz^n.$$

to order  $\lambda$ . These are initially valid for  $i = 1, 2$  but must hold in this form for  $i = 1, 2, 3$  by rotational symmetry of both the Poisson bracket and the Levi-Civita connection. One also finds from the algebra that  $z^m \bullet dz^m = 0$  (sum over  $m = 1, 2, 3$ ) at order  $\lambda$  on differentiating the radius 1 relation. Here  $\Omega^1$  is a projective module with  $dz^i$  as a redundant set of generators and a relation. We also have

$$\{dz^i, dz^j\}_\bullet = \lambda(3z^i z^j - \delta_{ij}) \text{Vol}$$

to order  $\lambda$  as derived in [10] for  $i = 1, 2$  and which then holds for  $i = 1, 2, 3$ . This can also be derived by applying  $d$  to the bimodule relations and using  $dz^i \wedge dz^j = \epsilon^{ij}_k z^k \text{Vol}$  at the classical level on the unit sphere. We will also use the antisymmetric lift  $\widetilde{\text{Vol}} = \frac{1}{2}(z^3)^{-1}(dz^1 \otimes dz^2 - dz^2 \otimes dz^1)$  at the classical level. The classical sphere metric  $g_{\mu\nu}$  is given in [10] in the  $z^1, z^2$  coordinates but we can also write it as

$$g = \sum_{i=1}^3 dz^i \otimes dz^i$$

Similarly, the inverse metric and metric inner product are

$$g^{\mu\nu} = \delta_{\mu\nu} - z^\mu z^\nu, \quad (dz^i, dz^j) = \delta_{ij} - z^i z^j$$

for  $\mu, \nu = 1, 2$ , which extends as the second equality for  $i, j = 1, 2, 3$ . The sphere is 2-dimensional so only two of the  $z^i$  are independent in any coordinate patch but the expressions themselves are rotationally invariant in terms of all three.

The work [10] also computes the quantum metric and quantum Levi-Civita connection at order  $\lambda$ . We have

$$\begin{aligned}
g_1 &= g_{\mu\nu} dz^\mu \otimes_1 dz^\nu - \frac{\lambda}{2(z^3)^2} dz^3 \otimes_1 \epsilon_{3ij} z^i dz^j + \lambda \widetilde{\text{Vol}} \\
&= g_{\mu\nu} dz^\mu \otimes_1 dz^\nu + \frac{\lambda}{2(z^3)^2} \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j) \\
\nabla_1 dz^\mu &= -z^\mu \bullet g_1 = -\widehat{\Gamma}^\mu_{\alpha\beta} dz^\alpha \otimes_1 dz^\beta - \lambda z^\mu \widetilde{\text{Vol}} + \frac{\lambda}{2} \left( dz^3 \otimes_1 (\epsilon^{\mu\beta} g_{\beta\gamma} + \frac{z^\mu z^\beta}{(z^3)^2} \epsilon_{\beta\gamma}) dz^\gamma \right) \\
&= -\widehat{\Gamma}^\mu_{\alpha\beta} dz^\alpha \otimes_1 dz^\beta - \frac{\lambda}{2(z^3)^2} (\epsilon_{3ij} z^\mu z^3 dz^i \otimes_1 dz^j - \epsilon^\mu_{\nu 3} dz^3 \otimes_1 dz^\nu)
\end{aligned}$$

where we massaged the formulae in [10]. The classical Christoffel symbols are  $\widehat{\Gamma}^\mu_{\alpha\beta} = z^\mu g_{\alpha\beta}$ .

If we work with coefficients  $\tilde{g}_{ij}$  in the middle for the metric then the given quantum metric corresponds to the correction term

$$h = \frac{(2 - (z^3)^2)}{(z^3)^3} \epsilon_{3ij} dz^i \otimes_1 dz^j = \frac{2(2 - (z^3)^2)}{(z^3)^2} \widetilde{\text{Vol}}$$

which we see is antisymmetric. For the inverse metric we have from (2.13) that

$$(dz^i, dz^j)_1 = g^{ij} + \frac{\lambda}{2} \epsilon_{ijk} z^k$$

to order  $\lambda$  when  $i, j = 1, 2$  but which extends to  $i, j = 1, 2, 3$  with  $g^{ij} = \delta_{ij} - z^i z^j$ . For the connection it is a nice check that the formula in lemma 2.2 gives the same answer for  $\nabla_1$ . Then we can calculate the quantum Riemann tensor from (2.17) or directly from the above formulae for  $\nabla_1$ .

$$\begin{aligned}
\text{Riem}_1(dz^\alpha) &= (d \otimes_1 \text{id} - (\wedge_1 \otimes_1 \text{id})(\text{id} \otimes_1 \nabla_1)) \nabla_1(dz^\alpha) \\
&= - (d(\Gamma_1^\alpha_{\mu\beta}) \wedge dz^\mu \otimes_1 dz^\beta + \Gamma_1^\alpha_{\mu\gamma} dz^\mu \wedge_1 \Gamma_1^\gamma_{\nu\beta} dz^\nu \otimes_1 dz^\beta)
\end{aligned}$$

which can be broken down into three terms as follows

(i) The first term gives

$$\begin{aligned}
d(\Gamma_1^\alpha_{\mu\beta}) \wedge dz^\mu \otimes_1 dz^\beta &= -\widehat{\Gamma}^\alpha_{\mu\beta, \nu} dz^\nu \wedge dz^\mu \otimes_1 dz^\beta - \frac{\lambda}{2} \partial_\mu \left( \frac{z^\alpha}{z^3} \right) \epsilon_{3\nu\beta} dz^\mu \wedge dz^\nu \otimes_1 dz^\beta \\
&= -\widehat{\Gamma}^\alpha_{\mu\beta, \nu} dz^\nu \wedge dz^\mu \otimes_1 dz^\beta - \frac{\lambda}{2(z^3)^2} \epsilon_{3\nu\beta} (z^3 dz^\alpha - z^\alpha dz^3) \wedge dz^\nu \otimes_1 dz^\beta \\
&= -\widehat{\Gamma}^\alpha_{\mu\beta, \nu} dz^\nu \wedge dz^\mu \otimes_1 dz^\beta - \frac{\lambda}{2z^3} (z^3 \text{Vol} \otimes_1 dz^\alpha - z^\alpha \text{Vol} \otimes_1 dz^3)
\end{aligned}$$

The last step comes from expanding the expression in the previous line and simplifying, this will prove useful in comparing to the other terms.

(ii) Expanding  $\wedge_1$  gives a further two terms at  $\mathcal{O}(\lambda)$ . But first, using the formula for the classical Christoffel symbols and metric compatibility note that

$$\nabla_\alpha \widehat{\Gamma}^\nu_{\mu\nu} dz^\nu = \nabla_\alpha z^\nu g_{\mu\nu} dz^\nu = (\delta^\nu_\alpha g_{\mu\nu} + z^\nu z_\nu g_{\alpha\mu}) dz^\nu.$$

Now consider

$$\begin{aligned}
& \omega^{\eta\zeta} \nabla_\eta \hat{\Gamma}^\alpha_{\mu\gamma} dz^\mu \wedge \nabla_\zeta \hat{\Gamma}^\gamma_{\nu\beta} dz^\nu \otimes_1 dz^\beta \\
&= \omega^{\eta\zeta} (\delta^\alpha_\eta g_{\mu\gamma} + z^\alpha z_\gamma g_{\eta\mu}) dz^\mu \wedge (\delta^\gamma_\zeta g_{\nu\beta} + z^\gamma z_\beta g_{\zeta\nu}) dz^\nu \otimes_1 dz^\beta \\
&= \omega^{\alpha\gamma} g_{\mu\gamma} g_{\nu\beta} dz^\mu \wedge dz^\nu \otimes_1 dz^\beta \\
&= \frac{1}{z^3} (\epsilon^\alpha_{\mu 3} + \epsilon_{3\mu\gamma} z^\alpha z^\gamma) g_{\nu\beta} dz^\mu \wedge z^\nu \otimes_1 dz^\beta = \text{Vol} \otimes_1 dz^\alpha
\end{aligned}$$

where the cancellations in the second line result from the antisymmetry of  $\mu, \nu$  and  $\eta, \zeta$ . For the second term use

$$H^{\mu\nu} = \frac{1}{2} (z^\mu z^\nu - \delta^{\mu\nu}) \text{Vol}$$

giving

$$\begin{aligned}
\hat{\Gamma}^\alpha_{\mu\gamma} \hat{\Gamma}^\gamma_{\nu\beta} H^{\mu\nu} \otimes_1 dz^\beta &= \frac{1}{2} z^\alpha z^\gamma g_{\mu\gamma} g_{\nu\beta} (z^\mu z^\nu - \delta^{\mu\nu}) \text{Vol} \otimes_1 dz^\beta \\
&= \frac{1}{2} z^\alpha z^\nu \left( \frac{(z^1)^2 + (z^2)^2}{(z^3)^4} - \frac{1}{(z^3)^4} \right) \delta_{\nu\beta} \text{Vol} \otimes_1 dz^\beta \\
&= -\frac{z^\alpha}{2z^3} \text{Vol} \otimes_1 dz^3
\end{aligned}$$

Combining these two (remembering to add an overall 1/2 to the first) results in

$$\begin{aligned}
& \hat{\Gamma}^\alpha_{\mu\gamma} dz^\mu \wedge_1 \hat{\Gamma}^\gamma_{\nu\beta} dz^\nu \otimes_1 dz^\beta \\
&= \hat{\Gamma}^\alpha_{\mu\gamma} \hat{\Gamma}^\gamma_{\nu\beta} dz^\mu \wedge dz^\nu \otimes_1 dz^\beta + \frac{\lambda}{2z^3} (z^3 \text{Vol} \otimes_1 dz^\alpha - z^\alpha \text{Vol} \otimes_1 dz^3)
\end{aligned}$$

(iii) The last term involves the  $\mathcal{O}(\lambda)$  of  $\Gamma_1^\alpha{}_{\mu\gamma} \Gamma_1^\gamma{}_{\nu\beta} dz^\mu \wedge dz^\nu \otimes_1 dz^\beta$  and is

$$z^\gamma g_{\nu\beta} (z^\alpha z^3 \epsilon_{3\mu\gamma} dz^\mu - \epsilon^\alpha_{\gamma 3} dz^3) \wedge dz^\nu \otimes_1 dz^\beta + z^\alpha g_{\mu\gamma} dz^\mu \wedge (z^\gamma z^3 \epsilon_{3\nu\beta} dz^\nu - \epsilon^\gamma_{\beta 3} dz^3) \otimes_1 dz^\beta.$$

The second term, which given in components is

$$z^\alpha g_{\mu\gamma} (z^\gamma z^3 \epsilon_{3\nu\beta} + \frac{1}{z^3} \epsilon^\gamma_{\beta} \delta_{\delta\nu} z^\delta)$$

can be shown to be symmetric in  $\mu, \nu$  and therefore vanishes, whereas the first can be expanded and simplified to give

$$z^\gamma g_{\nu\beta} (z^\alpha z^3 \epsilon_{3\mu\gamma} dz^\mu - \epsilon^\alpha_{\gamma 3} dz^3) \wedge dz^\nu \otimes_1 dz^\beta = -\frac{1}{(z^3)^2} (1 - (z^3)^2) \text{Vol} \otimes_1 dz^\alpha - 2 \frac{z^\alpha}{z^3} \text{Vol} \otimes_1 dz^3$$

Now, taking together the above terms gives the semiclassical Riemann tensor as

$$\text{Riem}_1(dz^\alpha) = -\frac{1}{2} \hat{R}^\alpha_{\beta\mu\nu} dx^\mu \wedge dx^\nu \otimes_1 dx^\beta + \frac{\lambda}{2(z^3)^2} (1 + (z^3)^2) \text{Vol} \otimes_1 dz^\alpha$$

Where the classical Riemann tensor is  $\hat{R}^\alpha_{\gamma\mu\nu} dz^\mu \wedge dz^\nu \otimes dz^\gamma = dz^\alpha \wedge g$ . This is the same result as the general tensorial calculation using (2.18), as a useful check.

For the Ricci tensor, the form of the quantum lift from proposition 2.4 is

$$i_1(dz^\mu \wedge dz^\nu) = \frac{1}{2} (dz^\mu \otimes_1 dz^\nu - dz^\nu \otimes_1 dz^\mu) + \lambda I(dz^\mu \wedge dz^\nu)$$

The functorial choice here comes out as  $I(dz^\mu \wedge dz^\nu) = 0$ , but we leave this general. In 2D the lift map has three independent components which, in tensor notation, we parametrize as  $\alpha := I^{12}_{11}$ ,  $\beta := I^{12}_{22}$  and  $\gamma := I^{12}_{12}$ , with the remaining components being related by symmetry. Then the tensorial formula (2.21) gives us

$$\begin{aligned} \text{Ricci}_1 = & -\frac{1}{2}g_1 - \frac{3\lambda}{2}\widetilde{\text{Vol}} \\ & - \frac{\lambda}{(z^3)^2} ((\alpha z^1 z^2 + \gamma((z^2)^2 - 1))dz^1 \otimes_1 dz^1 - (\beta z^1 z^2 + \gamma((z^1)^2 - 1))dz^2 \otimes_1 dz^2 \\ & + (\gamma z^1 z^2 + \alpha((z^1)^2 - 1))dz^1 \otimes_1 dz^2 - (\gamma z^1 z^2 + \beta((z^2)^2 - 1))dz^2 \otimes_1 dz^1) \end{aligned}$$

Next, following our general method, we fix  $I$  so that  $\wedge_1 \text{Ricci}_1 = 0$ , i.e. quantum symmetric. This results in the constraint

$$\gamma = -\frac{1}{4z^1 z^2} (3z^3 + 2\alpha((z^1)^2 - 1) + 2\beta((z^2)^2 - 1))$$

with  $\alpha$  and  $\beta$  undetermined. We also want  $\text{Ricci}_1$  to be hermitian or ‘real’ in the sense  $\text{flip}(* \otimes *)\text{Ricci}_1 = \text{Ricci}_1$  which already holds for  $-\frac{1}{2}g_1$ . Since  $\lambda$  is imaginary this requires the matrix of coefficients in the order  $\lambda$  terms displayed above to be antisymmetric as all tensors are real. This imposes three more constraints which are fortunately not independent and give us a unique suitable lift, namely with

$$\alpha = \frac{3}{4z^3}(1 - (z^2)^2), \quad \beta = \frac{3}{4z^3}(1 - (z^1)^2), \quad \gamma = \frac{3}{4} \frac{z^1 z^2}{z^3}.$$

The result (and similarly in any rotated coordinate chart) is

$$i_1(dz^1 \wedge dz^2) = \frac{1}{2} (dz^1 \otimes_1 dz^2 - dz^2 \otimes_1 dz^1) - \frac{3\lambda}{4z^3}g$$

$$\text{Ricci}_1 = -\frac{1}{2}g_1$$

where the latter in our conventions is analogous to the classical case. And from this or from (2.23) we get the quantum scalar curvature

$$S_1 = -\frac{1}{2}\widehat{S}, \quad \widehat{S} = \widehat{R}_{\mu\nu}g^{\mu\nu} = 2$$

the same as classically in our conventions, so this has no corrections at order  $\lambda$ . As remarked in the general theory, the quantum Ricci scalar is independent of the choice of lift  $I$ .

We also find no correction to the Laplacian at order  $\lambda$  since the classical Ricci tensor is proportional to the metric hence the contraction in theorem 2.3 gives  $\omega^{\alpha\beta}(\widehat{\nabla}_\beta df)_\alpha$  which factors through  $\widehat{\nabla} \wedge df = 0$  due to zero torsion of the Levi-Civita connection.

We close with some other comments about the model. In fact the parameter  $\lambda$  in this model is dimensionless and if we want to have the usual finite-dimensional ‘spin  $j$ ’ representations of our algebra then we need

$$\lambda = \imath/\sqrt{j(j+1)}$$

for some natural number  $j$  as a quantisation condition on the parameter. Our reality conventions require  $\lambda$  imaginary. It is also known from [7] that this differential algebra arises from twisting by a cochain at least to order  $\lambda^2$  *but* in such a way that the twisting also induces the correct differential structure at order  $\lambda$ , i.e. as given by the Levi-Civita connection. We take  $U(so_{1,3})$  with generators and relations

$$[M_i, M_j] = \epsilon_{ijk} M_k, \quad [M_i, N_j] = \epsilon_{ijk} N_k, \quad [N_i, N_j] = -\epsilon_{ijk} M_k$$

acting on the classical  $z^i$  (i.e. converting [7] to the coordinate algebra) as,

$$M_i \triangleright z^j = \epsilon_{ijk} z^k, \quad N_i \triangleright z^j = z^i z^j - \delta_{ij}.$$

This is the action of  $so_{1,3}$  on the ‘sphere at infinity’. The cochain we need is then [7]

$$F^{-1} = 1 + \lambda f + \frac{\lambda^2}{2} f^2 + \dots, \quad f = \frac{1}{2} M_i \otimes N_i$$

where the higher terms are conjectured to exist in such a way that the algebra remains associative at all orders (and gives the quantisation of  $S^2$  as a quotient of  $U(su_2)$ ). On the other hand cochain twisting extends the differential calculus to all orders as a graded quasi-algebra in the sense of [8]. Specifically, if we start with the classical algebra and exterior algebra on the sphere, the deformed products are

$$z^i \bullet z^j = (F^{-1} \triangleright z^i)(F^{-2} \triangleright z^j) = z^i z^j + \frac{\lambda}{2} \epsilon_{ijk} z^k$$

$$z^i \bullet dz^j = (F^{-1} \triangleright z^i) d(F^{-2} \triangleright z^j) = z^i dz^j + \frac{\lambda}{2} z^j \epsilon_{imn} z^m dz^n$$

$$dz^j \bullet z^i = (F^{-1} \triangleright dz^j) d(F^{-2} \triangleright z^i) = (dz^j) z^i - \frac{\lambda}{2} z^i \epsilon_{jmn} z^m dz^n - \frac{\lambda}{2} \epsilon_{ijm} dz^m$$

to order  $\lambda$ , giving relations

$$[z^i, dz^j] \bullet = \frac{\lambda}{2} ((z^i \epsilon_{jmn} + z^j \epsilon_{imn}) z^m dz^n + \epsilon_{ijm} dz^m) = \lambda z^j \epsilon_{jmn} z^m dz^n$$

in agreement with the quantisation of the calculus by the Levi-Civita connecton. For the last step we let

$$w^i = \epsilon^i_{jk} z^j dz^k.$$

and note that classically  $z^i w^j \epsilon_{ijk} = -dz^k$  using the differential of the sphere relation and hence  $z^i w^j - z^j w^i = -\epsilon_{ijk} dz^k$ . This twisting result in [7] is in contrast to other cochain twist or deformation theory quantisations such as in [32], which consider only the coordinate algebra. It means that although the differential calculus is not associative at order  $\lambda^2$ , corresponding to the curvature of the sphere, different brackets are related via an associator and hence strictly controlled. One can then twist other aspects of the noncommutative geometry using the formalism of [8], see also more recently [3].

To get a sense of how these equations fit together even though nonassociative, we work now in the quantum algebra *so from now till the end of the section all products are deformed ones*. We have the commutation relations

$$[z^i, z^j] = \lambda \epsilon^{ij}_k z^k, \quad [z^i, dz^j] = \lambda w^i z^j$$

to order  $\lambda$ . Then, if we apply  $d$  to the first relation we have

$$\begin{aligned}\lambda\epsilon_{ijk}dz^k &= [dz^i, z^j] + [z^i, dz^j] = \lambda(w^i z^j - w^j z^i) = \lambda\epsilon_{jik}\epsilon_{kmn}z^m w^n \\ &= -\lambda\epsilon_{ijk}\epsilon_{kmn}z^m \epsilon_{nab}z^a dz^b = -\lambda\epsilon_{ijk}(\delta_a^k \delta_b^m - \delta_b^k \delta_a^m)z^m z^a dz^b = \lambda\epsilon_{ijk}dz^k - \lambda z^m \epsilon_{ijk}z^k dz^m\end{aligned}$$

which confirms that  $\sum z^m dz^m = O(\lambda)$  (which is to be expected since it is zero classically). In fact we only need the commutation relations for  $i, j = 1, 2$  to arrive at this deduction. Moreover,

$$0 = d\left(\sum_m z^m dz^m\right) = 2z^m dz^m - \lambda w^m z^m + [dz^3, z^3] + \lambda w^3 z^3$$

and  $w^m z^m = O(\lambda)$  since zero classically, which tells us that

$$[z^3, dz^3] = \lambda w^3 z^3 + 2z^m dz^m.$$

Hence  $z^m dz^m = 0$  at order  $\lambda$  if the  $z^3$  commutation relations hold as claimed. In fact, assuming only the  $i, j = 1, 2$  commutation relations one can deduce (so long as  $z^3$  is invertible) that  $[z^3, dz^j] = \lambda w^3 z^j$  for  $j = 1, 2$  by looking at  $[(z^3)^2, dz^i] = 2z^3[z^3, dz^i]$  on the one hand and using the radius relation on the other hand. From this and  $\lambda\epsilon_{i3k}dz^k = [dz^i, z^3] + [z^i, dz^3]$  we deduce that  $[z^i, dz^3] = \lambda w^i z^3$  as claimed. Then by the same calculation as for the  $[z^3, dz^i]$  relation we can deduce  $[z^3, dz^3] = \lambda w^3 z^3$  as well. Thus, we have internal consistency of the quantum algebra relations even if we do not have associativity of the relations involving the  $dz^i$ .

### 3. Semiquantum FLRW model

We will use both Cartesian and spatially polar coordinates  $t, r, \theta, \phi$  whereby  $d^2\Omega = \sin^2(\theta)d\phi \otimes d\phi$  is the unit sphere metric. It is already known from [10] that for a bivector  $\omega$  to be rotationally invariant leads in polars to

$$\omega^{23} = \frac{f(t, r)}{\sin \theta} = -\omega^{32}, \quad \omega^{01} = g(t, r) = -\omega^{10} \quad (3.1)$$

for some functions  $f, g$  and other components zero. Our approach is to solve (2.2) for  $S$  using the above form of  $\omega$  and  $\hat{\nabla}$  for the chosen metric, which in the present section is the spatial flat FLRW one [19]

$$g = -dt \otimes dt + a(t)^2(dr \otimes dr + r^2 d^2\Omega)$$

with

$$\begin{aligned}\hat{\Gamma}_{11}^0 &= \dot{a}a, & \hat{\Gamma}_{22}^0 &= \dot{a}ar^2, & \hat{\Gamma}_{33}^0 &= \dot{a}ar^2 \sin^2(\theta), \\ \hat{\Gamma}_{01}^1 &= \frac{\dot{a}}{a}, & \hat{\Gamma}_{22}^1 &= -r, & \hat{\Gamma}_{33}^1 &= -r \sin^2(\theta) \\ \hat{\Gamma}_{02}^2 &= \frac{\dot{a}}{a}, & \hat{\Gamma}_{21}^2 &= \frac{1}{r}, & \hat{\Gamma}_{33}^2 &= -\sin(\theta) \cos(\theta) \\ \hat{\Gamma}_{03}^3 &= \frac{\dot{a}}{a}, & \hat{\Gamma}_{31}^3 &= \frac{1}{r}, & \hat{\Gamma}_{23}^3 &= \cot(\theta)\end{aligned}$$

Remarkably, if  $\omega$  is generic in the sense that the functions  $a, f, g$  are algebraically independent and invertible then it turns out that one can next solve the Poisson-compatibility condition (2.2) for  $S$  *uniquely* using computer algebra. This is relevant if we drop the requirement (2.3) that  $\omega$  obeys the Jacobi identity which is to say if we allow the coordinate algebra to be nonassociative at order  $\lambda^2$  and if we drop (2.14) which is to say we allow a possible quantum effect where  $\nabla_1 g_1 = O(\lambda)$  in its antisymmetric part. Such a theory appears quite natural for this reason, but for the present purposes we do want to go further and impose (2.3) as well as the condition (2.14) for the existence of a fully quantum Levi-Civita connection.

**Proposition 3.1.** *In the FLRW spacetime with spherically symmetric Poisson tensor, a Poisson-compatible connection obeying (2.14) and (2.3) requires up to normalisation that  $g(r, t) = 0$  and  $f(r, t) = 1$ . The contorsion tensor in this case is*

$$S_{022} = a\dot{a}r^2, \quad S_{122} = a^2r, \quad S_{033} = a\dot{a}r^2 \sin^2(\theta), \quad S_{133} = a^2r \sin^2(\theta)$$

$$S_{120} = S_{123} = S_{223} = S_{320} = S_{130} = S_{132} = S_{230} = S_{233} = 0$$

up to the outer antisymmetry of  $S_{\mu\nu\gamma}$ . The remaining components  $S_{\mu 0\nu}$ ,  $S_{\mu 1\nu}$  are undetermined but are irrelevant to the combination  $\omega^{\alpha\beta}\nabla_\beta$  (the contravariant connection), which is uniquely determined.

**Proof.** As already noted in [10] for  $\omega$  of the rotationally invariant form (3.1) to obey (2.3) comes down to

$$g\partial_t f = g\partial_r f = 0 \quad (3.2)$$

which tells us that either  $f = k$  a constant or  $g = 0$ . We examine the former case, then the Poisson compatibility condition (2.2) becomes

$$S_{201} = 0, \quad S_{301} = 0, \quad S_{001}g - \partial_r g = 0, \quad S_{314} = 0, \quad S_{233} = 0, \quad S_{322} = 0$$

$$S_{031}kr^2a^2 + S_{112}g \sin(\theta) = 0, \quad r^2a^2k \sin(\theta)S_{021} + S_{311}g = 0, \quad a\dot{a} \sin(\theta)g + kS_{231} = 0$$

$$S_{203}g - r^2 \sin(\theta)(ra^2 - S_{231}) = 0, \quad S_{002}a^2g + S_{112}g = 0, \quad \sin(\theta)g + S_{230}kr = 0$$

$$kr^4a^2 \sin(\theta)a\dot{a} + S_{312}g + kS_{022} = 0, \quad a^2r^3k \sin^2(\theta) + S_{203}g + kS_{331} = 0$$

$$r^4a^3 \sin^2(\theta)S_{033} - S_{312} = 0, \quad \sin(\theta)g - rkS_{320} = 0, \quad kr^2S_{031} - S_{002}g \sin(\theta) = 0$$

$$a^2S_{003} - S_{322} = 0, \quad S_{021}r^2 \sin(\theta) + S_{003} = 0, \quad 2ka\dot{a}r^2 \sin(\theta) - S_{022} \sin(\theta) - kS_{033} = 0$$

$$ga\dot{a} \sin(\theta) - kS_{321} = 0, \quad r^2a^2\partial_t g + r^2a\dot{a}g + S_{012} = 0, \quad (2ra^2 - S_{122})k \sin^2(\theta) + S_{331} = 0.$$

This is not enough to determine all the components of  $S$  and hence  $\nabla$  but determines enough of them for the Ricci 2-form to be uniquely determined for  $k \neq 0$ , as

$$\mathcal{R}_{01} = \frac{1}{r^2} (5r^2a\dot{a}\partial_t g - \dot{a}^2 + gr^2a\dot{a} + \partial_t^2 gr^2a\dot{a} - \partial_r^2 gr^2 - 2r\partial_r g + 6g)$$

$$\mathcal{R}_{23} = \frac{1}{kr^2} \sin(\theta) (k^2r^4a^2 + ga^2r^2 - g^2).$$

We can now impose the Levi-Civita condition, using the *Physics* package in *Maple*, to expand (2.14) and solve for  $g$  simultaneously with the above requirements of (2.2) (details



omitted). This results in  $g = 0$  as the only unique solution permitting Poisson compatibility and a quantum Levi-Civita connection for  $f$  a (nonzero) constant. The case  $f = 0$  also has this conclusion and we exclude this so as to exclude the unquantized case  $\omega = 0$  in our analysis. We now go back and examine the second case, setting  $g = 0$  and leaving  $f$  arbitrary. Now (2.2) includes

$$\partial_r f = 0, \quad \partial_t f = 0$$

independently of the contorsion tensor. Hence this takes us back to  $g = 0$  and  $f$  constant again. We can absorb the latter constant in  $\lambda$ , i.e. we take  $k = 1$  up to the overall normalisation of  $\omega$ . Then the above-listed content of (2.2) setting  $k = 1$  and  $g = 0$  gives us the values of  $S$  and 12 undetermined components as stated.

In the process above we also solved (2.14) so this holds for the stated  $S$  with  $f = 1$  and  $g = 0$ . As this depended on a Maple solution, we check it analytically, setting

$$Q_{\gamma\mu\nu} = \omega^{\alpha\beta} g_{\rho\sigma} S^\sigma{}_{\beta\nu} (R^\rho{}_{\mu\gamma\alpha} + \nabla_\alpha S^\rho{}_{\gamma\mu}) - \omega^{\alpha\beta} g_{\rho\sigma} S^\sigma{}_{\beta\mu} (R^\rho{}_{\nu\gamma\alpha} + \nabla_\alpha S^\rho{}_{\gamma\nu}) \quad (3.3)$$

while from the above with  $k = 1$ ,  $g = 0$ , the Ricci two-form for our solution is

$$\mathcal{R} = -\frac{1}{2}a^2 r^2 \sin(\theta) d\theta \wedge d\phi \quad (3.4)$$

and is independent of the undetermined components. This allows us to compute  $\hat{\nabla}_0 \mathcal{R}_{\mu\nu} = \hat{\nabla}_1 \mathcal{R}_{\mu\nu} = 0$  as well as

$$\hat{\nabla}_2 \mathcal{R}_{\mu\nu} = a \sin(\theta) \begin{pmatrix} 0 & 0 & -\dot{a}r & 0 \\ 0 & 0 & -a & 0 \\ \dot{a}r & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\nabla}_3 \mathcal{R}_{\mu\nu} = a \sin(\theta) \begin{pmatrix} 0 & 0 & \dot{a}r & 0 \\ 0 & 0 & a & 0 \\ -\dot{a}r & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Further calculation yields  $Q_{0\mu\nu} = Q_{1\mu\nu} = 0$  and

$$Q_{2\mu\nu} = a \sin(\theta) \begin{pmatrix} 0 & 0 & -\dot{a}r & 0 \\ 0 & 0 & -a & 0 \\ \dot{a}r & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_{3\mu\nu} = a \sin(\theta) \begin{pmatrix} 0 & 0 & \dot{a}r & 0 \\ 0 & 0 & a & 0 \\ -\dot{a}r & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Substituting into (2.14) we see that this holds in the form  $\frac{1}{2}Q_{\gamma\alpha\beta} - \frac{1}{2}\hat{\nabla}_\gamma \mathcal{R}_{\mu\nu} = 0$ .  $\square$

Thus we see that if we want the Poisson bracket to obey the Jacobi identity so as to keep an associative coordinate algebra and if we want a full quantum Levi-Civita connection without an  $O(\lambda)$  correction to the antisymmetric part of the quantum metric compatibility tensor  $\nabla_1 g_1$ , then rotational invariance forces us to a model in which time is central and in which the other commutation relations are also determined uniquely from  $\omega^{\alpha\beta} \nabla_\beta$ . To work these out it is convenient (though not essential) to work with the angular variables in terms of  $z^i = x^i/r$  as redundant unit sphere variables at each  $r, t$ , with  $dz^i = \frac{1}{r} dx^i - \frac{x^i}{r^2} dr$ . Now, using the contorsion tensor above, the Christoffel symbols of the ‘quantising’ connection come out as

$$\begin{aligned}
\Gamma_{\mu\nu}^0 &= \begin{pmatrix} 0 & -S^0_{01} & -S^0_{02} & -S^0_{03} \\ 0 & a\dot{a} - S^0_{11} & -S^0_{12} & -S^0_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Gamma_{\mu\nu}^1 &= \begin{pmatrix} S^1_{00} & \frac{\dot{a}}{a} & S^1_{02} & S^1_{03} \\ \frac{\dot{a}}{a} + S^1_{10} & 0 & S^1_{12} & S^1_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\Gamma_{\mu\nu}^2 &= \begin{pmatrix} S^2_{00} & S^2_{01} & \frac{\dot{a}}{a} & S^2_{03} \\ S^2_{10} & S^2_{11} & r^{-1} & S^2_{12} \\ 0 & 0 & \frac{z^1(1-(z^2)^2)}{(z^3)^2} & \frac{(z^1)^2 z^2}{(z^3)^2} \\ 0 & 0 & \frac{(z^1)^2 z^2}{(z^3)^2} & \frac{z^1(1-(z^1)^2)}{(z^3)^2} \end{pmatrix}, & \Gamma_{\mu\nu}^3 &= \begin{pmatrix} S^3_{00} & S^3_{01} & S^3_{03} & \frac{\dot{a}}{a} \\ S^3_{10} & S^3_{11} & S^3_{12} & r^{-1} \\ 0 & 0 & \frac{z^2(1-(z^2)^2)}{(z^3)^2} & \frac{z^1(z^2)^2}{(z^3)^2} \\ 0 & 0 & \frac{z^1(z^2)^2}{(z^3)^2} & \frac{z^2(1-(z^1)^2)}{(z^3)^2} \end{pmatrix}
\end{aligned} \tag{3.5}$$

while the bimodule relations are independent of the undetermined components of  $S$  and come out as

$$[z^i, z^j] = \lambda \epsilon^{ij}_k z^k, \quad [z^i, dz^j] = \lambda z^j \epsilon^i_{mn} z^m dz^n.$$

Our quantum algebra at order  $\lambda$  is thus classical in the  $r, t$  directions and a standard fuzzy sphere as in section 2.5 in the angular ones. We also have

$$[r, x^i] = 0, \quad [r, dx^i] = 0, \quad [x^i, dr] = 0$$

so that  $r, t, dt, dr$  are all central. The undetermined contorsion components do not enter these relations from (2.5) because only  $\omega^{23}$  is nonzero so contraction with the Christoffel symbols selects only the  $\Gamma^\mu_{2\nu}$  and  $\Gamma^\mu_{3\nu}$  components which depend on only the corresponding  $S$  components.

For the rest of this section, for the sake of brevity, we shall concentrate on the case where the undetermined and irrelevant  $S$  components are all set to zero, returning later when analyzing general spherically symmetric metrics to see what happens when these are included. For the record, changing to Cartesians, the nonzero bimodule relations are

$$[x^i, x^j] = \lambda r \epsilon^{ij}_k x^k, \quad [x^i, \Omega^j] = \frac{\lambda}{r} x^j \epsilon^i_{mn} x^m \Omega^n$$

by letting  $dz^i = \Omega^i/r$  while our choice of the undetermined contorsion tensor components allows us to write down a nice expression for the ‘quantising’ connection

$$\Gamma^i_{jk} = -\frac{x^m}{r^2} \epsilon^i_{kn} \epsilon^n_{jm}, \quad \Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i_j, \quad \Gamma^i_{j0} = 0$$

The torsion comes out as

$$T^i_{jk} = \frac{x^m}{r^2} \epsilon^i_{mn} \epsilon^n_{jk}, \quad T^i_{0j} = \frac{\dot{a}}{a} \delta^i_j$$

and the Riemann, Ricci and scalar curvatures of the ‘quantising’ connection are

$$R^i_{jkl} = \frac{1}{r^2} \epsilon^i_{jm} \epsilon^m_{kl} + \frac{1}{r^4} (x_j x^m \epsilon^i_{mn} \epsilon^n_{kl} + x^j x^m \epsilon_{jmn} \epsilon^n_{kl}) \tag{3.6}$$

$$R_{ij} = \frac{1}{r^4} (\delta_{ij} r^2 - x_i x_j), \quad S = \frac{2}{a^2 r^2} \tag{3.7}$$

and it should be noted that  $R^0_{jkl} = R^i_{0kl} = R^i_{j0l} = 0$  and  $R_{i0} = R_{0i} = 0$ .

### 3.1. Construction of quantum metric and quantum Levi-Civita connection

Having solved for a Poisson bracket and Poisson compatible metric-compatible connection we are in a position to read off, according to the theory in [10], the full exterior algebra and the quantum metric to lowest order. First compute

$$H^{ij} = -\frac{1}{2r^3} (\epsilon_{nk}^i x_m x^j x^k - r^2 \epsilon_{nk}^i \delta_m^j x^k) dx^n \wedge dx^m \quad (3.8)$$

from which we get

$$\mathcal{R}_{mn} = \frac{a^2}{r} \epsilon_{mnk} x^k, \quad \mathcal{R} = \frac{a^2}{2r} \epsilon_{mnk} x^k dx^n \wedge dx^m \quad (3.9)$$

As with the curvature, all time components are equal to zero. From  $\Gamma$  and  $H^{ij}$  we have

$$dx^i \wedge_1 dx^j = dx^i \wedge dx^j + \frac{\lambda}{2r^3} (r^2 \epsilon_{nm}^i x^j + r^2 \epsilon_{mj}^i x_n + r^2 \epsilon_{nk}^i \delta_m^j x^k + \epsilon_{nk}^i x^k x^j x_m) dx^m \wedge dx^n$$

$$dr \wedge_1 dx^i = dr \wedge dx^i, \quad dt \wedge_1 dx^i = dt \wedge dx^i, \quad dx^i \wedge_1 dt = dx^i \wedge dt \quad (3.10)$$

$$\{dx^i, dx^j\}_1 = \frac{\lambda}{r^3} (r^2 \epsilon_{nm}^i x^j + r^2 \epsilon_{mj}^i x_n + r^2 \epsilon_{nk}^i \delta_m^j x^k + \epsilon_{nk}^i x^k x^j x_m) dx^m \wedge dx^n.$$

Similary, from  $\Gamma$  and  $\mathcal{R}$  we compute  $g_1$  from (2.9). Remarkably, the correction term  $\frac{\lambda}{2} \omega^{\alpha\beta} g_{\mu\rho} \Gamma_{\alpha\kappa}^\rho \Gamma_{\beta\nu}^\kappa$  exactly cancels the  $\lambda \mathcal{R}_{\mu\nu}$  so that  $g_{1\mu\nu} = g_{\mu\nu}$  and

$$g_1 = g_{\mu\nu} dx^\mu \otimes_1 dx^\nu \quad (3.11)$$

Moreover since the components  $g_{\mu\nu}$  depend only on time, we also have that  $g_{1\mu\nu} = \tilde{g}_{1\mu\nu}$ . It is a nice check to verify that  $\wedge_1(g_1) = 0$  is satisfied as it must from our general theory. The second version of the metric is subtly different and equality depends on the form of the FLRW metric. One can also compute

$$\omega^{\mu\nu} [\nabla_\mu, \nabla_\nu] T^\alpha_{\beta\gamma} = 0, \quad \omega^{\mu\nu} [\nabla_\mu, \nabla_\nu] S^\alpha_{\beta\gamma} = 0$$

$$\hat{\nabla}_i \mathcal{R}_{mn} = -\frac{a^2}{r^3} (\epsilon_{mnk} x^k x_i - r^2 \epsilon_{mni}) - \frac{a\dot{a}}{r} (\epsilon_{ink} x^k - \epsilon_{imk} x^k)$$

and see once again that (2.14) holds as it must by construction in proposition 3.1.

Hence a quantum Levi-Civita connection for  $g_1$  exists by the theory from [10] and from lemma 2.2 we find it to be

$$\nabla_1(dx^\ell) = \hat{\Gamma}_{\mu\nu}^\ell dx^\mu \otimes_1 dx^\nu$$

which, like the quantum metric earlier, keeps its undeformed coefficients in the coordinate basis *if* we keep all coefficients to the left and use  $\otimes_1$ . The theory in [10] ensures that this is quantum torsion free and quantum metric compatible as a bimodule connection with generalised braiding  $\sigma_1$  from (2.16) which computes as

$$\begin{aligned} \sigma_1(dx^a \otimes_1 dx^b) &= dx^b \otimes_1 dx^a + \frac{\lambda}{r^3} (\epsilon_k^{bk} x_m x_n + \epsilon_{kn}^b x^k x^a x_m + 2r^2 \epsilon_k^{ab} x_k \delta_{mn} \\ &\quad + 2r^2 \epsilon_{mk}^b x_k \delta_n^a - r^2 \epsilon_{mk}^a x^k \delta_m^b + r^2 \epsilon_{bck} x^k \delta^{ab}) dx^n \otimes_1 dx^m \\ \sigma_1(dt \otimes_1 dx^a) &= dx^a \otimes_1 dt \\ \sigma_1(dx^a \otimes_1 dt) &= dt \otimes_1 dx^a \end{aligned} \quad (3.12)$$

It is a reassuring but rather nontrivial check to verify directly from our results for  $\nabla_1, \sigma_1, g_1$  that  $\nabla_1 g_1 = 0$  as implied by the general theory in [10]. Lastly, we compute the quantum lift map from proposition 2.4 as

$$\begin{aligned} i_1(dx^a \wedge dx^b) &= \frac{1}{2} (dx^a \otimes_1 dx^b - dx^b \otimes_1 dx^a) \\ &\quad - \frac{\lambda}{4r} \epsilon^{ab}{}_m x_n (dx^m \otimes_1 dx^n + dx^n \otimes_1 dx^m) \\ i_1(dt \wedge dx^\alpha) &= \frac{1}{2} (dt \otimes_1 dx^\alpha - dx^\alpha \otimes_1 dt) \\ i_1(dx^\alpha \wedge dt) &= \frac{1}{2} (dx^\alpha \otimes_1 dt - dt \otimes_1 dx^\alpha) \end{aligned} \quad (3.13)$$

where we have taken the functorial choice  $I = 0$ .

### 3.2. Quantum Laplace operator and curvature tensors

We first observe that  $[dx^m, g_{mn}] = 0$  for the FLRW metric since either the coefficients  $g_{mn}$  depend only on  $t$  or are constant in our basis. Hence the inverse metric is simply  $(dx^a, dx^b)_1 = g^{ab}$  undeformed similarly to the coefficients of  $g_1$ , since then

$$\begin{aligned} (f \bullet dx^\alpha, g_{\mu\nu} \bullet dx^\mu)_1 \bullet dx^\nu &= (f \bullet dx^\alpha, dx^\mu \bullet g_{\mu\nu})_1 \bullet dx^\nu - (f \bullet dx^\alpha, [dx^\mu, g_{\mu\nu}])_1 \bullet dx^\nu \\ &= f \bullet (dx^\alpha, dx^\mu)_1 \bullet g_{\mu\nu} \bullet dx^\nu = f \bullet dx^\alpha \end{aligned}$$

as required, where we also need that  $\{g^{am}, g_{mn}\} = 0$  which holds for the FLRW metric. Similarly on the other side. It follows that the quantum dimension is the same as the classical dimension, namely 4, in our model. Similarly, because  $\nabla_1, g_1, (\cdot)_1$  also have their classical form, from theorem 2.3 we get that

$$\square_1 f = g^{\alpha\beta} \left( f_{,\alpha\beta} + f_{,\gamma} \hat{\Gamma}^\gamma{}_{\alpha\beta} \right)$$

is also undeformed on the underlying vector space. We used that  $\nabla_1$  is a left connection. We can also calculate the quantum Riemann tensor using (2.17) from which we see that corrections come from  $\wedge_1$ .

$$\begin{aligned} \text{Riem}_1(dx^i) &= (d \otimes \text{id}) \nabla_1 dx^i - \hat{\Gamma}^i{}_{\mu\gamma} dx^\mu \wedge_1 \hat{\Gamma}^\gamma{}_{\nu\alpha} dx^\nu \otimes_1 dx^\alpha \\ &= -\frac{1}{2} \hat{R}^i{}_{\alpha\mu\nu} dx^\mu \wedge dx^\nu \otimes_1 dx^\alpha - \frac{\lambda \dot{a}^2}{2r^3} \epsilon^i{}_{nk} x^k x_j x_m dx^m \wedge dx^n \otimes_1 dx^j \\ &\quad - \frac{\lambda \dot{a}^2}{2r} (\epsilon^i{}_{nm} x_j + \epsilon^i{}_{jm} x_n + \epsilon^i{}_{nk} \delta_{jm} x^k) dx^m \wedge dx^n \otimes_1 dx^j \\ \text{Riem}_1(dt) &= -\frac{1}{2} \hat{R}^0{}_{\alpha\mu\nu} dx^\mu \wedge dx^\nu \otimes_1 dx^\alpha \end{aligned}$$

Next step is to calculate  $\text{Ricci}_1$  which comes out as

$$\text{Ricci}_1 = -\frac{1}{2} \hat{R}_{\alpha\beta} dx^\beta \otimes_1 dx^\alpha$$

with no corrections to the coefficients in this form. The classical Ricci tensor for the Levi-Civita connection in our conventions is

$$\widehat{\text{Ricci}} = -\frac{1}{2} \left( (2\dot{a}^2 + a\ddot{a}) \delta_{ij} dx^i \otimes dx^j - 3 \frac{\ddot{a}}{a} dt \otimes dt \right)$$

and  $\text{Ricci}_1$  has the same form just with  $\otimes_1$ . The components again depend only on time, hence are central, which means that  $\rho = 0$  as well. It remains to verify that  $\wedge_1(\text{Ricci}_1) = 0$  as it should have the same quantum symmetry as  $g_1$ . So using (3.10), we first see that  $dt \wedge_1 dt = 0$  leaving (since the coefficients are time dependent they can be neglected here)

$$\delta_{ij} dx^i \wedge_1 dx^j = \frac{\lambda}{2r^3} \delta_{ij} (r^2 \epsilon^i_{nm} x^j + r^2 \epsilon^{ij}_m x_n + r^2 \epsilon^i_{nk} \delta^j_m x^k + \epsilon^i_{nk} x^k x^j x_m) dx^m \wedge dx^n = 0$$

From (2.23) we calculate the scalar curvature. Since neither the quantum metric or Ricci tensor have any semiclassical correction, it is straightforward to see that the same is true of the Ricci scalar, i.e.

$$S_1 = -\frac{1}{2} \widehat{S}, \quad \widehat{S} = \widehat{R}_{\mu\nu} g^{\mu\nu} = \frac{6}{a^2} (\ddot{a}a + \dot{a}^2). \quad (3.14)$$

## 4. Semiquantisation of spherically symmetric metrics

### 4.1. General analysis for the spherical case

In the previous section we saw that for a spherically symmetric Poisson tensor, demanding a compatible connection that also satisfied (2.14) results in a unique quantisation at order  $\lambda$  of the FLRW metric. Something similar for the Schwarzschild black hole in [10] suggests a general phenomenon for the spherically symmetric case. We prove in the present section that this is generically true. For the metric we choose a diagonal form

$$g = a^2(r, t) dt \otimes dt + b^2(r, t) dr \otimes dr + c^2(r, t) (d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi)$$

where  $a, b, c$  are arbitrary functional parameters. The Poisson tensor is taken to be the same as in section 3, once again parameterized by

$$\omega^{23} = \frac{f(t, r)}{\sin \theta} = -\omega^{32}, \quad \omega^{01} = g(t, r) = -\omega^{10}$$

The Christoffel symbols for the above metric are

$$\begin{aligned} \widehat{\Gamma}_{00}^0 &= \frac{\partial_t a}{a}, & \widehat{\Gamma}_{01}^0 &= \frac{\partial_r a}{a}, & \widehat{\Gamma}_{33}^0 &= -\frac{b \partial_t b \sin^2(\theta)}{a^2}, & \widehat{\Gamma}_{11}^0 &= -\frac{b \partial_t b}{a^2}, & \widehat{\Gamma}_{22}^0 &= -\frac{c \partial_t c}{a^2} \\ \widehat{\Gamma}_{00}^1 &= -\frac{a \partial_r a}{b^2}, & \widehat{\Gamma}_{11}^1 &= \frac{\partial_r b}{b}, & \widehat{\Gamma}_{33}^1 &= -\frac{b \partial_r b \sin^2(\theta)}{b^2}, & \widehat{\Gamma}_{01}^1 &= \frac{\partial_t b}{b}, & \widehat{\Gamma}_{22}^1 &= -\frac{c \partial_t c}{b^2} \\ \widehat{\Gamma}_{02}^2 &= \frac{\partial_t c}{c}, & \widehat{\Gamma}_{21}^2 &= \frac{\partial_r c}{c}, & \widehat{\Gamma}_{33}^2 &= -\sin(\theta) \cos(\theta) \\ \widehat{\Gamma}_{03}^3 &= \frac{\partial_t c}{c}, & \widehat{\Gamma}_{31}^3 &= \frac{\partial_r c}{c}, & \widehat{\Gamma}_{23}^3 &= \cot(\theta). \end{aligned} \quad (4.1)$$

**Theorem 4.1.** *For a generic spherically symmetric metric with functional parameters  $a, b, c$  and spherically symmetric Poisson tensor, the Poisson-compatibility (2.2) and the quantum Levi-Civita condition (2.14) require up to normalisation that  $g(r, t) = 0$  and  $f(r, t) = 1$  and the contorsion tensor components*

$$S_{022} = c\partial_t c, \quad S_{122} = c\partial_r c, \quad S_{033} = c\partial_t c \sin^2(\theta), \quad S_{133} = c\partial_r c \sin^2(\theta)$$

$$S_{120} = S_{123} = S_{223} = S_{320} = S_{130} = S_{132} = S_{230} = S_{233} = 0$$

*up to the outer antisymmetry of  $S_{\mu\nu\gamma}$ . The remaining components  $S_{\mu 0\nu}, S_{\mu 1\nu}$  are undetermined but do not affect  $\omega^{\alpha\beta}\nabla_\beta$ , which is unique. The relations of the quantum algebra are uniquely determined to  $O(\lambda)$  as those of the fuzzy sphere*

$$[z^i, z^j] = \lambda \epsilon^{ij}_k z^k, \quad [z^i, dz^j] = \lambda z^j \epsilon^i_{mn} z^m dz^n$$

*as in section 2.5 and*

$$[t, x^\mu] = [r, x^\mu] = 0, \quad [x^\mu, dt] = [x^\mu, dr] = 0$$

*so that  $t, r, dt, dr$  are central at order  $\lambda$ .*

**Proof.** The first part is very similar to the proof of proposition 3.1 but with more complicated expressions. We once again require that either constant  $f = k$  or  $g = 0$  for  $\omega$  to be Poisson. Taking first  $f = k$  and leaving  $g$  arbitrary gives the Poisson compatibility condition (2.2) as

$$S^1_{02} = 0, \quad S^3_{01} = 0, \quad S^3_{22} = 0, \quad S^3_{10} = 0, \quad S^0_{12} = 0, \quad S^3_{32} = 0,$$

$$gabS^0_{01} + ab\partial_r g + ga\partial_r b + gb\partial_r a = 0, \quad ab^2\partial_t g + abg\partial_t b + b^2g\partial_t a + a^3gS^0_{11} = 0$$

$$c^2S^3_{31} - b^2S^1_{22} + 2c\partial_r c = 0, \quad kc^2S^0_{31} + gb^2\sin(\theta)S^1_{12} = 0, \quad g\partial_t c \sin(\theta) - cS^1_{32} = 0,$$

$$a^2g\sin(\theta)S^3_{12} - ka^2S^0_{22} - kc\partial_t c = 0, \quad gb^2S^3_{02}\sin(\theta) + kb^2S^1_{22}\sin(\theta) - kc\partial_r c = 0,$$

$$S^1_{12} + S^0_{02} = 0, \quad kcS^0_{32} + g\partial_r c \sin(\theta) = 0, \quad g\sin(\theta)S^3_{11} + kS^0_{21} = 0,$$

$$kc\partial_r c + gb^2\sin(\theta)S^3_{02} + kc^2S^3_{31} = 0, \quad kd^3\sin(\theta)S^3_{21} - gb^2\partial_t c = 0,$$

$$c^2S^3_{30} + a^2S^0_{22}a^2 + 2c\partial_r c = 0, \quad gS^3_{00}b^2\sin(\theta) + ka^2S^0_{21} = 0, \quad kc^2S^0_{31} - gb^2S^0_{02} = 0$$

$$kc^3\sin(\theta)S^3_{20} - a^2g\partial_r c = 0, \quad kc\partial_t c - ga^2S^3_{12}\sin(\theta) + kc^2S^3_{30}, \quad a^2S^3_{11} - b^2S^3_{00} = 0$$

Once again, the above is enough to determine  $\mathcal{R}$  and can then be solved for  $g$  simultaneously with (2.14) using computer algebra (details omitted) assuming that  $a, b, c$  are generic in the sense of invertible and not enjoying any particular relations. The only solution is  $g(r, t) = 0$  as in the FLRW case. Now, starting over with  $g = 0$  and  $f$  arbitrary, Poisson compatibility (2.2) gives a number of constraints including

$$\partial_r f = 0, \quad \partial_t f = 0$$

which again forces us back to  $g = 0, f = k$  (which we set to be 1). Our above reduction of (2.2) setting  $g = 0$  and  $k = 1$  then gives the contorsion tensor is as stated and by construction we also solved (2.14).

Now we now check (2.14) for this solution directly and independently of the computer algebra (which then does not require  $a, b, c$  generic). For this, the generalised Ricci two-form comes out as

$$\mathcal{R} = -\frac{1}{2}c^2 \sin(\theta) d\theta \wedge d\phi$$

giving us  $\hat{\nabla}_0 \mathcal{R}_{\mu\nu} = \hat{\nabla}_1 \mathcal{R}_{\mu\nu} = 0$  as well as

$$\hat{\nabla}_2 \mathcal{R}_{\mu\nu} = c \sin(\theta) \begin{pmatrix} 0 & 0 & -\partial_t c & 0 \\ 0 & 0 & -\partial_r c & 0 \\ \partial_t c & \partial_r c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\nabla}_3 \mathcal{R}_{\mu\nu} = c \sin(\theta) \begin{pmatrix} 0 & 0 & -\partial_t c & 0 \\ 0 & 0 & -\partial_r c & 0 \\ \partial_t c & \partial_r c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Further calculation yields  $Q_{0\mu\nu} = Q_{1\mu\nu} = 0$  and

$$Q_{2\mu\nu} = c \sin(\theta) \begin{pmatrix} 0 & 0 & -\partial_t c & 0 \\ 0 & 0 & -\partial_r c & 0 \\ \partial_t c & \partial_r c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_{3\mu\nu} = c \sin(\theta) \begin{pmatrix} 0 & 0 & -\partial_t c & 0 \\ 0 & 0 & -\partial_r c & 0 \\ \partial_t c & \partial_r c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Substituting, see see that (2.14) holds in the form  $\frac{1}{2}Q_{\gamma\alpha\beta} - \frac{1}{2}\nabla_\gamma \mathcal{R}_{\mu\nu} = 0$ , where  $Q$  is the expression (3.3). This we have solved for the contorsion tensor obeying (2.2) and (2.14) for any  $a, b, c$  and this gives us  $\omega^{\alpha\beta} \nabla_\beta$  uniquely if these are generic.

Next we take the last two local coordinates  $z^1$  and  $z^2$  while identifying  $(z^3)^2 = 1 - (z^2)^2 - (z^1)^2$ . Then the Poisson tensor becomes

$$\omega = z^3 \left( \frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial z^2} - \frac{\partial}{\partial z^2} \otimes \frac{\partial}{\partial z^1} \right)$$

giving the coordinate algebra as stated. Since only  $\omega^{23} = -\omega^{32}$  is nonzero, we also have  $\{t, x^\mu\} = \{r, x^\mu\} = 0$ . The ‘quantising’ connection is

$$\nabla dt = -\frac{\partial_t a}{a} dt \otimes dt - \frac{b \partial_t b}{a^2} dr \otimes dr - \frac{\partial_r a}{a} (dr \otimes dt + dt \otimes dr) - S^0_{0\mu} dt \otimes dx^\mu - S^0_{1\mu} dr \otimes dx^\mu$$

$$\nabla dr = -\frac{a \partial_r a}{b^2} dt \otimes dt - \frac{\partial_t b}{b} dr \otimes dr - \frac{\partial_r b}{b} (dr \otimes dt + dt \otimes dr) - S^1_{0\mu} dt \otimes dx^\mu - S^1_{1\mu} dr \otimes dx^\mu$$

$$\nabla dz^i = -\frac{\partial_r c}{c} dr \otimes dz^i - \frac{\partial_t c}{c} dt \otimes dz^i - \delta_{ab} z^i dz^a \otimes dz^b - S^i_{0\mu} dt \otimes dx^\mu - S^i_{1\mu} dr \otimes dx^\mu$$

due to the Christoffel symbols

$$\begin{aligned}\Gamma^0_{\mu\nu} &= \begin{pmatrix} \frac{\partial_a a}{a} & \frac{\partial_a a}{a} S^0_{01} & S^0_{02} & S^0_{03} \\ \frac{\partial_r a}{a} & \frac{b\partial_r b}{a^2} + S^0_{11} & S^0_{12} & S^0_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Gamma^1_{\mu\nu} &= \begin{pmatrix} S^1_{00} + \frac{a\partial_r a}{b^2} & \frac{\partial_b b}{b} & S^1_{02} & S^1_{03} \\ \frac{\partial_b b}{b} + S^1_{10} & \frac{\partial_r b}{b} & S^1_{12} & S^1_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \Gamma^2_{\mu\nu} &= \begin{pmatrix} S^2_{00} & S^2_{01} & \frac{\partial_c c}{c} & S^2_{03} \\ S^2_{10} & S^2_{11} & \frac{\partial_c c}{c} & S^2_{12} \\ 0 & 0 & \frac{z^1(1-(z^2)^2)}{(z^3)^2} & \frac{(z^1)^2 z^2}{(z^3)^2} \\ 0 & 0 & \frac{(z^1)^2 z^2}{(z^3)^2} & \frac{z^1(1-(z^1)^2)}{(z^3)^2} \end{pmatrix}, & \Gamma^3_{\mu\nu} &= \begin{pmatrix} S^3_{00} & S^3_{01} & S^3_{03} & \frac{\partial_c c}{c} \\ S^3_{10} & S^3_{11} & S^3_{12} & \frac{\partial_c c}{c} \\ 0 & 0 & \frac{z^2(1-(z^2)^2)}{(z^3)^2} & \frac{z^1(z^2)^2}{(z^3)^2} \\ 0 & 0 & \frac{z^1(z^2)^2}{(z^3)^2} & \frac{z^2(1-(z^1)^2)}{(z^3)^2} \end{pmatrix}\end{aligned}$$

From (2.5) we immediately see that  $[t, dx^\mu] = [r, dx^\mu] = 0$ . Furthermore,

$$[z^1, dx^\mu] = -z^3 \Gamma^\mu_{3\beta} dx^\beta, \quad [z^2, dx^\mu] = z^3 \Gamma^\mu_{2\beta} dx^\beta$$

so we can read off from the Christoffel symbols that  $[x^\mu, dt] = [x^\mu, dr] = 0$ . Evaluating the nonzero terms gives

$$[z^1, dz^1] = -\frac{z^1}{z^3} (z^1 z^2 dz^1 - ((z^1)^2 - 1) dz^2), \quad [z^2, dz^2] = \frac{z^2}{z^3} (z^1 z^2 dz^2 - ((z^2)^2 - 1) dz^1)$$

$$[z^2, dz^1] = -\frac{z^2}{z^3} (z^1 z^2 dz^1 - ((z^1)^2 - 1) dz^2), \quad [z^1, dz^2] = \frac{z^1}{z^3} (z^1 z^2 dz^2 - ((z^2)^2 - 1) dz^1)$$

which upon using  $\sum_i z^i dz^i = 0$  becomes  $[z^i, dz^j] = \lambda z^j \epsilon^i_{mn} z^m dz^n$ .  $\square$

So we see that in generalizing the analysis, we recover the same bimodule structure as in the FLRW case and by extension, that of the fuzzy sphere in section 2.5. The noncommutativity is purely spatial and confined to spatial ‘spherical shells’; the surfaces of fuzzy spheres at each time and each classical radius  $r$ . We have checked directly in the proof of the theorem that this is a solution for all  $a, b, c$  while for generic  $a, b, c$  we showed that it is the only solution, i.e. we are forced into this form from our assumptions and spherical symmetry. There do in fact exist particular combinations of these metric functional parameters which permit alternative solutions for  $f$  and  $g$ . In fact we already saw an example in section 2.4 with the Bertotti–Robinson metric which had  $f = 0$  and  $g = -r$ . To see why this was allowed, we take a brief look at the Poisson compatibility condition (2.2) again, now with arbitrary  $f$  and  $g$  and note the particular constraint

$$S^0_{32}fc + g\partial_r c \sin(\theta) = 0, \quad -S^1_{32}fc + g\partial_r c \sin(\theta) = 0$$

It is clear that with  $c$  arbitrary, we cannot have  $f = 0$  without also having  $g = 0$ . However, allowing  $c = \text{constant}$  means we can also take  $f = 0$  and  $g$  nonzero, as is the case with the Bertotti–Robinson metric. This leads to a different contorsion tensor with a flat  $\nabla$  and in fact this exceptional model was solved using algebraic methods in [28] including the quantum Levi-Civita connection to all orders in  $\lambda$ .

Proceeding with our generic spherically symmetric metric, for brevity we define

$$F_1 = \frac{1}{a^3 b} (a^2 b \partial_r^2 a - a b^2 \partial_t^2 b + b^2 \partial_t b \partial_r a - a^2 \partial_r b \partial_t a), \quad F_2 = \frac{a^2}{b^2} F_1$$



$$\begin{aligned}
F_3 &= \frac{c}{a^3 b} (a \partial_t b \partial_r c - b a \partial_t \partial_r c + b \partial_t c \partial_r a), \quad F_4 = \frac{c}{a^3 b^2} (b^2 \partial_t a \partial_t c + a^2 \partial_r c \partial_r a - b^2 a \partial_t^2 c) \\
F_5 &= \frac{c}{b^3 a^2} (a^2 \partial_r b \partial_r c + b^2 \partial_t c \partial_t b - a^2 b \partial_r^2 c), \quad F_6 = \frac{1}{b^2 a^2} (b^2 a^2 + b^2 (\partial_t c)^2 - a^2 (\partial_r c)^2) \\
F_7 &= \frac{1}{a^2 b c} (a^2 b \partial_r^2 - a^2 \partial_r b \partial_r c - b^2 \partial_t b \partial_t c), \quad F_8 = \frac{1}{a b^2 c} (-b^2 a \partial_t^2 + a^2 \partial_r a \partial_r c + b^2 \partial_t a \partial_t c) \\
F_9 &= \frac{1}{a b c} (b \partial_r a \partial_t c + a \partial_t b \partial_r c - a b \partial_r \partial_t c)
\end{aligned}$$

in which terms the Riemann tensor for the Poisson-compatible ‘quantising’ connection comes out as

$$\text{Riem}(\text{dt}) = -F_1 \text{dr} \wedge \text{dt} \otimes \text{dr} + C(\text{dr}), \quad \text{Riem}(\text{dr}) = F_2 \text{dr} \wedge \text{dt} \otimes \text{dt} + C(\text{dr}),$$

$$\text{Riem}(\text{dz}^i) = \delta_{ab} \text{dz}^i \wedge \text{dz}^a \otimes \text{dz}^b + C(\text{dz}^i),$$

where we have collected in the tensor  $C$  all contributions coming from the undetermined components of the contorison tensor, namely

$$\begin{aligned}
C(\text{dx}^\iota) &= \nabla_\mu S^\iota_{0\alpha} \text{dx}^\mu \wedge \text{dt} \otimes \text{dx}^\alpha + \nabla_\mu S^\iota_{1\alpha} \text{dx}^\mu \wedge \text{dr} \otimes \text{dx}^\alpha \\
&\quad + (S^\iota_{0\kappa} S^\kappa_{\nu\alpha} + S^\iota_{0\alpha} S^0_{0\nu} + S^\iota_{1\alpha} S^1_{0\nu}) \text{dt} \wedge \text{dx}^\nu \otimes \text{dx}^\alpha \\
&\quad + (S^\iota_{1\kappa} S^\kappa_{\nu\alpha} + S^\iota_{0\alpha} S^0_{1\nu} + S^\iota_{1\alpha} S^1_{1\nu}) \text{dr} \wedge \text{dx}^\nu \otimes \text{dx}^\alpha \\
&\quad + (S^\iota_{0\alpha} S^0_{i\nu} + S^\iota_{1\alpha} S^1_{i\nu}) \text{dz}^i \wedge \text{dx}^\nu \otimes \text{dx}^\alpha
\end{aligned}$$

We also have the classical Ricci tensor for the Levi-Civita connection

$$\begin{aligned}
\widehat{\text{Ricci}} &= -\frac{1}{2} ((F_2 + 2F_8) \text{dt} \otimes \text{dt} - (F_1 + 2F_7) \text{dr} \otimes \text{dr} + F_9 (\text{dt} \otimes \text{dr} + \text{dr} \otimes \text{dt}) \\
&\quad - \frac{1}{(z^3)^2} (F_6 + F_5 - F_4) \delta_{ij} \text{dz}^i \otimes \text{dz}^j)
\end{aligned}$$

and, for later reference, the Einstein tensor

$$\begin{aligned}
\widehat{\text{G}} &= -\frac{1}{2} \left( \frac{a^2}{c^2} (F_5 + 2F_6) \text{dt} \otimes \text{dt} - \frac{a^2}{c^2} (F_6 - 2F_4) \text{dr} \otimes \text{dr} + F_9 (\text{dt} \otimes \text{dr} + \text{dr} \otimes \text{dt}) \right. \\
&\quad \left. - \frac{1}{(z^3)^2} (F_5 - F_4 - \frac{c^2}{b^2} F_1) \delta_{ij} \text{dz}^i \otimes \text{dz}^j \right)
\end{aligned}$$

Before continuing, we turn briefly to the quantity  $\omega^{\beta\alpha}(R^\iota_{\nu\mu\alpha} + S^\iota_{\mu\nu;\alpha})$  which appears in several formulas in section 2, most importantly, the quantum Levi-Civita connection condition (2.14). In particular, we note that it is surprisingly simple with the only nonzero components

$$\omega^{2\alpha}(R^2_{22\alpha} + S^2_{22;\alpha}) = \frac{z^1 z^2}{z^3}, \quad \omega^{2\alpha}(R^3_{22\alpha} + S^3_{22;\alpha}) = \frac{(z^2)^2 - 1}{z^3},$$

$$\begin{aligned}\omega^{3\alpha}(R^3_{23\alpha} + S^3_{32;\alpha}) &= \frac{z^2 z^1}{z^3}, & \omega^{3\alpha}(R^3_{23\alpha} + S^3_{32;\alpha}) &= \frac{(z^2)^2 - 1}{z^3}, \\ \omega^{2\alpha}(R^2_{32\alpha} + S^2_{32;\alpha}) &= \frac{1 - (z^1)^2}{z^3}, & \omega^{2\alpha}(R^3_{32\alpha} + S^3_{32;\alpha}) &= -\frac{z^1 z^2}{z^3}, \\ \omega^{3\alpha}(R^2_{33\alpha} + S^2_{33;\alpha}) &= \frac{1 - (z^1)^2}{z^3}, & \omega^{3\alpha}(R^3_{33\alpha} + S^3_{33;\alpha}) &= -\frac{z^1 z^2}{z^3}\end{aligned}$$

The undetermined components of  $S$  do not contribute. In general the ‘quantising’ connection always enters in combination with the Poisson tensor e.g.  $\omega^{\alpha\beta}\Gamma^\iota_{\beta\gamma}$  so the same argument as in the proof of theorem 4.1 applies and we do not see the undetermined components  $S_{\mu 1\nu}$  or  $S_{\mu 2\nu}$  in geometrically relevant expressions, as demonstrated by the generalized Ricci 2-form which now is

$$\mathcal{R} = -\frac{1}{2}c^2\epsilon_{mnk}z^k dz^m \wedge dz^n = -\frac{c^2}{z^3}dz^1 \wedge dz^2$$

From this we have the quantum wedge product

$$dt \wedge_1 dx^\mu = dt \wedge dx^\mu, \quad dx^\mu \wedge_1 dt = dx^\mu \wedge dt, \quad dr \wedge_1 dx^\mu = dr \wedge dx^\mu, \quad dx^\mu \wedge_1 dr = dx^\mu \wedge dr$$

$$dz^i \wedge_1 dz^j = dz^i \wedge dz^j + \frac{\lambda}{2}(3z^i z^j - \delta^{ij})dz^1 \wedge dz^2$$

$$\{dz^i, dz^j\}_1 = \lambda(3z^i z^j - \delta^{ij})dz^1 \wedge dz^2$$

Our next step is to calculate the quantum metric.

$$g_1 = g_{\mu\nu}dx^\mu \otimes_1 dx^\nu + \frac{\lambda c^2}{2(z^3)^2}\epsilon_{3ij}(z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j)$$

Working with metric components  $\tilde{g}_{ij}$  (in the middle) we get

$$h = \frac{c^2(2 - (z^3)^2)}{(z^3)^3}\epsilon_{3ij}dz^i \otimes_1 dz^j$$

Meanwhile, for the inverse metric with components  $\tilde{g}^{ij}$  we get

$$(dz^1, dz^2)_1 = g^{23} + \frac{\lambda}{2}\frac{z^3}{c^2}, \quad (dz^2, dz^1)_1 = g^{32} - \frac{\lambda}{2}\frac{z^3}{c^2}$$

$$(dt, dt)_1 = g^{00}, \quad (dr, dr)_1 = g^{11}, \quad (dz^1, dz^1)_1 = g^{22}, \quad (dz^2, dz^2)_1 = g^{33}$$

Now, lemma 2.2 gives the quantum connection as

$$\nabla_1(dt) = -\hat{\Gamma}^0_{\mu\nu}dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{2(z^3)^2}\frac{c\partial_t c}{a^2}\epsilon_{3ij}(z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j)$$

$$\nabla_1(dr) = -\hat{\Gamma}^1_{\mu\nu}dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2(z^3)^2}\frac{c\partial_r c}{b^2}\epsilon_{3ij}(z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j) \quad (4.2)$$

$$\nabla_1(dz^a) = -\hat{\Gamma}_{\mu\nu}^i dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} \left( \epsilon_{ijk} z^k dz^a \otimes_1 dz^j - \frac{1}{(z^3)^2} \epsilon^a_{i3} dz^3 \otimes_1 dz^i \right)$$

Lastly, we calculate the associated braiding. Its contributions at order  $\lambda$  are

$$\sigma_1(dz^i \otimes_1 dz^j) = dz^j \otimes_1 dz^i + \lambda (\epsilon_{abc} z^c z^i z^j + \epsilon^j_{ac} z^c \delta^i_b + \epsilon^{ij}_c z^c \delta_{ab}) dz^a \otimes_1 dz^b$$

when calculated using (2.16). Meanwhile, from proposition 2.4 quantum antisymmetric lift is

$$i_1(dx^\mu \wedge dx^\nu) = \frac{1}{2} (dx^\mu \otimes_1 dx^\nu - dx^\nu \otimes_1 dx^\mu) + \lambda I(dx^\mu \wedge dx^\nu) \quad (4.3)$$

The functorial choice gives  $I(dx^\mu \wedge dx^\nu) = 0$ , but we leave this general.

#### 4.2. Quantum Laplace operator and curvature tensor

Following from the previous section, we first calculate the Laplace operator. From theorem 2.3 we get that

$$\square f = g^{\alpha\beta} \left( f_{,\alpha\beta} + f_{,\gamma} \hat{\Gamma}^\gamma_{\alpha\beta} \right)$$

as with the flat FLRW metric, is undeformed in the underlying algebra. Then, (2.18) gives the quantum Riemann tensor as

$$\begin{aligned} \text{Riem}_1(dt) &= -\frac{1}{2} \hat{R}^0_{\alpha\mu\nu} dx^\mu \wedge dx^\nu \otimes_1 dx^\alpha - \frac{\lambda}{2(z^3)^2} (\epsilon_{3ij} z^3 (F_3 dt - F_4 dr) \wedge dz^i \otimes_1 dz^j \\ &\quad + \epsilon_{3ij} z^i (F_3 dt - F_4 dr) \wedge dz^j \otimes_1 dz^3) \end{aligned}$$

$$\begin{aligned} \text{Riem}_1(dr) &= -\frac{1}{2} \hat{R}^1_{\alpha\mu\nu} dx^\mu \wedge dx^\nu \otimes_1 dx^\alpha + \frac{\lambda}{2(z^3)^2} (\epsilon_{3ij} z^3 (F_5 dt - F_3 dr) \wedge dz^i \otimes_1 dz^j \\ &\quad + \epsilon_{3ij} z^i (F_5 dt - F_3 dr) \wedge dz^j \otimes_1 dz^3) \end{aligned}$$

$$\text{Riem}_1(dz^\mu) = -\frac{1}{2} \hat{R}^\mu_{\alpha\mu\nu} dx^\mu \wedge dx^\nu \otimes_1 dx^\alpha + \frac{\lambda F_6}{2(z^3)^2} (1 + (z^3)^3) dz^1 \wedge dz^2 \otimes_1 dz^\mu$$

Using the lift map (4.3) and the tensor formula (2.21) we get the quantum Ricci tensor as

$$\begin{aligned} \text{Ricci}_1 &= -\frac{1}{2} \hat{R}_{\mu\nu} dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{4(z^3)^2} (F_6 + F_5 - F_4) \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j) \\ &\quad - \frac{3\lambda}{4z^3} F_6 \epsilon_{3ij} dz^i \otimes_1 dz^j - \frac{\lambda}{2} \hat{R}^\alpha_{\gamma\eta\zeta} I^{\eta\zeta}_{\alpha\nu} dx^\nu \otimes_1 dx^\gamma \end{aligned}$$

Lastly, we fix  $I$  so that we have both  $\wedge_1(\text{Ricci}_1) = 0$  and  $\text{flip}(* \otimes *) \text{Ricci}_1 = \text{Ricci}_1$ . For the latter, it is easiest to consider the quantum Ricci tensor with components in the middle so that from section 2.2 we have

$$\begin{aligned} \rho &= -\frac{1}{4(z^3)^3} ((F_5 - F_4)(2 - (z^3)^2) + 2F_6(1 + (z^3)^2)) \epsilon_{3ij} dz^i \otimes_1 dz^j \\ &\quad - \frac{1}{2} \hat{R}^\alpha_{\gamma\eta\zeta} I^{\eta\zeta}_{\alpha\nu} dx^\nu \otimes_1 dx^\gamma \end{aligned}$$

where, for comparison,  $\rho = -\frac{1}{2}\rho_{\mu\nu}dx^\mu \otimes_1 dx^\nu$ . The reality condition, since the coefficients are real and  $\lambda$  imaginary, requires this to be antisymmetric. Also,

$$\wedge_1(\text{Ricci}_1) = -\frac{3\lambda}{2z^3}F_6dz^1 \wedge dz^2 - \frac{\lambda}{2}\widehat{R}^\alpha_{\gamma\eta\zeta}I^{\eta\zeta}_{\alpha\nu}dx^\nu \wedge dx^\gamma$$

Putting this together results in

$$\widehat{R}^\alpha_{\gamma\eta\zeta}I^{\eta\zeta}_{\alpha\nu}dx^\nu \otimes_1 dx^\gamma = -\frac{3}{2z^3}F_6\epsilon_{3ij}dz^i \otimes_1 dz^j$$

This answer for the contraction of the lift map with the Riemann tensor is unique, but the same is not true of the lift map itself and we are left with a large moduli of possible solutions with most components of  $I$  undetermined. We examine the simplest possible form by setting these to zero, leaving us with

$$i_1(dz^1 \wedge dz^2) = \frac{1}{2}(dz^1 \otimes_1 dz^2 - dz^2 \otimes_1 dz^1) - \frac{3\lambda}{4z^3}\delta_{ij}dz^i \otimes_1 dz^j$$

as the only part with an  $\mathcal{O}(\lambda)$  contribution and which is the same as for the fuzzy sphere seen previously. This results in

$$\rho = -\frac{1}{4(z^3)^3}(F_6 + F_5 - F_4)(2 - (z^3)^2)\epsilon_{3ij}dz^i \otimes_1 dz^j$$

which we note has the same structure as  $h$  for the quantum metric, but with different coefficients. The quantum Ricci tensor (with components on the left) is now

$$\text{Ricci}_1 = -\frac{1}{2}\widehat{R}_{\mu\nu}dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{4(z^3)^2}(F_6 + F_5 - F_4)\epsilon_{3ij}(z^3dz^i \otimes_1 dz^j - z^jdz^3 \otimes_1 dz^i)$$

The scalar curvature, using (2.23), has no corrections and comes out as

$$S_1 = -\frac{1}{2}\widehat{S}, \quad \widehat{S} = \widehat{R}_{\mu\nu}g^{\mu\nu} = \frac{2}{c^2}(F_6 + 2F_5 - 2F_4) - \frac{2}{b^2}F_1$$

Note that it depends only on  $t$  and  $r$  and is therefore central in the algebra. From proposition 2.1, the quantum dimension comes out as

$$\dim(M)_1 = \dim(M) - \lambda\omega^{\alpha\beta}g^{\mu\nu}_{,\alpha}\Gamma_{\nu\beta\mu} = 4$$

It might also be of interest to think about a quantum Einstein tensor. While a general theorem has not been established, we could consider a ‘naive’ construction by analogy to the classical expression. Since the quantum and classical dimensions are the same, we could take for example

$$G_1 = \text{Ricci}_1 - \frac{1}{2}S_1g_1$$

which has the same form as the classical case. This can be written as

$$G_1 = -\frac{1}{2}G_{1\mu\nu}dx^\mu \otimes_1 dx^\nu = -\frac{1}{2}dx^\mu \bullet \widetilde{G}_{1\mu\nu} \otimes_1 dx^\nu$$

where  $\tilde{G}_{1\mu\nu} = G_{1\mu\nu} - \lambda\omega^{\alpha\beta}\hat{G}_{\gamma\nu,\alpha}\Gamma^\gamma_{\beta\mu}$  and as previously the hat denotes that this is for the Levi-Civita connection. Now since in our case  $S_1$  is purely classical and central, this can be expressed in component form as

$$\tilde{G}_{1\mu\nu} = \widetilde{\tilde{R}}_{1\mu\nu} - \frac{1}{2}\hat{S}\tilde{g}_{\mu\nu}$$

following the same pattern as how the components of  $\text{Ricci}_1$  and  $g_1$  are written. If we write  $\tilde{G}_{1\mu\nu} = \tilde{G}_{\mu\nu} + \lambda\Sigma_{\mu\nu}$  then

$$\Sigma = \rho - \frac{1}{4}\hat{S}h = \frac{1}{4(z^3)^3}(F_5 - F_4 - \frac{c^2}{b^2}F_1)(2 - (z^3)^2)\epsilon_{3ij}dz^i \otimes_1 dz^j$$

where  $\Sigma = -\frac{1}{2}\Sigma_{\mu\nu}dx^\mu \otimes_1 dx^\nu$  and is manifestly antisymmetric corresponding to  $\text{flip}(* \otimes *)G_1 = G_1$ . Indeed,  $G_1$  is both quantum symmetric and obeys the reality condition since  $\text{Ricci}_1$ ,  $g_1$  do, and  $\hat{S}$  is real and central.

With the results of this section, we can calculate the quantum geometry for all metrics of the form (4.1) simply by choosing appropriate parameters for  $a$ ,  $b$  and  $c$ .

#### 4.3. FLRW metric case

Comparing the above results with those of the FLRW metric in section 3, there is a disparity that previously the quantum metric appeared undeformed while now it has a quantum correction. We resolve this here. We first specialise the general theory above to the FLRW metric

$$g = -dt \otimes dt + a^2(t)(dr \otimes dr + r^2\delta_{ij}dz^i \otimes dz^j)$$

where we identify the parameters

$$a(r, t) = 1, \quad b(r, t) = a(t), \quad c(r, t) = ra(t).$$

This gives us the ‘quantising’ connection up to undetermined but irrelevant contorsion tensor components (which are set to zero for simplicity)

$$\nabla(dt) = -a\dot{a}dr \otimes dr, \quad \nabla(dr) = -a\dot{a}(dr \otimes dt + dt \otimes dr)$$

$$\nabla(dz^i) = -\frac{1}{r}dr \otimes dz^i - \frac{\dot{a}}{a}dt \otimes dz^i - \delta_{ab}z^i dz^a \otimes dz^b$$

Meanwhile, for the classical Ricci tensor of the Levi-Civita connection we have

$$\widehat{\text{Ricci}} = -\frac{1}{2}\left(-3\frac{\ddot{a}}{a}dt \otimes dt + (2\dot{a}^2 + a\ddot{a})(dr \otimes dr + r^2\delta_{ij}dz^i \otimes dz^j)\right)$$

and the curvature scalar is as in (3.14). Now, the quantum metric comes out as

$$g_1 = g_{\mu\nu}dx^\mu \otimes_1 dx^\nu + \frac{\lambda r^2}{2(z^3)^2}\epsilon_{3ij}(z^3 dz^i \otimes_1 dz^j - dz^3 \otimes_1 z^i dz^j) \quad (4.4)$$

where  $x^\mu$  refers to coordinates  $t, r, z^1, z^2$  as we used polar coordinates. Equivalently,  $\tilde{g}_{ij}$  (where the components are in the middle) has quantum correction

$$h = \frac{a(t)^2 r^2 (2 - (z^3)^2)}{(z^3)^3} \epsilon_{3ij} dz^i \otimes_1 dz^j \quad (4.5)$$

For the inverse metric with components  $\tilde{g}^{ij}$  we obtain

$$(dz^1, dz^2)_1 = g^{23} + \frac{\lambda}{2} \frac{z^3}{r^2}, \quad (dz^2, dz^1)_1 = g^{32} - \frac{\lambda}{2} \frac{z^3}{r^2}$$

$$(dt, dt)_1 = g^{00}, \quad (dr, dr)_1 = g^{11}, \quad (dz^1, dz^1)_1 = g^{22}, \quad (dz^2, dz^2)_1 = g^{33}$$

The quantum connection is

$$\nabla_1(dt) = -\hat{\Gamma}^0_{\mu\nu} dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{2(z^3)^2} a \dot{a} r^2 \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j)$$

$$\nabla_1(dr) = -\hat{\Gamma}^1_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2(z^3)^2} \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j)$$

$$\nabla_1(dz^a) = -\hat{\Gamma}^i_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} \left( \epsilon_{ijk} z^k z^a dz^i \otimes_1 dz^j - \frac{1}{(z^3)^2} \epsilon^a_{i3} dz^3 \otimes_1 dz^i \right)$$

Then, by computing  $(F_6 + F_5 - F_4) = r^2(2\dot{a}^2 + a\ddot{a})$  the quantum Ricci tensor is

$$\text{Ricci}_1 = \hat{R}_{\mu\nu} dx^\mu \otimes_1 dx^\nu - \frac{\lambda r^2}{4(z^3)^2} (2\dot{a}^2 + a\ddot{a}) \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j)$$

With components in the middle, this comes out as

$$\rho = -\frac{1}{4(z^3)^3} r^2 (2\dot{a}^2 + a\ddot{a}) (2 - (z^3)^2) \epsilon_{3ij} dz^i \otimes_1 dz^j$$

and in either case  $S_1 = -\frac{1}{2}\hat{S}$  in our conventions.

Now these results appear at first sight to be at odds with section 3 since there the quantum metric from (3.11) looks the same as classical when written in Cartesian coordinates. We first write it in terms of  $dz^i$  by writing  $dx^i = r dz^i - z^i dr$  and note that since  $z^i dr = z^i \bullet dr$ , we can take such  $z^i$  terms to the other side of  $\otimes_1$ . Since also  $dz^i \bullet z^i = O(\lambda^2)$  (sum over  $i$ ), we find

$$g_1 = -dt \otimes_1 dt + a^2(t)(dr \otimes_1 dr + r^2 \delta_{ij} dz^i \otimes_1 dz^j)$$

This begins to look like (4.4) but note that only  $z^1, z^2$  (say) are coordinates with  $z^3$  a function of them. In particular,

$$dz^3 = -(z^3)^{-1} \bullet (z^1 \bullet dz^1 + z^2 \bullet dz^2) = -(z^3)^{-1} (z^1 dz^1 + z^2 dz^2) - \frac{\lambda}{2} \epsilon_{3ij} z^i dz^j$$

would be needed to reduce to the form of (4.4) where the first term has only  $dz^a \otimes_1 dz^b$  for  $a, b = 1, 2$ . Equivalently, we show that we have the same  $\tilde{g}_{\mu\nu}$ . Considering only the angular part  $\delta_{ij} dz^i \otimes_1 dz^j = dz^1 \otimes_1 dz^1 + dz^2 \otimes_1 dz^2 + dz^3 \otimes_1 dz^3$  and examining the last term more closely (sum over repeated indices understood)

$$\begin{aligned}
dz^3 \otimes_1 dz^3 &= (z^3)^{-1} \bullet z^a \bullet dz^a \otimes_1 (z^3)^{-1} \bullet z^b \bullet dz^b \\
&= \left( \frac{z^a}{z^3} + \frac{\lambda}{2} \epsilon_{ac} z^c \right) \bullet dz^a \otimes_1 \left( \frac{z^b}{z^3} + \frac{\lambda}{2} \epsilon_{bd} z^d \right) \bullet dz^b \\
&= dz^a \bullet \left( \frac{z^a}{z^3} + \frac{\lambda}{2} \epsilon_{ac} z^c \right) \bullet \left( \frac{z^b}{z^3} + \frac{\lambda}{2} \epsilon_{bd} z^d \right) \otimes_1 dz^b + \left[ \frac{z^a}{z^3}, dz^a \right] \bullet \frac{z^b}{z^3} \otimes_1 dz^b \\
&= dz^a \bullet \frac{z^a z^b}{(z^3)^2} \otimes_1 dz^b + \frac{\lambda}{2} \left( dz^a \frac{1}{(z^3)^3} (\epsilon_{ab} + z^3 \epsilon_{bd} z^a z^d + z^3 \epsilon_{ac} z^b z^c) \otimes_1 dz^b \right. \\
&\quad \left. - dz^a \frac{1}{(z^3)^3} \epsilon_{cb} z^c z^a \otimes_1 dz^b \right) \\
&= dz^a \bullet \frac{z^a z^b}{(z^3)^2} \otimes_1 dz^b + \frac{\lambda}{2} dz^a \frac{(2 - (z^3)^2)}{(z^3)^3} \epsilon_{ab} \otimes_1 dz^b
\end{aligned}$$

The  $\bullet$  in the first term is left unevaluated so as to obtain  $\tilde{g}_{ij}$  and we clearly see that we now have the same semiclassical correction  $h$  as in (4.5). We can perform a similar calculation for the quantum Ricci tensor in section 3.2, making the same coordinate transformation as for the metric

$$\text{Ricci}_1 = -\frac{1}{2} \left( -3 \frac{\ddot{a}}{a} dt \otimes_1 dt + (2\dot{a}^2 + a\ddot{a}) (dr \otimes_1 dr + r^2 \delta_{ij} dz^i \otimes_1 dz^j) \right)$$

Indeed, since  $t$  and  $r$  are central in the algebra, the procedure is simply a repeat of that for the metric and clearly results in the same  $\rho$  as above. Thus we obtain the same results as in section 3 but only after allowing for the change of variables in the noncommutative algebra and  $\otimes_1$ .

#### 4.4. Schwarzschild metric

We now look at some examples of well known metrics that fit the above analysis. For the first, we reexamine the Schwarzschild metric case in [10, section 7.2]. There it was found that (as we would now expect), the quantum Levi-Civita condition is satisfied for a spherically symmetric Poisson tensor. A difference however, is that in [10] the torsion tensor was restricted to being rotationally invariant. By contrast, no such assumption is made here yet we are still led to a unique (con)torsion from theorem 4.1 up to undetermined components which we show do not enter into the quantum metric, quantum connection etc. Here

$$g = - \left( 1 - \frac{r_S}{r} \right) dt \otimes dt + \left( 1 - \frac{r_S}{r} \right)^{-1} dr \otimes dr + r^2 \delta_{ij} dz^i \otimes dz^j$$

so our three functional parameters are

$$a(r, t) = \left( 1 - \frac{r_S}{r} \right)^{\frac{1}{2}}, \quad b(r, t) = \left( 1 - \frac{r_S}{r} \right)^{-\frac{1}{2}}, \quad c(r, t) = r$$

giving the ‘quantising’ connection up to undetermined but irrelevant contorsion tensor components (which are set to zero for simplicity)

$$\nabla(dt) = - \left( 1 - \frac{r_S}{r} \right)^{-\frac{1}{2}} \frac{r_S}{r^2} (dr \otimes dt + dt \otimes dr)$$

$$\nabla(dr) = -\left(1 - \frac{r_S}{r}\right)^{\frac{3}{2}} \frac{r_S}{r^2} dt \otimes dt + \left(1 - \frac{r_S}{r}\right)^{-\frac{1}{2}} \frac{r_S}{r^2} dr \otimes dr$$

$$\nabla(dz^i) = -\frac{1}{r} dr \otimes dz^i - \delta_{ab} z^i dz^a \otimes dz^b$$

As a check, by transforming into spherical polars and likewise neglecting the irrelevant components, we can recover the ‘quantising’ connection in [10]. In particular

$$\nabla(d\theta) = -\frac{1}{r} dr \otimes d\theta + \cos(\theta) \sin(\theta) d\phi \otimes d\phi$$

$$\nabla(d\phi) = -\frac{1}{r} dr \otimes d\phi - \cot(\theta)(d\theta \otimes d\phi + d\phi \otimes d\theta)$$

which agrees with [10]. Obviously, the classical Ricci tensor for the Levi-Civita connection vanishes for the Schwarzschild metric, likewise for the curvature scalar.

Now, the quantum metric comes out as

$$g_1 = g_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda r^2}{2(z^3)^2} \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - dz^3 \otimes_1 z^i dz^j)$$

While  $\tilde{g}_{ij}$  (components in the middle) has quantum correction

$$h = \frac{r^2(2 - (z^3)^2)}{(z^3)^3} \epsilon_{3ij} dz^i \otimes_1 dz^j$$

For the inverse metric with components  $\tilde{g}^{ij}$  we get

$$(dz^1, dz^2)_1 = g^{23} + \frac{\lambda}{2} \frac{z^3}{r^2}, \quad (dz^2, dz^1)_1 = g^{32} - \frac{\lambda}{2} \frac{z^3}{r^2}$$

$$(dt, dt)_1 = g^{00}, \quad (dr, dr)_1 = g^{11}, \quad (dz^1, dz^1)_1 = g^{22}, \quad (dz^2, dz^2)_1 = g^{33}$$

The quantum Levi-Civita connection is

$$\nabla_1(dt) = -\hat{\Gamma}_{\mu\nu}^0 dx^\mu \otimes_1 dx^\nu$$

$$\nabla_1(dr) = -\hat{\Gamma}_{\mu\nu}^1 dx^\mu \otimes_1 dx^\nu + \frac{\lambda r}{2(z^3)^2} \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j)$$

$$\nabla_1(dz^a) = -\hat{\Gamma}_{\mu\nu}^i dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} \left( \epsilon_{ijk} z^k z^a dz^j \otimes_1 dz^i - \frac{1}{(z^3)^2} \epsilon^a_{i3} dz^3 \otimes_1 dz^i \right)$$

Meanwhile, from calculating the parameter  $F_6 + F_5 - F_4 = 0$ , we see that analogous to the classical case, the quantum Ricci tensor also vanishes

$$\text{Ricci}_1 = 0, \quad \rho = 0, \quad S_1 = 0.$$

#### 4.5. Bertotti–Robinson metric with fuzzy spheres

Another interesting example is the Bertotti–Robinson metric, discussed in the context of a different differential algebra in section 2.4. In order to draw a comparison between this case and the previous one, we define our metric as



$$g = -a^2 r^{2\alpha} dt \otimes dt + b^2 r^{-2} dr \otimes dr + c^2 \delta_{ij} dz^i \otimes dz^j$$

To chime with the conventions in this section, we relabel the constant terms and, compared to the metric in section 2.4, the off diagonal component is zero (either by diagonalising or setting the corresponding coefficient to zero). So our three functional parameters are

$$a(r, t) = ar^\alpha, \quad b(r, t) = br^{-1}, \quad c(r, t) = c$$

As explained after theorem 4.1, the theorem in this case does not give a unique quantum geometry but does give one. Dropping the undetermined and irrelevant contorsion components, the ‘quantising’ Poisson-connection comes out as

$$\nabla(dt) = -\frac{\alpha}{r}(dt \otimes dr + dr \otimes dt), \quad \nabla(dr) = -\alpha r^{3\alpha} \frac{a^2}{c^2} dt \otimes dt + \frac{1}{r} dr \otimes dr$$

$$\nabla(dz^i) = -\delta_{ab} z^i dz^a \otimes dz^b$$

This is markedly different from that in section 2.4 (apart from the different choice of coordinates), in particular with regard to the bimodule relations since previously  $t$  was not central. We also have the Ricci tensor for the Levi-Civita connection

$$\widehat{\text{Ricci}} = -\frac{1}{2} \left( \alpha^2 r^{2\alpha} \frac{a^2}{b^2} dt \otimes dt - \frac{\alpha^2}{r^2} dr \otimes dr + \delta_{ij} dz^i \otimes dz^j \right)$$

with the corresponding scalar curvature

$$\widehat{S} = \widehat{R}_{\mu\nu} g^{\mu\nu} = \frac{2}{c^2} - \frac{2\alpha^2}{b^2}$$

The quantum metric is

$$g_1 = g_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda c^2}{2(z^3)^2} \epsilon_{3ij} (z^3 dz^i \otimes_1 dz^j - z^i dz^3 \otimes_1 dz^j)$$

While  $\tilde{g}_{ij}$  (components in the middle) has the deformation term

$$h = \frac{c^2(2 - (z^3)^2)}{(z^3)^3} \epsilon_{3ij} dz^i \otimes_1 dz^j$$

For the inverse metric with components  $\tilde{g}^{ij}$  we get

$$(dz^1, dz^2)_1 = g^{23} + \frac{\lambda}{2} \frac{z^3}{c^2}, \quad (dz^2, dz^1)_1 = g^{32} - \frac{\lambda}{2} \frac{z^3}{c^2}$$

$$(dt, dt)_1 = g^{00}, \quad (dr, dr)_1 = g^{11}, \quad (dz^1, dz^1)_1 = g^{22}, \quad (dz^2, dz^2)_1 = g^{33}$$

Now, the quantum connection is

$$\nabla_1(dt) = -\widehat{\Gamma}^0_{\mu\nu} dx^\mu \otimes_1 dx^\nu, \quad \nabla_1(dr) = -\widehat{\Gamma}^1_{\mu\nu} dx^\mu \otimes_1 dx^\nu$$

$$\nabla_1(dz^a) = -\widehat{\Gamma}^i_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} \left( \epsilon_{ijk} z^k z^a dz^j \otimes_1 dz^i - \frac{1}{(z^3)^2} \epsilon^a_{i3} dz^3 \otimes_1 dz^i \right)$$

Again, calculating the parameter  $F_6 + F_5 - F_4 = 1$ , the quantum Ricci tensor is

$$\text{Ricci}_1 = -\frac{1}{2}\widehat{R}_{\mu\nu}dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{4(z^3)^2}\epsilon_{3ij}(z^3dz^i \otimes_1 dz^j - z^jdz^3 \otimes_1 dz^i)$$

With components in the middle, this comes out as

$$\rho = -\frac{1}{4(z^3)^3}(2 - (z^3)^2)\epsilon_{3ij}dz^i \otimes_1 dz^j$$

and in either case  $S_1 = -\frac{1}{2}\widehat{S}$  in our conventions.

## 5. Conclusions

In this paper we simplified and extended the study of Poisson–Riemannian geometry introduced in [10] to include a formula for the quantum Laplace–Beltrami operator at semiclassical order (theorem 2.3) and we also looked at the lifting map needed to define a reasonable Ricci tensor in a constructive approach to that. Our second main piece of analysis was theorem 4.1 for spherically symmetric Poisson tensors on spherically symmetric spacetimes. We found that if the metric components are sufficiently generic (in particular the coefficient of the angular part of the metric is not constant) then any quantisation has to have  $t, dt, r, dr$  central and nonassociative fuzzy spheres [7] at each value of time and radius. We also found that the Laplace–Beltrami operator has no corrections at order  $\lambda$ . Key to the startling rigidity here was condition (2.14) from [10] needed for the existence of a quantum Levi-Civita connection  $\nabla_1$ . Hence if one wanted to avoid this conclusion then [10] says that we *can* drop (2.14) and still have a canonical  $\nabla_1$  and now with a larger range of spherically symmetric models but with a new physical effect of  $\nabla_1 g_1 = O(\lambda)$ . One can also drop our other assumption in the analysis that  $\omega$  obeys (2.3) for the Jacobi identity. In that generic (nonassociative algebra) context we noted that spherical symmetry and Poisson-compatibility leads to a unique contorsion tensor, while imposing the Jacobi identity leads to half the modes of  $S$  being undetermined but in such a way that the contravariant connection  $\omega^{\alpha\beta}\nabla_\beta$  more relevant to the quantum geometry is still unique. This suggests an interesting direction for the general theory.

The paper also included detailed calculations of the quantum metric, quantum Levi-Civita connection and quantum Laplacian for a number of models, some of them, such as the FLRW, Schwarzschild and the time-central Bertotti–Robinson model being covered by theorem 4.1. The important case of the FLRW model was first solved directly in Cartesian coordinates both as a warm up and as an independent check of the main theorem (the needed quantum change of coordinates was provided in section 4.3). Two models not covered by our analysis of spherical symmetry are the 2D bicrossproduct model for which most of the algebraic side of the quantum geometry but not the quantum Laplacian was already found in [9], and the non-time central but spherically symmetric Bertotti–Robinson model for which the full quantum geometry was already found in [28] (this case is not excluded by theorem 4.1 since the coefficient of the angular metric is constant). In both cases the quantum spacetime algebra is the much-studied Majid–Ruegg spacetime  $[x_i, t] = \lambda x_i$  in [26]. The non-time central Bertotti–Robinson model quantises  $S^{n-1} \times dS_2$  and the quantum Laplacian in section 2.4 is quite similar to the old ‘Minkowski spacetime’ Laplacian for this spacetime algebra which has previously led to variable speed of light [1] in that, provided wave functions are normal ordered, one of the double-differentials becomes a finite-difference (the main difference from [1] is that this time there is an actual quantum geometry forcing the classical metric not

to be flat [28]). However, when we analysed this within Poisson–Riemannian geometry we found no order  $\lambda$  correction to the quantum Laplacian. We traced this to the formula for the bullet product in Poisson–Riemannian geometry in [10] being realised on the classical space by an antisymmetric deformation, which is analogous to Weyl-ordered rather than left or right normal ordered functions in the noncommutative algebra being identified with classical ones. Our conclusion then is that order  $\lambda$  predictions from such models [1] were an artefact of the hypothesised normal ordering assumption and that theorem 2.3 is a more stringent test within the paradigm of Poisson–Riemannian geometry. We should not then be too surprised that order  $\lambda$  corrections are more rare than one might naively have expected from the formula in theorem 2.3. The 2D bicrossproduct model in section 2.3 *does* however have an order  $\lambda$  deformation to the quantum Laplacian even within Poisson–Riemannian geometry and we were able to solve the deformed massless wave equation at order  $\lambda$  using Kummer functions (i.e. it is effectively the Kummer equation). This behaviour is reminiscent of the minimally coupled black hole in the wave operator approach of [25] without yet having a general framework for the physical interpretation of the order  $\lambda$  deformations obtained from Poisson–Riemannian geometry.

It is even less clear at the present time how to draw physical conclusions from our formulae for the quantum metric  $g_1$  and quantum Ricci tensor  $\text{Ricci}_1$ . In the FLRW model for example we found that  $g_1$  looks identical to the classical metric but of course as an element of the quantum tensor product  $\Omega^1 \otimes_1 \Omega^1$ . The physical understanding of how quantum tensors relate to classical ones is suggested here as a topic of further work. Another topic on which we made only a tentative comment at the end of section 4.2, is what should be the quantum Einstein tensor. Its deformation could perhaps be reinterpreted as an effective change to the stress energy tensor. This is another direction for further work.

## Appendix. Match up with the algebraic bicrossproduct model

This is a supplement to section 2.3 in which we will verify that the semiclassical theory obtained by our tensor calculus formulae agrees with the order  $\lambda$  part of the full quantum geometry found for this model by algebraic means in [9]. This provides a completely independent check of the main formulae in section 2.1.

We let  $\nu = r \bullet dt - t \bullet dr = \nu + \frac{\lambda}{2} dr$  and the (full) quantum metric, inverse quantum metric and quantum Levi-Civita connection in [9] are

$$g_1 = (1 + b\lambda^2)dr \otimes_1 dr + b\nu \otimes_1 \nu - b\lambda\nu \otimes_1 dr$$

$$(\nu, \nu)_1 = b^{-1}, \quad (dr, \nu)_1 = 0, \quad (\nu, dr)_1 = \frac{\lambda}{1 + b\lambda^2}, \quad (dr, dr)_1 = \frac{1}{1 + b\lambda^2}$$

$$\nabla_1 dr = \frac{8b}{r(4 + 7b\lambda^2)}\nu \otimes_1 \nu - \frac{12b\lambda}{r(4 + 7b\lambda^2)}\nu \otimes_1 dr$$

$$\nabla_1 \nu = -\frac{4b\lambda}{r(4 + 7b\lambda^2)}\nu \otimes_1 \nu - \frac{8(1 + b\lambda^2)}{r(4 + 7b\lambda^2)}\nu \otimes_1 dr$$

(there is a typo in the coefficient of  $\beta'$  in [9]).

We note immediately by expanding  $g_1$  to  $O(\lambda)$  and changing from  $\nu$  to  $\nu$  that

$$\begin{aligned}
g_1 &= dr \otimes_1 dr + bv \otimes_1 v + b \frac{\lambda}{2} (dr \otimes_1 v - v \otimes_1 dr) \\
&= g_{\mu\nu} dx^\mu \otimes_1 dx^\nu + b \frac{\lambda}{2} dr \otimes_1 v - \lambda bv \otimes_1 dr \\
&= g_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} b t dr \otimes_1 dr + \frac{\lambda}{2} b r dr \otimes_1 dt - \lambda b r dt \otimes_1 dr
\end{aligned}$$

the same as we obtained from (2.9). We used  $r dt = r \bullet dt - \frac{\lambda}{2} dr$  to move all coefficients to the left through  $\otimes_1$  in order to make this comparison. We can do a similar trick with  $r dt = (dt) \bullet r + \frac{\lambda}{2} dr$  to put the coefficients in the middle, giving

$$\begin{aligned}
g_1 &= dr \otimes_1 dr + b(dt) \bullet r^2 \otimes_1 dt + b(dr) \bullet t^2 \otimes_1 dr - b(dr) \bullet t \bullet r \otimes_1 dt \\
&\quad - b(dt) \bullet r \bullet t \otimes_1 dr + b \lambda (dr \otimes_1 v - v \otimes_1 dr)
\end{aligned}$$

so that

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{\lambda}{2} \begin{pmatrix} 0 & -3br \\ 3br & 0 \end{pmatrix}.$$

which agrees with  $\tilde{g}_{\mu\nu}$  implied by  $h_{\mu\nu} = -3br\epsilon_{\mu\nu}$  as computed from (2.12).

Similarly, working to  $O(\lambda)$ , the quantum connection from [10, proposition 7.1] is

$$\begin{aligned}
\nabla_1 dr &= \frac{2b}{r} (v \otimes_1 v - \lambda v \otimes_1 dr) \\
&= \frac{2b}{r} v \otimes_1 (r \bullet dt - t \bullet dr - \frac{\lambda}{2} dr) - \lambda \frac{2b}{r} v \otimes_1 dr \\
&= 2bv \otimes_1 dt - 2bv(r^{-1} \bullet t) \otimes_1 dr - \lambda \frac{3b}{r} v \otimes_1 dr \\
&= -\hat{\Gamma}_{\mu\nu}^1 dx^\mu \otimes_1 dx^\nu - 2\lambda b(dt - r^{-1} t dr) \otimes_1 dr
\end{aligned}$$

in the same manner as the computation of  $v \otimes_1 v$  for the metric, which agrees with the correction in  $\Gamma_1^1{}_{\mu\nu}$  of

$$\begin{aligned}
&\frac{\lambda}{2} \omega^{\alpha\beta} \hat{\Gamma}_{\mu 0, \alpha}^1 \Gamma_{\beta\nu}^0 - \frac{\lambda}{2} \omega^{\alpha\beta} \hat{\Gamma}_{00}^1 \Gamma_{\alpha\mu}^0 \Gamma_{\beta\nu}^0 - \frac{\lambda}{2} \omega^{01} S^1{}_{0\kappa} \nabla_1 S^\kappa{}_{\mu\nu} \\
&= -\frac{\lambda}{2} \hat{\Gamma}_{\mu 0, \nu}^1 - \lambda b \epsilon_{\mu\nu} - \frac{\lambda}{2} r \hat{\Gamma}_{0\kappa}^1 \nabla_1 S^\kappa{}_{\mu\nu} = 2\lambda b \begin{pmatrix} 0 & 1 \\ 0 & -r^{-1}t \end{pmatrix}
\end{aligned}$$

in lemma 2.2. Here

$$\nabla_1 S^0{}_{\mu\nu} = \begin{pmatrix} -\frac{2bt}{r} & \frac{2(1+br^2)}{r^2} \\ \frac{2bt^2}{r^2} & -\frac{2t(1+br^2)}{r^3} \end{pmatrix}, \quad \nabla_1 S^1{}_{\mu\nu} = \begin{pmatrix} -2b & \frac{2bt}{r} \\ \frac{2bt}{r} & -\frac{2br^2}{r^2} \end{pmatrix}$$

where  $\nabla_1$  in this context means with respect to  $r$ . Similarly, the semiquantum connection  $\nabla_1 v = -\frac{2}{r}(v \otimes_1 dr + b \lambda v \otimes_1 v)$  implies

$$\begin{aligned}
\nabla_1 dt &= \nabla_1(r^{-1} \bullet v + (r^{-1}t) \bullet dr) = dr^{-1} \otimes_1 v + d(r^{-1}t) \otimes_1 dr + r^{-1} \bullet \nabla_1 v + (r^{-1}t) \bullet \nabla_1 dr \\
&= -r^{-2} dr \otimes_1 (r \bullet dt - t \bullet dr - \frac{\lambda}{2} dr) - r^{-2} t dr \otimes_1 dr + r^{-1} dt \otimes_1 dr \\
&\quad - 2r^{-2}(v \otimes_1 dr + b\lambda v \otimes_1 v) + 2b(r^{-1}t) \bullet r^{-1}(v \otimes_1 v - \lambda v \otimes_1 dr) \\
&= -r^{-1} dr \otimes_1 dt + (r^{-2} \bullet t) dr \otimes_1 dr + \frac{\lambda}{2} r^{-2} dr \otimes_1 dr - r^{-2} t dr \otimes_1 dr + r^{-1} dt \otimes_1 dr \\
&\quad - 2r^{-2} v \otimes_1 dr - \lambda br^{-2} v \otimes_1 v + 2br^{-2} tv \otimes_1 (r \bullet dt - t \bullet dr - \frac{\lambda}{2} dr) - \lambda 2br^{-2} tv \otimes_1 dr \\
&= -r^{-1} dr \otimes_1 dt + r^{-2} t dr \otimes_1 dr - \frac{\lambda}{2} r^{-2} dr \otimes_1 dr - r^{-2} t dr \otimes_1 dr + r^{-1} dt \otimes_1 dr \\
&\quad - 2r^{-2} v \otimes_1 dr - \lambda br^{-2} v \otimes_1 v + 2b(r^{-2}t) \bullet rv \otimes_1 dt - 2b(r^{-2}t) \bullet tv \otimes_1 dr - \lambda 3br^{-2} tv \otimes_1 dr \\
&= -\hat{\Gamma}_{\mu\nu}^0 dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{2} r^{-2} dr \otimes_1 dr - \lambda br^{-2} v \otimes_1 v - \lambda br^{-1} v \otimes_1 dt - \lambda br^{-2} tv \otimes_1 dr \\
&= -\hat{\Gamma}_{\mu\nu}^0 dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{2} r^{-2} dr \otimes_1 dr - 2\lambda b(dt - r^{-1}t dr) \otimes_1 dt
\end{aligned}$$

which agrees with the correction to  $\Gamma_1^0{}_{\mu\nu}$  of

$$\begin{aligned}
&\frac{\lambda}{2} \omega^{\alpha\beta} \hat{\Gamma}_{\mu 0, \alpha}^0 \Gamma_{\beta\nu}^0 - \frac{\lambda}{2} \omega^{\alpha\beta} \hat{\Gamma}_{00}^0 \Gamma_{\alpha\mu}^0 \Gamma_{\beta\nu}^0 - \frac{\lambda}{2} \omega^{01} S_{0\kappa}^0 \nabla_1 S_{\mu\nu}^\kappa \\
&= -\frac{\lambda}{2} \hat{\Gamma}_{\mu 0, \nu}^0 - \lambda br^{-1} t \epsilon_{\mu\nu} - \frac{\lambda}{2} r \hat{\Gamma}_{0\kappa}^0 \nabla_1 S_{\mu\nu}^\kappa = \lambda \begin{pmatrix} 2b & 0 \\ -\frac{2bt}{r} & \frac{1}{2r^2} \end{pmatrix}
\end{aligned}$$

in lemma 2.2.

Next note that because  $\nabla v = \nabla dr = 0$ , we do not have any corrections to products with these basic 1-forms and this allows us to equally well write

$$df = (\partial_r f) \bullet dr + (\partial_v f) \bullet v$$

with the classical derivatives if we use this basis. Then working to  $O(\lambda)$ ,

$$\begin{aligned}
\Box_1 f &= (,)_1 \nabla_1 df = (,)_1 ((\partial_r f) \bullet \nabla_1 dr + (\partial_v f) \bullet \nabla_1 v + d\partial_r f \otimes_1 dr + d\partial_v f \otimes_1 v) \\
&= (,)_1 ((\partial_r f) \bullet \frac{2b}{r} v \otimes_1 v - (\partial_r f) \bullet \frac{2b\lambda}{r} v \otimes_1 dr - (\partial_v f) \bullet \frac{2}{r} v \otimes_1 dr - (\partial_v f) \bullet \frac{2b\lambda}{r} v \otimes_1 v \\
&\quad + (\partial_r^2 f) dr \otimes_1 dr + (\partial_v \partial_r f) v \otimes_1 dr + (\partial_r \partial_v f) dr \otimes_1 v + (\partial_v^2 f) v \otimes_1 v) \\
&= (\partial_r f) \frac{2b}{r} b^{-1} + \frac{\lambda}{2} \partial_v \partial_r f 2bb^{-1} - (\partial_v f) \frac{2}{r} \frac{\lambda}{2} - (\partial_v f) \frac{2b\lambda}{r} b^{-1} \\
&\quad + (\partial_r^2 f) + (\partial_v \partial_r f) \frac{\lambda}{2} - (\partial_r \partial_v f) \frac{\lambda}{2} + (\partial_v^2 f) b^{-1} \\
&= \Box f + \lambda \left( -\frac{3}{r} \partial_v f + \frac{1}{2} [\partial_v, \partial_r] f + \partial_v \partial_r f \right) \\
&= \Box f + \lambda \left( \frac{1}{r} \frac{\partial}{\partial t} \frac{\partial}{\partial r} f + \frac{t}{r^2} \frac{\partial^2}{\partial t^2} f - \frac{1}{r^2} \frac{\partial}{\partial t} f \right)
\end{aligned}$$

where we used  $h \bullet r^{-1} = hr^{-1} + \frac{\lambda}{2} \partial_v h$  for any function  $h$  and, to  $O(\lambda)$ ,

$$(v, v)_1 = b^{-1}, \quad (dr, v)_1 = -\frac{\lambda}{2}, \quad (v, dr)_1 = \frac{\lambda}{2}, \quad (dr, dr)_1 = 1.$$

This agrees with the quantum Laplacian to order  $\lambda$  obtained in section 2.3.

In this model we can in fact write down the full quantum Laplacian in noncommutative geometry in the setting of [9] just as easily and we do this now as it was not done in that work. We again write

$$df = (\partial_r f) \bullet dr + (\partial_v f) \bullet \nu$$

where  $\partial_r, \partial_v$  are now quantised versions of the ones before and are derivations of the noncommutative algebra since  $dr, \nu$  are central. They obey

$$\partial_v f(r) = 0, \quad \partial_v f(t) = r^{-1} \bullet \partial_0 f, \quad \partial_r f(r) = f', \quad \partial_r f(t) = -r^{-1} \bullet t \partial_0 f$$

as easily found using the derivation rule, the values on  $r, t$  and the relations  $t \bullet r^{-1} = r^{-1} \bullet (t + \lambda)$  in the algebra. Here  $\partial_0 f(t) = \lambda^{-1}(f(t + \lambda) - f(t))$  is a finite difference. Then for any  $f$  in the algebra, one can compute  $(\cdot)_1 \nabla_1 df$  using the full expressions above to obtain

$$\square_1 f = \frac{(8 - 6b\lambda^2)\partial_r f - \lambda(8 + 4b\lambda^2)\partial_v f}{4 + 7b\lambda^2} \bullet r^{-1} + \frac{\partial_r^2 f + \lambda \partial_v \partial_r f}{1 + b\lambda^2} + b^{-1} \partial_v^2 f$$

for  $\nabla_1$  the quantum Levi-Civita connection stated above for this model.

Finally, the work [9] already contained the full quantum Ricci as proportional to the quantum metric. The first ingredient for this is the quantum Riemann tensor in [9] and expanding this gives

$$\begin{aligned} \text{Riem}_1(dr) &= -2b \frac{1}{r^2} \bullet \nu \wedge_1 dr \otimes_1 (r \bullet dt - t \bullet dr) + \frac{7b\lambda}{r} dt \wedge dr \otimes dr \\ &= -2b dt \wedge_1 dr \otimes_1 dt + 2b \frac{1}{r^2} \bullet t \bullet r \bullet (dt \wedge_1 dr) \otimes_1 dr + \frac{7b\lambda}{r} dt \wedge dr \otimes dr \\ &= -2b dt \wedge dr \otimes_1 dt + 2b \left( \frac{t}{r} + \frac{\lambda}{2r^2} \{t, r\} + \frac{\lambda}{2} \left\{ \frac{1}{r^2}, t \right\} r \right) \bullet (dt \wedge dr) \otimes_1 dr \\ &\quad + \frac{7b\lambda}{r} dt \wedge dr \otimes dr \\ &= -2b dt \wedge dr \otimes_1 dt + 2b \left( \frac{t}{r} \right) \bullet (dt \wedge dr) \otimes_1 dr + \frac{4b\lambda}{r} dt \wedge dr \otimes dr \\ &= -\frac{1}{2} \widehat{R}^1_{\beta\mu\nu} dx^\mu \wedge dx^\nu \otimes_1 dx^\beta + 5\lambda \frac{b}{r} dt \wedge dr \otimes_1 dr + O(\lambda^2). \end{aligned}$$

where we use  $\nu, dr$  central for the second equality, then  $dt \wedge_1 dr = dt \wedge dr$  and  $\nabla_1(dt \wedge dr) = -r^{-1} dt \wedge dr$ . There is a similar formula for  $\text{Riem}_1(dt)$  obtained from  $\text{Riem}_1(\nu) = r \bullet \text{Riem}_1(dt) - t \bullet \text{Riem}_1(dr)$  given in [9] and expanding. Thus the curvature agrees with (2.24) obtained from our tensor formulae.

Next the lifting map  $i_1$  was given by the method in [9] uniquely (by the time the reality property is included) as,

$$i_1(\nu \wedge_1 dr) = \frac{1}{2}(\nu \otimes_1 dr - dr \otimes_1 \nu) + \frac{7\lambda}{4} g_1 + O(\lambda^2)$$

according to the order  $\lambda$  part of the full calculation in [9, section 6.2.1] (the 9/4 in [9, equation (5.21)] was an error and should be 7/4). It was then shown in [9] that  $\text{Ricci}_1 = g_1/r^2$ . Expanding the quantum metric from [9] as recalled above, the quantum Ricci is to  $O(\lambda^2)$ ,

$$\begin{aligned}
\text{Ricci}_1 &= \frac{1}{r^2} \bullet (dr \otimes_1 dr + br \bullet \nu \otimes_1 dt - bt \bullet \nu \otimes_1 dr - b\lambda \nu \otimes_1 dr) \\
&= \frac{1}{r^2} dr \otimes_1 dr + \frac{b}{r} (v + \frac{\lambda}{2} dr) \otimes_1 dt - (\frac{b}{r^2} \bullet t) (v + \frac{\lambda}{2} dr) \otimes_1 dr - \frac{b}{r^2} \lambda \nu \otimes_1 dr \\
&= -\frac{1}{2} \hat{R}_{\mu\nu} dx^\mu \otimes_1 dx^\nu - \frac{\lambda}{2} \frac{bt}{r^2} dr \otimes_1 dr + \frac{\lambda}{2} \frac{b}{r} dr \otimes_1 dt \\
&= -\frac{1}{2} \hat{R}_{\mu\nu} dx^\mu \otimes_1 dx^\nu + \frac{\lambda}{2} \frac{b}{r^2} dr \otimes_1 v + O(\lambda^2).
\end{aligned}$$

where the second equality uses  $\nabla dr = \nabla v = 0$  so that bullet products with these are classical products. For the third equality we used  $\{\frac{1}{r^2}, t\} = -\frac{2}{r^2}$  from the  $\bullet$  which cancels the last term. We obtain exactly this answer by calculation from (2.21).

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## Conclusion

### 5.1 Summary

At last, we conclude this thesis with a brief summary of results and considerations for further research. The overall aim has been to consider and investigate the appearance of non-locality in gravity within different formalisms. Our approach has been to investigate quantum gravity effects using approaches based on theory techniques.

In the first paper [2](#) we considered a form of nonlocality which emerges from a basic combination of quantum field theory with general relativity. Using the known result of the re-summed graviton propagator, it was shown to modify the scattering amplitude in a scalar field theory. It was argued that one could equally consider these modifications resulting from higher order effective operators which include the nonlocal term  $\ln(-\Box/\mu^2)$ . After showing that this operator could be resolved in a way that conserves causality, the resulting effective field theory was applied to cosmology, taking the scalar field as an inflaton. Here we demonstrated that the nonlocality has an effect on the cosmic microwave background in that it causes deviation from the standard prediction for the speed of sound. This predicted effects of order  $\sim NG_N^2$  dependant on the number of particles. Though small, depending on  $N$ , it these may be measurable with future experiments.

From this point on we changed tact and considered noncommutative geometry as a constructive approach to the problem of quantum gravity. It shares with the previous sections the feature of nonlocality, but this time in a theory where the notion of fundamental length is ‘built in’ in the form of a deformation parameter. The second paper [3](#) made the first foray into this subject at the level of effective field theory. The quantization was based on the canonical or Heisenberg algebra and applied to general relativity. Here we used the fact that the symmetries allowed by the algebra restrict us to considering unimodular gravity and showed that a homogeneous field does not result in corrections to the slow roll parameters for inflation. However, it was noted that perturbations to the field need not be

homogeneous. Using that fact that unimodular and standard general relativity are equivalent, it was argued how previous results on the power spectrum of the cosmic microwave background apply to our model, giving bounds on the parameter of noncommutativity.

The last paper [4](#) examined noncommutative geometry beyond the level of an effective field theory. Taking the view that quantum gravity ought to be described by quantum geometry, it represented to first attempt to apply the recently developed formalism of Poisson-Riemannian geometry to a physical metric. We also introduced the Laplace-Beltrami operator at the semiclassical level. Principally, it was demonstrated that generic spherically symmetric spacetimes have a unique quantization to first order in deformation and that there exists a quantum Levi-Civita connection. As a result we were able to obtain expressions for the Riemann, Ricci and metric tensors as well a Ricci scalar and inverse metric to the same order. From this was shown how the result applies to commonly used physical metrics e.g. FLRW.

## 5.2 Future Study

With that all being said, the obvious question seems to be “now what?”. It seems more appropriate to look at the individual topics and see what questions are still open.

The first paper is arguably rather comprehensive. It takes an established result and examines its consequences in a physical setting. The conclusion drawn from this is that while there is an effect, it is outside the reach of current experiments. A further avenue for research would be to investigate similar nonlocal operators on different spacetimes as in, for example, [\[38, 39\]](#).

The final paper leaves open questions in regard to physical predictions. As stated in the conclusion, it is difficult to extract any physical data from the model presented. A simple approach would have been to examine  $\mathcal{O}(\lambda)$  contributions to scalar objects like  $\square_1$  or  $S_1$ , however these were shown to vanish. Meanwhile, quantum tensors like  $g_1$  and  $\text{Ricci}_1$  do have first order corrections, but how to relate these classical tensors is not clear. Understanding this relation would be necessary in order to extract physical data and is still outstanding.

An alternative possibility for examining physical implications is presented at the end of

section 2.5 of the last paper. Here we show that to first order in deformation, the algebra is equivalent to a particular cochain twist for a fuzzy nonassociative sphere. It would therefore be possible to extend the quantization to  $\mathcal{O}(\lambda^2)$  by twisting, possibly yielding non-zero contributions to the quantum Laplace operator or Ricci scalar. In the case of the former, this would allow for the extraction of physical predictions from the theory. The drawback of this procedure is the loss of generality provided by Poisson-Riemannian, which describes some general deformation and is agnostic about the procedure through which it is obtained.

One can also envisage further development of Poisson-Riemannian geometry itself. We introduced the Laplace-Beltrami operator and suggested a form for a quantum Einstein tensor. The latter is however a hypothesis that applies only to the case of spherical metrics. A general quantum Einstein tensor is still an open question in Poisson-Riemannian and noncommutative geometry in general with first steps being taken in [\[40\]](#)

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## Algebras

In this section we define several notions and concepts along with their notation used throughout this thesis. So first define a *group* which is the double  $(G, \circ)$  consisting of the set  $G$  and composition  $\circ : G \times G \rightarrow G$  satisfying for all  $f, g, h \in G$

- Closure:  $f \circ g \in G$
- Associativity:  $(f \circ g) \circ h = f \circ (g \circ h)$
- Identity: There exists an element  $e \in G$  so that  $e \circ f = f \circ e = e$
- Inverse: For any element  $f$  there exists  $f^{-1} \in G$  so that  $f \circ f^{-1} = f^{-1} \circ f = e$

In particular, we can have additive group structure  $(G, +)$  where the identity is usually denoted by 0. Groups of this sort are *abelian*, i.e. the composition is commutative  $f + g = g + f$ . Another common case is a multiplicative group  $(G, \cdot)$  where the unit is usually denoted as 1. This can be, but generally isn't, abelian e.g. the group of  $n \times n$  real matrices  $GL(n)$ . Note that relaxing the requirement for an inverse gives a *semigroup*.

Next, define a *field* as the triple  $(k, \cdot, +)$  where  $k$  is a set on which is defined a product (or multiplication) and sum (or addition). Generally it requires

- $(k, +)$  an additive abelian group
- $(k - \{0\}, \cdot)$  a multiplicative abelian group
- For  $a, b, c \in k$  compatibility between the group structures in the form  $a \cdot (b + c) = a \cdot b + a \cdot c$

If we relax the conditions on the multiplicative structure so that it is merely a semigroup and/or not abelian,  $k$  is referred to as a *ring*.

A *vector space* is the triple  $(V, +, k)$  where  $k$  is a field and  $(V, +)$  an abelian group. The group  $k^* \equiv k - \{0\}$  acts multiplicatively on  $V$ , that is it has an action  $\cdot : k^* \times V \rightarrow V$  which



is compatible with the additive structure of the vector space so  $a \cdot (v + w) = a \cdot v + a \cdot w$  for  $a \in k^*$  and  $v, w \in V$ .

Now, if  $(V, +, k)$  is a vector space, there is also a *dual vector space*  $V^* = \text{Lin}_k(V, k)$  of  $k$ -linear maps from  $V$  to  $k$ . Taking  $a, b \in k$  and  $\phi \in V^*$ , this means we have  $\phi(av + bw) = a\phi(v) + b\phi(w)$ . We can then define  $\phi(v) = v(\phi)$  by which we have  $V^{**} \subseteq V$ .

Taking two vector spaces  $(V, +, k)$  and  $(W, +, k)$  over the same field  $k$  we can form the *tensor product* vector space  $(V \otimes W, +, k)$ . Taking  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$ , in general elements in the product can be written  $v \otimes w \in V \otimes W$  and have the defining relationships

- $v_1 \otimes w_1 + v_2 \otimes w_2 = v_2 \otimes w_2 + v_1 \otimes w_1$
- $v_1 a \otimes w_1 = v_1 \otimes a w_1$
- $v_1 \otimes (w_1 + w_2) = v_1 \otimes w_1 + v_1 \otimes w_2$
- $(v_1 + v_2) \otimes w_1 = v_1 \otimes w_1 + v_2 \otimes w_1$

where as before  $a \in k$ . With respect to the dual vector spaces  $V^*, W^*$  we also have that  $V^* \otimes W^* \subseteq (V \otimes W)^*$ . Furthermore, note that  $k$  can be thought of as a vector field over itself and by taking  $a \otimes v_1 \rightarrow av_1$  and  $v_1 \otimes a \rightarrow v_1 a$  we have the isomorphisms  $k \otimes V \cong V \cong V \otimes k$ .

Finally, an *algebra* is  $(A, \cdot, +, k)$  where  $(A, \cdot, +)$  is a ring and  $k$  a field. It carries a multiplicative action of  $k$  on  $A$  which is compatible with both the additive structure of  $A$  as well as its own multiplicative one. So, in a addition to  $A$  being a vector space over  $k$ , it also satisfies  $b(vw) = (bv)w = v(bw)$  for all  $b \in k$  and  $v, w \in A$ . There is also a tensor product of algebras so for  $(A, \cdot, +, k)$  and  $(B, \cdot, +, k)$ , we can define  $(A \otimes B, \cdot, +, k)$ . Then for  $v_1, v_2 \in A$  and  $w_1, w_2 \in B$  the product is  $(v_1 \otimes w_1) \cdot (v_2 \otimes w_2) = v_1 \cdot v_2 \otimes w_1 \cdot w_2$ .

$$\begin{array}{ccc}
 & A \otimes A \otimes A & \\
 \cdot \otimes \text{id} \swarrow & & \searrow \text{id} \otimes \cdot \\
 A \otimes A & & A \otimes A \\
 & \searrow \cdot & \swarrow \cdot \\
 & A &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & A \otimes A & \\
 \eta \otimes \text{id} \nearrow & \downarrow \cdot & \nwarrow \text{id} \otimes \eta \\
 k \otimes A & = & A = A \otimes k
 \end{array}
 \tag{A.0.1}$$

In addition to all of the above, we want the algebra  $(A, \cdot, +, k)$  to be associative with respect to the product, which is of course compatible with  $k$ . This is most simply expressed to

viewing the product as a  $k$ -linear map  $\cdot : A \otimes A \rightarrow A$  and then demanding that the first diagram [A.0.1](#) commute (as explained in [B](#)). On top of this we also want a unit  $1_A \in A$  which can be defined using the linear map  $\eta_v : k \rightarrow A$  which acts as  $\eta_v(k) = kv$ . Obviously  $\eta_v(1) = v$  so that for  $\eta \equiv \eta_{1_A}$ , the two remaining diagrams in [A.0.1](#) commute.

## Monoidal Categories

Here we aim to give a basic definition of categories, in particular monoidal categories, as used in this thesis. This is by not supposed to be an exhaustive introduction to the subject, but simply a brief overview and definitions of the terms used in the text.

Generally, a *category*  $\mathcal{C}$  consists of

- A collection  $\text{ob}(\mathcal{C})$  called objects
- A collection  $\text{hom}(\mathcal{C})$  of morphisms or maps between objects

Take  $a, b, c \in \text{ob}(\mathcal{C})$ . Every morphism has a source object  $a$  and target object  $b$  so the set of morphisms between any two objects is denoted  $\text{hom}(a, b)$ . One can construct the composition of morphisms  $\text{hom}(a, b) \times \text{hom}(b, c) \rightarrow \text{hom}(a, c)$  by taking  $f \in \text{hom}(a, b)$  and  $g \in \text{hom}(b, c)$ , then  $g \circ f \in \text{hom}(a, c)$ . There also exists for each object an identity morphism so that  $\text{id}_a \in \text{hom}(a, a)$  maps objects to itself. Now consider additionally  $h \in \text{hom}(c, d)$ . We then require that all morphisms in  $\text{hom}(\mathcal{C})$  satisfy

- Associativity:  $f \circ (g \circ h) = (f \circ g) \circ h$
- Identity:  $f \circ \text{id}_a = f$  and  $\text{id}_b \circ f = f$

The language of category theory is that of commutative diagrams. For example, the composition of morphisms mentioned above can be expressed as

$$\begin{array}{ccc}
 a & \xrightarrow{g \circ f} & c \\
 & \searrow f \quad \nearrow g & \\
 & b &
 \end{array}
 \tag{B.0.1}$$

and the diagram is said to commute. Now, say we take the morphisms  $f \in \text{hom}(a, b)$ ,  $g \in \text{hom}(a, c)$ ,  $h \in \text{hom}(c, d)$  and  $k \in \text{hom}(b, d)$  then this can similarly be expressed in

diagrammatic form as

$$\begin{array}{ccc}
 a & \xrightarrow{f} & b \\
 \downarrow g & & \downarrow k \\
 c & \xrightarrow{k} & d
 \end{array} \tag{B.0.2}$$

If this diagram commutes we have  $k \circ f = h \circ g$ . This representation is useful to express abstract relations between quantities and makes it easy to keep track of them.

We can also define a *functor*, a map between different categories. Take two categories  $\mathcal{C}$  and  $\mathcal{D}$  and a functor so that  $F : \mathcal{C} \rightarrow \mathcal{D}$ . This has the following properties

- To each  $a \in \text{ob}(\mathcal{C})$  it associates an object in  $\mathcal{D}$  so that  $F(a) \in \text{ob}(\mathcal{D})$
- To each  $f \in \text{hom}(\mathcal{C})$  it associates a morphism in  $\mathcal{D}$  so that  $F(f) \in \text{hom}(\mathcal{D})$

The functor preserves the identity  $F(\text{id}_a) = \text{id}_{F(a)}$  and composition  $F(f \circ g) = F(f) \circ F(g)$

If  $\mathcal{C}$  is a *monoidal category*, it possess additional structures to the ones shown above.

- A functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  called the tensor product
- A unit object  $\mathbf{1} \in \text{ob}(\mathcal{C})$
- A morphism  $\alpha : \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$  called the associator
- Two morphisms, the left unitor  $\rho : \mathbf{1} \otimes \mathcal{C} \rightarrow \mathcal{C}$  and right unitor  $\lambda : \mathcal{C} \otimes \mathbf{1} \rightarrow \mathcal{C}$

In the first condition, the tensor product is formally a map from a product category, hence it is described as a functor. For our purposes it is enough to note that the tensor product of two objects is itself an object in  $\mathcal{C}$ , similar to the axiom for group multiplication. We required two coherence conditions to hold. The first is expressed in the so-called triangle diagram which we require to commute

$$\begin{array}{ccc}
 (a \otimes \mathbf{1}) \otimes b & \xrightarrow{\alpha_{a, \mathbf{1}, b}} & a \otimes (\mathbf{1} \otimes b) \\
 \searrow \rho \otimes \text{id}_b & & \swarrow \text{id}_a \otimes \lambda \\
 & a \otimes b &
 \end{array} \tag{B.0.3}$$

An important property of the tensor product is that it is associative up to a natural isomorphism given by the associator. The necessary compatibility condition requires the following diagram to commute.

$$\begin{array}{ccc}
 & (a \otimes b) \otimes (c \otimes d) & \\
 \alpha_{a \otimes b, c, d} \nearrow & & \searrow \alpha_{a, b, c \otimes d} \\
 ((a \otimes b) \otimes c) \otimes d & & a \otimes (b \otimes (c \otimes d)) \\
 \downarrow \alpha_{a, b, c} \otimes \text{id}_d & & \uparrow \text{id}_a \otimes \alpha_{b, c, d} \\
 (a \otimes (b \otimes c)) \otimes d & \xrightarrow{\alpha_{a, b \otimes c, d}} & a \otimes ((b \otimes c) \otimes d)
 \end{array} \tag{B.0.4}$$

Lastly, we note that in the case presented in 1.3.4, the associator is trivial i.e.  $\alpha = \text{id} \otimes \text{id} \otimes \text{id}$  as well as  $\lambda = \rho$ .