Rotationally-Symmetric Solutions to a Nonlinear Elliptic System Under an Incompressibility Constraint and Related Problems

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Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another university for the award of any other degree.

George Morrison

Abstract

In this thesis we study the nonlinear elliptic system

$$\mathbf{EL}[u;\Omega,\varphi] := \begin{cases} \mathscr{L}[u] = \nabla\mathscr{P} & \text{in } \Omega, \\ \det\nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

for $\mathscr{P} = \mathscr{P}(x)$ a hydrostatic pressure (Lagrange multiplier) related to the incompressibility constraint $\det \nabla u = 1$ and

$$\mathscr{L}[u] := (\nabla u)^t \{\mathrm{div}[F_\xi(|x|, |u|^2, |\nabla u|^2)\nabla u] - F_s(|x|, |u|^2, |\nabla u|^2)u\}.$$

Here $F = F(r, s, \xi)$ is a sufficiently regular and suitably convex function and we take $\varphi \equiv x$, hence $\mathbf{EL}[u;\Omega,\varphi]$ is the Euler-Lagrange equation associated to the energy functional

$$\mathbb{F}[u;\Omega] := \int_\Omega F(|x|, |u|^2, |\nabla u|^2)\, dx.$$

The goal throughout is to classify solutions of $\mathbf{EL}[u;\Omega,\varphi]$ (that is, critical points of the energy $\mathbb{F}[u;\Omega]$) for two classes of geometrically-motivated maps. The first of which are *generalised twists* $u(x) = \mathbf{Q}(|x|)x$ for $\mathbf{Q}$ an $\mathbf{SO}(n)$-valued curve and the second are *whirls* $u(x) = \mathbf{Q}(\varrho)x$ for $\varrho = (\rho_1, \dots, \rho_N)$ a vector of 2-plane radial variables where $\mathbf{Q}$ has a more complex structure.

By relaxing the variational context we also consider a more general system where we set $F_\xi = \mathsf{A}$ and $F_s = -\mathsf{B}$ in $\mathscr{L}[u]$ above, for suitable functions $\mathsf{A}(r, s, \xi)$ and $\mathsf{B}(r, s, \xi)$ bearing no relationship to one-another (i.e. $\mathsf{A}_s \neq -\mathsf{B}_\xi$ necessarily). It is seen that many of the results derived in the variational setting have analogies in this more general framework.

Along with the analysis of the structure and irrotationality of the vector field $\mathscr{L}[u]$ and ultimate solution of the PDEs in u above we also study a series of isotropic ODEs for the $\mathbf{SO}(n)$-valued curves $\mathbf{Q}(r)$ over this compact Lie group, specifically considering geodesic-type solutions $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ for some $\mathscr{G}$ of class $\mathscr{C}^2$ and $\mathbf{H} \in \mathfrak{so}(n)$, the Lie algebra of skew-symmetric matrices.

We establish the existence of a countably infinite scale of twist and whirl solutions to $\mathbf{EL}[u;\Omega,\varphi]$ in even dimensions. By analysing the curl-free structure of the vector field $\mathscr{L}[u]$ we introduce a discriminant term

$$\Delta_F := \frac{2[(n+1)F_\xi + 2r^2F_{\xi\xi}|\mathbf{Q}^t\dot{\mathbf{Q}}\theta|^2 + 2r^2F_{s\xi}][F_\xi + r^2F_{\xi\xi}|\mathbf{Q}^t\dot{\mathbf{Q}}\theta|^2] + rF_\xi F_{r\xi}}{r^2(F_\xi + 2r^2F_{\xi\xi}|\mathbf{Q}^t\dot{\mathbf{Q}}\theta|^2)}$$

upon the assumption that $\mathbf{Q}$ solves a given ODE. This discriminant is derived formally in Chapter 5. Remarkably, if $\Delta_F \equiv 0$ then we have a previously unknown countably infinite scale of solutions to $\mathbf{EL}[u;\Omega,\varphi]$ in odd dimensions as well as even. This original result is made possible by the dependence of F on r and s; if $F = F(\xi)$ then Δ_F is nowhere zero and the only solution in odd dimensions is the trivial map $u \equiv x$.

One particular Lagrangian studied in detail is $F(r,s,\xi) = h(r,s)\xi$, which corresponds to a weighted Dirichlet energy when substituted into $\mathbb{F}[u;\Omega]$. Here a necessary and sufficient condition for the above discriminant to vanish is $2(n+1)h(r,s) + rh_r(r,s) + 4r^2h_s(r,s) \equiv 0$. The additional benefit of studying this Lagrangian is that solutions of $\mathbf{EL}[u;\Omega,\varphi]$ can be explicitly described and they admit a geodesic-type twist path $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$.

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George Morrison,
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Contents

Chapter 1

Introduction

The principal object of study in this thesis is the nonlinear elliptic system subject to an incompressibility constraint on its solution, explicitly

$$\mathbf{EL}[u;\Omega,\varphi] := \begin{cases} \mathscr{L}[u] = \nabla\mathscr{P} & \text{in } \Omega, \\ \det\nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{1.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary, $u = (u_1, \dots, u_n)$ is a map on Ω (and into $\mathbb{R}^n$), $\mathscr{P} = \mathscr{P}(x)$ is an *a priori* unknown hydrostatic pressure field (Lagrange multiplier) and $\mathscr{L}[u]$ is the second-order partial differential operator

$$\mathscr{L}[u] := (\nabla u)^t\{\operatorname{div}[F_\xi(|x|, |u|^2, |\nabla u|^2)\nabla u] - F_s(|x|, |u|^2, |\nabla u|^2)u\}. \tag{1.0.2}$$

It can be seen (and is derived formally in Appendix B) that this system arises as the Euler-Lagrange equation associated to the variational energy functional

$$\mathbb{F}[u;\Omega] := \int_\Omega F(|x|, |u|^2, |\nabla u|^2)\, dx \tag{1.0.3}$$

over the admissible space of incompressible p-Sobolev maps (with $p \geq 1$)

$$\mathscr{A}^p_\varphi := \left\{u \in W^{1,p}(\Omega, \mathbb{R}^n) : \det\nabla u = 1,\, u|_{\partial\Omega} = \varphi\right\}, \tag{1.0.4}$$

where $\nabla u = [\partial u_i/\partial x_j : 1 \leq i, j \leq n]$ is an $n \times n$ matrix field. For notation and background on the theory of Sobolev spaces see [2, 33, 44, 59, 81] and [90]. The

Lagrangian $F = F(r, s, \xi)$ in (1.0.3) is subject to suitable bounds and regularity, coercivity and convexity constraints and, in (1.0.2), $F_r = F_r(r, s, \xi)$, $F_s = F_s(r, s, \xi)$ and $F_\xi = F_\xi(r, s, \xi)$ denote the derivatives of $F = F(r, s, \xi)$ with respect to the first, second and third variables respectively. The divergence in (1.0.2) acts on the matrix field $F_\xi(|x|, |u|^2, |\nabla u|^2)\nabla u$ row-wise and throughout the text we take the boundary data $\varphi \equiv x$ in (1.0.1), which is interpreted in the sense of traces.

Various classes of Lagrangians pertaining to this problem are well established in the literature including, for example, $F(r, s, \xi) = (\xi/s)^{n/2}$ which corresponds to an energy in the theory of maps with bounded distortion (*c.f.* [5, 62, 71]) or $F(r, s, \xi) = h(r, s)\xi$ which corresponds to a weighted version of the classical Dirichlet energy. The latter of these two examples is studied in great detail in this thesis.

In terms of motivation, this system is in the canon of nonlinear elasticity. The bounded domain $\Omega \subset \mathbb{R}^n$ represents some hyperelastic body with the map u representing a *deformation* of Ω. Normally in the context of elasticity $n = 2$ or 3 but throughout the thesis we consider $n \geq 2$ to allow for more interesting and general mathematical situations. The energy $\mathbb{F}[u; \Omega]$ is the total elastic energy under consideration where the Lagrangian $F = F(r, s, \xi)$ represents the stored energy function for which we principally consider *isotropic* examples. We impose the incompressibility constraint $\det \nabla u = 1$ throughout which, together with the boundary condition $u \equiv x$ and the appropriate regularity or integrability of the deformation u, ensures that u preserves the volume of Ω. The term $\mathscr{P}$ appearing on the right-hand side of the PDE governing (1.0.1) enters in the derivation of the system as a Lagrange multiplier [*c.f.* (B.0.4)] and is referred to as the *hydrostatic pressure* field. For much more background on the theory of nonlinear elasticity and its applications see [3, 9, 10, 14, 24, 25, 54, 64] and the references therein.

Given this setup our goal is to find, among all volume-preserving deformations of Ω in the admissible space $\mathscr{A}^p_\varphi$, those which arise as critical points of the energy functional $\mathbb{F}[u; \Omega]$ and as such are equilibria of this total elastic energy. We restrict our study to the case that Ω is a hyperelastic incompressible annulus, that is $\Omega = \mathbb{X}^n = \mathbb{X}^n[a, b] := \{x \in \mathbb{R}^n : a < |x| < b\}$ with $0 < a < b < \infty$, and u is one of two classes of geometrically-motivated maps bearing some inherent symmetries. These are *generalised twist* and *whirls*, which both enjoy some natural rotational invariance as will be seen. All such geometric assumptions are treated fully in the introductions to the respective chapters.

This thesis is comprised of five technical chapters each containing an introduction of its own detailing the preliminaries needed. Given this, the discussion here will be kept short; we will briefly outline the highlights and interconnectivity of the chapters in broad strokes, deferring the technical details to the main body of the text.

Beginning with Chapter 2, in which the method and results are in line with those from the work in progress [67], we study the system $\mathbf{EL}[u;\Omega,\varphi]$ for a generalised twist $u(x) = \mathbf{Q}(|x|)x$ under the assumption that $\mathbf{Q} = \mathbf{Q}(r)$, an $\mathbf{SO}(n)$-valued curve called the *twist path*, solves a given ODE. Here we see that, upon the additional assumption that $\|\dot{\mathbf{Q}}\theta\|_{L^1(a,b)}$ is independent of θ, the system (1.0.1) admits an infinitude of solutions (indexed by the integers $\mathbb{Z}$) in even spatial dimensions, whereas, in great contrast, the only solution in odd dimensions is the identity map $u \equiv x$. These solutions in even dimensions necessarily admit geodesic-type twist paths $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ for $\mathbf{H}$ an appropriate $n \times n$ skew-symmetric matrix and $\mathscr{G}$ the solution of a given two-point boundary value problem. We also consider the particular Lagrangian $F(r,s,\xi) = h(r,s)\xi$ for some positive $\mathscr{C}^2$ function $h = h(r,s)$. By analysing the curl of the resulting vector field $\mathscr{L}[u]$ we extract a discriminant term and we see that the non-vanishing of this discriminant is a necessary condition for the triviality of solutions in odd dimensions. Here and for future reference, this discriminant is given by

$$\Delta_h := \frac{2(n+1)h(r,r^2) + rh_r(r,r^2) + 4r^2h_s(r,r^2)}{r^2}, \tag{1.0.5}$$

where h_r, h_s denote the derivatives of $h = h(r,s)$ in the first and second variables respectively. Colloquially we will say throughout the thesis that if this particular discriminant vanishes, that is $\Delta_h \equiv 0$, a necessary and sufficient condition for which is $2(n+1)h(r,r^2) + rh_r(r,r^2) + 4r^2h_s(r,r^2) \equiv 0$, then the "$h$-condition" holds. We note that there is an emerging literature devoted to the study of generalised twists in the context of nonlinear elasticity; see, for example, [30, 31, 61, 63, 66, 74, 75, 88].

Chapter 3 then constitutes a deeper study of the Lagrangian $F(r,s,\xi) = h(r,s)\xi$ first seen above, both for generalised twists as well as whirl maps $u(x) = \mathbf{Q}(\varrho)x$, for ϱ a suitable vector of 2-plane radial variables. When substituting such Lagrangians into the energy integral (1.0.3), this corresponds to a *weighted* version of the Dirichlet energy, with a positive weight function $h = h(r, s)$. Remarkably, it is seen that any whirl solution reduces to a generalised twist - that is, with a slight abuse of notation, $\mathbf{Q}(\varrho) = \mathbf{Q}(\|\varrho\|) = \mathbf{Q}(r)$ - and all such solutions are classified explicitly. In this chapter it is now verified that, if the h-condition holds (*i.e.* the discriminant Δ_h above vanishes) then there is an additional infinite class of solutions to the system (1.0.1) in odd spatial dimensions as well as even.

In Chapter 4 we consider the same questions posed for generalised twists in the first two chapters but now in a non-variational context. This is based on the published work [65]. We take the system (1.0.1) now governed by the PDE $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$, where

$$\mathscr{L}[u; \mathsf{A}, \mathsf{B}] := (\nabla u)^t \{\mathrm{div}[\mathsf{A}(|x|, |u|^2, |\nabla u|^2)\nabla u] + \mathsf{B}(|x|, |u|^2, |\nabla u|^2)u\}, \quad (1.0.6)$$

for $\mathsf{A} = \mathsf{A}(r, s, \xi)$, $\mathsf{B} = \mathsf{B}(r, s, \xi)$ suitably regular real-valued functions. Clearly if $\mathsf{A} = F_\xi$ and $\mathsf{B} = -F_s$, we return to the variational setting where the system $\mathbf{EL}[u; \Omega, \varphi]$ arises as the Euler-Lagrange equation associated to $\mathbb{F}[u; \Omega]$, but we stress that no such assumptions are made here. This short chapter exclusively considers geodesic-type twist loops and we observe the same phenomena as previously. That is, we have a countably infinite class of solutions to (1.0.1)-(1.0.6) in even spatial dimensions yet no nontrivial solutions in odd dimensions. Upon considering $\mathsf{A}(r, s, \xi) = h(r, s)$, $\mathsf{B}(r, s, \xi) = -h_s(r, s)\xi$ to mimic the weighted Dirichlet scenario, we see the introduction of a discriminant term and an infinite class of solutions in odd dimensions as well as even, given the vanishing of this discriminant, in line with what we have already observed in the variational setting. Here a necessary and sufficient condition for the vanishing of the discriminant is that $2(n+1)h(r, r^2) + rh_r(r, r^2) + 2r^2[h_s(r, r^2) - g(r, r^2)] \equiv 0$, which is directly comparable with the h-condition introduced previously.

We return to the variational context in Chapter 5 and, as in Chapter 2, work under the assumption that the twist path $\mathbf{Q}$ solves a given ODE in r, this one being slightly relaxed. In fact we conduct a study of the relationship between three ODEs for $\mathbf{Q}$, one arising as an Euler-Lagrange equation related to a restricted energy functional, before solving the system $\mathbf{EL}[u; \Omega, \varphi]$. By enforcing the assumption that $\mathbf{Q}$ solves a 'weaker' ODE than in Chapter 2, a discriminant is considered for the full Lagrangian $F = F(r, s, \xi)$ as opposed to the (restricted) weighted Dirichlet case, where no dependence on the third argument is required. This, as before, permits an infinitude of solutions in odd as well as even dimensions whenever this discriminant vanishes.

Finally in Chapter 6 we consider only whirl maps and return to the non-variational context - that is we solve (1.0.1)-(1.0.6). In analogy to the study of various ODEs for $\mathbf{Q}$ the work here depends intimately on a divergence-free system for a vector of $\mathscr{C}^2$ functions $f = (f_1, \dots, f_d)$ apparent in the description of the twist path $\mathbf{Q}$, which here we assume takes values on the maximal torus of block-diagonal matrices $\mathbb{T} \subset \mathbf{SO}(n)$. For more on the theory of Lie groups see [4, 27, 29, 35, 45, 47, 73, 80] and further references therein. We conduct a study of the PDE $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$ in three and four spatial dimensions by hand upon introducing a polar coordinate system. The chapter closes with a full n-dimensional analysis in the case where the function $\mathsf{A}(r, s, \xi) = h(r, s)$. Results here are consistent with the rest of the text and it is the component-wise analysis in low spatial dimensions which is novel.

Four appendices complement the main body of the text and gather together many key results used repeatedly throughout. Appendix A is a collection of key identities pertaining to generalised twists and whirls and their gradients. In particular there are two results devoted to proving that both classes of maps, that is twists and whirls, satisfy the incompressibility constraint $\det \nabla u = 1$. Appendix B is then a short yet formal derivation of the Euler-Lagrange equation (1.0.1)-(1.0.2) using the Lagrange multiplier method and considering an unconstrained energy.

In Appendices C and D we give a series of results proving, respectively, the existence and/or uniqueness of solutions to certain differential equations considered throughout the thesis and some curl-free results for generic vector fields, along with necessary and sufficient conditions under which these vector fields constitute gradients, a stronger property than irrotationality.

Chapter 2

Generalised Twists as Solutions to the Nonlinear System $\mathscr{L}[u] = \nabla \mathscr{P}$

In this chapter we address questions on the existence and multiplicity of a class of geometrically-motivated mappings serving as solutions to the nonlinear system in variation:

$$\begin{cases} (\nabla u)^t \left\{ \operatorname{div} [F_\xi \nabla u] - F_s u \right\} = \nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, $F = F(r, s, \xi)$ is a sufficiently smooth Lagrangian and $F_s = F_s(|x|, |u|^2, |\nabla u|^2)$ and $F_\xi = F_\xi(|x|, |u|^2, |\nabla u|^2)$ with F_s and F_ξ denoting the derivatives of F with respect to the second and third variables respectively. Furthermore $\mathscr{P} = \mathscr{P}(x)$ is an *a priori* unknown hydrostatic pressure resulting from the incompressibility constraint $\det \nabla u = 1$ and for convenience the boundary map φ is taken throughout as the identity. Of particular interest is when $\Omega = \mathbb{X}^n[a, b]$ is a symmetric finite annulus and $u = r\mathbf{Q}(r)\theta$ is an incompressible twist mapping with a twist path $\mathbf{Q} \in \mathscr{C}([a, b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a, b[, \mathbf{SO}(n))$. Here we prove that when the spatial dimension n is *even* the above system admits a countably infinite family of incompressible twisting solutions of different topological types whereas in sharp contrast when n is *odd* we have only the trivial solution $u \equiv x$.

2.1 Statement of the Result

Let $\Omega \subset \mathbb{R}^n$ (with $n \geq 2$) be a bounded domain and consider the variational integral

$$\mathbb{F}[u; \Omega] := \int_\Omega F(|x|, |u|^2, |\nabla u|^2)\, dx. \tag{2.1.1}$$

Here $F = F(r, s, \xi)$ is a twice continuously differentiable Lagrangian that is assumed to be bounded from below, coercive and to have a polynomial growth at infinity whilst being uniformly convex and monotone increasing in the third variable (see below for a precise formulation of the assumptions on F). The goal is then to seek extremisers (equivalently critical points) of $\mathbb{F}$ over the space of admissible weakly differentiable *incompressible* Sobolev mappings defined by

$$\mathscr{A}^p_\varphi(\Omega) := \Big\{ u \in W^{1,p}(\Omega, \mathbb{R}^n) : \det \nabla u = 1 \ \textit{a.e.} \text{ in } \Omega,\, u = \varphi \text{ on } \partial\Omega \Big\}, \tag{2.1.2}$$

where $1 \leq p < \infty$ is fixed. Note that the boundary mapping $\varphi \in \mathscr{C}(\partial\Omega, \mathbb{R}^n)$ is taken throughout to be $\varphi \equiv x$ whilst the last condition in (2.1.2) asserts that $u \equiv \varphi$ on $\partial\Omega$ in the sense of traces. Furthermore ∇u here denotes the gradient of u, an $n \times n$ matrix-field in Ω, with $\det \nabla u$ denoting the Jacobian determinant of u, also known as the deformation gradient. The Euler-Lagrange equation associated with the energy functional (2.1.1) over the space of admissible mappings $\mathscr{A}^p_\varphi(\Omega)$ can be formulated as[1]

$$\mathbf{EL}[u; \Omega, \varphi] = \begin{cases} \mathscr{L}[u] = \nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u \equiv \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.1.3}$$

where the differential operator $\mathscr{L} = \mathscr{L}[u]$ here is given explicitly by

$$\mathscr{L}[u] := (\nabla u)^t \left\{ \operatorname{div} \left[F_\xi(|x|, |u|^2, |\nabla u|^2) \nabla u \right] - F_s(|x|, |u|^2, |\nabla u|^2) u \right\}. \tag{2.1.4}$$

Here F_s and F_ξ denote the derivatives of the Lagrangian F with respect to the second and third variables respectively and $\mathscr{P} = \mathscr{P}(x)$ in (2.1.3) is an *a priori* unknown hydrostatic pressure resulting from the incompressibility constraint $\det \nabla u = 1$. Note firstly that by virtue of the incompressibility constraint we have $(\operatorname{cof} \nabla u)^{-1} = (\nabla u)^t$ which will be used repeatedly below and secondly that

[1]See Appendix B for a derivation of this system. Note that the identity mapping $u \equiv x$ is always a solution to this system in view of the vector field $\mathscr{L}[u \equiv x] = \nabla[F_\xi] - F_s x$ with $F_\xi = F_\xi(r, r^2, n)$, $F_s = F_s(r, r^2, n)$ being a gradient field in Ω.

the divergence operator in $\mathscr{L}[u]$ is understood to act row-wise on the matrix field $F_\xi(|x|, |u|^2, |\nabla u|^2)\nabla u$.

We confine to the case where the domain is a bounded, rotationally-symmetric annulus $\Omega = \mathbb{X}^n[a,b] := \{x \in \mathbb{R}^n : a < |x| < b\}$ with $0 < a < b < \infty$ and the extremiser u is a twist on $\mathbb{X}^n$, that is, a continuous self-mapping of the closed annulus onto itself agreeing with the identity on $\partial\mathbb{X}^n$ and admitting, in spherical-polar coordinates, the representation

$$u : x = (r, \theta) \mapsto (r, \mathbf{Q}(r)\theta), \qquad a \le r \le b, \quad r = |x|, \ \theta = x|x|^{-1}, \quad x \in \mathbb{X}^n. \tag{2.1.5}$$

For obvious geometric reasons the mapping $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n))$ is referred to as the twist *path* (or in the event $\mathbf{Q}(a) = \mathbf{Q}(b)$ the twist *loop*) associated with the twist u. The main result of this chapter is a multiplicity result in even dimensions for solutions of the nonlinear system (2.1.3)-(2.1.4) in the form of twist mappings and is formulated in the following theorem. Note that here and later we write $\mathbf{J}_n = diag(\mathbf{J}, ..., \mathbf{J})$ for n even and $\mathbf{J}_n = diag(\mathbf{J}, ..., \mathbf{J}, 0)$ for n odd where $\mathbf{J}$ is the constant 2×2 skew-symmetric matrix of rotation by angle $\pi/2$ [*cf.* (2.3.4)]. We also denote by $\exp\{\cdot\}$ the exponential map of the compact Lie group $\mathbf{SO}(n)$ whose domain is the Lie algebra $\mathfrak{so}(n)$ of skew-symmetric matrices.

For the sake of future reference and clarity we assume throughout that $F = F(r, s, \xi)$ is a twice continuously differentiable Lagrangian, that is, $F \in \mathscr{C}^2(U)$ where $U = U(\mathbb{X}^n[a,b]) = [a,b] \times]0,\infty[\times]0,\infty[\subset \mathbb{R}^3$. We assume that there exists some $c_0 \in \mathbb{R}$ such that $F(r,s,\xi) \ge c_0$ for all $(r,s,\xi) \in U$ and that for every compact set $K \subset]0,\infty[$ there are constants $c_1 = c_1(K), c_2 = c_2(K) > 0$ such that, for $p > 1$,

$$\begin{aligned} |F_\xi(r,s,\zeta^2)\zeta| &\le c_2|\zeta|^{p-1}, \qquad \forall (r,s,\zeta^2) \in U, \ \text{with } s \in K, \\ c_0 + c_1|\zeta|^p \le F(r,s,\zeta^2) &\le c_2|\zeta|^p, \qquad \forall (r,s,\zeta^2) \in U, \ \text{with } s \in K. \end{aligned}$$

In particular $\mathbb{F}$ is well-defined and bounded from below (yet not necessarily finite everywhere) on $\mathscr{A}^p_\varphi(\mathbb{X}^n)$. As for convexity all we assume is that $F_\xi > 0$, $F_{\xi\xi} \ge 0$ and that the twice continuously differentiable function $\zeta \mapsto F(r, r^2, n + r^2\zeta^2)$ is uniformly convex in ζ for all $a \le r \le b$ and $\zeta \in \mathbb{R}$.

Main Theorem. *For $n \ge 2$ even, the nonlinear system* (2.1.3) *has an infinite family of incompressible twisting solutions $u = u(x; m)$ (with $m \in \mathbb{Z}$) of class*

$\mathscr{C}^2$ admitting the representation

$$\begin{aligned} u(x;m) &= r\mathbf{Q}(r;m)\theta = r\exp\{\mathscr{G}(r;m)\mathbf{H}\}\theta \qquad\qquad r = |x|, \quad \theta = x|x|^{-1}, \\ &= r\mathbf{P}diag(\mathcal{R}[\mathscr{G}](r;m), \dots, \mathcal{R}[\mathscr{G}](r;m))\mathbf{P}^t\theta, \qquad x \in \overline{\mathbb{X}^n}, \end{aligned} \tag{2.1.6}$$

with $\mathbf{H} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$ and arbitrary $\mathbf{P} \in \mathbf{O}(n)$.[2] *Here the angle of rotation function $\mathscr{G} = \mathscr{G}(r;m) \in \mathscr{C}^2[a,b]$ is the unique solution to the two point boundary value problem*

$$\begin{cases} \dfrac{d}{dr}\left[r^{n+1}F_\xi\left(r, r^2, n + r^2\dot{\mathscr{G}}^2\right)\dot{\mathscr{G}}\right] = 0, \qquad a < r < b, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2m\pi, \end{cases} \tag{2.1.7}$$

while the twist loop $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\} = \mathbf{P}diag(\mathcal{R}[\mathscr{G}](r), \dots, \mathcal{R}[\mathscr{G}](r))\mathbf{P}^t$ and each diagonal block $\mathcal{R}[\mathscr{G}]$ is an $\mathbf{SO}(2)$ rotation matrix by angle $\mathscr{G}$ [see (2.3.4)].

2.2 An Euler-Lagrange Equation for the Twist Path $\mathbf{Q}(r)$

Our goal is to seek and describe solutions to the nonlinear system (2.1.3) that take the specific geometric form $u : (r, \theta) \mapsto (r, \mathbf{Q}(r)\theta)$. First we direct the reader to Appendix A where we gather numerous important identities relating to generalised twists and their gradients. Given in particular those identities gathered in Proposition A.0.3 we can proceed by restricting the energy functional (2.1.1) to the subclass of generalised twists $u = r\mathbf{Q}(r)\theta$ hence obtaining a formulation in terms of the associated twist loops $\mathbf{Q} = \mathbf{Q}(r)$. Indeed referring to (2.1.1) we can write

$$\begin{aligned} \mathbb{F}[u; \mathbb{X}^n] &= \int_{\mathbb{X}^n} F(|x|, |u|^2, |\nabla u|^2)\,dx \\ &= \int_a^b \int_{\mathbb{S}^{n-1}} r^{n-1}F(r, r^2, n + r^2|\dot{\mathbf{Q}}\theta|^2)\,d\mathcal{H}^{n-1}(\theta)\,dr \\ &= \int_a^b E(r, \dot{\mathbf{Q}})r^{n-1}dr =: \mathbb{E}[\mathbf{Q}; (a,b)], \end{aligned} \tag{2.2.1}$$

[2] Note that for even n any such $\mathbf{H}$ is a *skew-symmetric* square root of $-\mathbf{I}_n$. For odd n there is no such root.

where the reduced energy functional $\mathbb{E}$ has the Lagrangian $E = E(r, \mathbf{A})$ given by

$$E(r, \mathbf{A}) := \int_{\mathbb{S}^{n-1}} F(r, r^2, n + r^2 |\mathbf{A}\theta|^2)\, d\mathcal{H}^{n-1}(\theta) \tag{2.2.2}$$

for $a \le r \le b$ and $\mathbf{A}$ in the tangent space of $\mathbf{SO}(n)$, *i.e.*, the Lie algebra of $n \times n$ skew-symmetric matrices. As a matter of fact a basic inspection shows that definition (2.2.2) lends itself to an immediate generalisation in that $\mathbf{A}$ can be taken from the full space of $n \times n$ matrices. Thus when necessary we speak of E in this extended sense. The above formulation now prompts us to introduce the class of admissible twist loops, and subsequently search for extremising loops for the energy $\mathbb{E}$ from within this class. Towards this end we set

$$\mathscr{B}^p_{\mathbf{I}_n} = \mathscr{B}^p_{\mathbf{I}_n}(a, b) := \Big\{ \mathbf{Q} \in W^{1,p}(a, b; \mathbf{SO}(n)) : \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n \Big\}. \tag{2.2.3}$$

In search of extremising loops for this reduced energy we now proceed on to formulating the associated Euler-Lagrange equation. This as will be seen is a particular case of the following result.

Proposition 2.2.1. *Let $L = L(r, \eta, \zeta)$ be a sufficiently smooth Lagrangian and $\mathbf{Q} \in \mathscr{B}^p_{\mathbf{I}_n}(a, b)$ an extremal of class $\mathscr{C}^1$ of the energy integral*

$$\mathbb{L}[\mathbf{Q}; a, b] := \int_a^b L(r, \mathbf{Q}, \dot{\mathbf{Q}})\, dr, \qquad \dot{\mathbf{Q}} = \frac{d\mathbf{Q}}{dr}. \tag{2.2.4}$$

Then $\mathbf{Q}$ satisfies $\mathscr{E}_L[\mathbf{Q}; a, b] = 0$ where $\mathscr{E}_L$ denotes the second-order differential operator

$$\mathscr{E}_L[\mathbf{Q}; a, b] = -\frac{d}{dr}\left[L_\zeta \mathbf{Q}^t - \mathbf{Q} L^t_\zeta \right] + L_\eta \mathbf{Q}^t - \mathbf{Q} L^t_\eta + L_\zeta \dot{\mathbf{Q}}^t - \dot{\mathbf{Q}} L^t_\zeta. \tag{2.2.5}$$

Here $L_\eta = L_\eta(r, \mathbf{Q}, \dot{\mathbf{Q}})$ and $L_\zeta = L_\zeta(r, \mathbf{Q}, \dot{\mathbf{Q}})$ with the subscripts denoting the derivatives of L with respect to the second and third arguments respectively.

Proof. Let $\mathbf{Q}$ be as described and consider the one parameter family of variations $\mathbf{Q}_\varepsilon$ with $\varepsilon \in \mathbb{R}$ defined by

$$\mathbf{Q}_\varepsilon := \mathbf{Q} + \varepsilon(\mathbf{F} - \mathbf{F}^t)\mathbf{Q}, \tag{2.2.6}$$

where $\mathbf{F} \in \mathscr{C}^\infty_0(]a, b[, \mathbb{M}^{n\times n})$. Then it can be seen that up to the first order in ε, the variations $\mathbf{Q}_\varepsilon$ are in $\mathbf{SO}(n)$: $\mathbf{Q}_\varepsilon \mathbf{Q}^t_\varepsilon = \mathbf{I}_n + \mathrm{O}(\varepsilon^2) = \mathbf{Q}^t_\varepsilon \mathbf{Q}_\varepsilon$ and so for the purpose of the first variation of energy extremality of $\mathbf{Q}$ gives

$$\frac{d}{d\varepsilon} \mathbb{L}[\mathbf{Q}_\varepsilon; a, b] \Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_a^b L(r, \mathbf{Q}_\varepsilon, \dot{\mathbf{Q}}_\varepsilon)\, dr \Big|_{\varepsilon=0} = 0. \tag{2.2.7}$$

Now by a basic and straightforward differentiation it is evident that

$$\dot{\mathbf{Q}}_\varepsilon = d\mathbf{Q}_\varepsilon/dr = \dot{\mathbf{Q}} + \varepsilon[(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t)\mathbf{Q} + (\mathbf{F} - \mathbf{F}^t)\dot{\mathbf{Q}}].$$

As such we are tasked with solving

$$\begin{aligned}\frac{d}{d\varepsilon}\mathbb{L}[\mathbf{Q}_\varepsilon; a, b]\Big|_{\varepsilon=0} = \int_a^b \Big\{ &\Big\langle L_\eta(r, \mathbf{Q}, \dot{\mathbf{Q}}), (\mathbf{F} - \mathbf{F}^t)\mathbf{Q} \Big\rangle \\ &+ \Big\langle L_\zeta(r, \mathbf{Q}, \dot{\mathbf{Q}}), \Big[(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t)\mathbf{Q} + (\mathbf{F} - \mathbf{F}^t)\dot{\mathbf{Q}}\Big] \Big\rangle \Big\} dr = 0.\end{aligned} \tag{2.2.8}$$

Suppressing the arguments in the Lagrangian and its derivatives for brevity, a rearrangement of terms gives

$$\int_a^b \Big\langle -\frac{d}{dr}\left(L_\zeta \mathbf{Q}^t\right) + L_\zeta \dot{\mathbf{Q}}^t + L_\eta \mathbf{Q}^t,\ \mathbf{F} - \mathbf{F}^t \Big\rangle\, dr = 0 \tag{2.2.9}$$

and so the conclusion follows by noting the arbitrariness of $\mathbf{F} \in \mathscr{C}_0^\infty(]a, b[, \mathbb{M}^{n\times n})$, the skew-symmetry of the matrix field $\mathbf{F} - \mathbf{F}^t$ and the fundamental lemma of the calculus of variations. $\square$

We now consider the particular case of (2.2.1) above where the Lagrangian is given by $L(r, \mathbf{Q}, \dot{\mathbf{Q}}) = E(r, \dot{\mathbf{Q}})r^{n-1}$, with $E(r, \dot{\mathbf{Q}})$ as defined by (2.2.2). It is clear that here $L_\eta \equiv 0$ and so in this case

$$\mathscr{E}_L[\mathbf{Q}; a, b] = -\frac{d}{dr}\left[L_\zeta \mathbf{Q}^t - \mathbf{Q}^t L_\zeta^t\right] + L_\zeta \dot{\mathbf{Q}}^t - \dot{\mathbf{Q}} L_\zeta^t. \tag{2.2.10}$$

Moreover by a further reference to (2.2.2) we see that

$$L_\zeta(r, \dot{\mathbf{Q}}) = 2\int_{\mathbb{S}^{n-1}} r^{n+1} F_\xi(r, r^2, n + r^2|\dot{\mathbf{Q}}\theta|^2)\dot{\mathbf{Q}}\theta \otimes \theta\, d\mathcal{H}^{n-1}(\theta), \tag{2.2.11}$$

and so in particular it follows that $L_\zeta \dot{\mathbf{Q}}^t - \dot{\mathbf{Q}} L_\zeta^t = 0$. As such we have proved the following statement.

Corollary 2.2.2. *The Euler-Lagrange equation associated with the energy integral $\mathbb{E} = \mathbb{E}[\mathbf{Q}; a, b]$ defined by* (2.2.1) *over the space of admissible twist loops $\mathscr{B}^p_{\mathbf{I}_n}(a, b)$ has the formulation*

$$\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\Big\{ r^{n+1} F_\xi(r, r^2, n + r^2|\dot{\mathbf{Q}}\theta|^2)\left[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta\right] \Big\} d\mathcal{H}^{n-1}(\theta) = 0, \tag{2.2.12}$$

for $a < r < b$.

2.3 The Totally Integrable Case $F = h(r, s)\xi$

Before proceeding onto the solution and implications of the Euler-Lagrange equation (2.2.12), we pause briefly to discuss an important and illustrative case. Indeed here we take the integrand $F(r, s, \xi) = h(r, s)\xi$ for some strictly positive $h \in \mathscr{C}^2([a, b]\times]0, \infty[)$ with (2.1.1) then being a *weighted* form of the Dirichlet energy. For this choice of integrand the reduced Euler-Lagrange equation (2.2.12), noting $F_\xi \equiv 1$, can be written as

$$\frac{d}{dr}\left\{ r^{n+1} h(r, r^2) \int_{\mathbb{S}^{n-1}} \left[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta \right] \, d\mathcal{H}^{n-1}(\theta) \right\} = 0. \qquad (2.3.1)$$

Upon evaluating the spherical integral, *e.g.*, by using the divergence theorem, it is seen that the above leads to the second order ODE:

$$\frac{d}{dr}\left\{ r^{n+1} h(r, r^2) \dot{\mathbf{Q}}\mathbf{Q}^t \right\} = 0, \qquad a < r < b. \qquad (2.3.2)$$

Integrating once gives $r^{n+1}h(r, r^2)\dot{\mathbf{Q}}\mathbf{Q}^t = \mathbf{H}$ where $\mathbf{H}$ is a constant $n \times n$ skew-symmetric matrix. This by noting the boundary conditions on the twist path as required by $\mathbf{Q} \in \mathscr{B}^2_{\mathbf{I}_n}(a, b)$ (see (2.2.3) with $p = 2$) has the general solution $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$, with

$$\mathscr{H}(r) = \frac{\mathsf{H}(r)}{\mathsf{H}(b)}, \qquad \mathsf{H}(r) = \int_a^r \frac{ds}{s^{n+1} h(s, s^2)}, \qquad a \le r \le b. \qquad (2.3.3)$$

We see from the above that $\mathscr{H}(a) = 0$ and $\mathscr{H}(b) = 1$ so the boundary condition for the twist path $\mathbf{Q}(a) = \mathbf{I}_n$ is immediately satisfied. Depending on whether the dimension n is even or odd, the skew-symmetric matrix $\mathbf{H}$ can be orthogonally diagonalised and written as $\mathbf{H} = \mathbf{P}diag(c_1\mathbf{J}, \dots, c_k\mathbf{J})\mathbf{P}^t$ when $n = 2k$, and $\mathbf{H} = \mathbf{P}diag(c_1\mathbf{J}, \dots, c_{k-1}\mathbf{J}, 0)\mathbf{P}^t$ when $n = 2k - 1$. Here $\mathbf{P} \in \mathbf{O}(n)$ and the scalars $c_1, \dots, c_k$ are all real – in fact, the eigenvalues of $\mathbf{H}$ are seen to be $\pm ic_j$ with $1 \le j \le k$ when $n = 2k$, and $0, \pm ic_j$ with $1 \le j \le k - 1$ when $n = 2k - 1$. Furthermore the 2×2 matrices $\mathbf{J}$ and $\mathcal{R}$ are given respectively by

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{R}[t] = \exp\{t\mathbf{J}\} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \qquad (2.3.4)$$

It is thus seen that

$$\mathbf{Q}(b) = \mathbf{I}_n \iff \exp\{\mathscr{H}(b)\mathbf{H}\} = \mathbf{I}_n \iff \exp\{\mathbf{H}\} = \mathbf{I}_n \qquad (2.3.5)$$

and plainly this last identity holds *iff* $c_j \in 2\mathbb{Z}\pi$ for all $1 \le j \le k$. This therefore characterises all solutions to (2.3.2) in $\mathscr{B}^2_{\mathbf{I}_n}(a,b)$ as $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ with $\mathscr{H}$ as in (2.3.3) and $\mathbf{H}$ as just described.

Now moving forward onto evaluating the action of the differential operator $\mathscr{L}$ on the twist map u with twist path $\mathbf{Q} = \mathbf{Q}(r)$ we first note that here

$$\begin{aligned}\mathscr{L}[u] &= (\nabla u)^t \left\{\mathrm{div}\left[h(r,|u|^2)\nabla u\right] - h_s(r,|u|^2)|\nabla u|^2 u\right\} \\ &= (\nabla u)^t \left\{\nabla u \nabla[h(r,|u|^2)] + h(r,|u|^2)\Delta u - h_s(r,|u|^2)|\nabla u|^2 u\right\} \qquad (2.3.6)\end{aligned}$$

and so upon differentiation, substitution for u and noting $|u|^2 = r^2$ we can write, with reference to Proposition A.0.3,

$$\begin{aligned}\mathscr{L}[u] =& (\nabla u)^t \left\{[h_r(r,r^2) + 2rh_s(r,r^2)]\nabla u\,\theta + h(r,r^2)\Delta u - h_s(r,r^2)|\nabla u|^2 u\right\} \\ =& (\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta)\Big\{[h_r(r,r^2) + 2rh_s(r,r^2)](\mathbf{Q} + r\dot{\mathbf{Q}}) \\ &+ h(r,r^2)\left[(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}\right] - rh_s(r,r^2)(n + r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}\Big\}\theta. \qquad (2.3.7)\end{aligned}$$

Expanding (2.3.2) by direct differentiation and using $\mathbf{Q}^t[\mathrm{LHS}\,(2.3.2)]\mathbf{Q} = 0$ the above simplifies to

$$\begin{aligned}\mathscr{L}[u] =& [h_r(r,r^2) + 2rh_s(r,r^2)]\theta \\ &+ [r^2 h_r(r,r^2) + r^3 h_s(r,r^2) + (n+1)rh(r,r^2)]|\dot{\mathbf{Q}}\theta|^2\theta \\ &+ [r^2 h(r,r^2)\langle\dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle - nrh_s(r,r^2)]\theta - rh(r,r^2)\mathbf{Q}^t\dot{\mathbf{Q}}\dot{\mathbf{Q}}^t\mathbf{Q}\theta. \qquad (2.3.8)\end{aligned}$$

Referring to the Euler-Lagrange equation (2.1.3) we now need to verify $\mathscr{L}[u] = \nabla\mathscr{P}$. Clearly here the first two terms in (2.3.8) form $\nabla h(|x|,|x|^2)$ whilst upon substituting $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ with $\dot{\mathbf{Q}} = \dot{\mathscr{H}}\mathbf{H}\mathbf{Q}$ and $\ddot{\mathbf{Q}} = (\ddot{\mathscr{H}}\mathbf{H} + \dot{\mathscr{H}}^2\mathbf{H}^2)\mathbf{Q}$ it is plain to see that $\langle\dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle = \dot{\mathscr{H}}\ddot{\mathscr{H}}|\mathbf{H}\theta|^2$ and $\mathbf{Q}^t\dot{\mathbf{Q}}\dot{\mathbf{Q}}^t\mathbf{Q} = -\dot{\mathscr{H}}^2\mathbf{H}^2$. Therefore we have

$$\begin{aligned}\mathscr{L}[u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta] =& \nabla h(|x|,|x|^2) \\ &+ [r^2 h_r(r,r^2) + r^3 h_s(r,r^2) + (n+1)rh(r,r^2)]\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta \\ &+ [r^2 h(r,r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}}|\mathbf{H}\theta|^2 - nrh_s(r,r^2)]\theta \\ &+ rh(r,r^2)\dot{\mathscr{H}}^2\mathbf{H}^2\theta. \qquad (2.3.9)\end{aligned}$$

Note that the term $-rnh_s(r,r^2)\theta = \nabla\mathsf{a}(|x|)$ for an appropriate primitive term a since it is a function of r alone. Then by a further application of the ODE

(2.3.2) for $\mathbf{Q} = \exp\{\mathscr{H}(r)\mathbf{H}\}$ this expression significantly reduces to

$$\begin{aligned}\mathscr{L}[u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta] =& \nabla h(|x|, |x|^2) - r^3 h_s(r, r^2)\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta \\ &+ \nabla \mathsf{a}(|x|) + rh(r, r^2)\dot{\mathscr{H}}^2\mathbf{H}^2\theta. \end{aligned} \tag{2.3.10}$$

Now by an application of Lemma D.0.1 (see Appendix D) to the vector field $U(x) := \mathscr{L}[u] - \nabla[h(|x|, |x|^2) + \mathsf{a}(|x|)] = rh(r, r^2)\dot{\mathscr{H}}^2\mathbf{H}^2\theta - r^3 h_s(r, r^2)\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta$ with $\mathscr{A}(r) = -h_s(r, r^2)\dot{\mathscr{H}}^2$, $\mathscr{B}(r) = h(r, r^2)\dot{\mathscr{H}}^2$ we have

$$2\mathscr{A} + \dot{\mathscr{B}}/r = \frac{1}{r}\left[h_r(r, r^2)\dot{\mathscr{H}}^2 + 2h(r, r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}}\right]. \tag{2.3.11}$$

We can again apply the ODE (2.3.2) to the above to lose the second derivative in $\mathscr{H}$. After this rearrangement we see that $2\mathscr{A} + \dot{\mathscr{B}}/r \not\equiv 0$ *iff* $rh_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2 h_s(r, r^2) \not\equiv 0$ on $]a, b[$ (*c.f.* Lemma D.0.1 for notation). Under this assumption we have $\operatorname{curl} U(x) = 0 \iff \mathbf{H}^2 = -c^2\,\mathbf{I}_n$. This therefore leads to the conclusion $|c_1|^2 = \cdots = |c_k|^2 = c^2$ when $n = 2k$, and $|c_1| = \cdots = |c_{k-1}| = 0$ when $n = 2k - 1$. Finally setting $c = 2m\pi$ with $m \in \mathbb{Z}$ ($m = 0$ when n odd) the boundary condition $\mathbf{Q}(b) = \mathbf{I}_n$ is also seen to be satisfied. In conclusion, we see that here the *reduced* Euler-Lagrange equation (the ODE) versus the *full* Euler-Lagrange equation (the PDE) associated with the energy integrand $F = h(r, s)\xi$ have the following contrasting consequences:

- (ODE I) From (2.3.1)-(2.3.2) we have:

  $$(2.3.2) \iff \mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}, \qquad \mathbf{H}^t = -\mathbf{H}, \tag{2.3.12}$$

  where $\mathscr{H} = \mathscr{H}(r)$ is as in (2.3.3).

- (PDE) From $\mathscr{L}[u] = \nabla\mathscr{P}$ we have:

  $$\begin{aligned}&\operatorname{curl}\{U(x) = \mathscr{L}[u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta] - \nabla[h(|x|, |x|^2) + \mathsf{a}(|x|)]\} = 0 \\ &\iff \mathbf{H} = \begin{cases} 2m\pi\mathbf{P}\mathbf{J}_n\mathbf{P}^t & n \text{ even} \\ 0 & n \text{ odd}\end{cases}\end{aligned} \tag{2.3.13}$$

  and so $u \equiv x$ (for n odd) and $u = r\mathbf{P}\exp\{2m\pi\mathscr{H}(r)\mathbf{J}_n\}\mathbf{P}^t\theta$ (for n even). Here $\mathbf{J}_n = diag(\mathbf{J}, ..., \mathbf{J})$ with $\mathbf{J}$ as in (2.3.4).[3]

[3] Note that when $n = 2k$ from $\mathbf{H}^2 = -c^2\mathbf{I}_n$ it follows that $c_1, \ldots c_k \in \{\pm c\}$. By adjusting $\mathbf{P} \in \mathbf{O}(n)$, however, we can arrange and assume without loss of generality that indeed $c_1 = \cdots = c_k = c$.

- (ODE II) As a further observation note that upon considering the strengthened form of the ODE (2.3.1) obtained by discarding the spherical integration and instead assuming

$$\frac{d}{dr}\left\{r^{n+1}h(r,r^2)\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\right\}=0,\quad a<r<b,\quad \forall\,\theta\in\mathbb{S}^{n-1}, \tag{2.3.14}$$

it follows firstly from (2.3.12) that any solution here has the form $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ and subsequently upon noting $\mathbf{QH} = \mathbf{HQ}$ and invoking Lemma 5.2.2 that

$$\begin{aligned}
(2.3.14) \iff & \frac{d}{dr}\left\{r^{n+1}h(r,r^2)\dot{\mathscr{H}}(r)[\mathbf{HQ}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\mathbf{HQ}\theta]\right\}=0\\
\iff & \frac{d}{dr}\left\{r^{n+1}h(r,r^2)\dot{\mathscr{H}}(r)\right\}[\mathbf{HQ}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\mathbf{HQ}\theta]\\
& +\left\{r^{n+1}h(r,r^2)\dot{\mathscr{H}}(r)^2\right\}[\mathbf{H}^2\mathbf{Q}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\mathbf{H}^2\mathbf{Q}\theta]=0\\
\iff & \mathbf{Q}\left[\mathbf{H}^2\theta\otimes\theta-\theta\otimes\mathbf{H}^2\theta\right]\mathbf{Q}^t=0 \iff \mathbf{H}^2=-c^2\mathbf{I}_n.
\end{aligned} \tag{2.3.15}$$

It is therefore seen that this strengthened version of the Euler-Lagrange equation (2.2.12) imposes the same restriction on the twist paths $\mathbf{Q} = \mathbf{Q}(r)$ as does the curl-free condition in the PDE. This stronger form of the ODE (2.2.12) and its curious implications will be discussed further in the next section.

2.4 Extremising Twist Paths as Scaled Geodesics on the Lie Group SO(n)

One of the main features of the Euler-Lagrange equation (2.2.12) is the presence of the spherical integral which, unlike the case with the weighted Dirichlet energy considered in the last section [*see* (2.3.2)], prevents one from reducing the equation to a directly integrable ODE in the radial variable and thus obtaining an explicit representation of the solutions as in (2.3.3). Motivated by the discussion in the previous section we start here by first considering solutions to (2.2.12) in the form $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ where $\mathscr{G} = \mathscr{G}(r)$ is a suitable function in $\mathscr{C}^2[a,b]$ and $\mathbf{H}$ is the constant $n\times n$ skew-symmetric matrix with $\mathbf{H} = \mathbf{PJ}_n\mathbf{P}^t$. Here and below $\mathbf{J}_n = diag(\mathbf{J},...,\mathbf{J})$ when n is even and $\mathbf{J}_n = diag(\mathbf{J},...,\mathbf{J},0)$

when n is odd, where $\mathbf{J} = \mathcal{R}[\pi/2]$ as in (2.3.4). Starting with the n even case where $|\dot{\mathbf{Q}}\theta|^2 = \dot{\mathscr{G}}^2|\mathbf{H}\theta|^2 = \dot{\mathscr{G}}^2$ and writing $F_\xi = F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)$ for short it is readily seen that

$$\begin{aligned}
\text{LHS (2.2.12)} &= \frac{d}{dr}\int_{\mathbb{S}^{n-1}} \left\{ r^{n+1} F_\xi [\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta] \right\} d\mathcal{H}^{n-1}(\theta) \\
&= \frac{d}{dr}\left\{ r^{n+1} F_\xi \dot{\mathscr{G}} \int_{\mathbb{S}^{n-1}} [\mathbf{HQ}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{HQ}\theta] \, d\mathcal{H}^{n-1}(\theta) \right\} \\
&= \frac{d}{dr}\left\{ r^{n+1} F_\xi \dot{\mathscr{G}} \omega_n [\mathbf{HQQ}^t - \mathbf{Q}(\mathbf{HQ})^t] \right\} \\
&= \frac{d}{dr}\left\{ r^{n+1} F_\xi \dot{\mathscr{G}} \right\} (2\omega_n \mathbf{H}). && (2.4.1)
\end{aligned}$$

As such in even dimensions a twist path $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ is a solution to the Euler-Lagrange equation (2.2.12) provided that the angle of rotation function $\mathscr{G}$ satisfies the second-order ODE

$$\frac{d}{dr}\left[r^{n+1} F_\xi \left(r, r^2, n + r^2 \dot{\mathscr{G}}^2 \right) \dot{\mathscr{G}} \right] = 0, \qquad a < r < b. \tag{2.4.2}$$

Now rather than following the route leading to (2.3.13) based on an analysis and verification of the PDE (2.1.3)-(2.1.4) and the curl-free condition on the vector field $\mathscr{L}[u = r\mathbf{Q}(r)\theta]$, in what follows we focus instead on the the ODE (2.2.12) and show that by a natural strengthening of (2.2.12) and invoking an interesting observation regarding geodesics on $\mathbf{SO}(n)$, the twist paths $\mathbf{Q} = \mathbf{Q}(r)$ serving as solutions here must have exactly the form and structure alluded to above. Towards this end it is readily seen that a stronger condition implying (2.2.12) is the strengthened ODE:

$$\frac{d}{dr}\left\{ r^{n+1} F_\xi \left(r, r^2, n + r^2 |\dot{\mathbf{Q}}\theta|^2 \right) \left[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta \right] \right\} = 0, \qquad a < r < b, \tag{2.4.3}$$

for *all* $\theta \in \mathbb{S}^{n-1}$. That $\mathbf{Q} = \exp\{\mathscr{G}\mathbf{H}\}$ with $\mathscr{G}$ satisfying (2.4.2) is still a solution to this stronger form of (2.2.12) follows by noting that here $\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta = \dot{\mathscr{G}}\mathbf{HQ}\theta \otimes \mathbf{Q}\theta$ and $\ddot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta = \ddot{\mathscr{G}}\mathbf{HQ}\theta \otimes \mathbf{Q}\theta - \dot{\mathscr{G}}^2 \mathbf{Q}\theta \otimes \mathbf{Q}\theta$. Hence for n even by substitution and a straightforward differentiation starting from (2.2.12) we have

$$\begin{aligned}
\text{LHS } (2.4.3) &= \frac{d}{dr}\left\{ r^{n+1}F_\xi\left(r, r^2, n+r^2|\dot{\mathbf{Q}}\theta|^2\right)\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\right\} \\
&= \left\{\frac{d}{dr}\left[r^{n+1}F_\xi\left(r, r^2, n+r^2\dot{\mathscr{G}}^2\right)\right]\dot{\mathscr{G}} + r^{n+1}F_\xi\left(r, r^2, n+r^2\dot{\mathscr{G}}^2\right)\ddot{\mathscr{G}}\right\}\times \\
&\quad\times(\mathbf{HQ}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\mathbf{HQ}\theta) \\
&= \left\{\frac{d}{dr}\left[r^{n+1}F_\xi\left(r, r^2, n+r^2\dot{\mathscr{G}}^2\right)\dot{\mathscr{G}}\right]\right\}(\mathbf{HQ}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\mathbf{HQ}\theta) \\
&= 0, \qquad (2.4.4)
\end{aligned}$$

as claimed. Now moving forward note that for a twist path $\mathbf{Q}\in\mathscr{C}^1([a,b],\mathbf{SO}(n))$ the integral

$$I(\mathbf{Q},\theta) = \int_a^b |\dot{\mathbf{Q}}\theta|\,dr \qquad (2.4.5)$$

represents the length of the curve $\gamma\in\mathscr{C}^1([a,b],\mathbb{S}^{n-1})$ given by $\gamma(r) = \mathbf{Q}(r)\theta$. Evidently for n even if $\mathbf{Q} = \exp\{\mathscr{G}\mathbf{H}\}$ with $\mathbf{H} = \mathbf{PJ}_n\mathbf{P}^t$ then this integral is independent of θ. We are now in a position to prove the following result.

Theorem 2.4.1. *Assume $\mathbf{Q}\in\mathscr{C}^1([a,b],\mathbf{SO}(n))\cap\mathscr{C}^2(]a,b[,\mathbf{SO}(n))$ with $\mathbf{Q}(a) = \mathbf{I}_n$ and $\mathbf{Q}(b) = \mathbf{I}_n$ satisfies (2.4.3). Assume additionally that the integral $I(\mathbf{Q},\theta)$ given by (2.4.5) is independent of θ. Then depending on the dimension n being even or odd we have the following description of $\mathbf{Q}$:*

- *n even: There exists $m\in\mathbb{Z}$ and $\mathbf{P}\in\mathbf{O}(n)$ such that*

$$\begin{aligned}
\mathbf{Q}(r) &= \exp\{\mathscr{G}(r;m)\mathbf{PJ}_n\mathbf{P}^t\} \\
&= \mathbf{P}\,diag(\mathcal{R}[\mathscr{G}](r;m),\dots,\mathcal{R}[\mathscr{G}](r;m))\mathbf{P}^t, \qquad a\le r\le b, \qquad (2.4.6)
\end{aligned}$$

  where $\mathbf{J}_n = diag(\mathbf{J},\dots,\mathbf{J})$, $\mathbf{J}$ and $\mathcal{R}$ are as in (2.3.4) and $\mathscr{G} = \mathscr{G}(r;m)\in\mathscr{C}^2[a,b]$ is the unique solution to the two point boundary value problem

$$\begin{cases}
\dfrac{d}{dr}\left[r^{n+1}F_\xi\left(r, r^2, n+r^2\dot{\mathscr{G}}^2\right)\dot{\mathscr{G}}\right] = 0, \qquad a<r<b, \\
\mathscr{G}(a) = 0, \\
\mathscr{G}(b) = 2m\pi.
\end{cases} \qquad (2.4.7)$$

- *n odd: $\mathbf{Q}\equiv\mathbf{I}_n$.*

Proof. Since $I(\mathbf{Q},\theta)=0$ implies $|\dot{\mathbf{Q}}\theta|=0$ and hence $\mathbf{Q}\equiv\mathbf{I}_n$, in the rest of the proof we assume $I(\mathbf{Q},\theta)>0$. Now we start by observing that if $\mathbf{Q}$ is a solution to (2.4.3) for every θ, then it also satisfies the equation

$$\frac{d}{dr}\left[r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\dot{\mathbf{Q}}\theta\right]+r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|^2\mathbf{Q}\theta=0. \tag{2.4.8}$$

Indeed starting from the left and writing $F_\xi=F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)$ we have

$$\begin{aligned}\frac{d}{dr}\left\{r^{n+1}F_\xi\dot{\mathbf{Q}}\theta\right\}=&\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\mathbf{Q}\theta\right\}\\ =&\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}\mathbf{Q}\theta\\ &+\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}\dot{\mathbf{Q}}\theta=-r^{n+1}F_\xi|\dot{\mathbf{Q}}\theta|^2\mathbf{Q}\theta,\end{aligned} \tag{2.4.9}$$

where in deducing the last equality we have used (2.4.3). Let us now introduce the integral

$$\mathscr{F}(r,\theta):=\int_a^r|\dot{\mathbf{Q}}(s)\theta|ds,\qquad a\le r\le b,\quad \theta\in\mathbb{S}^{n-1}. \tag{2.4.10}$$

Then testing (2.4.7) against $\mathscr{F}$ and using (2.4.8) by way of differentiating and then taking the inner product with $\dot{\mathbf{Q}}\theta$ we can write with $F_\xi=F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)$ as above and upon noting $\dot{\mathscr{F}}^2=|\dot{\mathbf{Q}}\theta|^2$,

$$\begin{aligned}\frac{d}{dr}\left\{r^{n+1}F_\xi|\dot{\mathbf{Q}}\theta|\right\}&=\frac{d}{dr}\left\{r^{n+1}F_\xi\right\}|\dot{\mathbf{Q}}\theta|+r^{n+1}F_\xi\frac{\langle\ddot{\mathbf{Q}}\theta,\dot{\mathbf{Q}}\theta\rangle}{|\dot{\mathbf{Q}}\theta|}\\ &=-r^{n+1}F_\xi\langle\dot{\mathbf{Q}}\theta,\mathbf{Q}\theta\rangle|\dot{\mathbf{Q}}\theta|=0,\end{aligned} \tag{2.4.11}$$

where the last identity uses the skew-symmetry of $\mathbf{Q}^t\dot{\mathbf{Q}}$. Note that this argument shows that, as a function of r, $r^{n+1}F_\xi|\dot{\mathbf{Q}}\theta|$ is a positive constant on any interval on which $|\dot{\mathbf{Q}}\theta|$ is non-zero and so a basic continuity argument implies that either $|\dot{\mathbf{Q}}\theta|\equiv 0$ on $[a,b]$ or $|\dot{\mathbf{Q}}\theta|>0$ on $[a,b]$. Furthermore it also shows that $\mathscr{F}(r,\theta)$ is a (*non-zero*) solution to the ODE in (2.4.7) for every fixed $\theta\in\mathbb{S}^{n-1}$.

Now this solution satisfies the end-point conditions $\mathscr{F}(a)=0$ and $\mathscr{F}(b)=I(\mathbf{Q},\theta)>0$ where the latter by assumption is independent of θ. We next aim to show that these together imply that $\mathscr{F}(r,\theta)$ is independent of θ. To this end we first note that solutions to (2.4.7) are extremisers over $\mathscr{D}^p_m(a,b)=\{\mathscr{G}\in W^{1,p}(a,b):\mathscr{G}(a)=0,\mathscr{G}(b)=2\pi m\}$ of the energy

$$\mathscr{G}\mapsto\int_a^b F\left(r,r^2,n+r^2\dot{\mathscr{G}}^2\right)r^{n-1}\,dr. \tag{2.4.12}$$

It is straightforward to verify that this functional is strictly convex (due to the assumptions on F: $F_\xi > 0$ and F being uniformly convex in ξ). Therefore, using standard results, solutions to (2.4.7) are the unique minimisers of this energy functional with respect to their own boundary conditions. This implies that as $\mathscr{F}(r,\theta)$ solves the ODE in (2.4.7) for all θ and the end-point conditions on $\mathscr{F}$, *i.e.*, at $r = a$ and $r = b$ are independent of θ, by the stated uniqueness of minimisers, the function $\mathscr{F}(r,\theta)$ must also be independent of θ. Now returning to the ODE in (2.4.7) it follows after integrating once that any solution $\mathscr{G} = \mathscr{G}(r)$ satisfies

$$r^{n+1} F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}} \equiv c, \qquad a < r < b, \tag{2.4.13}$$

for a suitable constant $c \in \mathbb{R}$. Thus as $F_\xi > 0$, all non-zero solutions to (2.4.7), in particular $\mathscr{F}$, are *strictly* monotone and hence invertible. Let $\mathscr{F}^{-1}(s) = r(s)$ and $\mathbf{Q}(r(s)) = \mathbf{K}(s)$ for $\mathbf{K} \in \mathscr{C}^2(]0,l[, \mathbf{SO}(n)) \cap \mathscr{C}([0,l], \mathbf{SO}(n))$ where $l = \mathscr{F}(b)$. Then writing $\mathbf{Q}(r) = \mathbf{K}(\mathscr{F}(r))$ we have $\dot{\mathbf{Q}} = \mathbf{K}'\dot{\mathscr{F}}$ (where prime denotes d/ds). Hence starting from (2.4.8) we can write, with $F_\xi = F_\xi(r, r^2, n + r^2\dot{\mathscr{F}}^2)$ for short,

$$\frac{d}{dr}\left[r^{n+1} F_\xi \dot{\mathbf{Q}}\theta\right] + r^{n+1} F_\xi |\dot{\mathbf{Q}}\theta|^2 \mathbf{Q}\theta = 0. \tag{2.4.14}$$

This upon substitution and a change of variables with $d/dr = \dot{\mathscr{F}} d/ds$ gives

$$\frac{d}{ds}\left[r^{n+1} F_\xi \dot{\mathscr{F}} \mathbf{K}'\theta\right] + r^{n+1} F_\xi \dot{\mathscr{F}} |\mathbf{K}'\theta|^2 \mathbf{K}\theta = c\left[\mathbf{K}'' + |\mathbf{K}'\theta|^2 \mathbf{K}\right]\theta = 0, \tag{2.4.15}$$

that is the geodesic equation on the unit sphere for $\gamma(s) = \mathbf{K}(s)\theta$. We need to solve this for $\mathbf{K} = \mathbf{K}(s)$ subject to $|\mathbf{K}'\theta|^2 = |\dot{\mathbf{Q}}\theta|^2/\dot{\mathscr{F}}^2 = 1$.

Indeed by taking the ansatz $\mathbf{K}(s) = \exp\{s\mathbf{A}\}$ for a constant $n \times n$ skew-symmetric matrix $\mathbf{A}$ we have $[\mathbf{A}^2 + \mathbf{I}_n]\mathbf{K} = 0$. For n odd this has no solution (with $I(\mathbf{Q}, \theta) > 0$) whilst for n even it gives $\mathbf{A} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$. It now follows at once that $\mathbf{Q}$ has the form described in the theorem, that is, for n odd $\mathbf{Q}(r) \equiv \mathbf{I}_n$ and for n even firstly

$$\mathbf{K}(s) = \mathbf{P}diag(\mathcal{R}[s], \ldots, \mathcal{R}[s])\mathbf{P}^t, \qquad 0 \leq s \leq l, \tag{2.4.16}$$

with $l = 2m\pi$ so that $\mathbf{K}(0) = \mathbf{K}(l) = \mathbf{I}_n$ and then

$$\mathbf{Q}(r) = \mathbf{K}(\mathscr{F}(r)) = \mathbf{P}diag(\mathcal{R}[\mathscr{F}](r), \ldots, \mathcal{R}[\mathscr{F}](r))\mathbf{P}^t, \tag{2.4.17}$$

where $\mathscr{F}$ is a solution to (2.4.7) with $\mathscr{F}(a) = 0$, $\mathscr{F}(b) = 2m\pi$. $\square$

2.5 Return to the Nonlinear System (2.1.3)-(2.1.4)

We next move on to contemplating the task of obtaining and characterising the twisting solutions u to the resulting nonlinear system

$$\begin{cases} \mathscr{L}[u] = \nabla\mathscr{P} & \text{in } \Omega, \\ \det\nabla u = 1 & \text{in } \Omega, \\ u \equiv x & \text{on } \partial\Omega, \end{cases} \tag{2.5.1}$$

where this system is derived in Appendix B and the differential operator $\mathscr{L}[u]$ is ultimately described by (B.0.6). Since the primary task here is to seek twisting solutions to the system (2.5.1) we proceed by first referring to Corollary A.0.5 which lists some key identities related to generalised twists for the specific choice of twist paths $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{A}\}$ with $\mathscr{G}(r)$ ($a \le r \le b$) a sufficiently regular angle of rotation function and $\mathbf{A}$ a fixed $n \times n$ skew-symmetric matrix. We need to discern which corresponding generalised twists $u = r\mathbf{Q}(r)\theta$ serve as solutions to the Euler-Lagrange equation (2.5.1) and this is resolved in even dimensions in the following result. We note that since the only twist loop $\mathbf{Q} = \mathbf{Q}(r)$ which, in odd dimensions, satisfies the assumptions laid out in Theorem 2.4.1 is the constant matrix $\mathbf{Q} \equiv \mathbf{I}_n$ it is necessary that the only generalised twist solution to (2.5.1) in odd dimensions is the identity map $u \equiv x$.

Theorem 2.5.1. *Let $n \ge 2$ be even and suppose $\mathscr{G} \in \mathscr{C}^2[a,b]$ is a solution to the boundary value problem* (2.4.7). *Let $\mathbf{H} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$ for some $\mathbf{P} \in \mathbf{O}(n)$ and put*

$$\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\} = \mathbf{P}diag(\mathcal{R}[\mathscr{G}](r), ..., \mathcal{R}[\mathscr{G}](r))\mathbf{P}^t, \qquad a \le r \le b \tag{2.5.2}$$

where $\mathbf{J}$, $\mathcal{R} \in \mathbf{SO}(2)$ are given by (2.3.4)*. Then u solves* (2.5.1)*; that is $\mathscr{L}[u = r\mathbf{Q}(r)\theta]$ is a gradient field in $\mathbb{X}^n[a,b]$. Specifically, $\mathscr{L}[u] = \nabla\mathscr{P}$ where*

$$\mathscr{P} = F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2) - G(r) \tag{2.5.3}$$

up to an additive constant where $\nabla G = r[\dot{\mathscr{G}}^2 F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2) + F_s(r, r^2, n + r^2\dot{\mathscr{G}}^2)]\theta$.

Proof. We use the description of the vector field $\mathscr{L}[u = r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta]$ as in (A.0.8) in Corollary A.0.5 along with the substitution $\mathbf{A} = \mathbf{H} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$. In this case, basic calculations give $\dot{\mathbf{Q}} = \dot{\mathscr{G}}\mathbf{H}\mathbf{Q}$, $\ddot{\mathbf{Q}} = (\ddot{\mathscr{G}}\mathbf{H} - \dot{\mathscr{G}}^2\mathbf{I}_n)\mathbf{Q}$, by virtue of

$\mathbf{H}^2 = -\mathbf{I}_n$ and $|\theta^\star|^2 = 1$ where $\theta^\star = \mathbf{H}\theta$ and $\langle\theta^\star, \theta\rangle = 0$. This gives us

$$\begin{aligned}\mathscr{L}[u] =& \mathscr{L}[r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta] = F_{\xi\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)\times \\ &\times (2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}})(\theta + r\dot{\mathscr{G}}\mathbf{H}\theta + r^2\dot{\mathscr{G}}^2\theta) \\ &+ 2rF_{s\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)(\theta + r\dot{\mathscr{G}}\mathbf{H}\theta + r^2\dot{\mathscr{G}}^2\theta) \\ &+ F_{r\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)(\theta + r\dot{\mathscr{G}}\mathbf{H}\theta + r^2\dot{\mathscr{G}}^2\theta) \\ &+ F_\xi\left\{\left[(n+1)\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\right]\mathbf{H}\theta + \left[r(n+1)\dot{\mathscr{G}}^2 - r\dot{\mathscr{G}}^2 + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\right]\theta\right\} \\ &- rF_s(r, r^2, n + r^2\dot{\mathscr{G}}^2)\theta. \end{aligned} \tag{2.5.4}$$

Observe that here we can write $\mathscr{L}[u] = \mathscr{A}(r)\theta + \mathscr{B}(r)\mathbf{H}\theta$, where the factors $\mathscr{A}$ and $\mathscr{B}$ are respectively given by

$$\begin{aligned}\mathscr{A}(r) :=& F_{\xi\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)(1 + r^2\dot{\mathscr{G}}^2)(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}) \\ &+ 2rF_{s\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)(1 + r^2\dot{\mathscr{G}}^2) + F_{r\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)(1 + r^2\dot{\mathscr{G}}^2) \\ &+ F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\left[r(n+1)\dot{\mathscr{G}}^2 - r\dot{\mathscr{G}}^2 + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\right] - rF_s(r, r^2, n + r^2\dot{\mathscr{G}}^2), \end{aligned} \tag{2.5.5}$$

and

$$\begin{aligned}\mathscr{B}(r) :=& rF_{\xi\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}) \\ &+ 2r^2F_{s\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}} + rF_{r\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}} \\ &+ F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)[(n+1)\dot{\mathscr{G}} + r\ddot{\mathscr{G}}]. \end{aligned} \tag{2.5.6}$$

Since $\mathscr{G}(r)$ by assumption is a solution to the ODE in (2.4.7) it can be seen that $\mathscr{B}(r) = 0$, and $\mathscr{A}(r)$ can be reduced to

$$\begin{aligned}\mathscr{A}(r) =& F_{\xi\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2)(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}) \\ &+ 2rF_{s\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2) + F_{r\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2) \\ &- r\dot{\mathscr{G}}^2F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2) - rF_s(r, r^2, n + r^2\dot{\mathscr{G}}^2) \end{aligned} \tag{2.5.7}$$

and subsequently this gives

$$\begin{aligned}\mathscr{A}(r)\theta =& \nabla F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2) - \\ & r[\dot{\mathscr{G}}^2F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2) + F_s(r, r^2, n + r^2\dot{\mathscr{G}}^2)]\theta. \end{aligned} \tag{2.5.8}$$

It is evident that $r[\dot{\mathscr{G}}^2F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2) + F_s(r, r^2, n + r^2\dot{\mathscr{G}}^2)]\theta$ is a gradient; indeed here $\mathscr{L}[u] = \nabla\mathscr{P}$ with $\mathscr{P}$ as in (2.5.3). It is easily seen that the boundary condition $u \equiv x$ in (2.5.1) follows from $\mathscr{G}(a) = 0$, $\mathscr{G}(b) = 2m\pi$ and the fact that u satisfies the incompressibility constraint $\det\nabla u = 1$ is proved in Proposition A.0.4. $\square$

It remains to prove the Main Theorem as stated earlier in the chapter.

Proof. (*Main Theorem*) With the above propositions and lemmas at our disposal we can now move on to completing the proof of the main theorem as presented in the first section of this chapter. Indeed all that remains is to prove that for each $m \in \mathbb{Z}$ the boundary value problem

$$\mathbf{BVP}[\mathscr{G}; F_\xi] := \begin{cases} \dfrac{d}{dr}\left[r^{n+1} F_\xi\left(r, r^2, n + r^2 \dot{\mathscr{G}}^2\right) \dot{\mathscr{G}}\right] = 0, \quad a < r < b, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2m\pi, \end{cases} \tag{2.5.9}$$

has a unique solution $\mathscr{G} = \mathscr{G}(r; m)$ in $\mathscr{C}^2[a, b]$. For this we refer to Proposition C.0.1 with the specific choice of $\mathsf{A}(r, s, \xi) = F_\xi(r, s, \xi)$. $\square$

Chapter 3

The h-Condition and Consideration of Weighted Dirichlet Type Lagrangians

In this chapter we address questions on the existence and multiplicity of solutions to the nonlinear elliptic system in divergence form

$$\begin{cases} \operatorname{div}\left[h(|x|,|u|^2)\nabla u\right] - h_s(|x|,|u|^2)|\nabla u|^2 u = [\operatorname{cof}\nabla u]\nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $h = h(r,s) > 0$, $\mathscr{P} = \mathscr{P}(x)$ is an *a priori* unknown hydrostatic pressure field and φ is a suitable boundary map. Most notably, for a finite symmetric annulus we prove the existence of an infinite scale of topologically distinct twisting solutions to the system by way of analysing an associated reduced energy, the resulting Euler-Lagrange equation and a structure theorem for curl-free vector fields generated by skew-symmetric matrices. An "*h-condition*" capturing a contrasting and surprising behaviour in the nature and multiplicity of twisting solutions is introduced and exploited. Other classes of solutions with 2-plane symmetries are examined and relations to closed geodesics on the Lie group $\mathbf{SO}(n)$ in the form $\mathbf{Q}(r) = \exp\{f(r)\mathbf{H}\}$ are explored and discussed.

3.1 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain and consider the variational energy integral

$$\mathbb{F}[u;\Omega] := \int_\Omega F(x,u,\nabla u)\,dx, \tag{3.1.1}$$

where the Lagrangian is of *weighted* Dirichlet type $F(x,u,\nabla u) = h(|x|,|u|^2)|\nabla u|^2$ with $h = h(r,s) > 0$ of class $\mathscr{C}^2$ and u in the space of incompressible Sobolev maps $\mathscr{A}_\varphi(\Omega) := \{u \in W^{1,2}(\Omega,\mathbb{R}^n) : \det \nabla u = 1 \text{ } a.e. \text{ in } \Omega,\ u \equiv \varphi \text{ on } \partial\Omega\}$. Here ∇u denotes the gradient of u, an $n \times n$ matrix field in Ω required to satisfy the pointwise incompressibility constraint $\det \nabla u = 1$ in Ω (hence the algebraic identity $\operatorname{cof} \nabla u = (\nabla u)^{-t}$). To avoid unnecessary technicalities and to fix ideas φ is taken as the identity map $\varphi \equiv x$ and boundary values are interpreted in the sense of traces. Now extremisers (or equivalently critical points) of this energy over the admissible space $\mathscr{A}_\varphi(\Omega)$ can be seen, *e.g.*, using the Lagrange multiplier method (*c.f.* Appendix B), to satisfy the nonlinear system

$$\begin{cases} \mathscr{L}_h[u] = \nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.1.2}$$

where $\mathscr{P} = \mathscr{P}(x)$ is an *a priori* unknown hydrostatic pressure corresponding to the incompressibility constraint – the Lagrange multiplier – and the action of the differential operator $\mathscr{L}_h$ is given by

$$\begin{aligned} \mathscr{L}_h[u] &= (\operatorname{cof} \nabla u)^{-1}\{\operatorname{div}\left[h(r,|u|^2)\nabla u\right] - h_s(r,|u|^2)|\nabla u|^2 u\} \\ &= (\nabla u)^t\{h_r(r,|u|^2)\nabla u\theta + h_s(r,|u|^2)\nabla u \nabla |u|^2\} \\ &\quad + h(r,|u|^2)(\nabla u)^t \Delta u - h_s(r,|u|^2)|\nabla u|^2(\nabla u)^t u. \end{aligned} \tag{3.1.3}$$

Here $r = |x|$, $\theta = x|x|^{-1}$ and $h_r = h_r(r,s)$ and $h_s = h_s(r,s)$ are the derivatives of the weight function h in the first and second arguments respectively. As a result of this formulation, it is evident that if u is a solution to this system, then necessarily $\operatorname{curl} \mathscr{L}_h[u] = \operatorname{curl} \nabla \mathscr{P} \equiv 0$ in Ω, that is,

$$\begin{aligned} \operatorname{curl} \mathscr{L}_h[u] = \operatorname{curl} \Big\{ &(\nabla u)^t[h_r(r,|u|^2)\nabla u\theta + h_s(r,|u|^2)\nabla u \nabla |u|^2] \\ &+ h(r,|u|^2)(\nabla u)^t \Delta u - h_s(r,|u|^2)|\nabla u|^2(\nabla u)^t u \Big\} \equiv 0. \end{aligned} \tag{3.1.4}$$

However note that this condition, unless $\Omega \subset \mathbb{R}^n$ has a particular homology, would not on its own imply that the vector field $\mathscr{L}_h[u]$ is a gradient field, here, $\nabla \mathscr{P}$.

Throughout this chapter we specialise to the geometric set up where $\Omega = \mathbb{X}^n = \mathbb{X}^n[a,b] := \{x \in \mathbb{R}^n : a < |x| < b\}$ is a finite symmetric annulus with $b > a > 0$, $\varphi \equiv x$, namely, the identity map and $h \in \mathscr{C}^2([a,b]\times]0,\infty[)$ satisfies $h > 0$. In this context by a generalised twist we understand a map $u \in \mathscr{C}(\overline{\mathbb{X}^n}, \overline{\mathbb{X}^n})$, which, in spherical coordinates, admits the representation

$$u : (r,\theta) \mapsto (r, \mathbf{Q}(|x|)\theta), \qquad r = |x|,\, \theta = x|x|^{-1}. \tag{3.1.5}$$

The curve $\mathbf{Q} \in \mathscr{C}([a,b]; \mathbf{SO}(n))$ is referred to as the twist path associated with u. In order to ensure $u \equiv x$ on $\partial\Omega = \partial\mathbb{X}^n$ we set $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ where $\mathbf{I}_n$ is the $n \times n$ identity matrix. In this event the twist path is a closed curve in $\mathbf{SO}(n)$ and as such we refer to it as the twist *loop* associated with u. Our aim is to establish the existence of an infinitude of twisting solutions to the nonlinear system (3.1.2)-(3.1.3) by appropriately formulating the action of $\mathscr{L}_h$ on sufficiently regular twists u and solving the resulting PDE. Over the course of the chapter it will become apparent that certain closed (*scaled*) geodesics of the compact Lie group $\mathbf{SO}(n)$ in the form $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ $(a \le r \le b)$ will play a prominent role by serving as the twist loops for the sought twisting solutions u to (3.1.2)-(3.1.3). Here $\mathscr{G} \in \mathscr{C}^2[a,b]$ is in turn a solution to a two point boundary value problem and $\mathbf{H}$ is a suitable skew-symmetric matrix in the Lie algebra $\mathfrak{so}(n)$.

We remark here that a solution to this system is $u \equiv x$. Indeed, upon substitution, (3.1.3) reduces to $\mathscr{L}_h[u \equiv x] = [h_r(r,r^2) + 2rh_s(r,r^2) - rnh_s(r,r^2)]\theta = \nabla\mathscr{P}$. The left-hand side here can be written as $s(r)\theta$ and as such is the gradient of some appropriate primitive function $s(r)\theta = \nabla S(|x|)$ that depends on the radial variable alone.

The second class of symmetric maps we consider as solutions to (3.1.2)-(3.1.3) and extremisers of (3.1.1) are the so-called whirl maps (or whirls for simplicity). These are maps $u \in \mathscr{C}(\overline{\mathbb{X}^n}, \overline{\mathbb{X}^n})$ of the form

$$u : (r,\theta) \mapsto r\mathbf{Q}(\rho_1,\ldots,\rho_N)\theta, \qquad r = |x|, \theta = x|x|^{-1}. \tag{3.1.6}$$

Here $x \in \overline{\mathbb{X}^n}$ and we denote by $\varrho = \varrho(x)$ the 2-plane radial variables $(\rho_1,\ldots,\rho_N)$, defined, depending on the dimension n being even or odd, as follows:

(*i*) If $n = 2d$ set $N = d$ and

$$\rho_j = \sqrt{x_{2j-1}^2 + x_{2j}^2}, \quad 1 \le j \le d. \tag{3.1.7}$$

(*ii*) If $n = 2d + 1$ set $N = d + 1$ and

$$\rho_j = \begin{cases} \sqrt{x_{2j-1}^2 + x_{2j}^2}, & 1 \le j \le d, \\ x_n, & j = d + 1. \end{cases} \tag{3.1.8}$$

It is seen that for $x \in \mathbb{X}^n$ the vector $\varrho = \varrho(x)$ lies in the semi-annular domain $\mathbb{A}_n$ where $\mathbb{A}_n := \{\varrho \in \mathbb{R}_+^d : a < \|\varrho\| < b\}$ when $n = 2d$ and $\mathbb{A}_n := \{\varrho \in \mathbb{R}_+^d \times \mathbb{R} : a < \|\varrho\| < b\}$ when $n = 2d + 1$ with $\|\varrho\| = \sqrt{\rho_1^2 + ... + \rho_N^2}$ for the norm of ϱ. Using this notation, we require in (3.1.6) that $\mathbf{Q} \in \mathscr{C}(\overline{\mathbb{A}_n}, \mathbf{SO}(n))$. For future reference, we will denote the three boundary segments of $\mathbb{A}_n$ as

$$\begin{aligned} (\partial\mathbb{A}_n)_a &= \{\rho \in \partial\mathbb{A}_N : \|\varrho\| = a\}, \\ (\partial\mathbb{A}_n)_b &= \{\rho \in \partial\mathbb{A}_N : \|\varrho\| = b\}, \\ \Gamma_n &= \partial\mathbb{A}_n \setminus \{(\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b\}. \end{aligned} \tag{3.1.9}$$

We impose that the matrix-valued map $\mathbf{Q}$ must take values on the maximal torus $\mathbb{T}$ of $\mathbf{SO}(n)$ consisting of 2×2 block-diagonal rotation matrices and for definiteness we specifically consider $\mathbf{Q} = \mathbf{Q}(\varrho)$ in the form

$$\mathbf{Q}(\rho_1, ..., \rho_N) = \begin{cases} diag(\mathcal{R}[f_1], \dots, \mathcal{R}[f_d]) & n = 2d, \\ diag(\mathcal{R}[f_1], \dots, \mathcal{R}[f_d], 1) & n = 2d + 1. \end{cases} \tag{3.1.10}$$

Here $\mathcal{R}$ is a 2×2 rotation matrix defined via (3.3.2) and the functions $f_j \in \mathscr{C}(\overline{\mathbb{A}_n})$ for all $1 \le j \le d$ satisfy $f_j \equiv 0$ on $(\partial\mathbb{A}_n)_a$ and $f_j \equiv 2m_j\pi$ on $(\partial\mathbb{A}_n)_b$. Note that $x \in \partial\mathbb{X}_a^n = \{|x| = a\} \iff \varrho(x) \in (\partial\mathbb{A}_n)_a$ and $x \in \partial\mathbb{X}_b^n = \{|x| = b\} \iff \varrho(x) \in (\partial\mathbb{A}_n)_b$. The functions $f_j = f_j(\varrho)$ and hence the map $\mathbf{Q} = \mathbf{Q}(\varrho)$ are left free on the flat part of the boundary Γ_n.

3.2 Identities Related to the Action $\mathscr{L}_h[u]$

In this section the action $\mathscr{L}_h[u = r\mathbf{Q}(r)\theta]$ is formally derived and its consequences on, in particular, the twist path $\mathbf{Q}(r)$, is analysed. We first refer to Proposition A.0.3 where many properties pertaining to the kinematics of generalised twists are derived, which are used in this section and throughout the chapter. With these identities at hand we can obtain an explicit representation for the action $\mathscr{L}_h[u]$ as given by (3.1.3) for a sufficiently regular twist $u = r\mathbf{Q}(r)\theta$. That is,

$$\begin{aligned}\mathscr{L}_h[u] =& (\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta)\{[h_r(r,r^2) + 2rh_s(r,r^2)](\mathbf{Q} + r\dot{\mathbf{Q}}) \\ &+ h(r,r^2)[(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}] - rh_s(r,r^2)(n + r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}\}\theta. \qquad (3.2.1)\end{aligned}$$

Expanding this further and introducing the skew-symmetric matrix field $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ we can write, for $a < r < b$,

$$\begin{aligned}\mathscr{L}_h[u] =& [h_r(r,r^2) + 2rh_s(r,r^2)](\theta + r\mathbf{A}\theta + r^2|\mathbf{A}\theta|^2\theta) \\ &+ h(r,r^2)[(n+1)\mathbf{A}\theta + r(\dot{\mathbf{A}} + \mathbf{A}^2)\theta \\ &+ r(n+1)|\mathbf{A}\theta|^2\theta + r^2\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\theta] \\ &- rh_s(r,r^2)(n + r^2|\mathbf{A}\theta|^2)\theta. \qquad (3.2.2)\end{aligned}$$

The above description follows upon noting $|\dot{\mathbf{Q}}\theta|^2 = |\mathbf{A}\theta|^2$, $\mathbf{Q}^t\ddot{\mathbf{Q}} = \dot{\mathbf{A}} + \mathbf{A}^2$ and $\langle\dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle = \langle\mathbf{Q}^t\dot{\mathbf{Q}}\theta, \mathbf{Q}^t\ddot{\mathbf{Q}}\theta\rangle = \langle\mathbf{A}\theta, (\dot{\mathbf{A}} + \mathbf{A}^2)\theta\rangle = \langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle + \langle\mathbf{A}\theta, \mathbf{A}^2\theta\rangle = \langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle$ in view of $\mathbf{A}$ being skew-symmetric. Now a straightforward inspection shows that we can write $\mathscr{L}_h[u]$ in the alternative and more suggestive form

$$v := \mathscr{L}_h[u] = \mathscr{A}(r,\theta)\theta + rh(r,r^2)\mathbf{A}^2\theta + \frac{1}{r^n}\frac{d}{dr}\left[r^{n+1}h(r,r^2)\mathbf{A}\right]\theta, \qquad (3.2.3)$$

where $\mathscr{A}(r,\theta)$ denotes the scalar-valued function

$$\begin{aligned}\mathscr{A}(r,\theta) :=& [h_r(r,r^2) + 2rh_s(r,r^2)](1 + r^2|\mathbf{A}\theta|^2) \qquad (3.2.4) \\ &+ rh(r,r^2)[(n+1)|\mathbf{A}\theta|^2 + r\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle] - rh_s(r,r^2)(n + r^2|\mathbf{A}\theta|^2).\end{aligned}$$

Similarly upon introducing the skew-symmetric matrix $\mathbf{B} = \dot{\mathbf{Q}}\mathbf{Q}^t$ we can write $\mathscr{L}_h[u] = \mathbf{Q}^t w(x)\mathbf{Q}$ with

$$w := \mathscr{B}(r,\theta)\theta + rh(r,r^2)\mathbf{B}^2\theta + \frac{1}{r^n}\frac{d}{dr}\left[r^{n+1}h(r,r^2)\mathbf{B}\right]\theta, \qquad (3.2.5)$$

where $\mathscr{B}(r,\theta)$ is the scalar-valued function

$$\begin{aligned}\mathscr{B}(r,\theta) :=& [h_r(r,r^2) + 2rh_s(r,r^2)](1 + r^2|\mathbf{B}\theta|^2) \qquad (3.2.6) \\ &+ rh(r,r^2)[(n+1)|\mathbf{B}\theta|^2 + r\langle\mathbf{B}\theta, \dot{\mathbf{B}}\theta\rangle] - rh_s(r,r^2)(n + r^2|\mathbf{B}\theta|^2).\end{aligned}$$

We proceed with (3.2.3)-(3.2.4) and note that in order for $u = r\mathbf{Q}(r)\theta$ to furnish a solution to (3.1.2)-(3.1.3) it is required that $\mathscr{L}_h[u] = \nabla\mathscr{P}$. Thus by enforcing (3.2.2) to be a gradient it must necessarily be that

$$\int_0^{2\pi} \langle \mathscr{L}_h[u](r\gamma(t)), \gamma'(t)\rangle \, dt = \int_0^{2\pi} \langle v(r\gamma(t)), \gamma'(t)\rangle \, dt = \int_0^{2\pi} \frac{d}{dt}\mathscr{P}(\gamma(t)) \, dt = 0 \tag{3.2.7}$$

(with prime denoting d/dt) where $\gamma = \gamma(t) \in \mathscr{C}^1([0, 2\pi], \mathbb{S}^{n-1})$ is *closed* and $x = r\gamma(t)$ with $a < r < b$ fixed. Henceforth we assume this to be true and look to recover information on $\mathbf{A}$. Indeed specialising to $\theta = \gamma(t)$ as above and using (3.2.3) we can expand the integrand in the left-hand side of (3.2.7) as

$$\begin{aligned} \langle \mathscr{L}_h[u](r\gamma(t)), \gamma'(t)\rangle = \langle v(r\gamma(t)), \gamma'(t)\rangle = \mathscr{A}(r, \theta)\langle \gamma(t), \gamma'(t)\rangle \\ + rh(r, r^2)\langle \mathbf{A}^2\gamma(t), \gamma'(t)\rangle + \frac{1}{r^n}\left\langle \frac{d}{dr}\left[r^{n+1}h(r, r^2)\mathbf{A}\right]\gamma(t), \gamma'(t)\right\rangle. \end{aligned} \tag{3.2.8}$$

Since γ is a curve on the unit sphere we have $\langle \gamma, \gamma'\rangle = 0$ and subsequently (3.2.7)-(3.2.8) under the assumption $v = \mathscr{L}_h[u] = \nabla\mathscr{P}$ simplifies to

$$\begin{aligned} \int_0^{2\pi} \langle \mathscr{L}_h[u](r\gamma(t)), \gamma'(t)\rangle \, dt = \int_0^{2\pi} \langle \mathbf{E}(r)\gamma(t), \gamma'(t)\rangle \, dt \\ + \frac{1}{r^n}\int_0^{2\pi} \langle \mathbf{F}(r)\gamma(t), \gamma'(t)\rangle \, dt = \int_0^{2\pi} \frac{d}{dt}\mathscr{P}(\gamma(t)) \, dt = 0, \end{aligned} \tag{3.2.9}$$

where $\mathbf{E} = rh(r, r^2)\mathbf{A}^2$ and $\mathbf{F} = d/dr[r^{n+1}h(r, r^2)\mathbf{A}]$ are symmetric and skew-symmetric matrix fields on $]a, b[$ respectively.

Lemma 3.2.1. *Let $\mathbf{E}$ be a symmetric $n \times n$ matrix and $\gamma = \gamma(t) \in \mathscr{C}^1([0, 2\pi], \mathbb{S}^{n-1})$ be a closed curve. Then*

$$\int_0^{2\pi} \langle \mathbf{E}\gamma(t), \gamma'(t)\rangle \, dt = 0, \qquad \gamma' = \frac{d}{dt}\gamma(t). \tag{3.2.10}$$

Proof. As γ is closed and $\mathbf{E}$ is symmetric this follows by integrating the identity $d/dt\langle \mathbf{E}\gamma, \gamma\rangle = \langle \mathbf{E}\gamma', \gamma\rangle + \langle \mathbf{E}\gamma, \gamma'\rangle = 2\langle \mathbf{E}\gamma, \gamma'\rangle$ noting $\langle \mathbf{E}\gamma, \gamma\rangle|_{t=2\pi} = \langle \mathbf{E}\gamma, \gamma\rangle|_{t=0}$. □

Utilising this lemma the integral involving the symmetric matrix field $\mathbf{E} = rh(r, r^2)\mathbf{A}^2$ in (3.2.9) vanishes and so, summarising, assuming $v = \mathscr{L}_h[u] = \nabla\mathscr{P}$, we have

$$\int_0^{2\pi} \langle \mathscr{L}_h[u](r\gamma(t)), \gamma'(t)\rangle \, dt = \frac{1}{r^n}\int_0^{2\pi} \left\langle \frac{d}{dr}\left[r^{n+1}h(r, r^2)\mathbf{A}\right]\gamma(t), \gamma'(t)\right\rangle dt = 0 \tag{3.2.11}$$

for every closed curve $\gamma \in \mathscr{C}^1([0, 2\pi], \mathbb{S}^{n-1})$. Now we turn into dealing with the skew-symmetric matrix field $\mathbf{F}$.

Lemma 3.2.2. *Let* $\mathbf{F}$ *be an* $n \times n$ *skew-symmetric matrix and let* $\gamma = \mathbf{PR}\varrho$ *with* $\mathbf{P}, \mathbf{R} \in \mathbf{O}(n)$ *and* $\varrho \in \mathscr{C}^\infty([0, 2\pi], \mathbb{S}^{n-1})$ *the closed curved given by*

$$\varrho(t) = \begin{cases} \rho_1 = \sin t \, \sin \phi_2 \, \sin \phi_3 \ldots \sin \phi_{n-1}, \\ \rho_2 = \cos t \, \sin \phi_2 \, \sin \phi_3 \ldots \sin \phi_{n-1}, \\ \rho_3 = \cos \phi_2 \, \sin \phi_3 \ldots \sin \phi_{n-1}, \\ \vdots \\ \rho_{n-1} = \cos \phi_{n-2} \, \sin \phi_{n-1}, \\ \rho_n = \cos \phi_{n-1}. \end{cases} \tag{3.2.12}$$

Here $\phi_\ell \in [0, \pi]$ *for all* $2 \leq \ell \leq n-1$ *and denoting by* $(e_k : 1 \leq k \leq n)$ *the standard basis of* $\mathbb{R}^n$, $\mathbf{R} = \mathbf{R}(i,j)$ *is the orthogonal transformation swapping the pair of basis vectors* (e_1, e_2) *with* (e_i, e_j) $(1 \leq i < j \leq n)$ *and leaving the rest fixed. Then*

$$\int_0^{2\pi} \langle \mathbf{F}\gamma(t), \gamma'(t) \rangle \, dt = 0 \iff \mathbf{F} = 0. \tag{3.2.13}$$

Proof. First note that any skew-symmetric matrix $\mathbf{F}$ can be orthogonally diagonalised, that is, $\mathbf{F} = \mathbf{PDP}^t$, where $\mathbf{P} \in \mathbf{O}(n)$ and $\mathbf{D} = diag(d_1\mathbf{J}, \ldots, d_k\mathbf{J})$ if the dimension $n = 2k$ is even or $\mathbf{D} = diag(d_1\mathbf{J}, \ldots, d_{k-1}\mathbf{J}, 0)$ if $n = 2k-1$ is odd. Here $\mathbf{J}$ is the 2×2 rotation matrix by angle $\pi/2$ [*c.f.* (3.3.2)]. Now upon setting $\gamma = \mathbf{PR}\varrho$ as per the statement of the lemma the integral in (3.2.13) becomes

$$\int_0^{2\pi} \langle \mathbf{F}\gamma(t), \gamma'(t) \rangle \, dt = \int_0^{2\pi} \langle \mathbf{D}\omega(t), \omega'(t) \rangle dt = 0 \tag{3.2.14}$$

where $\omega := \mathbf{R}\varrho$ and so to prove Lemma 3.2.2 it is sufficient only to show that the above integral equality implies $\mathbf{D} = 0$. We proceed in a component-wise fashion and substituting this ω into (3.2.14) we have that $\omega'(t) = \mathbf{R}\varrho'(t) = \mathbf{R}(\rho_2, -\rho_1, 0, \ldots, 0)$, and so it can be verified that

$$\int_0^{2\pi} \langle \mathbf{D}\omega, \omega' \rangle \, dt = 2\pi(\rho_1^2 + \rho_2^2)\mathbf{D}_{ij} \tag{3.2.15}$$

with $\mathbf{D}_{ij} \in \mathbb{R}$. As such, if the identity on the left-hand side of the equation above is zero, it follows that $\mathbf{D} = 0$ and the identity (3.2.13) is verified. $\square$

Recalling (3.2.11), the quantity $\mathbf{F} = d/dr[r^{n+1}h(r, r^2)\mathbf{A}]$ is a skew-symmetric matrix field and so we can use this most recent lemma to confirm that if the equality (3.2.11) holds, it must be that $\mathbf{F}$ as prescribed here is identically zero. To summarise, if we assume $\mathscr{L}_h[u]$ is a gradient, that is, u solves (3.1.2)-(3.1.3), then setting $\mathscr{L}_h[u] = v(x)$ in (3.2.2) with the parametrisation $x = r\theta = r\gamma(t)$ for some closed curve $\gamma = \mathbf{P}\varrho \in \mathscr{C}^\infty([0, 2\pi], \mathbb{S}^{n-1})$, then the integral equality (3.2.7) holds. However, we have seen that this immediately reduces to (3.2.11), which then holds *iff* $d/dr[r^{n+1}h(r, r^2)\mathbf{A}] = 0$ on $]a, b[$. As such we have proved the following result.

Theorem 3.2.3. *Let $u = r\mathbf{Q}(r)\theta$ be a generalised twist on $\mathbb{X}^n[a, b]$ with twist path $\mathbf{Q} \in \mathscr{C}([a, b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a, b[, \mathbf{SO}(n))$. Then if u satisfies $\mathscr{L}_h[u] = \nabla\mathscr{P}$ for some hydrostatic pressure field $\mathscr{P}$, then $\mathbf{Q}$ satisfies the ODE*

$$\frac{d}{dr}\left\{r^{n+1}h(r, r^2)\mathbf{Q}^t\frac{d\mathbf{Q}}{dr}\right\} = 0, \qquad a < r < b. \tag{3.2.16}$$

Remark 3.2.4. The above ODE can itself be interpreted as an Euler-Lagrange equation. Indeed restricting the $\mathbb{F}[u; \mathbb{X}^n]$ energy to the class of generalised twists and substituting for $|\nabla u|^2$ from (iii) in Proposition A.0.3 we can write

$$\begin{aligned}
\mathbb{F}[r\mathbf{Q}(r)\theta; \mathbb{X}^n] &= \int_{\mathbb{S}^{n-1}}\int_a^b h(r, r^2)(n + r^2|\dot{\mathbf{Q}}\theta|^2)r^{n-1}\, dr d\mathcal{H}^{n-1}(\theta) \\
&= \omega_n\left[n^2\int_a^b h(r, r^2)r^{n-1}dr + \int_a^b h(r, r^2)|\dot{\mathbf{Q}}|^2 r^{n+1}\, dr\right] \\
&= n^2\omega_n||h(r, r^2)r^{n-1}||_{L^1(a,b)} + \omega_n\mathbb{E}[\mathbf{Q}; a, b].
\end{aligned} \tag{3.2.17}$$

It can then be seen that the Euler-Lagrange equation associated to the restricted energy $\mathbb{E}[\mathbf{Q}; a, b]$ over the space of admissible loops $\mathscr{B}_{\mathbf{I}_n}(a, b) := \{\mathbf{Q} \in W^{1,2}(]a, b[; \mathbf{SO}(n)) : \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n\}$ is precisely the second order ODE in the twist loop $\mathbf{Q} = \mathbf{Q}(r)$:[4]

$$\frac{d}{dr}\left[r^{n+1}h(r, r^2)\frac{d\mathbf{Q}}{dr}\mathbf{Q}^t\right] = \mathbf{Q}\frac{d}{dr}\left[r^{n+1}h(r, r^2)\mathbf{Q}^t\frac{d\mathbf{Q}}{dr}\right]\mathbf{Q}^t = 0. \tag{3.2.18}$$

See also Corollary 2.2.2 and the surrounding discussion.

3.3 Geodesic Solutions of the ODE (3.2.16) and the Energy-Length Identity

In this section we resolve the boundary value problem associated with the ODE (3.2.16) over the space of admissible loops $\mathscr{B}_{\mathbf{I}_n}(a, b)$ introduced above. A first

[4] Note that for any $a = a(r)$ of class $\mathscr{C}^1$ we have $d/dr[a(r)\dot{\mathbf{Q}}\mathbf{Q}^t] = d/dr[a(r)\mathbf{Q}\mathbf{Q}^t\dot{\mathbf{Q}}\mathbf{Q}^t] = \mathbf{Q}d/dr[a(r)\mathbf{Q}^t\dot{\mathbf{Q}}]\mathbf{Q}^t + a(r)[\dot{\mathbf{Q}}\mathbf{Q}^t\dot{\mathbf{Q}}\mathbf{Q}^t + \mathbf{Q}\mathbf{Q}^t\dot{\mathbf{Q}}\dot{\mathbf{Q}}^t] = \mathbf{Q}d/dr[a(r)\mathbf{Q}^t\dot{\mathbf{Q}}]\mathbf{Q}^t$ by virtue of the orthogonality of $\mathbf{Q}$ and the skew-symmetry of $\mathbf{Q}^t\dot{\mathbf{Q}}$.

integration yields $r^{n+1}h(r,r^2)\mathbf{Q}^t\dot{\mathbf{Q}} = \mathbf{H}$ for a constant skew-symmetric matrix $\mathbf{H}$. When combined with the left boundary condition $\mathbf{Q}(a) = \mathbf{I}_n$ this first order ODE is seen to have the general solution $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ where the *profile* $\mathscr{H} \in \mathscr{C}^2[a,b]$ is given by

$$\mathscr{H}(r) = \frac{\mathsf{H}(r)}{\mathsf{H}(b)}, \qquad \mathsf{H}(r) = \int_a^r \frac{ds}{s^{n+1}h(s,s^2)}. \tag{3.3.1}$$

Indeed it is evident from the above that $\mathscr{H}(a) = 0$, $\mathscr{H}(b) = 1$, and so $\mathbf{Q}(a) = \mathbf{I}_n$ is immediately satisfied. Anticipating on the right boundary condition $\mathbf{Q}(b) = \mathbf{I}_n$, we can first proceed by orthogonally diagonalising $\mathbf{H}$ by writing $\mathbf{H} = \mathbf{P}diag(c_1\mathbf{J},\dots,c_k\mathbf{J})\mathbf{P}^t$ when $n = 2k$ and $\mathbf{H} = \mathbf{P}diag(c_1\mathbf{J},\dots,c_{k-1}\mathbf{J},0)\mathbf{P}^t$ when $n = 2k-1$. Note that here the string of scalars $c_1,\dots,c_k$ with $c_k = 0$ in odd dimensions are all real – in fact, the eigenvalues of $\mathbf{H}$ are $\pm ic_j$ when n is even, and $\pm ic_j, 0$ when n is odd. Furthermore $\mathbf{P} \in \mathbf{O}(n)$, and the 2×2 matrices $\mathbf{J}$ and $\mathcal{R}$ are given by

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{R}[t] = \exp\{t\mathbf{J}\} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{3.3.2}$$

We will now verify the boundary condition at $\mathbf{Q}(b) = \mathbf{I}_n$ in even and odd dimensions independently.

- ($n = 2k$) Here we write

$$\begin{aligned}\mathbf{Q}(b) &= \exp\{\mathscr{H}(b)\mathbf{H}\} = \exp\{\mathbf{P}diag(c_1\mathbf{J},\dots,c_k\mathbf{J})\mathbf{P}^t\} \\ &= \mathbf{P}diag(\mathcal{R}[c_1],\dots,\mathcal{R}[c_k])\mathbf{P}^t \\ &= \mathbf{I}_n \iff c_j = 2m_j\pi, \qquad m_j \in \mathbb{Z} \qquad \forall\, 1 \le j \le k.\end{aligned}$$

- ($n = 2k-1$) Here we write

$$\begin{aligned}\mathbf{Q}(b) &= \exp\{\mathscr{H}(b)\mathbf{H}\} = \exp\{\mathbf{P}diag(c_1\mathbf{J},\dots,c_{k-1}\mathbf{J},0)\mathbf{P}^t\} \\ &= \mathbf{P}diag(\mathcal{R}[c_1],\dots,\mathcal{R}[c_{k-1}],1)\mathbf{P}^t \\ &= \mathbf{I}_n \iff c_j = 2m_j\pi, \qquad m_j \in \mathbb{Z}, \qquad \forall\, 1 \le j \le k-1.\end{aligned}$$

With this boundary condition being satisfied the solutions $\mathbf{Q} = \mathbf{Q}(r;\mathsf{m})$ to (3.2.16) in $\mathscr{B}_{\mathbf{I}_n}(a,b)$ with $\mathsf{m} \in \mathbb{Z}^k$ are given by $\mathbf{Q}(r;\mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$, where $\mathscr{H}(r)$ is given by (3.3.1) and for the skew-symmetric matrix $\mathbf{H} = \mathbf{H}(\mathsf{m})$

we have

$$\mathbf{H}(\mathsf{m}) = \begin{cases} \mathbf{P} diag(2m_1\pi\mathbf{J}, \dots, 2m_k\pi\mathbf{J})\mathbf{P}^t & n = 2k, \\ \mathbf{P} diag(2m_1\pi\mathbf{J}, \dots, 2m_{k-1}\pi\mathbf{J}, 0)\mathbf{P}^t & n = 2k-1. \end{cases} \tag{3.3.3}$$

Here we remark that the resulting twist loops $\mathbf{Q} = \mathbf{Q}(r; \mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$ are *closed* scaled geodesics based at $\mathbf{I}_n$ on the compact Lie group $\mathbf{SO}(n)$ with the skew-symmetric matrix $\mathbf{H}$ in the Lie algebra $\mathfrak{so}(n)$ presenting the tangent at the origin to the geodesic and the matrix exponential being the canonical exponential map from the Lie algebra $\mathfrak{so}(n)$ to the Lie group $\mathbf{SO}(n)$. Let us finish off by computing explicitly the $\mathbb{E}$-energy as in (3.2.17) for solutions $\mathbf{Q}(r; \mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$ to (3.2.16) as described above and compare it to the length $\mathsf{L}[\mathbf{Q}]$ where, as standard,

$$\mathsf{L}[\mathbf{Q}] := \int_a^b |\dot{\mathbf{Q}}(r)|\, dr = \int_a^b \sqrt{tr[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}]}\, dr. \tag{3.3.4}$$

Proposition 3.3.1. *Let $\mathbf{Q} = \mathbf{Q}(r)$ be an extremising twist path for the restricted energy $\mathbb{E}$ as in* (3.2.17)*, that is, a solution to the ODE* (3.2.16) *in the admissible loop space $\mathscr{B}_{\mathbf{I}_n}(a, b)$. Then we have the energy-length identity:*

$$\mathbb{E}[\mathbf{Q}] = \frac{\mathsf{L}^2[\mathbf{Q}]}{\|s^{-(n+1)}/h(s, s^2)\|_{L^1(a,b)}}, \tag{3.3.5}$$

where $h(s, s^2)$ is the weight function in the Lagrangian and $\mathsf{L}[\mathbf{Q}]$ is defined via (3.3.4)*.*

Proof. Given that $\mathbf{Q} = \mathbf{Q}(r; \mathsf{m})$ is a solution to (3.2.16) in $\mathscr{B}_{\mathbf{I}_n}(a, b)$ we have seen that $\mathbf{Q} = \mathbf{Q}(r; \mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$ with $\mathscr{H}(r)$ given explicitly by (3.3.1) and the skew-symmetric matrix $\mathbf{H}$ as in (3.3.3). We begin by computing the energy and see that

$$\mathbb{E}[\mathbf{Q}; a, b] = \mathbb{E}[\exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}; a, b] = \int_a^b h(r, r^2)\dot{\mathscr{H}}^2|\mathbf{H}|^2 r^{n+1} dr. \tag{3.3.6}$$

Now $\dot{\mathscr{H}}(r) = [\mathsf{H}(b) r^{n+1} h(r, r^2)]^{-1}$ and $|\mathbf{H}|^2 = 8\pi^2\|\mathsf{m}\|^2$ with $\|\mathsf{m}\|^2 = \sum_{i=1}^k m_i^2$ (recall $m_k = 0$ when the dimension n is odd). Substituting $\dot{\mathscr{H}}$ and $|\mathbf{H}|^2$ into (3.3.6) gives

$$\mathbb{E}\left[\exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}; a, b\right] = \frac{8\pi^2}{\mathsf{H}^2(b)}\|\mathsf{m}\|^2 \int_a^b \frac{r^{-(n+1)}}{h(r, r^2)} dr = \frac{8\pi^2}{\mathsf{H}(b)}\|\mathsf{m}\|^2. \tag{3.3.7}$$

Next turning to the length, noting that weight function $h > 0$ on $[a, b]$ and so $\dot{\mathscr{H}} > 0$ on $[a, b]$, we can write

$$\begin{aligned}\mathsf{L}[\mathbf{Q}] = \mathsf{L}[\exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}] &= 2\sqrt{2}\pi\|\mathsf{m}\| \int_a^b |\dot{\mathscr{H}}(r)|\, dr \\ &= 2\sqrt{2}\pi\|\mathsf{m}\|[\mathscr{H}(b) - \mathscr{H}(a)] = 2\sqrt{2}\pi\|\mathsf{m}\|. \end{aligned} \tag{3.3.8}$$

The result now follows by comparing the two and squaring the length. □

3.4 Generalised Twists as Solutions to $\mathscr{L}_h[u] = \nabla\mathscr{P}$ and the h-Condition

Let us now turn to the differential operator $\mathscr{L}_h$ given by (3.1.3) and seek solutions to the system (3.1.2)-(3.1.3) in the form of incompressible twists $u = r\mathbf{Q}(r)\theta$. As here necessarily the twist loop $\mathbf{Q}$ solves the second-order ODE $d/dr[r^{n+1}h(r, r^2)\mathbf{Q}^t\dot{\mathbf{Q}}] = 0$, for $a < r < b$, by virtue of what was discussed in the previous section, we have $\mathbf{Q}(r; \mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$ and thus with $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$, $\mathbf{A}^2 = \dot{\mathscr{H}}^2\mathbf{H}^2$. As such, the aforementioned ODE in this context is given by

$$\frac{d}{dr}\left[r^{n+1}h(r, r^2)\dot{\mathscr{H}}\right] = 0, \qquad a < r < b. \tag{3.4.1}$$

The action of $\mathscr{L}_h$ as (3.1.3) on u then reduces to[5]

$$\begin{aligned}\mathscr{L}_h[u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta] = &\nabla h(|x|, |x|^2) - nrh_s(r, r^2)\theta \\ &- r^3h_s(r, r^2)\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta + rh(r, r^2)\dot{\mathscr{H}}^2\mathbf{H}^2\theta.\end{aligned} \tag{3.4.2}$$

The first two terms in the above expression are clearly gradients and so we can apply Proposition D.0.2 to the remainder with $\mathscr{A}(r, |\mathbf{H}x|^2) = -h_s(r, r^2)\dot{\mathscr{H}}^2|\mathbf{H}x|^2$ and $\mathscr{B}(r, |\mathbf{H}x|^2) = h(r, r^2)\dot{\mathscr{H}}^2$. Then with $z = |\mathbf{H}x|^2$

$$\begin{aligned}2\mathscr{A}_z + \mathscr{B}_r/r &= \frac{1}{r}\left[h_r(r, r^2)\dot{\mathscr{H}}^2 + 2h(r, r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}}\right] \\ &= -\left\{2\frac{n+1}{r^2}h(r, r^2) + \frac{1}{r}h_r(r, r^2) + 4h_s(r, r^2)\right\}\dot{\mathscr{H}}^2,\end{aligned} \tag{3.4.3}$$

the second equality following by an application of (3.4.1). For the sake of future reference it is convenient to introduce the notation $\mathscr{F}(r) = rh_r(r, r^2) + 2(n + 1)h(r, r^2) + 4r^2h_s(r, r^2)$ with $a \le r \le b$ (colloquially we will say if $\mathscr{F}(r) \equiv 0$

[5] See the discussion leading up to the identity (2.3.10) for further details.

then the "*h-condition*" holds). Then if $\mathscr{F}(r) \not\equiv 0$ on $]a,b[$ by invoking the first part of Proposition D.0.2 with $\mathscr{A}_z + \mathscr{B}_r/r \not\equiv 0$ we have, with $U(x) := \mathscr{L}_h[u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta] - \nabla h(|x|,|x|^2) + nrh_s(r,r^2)\theta$,

$$\operatorname{curl} U(x) = 0 \iff \mathbf{H}^2 = -c^2\mathbf{I}_n \iff \mathscr{L}_h[u] \text{ is a gradient.}$$

This, given the orthogonal diagonalisation of the skew-symmetric matrix $\mathbf{H}$ as in (D.0.6), leads us to conclude $|c_1|^2 = \cdots = |c_k|^2 =: c^2$ when $n = 2k$ is even and $|c_1|^2 = \cdots = |c_{k-1}|^2 = |c_k|^2 = 0$ when $n = 2k-1$ is odd. Now regarding the boundary conditions, evidently $\mathbf{Q}(a) = \mathbf{I}_n$ as a result of $\mathscr{H}(a) = 0$ and so in order to satisfy $\mathbf{Q}(b) = \mathbf{I}_n$ we first note that $\mathbf{Q}(b) = \exp\{\mathscr{H}(b)\mathbf{H}\}$ and so when n is odd $\mathbf{Q} \equiv \mathbf{I}_n$ and when n is even necessarily $\mathbf{Q} = \mathbf{Q}(r;m) = \exp\{2m\pi\mathscr{H}(r)\mathbf{P}\mathbf{J}_n\mathbf{P}^t\}$, where $m \in \mathbb{Z}$ and $\mathbf{J}_n = diag(\mathbf{J},\dots,\mathbf{J})$ with $\mathbf{J}$ as in (3.3.2).

Next when $\mathscr{F}(r) \equiv 0$ on $]a,b[$ the corresponding vector field $\mathscr{L}_h[u]$ is still a gradient by the second part of Proposition D.0.2 but now with no further restrictions on the skew-symmetric matrix $\mathbf{H}$. Indeed referring to (3.4.2) consider

$$\begin{aligned} U(x) &= \mathscr{L}_h[u] - \nabla h(|x|,|x|^2) + nrh_s(r,r^2)\theta \\ &= -r^3 h_s(r,r^2)\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta + rh(r,r^2)\dot{\mathscr{H}}^2\mathbf{H}^2\theta. \end{aligned} \tag{3.4.4}$$

To show that U is a gradient and hence $u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta$ is a solution to $\mathscr{L}_h[u] = \nabla\mathscr{P}$ it suffices to show that there exists $f = f(r,z)$ such that

$$\nabla f(|x|,|\mathbf{H}x|^2) = f_r(r,|\mathbf{H}x|^2)\theta - 2f_z(r,|\mathbf{H}x|^2)\mathbf{H}^2x = U(x), \qquad x \in \mathbb{X}^n.$$

Upon referring to (3.4.4) this in particular means that we must have

$$f_r(r,z) = -z\dot{\mathscr{H}}^2 rh_s(r,r^2), \qquad f_z(r,z) = -\frac{1}{2}h(r,r^2)\dot{\mathscr{H}}^2.$$

Naturally for this to be so it is necessary to have $\partial_z f_r - \partial_r f_z \equiv 0$ which is seen to hold (suppressing the arguments for brevity) by virtue of

$$\begin{aligned} \partial_z f_r - \partial_r f_z &= -rh_s(r,r^2)\dot{\mathscr{H}}^2 + [h_r(r,r^2) + 2rh_s(r,r^2)]\frac{\dot{\mathscr{H}}^2}{2} + h(r,r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}} \\ &= \frac{1}{2}h_r(r,r^2)\dot{\mathscr{H}}^2 + h(r,r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}} = -\frac{\dot{\mathscr{H}}^2}{2r}\mathscr{F}(r), \end{aligned} \tag{3.4.5}$$

where the last identity follows by invoking the ODE (3.4.1) for $\mathscr{H}$. Now referring to the above it is enough to set

$$f(r,z) := -\frac{1}{2}h(r,r^2)\dot{\mathscr{H}}^2 z + g(r), \qquad a \le r \le b, \quad z \ge 0, \tag{3.4.6}$$

with $g \in \mathscr{C}^1[a,b]$ to be determined. From this we can directly compute

$$f_r(r,z) = -\frac{1}{2}\left\{[h_r(r,r^2) + 2rh_s(r,r^2)]\dot{\mathscr{H}}^2 + 2h(r,r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}}\right\}z + \dot{g}(r) \quad (3.4.7)$$

and so $\dot{g}(r) = z\dot{\mathscr{H}}[1/2h_r\dot{\mathscr{H}} + h\ddot{\mathscr{H}}] = 0$ by arguing as in (3.4.5). Thus $g \equiv c$ in (3.4.6) for some $c \in \mathbb{R}$ and therefore $U = \nabla f(|x|, |\mathbf{H}x|^2)$.

We summarise that, in comparing the full Euler-Lagrange equation with the version restricted to the space of twist paths, we have the following implications:

- (ODE) The twist path $\mathbf{Q} = \mathbf{Q}(r)$ with $\mathbf{Q}(a) = \mathbf{I}_n$ solves (3.2.16) *if and only if* for $a \le r \le b$

$$\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}, \qquad \mathbf{H}^t = -\mathbf{H}, \tag{3.4.8}$$

  where the profile $\mathscr{H} = \mathscr{H}(r)$ is as given explicitly by (3.3.1).

- (PDE I) If $\mathscr{F}(r) = rh_r(r,r^2) + 2(n+1)h(r,r^2) + 4r^2h_s(r,r^2) \not\equiv 0$ on $]a,b[$ then for $u = r\mathbf{Q}(r)\theta$ with $u \equiv \varphi$ on $\partial\mathbb{X}^n_a = \{|x| = a\}$ we have

$$\mathscr{L}_h[u] = \nabla\mathscr{P} \iff \mathbf{Q} \text{ is as in (3.4.8) with}$$
$$\mathbf{H} = \begin{cases} c\mathbf{P}\mathbf{J}_n\mathbf{P}^t & n \text{ even}, \\ 0 & n \text{ odd}, \end{cases} \tag{3.4.9}$$

  for some real constant c where referring to the diagonalisation of $\mathbf{H}$: $|c_1| = ... = |c_k| = |c|$, $\mathbf{P} \in \mathbf{O}(n)$ and $\mathbf{J}_n = (\mathbf{J}, ..., \mathbf{J})$ with $\mathbf{J}$ as in (3.3.2).

- (PDE II) If $\mathscr{F}(r) = rh_r(r,r^2) + 2(n+1)h(r,r^2) + 4r^2h_s(r,r^2) \equiv 0$ on $]a,b[$ then for $u = r\mathbf{Q}(r)\theta$ with $u \equiv \varphi$ on $\partial\mathbb{X}^n_a = \{|x| = a\}$ we have

$$\mathscr{L}_h[u] = \nabla\mathscr{P} \iff \mathbf{Q} \text{ is as in (3.4.8) with } \mathbf{H}^t = -\mathbf{H}. \tag{3.4.10}$$

Thus, unlike the case in (3.4.9), no further restriction on the skew-symmetric matrix $\mathbf{H}$ is needed if the h-condition holds, that is $rh_r(r,r^2)+2(n+1)h(r,r^2)+4r^2h_s(r,r^2) \equiv 0$. Now by taking into account both the boundary conditions $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ and the subsequent necessary adjustments on $\mathbf{H}$ we arrive at the following statement on twist solutions to (3.1.2)-(3.1.3).

Theorem 3.4.1. *Let $u = r\mathbf{Q}(r)\theta$ be a generalised twist on $\mathbb{X}^n[a,b]$ with twist loop $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathbf{SO}(n))$ satisfying $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$. Then u is a solution to the nonlinear Euler-Lagrange system* (3.1.2)-(3.1.3) *iff $\mathbf{Q}$ is as described below.*

1. ($\mathscr{F} = rh_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2 h_s(r, r^2) \not\equiv 0$ *on* $]a, b[$) *Here depending on the dimension* n *being even or odd we have*

   (i) n *even:* $\mathbf{Q} = \mathbf{Q}(r; m) = \exp\{\mathscr{H}(r)\mathbf{H}(m)\}$ ($a \le r \le b$) *with* $\mathbf{H}(m) = 2m\pi\mathbf{P}\mathbf{J}_n\mathbf{P}^t$ *where* $\mathbf{P} \in \mathbf{O}(n)$, $m \in \mathbb{Z}$ *and* $\mathbf{J}_n = diag(\mathbf{J}, \dots, \mathbf{J})$ *with* $\mathbf{J}$ *as in* (3.3.2).

   (ii) n *odd:* $\mathbf{H} \equiv 0$ *leading to* $\mathbf{Q} \equiv \mathbf{I}_n$. *Hence the identity map* $u \equiv x$ *is the only twisting solution of* (3.1.2)-(3.1.3).

2. ($\mathscr{F} = rh_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2 h_s(r, r^2) \equiv 0$ *on* $]a, b[$) *Here* $\mathbf{Q} = \mathbf{Q}(r; \mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$ ($a \le r \le b$) *with* $\mathbf{H}(\mathsf{m}) = \mathbf{P}diag(2m_1\pi\mathbf{J}, ..., 2m_k\pi\mathbf{J})\mathbf{P}^t$ *when* $n = 2k$ *and* $\mathbf{H}(\mathsf{m}) = \mathbf{P}diag(2m_1\pi\mathbf{J}, ..., 2m_{k-1}\pi\mathbf{J}, 0)\mathbf{P}^t$ *when* $n = 2k-1$. *Moreover* $\mathbf{P} \in \mathbf{O}(n)$ *and* $\mathsf{m} \in \mathbb{Z}^k$.

The energy of an extremising twist can now be explicitly calculated by taking advantage of the above characterisation of its twist path, specifically, $\mathbf{Q}(r; \mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$ with $\mathbf{H}(\mathsf{m})$ as in Theorem 3.4.1 part 2 (note that part 1 of the theorem is essentially a special case of this). We first observe that for $u \equiv x$,

$$\mathbb{F}[u \equiv x; \mathbb{X}^n] = \int_{\mathbb{X}^n} h(|x|, |u|^2)|\nabla u|^2 \, dx = n^2\omega_n \int_a^b h(r, r^2) r^{n-1} \, dr. \tag{3.4.11}$$

Next upon noting $|\nabla u|^2 = n + r^2|\dot{\mathbf{Q}}\theta|^2 = n + r^2|\dot{\mathscr{H}}(r)\mathbf{H}\theta|^2$ and with $\mathbf{H}$ and $\mathsf{m} = (m_1, \dots, m_k)$ as above

$$\begin{aligned} \mathbb{F}[u; \mathbb{X}^n] &= \int_a^b \int_{\mathbb{S}^{n-1}} h(r, r^2)(n + r^2\dot{\mathscr{H}}^2(r)|\mathbf{H}\theta|^2) r^{n-1} \, dr d\mathcal{H}^{n-1}(\theta) \\ &= \omega_n \left[\int_a^b n^2 h(r, r^2) r^{n-1} dr + \int_a^b h(r, r^2)\dot{\mathscr{H}}^2(r)|\mathbf{H}|^2 r^{n+1} \, dr \right] \\ &= \mathbb{F}[u \equiv x; \mathbb{X}^n] + 8\pi^2\omega_n ||\mathsf{m}||^2 \int_a^b h(r, r^2)\dot{\mathscr{H}}^2(r) r^{n+1} \, dr. \end{aligned} \tag{3.4.12}$$

Recalling (3.3.6)-(3.3.7) (and noting that in the above we have used $|\mathbf{H}|^2 = 8\pi^2||\mathsf{m}||^2$), we see that the value of the integral above is $1/\mathsf{H}(b)$ and so

$$\mathbb{F}[u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta; \mathbb{X}^n] - \mathbb{F}[x; \mathbb{X}^n] = \frac{8\pi^2\omega_n}{\mathsf{H}(b)} ||\mathsf{m}||^2. \tag{3.4.13}$$

In particular it is seen that the energy diverges quadratically in m as $||\mathsf{m}|| \nearrow \infty$. Now taking the length of the twist loop $\mathbf{Q}$ given by $\mathsf{L}[\mathbf{Q}] = 2\sqrt{2}\pi||\mathsf{m}||$ (*cf.* (3.3.8) for details), upon comparing this length with the energy (3.4.13) we have the energy-length identity

$$\mathbb{F}[u;\mathbb{X}^n] - \mathbb{F}[x;\mathbb{X}^n] = \frac{\omega_n}{\mathsf{H}(b)}\mathsf{L}^2[\mathbf{Q}] = \omega_n\mathbb{E}[\mathbf{Q}], \tag{3.4.14}$$

for extremising twists $u = r\mathbf{Q}(r;\mathsf{m})\theta$ where $\mathbf{Q}(r;\mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$. Here $\mathbb{E}$ is the twist path energy as in (3.3.6).

We close the section by returning to the context of Theorem 3.4.1 and giving, for the sake of illustration, a class of energy integrals that satisfy the h-condition $rh_r(r,r^2)+2(n+1)h(r,r^2)+4r^2h_s(r,r^2) \equiv 0$ and for which the associated Euler-Lagrange system admits an infinitude of nontrivial twisting solutions regardless of n being even or odd.

Example 3.4.2. Consider $h(r,s) = \mathsf{a}(r)\mathsf{b}(s) = r^\alpha s^\beta$ for real α, β, $a \le r \le b$ and $s > 0$. Then $rh_r(r,r^2) + 2(n+1)h(r,r^2) + 4r^2h_s(r,r^2) \equiv 0 \iff r\dot{\mathsf{a}}\mathsf{b} + 2(n+1)\mathsf{a}\mathsf{b} + 4r^2\mathsf{a}\dot{\mathsf{b}} \equiv 0$, that is, $\alpha + 2(n+1) + 4\beta = 0$ and the energy (3.1.1) can be rewritten as

$$\mathbb{G}_\beta[u;\mathbb{X}^n] = \int_{\mathbb{X}^n} \frac{|u|^{2\beta}|\nabla u|^2}{|x|^{2(n+1)+4\beta}}\,dx. \tag{3.4.15}$$

Note that by linearity any finite sum $h(r,s) = \sum_j c_j r^{\alpha_j}s^{\beta_j}$ with $c_j > 0$ and $\alpha_j+2(n+1)+4\beta_j = 0$ still verifies $rh_r(r,r^2)+2(n+1)h(r,r^2)+4r^2h_s(r,r^2) \equiv 0$. Of course these are by no means the only functions $h > 0$ satisfying the latter.

By Theorem 3.4.1 for each $\mathsf{m} = (m_1,...,m_k) \in \mathbb{Z}^k$ there exists a generalised twist $u(x;\mathsf{m}) = r\exp\{\mathscr{H}(r)\mathbf{H}(m)\}\theta$ with profile $\mathscr{H} = \mathscr{H}(r)$ as in (3.3.1) and $\mathbf{H}$ a suitable skew-symmetric matrix as described such that u is a solution to the resulting system (3.1.2)-(3.1.3). Thus, so long as $\mathscr{F} \equiv 0$, as, e.g. is the case for the above energies, then regardless of n being even or odd the system has an infinite scale of topologically distinct twisting solutions, in total contrast to when $\mathscr{F} \not\equiv 0$ where by Theorem 3.4.1 there are no nontrivial solutions for n odd.

3.5 Whirl Maps $u = \mathbf{Q}(\varrho)x$ and the Restricted $\mathbb{H}_\ell$ Energies

We now aim to seek solutions to the system (3.1.2)-(3.1.3) from amongst whirl maps. Recall from the Introduction and opening section of this chapter that these are continuous self-maps of the closed symmetric annulus $\overline{\mathbb{X}^n}$ onto itself in the form $u : (r, \theta) \mapsto r\mathbf{Q}(\rho_1, \dots, \rho_N)\theta$, where $r = |x|$, $\theta = x|x|^{-1}$ and $x \in \overline{\mathbb{X}^n}$. The vector of 2-plane radial variables $\varrho = (\rho_1, ..., \rho_N)$ is defined, depending on n being even or odd, as:

(*i*) n even: $N = d = n/2$ and $\rho_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2} \,\forall\, 1 \le j \le N$,

(*ii*) n odd: $N = d+1 = (n+1)/2$ and $\rho_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2} \,\forall\, 1 \le j \le d$, $\rho_N = x_n$.

Here $x \in \mathbb{X}^n \iff \varrho \in \mathbb{A}_n$ where $\mathbb{A}_n = \{\varrho \in \mathbb{R}_+^N : a < \|\varrho\| < b\}$ for $n = 2N$ and $\mathbb{A}_N = \{\varrho \in \mathbb{R}_+^{N-1} \times \mathbb{R} : a < \|\varrho\| < b\}$ for $n = 2N - 1$.

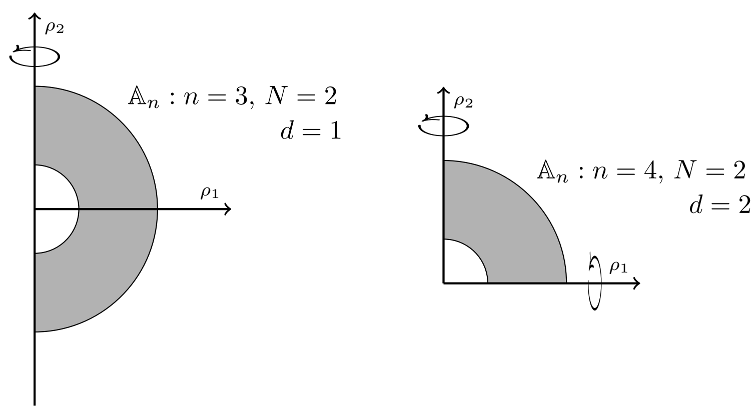

Figure 1: The contrasting symmetries in the semi-annular region $\mathbb{A}_n$ associated with $\mathbb{X}^n$ for n odd versus n even.

We now refer the reader to Proposition A.0.6 in Appendix A, which gathers numerous key calculus identities for whirl maps as described here. Along with the following Lemma A.0.7 there we also prove that all such whirl maps satisfy the incompressibility constraint $\det \nabla u = 1$. With these identities at hand the proof of the following result is immediate. Before proceeding we remark that, given the definition of a twist loop $\mathbf{Q}$ related to a whirl map $u = \mathbf{Q}(\varrho)x$ as in (3.1.10) we have the alternative definition $\mathbf{Q}(\varrho) = \exp\{\mathbf{H}(\varrho)\}$ where $\mathbf{H} : \overline{\mathbb{A}_n} \to \mathfrak{so}(n)$ is given by

$$\mathbf{H}(\varrho) = \begin{cases} diag(f_1\mathbf{J}, \dots, f_d\mathbf{J}) & n = 2d, \\ diag(f_1\mathbf{J}, \dots, f_d\mathbf{J}, 0) & n = 2d + 1 \end{cases} \tag{3.5.1}$$

for $\mathbf{J}$ as defined in (3.3.2) and $f_j \in \mathscr{C}(\overline{\mathbb{A}_n})$ for all $1 \le j \le d$ such $f_j \equiv 0$ on $(\partial\mathbb{A}_n)_a$ and $f_j \equiv 2m_j\pi$ on $(\partial\mathbb{A}_n)_b$.

Proposition 3.5.1. *Let* $u = r\mathbf{Q}(\varrho)\theta$ *with* $\mathbf{Q} \in \mathscr{C}^2(\mathbb{A}_n, \mathbf{SO}(n)) \cap \mathscr{C}(\overline{\mathbb{A}_n}, \mathbf{SO}(n))$ *given by* $\mathbf{Q} = \exp\{\mathbf{H}(\varrho)\}$ *for* $\mathbf{H}$ *defined by* (3.5.1)*. Then the action* $\mathscr{L}_h[u]$ *for* $\mathscr{L}_h$ *as in* (3.1.3) *can be formulated as*

$$\begin{aligned}\mathscr{L}_h[u] = \left(\mathbf{I}_n + \sum_{\ell=1}^{N} \nabla\rho_\ell \otimes \mathbf{H}_{,\ell}x\right)\bigg\{[h_r + 2rh_s]\left(\mathbf{I}_n + \sum_{\ell=1}^{N} \mathbf{H}_{,\ell}x \otimes \nabla\rho_\ell\right)\theta \\ + h\sum_{\ell=1}^{N}\left[\mathbf{H}_{,\ell\ell}x + \Delta\rho_\ell \mathbf{H}_{,\ell}x + 2\mathbf{H}_{,\ell}\nabla\rho_\ell\right] - rh_s\left(n + \sum_{\ell=1}^{N}|\mathbf{H}_{,\ell}x|^2\right)\theta\bigg\}.\end{aligned} \tag{3.5.2}$$

Here, $h = h(r, r^2)$, $h_r = h_r(r, r^2)$ *and* $h_s = h_s(r, r^2)$ *and* $\mathbf{H}_{,\ell}$, $\mathbf{H}_{,\ell\ell}$ *denote the first and second derivatives of* $\mathbf{H}$ *with respect to* ρ_ℓ*.*

Given the description of $|\nabla u|^2$ in Proposition A.0.6 [*c.f.* (A.0.15)], we now proceed by restricting the energy functional $\mathbb{F}[u; \mathbb{X}^n]$ to the class of whirls $u = r\mathbf{Q}(\varrho)\theta$ with the map $\mathbf{Q} = \mathbf{Q}(\varrho)$ as in (3.5.1) thus writing

$$\begin{aligned}\mathbb{F}[u = r\mathbf{Q}(\varrho)\theta; \mathbb{X}^n] &= \int_{\mathbb{X}^n} h(|x|, |x|^2)|\nabla[r\mathbf{Q}(\varrho)\theta]|^2\,dx \\ &= \int_{\mathbb{X}^n} h(|x|, |x|^2)\left(n + \sum_{\ell=1}^{N}|\mathbf{H}_{,\ell}x|^2\right)dx.\end{aligned} \tag{3.5.3}$$

By changing the variables of integration (3.5.3) can be reformulated as

$$\begin{aligned}\mathbb{F}[u = r\mathbf{Q}(\varrho)\theta; \mathbb{X}^n] - n^2\omega_n\int_a^b h(r, r^2)r^{n-1}\,dr = \\ = (2\pi)^d\int_{\mathbb{A}_n} h(\|\varrho\|, \|\varrho\|^2)\sum_{\ell=1}^{d}\rho_\ell^2|\nabla_{\mathbb{A}}f_\ell|^2\prod_{j=1}^{d}\rho_j\,d\varrho =: (2\pi)^d\sum_{\ell=1}^{d}\mathbb{H}_\ell[f_\ell; \mathbb{A}_n],\end{aligned} \tag{3.5.4}$$

where we have used the identity (A.0.16). Regarding the Jacobian of this transformation, note that we hereafter set

$$\prod_{j=1}^{d}\rho_j = \omega(\rho_1, \dots, \rho_d; d) =: \omega(\varrho; d). \tag{3.5.5}$$

Therefore when $n = 2d$, $N = d$ the above product features all $\rho_1, \dots, \rho_N$, whereas when $n = 2d+1$, $N = d+1$ the product features $\rho_1, \dots, \rho_{N-1}$. As such, the restricted energy $\mathbb{H}_\ell$ (with $1 \le \ell \le d$) can be expressed as

$$\mathbb{H}_\ell[f; \mathbb{A}_n] := \int_{\mathbb{A}_n} h(\|\varrho\|, \|\varrho\|^2)|\nabla_{\mathbb{A}}f|^2\rho_\ell^2\omega(\varrho; d)\,d\varrho, \qquad \|\varrho\|^2 = \sum_{j=1}^{N}\rho_j^2. \tag{3.5.6}$$

3.6 The Euler-Lagrange Equation Associated to $\mathbb{H}_\ell[f;\mathbb{A}_n]$

When considering the restricted energy functional $\mathbb{H}_\ell[f;\mathbb{A}_n]$ $(1 \le \ell \le d)$ given by (3.5.6), we have the functions $f = f(\rho_1, ..., \rho_N)$ in the admissible space

$$\mathscr{D}(\mathbb{A}_n) = \bigcup_{m\in\mathbb{Z}} \mathscr{D}_m(\mathbb{A}_n), \tag{3.6.1}$$

where, for each integer $m \in \mathbb{Z}$ we have set

$$\mathscr{D}_m(\mathbb{A}_n) := \left\{ f \in W^{1,2}(\mathbb{A}_n) : f = 0 \text{ on } (\partial\mathbb{A}_n)_a,\ f = 2m\pi \text{ on } (\partial\mathbb{A}_n)_b \right\}. \tag{3.6.2}$$

Intending to solve the system (3.1.2)-(3.1.3) we proceed onto extremising the restricted energy $\mathbb{H}_\ell[f;\mathbb{A}_n]$ over the space $\mathscr{D}_m(\mathbb{A}_n)$. Now recalling the three-part decomposition of $\partial\mathbb{A}_n$ given by (3.1.9), the Euler-Lagrange equation associated with $\mathbb{H}_\ell$ over $\mathscr{D}(\mathbb{A}_n)$ is seen to be (with $1 \le \ell \le d$, $m \in \mathbb{Z}$)

$$\mathbf{BVP}[f;\mathsf{m}] = \begin{cases} \mathrm{div}_{\mathbb{A}}\left[h(\|\varrho\|, \|\varrho\|^2)\rho_\ell^2\omega(\varrho;d)\nabla_{\mathbb{A}} f\right] = 0 & \text{in } \mathbb{A}_n, \\ f = 0 & \text{on } (\partial\mathbb{A}_n)_a, \\ f = 2m\pi & \text{on } (\partial\mathbb{A}_n)_b, \\ h(\|\varrho\|, \|\varrho\|^2)\rho_\ell^2\omega(\varrho;d)\partial_\nu f = 0 & \text{on } \Gamma_n. \end{cases} \tag{3.6.3}$$

Here $\partial_\nu f = \nabla_{\mathbb{A}} f \cdot \nu$ with ν being the unit outward normal field on Γ_n and ρ_ℓ is the ℓ^{th} component of the vector ϱ. See Theorem 6.2.1 for an analogous result in a more general context.

We now aim to show that this system has a unique solution in the form $f = f(\varrho;m) = \mathscr{G}(\|\varrho\|;m)$ for a suitable $\mathscr{G} = \mathscr{G}(r;m) \in \mathscr{C}^2[a,b]$, as a matter of fact,

$$f(\varrho;m) := 2m\pi\frac{\mathsf{H}\left(\|\varrho\|\right)}{\mathsf{H}(b)}, \qquad \mathsf{H}(t) := \int_a^t \frac{ds}{s^{n+1}h(s,s^2)}. \tag{3.6.4}$$

Towards this end, it is first easy to verify that the boundary conditions hold by virtue of $\mathsf{H}(a) = 0$ and the scaling at $||\varrho|| = b$. Furthermore by direct differentiation we have

$$\frac{\partial f}{\partial \rho_i} = 2m\pi\frac{\dot{\mathsf{H}}(r)}{\mathsf{H}(b)}\frac{\rho_i}{r} = \frac{2m_i\pi}{\mathsf{H}(b)}\frac{\rho_i}{r^{n+2}h(r,r^2)}, \qquad 1 \le i \le N. \tag{3.6.5}$$

Now, specialising first to even dimensions $n = 2d$, $N = d$, we see that

$$\begin{aligned}
\mathrm{div}_{\mathbb{A}}\left[h(r,r^2)\rho_\ell^2\omega(\varrho;d)\nabla_{\mathbb{A}}f\right] &= \sum_{i=1}^{N}\frac{\partial}{\partial\rho_i}\frac{2m_i\pi}{\mathsf{H}(b)}\left(h(r,r^2)\frac{\rho_i\rho_\ell^2}{r^{n+2}h(r,r^2)}\omega(\varrho;d)\right)\\
&= \frac{2m_i\pi}{\mathsf{H}(b)}\sum_{i=1}^{d}\left(\frac{\rho_\ell^2}{r^{n+2}}\omega(\varrho;d)-(n+2)\frac{\rho_i^2\rho_\ell^2}{r^{n+4}}\omega(\varrho;d)\right.\\
&\qquad\left.+2\frac{\rho_i\rho_\ell\delta_{i\ell}}{r^{n+2}}\omega(\varrho;d)+\frac{\rho_i\rho_\ell^2}{r^{n+2}}\frac{\omega(\varrho;d)}{\rho_i}\right)\\
&= \frac{1}{r^{n+2}}\frac{2m_i\pi}{\mathsf{H}(b)}\rho_\ell\Big[d\rho_\ell-(2d+2)\rho_\ell+2\rho_\ell+d\rho_\ell\Big]\omega(\varrho;d)=0.
\end{aligned} \tag{3.6.6}$$

Next for $n = 2d + 1$, $N = d + 1$ we proceed similarly but recall that $\rho_N = x_n$. For the first $\rho_1, \ldots, \rho_d$ terms in the divergence we use (3.6.6) above which gives

$$\begin{aligned}
\frac{2m_i\pi}{\mathsf{H}(b)}\sum_{i=1}^{d}\frac{\partial}{\partial\rho_i}\left(\frac{\rho_i\rho_\ell^2}{r^{n+2}}\omega(\varrho;d)\right) &= \\
&= \frac{1}{r^{n+2}}\frac{2m_i\pi}{\mathsf{H}(b)}\rho_\ell\left[d\rho_\ell-\frac{(2d+3)}{r^2}\rho_\ell\sum_{i=1}^{d}\rho_i^2+2\rho_\ell+d\rho_\ell\right]\omega(\varrho;d).
\end{aligned} \tag{3.6.7}$$

To this we add the N^{th} term in the divergence sum, which is seen to be

$$\frac{2m_i\pi}{\mathsf{H}(b)}\frac{\partial}{\partial\rho_N}\left(\frac{\rho_N\rho_\ell^2}{r^{n+2}}\omega(\varrho;d)\right)=\frac{1}{r^{n+2}}\frac{2m_i\pi}{\mathsf{H}(b)}\rho_\ell\left(\rho_\ell-\frac{(2d+3)}{r^2}\rho_\ell\rho_N^2\right)\omega(\varrho;d). \tag{3.6.8}$$

Coupling this with (3.6.7) therefore gives

$$\begin{aligned}
\mathrm{div}_{\mathbb{A}}\Big[h(r,r^2)\rho_\ell^2\omega(\varrho;d)\nabla_{\mathbb{A}}f\Big] &= \frac{2m_i\pi}{\mathsf{H}(b)}\sum_{i=1}^{N}\frac{\partial}{\partial\rho_i}\left(\frac{\rho_i\rho_\ell^2}{r^{n+2}}\omega(\varrho;d)\right)\\
&= \frac{1}{r^{n+2}}\frac{2m_i\pi}{\mathsf{H}(b)}\rho_\ell\Big[d\rho_\ell-(2d+3)\rho_\ell+2\rho_\ell+(d+1)\rho_\ell\Big]\omega(\varrho;d)=0.
\end{aligned} \tag{3.6.9}$$

Theorem 3.6.1. *Let $f(\varrho, m) = 2m\pi\mathsf{H}(||\varrho||)/\mathsf{H}(b)$ with H as in* (3.6.4) *and $m \in \mathbb{Z}$. Then f in $\mathscr{C}^2(\overline{\mathbb{A}_n})$ is the* unique *solution to the system* (3.6.3) *and the unique minimiser of the restricted energy $\mathbb{H}_\ell[f;\mathbb{A}_n]$ over $\mathscr{D}_m(\mathbb{A}_n)$.*

Proof. The proof is immediate upon the above calculations and Proposition C.0.2 adapted to the special case of $\mathsf{A}(r, s, \xi) = h(r, s)\xi$. □

3.7 Whirls as Solutions to the System $\mathscr{L}_h[u] = \nabla \mathscr{P}$

Recall the assumption that for a whirl map $u = r\mathbf{Q}(\varrho)\theta$ the twist path $\mathbf{Q} = \mathbf{Q}(\varrho) \in \mathscr{C}(\overline{\mathbb{A}_n}, \mathbf{SO}(n))$ must take values on the maximal torus $\mathbb{T} \subset \mathbf{SO}(n)$ of block diagonal 2×2 planar rotations: $\mathbb{T} = \{diag(\mathcal{R}[\phi_1], ..., \mathcal{R}[\phi_d]) : \phi_1, ..., \phi_d \in \mathbb{R}\}$ when $n = 2d$ and $\mathbb{T} = \{diag(\mathcal{R}[\phi_1], ..., \mathcal{R}[\phi_d], 1) : \phi_1, ..., \phi_d \in \mathbb{R}\}$ when $n = 2d+1$. This means that with $f_\ell = f_\ell(\varrho, m_\ell) \in \mathscr{C}(\overline{\mathbb{A}_n})$ $(1 \le \ell \le d)$ satisfying $f_\ell \equiv 0$ on $(\partial\mathbb{A}_n)_a$ and $f_\ell \equiv 2m_\ell\pi$ on $(\partial\mathbb{A}_n)_b$ we can write [*cf.* (3.1.10) and (3.5.1)]

$$\mathbf{Q}(\varrho) = \mathbf{Q}(\varrho; \mathsf{m}) = \begin{cases} \exp\{diag(f_1\mathbf{J}, \dots, f_d\mathbf{J})\} & n = 2d, \\ \exp\{diag(f_1\mathbf{J}, \dots, f_d\mathbf{J}, 0)\} & n = 2d+1, \end{cases} \tag{3.7.1}$$

where $\mathsf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$. In seeking solutions to (3.1.2)-(3.1.3) in the form of whirls, using results in the previous section, specifically Theorem 3.6.1, we must further specialise to $f_\ell(\varrho; m_\ell) = 2m_\ell\pi\mathscr{H}(||\varrho||)$ for $1 \le \ell \le d$, with $\mathscr{H}(r)$ as in (3.3.1) and $\mathbf{J}$ being the 2×2 matrix $\mathcal{R}[\pi/2]$ given by (3.3.2). Our goal here is to show, by analysing the PDE $\mathscr{L}_h[u] = \nabla\mathscr{P}$, that a necessary and sufficient condition for these whirls to be solutions to (3.1.2)-(3.1.3) is $f_\ell \in \{\pm 2m\pi\mathscr{H}(||\varrho||)\}$ for $1 \le \ell \le d$, *i.e.*, $m_\ell \in \{\pm m\}$ with $m \in \mathbb{Z}$ when $rh_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2h_s(r, r^2) \not\equiv 0$. This means that for a whirl to furnish a solution to the system (3.1.2)-(3.1.3), here, up to a sign, the functions f_ℓ, or equivalently, the integers m_ℓ must all be equal. In contrast when $rh_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2h_s(r, r^2) \equiv 0$ (*i.e.* the h-condition holds) no such restriction on f_ℓ or m_ℓ is needed. A more precise formulation of this is given below.

Theorem 3.7.1. *Suppose $u = r\mathbf{Q}(\varrho)\theta$ is a whirl map with $\mathbf{Q} \in \mathscr{C}^2(\overline{\mathbb{A}_n}, \mathbf{SO}(n))$ satisfying $\mathbf{Q}(\varrho) = \mathbf{I}_n$ for $\varrho \in (\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b$. Then u is a solution to the nonlinear system* (3.1.2)-(3.1.3) *if and only if $\mathbf{Q} = \mathbf{Q}(\varrho; \mathsf{m})$ is as described below.*

1. $(rh_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2h_s(r, r^2) \not\equiv 0$ *on* $]a, b[)$ *Here, depending on the dimension n being even or odd, we have*

   (i) *n even: $|m_1| = \cdots = |m_d|$ and subsequently*

$$\begin{aligned} \mathbf{Q}(\varrho; \mathsf{m}) &= \exp\{diag(2m_1\pi\mathscr{H}(||\varrho||)\mathbf{J}, \dots, 2m_d\pi\mathscr{H}(||\varrho||)\mathbf{J})\}, \quad \varrho \in \overline{\mathbb{A}_n} \\ &= diag(\mathcal{R}[2m_1\pi\mathscr{H}(||\varrho||)], \dots, \mathcal{R}[2m_d\pi\mathscr{H}(||\varrho||)]), \end{aligned} \tag{3.7.2}$$

with $\mathcal{R}$ *and* $\mathbf{J}$ *defined by* (3.3.2).

(*ii*) n *odd:* $m_1 = \cdots = m_d = 0$ *and therefore* $\mathbf{Q} \equiv \mathbf{I}_n$.

2. $(rh_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2 h_s(r, r^2) \equiv 0$ *on* $]a, b[)$ *Here* $\mathbf{Q}(\varrho) = \exp\{\mathscr{H}(||\varrho||)\mathbf{H}\}$ *for all* $\varrho \in \overline{\mathbb{A}_n}$ *where* $\mathbf{H} = diag(2\pi m_1 \mathbf{J}, ..., 2\pi m_d \mathbf{J})$ *when* $n = 2d$ *and* $\mathbf{H} = diag(2\pi m_1 \mathbf{J}, ..., 2\pi m_d \mathbf{J}, 0)$ *when* $n = 2d+1$. *In this case there is no restriction on* m.

Before proceeding with the proof we pause briefly to take a closer look at the identities in Proposition A.0.6 when, with a slight abuse of notation, $\mathbf{Q}(\varrho) = \mathbf{Q}(r)$ where $r = \|\varrho\| = \sqrt{\rho_1^2 + ... + \rho_N^2}$ (see also Proposition A.0.3). Beginning with the gradient we note that

$$\mathbf{Q}_{,\ell} = \frac{\partial \mathbf{Q}(\|\varrho\|)}{\partial \rho_\ell} = \frac{\rho_\ell}{r}\dot{\mathbf{Q}}(r), \qquad \sum_{\ell=1}^{N} \rho_\ell \nabla \rho_\ell = \nabla ||\varrho||^2/2 = x, \tag{3.7.3}$$

by virtue of $r = ||\varrho||$ and therefore

$$\nabla u = \mathbf{Q} + \dot{\mathbf{Q}}\theta \otimes \sum_{\ell=1}^{N} \rho_\ell \nabla \rho_\ell = \mathbf{Q} + \dot{\mathbf{Q}}\theta \otimes x = \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta. \tag{3.7.4}$$

In particular it follows from this that $|\nabla u|^2 = n + r^2|\dot{\mathbf{Q}}\theta|^2$ as in Proposition A.0.3. Finally for the Laplacian Δu we first note that $\Delta \rho_\ell = 1/\rho_\ell$ except for n odd and $\ell = N$ where $\Delta \rho_N = 0$ and

$$\mathbf{Q}_{,\ell\ell} = \frac{\rho_\ell^2}{r^2}\ddot{\mathbf{Q}} + \frac{1}{r}\frac{r^2 - \rho_\ell^2}{r^2}\dot{\mathbf{Q}}, \tag{3.7.5}$$

therefore giving, using (*iii*) in Proposition A.0.6,

$$\begin{aligned}\Delta u &= \sum_{\ell=1}^{N} \left[\left(\frac{\rho_\ell^2}{r^2}\ddot{\mathbf{Q}} + \frac{r^2 - \rho_\ell^2}{r^3}\dot{\mathbf{Q}} \right) x + \frac{\rho_\ell}{r}\Delta\rho_\ell \dot{\mathbf{Q}} x + 2\frac{\rho_\ell}{r}\dot{\mathbf{Q}}\nabla\rho_\ell \right] \\ &= \begin{cases} r\ddot{\mathbf{Q}}\theta + (N-1)\dot{\mathbf{Q}}\theta + N\dot{\mathbf{Q}}\theta + 2\dot{\mathbf{Q}}\theta, & n \text{ even}, \\ r\ddot{\mathbf{Q}}\theta + (N-1)\dot{\mathbf{Q}}\theta + (N-1)\dot{\mathbf{Q}}\theta + 2\dot{\mathbf{Q}}\theta, & n \text{ odd} \end{cases} \\ &= r\ddot{\mathbf{Q}}\theta + (n+1)\dot{\mathbf{Q}}\theta \end{aligned} \tag{3.7.6}$$

in view of $N = n/2$ when n is even and $N = (n+1)/2$ when n is odd. With these identities at hand we present the proof of Theorem 3.7.1.

Proof. Let $u = r\mathbf{Q}(\varrho)\theta$ be a whirl map as described. Then by the existence and uniqueness result in the previous section on the extremisers $f_1, ..., f_d$ to the restricted energies $\mathbb{H}_\ell$ it is plain that these functions and hence $\mathbf{Q}$ depend only on $r = ||\varrho||$. In fact from the explicit description of $f_1, ..., f_d$ in Theorem 3.6.1 we have $\mathbf{Q}(\varrho) = \exp\{\mathscr{H}(||\varrho||)\mathbf{H}\}$ with $\mathbf{H} = diag(2m_1\pi\mathbf{J}, ..., 2m_d\pi\mathbf{J})$ for $n = 2d$ even and $\mathbf{H} = diag(2m_1\pi\mathbf{J}, ..., 2m_d\pi\mathbf{J}, 0)$ for $n = 2d + 1$ odd and moreover that the profile $\mathscr{H} = \mathscr{H}(r)$ solves the ODE (3.4.1). Thus starting from the formulation of the action $\mathscr{L}_h[u = r\mathbf{Q}(\varrho)\theta]$ as in (3.5.2) we have

$$\begin{aligned}
\mathscr{L}_h[u = \mathbf{Q}(\varrho)x] = {} & [h_r(r, r^2) + 2rh_s(r, r^2)] \times \\
& \times \left(\mathbf{I}_n + \sum_{\ell=1}^{N} \nabla\rho_\ell \otimes \mathbf{H}_{,\ell}x\right)\left(\mathbf{I}_n + \sum_{\ell=1}^{N} \mathbf{H}_{,\ell}x \otimes \nabla\rho_\ell\right)\theta \\
& + h(r, r^2)\left(\mathbf{I}_n + \sum_{\ell=1}^{N} \nabla\rho_\ell \otimes \mathbf{H}_{,\ell}x\right) \sum_{\ell=1}^{N} [\mathbf{H}_{,\ell\ell}x + \Delta\rho_\ell\mathbf{H}_{,\ell}x + 2\mathbf{H}_{,\ell}\nabla\rho_\ell] \\
& - rh_s(r, r^2)\left(n + \sum_{\ell=1}^{N} |\mathbf{H}_{,\ell}x|^2\right)\left(\mathbf{I}_n + \sum_{\ell=1}^{N} \nabla\rho_\ell \otimes \mathbf{H}_{,\ell}x\right)\theta
\end{aligned} \tag{3.7.7}$$

and so substituting for $\mathbf{Q}$ by invoking the discussion prior to the proof and the formulation above leads to

$$\begin{aligned}
\mathscr{L}_h[u = r\mathbf{Q}(\varrho)\theta] = {} & (\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta)\Big\{[h_r(r, r^2) + 2rh_s(r, r^2)](\mathbf{Q}\theta + r\dot{\mathbf{Q}}\theta) \\
& + h(r, r^2)\left[r\ddot{\mathbf{Q}} + (n + 1)\dot{\mathbf{Q}}\right]\theta - rh_s(r, r^2)(n + r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}\theta\Big\} \\
= {} & \nabla h(|x|, |x|^2) - nrh_s(r, r^2)\theta \\
& + \left[r^2h_r(r, r^2) + r^3h_s(r, r^2) + r(n + 1)h(r, r^2)\right]\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta \\
& + r^2h(r, r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}}|\mathbf{H}\theta|^2\theta + rh(r, r^2)\dot{\mathscr{H}}^2\mathbf{H}^2\theta.
\end{aligned} \tag{3.7.8}$$

By an application of the ODE (3.4.1) for $\mathscr{H}$ we see a significant simplification, that is

$$\begin{aligned}
\mathscr{L}_h[u = r\mathbf{Q}(\varrho)\theta] = {} & \nabla h(|x|, |x|^2) - nrh_s(r, r^2)\theta - r^3h_s(r, r^2)\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta \\
& + rh(r, r^2)\dot{\mathscr{H}}^2\mathbf{H}^2\theta.
\end{aligned} \tag{3.7.9}$$

Note that this is precisely the same formulation as in the case of generalised twists - see (3.4.2) and Theorem 3.4.1.

Finally returning to $\mathscr{L}_h[u = r\mathbf{Q}(\varrho)\theta] = \nabla\mathscr{P}$, Proposition D.0.2 leads to the conclusion that if $rh_r(r, r^2) + 2(n + 1)h(r, r^2) + 4r^2h_s(r, r^2) \not\equiv 0$ then $\operatorname{curl} U = \mathscr{L}_h[u] - \nabla h + rnh_s = 0 \iff |f_1|^2 = \cdots = |f_d|^2$ when $n = 2d$ is even and

$|f_1|^2 = \cdots = |f_d|^2 = 0$ when $n = 2d+1$ is odd. As such this gives $m_\ell = 0$ for all $1 \le \ell \le d$ when $n = 2d+1$ is odd and $|m_1| = \cdots = |m_d| := |m|$ when $n = 2d$ and so $f_\ell \in \{\pm 2m\pi \mathscr{H}(||\varrho||)\}$ for all $1 \le \ell \le d$. If $rh_r(r, r^2) + 2(n+1)h(r, r^2) + 4r^2 h_s(r, r^2) \equiv 0$ we have, again by Proposition D.0.2, that $\mathscr{L}_h[u = r\mathbf{Q}(\varrho)\theta]$ is curl-free as well as a gradient with no restriction on m_ℓ. $\square$

Chapter 4

The Non-Variational System $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$ and the Discriminant $\Delta(h, g)$

In this chapter we consider the second order nonlinear elliptic PDE in divergence form given by

$$\mathrm{div}[\mathsf{A}(|x|, |u|^2, |\nabla u|^2)\nabla u] + \mathsf{B}(|x|, |u|^2, |\nabla u|^2)u = [\mathrm{cof}\, \nabla u]\nabla \mathscr{P},$$

where the unknown vector field u satisfies the pointwise incompressibility constraint $\det \nabla u = 1$ along with suitable boundary conditions and $\mathscr{P} = \mathscr{P}(x)$ is an *a priori* unknown hydrostatic pressure field. Here, $\mathsf{A} = \mathsf{A}(r, s, \xi)$ and $\mathsf{B} = \mathsf{B}(r, s, \xi)$ are sufficiently regular scalar functions satisfying natural structural properties. Most notably, in the case of a finite symmetric annulus, we prove the existence of a countably infinite scale of topologically distinct twisting solutions to the system in all even dimensions. In sharp contrast in odd dimensions the only twisting solution is the map $u \equiv x$. We study a related class of systems by introducing the novel notion of a *discriminant*. Using this, a complete and *explicit* characterisation of all twisting solutions for $n \geq 2$ is given and a curious dichotomy in the behaviour of the system and its solutions is captured and analysed.

4.1 Statement of the Result

We consider a second order nonlinear system in divergence form in a bounded domain $\Omega \subset \mathbb{R}^n$ subject to a pointwise incompressibility constraint:

$$\begin{cases} \mathrm{div}[\mathsf{A}(|x|, |u|^2, |\nabla u|^2)\nabla u] + \mathsf{B}(|x|, |u|^2, |\nabla u|^2)u = [\mathrm{cof}\, \nabla u]\nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Here $\mathscr{P}$ is an *a priori* unknown hydrostatic pressure field corresponding to the pointwise constraint $\det \nabla u = 1$ and to avoid unnecessary technicalities and fix ideas φ is taken throughout as the identity map, that is, $\varphi \equiv x$. Moreover, $\mathsf{A} = \mathsf{A}(r, s, \xi)$ and $\mathsf{B} = \mathsf{B}(r, s, \xi)$ are real-valued functions of classes $\mathscr{C}^1$ and $\mathscr{C}$ respectively with A being positive, monotone in the third variable and having a suitable growth (see below for a formulation of the assumptions on A and B). The divergence operator acts row-wise on the matrix field $\mathsf{A}(|x|, |u|^2, |\nabla u|^2)\nabla u$. By taking advantage of the incompressibility constraint $\det \nabla u = 1$ and thus the algebraic identity $\mathrm{cof}\, \nabla u = (\nabla u)^{-t}$ the above system can be reformulated as

$$\begin{cases} \mathscr{L}[u; \mathsf{A}, \mathsf{B}] := \nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

where the second order differential operator $\mathscr{L} = \mathscr{L}[u; \mathsf{A}, \mathsf{B}]$ here is given by

$$\begin{aligned} \mathscr{L}[u; \mathsf{A}, \mathsf{B}] &= (\nabla u)^t \mathrm{div}[\mathsf{A}(|x|, |u|^2, |\nabla u|^2)\nabla u] + \mathsf{B}(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t u \\ &= \mathsf{A}(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t \Delta u + (\nabla u)^t \nabla u\, \nabla \mathsf{A}(|x|, |u|^2, |\nabla u|^2) \\ &\quad + \mathsf{B}(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t u. \end{aligned} \tag{4.1.2}$$

As a result of this formulation, it is evident that if u is a solution to this system, then necessarily $\mathrm{curl}\, \mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \mathrm{curl}\, \nabla \mathscr{P} \equiv 0$ in Ω, that is,

$$\mathrm{curl}\, \left[(\nabla u)^t \mathrm{div}[\mathsf{A}(|x|, |u|^2, |\nabla u|^2)\nabla u] + \mathsf{B}(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t u\right] \equiv 0. \tag{4.1.3}$$

However note that this condition, unless $\Omega \subset \mathbb{R}^n$ has a particular homology, would not on its own imply that the vector field $\mathscr{L}[u; \mathsf{A}, \mathsf{B}]$ is a gradient field, here, $\nabla \mathscr{P}$. Note also that if $\mathsf{A}(r, s, \xi) = F_\xi(r, s, \xi)$ and $\mathsf{B}(r, s, \xi) = -F_s(r, s, \xi)$ for some class $\mathscr{C}^2$ Lagrangian $F = F(r, s, \xi)$ (hence in particular $\mathsf{A}_s + \mathsf{B}_\xi \equiv 0$)

then the system (4.1.1)-(4.1.2) is precisely the Euler-Lagrange equation associated with the variational energy integral ($p \geq 1$)

$$\mathbb{F}[u; \Omega] = \int_\Omega F(|x|, |u|^2, |\nabla u|^2)\, dx, \qquad u \in \mathscr{A}^p_\varphi(\Omega), \tag{4.1.4}$$

where $\mathscr{A}^p_\varphi(\Omega) = \{u \in W^{1,p}(\Omega; \mathbb{R}^n) : \det \nabla u = 1 \ a.e. \text{ in } \Omega \text{ and } u = \varphi \text{ on } \partial\Omega\}$. Although the existence of an energy integral can largely facilitate the analysis, we emphasise that the assumptions put in place on A and B here are more general and do not assume or associate an energy or a variational structure to the system.[6] For more on the background formulation and applications of the system (4.1.1)-(4.1.2) in particular to geometry, function theory, mechanics and nonlinear elasticity *see* [5, 6, 10, 13, 24, 48] and the references therein.

Throughout this chapter we specialise to the geometric set up where $\Omega = \mathbb{X}^n = \mathbb{X}^n[a, b] := \{x \in \mathbb{R}^n : a < |x| < b\}$ is a finite symmetric annulus with $b > a > 0$ and $\varphi \equiv x$, the identity map. In this context, by a generalised twist (or simply *twist*) we understand a map $u \in \mathscr{C}(\overline{\mathbb{X}^n}, \overline{\mathbb{X}^n})$ which in spherical coordinates admits the representation

$$u : (r, \theta) \mapsto (r, \mathbf{Q}(r)\theta), \qquad r = |x|, \theta = x|x|^{-1}, \quad x \in \overline{\mathbb{X}^n}. \tag{4.1.5}$$

Here $\mathbf{Q} = \mathbf{Q}(r) \in \mathscr{C}([a, b], \mathbf{SO}(n))$ is called the twist *path* associated with u. Now, in order to ensure $u = \varphi$ on $\partial\Omega = \partial\mathbb{X}^n$ we set $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$. In this event, the twist path forms a closed curve in $\mathbf{SO}(n)$ based at $\mathbf{I}_n$ called the twist *loop* that in turn represents an element of the fundamental group $\pi_1(\mathbf{SO}(n)) \cong \mathbb{Z}_2$ ($n \geq 3$) and $\mathbb{Z}$ ($n = 2$). Our aim here is to prove the existence of multiple twisting solutions to (4.1.1)-(4.1.2) by carefully formulating the action of $\mathscr{L}$ on sufficiently regular twists and then specialising to those having a geodesic twist loop $\mathbf{Q}(r) = \exp\{f(r)\mathbf{H}\}$ for suitable choices of $f = f(r)$ and $\mathbf{H} \in \mathfrak{so}(n)$. It is quite remarkable that here, despite the form of $\mathscr{L}[u; \mathsf{A}, \mathsf{B}]$, in the construction of multiple twist solutions we can take advantage of an arising variational structure on A whilst separating the roles of A, B and only reuniting them again at the last stage of the argument when enforcing the curl-free condition on $\mathscr{L}[u; \mathsf{A}, \mathsf{B}]$.

For the sake of future reference let us proceed by describing the assumptions. Indeed we assume throughout that $\mathsf{A} = \mathsf{A}(r, s, \xi)$, $\mathsf{B} = \mathsf{B}(r, s, \xi)$ are of classes

[6] Despite this, our results even in the variational context are new and of interest. Among the many important examples here one can refer to the cases $F(r, s, \xi) = h(r, s)\xi$ with $h > 0$ and of class $\mathscr{C}^2$ where the resulting $\mathbb{F}$ is a weighted Dirichlet type energy and $F(r, s, \xi) = (\xi/s)^{n/2}$ ($n \geq 2$) where the resulting $\mathbb{F}$ is the classical distortion energy (*see*, e.g., [5, 6, 46, 61, 62]).

$\mathscr{C}^1(U)$, $\mathscr{C}(U)$ respectively where $U = U(\mathbb{X}^n[a,b]) = [a,b]\times]0,\infty[\times]0,\infty[\subset \mathbb{R}^3$ and that A is strictly positive, i.e., $\mathsf{A}(r,s,\xi) > 0$, monotone in the ξ variable, *i.e.* $\mathsf{A}_\xi(r,s,\xi) \geq 0$ for all $(r,s,\xi) \in U$ and finally that for every compact set $K \subset]0,\infty[$ there are constants $c_1 = c_1(K), c_2 = c_2(K) > 0$ such that

$$c_1|\zeta|^{p-1} \leq \mathsf{A}(r,s,\zeta^2)|\zeta| \leq c_2|\zeta|^{p-1}, \qquad \forall(r,s,\zeta^2) \in U, s \in K, p > 1. \tag{4.1.6}$$

Naturally the (*untwisting*) identity map $u \equiv x$ – corresponding to the constant twist loop $\mathbf{Q} \equiv \mathbf{I}_n$ – is always a solution to the system (4.1.1) for a suitable hydrostatic pressure field $\mathscr{P}$ as is seen by substitution:

$$\begin{aligned}\mathscr{L}[u \equiv x; \mathsf{A}, \mathsf{B}] &= (\nabla u)^t \mathrm{div}[\mathsf{A}(|x|, |u|^2, |\nabla u|^2)\nabla u] + \mathsf{B}(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t u \\ &= \mathrm{div}[\mathsf{A}(|x|, |x|^2, n)\mathbf{I}_n] + \mathsf{B}(|x|, |x|^2, n)x \\ &= \nabla \mathsf{A}(|x|, |x|^2, n) + \mathsf{B}(|x|, |x|^2, n)x = \nabla \mathscr{P}.\end{aligned} \tag{4.1.7}$$

In this chapter we show that, interestingly and somewhat unexpectedly, in all *even* dimensions $n \geq 2$ there is a further infinite family of topologically distinct twisting solutions to this system as formulated below.

Main Theorem. *Let $n \geq 2$ be even and A, B as above. Then for each $m \in \mathbb{Z}$ there exists a generalised twist $u = u(x; m)$ of class $\mathscr{C}^2$ serving as a solution to the nonlinear system* (4.1.1)-(4.1.2). *More specifically $u(x; m) = r\mathbf{Q}(r; m)\theta$ is a generalised twist with twist path $\mathbf{Q} = \mathbf{Q}(r; m)$ given explicitly by*

$$\begin{aligned}\mathbf{Q}(r; m) &= \exp\{\mathscr{G}(r; m)\mathbf{H}\}, \qquad a \leq r \leq b, \quad m \in \mathbb{Z} \\ &= \mathbf{P} diag(\mathcal{R}[\mathscr{G}](r; m), \ldots, \mathcal{R}[\mathscr{G}](r; m))\mathbf{P}^t,\end{aligned} \tag{4.1.8}$$

where $\mathscr{G} = \mathscr{G}(r; m)$ is the unique solution to the boundary value problem

$$\mathbf{BVP}[\mathscr{G}; \mathsf{A}] = \begin{cases} \dfrac{d}{dr}\left[r^{n+1}\mathsf{A}(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\right] = 0, & a < r < b, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2m\pi. \end{cases} \tag{4.1.9}$$

Moreover $\mathbf{H}$ is the $n \times n$ skew-symmetric matrix $\mathbf{H} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$ with $\mathbf{P} \in \mathbf{O}(n)$ arbitrary, $\mathbf{J}_n = diag(\mathbf{J}, ..., \mathbf{J})$ and $\mathbf{J}$ as in (4.2.2)*. Furthermore, $\mathscr{P}$ represents a hydrostatic pressure associated with u and is given by*

$$\mathscr{P}(x; m) = \mathsf{A}(r, r^2, n + r^2\dot{\mathscr{G}}^2) + \mathsf{S}(r), \qquad r = |x|, \quad x \in \overline{\mathbb{X}^n}, \tag{4.1.10}$$

where $\nabla \mathsf{S} = r[\mathsf{B}(r, r^2, n + r^2\dot{\mathscr{G}}^2) - \dot{\mathscr{G}}^2\mathsf{A}(r, r^2, n + r^2\dot{\mathscr{G}}^2)]\theta$.

4.2 Construction of Countably Infinitely Many Solutions to the Nonlinear System (4.1.1)-(4.1.2)

First we refer the reader to Proposition A.0.3 in Appendix A to recall some of the key identities pertaining to a generalised twist $u(x) = \mathbf{Q}(|x|)x$ and its gradient ∇u. We state here (and for reference throughout this chapter) that the action of the partial differential operator $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ on a generalised twist is

$$\begin{aligned}\mathscr{L}[u;\mathsf{A},\mathsf{B}] &= (\mathbf{Q}^t + r\theta\otimes\dot{\mathbf{Q}}\theta)\times \\ &\times\Big\{\mathsf{A}_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)(\mathbf{Q}+r\dot{\mathbf{Q}}\theta\otimes\theta)(2r|\dot{\mathbf{Q}}\theta|^2\theta + r^2\nabla|\dot{\mathbf{Q}}\theta|^2) \\ &+ 2r\mathsf{A}_s(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)(\mathbf{Q}\theta + r\dot{\mathbf{Q}}\theta) + \mathsf{A}_r(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)(\mathbf{Q}\theta + r\dot{\mathbf{Q}}\theta) \\ &+ \mathsf{A}(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\left[(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}\right]\theta + r\mathsf{B}(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}\theta\Big\},\end{aligned} \tag{4.2.1}$$

which follows by direct substitution. In this section we specialise to generalised twists u whose loops $\mathbf{Q} = \mathbf{Q}(r)$ $(a \le r \le b)$ are suitably scaled geodesics on the compact Lie group $\mathbf{SO}(n)$ based at $\mathbf{I}_n$ with n even. For this we take $\mathbf{Q} = \exp\{\mathscr{G}(r)\mathbf{H}\}$ for a suitable $\mathscr{G} \in \mathscr{C}^2[a,b]$ [*cf.* (4.2.9)] and $\mathbf{H}$ the $n\times n$ skew-symmetric matrix $\mathbf{H} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$, with $\mathbf{P}\in\mathbf{O}(n)$ arbitrary and $\mathbf{J}_n = diag(\mathbf{J},...,\mathbf{J})$ where

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{R}[t] = \exp\{t\mathbf{J}\} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{4.2.2}$$

It is seen that here $\dot{\mathbf{Q}} = \dot{\mathscr{G}}\mathbf{H}\mathbf{Q}$, $\ddot{\mathbf{Q}} = (\ddot{\mathscr{G}}\mathbf{H} - \dot{\mathscr{G}}^2\mathbf{I}_n)\mathbf{Q}$ and since the dimension n is taken even, $|\dot{\mathbf{Q}}\theta|^2 = \dot{\mathscr{G}}^2|\mathbf{H}\theta|^2 = \dot{\mathscr{G}}^2$. Moreover, as $\mathbf{Q}$ is based at $\mathbf{I}_n$, this forces the angle of rotation function $\mathscr{G}$ to take (without loss of generality) the boundary values $\mathscr{G}(a) = 0$ and $\mathscr{G}(b) = 2m\pi$ for some $m\in\mathbb{Z}$. Under this set of assumptions it is seen that the action of the differential operator $\mathscr{L}$ on the twist map u can be formulated as

$$\begin{aligned}
&\mathscr{L}[u = r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta;\mathsf{A},\mathsf{B}] = (\mathbf{I}_n + r\dot{\mathscr{G}}\theta\otimes\mathbf{H}\theta)\times \\
&\quad\times\Big\{\mathsf{A}_\xi(r,r^2,n+r^2\dot{\mathscr{G}}^2)(\mathbf{I}_n + r\dot{\mathscr{G}}\mathbf{H}\theta\otimes\theta)(2r\dot{\mathscr{G}}^2\theta + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\theta) \qquad (4.2.3)\\
&+2r\mathsf{A}_s(r,r^2,n+r^2\dot{\mathscr{G}}^2)(\mathbf{I}_n + r\dot{\mathscr{G}}\mathbf{H})\theta + \mathsf{A}_r(r,r^2,n+r^2\dot{\mathscr{G}}^2)(\mathbf{I}_n + r\dot{\mathscr{G}}\mathbf{H})\theta \\
&+\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\Big[(n+1)\dot{\mathscr{G}}\mathbf{H} + r(\ddot{\mathscr{G}}\mathbf{H} - \dot{\mathscr{G}}^2\mathbf{I}_n)\Big]\theta + r\mathsf{B}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\theta\Big\}.
\end{aligned}$$

Upon taking into account the necessary cancellations and after rearranging terms this action can be written in the form

$$\mathscr{L}[u = r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta;\mathsf{A},\mathsf{B}] = \mathscr{A}(r)\theta + \mathscr{B}(r)\mathbf{H}\theta, \qquad (4.2.4)$$

where the scalar functions $\mathscr{A} = \mathscr{A}(r)$ and $\mathscr{B} = \mathscr{B}(r)$ here are given by

$$\begin{aligned}
\mathscr{A}(r) :=\, &\mathsf{A}_\xi(r,r^2,n+r^2\dot{\mathscr{G}}^2)(1+r^2\dot{\mathscr{G}}^2)(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}) \qquad (4.2.5)\\
&+2r\mathsf{A}_s(r,r^2,n+r^2\dot{\mathscr{G}}^2)(1+r^2\dot{\mathscr{G}}^2) + \mathsf{A}_r(r,r^2,n+r^2\dot{\mathscr{G}}^2)(1+r^2\dot{\mathscr{G}}^2)\\
&+\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\Big[r(n+1)\dot{\mathscr{G}}^2 - r\dot{\mathscr{G}}^2 + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\Big] + r\mathsf{B}(r,r^2,n+r^2\dot{\mathscr{G}}^2),
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{B}(r) :=\, &r\mathsf{A}_\xi(r,r^2,n+r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}})\\
&+2r^2\mathsf{A}_s(r,r^2,n+r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}} + r\mathsf{A}_r(r,r^2,n+r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\\
&+\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)[(n+1)\dot{\mathscr{G}} + r\ddot{\mathscr{G}}]. \qquad (4.2.6)
\end{aligned}$$

Focusing on these coefficients of $\mathscr{L}[u = r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta;\mathsf{A},\mathsf{B}]$ it can be seen further that

$$\begin{aligned}
\mathscr{A}(r) = &\frac{\dot{\mathscr{G}}}{r^{n-1}}\frac{d}{dr}\Big[r^{n+1}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\Big] + \frac{d}{dr}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\\
&+r\Big[\mathsf{B}(r,r^2,n+r^2\dot{\mathscr{G}}^2) - \dot{\mathscr{G}}^2\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\Big], \qquad (4.2.7)
\end{aligned}$$

and similarly

$$\mathscr{B}(r) = \frac{1}{r^n}\frac{d}{dr}\Big[r^{n+1}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\Big]. \qquad (4.2.8)$$

As suggested by the above formulation, we now proceed by choosing the angle of rotation function $\mathscr{G} \in \mathscr{C}^2[a,b]$ to be a solution to the second order ODE associated with $\mathsf{A} = \mathsf{A}(r,s,\xi)$:

$$\frac{d}{dr}\Big[r^{n+1}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\Big] = 0, \qquad a<r<b, \qquad (4.2.9)$$

supplemented by the boundary conditions $\mathscr{G}(a) = 0$ and $\mathscr{G}(b) = 2m\pi$ (the existence of solutions $\mathscr{G}$ with the required degree of regularity is established in Proposition C.0.1). Referring to the description of the action $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ in (4.2.4) together with the above formulations of $\mathscr{A}(r)$, $\mathscr{B}(r)$, in light of the ODE (4.2.9) it is now seen that,

$$\begin{aligned}
\mathscr{L}[u;\mathsf{A},\mathsf{B}] &= \mathscr{L}[r\exp\{\mathscr{G}(r;m)\mathbf{H}\}\theta;\mathsf{A},\mathsf{B}] = \mathscr{A}(r)\theta + \mathscr{B}(r)\mathbf{H}\theta \\
&= \Bigg\{ \frac{\dot{\mathscr{G}}}{r^{n-1}}\frac{d}{dr}\left[r^{n+1}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\right] + \frac{d}{dr}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2) \\
&\quad + r\left[\mathsf{B}(r,r^2,n+r^2\dot{\mathscr{G}}^2) - \dot{\mathscr{G}}^2\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\right]\Bigg\}\theta \\
&\quad + \frac{1}{r^n}\frac{d}{dr}\left[r^{n+1}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\right]\mathbf{H}\theta \qquad (4.2.10)\\
&= \left[\frac{d}{dr}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2) + r\mathsf{B}(r,r^2,n+r^2\dot{\mathscr{G}}^2) - r\dot{\mathscr{G}}^2\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\right]\theta.
\end{aligned}$$

We are now in a position to prove the Main Theorem of this chapter.

Proof. (Main Theorem) Recalling the description of $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ from (4.2.10) and with $\mathscr{G} = \mathscr{G}(r;m)$ as above all that remains is to show that the vector field

$$\begin{aligned}
v := \mathscr{L}[u;\mathsf{A},\mathsf{B}] &= (\nabla u)^t \mathrm{div}[\mathsf{A}(|x|,|u|^2,|\nabla u|^2)\nabla u] + \mathsf{B}(|x|,|u|^2,|\nabla u|^2)(\nabla u)^t u \\
&= \bigg[\frac{d}{dr}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2) + r\mathsf{B}(r,r^2,n+r^2\dot{\mathscr{G}}^2) \\
&\quad - r\dot{\mathscr{G}}^2\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\bigg]\theta, \qquad (4.2.11)
\end{aligned}$$

is a gradient field in $\mathbb{X}^n$, that is, $v = \nabla\mathscr{P}$. Towards this end it is firstly seen that $d/dr\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\theta = \nabla\mathsf{A}(|x|,|x|^2,n+|x|^2\dot{\mathscr{G}}^2)$ and secondly upon writing the remainder as $r[\mathsf{B}(r,r^2,n+r^2\dot{\mathscr{G}}^2) - \dot{\mathscr{G}}^2\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)]\theta = s(r)\theta$ for some $s \in \mathscr{C}[a,b]$ we have $r[\mathsf{B}(r,r^2,n+r^2\dot{\mathscr{G}}^2) - \dot{\mathscr{G}}^2\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\theta = \nabla\mathsf{S}(|x|)$ for a suitable primitive S of s. This therefore shows that the two segments of v are both gradients and hence completes the proof. □

4.3 The Case $\mathsf{A} = h(r,s)$, $\mathsf{B} = g(r,s)\xi$ and the Discriminant $\Delta(h,g)$

In this section we consider a particular case of the system (4.1.1)-(4.1.2) where, quite remarkably, all the twist solutions with twist loops of class $\mathscr{C}^2$ can be explicitly computed and described. Here we take the differential operator $\mathscr{L} = \mathscr{L}[u; \mathsf{A}, \mathsf{B}]$ with $\mathsf{A} = h(r, s)$, $\mathsf{B} = g(r, s)\xi$, that is,[7]

$$\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = (\nabla u)^t \left\{ \mathrm{div}[h(r, |u|^2)\nabla u] + g(r, |u|^2)|\nabla u|^2 u \right\}, \tag{4.3.1}$$

where $h = h(r, s) > 0$ and $g = g(r, s)$ are of classes $\mathscr{C}^2$ and $\mathscr{C}$ respectively. Taking a twist $u = r\mathbf{Q}(r)\theta$ with twist loop $\mathbf{Q} \in \mathscr{C}^2(]a, b[, \mathbf{SO}(n)) \cap \mathscr{C}([a, b], \mathbf{SO}(n))$ we can then write

$$\begin{aligned} \mathscr{L}[u; \mathsf{A}, \mathsf{B}] &= (\nabla u)^t \left\{ \nabla u \nabla [h(r, |u|^2)] + h(r, |u|^2)\Delta u + g(r, |u|^2)|\nabla u|^2 u \right\} \\ &= (\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta) \Big\{ [h_r(r, r^2) + 2rh_s(r, r^2)](\mathbf{Q} + r\dot{\mathbf{Q}}) \\ &\quad + h(r, r^2) \left[(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}} \right] + rg(r, r^2)(n + r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q} \Big\} \theta, \end{aligned} \tag{4.3.2}$$

or equivalently and upon rearranging terms

$$\begin{aligned} \mathscr{L}[u; \mathsf{A}, \mathsf{B}] = & [h_r(r, r^2) + 2rh_s(r, r^2) + nrg(r, r^2)]\theta \\ & + [r^2 h_r(r, r^2) + 2r^3 h_s(r, r^2) + (n+1)rh(r, r^2) + r^3 g(r, r^2)]|\dot{\mathbf{Q}}\theta|^2\theta \\ & + [(n+1)h(r, r^2)\mathbf{Q}^t\dot{\mathbf{Q}} + rh(r, r^2)\mathbf{Q}^t\ddot{\mathbf{Q}} + r^2 h(r, r^2)\langle \dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta \rangle \mathbf{I}_n \\ & + r[h_r(r, r^2) + 2rh_s(r, r^2)]\mathbf{Q}^t\dot{\mathbf{Q}}]\theta. \end{aligned} \tag{4.3.3}$$

It is seen without difficulty that the action $\mathscr{L}[u; \mathsf{A}, \mathsf{B}]$ above can be given an alternative and more suggestive reformulation

$$\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \mathsf{F}(r, \theta)\theta + \frac{1}{r^n}\frac{d}{dr}\left[r^{n+1} h(r, r^2)\mathbf{Q}^t\dot{\mathbf{Q}} \right]\theta - rh(r, r^2)\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta, \tag{4.3.4}$$

where $\mathsf{F} = \mathsf{F}(r, \theta)$ is the scalar-valued function defined by

$$\begin{aligned} \mathsf{F}(r, \theta) = & h_r(r, r^2) + 2rh_s(r, r^2) + nrg(r, r^2) + r^2 h(r, r^2)\langle \dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta \rangle \\ & + \left[r^2 h_r(r, r^2) + 2r^3 h_s(r, r^2) + (n+1)rh(r, r^2) + r^3 g(r, r^2) \right] |\dot{\mathbf{Q}}\theta|^2. \end{aligned} \tag{4.3.5}$$

Now starting from $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$ upon tensorisation and integration over the sphere we obtain for $a < r < b$ (*c.f.* Proposition 5.2.3)

$$\int_{\mathbb{S}^{n-1}} \mathscr{L}[u; \mathsf{A}, \mathsf{B}] \otimes \theta - \theta \otimes \mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \int_{\mathbb{S}^{n-1}} \nabla \mathscr{P} \otimes \theta - \theta \otimes \nabla \mathscr{P} = 0. \tag{4.3.6}$$

[7] Note that $g(r, s) = -h_s(r, s)$ corresponds to the variational case $F(r, s, \xi) = h(r, s)\xi$ where $\mathsf{A} = F_\xi = h(r, s)$ and $\mathsf{B} = -F_s = g(r, s)\xi$. However we do not make such assumption here.

Now upon substituting from (4.3.4), noting $\mathsf{F}\theta \otimes \theta - \theta \otimes \mathsf{F}\theta \equiv 0$ [*cf.* (4.3.5)] together with the fact that $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta$ integrates to zero over the sphere because of the symmetry of the matrix $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}$, it follows at once that for $a < r < b$,

$$\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\left\{r^{n+1}h(r,r^2)\left[\mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \mathbf{Q}^t\dot{\mathbf{Q}}\theta\right]\right\} d\mathcal{H}^{n-1}(\theta) = 0. \qquad (4.3.7)$$

Therefore by virtue of the skew-symmetry of $\mathbf{Q}^t\dot{\mathbf{Q}}$, after evaluating the above integral, we arrive at

$$\int_{\mathbb{S}^{n-1}} \mathscr{L}[u; \mathsf{A}, \mathsf{B}] \otimes \theta - \theta \otimes \mathscr{L}[u; \mathsf{A}, \mathsf{B}] = 2\frac{\omega_n}{r^n}\frac{d}{dr}\left[r^{n+1}h(r,r^2)\mathbf{Q}^t\dot{\mathbf{Q}}\right] = 0. \qquad (4.3.8)$$

Thus, summarising, we have shown that if $\mathscr{L}[u = r\mathbf{Q}(r)\theta; \mathsf{A}, \mathsf{B}] = \nabla\mathscr{P}$ then the twist path $\mathbf{Q} = \mathbf{Q}(r)$ must satisfy the ODE on the right in (4.3.8), which is the counterpart of (4.2.9) in this context. By an easy inspection this ODE is now seen to be completely integrable and with the choice of boundary conditions $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ to have the explicit solutions

$$\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}, \qquad a \le r \le b, \qquad (4.3.9)$$

where the profile function $\mathscr{H}$ is given by

$$\mathscr{H}(r) = \frac{\mathsf{H}(r)}{\mathsf{H}(b)}, \qquad \mathsf{H}(r) = \int_a^r \frac{ds}{s^{n+1}h(s,s^2)}. \qquad (4.3.10)$$

Moreover the skew-symmetric matrix $\mathbf{H}$ is given in block diagonalised form as

$$\mathbf{H} = \begin{cases} \mathbf{P}\,diag(2m_1\pi\mathbf{J}, \dots, 2m_k\pi\mathbf{J})\mathbf{P}^t, & n = 2k, \\ \mathbf{P}\,diag(2m_1\pi\mathbf{J}, \dots, 2m_{k-1}\pi\mathbf{J}, m_k)\mathbf{P}^t, & n = 2k-1. \end{cases} \qquad (4.3.11)$$

Here $m_1, \dots, m_k \in \mathbb{Z}$ with $m_k = 0$ when $n = 2k-1$, $\mathbf{P} \in \mathbf{O}(n)$ and $\mathbf{J}$ is as in (4.2.2). Now taking $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ with $\dot{\mathbf{Q}} = \dot{\mathscr{H}}\mathbf{H}\mathbf{Q}$, $\ddot{\mathbf{Q}} = \ddot{\mathscr{H}}\mathbf{H}\mathbf{Q} + \dot{\mathscr{H}}^2\mathbf{H}^2\mathbf{Q}$ and $\langle\dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle = \dot{\mathscr{H}}\ddot{\mathscr{H}}|\mathbf{H}\mathbf{Q}\theta|^2 + \dot{\mathscr{H}}^3\langle\mathbf{H}\mathbf{Q}\theta, \mathbf{H}^2\mathbf{Q}\theta\rangle = \dot{\mathscr{H}}\ddot{\mathscr{H}}|\mathbf{H}\theta|^2$ where the last equality uses the skew-symmetry of $\mathbf{H}$ and the fact that $\mathbf{H}$ and $\mathbf{Q}$ commute the formulation of the action of $\mathscr{L}$ on u reduces to

$$\begin{aligned} \mathscr{L}[u; \mathsf{A}, \mathsf{B}] = {} & [h_r(r,r^2) + 2rh_s(r,r^2) + nrg(r,r^2)]\theta \\ & + r^3g(r,r^2)\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta + rh(r,r^2)\dot{\mathscr{H}}^2\mathbf{H}^2\theta, \end{aligned} \qquad (4.3.12)$$

where in the above we have reduced terms based on the assumption that $\mathbf{Q} = \exp\{\mathscr{H}(r)\mathbf{H}\}$ solves the ODE (4.3.8). We are now in a position to apply Proposition D.0.1 [*see* Appendix D] to the vector field

$$U(x) = \mathscr{L}[u;\mathsf{A},\mathsf{B}] - \nabla[h(|x|,|x|^2) + n\mathsf{g}(|x|)] = \mathscr{A}(r)|\mathbf{H}x|^2x + \mathscr{B}(r)\mathbf{H}^2x, \tag{4.3.13}$$

where $\mathsf{g} = \mathsf{g}(r)$ is a primitive of $rg(r,r^2)$ and

$$\mathscr{A}(r) = g(r,r^2)\dot{\mathscr{H}}^2, \qquad \mathscr{B}(r) = h(r,r^2)\dot{\mathscr{H}}^2. \tag{4.3.14}$$

Indeed by writing $\Delta(h,g) = -(2\mathscr{A} + \dot{\mathscr{B}}/r)/\dot{\mathscr{H}}^2$ we have that if $\Delta \equiv 0$ then U is a gradient field and if $\Delta \not\equiv 0$ then U is a gradient field *if and only if*, referring to (4.3.11), $|m_1| = \cdots = |m_k|$. Note that by a basic calculation and making use of the ODE (4.3.8) the discriminant Δ here can be more explicitly described as[8]

$$\Delta(h,g) = \frac{2(n+1)h + rh_r + 2r^2(h_s - g)}{r^2}. \tag{4.3.15}$$

We have therefore proved the following theorem that captures a contrast in the behaviour of the system and its twisting solutions.

Theorem 4.3.1. *Let $\mathbf{Q} \in \mathscr{C}^2(]a,b[,\mathbf{SO}(n)) \cap \mathscr{C}([a,b],\mathbf{SO}(n))$ be a twist loop based at $\mathbf{I}_n$ and consider the differential operator $\mathscr{L} = \mathscr{L}[u;\mathsf{A},\mathsf{B}]$ with A,B as above and $\Delta(h,g)$ as in (4.3.15). Then $\mathscr{L}[u = r\mathbf{Q}(r)\theta;\mathsf{A},\mathsf{B}] = \nabla\mathscr{P}$ if and only if $\mathbf{Q}$ is as described below.*

1. $\Delta(h,g) \not\equiv 0$*: Here depending on n being even or odd we have*
   - *(i) n even. $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ $(a \le r \le b)$ with $\mathbf{H} = 2m\pi\mathbf{P}\mathbf{J}_n\mathbf{P}^t$ where $\mathbf{P} \in \mathbf{O}(n)$, $m \in \mathbb{Z}$ and $\mathbf{J}_n = diag(\mathbf{J},\dots,\mathbf{J})$.*
   - *(ii) n odd. $\mathbf{H} \equiv 0$ and thus $\mathbf{Q} \equiv \mathbf{I}_n$. Hence the identity map $u \equiv x$ is the only twisting solution to* (4.1.1).
2. $\Delta(h,g) \equiv 0$*: Here $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ $(a \le r \le b)$ with $\mathbf{H}$ as in* (4.3.11) *and with no further restrictions on $m_1,\dots,m_k$.*

[8] Compare this with the similar and illustrative examples (5.5.5) and (6.4.14).

Chapter 5

A Lagrangian Discriminant on Critical Loops Associated with $\mathbf{curl}\, \mathscr{L}[u] = 0$

The aim of this chapter is to comprehensively solve the nonlinear elliptic system in variation

$$\begin{cases} \mathrm{div}[F_\xi \nabla u] - F_s u = [\mathrm{cof}\, \nabla u] \nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u \equiv \varphi & \text{on } \partial\Omega, \end{cases}$$

where $F_\xi = F_\xi(|x|, |u|^2, |\nabla u|^2)$, $F_s = F_s(|x|, |u|^2, |\nabla u|^2)$ and $F = F(r, s, \xi)$ is a twice continuously differentiable Lagrangian. Here $\mathscr{P} = \mathscr{P}(x)$ is a hydrostatic pressure field associated with the incompressibility constraint $\det \nabla u = 1$ and φ is a prescribed boundary map. In the geometric setting where the domain is a bounded symmetric annulus, we connect the system to a class of isotropic ODEs on the Lie group $\mathbf{SO}(n)$ and establish the existence of an infinite scale of topologically distinct geodesic type solutions to these ODEs in the form $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$, with suitable profile $\mathscr{G} = \mathscr{G}(r)$ and $\mathbf{H} \in \mathfrak{so}(n)$. Passing to the full system next, a Lagrangian discriminant capturing the irrotationality of the vector field $\mathscr{L}[u] = (\nabla u)^t \{\mathrm{div}[F_\xi \nabla u] - F_s u\}$ is introduced and exploited. A set of contrasting behaviours of the system and its solutions are then singled out and discussed by a detailed analysis of the associated discriminants.

5.1 Preliminaries and Outline

For $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ a bounded domain with a sufficiently smooth boundary consider the variational energy functional

$$\mathbb{F}[u;\Omega] = \int_\Omega \mathscr{W}(x, u, \nabla u)\, dx \tag{5.1.1}$$

where $\mathscr{W} = \mathscr{W}(x, u, \mathbf{F})$ is a twice continuously differentiable Lagrangian and u is taken in the admissible space of incompressible Sobolev maps (with suitable $p \geq 1$)

$$\mathscr{A}^p_\varphi(\Omega) := \left\{ u \in W^{1,p}(\Omega, \mathbb{R}^n) : \det\nabla u = 1,\ u \equiv \varphi \text{ on } \partial\Omega \right\}. \tag{5.1.2}$$

In the above formulation $\varphi \in \mathscr{C}^1(\partial\Omega, \mathbb{R}^n)$ is a fixed boundary map and the incompressibility constraint in (5.1.2) is assumed to hold pointwise, that is, *a.e.* in Ω. The Euler-Lagrange equation associated with (5.1.1)-(5.1.2) can be obtained formally by using the so-called Lagrange multiplier method and is given by the nonlinear system (*see*, e.g., [3, 14, 24, 65] for more)

$$\begin{cases} \operatorname{div}[\mathscr{W}_\mathbf{F}(x, u, \nabla u)] - \mathscr{W}_u(x, u, \nabla u) = [\operatorname{cof} \nabla u]\nabla \mathscr{P} & \text{in } \Omega, \\ \det\nabla u = 1 & \text{in } \Omega, \\ u \equiv \varphi & \text{on } \partial\Omega. \end{cases} \tag{5.1.3}$$

Here $\mathscr{P} = \mathscr{P}(x)$ is an *a priori* unknown Lagrange multiplier, often called the hydrostatic pressure field, associated with the incompressibility constraint on u and the divergence operator acts on the matrix field $\mathscr{W}_\mathbf{F}$ row-wise. Moreover cof denotes the usual cofactor matrix that, thanks to the incompressibility constraint, we have $[\operatorname{cof} \nabla u]^{-1} = (\nabla u)^t$ while the boundary condition $u \equiv \varphi$ on $\partial\Omega$ in (5.1.3) is understood in the sense of traces.

Motivated by consideration of rotational symmetry in solutions to the system (5.1.3), in this paper we specialise entirely to *isotropic* Lagrangians of the form $\mathscr{W}(x, u, \mathbf{F}) = F(r, s, \xi)/2$ with $(r, s, \xi) = (|x|, |u|^2, |\mathbf{F}|^2)$ and subject to suitable convexity and monotonicity assumptions on F in the ξ variable. Here (5.1.3) can be written as

$$\begin{cases} \mathscr{L}[u] = \nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u \equiv \varphi & \text{on } \partial\Omega, \end{cases} \tag{5.1.4}$$

with $\mathscr{L}$ denoting the second-order differential operator given explicitly by

$$\begin{aligned} \mathscr{L}[u] :=& (\nabla u)^t \{ \mathrm{div}[F_\xi(|x|, |u|^2, |\nabla u|^2)\nabla u] - F_s(|x|, |u|^2, |\nabla u|^2) u \} \\ =& \nabla F_\xi(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t \nabla u + F_\xi(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t \Delta u \\ &- F_s(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t u. \end{aligned} \tag{5.1.5}$$

For the sake of clarity note that by a (*classical*) solution we hereafter mean a pair $(u, \mathscr{P})$ with u of class $\mathscr{C}^2(\Omega, \mathbb{R}^n) \cap \mathscr{C}(\overline{\Omega}, \mathbb{R}^n)$ and $\mathscr{P}$ of class $\mathscr{C}^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$ such that (5.1.4) holds in a pointwise sense in Ω.[9] Now proceeding forward and arguing either formally and in a distributional sense, or classically, upon assuming further differentiability on $\mathscr{L}$, it is seen from (5.1.4)-(5.1.5) that $\mathrm{curl}\,\mathscr{L}[u] = \mathrm{curl}\,\nabla \mathscr{P} \equiv 0$ in Ω, that is

$$\begin{aligned} \mathrm{curl} \Big\{ & \nabla F_\xi(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t \nabla u + F_\xi(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t \Delta u \\ & - F_s(|x|, |u|^2, |\nabla u|^2)(\nabla u)^t u \Big\} \equiv 0. \end{aligned} \tag{5.1.6}$$

However, unless Ω has a particular homology, this is clearly not a sufficient condition for $\mathscr{L}[u]$ to be a gradient field, here specifically $\nabla \mathscr{P}$.

Throughout the chapter we specialise to the geometric set up where $\Omega = \mathbb{X}^n = \mathbb{X}^n[a, b] := \{x \in \mathbb{R}^n : a < |x| < b\}$ is a bounded symmetric annulus with $b > a > 0$, and $\varphi \equiv x$, *i.e.*, the identity map. In this context by a generalised twist (or *twist* for brevity) we understand a map $u \in \mathscr{C}(\overline{\mathbb{X}^n}, \overline{\mathbb{X}^n})$ that admits the representation

$$u : (r, \theta) \mapsto (r, \mathbf{Q}(r)\theta), \qquad r = |x|, \theta = x|x|^{-1}, \quad x \in \overline{\mathbb{X}^n}. \tag{5.1.7}$$

The curve $\mathbf{Q} \in \mathscr{C}([a, b], \mathbf{SO}(n))$ here is called the twist path associated with u. Moreover in order to ensure $u \equiv x$ on $\partial\Omega = \partial\mathbb{X}^n$ we set $\mathbf{Q}(a) = \mathbf{Q}(b) =$

[9] The pair $(u^\star, \mathscr{P}^\star)$ with $u^\star \equiv x$ the identity map and $\mathscr{P}^\star$ as below is a solution to (5.1.4). As a matter of fact from $\nabla u^\star = \mathbf{I}_n$ we have $\mathscr{L}[u^\star] = \mathrm{div}[F_\xi(|x|, |x|^2, n)\mathbf{I}_n] - F_s(|x|, |x|^2, n)x = \nabla F_\xi(|x|, |x|^2, n) - F_s(|x|, |x|^2, n)x$ and so $\mathscr{L}[u^\star] = \nabla \mathscr{P}^\star$ with $\mathscr{P}^\star = F_\xi(|x|, |x|^2, n) - G(|x|)$ where $G = G(r)$ is a primitive of $g(r) = rF_s(r, r^2, n)$.

$\mathbf{I}_n$. In this event the twist path forms a closed curve in $\mathbf{SO}(n)$, based at $\mathbf{I}_n$, called the twist *loop*, that in turn represents an element of the fundamental group $\pi_1(\mathbf{SO}(n)) \cong \mathbb{Z}_2$ $(n \geq 3)$ and $\mathbb{Z}$ $(n = 2)$. Our aim is to establish the existence of an infinitude of twisting solutions to the nonlinear system (5.1.4) by appropriately formulating the action of $\mathscr{L}$ on sufficiently regular twists u and solving the resulting PDE.

The first major thread of this chapter comprising Sections 5.2-5.3 focuses on a study of three interrelated ODEs considered over the Lie group $\mathbf{SO}(n)$ and formulated for twist loops $\mathbf{Q} = \mathbf{Q}(r)$ in the space

$$\mathscr{B}^p_{\mathbf{I}_n}(a,b) := \left\{\mathbf{Q} \in W^{1,p}(a,b;\mathbf{SO}(n)) : \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n\right\}. \tag{5.1.8}$$

The first ODE in the list can be seen to arise as the Euler-Lagrange equation associated to a restricted energy functional (*c.f.* Remark 5.2.6) and is given on the interval $a < r < b$ by

$$\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\left[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta\right]\right\} d\mathcal{H}^{n-1}(\theta) = 0. \tag{5.1.9}$$

This ODE can also be extracted directly from the PDE $\mathscr{L}[u] = \nabla\mathscr{P}$ as shown in Proposition 5.2.4 and as such serves as a necessary condition on the twist path for any twist solution to the system (5.1.4). By discarding the spherical integral we will also consider the strengthened, *pointwise*, equation

$$\frac{d}{dr}\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\left[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta\right]\right\} = 0, \qquad a < r < b. \tag{5.1.10}$$

Naturally any solution to (5.1.10) is by default a solution to (5.1.9) but not *vice versa* due to the strengthening of the integral constraint in (5.1.9) to a pointwise one in (5.1.10) (*see, e.g.*, Theorem 5.3.1). The third and final equation of interest (for which we observe close links to the previous two and the system (5.1.4)) comprises

$$\frac{d}{dr}\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}^t\dot{\mathbf{Q}}\right\} = 0, \qquad a < r < b. \tag{5.1.11}$$

This ODE and its solutions will play a central role in constructing twist solutions to the system (5.1.4) as well as the analysis of irrotationality of the vector field $\mathscr{L}[u]$. Indeed for any twist u whose twist path $\mathbf{Q}$ is a solution to the ODE (5.1.11) we have curl$\mathscr{L}[u] = \Delta_F[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta]$, where Δ_F –

hereafter called the Lagrangian discriminant associated with the Lagrangian F (*c.f.* Theorem 5.4.2) – is given explicitly by

$$\Delta_F = \frac{2[(n+1)F_\xi + 2(\xi-n)F_{\xi\xi} + 2r^2F_{s\xi}][F_\xi + (\xi-n)F_{\xi\xi}] + rF_\xi F_{r\xi}}{r^2[F_\xi + 2F_{\xi\xi}(\xi-n)]}. \tag{5.1.12}$$

Our analysis shows that the vanishing *vs.* non-vanishing of this discriminant along a solution $\mathbf{Q}=\mathbf{Q}(r)$ to (5.1.11) has interesting and grave implications on the structure and form of the resulting twist $u=r\mathbf{Q}(r)\theta$ being a solution to (5.1.4). In Section 5.5 we take up this specific task and look at how the explicit structure of the Lagrangian F can affect the vanishing or non-vanishing of Δ_F. A remarkable feature here is that if F has no *joint* (r,s) and ξ dependence then, subject to the mere monotonicity and convexity assumptions on F, the discriminant is always *strictly* positive and hence nowhere vanishing. In contrast, in the simplest case of *joint* (r,s) and ξ dependence examples will be given to show that the discriminant can completely vanish and hence a totally new set of geodesic type solutions to (5.1.11) will emerge as twist solutions to the system (5.1.4).

For the sake of future reference we assume that $F\in\mathscr{C}^2(U)$ where $U=[a,b]\times]0,\infty[\times]0,\infty[\subset\mathbb{R}^3$ and that $F_\xi>0$, $F_{\xi\xi}\geq 0$. Moreover we assume that the twice continuously differentiable function $\zeta\mapsto F(r,r^2,n+r^2\zeta^2)$ is uniformly convex in ζ for all $a\leq r\leq b$ and $\zeta\in\mathbb{R}$. Regarding bounds and coercivity we assume that F is bounded from below: $F(r,s,\xi)\geq c_0$ for some $c_0\in\mathbb{R}$ and that for all $(r,s,\xi)\in U$ and every compact $K\subset]0,\infty[$ there exist constants $c_1,c_2>0$ depending on K such that for some $p>1$:

$$|F_\xi(r,s,\zeta^2)\zeta| \leq c_2|\zeta|^{p-1} \qquad \forall\,(r,s,\zeta^2)\in U:\ s\in K, \tag{5.1.13}$$

$$c_0 + c_1|\zeta|^p \leq F(r,s,\zeta^2) \leq c_2|\zeta|^p \qquad \forall\,(r,s,\zeta^2)\in U:\ s\in K. \tag{5.1.14}$$

Let us finish off this introduction with a brief description of the plan of the chapter. In Section 5.2 we focus on the three ODEs listed above and investigate their relationship to one another and to the PDE $\mathscr{L}[u]=\nabla\mathscr{P}$. In Section 5.3 we take a closer look at Lagrangians of the weighted Dirichlet type $F(r,s,\xi)=h(r,s)\xi$ and readdress the inter-relationship between these three ODEs in that context. Interestingly, a complete and explicit representation of all solutions as well as an exact relationship between the ODEs can be given here (*c.f.* Theorem 5.3.1). The highlight of Section 5.4 is the computation of the curl of the vector field $\mathscr{L}[u]$ which then leads to the formulation (5.1.12) of the Lagrangian discriminant Δ_F. This notion and its applications are then exploited further in Section 5.5 in the context of Lagrangian form and structure. The chapter ends by a return to the system (5.1.4) and a complete classification of its multiple twisting solutions.

5.2 The Action $\mathscr{L}[u]$ on Generalised Twists and Interrelation of Differential Operators

This section is principally concerned with a study of the differential operator $\mathscr{L}[u]$ defined by (5.1.5) and its relation to the ODEs (5.1.9)-(5.1.11). The first result here gives an explicit representation of the action $\mathscr{L}[u]$ when u is a sufficiently regular generalised twist.

Proposition 5.2.1. *Let* $u = r\mathbf{Q}(r)\theta$ *with* $\mathbf{Q} \in \mathscr{C}^2(]a,b[,\mathbf{SO}(n)) \cap \mathscr{C}([a,b],\mathbf{SO}(n))$. *Then the action of the differential operator* $\mathscr{L}$ *on* u *can be described as*

$$\begin{aligned}\mathscr{L}[u] =& \nabla F_\xi(|x|,|x|^2,n+|\dot{\mathbf{Q}}x|^2) + \frac{1}{r^n}\frac{d}{dr}\left[r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}^t\dot{\mathbf{Q}}\theta\right] \\ &- rF_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta + \mathscr{A}(r,\theta)\theta, \end{aligned} \tag{5.2.1}$$

where $\mathscr{A} = \mathscr{A}(r,\theta)$ *is the scalar-valued function given by*

$$\begin{aligned}\mathscr{A}(r,\theta) =& r^2F_{\xi\xi}(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|^2(2r|\dot{\mathbf{Q}}\theta|^2\theta + r^2\nabla|\dot{\mathbf{Q}}\theta|^2) \\ &+ 2r^3F_{s\xi}(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|^2 + r^2F_{r\xi}(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|^2 \\ &+ rF_\xi[(n+1)|\dot{\mathbf{Q}}\theta|^2 + r\langle\dot{\mathbf{Q}}\theta,\ddot{\mathbf{Q}}\theta\rangle] - rF_s(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2) \end{aligned} \tag{5.2.2}$$

and $\dot{\mathbf{Q}} = d\mathbf{Q}(r)/dr$, $\ddot{\mathbf{Q}} = d^2\mathbf{Q}(r)/dr^2$.

Proof. First, the identity (A.0.7) describes precisely the action $\mathscr{L}$ on a generalised twist as in the statement of the theorem as

$$\begin{aligned}\mathscr{L}[u] =& (\mathbf{Q}^t + r\theta\otimes\dot{\mathbf{Q}}\theta)\Big\{F_{\xi\xi}(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)(\mathbf{Q}+r\dot{\mathbf{Q}}\theta\otimes\theta)\times \\ &\times(2r|\dot{\mathbf{Q}}\theta|^2\theta + r^2\nabla[|\dot{\mathbf{Q}}\theta|^2]) \\ &+ [2rF_{s\xi}(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2) + F_{r\xi}(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)](\mathbf{Q}+r\dot{\mathbf{Q}}\theta\otimes\theta)\theta \\ &+ F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)[(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}]\theta - rF_s(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}\theta\Big\}.\end{aligned}$$

Upon multiplying through $(\nabla u)^t = \mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta$ and making the appropriate rearrangements the identity (5.2.1) follows at once. $\square$

We remark that the fact that a generalised twist $u = r\mathbf{Q}(r)\theta$ as in the statement of the theorem above satisfies the incompressibility constraint $\det \nabla u = 1$ is established in Proposition A.0.4. Before proceeding further, we briefly digress to give a result which will be used in various forms in the chapter. Towards this end let us introduce the notation $\mathsf{S}_{\mathbf{F}}[\theta]$ for the tensor product

$$\mathsf{S}_{\mathbf{F}}[\theta] := \mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta, \tag{5.2.3}$$

where $\mathbf{F}$ is a fixed $n \times n$ matrix and $\theta \in \mathbb{S}^{n-1}$.

Lemma 5.2.2. $\mathsf{S}_{\mathbf{F}}[\theta] \equiv 0$ *for all* $\theta \in \mathbb{S}^{n-1}$ *iff* $\mathbf{F} = \mathsf{f}\mathbf{I}_n$ *for some* $\mathsf{f} \in \mathbb{R}$.

Thus the tensor $\mathbf{F}\theta \otimes \theta$ is skew-symmetric for all unit vectors θ *iff* the matrix $\mathbf{F}$ is a multiple of the identity matrix. Now before giving the proof of the lemma, it is instructive to note that upon integrating $\mathsf{S}_{\mathbf{F}}[\theta]$ over the sphere, we have

$$\int_{\mathbb{S}^{n-1}} \mathsf{S}_{\mathbf{F}}[\theta]\, d\mathcal{H}^{n-1}(\theta) = \omega_n(\mathbf{F} - \mathbf{F}^t), \tag{5.2.4}$$

where ω_n is the volume of the unit n-ball. Thus in contrast to the more stringent conclusion in Lemma 5.2.2, the spherical *integral* of $\mathsf{S}_{\mathbf{F}}[\theta]$ in (5.2.4) only sees the skew-symmetric part of $\mathbf{F}$, and in particular vanishes *iff* $\mathbf{F}$ is symmetric.

Proof. From $\mathsf{S}_{\mathbf{F}}[\theta] \equiv 0$ for $\theta \in \{e_1, ..., e_n\}$ – the standard basis of $\mathbb{R}^n$ – it follows that $\mathbf{F}$ is diagonal, that is, $\mathbf{F} = diag(\mathsf{f}_1, \dots, \mathsf{f}_n)$ for $(\mathsf{f}_i)_{i=1}^n \subset \mathbb{R}$. Hence the condition $\mathsf{S}_{\mathbf{F}}[\theta] \equiv 0$ reduces to $\theta_i\theta_j(\mathsf{f}_i - \mathsf{f}_j) \equiv 0$ for all $1 \le i, j \le n$. As such $\mathsf{f}_1 = \cdots = \mathsf{f}_n =: \mathsf{f}$ for some $\mathsf{f} \in \mathbb{R}$. Conversely and trivially if $\mathbf{F} = \mathsf{f}\mathbf{I}_n$ then $\mathsf{S}_{\mathbf{F}}[\theta] = \mathsf{f}[\theta \otimes \theta - \theta \otimes \theta] \equiv 0$ for all $\theta \in \mathbb{S}^{n-1}$. $\square$

If we introduce the notation $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$, then we can write the action $\mathscr{L}[u]$ for a twist u, as

$$\begin{aligned}\mathscr{L}[u] =& \nabla F_\xi(|x|, |x|^2, n + |\mathbf{A}x|^2) + \frac{1}{r^n}\frac{d}{dr}\left[r^{n+1}F_\xi(r, r^2, n + r^2|\mathbf{A}\theta|^2)\mathbf{A}\right]\theta \\ &+ rF_\xi(r, r^2, n + r^2|\mathbf{A}\theta|^2)\mathbf{A}^2\theta + \mathscr{A}(r,\theta)\theta.\end{aligned} \tag{5.2.5}$$

In this case $\mathscr{A}=\mathscr{A}(r,\theta)$ in turn can be reformulated as

$$\begin{aligned}\mathscr{A}(r,\theta)=&r^2F_{\xi\xi}(r,r^2,n+r^2|\mathbf{A}\theta|^2)|\mathbf{A}\theta|^2\nabla(r^2|\mathbf{A}\theta|^2)\\&+2r^3F_{s\xi}(r,r^2,n+r^2|\mathbf{A}\theta|^2)|\mathbf{A}\theta|^2+r^2F_{r\xi}(r,r^2,n+r^2|\mathbf{A}\theta|^2)|\mathbf{A}\theta|^2\\&+rF_\xi(r,r^2,n+r^2|\mathbf{A}\theta|^2)[(n+1)|\mathbf{A}\theta|^2+r\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle]\\&-rF_s(r,r^2,n+r^2|\mathbf{A}\theta|^2).\end{aligned}\tag{5.2.6}$$

This follows by noting the identities $\mathbf{A}^2=\mathbf{Q}^t\dot{\mathbf{Q}}\mathbf{Q}^t\dot{\mathbf{Q}}=-\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}$ and $\langle\dot{\mathbf{Q}}\theta,\ddot{\mathbf{Q}}\theta\rangle=\langle\mathbf{A}\theta,(\dot{\mathbf{A}}+\mathbf{A}^2)\theta\rangle=\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle$. Moving forward we now wish to emphasise and further study the resulting commutator-like relation (henceforth abbreviating the arguments of F and any of its derivatives)

$$\begin{aligned}\mathscr{L}[u]\otimes\theta-\theta\otimes\mathscr{L}[u]=&\nabla F_\xi\otimes\theta-\theta\otimes\nabla F_\xi+rF_\xi(\mathbf{A}^2\theta\otimes\theta-\theta\otimes\mathbf{A}^2\theta)\\&+\frac{1}{r^n}\frac{d}{dr}\left[r^{n+1}F_\xi\{\mathbf{A}\theta\otimes\theta-\theta\otimes\mathbf{A}\theta\}\right],\end{aligned}\tag{5.2.7}$$

which holds thanks to the pointwise identity $\mathscr{A}(r,\theta)\theta\otimes\theta-\theta\otimes\mathscr{A}(r,\theta)\theta=0$.

Proposition 5.2.3. *Let $\mathscr{P}\in\mathscr{C}^1(U)$ with $U\subset\mathbb{R}^n$ an open neighbourhood of the unit sphere $\mathbb{S}^{n-1}$. Then*

$$\int_{\mathbb{S}^{n-1}}[\nabla\mathscr{P}\otimes\theta-\theta\otimes\nabla\mathscr{P}]\,d\mathcal{H}^{n-1}(\theta)=0.\tag{5.2.8}$$

Proof. Firstly by restricting to the unit surface of the sphere and splitting the gradient into a normal and tangential part in the usual way, we can write

$$\nabla\mathscr{P}=(\mathbf{I}_n-\theta\otimes\theta)\nabla\mathscr{P}+\langle\nabla\mathscr{P},\theta\rangle\theta=\nabla_T\mathscr{P}+\nabla_N\mathscr{P}.\tag{5.2.9}$$

It is seen that $\nabla_N\mathscr{P}\otimes\theta-\theta\otimes\nabla_N\mathscr{P}=0$ and so to establish (5.2.8) it suffices to justify the integral identity

$$\int_{\mathbb{S}^{n-1}}[\nabla_T\mathscr{P}\otimes\theta-\theta\otimes\nabla_T\mathscr{P}]\,d\mathcal{H}^{n-1}(\theta)=0.\tag{5.2.10}$$

Now by a direct differentiation it is evident that $\nabla_T(\mathscr{P}\theta)=\theta\otimes\nabla_T\mathscr{P}+\mathscr{P}\nabla_T\theta$, and so referring to (5.2.10) we can write

$$\begin{aligned}\nabla_T\mathscr{P}\otimes\theta-\theta\otimes\nabla_T\mathscr{P}=&[\nabla_T(\mathscr{P}\theta)-\mathscr{P}\nabla_T\theta]^t-[\nabla_T(\mathscr{P}\theta)-\mathscr{P}\nabla_T\theta]\\=&[\nabla_T(\mathscr{P}\theta)]^t-[\nabla_T(\mathscr{P}\theta)].\end{aligned}\tag{5.2.11}$$

Here in deducing the second identity we have taken into account the symmetry $\nabla_T\theta=[\nabla_T\theta]^t=\mathbf{I}_n-\theta\otimes\theta$. The conclusion now follows by integrating (5.2.11) and invoking the divergence theorem on the sphere with $\partial\mathbb{S}^{n-1}=\{\emptyset\}$. □

Proposition 5.2.4. *If $\mathscr{L}[u]=\nabla\mathscr{P}$ holds for a generalised twist $u=r\mathbf{Q}(r)\theta$ with $\mathbf{Q}\in\mathscr{C}^2(]a,b[,\mathbf{SO}(n))\cap\mathscr{C}([a,b],\mathbf{SO}(n))$ then the twist path $\mathbf{Q}$ satisfies the ODE*

$$\frac{d}{dr}\left\{\int_{\mathbb{S}^{n-1}} r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]d\mathcal{H}^{n-1}(\theta)\right\}=0. \tag{5.2.12}$$

Proof. Using the integral identity in the previous proposition and assuming that u satisfies $\mathscr{L}[u]=\nabla\mathscr{P}$ we have, upon referring to (5.2.7) with $\mathbf{A}=\mathbf{Q}^t\dot{\mathbf{Q}}$,

$$\begin{aligned}
0&=\int_{\mathbb{S}^{n-1}}\{\nabla\mathscr{P}\otimes\theta-\theta\otimes\nabla\mathscr{P}\}\,d\mathcal{H}^{n-1}(\theta)\\
&=\int_{\mathbb{S}^{n-1}}\{\mathscr{L}[u]\otimes\theta-\theta\otimes\mathscr{L}[u]\}\,d\mathcal{H}^{n-1}(\theta)\\
&=\int_{\mathbb{S}^{n-1}}\Big\{rF_\xi\left[\mathbf{A}^2\theta\otimes\theta-\theta\otimes\mathbf{A}^2\theta\right]\\
&\qquad+\frac{1}{r^n}\frac{d}{dr}\left\{r^{n+1}F_\xi\left[\mathbf{A}\theta\otimes\theta-\theta\otimes\mathbf{A}\theta\right]\right\}\Big\}d\mathcal{H}^{n-1}(\theta),
\end{aligned}\tag{5.2.13}$$

where we have written $F_\xi=F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)$ for brevity. Now considering the second term in the last integral in (5.2.13) we can write

$$\begin{aligned}
\mathscr{I}&=\frac{d}{dr}\left\{r^{n+1}F_\xi\left[\mathbf{Q}^t\dot{\mathbf{Q}}\theta\otimes\theta-\theta\otimes\mathbf{Q}^t\dot{\mathbf{Q}}\theta\right]\right\}\\
&=\frac{d}{dr}\left\{r^{n+1}F_\xi\mathbf{Q}^t\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\mathbf{Q}\right\}\\
&=\mathbf{Q}^t\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}\mathbf{Q}\\
&\quad+\dot{\mathbf{Q}}^tr^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\mathbf{Q}+\mathbf{Q}^tr^{n+1}F_\xi\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\dot{\mathbf{Q}}\\
&=\mathscr{I}_1+\mathscr{I}_2+\mathscr{I}_3.
\end{aligned}\tag{5.2.14}$$

Now the sum of the two terms $\mathscr{I}_2$ and $\mathscr{I}_3$ in the last line in (5.2.14) is seen to simplify to:

$$\begin{aligned}
\mathscr{I}_2+\mathscr{I}_3&=\dot{\mathbf{Q}}^tr^{n+1}F_\xi\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\mathbf{Q}\\
&\quad+\mathbf{Q}^tr^{n+1}F_\xi\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\dot{\mathbf{Q}} \qquad (5.2.15)\\
&=r^{n+1}F_\xi\left\{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta-\dot{\mathbf{Q}}^t\mathbf{Q}\theta\otimes\mathbf{Q}^t\dot{\mathbf{Q}}\theta+\mathbf{Q}^t\dot{\mathbf{Q}}\theta\otimes\dot{\mathbf{Q}}^t\mathbf{Q}\theta-\theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right\}
\end{aligned}$$

where we have used the orthogonality of $\mathbf{Q}$. Now in view of $\dot{\mathbf{Q}}^t\mathbf{Q}$ being skew-symmetric, the middle two terms in (5.2.15) cancel and so returning to (5.2.14)

we have

$$\begin{aligned}\mathscr{I} = \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 &= \frac{d}{dr}\left\{r^{n+1}F_\xi\left[\mathbf{Q}^t\dot{\mathbf{Q}}\theta\otimes\theta - \theta\otimes\mathbf{Q}^t\dot{\mathbf{Q}}\theta\right]\right\}\\ &= \mathbf{Q}^t\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}\mathbf{Q}\\ &\quad + r^{n+1}F_\xi\left[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta - \theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right]. \end{aligned} \tag{5.2.16}$$

Now since by a direct calculation $\mathbf{A}^2 = -\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}$ upon referring to and substituting into (5.2.13) we have

$$\begin{aligned}\mathrm{RHS}(5.2.13) = &\int_{\mathbb{S}^{n-1}}\left\{-rF_\xi\left[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta - \theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right]\right.\\ &+\frac{1}{r^n}\mathbf{Q}^t\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}\mathbf{Q}\\ &\left.+rF_\xi\left[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta - \theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right]\right\}d\mathcal{H}^{n-1}(\theta)\\ =&\frac{1}{r^n}\mathbf{Q}^t\left\{\int_{\mathbb{S}^{n-1}}\frac{d}{dr}[r^{n+1}F_\xi\{\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\}]d\mathcal{H}^{n-1}(\theta)\right\}\mathbf{Q} = 0\end{aligned}$$

which is the required conclusion. □

Remark 5.2.5. Consider the second order tensor quantity $\mathscr{L}[u]\otimes\theta - \theta\otimes\mathscr{L}[u]$ associated with an arbitrary $u \in \mathscr{C}^2(\overline{\mathbb{X}^n}, \mathbb{R}^n)$. Then referring to the formulation (5.2.7) and the calculations (5.2.14)-(5.2.16) in the proof of Proposition 5.2.4 it is seen that for a twist $u = r\mathbf{Q}(r)\theta$ with a twice continuously differentiable twist path $\mathbf{Q}$ we have

$$\begin{aligned}\mathscr{L}[u]\otimes\theta - \theta\otimes\mathscr{L}[u] &= \nabla F_\xi\otimes\theta - \theta\otimes\nabla F_\xi \\ &\quad + \mathbf{Q}^t\frac{1}{r^n}\frac{d}{dr}\left\{r^{n+1}F_\xi\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\right\}\mathbf{Q}.\end{aligned} \tag{5.2.17}$$

Remark 5.2.6. Interestingly, the ODE (5.2.12) has a variational character resulting from restricting the energy functional $\mathbb{F}$ to the class of admissible twists. Indeed recalling (5.1.1) and the description of $|\nabla u|^2$ in Proposition A.0.3 we can write

$$\begin{aligned}\mathbb{F}[u = r\mathbf{Q}(r)\theta; \mathbb{X}^n] &= \int_{\mathbb{S}^{n-1}}\int_a^b F(r, r^2, n + r^2|\dot{\mathbf{Q}}\theta|^2)r^{n-1}\,dr d\mathcal{H}^{n-1}(\theta)\\ &= \int_a^b E(r, \dot{\mathbf{Q}})r^{n-1}\,dr =: \mathbb{E}[\mathbf{Q}; a, b],\end{aligned} \tag{5.2.18}$$

where the Lagrangian $E=E(r,\mathbf{X})$ with $a\le r\le b$ and $\mathbf{X}$ is a skew-symmetric $n\times n$ matrix is the spherical integral of $F(r,r^2,n+r^2|\mathbf{X}\theta|^2)$. The Euler-Lagrange equation associated with the restricted energy $\mathbb{E}$ over the space of $\mathbf{SO}(n)$-valued fixed end-point twist paths $\mathbf{Q}=\mathbf{Q}(r)$ is then easily seen to coincide with (5.2.12). This is also formally recovered in Corollary 2.2.12.

Proposition 5.2.7. *Assume* $\mathscr{L}[u]=\nabla\mathscr{P}$ *for a generalised twist* $u=r\mathbf{Q}(r)\theta$ *with* $\mathbf{Q}\in\mathscr{C}^2(]a,b[,\mathbf{SO}(n))\cap\mathscr{C}([a,b],\mathbf{SO}(n))$. *If the quantity* $|\dot{\mathbf{Q}}\theta|$ *depends only on the radial variable* r*, that is, is independent of the spherical variable* θ*, then* $\mathbf{Q}$ *satisfies* (5.1.11). *Indeed, under the latter condition,* (5.1.9) *and* (5.1.11) *are equivalent.*

Proof. First if $\mathscr{L}[u]=\nabla\mathscr{P}$ then by Proposition 5.2.4 the twist path $\mathbf{Q}$ satisfies (5.1.9). Now if $|\dot{\mathbf{Q}}\theta|$ is additionally independent of θ (*i.e.*, is a function of r alone) then starting with the integral on the left of (5.1.9) followed by an application of the divergence theorem we can write

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}\frac{d}{dr}&\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\right\}d\mathcal{H}^{n-1}(\theta)=\\ &=\frac{d}{dr}\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\int_{\mathbb{S}^{n-1}}\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\,d\mathcal{H}^{n-1}(\theta)\right\}\\ &=2\omega_n\frac{d}{dr}\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\dot{\mathbf{Q}}\mathbf{Q}^t\right\}=2\omega_n\mathbf{Q}\mathrm{LHS}(5.1.11)\mathbf{Q}^t,\end{aligned}\tag{5.2.19}$$

where we have used the fact that

$$\begin{aligned}\frac{d}{dr}&\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\dot{\mathbf{Q}}\mathbf{Q}^t\right\}=\\ &=\mathbf{Q}\frac{d}{dr}\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}^t\dot{\mathbf{Q}}\right\}\mathbf{Q}^t\end{aligned}\tag{5.2.20}$$

as in (3.2.18). As such it is immediately seen that under the stated independence condition the twist path $\mathbf{Q}=\mathbf{Q}(r)$ solves (5.1.9) *iff* it solves (5.1.11). $\square$

An instructive example is the geodesic twist path $\mathbf{Q}(r)=\exp\{\mathscr{G}(r)\mathbf{H}\}$ where $\mathscr{G}=\mathscr{G}(r)\in\mathscr{C}^2[a,b]$ and $\mathbf{H}$ is a suitable $n\times n$ skew-symmetric matrix. Indeed here a basic calculation gives $|\dot{\mathbf{Q}}\theta|=|\dot{\mathscr{G}}\mathbf{H}\theta|$ and therefore when n is even, upon taking $\mathbf{H}=\mathbf{P}diag(\mathbf{J},\dots,\mathbf{J})\mathbf{P}^t$ with $\mathbf{P}\in\mathbf{O}(n)$ and

$$\mathbf{J}=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad\mathcal{R}[\zeta]=\exp\{\zeta\mathbf{J}\}=\begin{pmatrix}\cos\zeta&-\sin\zeta\\ \sin\zeta&\cos\zeta\end{pmatrix},\tag{5.2.21}$$

we have $\mathbf{H}^2 = -\mathbf{I}_n$ and so as a result $|\dot{\mathbf{Q}}\theta| = |\dot{\mathscr{G}}|$ is independent of θ. When n is odd by contrast there is no skew-symmetric matrix $\mathbf{H}$ satisfying $\mathbf{H}^2 = -\mathbf{I}_n$ (due to the presence of at least one zero eigenvalue for $\mathbf{H}$) and as such here $|\mathbf{H}\theta|$ is independent of θ *iff* $\mathbf{H} = 0$ and so $\mathbf{Q} \equiv \mathbf{I}_n$.

Discussing further the relationship between the ODEs (5.1.9) and (5.1.11), for a general Lagrangian F and for a twice continuously differentiable twist path $\mathbf{Q} = \mathbf{Q}(r)$ (with no assumption on $|\dot{\mathbf{Q}}\theta|$), it can be seen that (5.1.11) $\implies$ (5.1.9) as follows. Indeed upon writing $\mathscr{M}$ for the operator

$$\mathscr{M}[\mathbf{Q}] := \frac{d}{dr}\left[r^{n+1}F_\xi(r, r^2, n + r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}^t\dot{\mathbf{Q}}\right] \tag{5.2.22}$$

it follows from (5.1.11) that $\mathscr{M}[\mathbf{Q}]\theta \otimes \theta - \theta \otimes \mathscr{M}[\mathbf{Q}]\theta = 0$ for all $a < r < b$ and $|\theta| = 1$. Noting that the tensor quantity on the left here is exactly the expression on the right of the first line in (5.2.16), by integrating over the unit sphere we can write,

$$\begin{aligned}
0 &= \int_{\mathbb{S}^{n-1}} (\mathscr{M}[\mathbf{Q}]\theta \otimes \theta - \theta \otimes \mathscr{M}[\mathbf{Q}]\theta)\, d\mathcal{H}^{n-1}(\theta) \\
&= \mathbf{Q}^t\left[\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\left\{r^{n+1}F_\xi\left[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta\right]\right\} d\mathcal{H}^{n-1}(\theta)\right]\mathbf{Q} \\
&\quad + r^{n+1}\int_{\mathbb{S}^{n-1}} F_\xi\left[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right] d\mathcal{H}^{n-1}(\theta).
\end{aligned} \tag{5.2.23}$$

Now as $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}$ is a symmetric matrix field on (a, b) the second integral on the right here vanishes by (5.2.4) and so $\mathbf{Q}$ satisfies (5.1.9) (see also Section 5.3 and Theorem 5.3.1 for related results on a particular class of Lagrangians).

Moving forward and in line with Proposition 5.2.7 we next give a result on the equivalence of the ODEs (5.1.10) and (5.1.11) by introducing the L-norm associated with a differentiable twist path $\mathbf{Q} = \mathbf{Q}(r)$ (with $a \le r \le b$) by setting

$$\mathsf{L}(\mathbf{Q}, \theta) = \|\dot{\mathbf{Q}}\theta\|_{L^1(a,b)} = \int_a^b \sqrt{\langle\dot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\rangle}\, dr. \tag{5.2.24}$$

Theorem 5.2.8. *Let $\mathbf{Q} \in \mathscr{C}^2(]a, b[, \mathbf{SO}(n)) \cap \mathscr{C}^1([a, b], \mathbf{SO}(n))$ satisfy the end-point conditions $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ and suppose that the L-norm (5.2.24) is independent of θ. The following are equivalent:*

(*i*) $\mathbf{Q}$ *solves* (5.1.10),

(*ii*) $\mathbf{Q}$ *solves* (5.1.11),

(*iii*) (*Dimensional dichotomy*) *Depending on n being even or odd* $\mathbf{Q}$ *admits the factorisation:*

- $(n=2k)$ *For* $m \in \mathbb{Z}$ *and* $\mathbf{P} \in \mathbf{O}(n)$

$$\begin{aligned}\mathbf{Q} = \mathbf{Q}(r;m) &= \exp\{\mathscr{G}(r;m)\mathbf{P}\mathbf{J}_k\mathbf{P}^t\} \\ &= \mathbf{P} diag(\mathcal{R}[\mathscr{G}](r;m),\dots,\mathcal{R}[\mathscr{G}](r;m))\mathbf{P}^t,\end{aligned} \tag{5.2.25}$$

where $\mathbf{J}_k = diag(\mathbf{J},\dots,\mathbf{J})$ *with* $\mathbf{J}$ *and* $\mathcal{R}$ *as in* (5.2.21) *and* $\mathscr{G} \in \mathscr{C}^2[a,b]$ *the unique solution of the boundary value problem*

$$\begin{cases} \dfrac{d}{dr}\left[r^{n+1}F_\xi\left(r,r^2,n+r^2\dot{\mathscr{G}}^2\right)\dot{\mathscr{G}}\right]=0, \\ \mathscr{G}(a)=0, \\ \mathscr{G}(b)=2m\pi. \end{cases} \tag{5.2.26}$$

- $(n=2k-1)$ $\mathbf{Q} \equiv \mathbf{I}_n$.

Proof. Throughout the proof we will assume that $\mathsf{L}(\mathbf{Q},\theta)>0$ as by assumption $\mathsf{L}(\mathbf{Q},\theta)=0$ *iff* $\mathbf{Q}\equiv\mathbf{I}_n$ in which case the above equivalences are trivially true. We first justify the implication $(ii) \implies (iii)$. To this end we introduce the function

$$\mathscr{G}(r,\theta,\mathbf{Q}) := \int_a^r |\dot{\mathbf{Q}}(s)\theta|\,ds, \qquad a \le r \le b, \tag{5.2.27}$$

with $\dot{\mathscr{G}}(r) = d/dr[\mathscr{G}(r,\theta,\mathbf{Q})] = |\dot{\mathbf{Q}}(r)\theta|$. Given that (5.1.11) holds we can write

$$\frac{d}{dr}\left[r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|^2\right] - r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\langle\dot{\mathbf{Q}}\theta,\ddot{\mathbf{Q}}\theta\rangle = 0 \tag{5.2.28}$$

and therefore by a straightforward differentiation

$$\begin{aligned} 0 &= \frac{d}{dr}\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|\right\}|\dot{\mathbf{Q}}\theta| \\ &\quad + r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|\frac{\langle\dot{\mathbf{Q}}\theta,\ddot{\mathbf{Q}}\theta\rangle}{|\dot{\mathbf{Q}}\theta|} \\ &\quad - r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)\langle\dot{\mathbf{Q}}\theta,\ddot{\mathbf{Q}}\theta\rangle \\ &= \frac{d}{dr}\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|\right\}|\dot{\mathbf{Q}}\theta|. \end{aligned} \tag{5.2.29}$$

The above calculation shows that for a fixed θ, as a function of r, $r^{n+1}F_\xi|\dot{\mathbf{Q}}\theta| = c$ on any interval on which $|\dot{\mathbf{Q}}\theta|$ is non-zero and so a basic continuity argument implies that either $|\dot{\mathbf{Q}}\theta| \equiv 0$ on $[a, b]$ *or* $|\dot{\mathbf{Q}}\theta| > 0$ on $[a, b]$. Since we are avoiding the former case then $\mathscr{G}(r, \theta, \mathbf{Q})$ solves (5.2.26) on the whole interval $[a, b]$ for every fixed $\theta \in \mathbb{S}^{n-1}$. Now we see that $\mathscr{G}(a, \theta, \mathbf{Q}) = 0$ and $\mathscr{G}(b, \theta, \mathbf{Q}) = \mathsf{L}(\mathbf{Q}, \theta)$ which is independent of θ by assumption. Solutions of (5.2.26) are plainly extremisers over $\mathscr{D}^p_m(a, b) = \{\mathscr{G} \in W^{1,p}(a, b) : \mathscr{G}(a) = 0, \mathscr{G}(b) = 2m\pi\}$ of the energy

$$\mathbb{I} : \mathscr{G} \mapsto \int_a^b F(r, r^2, n + r^2\dot{\mathscr{G}}^2)\, r^{n-1}\, dr, \qquad \mathscr{G} \in \mathscr{D}^p_m(a, b). \tag{5.2.30}$$

As the functional $\mathbb{I}$ in (5.2.30) is strictly convex (note that $F_\xi > 0$ and $F_{\xi\xi} \geq 0$) using a standard convexity argument it follows that solutions to (5.2.26) are the unique minimisers of this energy functional with respect to their own boundary conditions. Since $\mathscr{G}$ has been shown to be independent of θ at its end-points it follows that $\mathscr{G}(r, \theta, \mathbf{Q}) = \mathscr{G}(r, \mathbf{Q})$ is independent of θ for all $a \leq r \leq b$.

Next as $F_\xi > 0$ it is evident that all (non-zero) solutions of (5.2.26) are strictly monotone and hence invertible on $[a, b]$. Now put $r(s) = \mathscr{G}^{-1}(s)$ for the inverse and write $\mathbf{Q}(r(s)) = \mathbf{K}(s)$ for $\mathbf{K} \in \mathscr{C}^2(]0, l[, \mathbf{SO}(n)) \cap \mathscr{C}([0, l], \mathbf{SO}(n))$ where $l = \mathscr{G}(b)$. Then $\mathbf{Q}(r) = \mathbf{K}(\mathscr{G}(r))$ so that $\dot{\mathbf{Q}} = \mathbf{K}'\dot{\mathscr{G}}$ (with prime denoting d/ds). Returning to (5.1.11) we have after a change of variables

$$\frac{d}{ds}\left\{r^{n+1}F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\mathbf{K}^t\mathbf{K}'\right\} = 0 \iff c\frac{d}{ds}\mathbf{K}^t\mathbf{K}' = 0. \tag{5.2.31}$$

Here we have written $r^{n+1}F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}} = c$ (with $c \neq 0$) as a result of $\mathscr{G}$ being a solution to (5.2.26). Now taking $\mathbf{K}(s) = \exp\{s\mathbf{H}\}$ with $\mathbf{H}$ a constant skew-symmetric matrix to be specified we have $\mathbf{K}^t\mathbf{K}' = \mathbf{H}$ and so $\mathbf{K}$ solves (5.2.31). As $s(r) = \mathscr{G}(r)$ this translates to $\mathbf{Q}(r; m) = \exp\{\mathscr{G}(r; m)\mathbf{H}\}$ with $\mathscr{G}$ solving (5.2.26). Calculating the L-norm it is seen that

$$\mathsf{L}[\mathbf{Q} = \exp\{\mathscr{G}(r; m)\mathbf{H}\}, \theta] = \int_a^b |\dot{\mathscr{G}}(r; m)\mathbf{H}\theta|\, dr = |\mathbf{H}\theta| \int_a^b |\dot{\mathscr{G}}(r; m)|\, dr \tag{5.2.32}$$

and so by inspection $\mathsf{L}[\mathbf{Q}, \theta]$ is independent of θ *iff* $|\mathbf{H}\theta|$ is independent of θ. Now consider orthogonally diagonalising $\mathbf{H}$, i.e., writing $\mathbf{H} = \mathbf{P}diag(c_1\mathbf{J}, \dots, c_k\mathbf{J})\mathbf{P}^t$ for $n = 2k$ and $\mathbf{H} = \mathbf{P}diag(c_1\mathbf{J}, \dots, c_{k-1}\mathbf{J}, c_k)\mathbf{P}^t$ for $n = 2k-1$ with $\mathbf{P} \in \mathbf{O}(n)$, $\mathbf{J}$ as in (5.2.21) and $c_1, \dots c_k$ suitable real constants, in fact, here $\pm ic_j$ ($1 \leq j \leq k$) being the eigenvalues of $\mathbf{H}$ (note that $c_k = 0$ for n odd). By a direct calculation it is then seen that $|\mathbf{H}\theta|$ is independent of θ *iff* $c_1, \dots, c_k$ are equal modulo sign, that is, $|c_1| = \cdots = |c_n| =: |c|$. When n is odd this gives $c = 0$ and hence $\mathbf{Q} \equiv \mathbf{I}_n$ and when n is even by adjusting $\mathbf{P} \in \mathbf{O}(n)$ if necessary this gives, without loss of generality, $c_1 = \cdots = c_k = c$.

Now moving on to the endpoint conditions on $\mathbf{Q}$ (and only in the n even case as for n odd $\mathbf{Q} \equiv \mathbf{I}_n$) it is firstly seen from $\mathscr{G}(a; m) = 0$ that $\mathbf{Q}(a) = \exp\{0\} = \mathbf{I}_n$ and from $\mathscr{G}(b; m) = 2m\pi$ that $\mathbf{Q}(b) = \exp\{c2m\pi\mathbf{P} diag(\mathbf{J}, \dots, \mathbf{J})\mathbf{P}^t\}$. Thus to satisfy $\mathbf{Q}(b) = \mathbf{I}_n$ it suffices to take $c = 1$ and then (5.2.25) follows.

The implication $(i) \implies (iii)$ is precisely the content of Theorem 2.4.1 and as such here we provide just a sketch of a proof and refer to the aforementioned theorem for further details. First assume that $\mathbf{Q}$ solves the ODE (5.1.10). Multiplying this equation by $\mathbf{Q}\theta$ and using the observation $[\mathbf{Q}\theta \otimes \ddot{\mathbf{Q}}\theta]\mathbf{Q}\theta = -|\dot{\mathbf{Q}}\theta|^2\mathbf{Q}\theta$ it follows that $\mathbf{Q}$ satisfies

$$\frac{d}{dr}\left[r^{n+1}F_\xi(r, r^2, n + r^2|\dot{\mathbf{Q}}\theta|^2)\dot{\mathbf{Q}}\theta\right] + r^{n+1}F_\xi(r, r^2, n + r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|^2\mathbf{Q}\theta = 0. \tag{5.2.33}$$

Thus upon writing $F_\xi = F_\xi(r, r^2, n+r^2|\dot{\mathbf{Q}}\theta|^2)$ for short it follows from a straightforward calculation that

$$\begin{aligned}\frac{d}{dr}\left\{r^{n+1}F_\xi|\dot{\mathbf{Q}}\theta|\right\} &= \frac{d}{dr}\left\{r^{n+1}F_\xi\right\}|\dot{\mathbf{Q}}\theta| + r^{n+1}F_\xi\frac{\langle\ddot{\mathbf{Q}}\theta, \theta\rangle}{|\dot{\mathbf{Q}}\theta|} \\ &= -r^{n+1}F_\xi\langle\dot{\mathbf{Q}}\theta, \mathbf{Q}\theta\rangle|\dot{\mathbf{Q}}\theta| = 0,\end{aligned} \tag{5.2.34}$$

where the final inequality holds true by virtue of $\mathbf{Q}^t\dot{\mathbf{Q}}$ being skew-symmetric. Now recalling $\mathscr{G}$ as defined in (5.2.27) the above shows that $\mathscr{G}$ solves the ODE in (5.2.26). Thus a similar convexity argument as in the previous part shows that $\mathscr{G}$ is independent of θ for $a \le r \le b$. With $\mathbf{K}$ as before, the equation (5.2.33) after a change of variables becomes

$$\frac{d}{ds}\left[r^{n+1}F_\xi\dot{\mathscr{G}}\mathbf{K}'\theta\right] + r^{n+1}F_\xi\dot{\mathscr{G}}|\mathbf{K}'\theta|^2\mathbf{K}\theta = c\left[\mathbf{K}'' + |\mathbf{K}'\theta|^2\mathbf{K}\right]\theta = 0, \tag{5.2.35}$$

which is precisely the geodesic equation on the sphere for $\gamma(s) = \mathbf{K}(s)\theta$. Writing $\mathbf{K}(s) = \exp\{s\mathbf{H}\}$ as before this gives $[\mathbf{H}^2 + \mathbf{I}_n]\mathbf{K} = 0$ which has no solutions in odd dimensions if $\mathsf{L}(\mathbf{Q}, \theta) > 0$ whilst for n even we recover $\mathbf{H} = \mathbf{P}\mathbf{J}_k\mathbf{P}^t$ and the concluding description of $\mathbf{Q}$ follows.

We have shown that either of (i) or (ii) implies (iii) and therefore all that remains is to show the converse, namely, that (iii) implies (i) and (ii). Towards this end. first observe that if the dimension n is odd, then $\mathbf{Q} \equiv \mathbf{I}_n$ and both

(5.1.10) or (5.1.11) trivially hold. For n even and with $\mathbf{Q}$ as in (5.2.25) we have $\dot{\mathbf{Q}} = \dot{\mathscr{G}}\mathbf{H}\mathbf{Q}$ and so starting with (5.1.11) we can write

$$\text{LHS}(5.1.11) = \frac{d}{dr}\left\{r^{n+1}F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\right\}\mathbf{H} = 0 \tag{5.2.36}$$

which holds since $\mathscr{G}$ is taken as a solution of (5.2.26). Likewise regarding (5.1.10) we can write

$$\begin{aligned}\text{LHS}(5.1.10) =& \frac{d}{dr}\left\{r^{n+1}F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\left[\mathbf{H}\mathbf{Q}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{H}\mathbf{Q}\theta\right]\right\}\\ =& \frac{d}{dr}\left\{r^{n+1}F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\right\}\left[\mathbf{H}\mathbf{Q}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{H}\mathbf{Q}\theta\right]\\ &+ r^{n+1}F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}^2\left[\mathbf{H}^2\mathbf{Q}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{H}^2\mathbf{Q}\theta\right] = 0,\end{aligned} \tag{5.2.37}$$

with the final equality in (5.2.37) holding as a consequence of $\mathscr{G}$ being a solution to (5.2.26) and the identity $\mathbf{H}^2 = -\mathbf{I}_n$. As such we have shown that (iii) implies both (i) and (ii) and so the proof is complete. $\square$

Remark 5.2.9. Consider the quantity $\mathsf{S}_{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}}[\theta] = \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta$ as in (5.2.3) and appearing in the proof of Proposition 5.2.4 [*c.f.* (5.2.16)]. Then by Lemma 5.2.2 we have

$$\mathsf{S}_{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}}[\theta] \equiv 0 \iff \dot{\mathbf{Q}}^t\dot{\mathbf{Q}} = \sigma(r)\mathbf{I}_n \tag{5.2.38}$$

for some non-negative $\sigma = \sigma(r)$ with $a \le r \le b$. This being so $|\dot{\mathbf{Q}}\theta|^2 = \sigma(r)$ and hence the L-norm (5.2.24) is independent of θ. As such if $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta - \theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta = 0$ for a $\mathbf{Q}$ of class $\mathscr{C}^2$ that solves either of the ODEs (5.1.10) or (5.1.11) then $\mathbf{Q}$ is as described in part (iii) of Theorem 5.2.8.

5.3 The Lagrangian $F(r, s, \xi) = h(r, s)\xi$: The ODEs (5.1.9) and (5.1.10) *vs.* (5.1.11) on $\mathbf{SO}(n)$

In this section we take a closer look at the ODEs (5.1.10) and (5.1.11) with the aim of discussing the possible relationship between the two. Here we specialise to Lagrangians F of the type $F(r, s, \xi) = h(r, s)\xi$ with $h > 0$ of class $\mathscr{C}^2$ where the resulting ODEs are completely integrable and one can obtain explicit solutions for the twist path $\mathbf{Q} = \mathbf{Q}(r)$. Note that upon substituting this Lagrangian into the energy integral (5.1.1) the resulting functional takes the form of a weighted Dirichlet energy integral whose restriction to twists u takes the form

$$\begin{aligned}\mathbb{F}[u=r\mathbf{Q}(r)\theta;\mathbb{X}^n] &= \int_{\mathbb{X}^n} h(r,|u|^2)|\nabla u|^2\,dx \\ &= \int_a^b\int_{\mathbb{S}^{n-1}} h(r,r^2)[n+r^2|\dot{\mathbf{Q}}\theta|^2]r^{n-1}\,drd\mathcal{H}^{n-1}(\theta). \end{aligned} \tag{5.3.1}$$

Starting from the ODE (5.1.11) it is seen that in this case with $F_\xi = h$ we have

$$\frac{d}{dr}\left[r^{n+1}h(r,r^2)\mathbf{Q}^t\dot{\mathbf{Q}}\right]=0, \qquad a<r<b. \tag{5.3.2}$$

This ODE subject to the endpoint condition $\mathbf{Q}(a)=\mathbf{I}_n$ as required by (5.1.8) (with $p=2$) admits the specific solutions $\mathbf{Q}(r)=\exp\{\mathscr{H}(r)\mathbf{H}\}$ with $\mathbf{H}\in\mathfrak{so}(n)$ arbitrary and $\mathscr{H}\in\mathscr{C}^2[a,b]$ given explicitly by

$$\mathscr{H}(r)=\frac{\mathsf{H}(r)}{\mathsf{H}(b)}, \qquad \mathsf{H}(r)=\int_a^r\frac{ds}{s^{n+1}h(s,s^2)}, \qquad a<r<b. \tag{5.3.3}$$

By virtue of $\mathscr{H}(a)=0$ the endpoint condition $\mathbf{Q}(a)=\mathbf{I}_n$ is trivially satisfied. Anticipating on the other endpoint condition, we again proceed by orthogonally diagonalising $\mathbf{H}$, that is, writing $\mathbf{H}=\mathbf{P}diag(c_1\mathbf{J},\dots,c_k\mathbf{J})\mathbf{P}^t$ for $n=2k$ and $\mathbf{H}=\mathbf{P}diag(c_1\mathbf{J},\dots,c_{k-1}\mathbf{J},c_k)\mathbf{P}^t$ for $n=2k-1$ (with $c_k=0$ in the odd case). In order to satisfy $\mathbf{Q}(b)=\mathbf{I}_n$ we observe that $c_j\in 2\mathbb{Z}\pi$ for all $1\le j\le k$ and so for $\mathbf{Q}=\exp\{\mathscr{H}(r)\mathbf{H}\}$ in (5.1.8) to solve (5.3.2) $\mathbf{H}$ must have the form

$$\mathbf{H}=\begin{cases}\mathbf{P}diag(2m_1\pi\mathbf{J},\dots,2m_k\pi\mathbf{J})\mathbf{P}^t & n=2k,\\ \mathbf{P}diag(2m_1\pi\mathbf{J},\dots,2m_{k-1}\pi\mathbf{J},0)\mathbf{P}^t & n=2k-1,\end{cases} \tag{5.3.4}$$

with $\mathbf{J}$ as in (5.2.21) and $m_1,\dots,m_k\in\mathbb{Z}$. Now the ODE (5.1.10) for the choice of Lagrangian $F=h(r,s)\xi$ is seen to be

$$\frac{d}{dr}\left\{r^{n+1}h(r,r^2)\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\right\}=0, \quad a<r<b. \tag{5.3.5}$$

Integrating (5.3.5) over the sphere and using the divergence theorem gives (5.3.2). It thus follows that any solution $\mathbf{Q}=\mathbf{Q}(r)$ here must also solve (5.3.5) and so by the above discussion must have the form $\mathbf{Q}(r)=\exp\{\mathscr{H}(r)\mathbf{H}\}$. Now with this at hand it is seen upon substitution in (5.3.5) that

$$\begin{aligned}0&=\frac{d}{dr}\left\{r^{n+1}h(r,r^2)\dot{\mathscr{H}}(r)\left[\mathbf{H}\mathbf{Q}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\mathbf{H}\mathbf{Q}\theta\right]\right\}\\ &=\frac{d}{dr}\left\{r^{n+1}h(r,r^2)\dot{\mathscr{H}}\right\}\left[\mathbf{H}\mathbf{Q}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\mathbf{H}\mathbf{Q}\theta\right]\\ &\quad+r^{n+1}h(r,r^2)\dot{\mathscr{H}}^2\left[\mathbf{H}^2\mathbf{Q}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\mathbf{H}^2\mathbf{Q}\theta\right].\end{aligned} \tag{5.3.6}$$

This clearly holds if $d/dr[r^{n+1}h(r,r^2)\dot{\mathscr{H}}] = 0$ (corresponding to (5.3.2) with $\mathbf{Q} = \exp\{\mathscr{H}(r)\mathbf{H}\}$) and $\mathbf{H}^2\mathbf{Q}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{H}^2\mathbf{Q}\theta \equiv 0$. Regarding this second condition a rearrangement yields $\mathbf{Q}[\mathbf{H}^2\theta \otimes \theta - \theta \otimes \mathbf{H}^2\theta]\mathbf{Q}^t \equiv 0$ (note that $\mathbf{H}$ and $\mathbf{Q}$ commute), so by Lemma 5.2.2 it is necessary and sufficient here to have $\mathbf{H}^2 = c\mathbf{I}_n$ for $c \in \mathbb{R}$ or translating into the scalars $c_1, \dots, c_k$ to have $m_1 = \cdots = m_k =: m$ (by a suitable adjustment of $\mathbf{P} \in \mathbf{O}(n)$ if needed as seen before). Consideration of n even and odd separately by noting that $m = 0$ for n odd, hence $\mathbf{H} \equiv 0$, and $c = 2m\pi$ with $m \in \mathbb{Z}$ arbitrary for n even leads to the following statement.

Theorem 5.3.1. *Consider the Lagrangian $F(r,s,\xi) = h(r,s)\xi$ with $h > 0$ and of class $\mathscr{C}^1([a,b]\times]0,\infty[)$ along with the ODEs* (5.1.9)-(5.1.11) *on the compact Lie group $\mathbf{SO}(n)$ together with the endpoint conditions $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$. Then:*

- (i) *The ODEs* (5.1.9) *and* (5.1.11) *are equivalent.*
- (ii) *Every solution to* (5.1.9) *and* (5.1.11) *in $\mathscr{B}^2_{\mathbf{I}_n}(a,b)$ has the form*

$$\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}, \qquad a \le r \le b, \tag{5.3.7}$$

  with $\mathscr{H} = \mathscr{H}(r)$ as in (5.3.3) *and* $\mathbf{H}$ *as in* (5.3.4) *with $m_1, \dots m_k \in \mathbb{Z}$.*
- (iii) *Every solution to* (5.1.10) *in $\mathscr{B}^2_{\mathbf{I}_n}(a,b)$ is as in (ii) above subject to additionally having $m_j \in \{\pm m\}$ for $1 \le j \le k$ with $m \in \mathbb{Z}$ when $n = 2k$ and $\mathbf{H} \equiv 0$, that is, $m = 0$ when $n = 2k - 1$.*

Thus in particular it follows from this theorem that these two sets of ODEs are *not* equivalent; in fact, the ODEs (5.1.9) and (5.1.11) have a much wider solution set due to there being no constraint on the choice of integer $m_1, \dots, m_k$ contrary to (5.1.10) where apart from a sign the latter integers all have to coincide, that is, $|m_1| = \cdots = |m_k|$. In particular for n odd this has the severe consequence that $\mathbf{H} = 0$ and hence $\mathbf{Q} \equiv \mathbf{I}_n$.

5.4 Irrotationality of the Vector Field $\mathscr{L}[u]$ and the Lagrangian Discriminant Δ_F

The starting aim of this section is the pivotal step of computing the curl of the vector field $\mathscr{L}[u]$ as a key ingredient in solving the system (5.1.4), specifically, the PDE $\mathscr{L}[u]=\nabla\mathscr{P}$. Evaluating the action $\mathscr{L}[u]$ for $u=r\mathbf{Q}(r)\theta$, upon taking $\mathscr{L}[u]$ as in the proof of Proposition 5.2.1 and multiplying through the term $[\mathrm{cof}\,\nabla u]^{-1}=[\nabla u]^t$ and abbreviating the arguments of $F=F(r,r^2,n+r^2|\dot{\mathbf{Q}}\theta|^2)$ and any of its derivatives for the sake of brevity, we obtain

$$\begin{aligned}\mathscr{L}[u]=&F_{\xi\xi}[\mathbf{I}_n+r(\mathbf{Q}^t\dot{\mathbf{Q}}\theta\otimes\theta+\theta\otimes\mathbf{Q}^t\dot{\mathbf{Q}}\theta)+r^2|\dot{\mathbf{Q}}\theta|^2\theta\otimes\theta]\nabla(r^2|\dot{\mathbf{Q}}\theta|^2)\\&+[2rF_{s\xi}+F_{r\xi}](\theta+r\mathbf{Q}^t\dot{\mathbf{Q}}\theta+r^2|\dot{\mathbf{Q}}\theta|^2\theta)\\&+F_\xi[(n+1)\mathbf{Q}^t\dot{\mathbf{Q}}+r\mathbf{Q}^t\ddot{\mathbf{Q}}+r(n+1)|\dot{\mathbf{Q}}\theta|^2\mathbf{I}_n+r^2\langle\dot{\mathbf{Q}}\theta,\ddot{\mathbf{Q}}\theta\rangle\mathbf{I}_n]\theta-rF_s\theta.\end{aligned}\tag{5.4.1}$$

Now using $\mathbf{A}=\mathbf{Q}^t\dot{\mathbf{Q}}$ as before for convenience we can rewrite the above as

$$\begin{aligned}\mathscr{L}[u]=&F_{\xi\xi}[\mathbf{I}_n+r(\mathbf{A}\theta\otimes\theta+\theta\otimes\mathbf{A}\theta)+r^2|\mathbf{A}\theta|^2\theta\otimes\theta]\nabla(r^2|\mathbf{A}\theta|^2)\\&+[2rF_{s\xi}+F_{r\xi}](\theta+r\mathbf{A}\theta+r^2|\mathbf{A}\theta|^2\theta)\\&+F_\xi[(n+1)\mathbf{A}+r(\dot{\mathbf{A}}+\mathbf{A}^2)+r(n+1)|\mathbf{A}\theta|^2\mathbf{I}_n\\&+r^2\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle\mathbf{I}_n]\theta-rF_s\theta.\end{aligned}\tag{5.4.2}$$

Next a straightforward differentiation and recalling the skew-symmetry of $\mathbf{A}$ gives the identity

$$\nabla|\mathbf{A}x|^2=\nabla(r^2|\mathbf{A}\theta|^2)=\frac{d}{dr}(r^2|\mathbf{A}\theta|^2)\theta-2r\mathbf{A}^2\theta-2r|\mathbf{A}\theta|^2\theta,\tag{5.4.3}$$

which then results in $\nabla F_\xi=F_{\xi\xi}\nabla(r^2|\mathbf{A}\theta|^2)+2rF_{s\xi}\theta+F_{r\xi}\theta$. As a consequence we can write the differential operator action $\mathscr{L}[u]$ in (5.4.2) as

$$\begin{aligned}\mathscr{L}[u]=&\nabla F_\xi+\frac{1}{r^n}\frac{d}{dr}\left[r^{n+1}F_\xi\mathbf{A}\right]\theta+\frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n+1}F_\xi|\mathbf{A}\theta|^2\right]\theta\\&+rF_\xi\mathbf{A}^2\theta-r^2\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle F_\xi\theta-rF_s\theta.\end{aligned}\tag{5.4.4}$$

Now the main advantage of the above formulation lies in the fact that it contains representations of the ODEs encountered earlier in Section 5.2 and so further links the system (5.1.4) and the PDE $\mathscr{L}_F[u]=\nabla\mathscr{P}$ to the three classes of ODEs studied earlier. In fact from this point on we shall assume that the twist path $\mathbf{Q}=\mathbf{Q}(r)$ associated with the twist u is a solution to the ODE (5.1.11), that is, (5.4.5) below, and aim to reformulate the action $\mathscr{L}[u]$ under the assumption

$$\frac{d}{dr}\left\{r^{n+1}F_\xi(r,r^2,n+r^2|\mathbf{A}\theta|^2)\mathbf{A}\right\}=0,\qquad\mathbf{A}=\mathbf{Q}^t\dot{\mathbf{Q}},\qquad a<r<b.\tag{5.4.5}$$

Note that this ODE is formally stronger than (5.1.9) (which arises as a necessary condition for twist solutions to $\mathscr{L}[u]=\nabla\mathscr{P}$) but it has the advantage of being more natural, pointwise and directly embedded in the operator action $\mathscr{L}[u]$. Furthermore, it is equivalent to (5.1.9) in many cases of interest or subject to additional conditions as discussed before. *See* Proposition 5.2.4, Proposition 5.2.7, the accompanying discussion and Theorem 5.2.8. *See* also Theorem 5.3.1.

Proposition 5.4.1. *Let* $\mathbf{Q} \in \mathscr{C}^2(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}([a,b], \mathbf{SO}(n))$ *assume that* $\mathbf{Q}$ *satisfies the ODE* (5.4.5)*. Then with* $u = r\mathbf{Q}(r)\theta$*, the action* $\mathscr{L}[u]$ *reduces to*

$$\mathscr{L}[u] = \nabla F_\xi + rF_\xi \mathbf{A}^2\theta - rF_s\theta, \tag{5.4.6}$$

where $F_s = F_s(r, r^2, n + r^2|\dot{\mathbf{Q}}\theta|^2)$ *and* $F_\xi = F_\xi(r, r^2, n + r^2|\dot{\mathbf{Q}}\theta|^2)$ *denote the derivatives of the function* $F = F(r, s, \xi)$ *in the second and third variables respectively.*

Proof. Given (5.4.5) it is seen upon taking the inner product $\langle \mathrm{LHS}(5.4.5)\theta, \mathbf{A}\theta\rangle = 0$ that

$$\begin{aligned} 0 = \frac{1}{r^n}\left\langle \frac{d}{dr}\left[r^{n+1}F_\xi \mathbf{A}\right]\theta, \mathbf{A}\theta\right\rangle = &\Big\{(n+1)F_\xi|\mathbf{A}\theta|^2 + rF_{r\xi}|\mathbf{A}\theta|^2 \\ &+2r^2F_{s\xi}|\mathbf{A}\theta|^2 + rF_{\xi\xi}\frac{d}{dr}(r^2|\mathbf{A}\theta|^2)|\mathbf{A}\theta|^2 + rF_\xi\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\Big\} \\ =&\frac{1}{r^n}\frac{d}{dr}\left[r^{n+1}F_\xi|\mathbf{A}\theta|^2\right] - rF_\xi\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle \end{aligned} \tag{5.4.7}$$

and so upon rearranging we get $r^{-n}d/dr[r^{n+1}F_\xi|\mathbf{A}\theta|^2] = rF_\xi\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle$. This being so and referring to (2.1.4) we have the result. $\square$

We now come to the main aim of this section, namely, given the formulation of $\mathscr{L}[u]$ for a twist u with a twist path $\mathbf{Q}$ satisfying the ODE (5.4.5), to compute its curl, and discuss the irrotationality of the action $\mathscr{L}[u]$, *i.e.*, it being curl-free.

Theorem 5.4.2. *Let* $u = r\mathbf{Q}(r)\theta$ *with* $\mathbf{Q} \in \mathscr{C}^2(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}([a,b], \mathbf{SO}(n))$ *and assume that* $\mathbf{Q}$ *is a solution to the ODE* (5.4.5)*. Then*

$$\mathrm{curl}\,(\mathscr{L}[u] - \nabla F_\xi) = -\Delta_F\left[\mathbf{A}^2x \otimes x - x \otimes \mathbf{A}^2x\right], \tag{5.4.8}$$

where the Lagrangian discriminant Δ_F *is given by*

$$\Delta_F = \frac{2[(n+1)F_\xi + 2r^2F_{\xi\xi}|\mathbf{A}\theta|^2 + 2r^2F_{s\xi}][F_\xi + r^2F_{\xi\xi}|\mathbf{A}\theta|^2] + rF_\xi F_{r\xi}}{r^2(F_\xi + 2r^2F_{\xi\xi}|\mathbf{A}\theta|^2)}. \tag{5.4.9}$$

Thus if the discriminant Δ_F *is nowhere zero in* $\mathbb{X}^n$ *then* $\mathrm{curl}\,(\mathscr{L}[u] - \nabla F_\xi) \equiv 0$ *in* $\mathbb{X}^n$ *iff* $-\mathbf{A}^2 = \alpha(r)\mathbf{I}_n$ *for some non-negative* $\alpha \in \mathscr{C}^1]a,b[\cap\mathscr{C}[a,b]$*.*

Proof. Referring to the formulation (5.4.6), we proceed by calculating the curl of the $\mathscr{C}^1$ vector field $v := \mathscr{L}[u] - \nabla F_\xi$, specifically,

$$v = F_\xi(|x|, |x|^2, n + |\mathbf{A}x|^2)\mathbf{A}^2 x - F_s(|x|, |x|^2, n + |\mathbf{A}x|^2)x, \qquad x \in \mathbb{X}^n. \quad (5.4.10)$$

Setting $\mathbf{F} = \mathbf{A}^2$ and noting $[\operatorname{curl} v]_{ij} = v_{i,j} - v_{j,i}$ for $1 \le i, j \le n$ we have from (5.4.10)

$$v_{i,j} = r\nabla_j F_\xi [\mathbf{F}\theta]_i + rF_\xi[\dot{\mathbf{F}}\theta]_i\theta_j + F_\xi \mathbf{F}_{ij} - r\nabla_j F_s \theta_i - F_s\delta_{ij} \qquad (5.4.11)$$

where $\nabla_j = \partial/\partial x_j$, $F_s = F_s(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle)$ and $F_\xi = F_\xi(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle)$ since $|\mathbf{A}\theta|^2 = -\langle \mathbf{F}\theta, \theta\rangle$ and in a similar way

$$v_{j,i} = r\nabla_i F_\xi [\mathbf{F}\theta]_j + rF_\xi[\dot{\mathbf{F}}\theta]_j\theta_i + F_\xi \mathbf{F}_{ji} - r\nabla_i F_s \theta_j - F_s\delta_{ji}. \qquad (5.4.12)$$

Now recalling that $\mathbf{F} = \mathbf{A}^2 = -\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}$ is a symmetric matrix field ($\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ is skew-symmetric) it follows from (5.4.11)-(5.4.12) after taking into account the appropriate cancellations and changing to tensor notation that

$$\begin{aligned} \operatorname{curl} v =& \mathbf{F}x \otimes \nabla F_\xi - \nabla F_\xi \otimes \mathbf{F}x + \frac{F_\xi}{r}\left(\dot{\mathbf{F}}x \otimes x - x \otimes \dot{\mathbf{F}}x\right) \\ & + \nabla F_s \otimes x - x \otimes \nabla F_s. \end{aligned} \qquad (5.4.13)$$

By using (5.4.3) it is seen that the gradient terms above are given respectively by

$$\begin{aligned} \nabla F_\xi =& \nabla F_\xi(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle) = F_{r\xi}(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle)\theta \\ & + 2rF_{s\xi}(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle)\theta \\ & - F_{\xi\xi}(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle)[2r\mathbf{F} + r^2\langle \dot{\mathbf{F}}\theta, \theta\rangle \mathbf{I}_n]\theta \end{aligned}$$

and similarly

$$\begin{aligned} \nabla F_s =& \nabla F_s(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle) = F_{rs}(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle)\theta \\ & + 2rF_{ss}(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle)\theta \\ & - F_{s\xi}(r, r^2, n - r^2\langle \mathbf{F}\theta, \theta\rangle)[2r\mathbf{F} + r^2\langle \dot{\mathbf{F}}\theta, \theta\rangle \mathbf{I}_n]\theta. \end{aligned}$$

Therefore the contribution of these two gradients in (5.4.13) is respectively given by

$$\begin{aligned} \mathbf{F}x \otimes \nabla F_\xi - \nabla F_\xi \otimes \mathbf{F}x =& [r^{-1}F_{r\xi} + 2F_{s\xi} - rF_{\xi\xi}\langle \dot{\mathbf{F}}\theta, \theta\rangle](\mathbf{F}x \otimes x - x \otimes \mathbf{F}x) \\ & - 2F_{\xi\xi}[\mathbf{F}x \otimes \mathbf{F}x - \mathbf{F}x \otimes \mathbf{F}x] \end{aligned} \qquad (5.4.14)$$

and likewise

$$\begin{aligned}\nabla F_s \otimes x - x \otimes \nabla F_s = &[r^{-1}F_{rs} + 2F_{ss} - rF_{s\xi}\langle \dot{\mathbf{F}}\theta, \theta\rangle](x \otimes x - x \otimes x) \\ &- 2F_{s\xi}(\mathbf{F}x \otimes x - x \otimes \mathbf{F}x),\end{aligned} \tag{5.4.15}$$

by virtue of the trivial calculations. Therefore by substituting the simplified expressions (5.4.14) and (5.4.15) back into (5.4.13) this gives

$$\operatorname{curl} v = (r^{-1}F_{r\xi} - rF_{\xi\xi}\langle \dot{\mathbf{F}}\theta, \theta\rangle)[\mathbf{F}x \otimes x - x \otimes \mathbf{F}x] + r^{-1}F_\xi[\dot{\mathbf{F}}x \otimes x - x \otimes \dot{\mathbf{F}}x]. \tag{5.4.16}$$

Now since the ODE (5.4.5) is assumed to hold, upon noting that $\dot{\mathbf{F}} = \mathbf{A}\dot{\mathbf{A}} + \dot{\mathbf{A}}\mathbf{A}$ a rearrangement of the equation $\mathbf{A}[\text{LHS}(5.4.5)] + [\text{LHS}(5.4.5)]\mathbf{A} = 0$ for $\dot{\mathbf{F}}$ yields the identity

$$\dot{\mathbf{F}} = -2\left[\frac{n+1}{r} + \frac{\partial_r(F_\xi)}{F_\xi}\right]\mathbf{F}, \tag{5.4.17}$$

where we have written

$$\partial_r(F_\xi) = F_{r\xi} + 2rF_{s\xi} - F_{\xi\xi}\frac{d}{dr}(r^2\langle \mathbf{F}\theta, \theta\rangle). \tag{5.4.18}$$

Substituting $\dot{\mathbf{F}}$ into (5.4.16) gives us $\operatorname{curl} v = -\Delta_F\,[\mathbf{F}x \otimes x - x \otimes \mathbf{F}x]$ where

$$\Delta_F := \frac{2}{r}\left[\frac{n+1}{r} + \frac{\partial_r(F_\xi)}{F_\xi}\right]\left(F_\xi - r^2F_{\xi\xi}\langle \mathbf{F}\theta, \theta\rangle\right) - \frac{1}{r}F_{r\xi}. \tag{5.4.19}$$

We now aim at simplifying Δ_F by expanding the expression $\partial_r(F_\xi)$ in (5.4.18). Towards this end we first note that $d/dr(r^2\langle \mathbf{F}\theta, \theta\rangle) = 2r\langle \mathbf{F}\theta, \theta\rangle + r^2\langle \dot{\mathbf{F}}\theta, \theta\rangle$. Thus substituting in $\dot{\mathbf{F}}$ as per (5.4.17) gives

$$\begin{aligned}\frac{d}{dr}(r^2\langle \mathbf{F}\theta, \theta\rangle) &= 2\left[r\langle \mathbf{F}\theta, \theta\rangle - r^2\left(\frac{n+1}{r} + \frac{\partial_r(F_\xi)}{F_\xi}\right)\langle \mathbf{F}\theta, \theta\rangle\right] \\ &= -2r\left[n + r\frac{\partial_r(F_\xi)}{F_\xi}\right]\langle \mathbf{F}\theta, \theta\rangle.\end{aligned} \tag{5.4.20}$$

Now recalling that $-\langle \mathbf{F}\theta, \theta\rangle = |\mathbf{A}\theta|^2$, upon substituting the above into (5.4.18), we have

$$\partial_r(F_\xi) = F_{r\xi} + 2rF_{s\xi} - 2rF_{\xi\xi}\left[n + r\frac{\partial_r(F_\xi)}{F_\xi}\right]|\mathbf{A}\theta|^2, \tag{5.4.21}$$

or upon rearranging

$$\partial_r(F_\xi) = \frac{F_\xi[F_{r\xi} + 2rF_{s\xi} - 2rnF_{\xi\xi}|\mathbf{A}\theta|^2]}{F_\xi + 2r^2F_{\xi\xi}|\mathbf{A}\theta|^2}. \tag{5.4.22}$$

Substituting this into (5.4.19) followed by some basic manipulations yields (5.4.9). For the final remark, note that if Δ_F is nowhere zero in $\mathbb{X}^n$ then curl$v \equiv 0$ *iff* $\mathbf{A}^2 x \otimes x - x \otimes \mathbf{A}^2 x \equiv 0$ and an application of Lemma 5.2.2 gives $\mathbf{A}^2 = -\alpha \mathbf{I}_n$ for some non-negative α as in the statement of the theorem upon observing that $\mathbf{A}^2$ is nonpositive-definite. □

5.5 Examples of Vanishing *vs.* Non-Vanishing Discriminants Δ_F

The purpose of this short section is to study the Lagrangian discriminant Δ_F as given by (5.4.9) in some enlightening cases, for different families of Lagrangians F, and to verify the remarkable fact that in many cases of interest the vanishing or non-vanishing of this discriminant is more a structural property associated with the Lagrangian than the assumed twist path solution to (5.4.5) along which Δ_F is being considered. This considerably simplifies the verification of the assumption in Theorem 5.4.2 regarding the behaviour of Δ_F and facilitates the discussion of solvability of the PDE $\mathscr{L}[u] = \nabla \mathscr{P}$ and the system (5.1.4) for twists.

Towards this end, recall that $F = F(r, s, \xi)$ is a Lagrangian of class $\mathscr{C}^2$ that is convex and monotone increasing in the third variable, specifically, $F_\xi > 0$ and $F_{\xi\xi} \geq 0$.

- $F(r, s, \xi) = F(\xi)$: In this case the mixed partial derivatives $F_{r\xi}$, $F_{s\xi}$ vanish completely and therefore
$$\Delta_F = \frac{2[(n+1)F_\xi + 2r^2 F_{\xi\xi}|\mathbf{A}\theta|^2][F_\xi + r^2 F_{\xi\xi}|\mathbf{A}\theta|^2]}{r^2(F_\xi + 2r^2 F_{\xi\xi}|\mathbf{A}\theta|^2)}. \tag{5.5.1}$$
It is clear, thanks to the assumptions $F_\xi > 0$, $F_{\xi\xi} \geq 0$ on the Lagrangian F that the above discriminant is always strictly positive, that is, $\Delta_F > 0$ in $\mathbb{X}^n$ and so without any further assumption we have curl$\,(\mathscr{L}[u] - \nabla F_\xi) \equiv 0$ in $\mathbb{X}^n$ *iff* $\mathbf{A}^2\theta \otimes \theta - \theta \otimes \mathbf{A}^2\theta \equiv 0$ or *iff* $-\mathbf{A}^2 = \alpha \mathbf{I}_n$ (see also the comments following Theorem 5.6.2).

- $F(r, s, \xi) = G(\xi) + H(r, s)$: Here, despite the explicit (r, s) dependence in the Lagrangian, the mixed derivatives $F_{r\xi}$, $F_{s\xi}$ vanish completely again and a straightforward calculation gives

$$\Delta_F = \frac{2[(n+1)G_\xi + 2r^2 G_{\xi\xi}|\mathbf{A}\theta|^2][G_\xi + r^2 G_{\xi\xi}|\mathbf{A}\theta|^2]}{r^2(G_\xi + 2r^2 G_{\xi\xi}|\mathbf{A}\theta|^2)}. \tag{5.5.2}$$

The Lagrangian discriminant here is exactly the same as (5.5.1) and so the implication of the irrotationality of $\mathscr{L}[u] - \nabla F_\xi$ on the twist path $\mathbf{Q}$ is the same. Thus the overall conclusion is unchanged by this type of dependence on (r, s).

- $F(r, s, \xi) = F(r, \xi)$ *vs.* $F(r, s, \xi) = F(s, \xi)$: Let us now consider cases where the Lagrangian F has a "*joint*" (r, s) and ξ dependence. Assuming first that $F = F(r, \xi)$, i.e., no s-dependence, then (5.4.9) becomes

$$\Delta_F = \frac{2[(n+1)F_\xi + 2r^2 F_{\xi\xi}|\mathbf{A}\theta|^2][F_\xi + r^2 F_{\xi\xi}|\mathbf{A}\theta|^2] + rF_\xi F_{r\xi}}{r^2(F_\xi + 2r^2 F_{\xi\xi}|\mathbf{A}\theta|^2)}. \tag{5.5.3}$$

Likewise assuming $F(r, s, \xi) = F(s, \xi)$, i.e., no r-dependence, then Δ_F becomes

$$\Delta_F = \frac{2[(n+1)F_\xi + 2r^2 F_{\xi\xi}|\mathbf{A}\theta|^2 + 2r^2 F_{s\xi}][F_\xi + r^2 F_{\xi\xi}|\mathbf{A}\theta|^2]}{r^2(F_\xi + 2r^2 F_{\xi\xi}|\mathbf{A}\theta|^2)}. \tag{5.5.4}$$

In both these cases it is seen that, despite the presence of positive terms, the effect of the mixed derivative terms $F_{r\xi}$ or $F_{s\xi}$ can – and in general will – result in the discriminant changing sign or even vanishing completely. Thus unlike the first two set of examples, in neither of the cases above, can it be deduced that the Lagrangian discriminant Δ_F is nowhere zero. As a matter of fact, to complement the previous examples, here, one can give examples of Lagrangians F, where $\Delta_F \equiv 0$ in $\mathbb{X}^n$ (see below in particular (5.5.5)-(5.5.6) and the comments following Theorem 5.6.2).

- $F(r, s, \xi) = h(r, s)\xi$ with $h > 0$:[10] In this case we have $F_\xi = h(r, s)$ with $F_{r\xi} = h_r(r, s)$, $F_{s\xi} = h_s(r, s)$ and $F_{\xi\xi} \equiv 0$. As a result here Δ_F becomes

$$\begin{aligned}\Delta_F &= \frac{2(n+1)h^2(r, r^2) + 4r^2 h(r, r^2)h_s(r, r^2) + rh(r, r^2)h_r(r, r^2)}{r^2 h(r, r^2)} \\ &= \frac{2(n+1)h(r, r^2) + rh_r(r, r^2) + 4r^2 h_s(r, r^2)}{r^2}.\end{aligned} \tag{5.5.5}$$

In particular if h is such that $rh_r(r, r^2) + 4r^2 h_s(r, r^2) + 2(n+1)h(r, r^2) \equiv 0$, then $\Delta_F \equiv 0$ in $\mathbb{X}^n$. This has the interesting consequence that for such h

[10]See Section 5.3 for more on these weighted Dirichlet type Lagrangians.

we can have $\operatorname{curl}(\mathscr{L}_F[u] - \nabla F_\xi) \equiv 0$ in $\mathbb{X}^n$ along a twist $u = r\mathbf{Q}(r)\theta$ with $\mathbf{Q}$ a solution to (5.4.5) without $\mathbf{Q}$ necessarily having to satisfy $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} \equiv \alpha(r)\mathbf{I}_n$ (compare with Theorem 5.3.1). Thus unlike the case where $F = F(\xi)$ [*c.f.* (5.5.1)] if the Lagrangian F has a "*joint*" (r, s) and ξ dependence then Δ_F need not be everywhere nonzero in $\mathbb{X}^n$.

To elaborate further on this last point and example consider specifically the choice $h(r, s) = r^{-\alpha}s^{-\beta}$ for α, β real constants (note that with $0 < a \le r \le b$ and $s > 0$ this function h is of class $\mathscr{C}^2$ and strictly positive over $[a, b]\times]0, \infty[$). Then $rh_r(r, r^2) + 4r^2h_s(r, r^2) + 2(n + 1)h(r, r^2) \equiv 0$ *iff* $f(\alpha, \beta) = 0$ where

$$f(\alpha, \beta) := \alpha + 4\beta - 2(n + 1), \quad (\alpha, \beta) \in \mathbb{R}^2. \tag{5.5.6}$$

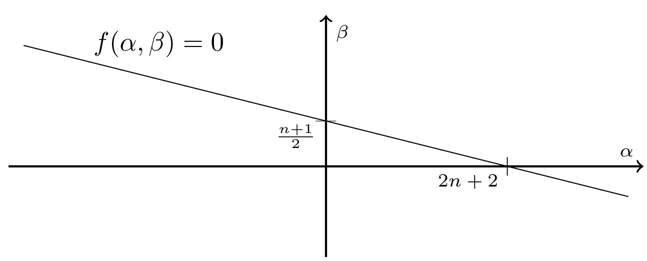

Figure 2: The line $f(\alpha, \beta) = 0$ in the (α, β)-plane with f defined by (5.5.6).

By linearity of the condition on h above, if we take any finite sequence of constants $c_1, \dots, c_N > 0$ as well as $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N$ in $\mathbb{R}$ then $h = h(r, s)$ obtained as the finite sum

$$h(r, s) = \sum_{j=1}^{N} c_j r^{-\alpha_j} s^{-\beta_j} \tag{5.5.7}$$

still satisfies the condition $rh_r(r, r^2) + 4r^2h_s(r, r^2) + 2(n+1)h(r, r^2) \equiv 0$ on $[a, b]$ provided that $f(\alpha_j, \beta_j) = \alpha_j + 4\beta_j - 2(n+1) = 0$ for each $1 \le j \le N$. Of course the class of $h > 0$ satisfying $rh_r(r, r^2) + 4r^2h_s(r, r^2) + 2(n + 1)h(r, r^2) \equiv 0$ is much broader.

Motivated by Theorem 5.2.8 if we take the twist path $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ with $\mathscr{G} \in \mathscr{C}^2[a, b]$ and $\mathbf{H}$ a constant $n\times n$ skew-symmetric matrix then, assuming that the ODE

$$\frac{d}{dr}\left[r^{n+1}F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2)\dot{\mathscr{G}}\right] = 0 \tag{5.5.8}$$

holds, by Theorem 5.4.2 we recover the discriminant

$$\Delta_F = \frac{2[(n+1)F_\xi + 2r^2\dot{\mathscr{G}}^2 F_{\xi\xi}|\mathbf{H}\theta|^2 + 2r^2 F_{s\xi}][F_\xi + r^2\dot{\mathscr{G}}^2 F_{\xi\xi}|\mathbf{H}\theta|^2] + rF_\xi F_{r\xi}}{r^2(F_\xi + 2r^2\dot{\mathscr{G}}^2 F_{\xi\xi}|\mathbf{H}\theta|^2)}. \tag{5.5.9}$$

It is possible to simplify the expression of Δ_F significantly, but first it is convenient to note that, with $\mathbf{A} = \dot{\mathscr{G}}\mathbf{H}$, the curl of the corresponding vector field $\mathscr{L}[u] - \nabla F_\xi$ is, by (5.4.8),

$$\mathrm{curl}\,(\mathscr{L}[u = r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta] - \nabla F_\xi) = -\dot{\mathscr{G}}^2\Delta_F[\mathbf{H}^2 x\otimes x - x\otimes\mathbf{H}^2 x]. \tag{5.5.10}$$

We proceed with calculating $\dot{\mathscr{G}}^2\Delta_F$ and towards this end we first observe that by multiplying (5.5.8) by a factor of $r^{-n}\dot{\mathscr{G}}F_\xi$ we have the identity

$$\begin{aligned} r\dot{\mathscr{G}}^2F_\xi F_{r\xi} = -\Big\{&(n+1)\dot{\mathscr{G}}^2F_\xi^2 + 2r^2\dot{\mathscr{G}}^2F_\xi F_{s\xi} + 2r^3\dot{\mathscr{G}}^3\ddot{\mathscr{G}}F_\xi F_{\xi\xi}|\mathbf{H}\theta|^2 \\ &+ 2r^2\dot{\mathscr{G}}^4F_\xi F_{\xi\xi}|\mathbf{H}\theta|^2 + r\dot{\mathscr{G}}\ddot{\mathscr{G}}F_\xi^2\Big\}. \end{aligned} \tag{5.5.11}$$

Now considering the numerator $\mathscr{I}$ of $\dot{\mathscr{G}}^2\Delta_F$ with Δ_F as in (5.5.9), upon using (5.5.11), we can write

$$\begin{aligned} \mathscr{I} =& 2\dot{\mathscr{G}}^2\left[(n+1)F_\xi + 2r^2\dot{\mathscr{G}}^2F_{\xi\xi}|\mathbf{H}\theta|^2 + 2r^2F_{s\xi}\right] \\ &\times\left[F_\xi + r^2\dot{\mathscr{G}}^2F_{\xi\xi}|\mathbf{H}\theta|^2\right] + r\dot{\mathscr{G}}^2F_\xi F_{r\xi} \\ =& (n+1)\dot{\mathscr{G}}^2F_\xi^2 + 2r^2(n+1)\dot{\mathscr{G}}^4F_\xi F_{\xi\xi}|\mathbf{H}\theta|^2 + 2r^2\dot{\mathscr{G}}^4F_\xi F_{\xi\xi}|\mathbf{H}\theta|^2 \\ &+ 4r^4\dot{\mathscr{G}}^6F_{\xi\xi}^2|\mathbf{H}\theta|^4 + 2r^2\dot{\mathscr{G}}^2F_\xi F_{s\xi} + 4r^4\dot{\mathscr{G}}^4F_{s\xi}F_{\xi\xi}|\mathbf{H}\theta|^2 \\ &- 2r^3\dot{\mathscr{G}}^3\ddot{\mathscr{G}}F_\xi F_{\xi\xi}|\mathbf{H}\theta|^2 - r\dot{\mathscr{G}}\ddot{\mathscr{G}}F_\xi^2 \\ =& \left[(n+1)\dot{\mathscr{G}}^2F_\xi + 2r^2\dot{\mathscr{G}}^2F_{s\xi} + 2r^2\dot{\mathscr{G}}^4F_{\xi\xi}|\mathbf{H}\theta|^2 - r\dot{\mathscr{G}}\ddot{\mathscr{G}}F_\xi\right] \\ &\times\left[F_\xi + 2r^2\dot{\mathscr{G}}^2F_{\xi\xi}|\mathbf{H}\theta|^2\right]. \end{aligned} \tag{5.5.12}$$

Hence returning to the discriminant Δ_F as expressed by (5.5.9) and with the numerator $\mathscr{I}$ of $\dot{\mathscr{G}}^2\Delta_F$ as calculated above we have

$$\dot{\mathscr{G}}^2\Delta_F = \frac{1}{r^2}\left[(n+1)\dot{\mathscr{G}}^2F_\xi + 2r^2\dot{\mathscr{G}}^2F_{s\xi} + 2r^2\dot{\mathscr{G}}^4F_{\xi\xi}|\mathbf{H}\theta|^2 - r\dot{\mathscr{G}}\ddot{\mathscr{G}}F_\xi\right]. \tag{5.5.13}$$

A further application of the ODE (5.5.8) now leads to the following statement.

Proposition 5.5.1. *Assume the twist path $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ with $\mathscr{G} \in \mathscr{C}^2[a,b]$ and $\mathbf{H}$ skew-symmetric satisfies* (5.5.8). *Then the Lagrangian discriminant Δ_F along $\mathbf{Q}$ admits the formulation*

$$\Delta_F = \frac{1}{r^2}\left[2(n+1)F_\xi + rF_{r\xi} + 4r^2F_{s\xi} + 2r^2\dot{\mathscr{G}}(2\dot{\mathscr{G}} + r\ddot{\mathscr{G}})F_{\xi\xi}|\mathbf{H}\theta|^2\right]. \tag{5.5.14}$$

As a direct application of the above consider again $F(r,s,\xi)=h(r,s)\xi$. Then substitution in (5.5.14) gives $\Delta_F=[2(n+1)h(r,r^2)+rh_r(r,r^2)+4r^2h_s(r,r^2)]/r^2$ in full agreement with (5.5.5).

5.6 The Nonlinear System (5.1.4) and the Infinite Twist Solutions to the PDE $\mathscr{L}[u]=\nabla\mathscr{P}$

Having discussed the three families of ODEs (5.1.9)-(5.1.11), their solvability for twist paths $\mathbf{Q}=\mathbf{Q}(r)$ and their relationships to one-another, in this final section of the chapter we return to the nonlinear system

$$\begin{cases} \mathscr{L}[u]=\nabla\mathscr{P} & \text{in } \mathbb{X}^n, \\ \det\nabla u=1 & \text{in } \mathbb{X}^n, \\ u\equiv x & \text{on } \partial\mathbb{X}^n, \end{cases} \tag{5.6.1}$$

with

$$\mathscr{L}[u]=(\nabla u)^t\left\{\operatorname{div}[F_\xi(|x|,|u|^2,|\nabla u|^2)\nabla u]-F_s(|x|,|u|^2,|\nabla u|^2)u\right\} \tag{5.6.2}$$

and address questions of existence and multiplicity of twist solutions $u=r\mathbf{Q}(r)\theta$ to the system. Recall that here the starting assumption is that the twist path $\mathbf{Q}=\mathbf{Q}(r)$ is a solution to (5.1.11). As a result the differential action $\mathscr{L}[u]$ admits the formulation in Proposition 5.4.1 and the curl of the vector field $\mathscr{L}[u]-\nabla F_\xi$ factorises into a product entailing the Lagrangian discriminant Δ_F as in (5.4.9) and the tensor field $[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}x\otimes x-x\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}x]$ for $x\in\mathbb{X}^n$ (*c.f.* Theorem 5.4.2 for details). The next theorem describes the implications of the PDE on the twist path disregarding any boundary conditions.

Theorem 5.6.1. *Let* $u=r\mathbf{Q}(r)\theta$ *with* $\mathbf{Q}\in\mathscr{C}^2(]a,b[,\mathbf{SO}(n))\cap\mathscr{C}([a,b],\mathbf{SO}(n))$. *Assume that* $\mathbf{Q}$ *satisfies the ODE* (5.1.11) *and that the Lagrangian discriminant* Δ_F *is nowhere zero in* $\mathbb{X}^n$. *Then the following are equivalent:*

- (*i*) $\mathscr{L}[u]=\nabla\mathscr{P}$ *in* $\mathbb{X}^n$,
- (*ii*) $\operatorname{curl}(\mathscr{L}[u]-\nabla F_\xi)=0$ *in* $\mathbb{X}^n$,
- (*iii*) $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}=\alpha(r)\mathbf{I}_n$ *for some non-negative* $\alpha\in\mathscr{C}^1]a,b[\cap\mathscr{C}[a,b]$,

Furthermore under any of these assumptions $\nabla\mathscr{P}$ *in* (i) *takes the specific form*

$$\begin{aligned}\nabla\mathscr{P}=&\nabla F_\xi(|x|,|x|^2,n+|x|^2\alpha(|x|))-F_\xi(|x|,|x|^2,n+|x|^2\alpha(|x|))\alpha(|x|)x\\&-F_s(|x|,|x|^2,n+|x|^2\alpha(|x|))x.\end{aligned}\tag{5.6.3}$$

Proof. If $\mathbf{Q}$ satisfies (5.1.11) and Δ_F is nowhere zero in $\mathbb{X}^n$ then the implication $(ii)\Longrightarrow(iii)$ follows from Theorem 5.4.2. Substituting (iii) into the vector field v as given by (5.4.10) yields (recall that $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}=-\mathbf{A}^2$ for $\mathbf{A}=\mathbf{Q}^t\dot{\mathbf{Q}}$):

$$v(x)=-F_\xi(|x|,|x|^2,n+|x|^2\alpha(|x|))\alpha(|x|)x-F_s(|x|,|x|^2,n+|x|^2\alpha(|x|))x.\tag{5.6.4}$$

Now let $\Phi=\Phi(r)$ be a primitive for $f(r):=-r[F_\xi\alpha+F_s]$, that is, $\Phi'=f$. Then $v=\nabla\Phi(|x|)$ and so $\mathscr{L}[u]=\nabla F_\xi+v=\nabla\mathscr{P}$ with $\nabla\mathscr{P}$ as per (5.6.3). This justifies $(iii)\Longrightarrow(i)$. Next assume (i). Then referring to (5.4.6), $v=\mathscr{L}[u]-\nabla F_\xi$ is a $\mathscr{C}^1$ gradient field in $\mathbb{X}^n$ and so as a result its curl vanishes. This gives (ii). □

Theorem 5.6.2. *Let* $u=r\mathbf{Q}(r)\theta$ *where* $\mathbf{Q}\in\mathscr{C}^2(]a,b[,\mathbf{SO}(n))\cap\mathscr{C}^1([a,b],\mathbf{SO}(n))$ *and* $\mathbf{Q}(a)=\mathbf{Q}(b)=\mathbf{I}_n$. *Assume that* $\mathbf{Q}$ *satisfies the ODE* (5.1.11) *and that* Δ_F *is nowhere zero in* $\mathbb{X}^n$. *Then the following are equivalent:*

(i) *u is a solution to the system* (5.6.1)-(5.6.2).

(ii) *Depending on n being even or odd* $\mathbf{Q}$ *has the representation:*

- *n even: There exists* $\mathbf{P}\in\mathbf{O}(n)$ *and* $m\in\mathbb{Z}$ *such that*

$$\mathbf{Q}=\mathbf{Q}(r;m)=\mathbf{P}diag(\mathcal{R}[\mathscr{G}](r;m),\dots,\mathcal{R}[\mathscr{G}](r;m))\mathbf{P}^t,$$

  where $\mathscr{G}=\mathscr{G}(r;m)\in\mathscr{C}^2[a,b]$ *is the unique solution to the two point boundary value problem* (5.2.26) *and* $\mathcal{R}$ *is as defined by* (5.2.21).

- *n odd:* $\mathbf{Q}\equiv\mathbf{I}_n$ *on* $[a,b]$ *and hence* $u\equiv x$.

Proof. In view of the twist path $\mathbf{Q}=\mathbf{Q}(r)$ being a solution to the ODE (5.1.11) and Δ_F being nowhere zero in $\mathbb{X}^n$ from Theorem 5.6.1 we have:

$$\begin{aligned}\mathscr{L}[u]=\nabla\mathscr{P}\iff&\operatorname{curl}(\mathscr{L}[u]-\nabla F_\xi)=0\\\iff&\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta-\theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta=0\\\iff&\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}=\alpha(r)\mathbf{I}_n.\end{aligned}$$

The equivalence $(i)\iff(ii)$ now follows from Theorem 5.2.8 upon noting that by the last identity $|\dot{\mathbf{Q}}\theta|^2$ is independent of θ. The proof is thus complete. □

Let us finish the chapter by presenting two examples depicting contrasting behaviour of the discriminant and the implication it bears on the associated twist solutions to the system (5.1.4).

Firstly, assume $F(r,s,\xi) = F(\xi)$. Then by (5.5.1), Δ_F is strictly positive and hence nowhere zero in $\mathbb{X}^n$. If $\mathbf{Q} \in \mathscr{C}^2(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}^1([a,b], \mathbf{SO}(n))$ satisfying $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ is a solution to the ODE (5.1.11) then the associated twist $u = r\mathbf{Q}(r)\theta$ is a solution to the system (5.1.4) *iff* $\mathbf{Q}$ is as described in part (ii) of Theorem 5.6.2, that is, $\mathbf{Q} \equiv \mathbf{I}_n$ for n odd or $\mathbf{Q}$ is a geodesic loop of the form $\mathbf{Q}(r) = \mathbf{P}\exp\{\mathscr{G}(r) diag(\mathbf{J}, \dots, \mathbf{J})\}\mathbf{P}^t$ with $\mathscr{G}$ a solution to (5.2.26) and $\mathbf{P} \in \mathbf{O}(n)$ for n even. By (5.5.2) the same conclusion holds for Lagrangians of the form $F(r,s,\xi) = F(\xi) + G(r,s)$.

In sharp contrast consider next $F(r,s,\xi) = h(r,s)\xi$ as in Section 5.3. Then as seen $r^2\Delta_F = 2(n+1)h(r,r^2) + 4r^2 h_s(r,r^2) + r h_r(r,r^2)$ for $a \le r \le b$ [*cf.* (5.5.5)]. By Theorem 5.3.1 any solution to the ODE (5.1.11) satisfying $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ has the form $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ with $\mathscr{H}$ as in (5.3.3) and $\mathbf{H}$ as in (5.3.4). For the system (5.1.4) with $\mathscr{L}_F[u] = (\nabla u)^t\{\mathrm{div}[h(|x|,|u|^2)\nabla u] - h_s(|x|,|u|^2)|\nabla u|^2 u\}$, however, we have the following:[11]

- If $2(n+1)h(r,r^2) + 4r^2 h_s(r,r^2) + r h_r(r,r^2) \not\equiv 0$ over $[a,b]$: Every twist solution to (5.1.4) of class $\mathscr{C}^2$ has the form $u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta$ with $\mathbf{H}$ given by

$$\mathbf{H} = \begin{cases} 2m\pi \mathbf{P} diag(\mathbf{J}, \dots, \mathbf{J})\mathbf{P}^t & n = 2k, \\ 0 & n = 2k-1. \end{cases} \tag{5.6.5}$$

- If $2(n+1)h(r,r^2) + 4r^2 h_s(r,r^2) + r h_r(r,r^2) \equiv 0$ over $[a,b]$: Every twist solution to (5.1.4) of class $\mathscr{C}^2$ has the form $u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta$ with

$$\mathbf{H} = \begin{cases} \mathbf{P} diag(2m_1\pi\mathbf{J}, \dots, 2m_k\pi\mathbf{J})\mathbf{P}^t & n = 2k, \\ \mathbf{P} diag(2m_1\pi\mathbf{J}, \dots, 2m_{k-1}\pi\mathbf{J}, 2m_k\pi)\mathbf{P}^t & n = 2k-1. \end{cases} \tag{5.6.6}$$

  Here $\mathbf{P} \in \mathbf{O}(n)$, $\mathbf{J}$ is as in (5.2.21) and $m_1, \dots, m_k \in \mathbb{Z}$ are *arbitrary* with only $m_k = 0$ in odd dimensions.

[11] For details and more see Chapter 3.

Chapter 6

Whirl Maps as Solutions to $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$ in Low Dimensions

In this chapter we study a non-variational, nonlinear and in divergence form PDE

$$\operatorname{div}[\mathsf{A}(|x|, |u|^2, |\nabla u|^2)\nabla u] + \mathsf{B}(|x|, |u|^2, |\nabla u|^2)u = [\operatorname{cof} \nabla u]\nabla \mathscr{P}$$

where the solution $u : \Omega \to \mathbb{R}^n$ is subject to suitable boundary conditions as well as the incompressibility constraint $\det \nabla u = 1$ *almost everywhere* in $\Omega \subset \mathbb{R}^n$. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\mathsf{A} = \mathsf{A}(r, s, \xi)$ and $\mathsf{B} = \mathsf{B}(r, s, \xi)$ are continuous scalar-valued functions satisfying suitable growth at infinity. We solve this specifically for whirl maps $u(x) = \mathbf{Q}(\varrho)x$ where $\mathbf{Q}$ is an $\mathbf{SO}(n)$-valued map taking values in the maximal torus of block-diagonal rotation matrices and $\varrho = (\rho_1, \dots, \rho_N)$ a suitable vector of two-plane radial variables. We focus on low spatial dimensions $n = 2, 3, 4$ by implementing a polar coordinate system to re-frame the analysis in a novel way. We study in particular the case when the function $\mathsf{A}(r, s, \xi) = h(r, s)$ shows no dependence on the third variable and throughout the text consider the notion of a *discriminant* $\Delta(\mathsf{A}, \mathsf{B})$ which has a significant influence on the solution set of the PDE above.

6.1 Preliminaries and Motivation

This chapter is concerned with the solvability of the nonlinear and non-variational elliptic system

$$\begin{cases}\mathrm{div}[\mathsf{A}(|x|,|u|^2,|\nabla u|^2)\nabla u]+\mathsf{B}(|x|,|u|^2,|\nabla u|^2)u=[\mathrm{cof}\,\nabla u]\nabla\mathscr{P} & \text{in } \Omega,\\ \det\nabla u=1 & \text{in } \Omega,\\ u=\varphi & \text{on } \partial\Omega,\end{cases} \tag{6.1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$ and $\mathscr{P}=\mathscr{P}(x)$ is *a priori* unknown. The real-valued functions $\mathsf{A}=\mathsf{A}(r,s,\xi)$ and $\mathsf{B}=\mathsf{B}(r,s,\xi)$ are, respectively, of class $\mathscr{C}^1$ and $\mathscr{C}^0$ with A being strictly positive and monotone increasing in the third variable. We consider the admissible class of solutions

$$\mathscr{A}^p_\varphi(\Omega):=\{u\in W^{1,p}(\Omega,\mathbb{R}^n):\det\nabla u=1,u|_{\partial\Omega}=\varphi\},\qquad p\geq 1. \tag{6.1.2}$$

These therefore are the incompressible p-Sobolev maps and $\mathsf{A}(|x|,|u|^2,|\nabla u|^2)\nabla u$ defines a matrix field upon which the divergence in the first line of (6.1.1) acts row-wise. The boundary data is interpreted in the sense of traces and for simplicity we take throughout the chapter $\varphi\equiv x$ to be the identity. We impose the pointwise incompressibility constraint $\det\nabla u=1$ *a.e.* in Ω which leads to the algebraic identity $[\mathrm{cof}\,\nabla u]^{-1}=(\nabla u)^t$. This being so the PDE governing (6.1.1) will be written as $\mathscr{L}[u;\mathsf{A},\mathsf{B}]=\nabla\mathscr{P}$,[12] where

$$\mathscr{L}[u;\mathsf{A},\mathsf{B}]:=(\nabla u)^t\left\{\mathrm{div}[\mathsf{A}(|x|,|u|^2,|\nabla u|^2)\nabla u]+\mathsf{B}(|x|,|u|^2,|\nabla u|^2)u\right\}. \tag{6.1.3}$$

We can pose the system in a variational setting where $\mathsf{A}(r,s,\xi)=F_\xi(r,s,\xi)$, $\mathsf{B}(r,s,\xi)=-F_s(r,s,\xi)$ for $F=F(r,s,\xi)$ a Lagrangian of class $\mathscr{C}^2$ where F_s, F_ξ denote its derivatives in the second and third arguments respectively. This being so the system (6.1.1) arises as the Euler-Lagrange equation associated with the energy functional

$$\mathbb{F}[u;\Omega]:=\int_\Omega F(|x|,|u|^2,|\nabla u|^2)\,dx \tag{6.1.4}$$

over the admissible space of incompressible p-Sobolev maps $\mathscr{A}^p_\varphi(\Omega)$. In this case the unknown $\mathscr{P}$ appearing in (6.1.1) enters the system as a Lagrange multiplier.

[12] One can immediately see that the identity map $u\equiv x$ is a solution of the above, indeed by substitution we see that $\mathscr{L}[u\equiv x;\mathsf{A},\mathsf{B}]=\nabla\mathsf{A}(|x|,|x|^2,n)+\mathsf{B}(r,r^2,n)x=\nabla\mathscr{P}$ for $\mathscr{P}=\mathsf{A}(|x|,|x|^2,n)+G(|x|)$ such that $\nabla G(|x|)=\mathsf{B}(r,r^2,n)x$.

Such an exposition has its roots in the theory of nonlinear elasticity, wherein the map u plays the role of a volume-preserving deformation of a bounded domain $\Omega \subset \mathbb{R}^n$. In this setting the Lagrange multiplier $\mathscr{P}$ describes a hydrostatic pressure field. Despite this construction we emphasise that in this chapter we do not assume that the system (6.1.1) arises from the variational energy (6.1.4) and that the assumptions to be imposed on the functions $\mathsf{A}(r,s,\xi)$ and $\mathsf{B}(r,s,\xi)$ will be enough to facilitate a detailed analysis in a more general setting.

Let us now describe the geometric setup in this chapter. We will seek solutions of (6.1.1) exhibiting certain symmetries and as such we first restrict the spatial variable x to the n-dimensional generalised annulus $\mathbb{X}^n = \mathbb{X}^n[a,b] \subset \mathbb{R}^n$ defined by $\mathbb{X}^n := \{x \in \mathbb{R}^n : a < x < b\}$ with $0 < a < b < \infty$. Regarding the assumptions on the functions A and B we have $\mathsf{A} = \mathsf{A}(r,s,\xi) \in \mathscr{C}^1(U)$, $\mathsf{B} = \mathsf{B}(r,s,\xi) \in \mathscr{C}(U)$ where $U = U(\mathbb{X}^n[a,b]) =]a,b[\times]0,\infty[\times]0,\infty[\subset \mathbb{R}^3$. For every compact set $K \subset]0,\infty[$ there exist positive constants $c_1 = c_1(K)$, $c_2 = c_2(K)$ such that

$$c_1|\zeta|^{p-1} \le \mathsf{A}(r,s,\zeta^2)|\zeta| \le c_2|\zeta|^{p-1} \quad \forall\, (r,s,\xi) \in U : s \in K,\, p > 1.$$

We assume that the function A is strictly positive and monotone increasing in the third variable, that is $\mathsf{A}_\xi(r,s,\xi) \ge 0$ for all $(r,s,\xi) \in U$.

In terms of admissible solutions to the system (6.1.1) we consider exclusively *whirl* maps $u \in \mathscr{C}^2(\overline{\mathbb{X}^n}, \overline{\mathbb{X}^n})$. These are continuous self-maps over the closure of $\mathbb{X}^n$ given by

$$u(x) = \mathbf{Q}(\rho_1, \dots, \rho_N)x. \tag{6.1.5}$$

Here the vector of 2-planar radial variables $\varrho = (\rho_1, \dots, \rho_N)$ is described, depending on whether the underlying spatial dimension n is even or odd, as follows.

- If the dimension $n = 2d$ is even then we set $N = d$ and for each $1 \le j \le d$ we define $\zeta_j := (x_{2j-1}, x_{2j})$. Then $\rho_j = \|\zeta_j\|$ and $x \in \mathbb{X}^n \iff \varrho \in \mathbb{A}_n$ where
$$\mathbb{A}_n := \{\varrho \in \mathbb{R}^d_+ : a < \|\varrho\| < b\}. \tag{6.1.6}$$
- If $n = 2d+1$ is odd then set $N = d+1$. The variables $\rho_1, \dots, \rho_d$ are given by $\rho_j = \|\zeta_j\|$ and we set $\rho_{d+1} := x_n$. Then $x \in \mathbb{X}^n \iff \varrho \in \mathbb{A}_n$ where, similarly to the above,
$$\mathbb{A}_n := \{\varrho \in \mathbb{R}^d_+ \times \mathbb{R} : a < \|\varrho\| < b\}. \tag{6.1.7}$$

For illustrative purposes we describe the setting in dimensions $n=3$ and $n=4$ here; in the former case the vector ϱ is described by $\varrho=(\rho,z)$ with $\rho=(x_1^2+x_2^2)^{1/2}$, $z=x_3$. Similarly When $n=4$ we have $\varrho=(\rho_1,\rho_2)$ with $\rho_1=(x_1^2+x_2^2)^{1/2}$, $\rho_2=(x_3^2+x_4^2)^{1/2}$. The semi-annular region $\mathbb{A}_n$ defined in odd and even dimensions above admits the boundary

$$\partial\mathbb{A}_n:=(\partial\mathbb{A}_n)_a\cup(\partial\mathbb{A}_n)_b\cup\Gamma_n, \tag{6.1.8}$$

where $(\partial\mathbb{A}_n)_a=\{\varrho\in\partial\mathbb{A}_n:\|\varrho\|=a\}$ and similarly $(\partial\mathbb{A}_n)_b=\{\varrho\in\partial\mathbb{A}_n:\|\varrho\|=b\}$. Γ_n is a disconnected set simply defined as $\Gamma_n=\partial\mathbb{A}_n\setminus\{(\partial\mathbb{A}_n)_a\cup(\partial\mathbb{A}_n)_b\}$.

The map $\mathbf{Q}$ takes values in the compact Lie group of rotation matrices $\mathbf{SO}(n)$ and given the assumed boundary data $u\equiv x$ we will impose the condition $\mathbf{Q}(\varrho)=\mathbf{I}_n$ for $\varrho\in(\partial\mathbb{A}_n)_a\cup(\partial\mathbb{A}_n)_b$ and where $\mathbf{I}_n$ is the $n\times n$ identity matrix. As such $\mathbf{Q}$ manifests a closed loop in $\mathbf{SO}(n)$ with initial and terminal point at $\mathbf{I}_n$. By considering symmetries it is necessary to assume that the matrix map $\mathbf{Q}$ takes values in the maximal torus $\mathbb{T}$ of $\mathbf{SO}(n)$ consisting of 2×2 block-diagonal rotation matrices and we specifically consider loops of the form $\mathbf{Q}(\varrho)=\exp\{\mathbf{H}(\varrho)\}$, where $\mathbf{H}:\overline{\mathbb{A}_n}\to\mathfrak{so}(n)$ is given by

$$\mathbf{H}(\varrho)=\begin{cases} diag(f_1\mathbf{J},\dots,f_d\mathbf{J}) & n=2d,\\ diag(f_1\mathbf{J},\dots,f_d\mathbf{J},0) & n=2d+1.\end{cases} \tag{6.1.9}$$

Here the matrix $\mathbf{J}$ describes a counterclockwise rotation by an angle of $\pi/2$ as per (6.3.2) and the functions $f_\ell\in\mathscr{C}(\overline{\mathbb{A}_n})$ for all $1\le\ell\le d$ satisfy $f_\ell\equiv0$ on $(\partial\mathbb{A}_n)_a$ and $f_\ell\equiv2m_\ell\pi$ on $(\partial\mathbb{A}_n)_b$ for $m_\ell\in\mathbb{Z}$. We see that the matrix $\mathbf{H}$ takes values in $\mathfrak{so}(n)$, which is the space of $n\times n$ skew-symmetric matrices and the Lie algebra associated to the Lie group $\mathbf{SO}(n)$. As such, in the definition $\mathbf{Q}(\varrho)=\exp\{\mathbf{H}(\varrho)\}$, exp denotes the matrix exponential which serves as the canonical exponential map from $\mathfrak{so}(n)$ to $\mathbf{SO}(n)$. Any necessary preliminaries on the theory of Lie groups and associated matters can be found in [27, 35, 45, 47, 73] and references therein.

To avoid confusion throughout this chapter we will denote any calculus operations undertaken with respect to the ϱ variables with a subscript $\mathbb{A}$, for example $\mathrm{div}_{\mathbb{A}}=\sum_{j=1}^N\partial_{\rho_j}$. If subscripts are ever omitted it should be clear from context that we are working with respect to the x variables, as for example in the statement of the main Euler-Lagrange system in (6.1.1). At times, however we will employ the subscript $\mathbb{X}$ when operating with respect to the variables $x_1,\dots,x_n$ to avoid ambiguity.

Part of our strategy will be to consider the following boundary value problem associated to the angle of rotation functions $f = (f_1, \dots, f_d)$ where, for each $1 \le \ell \le d$, we have

$$\begin{cases} \operatorname{div}_{\mathbb{A}} \left[\mathsf{A}\left(\|\varrho\|, \|\varrho\|^2, n + \sum_{\ell=1}^{d} \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2\right) \rho_l^2 \omega(\varrho; d) \nabla_{\mathbb{A}} f_\ell\right] = 0 & \text{in } \mathbb{A}_n, \\ f \equiv 0 & \text{on } (\partial\mathbb{A}_n)_a, \\ f \equiv 2\mathsf{m}\pi & \text{on } (\partial\mathbb{A}_n)_b, \\ \mathsf{A}\left(\|\varrho\|, \|\varrho\|^2, n + \sum_{\ell=1}^{d} \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2\right) \rho_l^2 \omega(\varrho; d) \partial_\nu f_\ell = 0 & \text{on } \Gamma_n. \end{cases} \tag{6.1.10}$$

Here $\mathsf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, $\omega(\varrho; d) = \rho_1 \dots \rho_d$ as defined in (6.2.1) and ∂_ν is the outward-pointing unit normal on the flat part of the boundary, Γ_n. This restricted system arises naturally in a variational context as an Euler-Lagrange equation of its own and is derived in the following section. We consider this system to be of independent interest but the role it plays in the chapter will be crucial. Its unique solution $f = (f_1, \dots, f_d)$ will be substituted into the description of a whirl map as in (6.1.9) and it is for such whirls that we aim to solve the full system (6.1.1).

In Section 6.3 we conduct a full analysis of this and the resulting system (6.1.1) when $n = 3, 4$. Here $N = 2$, so by introducing polar coordinates (r, θ) where $a \le r \le b$ and the range of θ depends on the underlying dimension n we solve the Euler-Lagrange equation above as well as the full system (6.1.1) in a novel way. Solutions here depend acutely on a *discriminant* term $\Delta(\mathsf{A}, \mathsf{B})$ [*see* the explicit examples (6.3.18) and (6.3.40)] extracted upon studying the irrotationality of the vector field $\mathscr{L}[u; \mathsf{A}, \mathsf{B}]$.[13] By way of motivating the introduction of this discriminant we see in Theorem 6.3.1 that when $\Delta(\mathsf{A}, \mathsf{B}) \not\equiv 0$ over $\mathbb{A}_n$ the only solution of $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla\mathscr{P}$ when the dimension $n = 3$ is the identity map $u \equiv x$, whereas if $\Delta(\mathsf{A}, \mathsf{B}) \equiv 0$ there is, in great contrast, an infinitude of admissible solutions to this PDE.

Section 6.4 then considers a particular variant of the system (6.1.1) where

[13] Recall that all gradient fields are necessarily curl-free, so if a whirl map u should act as a solution to the PDE (6.1.3) we see that $\operatorname{curl} \mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \operatorname{curl} \nabla\mathscr{P} \equiv 0$.

$\mathsf{A}(r, s, \xi) = h(r, s)$ for some $0 < h \in \mathscr{C}^2([a, b] \times \mathbb{R})$. That is we consider

$$\begin{cases} (\nabla u)^t \{\operatorname{div}[h(|x|, |u|^2)\nabla u] + \mathsf{B}(|x|, |u|^2, |\nabla u|^2)u\} = \nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{6.1.11}$$

In this particular setting it can be seen that the picture simplifies and we are able to classify explicitly all solutions u to the PDE $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$ in all spatial dimensions $n \geq 2$. Here too, when searching for solutions of the full PDE we work with a *discriminant* term $\Delta(\mathsf{A}, \mathsf{B})$ depending on the functions A and B and their derivatives as well as the underlying spatial dimension n. In this n-dimensional context and with $\mathsf{A}(r, s, \xi) = h(r, s)$ this admits the description [*c.f.* (6.4.14)]

$$\Delta(h, \mathsf{B}) := 2\frac{(n+1)}{r^2}h(r, r^2) + \frac{1}{r}h_r(r, r^2) + 2[h_s(r, r^2) - B_\xi(r, r^2, n + \dot{\mathscr{H}}^2|\mathbf{H}x|^2)]. \tag{6.1.12}$$

This system is a generalisation of that considered in Section 4.3 for generalised twist maps $u(x) = \mathbf{Q}(|x|)x$ where $\mathsf{B}(r, s, \xi) = g(r, s)\xi$ and also acts as a non-variational analogy to the weighted Dirichlet setting considered widely throughout the text and principally in Chapter 3.

6.2 The Variational System BVP$[f; \mathscr{U}, \mathsf{m}]$

This section is devoted to the derivation of the system given in (6.1.10) by variational methods and a brief study of its solution and impact on the PDE $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$ when the spatial dimension $n = 2$. We begin by introducing some notational conventions used here and throughout the chapter. We consider the vector of functions $f = (f_1, \dots, f_d)$ appearing in (6.1.9) taken from the admissible space $\mathscr{B}^p_{\mathsf{m}}(\mathbb{A}_n) := \{f = (f_1, \dots, f_d) \in W^{1,p}(\mathbb{A}_n, \mathbb{R}^d) : f \equiv 0 \text{ on } (\partial\mathbb{A}_n)_a,\ f \equiv 2\mathsf{m}\pi \text{ on } (\partial\mathbb{A}_n)_b,\ p \geq 1\}$ for $\mathsf{m} \in \mathbb{Z}^d$. Moreover we will employ the abbreviated notation $\omega(\varrho; d)$ to symbolise

$$\omega(\varrho; d) = \omega(\rho_1, \dots, \rho_d; d) := \prod_{j=1}^{d} \rho_j, \tag{6.2.1}$$

with $\omega(\varrho; d) > 0$ in $\mathbb{A}_n$.[14]

[14]Note that the dimension $n = 2d$ is even $\omega(\varrho; d)$ features all components $\rho_1, \dots, \rho_N$ of the vector ϱ, whereas when $n = 2d + 1$ is odd, $\omega(\varrho; d)$ only accounts for $\rho_1, \dots, \rho_{N-1}$.

To motivate ideas here we may think of the scalar-valued function $\mathsf{A}(r, s, \xi)$ appearing in (6.1.1) as a derivative of some Lagrangian $E = E(r, s, \xi)$ in the third variable, in the sense that

$$E(r, s, \xi) = \int_0^\xi \mathsf{A}(r, s, \zeta)\, d\zeta, \qquad a \le r \le b,\ s > 0,\ \xi > 0, \tag{6.2.2}$$

so that $E_\xi(r, s, \xi) = \mathsf{A}(r, s, \xi)$. If E is the Lagrangian of some energy functional as in

$$\mathbb{E}[f; \mathbb{A}_n] := \int_{\mathbb{A}_n} E\left(\|\varrho\|, \|\varrho\|^2, n + \sum_{\ell=1}^d \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2\right) \omega(\varrho; d)\, d\varrho, \tag{6.2.3}$$

then the following result holds.

Theorem 6.2.1. *Consider the variational energy functional $\mathbb{E}[f; \mathbb{A}_n]$ where $F_\xi(r, s, \xi) = \mathsf{A}(r, s, \xi)$ as in (6.2.2). Then the Euler-Lagrange equation associated to $\mathbb{E}[f; \mathbb{A}_n]$ over the admissible space*

$$\mathscr{B}(\mathbb{A}_n; \mathsf{m}) = \bigcup_{\mathsf{m} \in \mathbb{Z}^d} \mathscr{B}^p_{\mathsf{m}}(\mathbb{A}_n), \tag{6.2.4}$$

is the system

$$\mathbf{BVP}[f; \mathscr{U}, \mathsf{m}] = \begin{cases} \mathrm{div}_{\mathbb{A}} \mathscr{U}(\varrho, \nabla_{\mathbb{A}} f) = 0 & \text{in } \mathbb{A}_n, \\ f \equiv 0 & \text{on } (\partial \mathbb{A}_n)_a, \\ f \equiv 2\mathsf{m}\pi & \text{on } (\partial \mathbb{A}_n)_b, \\ \mathscr{U}(\varrho, \nabla_{\mathbb{A}} f)\nu = 0 & \text{on } \Gamma_n, \end{cases} \tag{6.2.5}$$

where $\mathscr{U}$ is the $d \times N$-dimensional matrix field with row components $\mathscr{U}_\ell$ given by

$$\mathscr{U}_\ell(\varrho, \nabla_{\mathbb{A}} f) := \mathsf{A}\left(\|\varrho\|, \|\varrho\|^2, n + \sum_{\ell=1}^d \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2\right) \rho_\ell^2 \omega(\varrho; d) \nabla_{\mathbb{A}} f_\ell. \tag{6.2.6}$$

Here the divergence in the first line acts row-wise whilst in the Neumann boundary condition ν is the outward-pointing unit normal vector to Γ_n. Regarding the arguments of $\mathsf{A}(r, s, \xi) = \mathsf{A}(|x|, |u|^2, |\nabla u|^2)$ in (6.2.6) we see that $|x| = \|\varrho\|$ and $|u| = |\mathbf{Q}x| = |x| = \|\varrho\|$. For the third argument we use the description of $|\nabla u|^2$ for $u = \mathbf{Q}(\varrho)x$ as appearing in Proposition A.0.6. We remark that the uniqueness of any solution $f \in \mathscr{C}^2(\overline{\mathbb{A}_n}, \mathbb{R}^d)$ to the system $\mathbf{BVP}[f; \mathscr{U}, \mathsf{m}]$ is established in Proposition C.0.2.

We can now explicitly describe solutions to the system (6.2.5)-(6.2.6) when the spatial dimension n is even and the vector m admits equal entries, that is, $m_1 = \cdots = m_d =: m$ for $m \in \mathbb{Z}$. In this case we see that for each $1 \leq \ell \leq d$ the angle of rotation function f_ℓ satisfies $f_\ell(a) = 0$, $f_\ell(b) = 2m\pi$, thus we have equality of boundary conditions for the functions $f_1, \ldots f_d$ which is not *a priori* assumed in general.

Theorem 6.2.2. *For $n = 2d$ even consider the system* (6.2.5)-(6.2.6) *with $m_1 = \cdots = m_d =: m \in \mathbb{Z}$. This admits the solution $f(\varrho; \mathsf{m}) = (f_1, \ldots, f_d)$ where for each $1 \leq \ell \leq d$ we have $f_\ell = f_\ell(\varrho; m_\ell) = \mathscr{G}(\|\varrho\|; m)$ such that $\mathscr{G} \in \mathscr{C}^2[a, b]$ solves the two-point boundary value problem*

$$\begin{cases} \dfrac{d}{dr}\left[r^{n+1}\mathsf{A}(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\right] = 0, \quad a < r < b, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2m\pi. \end{cases} \tag{6.2.7}$$

We note that the existence *and* uniqueness of solutions to (6.2.7) with the required $\mathscr{C}^2$-regularity is established in Proposition C.0.1.

Proof. We begin by verifying the divergence-free statement in the first line of (6.2.5) and, taking $f_\alpha(\varrho; m_\alpha) = \mathscr{G}(\|\varrho\|; m) = \mathscr{G}(r; m)$ for any $1 \leq \alpha, i \leq d$ we have

$$\frac{\partial f_\alpha}{\partial \rho_i} = \dot{\mathscr{G}}\frac{\rho_i}{r} \implies \sum_{\ell=1}^{d} \rho_\ell^2 |\nabla_{\mathbb{A}} f_\alpha|^2 = \sum_{\ell=1}^{d} \rho_\ell^2 \dot{\mathscr{G}}^2 \frac{\rho_\ell^2}{r^2} = r^2\dot{\mathscr{G}}^2. \tag{6.2.8}$$

This being so we must compute the divergence (for all $1 \leq \alpha \leq d$)

$$\begin{aligned} \operatorname{div}_{\mathbb{A}} \mathscr{U}_\alpha(\varrho, \nabla_{\mathbb{A}} f) &= \operatorname{div}_{\mathbb{A}} \left[\mathsf{A}\left(r, r^2, n + \sum_{\ell=1}^{d} \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2\right) \rho_\alpha^2 \omega(\varrho; d) \nabla_{\mathbb{A}} f_\alpha\right] \\ &= \sum_{i=1}^{d} \frac{\partial}{\partial \rho_i}\left[\mathsf{A}(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\frac{\rho_i\rho_\alpha^2}{r}\omega(\varrho; d)\right]. \end{aligned} \tag{6.2.9}$$

Upon abbreviating the arguments of $\mathsf{A} = \mathsf{A}(r, r^2, n + r^2\dot{\mathscr{G}}^2)$ and an application of the product rule (6.2.9) becomes

$$
\begin{aligned}
\sum_{i=1}^{d} \frac{\partial}{\partial \rho_i} \left[\mathsf{A}\dot{\mathscr{G}} \frac{\rho_i \rho_\alpha^2}{r} \, \omega(\varrho; d) \right] = \sum_{i=1}^{d} \Bigg[& \dot{\mathsf{A}}\dot{\mathscr{G}} \frac{\rho_i^2 \rho_\alpha^2}{r^2} \omega(\varrho; d) + \mathsf{A}\ddot{\mathscr{G}} \frac{\rho_i^2 \rho_\alpha^2}{r^2} \omega(\varrho; d) \\
& - \mathsf{A}\dot{\mathscr{G}} \frac{\rho_i^2 \rho_\alpha^2}{r^2} \omega(\varrho; d) + \mathsf{A}\dot{\mathscr{G}} \frac{\rho_\alpha^2}{r} \omega(\varrho; d) \\
& + 2\mathsf{A}\dot{\mathscr{G}} \frac{\rho_i \rho_\alpha}{r} \delta_{i\alpha} \omega(\varrho; d) + \mathsf{A}\dot{\mathscr{G}} \frac{\rho_i \rho_\alpha^2}{r} \prod_{\substack{j=1 \\ j \neq i}}^{d} \rho_j \Bigg] \\
= {} & \frac{\rho_\alpha^2}{r} \omega(\varrho; d) \left\{ r\dot{\mathsf{A}}\dot{\mathscr{G}} + r\mathsf{A}\ddot{\mathscr{G}} + (2d+1)\mathsf{A}\dot{\mathscr{G}} \right\} \\
= {} & \frac{\rho_\alpha^2}{r^{n+1}} \omega(\varrho; d) \frac{d}{dr} \left[r^{n+1} \mathsf{A}(r, r^2, n + r^2 \dot{\mathscr{G}}^2) \dot{\mathscr{G}} \right].
\end{aligned} \tag{6.2.10}
$$

This being so, under the assumption that $\mathscr{G}$ solves the ODE governing (6.2.7) we see that

$$
\mathrm{div}_{\mathbb{A}} \left[\mathsf{A}\left(r, r^2, n + \sum_{\ell=1}^{d} \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2 \right) \rho_\alpha^2 \omega(\varrho; d) \nabla_{\mathbb{A}} f_\alpha \right] = 0,
$$

verifying the divergence-free statement. Regarding the boundary conditions, if $\mathscr{G}$ is a solution of (6.2.7) as claimed then $\mathscr{G}(a; m) = 0$, $\mathscr{G}(b; m) = 2m\pi$ so $f \equiv 0$ on $(\partial\mathbb{A}_n)_a$ and $f \equiv 2m\pi$ on $(\partial\mathbb{A}_n)_b$ as required. □

We close the section with an explicit solution of the system $\mathbf{BVP}[f; \mathscr{U}, \mathsf{m}]$ when $n = 2$. In this case there is a single angle of rotation function f which depends on $\varrho = (x_1^2 + x_2^2)^{1/2} = r$. As such the PDE governing the system becomes an ODE in r, which leads to the boundary value problem

$$
\begin{cases}
\dfrac{d}{dr} \left[r^3 \mathsf{A}(r, r^2, 2 + r^2 \dot{f}^2) \dot{f} \right] = 0, \quad a < r < b, \\
f(a) = 0, \\
f(b) = 2m\pi.
\end{cases} \tag{6.2.11}
$$

We know from the previous theorem that this admits a unique solution $f \in \mathscr{C}^2[a, b]$. In this context we illustrate how the PDE $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla\mathscr{P}$ simplifies upon the assumption that f solves (6.2.11). First, given (6.1.9), we have the description of a whirl map $u = \mathbf{Q}x$ as $u(x) = (x_1 \cos f - x_2 \sin f, x_1 \sin f + x_2 \cos f)$. It is computed that $\nabla u = \mathbf{Q} + \dot{f}/r[-(x_1 \sin f + x_2 \cos f), x_1 \cos f -$

$x_2 \sin f] \otimes x$, from which we have, writing $\mathsf{A} = \mathsf{A}(r, r^2, 2 + r^2\dot{f}^2)$,

$$\begin{aligned} \mathrm{div}[\mathsf{A}\nabla u] = & \left[\frac{1}{r}\dot{\mathsf{A}} - \dot{f}^2\mathsf{A}\right] \begin{bmatrix} x_1 \cos f - x_2 \sin f \\ x_1 \sin f + x_2 \cos f \end{bmatrix} \\ & - \left[\dot{f}\dot{\mathsf{A}} + \left(\ddot{f} + 3/r\dot{f}\right)\mathsf{A}\right] \begin{bmatrix} x_1 \sin f + x_2 \cos f \\ -x_1 \cos f + x_2 \sin f \end{bmatrix}, \end{aligned} \tag{6.2.12}$$

where

$$\dot{\mathsf{A}} = \frac{d}{dr}\mathsf{A}(r, r^2, 2 + r^2\dot{f}^2) = \mathsf{A}_r + 2r\mathsf{A}_s + 2(r\dot{f}^2 + r^2\dot{f}\ddot{f})\mathsf{A}_\xi, \tag{6.2.13}$$

with A_r, A_s, A_ξ denoting the derivatives of $\mathsf{A} = \mathsf{A}(r, s, \xi)$ in the first, second and third variables respectively. From this it follows that

$$\begin{aligned} \mathscr{L}[u;\mathsf{A},\mathsf{B}] &= (\nabla u)^t\{\mathrm{div}[\mathsf{A}(r, r^2, 2 + r^2\dot{f}^2)\nabla u] + \mathsf{B}(r, r^2, 2 + r^2\dot{f}^2)u\} \\ &= \nabla\mathsf{A}(|x|, |x|^2, 2 + |x|^2\dot{f}^2) + \mathsf{B}(r, r^2, 2 + r^2\dot{f}^2)x \\ &\quad + \left[r\dot{\mathsf{A}}\dot{f}^2 + \mathsf{A}(r\dot{f}\ddot{f} + 2\dot{f}^2)\right] x + \left[\dot{\mathsf{A}}\dot{f} + \mathsf{A}(\ddot{f} + 3/r\dot{f})\right] x^\perp. \end{aligned} \tag{6.2.14}$$

Here we use the notation $x = (x_1, x_2)$ and $x^\perp = (-x_2, x_1)$ for the vector orthogonal to x. We can now introduce the ODE in (6.2.11) and hence rewrite the above as

$$\begin{aligned} \mathscr{L}[u;\mathsf{A},\mathsf{B}] = & \nabla\mathsf{A}(|x|, |x|^2, 2 + |x|^2\dot{f}^2) + \mathsf{B}(r, r^2, 2 + r^2\dot{f}^2)x \\ & + \frac{\dot{f}}{r^2}\frac{d}{dr}\left[r^3\mathsf{A}(r, r^2, 2 + r^2\dot{f}^2)\dot{f}\right] x - \dot{f}^2\mathsf{A}(r, r^2, 2 + r^2\dot{f}^2)x \\ & + \frac{1}{r^3}\frac{d}{dr}\left[r^3\mathsf{A}(r, r^2, 2 + r^2\dot{f}^2)\dot{f}\right] x^\perp. \end{aligned} \tag{6.2.15}$$

As such, if we assume that the function f is a solution of (6.2.11) then the above reduces to

$$\begin{aligned} \mathscr{L}[u;\mathsf{A},\mathsf{B}] = & \nabla\mathsf{A}(|x|, |x|^2, 2 + |x|^2\dot{f}^2) \\ & + \mathsf{B}(r, r^2, 2 + r^2\dot{f}^2)x - \dot{f}^2\mathsf{A}(r, r^2, 2 + r^2\dot{f}^2)x \end{aligned} \tag{6.2.16}$$

and as such is a gradient field. That is $\mathscr{L}[u;\mathsf{A},\mathsf{B}] = \nabla\mathscr{P}$ with $\mathscr{P} = \mathsf{A} + G$ such that $\nabla G = r[\mathsf{B} - \dot{f}^2\mathsf{A}]\theta$. We conclude that for any solution f to (6.2.11) the PDE $\mathscr{L}[u;\mathsf{A},\mathsf{B}] = \nabla\mathscr{P}$ holds, and in particular we have infinitely many solutions indexed by the integers of the form

$$u(x; m) = \begin{bmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6.2.17}$$

where, for each $m \in \mathbb{Z}$, $f = f(r; m)$ is a solution of the system (6.2.11).

6.3 Derivation of the Vector Field $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ in Three and Four Dimensions

Following on from the discussion for $n=2$, here we formulate explicitly the vector field $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ and classify all whirl solutions to (6.1.1) when the spatial dimension $n=3,4$ in a component-wise fashion and with a consideration of the restricted Euler-Lagrange system $\mathbf{BVP}[f;\mathscr{U},\mathsf{m}]$. We initially work with respect to the ϱ variables before switching to a polar coordinate system which will facilitate a deeper analysis.

6.3.1 The System $\mathbf{BVP}[f;\mathscr{U},\mathsf{m}]$ with $d=1$ and $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ for $n=3$

Beginning in three spatial dimensions we first explore the restricted Euler-Lagrange system $\mathbf{BVP}[f;\mathscr{U},\mathsf{m}]$. We have the indices $d=1$ and $N=2$, so consider a single angle of rotation function $f(\varrho)$ with $\varrho=(\rho,z)$, $\rho=(x_1^2+x_2^2)^{1/2}$, $z=x_3$. The system reduces to a single PDE, denoted $\Lambda=0$, where we introduce

$$\Lambda:=\mathrm{div}_{\mathbb{A}}\left[\mathsf{A}(\|\varrho\|,\|\varrho\|^2,3+\rho^2|\nabla_{\mathbb{A}}f|^2)\rho^3\nabla_{\mathbb{A}}f\right]=\partial_\rho\left[\mathsf{A}\rho^3\partial_\rho f\right]+\partial_z\left[\mathsf{A}\rho^3\partial_z f\right]. \tag{6.3.1}$$

Here we have $\mathbf{Q}=\mathrm{diag}(\mathcal{R}[f],1)$ for $\mathcal{R}$, $\mathbf{J}\in\mathbf{SO}(2)$ defined by

$$\mathcal{R}[\alpha]=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix},\qquad \mathbf{J}=\mathcal{R}[\pi/2]=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{6.3.2}$$

Hence u is the 3-vector $u=[x_1\cos f-x_2\sin f, x_1\sin f+x_2\cos f, x_3]$ and for the gradient $\nabla u=\partial u_i/\partial x_j\,:\,1\le i,j\le 3$ we compute

$$\nabla u=\begin{bmatrix}\mathcal{R}[f] & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}-(x_1\sin f+x_2\cos f)\\ x_1\cos f-x_2\sin f\\ 0\end{bmatrix}\otimes\begin{bmatrix}\dfrac{x_1}{\rho}\partial_\rho f\\ \dfrac{x_2}{\rho}\partial_\rho f\\ \partial_z f\end{bmatrix} \tag{6.3.3}$$

and we note that we have used the chain rule identities

$$\frac{\partial}{\partial x_i}=\frac{\partial}{\partial\rho}\frac{x_i}{\rho}\ \text{for } i=1,2,\qquad \frac{\partial}{\partial x_3}=\frac{\partial}{\partial z}. \tag{6.3.4}$$

From this it follows that

$$\begin{aligned}\operatorname{div}[\mathsf{A}(\|\varrho\|,\|\varrho\|^2,3+\rho^2|\nabla_{\mathbb{A}}f|^2)\nabla u] =& \left\{\frac{1}{\rho}\partial_\rho\mathsf{A}-\mathsf{A}|\nabla_{\mathbb{A}}f|^2\right\}\begin{bmatrix} x_1\cos f - x_2\sin f\\ x_1\sin f + x_2\cos f\\ 0\end{bmatrix}\\ &-\left\{\partial_\rho f\partial_\rho\mathsf{A}+\partial_z f\partial_z\mathsf{A}+\mathsf{A}\left[3\frac{\partial_\rho f}{\rho}+\Delta_{\mathbb{A}}f\right]\right\}\begin{bmatrix} x_1\sin f + x_2\cos f\\ -x_1\cos f + x_2\sin f\\ 0\end{bmatrix}\\ &+\partial_z\mathsf{A}\,e_3. \end{aligned} \tag{6.3.5}$$

By definition we then have, for $\mathscr{L}[u;\mathsf{A},\mathsf{B}] = (\nabla u)^t \times \{\operatorname{div}[\mathsf{A}(\|\varrho\|,\|\varrho\|^2,3+\rho^2|\nabla_{\mathbb{A}}f|^2)\nabla u]+\mathsf{B}(\|\varrho\|,\|\varrho\|^2,3+\rho^2|\nabla_{\mathbb{A}}f|^2)u\}$,

$$\begin{aligned}\mathscr{L}[u;\mathsf{A},\mathsf{B}] =& \,diag\left(\left[\frac{\partial_\rho f}{\rho^2}\Lambda+\frac{1}{\rho}\partial_\rho\mathsf{A}-|\nabla_{\mathbb{A}}f|^2\mathsf{A}\right]\mathbf{I}_2,\left[\frac{\partial_z f}{\rho}\Lambda+\partial_z\mathsf{A}\right]\right)x\\ &+\frac{\Lambda}{\rho^3}\,[\Pi x]^\perp+\mathsf{B}x \end{aligned} \tag{6.3.6}$$

where, recall, Λ is the differential operator defined in (6.3.1). In the above $\Pi x = (x_1,x_2,0)$ denotes the projection of x in the (x_1,x_2) hyperplane with $[\Pi x]^\perp = (-x_x,x_1,0)$. From this we see that if $\Lambda=0$ holds then our vector field under consideration reduces to

$$\mathscr{L}[u;\mathsf{A},\mathsf{B}] = diag\left(\left[\frac{1}{\rho}\partial_\rho\mathsf{A}-|\nabla_{\mathbb{A}}f|^2\mathsf{A}\right]\mathbf{I}_2,\partial_z\mathsf{A}\right)x+\mathsf{B}x. \tag{6.3.7}$$

We remark that, by the chain rule, $\nabla_{\mathbb{X}}\mathsf{A}(\|\varrho\|,\|\varrho\|^2,3+\rho^2|\nabla_{\mathbb{A}}f|^2) = 1/\rho\partial_\rho\mathsf{A}\Pi x+\partial_z\mathsf{A}e_3$, hence it is possible to rewrite the above as

$$\begin{aligned}\mathscr{L}[u;\mathsf{A},\mathsf{B}] =& \nabla_{\mathbb{X}}\mathsf{A}(\|\varrho\|,\|\varrho\|^2,3+\rho^2|\nabla_{\mathbb{A}}f|^2)\\ &-|\nabla_{\mathbb{A}}f|^2\mathsf{A}(\|\varrho\|,\|\varrho\|^2,3+\rho^2|\nabla_{\mathbb{A}}f|^2)\Pi x\\ &+\mathsf{B}(\|\varrho\|,\|\varrho\|^2,3+\rho^2|\nabla_{\mathbb{A}}f|^2)x. \end{aligned} \tag{6.3.8}$$

In order for the main PDE $\mathscr{L}[u;\mathsf{A},\mathsf{B}]=\nabla\mathscr{P}$ to be solved it is necessary that the vector field $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ be irrotational. We compute the curl of the vector field $U(x) := \mathscr{L}[u;\mathsf{A},\mathsf{B}]-\nabla_{\mathbb{X}}\mathsf{A} = \mathsf{B}x-|\nabla_{\mathbb{A}}f|^2\mathsf{A}\Pi x$ in dimension $n=3$ as[15]

$$\operatorname{curl} U = \nabla_{\mathbb{X}}\times U = \left\{\partial_z\left[\mathsf{B}-|\nabla_{\mathbb{A}}f|^2\mathsf{A}\right]-\frac{z}{\rho}\partial_\rho\mathsf{B}\right\}\begin{bmatrix} x_1\\ -x_2\\ 0\end{bmatrix}. \tag{6.3.9}$$

[15] Observe that $\operatorname{curl}\nabla_{\mathbb{X}}\mathsf{A}\equiv 0$ which is why we subtract it from the vector field $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ and study the curl of the reduced vector field U.

We now introduce a polar coordinate system in order to go deeper into the analysis and with $\rho = (x_1^2 + x_2^2)^{1/2}$ and $z = x_3$ we set $\rho = r\cos\theta$, $z = r\sin\theta$. Then we have transformed the analysis to the rectangular domain $\mathscr{R}_3 := \{(r,\theta) \in \mathbb{R}^2 : a \le r \le b, -\pi/2 \le \theta \le \pi/2\}$, the subscript 3 referring to the underlying spatial dimension $n = 3$ (*see* Figure 3).

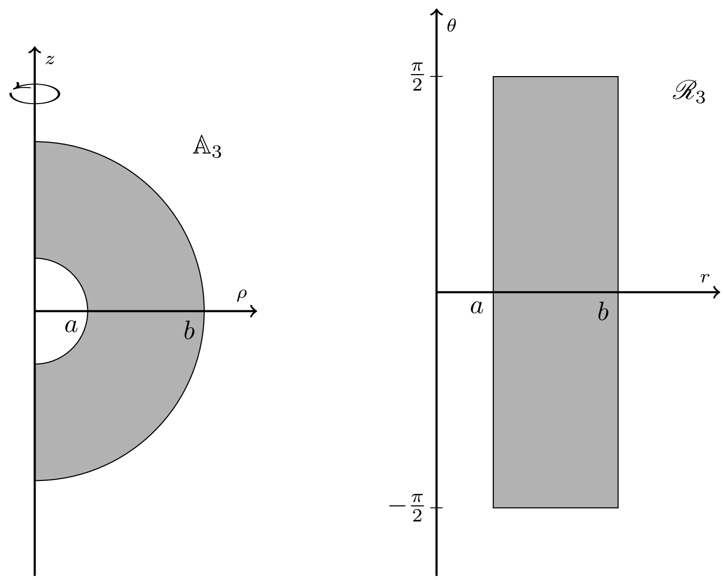

Figure 3: The 2-dimensional domains $\mathbb{A}_3$ and $\mathscr{R}_3$ defined in the (ρ, z) and (r, θ) planes respectively.

By the chain rule we have the following:

$$\frac{\partial}{\partial \rho} = \frac{\rho}{r}\frac{\partial}{\partial r} - \frac{z}{r^2}\frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial z} = \frac{z}{r}\frac{\partial}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial \theta}. \tag{6.3.10}$$

With this we first re-express the divergence-free PDE $\Lambda = 0$ with Λ defined by (6.3.1) as

$$\begin{aligned} &\cos\theta\partial_r \left[r^3\cos^3\theta \mathsf{A} \left(\cos\theta\partial_r f - \frac{\sin\theta}{r}\partial_\theta f \right) \right] \\ &\quad - \frac{\sin\theta}{r}\partial_\theta \left[r^3\cos^3\theta \mathsf{A} \left(\cos\theta\partial_r f - \frac{\sin\theta}{r}\partial_\theta f \right) \right] \\ &\quad + \sin\theta\partial_r \left[r^3\cos^3\theta \mathsf{A} \left(\sin\theta\partial_r f + \frac{\cos\theta}{r}\partial_\theta f \right) \right] \\ &\quad + \frac{\cos\theta}{r}\partial_\theta \left[r^3\cos^3\theta \mathsf{A} \left(\sin\theta\partial_r f + \frac{\cos\theta}{r}\partial_\theta f \right) \right] = 0, \end{aligned} \tag{6.3.11}$$

where $\mathsf{A} = \mathsf{A}(r, r^2, 3 + r^2 \cos^2\theta |\nabla_{\mathbb{A}} f|^2)$ and $f = f(r,\theta)$. By coupling this with the boundary conditions present in (6.2.5) and a simplification of the PDE above the full system under consideration becomes

$$\begin{cases} \partial_r \left[r^4 \cos^3\theta \mathsf{A} \partial_r f \right] + \partial_\theta \left[r^2 \cos^3\theta \mathsf{A} \partial_\theta f \right] = 0 & (r,\theta) \in \mathscr{R}_3, \\ f = 0 & r = a, \\ f = 2m\pi & r = b \\ r^2 \cos^3\theta \mathsf{A} [r\cos\theta \partial_r f - \sin\theta \partial_\theta f] = 0 & \theta = \pm\pi/2. \end{cases} \tag{6.3.12}$$

Furthermore, upon re-evaluating the arguments of A and B we can rewrite the reduced vector field $\mathscr{L}[u; \mathsf{A}, \mathsf{B}]$ as in (6.3.8) as

$$\begin{aligned} \mathscr{L}[u; \mathsf{A}, \mathsf{B}] = & \nabla_{\mathbb{X}} \mathsf{A}(r, r^2, 3 + r^2 \cos^2\theta |\nabla_{\mathbb{A}} f|^2) \\ & - |\nabla_{\mathbb{A}} f|^2 \mathsf{A}(r, r^2, 3 + r^2 \cos^2\theta |\nabla_{\mathbb{A}} f|^2) \Pi x \\ & + \mathsf{B}(r, r^2, 3 + r^2 \cos^2\theta |\nabla_{\mathbb{A}} f|^2) x. \end{aligned} \tag{6.3.13}$$

We now return to the curl of the vector field $U(x) = \mathscr{L}[u; \mathsf{A}, \mathsf{B}] - \nabla_{\mathbb{X}} \mathsf{A} = \mathsf{B}x - |\nabla_{\mathbb{A}} f|^2 \mathsf{A} \Pi x$, computed in the (ρ, z)-coordinate system in (6.3.9). By using the chain rule as in (6.3.10) we see that

$$\operatorname{curl} U = \nabla_{\mathbb{X}} \times U = \left\{ \partial_z \left[\mathsf{B} - |\nabla_{\mathbb{A}} f|^2 \mathsf{A} \right] - \frac{z}{\rho} \partial_\rho \mathsf{B} \right\} \begin{bmatrix} x_1 \\ -x_2 \\ 0 \end{bmatrix} \tag{6.3.14}$$

$$= \left\{ -\sin\theta \partial_r \left(|\nabla_{\mathbb{A}} f|^2 \mathsf{A} \right) + \frac{\partial_\theta \mathsf{B}}{r\cos\theta} - \frac{\cos\theta}{r} \partial_\theta \left(|\nabla_{\mathbb{A}} f|^2 \mathsf{A} \right) \right\} \begin{bmatrix} x_1 \\ -x_2 \\ 0 \end{bmatrix}.$$

We simplify the picture and analyse this curl further by assuming that the function f, which acts a solution to the divergence-free equation (6.3.11), depends on the radial variable r alone, in which case $\partial_\theta f \equiv 0$ and, by (6.3.12), $f = f(r)$ satisfies the equation

$$\frac{d}{dr} \left[r^4 \mathsf{A}(r, r^2, 3 + r^2 \cos^2\theta \dot{f}^2) \dot{f} \right] = 0, \tag{6.3.15}$$

since $|\nabla_{\mathbb{A}} f|^2 = \dot{f}^2$ in this case. Furthermore the curl of the vector field U as calculated in (6.3.14) becomes

$$\operatorname{curl} U = -\dot{f}\sin\theta\left\{2\ddot{f}\mathsf{A}+\dot{f}\left[\mathsf{A}_r+2r\mathsf{A}_s\right]+2r^2\dot{f}^2\ddot{f}\cos^2\theta\mathsf{A}_\xi+2r\dot{f}\mathsf{B}_\xi\right\}\begin{bmatrix}x_1\\-x_2\\0\end{bmatrix}, \tag{6.3.16}$$

where $\mathsf{A}_r=\mathsf{A}_r(r,s,\xi)$ denotes the derivative of A in the first variable with similar definitions holding for A_s, A_ξ and B_ξ. If we apply the ODE (6.3.15) to the coefficient of the curl above we have, after a rearrangement,

$$\operatorname{curl} U = \dot{f}^2\sin\theta\left\{\frac{8}{r}\mathsf{A}+\mathsf{A}_r+2r\left[\mathsf{A}_s-\mathsf{B}_\xi\right]+2r\dot{f}\cos^2\theta\mathsf{A}_\xi\left[2\dot{f}+r\ddot{f}\right]\right\}\begin{bmatrix}x_1\\-x_2\\0\end{bmatrix}. \tag{6.3.17}$$

We now introduce the notation

$$\Delta_3(\mathsf{A},\mathsf{B}) := \frac{8}{r^2}\mathsf{A}+\frac{1}{r}\mathsf{A}_r+2[\mathsf{A}_s-\mathsf{B}_\xi]+2\dot{f}\cos^2\theta\mathsf{A}_\xi\left[2\dot{f}+r\ddot{f}\right] \tag{6.3.18}$$

which we refer to as a *discriminant* term, extracted from a study of the irrotationality of the vector field $U(x)=\mathscr{L}[u;\mathsf{A},\mathsf{B}]-\nabla_{\mathbb{X}}\mathsf{A}$. This captures a startling distinction in the cardinality of solution sets to the PDE $\mathscr{L}[u;\mathsf{A},\mathsf{B}]=\nabla\mathscr{P}$ depending on whether or not this discriminant vanishes, as is highlighted in the following result. Recall the description of the twist loop $\mathbf{Q}=\exp\{\mathbf{H}(\varrho)\}$ with $\mathbf{H}$ as in (6.1.9).

Theorem 6.3.1. *Let $n=3$ and for all $m\in\mathbb{Z}$ take $f=f(r;m)\in\mathscr{C}^2[a,b]$ a solution of (6.3.15) satisfying $f(a)=0$, $f(b)=2m\pi$. Consider the vector field $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ be defined by (6.3.13) and the quantity $\Delta_3(\mathsf{A},\mathsf{B})$ given by (6.3.18). Then a whirl map $u=\operatorname{diag}(\mathcal{R}[f],1)x$ solves the PDE $\mathscr{L}[u;\mathsf{A},\mathsf{B}]=\nabla\mathscr{P}$ in one of the following cases.*

- *If $\Delta_3(\mathsf{A},\mathsf{B})\not\equiv 0$ over $\mathscr{R}_3$ then for any $m\in\mathbb{Z}$ $f(r;m)\equiv 0$ leading to $u\equiv x$ and $\mathscr{P}=\mathsf{A}+G(|x|)$ such that $\nabla G=r\mathsf{B}(r,r^2,3)\theta$.*
- *If $\Delta_3(\mathsf{A},\mathsf{B})\equiv 0$ over $\mathscr{R}_3$ then there is no restriction on $f=f(r;m)$ and for each $m\in\mathbb{Z}$ there exists a corresponding whirl map $u(x;m)=\operatorname{diag}(\mathcal{R}[f(r;m)],1)x$ which solves $\mathscr{L}[u;\mathsf{A},\mathsf{B}]=\nabla\mathscr{P}$, with*

$$\mathscr{P}=\mathsf{A}(|x|,|x|^2,3+|x|^2\cos^2\theta\dot{f}^2)+\int_0^r s\mathsf{B}(r,r^2,3+r^2\cos^2\theta\dot{f}^2)\,ds.$$

Proof. If $\Delta_3(\mathsf{A}, \mathsf{B}) \not\equiv 0$ then, supposing $\operatorname{curl} U \equiv 0$ with $U = \mathscr{L}[u; \mathsf{A}, \mathsf{B}] - \nabla_{\mathbb{X}} \mathsf{A}$, which holds whenever $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$, it follows from (6.3.17) that $\dot{f}(R) = 0$ for some fixed $a < R < b$. Upon evaluating (6.3.15) we see that

$$\dot{f} = \frac{c}{r^4 \mathsf{A}(r, r^2, 3 + r^2 \cos^2 \theta \dot{f}^2)}, \qquad c \in \mathbb{R},\ a < r < b \tag{6.3.19}$$

and since $\mathsf{A} > 0$ by assumption we conclude that $\dot{f}$ does not change sign. Therefore if $f(R) = 0$ it follows that $c = 0$ and hence $f \equiv 0$ over $a \leq r \leq b$ by virtue of the boundary condition $f(a) = 0$. Consequently, with $\mathcal{R}$ defined by (6.3.2), we have $\mathcal{R}[0] = \mathbf{I}_2$ and for u as in the statement of the theorem it is easily seen that $u \equiv x$ is the only solution to the PDE $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$. The explicit description of the pressure field $\mathscr{P}$ follows by a direct substitution.

Alternatively if $\Delta_3(\mathsf{A}, \mathsf{B}) \equiv 0$ then $\operatorname{curl} U \equiv 0$ and it is possible that $\dot{f}$ is nowhere zero over $a < r < b$. For the corresponding whirl map $u = \operatorname{diag}(\mathcal{R}[f], 1)x$, it follows by Proposition D.0.2 that the corresponding vector field $\mathscr{L}[u; \mathsf{A}, \mathsf{B}]$ still constitutes a gradient and thus in turn we have a solution of the PDE $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$. Indeed, first set $\mathbf{H} = diag(\mathbf{J}, 0)$ where $\mathbf{J} = \mathcal{R}[\pi/2]$. Then, adopting the notation used in Proposition D.0.2 we have $U(x) = \mathscr{A}(r, z)x + \mathscr{B}(r, z)\mathbf{H}^2 x$ where $z = |\mathbf{H}x|^2$ and the scalar-valued functions $\mathscr{A} = \mathscr{A}(r, z)$ and $\mathscr{B} = \mathscr{B}(r, z)$ are given respectively by $\mathscr{A}(r, z) = \mathsf{B}(r, r^2, 3 + \dot{f}^2 z)$, $\mathscr{B}(r, z) = \dot{f}^2 \mathsf{A}(r, r^2, 3 + \dot{f}^2 z)$. Then we have

$$2\mathscr{A}_z(r, z) + \frac{\mathscr{B}_r(r, z)}{r} = 2\mathsf{B}_\xi \dot{f}^2 + \frac{\mathsf{A}_r}{r} \dot{f}^2 + 2\mathsf{A}_s \dot{f}^2 + \frac{2}{r} \mathsf{A}_\xi \dot{f}^3 \ddot{f} z + \frac{2}{r} \dot{f} \ddot{f} \mathsf{A}, \tag{6.3.20}$$

or by an application of the ODE (6.3.15) it follows that $2\mathscr{A}_z + \mathscr{B}_r / r \equiv 0$ given that $\Delta_3 \equiv 0$. Next define

$$\psi(r, z) := \int_0^r s \mathsf{B}(s, s^2, 3 + s^2 \cos^2 \theta \dot{f}^2)\, ds. \tag{6.3.21}$$

From this we see that

$$\begin{aligned} \nabla \psi(|x|, |\mathbf{H}x|^2) &= \psi_r(r, z) x / r - 2\psi_z(r, z) \mathbf{H}^2 x \\ &= \mathsf{B}(r, r^2, 3 + \dot{f}^2 z) x + \dot{f}^2 \mathsf{A}(r, r^2, 3 + \dot{f}^2 z) \mathbf{H}^2 x = U(x), \end{aligned} \tag{6.3.22}$$

since $-2\psi_z \mathbf{H}^2 x = -2[\int_0^r s \dot{f}^2 \mathsf{B}_\xi(s, s^2, 3 + \dot{f}^2 z)\, ds] \mathbf{H}^2 x = -\dot{f}^2 \mathsf{A}(r, r^2, 3 + \dot{f}^2 z) \Pi x$ by virtue of the identity $2\mathscr{A}_z + \mathscr{B}_r / r \equiv 0$. It follows that $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$ with $\mathscr{P} = \mathsf{A} + \psi$. $\square$

6.3.2 The Divergence-Free System $\Lambda_1 = \Lambda_2 = 0$ and $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ for $n = 4$

Next we go through the same procedure when the spatial dimension $n = 4$ with $d = N = 2$. First the system $\mathbf{BVP}[f;\mathscr{U},\mathsf{m}]$ leads to the pair of PDEs $\Lambda_1 = 0$, $\Lambda_2 = 0$, where

$$\begin{aligned}\Lambda_1 :=& \, \partial_{\rho_1}\left[\mathsf{A}(\|\varrho\|,\|\varrho\|^2, 4+\rho_1^2|\nabla_{\mathbb{A}}f_1|^2+\rho_2^2|\nabla_{\mathbb{A}}f_2|^2)\rho_1^3\rho_2\partial_{\rho_1}f_1\right]\\ &+\partial_{\rho_2}\left[\mathsf{A}(\|\varrho\|,\|\varrho\|^2, 4+\rho_1^2|\nabla_{\mathbb{A}}f_1|^2+\rho_2^2|\nabla_{\mathbb{A}}f_2|^2)\rho_1^3\rho_2\partial_{\rho_2}f_1\right], \end{aligned} \tag{6.3.23}$$

$$\begin{aligned}\Lambda_2 :=& \, \partial_{\rho_1}\left[\mathsf{A}(\|\varrho\|,\|\varrho\|^2, 4+\rho_1^2|\nabla_{\mathbb{A}}f_1|^2+\rho_2^2|\nabla_{\mathbb{A}}f_2|^2)\rho_1\rho_2^3\partial_{\rho_1}f_2\right]\\ &+\partial_{\rho_2}\left[\mathsf{A}(\|\varrho\|,\|\varrho\|^2, 4+\rho_1^2|\nabla_{\mathbb{A}}f_1|^2+\rho_2^2|\nabla_{\mathbb{A}}f_2|^2)\rho_1\rho_2^3\partial_{\rho_2}f_2\right]. \end{aligned} \tag{6.3.24}$$

Here each f_i, $i = 1, 2$ depends on $\varrho = (\rho_1, \rho_2) = (\|(x_1, x_2)\|, \|(x_3, x_4)\|)$. In this setting $\mathbf{Q} = \mathbf{Q}(\varrho)$ is the block-diagonal matrix $\mathbf{Q} = \mathrm{diag}(\mathcal{R}[f_1], \mathcal{R}[f_2])$ for $\mathcal{R}$ as in (6.3.2), hence our whirl map u is given in components by

$$u(x) = \mathbf{Q}x = \begin{bmatrix} x_1\cos f_1 - x_2\sin f_1 \\ x_1\sin f_1 + x_2\cos f_1 \\ x_3\cos f_2 - x_4\sin f_2 \\ x_3\sin f_2 + x_4\cos f_2 \end{bmatrix}. \tag{6.3.25}$$

We compute the gradient of u as the 4×4 matrix

$$\begin{aligned}\nabla u =& \begin{bmatrix} \mathcal{R}[f_1] & 0 \\ 0 & \mathcal{R}[f_2] \end{bmatrix} + \begin{bmatrix} -\partial_{\rho_1}f_1(x_1\sin f_1 + x_2\cos f_1) \\ \partial_{\rho_1}f_1(x_1\cos f_1 - x_2\sin f_1) \\ -\partial_{\rho_1}f_2(x_3\sin f_2 + x_4\cos f_2) \\ \partial_{\rho_1}f_2(x_3\cos f_2 - x_4\sin f_2) \end{bmatrix} \otimes \begin{bmatrix} x_1/\rho_1 \\ x_2/\rho_1 \\ 0 \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} -\partial_{\rho_2}f_1(x_1\sin f_1 + x_2\sin f_1) \\ \partial_{\rho_2}f_1(x_1\cos f_1 - x_2\sin f_1) \\ -\partial_{\rho_2}f_2(x_3\sin f_2 + x_4\cos f_2) \\ \partial_{\rho_2}f_2(x_3\cos f_2 - x_4\sin f_2) \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ x_3/\rho_2 \\ x_4/\rho_2 \end{bmatrix}. \end{aligned} \tag{6.3.26}$$

By the chain rule it is then computed that

$$
\begin{aligned}
&\mathrm{div}[\mathsf{A}\nabla u] = \mathrm{div}[\mathsf{A}(\|\varrho\|, \|\varrho\|^2, 4 + \rho_1^2|\nabla_{\mathbb{A}} f_1|^2 + \rho_2^2|\nabla_{\mathbb{A}} f_2|^2)\nabla u] \\
&= diag\Bigg(\left[\frac{1}{\rho_1}\partial_{\rho_1}\mathsf{A} - \mathsf{A}|\nabla_{\mathbb{A}} f_1|^2\right]\mathbf{I}_2, \left[\frac{1}{\rho_2}\partial_{\rho_2}\mathsf{A} - \mathsf{A}|\nabla_{\mathbb{A}} f_2|^2\right]\mathbf{I}_2\Bigg) \times \\
&\quad \times \begin{bmatrix} x_1\cos f_1 - x_2 \sin f_1 \\ x_1 \sin f_1 + x_2\cos f_1 \\ x_3\cos f_2 - x_4\sin f_2 \\ x_3\sin f_2 + x_4\cos f_2 \end{bmatrix} \\
&\quad - diag\Bigg(\bigg\{\partial_{\rho_1} f_1 \partial_{\rho_1}\mathsf{A} + \partial_{\rho_2} f_1\partial_{\rho_2}\mathsf{A} + \mathsf{A}\left[3\frac{\partial_{\rho_1} f_1}{\rho_1} + \frac{\partial_{\rho_2} f_1}{\rho_2} + \Delta_{\mathbb{A}} f_1\right]\bigg\}\mathbf{I}_2, \\
&\quad \bigg\{\partial_{\rho_1} f_2 \partial_{\rho_1}\mathsf{A} + \partial_{\rho_2} f_2\partial_{\rho_2}\mathsf{A} + \mathsf{A}\left[3\frac{\partial_{\rho_2} f_2}{\rho_2} + \frac{\partial_{\rho_1} f_2}{\rho_1} + \Delta_{\mathbb{A}} f_2\right]\bigg\}\mathbf{I}_2\Bigg) \times \\
&\quad \times \begin{bmatrix} x_1 \sin f_1 + x_2\cos f_1 \\ -x_1\cos f_1 + x_2\sin f_1 \\ x_3\sin f_2 + x_4\cos f_2 \\ -x_3\cos f_2 + x_4\sin f_2 \end{bmatrix}.
\end{aligned} \tag{6.3.27}
$$

With $\mathscr{L}[u;\mathsf{A},\mathsf{B}] = (\nabla u)^t\{\mathrm{div}[\mathsf{A}\nabla u] + \mathsf{B}u\}$ we then have

$$
\begin{aligned}
\mathscr{L}[u;\mathsf{A},\mathsf{B}] = diag\Bigg(&\left[\frac{1}{\rho_1}\partial_{\rho_1}\mathsf{A} + \frac{\partial_{\rho_1} f_1}{\rho_1^2\rho_2}\Lambda_1 - |\nabla_{\mathbb{A}} f_1|^2\mathsf{A}\right]\mathbf{I}_2, \\
&\left[\frac{1}{\rho_2}\partial_{\rho_2}\mathsf{A} + \frac{\partial_{\rho_2} f_2}{\rho_1\rho_2^2}\Lambda_2 - |\nabla_{\mathbb{A}} f_2|^2\mathsf{A}\right]\mathbf{I}_2\Bigg)x \\
&+ diag\left(\frac{1}{\rho_1^3\rho_2}\Lambda_1\mathbf{I}_2, \frac{1}{\rho_1\rho_2^3}\Lambda_2\mathbf{I}_2\right)x^\perp + \mathsf{B}x,
\end{aligned} \tag{6.3.28}
$$

where $x^\perp = (-x_2, x_1, -x_4, x_3)$. In particular if the divergence-free system arising from $\mathbf{BVP}[f;\mathscr{U},\mathsf{m}]$ holds (*i.e.* $\Lambda_1 = \Lambda_2 = 0$) then the above simplifies to

$$
\mathscr{L}[u;\mathsf{A},\mathsf{B}] = diag\Bigg(\left[\frac{1}{\rho_1}\partial_{\rho_1}\mathsf{A} - |\nabla_{\mathbb{A}} f_1|^2\mathsf{A}\right]\mathbf{I}_2, \left[\frac{1}{\rho_2}\partial_{\rho_2}\mathsf{A} - |\nabla_{\mathbb{A}} f_2|^2\mathsf{A}\right]\mathbf{I}_2\Bigg)x + \mathsf{B}x, \tag{6.3.29}
$$

where the arguments of A (and, by analogy, B) are $\mathsf{A} = \mathsf{A}(\|\varrho\|, \|\varrho\|^2, 4 + \rho_1^2|\nabla_{\mathbb{A}} f_1|^2 + \rho_2^2|\nabla_{\mathbb{A}} f_2|^2)$. Upon noticing that $\nabla_{\mathbb{X}}\mathsf{A}(\|\varrho\|, \|\varrho\|^2, 4 + \rho_1^2|\nabla_{\mathbb{A}} f_1|^2 + \rho_2^2|\nabla_{\mathbb{A}} f_2|^2) = diag(1/\rho_1\partial_{\rho_1}\mathsf{A}\mathbf{I}_2, 1/\rho_2\partial_{\rho_2}\mathsf{A}\mathbf{I}_2)x$ we can write

$$
\begin{aligned}
\mathscr{L}[u\,;\mathsf{A},\mathsf{B}] &= \nabla_{\mathbb{X}}\mathsf{A}(\|\varrho\|, \|\varrho\|^2, 4 + \rho_1^2|\nabla_{\mathbb{A}} f_1|^2 + \rho_2^2|\nabla_{\mathbb{A}} f_2|^2) \\
&\quad - \mathsf{A}(\|\varrho\|, \|\varrho\|^2, 4 + \rho_1^2|\nabla_{\mathbb{A}} f_1|^2 + \rho_2^2|\nabla_{\mathbb{A}} f_2|^2)\, diag\left(|\nabla_{\mathbb{A}} f_1|^2\mathbf{I}_2, |\nabla_{\mathbb{A}} f_2|^2\mathbf{I}_2\right)x \\
&\quad + \mathsf{B}(\|\varrho\|, \|\varrho\|^2, 4 + \rho_1^2|\nabla_{\mathbb{A}} f_1|^2 + \rho_2^2|\nabla_{\mathbb{A}} f_2|^2)x.
\end{aligned} \tag{6.3.30}
$$

Similarly to when $n = 3$ we will be interested in computing the curl of the vector field $U(x) := \mathscr{L}[u;\mathsf{A},\mathsf{B}] - \nabla_{\mathbb{X}}\mathsf{A} = \mathsf{B}x - diag(|\nabla_{\mathbb{A}} f_1|^2\mathbf{I}_2, |\nabla_{\mathbb{A}} f_2|^2\mathbf{I}_2)\mathsf{A}x$. For each $1 \le i, j, \le 4$ we have

$$\begin{aligned}\operatorname{curl} U &:= \partial_{x_j} U_i - \partial_{x_i} U_j \\ &= \left\{ \frac{1}{\rho_{s(j)}} \partial_{\rho_{s(j)}} \left[\mathsf{B} - |\nabla_{\mathbb{A}} f_{s(i)}|^2 \mathsf{A}\right] - \frac{1}{\rho_{s(i)}} \partial_{\rho_{s(i)}} \left[\mathsf{B} - |\nabla_{\mathbb{A}} f_{s(j)}|^2 \mathsf{A}\right] \right\} x_i x_j,\end{aligned} \tag{6.3.31}$$

where $s(k) = \lfloor (k+1)/2 \rfloor$ for $1 \le k \le 4$. As with $n = 3$ we introduce a polar coordinate system here and set $\rho_1 = r\cos\theta$, $\rho_2 = r\sin\theta$ and consider the rectangular domain $\mathscr{R}_4 := \{(r,\theta) \in \mathbb{R}^2 : a \le r \le b,\ 0 \le \theta \le \pi/2\}$ (*see* Figure 4).

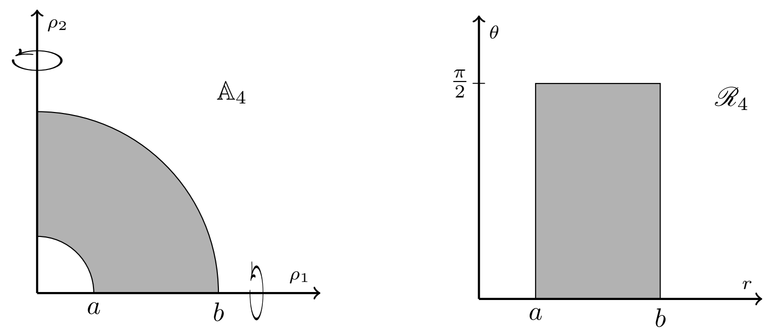

Figure 4: The 2-dimensional domains $\mathbb{A}_4$ and $\mathscr{R}_4$ defined in the (ρ_1, ρ_2) and (r, θ) planes respectively.

For the divergence-free system $\Lambda_1 = \Lambda_2 = 0$ the differential operators are defined in polar coordinates by

$$\begin{aligned}\Lambda_1 = {} & \cos\theta\partial_r \left[r^4 \sin\theta\cos^3\theta\mathsf{A}\left(\cos\theta\partial_r f_1 - \frac{\sin\theta}{r}\partial_\theta f_1\right)\right] \\ & - \frac{\sin\theta}{r}\partial_\theta \left[r^4 \sin\theta\cos^3\theta\mathsf{A}\left(\cos\theta\partial_r f_1 - \frac{\sin\theta}{r}\partial_\theta f_1\right)\right] \\ & + \sin\theta\partial_r \left[r^4 \sin\theta\cos^3\theta\mathsf{A}\left(\sin\theta\partial_r f_1 + \frac{\cos\theta}{r}\partial_\theta f_1\right)\right] \\ & + \frac{\cos\theta}{r}\partial_\theta \left[r^4 \sin\theta\cos^3\theta\mathsf{A}\left(\sin\theta\partial_r f_1 + \frac{\cos\theta}{r}\partial_\theta f_1\right)\right],\end{aligned} \tag{6.3.32}$$

$$\begin{aligned}\Lambda_2 = {} & \cos\theta\partial_r\left[r^4\sin^3\theta\cos\theta\mathsf{A}\left(\cos\theta\partial_r f_2 - \frac{\sin\theta}{r}\partial_\theta f_2\right)\right] \\ & - \frac{\sin\theta}{r}\partial_\theta\left[r^4\sin^3\theta\cos\theta\mathsf{A}\left(\cos\theta\partial_r f_2 - \frac{\sin\theta}{r}\partial_\theta f_2\right)\right] \\ & + \sin\theta\partial_r\left[r^4\sin^3\theta\cos\theta\mathsf{A}\left(\sin\theta\partial_r f_2 + \frac{\cos\theta}{r}\partial_\theta f_2\right)\right] \\ & + \frac{\cos\theta}{r}\partial_\theta\left[r^4\sin^3\theta\cos\theta\mathsf{A}\left(\sin\theta\partial_r f_2 + \frac{\cos\theta}{r}\partial_\theta f_2\right)\right]. \end{aligned} \tag{6.3.33}$$

In the above we have $\mathsf{A} = \mathsf{A}(r, r^2, 4+r^2[\cos^2\theta|\nabla_{\mathbb{A}} f_1|^2+\sin^2\theta|\nabla_{\mathbb{A}} f_2|^2])$. Upon an expansion and subsequent simplification of the equations $\Lambda_1 = 0$, $\Lambda_2 = 0$ with Λ_1, Λ_2 as above and referring to the system (6.2.5) we introduce the relevant boundary conditions and are thus required to simultaneously solve the systems

$$\begin{cases} \partial_r\left[r^5\sin\theta\cos^3\theta\mathsf{A}\partial_r f_1\right] + \partial_\theta\left[r^3\sin\theta\cos^3\theta\mathsf{A}\partial_\theta f_1\right] = 0 & (r,\theta)\in\mathscr{R}_4, \\ f_1 = 0 & r = a, \\ f_1 = 2m_1\pi & r = b, \\ r^3\sin\theta\cos^3\theta\mathsf{A}[r\cos\theta\partial_r f_1 - \sin\theta\partial_\theta f_1] = 0 & \theta = \pi/2 \\ r^3\sin\theta\cos^3\theta\mathsf{A}[\sin\theta\partial_r f_1 + r\cos\theta\partial_\theta f_1] = 0, & \theta = 0 \end{cases} \tag{6.3.34}$$

and

$$\begin{cases} \partial_r\left[r^5\sin^3\theta\cos\theta\mathsf{A}\partial_r f_2\right] + \partial_\theta\left[r^3\sin^3\theta\cos\theta\mathsf{A}\partial_\theta f_2\right] = 0 & (r,\theta)\in\mathscr{R}_4, \\ f_2 = 0 & r = a, \\ f_2 = 2m_2\pi & r = b, \\ r^3\sin^3\theta\cos\theta\mathsf{A}[r\cos\theta\partial_r f_2 - \sin\theta\partial_\theta f_2] = 0 & \theta = \pi/2 \\ r^3\sin^3\theta\cos\theta\mathsf{A}[r\sin\theta\partial_r f_2 + \cos\theta\partial_\theta f_2] = 0 & \theta = 0. \end{cases} \tag{6.3.35}$$

Regarding the curl of the vector field $U(x)$ as in (6.3.31) we have, for $1 \leq i, j, \leq 4$,

$$\operatorname{curl} U = -\overline{\Delta}\begin{bmatrix} 0 & 0 & x_1x_3 & x_1x_4 \\ 0 & 0 & x_2x_3 & x_2x_4 \\ -x_1x_3 & -x_2x_3 & 0 & 0 \\ -x_1x_4 & -x_2x_4 & 0 & 0 \end{bmatrix} \tag{6.3.36}$$

where

$$\begin{aligned}\overline{\Delta} := &\frac{1}{r}\left[\partial_r\left(|\nabla_{\mathbb{A}} f_2|^2\mathsf{A}\right) - \partial_r\left(|\nabla_{\mathbb{A}} f_1|^2\mathsf{A}\right)\right] + \frac{1}{r^2}\partial_\theta \mathsf{B}\left[\cot\theta + \tan\theta\right] \\ &- \frac{1}{r^2}\left[\cot\theta\partial_\theta\left(|\nabla_{\mathbb{A}} f_1|^2\mathsf{A}\right) + \tan\theta\partial_\theta\left(|\nabla_{\mathbb{A}} f_2|^2\mathsf{A}\right)\right]. \qquad (6.3.37)\end{aligned}$$

If we denote the constant matrix in (6.3.36) as $\mathbf{C}$ then in the instance that $f_1 = f_1(r)$, $f_2 = f_2(r)$ we have

$$\begin{aligned}\operatorname{curl} U = &-\Bigg\{\frac{2}{r}\mathsf{A}\left[\dot{f}_2\ddot{f}_2 - \dot{f}_1\ddot{f}_1\right] \qquad (6.3.38)\\ &+ \frac{1}{r}\left(\mathsf{A}_r + 2r[\mathsf{A}_s + \mathsf{B}_\xi] + 2r^2\mathsf{A}_\xi[\cos^2\theta\dot{f}_1\ddot{f}_1 + \sin^2\theta\dot{f}_2\ddot{f}_2]\right)\left[\dot{f}_2^2 - \dot{f}_1^2\right]\Bigg\}\mathbf{C}.\end{aligned}$$

By an application of the ODEs in r governing (6.3.41) below to this coefficient we have the rearrangement

$$\begin{aligned}\operatorname{curl} U = &\left[\dot{f}_2^2 - \dot{f}_1^2\right]\Bigg\{\frac{10}{r^2}\mathsf{A} + \frac{1}{r}\mathsf{A}_r + 2\left[\mathsf{A}_s - \mathsf{B}_\xi\right] \qquad (6.3.39)\\ &+ 2\mathsf{A}_\xi\left[\cos^2\theta\dot{f}_1\left(2\dot{f}_1 + r\ddot{f}_1\right) + \sin^2\theta\dot{f}_2\left(2\dot{f}_2 + r\ddot{f}_2\right)\right]\Bigg\}\mathbf{C},\end{aligned}$$

which leads us to introduce the notation

$$\begin{aligned}\Delta_4(\mathsf{A},\mathsf{B}) := &\frac{1}{r^2}\Bigg\{10\mathsf{A} + r\mathsf{A}_r + 2r^2\left[\mathsf{A}_s - \mathsf{B}_\xi\right] \qquad (6.3.40)\\ &+ 2r^2\mathsf{A}_\xi\left[\cos^2\theta\dot{f}_1\left(2\dot{f}_1 + r\ddot{f}_1\right) + \sin^2\theta\dot{f}_2\left(2\dot{f}_2 + r\ddot{f}_2\right)\right]\Bigg\}.\end{aligned}$$

This is the requisite discriminant term when the dimension $n = 4$, analogous to the identity (6.3.18), which again has a significant effect on the solution sets of the PDE $\mathscr{L}[u;\mathsf{A},\mathsf{B}] = \nabla\mathscr{P}$. Before presenting this result we remark that the divergence-free systems (6.3.34)-(6.3.35) in the case that the functions f_1, f_2 have no θ-dependence become, for $i = 1, 2$,

$$\begin{cases}\dfrac{d}{dr}\left[r^5\mathsf{A}(r, r^2, 4 + r^2[\cos^2\theta\dot{f}_1^2 + \sin^2\dot{f}_2^2])\dot{f}_i\right] = 0 & a < r < b,\\ f_i = 0 & r = a,\\ f_i = 2m_i\pi & r = b,\end{cases} \qquad (6.3.41)$$

where $m_i \in \mathbb{Z}$ for $i = 1, 2$.

Theorem 6.3.2. *For $n = 4$ let $u = diag(\mathcal{R}[f_1], \mathcal{R}[f_2])x$ for $\mathcal{R}$ defined by* (6.3.2) *and $f_1 = f_1(r; m_1)$, $f_2 = f_2(r; m_2) \in \mathscr{C}^2[a,b]$ serving as solutions of the systems* (6.3.41) *for $i = 1$, 2. Assume that $\Delta_4(\mathsf{A},\mathsf{B}) \not\equiv 0$ over $\mathscr{R}_4$ for $\Delta_4(\mathsf{A},\mathsf{B})$ defined by* (6.3.40). *Then the following are equivalent.*

- *u solves the PDE $\mathscr{L}[u;\mathsf{A},\mathsf{B}] = \nabla\mathscr{P}$.*
- *We have $f_1(r;m_1) = f_2(r;m_2) =: F(r;m)$ such that, for each $m \in \mathbb{Z}$, $F \in \mathscr{C}^2[a,b]$ is the unique solution of the two-point boundary value problem*

$$\begin{cases} \dfrac{d}{dr}\left[r^5\mathsf{A}(r,r^2,4+r^2\dot{F}^2)\dot{F}\right] & = 0 \\ F(a) = 0, \\ F(b) = 2m\pi. \end{cases} \tag{6.3.42}$$

Furthermore in this case we have the explicit description of the pressure field

$$\mathscr{P} = \mathsf{A}(|x|,|x|^2,4+r^2\dot{F}^2) + G(|x|) \tag{6.3.43}$$

and G satisfies $\nabla G(|x|) = \mathsf{B}(r,r^2,4+r^2\dot{F}^2)x - \dot{F}^2\mathsf{A}(r,r^2,4+r^2\dot{F}^2)x$.

As such we see that when $\Delta_4(\mathsf{A},\mathsf{B}) \not\equiv 0$ we are forced into equality of boundary conditions for the angle of rotation functions f_1, f_2 which leads to their being equal too, thus we are in the setting of Theorem 6.2.2. Note also that the two-point boundary value problem (6.3.42) is precisely the analogy of (6.2.7).

Proof. In the first instance it can be seen from (6.3.39) that if $\Delta_4(\mathsf{A},\mathsf{B}) \not\equiv 0$ then $\operatorname{curl}\mathscr{L}[u;\mathsf{A},\mathsf{B}] \equiv 0$ *if and only if* $\dot{f}_1^2 = \dot{f}_2^2 \implies f_1 = f_2$ up to a constant. By evaluating the boundary condition imposed at $r = a$ it is clear that $f_1(a) - f_2(a) = 0$ and so by continuity and the fact that f_1 and f_2 do not change sign, since, for $i = 1,2$ and with $\mathsf{A} > 0$,

$$\dot{f}_i = \frac{c}{r^5\mathsf{A}(r,r^2,4+r^2[\cos^2\theta\dot{f}_1^2 + \sin^2\theta\dot{f}_2^2])}, \qquad c \in \mathbb{R},\ a < r < b, \tag{6.3.44}$$

we have $f_1 \equiv f_2$ and in particular $m_1 = m_2 =: m \in \mathbb{Z}$. Call this common function $F = F(r;m)$ and by adding the equations in the system (6.3.41), we have that F solves (6.3.42). The existence and uniqueness of a solution to this system follows from Lemma C.0.1 (*see* also Theorem 6.2.2). Clearly if the above holds then we see by substitution $u(x) = diag(\mathcal{R}[F],\mathcal{R}[F])x$ and

$$\begin{aligned} \mathscr{L}[u;\mathsf{A},\mathsf{B}] = & \nabla\mathsf{A}(r,r^2,4+r^2\dot{F}^2) - \dot{F}^2\mathsf{A}(r,r^2,4+r^2\dot{F}^2)x \\ & + \mathsf{B}(r,r^2,4+r^2\dot{F}^2)x, \end{aligned} \tag{6.3.45}$$

thus the result is proved. $\square$

6.4 A Particular System with $\mathsf{A}=h(r,s)$ and the Influence of the Discriminant $\Delta(h,\mathsf{B})$

In this final section we return to a general n-dimensional setting and fix $\mathsf{A}(r,s,\xi)=h(r,s)$[16] as a radial function (with $s=r^2$) and explicitly solve the system

$$\begin{cases} (\nabla u)^t\{\mathrm{div}[h(|x|,|x|^2)\nabla u]+\mathsf{B}(|x|,|x|^2,|\nabla u|^2)u\}=\nabla\mathscr{P} & \text{in } \mathbb{X}^n,\\ \det\nabla u=1 & \text{in } \mathbb{X}^n,\\ u\equiv x & \text{on } \partial\mathbb{X}^n, \end{cases} \tag{6.4.1}$$

Here $h(r,s)>0$ and we begin this section with a study of the system analogous to (6.2.5)-(6.2.6) in this setting. Specifically for each $1\le\ell\le d$ we aim to solve

$$\mathbf{BVP}[h;f_\ell,m_\ell]=\begin{cases} \mathrm{div}_{\mathbb{A}}\left[h(\|\varrho\|,\|\varrho\|^2)\rho_\ell^2\omega(\varrho;d)\nabla_{\mathbb{A}}f_\ell\right]=0 & \text{in } \mathbb{A}_n,\\ f_\ell\equiv 0 & \text{on } (\partial\mathbb{A}_n)_a,\\ f_\ell\equiv 2m_\ell\pi & \text{on } (\partial\mathbb{A}_n)_b,\\ h(\|\varrho\|,\|\varrho\|^2)\rho_l^2\omega(\varrho;d)\partial_\nu f_\ell=0 & \text{on } \Gamma_n, \end{cases} \tag{6.4.2}$$

over the admissible space (6.2.4) with $p=2$. Here $m_\ell\in\mathbb{Z}$ and $\omega(\varrho;d)=\rho_1\dots\rho_d$.

As a first remark observe that, unlike the system (6.2.5), the above depends on only the ℓ-th component of both the vectors f and ϱ and as such the system decouples and we can solve it for each fixed $1\le\ell\le d$ individually. This system is in fact precisely (3.6.3) previously studied in Chapter 3 in the context of the weighted Dirichlet energy. We repeat the result establishing its unique solution but defer the proof to Theorem 3.6.1 and the calculations preceding it.

Theorem 6.4.1. *For all $n\ge2$ and each $1\le\ell\le d$ the system (6.4.2) admits the unique solution $f_\ell(\varrho;m_\ell)=\mathscr{G}(r,m_\ell):=2m_\ell\pi\mathscr{H}(r)$, where $\mathscr{H}(r)\in\mathscr{C}^2[a,b]$ is defined by*

$$\mathscr{H}(r)=\frac{\mathsf{H}(r)}{\mathsf{H}(b)},\qquad \mathsf{H}(r):=\int_a^r\frac{ds}{s^{n+1}h(s,s^2)}. \tag{6.4.3}$$

[16] Note that if, additionally, $\mathsf{B}(r,s,\xi)=-h_s(r,s)\xi$ this corresponds to the variational case $F(r,s,\xi)=h(r,s)\xi$, which, if substituted into the energy functional (6.1.4), corresponds to a weighted form of the classical Dirichlet energy. This system has been considered many times throughout the thesis and we refer to Chapter 3 for a thorough analysis.

It is easily verified that for all $1 \le \ell \le d$ the function $f_\ell = \mathscr{G}$ defined above is the unique solution to the boundary value problem

$$\begin{cases} \dfrac{d}{dr}\left[r^{n+1}h(r,r^2)\dot{\mathscr{G}}\right] = 0, \quad a < r < b, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2m_\ell\pi, \end{cases} \tag{6.4.4}$$

which is the counterpart of (6.2.7) for the choice of $\mathsf{A}(r,s,\xi) = h(r,s)$.

As a consequence of the above result we henceforth only consider twist loops $\mathbf{Q}$ which depend on $\|\varrho\| = r$ and take the precise form $\mathbf{Q}(r;\mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$ with $\mathscr{H}$ as defined in (6.4.3) and $\mathbf{H}$ the constant $n\times n$ skew-symmetric matrix given by

$$\mathbf{H}(\mathsf{m}) = \begin{cases} diag(2m_1\pi\mathbf{J},\dots,2m_d\pi\mathbf{J}), & n = 2d, \\ diag(2m_1\pi\mathbf{J},\dots,2m_d\pi\mathbf{J},0), & n = 2d+1, \end{cases} \tag{6.4.5}$$

with $\mathbf{J}$ as defined in (6.3.2). We now wish to formulate an appropriate description of the vector field $\mathscr{L}[u;\mathsf{A},\mathsf{B}]$ given that the system $\mathbf{BVP}[h;f_\ell,m_\ell]$ is satisfied for all $1 \le \ell \le d$ and that $\mathbf{Q}(r;\mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}(\mathsf{m})\}$ is as described above. Recall the definition

$$\begin{aligned} \mathscr{L}[u;h,\mathsf{B}] &= (\nabla u)^t\{\mathrm{div}[h(|x|,|x|^2)\nabla u] + \mathsf{B}(r,|u|^2,|\nabla u|^2)u\} \qquad (6.4.6)\\ &= (\nabla u)^t\{[h_r(r,r^2)+2rh_s(r,r^2)]\nabla u\theta + h(r,r^2)\Delta u + \mathsf{B}(r,r^2,|\nabla u|^2)u\}. \end{aligned}$$

In the instance that $\mathbf{Q}(\varrho) = \mathbf{Q}(\|\varrho\|)$ we have[17]

$$\nabla u = \mathbf{Q} + \dot{\mathbf{Q}}\theta \otimes \sum_{\ell=1}^{N} \rho_\ell \nabla\rho_\ell = \mathbf{Q} + \dot{\mathbf{Q}}\theta\otimes x = \mathbf{Q} + r\dot{\mathbf{Q}}\theta\otimes\theta. \tag{6.4.7}$$

In particular it follows from this that $|\nabla u|^2 = n + r^2|\dot{\mathbf{Q}}\theta|^2 = n + r^2\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2$ with $\mathbf{H}$ defined by (6.4.5). For the Laplacian Δu we first note that $\Delta\rho_\ell = 1/\rho_\ell$ except for n odd and $\ell = N$ where $\Delta\rho_N = 0$ and

$$\mathbf{Q}_{,\ell\ell} = \frac{\rho_\ell^2}{r^2}\ddot{\mathbf{Q}} + \frac{1}{r}\frac{r^2-\rho_\ell^2}{r^2}\dot{\mathbf{Q}}. \tag{6.4.8}$$

[17]Recall the identities (3.7.4)-(3.7.6) and the collection in Proposition A.0.3.

It follows that

$$
\begin{aligned}
\Delta u &= \sum_{\ell=1}^{N} [\mathbf{Q}_{,\ell\ell} x + \Delta \rho_\ell \mathbf{Q}_{,\ell} x + 2\mathbf{Q}_{,\ell} \nabla \rho_\ell] \\
&= \sum_{\ell=1}^{N} \left[\left(\frac{\rho_\ell^2}{r^2} \ddot{\mathbf{Q}} + \frac{r^2 - \rho_\ell^2}{r^3} \dot{\mathbf{Q}} \right) x + \frac{\rho_\ell}{r} \Delta \rho_\ell \dot{\mathbf{Q}} x + 2 \frac{\rho_\ell}{r} \dot{\mathbf{Q}} \nabla \rho_\ell \right] \\
&= r \ddot{\mathbf{Q}} \theta + (n+1) \dot{\mathbf{Q}} \theta. \qquad (6.4.9)
\end{aligned}
$$

With these identities at hand we recover

$$
\begin{aligned}
\mathscr{L}[u; h, \mathsf{B}] &= (\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta) \Big\{ [h_r(r, r^2) + 2r h_s(r, r^2)](\mathbf{Q}\theta + r\dot{\mathbf{Q}}\theta) \\
&\quad + h(r, r^2) \left[r\ddot{\mathbf{Q}} + (n+1)\dot{\mathbf{Q}} \right] \theta + r\mathsf{B}(r, r^2, n + r^2 |\dot{\mathbf{Q}}\theta|^2) \mathbf{Q}\theta \Big\} \\
&= \nabla h(|x|, |x|^2) + \left[r^2 h_r(r, r^2) + 2r^3 h_s(r, r^2) + r(n+1) h(r, r^2) \right] \dot{\mathscr{H}}^2 |\mathbf{H}\theta|^2 \theta \\
&\quad + r^2 h(r, r^2) \dot{\mathscr{H}} \ddot{\mathscr{H}} |\mathbf{H}\theta|^2 \theta + r h(r, r^2) \dot{\mathscr{H}}^2 \mathbf{H}^2 \theta + r\mathsf{B}(r, r^2, n + r^2 \dot{\mathscr{H}}^2 |\mathbf{H}\theta|^2) \theta. \qquad (6.4.10)
\end{aligned}
$$

By an application of the ODE governing (6.4.4) (note that, as a scalar multiple of $\mathscr{G}$, the function $\mathscr{H}$ also solves this ODE) the above simplifies to

$$
\mathscr{L}[u; h, \mathsf{B}] = \nabla h(|x|, |x|^2) + h(r, r^2) \dot{\mathscr{H}}^2 \mathbf{H}^2 x + \mathsf{B}(r, s, \xi) x. \qquad (6.4.11)
$$

Anticipating on solving the PDE $\mathscr{L}[u; h, \mathsf{B}] = \nabla \mathscr{P}$ we next compute the curl of the vector field

$$
\begin{aligned}
U(x) &= \mathscr{L}[u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta; h, \mathsf{B}] - \nabla h(|x|, |x|^2) \\
&= \mathsf{B}(r, r^2, n + \dot{\mathscr{H}}^2 |\mathbf{H}x|^2) x + h(r, r^2) \dot{\mathscr{H}}^2 \mathbf{H}^2 x. \qquad (6.4.12)
\end{aligned}
$$

We now apply Proposition D.0.2 to this vector field where, given the description of $\mathbf{H}$ as in (6.4.5), we set $c_\ell = 2m_\ell \pi$, $1 \le \ell \le d$ and the scalar-valued functions $\mathscr{A}(r, z)$ and $\mathscr{B}(r, z)$, $z = |\mathbf{H}x|^2$ therein are given by $\mathscr{A}(r, z) = \mathsf{B}(r, r^2, n + \dot{\mathscr{H}}^2 z)$, $\mathscr{B}(r, z) = h(r, r^2) \dot{\mathscr{H}}^2$. Then, after an application of the ODE (6.4.4), we have

$$
2\mathscr{A}_z(r, z) + \frac{\mathscr{B}_r(r, z)}{r} = -\dot{\mathscr{H}}^2 \left[\frac{2(n+1)h + r h_r + 2r^2 [h_s - \mathsf{B}_\xi]}{r^2} \right] \qquad (6.4.13)
$$

and, with

$$
\Delta(h, \mathsf{B}) := 2\frac{(n+1)}{r^2} h(r, r^2) + \frac{1}{r} h_r(r, r^2) + 2[h_s(r, r^2) - \mathsf{B}_\xi(r, r^2, n + \dot{\mathscr{H}}^2 |\mathbf{H}x|^2)] \qquad (6.4.14)
$$

we have the explicit description

$$[\mathrm{curl}\, U]_{ij} = 4\pi^2 \dot{\mathscr{H}}^2 \Delta(h, \mathsf{B}) \left(m_{s(i)}^2 - m_{s(j)}^2 \right) x_i x_j, \tag{6.4.15}$$

where $s(k) = \lfloor (k+1)/2 \rfloor$ for all $1 \leq k \leq n$. This leads us to the following result concerning solutions of the PDE $\mathscr{L}[u; h, \mathsf{B}] = \nabla \mathscr{P}$. Compare this with Theorem 3.7.1, which is a special case of the below.

Theorem 6.4.2. *Let $\mathbf{Q} \in \mathscr{C}^2(]a, b[, \mathbf{SO}(n)) \cap \mathscr{C}([a, b], \mathbf{SO}(n))$ satisfy $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ and be given by $\mathbf{Q}(\|\varrho\|; \mathsf{m}) = \exp\{\mathscr{H}(\|\varrho\|)\mathbf{H}(\mathsf{m})\}$ for $\mathscr{H} \in \mathscr{C}^2[a, b]$ defined by* (6.4.3) *and $\mathbf{H} = \mathbf{H}(\mathsf{m})$ as given by* (6.4.5)*. Moreover consider the vector field $\mathscr{L}[u; h, \mathsf{B}]$ as in* (6.4.11) *and $\Delta(h, \mathsf{B})$ as in* (6.4.14)*. Then the whirl map $u = \mathbf{Q}(\|\varrho\|)x$ solves the system* (6.4.1) *iff one of the following hold.*

1. *If $\Delta(h, \mathsf{B}) \not\equiv 0$ over $\mathbb{A}_n$ then, depending on the dimension n being even or odd, we have*

   (*i*) *n even: Here $\mathbf{H} = \mathbf{H}(\mathsf{m}) = diag(2m_1\pi\mathbf{J}, \dots, 2m_k\pi\mathbf{J})$ with $|m_1| = \cdots = |m_k|$.*

   (*ii*) *n odd: Here $\mathbf{H} \equiv 0$ necessarily leading to $\mathbf{Q} \equiv \mathbf{I}_n$ and $u \equiv x$ the only solution of* (6.4.1)*.*

2. *If $\Delta(h, \mathsf{B}) \equiv 0$ over $\mathbb{A}_n$ then $\mathbf{Q}(\|\varrho\|) = \exp\{\mathscr{H}(\|\varrho\|)\mathbf{H}(\mathsf{m})\}$ as in the statement of the theorem with no restriction on the integers m_i, $1 \leq i \leq k$.*

Proof. First suppose $\mathscr{L}[u; \mathsf{A}, \mathsf{B}] = \nabla \mathscr{P}$ holds. Then $\mathrm{curl}\, \mathscr{L}[u; \mathsf{A}, \mathsf{B}] \equiv 0 \implies \mathrm{curl}\, U(x) \equiv 0$ for U as defined by (6.4.12). If $\Delta(h, \mathsf{B}) \not\equiv 0$ over $\mathbb{A}_n$ then, by (D.0.10), we see that $m_{s(i)}^2 = m_{s(j)}^2$ for all $1 \leq i, j \leq k$, in which case $|m_1| = \cdots = |m_k| =: m$. Since $m_k = 0$ when the dimension n is odd we conclude $m = 0$ in this case and 1 follows. If, however, $\Delta(h, \mathsf{B}) \equiv 0$ over $\mathbb{A}_n$ then $\mathrm{curl}\, U(x) \equiv 0$ with no further assumptions required on the integers $m_1, \dots, m_k$ and we conclude as in part 2.

Conversely, it follows from Proposition D.0.2 that in either of the cases *1* and *2* above the resulting vector field $\mathscr{L}[u = \mathbf{Q}(\|\varrho\|; \mathsf{m})x; h, \mathsf{B}]$ constitutes a gradient and hence the PDE governing (6.4.1) is satisfied. Moreover the incompressibility constraint $\det \nabla u = 1$ is a result obtained in Lemma A.0.7 and the discussion preceding it (*see* also Proposition A.0.3). It remains to verify the boundary conditions and first we remark that, with $\mathscr{H}$ given by (6.4.3) we have $\mathscr{H}(a) = 0 \implies \mathbf{Q}(a) = \mathbf{I}_n$ and $u|_{\|\varrho\|=a} = x$. Finally with $\mathscr{H}(b) = 1$ we see that, in any case above, $\mathbf{Q}(b) = \exp\{\mathbf{H}(\mathsf{m})\} = \mathbf{I}_n \implies u|_{\|\varrho\|=a} = x$. This completes the proof. $\square$

It is easily seen that $\Delta(h, \mathsf{B}) \equiv 0$ if, given any $0 < h : [a, b] \times]0, \infty[\to \mathbb{R}$, the corresponding function $\mathsf{B} = \mathsf{B}(r, s, \xi)$ satisfies

$$\mathsf{B}_\xi(r, r^2, n + \dot{\mathscr{H}}^2 |\mathbf{H}x|^2) = 2(n+1)h(r, r^2) + rh_r(r, r^2) + 2r^2 h_s(r, r^2). \quad (6.4.16)$$

If we are in the variational context with $\mathsf{B}(r, s, \xi) = -h_s(r, s)\xi$ then $\Delta(h, \mathsf{B}) \equiv 0$ *if and only if* the "h-condition" holds, that is

$$2(n+1)h(r, r^2) + rh_r(r, r^2) + 4r^2 h_s(r, r^2) \equiv 0, \qquad a < r < b. \quad (6.4.17)$$

One such function $h = h(r, s)$ which satisfies the above is $h(r, s) = r^{-\alpha} s^{-\beta}$ for α, $\beta \in \mathbb{R}$ (*see* Figure 2). Then the h-condition holds *if and only if* $\alpha + 4\beta - 2(n+1) = 0$. Of course the class of functions h for which $\Delta(h, -h_s) \equiv 0$ is much larger still.

Appendix A

Key Identities

This first appendix gathers all the necessary key calculus identities pertaining to generalised twists $u(x) = \mathbf{Q}(|x|)x$ and whirls $u(x) = \mathbf{Q}(\varrho)x$ which are used liberally throughout the main body of the text. The first result is relevant to twists inasmuch as $\mathbf{Q} = \mathbf{Q}(r)$, but in fact applies to a wider class of maps, as is seen below. Indeed they adapt to the generalised twist case as a simple corollary.

Proposition A.0.1. *Let $v \in \mathscr{C}^2(\overline{\mathbb{X}^n}, \overline{\mathbb{X}^n})$ and $u = \mathbf{Q}(|x|)v(x)$ for some twist path $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathbf{SO}(n))$. Then the following identities hold.*

(i) $\nabla u = \mathbf{Q}\nabla v + \dot{\mathbf{Q}}v \otimes \theta$,

(ii) $|\nabla u|^2 = |\nabla v|^2 + |\dot{\mathbf{Q}}v|^2 + 2\,\langle \mathbf{Q}^t\dot{\mathbf{Q}}v, \nabla v\,\theta\rangle$,

(iii) $\Delta u = 2\dot{\mathbf{Q}}\nabla v\,\theta + \mathbf{Q}\Delta v + \ddot{\mathbf{Q}}v + \dfrac{n-1}{r}\dot{\mathbf{Q}}v$,

(iv) $\det \nabla u = \det \nabla v + \langle \mathbf{Q}^t\dot{\mathbf{Q}}v, [\operatorname{cof} \nabla v]\theta\rangle$, *whenever* $\det \nabla v(x) \neq 0$.

In the above $r = |x|$ with $a \le r \le b$ and $\theta = x|x|^{-1}$. Furthermore with La-

grangian $F = F(r, s, \xi)$ we have

$$\begin{aligned}\operatorname{div}[F_\xi(r, |u|^2, |\nabla u|^2)\nabla u] = F_{\xi\xi}(r, |u|^2, |\nabla u|^2)(\mathbf{Q}\nabla v + \dot{\mathbf{Q}}v \otimes \theta)\times \\ \times [\nabla(|\nabla v|^2) + \nabla(|\dot{\mathbf{Q}}v|^2) + 2\nabla\langle \mathbf{Q}^t\dot{\mathbf{Q}}v, \nabla v\,\theta\rangle] \\ + F_{s\xi}(r, |u|^2, |\nabla u|^2)(\mathbf{Q}\nabla v + \dot{\mathbf{Q}}v \otimes \theta)\nabla(|v|^2) \\ + F_{r\xi}(r, |u|^2, |\nabla u|^2)(\mathbf{Q}\nabla v + \dot{\mathbf{Q}}v \otimes \theta)\theta \\ + F_\xi(r, |u|^2, |\nabla u|^2)\left[2\dot{\mathbf{Q}}\nabla v\,\theta + \mathbf{Q}\Delta v + \ddot{\mathbf{Q}}v + \frac{n-1}{r}\dot{\mathbf{Q}}v\right].\end{aligned}$$

Proof. The first identity follows by a straightforward differentiation. Indeed proceeding directly we can write

$$\nabla u = \mathbf{Q}\nabla v + \nabla\mathbf{Q}(|x|)v = \mathbf{Q}\nabla v + \dot{\mathbf{Q}}v \otimes \theta = \mathbf{Q}(\nabla v + \mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta). \tag{A.0.1}$$

Proceeding immediately from this on to (iv), using the description of ∇u as given in (i) above, the assumed invertibility of ∇v and the fact that determinant is a quasiaffine function on the space of $n \times n$ matrices (*c.f.* [60]) – as a result of which $\det(\mathbf{I}_n + \zeta \otimes \xi) = 1 + \langle \zeta, \xi\rangle$ for any $\zeta, \xi \in \mathbb{R}^n$ – it follows at once that

$$\begin{aligned}\det\nabla u &= \det\mathbf{Q} \times \det(\nabla v + \mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta) \\ &= \det\nabla v\left[1 + \langle(\nabla v)^{-1}\mathbf{Q}^t\dot{\mathbf{Q}}v, \theta\rangle\right] \\ &= \det\nabla v + \langle \mathbf{Q}^t\dot{\mathbf{Q}}v, [\operatorname{cof}\nabla v]\theta\rangle.\end{aligned} \tag{A.0.2}$$

Next for (ii) using the description of the Hilbert-Schmidt norm of the matrix field ∇u we can write

$$\begin{aligned}|\nabla u|^2 &= tr\{[\nabla u]^t[\nabla u]\} \\ &= tr\left\{([\nabla v]^t\mathbf{Q}^t + \theta \otimes \dot{\mathbf{Q}}v)(\mathbf{Q}[\nabla v] + \dot{\mathbf{Q}}v \otimes \theta)\right\} \\ &= tr\left\{[\nabla v]^t[\nabla v] + [\nabla v]^t\mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta + \theta \otimes [\nabla v]^t\mathbf{Q}^t\dot{\mathbf{Q}}v + (\theta \otimes \dot{\mathbf{Q}}v)(\dot{\mathbf{Q}}v \otimes \theta)\right\} \\ &= |\nabla v|^2 + 2\langle \mathbf{Q}^t\dot{\mathbf{Q}}v, \nabla v\,\theta\rangle + |\dot{\mathbf{Q}}v|^2.\end{aligned} \tag{A.0.3}$$

Likewise for (iii) by taking the divergence of ∇u as given by (i), we compute the Laplacian Δu to be

$$\Delta u = \operatorname{div}(\mathbf{Q}\nabla v + \dot{\mathbf{Q}}v \otimes \theta) = 2\dot{\mathbf{Q}}\nabla v\theta + \mathbf{Q}\Delta v + \ddot{\mathbf{Q}}v + \frac{n-1}{r}\dot{\mathbf{Q}}v. \tag{A.0.4}$$

The final identity then follows by direct differentiation and use of the chain rule:

$$\begin{aligned}\operatorname{div}[F_\xi(r,|u|^2,|\nabla u|^2)\nabla u] =& F_{\xi\xi}(r,|u|^2,|\nabla u|^2)\nabla u\nabla(|\nabla u|^2)\\&+F_{s\xi}(r,|u|^2,|\nabla u|^2)\nabla u\nabla(|u|^2)\\&+F_{r\xi}(r,|u|^2,|\nabla u|^2)\nabla u\theta+F_\xi(r,|u|^2,|\nabla u|^2)\Delta u.\end{aligned}\tag{A.0.5}$$

Noting that $|v|^2 = |u|^2$ we now have all the identities to complete the expression above and the result follows. □

Proposition A.0.2. *Let $v \in \mathscr{C}^2(\overline{\mathbb{X}^n}, \overline{\mathbb{X}^n})$ and $u = \mathbf{Q}(|x|)v(x)$ for some twist path $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathbf{SO}(n))$. Then the vector field $\mathscr{L}[u] = (\nabla u)^t\{\operatorname{div}[F_\xi(|x|,|u|^2,|\nabla u|^2)\nabla u] - F_s(|x|,|u|^2,|\nabla u|^2)u\}$ is described by*

$$\begin{aligned}\mathscr{L}[u] =& \Big[(\nabla v)^t\mathbf{Q}^t+\theta\otimes\dot{\mathbf{Q}}v\Big]\Big\{F_{\xi\xi}(r,|u|^2,|\nabla u|^2)(\mathbf{Q}\nabla v+\dot{\mathbf{Q}}v\otimes\theta)\Big[\nabla(|\nabla v|^2)\\&+\nabla(|\dot{\mathbf{Q}}v|^2)+2\nabla\langle\mathbf{Q}^t\dot{\mathbf{Q}}v,\nabla v\,\theta\rangle\Big]+F_{s\xi}(r,|u|^2,|\nabla u|^2)\times\\&\times(\mathbf{Q}\nabla v+\dot{\mathbf{Q}}v\otimes\theta)\nabla(|v|^2)+F_{r\xi}(r,|u|^2,|\nabla u|^2)(\mathbf{Q}\nabla v+\dot{\mathbf{Q}}v\otimes\theta)\theta\\&+F_\xi(r,|u|^2,|\nabla u|^2)\Big[2\dot{\mathbf{Q}}\nabla v\,\theta+\mathbf{Q}\Delta v+\ddot{\mathbf{Q}}v+\frac{n-1}{r}\dot{\mathbf{Q}}v\Big]\\&-F_s(r,|u|^2,|\nabla u|^2)\mathbf{Q}(r)v\Big\}.\end{aligned}\tag{A.0.6}$$

Proof. The result is a direct consequence of the definition of $\mathscr{L}[u]$ as in the statement of the result and the relevant identities in Proposition A.0.1. □

Proposition A.0.3. *Let $u = r\mathbf{Q}(r)\theta$ be a generalised twist with a twice continuously differentiable twist path $\mathbf{Q}$, that is, $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathbf{SO}(n))$. Then, with $r = |x|$, $\theta = x|x|^{-1}$, the following identities hold:*

(*i*) $\nabla u = \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta$,

(*ii*) $|\nabla u|^2 = n + r^2|\dot{\mathbf{Q}}\theta|^2$,

(*iii*) $\Delta u = [(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}]\theta$.

Consequently, the action of the operator $\mathscr{L} = \mathscr{L}[u]$ can be written as

$$\begin{aligned}\mathscr{L}[u] =& (\nabla u)^t\left\{\operatorname{div}\left[F_\xi(|x|,|u|^2,|\nabla u|^2)\nabla u\right]-F_s(|x|,|u|^2,|\nabla u|^2)u\right\}\\=&(\mathbf{Q}^t+r\theta\otimes\dot{\mathbf{Q}}\theta)\Big[F_{\xi\xi}(r,r^2,|\nabla u|^2)(\mathbf{Q}+r\dot{\mathbf{Q}}\theta\otimes\theta)(2r|\dot{\mathbf{Q}}\theta|^2\theta+r^2\nabla|\dot{\mathbf{Q}}\theta|^2)\\&+2rF_{s\xi}(r,r^2,|\nabla u|^2)(\mathbf{Q}\theta+r\dot{\mathbf{Q}}\theta)+F_{r\xi}(r,r^2,|\nabla u|^2)(\mathbf{Q}\theta+r\dot{\mathbf{Q}}\theta)\\&+F_\xi(r,r^2,|\nabla u|^2)[(n+1)\dot{\mathbf{Q}}+r\ddot{\mathbf{Q}}]\theta-rF_s(r,r^2,|\nabla u|^2)\mathbf{Q}\theta\Big].\end{aligned}\tag{A.0.7}$$

Proof. The proof is a direct consequence of Proposition A.0.1 and Proposition A.0.2 upon setting $v \equiv x$ and noting that for a generalised twist u as above we have $|u|^2 = |r\mathbf{Q}(r)\theta|^2 = r^2$. $\square$

We proceed to show that any generalised twist $u(x) = \mathbf{Q}(|x|)x$ satisfies the incompressibility constraint $\det \nabla u = 1$.

Proposition A.0.4. *Let $u = r\mathbf{Q}(r)\theta$ with $\mathbf{Q} \in \mathscr{C}^1(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}([a,b], \mathbf{SO}(n))$. Then u is incompressibile.*

Proof. From Proposition A.0.3 above we have $\nabla u = \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta$. Hence $\det \nabla u = \det[\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta] = \det[\mathbf{I}_n + r\mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta] = 1$ where in concluding the second equality we have used $\det[\mathbf{I}_n + \zeta \otimes \xi] = 1 + \langle \zeta, \xi \rangle$ for $\zeta, \xi \in \mathbb{R}^n$ resulting from the rank-one affine property of the determinant function and $\langle \mathbf{Q}^t\dot{\mathbf{Q}}\theta, \theta \rangle = 0$ resulting from $\mathbf{Q}^t\dot{\mathbf{Q}}$ being skew-symmetric. $\square$

The next corollary lists the analogous results in the instance when the twist path $\mathbf{Q} = \mathbf{Q}(r)$ manifests a geodesic in the compact Lie group $\mathbf{SO}(n)$. That is $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{A}\}$ for a twice continuously-differentiable function $\mathscr{G}$ (this regularity is not in general a requirement but will be seen to be necessary for the purposes of the following result) and $\mathbf{A}$ is a constant $n \times n$ skew-symmetric matrix.

Corollary A.0.5. *Let $u = r\mathbf{Q}(r)\theta$ be a generalised twist with the twist path $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{A}\}$ for some $\mathscr{G} \in \mathscr{C}^2([a,b])$ and $n \times n$ skew-symmetric matrix $\mathbf{A}$. Then with $r = |x|, \theta = x|x|^{-1}$ and $\theta^\star = \mathbf{A}\theta$ the following identities hold:*

(i) $\nabla u = \mathbf{Q}(\mathbf{I}_n + r\dot{\mathscr{G}}\theta^\star \otimes \theta)$,

(ii) $|\nabla u|^2 = n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2$,

(iii) $\Delta u = \mathbf{Q}\left[(n+1)\dot{\mathscr{G}}\theta^\star + r\ddot{\mathscr{G}}\theta^\star + \dot{\mathscr{G}}^2\mathbf{A}\theta^\star\right]$,

(iv) $\det \nabla u = \det[\mathbf{Q}(\mathbf{I}_n + r\dot{\mathscr{G}}\theta^\star \otimes \theta)] = 1$.

In particular for the Lagrangian $F = F(|x|, |u|^2, |\nabla u|^2)$ we have

$$
\begin{aligned}
&\mathscr{L}[u] = \mathscr{L}[r\exp\{\mathscr{G}(r)\mathbf{A}\}\theta] = F_{\xi\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2)\times \\
&\times \left[\left(\mathbf{I}_n + r\dot{\mathscr{G}}(\theta^\star \otimes \theta + \theta \otimes \theta^\star) + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\theta \otimes \theta\right) \left(2r\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\nabla[\dot{\mathscr{G}}^2|\theta^\star|^2]\right) \right] \\
&+ \left[2rF_{s\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2) + F_{r\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2)\right]\times \\
&\times (\theta + r\dot{\mathscr{G}}\theta^\star + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\theta) \\
&+ F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2)\Big[(n+1)\dot{\mathscr{G}}\theta^\star + r(\ddot{\mathscr{G}}\theta^\star + \dot{\mathscr{G}}^2\mathbf{A}\theta^\star) \\
&+ r(n+1)\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\theta^\star|^2\theta\Big] - rF_s(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2)\theta. \qquad \text{(A.0.8)}
\end{aligned}
$$

Proof. The first four identities follow immediately from Proposition A.0.3 upon noting $\dot{\mathbf{Q}} = \dot{\mathscr{G}}\mathbf{A}\mathbf{Q}$, $\ddot{\mathbf{Q}} = (\ddot{\mathscr{G}}\mathbf{A} + \dot{\mathscr{G}}^2\mathbf{A}^2)\mathbf{Q}$ and $|\dot{\mathbf{Q}}\theta|^2 = \dot{\mathscr{G}}^2\langle\mathbf{A}\theta, \mathbf{A}\theta\rangle = \dot{\mathscr{G}}^2|\theta^\star|^2$. We also have that $\mathbf{Q}$ and $\mathbf{A}$ commute and $\langle\theta, \theta^\star\rangle = 0$ since $\mathbf{A}$ is a skew-symmetric matrix. Now to finish off the proof a further reference to Proposition A.0.3 gives

$$
\begin{aligned}
\mathscr{L}[u] = {} & (\mathbf{I}_n + r\dot{\mathscr{G}}\theta \otimes \theta^\star)\mathbf{Q}^t\times \\
& \times \Big\{\mathbf{Q}F_{\xi\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2)(\mathbf{I}_n + r\dot{\mathscr{G}}\theta^\star \otimes \theta)(2r\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\nabla[\dot{\mathscr{G}}^2|\theta^\star|^2]) \\
& + 2r\mathbf{Q}F_{s\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2)(\theta + r\dot{\mathscr{G}}\theta^\star) \\
& + \mathbf{Q}F_{r\xi}(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2)(\theta + r\dot{\mathscr{G}}\theta^\star) \\
& + \mathbf{Q}F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2)\left[(n+1)\dot{\mathscr{G}}\theta^\star + r\ddot{\mathscr{G}}\theta^\star + r\dot{\mathscr{G}}^2\mathbf{A}\theta^\star\right] \\
& - r\mathbf{Q}F_s(r, r^2, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2)\theta\Big\}. \qquad \text{(A.0.9)}
\end{aligned}
$$

Multiplying the factor of $(\mathbf{I}_n + r\dot{\mathscr{G}}\theta \otimes \theta^\star)$ through and using the orthogonality of the matrix function $\mathbf{Q}$ yields the desired result. □

We now present a result similar in nature to those above but now pertaining to whirl maps $u(x) = \mathbf{Q}(\varrho)x$ with twist path $\mathbf{Q}$ restricted to the maximal torus $\mathbb{T} \subset \mathbf{SO}(n)$ and given explicitly by $\mathbf{Q}(\varrho) = \exp\{\mathbf{H}(\varrho)\}$, where $\mathbf{H} : \overline{\mathbb{A}_n} \to \mathfrak{so}(n)$ is given by

$$
\mathbf{H}(\varrho) = \begin{cases} diag(f_1\mathbf{J}, \ldots, f_d\mathbf{J}) & n = 2d, \\ diag(f_1\mathbf{J}, \ldots, f_d\mathbf{J}, 0) & n = 2d+1. \end{cases} \qquad \text{(A.0.10)}
$$

Here $\mathbf{J} = \mathcal{R}[\pi/2]$ defines a rotation by angle $\pi/2$ as in, for example, (2.3.4) and the functions $f_\ell \in \mathscr{C}(\overline{\mathbb{A}_n})$ for all $1 \le \ell \le d$ satisfy $f_\ell \equiv 0$ on $(\partial\mathbb{A}_n)_a$ and $f_\ell \equiv 2m_\ell\pi$ on $(\partial\mathbb{A}_n)_b$ for $m_\ell \in \mathbb{Z}$. Recall moreover that $\mathbb{A}_n$ is the semi-annular domain defined independently depending on whether the underlying spatial dimension n is odd or even [*see*, for example, (6.1.6)-(6.1.7)].

Proposition A.0.6. *Let* $u(x) = \mathbf{Q}(\varrho)x$ *with* $\mathbf{Q} \in \mathscr{C}(\mathbb{A}_n, \mathbf{SO}(n)) \cap \mathscr{C}^2(\mathbb{A}_n, \mathbf{SO}(n))$ *given by* (A.0.10)*. Then the following identities hold.*

(i) $\nabla u = \mathbf{Q}\left(\mathbf{I}_n + \sum_{\ell=1}^{N} \mathbf{H}_{,\ell}x \otimes \nabla\rho_\ell\right),$

(ii) $|\nabla u|^2 = n + \sum_{\ell=1}^{d} \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2,$

(iii) $\det\nabla u = 1,$

(iv) $\Delta u = \mathbf{Q}\sum_{\ell=1}^{N} \left[\mathbf{H}_{,\ell\ell}x + \Delta\rho_\ell \mathbf{H}_{,\ell}x + 2\mathbf{H}_{,\ell}\nabla\rho_\ell\right].$

Here $\mathbf{H}_{,\ell}$ *and* $\mathbf{H}_{,\ell\ell}$ *denote the first and second derivatives of the skew-symmetric matrix* $\mathbf{H}$ *as defined by* (A.0.10) *with respect to* ρ_ℓ *and throughout the statement and proof* $\nabla_{\mathbb{A}}$ *denotes the gradient taken with respect to the variables* $\rho_1, \dots, \rho_N$*.*

Proof. We begin by noting that, with $\mathbf{Q} = \exp\{\mathbf{H}(\varrho)\}$ we have $\mathbf{Q}_\ell = \mathbf{H}_{,\ell}\mathbf{Q}$ and $\mathbf{Q}_{,\ell\ell} = (\mathbf{H}_{,\ell\ell} + \mathbf{H}_{,\ell}^2)\mathbf{Q}$. Moreover since $\mathbf{Q}$ takes values on the maximal torus $\mathbb{T}$ os $\mathbf{SO}(n)$ it follows that $\mathbf{Q}$ and $\mathbf{H}$ (along with any of its derivatives) commute. The first identity is the consequence of a straightforward differentiation via the product rule. Indeed

$$\nabla u = \mathbf{Q} + \sum_{\ell=1}^{N} \mathbf{Q}_{,\ell}x \otimes \nabla\rho_\ell = \mathbf{Q}\left(\mathbf{I}_n + \sum_{\ell=1}^{N} \mathbf{H}_{,\ell}x \otimes \nabla\rho_\ell\right). \tag{A.0.11}$$

Jumping to identity *(iii)* we have

$$\det\nabla u = \det\left(\mathbf{I}_n + \sum_{\ell=1}^{N} \mathbf{H}_{,\ell}x \otimes \nabla\rho_\ell\right), \tag{A.0.12}$$

since $\det\mathbf{Q} = 1$. To verify the incompressibility constraint $\det\nabla u = 1$ we will require, for any $1 \le i, j \le N$, the identity $\langle \mathbf{H}_{,i}x, \nabla\rho_j\rangle = 0$. To justify this first observe that

$$\mathbf{H}_{,\ell} = \begin{cases} diag(\partial_\ell f_1 \mathbf{J}, \dots, \partial_\ell f_d \mathbf{J}) & n = 2d, \\ diag(\partial_\ell f_1 \mathbf{J}, \dots, \partial_\ell f_d \mathbf{J}, 0) & n = 2d+1, \end{cases} \tag{A.0.13}$$

where $\partial_\ell f_k = \partial_{\rho_\ell} f_k$ and $\mathbf{J} = \mathcal{R}[\pi/2]$. Furthermore for any $1 \le k \le d$ let $\zeta_k = (x_{2k-1}, x_{2k})$ in any dimension and $\zeta_{d+1} = x_n$ if the dimension $n = 2d+1$ is odd. Then it is seen that $\nabla \rho_k = (0, 0, \ldots, \zeta_k/\rho_k, 0, \ldots, 0)$, so in particular $\langle \nabla\rho_j, \nabla\rho_k\rangle = \delta_{jk}$. We also see that

$$\mathbf{H}_{,\ell} x = \begin{cases} diag(\partial_\ell f_1 \mathbf{J}\zeta_1, \ldots, \partial_\ell f_d \mathbf{J}\zeta_d) & n = 2d, \\ diag(\partial_\ell f_1 \mathbf{J}\zeta_1, \ldots, \partial_\ell f_d \mathbf{J}\zeta_d, 0) & n = 2d+1, \end{cases} \tag{A.0.14}$$

upon which $\langle \mathbf{H}_{,\ell} x, \nabla\rho_k\rangle = \langle \partial_\ell f_k \mathbf{J}\zeta_k, \zeta_k/\rho_k\rangle = 0$ since $\mathbf{J}$ is skew-symmetric. Now returning to (A.0.12) we see that an application of Lemma A.0.7 below to the strings of vectors $a_i = \mathbf{H}_{,i} x$ and $b_i = \nabla\rho_i$ $(1 \le i \le N)$ we conclude $\det\nabla u = 1$.

For identity (ii) we employ the Hilbert-Schmidt description of the norm which gives

$$\begin{aligned}
|\nabla u|^2 &= tr\{\nabla u (\nabla u)^t\} \\
&= tr\Bigg\{\Bigg(\mathbf{Q} + \sum_{\ell=1}^N \mathbf{Q}_{,\ell} x \otimes \nabla\rho_\ell\Bigg)\Bigg(\mathbf{Q}^t + \sum_{\ell=1}^N \nabla\rho_\ell \otimes \mathbf{Q}_{,\ell} x\Bigg)\Bigg\} \\
&= tr\Bigg\{\mathbf{I}_n + \sum_{\ell=1}^N \mathbf{Q}\nabla\rho_\ell \otimes \mathbf{Q}_{,\ell} x + \sum_{\ell=1}^N \mathbf{Q}_{,\ell} x \otimes \mathbf{Q}\nabla\rho_\ell + \sum_{\ell=1}^N \mathbf{Q}_{,\ell} x \otimes \mathbf{Q}_{,\ell} x\Bigg\} \\
&= n + \sum_{\ell=1}^N |\mathbf{Q}_{,\ell} x|^2 = n + \sum_{\ell=1}^N |\mathbf{H}_{,\ell}\mathbf{Q} x|^2 = n + \sum_{\ell=1}^N |\mathbf{H}_{,\ell} x|^2,
\end{aligned} \tag{A.0.15}$$

as $|\mathbf{Q}| = 1$. It is then easily seen that

$$\sum_{\ell=1}^N |\mathbf{H}_{,\ell} x|^2 = \sum_{\ell=1}^d \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2 \tag{A.0.16}$$

and identity (ii) follows. Finally for the Laplacian we have that $\Delta\rho_\ell = 1/\rho_\ell$ (except when n is odd and $\ell = N$, in which case $\Delta\rho_n = 0$), then taking the divergence of identity (i) yields the result. □

We state a lemma which was used in the course of Proposition A.0.6 to verify the incompressibility constraint $\det\nabla u = 1$, here presented in a general context.

Lemma A.0.7. *Let $(a_i)_{i=1}^k$ and $(b_i)_{i=1}^k$ be strings of mutually orthogonal vectors in $\mathbb{R}^n$ satisfying $\langle a_j, b_l\rangle = 0$ for all $1 \le j, l \le k$. Then*

$$\det\Bigg(\mathbf{I}_n + \sum_{i=1}^k a_i \otimes b_i\Bigg) = 1. \tag{A.0.17}$$

Proof. We prove the result by induction on i. For $i = 1$ it is immediate that $\det(\mathbf{I}_n + a_1 \otimes b_1) = 1 + \langle a_1, b_1 \rangle = 1$ by the rank-one affine property of the determinant function and since a_1 and b_1 are orthogonal by assumption. Now assume (A.0.17) holds for a fixed $j \in \mathbb{N}$, that is $\det A_j = 1$ where we have defined

$$A_j := \mathbf{I}_n + \sum_{i=1}^{j} a_i \otimes b_i. \tag{A.0.18}$$

Then $A_j^{-1} = \mathbf{I}_n - \sum_{i=1}^{j} a_i \otimes b_i$ and observe that

$$\det \left(\mathbf{I}_n + \sum_{i=1}^{j+1} a_i \otimes b_i \right) = \left(1 + \langle b_{j+1}, A_j^{-1} a_{j+1} \rangle\right) \det A_j. \tag{A.0.19}$$

Since $\det A_j = 1$ and $\langle a_i, b_i \rangle = 0$ we see that $\det(\mathbf{I}_n + \sum_{i=1}^{j+1} a_i \otimes b_i) = 1 + \langle b_{j+1}, a_{j+1} \rangle = 1$ which is the required conclusion for $i = j + 1$. $\square$

Appendix B

Derivation of the Euler-Lagrange Equation $\mathscr{L}[u] = \nabla \mathscr{P}$

In this short appendix we briefly outline the derivation of the Euler-Lagrange system

$$\begin{cases} (\nabla u)^t \left\{ \operatorname{div}\left[F_\xi \nabla u\right] - F_s u \right\} = \nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{B.0.1}$$

associated to the variational integral

$$\mathbb{F}[u; \Omega] := \int_\Omega F(|x|, |u|^2, |\nabla u|^2)\, dx \tag{B.0.2}$$

over the space of incompressible p-Sobolev mappings (with $p > 1$)

$$\mathscr{A}^p_\varphi(\Omega) := \left\{ u \in W^{1,p}(\Omega, \mathbb{R}^n) : \det \nabla u = 1,\, u|_{\partial\Omega} = \varphi \right\}. \tag{B.0.3}$$

Using the method of Lagrange multipliers we take the unconstrained energy functional

$$\mathbb{E}[u; \Omega] := \int_\Omega \left\{ F(|x|, |u|^2, |\nabla u|^2) - 2\mathscr{P}(x)[\det \nabla u - 1] \right\} dx, \tag{B.0.4}$$

where $\mathscr{P} = \mathscr{P}(x)$ is a suitable and *a priori* unknown Lagrange multiplier. Note in particular that $\mathbb{E}[u; \Omega] = \mathbb{F}[u; \Omega]$ whenever $u \in \mathscr{A}^p_\varphi(\Omega)$. Now fix $u \in \mathscr{A}^p_\varphi(\Omega)$

of class $\mathscr{C}^2$ satisfying $(|x|, |u|^2, |\nabla u|^2) \in U = ([a,b], \times]0, \infty[\times]0, \infty[) \subset \mathbb{R}^3$ for all $x \in \Omega$ and for $\phi \in \mathscr{C}_c^\infty(\Omega, \mathbb{R}^n)$ and $\varepsilon \in \mathbb{R}$ put $u_\varepsilon = u + \varepsilon\phi$. Then by a basic compactness argument for ε sufficiently small $(|x|, |u_\varepsilon|^2, |\nabla u_\varepsilon|^2) \in U$ for all $x \in \Omega$ and therefore by examining the first-order condition $d/d\varepsilon(\mathbb{E}[u_\varepsilon, \Omega])|_{\varepsilon=0} = 0$ we can write

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\varepsilon}\mathbb{E}[u_\varepsilon; \Omega]\bigg|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_\Omega \left\{\frac{1}{2}F(|x|, |u_\varepsilon|^2, |\nabla u_\varepsilon|^2) - \mathscr{P}(x)\left[\det\nabla u_\varepsilon - 1\right]\right\} dx\bigg|_{\varepsilon=0} \\
&= \int_\Omega \left\{F_s\langle u, \phi\rangle + F_\xi\langle \nabla u, \nabla\phi\rangle - \mathscr{P}(x)\langle \operatorname{cof}\nabla u, \nabla\phi\rangle\right\} dx \\
&= \int_\Omega \langle F_s u - \operatorname{div}(F_\xi \nabla u) + [\operatorname{cof}\nabla u]\nabla\mathscr{P} + \mathscr{P}\operatorname{div}[\operatorname{cof}\nabla u], \phi\rangle\, dx \\
&= \int_\Omega \langle F_s u - \operatorname{div}(F_\xi \nabla u) + [\operatorname{cof}\nabla u]\nabla\mathscr{P}, \phi\rangle\, dx = 0. \qquad \text{(B.0.5)}
\end{aligned}$$

Here $F_s = F_s(|x|, |u|^2, |\nabla u|^2)$, $F_\xi = F_\xi(|x|, |u|^2, |\nabla u|^2)$ where F_s, F_ξ denote the derivatives of F with respect to the second and third arguments respectively. The last line uses the Piola identity which gives $\operatorname{div}\operatorname{cof}\nabla u = 0$, whilst the divergence operator is understood to act on the matrix field $F_\xi(|x|, |u|^2, |\nabla|^2)\nabla u$ row-wise. Now the arbitrariness of $\phi \in \mathscr{C}_c^\infty(\Omega, \mathbb{R}^n)$ gives the Euler-Lagrange equation

$$\begin{aligned}
\mathscr{L}[u] &= (\nabla u)^t \left\{\operatorname{div}\left[F_\xi(|x|, |u|^2, |\nabla u|^2)\nabla u\right] - F_s(|x|, |u|^2, |\nabla u|^2)u\right\} \\
&= (\nabla u)^t \Big\{F_\xi(|x|, |u|^2, |\nabla u|^2)\Delta u + \nabla u \nabla F_\xi(|x|, |u|^2, |\nabla u|^2) \\
&\qquad - F_s(|x|, |u|^2, |\nabla u|^2)u\Big\} = \nabla\mathscr{P}. \qquad \text{(B.0.6)}
\end{aligned}$$

Here we have used the identity $(\operatorname{cof}\nabla u)^{-1} = (\nabla u)^t$ which holds by virtue of the fact that $\det\nabla u = 1$. The term $\mathscr{P}$, which entered the system as a Lagrange multiplier associated to the unconstrained energy functional (B.0.4), is, in the context of nonlinear elasticity, referred to as the *hydrostatic pressure* term associated to u which itself is interpreted as a volume-preserving deformation of the body $\Omega \subset \mathbb{R}^n$.

Appendix C

Some Existence and Uniqueness Results

We now gather two important results regarding the existence and/or uniqueness of solutions to differential equations used throughout the main body of the thesis. The first deals with the uniqueness of a boundary value problem whose solution features prominently throughout the text as the profile of a geodesic twist path $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$. This appears both in the variational (Chapters 2, 3 and 5) and non-variational (Chapters 4 and 6) context and we give the result here in the more general non-variational context despite it being motivated, naturally, by variational methods.

Proposition C.0.1. *For each $m \in \mathbb{Z}$ there exists a unique solution $\mathscr{G} = \mathscr{G}(r;m) \in \mathscr{C}^2[a,b]$ to the two point boundary value problem*

$$\mathbf{BVP}[\mathscr{G};\mathsf{A}] = \begin{cases} \dfrac{d}{dr}\left[r^{n+1}\mathsf{A}(r,r^2,n+r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}\right] = 0, & a < r < b, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2m\pi. \end{cases} \tag{C.0.1}$$

Proof. It is not difficult to see that the system $\mathbf{BVP}[\mathscr{G};\mathsf{A}]$ above is the Euler-Lagrange equation associated with the energy functional

$$\mathscr{G} \mapsto \int_a^b F(r,r^2,n+r^2\dot{\mathscr{G}}^2)r^{n-1}\,dr, \tag{C.0.2}$$

when the Lagrangian $F = F(r,s,\xi)$ of class $\mathscr{C}^2$ is chosen as a derivative of $\mathsf{A}(r,s,\xi)$, specifically, in the sense that $F_\xi(r,s,\xi) = \mathsf{A}(r,s,\xi)$. Thus hereafter we set

$$F(r,s,\xi)=\int_0^\xi \mathsf{A}(r,s,\zeta)\,d\zeta, \qquad a\le r\le b, \quad s>0,\, \xi>0. \tag{C.0.3}$$

Now the assumptions on A result in F being uniformly convex and monotone increasing in ξ. Thus in particular the twice continuously differentiable function $\zeta\mapsto F(r,r^2,n+r^2\zeta^2)$ here is uniformly convex in ζ for all $a\le r\le b$ and $\zeta\in\mathbb{R}$. Furthermore the growth and coercivity of F follows from the similar assumptions set earlier on A and so we have $F_\xi(r,s,\xi)>0$, $F_{\xi\xi}(r,s,\xi)=\mathsf{A}_\xi(r,s,\xi)\ge 0$ and $c_0+c_1|\xi|^{p/2}\le F(r,s,\xi)\le c_2|\xi|^{p/2}$ with $p>1$.

Minimising (C.0.2) over

$$\mathscr{B}^p_m(a,b)=\{\mathscr{G}\in W^{1,p}(a,b):\mathscr{G}(a)=0,\mathscr{G}(b)=2m\pi\} \tag{C.0.4}$$

and applying the direct methods of the calculus of variations now results in the existence of a minimiser $\mathscr{G}^\star$. The $\mathscr{C}^2$ regularity of $\mathscr{G}^\star$ follows by invoking the Tonelli-Hilbert-Weierstrass differentiability theorem (*cf.*, e.g., [21] pp. 57-62) and the uniqueness of minimiser follows from the uniform convexity of F in the ξ-variable and a basic convexity argument. □

We now present an existence result used in Chapters 3 and 6 for the Euler-Lagrange system

$$\mathbf{BVP}[f;\mathscr{U},\mathsf{m}]=\begin{cases}\operatorname{div}_{\mathbb{A}}\mathscr{U}(\varrho,\nabla_{\mathbb{A}}f)=0 & \text{in } \mathbb{A}_n,\\ f\equiv 0 & \text{on } (\partial\mathbb{A}_n)_a,\\ f\equiv 2\mathsf{m}\pi & \text{on } (\partial\mathbb{A}_n)_b,\\ \mathscr{U}(\varrho,\nabla_{\mathbb{A}}f)\nu=0 & \text{on } \Gamma_n.\end{cases} \tag{C.0.5}$$

For definiteness the semi-annular domain $\mathbb{A}_n$ and its boundary segments is described formally in (6.1.6), (6.1.7) and (6.1.8). Moreover $\varrho=(\rho_1,\dots,\rho_N)$ is the vector of 2-plane radial variables as used frequently throughout Chapters 3 and 6. In the above $\mathscr{U}$ is the $d\times N$-dimensional matrix field with row components $\mathscr{U}_\ell$ given by

$$\mathscr{U}_\ell(\varrho,\nabla_{\mathbb{A}}f):=\mathsf{A}\left(\|\varrho\|,\|\varrho\|^2,n+\sum_{\ell=1}^d\rho_\ell^2|\nabla_{\mathbb{A}}f_\ell|^2\right)\rho_\ell^2\omega(\varrho;d)\nabla_{\mathbb{A}}f_\ell, \tag{C.0.6}$$

where $\|\varrho\|=(\rho_1^2+\cdots+\rho_N^2)^{1/2}=|x|=r$. Also note that the third argument in A above is precisely the quantity $|\nabla u|^2$ for a whirl map $u=\mathbf{Q}(\varrho)x$, as derived in Proposition A.0.6. We thus think of the function $\mathsf{A}=\mathsf{A}(r,s,\xi)$ above as $\mathsf{A}=\mathsf{A}(r,r^2,|\nabla u|^2)$.

Proposition C.0.2. *Consider the matrix field* $\mathscr{U}(\varrho, \nabla_{\mathbb{A}} f)$ *defined row-wise by* (C.0.6). *Assume that the function* A *is strictly positive and is monotone increasing in the third variable. Then for each* $\mathsf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$ *the solution* $f \in \mathscr{C}^2(\overline{\mathbb{A}_n}, \mathbb{R}^d)$ *of the system* $\mathbf{BVP}[f; \mathscr{U}, \mathsf{m}]$ *is unique.*

Proof. As with the previous result in this appendix it is necessary to motivate ideas by taking the scalar-valued function $\mathsf{A}(r, s, \xi)$ as a derivative of some Lagrangian $F = F(r, s, \xi)$ in the third variable as in (C.0.3). If F is the Lagrangian of some energy functional as in

$$\mathbb{F}[f; \mathbb{A}_n] := \int_{\mathbb{A}_n} F\left(\|\varrho\|, \|\varrho\|^2, n + \sum_{\ell=1}^{d} \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2\right) \omega(\varrho; d)\, d\varrho, \tag{C.0.7}$$

then the system (C.0.5) is precisely the Euler-Lagrange equation associated to this functional above. The assumptions on A imply that F is convex, strictly increasing in ξ for $\xi > 0$ while F_ξ is increasing for $\xi > 0$.

To prove uniqueness, this uniform convexity first tells us that solutions of (C.0.5) are the minimisers of the energy $\mathbb{F}$ with respect to their own boundary conditions and conversely minimisers of $\mathbb{F}$ are solutions of the Euler-Lagrange system (C.0.5). Fix an assumed solution f of (C.0.5) and set $g = f + \varphi$ for $g \in \mathscr{B}_p[\mathbb{A}_n; \mathsf{m}] := \{f = (f_1, \dots, f_d) \in W^{1,p}(\overline{\mathbb{A}_n}, \mathbb{R}^d) : f \equiv 0 \text{ on } (\partial\mathbb{A}_n)_a,\ f \equiv 2\mathsf{m}\pi \text{ on } (\partial\mathbb{A}_n)_b,\ p > 1,\ \mathsf{m} \in \mathbb{Z}^d\}$ and $\varphi \in W_0^{1,p}(\overline{\mathbb{A}_n}, \mathbb{R}^d)$. We use the convexity inequality

$$F(\|\varrho\|, \|\varrho\|^2, \zeta_2) - F(\|\varrho\|, \|\varrho\|^2, \zeta_1) \geq F_\xi(\|\varrho\|, \|\varrho\|^2, \zeta_1)(\zeta_2 - \zeta_1), \quad \zeta_1, \zeta_2 \in \mathbb{R},$$

specifically with the choices of $\zeta_1 := n + \sum \rho_\ell^2 |\nabla_{\mathbb{A}} f_\ell|^2$, $\zeta_2 := n + \sum \rho_\ell^2 |\nabla_{\mathbb{A}} g_\ell|^2$ for $1 \leq \ell \leq d$. By the divergence theorem and the convexity inequality above

we have

$$
\begin{aligned}
\mathsf{D}_{\mathbb{F}}[f,g] := &\,\mathbb{F}[g;\mathbb{A}_n] - \mathbb{F}[f,\mathbb{A}_n] \\
&\geq \int_{\mathbb{A}_n} F_\xi\left(\|\varrho\|,\|\varrho\|^2,\zeta_1\right)\sum_{\ell=1}^{d}\rho_\ell^2(|\nabla_{\mathbb{A}} g_\ell|^2 - |\nabla_{\mathbb{A}} f_\ell|^2)\omega(\varrho;d)\,d\varrho \\
&\geq -2\sum_{\ell=1}^{d}\int_{\mathbb{A}_n}\mathrm{div}_{\mathbb{A}}\left[F_\xi\left(\|\varrho\|,\|\varrho\|^2,\zeta_1\right)\rho_\ell^2\nabla_{\mathbb{A}} f_\ell\,\omega(\varrho;d)\right]\varphi_\ell\,d\varrho \\
&\quad + 2\sum_{\ell=1}^{d}\int_{\Gamma_n}\left[F_\xi\left(\|\varrho\|,\|\varrho\|^2,\zeta_1\right)\rho_\ell^2\omega(\varrho;d)\partial_\nu f_\ell\right]\varphi_\ell\,d\varrho \\
&\quad + \int_{\mathbb{A}_n} F_\xi\left(\|\varrho\|,\|\varrho\|^2,\zeta_1\right)\sum_{\ell=1}^{d}\rho_\ell^2|\nabla_{\mathbb{A}}\varphi_\ell|^2\omega(\varrho;d)d\varrho. \qquad \text{(C.0.8)}
\end{aligned}
$$

Since $F_\xi > 0$ by assumption we have $\mathsf{D}_{\mathbb{F}} \geq 0$, with equality if and only if $\varphi \equiv 0$, in which case $f \equiv g$. Thus any solution of $\mathbf{BVP}[f;\mathscr{U},\mathsf{m}]$ is unique. □

Appendix D

A Collection of Curl-Free Results

In this final appendix we present two results on the curl-free and gradient structure of certain vector fields used throughout the main body of the thesis. They are both similar in nature but carry somewhat different assumptions which makes their application throughout the text distinct.

Proposition D.0.1. *Let $\mathscr{A} = \mathscr{A}(r), \mathscr{B} = \mathscr{B}(r) \in \mathscr{C}^1(]a, b[)$ and suppose that $\mathbf{H}$ is the constant $n \times n$ skew-symmetric matrix given by $\mathbf{H} = diag(\alpha_1 \mathbf{J}, ..., \alpha_k \mathbf{J})$ when $n = 2k$ and $\mathbf{H} = diag(\alpha_1 \mathbf{J}, ..., \alpha_{k-1} \mathbf{J}, \alpha_k)$ when $n = 2k - 1$ where $\alpha_1, ..., \alpha_k$ are real constants and $\mathbf{J}$ is as in, for example,* (2.3.4)*. Consider the vector field*

$$U(x) = \mathscr{A}(|x|)|\mathbf{H}x|^2 x + \mathscr{B}(|x|)\mathbf{H}^2 x, \qquad x \in \mathbb{X}^n[a, b]. \tag{D.0.1}$$

Then the following hold.

- *If $2\mathscr{A} + \dot{\mathscr{B}}/r \equiv 0$ in $\mathbb{X}^n$ then $U = -\nabla[\mathscr{B}(r)|\mathbf{H}x|^2/2]$.*
- *If $2\mathscr{A} + \dot{\mathscr{B}}/r \not\equiv 0$ in $\mathbb{X}^n$ then $\operatorname{curl} U \equiv 0$ in $\mathbb{X}^n$ iff $|\alpha_1| = ... = |\alpha_k| =: |\alpha|$, that is, $\mathbf{H}^2 = -\alpha^2 \mathbf{I}_n$. In this case U is again a gradient field in $\mathbb{X}^n$.*

Proof. For $1 \le j \le n$ set $s(j) = \lfloor (j + 1)/2 \rfloor$. Then $1 \le s(j) \le k$ and $s(n) = k$.

Put $V(x) = \mathscr{A}(|x|)|\mathbf{H}x|^2 x$ and $W(x) = \mathscr{B}(|x|)\mathbf{H}^2 x$. Then

$$\begin{aligned}V(x) &= \mathscr{A}(r)\begin{cases}[\alpha_1^2(x_1^2+x_2^2)+...+\alpha_k^2(x_{n-1}^2+x_n^2)]x & n=2k\\ [\alpha_1^2(x_1^2+x_2^2)+...+\alpha_{k-1}^2(x_{n-2}^2+x_{n-1}^2)+\alpha_k^2 x_n^2]x & n=2k-1\end{cases}\\ &= \mathscr{A}(r)\left[\sum_{l=1}^{n}\alpha_{s(l)}^2 x_l^2\right]x,\end{aligned}$$

and likewise

$$W(x) = -\mathscr{B}(r)\begin{cases}[\alpha_1^2 x_1, \alpha_1^2 x_2, ..., \alpha_k^2 x_{n-1}, \alpha_k^2 x_n]^t & n=2k\\ [\alpha_1^2 x_1, \alpha_1^2 x_2, ..., \alpha_{k-1}^2 x_{n-2}, \alpha_{k-1}^2 x_{n-1}, \alpha_k^2 x_n]^t & n=2k-1.\end{cases}$$

Now computing the curl of the vector fields V, W directly a straightforward differentiation gives

$$\begin{aligned}[\operatorname{curl} V]_{ij} =& [\mathscr{A}(|x|)|\mathbf{H}x|^2 x]_{i,j} - [\mathscr{A}(|x|)|\mathbf{H}x|^2 x]_{j,i}\\ =& \left(\dot{\mathscr{A}}(r)\frac{x_j x_i}{r} + \mathscr{A}(r)\delta_{ij}\right)\left[\sum_{l=1}^{n}\alpha_{s(l)}^2 x_l^2\right] + 2\mathscr{A}(r)\alpha_{s(j)}^2 x_j x_i\\ &- \left\{\left(\dot{\mathscr{A}}(r)\frac{x_i x_j}{r} + \mathscr{A}(r)\delta_{ji}\right)\left[\sum_{l=1}^{n}\alpha_{s(l)}^2 x_l^2\right] + 2\mathscr{A}(r)\alpha_{s(i)}^2 x_i x_j\right\}\\ =& -2\mathscr{A}(r)\left(\alpha_{s(i)}^2 - \alpha_{s(j)}^2\right)x_i x_j,\end{aligned} \tag{D.0.2}$$

and in a similar way

$$\begin{aligned}[\operatorname{curl} W]_{ij} &= [\mathscr{B}(|x|)\mathbf{H}^2 x]_{i,j} - [\mathscr{B}(|x|)\mathbf{H}^2 x]_{j,i}\\ &= -\dot{\mathscr{B}}(r)\frac{x_j x_i}{r}\alpha_{s(i)}^2 - \mathscr{B}(r)\alpha_{s(i)}^2\delta_{ij} + \dot{\mathscr{B}}(r)\frac{x_i x_j}{r}\alpha_{s(j)}^2 + \mathscr{B}(r)\alpha_{s(j)}^2\delta_{ij}\\ &= -\dot{\mathscr{B}}(r)\frac{x_i x_j}{r}\left(\alpha_{s(i)}^2 - \alpha_{s(j)}^2\right).\end{aligned} \tag{D.0.3}$$

By combining the two we now obtain

$$[\operatorname{curl} U]_{ij} = -\left(2\mathscr{A}(r) + \dot{\mathscr{B}}(r)/r\right)\left(\alpha_{s(i)}^2 - \alpha_{s(j)}^2\right)x_i x_j, \tag{D.0.4}$$

and so subject to $2\mathscr{A} + \dot{\mathscr{B}}/r \not\equiv 0$ we have that $\operatorname{curl} U \equiv 0 \implies \alpha_1^2 = ... = \alpha_k^2$. Conversely if $\alpha_1^2 = ... = \alpha_k^2 =: \alpha^2$ then $U = \alpha^2(r^2\mathscr{A}(r) - \mathscr{B}(r))x$ is a gradient field in $\mathbb{X}^n[a,b]$ and thus curl-free. If $2\mathscr{A} + \dot{\mathscr{B}}/r \equiv 0$ observe that

$$\begin{aligned}\nabla[\mathscr{B}(r)|\mathbf{H}x|^2] &= \dot{\mathscr{B}}(r)|\mathbf{H}x|^2 x/r + 2\mathscr{B}(r)\mathbf{H}^t\mathbf{H}x\\ &= -2\left[\mathscr{A}(r)|\mathbf{H}x|^2 + \mathscr{B}(r)\mathbf{H}^2 x\right],\end{aligned} \tag{D.0.5}$$

so $U = -\nabla[\mathscr{B}(r)|\mathbf{H}x|^2/2]$ as required. □

The next result is similar in nature but more general in its scope and used directly in Chapters 3 and 6.

Proposition D.0.2. *Let $\mathscr{A} = \mathscr{A}(r,z)$, $\mathscr{B} = \mathscr{B}(r,z) \in \mathscr{C}^1(]a,b[\times\mathbb{R},\mathbb{R})$ and let $\mathbf{H}$ be the constant $n \times n$ skew-symmetric matrix given by*

$$\mathbf{H} = \begin{cases} (c_1\mathbf{J}, \dots, c_k\mathbf{J}) & \text{when } n = 2k, \\ (c_1\mathbf{J}, \dots, c_{k-1}\mathbf{J}, c_k) & \text{when } n = 2k-1. \end{cases} \tag{D.0.6}$$

Here $(c_j : 1 \le j \le k) \subset \mathbb{R}$ and $\mathbf{J}$ is given by, for example, (2.3.4). *Consider the vector field U defined by*

$$U(x) = \mathscr{A}(|x|, |\mathbf{H}x|^2)x + \mathscr{B}(|x|, |\mathbf{H}x|^2)\mathbf{H}^2 x, \qquad x \in \mathbb{X}^n, \tag{D.0.7}$$

and let $\mathscr{F}(r,z) := 2\mathscr{A}_z + \mathscr{B}_r/r$ with $z = |\mathbf{H}x|^2$ where $\mathscr{A}_z$ denotes the derivative of $\mathscr{A} = \mathscr{A}(r,z)$ in the second variable and $\mathscr{B}_r$ denotes the derivative of $\mathscr{B} = \mathscr{B}(r,z)$ in the first variable. Then the following hold:

- *If $\mathscr{F} \not\equiv 0$ in $\mathbb{X}^n$, then*

$$\operatorname{curl} U \equiv 0 \text{ in } \mathbb{X}^n \iff |c_1| = \cdots = |c_k| := c \iff \mathbf{H}^2 = -c^2\mathbf{I}_n. \tag{D.0.8}$$

- *If $\mathscr{F} \equiv 0$ in $\mathbb{X}^n$ then $\operatorname{curl} U \equiv 0$ in $\mathbb{X}^n$ with no further restriction on $\mathbf{H}$.*

In either case the vector field U is a gradient field in $\mathbb{X}^n$.

Proof. First we calculate $\operatorname{curl} U$ where for the sake of convenience we split the vector field U as $V + W$ with $V = \mathscr{A}(|x|, |\mathbf{H}x|^2)x$, $W = \mathscr{B}(|x|, |\mathbf{H}x|^2)\mathbf{H}^2 x$. We also write

$$|\mathbf{H}x|^2 = \sum_{l=1}^{n} c_{s(l)}^2 x_l^2,$$

where, as in the previous result, $s(l) = \lfloor (l+1)/2 \rfloor$ for all $1 \le l \le n$. This being so, $|\mathbf{H}x|^2 = c_1^2(x_1^2 + x_2^2) + \cdots + c_k^2(x_{n-1}^2 + x_n^2)$ when $n = 2k$ is even, and $|\mathbf{H}x|^2 = c_1^2(x_1^2 + x_2^2) + \cdots + c_{k-1}^2(x_{n-2}^2 + x_{n-1}^2) + c_k^2 x_n^2$ when $n = 2k-1$ is odd. As a result W can be written as

$$\begin{aligned} W(x) &= \mathscr{B}(|x|, |\mathbf{H}x|^2)\mathbf{H}^2 x \qquad\qquad (\text{D.0.9}) \\ &= -\mathscr{B}(|x|, |\mathbf{H}x|^2) \begin{cases} [c_1^2 x_1, c_1^2 x_2, \dots c_k^2 x_{n-1}, c_k^2 x_n]^t & n = 2k \\ [c_1^2 x_1, c_1^2 x_2, \dots, c_{k-1}^2 x_{n-2}, c_{k-1}^2 x_{n-1}, c_k^2 x_n]^t & n = 2k-1. \end{cases} \end{aligned}$$

We now proceed on to computing the curls of the vector fields V and W respectively. Denoting by $\mathscr{A}_r$, $\mathscr{A}_z$ the derivatives in the first and second variables of $\mathscr{A} = \mathscr{A}(r, z)$ a straightforward differentiation gives

$$\begin{aligned}[\mathrm{curl} V]_{ij} =& V_{i,j} - V_{j,i} = [\mathscr{A}(|x|, |\mathbf{H}x|^2)x]_{i,j} - [\mathscr{A}(|x|, |\mathbf{H}x|^2)x]_{j,i} \\ =& \mathscr{A}_r(r, |\mathbf{H}x|^2)\frac{x_j x_i}{r} + 2\mathscr{A}_z(r, |\mathbf{H}x|^2)c^2_{s(j)}x_j x_i + \mathscr{A}(r, |\mathbf{H}x|^2)\delta_{ij} \\ & - \left[\mathscr{A}_r(r, |\mathbf{H}x|^2)\frac{x_i x_j}{r} + 2\mathscr{A}_z(r, |\mathbf{H}x|^2)c^2_{s(i)}x_i x_j + \mathscr{A}(r, |\mathbf{H}x|^2)\delta_{ij}\right] \\ =& -2\mathscr{A}_z(r, |\mathbf{H}x|^2)\left(c^2_{s(i)} - c^2_{s(j)}\right)x_i x_j. \end{aligned} \tag{D.0.10}$$

Similarly for W with $\mathscr{B}_r$, $\mathscr{B}_z$ denoting the derivatives of $\mathscr{B} = \mathscr{B}(r, z)$ with respect to the first and second variables respectively we have

$$\begin{aligned}[\mathrm{curl}\, W]_{ij} &= W_{i,j} - W_{j,i} = [\mathscr{B}(|x|, |\mathbf{H}x|^2)\mathbf{H}^2 x]_{i,j} - [\mathscr{B}(|x|, |\mathbf{H}x|^2)\mathbf{H}^2 x]_{j,i} \\ &= -\left[\mathscr{B}_r(r, |\mathbf{H}x|^2)\frac{x_j x_i}{r}c^2_{s(i)} + 2\mathscr{B}_z(r, |\mathbf{H}x|^2)c^2_{s(j)}c^2_{s(i)}x_j x_i + \mathscr{B}(r, |\mathbf{H}x|^2)c^2_{s(i)}\delta_{ij}\right] \\ &\quad + \left[\mathscr{B}_r(r, |\mathbf{H}x|^2)\frac{x_i x_j}{r}c^2_{s(j)} + 2\mathscr{B}_z(r, |\mathbf{H}x|^2)c^2_{s(i)}c^2_{s(j)}x_i x_j + \mathscr{B}(r, |\mathbf{H}x|^2)c^2_{s(j)}\delta_{ij}\right] \\ &= -\mathscr{B}_r(r, |\mathbf{H}x|^2)\frac{x_i x_j}{r}\left(c^2_{s(i)} - c^2_{s(j)}\right). \end{aligned} \tag{D.0.11}$$

By combining (D.0.10) and (D.0.11) we thus obtain

$$[\mathrm{curl}\, U]_{ij} = -\left(2\mathscr{A}_z(|x|, |\mathbf{H}x|^2) + \frac{\mathscr{B}_r(|x|, |\mathbf{H}x|^2)}{r}\right)\left(c^2_{s(i)} - c^2_{s(j)}\right)x_i x_j. \tag{D.0.12}$$

From this it follows that if $\mathscr{F} \not\equiv 0$ in $\mathbb{X}^n$ then $\mathrm{curl}\, U \equiv 0$ in $\mathbb{X}^n$ if and only if $c_1^2 = \cdots = c_k^2$. (Note that firstly $\mathscr{F}$ is a continuous function of x and so if it does not vanish at a point then it does not vanishes in a neighbourhood of the point and secondly that the factors $x_i x_j$ vanish only on the coordinate hyperplanes.) Likewise if $\mathscr{F} \equiv 0$ in $\mathbb{X}^n$ then $\mathrm{curl}\, U \equiv 0$ in $\mathbb{X}^n$ with no impositions to be made on $c_1, \ldots, c_k$. This proves the first part of the result.

We need to prove that U is a gradient in either case. First suppose that $\mathscr{F} \not\equiv 0$ and $c_1^2 = \cdots = c_k^2$. In this case $U(x) = [\mathscr{A}(r, c^2r^2) - c^2\mathscr{B}(r, c^2r^2)]x = s(r)x$ and this is clearly a gradient in $\mathbb{X}^n$. Next suppose that $\mathscr{F} \equiv 0$. We claim that $U(x) = \nabla f(|x|, |\mathbf{H}x|^2)$ for a suitable choice of $f = f(r, z)$. Indeed assuming this to be the case, by direct differentiation we have,

$$\begin{aligned}\nabla f(|x|, |\mathbf{H}x|^2) &= f_r(r, |\mathbf{H}x|^2)\theta - 2rf_z(r, |\mathbf{H}x|^2)\mathbf{H}^2\theta \\ &= \mathscr{A}(r, |\mathbf{H}x|^2)x + \mathscr{B}(r, |\mathbf{H}x|^2)\mathbf{H}^2 x = U(x) \end{aligned} \tag{D.0.13}$$

provided that we set $f_r(r,z) = r\mathscr{A}(r,z)$ and $f_z(r,z) = -\mathscr{B}(r,z)/2$. Now let $\mathscr{R} = \{(r,z) : r = |x|,\, z = |\mathbf{H}x|^2 \text{ with } x \in \mathbb{X}^n\}$. Then $\mathscr{R} \subset [a,b] \times \mathbb{R}$ is seen to be simply-connected; in fact, denoting by $\underline{c}, \overline{c} \geq 0$ the minimum and maximum eigenvalues $c_1^2, \dots, c_k^2$ of the diagonal matrix $\mathbf{H}^t\mathbf{H}$ [*see* (D.0.6)] we have that $\mathscr{R} = \{(r,z) : a < r < b, \underline{c}r^2 \leq z \leq \overline{c}r^2\}$. Next since $\mathscr{F} \equiv 0$ in $\mathscr{R}$ we have

$$\partial_r f_z(r,z) - \partial_z f_r(r,z) = -\left(r\mathscr{A}_z(r,z) + \frac{1}{2}\mathscr{B}_r(r,z)\right) = 0, \quad (r,z) \in \mathscr{R}, \tag{D.0.14}$$

and this therefore justifies the existence of a primitive $f \in \mathscr{C}^2(\mathscr{R})$ as required. Indeed to describe f more explicitly consider setting

$$f(r,z) = \int_a^r s\mathscr{A}(s,z)\,ds + g(z), \qquad (r,z) \in \mathscr{R}, \tag{D.0.15}$$

with $g = g(z)$ to be determined below. Then $f_r(r,z) = r\mathscr{A}(r,z)$ and to fix g it suffices to set

$$f_z(r,s) = \int_a^r s\mathscr{A}_z(s,z)\,ds + g'(z) = -\frac{1}{2}\mathscr{B}(r,z) = -\frac{1}{2}\left\{\int_a^r \mathscr{B}_r(s,z)\,ds + \mathscr{B}(a,z)\right\}, \tag{D.0.16}$$

that is,

$$g'(z) = \int_a^r -\left\{s\mathscr{A}_z(s,z) + \frac{1}{2}\mathscr{B}_r(s,z)\right\} ds - \frac{1}{2}\mathscr{B}(a,z) = -\frac{1}{2}\mathscr{B}(a,z). \tag{D.0.17}$$

Thus upon choosing f as in (D.0.15) with g a primitive of $-\mathscr{B}(a,z)/2$ as above we have (D.0.13) and so U is a gradient in $\mathbb{X}^n$ as claimed. $\square$

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