Weakly coupled fixed points and interacting ultraviolet completions of vanilla quantum field theories;

or, better asymptotically safe than asymptotically sorry

Andrew David Bond

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Department of Physics and Astronomy,
University of Sussex

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

The work in this thesis has been completed in collaboration with Daniel F. Litim, and is comprised of the following papers:

- Andrew D. Bond, and Daniel F. Litim, 'Theorems for asymptotic safety of gauge theories', Eur. Phys. J. C77, 429 (2017), [Erratum: Eur. Phys.J.C77,no.8,525(2017)], arXiv:1608.00519 [hep-th],
- Andrew D. Bond, and Daniel F. Litim, 'Price of Asymptotic Safety', arXiv:1801.08527 [hep-th],
- Andrew D. Bond, and Daniel F. Litim, 'More asymptotic safety guaranteed', Phys. Rev. D97, 085008 (2018), arXiv:1707.04217 [hep-th],
- Andrew D. Bond, and Daniel F. Litim, 'Asymptotic safety guaranteed in supersymmetry', Phys. Rev. Lett. 119, 211601 (2017), arXiv:1709.06953 [hep-th].

I have led or corroborated all the original research presented in this thesis.

Signature:

Andrew David Bond

Ultraviolet completion is surprisingly my type, so it's a fact (10,6)

Secured spot making scale invariant theory (5,5)

UNIVERSITY OF SUSSEX

ANDREW DAVID BOND
DOCTOR OF PHILOSOPHY

WEAKLY COUPLED FIXED POINTS AND INTERACTING ULTRAVIOLET COMPLETIONS OF VANILLA QUANTUM FIELD THEORIES

SUMMARY

The renormalisation group is a crucial tool for understanding scale-dependent quantum field theories. Renormalisation group fixed points correspond to theories where scale invariance is restored at the quantum level, and may provide high- or low-energy limits for more general quantum field theories. In particular, those reached in the ultraviolet allow theories to be defined microscopically, a scenario known as asymptotic safety.

In this work I investigate fixed points of conventional four-dimensional, flat-space, perturbatively renormalisable, local quantum field theories. Focusing on weakly interacting fixed points the problem becomes amenable to perturbation theory. The approach is two-fold: on the one hand to understand general conditions for the existence of such fixed points, and on the other to construct theories which introduce new features compared to previous examples.

To understand perturbative fixed points, general calculations for theories of this type are exploited. It is established, for gauge theories, interacting fixed points may be nonzero in gauge couplings alone, or in gauge and Yukawa couplings. Deriving novel group theory bounds it is established that only the latter may possibly be ultraviolet. Additionally it is shown that theories without gauge interactions cannot possess weakly coupled fixed points, and the connexion between this fact and the impossibility of such theories being asymptotically free is highlighted.

Two explicit families of examples are presented: a theory with semisimple gauge group is analysed in detail, containing many new fixed points, a rich phase structure, and asymptotically safe regions of parameter space, and a separate supersymmetric model with an ultraviolet fixed point, providing the first known explicit example of an asymptotically safe supersymmetric gauge theory.

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Part I

Introduction

I. PRELUDE

The standard model of particle physics is one of the most successful descriptions of natural phenomena. This was highlighted particularly by the discovery of the Higgs boson [1–3], nearly fifty years after it had been first theorised [4–6], and was the final fundamental particle within the theory to be observed as well as the first apparently elementary scalar. This is of course not the end of the story, and there are several difficulties which point towards the need for new theories to ultimately supplant the standard model. In particular there are two main issues which will be of particular relevance to this work.

Firstly, the standard model is an effective theory, working well at the energies that can be probed currently in particle colliders, but being limited in the range of energy scales it can reach and remain predictive. If we investigate what happens to the couplings of the theory we find that they become arbitrarily large at high energies, meaning that we can no longer make sensible predictions. Ultimately if we wish to describe the full range of possible phenomena, it would be desirable to have a theory which is ultraviolet complete, meaning that it can make sensible predictions regardless of how high the energies we wish to explore are. Such a theory can really be viewed as fundamental, as it makes testable claims about how physics behaves at arbitrarily small length scales.

The other important issue unaddressed by the standard model is gravity. This of course would not necessarily be a problem if we could find a theory of quantum gravity which could work consistently with the standard model. However, this is by no means an easy task. Although both the standard model and general relativity work very well at describing physics in their respective areas of domain, to be able to understand what happens when both gravitational and quantum effects are relevant, such as in the vicinity of black holes, requires a theory which incorporates both.

The perturbative approach to quantise general relativity suffers from the problem that, owing to the negative mass dimension of Newton's coupling, the theory is perturbatively non-renormalisable, leading ultimately to an infinite number of free parameters in order to deal with divergent integrals in successive orders of the loop expansion. The theory may still be regarded as an effective theory [7–9], but a fundamental theory necessitates a different approach, in some form or other. There are a vast range of ideas to tackle this issue, many of which alter the theory quite distinctly from the perturbative field theory approach, such as for example string theory (see e.g. [10, 11]), which modifies the degrees of freedom from point-like particles to extended objects, as well as changing the dimensionality of spacetime.

One approach to constructing quantum field theories that could potentially address both of these issues is that of *asymptotic safety*. A theory which is asymptotically safe possesses an ultraviolet (UV) fixed point of the renormalisation group (RG), which in general will be interacting. The theory is defined on some renormalisation group trajectory, which reaches the UV fixed point asymptotically at high energies. In this way the couplings of the theory remain finite. As long as the space of such trajectories is finite dimensional, there will be only be finitely many free parameters, and thus the theory can become predictive [12, 13]; such a theory is then said to be asymptotically safe [14, 15].

Asymptotic safety may arise in a variety of different scenarios. Firstly, one can view asymptotic freedom as a special case, where the ultraviolet fixed point is in fact non-interacting. Originally discovered in non-Abelian gauge theories with only a gauge coupling [16, 17], it can also be extended to theories with Yukawa and scalar quartic couplings driven by the gauge sector, provided certain conditions on the theory parameters hold [18–20]. Gravity close to its critical dimensionality of 2 develops a perturbative ultraviolet fixed point [15, 21, 22], which is a strong motivation for four dimensional studies. Other approaches to gravity utilise large-N techniques [23–25]. The four-fermion Gross-Neveu model [26] similarly can have an ultraviolet fixed point perturbatively both by similarly being close to its critical dimension [27, 28], or at fixed dimensionality with a large number of fields [29]. Non-Abelian gauge theories similarly near to four dimensions may become asymptotically safe [30, 31]. It may also occur in scalar field theories, such as the three-dimensional non-linear sigma model [32] and ϕ^4 theory in non-commutative spacetime [33, 34].

As an approach to quantum gravity, in asymptotic safety the theory is described as an ordinary four-dimensional quantum field theory, much like the standard model. As gravity is perturbatively nonrenormalisable, if a UV fixed point exists it must necessarily be nonperturbative, which poses some calculational difficulty. However, functional techniques based on exact renormalisation group equations [35–37] allow such regimes to be probed in various levels of approximation. There have been a huge number of studies made to provide evidence for the existence for such a fixed point, such as those starting from Einstein-Hilbert theories [38–40], including terms in the action which are functions of the Ricci scalar [41–46] as well as further gravitational invariants [47–49], and including the effects of matter coupled to gravity [50–52]; for a small recent review on progress see [53]. The nonperturbative nature of the problem however means that a full understanding is an extremely difficult endeavour.

A recent stream of research that has garnered some interest is concerned with asymptotic safety in the setting of four-dimensional gauge-Yukawa theories, building upon the recent example of a simple example model which may develop an interacting ultraviolet fixed point in a controlled perturbative setting [54, 55]. As well as being directly applicable to exploration of asymptotically safe extensions of the standard model [56, 57], this example provides means to study theories with ultraviolet fixed points which are amenable to the tools of perturbation theory, that may be able to give useful insight to asymptotically safe theories in general, ultimately perhaps aiding understanding of nonperturbative theories such as gravity. Various aspects of this example model have been studied further, such as vacuum stability [58], including masses and symmetry breaking [59], the effects of higher dimensional operators [60, 61], effects beyond leading order [62] or away from four dimensions [63] as well as more formal considerations [64–66].

The works comprising this thesis will focus on addressing two issues related to weakly coupled fixed points in gauge-Yukawa theories. Firstly, a full categorisation of all possible types of perturbative fixed point for four-dimensional local quantum field theories with up to spin-1 degrees of freedom, along with an understanding of the ingredients required for them to be ultraviolet. The advantage of weak coupling means both that canonical power counting is still valid owing to small anomalous dimensions, and that results are reliable, with higher order corrections having strictly subleading effects. Ultimately the origin of these interacting fixed points stems from the inclusion of non-Abelian gauge fields in a way which is intimately connected to the way in which theories may become asymptotically free. Secondly, the set of perturbatively asymptotically safe examples is extended in two directions, using large-N theories which allow exact perturbative control. The first provides a semisimple setting and is analysed in detail, offering a wide variety of possible phase structures, including novelly an asymptotically safe regime where the theory has lost asymptotic freedom. The second example demonstrates the mechanism at work in a supersymmetric theory, demonstrating explicitly that weakly coupled ultraviolet fixed points are not incompatible with the constraints of supersymmetry.

Before moving on to the main content we will provide a short review of some preliminaries that will be of use for the rest of the works.

II. PRELIMINARIES

A. Renormalisation group

In order to understand scale dependence of quantum field theories, it is important to keep in mind how we define these theories. Usually we will talk in the language of the corresponding classical theory, described by some classical action S, expressed as the spacetime integral of a local Lagrangian $\mathcal{L}$,

$$S = \int d^4x \mathcal{L} \,, \tag{I.1}$$

which itself can be expanded in terms of a series of operators $\mathcal{O}_i$ which are built from spacetime derivatives and local fields,

$$\mathcal{L} = \sum_i \lambda_i \mathcal{O}_i \,, \tag{I.2}$$

where the corresponding coefficients λ_i are the couplings of the theory. To describe the quantum theory we may define the partition function via the path integral

$$Z[\mathcal{J}] = \int \mathcal{D}\varphi e^{i\left(S+\int d^4x \mathcal{J}(x)\varphi(x)\right)} \,. \tag{I.3}$$

For notational convenience we will consider a theory with just a single scalar field $\varphi(x)$. Taking appropriate derivatives of this gives us the correlation functions from which we can get observables.

The couplings λ_i determine the strength of the interactions encoded in the various operators $\mathcal{O}_i$, which in general will be all possible terms built from the fields of the theory which obey the symmetries. The procedure of quantisation, however, will in general alter the features of our original, classical theory, and in particular, symmetries of the classical theory are no longer guaranteed to be symmetries of the full quantum theory. Even if we begin with a scale invariant Lagrangian, we should not necessarily expect a scale invariant quantum theory. Symmetries broken by quantum effects are said to be anomalous, and the theory is said to possess an anomaly. This is not a problem, provided we do not rely on the symmetry for consistency. For constructing theories of gauge fields, it is important that gauge symmetries are not anomalous, as otherwise gauge invariance will no longer be preserved at the quantum level. This can be ensured by making sure that suitable matter content is included in the theory.

To understand the effect of high energy fluctuations on the theory, let's consider a theory which is cut off at some high scale Λ. Consider the partition function, at zero source for notational ease, given by the path integral

$$\int_{p<\Lambda} \mathcal{D}\varphi e^{-S[\varphi]} \,, \tag{I.4}$$

where we only integrate those field modes with energy below Λ, and we have Wick rotated to Euclidean space to enforce the cutoff. We choose some energy scale $k < \Lambda$, and split the field into those modes φ_l with energy less than k, and those modes φ_h with energy greater than k, so that $\varphi = \varphi_l + \varphi_h$. We then integrate over the high scale modes,

$$\int_{p<k} \mathcal{D}\varphi_l \int_{k<p<\Lambda} \mathcal{D}\varphi_h e^{iS[\varphi_l+\varphi_h]} = \int_{p<k} \mathcal{D}\varphi_l e^{iS'[\varphi_l]} \,, \tag{I.5}$$

and we are left with something that looks very much like our original path integral. The difference is that we are now only integrating over modes which have momentum below k, as we have integrated out the high energy fluctuations. The effect of these are encoded in the new action S'_k, which in general will be distinct from S. We may again expand S'_k as in (I.2), as they are a basis set of operators, except we should now expect that the coefficients, namely the couplings λ_i, will be different. If we split the field modes at some new value $\tilde{k}$, we end up with a new action $S'_{\tilde{k}}$, and we can account for the change in scale by changing the values of the couplings. In this way, the couplings we consider from integrating out the high energy fluctuations become scale-dependent quantities.

Continuing this idea to the point where the change in scale becomes infinitesimal we end up with the renormalisation group equations

$$\partial_t \lambda_i = \beta_i(\lambda)\,, \tag{I.6}$$

where we consider the derivative with respect to so-called 'renormalisation group time',

$$t = \log(k/\Lambda)\,, \tag{I.7}$$

where k is the renormalisation scale, and Λ is our arbitrary reference scale. Note that a change in reference scale simply corresponds to a shift in t — if we consider some new t' where we use a reference scale Λ', then we have

$$t' = \log(k/\Lambda') = \log(k/\Lambda) + \log(\Lambda/\Lambda') = t + \log(\Lambda/\Lambda')\,. \tag{I.8}$$

The beta functions appearing in the right hand side of (I.6) encode all of the information about the scale variation of the coupling λ_i, but will in general depend on all of the couplings in the theory. The calculation of beta functions is in general a difficult task, and except in certain special cases cannot be done exactly. In the works contained here we shall be dealing with approximations to beta functions which are calculated perturbatively, as a power series in the couplings.

Now in general solutions to (I.6) can have a range of possible behaviours. As a simple example, consider a beta function given by

$$\beta_\lambda = A\lambda^2\,, \qquad\qquad A > 0\,, \tag{I.9}$$

such as is found in the lowest perturbative approximation in ϕ^4 theory, or quantum electrodynamics. If we naïvely solve this equation we find the behaviour of the coupling with scale to be

$$\lambda(t) = \frac{\lambda_0}{1 - \lambda_0 A t}\,, \tag{I.10}$$

with $\lambda_0 \equiv \lambda(t = 0) > 0$. This solution runs into problems at the Landau pole $t = 1/(\lambda_0 A) > 0$ as the coupling becomes infinitely large at this point, and we may no longer make predictions beyond this energy scale [1].

One way to make sure that the interactions of our theory are well behaved at the highest scales is to find trajectories which asymptote towards a fixed (and physically acceptable)

[1] Of course if (I.9) is simply an approximation from perturbation theory, what this signals is that a perturbative analysis is no longer a good approximation, and to understand what happens we must understand the theory non-perturbatively.

value [2] in all couplings as t becomes arbitrarily large, i.e.

$$\lim_{t\to\infty} \lambda_i(t) = \lambda_i^* \,. \tag{I.11}$$

To understand what happens at this point it is convenient to introduce a ratio of identical exponential functions,

$$\lim_{t\to\infty} \lambda_i(t) = \lim_{t\to\infty} \frac{\lambda_i(t)e^t}{e^t} \,. \tag{I.12}$$

The denominator will in the limit tend to $+\infty$, and the numerator will either do the same, or tend to some finite value if the coupling tends to a zero fixed point at just the right speed to counterbalance the exponential. Nevertheless in either case a general version of L'Hôpital's rule applies, and the limit will be equal to the limit of the expression we get by taking the derivative separately of the numerator and of the denominator,

$$\lim_{t\to\infty} \frac{\lambda_i(t)e^t}{e^t} = \lim_{t\to\infty} \frac{(\lambda_i(t) + \partial_t\lambda_i(t))e^t}{e^t} = \lambda_i^* + \lim_{t\to\infty} \partial_t\lambda_i(t) \,. \tag{I.13}$$

By the renormalisation group equations (I.6) the derivative of the coupling is given by the corresponding beta function, and as long as this is continuous at the limiting value (I.11), it must satisfy

$$\beta_i(\lambda^*) = 0 \,. \tag{I.14}$$

This is very useful, as it means that points in coupling space which trajectories asymptote to are fixed points of the renormalisation group equations (I.6), where the theory is scale invariant at the quantum level. From a practical standpoint this is very attractive as finding solutions to (I.14) is generally an easier task than solving the full set of equations.

We may also look at the problem from the other direction — suppose we have found some fixed point, satisfying (I.14) for all of the couplings of our theory. How are we to determine whether in fact there are any trajectories reaching it in the large t limit? To do so we define a new set of variables which describe deviations from the fixed point value

$$\delta\lambda_i(t) \equiv \lambda_i(t) - \lambda_i^* \,, \tag{I.15}$$

which will necessarily follow the same running as the $\lambda(i)$ themselves,

$$\partial_t\delta\lambda_i = \partial_t\lambda_i = \beta_i(\lambda) \,. \tag{I.16}$$

We express the beta functions in terms of the variations by performing a Taylor expansion

[2] Other behaviours, such as limit cycles, may also be possible, but these lie outside the scope of these works.

around the fixed point values

$$\beta_i(\lambda) = \beta_i(\lambda^*) + \left.\frac{\partial \beta_i}{\partial \lambda_j}\right|_{\lambda=\lambda^*} \delta\lambda_j + \cdots = M_{ij}\delta\lambda_j + O(\delta\lambda^2)\,, \tag{I.17}$$

where the constant piece vanishes by definition of the fixed point, and linear piece is given in terms of the *stability matrix*, defined by

$$M_{ij} \equiv \left.\frac{\partial \beta_i}{\partial \lambda_j}\right|_{\lambda=\lambda^*} . \tag{I.18}$$

We are interested in behaviour arbitrarily close to the fixed point, so that we may neglect terms quadratic in the deviation.

Now let us suppose that the matrix M is diagonalisable [3], so that there is some change of basis with corresponding matrix P

$$M = PDP^{-1}\,, \tag{I.19}$$

so that D is purely diagonal,

$$D_{ij} = \theta_i \delta_{ij}\,. \tag{I.20}$$

We can then transform our dynamical system

$$\partial_t \vec{\delta\lambda} = M\vec{\delta\lambda} = PDP^{-1}\vec{\delta\lambda} \tag{I.21}$$

by moving to a new set of variables

$$\vec{a} \equiv P^{-1}\vec{\delta\lambda} \tag{I.22}$$

which diagonalise our set of equations

$$\partial_t \vec{a} = D\vec{a}\,, \tag{I.23}$$

so that they decouple, and we have a set of component differential equations

$$\partial_t a_i = \theta_i a_i\,. \tag{I.24}$$

These are readily solved to yield the general solutions

$$a_i(t) = c_i e^{\theta_i t}\,, \tag{I.25}$$

for some set of arbitrary integration constants c_i.

[3] If this is not the case then we can still follow a similar line of reasoning by bringing it into Jordan normal form — it simply means the correpsonding solutions will be altered in form. As this technical complication adds nothing to the discussion we will not consider it any further here.

Now if we write P in terms of column vectors

$$P = (\vec{v}^{\,(1)}, \vec{v}^{\,(2)}, \vec{v}^{\,(3)}, \dots)\,, \tag{I.26}$$

we can explicitly translate the solution (I.25) back to our original variables, yielding

$$\delta\lambda_i(t) = \sum_j P_{ij} a_j(t) = \sum_j c_j v_i^{(j)} e^{\theta_j t}\,. \tag{I.27}$$

To determine how the $\vec{v}^{\,(i)}$ relate to our system more clearly we can see by right-multiplying (I.19) by P that they will be right eigenvectors for M,

$$M\vec{v}^{\,(i)} = \theta_i \vec{v}^{\,(i)}\,. \tag{I.28}$$

Thus to fully determine the linearised solution it suffices to calculate the eigenvalues and eigenvectors of the stability matrix M.

Now naïvely we have a free parameter c_i for each coupling, which would appear to cause an issue for the predictivity of our theory. However we are only dealing with trajectories which come arbitrarily close to the fixed point in the ultraviolet, meaning that we need $\delta\lambda_i(t \to \infty) = 0$ for all possible couplings. We can see that if $\mathrm{Re}(\theta_j) < 0$ then the corresponding term appearing in the solution (I.27) will indeed vanish in this limit, and we can say nothing further about the parameter c_j — we describe the direction of the corresponding eigenvector as relevant. If instead $\mathrm{Re}(\theta_j) > 0$ the corresponding term will blow up at large scales, and so the only way trajectories will be able reach the fixed point is if $c_j = 0$ — such directions are said to be irrelevant,

$$c_j = \begin{cases} \text{free parameter}\,, & \mathrm{Re}(\theta_j) < 0\,, \\ 0\,, & \mathrm{Re}(\theta_j) > 0\,. \end{cases} \tag{I.29}$$

Any couplings for which $\mathrm{Re}(\theta_j) = 0$ are said to be marginal. For these directions the linearisation is not a sufficient tool to understand the limiting behaviour, and more detailed analysis is required.

We then have another important condition for our fixed point to be able to describe the ultraviolet limit of a theory in a meaningful way — we require the number of relevant directions to be finite. If this is the case, then we have only finitely many free parameters. A series of measurements can in principle be made to determine these, after which our theory will be fully predictive. If we had infinitely many free parameters we would never reach the point of predictivity, as we could keep adjusting our unspecified constants to match the results. If our theory possesses an ultraviolet fixed point with a finite number of relevant directions it is said to be *asymptotically safe*.

In general, the ethos of the renormalisation group approach is that when writing down terms in the action, one must write down all terms compatible with the symmetries of the theory. In general this will be an infinite set of potential operators. Take for example a theory of a single real scalar field ϕ in four dimensions, which has a $\mathbb{Z}_2$ symmetry $\phi \to -\phi$. When encountering such a theory in textbooks one usually only considers the potential up to the quartic term

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4\,, \tag{I.30}$$

where the numerical prefactors of the couplings are purely conventional. However, in a renormalisation group setting one must also consider higher polynomial terms

$$\Delta V(\phi) = c_6\phi^6 + c_8\phi^8 + c_{10}\phi^{10} + \dots\,, \tag{I.31}$$

as all terms of the form ϕ^{2n} for natural numbers n are perfectly compatible with the $\mathbb{Z}_2$ symmetry of the theory. The situation is even more complicated for theories with more fields or symmetries. For example, the standard model has over 50 operators at dimension 6 alone [67]. At any rate, finding fixed points for the entire theory will involve solving the fixed point equation for the full, infinite, set of couplings. It may seem that determining the number of relevant directions will be a daunting task, but fortunately with weakly coupled fixed points we have a shortcut.

In dimension four, fields carry mass dimension. The action overall must be dimensionless, meaning that terms in the Lagrangian must scale with four powers of mass (equivalently inverse length), to compensate the measure of the spacetime integral. The kinetic terms set the scaling of fields. For example, as derivatives carry mass dimension one, a term $(\partial_\mu\phi)^2$ will have mass dimension four only if ϕ carries mass dimension one.

At a fixed point, by definition, there is no meaningful notion of scale. As couplings are in general dimensionful quantities, to keep the action dimensionless, it may at first seem that such couplings cannot possess interacting fixed points. However, the quantities that should be considered are really the dimensionless couplings, which are simply the couplings defined by scaling out appropriate powers of the RG scale μ. If we have some coupling X with classical mass dimension d_X,

$$[X] = d_X\,, \tag{I.32}$$

then the quantity

$$\tilde{X} \equiv \mu^{-d_X}X \tag{I.33}$$

will be dimensionless, and it is this dimensionless coupling which will attain (or not) a fixed point. To find the beta function for the dimensionless coupling we may take the

t-derivative of equation (I.33), so that we have

$$\mu\partial_\mu \tilde{X} = \mu^{-d_X} \left(-d_X X + \mu\partial_\mu X\right) , \tag{I.34}$$

which means that

$$\beta_{\tilde{X}} = -d_X \tilde{X} + \mu^{-d_X} \beta_X . \tag{I.35}$$

The second term is merely a factor which scales the dimensionful beta function into a dimensionless quantity, but the first term represents a structural difference — a linear piece of the beta function, with a coefficient given by the classical mass dimension of the coupling in question. One may view this as the 'tree-level' contribution to the running of the dimensionless coupling.

Now in the vicinity of fixed points of theories in the general case, the tree-level piece in equation (I.35) will not directly tell us much, as if the couplings are strong, the second piece of the equation will become important, and govern whether we can reach the fixed point in a given direction in the UV or in the infrared (IR). However, if we restrict ourselves to the case where the fixed point value $\tilde{X}^*$ is small,

$$|X^*| \ll 1 , \tag{I.36}$$

the tree-level term will tell us something very important about the flow of the coupling $\tilde{X}$ in the vicinity of the fixed point.

Let us suppose we have a perturbative expansion of the beta function for X, which we scale to be dimensionless and write in terms of $\tilde{X}$, [4]

$$\mu^{-d_X} \beta_X = a\tilde{X}^2 + b\tilde{X}^3 + c\tilde{X}^4 + \dots . \tag{I.37}$$

The dimensionless beta function (I.35) will take the form

$$\beta_{\tilde{X}} = -d_X \tilde{X} + a\tilde{X}^2 + b\tilde{X}^3 + c\tilde{X}^4 + \dots . \tag{I.38}$$

If we now consider the linearisation around the fixed point, we see that the corresponding eigenvalue will be determined by the derivative

$$\frac{\partial \beta_{\tilde{X}}}{\partial \tilde{X}} = -d_X + O(\tilde{X}) , \tag{I.39}$$

so that we have [5]

$$\theta_X \approx -d_X , \tag{I.40}$$

[4] In general the beta function for X will also depend on the other couplings of the theory, however they will not affect the reasoning here, and so we neglect them for clarity.

[5] In general systems with multiple couplings eigenvalues will not correspond directly to individual couplings, but will be an admixture of various couplings.

Put together with (I.29) this means that the sign of the classical dimension d_X of the coupling tells us about the relevance of the coupling — if it has positive mass dimension, then we have an additional dimension of UV critical surface, parameterised by the corresponding integration constant. If, however the mass dimension is negative, we do not have an additional free parameter, and the trajectory for this coupling will be fixed by those of the other couplings of the theory. This is good news, as in order for our theory to be predictive, we must have only a finite number of free parameters.

As in four dimensions fields have positive mass dimension, the set of couplings with positive mass dimension will be a finite set (in the example above it will be precisely the coupling m^2), while the infinite set of couplings which have negative mass dimension will not correspond to free parameters for trajectories which reach the fixed point in the UV (these are the couplings $c_6, c_8, c_{10}, \ldots$ in the above example). The only couplings whose relevancy are not fixed will be the classically marginal couplings which have vanishing mass dimension (λ in the above example), where this approximation will not be useful, and quantum effects become the primary driver of their UV behaviour. In these works we will only be dealing with these classically marginal couplings, which will consist of a finite set for any given theory.

B. Beta functions of vanilla quantum field theories

In this work the theories considered will be four-dimensional, flat-space, perturbatively renormalisable, local quantum field theories, in some sense very 'vanilla' quantum field theories, that may frequently be found as examples in introductory textbooks. Theories of this type may be written down in full generality, and such theories will have a Lagrangian of the form

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\, a} + \frac{1}{2}D_\mu\Phi_A D^\mu\Phi_A + i\bar{\Psi}_J\sigma^\mu D_\mu\Psi_J - \frac{1}{2}\left(Y^A_{JK}\Psi_J\Psi_K\Phi_A + \text{h.c.}\right)\\ &- \frac{1}{4!}\lambda_{ABCD}\Phi_A\Phi_B\Phi_C\Phi_D + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{h.o.}}\,, \end{aligned} \tag{I.41}$$

where we leave implicit the details of the gauge fixing terms ($\mathcal{L}_{\text{g.f.}}$) and the ghost terms ($\mathcal{L}_{\text{ghost}}$), as the details of these will not play any relevant role. Similarly we leave implicit terms with massive couplings ($\mathcal{L}_{\text{mass}}$) as well as the higher order terms ($\mathcal{L}_{\text{h.o.}}$), with couplings of negative massive dimension. From (I.40) we know that the couplings with positive mass dimension (which are necessarily finite in number) will give rise to relevant directions, whilst the infinite set of couplings with negative mass dimension from higher order terms will all be irrelevant at a weak fixed point. As such we will only be concerned with the classically marginal couplings, whose relevancy will require further investigation.

The Lagrangian (I.41) is written in terms of real scalar fields Φ_A, Weyl fermions Ψ_J, which are each in some (in general reducible) representation of the gauge group $\mathcal{G}$. The covariant derivative terms for the matter fields are

$$D_\mu \Psi_J \equiv \left(\delta_{JK}\partial_\mu - igt^a_{JK}A^a_\mu\right)\Psi_K\,, \tag{I.42}$$
$$D_\mu \Phi_A \equiv \left(\delta_{AB}\partial_\mu - ig\theta^a_{AB}A^a_\mu\right)\Phi_B\,, \tag{I.43}$$

with A^a_μ the gauge boson fields. The sets of matrices t^a and θ^a both form representations of the Lie algebra of $\mathcal{G}$, corresponding to the fermion and scalar fields respectively, and are Hermitian as the group representations are unitary. Additionally, as we are considering the scalar fields as decomposed into real representations, the matrices θ^a will be purely imaginary, and thus antisymmetric. The gauge fields enter the Lagrangian additionally dynamically via the field-strength tensor

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu\,, \tag{I.44}$$

with f^{abc} the structure constants for $\mathcal{G}$. Here μ, ν are spacetime indices, a, b, c are adjoint Lie algebra indices, $J, K, \ldots$ represent general flavour and colour indices for fermions, and $A, B, \ldots$ general scalar indices. Throughout this section repeated indices are taken to be summed over, unless indicated otherwise. The matrices Y^A_{JK} are symmetric in their fermion indices, and the quartic tensor λ_{ABCD} is completely symmetric.

1. Lie algebras and invariance

We will here note a small amount of information relating to Lie algebras, which will be relevant for the renormalisation group running involving gauge couplings. We will use T^a for a generic basis set of representation matrices of the Lie algebra of $\mathcal{G}$, in the representation R, where we do not wish to distinguish between t^a and θ^a. By definition they satisfy

$$[T^a, T^b] = if^{abc}T^c\,. \tag{I.45}$$

The structure constants f^{abc} are by definition antisymmetric in the first two indices, but we can in fact choose a basis for the T^a such that they are antisymmetric in all indices as follows. As the matrices T^a are Hermitian, the trace of the product any of two of them forms an inner product [6]. Then the Gram matrix, formed of taking the inner product of each pair of basis elements in turn,

$$\mathrm{Tr}(T^aT^b) = M^{ab}\,, \tag{I.46}$$

[6] In general for the vector space of $m \times n$ complex matrices, the product $\langle A, B\rangle \equiv \mathrm{Tr}(A^\dagger B)$ forms an inner product on the space. If we have Hermitian matrices, then this simplifies to the trace of the product of elements.

is necessarily positive definite. This means we may diagonalise it via an appropriate choice of T^a's so that we have only positive entries on the diagonal, and as we may freely rescale the individual T^a's we can in fact have it proportional to the identity, so that we have

$$\mathrm{Tr}(T^aT^b) = S_2^R\delta^{ab}\,. \tag{I.47}$$

The proportionality constant S_2^R is sometimes referred to as the Dynkin index for the representation R [7]. This relation fixes for us the choice of basis and normalisation [8]. By contracting this with if^{cda} and relabelling indices, we have the relationship

$$f^{abc} = \frac{-i}{S_2^R}\mathrm{Tr}\left([T^a,T^b]T^c\right)\,, \tag{I.48}$$

from which the cyclic property of the trace implies that $f^{abc} = f^{bca} = f^{cab}$, and thus the structure constants in this basis are completely antisymmetric. We will always assume we are using such a basis.

Now let's consider the *quadratic Casimir* matrix defined by

$$C_2^R \equiv T^aT^a\,. \tag{I.49}$$

We may take the commutator of this with an arbitrary basis element T^b,

$$[C_2^R,T^b] = [T^aT^a,T^b] = T^a[T^a,T^b] + [T^a,T^b]T^a = if^{abc}(T^aT^c + T^cT^a) = 0\,, \tag{I.50}$$

which vanishes as we are contracting a symmetric object with an antisymmetric one. Therefore this matrix is an invariant of the algebra, and is thus proportional to the identity for each irreducible representation R. We shall also use C_2^R to denote the proportionality constant,

$$\mathbf{C_2^R} = C_2^R\mathbb{1}\,, \tag{I.51}$$

which we shall also refer to as the quadratic Casimir, or simply Casimir for the representation R with context dictating which we are discussing [9]. This is related to the Dynkin index by considering the expression

$$\mathrm{Tr}(T^aT^a) = C_2^R\mathrm{Tr}(\mathbb{1}) = S_2^R\delta^{aa}\,, \tag{I.52}$$

which we may obtain from (I.47) by tracing over the adjoint indices a, b, or from (I.51) by

[7] Note that by definition, this is well-defined even for a reducible representation, in which case it is simply the sum of Dynkin indices for each of its irreducible constituents.

[8] Popular choices are $S_2^R = \frac{1}{2}$ or $S_2^R = 1$ for the fundamental representation. Any choice is purely for convenience, and will not affect the physics.

[9] Here we see that although the quadratic matrix (I.49) is defined for arbitrary representations, it only makes sense to talk about proportionality coefficients C_2^R in (I.51) for irreducible representations.

tracing over the implicit representation indices. Equating the two we get that

$$S_2^R d_{\mathcal{G}} = C_2^R d_R \,, \tag{I.53}$$

with $d_{\mathcal{G}}$ the dimension of $\mathcal{G}$, and d_R the dimension of the representation R. Necessarily this means that the Dynkin index and quadratic Casimir coincide for the adjoint representation.

Now we consider some properties of the Yukawa matrices Y^A_{JK}. We need the Lagrangian (I.41) to be invariant under gauge transformations [10]. We may perform a transformation on our fields

$$\Psi_J \to \Psi'_J = U^F_{JK}\Psi_K \,, \qquad\qquad \Phi_A \to \Phi'_A = U^S_{AB}\Psi_B \,, \tag{I.54}$$

where the unitary transformations are related to the corresponding Lie algebra elements as

$$U^F = e^{i\alpha^a t^a} \,, \qquad\qquad U^S = e^{i\alpha^a \theta^a} \,, \tag{I.55}$$

for some choice of parameter α^a. Under (I.54), the Yukawa term in(I.41) transforms as

$$Y^A_{JK}\Phi_A\Psi_J\Psi_K \to \left(Y^A_{JK}U^S_{AB}U^F_{JL}U^F_{KM}\right)\Phi_B\Psi_L\Psi_M \,. \tag{I.56}$$

As this term must be invariant under such a transformation, we require that

$$\left(Y^A_{JK}U^S_{AB}U^F_{JL}U^F_{KM}\right) = Y^B_{LM} \,. \tag{I.57}$$

Expanding this in terms in terms of the Lie algebra elements using (I.55), examining the $O(\alpha)$ piece leads us to the relationship

$$Y^A_{JL}t^a_{LK} + \bar{t^a}_{JL}Y^A_{LK} = \theta^a_{AB}Y^B_{JK} \,, \tag{I.58}$$

where an overline indicates complex conjugation. If the fermions are in a real representation, so that $\bar{t^a} = -t^a$, then the left-hand-side of this relationship becomes a commutator, and equation (I.58) bears a resemblance to the Lie algebra relationship (I.45) with the components of the scalar representation matrices θ^a playing the role of the structure constants, an analogy which will prove occasionally useful. The analogue of the quadratic Casimir (I.49) is

$$\mathbf{Y_2^F} \equiv \overline{\mathbf{Y}^A}\mathbf{Y}^A \,, \tag{I.59}$$

where we use bold to indicate they should be understood as matrices acting on fermion

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[10] and of course any global symmetries we may have also.

indices, and of (I.47) is

$$Y_2^{S\,AB} \equiv \frac{1}{2}\mathrm{Tr}\left(\mathbf{Y}^{A\dagger}\mathbf{Y}^B + \mathbf{Y}^{B\dagger}\mathbf{Y}^A\right) . \tag{I.60}$$

Both of these quantities appear in the beta functions for couplings, via corrections to fermion and scalar legs respectively. Taking the commutator of (I.59) with the fermion representation matrices, and using the invariance relation (I.58) and its complex conjugate we find that

$$[\mathbf{Y_2^F}, \mathbf{t}^a] = \overline{\mathbf{Y}^A}\mathbf{Y}^A\mathbf{t}^a - \mathbf{t}^a\overline{\mathbf{Y}^A}\mathbf{Y}^A = \overline{\mathbf{Y}^A}\left(\theta^a_{AB}\mathbf{Y}^B - \mathbf{t}^a\mathbf{Y}^A\right) - \left(-\theta^a_{AB}\overline{\mathbf{Y}^B} - \overline{\mathbf{Y}^A}\overline{\mathbf{t}^a}\right)\mathbf{Y}^A \tag{I.61}$$

$$= \theta^a_{AB}\left(\overline{\mathbf{Y}^A}\mathbf{Y}^B + \overline{\mathbf{Y}^B}\mathbf{Y}^A\right) = 0\,, \tag{I.62}$$

where the last expression vanishes owing to the fact that the scalar representation matrices θ^a are antisymmetric in the representation indices. Therefore, as with the Casimir element, $\mathbf{Y_2^F}$ will be proportional to the identity on each irreducible fermion representation. Similarly if we take the commutator of the combination (I.60) with a scalar representation matrix, with respect to the scalar indices, upon using (I.58) we similarly find

$$Y^S_{2\,AC}\theta^a_{CB} - \theta^a_{AC}Y^S_{2\,CB} = 0\,, \tag{I.63}$$

so that this also takes constant values on each irreducible scalar representation.

We can similarly derive an invariance relation for the quartic tensor λ_{ABCD},

$$\theta^a_{EA}\lambda_{EBCD} + \theta^a_{EB}\lambda_{AECD} + \theta^a_{EC}\lambda_{ABED} + \theta^a_{ED}\lambda_{ABCE} = 0\,, \tag{I.64}$$

which we can use in an analogous fashion to show that objects such as

$$\Lambda^2_{AB} \equiv \tfrac{1}{6}\lambda_{ACDE}\lambda_{BCDE}\,, \tag{I.65}$$

which appears for example in the two-loop Yukawa beta function, commute with scalar representation matrices and therefore take constant values for each irreducible scalar representation.

2. *Form of low order beta functions*

Now we can look at expressions for the lowest order beta functions for this most general theory. The gauge beta function takes the form [16, 17, 68–74] [11]

$$\beta_g = \tfrac{1}{2}g^3(-B + Cg^2 - 2Y_4^F)\,, \tag{I.66}$$

[11] Here and throughout this section we will suppress loop factors 4π for clarity — effectively absorbing them into the definition of the couplings.

The factor $\frac{1}{2}$ is purely conventional to make it easier when dealing with the squared coupling $\alpha = g^2/(4\pi)^2$. The pure gauge coefficients are given by

$$B = \tfrac{22}{3}C_2^{\mathcal{G}} - \tfrac{4}{3}S_2^F - \tfrac{1}{3}S_2^S\,, \tag{I.67}$$

$$C = -\tfrac{68}{3}(C_2^{\mathcal{G}})^2 + (4C_2^F + \tfrac{20}{3}C_2^{\mathcal{G}})S_2^F + (4C_2^S + \tfrac{2}{3}C_2^{\mathcal{G}})S_2^S\,, \tag{I.68}$$

where F and S are the fermion and scalar gauge representations respectively [12]. The Yukawa contribution term is defined by

$$Y_4^F \equiv \mathrm{Tr}(\mathbf{C_2^F Y_2^F})/d_{\mathcal{G}}\,. \tag{I.69}$$

The running of the Yukawa matrices to one-loop is [18, 75, 76]

$$\beta^A = \tfrac{1}{2}(\overline{\mathbf{Y_2^F}}\mathbf{Y}^A + \mathbf{Y}^A\mathbf{Y_2^F}) + Y_{2\,AB}^S\mathbf{Y}^B + 2\mathbf{Y}^B\overline{\mathbf{Y}^A}\mathbf{Y}^B - 3g^2(\mathbf{C_2^F}\mathbf{Y}^A + \mathbf{Y}^A\mathbf{C_2^F})\,. \tag{I.70}$$

and the quartic tensor beta function takes the form [18, 76, 77]

$$\beta_{ABCD} = \Lambda^2_{ABCD} + \Lambda^Y_{ABCD} - 4H_{ABCD} - 3g^2\Lambda^S_{ABCD} + 3g^4A_{ABCD}\,. \tag{I.71}$$

Here the various contributions are defined by

$$\Lambda^Y_{ABCD} \equiv \tfrac{1}{6}\sum_{(ABCD)} Y^S_{2\,AE}\lambda_{EBCD}\,, \qquad \Lambda^S_{ABCD} \equiv \tfrac{1}{6}\sum_{(ABCD)} C^S_{2\,AE}\lambda_{EBCD}\,, \tag{I.72}$$

$$\Lambda^2_{ABCD} \equiv \tfrac{1}{8}\sum_{(ABCD)} \lambda_{ABEF}\lambda_{EFCD}\,, \qquad H_{ABCD} \equiv \tfrac{1}{4}\sum_{(ABCD)} \mathrm{Tr}(\overline{\mathbf{Y}^A}\mathbf{Y}^B\overline{\mathbf{Y}^C}\mathbf{Y}^D)\,, \tag{I.73}$$

$$A_{ABCD} \equiv \tfrac{1}{8}\sum_{(ABCD)} \{\theta^a,\theta^b\}_{AB}\{\theta^a,\theta^b\}_{CD}\,, \tag{I.74}$$

where summation is over all permutations of the indices indicated in parentheses, and the numerical prefactors mean that each unique configuration is counted only once.

Fully general expressions have additionally been computed up to two-loops in the Yukawa and quartic couplings [74–77], and three-loop in the gauge couplings [78], featuring other, similar structures, although are far more complicated owing to the wider variety of potential topologies on offer. We shall not need to investigate higher order terms in these works as the important physics will be captured at the orders expressed here, and so we will not need to refer to explicit expressions for higher loop orders.

[12] In fact if these representations are reducible, the terms appearing in C imply a sum over individual irreducible representations.

C. Calculating beta functions

Typically when dealing with specific models, expressing Lagrangians in the form (I.41) is not the most convenient way to write them down, and in practice is rarely done. It is helpful therefore to have a simple way of translating between the notations most convenient for a particular example model, and the general formalism (I.41), to aid in evaluating the expressions(II.1),(I.70) and (I.71).

We wish to move between general descriptions, which package fermion fields into a single field Ψ, with a single index J containing all flavour and colour information, and those of a specific model, where typically one will have several fermion fields with a variety of indices. For example consider the standard model, which has fermion fields $q_{\alpha_3\alpha_2 i}, u_{\alpha_3 i}, d_{\alpha_3 i}, l_{\alpha_2 i}, e_i$, with α_3, α_2, i representing fundamental $SU(3), SU(2)$ and generation indices respectively. There are 45 distinct component fermion fields, which we could list in some order to package into Ψ, for example

$$\Psi_J = (q_{111}, q_{211}, q_{311}, q_{121}, \ldots, u_{11}, u_{21}, \ldots,) \, , \tag{I.75}$$

by choosing some particular mapping between the standard model indices and the general index J. However, we will find it more convenient to use an approach which does not need to make reference to an explicit encoding.

We use the structure-delta formalism developed in [79]. The structure deltas $\Delta^{\psi}_{J;\{a\}}$ are objects defined implicitly through the relations

$$\psi_{\{a\}} = \Delta^{\psi}_{J;\{a\}} \Psi_J \, , \qquad\qquad \phi_{\{b\}} = \Delta^{\phi}_{A;\{b\}} \Phi_A \, . \tag{I.76}$$

They are bookkeeping devices which implicitly map between general index sets $A, J, \ldots$, and model-specific index sets $\{a\}, \{b\}, \ldots$, without the need to explicitly state an encoding. ψ, ϕ are labels for different fields in the particular model of interest. They obey the fundamental relation

$$\Delta^{\psi}_{J;a_1 b_1 c_1 \ldots} \Delta^{\psi'}_{J;d_1 e_1 f_1 \ldots} = \begin{cases} \delta_{a_1 d_1} \delta_{b_1 e_1} \delta_{c_1 f_1} \ldots, & \psi = \psi' \\ 0, & \psi \neq \psi' \end{cases} , \tag{I.77}$$

$$\Delta^{\phi}_{A;\alpha_1 \beta_1 \ldots} \Delta^{\phi'}_{A;\gamma_1 \varepsilon_1 \ldots} = \begin{cases} \delta_{\alpha_1 \gamma_1} \delta_{\beta_1 \varepsilon_1} \ldots, & \phi = \phi' \\ 0, & \phi \neq \phi' \end{cases} , \tag{I.78}$$

This means that we can define the objects

$$\Delta^{\psi}_{J;a_1 b_1 c_1 \ldots} \Delta^{\psi}_{K;a_1 b_1 c_1 \ldots} = \mathbb{1}^{\psi}_{JK} \, , \tag{I.79}$$

$$\Delta^{\phi}_{A;\alpha_1 \beta_1 \ldots} \Delta^{\phi}_{B;\alpha_1 \beta_1 \ldots} = \mathbb{1}^{\phi}_{AB} \, , \tag{I.80}$$

which then act as projection operators for the relevant field subspaces,

$$\mathbb{1}^{\psi}_{JK}\Delta^{\psi'}_{K;a_1b_1c_1\dots} = \begin{cases} \Delta^{\psi}_{J;a_1b_1c_1\dots}, & \psi = \psi' \\ 0, & \psi \neq \psi' \end{cases}, \tag{I.81}$$

$$\mathbb{1}^{\phi}_{AB}\Delta^{\phi'}_{B;\alpha_1\beta_1\dots} = \begin{cases} \Delta^{\phi}_{A;\alpha_1\beta_1\dots}, & \phi = \phi' \\ 0, & \phi \neq \phi' \end{cases}, \tag{I.82}$$

We will use $\mathbb{1}^F, \mathbb{1}^S$ to refer to the identity acting on the entire fermion and scalar field spaces respectively.

As an example of how this is used in practice, we will make a few calculations in an explicit model. Specifically we take the model considered in [55]. The model has an $SU(N_c)$ gauge symmetry, with N_f Weyl fermions ζ in the fundamental representation, and N_f Weyl fermions χ in the antifundamental. Additionally there are N_f^2 gauge singlet complex scalars. The Yukawa and quartic interactions obey an $SU(N_f) \times SU(N_f)$ chiral symmetry, which enter the Lagrangian via the terms

$$\mathcal{L}_{\text{int}} = -y\text{Tr}(\zeta H \chi) + \text{h.c.} - u\text{Tr}(H^\dagger H H^\dagger H) - v\text{Tr}(H^\dagger H)\text{Tr}(H^\dagger H)\,, \tag{I.83}$$

where traces are over flavour and colour indices. When considered in the Veneziano limit $N_c, N_f \to \infty$, for suitable values of the ratio, the theory can develop an interacting ultraviolet gauge-Yukawa fixed point, which is completely controllable within perturbation theory.

The first step to translating the terms in the Lagrangian of this theory into a general form is to write in the indices (for which we will use $\alpha, \beta, \dots$ for fundamental gauge indices, and $i, j, \dots$ without and with primes $'$ for left and right chiral indices respectively), and decompose the complex scalar field H in terms of a real scalar field h, so that (I.83) becomes

$$\mathcal{L}_{\text{int}} = -\frac{y}{\sqrt{2}}\mathfrak{o}^c\zeta_{i\alpha}h^c_{ij'}\chi_{j'\alpha} + \text{h.c.} - \frac{u}{4}\mathfrak{c}^c\mathfrak{o}^d\mathfrak{c}^e\mathfrak{o}^f h^c_{ij'}h^d_{il'}h^e_{kl'}h^f_{kj'} - \frac{v}{4}\mathfrak{c}^c\mathfrak{o}^d\mathfrak{c}^e\mathfrak{o}^f h^c_{ij'}h^d_{ij'}h^e_{kl'}h^f_{kl'} \tag{I.84}$$

As we have decomposed the scalars into real components we use the object $\mathfrak{o}^c$ and its conjugate $\mathfrak{c}^c$ to keep track of the complex structure. These are defined by

$$\mathfrak{o}^c = \begin{cases} 1, & c = 1 \\ i, & c = 2 \end{cases}, \qquad \mathfrak{c}^c \equiv \overline{\mathfrak{o}^c} = \begin{cases} 1, & c = 1 \\ -i, & c = 2 \end{cases}, \tag{I.85}$$

and satisfy

$$\mathfrak{o}^c\mathfrak{c}^d = \delta^{cd} - i\epsilon^{cd}\,, \qquad \epsilon^{12} = +1\,. \tag{I.86}$$

We will also make use of the combination

$$\Gamma^{cdef} \equiv \tfrac{1}{2}(\mathfrak{o}^c \mathfrak{c}^d \mathfrak{o}^e \mathfrak{c}^f + \mathfrak{c}^c \mathfrak{o}^d \mathfrak{c}^e \mathfrak{o}^f) = \delta^{cd}\delta^{ef} - \epsilon^{cd}\epsilon^{ef} = \Gamma^{defc} = \Gamma^{edcf} \tag{I.87}$$

which furthermore obeys

$$\Gamma^{ccef} = 2\delta^{ef}\,, \quad \Gamma^{cdcf} = 0\,, \quad \Gamma^{cdef}\Gamma^{cgeh} = 2\Gamma^{dgfh}\,, \quad \Gamma^{cdef}\Gamma^{cegh} = 0\,, \quad \Gamma^{cdef}\Gamma^{cdgh} = 2\Gamma^{efhg}\,. \tag{I.88}$$

The goal is to write the general Yukawa and quartic tensors Y^A_{JK}, λ_{ABCD}, in terms of y, u, v, and structure-deltas. We then use these to compute the expressions such as (I.59) for the beta functions of general couplings, and project onto the appropriate structures as necessary. For example, using the relation (I.76), we have

$$\mathcal{L}_y = -\frac{y\mathfrak{o}^c}{\sqrt{2}} h^c_{ij'} \zeta^{i\alpha}\chi^{j'\alpha} + \text{h.c.}\,, \tag{I.89}$$

$$= -\frac{y\mathfrak{o}^c}{\sqrt{2}} \tfrac{1}{2} \Delta^{h\ c}_{A;ij'} (\Delta^{\zeta}_{J;\alpha i}\Delta^{\chi}_{K;\alpha j'} + \Delta^{\zeta}_{K;\alpha i}\Delta^{\chi}_{J;\alpha j'})\Phi_A \Psi_J \Psi_K + \text{h.c.}\,, \tag{I.90}$$

where we have made sure to symmetrise over the fermions. From this we can read off the Yukawa matrices as

$$Y^A_{JK} = \frac{y\mathfrak{o}^c}{\sqrt{2}} \Delta^{h\ c}_{A;ij'} \left(\Delta^{\zeta}_{J;\alpha i}\Delta^{\chi}_{K;\alpha j'} + \Delta^{\chi}_{J;\alpha j'}\Delta^{\zeta}_{K;\alpha i}\right)\,. \tag{I.91}$$

Following a similar procedure we find the quartic tensors are

$$u_{ABCD} = \frac{u}{4} \sum_{(ABCD)} \Gamma^{cdef} \Delta^{h\ c}_{A;ij'} \Delta^{h\ d}_{B;il'} \Delta^{h\ e}_{C;kl'} \Delta^{h\ f}_{D;kj'}\,, \tag{I.92}$$

$$= u\Delta^{h\ c}_{A;ij'} \sum_{(BCD)} \Gamma^{cdef} \Delta^{h\ d}_{B;il'} \Delta^{h\ e}_{C;kl'} \Delta^{h\ f}_{D;kj'}\,, \tag{I.93}$$

$$v_{ABCD} = 2v(\mathbb{1}^h_{AB}\mathbb{1}^h_{CD} + \mathbb{1}^h_{AC}\mathbb{1}^h_{BD} + \mathbb{1}^h_{AD}\mathbb{1}^h_{BC})\,, \tag{I.94}$$

$$\lambda_{ABCD} = u_{ABCD} + v_{ABCD}\,, \tag{I.95}$$

where we recall the notation $\sum_{(\ldots)}$ means to sum over all distinct permutations of the indices in parentheses.

As an explicit example, we can calculate $\mathbf{Y_2^F}$, as defined in (I.59), for this model. We have that

$$Y^F_{2\,JK} \equiv \overline{Y^A_{JL}} Y^A_{LK}\,, \tag{I.96}$$

$$= \frac{y\mathfrak{c}^c}{\sqrt{2}} \Delta^{h\ c}_{A;ij'} \left(\Delta^{\zeta}_{J;\alpha i}\Delta^{\chi}_{L;\alpha j'} + \Delta^{\chi}_{J;\alpha j'}\Delta^{\zeta}_{L;\alpha i}\right) \frac{y\mathfrak{o}^d}{\sqrt{2}} \Delta^{h\ d}_{A;kl'} \left(\Delta^{\zeta}_{L;\beta k}\Delta^{\chi}_{K;\beta l'} + \Delta^{\chi}_{L;\beta l'}\Delta^{\zeta}_{K;\beta k}\right)\,. \tag{I.97}$$

Then using the fundamental relations (I.77) we find that

$$
\begin{aligned}
Y^{F}_{2\,JK} &= \overline{Y^{A}_{JL}} Y^{A}_{LK}\,, && \text{(I.98)}\\
&= \frac{y^2 \mathfrak{c}^{c} \mathfrak{d}^{d}}{2} \delta_{ik}\delta_{j'l'}\delta^{cd}\left(\Delta^{\zeta}_{J;\alpha i}\Delta^{\zeta}_{K;\beta k}\delta_{\alpha\beta}\delta_{j'l'} + \Delta^{\chi}_{J;\alpha j'}\Delta^{\chi}_{K;\beta l'}\delta_{\alpha\beta}\delta_{ik}\right)\,, && \text{(I.99)}\\
&= y^2\left(\Delta^{\zeta}_{J;\alpha i}\Delta^{\zeta}_{K;\alpha i}\delta_{j'j''} + \Delta^{\chi}_{J;\alpha j'}\Delta^{\chi}_{K;\alpha j'}\delta_{ii}\right) = y^2 N_f\left(\mathbb{1}^{\zeta}_{JK} + \mathbb{1}^{\chi}_{JK}\right) = y^2 N_f \mathbb{1}^{F}_{JK}\,, && \text{(I.100)}
\end{aligned}
$$

Similarly we find

$$
Y^{S}_{2\,AB} = y^2 N_c \mathbb{1}^{S}_{AB}\,, \qquad \mathbf{Y}^{B}\overline{\mathbf{Y}^{A}}\mathbf{Y}^{B} = \mathbf{0}\,, \tag{I.101}
$$

and with the fermion Casimir

$$
\mathbf{C_2^F} = C_2^N \mathbb{1}^{\mathbf{F}}\,, \qquad C_2^N = \tfrac{1}{2}(N_c - 1/N_c)\,, \tag{I.102}
$$

the Yukawa matrix beta function is given to one-loop by

$$
\partial_t \mathbf{Y}^A = \mathbf{Y}^A\left((N_f + N_c)y^2 - 6C_2^N g^2\right) \tag{I.103}
$$

which gives for the squared Yukawa coupling simply

$$
\partial_t(y^2) = 2y\partial_t y = 2y^2((N_f + N_c)y^2 - 6C_2^N g^2)\,. \tag{I.104}
$$

The additional relevant gauge factors are given by

$$
S_2^F = (2N_f)\tfrac{1}{2} = N_f\,, \qquad C_2^{\mathcal{G}} = N_c \qquad S_2^S = 0\,, \qquad Y_4^F = N_f^2 y^2\,, \tag{I.105}
$$

and the Yukawa contribution to the two-loop gauge beta function is encapsulated in the term

$$
Y_4^F \equiv \mathrm{Tr}(\mathbf{C_2^F}\mathbf{Y_2^F})/d_{\mathcal{G}} = y^2 N_f C_2^N \mathrm{Tr}(\mathbb{1}^{\mathbf{F}})/d_{\mathcal{G}} = y^2 N_f C_2^N (2N_f N_c)/d_{\mathcal{G}} = y^2 N_f^2\,, \tag{I.106}
$$

and as the gauge beta function is simply a scalar quantity we may straightforwardly write it down to two-loop as

$$
\beta_{g^2} = g^4\left(-\frac{4}{3}\left(\frac{11}{2}N_c - N_f\right) + \left(\left(4C_2^N + \frac{20}{3}N_c\right)N_f - \frac{68}{3}N_c^2\right)g^2 - 2N_f^2 y^2\right)\,. \tag{I.107}
$$

The quartic beta functions are much more tedious to compute owing to the various sums over permutations, which require some care. We may divide the pure quartic contributions into three distinct terms depending on which couplings contribute, as

$$\Lambda^2_{ABCD} \equiv \tfrac{1}{8} \sum_{(ABCD)} (u_{ABEF} + v_{ABEF})(u_{EFCD} + v_{EFCD}) \,, \tag{I.108}$$

$$= \tfrac{1}{8} \sum_{(ABCD)} u_{ABEF} u_{EFCD} + \tfrac{1}{8} \sum_{(ABCD)} 2 v_{ABEF} u_{EFCD} + \tfrac{1}{8} \sum_{(ABCD)} v_{ABEF} v_{EFCD} \,, \tag{I.109}$$

$$\equiv \Lambda^{2\,(uu)}_{ABCD} + 2\Lambda^{2\,(uv)}_{ABCD} + \Lambda^{2\,(uv)}_{ABCD} \,. \tag{I.110}$$

Let us concentrate on the middle term for demonstrative purposes. An individual summand takes the form

$$v_{ABEF} u_{EFCD} = 2v(\mathbb{1}^h_{AB}\mathbb{1}^h_{EF} + \mathbb{1}^h_{AE}\mathbb{1}^h_{BF} + \mathbb{1}^h_{AF}\mathbb{1}^h_{BE}) u_{EFCD} \,, \tag{I.111}$$

$$= 2vu_{CDEE}\mathbb{1}^h_{AB} + 4vu_{ABCD} \,. \tag{I.112}$$

We may simplify the first term from the expression (I.92) by making use of the first two expressions in (I.88),

$$u_{ABEE} = u\Delta^{h\ c}_{A;ij'} \sum_{(BEE)} \Gamma^{cdef} \Delta^{h\ d}_{B;il'} \Delta^{h\ e}_{E;kl'} \Delta^{h\ f}_{E;kj'} \,, \tag{I.113}$$

$$= u\Delta^{h\ c}_{A;ij'} \Gamma^{cdef} \left(2\Delta^{h\ d}_{B;il'} \Delta^{h\ e}_{E;kl'} \Delta^{h\ f}_{E;kj'} + 2\Delta^{h\ d}_{E;il'} \Delta^{h\ e}_{E;kl'} \Delta^{h\ f}_{B;kj'} \right) \,, \tag{I.114}$$

$$= 2u\Delta^{h\ c}_{A;ij'} \left(\Delta^{h\ d}_{B;il'} 2\delta^{cd} \delta_{kk} \delta_{l'j'} + 2\delta^{cf} \delta_{ik} \delta_{l'l'} \Delta^{h\ f}_{B;kj'} \right) \,, \tag{I.115}$$

$$= 8uN_F \mathbb{1}^h_{AB} \,, \tag{I.116}$$

so that

$$v_{ABEF} u_{EFCD} = 16 N_F uv \mathbb{1}^h_{AB} \mathbb{1}^h_{CD} + 4vu_{ABCD} \,. \tag{I.117}$$

Summing over all permutations we have

$$\sum_{(ABCD)} v_{ABEF} u_{EFCD} = 4! \left(\frac{16}{3} N_F uv (\mathbb{1}^h_{AB}\mathbb{1}^h_{CD} + \mathbb{1}^h_{AC}\mathbb{1}^h_{BD} + \mathbb{1}^h_{AD}\mathbb{1}^h_{BC}) + 4vu_{ABCD} \right) \,, \tag{I.118}$$

$$= 4! \left(\frac{8}{3} N_F uv_{ABCD} + 4vu_{ABCD} \right) \,, \tag{I.119}$$

and inserting the relevant prefactors leads to

$$2\Lambda^{2\,(uv)}_{ABCD} \equiv 2 \times \tfrac{1}{8} \sum_{(ABCD)} v_{ABEF} u_{EFCD} = 16 N_F uv_{ABCD} + 24 vu_{ABCD} \,. \tag{I.120}$$

We may follow similar calculations to do the same for each of the other quartic self-contributions, ultimately arriving at

$$\Lambda^2_{ABCD} = \left(8N_f^2u^2 + 24uv\right)\frac{u_{ABCD}}{u} + \left(4(N_f^2+4)v^2 + 16N_fuv + 12u^2\right)\frac{v_{ABCD}}{v} \,. \tag{I.121}$$

By projecting the contributions onto the separate quartic structures, we see which terms contribute to which quartic beta function. When combined with the Yukawa contributions,

$$H_{ABCD} = \frac{1}{2}N_c^2y^2\frac{u_{ABCD}}{u}\,, \qquad \Lambda^Y_{ABCD} = 4N_cy\lambda_{ABCD}\,, \tag{I.122}$$

and after projecting onto the relevant structures these yield the full one-loop quartic beta functions

$$\beta_u = 8N_f^2u^2 + 24uv + 4N_cuy - 4N_c^2y^2\,, \quad \beta_v = 4(N_f^2+4)v^2 + 16N_fuv + 12u^2 + 4N_cvy\,, \tag{I.123}$$

as the gauge-related contributions vanish

$$\Lambda^S_{ABCD} = 0\,, \qquad A_{ABCD} = 0\,, \tag{I.124}$$

due to the fact that the scalars of the theory are gauge singlets.

We may check that these results agree with [55], after rescaling the couplings of the theory with appropriate powers of N_f and N_c so as to take the Veneziano limit, and translating to the variables used there. We may similarly apply these techniques to translate two-loop beta functions for each coupling, as well as the three-loop contribution to the gauge coupling. Additionally doing the same for theories involving larger numbers of couplings only increases the computational load, and proceeds similarly.

D. Supersymmetry

A particular class of field theories that are warrant special consideration are those which possess supersymmetry. Although one can always write simple supersymmetric field theories in the language of ordinary gauge-Yukawa theories, to do so would mean that one would miss out on all the unique features which imbue supersymmetric theories with much of their power. We will be particularly interested in how the couplings run in such theories.

Supersymmetry is frequently introduced as a loophole to the Coleman-Mandula theorem [80], which under certain assumptions (which supersymmetry evades [81]) disallows mixing between the spacetime symmetry of the Poincaré group of relativity with any internal symmetries. From this angle one may be motivated to examine supersymmetric theories purely because they are part of the range of possibilities. However there are other possible motivations to particularly look at supersymmetric theories.

Firstly, supersymmetric field theories may be able to address some questions left unanswered by the standard model. Supersymmetry may offer a solution to the hierarchy problem of the Higgs mass in the standard model, by allowing partial cancellations of contributions to the Higgs mass between particles and their superpartners [82–85], which in the simplest settings should be accessible at energies similar to those accessible at the LHC. Additionally, many versions of supersymmetric theories quite naturally offer candidates for dark matter [86, 87], so that one may solve this additional problem 'for free'.

The second is from a more aesthetic theoretical standpoint. The fact that supersymmetry offers a symmetry between the bosonic and fermionic sectors of a theory allows for a direct unification of types of matter which are otherwise independent quantities. Additionally, supersymmetric theories have many remarkable properties which allow for unique phenomena and ease of computation in comparison to non-supersymmetric theories. These include amongst others all-orders results [88, 89], dualities [90–92], and remarkable and unexpected structures within scattering amplitudes of supersymmetric theories [93–95].

Regardless of the motivation, supersymmetric theories with field content up to spin-1 can be viewed as a particular subset of gauge-Yukawa theories, where certain relations between couplings are imposed. As such, general results for the fixed point structure gauge-Yukawa theories will apply just as well in these settings. However, being that they are far more greatly constrained, it is not obvious a priori that the conditions which allow a theory to develop an interacting weakly coupled UV fixed point should be compatible with a theory possessing supersymmetry, and in fact the straightforward supersymmetric version of the model [55] is no longer asymptotically safe [96]. Furthermore, there had been some indication that weakly coupled ultraviolet fixed points may be fundamentally impossible in supersymmetric theories [96, 97] [13].

Although we will not need the full details of how supersymmetry works to understand the fixed point structure, it is worth bearing in mind exactly what it means. It is a symmetry which is governed by a superalgebra — a generalisation of a Lie algebra where elements are given a Z_2 grading. The even elements correspond to an ordinary Lie algebra, and physically correspond to bosonic operators obeying commutation laws, whilst the new, odd, elements correspond to fermionic operators, necessarily carrying a representation of the bosonic part. Spacetime supersymmetry contains the Poincaré algebra within its bosonic part, and the fermionic operators, 'supercharges', carry spinor representations — this is how supersymmetry relates fields of different spin.

We may partly categorise supersymmetry algebras by the number of independent sets of supercharges we introduce, $\mathcal{N}$. Theories are constructed from superfields, which form representations of the super algebra, and we may examine the spin of the various component fields for each case. For $\mathcal{N} = 1$ there are two supermultiplets — the vector superfield which contains a spin-1 and a spin-$\frac{1}{2}$ components, which will be the gauge field in an adjoint representation of the gauge group, and a chiral superfield which consists of a spin-$\frac{1}{2}$

[13] We can evade the arguments leading to this conclusion as long as the infrared limit of the theory is not the Gaussian fixed point, and as long as we have a semisimple theory for which the Gaussian fixed point is not a complete infrared sink.

	non-SUSY			$\mathcal{N}=1$		$\mathcal{N}=2$		$\mathcal{N}=4$
	Vector A_μ	**Fermion** ψ	**Scalar** ϕ	**Vector** V_μ	**Chiral** Φ	**Vector** V_μ	**Hyper** Φ	**Vector** V_μ
A_μ	1			1		1		1
ψ		1		1	1	2	2	4
ϕ			1		2	2	4	6
g	A_μ, ψ or A_μ, ϕ			V_μ, Φ		V_μ, Φ		V_μ
y	ψ, ϕ			Φ				
λ	ϕ							
Chiral?	Yes			Yes		No		No
Running						One-loop exact		Conformal

Table 1. A summary of the spin of the components of the various superfields for different amounts of supersymmetry, with the non-supersymmetry case for comparison. Also shown are the fields involved in the different independent couplings, where applicable, as well as notes on the RG behaviour, and whether or not the theory may be chiral.

and a scalar, which constitutes the matter content, and may be in any representation. As well as the gauge coupling, we may have Yukawa-like couplings between three chiral superfields.

Theories with extended supersymmetry, $\mathcal{N} > 1$ are significantly more constrained, which has important consequences for their RG flow. $\mathcal{N} = 2$ theories similarly have a gauge multiplet and a matter field, although the matter is always vector-like, meaning the theory is never chiral. Additionally, the only coupling is the gauge coupling, whose running is exact at one-loop [98, 99], meaning that the only fixed point we can have is the Gaussian, which is either UV or IR free depending purely on the one-loop coefficient [14]. Maximally symmetric $\mathcal{N} = 4$ [15] is even more constrained, such that there is no freedom to choose any independent matter content, and in fact the theory is conformal regardless of the value of the gauge coupling, meaning that there is no RG running [98–101]. As such we will not be looking any further at theories with extended supersymmetry.

Table **1** summarises the components of different superfields for the various levels of supersymmetry, with the non-supersymmetry case for comparison. Additionally shown are the independent couplings of the theory, and the fields involved in the corresponding interaction.

As in the case of non-supersymmetric field theories, we may write down the most general $\mathcal{N} = 1$ supersymmetric theory. The quantity we will be most interested in is the

[14] Additionally if the one-loop term vanishes the theory will be finite.
[15] $\mathcal{N} = 3$ theories are identical to $\mathcal{N} = 4$ on shell. Theories with $\mathcal{N} > 4$ necessarily contain degrees of freedom higher than spin-1, so that they are theories of supergravity and as such lie outside the scope of this work.

superpotential

$$W(\Phi) = \frac{1}{6}Y^{IJK}\Phi_I\Phi_J\Phi_K + \frac{1}{2}M^{IJ}\Phi_I\Phi_J + L^I\Phi_I\,, \tag{I.125}$$

which gives rise to the non-gauge interactions of the theory. As in the case of non-supersymmetric field theories, we will in fact restrict only to dimensionless interactions, and so it will be only the first term of (I.125) which will be of concern. In many applications of supersymmetry it useful to formulate things in terms of chiral superfields Φ using the machinery of superspace, where Φ packages together a Weyl fermion ψ and scalar field ϕ, see table **1**, both in the same representation of the symmetry group of the theory. However, as we wish to discuss things from a gauge-Yukawa perspective, it will be more convenient for us to think in terms of the component fields ψ and ϕ. We may get the interaction part of the Lagrangian from the superpotential by considering it as a function of the scalar component field ϕ, as

$$W(\phi) = \frac{1}{6}Y^{IJK}\phi_I\phi_J\phi_K + W_{\text{mass}}\,, \tag{I.126}$$

by taking appropriate derivatives

$$W^I \equiv \frac{\delta W}{\delta\phi_I}\,, \qquad\qquad W^{ij} \equiv \frac{\delta^2 W}{\delta\phi_I\delta\phi_J}\,, \tag{I.127}$$

we have the non-gauge interaction part of the Lagrangian

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}W^{IJ}\psi_I\psi_J + \text{h.c.} - W^I\overline{W_I}\,, \tag{I.128}$$

which in terms of the component fields is simply

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}Y^{IJK}\phi_I\psi_J\psi_K + \text{h.c.} - \tfrac{1}{4}Y^{IJM}Y_{KLM}\phi_I\phi_J\overline{\phi^K\phi^L} + \mathcal{L}_{\text{mass}}\,, \tag{I.129}$$

where we have suppressed terms involving massive couplings. Lowering the indices on Y implies complex conjugation. From this we see that Y^{IJK} gives rise to a series of Yukawa interactions, as well as completely fixing the quartics of our theory.

Additionally, the theory contains a gaugino λ^a, which is the superpartner of the gauge field. For supersymmetry to be preserved, this must couple to the charged fermions and scalars of the theory, as the vector gauge field does, so that we additionally have interaction terms

$$\mathcal{L}_{\text{gaugino int.}} = -\sqrt{2}g\left(\overline{\phi}T^a\psi\right)\lambda^a + \text{h.c.}\,. \tag{I.130}$$

Here the components of the chiral multiplet are (ψ, ϕ), with representation matrices T^a. From a gauge-Yukawa perspective these look like Yukawa terms, coupling fermions to scalars. However, supersymmetry links these terms directly to the gauge coupling. Unsurprisingly, a side effect of supersymmetry mixing particles of different spin is that it also mixes the different types of couplings, which we ordinarily classify in terms of which spin particles are involved. Thus what in supersymmetric theories we consider to be the gauge coupling, in gauge-Yukawa language looks like a combination of gauge and Yukawa couplings, and what in supersymmetry we consider to be Yukawa couplings look through gauge-Yukawa eyes to be Yukawa and quartic couplings. This relation between couplings is preserved as couplings run due to supersymmetry. The fact that the scalar quartic is entirely determined by the Yukawa couplings simplifies considerations as we need only deal with gauge and Yukawa coupling, and as the potential comes from a term which is inherently positive, supersymmetry in fact guarantees that the scalar potential is bounded from below and thus we have a stable vacuum state — a fact that in non-supersymmetric theories will depend on the value of the couplings of the theory. This mixing between couplings helps to demonstrate why phenomena that may occur in non-supersymmetric theories are by no means guaranteed to carry over into the realm of supersymmetry.

For this general theory we may write down the low order beta functions, much as we did for non-supersymmetric theories in (I.66),(I.70) and (I.71). The gauge beta function takes a structurally similar form [73, 102]

$$\beta_g = \tfrac{1}{2} g^3 \left(-B_{\text{susy}} + C_{\text{susy}} g^2 - 2 Y_{4\,\text{susy}} \right) , \tag{I.131}$$

where we now have modified coefficients

$$B_{\text{susy}} = 6 C_2^{\mathcal{G}} - 2 S_2^R \,, \qquad C_{\text{susy}} = -12 (C_2^{\mathcal{G}})^2 + 4 (C_2^{\mathcal{G}} + 2 C_2^R) S_2^R \,, \tag{I.132}$$

and the Yukawa contribution is effectively unchanged

$$Y_{4\,\text{susy}} \equiv C^R_{2\,JK} \overline{Y^{KLM}} Y^{LMJ} / d_{\mathcal{G}} \,. \tag{I.133}$$

The Yukawa tensor itself runs as [103]

$$\beta^{IJK} = Y^{IJM} \gamma^K_M + Y^{JKM} \gamma^I_M + Y^{KIM} \gamma^J_M \,, \tag{I.134}$$

where γ is the anomalous dimension matrix of the chiral superfield. This is in fact an all-orders result [16], as supersymmetry ensures there are no direct, vertex, contributions to the running of Y^{IJK} — it only feels the effect of scale-dependence indirectly through the chiral superfields. We may put this running back on the usual perturbative footing by expanding the anomalous dimensions as perturbative power series, so that for example we have the one-loop expressions

$$\gamma^{(1)\,I}_{\;\;J} = \tfrac{1}{2} Y^{IMN} Y_{JMN} - 2 g^2 C^{R\,J}_{2\,I} \,. \tag{I.135}$$

[16] Provided that we are in a suitable class of renormalisation scheme. In this work we will only be dealing with low orders results, and so we may assume to be working in such a scheme throughout.

Now we should be able to get these same results be viewing the theory as a completely general field theory using the equations (I.66) and (I.70), if we recall that we will have a single fermion λ in the adjoint representation $\mathcal{G}$, the gaugino, as well as another ψ in a general representation R, which will be accompanied by two scalars in the same representation R. In fact, we may rewrite (I.132) to find that

$$B_{\text{susy}} = 6C_2^{\mathcal{G}} - 2S_2^R = \frac{22}{3}C_2^{\mathcal{G}} - \frac{4}{3}(C_2^{\mathcal{G}} + S_2^R) - \frac{1}{3}(2S_2^R) = B_{\text{gen}}(F = \mathcal{G} + R, S = 2R)\,, \tag{I.136}$$

where B_{gen} is the ordinary one-loop coefficient appearing in (I.66). This is exactly as it should be, as from the gauge-Yukawa perspective we have two fermions — the gaugino from the vector multiplet in $\mathcal{G}$ and one in a general representation R from the chiral multiplet, and the only scalars being the two from the chiral multiplet in representation R.

Similarly at two-loop we have

$$\begin{aligned} C_{\text{susy}} &= -12(C_2^{\mathcal{G}})^2 + 4(C_2^{\mathcal{G}} + 2C_2^R)S_2^R\,, \\ &= -\frac{68}{3}(C_2^{\mathcal{G}})^2 + (4C_2^{\mathcal{G}} + \frac{20}{3}C_2^{\mathcal{G}})C_2^{\mathcal{G}} + (4C_2^R + \frac{20}{3}C_2^{\mathcal{G}})S_2^R \\ &\quad + (4C_2^R + \frac{2}{3}C_2^{\mathcal{G}})2S_2^R - 4(C_2^R + C_2^{\mathcal{G}})S_2^R\,, \\ &= C_{\text{gen}}(F = \mathcal{G} + R, S = 2R) - 4(C_2^R + C_2^{\mathcal{G}})S_2^R\,. \end{aligned} \tag{I.137}$$

It may initially seem like this result is at odds with the gauge-Yukawa perspective, as we have a leftover piece. However, we must not neglect the contribution of the term (I.130), which contributes from this point of view as a Yukawa term. We may write the corresponding Yukawa matrices in the delta formalism as

$$Y_g{}^A_{JK} = g\mathfrak{c}^c T^a_{\alpha\beta}\Delta^{\phi\ c}_{A;\alpha}\left(\Delta^{\psi}_{J;\beta}\Delta^{\lambda}_{K;}{}^a + \Delta^{\lambda}_{J;}{}^a\Delta^{\psi}_{K;\beta}\right)\,. \tag{I.138}$$

It is then straightforward to calculate

$$Y^F_{2\,JK} = 2g^2\left(\Delta^{\psi}_{J;\beta}\Delta^{\psi}_{K;\delta}(T^aT^a)_{\beta\delta} + \Delta^{\lambda}_{J;}{}^a\Delta^{\lambda}_{K;}{}^b\,\mathrm{Tr}(T^aT^b)\right) = 2g^2(C_2^R\mathbb{1}^{\psi}_{JK} + S_2^R\mathbb{1}^{\lambda}_{JK})\,, \tag{I.139}$$

and as the full fermion Casimir is

$$\mathbf{C_2^F} = C_2^R\mathbb{1}^{\psi} + C_2^{\mathcal{G}}\mathbb{1}^{\lambda} \tag{I.140}$$

we have the invariant

$$Y^F_{g\,4} \equiv \mathrm{Tr}(\mathbf{C_2^F}\mathbf{Y_2^F})/d_G = 2g^2(C_2^R + C_2^{\mathcal{G}})S_2^R \tag{I.141}$$

so that we have the full two-loop relationship

$$g^2 C_{\text{susy}} = g^2 C_{\text{gen}}(F = \mathcal{G} + R, S = 2R) - 2Y^F_{g\,4}\,. \tag{I.142}$$

If we have Yukawa couplings in our supersymmetric theory additionally, these will contribute additionally at two-loop in the usual way.

An additional tool which is of particular benefit to the study of supersymmetric fixed points is the a-theorem, which in its weak form states that there is a function a of the couplings of a theory, which is always lower at the IR limit of a trajectory than at the UV limit,

$$\Delta a \equiv a_{UV} - a_{IR} > 0\,. \tag{I.143}$$

Although applicable to more general four-dimensional field theories [104–106], its utility for supersymmetric theories is related to the evaluation in these cases of the a-function.

Fixed points by definition correspond to scale-invariant theories, but in many cases this symmetry is in fact automatically extended to conformal symmetry. In the supersymmetric case, the symmetry then becomes superconformal. A novelty of the superconformal algebra, which does not occur in the straightforward supersymmetric extension of the Poincaré algebra, is that the bosonic part, includes not only the conformal algebra but an additional $U(1)$ piece, which is generally known as R-symmetry [17]. Its relevance in this context is that the function a evaluated at a superconformal fixed point is determined in terms of the charges of the chiral superfields of the theory under the $U(1)_R$ symmetry as [107, 108]

$$a = \frac{3}{32}\left(3\mathrm{Tr}(R^3) - \mathrm{Tr}(R)\right)\,. \tag{I.144}$$

This is all well and good provided that one knows the R-charges of fields in the theory. In general at a fixed point there can be multiple candidate $U(1)$ possibilities which are anomaly free and commute with all other symmetries of the theory. However, the $U(1)$ which coincides with the R-symmetry of the superconformal algebra may be determined from these through the technique of a-maximisation [109], which states that a candidate a-function defined as in (I.144) on the space of possible R-charges takes a local maximum at the correct value. However we will only be considering weak coupling in these works there is also access to the R-charges via a perturbative expansion, owing to the relationship between these and the anomalous dimensions of the chiral superfields for conformal field theories,

$$R = \frac{2}{3}(1+\gamma)\,, \tag{I.145}$$

[17] For extended supersymmetry the R-symmetry group will be larger, and in general non-Abelian.

and the loop expanded expressions (I.135).

E. In this work

This work consists of four papers which explore aspects of perturbative asymptotic safety from two angles. On one hand to explore general considerations to determine conditions for perturbative fixed points to exist, and in particular to be ultraviolet, which results in a full classification of all possible weakly coupled fixed points for local four-dimensional perturbatively renormalisable quantum field theories. The other approach focuses on the construction of explicit families of examples that extend the class of known theories with perturbative asymptotic safety, and which incorporate additional features.

Firstly in Part II we analyse the perturbative beta function of general gauge theories, by slowly building up complexity. We start with a theory with only a gauge coupling, and analyse fixed points of Caswell-Banks-Zaks types — interacting in the gauge coupling [18] alone. We derive a lower bound for the value of the quadratic Casimir of irreducible representations, measured in units of the Casimir for the adjoint, which is valid for all compact simple Lie algebras. The relevance of this bound is that it precludes the possibility that such a fixed point could be ultraviolet in nature. We then demonstrate that all weakly coupled fixed points in gauge theories must be of this type, or interacting gauge-Yukawa types, or products thereof, and the condition by which an interacting gauge-Yukawa fixed point may be ultraviolet.

Secondly in Part III we examine theories without gauge interactions. We revisit the Coleman-Gross theorem, which asserts that such theories may not be asymptotically free. The argument as originally stated applies implicitly to theories with only Dirac fermions — we generalise this through a non-trivial modification to allow the more general case of fermions being Weyl. We then establish that such theories may not possess any perturbative interacting fixed points, ultraviolet or otherwise, thus demonstrating the necessity of gauge fields for a theory to have a perturbative ultraviolet completion, interacting or otherwise. We highlight in particular the fact that these two facts are intertwined, stemming ultimately from the same relations.

In the third paper, Part IV, we analyse in detail a semisimple gauge-Yukawa model. We begin by analysing the generic structure of a class of models with two gauge and two Yukawa couplings, and is structured such that each Yukawa coupling may be associated to only one of the simple gauge factors. In the gauge-Yukawa sector this leads algebraically to nine distinct fixed points. We then focus on a particular family of minimal models, consisting of the 'square' of a simple asymptotically safe theory, augmented by a single messenger fermion charged under both gauge groups. This model is explored in some detail in a generalised Veneziano limit, and its entire effective two-dimensional parameter

[18] and possibly quartic couplings, but these only have subleading effects on the value of the fixed point.

space is analysed. At different parts of the space we find areas with distinct phenomena. These include having all nine fixed points physical, having theories where one gauge sector is free in the UV and the IR, but interacting between, areas where the theory can only be considered effective, and areas where the ultimate UV fixed point of the theory is interacting, and are thus asymptotically safe.

In Part V we look at a particular supersymmetric theory with semisimple gauge group, and a single Yukawa coupling. We outline the potential fixed points, and discuss in particular a parameter point where the Gaussian fixed point is destabilised, and the only viable semisimple ultraviolet completion of the theory is an interacting gauge-Yukawa fixed point, thus demonstrating the existence of supersymmetric perturbative asymptotic safety. We also discuss how this model satisfies compatibility with the a-theorem.

Finally we make some concluding remarks summarising key results and possible interesting directions for future work.

Part II

Theorems for Asymptotic Safety of Gauge Theories

Andrew D. Bond[1] and Daniel F. Litim[1]

[1]Department of Physics and Astronomy, U Sussex, Brighton, BN1 9QH, U.K.

We classify the weakly interacting fixed points of general gauge theories coupled to matter and explain how the competition between gauge and matter fluctuations gives rise to a rich spectrum of high- and low-energy fixed points. The pivotal role played by Yukawa couplings is emphasized. Necessary and sufficient conditions for asymptotic safety of gauge theories are also derived, in conjunction with strict no go theorems. Implications for phase diagrams of gauge theories and physics beyond the Standard Model are indicated.

1. Fixed points of the renormalisation group play an important role in quantum field theory and particle physics [12, 13]. Low-energy fixed points characterise continuous phase transitions and the dynamical breaking of symmetry. High-energy fixed points are central for the fundamental definition of quantum field theory. Important examples are provided by asymptotic freedom of non-abelian gauge theories [16, 17] where the high-energy fixed point is non-interacting. Gauge theories with complete asymptotic freedom, meaning asymptotic freedom for all of its couplings, are of particular interest in the search for extensions of the Standard Model [19]. Asymptotically free gauge theories can also display weakly coupled infrared (IR) fixed points [72, 110]. More recently, it was discovered that gauge theories can develop interacting ultraviolet (UV) fixed points [55], a scenario known as asymptotic safety. This intriguing new phenomenon, originally conjectured in the context of quantum gravity [15], offers the prospect for consistent UV completions of particle physics beyond the paradigm of asymptotic freedom [111].

In this Letter we classify all weakly interacting fixed points of general gauge theories coupled to matter in four space-time dimensions starting from first principles. Our motivation for doing so is twofold: Firstly, we want to understand in general terms whether and how the competition between gauge and matter field fluctuations gives rise to quantum scale invariance. We expect that insights into conformal windows of gauge theories will offer new directions for particle physics above the electroweak energy scale. Secondly, we are particularly interested in the dynamical origin for asymptotic safety in gauge theories and conditions under which it may arise. We also hope that insights into the inner working of asymptotic safety at weak coupling will offer clues for mechanisms of asymptotic safety at strong coupling [44, 112].

We pursue these questions in perturbation theory starting with pure gauge interactions and gradually adding in more gauge and matter couplings. We will find a rich spectrum of interacting high- and low-energy fixed points including necessary and sufficient conditions for their existence. Furthermore, we highlight the central importance of Yukawa couplings to balance gauge against matter fluctuations. We thereby also establish that the presence of scalar fields such as the Higgs are strict necessary conditions for asymptotic safety at weak coupling. Further key ingredients for our results are bounds on quadratic Casimirs which are derived for general Lie algebras, together with structural aspects of perturbation theory which are detailed as we proceed.

2. We begin our investigation of weakly coupled fixed points by considering (non-)abelian vector gauge theories with a simple gauge group $\mathcal{G}$ and gauge coupling g, interacting with spin-$\frac{1}{2}$ fermions or scalars or both. Throughout we scale loop factors into the definition of couplings and introduce $\alpha = g^2/(4\pi)^2$. The renormalisation group running of the gauge coupling up to two loop order in perturbation theory reads

$$\beta = -B\,\alpha^2 + C\,\alpha^3 + \mathcal{O}(\alpha^4)\,, \tag{II.1}$$

where $\beta \equiv d\alpha/d(\ln\mu)$, and μ denoting the RG momentum scale. The one and two loop coefficients in (II.1) are known for arbitrary field content and given in [16, 17, 68–70] and [71–73], respectively. In terms of the Dynkin index S_2^R and the quadratic Casimir C_2^R of quantum fields in some irreducible representation (irrep) R of the gauge group, they can be written as[19]

$$B = \frac{2}{3}\left(11C_2^{\mathcal{G}} - 2S_2^F - \frac{1}{2}S_2^S\right), \tag{II.2}$$

$$C = 2\left[\left(\frac{10}{3}C_2^{\mathcal{G}} + 2C_2^F\right)S_2^F + \left(\frac{1}{3}C_2^{\mathcal{G}} + 2C_2^S\right)S_2^S - \frac{34}{3}(C_2^{\mathcal{G}})^2\right]. \tag{II.3}$$

The terms involving $C_2^{\mathcal{G}}$ – the quadratic Casimir in the adjoint representation of the gauge group – arise due to the fluctuations of the gauge fields. The fluctuations of charged fermionic (F) or scalar (S) matter fields, if present, contribute to (II.1) via the terms proportional to the Dynkin index of their representation.

Gauge theories with (II.1) will always display the free Gaussian fixed point $\alpha_* = 0$. If $B > 0$ this is the well-known ultraviolet (UV) fixed point of asymptotic freedom [16, 17] such as in QCD. For $B < 0$, instead, the theory becomes free in the infrared (IR) such as in QED. In addition, (II.1) can also display an interacting fixed point

$$\alpha_* = \frac{B}{C} \tag{II.4}$$

which is perturbative if $\alpha_* \ll 1$ and physically acceptable provided that $B \cdot C > 0$. For $B \cdot C < 0$ the would-be fixed point reads $\alpha_* < 0$ and resides in an unphysical regime where the theory is sick even non-perturbatively [113], which may be viewed for instance as giving the kinetic term the wrong sign, leading to unitarity violation. Also, if $B < 0$ ($B > 0$), (II.4) corresponds to an interacting UV (IR) fixed point. We conclude that the availability and nature of interacting fixed points is encoded in the signs and magnitude of (II.2) and (II.3). From the explicit expressions, we observe that the pure gauge contributions to both the one and two loop terms are either negative (non-abelian) or vanishing (abelian). Conversely, terms originating from fermionic or scalar matter contribute positively. This means that with a sufficiently small amount of matter (including none), the gauge boson contributions dominate and we have $B > 0, C < 0$. On the other hand, for a sufficiently large amount of matter, the matter contributions dominate and we end up with $B \leq 0, C > 0$. The latter is trivially the case for abelian gauge groups whose quadratic Casimir vanishes identically, $C_2^{U(1)} = 0$. Weakly interacting fixed points are absent in either of these cases.

The question of what may happen when the pure gauge and matter contributions are of similar size is not immediately obvious. It has long been known that it is possible for theories to have $B, C > 0$, which are therefore asymptotically free and which, if $B \ll$

[19] Throughout, we treat fermions as Weyl and scalars as real.

C, can lead to a perturbative infrared Banks-Zaks fixed point [72, 110]. However, no examples have been found for which $B, C < 0$ and where the analagous fixed point would be ultraviolet. To see if such a scenario is possible in principle, we must examine the relative effects of matter on the one- and two-loop contributions. To that end, we resolve (II.2) for the adjoint Casimir and insert the result into the last term of (II.3) to find

$$C = \frac{2}{11}\Big[2S_2^F\left(11C_2^F + 7C_2^{\mathcal{G}}\right) + 2S_2^S\left(11C_2^S - C_2^{\mathcal{G}}\right) - 17B\,C_2^{\mathcal{G}}\Big]\,. \tag{II.5}$$

We make the following observations. The first term in (II.5) due to the fermions is manifestly positive-definite. The last term in (II.5) is positive-definite provided that $B < 0$. Hence, as has been noted by Caswell [72], fermionic matter alone cannot generate an asymptotically safe UV fixed point in perturbation theory. The middle term however, due to charged scalars, is not manifestly positive definite and it cannot be decided *prima facie* whether or not it may generate an interacting UV fixed point with $B < 0$ and $C < 0$.

3. In order to progress with the analysis of (II.5), we must find expressions for the smallest quadratic Casimir for any simple Lie algebra $\mathcal{G}$. Irreducible representations of simple Lie algebras are conveniently characterised by their highest weight Λ, which for a rank-n Lie algebra is an n-dimensional vector of non-negative integers, not all of which are zero.[20] This is due to the theorem of highest weight, which states that inequivalent irreps are in one-to-one correspondence with distinct highest weights. The Racah formula offers an explicit expression for the quadratic Casimir for any irrep R with highest weight Λ. It is given by

$$C_2(\Lambda) = \frac{1}{2}(\Lambda, \Lambda + 2\delta)\,, \tag{II.6}$$

where $(u, v) \equiv \sum_{ij} G_{ij}\, u^i\, v^j$ denotes the inner product of two highest weights, with $u = \sum_{i=1}^{n} u^i\,\Lambda_i$. The weight metrics $G \equiv (G_{ij})$ are known explicitly for any Lie algebra $\mathcal{G}$. Note that $(u, v) > 0$ for any two weights. The n-component vector δ in (II.6) denotes half the sum of the positive roots and reads $\delta = (1, 1, \ldots, 1)$ in the Dynkin basis (which we use exclusively). The normalisation factor $\frac{1}{2}$ in (II.6) is conventional.[21]

For any Lie algebra, the highest weight of irreps with the smallest quadratic Casimir must be one of the fundamental weights Λ_k (with $k \in \{1, \ldots, n\}$), whose components are defined as

$$(\Lambda_k)^i = \delta_k^i\,. \tag{II.7}$$

This can be understood as follows. Consider two highest weights Λ and λ, which may be used to construct a new irrep with highest weight $\Lambda + \lambda$. The bilinearity of the inner

[20] We are not interested in trivial representations given that uncharged fields cannot contribute to (II.1).

[21] In general, the quadratic Casimir is only defined up to a multiplicative constant for a given Lie algebra, and thus we are free to choose the overall normalisation.

product (II.6) then implies that

$$C_2(\Lambda + \lambda) > C_2(\Lambda) + C_2(\lambda) > C_2(\Lambda)\,. \tag{II.8}$$

It follows, trivially, that C_2 can be made arbitrarily large. To find the smallest C_2, however, (II.8) states that we only need to consider irreps whose highest weights have a single non-vanishing component. Assuming Λ to be one such weight and taking $\lambda = m\,\Lambda$ for some integer $m \geq 1$, (II.8) also states that we only need to consider highest weights where this single non-vanishing component takes the smallest non-vanishing value, which is unity. This establishes (II.7). Inserting (II.7) into (II.6), and denoting by G the weight metric of the gauge group $\mathcal{G}$, we find the quadratic Casimir in terms of the fixed index k as

$$C_2 = \frac{1}{2} G_{kk} + \sum_{i=1}^{n} G_{ki}\,. \tag{II.9}$$

It remains to identify the minima of (II.9) with respect to k for the four classical and the five exceptional Lie algebras separately, following the Cartan classification, starting with the rank-n classical Lie algebras A_n, B_n, C_n and D_n [114]. For $n \geq 1, 2, 3$ and 4 they correspond to the unique Lie algebras $\mathbf{su}(n{+}1), \mathbf{so}(2n{+}1), \mathbf{sp}(n)$ and $\mathbf{so}(2n)$, respectively. Explicit expressions for the weight metrics are summarised in [115]. For our purposes we write them in closed form as

$$\begin{aligned}
(G^{A_n})_{ij} &= \min(i,j) - \frac{ij}{n+1}\,,\\
(G^{B_n})_{ij} &= \frac{1}{2}\left[\min(i,j)(2 - \delta_{in} - \delta_{jn}) + \frac{n}{2}\delta_{in}\delta_{jn}\right]\,,\\
(G^{C_n})_{ij} &= \frac{1}{2}\min(i,j)\,,\\
(G^{D_n})_{ij} &= \frac{1}{2}\Big[\min(i,j)\,(2 - \delta_{in} - \delta_{jn} - \delta_{i,n-1} - \delta_{j,n-1}) + \frac{n}{2}\,(\delta_{i,n-1}\delta_{j,n-1} + \delta_{in}\delta_{jn})\\
&\qquad + \frac{1}{2}(n-2)\,(\delta_{i,n-1}\delta_{j,n} + \delta_{i,n}\delta_{j,n-1})\Big]\,.
\end{aligned} \tag{II.10}$$

For illustration, we consider explicitly the case for A_n, where $G_{kk} = k(n+1-k)/(n+1)$, which, combined with

$$\sum_{i=1}^{n} G_{ki} = \sum_{i=1}^{k} i + \sum_{i=k+1}^{n} k - \frac{k}{n+1}\sum_{i=1}^{n} i = \frac{1}{2}k\,(n+1-k)\,,$$

leads to the desired expression for $C_2(A_n)$ as stated in (II.11) below. Analogous, if slightly more tedious, intermediate steps for the other cases lead to the result

$$
\begin{aligned}
C_2(A_n) &= \frac{k}{2}\frac{(n+1-k)(n+2)}{n+1}\,, \\
C_2(B_n) &= \frac{1}{2}\left(k(2n+1-k) - \frac{1}{4}n(3+2n)\delta_{kn}\right)\,, \\
C_2(C_n) &= \frac{k}{2}\left(n+1-\frac{1}{2}k\right)\,, \\
C_2(D_n) &= \frac{1}{2}\left(k(2n-k) - \frac{n}{4}\left(2n-3+4k\right)\left(\delta_{k,n-1}+\delta_{kn}\right)\right)\,,
\end{aligned}
\tag{II.11}
$$

with k taking values between 1 and n. To find the global minima of the expressions (II.11) with respect to k, we proceed as follows. For A_n and C_n, the expressions are quadratic polynomials in k with negative k^2 coefficient, implying that its minima are achieved at the boundaries, meaning either $k = 1$ or $k = n$, or both. For B_n and D_n, additionally, the expressions are discontinuous for certain intermediate values of k (owing to the $\delta_{k,n-1}$ and δ_{kn} factors). This implies that global minima may additionally be achieved for integer values of k within the interval $(1, n)$. With this in mind, and after evaluating all possible cases, the final result for the smallest quadratic Casimir for the classical Lie algebras is found to be

$$
\begin{aligned}
\min\, C_2(A_n) &= \frac{n}{2}\frac{n+2}{n+1}\,, \\
\min\, C_2(B_n) &= \begin{cases} \frac{1}{8}n(2n+1) & \text{for} \quad n = 2,3 \\ n & \text{for} \quad n \geq 4 \end{cases}\,, \\
\min\, C_2(C_n) &= \frac{n}{2} + \frac{1}{4}\,, \\
\min\, C_2(D_n) &= n - \frac{1}{2}\,.
\end{aligned}
\tag{II.12}
$$

The five exceptional groups $E_{6,7,8}, F_4$, and G_2 have a fixed size, hence finding the smallest Casimir amounts to a simple minimisation. Using the appropriate expressions for the weight metrics [115], our results are summarised in Tab. **2** where, for convenience, we express (II.12) using the particle physics nomenclature for the gauge groups.

A few comments are in order: (i) For A_n either boundary is minimal, corresponding to the fundamental and anti-fundamental representation. (ii) For B_n the Casimir is minimal for $k = n$ (the fundamental spinor representation) provided $n = 2$ or 3, and for $k = 1$ (the fundamental vector representation) provided $n \geq 4$. (iii) For C_n and D_n, the Casimir is minimal for $k = 1$ (the fundamental vector representation). (iv) For D_4, three smallest Casimirs are achieved for $k = 1, 3$ and 4. This degeneracy is due to the fact that the Dynkin diagram for D_4 possesses a three-fold symmetry, and thus there is a triality between the fundamental vector and the two inequivalent spinor representations. (v) For the exceptional groups, we find that the smallest Casimir is unique, except for E_6. (vi) E_8 is the only group where the smallest Casimir is achieved for the adjoint representation

symmetry	range	min C_2	C_2(adj)	χ	irrep with smallest C_2
$\boldsymbol{SU(N)}$	$N \geq 2$	$\frac{N^2-1}{2N}$	N	$\boldsymbol{\frac{1}{2}\left(1-\frac{1}{N^2}\right)}$	fundamental $\boldsymbol{N}$ and $\overline{\boldsymbol{N}}$
$\boldsymbol{SO(N)}$	$3 \leq N \leq 7$	$\frac{1}{16}N(N-1)$	$N-2$	$\boldsymbol{\frac{N}{16}\frac{N-1}{N-2}}$	fundamental spinors $\mathbf{2^{\lceil N/2 \rceil -1}}$
	$N = 8$	$\frac{7}{2}$	6	$\boldsymbol{\frac{7}{12}}$	fundamental vector $\mathbf{8_v}$ and fundamental spinors $\mathbf{8_s}$, $\mathbf{8_c}$
	$N \geq 9$	$\frac{1}{2}(N-1)$	$N-2$	$\boldsymbol{\frac{N-1}{2(N-2)}}$	fundamental $\boldsymbol{N}$
$\boldsymbol{Sp(N)}$	$N \geq 1$	$\frac{1}{4}(2N+1)$	$N+1$	$\boldsymbol{\frac{2N+1}{4(N+1)}}$	fundamental $\boldsymbol{2N}$
$\boldsymbol{E_8}$		30	30	$\mathbf{1}$	adjoint **248**
$\boldsymbol{E_7}$		$\frac{57}{4}$	18	$\boldsymbol{\frac{19}{24}}$	fundamental **56**
$\boldsymbol{E_6}$		$\frac{26}{3}$	12	$\boldsymbol{\frac{13}{18}}$	fundamental **27** and $\overline{\mathbf{27}}$
$\boldsymbol{F_4}$		6	9	$\boldsymbol{\frac{2}{3}}$	fundamental **26**
$\boldsymbol{G_2}$		2	4	$\boldsymbol{\frac{1}{2}}$	fundamental **7**

Table 2. Summary of minimal Casimirs for the classical and exceptional Lie algebras along with the Casimir in the adjoint, their ratio χ, and the representations that attain the minimum. We notice that for D_4, corresponding to $SO(8)$, the Dynkin diagram has a three-fold symmetry leading to triality amongst the smallest Casimirs in the fundamental vector and spinor representations.

(which is also one of the fundamental representations). (*vii*) While the quadratic Casimir in general is a non-monotonic function of the dimensionality of the representation, our findings establish that the smallest Casimir always corresponds to those representations with the smallest dimension, which is always one of the fundamental representations.

Since the overall normalisation of quadratic Casimirs (II.6) can be chosen freely, it is useful to consider the ratio between the smallest quadratic Casimir and the Casimir in the adjoint,

$$\chi = \frac{\min\, C_2(R)}{C_2(\text{adj})}\,, \tag{II.13}$$

which is independent of the normalisation. Fig. **1** shows our results for χ for all simple Lie algebras. Evidently, χ is going to be bounded from above $\chi \leq 1$ because the adjoint representation always exists. The upper boundary is achieved for the exceptional group E_8. Furthermore, χ is also bounded from below,

$$\frac{3}{8} \leq \chi \leq 1\,. \tag{II.14}$$

The lower bound is achieved for the fundamental two-dimensional representation of $SU(2) \simeq SO(3) \simeq Sp(1)$, and for the two inequivalent two-dimensional representation of $SO(4)$. We observe that χ is an increasing function with N for $SU(N)$ and $Sp(N)$, interpolating between $\frac{3}{8}$ for small N and $\frac{1}{2}$ in the infinite-N limit. For $SO(N)$, we find

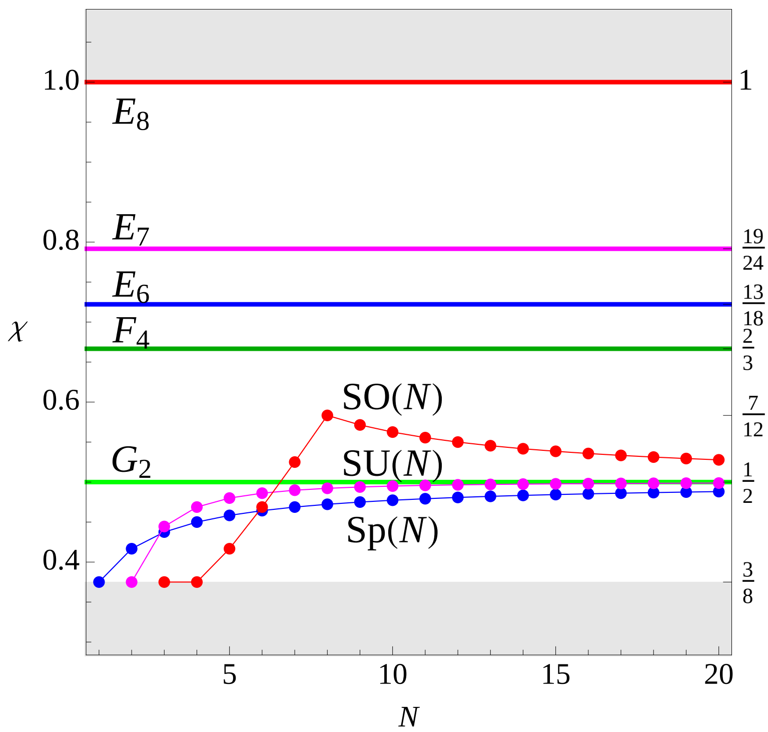

Figure 1. Shown is the ratio χ (II.13) – the smallest achievable quadratic Casimir in units of the Casimir in the adjoint – for all simple Lie algebras. The gray areas show the excluded domains. We observe that $\frac{3}{8} \leq \chi \leq 1$. The lower bound is achieved for the fundamental two-dimensional representation of $SU(2) \simeq SO(3) \simeq Sp(1)$, and for the two inequivalent two-dimensional representation of $SO(4)$. For the exceptional groups the smallest Casimir grows with the rank of the group. The upper bound is achieved for E_8. In all cases, the smallest quadratic Casimir is achieved for the irreducible representation of smallest dimensionality.

that χ grows from $\frac{3}{8}$ to its maximum $\frac{7}{12}$ at $N = 8$, from where it decays with increasing N towards $\frac{1}{2}$ from above. From the exceptional groups, only G_2 has a χ value close to those of the classical groups. All other exceptional groups have larger values for χ, which furthermore increases with the rank of the group.

4. We are now in a position to develop the central results of this work, summarised in Tab. **3** and Tab. **4**. We have observed in (II.5) that charged scalars potentially may turn the two loop coefficient C negative even if $B \leq 0$, provided that nontrivial scalar irreps are found with $C_2^S < \frac{1}{11} C_2^{\mathcal{G}}$. However, the result (II.13), (II.14) now firmly establishes that this is out of reach for any simple Lie algebra, owing to $C_2^S \geq \frac{3}{8} C_2^{\mathcal{G}}$. Moreover, we find that the two loop coefficient obeys

$$C \geq C_2^{\mathcal{G}} \left(\frac{89}{22} S_2^F + \frac{25}{22} S_2^S - \frac{34}{11} B \right) \tag{II.15}$$

for any non-abelian gauge theory. Hence, while it is possible to have B parametrically small such as in a Veneziano limit with suitably rescaled gauge coupling [116], the result (II.15) also shows that it is impossible to have both B and C parametrically small. Most importantly, we conclude that for any gauge theory with a vanishing or positive one loop coefficient for its gauge coupling's β function, the two loop coefficient is necessarily positive,

$$B \leq 0 \quad \Rightarrow \quad C > 0\,, \tag{II.16}$$

see (II.1). It is worth noting that (II.16) is not an equivalence: while $C < 0$ arises exclusively only if $B > 0$, the case $C > 0$ can arise irrespective of the sign of B [72, 110]. Consequently, Banks-Zaks fixed points are invariably IR fixed points. From the viewpoint of the asymptotic safety conjecture, our result (II.16) has the form of a no go theorem: within perturbation theory, irrespective of the matter content and in the absence of non gauge interactions, asymptotic safety cannot be realised for any four-dimensional simple non-abelian, or abelian, gauge theory.[22]

The result (II.16) straightforwardly generalises to matter fields in generic reducible representations under the gauge symmetry. In this case it suffices to replace terms involving Dynkin indices and matter Casimirs in the one and two loop coefficients by

$$S_2^R \to \sum_i S_2^{R_i}\,, \quad S_2^R\, C_2^R \to \sum_i S_2^{R_i}\, C_2^{R_i}\,, \tag{II.17}$$

where the sums run over the decomposition into irreducible representations of the fermionic ($R = F$) and scalar ($R = S$) matter fields. Applying (II.17) to the two loop coefficient (II.5), we find that all fermionic contributions remain manifestly positive definite, and that each summand of the scalar contributions is positive definite owing to (II.13), (II.14). We conclude that the no go theorem (II.16) holds true for general matter representations, as summarised in Tab. **3 b)**.

5. Turning to more general gauge interactions, we consider gauge theories with product gauge groups $\mathcal{G} \equiv \otimes_{a=1}^n \mathcal{G}_a$ and multiple gauge couplings α_a, each associated with a simple or abelian factor $\mathcal{G}_a$. We assume the presence of scalar and/or fermionic matter fields, some or all of which are charged under some or all of the gauge symmetries. In the absence of Yukawa interactions, the β functions for the gauge couplings up to two loops in perturbation theory are of the form

$$\beta_a = \alpha_a^2\,(-B_a + C_{ab}\,\alpha_b) + \mathcal{O}(\alpha^4)\,, \tag{II.18}$$

and $a, b = 1, \cdots, n$. The coefficients B_a and C_{aa} (no sum) are the standard one and two loop coefficients of the gauge coupling α_a as given in (II.2), (II.3). The new terms at two loop level are the off-diagonal contributions C_{ab} ($a \neq b$) which parametrise the $\mathcal{O}(\alpha_b)$ contributions to the renormalisation group flow of couplings α_a. Nontrivial mixing between two gauge couplings arises through matter fields which are charged under both

[22] Caswell has observed some time back that *"We do not expect to find a gauge theory of the above type* [meaning with (II.1)] *where β starts out positive and goes negative near enough to the origin for the zero to be valid in perturbation theory."* [72]. Our result (II.16) offers a general proof for Caswell's conjecture.

of these. The mixing terms can then be written as [117, 118]

$$C_{ab} = 4\left(C_2^{F_b} S_2^{F_a} + C_2^{S_b} S_2^{S_a}\right) \quad (a \neq b)\,. \tag{II.19}$$

The subscripts a, b on the Casimir or Dynkin index of the matter fields indicate the subgroup of $\mathcal{G}$. From (II.19) it follows that the mixing terms are manifestly non-negative ($C_{ab} \geq 0$) for any semi simple quantum gauge theory with or without abelian factors. The expression (II.19) has a straightforward generalisation for reducible representations. Furthermore, if the theory contains more than one abelian factor, the off-diagonal contributions take a slightly different form in the presence of kinetic mixing [119, 120]. In either of these cases, the mixing terms remain manifestly non-negative ($C_{ab} \geq 0$, $a \neq b$). Together with (II.16) for all diagonal entries, we find that

$$B_a \leq 0 \quad \Rightarrow \quad C_{ab} \geq 0 \quad \text{for all } \ b\,, \tag{II.20}$$

meaning that for every infrared free gauge group factor $\mathcal{G}_a$, the corresponding column of the two loop gauge contribution matrix (C_{ab}) is non-negative.

The result (II.20) has immediate implications for interacting fixed points of quantum field theories with (II.18), which, to leading order in perturbation theory, are given by all solutions of the linear equations

$$B_a = C_{ab}\,\alpha_b^*\,, \quad \text{subject to} \quad \alpha_b^* \geq 0\,. \tag{II.21}$$

Assuming that $B_a \leq 0$ for at least one of the subgroups $\mathcal{G}_a$, it follows from (II.20) that for (II.21) to have a solution, at least one of the fixed points α_b^* must take negative values. However, we have already explained that such solutions are inconsistent [113], and conclude that the theory cannot have physically acceptable interacting fixed points within the perturbative regime as soon as any of the gauge factors is infrared free ($B_a \leq 0$). In other words, the result (II.20) has the form of a no go theorem: asymptotic safety cannot be achieved for any semi-simple quantum gauge theory of the type (II.18) with or without abelian factors and irrespective of the matter content.

Reversing the line of reasoning, our findings also establish that physically-acceptable interacting fixed points in gauge theories with (II.18) and without Yukawa interactions can only be achieved if all gauge group factors are asymptotically free ($B_a > 0$), which excludes $U(1)$ factors straightaway, see Tab. **4 b)**. All weakly interacting fixed point solutions of (II.21) are necessarily IR fixed points of the Banks-Zaks type inasmuch as they arise from balancing one and two loop gauge field fluctuations. They also display a lesser number of relevant directions than the asymptotically free Gaussian UV fixed point meaning that UV-IR connecting trajectories exist which flow from the Gaussian down to any of the interacting fixed points.

Next, we investigate scalar and Yukawa-type matter couplings, and clarify whether these may help to generate weakly interacting fixed points.

6. Scalar self-interactions arise unavoidably in settings with charged scalars owing to the fluctuations of the gauge fields or in settings with uncharged scalars as long as these couple indirectly to the gauge fields through charged fermions and Yukawa interactions. Quartic scalar self interactions or cubic ones in a phase with spontaneous symmetry breaking renormalise the gauge couplings starting at the three loop (four loop) level in perturbation theory, provided the scalars are charged (uncharged) [121].

In the light of (II.16), to help generate an interacting fixed point in the gauge sector once $B \leq 0$, the scalar couplings would have to outweigh the one loop as well as the two loop gauge contributions. Even if the one loop term vanishes identically ($B = 0$), the result (II.14) together with (II.2), (II.5) and (II.15) establishes that the two loop gauge coefficient is strictly positive $C(B = 0) \geq C_{\min}$ and of order unity, with

$$C_{\min}/(C^{\mathcal{G}})^2 = 22\tfrac{1}{4}\,. \tag{II.22}$$

The absolute minimum (II.22) is achieved for $Sp(1), SU(2), SO(3)$ and $SO(4)$ gauge symmetries. The bound becomes slightly stronger with increasing N, reaching $C_{\min}/(C^{\mathcal{G}})^2 = 25$ for the classical Lie groups in the infinite N limit. For the exceptional groups G_2, F_4, E_6, E_7 and E_8 we find the increasingly stronger bounds $C_{\min}/(C^{\mathcal{G}})^2 = 25, 50\tfrac{2}{3}, 55\tfrac{5}{9}, 61\tfrac{2}{3}$ and 80, respectively. Notice also that for all gauge groups the minimum is achieved for charged fermions only. The presence of charged scalars systematically enhances $C > C_{\min}$. Thus, coming back to the scalar self interactions, even in the most favourable scenario where the one-loop coefficient vanishes and the gauge coupling is perturbatively small, a cancellation between the two loop gauge and the three or four loop scalar contributions requires scalar couplings of order unity owing to the lower bounds (II.15), (II.22).[23] Hence, the feasibility of such a scenario necessitates non-perturbatively large scalar couplings, outside the perturbative domain. We conclude that non-abelian gauge theories with any type of self interacting scalar matter, and with or without fermionic matter but without Yukawa interactions, cannot become asymptotically safe within perturbation theory. This result also completes the no go theorems stated in Tab. **3 b)** and **c)** in the presence of scalar matter.

7. Yukawa couplings are naturally present in settings with both scalar and fermionic matter fields [122], and contribute to the running of (some of) the gauge couplings provided that (some of) the fermions carry charges under (some of) the gauge groups. Scalars may or may not carry charges. Yukawa couplings are technically natural [123] and cannot be switched-on by fluctuations: the limit of vanishing Yukawa couplings constitutes an exact fixed point of the theory.

For concreteness we consider simple non-abelian or abelian gauge theories with the most general Yukawa interactions taking the form $\sim \frac{1}{2}(\mathbf{Y}^A)_{JL}\phi^A\,\psi_J\,\zeta\,\psi_L$ with $\zeta = \pm i\sigma_2$,

[23] For this estimate we have assumed that the relevant loop factor $(4\pi)^2$ is scaled into the definition of the scalar self-coupling, consistent with our conventions for the gauge and Yukawa couplings.

with Weyl indices suppressed. In perturbation theory the Yukawa couplings $\mathbf{Y}^A$ contribute to the renormalisation of the gauge coupling starting at the two loop level, and the beta function (II.1) is replaced by [74]

$$\beta = \alpha^2 \left(-B + C\,\alpha - 2\,Y_4\right) . \tag{II.23}$$

The Yukawa couplings enter through the new term $Y_4 = \mathrm{Tr}[\mathbf{C}_2^F\,\mathbf{Y}^A\,(\mathbf{Y}^A)^\dagger]/d(G)$, with $d(G)$ the dimension of the gauge group, $\mathbf{Y}^A$ the (matrix of) Yukawa couplings, $\mathbf{C}_2^F$ the matrix of quadratic Casimirs of the fermionic irreps, and the trace summing over all fermionic indices. Notice that we have scaled the loop factor of (4π) into the definition of $\mathbf{Y}^A$. The coefficients B and C are as in (II.2) and (II.3). In general, the matrix $\mathbf{C}_2^F$ is diagonal according to the fermionic irreps, implying that Y_4 is positive as long as (some of) the Yukawa couplings are non-vanishing. Positivity of Y_4 can be made manifest by rewriting it as

$$Y_4 = \sum_{AJL} S_2^{F_J}\,\left|(\mathbf{Y}^A)_{JL}\right|^2/d(F_J) \geq 0\,. \tag{II.24}$$

It follows that Yukawa couplings contribute with an overall negative sign to the running of gauge couplings, irrespective of the sign of the one loop gauge coefficient B. Assuming that the Yukawa couplings, and thus Y_4, take a fixed point of their own, interacting fixed points of (II.23) take the form (II.4) except that the one loop coefficient is effectively shifted $B \to B' = B + 2\,Y_4^*$, with

$$B' \geq B\,. \tag{II.25}$$

This Yukawa-induced shift has important implications. Most notably, in settings where the gauge sector is asymptotically non free ($B \leq 0$), the Yukawa contribution Y_4^* may effectively change the sign of the one loop coefficient ($B' > 0$), thereby enabling a viable interacting fixed point

$$\alpha_* = \frac{B'}{C}\,. \tag{II.26}$$

In more physical terms, for infrared free theories these findings state that the growth of the gauge coupling with energy, as dictated by the positive one and two loop gauge contributions (II.16), is invariably slowed down, and, as long as $B' > 0$, eventually brought to a halt by Yukawa interactions. In particular, the occurrence of a UV Landau pole in the gauge coupling can be avoided dynamically. As we have shown earlier, neither scalar self interactions nor further gauge couplings are able to negotiate a fixed point at weak coupling once $B \leq 0$. We therefore conclude that Yukawa interactions are the *only* type of interactions that can generate an interacting UV fixed point for *any* weakly coupled gauge theory.

In view of the above it is useful to investigate the Yukawa sector in more detail. To that end, we exploit the explicit flow for the Yukawa couplings $\boldsymbol{\beta}^A = d\mathbf{Y}^A/d\ln\mu$. At the leading non-trivial order in perturbation theory which is one loop, it takes the form [18, 124]

$$\boldsymbol{\beta}^A = \mathbf{E}^A(Y) - \alpha\,\mathbf{F}^A(Y)\,. \tag{II.27}$$

The terms $\mathbf{E}^A(Y)$, which are of cubic order in the Yukawa couplings, arise from fluctuations of the fermion and scalar fields and encode vertex and propagator corrections [124]. General expressions for $\mathbf{E}^A$ in the conventions adopted here are given in [75, 76]. The terms $\mathbf{F}^A(Y) = 3\{\mathbf{C}_2^F, \boldsymbol{Y}^A\}$ originate primarily from gauge field fluctuations and are (block-)diagonally proportional to $\boldsymbol{Y}^A$ following the fermion irreps [18]. Scalar self couplings contribute to (II.27) starting at two loop and can be neglected for sufficiently small couplings.

The nullcline condition $\boldsymbol{\beta}^A(Y,\alpha) = 0$ for the Yukawa couplings has two types of solutions. The Gaussian fixed point $\mathbf{Y}^A_* = 0$ always exists, because both $\mathbf{E}^A$ and $\mathbf{F}^A$ vanish individually for vanishing Yukawa couplings, whence $\boldsymbol{\beta}^A(Y=0,\alpha) = 0$. In addition, and provided that the gauge coupling is non-vanishing, the two terms in (II.27) can balance against each other. Dimensional analysis shows that the functions $\bar{\boldsymbol{\beta}}^A(C) \equiv \boldsymbol{\beta}^A(\sqrt{\alpha}\,C,\alpha)/\alpha^{3/2}$ are independent of the gauge coupling α, implying that Yukawa nullclines take the form

$$\mathbf{Y}^A_* = \frac{g}{4\pi}\,\mathbf{C}^A\,. \tag{II.28}$$

The "reduced" Yukawa couplings $\mathbf{C}^A$ are numerical matrices independent of the gauge coupling g which solve $\bar{\boldsymbol{\beta}}^A(C) = 0$, meaning $\boldsymbol{E}^A(C) = \mathbf{F}^A(C)$ for $\mathbf{C}^A \neq 0$. Evidently $\mathbf{C}^A = 0$ corresponds to the Gaussian.[24] The solutions (II.28) are promoted to genuine fixed points of the coupled system (II.23), (II.27) iff the gauge coupling simultaneously takes a real fixed point g_* (II.26). At the fixed point, perturbativity in the Yukawa couplings then follows parametrically from perturbativity in the gauge coupling.

Inserting the nullcline back into (II.23) we find that the Yukawa-induced terms are of order α^3 owing to (II.28). This establishes that the shifted one loop coefficient B' depends linearly on α through Y_4^*, meaning that (II.26) constitutes an implicit equation for α_*. The implicit dependences are resolved by accounting for the Yukawa contributions as, effectively, modifications of the two loop coefficient. We find

$$Y_4 = D \cdot \alpha \tag{II.29}$$

where the coefficient $D = \mathrm{Tr}[\mathbf{C}_2^F\,\mathbf{C}^A\,(\mathbf{C}^A)^\dagger]/d(G) \geq 0$ only depends on group theoretical

[24] For any nullcline $\mathbf{C}^A$ (II.28), $-\mathbf{C}^A$ and $\mathbf{C}^{A\dagger} = \mathbf{C}^{A\,*}$ are physically equivalent nullclines. In the literature one-loop nullclines are sometimes referred to as "fixed points" (for the reduced couplings) or "eigenvalue conditions" [125].

case	gauge group	matter	**Yukawa**	**asymptotic safety**	info
a)	simple	fermions in irreps	**No**	**No**	Ref. [72]
b)	simple or abelian	fermions, any rep	**No**	**No**	$(II.16)$
		scalars, any rep	**No**	**No**	$(II.16), (II.22)$
		fermions and scalars, any rep	**No**	**No**	$(II.16), (II.22)$
c)	semi-simple, with or without abelian factors	fermions, any rep	**No**	**No**	$(II.20)$
		scalars, any rep	**No**	**No**	$(II.20), (II.22)$
		fermions and scalars, any rep	**No**	**No**	$(II.20), (II.22)$
d)	simple or abelian	fermions and scalars, any rep	**Yes**	**Yes**	$(II.31), (II.38)$
e)	semi-simple, with or without abelian factors	fermions and scalars, any rep	**Yes**	**Yes**	$(II.34), (II.38)$

Table 3. Asymptotic safety in gauge theories coupled to matter with **a)** – **c)** stating strict no go theorems and **d)** – **e)** necessary and sufficient conditions.

weights and the reduced Yukawa couplings parametrising the nullcline, but not on the gauge coupling. The projection of the flow for the gauge coupling (II.23) along a hypersurface with $\boldsymbol{\beta}^A = 0$ then takes the form (II.1) except that the two loop gauge coefficient C is shifted into $C \to C' = C - 2\,D$. The shift term vanishes iff all Yukawa couplings vanish but is strictly negative otherwise, whence

$$C' \leq C\,. \tag{II.30}$$

This result makes it manifest that Yukawa contributions can dynamically lower the effective two loop coefficient, possibly avoiding the no go theorem (II.16). Furthermore, the shift (II.30) implies that interacting fixed points for the gauge coupling take the form (II.4) with $C \to C'$,

$$\alpha_* = \frac{B}{C'}\,. \tag{II.31}$$

We stress that the expressions (II.26) and (II.31) for the gauge coupling fixed point are equivalent and numerically identical. For practical purposes, however, the latter representation, if available, is preferred as it provides the fully resolved version of the former. Following on from our earlier discussion, the fixed points (II.31) are physical as long as $B \cdot C' > 0$, and perturbative if $|B| \ll |C'|$. If $B > 0$ and $C' > 0$, they constitute infrared fixed points of the theory, similar to Banks-Zaks fixed points except for the additional presence of Yukawa interactions. If $B < 0$ and $C' < 0$, they constitute interacting UV fixed points and qualify as asymptotically safe UV completions for the theory, see Tab. **4 c)** for a summary. No such weakly coupled UV completion can arise without Yukawa interactions.

We conclude that Yukawa couplings offer a dynamical mechanism to negotiate interacting fixed points in gauge theories. Most importantly, for asymptotically non-free gauge theories with $B \le 0$, they offer a *unique* mechanism to generate weakly interacting fixed points. The strict no go theorem (II.16) may then be circumnavigated under the auxiliary condition that the Yukawa-induced shift term comes out large enough for C' to turn negative. This result, summarised in Tab. **3‘d)**, thus takes the form of a necessary condition for asymptotic safety.

8. Our results are straightforwardly generalised to gauge-Yukawa theories with several abelian or non-abelian gauge group factors, assuming that some or all of the fermions are charged under some or all of the gauge groups, while the scalars may or may not be charged. The renormalisation of the gauge couplings then takes the form [74]

$$\beta_a = \alpha_a^2 \left(-B_a + C_{ab}\,\alpha_b - 2\,Y_{4,a}\right)\,, \tag{II.32}$$

where the two loop Yukawa contributions now arise through $Y_{4,a} = \mathrm{Tr}[\mathbf{C}_2^{F_a}\,\mathbf{Y}^A\,(\mathbf{Y}^A)^\dagger]/d(G_a) \geq 0$. As is evident from the explicit expression, the quadratic Casimir of the fermions takes the role of a projector to identify the contributions to the running of α_a. The running of the Yukawa couplings continues to be given by (II.27), except that further gauge field contributions turn the last term into a sum over gauge groups $\alpha\,\mathbf{F}^A \to \alpha_a\,\mathbf{F}_a^A$ with $\mathbf{F}_a^A(Y) = 3\{\mathbf{C}_2^{F_a}, \boldsymbol{Y}^A\}$ [75]. This modification leads to a larger variety of Yukawa nullclines, depending on which of the gauge couplings take vanishing or non-vanishing values at the fixed point. Provided that some or all of the Yukawa couplings take interacting fixed points they will contribute to the running of the gauge couplings (II.32) through $Y_{4,a}^* \geq 0$. Consequently, the gauge beta functions reduce to the form (II.18) except that the one loop coefficients are effectively shifted, $B_a \to B'_a = B_a + 2\,Y_{4,a}^*$, due to the fixed point in the Yukawa sector. Most importantly, we observe that

$$B'_a \geq B_a\,. \tag{II.33}$$

Equality holds true iff all Yukawa couplings take Gaussian values. The shift (II.33) implies that gauge coupling fixed points of the theory arise as the solutions of

$$B'_a = C_{ab}\,\alpha_b^*\,,\quad \text{subject to}\quad \alpha_b^* \geq 0\,. \tag{II.34}$$

Once more, this structure has important implications. Following on from our earlier discussion of (II.21), the fixed point condition (II.34) can have physical solutions iff all B'_a are positive. Due to (II.33) this is naturally the case as long as each gauge group factor is asymptotically free. The theory is then asymptotically free in all gauge factors with interacting fixed points of the Banks-Zaks and the gauge-Yukawa type, and combinations and products thereof. The decisive difference with (II.21) comes into its own for theories where some or all B_a are negative. Provided that the Yukawa-induced shift terms ensure that all B'_a become positive numbers even if one or several of the gauge factors are not asymptotically free, the fixed point condition (II.34) can have a variety of novel solutions, see Tab. **4 d)**. Such fixed points are genuinely of the gauge-Yukawa type, and furthermore constitute candidates for asymptotically safe UV completions of the theory. Also, no such fixed point can arise out of theories with (II.21), which once more highlights the pivotal role played by Yukawa interactions.

As a final remark, we note that the fixed point condition (II.34) still depends implicitly on the gauge couplings through B'_a, once Y_4 is evaluated on a nullcline. It is straightforward to resolve the implicit dependence provided that $Y_{4,a}$ takes the form

$$Y_{4,a} = D_{ab}\,\alpha_b \tag{II.35}$$

along Yukawa nullclines, in analogy to (II.29).[25] Continuity in each of the gauge couplings $\alpha_b \geq 0$ together with the non-negativity of $Y_{4,a}$ allows us to observe that the matrix (D_{ab}) is non-negative. The flow of the gauge couplings (II.32) is reduced to (II.18), except that the two loop term is shifted $C_{ab} \to C'_{ab} = C_{ab} - 2\,D_{ab}$ following (II.35). We conclude that the Yukawa contributions along nullclines effectively reduce the two loop gauge contributions to the renormalisation of gauge couplings. In this representation, the fixed point condition (II.34) turns into the equivalent form

$$B_a = C'_{ab}\,\alpha^*_b\,, \quad \text{subject to} \quad \alpha^*_b \geq 0\,. \tag{II.36}$$

For non-negative C'_{ab}, as has been shown above, interacting fixed points can only be realised if all gauge group factors are asymptotically free. Here, however, the matrix (C'_{ab}) is no longer required to be strictly non-negative, unlike the matrix (C_{ab}) of two loop gauge contributions, and the no go theorem (II.20) can be avoided owing to the Yukawa contributions. In view of the asymptotic safety conjecture, this completes our proof that charged fermions with charged or uncharged scalars and, most crucially, Yukawa interactions, constitute strictly necessary ingredients for interacting UV fixed points in general weakly coupled gauge theories, see Tab. **3 e)**.

9. Gauge-Yukawa fixed points necessitate scalar fields. Consequently, two auxiliary conditions arise: Firstly, the scalar sector must achieve a fixed point of its own, interacting or otherwise. Secondly, the scalar sector must admit a stable ground state. To appreciate that both of these requirements are non-empty, we consider the renormalisation group flow $\boldsymbol{\beta} = d\boldsymbol{\lambda}/d\ln\mu$ for the quartic scalar couplings $\boldsymbol{\lambda} = (\lambda_{ABCD})$ based on the interaction Lagrangean $\sim \frac{1}{4!}\lambda_{ABCD}\,\phi^A\phi^B\phi^C\phi^D$. To leading order the beta functions $\boldsymbol{\beta} = \boldsymbol{\beta}(\lambda, Y, \alpha)$ depend quadratically on the quartics, on the Yukawa and gauge couplings, and on group theoretical factors related to the gauge transformations of the scalars (if charged) [18]. Explicit expressions and generalisations for product gauge groups can be found in [76,

[25] The form (II.35) is evident if only one of the gauge couplings, say g_b, is non-vanishing. The nullcline takes the form $\mathbf{Y}^A_{b,*} = \frac{g_b}{4\pi}\,\mathbf{C}^A_b$, see (II.28), with $\boldsymbol{C}^A_b$ a solution of $\mathbf{E}^A(C) = \mathbf{F}^A_b(C)$, leading to $D_{ab} = \mathrm{Tr}[\mathbf{C}_2^{F_a}\,\mathbf{C}^A_b\,(\mathbf{C}^A_b)^\dagger]/d(G_a) \geq 0$. More generally, (II.35) holds true for any quantum field theory whose one loop Yukawa vertex corrections obey $\mathbf{Y}^B\mathbf{Y}^{\dagger A}\mathbf{Y}^B = \mathbf{Y}^A\,\mathrm{Tr}\,\mathbf{M}^{BC}(\mathbf{Y}^{\dagger B}\mathbf{Y}^C + \mathbf{Y}^{\dagger C}\mathbf{Y}^B)$ for some matrix $(\mathbf{M}^{BC})_{JL} = m^B_J\,\delta^{BC}\,\delta_{JL}$ which is block-diagonally proportional to the identity in field space with real m^B_J. In these cases the flow for the Yukawa couplings (II.27) are mapped explicitly onto closed flows for their squares $|(\mathbf{Y}^A)_{JK}|^2$ whose nullclines, and consequently $Y_{4,a}$ on nullclines, are linear functions of the squares of the gauge couplings, α_b. In theories with more complex Yukawa vertex corrections (*e.g.* Pati-Salam, trinification) the relation between $Y_{a,4}$ and α_b takes a more general form.

case	gauge group	Yukawa	parameter	interacting FPs	type	info
a)	simple	**No**	$B > 0$ and $C > 0$	Banks-Zaks	**IR**	Refs. [72, 110]
b)	semi-simple, no $U(1)$ factors	**No**	all $B_a > 0$	Banks-Zaks and products thereof	**IR**	soln of (II.21)
c)	simple	**Yes**	$B > 0$ and $C > 0 > C'$	Banks-Zaks	**IR**	Fig. **3**
	simple	**Yes**	$B > 0$ and $C > C' > 0$	BZ and GYs	**IR**	Fig. **4**
	simple or abelian	**Yes**	$B < 0$ and $C' < 0$	gauge-Yukawas	**UV/IR**	Fig. **5**
d)	semi-simple, with or without $U(1)$ factors	**Yes**	all $B'_a > 0$	BZs and GYs and products thereof	**UV/IR**	soln of (II.34)

Table 4. Summary of weakly interacting fixed points in gauge theories, detailing the availability of Banks-Zaks (BZ) or gauge-Yukawa (GY) type fixed points, or combinations and products thereof.

77]. Scalar self couplings are not technically natural [123] and can be switched-on by fluctuations of the fermions (due to the presence of Yukawa couplings) or by fluctuations of the gauge fields (if the scalars are charged), implying that $\boldsymbol{\beta}(\lambda = 0, Y, \alpha) \neq 0$ in general.

Next we turn to the scalar nullclines $\boldsymbol{\beta} = 0$, subject to $\boldsymbol{\beta}^A \to 0$. Using dimensional analysis, we observe that the functions $\bar{\boldsymbol{\beta}}(\bar{C}, C) \equiv \boldsymbol{\beta}(\alpha\,\bar{C}, \alpha\,C, \alpha)/\alpha^2$ are α-independent. The implicit solutions $\bar{\boldsymbol{C}}$ of the quadratic algebraic equations $\bar{\boldsymbol{\beta}}(\bar{C}(C), C) = 0$ provide us with

$$\boldsymbol{\lambda}_* = \alpha\,\bar{\boldsymbol{C}}\,. \tag{II.37}$$

The "reduced" scalar couplings $\bar{\boldsymbol{C}}$ are numerical tensors which depend on group theoretical factors and the reduced Yukawa couplings, but not explicitly on the gauge coupling. Since the quartics do not impact on the gauge-Yukawa flow (to leading order) it is immaterial for this analysis whether the gauge coupling is slowly running or sitting on a fixed point.

Qualitatively and quantitatively different types of solutions $\boldsymbol{\lambda}_*$ arise for all physically inequivalent Yukawa nullclines with $\boldsymbol{C}^A \neq 0$, and with $\boldsymbol{C}^A \to 0$. In either of these cases, owing to the quadratic nature of the defining equations, solutions (II.37) generically come up in inequivalent pairs $\bar{\boldsymbol{C}}_\pm$ per Yukawa nullcline with complex entries. Reality of quartic couplings is not automatically guaranteed and must be required as an auxiliary condition. Vacuum stability necessitates that $\boldsymbol{\lambda}_*$ is a positive-definite tensor.[26] This information is not encoded in the renormalisation group flow even if the scalar couplings come out real, meaning that the stability of the effective potential $V_{\rm eff}(\phi)$ provides an independent constraint. We therefore conclude that (II.37), subject to

$$\lambda^*_{ABCD} = \text{real}\,, \quad \text{and} \quad V_{\rm eff}(\phi) = \text{stable}\,, \tag{II.38}$$

[26] In the presence of flat directions, Coleman-Weinberg type resummations [126] for the leading logarithmic corrections of the effective potential will have to be invoked [70].

are mandatory auxiliary conditions for gauge theories with scalar matter to display a physically acceptable scalar sector, in addition to the conditions for free or interacting fixed points in the gauge or gauge-Yukawa sectors.

A few comments are in order: (i) Solutions of (II.38) with $\boldsymbol{C}^A \neq 0$ are mandatory for gauge-Yukawa fixed points and for asymptotic safety [55, 58]. Those with $\boldsymbol{C}^A = 0$ are mandatory for Banks-Zaks fixed points in the presence of scalar matter. (ii) Both of (II.38) must be imposed irrespective of the UV or IR nature of the underlying fixed point. (iii) If two solutions $\bar{\boldsymbol{C}}_\pm$ are physical, one of them is UV and the other IR relevant. (iv) Solutions to (II.37), (II.38) also control trajectories in the vicinity of free or interacting fixed points [125]. Those with $\boldsymbol{C}^A \neq 0$ entail that gauge, Yukawa, and scalar couplings run at the same rate and govern the approach to gauge-Yukawa fixed points. Those with $\boldsymbol{C}^A \to 0$ (referring to reduced Yukawa couplings which approach the Gaussian very rapidly $\boldsymbol{Y}^A(\alpha)/\sqrt{\alpha} \equiv \boldsymbol{C}^A(\alpha) \ll 1$) are relevant for asymptotically free theories to display complete asymptotic freedom, and for trajectories approaching Banks-Zaks fixed points. Scalar couplings then run into the Gaussian UV fixed point either alongside the gauge coupling, or faster $\boldsymbol{\lambda}_*(\alpha)/\alpha \ll 1$. The latter follows from the α-dependence of the reduced Yukawa couplings $\boldsymbol{C}^A(\alpha)$ which entails an implicit α-dependence for the quartics [18]. (iv) A method to find solutions in the limit $\boldsymbol{C}^A \to 0$ has been detailed in [127]. Physical solutions for the combined Yukawa and scalar nullclines with (II.38) exist and are known for a number of theories [128–131].[27]

This completes the derivation of necessary and sufficient conditions of existence for weakly interacting fixed points in general gauge theories coupled to matter.

10. Next, we return to the starting point of our investigation where we observed that the competition between gauge field and matter fluctuations, and hence the relative signs and size of the loop coefficient B and C (for theories with a simple gauge group) determines the fixed point structure. However, it has become clear that a third quantity, C', controlled by Yukawa interactions, plays an equally important role. To illustrate its impact, we turn to a brief discussion of weakly coupled gauge theories from the viewpoint of their phase diagrams. Four distinct cases arise: Besides the Gaussian fixed point, gauge theories either display none, the Banks-Zaks, gauge-Yukawa, or the Banks-Zaks and gauge-Yukawa fixed points, depending on the values for B, C, and C', see Tab. **4 c)**. The different phase diagrams are shown qualitatively in Figs. **2**–**5**, projected onto the (α, Y_4) plane.

Gauge theories with $B > 0$ and $C < 0$ have no weakly coupled fixed points. At weak coupling, the phase diagram solely displays asymptotic freedom and the Gaussian UV fixed point, Fig. **2**. The set of UV free trajectories emanating out of it are indicated by the red shaded area. Its upper boundary is provided by the Yukawa nullcline which also acts as an infrared attractor [132–136] due to the fact that the sign of (II.27) is always controlled by the gauge field fluctuations for small Yukawa couplings. On the scaling trajectory,

[27] See [55, 58] and [19, 20] for recent results in the context of asymptotic safety and asymptotic freedom, respectively.

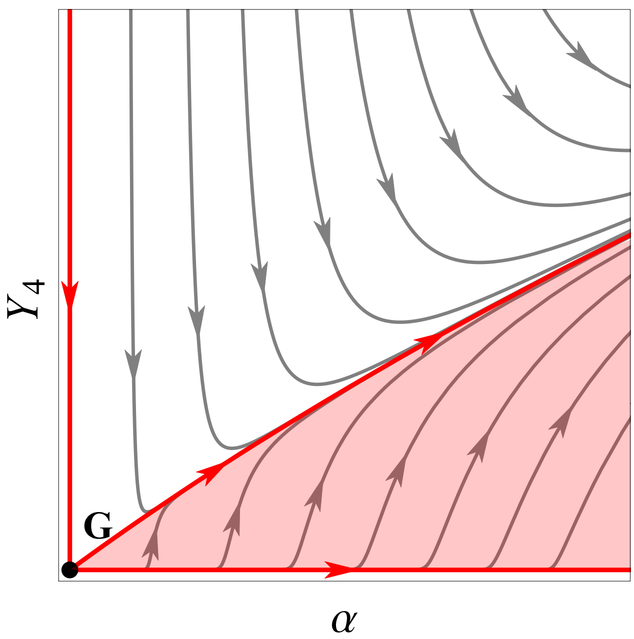

Figure 2. Phase diagram of gauge-Yukawa theories with $B > 0$ and $C < 0$ at weak coupling showing asymptotic freedom and the Gaussian UV fixed point (G). Arrows indicate the flow towards the IR. The red-shaded area covers the set of UV complete trajectories emanating from the Gaussian UV fixed point. The Yukawa nullcline acts on trajectories as an IR attractor.

the gauge, Yukawa and scalar couplings run at the same rate into the Gaussian UV fixed point [125]. UV free trajectories continue towards the domain of strong coupling where the theory is expected to display confinement and chiral symmetry breaking, or, possibly, a strongly coupled IR fixed point. On the other hand, above the Yukawa nullcline no trajectories are found which can reach the Gaussian in the UV. On such trajectories, the theory technically loses asymptotic freedom. Predictivity is then limited up to a finite UV scale, unless a strongly coupled UV fixed point materialises out of the blue.

Gauge theories with $B > 0$ and $C > 0 > C'$ additionally develop a Banks-Zaks fixed point (II.4) which is perturbative provided B/C is sufficiently small. Yukawa couplings are immaterial for this. Banks-Zaks fixed points are always weakly attractive in the gauge and strongly repulsive in the Yukawa direction. The former follows from asymptotic freedom together with (II.23), while the latter follows from (II.27) and $\partial \boldsymbol{F}^A/\partial \mathbf{Y}^B$ being non-negative and proportional to the gauge coupling times the sum of the quadratic Casimirs of the fermions attached to the vertex. Moreover, at weak coupling and close to the Banks-Zaks, the flow is always parametrically faster into the Y_4 than into the gauge direction. Consequently, the Bank-Zaks fixed point together with the Yukawa nullcline act as a strong infrared-attractive funnel for all trajectories emanating from the Gaussian UV fixed point, see Fig. **3**. This leads to low energy relations between the Yukawa and the gauge coupling dictated by (II.27) (at weak coupling), irrespective of their detailed UV origin.[28] Elsewise

[28] Exact examples are given by the gauge-Yukawa theories of [55] in the parameter range $0 < 11/2 - N_F/N_C \ll 1$.

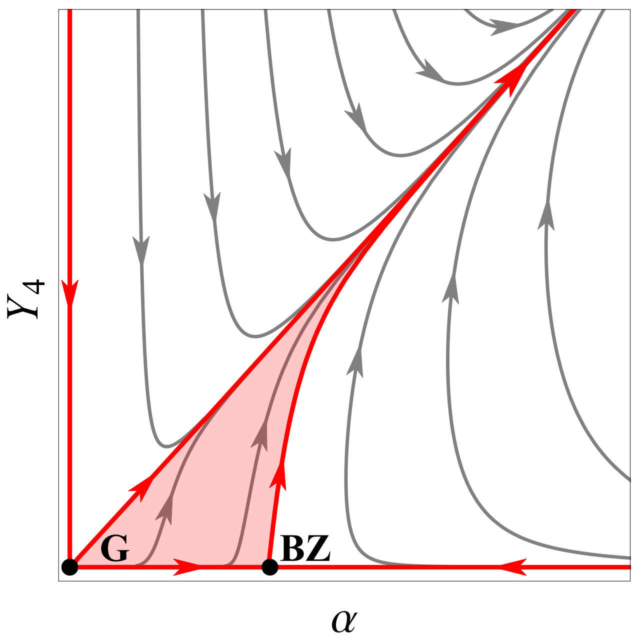

Figure 3. Phase diagram of gauge-Yukawa theories with $B > 0$ and $C > 0 > C'$ at weak coupling showing asymptotic freedom with the Gaussian and the Banks-Zaks fixed point (BZ). Notice the funnelling of all UV free trajectories towards the Yukawa nullcline as furthered by the Banks-Zaks fixed point.

the same discussion as in the previous example applies.

Progressing towards gauge theories with $B > 0$ and $C > C' > 0$ we now additionally observe a fully interacting gauge-Yukawa fixed point besides the Banks-Zaks, displayed in Fig. **4**. The main new effect in theories with $C' > 0$ as opposed to those with $C' < 0$ is that the funnelling of flow trajectories towards the IR attractive Yukawa nullcline comes to a halt, whereby couplings take an interacting IR fixed point (II.28), (II.31). Furthermore, the fixed point is genuinely attractive in both the gauge and the Yukawa directions.[29] The theory comes out more strongly coupled at the gauge-Yukawa than at the Banks-Zaks fixed point owing to (II.30). The gauge-Yukawa fixed point characterises a second order phase transition between a symmetric phase and a phase with spontaneous symmetry breaking where the scalars acquire a non-vanishing vacuum expectation value. Details of the phase transition becomes visible once mass terms are added, taking the role of temperature, with the scalar vacuum expectation values serving as order parameters. Spontaneous symmetry breaking may also entail the breaking of chiral symmetry via Yukawa couplings. Away from fixed points, the theory may display a number of further phenomena such as first order phase transitions, dimensional transmutation, decoupling, and confinement in the deep IR.[30]

Turning to simple or abelian gauge theories with $B < 0$ and $C' < 0$ we observe that asymptotic freedom is absent and the Gaussian has become an infrared fixed point. Also,

[29] In theories with several Yukawa couplings several gauge-Yukawa fixed point may arise of which at least one is fully IR attractive. See [137] for an explicit example with a single Yukawa coupling.

[30] Phenomenological aspects of IR gauge-Yukawa fixed points have been pioneered in [137, 138] (see also [139, 140]). Models with gauge-Yukawa fixed points have also been studied from the viewpoint of conformal field theory [141] and the a theorem [54].

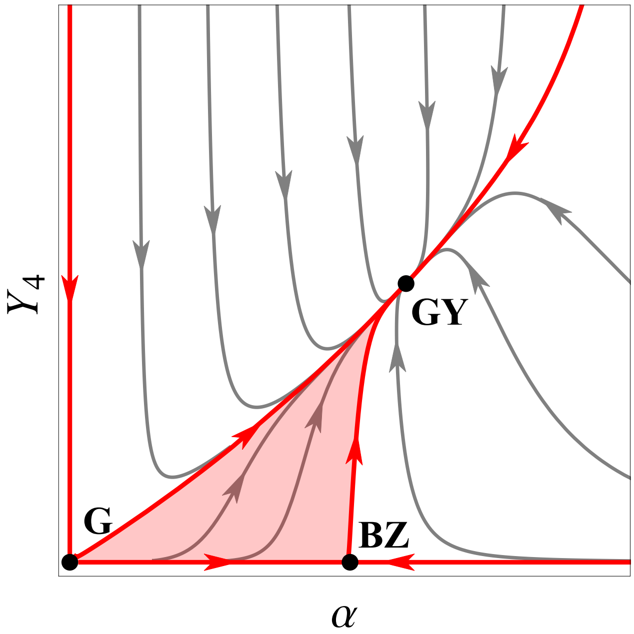

Figure 4. Fixed points and phase diagrams of gauge-Yukawa theories with $B > 0$ and $C > C' > 0$ at weak coupling showing asymptotic freedom with Gaussian, Banks-Zaks, and gauge-Yukawa fixed points (GY). Notice that the gauge-Yukawa fixed point attracts UV free trajectories emanating from the Gaussian.

it is impossible for this type of theories to have a Banks-Zaks fixed point owing to the no go theorem (II.16). However, the Yukawa interactions have turned the two loop coefficient $C > 0$ effectively into $C' < 0$ allowing for an interacting gauge-Yukawa fixed point (II.31) as displayed in Fig. **5**. This fixed point genuinely displays an attractive and a repulsive direction, the former being a consequence of the IR attractive nature of Yukawa nullclines, and the latter a consequence of infrared freedom in the gauge coupling. Moreover, it qualifies as an asymptotically safe fixed point owing to the two UV finite trajectories emanating out of it [55]. The weak coupling trajectory connects the interacting fixed point with the Gaussian in the infrared whereby the theory remains unconfined at all scales. The strong coupling trajectory, as in the previous cases, is expected to lead to confinement and chiral symmetry breaking, or conformal behaviour at low energies. Away from the Yukawa nullcline (which always coincides with the hypercritical surface of the gauge-Yukawa fixed point), no trajectories are found which can reach the gauge-Yukawa fixed point in the UV. On such trajectories, the theory technically loses asymptotic safety and predictivity is limited by a maximal UV scale unless a novel UV fixed point emerges at strong coupling.

As an aside, it is worth noticing a similarity between gauge-Yukawa theories with complete asymptotic freedom and a Banks-Zaks, and gauge-Yukawa theories with asymptotic safety, see Figs. **3** and **5**. In both cases, trajectories which escape from the UV fixed point region towards strong coupling in the IR are solely determined by the Yukawa nullcline. All settings predict IR relations between Yukawa and gauge couplings. In the former case

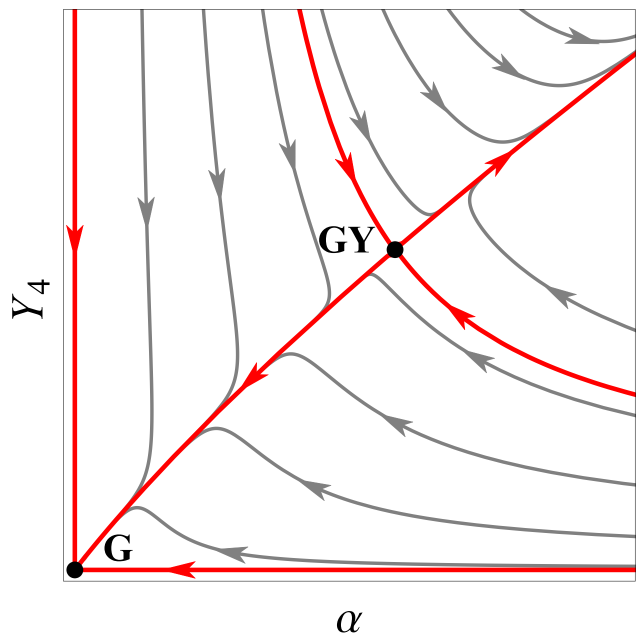

Figure 5. Fixed points and phase diagrams of gauge-Yukawa theories with $B < 0$ and $C' < 0$ at weak coupling showing asymptotic safety together with the Gaussian and gauge-Yukawa fixed points. Notice that the set of UV finite trajectories is confined to a hypercritical surface dictated by the Yukawa nullcline.

this arises due to a funnel effect while in the latter it follows from the unstable direction of the interacting UV fixed point. Without Banks-Zaks, IR relations may be avoided at the expense of substantial fine-tuning in the deep UV, see Fig. **2**.

The discussion of phase diagrams generalises to more complex settings. Gauge theories with several independent Yukawa couplings will lead to several parameters C', which, depending on their magnitudes, may generate several gauge-Yukawa fixed points. Phase diagrams will then display an enhanced structure owing to additional cross-over phenomena amongst the various fixed points. An even richer pattern arises for theories with product gauge groups, see Tab. **4 d)**. Here, the gauge loop coefficients B_a and C_{ab} together with the Yukawa-induced coefficients B'_a uniquely determine the fixed point structure at weak coupling. Evidently, for each gauge coupling individually our discussion based on the "diagonal" coefficients B, C and C' applies, meaning that parts of the enlarged phase diagrams materialise as "direct products" of those shown in Figs. **2**–**5**. As a novel addition, theories will also display "off-diagonal" Banks-Zaks and gauge-Yukawa fixed points as well as fully interacting products thereof, depending on the availability and structure of the solutions to (II.34).[31] Furthermore, each interacting fixed point naturally relates to a conformal window similar to those of QCD with fermionic matter. Some of the fixed points of (product) gauge theories offer UV conformal windows around fixed points with exact asymptotic safety at weak coupling. It is therefore natural to speculate that some such models may qualify as UV completions for the Standard Model of particle physics.

[31] See [142] for a recent example in semi-simple gauge theories without Yukawa couplings.

11. Finally, we briefly comment on interacting fixed points in supersymmetric QFTs. Supersymmetry imposes relations amongst gauge, Yukawa, and scalar couplings [143]. In general, quartic scalar selfinteractions are no longer independent. For theories with $N = 1$ supersymmetry without superpotentials, gauge beta functions remain of the form (II.1) at weak coupling. The signs of B and C depend on the matter content [73]. Gauge sectors can develop Banks-Zaks fixed points (II.4) which are always IR ($B > 0$) but never UV [97], fully consistent with our findings in non-supersymmetric theories (II.16), (II.20). An important difference arises once superpotentials (i.e. Yukawa couplings) are present. Owing to supersymmetry, Yukawas can only take weakly interacting fixed points provided at least one of the gauge sectors is asymptotically free [97]. This implies that asymptotic safety at weak coupling is out of reach for simple $N = 1$ supersymmetric gauge theories. Overall, weakly interacting fixed points are either absent, or of the Banks-Zaks, or of the gauge-Yukawa type, and phase diagrams of simple $4d$ gauge theories with $N = 1$ supersymmetry take the form Fig. **2** or Fig. **4**, while settings with Fig. **3** or Fig. **5** cannot be realised. For $N = 2$ supersymmetry, Yukawa couplings are no longer independent but related to the gauge coupling. Moreover, the running of the gauge coupling becomes one-loop exact with (II.1) and $C \equiv 0$ [98, 102]. Hence, $N = 2$ theories are either asymptotically free or infrared free and interacting fixed points cannot arise. In the limit where $B = 0$, the gauge coupling becomes exactly marginal leading to a line of fixed points [98]. The latter continues to hold true for maximally extended supersymmetry, $N = 4$ SYM, where the constraints from supersymmetry are so powerful that the theory does not flow under the RG, and any value of the gauge coupling corresponds to a fixed point.[32]

12. In summary, we have identified the interacting fixed points of four-dimensional gauge theories in the regime where gauge and matter fields remain good fundamental degrees of freedom. Low-energy fixed points are either of the Banks-Zaks or gauge-Yukawa type, or combinations and products thereof (Tab. **4**), offering a rich spectrum of phenomena including phase transitions and the spontaneous breaking of symmetry. We have also derived no go theorems together with necessary and sufficient conditions to guarantee asymptotic safety of general gauge theories (Tab. **3**). Interacting high-energy fixed points are invariably of the gauge-Yukawa type and require elementary scalar fields such as the Higgs. Hence, the findings of [55] were not a coincidence: rather, the dynamical mechanism to tame the notorious Landau poles of general infrared free gauge theories is *unique*, and, owing to the group-theoretical limitation (II.14), *exclusively* delivered through Yukawa interactions. We conclude that our findings open a window of opportunities towards perturbative UV completions of the Standard Model beyond the paradigm of asymptotic freedom.

[32] For further constraints on supersymmetric fixed points including at strong coupling, see [96, 97].

Part III

Price of asymptotic safety

Andrew D. Bond[1] and Daniel F. Litim[1]

[1]Department of Physics and Astronomy, U Sussex, Brighton, BN1 9QH, U.K.

All known examples of $4d$ quantum field theories with asymptotic freedom or asymptotic safety at weak coupling involve non-abelian gauge interactions. We demonstrate that this is not a coincidence: no weakly coupled fixed points, ultraviolet or otherwise, can be reliably generated in theories lacking gauge interactions. Implications for conformal field theory and phase transitions are indicated.

Introduction.— A turning point in the understanding of high-energy physics has been the discovery of asymptotic freedom in non-abelian gauge theories [16, 17]. It ensures that certain renormalisable quantum field theories remain predictive in the high-energy limit where couplings becomes free [18, 70, 125]. Non-abelian gauge fields are decisive for this to happen: without them, asymptotic freedom cannot be achieved in any theory involving Dirac fermions, photons, or scalars [124].

In the absence of asymptotic freedom, particle theories are generically plagued by divergences and a breakdown of predictivity in the high-energy limit. Some such theories, however, remain well-defined thanks to strict cancellations at the quantum level [55, 144]. This scenario, known as asymptotic safety, has originally been put forward to explain the quantum nature of gravity [15, 38, 40, 44, 111]. Thereby couplings achieve *interacting* high-energy fixed points under the renormalisation group [12]. For weakly coupled matter-gauge theories, theorems for interacting fixed points are available [144]. Proofs and examples of asymptotic safety cover ordinary [55, 62, 145], and supersymmetric gauge theories [146], and extensions of the Standard Model [56].

It appears that all known examples of four-dimensional particle theories with asymptotic freedom or asymptotic safety at weak coupling involve non-abelian gauge interactions. It is the purpose of this Letter to demonstrate that this is not a coincidence: no weakly interacting fixed points, ultraviolet or otherwise, can be reliably generated in theories lacking gauge interactions. Partial results in support of our claim have been made available in [124, 144]. Here, we provide the missing pieces which are, on the one hand, an extension of the Coleman-Gross theorem [124], and a no-go-theorem for weakly interacting fixed points in non-gauge theories, on the other. Taken together, non-abelian gauge interactions are the unique price for particle theories to remain strictly perturbative and predictive at asymptotically high energies, and to display weakly coupled fixed points at low energies.

Price of asymptotic freedom.— To establish our claim, we first revisit asymptotic freedom of general, renormalisable particle theories in four dimensions involving gauge fields, fermions, or scalars. Without loss of generality, we limit the analysis to the canonically marginal interactions which are the gauge, the Yukawa, and the scalar self-couplings $\{g_i, \mathbf{Y}^A_{IJ}, \lambda_{ABCD}\}$, respectively. We assume canonically normalised kinetic terms with gauge couplings g_i for each gauge factor. Our conventions for the most general Yukawa and scalar couplings are

$$\begin{aligned} L_{\text{Yuk.}} &= -\tfrac{1}{2}(\mathbf{Y}^A_{JK}\Phi^A\Psi_J\Psi_K + \text{h.c.})\,, \\ L_{\text{pot.}} &= -\tfrac{1}{4!}\lambda_{ABCD}\Phi^A\Phi^B\Phi^C\Phi^D\,, \end{aligned} \tag{III.1}$$

where Ψ_J denote Weyl fermions, and Φ^A real scalars. Matter fields may be charged under the gauge groups.

Next, we turn to quantum effects and the renormalisation group running of couplings. The point in coupling space where all couplings vanish, the free theory, is always a fixed point of the renormalisation group. Then, for any theory to be free at asymptotically high energies, the free fixed point must be ultraviolet and the beta functions negative for sufficiently small couplings,

$$\mu\,\partial_\mu\,(g, Y, \lambda) < 0\,, \tag{III.2}$$

with μ the renormalisation group scale. After scaling the loop factor into the couplings, as we shall consistently do throughout, the one-loop gauge beta functions is [16, 17, 70]

$$\mu\,\partial_\mu g_i = \left(-\tfrac{11}{3}C_2^{G_i} + \tfrac{2}{3}S_2^{F_i} + \tfrac{1}{6}S_2^{S_i}\right) g_i^3\,. \tag{III.3}$$

Non-abelian gauge fields contribute negatively and proportionally to the quadratic Casimir of the gauge group in the adjoint ($C_2^{G_i}$). Matter fields contribute positively and proportionally to their Dynkin indices (S_2). The key feature of non-abelian theories is that (III.3) may have either sign, depending on the matter fields, including (III.2). As is well-known, the latter is the origin for asymptotic freedom in general gauge theories with matter [18, 70, 125]. Next, we establish that (III.3) is also the unique origin for weakly interacting fixed points and asymptotic safety.

Coleman-Gross theorem revisited.— To clarify the role of gauge field fluctuations we first revisit the Coleman-Gross theorem [124]. It states that a non-gauge theory of scalars with or without Dirac fermions cannot become asymptotically free. To cover the most general setting, we extend the theorem towards Weyl fermions. It is convenient to view the Yukawa couplings as symmetric matrices $\mathbf{Y}^A$ in the fermion indices $(\mathbf{Y}^A)_{JK} \equiv \mathbf{Y}^A_{JK}$. Their running with momentum scale μ at the leading order in perturbation theory is given by [75, 76]

$$\mu\,\partial_\mu \mathbf{Y}^A = \tfrac{1}{2}\left(\overline{\mathbf{Y}_\mathbf{2}^{\boldsymbol{F}}}\,\mathbf{Y}^A + \mathbf{Y}^A\,\mathbf{Y}_\mathbf{2}^{\boldsymbol{F}}\right) + Y_2^{S\,AB}\,\mathbf{Y}^B + 2\mathbf{Y}^B\,\mathbf{Y}^{A\dagger}\,\mathbf{Y}^B\,, \tag{III.4}$$

where the bar denotes complex conjugation, and summation over repeated indices is implied. We have also introduced the quadratic combinations $\mathbf{Y_2}{}^{AB}_{JK} = \frac{1}{2}\left(\mathbf{Y}^{A\dagger}\mathbf{Y}^B + \mathbf{Y}^{B\dagger}\mathbf{Y}^A\right)_{JK}$ alongside $\mathbf{Y}_\mathbf{2}^{\boldsymbol{F}}{}_{JK} \equiv \mathbf{Y_2}{}^{AA}_{JK}$ and $Y_2^{S\,AB} \equiv \mathbf{Y_2}{}^{AB}_{JJ}$. The first and second term in (III.4) arise from the wave function renormalisation of the fermion and scalar propagators, whereas the last term stems from vertex corrections. We note that the Yukawa couplings and their flows (III.4) transform as tensors under a change of basis, *i.e.* general linear transformations of the fields which leave (III.1) invariant.

We shall now focus our attention on the flow for the sum of the squared absolute values of all Yukawa couplings,

$$\mu\,\partial_\mu \mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{A}) = \mathrm{Tr}[(\overline{\mathbf{Y}_{\mathbf{2}}^{\boldsymbol{F}}})^2] + \mathrm{Tr}[(\mathbf{Y}_{\mathbf{2}}^{\boldsymbol{F}})^2] + 4\mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{B}\mathbf{Y}^{A\dagger}\mathbf{Y}^{B}) \\ + \mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{B})\left[\mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{B}) + (A \leftrightarrow B)\right] \tag{III.5}$$

for if we are to have all Yukawa beta functions negative, then this combination must be negative as well. We emphasize that the flow (III.5) and the conclusions drawn from it are independent of the choice of field base. A lower bound for (III.5) follows by using that $\mathrm{Re}\, z^2 \leq z^* z$ for any complex number z, whence

$$\mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{B})\mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{B}) \leq \mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{B})\mathrm{Tr}(\mathbf{Y}^{B\dagger}\mathbf{Y}^{A})\,. \tag{III.6}$$

Next, we introduce the three real trace invariants $T_1 = \mathrm{Tr}[(\mathbf{Y}_{\mathbf{2}}^{\boldsymbol{F}})^2] = \mathrm{Tr}[(\overline{\mathbf{Y}_{2}^{\boldsymbol{F}}})^2]$, $T_2 = \mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{B}\mathbf{Y}^{A\dagger}\mathbf{Y}^{B})$, and $T_3 = \mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{B})\mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{B})$. By definition, T_2 may have either sign while $T_1, T_3 \geq 0$. In terms of these, and together with (III.6), we find that the Yukawa beta function (III.5) is bounded from below,

$$\mu\,\partial_\mu \mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{A}) \geq 2(T_1 + T_2) + 2(T_2 + T_3)\,. \tag{III.7}$$

Recalling that the Yukawa couplings are symmetric in the fermionic indices, we rearrange the sums as follows

$$\begin{aligned}
T_1 + T_2 &= \mathbf{Y}^{A\dagger}_{JK}\mathbf{Y}^{A}_{KL}\mathbf{Y}^{B\dagger}_{LM}\mathbf{Y}^{B}_{MJ} + \mathbf{Y}^{A\dagger}_{JK}\mathbf{Y}^{B}_{KL}\mathbf{Y}^{A\dagger}_{LM}\mathbf{Y}^{B}_{MJ}\,,\\
&= \mathbf{Y}^{A\dagger}_{JK}\mathbf{Y}^{B}_{MJ}\Big(\mathbf{Y}^{A}_{KL}\mathbf{Y}^{B\dagger}_{LM} + \mathbf{Y}^{B}_{KL}\mathbf{Y}^{A\dagger}_{LM}\Big)\,,\\
&= \tfrac{1}{2}\Big(\mathbf{Y}^{A\dagger}_{JK}\mathbf{Y}^{B}_{MJ} + \mathbf{Y}^{B\dagger}_{JK}\mathbf{Y}^{A}_{MJ}\Big)\Big(\mathbf{Y}^{A}_{KL}\mathbf{Y}^{B\dagger}_{LM} + \mathbf{Y}^{B}_{KL}\mathbf{Y}^{A\dagger}_{LM}\Big)\,,\\
&= 2\mathbf{Y}_{\mathbf{2}}{}^{AB}_{KM}\mathbf{Y}_{\mathbf{2}}{}^{AB}_{MK} = 2\overline{\mathbf{Y}_{\mathbf{2}}{}^{AB}_{MK}}\mathbf{Y}_{\mathbf{2}}{}^{AB}_{MK}\,,
\end{aligned} \tag{III.8}$$

$$\begin{aligned}
T_2 + T_3 &= \mathbf{Y}^{A}_{JK}\mathbf{Y}^{B\dagger}_{KL}\mathbf{Y}^{A}_{LM}\mathbf{Y}^{B\dagger}_{MJ} + \mathbf{Y}^{A}_{JK}\mathbf{Y}^{B\dagger}_{KJ}\mathbf{Y}^{A}_{LM}\mathbf{Y}^{B\dagger}_{ML}\,,\\
&= \mathbf{Y}^{A}_{JK}\mathbf{Y}^{A}_{LM}\left(\mathbf{Y}^{B\dagger}_{KJ}\mathbf{Y}^{B\dagger}_{ML} + \mathbf{Y}^{B\dagger}_{KL}\mathbf{Y}^{B\dagger}_{MJ}\right)\,,\\
&= \tfrac{1}{2}\Big(\mathbf{Y}^{A}_{KJ}\mathbf{Y}^{A}_{ML} + \mathbf{Y}^{A}_{KL}\mathbf{Y}^{A}_{MJ}\Big)\Big(\mathbf{Y}^{B\dagger}_{KJ}\mathbf{Y}^{B\dagger}_{ML} + \mathbf{Y}^{B\dagger}_{KL}\mathbf{Y}^{B\dagger}_{MJ}\Big)\,.
\end{aligned} \tag{III.9}$$

As is evidenced by the explicit expressions, both (III.8) and (III.9) are sums of absolute values squared and therefore manifestly semi-positive definite,

$$\begin{aligned}
T_1 + T_2 &\geq 0\,,\\
T_2 + T_3 &\geq 0\,.
\end{aligned} \tag{III.10}$$

Most importantly, the bounds (III.10) dictate positivity for the flow (III.7) close to the

Gaussian,

$$\mu\,\partial_\mu \mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{A}) \geq 0\,, \tag{III.11}$$

and establish that asymptotic freedom is unavailable. Had we substituted Weyl by Dirac fermions in (III.1), we would have found the lower bound $\mu\partial_\mu \mathrm{Tr}(\mathbf{Y}^{A\dagger}\mathbf{Y}^{A}) \geq 2T_1 + 4(T_2 + T_3)$, instead of (III.7). For theories with Dirac fermions only, the non-negativity of T_1 together with $T_2 + T_3 > 0$ is sufficient to conclude the absence of asymptotic freedom [124]. Clearly, the bounds for Weyl and Dirac fermions are inequivalent: while the former entail the latter, the converse is not true.

One might wonder whether scalar self-interactions may upset the conclusion. Scalar couplings contribute to the Yukawa beta function starting at two-loop order. Therefore, if they were to reliably generate asymptotic freedom, they must do so along a renormalisation group trajectory where they are parametrically larger than the Yukawa couplings. Assuming this to be the case, we can then ignore the Yukawa contribution to the running of the quartics. In other words, the scalar sector must become asymptotically free in its own right. This, however, is known to be impossible [124]. We reproduce here the line of reasoning as some of this is needed later.

To leading order in perturbation theory, a scalar theory with quartic interactions (III.1) has the beta function [18]

$$\beta_{ABCD} = \frac{1}{8}\sum_{\{ABCD\}} \lambda_{ABEF}\,\lambda_{EFCD}\,, \tag{III.12}$$

where $\beta_{ABCD} \equiv \mu\partial_\mu\lambda_{ABCD}$ with λ fully symmetric in its indices, and the sum running over all permutations. For clarity, in the following we shall write out any index sums explicitly. Vacuum stability requires that for each A we must have $\lambda_{AAAA} \geq 0$, or else the potential becomes unbounded in the ϕ_A direction. Together with (III.12) we have

$$\beta_{AAAA} \propto \sum_{B,C} \lambda_{AABC}\,\lambda_{AABC} \geq 0\,, \tag{III.13}$$

showing that vacuum stability is incompatible with asymptotic freedom, for which we would need this beta function to be negative, (III.2). Let us then switch off all such couplings identically, $\lambda_{AAAA}(\mu) = 0$. In this scenario, their flows and all couplings appearing on the right-hand-side of (III.13) have to vanish, or else a non-zero value for λ_{AAAA} is generated by fluctuations. Specifically, taking $B = C$ it follows that $\lambda_{AABB}(\mu) = 0$ at all scales, which again necessitates $\beta_{AABB} = 0$. Since these beta functions are the sums of squares,

$$\beta_{AABB} = \beta_{ABAB} \propto \sum_{C,D} \lambda_{ABCD}\,\lambda_{ABCD} \geq 0\,, \tag{III.14}$$

the pattern percolates: each and every coupling appearing on the right-hand-side vanishes, $\lambda_{ABCD}(\mu) = 0$, and the theory remains free at all scales [124]. Thus, we conclude that the Coleman-Gross theorem holds true for theories with Weyl fermions, and asymptotic freedom cannot be achieved without non-abelian gauge fields.

Price of interacting fixed points.— We are now in a position to discuss weakly interacting fixed points of general, renormalisable theories in four dimensions involving gauge fields, fermions, or scalars. At weak coupling, anomalous dimensions are small and canonical power counting remains applicable. It is then sufficient to establish weakly-coupled fixed points (g_*, Y_*, λ_*) for the canonically marginal couplings of the theory, which are the perturbatively-controlled solutions of

$$\mu\,\partial_\mu\,(g, Y, \lambda)\,|_* = 0\,, \tag{III.15}$$

other than the Gaussian, where at least some or all couplings are non-zero [62]. For general gauge theories, a full classification of weakly coupled fixed point solutions to (III.15) has been given in [144]. Perturbative fixed points are either free (the Gaussian), or interacting in the gauge sector (Caswell–Banks-Zaks fixed points) [72, 110], or simultaneously interacting in the gauge and the Yukawa sector (gauge-Yukawa fixed points). Fixed points may be partially or fully interacting, depending on whether some or all gauge couplings take non-zero values. Scalar self interactions must take free or interacting fixed points of their own, compatible with vacuum stability (Tab. **5**). Banks-Zaks fixed points are always infrared, yet the Gaussian and gauge-Yukawa fixed points can be infrared or ultraviolet. In particular, asymptotic safety at weak coupling arises solely via gauge-Yukawa fixed points [144].

We emphasize that all weakly-interacting fixed points of Tab. **5** are controlled by the one-loop gauge coefficient in (III.3). Its smallness, for suitable matter, ensures strict perturbativity [62]. We conclude that weakly interacting fixed points and asymptotic safety in general non-abelian gauge theories coupled to matter have the same dynamical origin as asymptotic freedom.

No-go-theorem for scalar-Yukawa fixed points.— In order to complete our claim, and inasmuch as asymptotic freedom cannot arise without non-abelian gauge fields, we finally must show that weakly interacting fixed points cannot arise in the absence of gauge interactions. To that end, we return to scalar-Yukawa theories with interaction Lagrangean (III.1). Assuming that Yukawa and scalar couplings are small, we must have $\mu\partial_\mu \mathbf{Y}^A|_* = 0$ at the leading non-trivial order in perturbation theory. Consequently, the bounds (III.10), (III.11) must be saturated. However, (III.10) only vanish for vanishing Yukawa couplings,

$$\mathbf{Y}^A_{JK} = 0\,. \tag{III.16}$$

This is understood as follows. Being a sum of absolute values squared, the expression

Case	Condition	Fixed Point
i)	$g_i = \mathbf{Y}^A_{JK} = \lambda_{ABCD} = 0$	Gaussian
ii)	some $g_i \neq 0$, all $\mathbf{Y}^A_{JK} = 0$	Banks-Zaks
iii)	some $g_i \neq 0$, some $\mathbf{Y}^A_{JK} \neq 0$	gauge-Yukawa

Table 5. Fixed points of general weakly interacting quantum field theories in four dimensions. In cases *ii*) and *iii*), scalar self-interactions, if present, must take fixed points λ^*_{ABCD} compatible with vacuum stability [144].

(III.8) vanishes if and only if each term in the final sum vanishes, $\mathbf{Y_2}{}^{AB}_{JK} = 0$. From the definition for $\mathbf{Y_2}$, and after contracting over scalar indices we find that the matrix $\mathbf{Y}^{\boldsymbol{F}}_{\mathbf{2}}$ also vanishes. Taking its trace $\mathbf{Y}^{\boldsymbol{F}}_{\mathbf{2}}{}_{JJ} = \overline{\mathbf{Y}^A_{JK}}\mathbf{Y}^A_{JK} = 0$ implies (III.16) and the vanishing of (III.8) and (III.9). We conclude that the only available fixed point in the Yukawa sector at one-loop, without gauge fields, is the Gaussian, and it must be infrared.

Once more, scalar couplings cannot upset this conclusion: scalar selfinteractions contribute to the running of Yukawas starting at two loop. In principle, they could balance the one-loop Yukawa terms provided they are parametrically larger while still remaining perturbative in their own right. For such a mechanism to be operative, some scalar quartics must take weakly interacting fixed points by themselves. Under this assumption we can safely neglect the parametrically smaller Yukawa contributions. Let us then pick A, B such that for some C, D we have $\lambda^*_{ABCD} \neq 0$. This implies the strict inequality

$$\sum_{C,D} \lambda^*_{ABCD}\, \lambda^*_{ABCD} > 0\,, \tag{III.17}$$

as this is a sum of squares of which at least one entry is non-zero. Combining (III.17) with (III.14) we conclude that the flows of λ_{ABAB} and λ_{AABB} are strictly positive, which is in conflict with (III.15), and the assumption of a weakly coupled fixed point in the scalar sector cannot be maintained. This establishes that the sole perturbatively-controlled fixed point is the Gaussian, which is invariably infrared. Ultimately, in any scalar-Yukawa theory the unavailability of weakly interacting fixed points and asymptotic safety has the same dynamical origin as the unavailability of asymptotic freedom.

Discussion.— In this Letter, we have investigated free or weakly interacting fixed points of $4d$ particle theories with gauge fields, fermions, or scalars. From the viewpoint of high-energy physics, our findings establish that asymptotic freedom and asymptotic safety are two sides of one and the same medal. Quantum fluctuations of matter fields alone, with or without photons, are incapable of generating a well-defined and predictive short-distance limit at weak coupling. Rather, the unique driver for viable ultraviolet completions – *i.e.* fixed points of the renormalisation group with asymptotic freedom or asymptotic safety – are the fluctuations of non-abelian gauge fields. Hence, non-abelian gauge fields are the price for *any* particle theory to remain strictly perturbative and predictive at asymptotically high energies. Similarly, from the viewpoint of low-energy physics, our results imply that weakly-coupled infrared fixed points and second order (quantum) phase transitions cannot arise without gauge fields. Thus, the universality class for any such phase transition contains gauge interactions.

Finally, our results also have implications for conformal field theories in four space-time dimensions. Here, conditions under which $4d$ scale invariance entails full conformal invariance are of particular interest [105, 106, 141, 147]. Using techniques related to the proof of the a-theorem [105, 106], it has been demonstrated that any relativistic and unitary four-dimensional theory that remains perturbative in the ultraviolet or infrared asymptotes to a conformal field theory [141]. All weakly interacting fixed points discussed in this work (Tab. **5**) belong to this category [144]. This leads to the important conclusion that elementary gauge fields are the price for interacting and strictly perturbative $4d$ conformal field theories.

Part IV

More asymptotic safety guaranteed

Andrew D. Bond[1] and Daniel F. Litim[1]

[1]Department of Physics and Astronomy, U Sussex, Brighton, BN1 9QH, U.K.

We study interacting fixed points and phase diagrams of simple and semi-simple quantum field theories in four dimensions involving non-abelian gauge fields, fermions and scalars in the Veneziano limit. Particular emphasis is put on new phenomena which arise due to the semi-simple nature of the theory. Using matter field multiplicities as free parameters, we find a large variety of interacting conformal fixed points with stable vacua and crossovers inbetween. Highlights include semi-simple gauge theories with exact asymptotic safety, theories with one or several interacting fixed points in the IR, theories where one of the gauge sectors is both UV free and IR free, and theories with weakly interacting fixed points in the UV and the IR limits. The phase diagrams for various simple and semi-simple settings are also given. Further aspects such as perturbativity beyond the Veneziano limit, conformal windows, and implications for model building are discussed.

I. INTRODUCTION

Asymptotic freedom is a key feature of non-Abelian gauge theories [16, 17]. It predicts that interactions weaken with growing energy due to quantum effects, thereby reaching a free ultraviolet (UV) fixed point under the renormalisation group. Asymptotic safety, on the other hand, stipulates that running couplings may very well asymptote into an interacting UV fixed point at highest energies [12, 15]. The most striking difference between asymptotically free and asymptotically safe theories relates to residual interactions in the UV. Canonical power counting is modified, whence establishing asymptotic safety in a reliable manner becomes a challenging task [44].

Rigorous results for asymptotic safety at weak coupling have been known for a long time for models including either scalars, fermions, gauge fields or gravitons, and away from their respective critical dimensionality [15, 21–25, 28–30, 148–150]. In these toy models asymptotic safety arises through the cancellation of tree level and leading order quantum terms. Progress has also been made to substantiate the asymptotic safety conjecture beyond weak coupling [44]. This is of particular relevance for quantum gravity where good evidence has arisen in a variety of different settings [38, 40, 49, 50, 52, 111, 112, 151–158].

An important new development in the understanding of asymptotic safety has been initiated in [55] where it was shown that certain four-dimensional quantum field theories involving $SU(N)$ gluons, quarks, and scalars can develop weakly coupled UV fixed points. Results have been extended beyond classically marginal interactions [60]. Structural insights into the renormalisation of general gauge theories have led to necessary and sufficient conditions for asymptotic safety, alongside strict no go theorems [144, 159]. Asymptotic safety invariably arises as a quantum critical phenomenon through cancellations at loop level for which all three types of elementary degrees of freedom — scalars, fermions and gauge fields — are required. Findings have also been extended to cover supersymmetry [146] and UV conformal windows [62]. Throughout, it is found that suitable Yukawa interactions are pivotal [144, 159].

In this paper, we are interested in fixed points of semi-simple gauge theories. Our primary motivation is the semi-simple nature of the Standard Model, and the prospect for asymptotically safe extensions thereof [56]. We are particularly interested in semi-simple theories where interacting fixed points and asymptotic safety can be established rigorously [144]. More generally, we also wish to understand how low- and high-energy fixed points are generated dynamically, what their features are, and whether novel phenomena arise owing to the semi-simple nature of the underlying gauge symmetry. Understanding the stability of a Higgs-like ground state at interacting fixed points is also of interest in view of the "near-criticality" of the Standard Model vacuum [160, 161].

We investigate these questions for quantum field theories with $SU(N_C)\times SU(N_c)$ local gauge symmetry coupled to massless fermionic and singlet scalar matter. Our models also have a global $U(N_\mathrm{F})_L \times U(N_\mathrm{F})_R \times U(N_\mathrm{f})_L \times U(N_\mathrm{f})_R$ flavour symmetry, and are characterised by up to nine independent couplings. Matter field multiplicities serve as free parameters. We obtain rigorous results from the leading orders in perturbation theory by adopting a Veneziano limit. We then provide a comprehensive classification of quantum field theories according to their UV and IR limits, their fixed points, and eigenvalue spectra. Amongst these, we find semi-simple gauge theories with exact asymptotic safety in the UV. We also find a large variety of theories with crossover- and low-energy fixed points. Further novelties include theories with inequivalent yet fully attractive IR conformal fixed points, theories with weakly interacting fixed points in both the UV and the IR, and massless theories with a non-trivial gauge sector which is UV free and IR free. We illustrate our results by providing general phase diagrams for simple and semi-simple gauge theories with and without Yukawa interactions.

The paper is organised as follows. General aspects of weakly interacting fixed points in $4d$ gauge theories are laid out in Sec. **II**, together with first results and expressions for universal exponents. In Sec. **III** we introduce concrete families of semi-simple gauge theories coupled to elementary singlet "mesons" and suitably charged massless fermions. Perturbative RG equations for all gauge, Yukawa and scalar couplings and masses in a Veneziano limit are provided to the leading non-trivial orders in perturbation theory. Sec. **16** presents our results for all interacting perturbative fixed points and their universal scaling exponents. Particular attention is paid to new effects which arise due to the semi-simple nature of the models. Sec. **V** provides the corresponding fixed points in the scalar sector. It also establishes stability of the quantum vacuum whenever a physical fixed point arises in the gauge sector. Using field multiplicities as free parameters, Sec. **VI** provides a complete classification of distinct models with asymptotic freedom or asymptotic safety in the UV, or without UV completions, together with their scaling in the deep IR. In Sec. **VII**, the generic phase diagrams for simple and semi-simple gauge theories with and without Yukawas are discussed. The phase diagrams, UV – IR transitions, and aspects of IR conformality are analysed in more depth for sample theories with asymptotic freedom and asymptotic safety. Further reaching topics such as exact perturbativity, extensions beyond the Veneziano limit, and conformal windows are discussed in Sec. **VIII**. Sec. **IX** closes with a brief summary.

II. FIXED POINTS OF GAUGE THEORIES

In this section, we discuss general aspects of interacting fixed points in semi-simple gauge theories which are weakly coupled to matter, with or without Yukawa interactions, following [144, 159]. We also introduce some notation and conventions.

A. Fixed points in perturbation theory

We are interested in the renormalisation of general gauge theories coupled to matter fields, with or without Yukawa couplings. The running of the gauge couplings $\alpha_i = g_i^2/(4\pi)^2$ with the renormalisation group scale μ is determined by the beta functions of the theory. Expanding them perturbatively up to two loop we have

$$\mu\partial_\mu \alpha_i \equiv \beta_i = \alpha_i^2(-B_i + C_{ij}\alpha_j - 2\,Y_{4,i}) + O(\alpha^4)\,, \tag{IV.1}$$

where a sum over gauge group factors j is implied. The one- and two-loop gauge contributions B_i and C_{ij} and the two-loop Yukawa contributions $Y_{4,i}$ are known for general gauge theories, see [74–77, 144] for explicit expressions. While B_i and C_{ii} may take either sign, depending on the matter content, the Yukawa contribution $Y_{4,i}$ and the off-diagonal gauge contributions C_{ij} $(i \neq j)$ are strictly positive in any quantum field theory. Scalar couplings do not play any role at this order in perturbation theory. The effect of Yukawa couplings is incorporated by projecting the gauge beta functions (IV.1) onto the Yukawa nullclines ($\beta_Y = 0$), leading to explicit expressions for $Y_{4,i}$ in terms of the gauge couplings g_j. Moreover, for many theories the Yukawa contribution along nullclines can be written as $Y_{4,i} = D_{ij}\,\alpha_j$ with $D_{ij} \geq 0$ [144]. We can then go one step further and express the net effect of Yukawa couplings as a shift of the two loop gauge contribution, $C_{ij} \to C'_{ij} = C_{ij} - 2D_{ij} \leq C_{ij}$. Notice that the shift will always be by some negative amount provided at least one of the Yukawa couplings is non-vanishing. It leads to the reduced gauge beta functions

$$\beta_i = \alpha_i^2(-B_i + C'_{ij}\alpha_j) + O(\alpha^4)\,. \tag{IV.2}$$

Fixed points solutions of (IV.2) are either free or interacting and $\alpha^* = 0$ for some or all gauge factors is always a self-consistent solution. Consequently, interacting fixed points are solutions to

$$B_i = C'_{ij}\,\alpha_j^*\,, \quad \text{subject to} \quad \alpha_i^* > 0\,, \tag{IV.3}$$

where only those rows and columns are retained where gauge couplings are interacting.

Next we discuss the role of Yukawa couplings for the fixed point structure. In the absence of Yukawa couplings, the two-loop coefficients remain unshifted $C'_{ij} = C_{ij}$. An immediate consequence of this is that any interacting fixed point must necessarily be IR. The reason is as follows: for an interacting fixed point to be UV, asymptotic freedom cannot be maintained for all gauge factors, meaning that some $B_i < 0$. However, as has been established in [144], $B_i \leq 0$ necessarily entails $C_{ij} \geq 0$ in any $4d$ quantum gauge theory. If the left hand side of (IV.3) is negative, if only for a single row, positivity of C_{ij} requires that some α_j^* must take negative values for a fixed point solution to arise.

This, however, is unphysical [113] and we are left with $B_i > 0$ for each i, implying that asymptotic freedom remains intact in all gauge sectors. Besides the Gaussian, the theory may have weakly interacting infrared Banks-Zaks fixed points in each gauge sector, as well as products thereof, which arise as solutions to (IV.3) with the unshifted coefficients.

In the presence of Yukawa couplings, the coefficients C'_{ij} can in general take either sign. This has far reaching implications. Firstly, the theory can additionally display gauge-Yukawa fixed points where both the gauge and the Yukawa couplings take interacting values. Most importantly, solutions to (IV.3) are then no longer limited to theories with asymptotic freedom. Instead, interacting fixed points can be infrared, ultraviolet, or of the crossover type. In general we may expect gauge-Yukawa fixed points for each independent Yukawa nullcline. In summary, perturbative fixed points are either (*i*) free and given by the Gaussian, or (*ii*) free in the Yukawa but interacting in the gauge sector (Banks-Zaks fixed points), or (*iii*) simultaneously interacting in the gauge and the Yukawa sector (gauge-Yukawa fixed points), or (*iv*) combinations and products of (*i*), (*ii*) and (*iii*). Banks-Zaks fixed points are always IR, while the Gaussian and gauge-Yukawa fixed points can be either UV or IR. Depending on the details of the theory and its Yukawa structure, either the Gaussian or one of the interacting gauge-Yukawa fixed points will arise as the "ultimate" UV fixed point of the theory and may serve to define the theory fundamentally [159].

The effect of scalar quartic self-couplings on the fixed point is strictly sub-leading in terms of the values of the fixed points, as they do not affect the running of gauge couplings at this order of perturbation theory. However, as to have a true fixed point we must acquire one in all couplings, they provide additional constraints on the physicality of candidate gauge-Yukawa fixed points, as we additionally require that the quartic couplings take fixed points which are both real-valued, and lead to a bounded potential which leads to a stable vacuum state.

B. Gauge couplings

Let us now consider a semi-simple gauge-Yukawa theory with non-Abelian gauge fields under the semi-simple gauge group $\mathcal{G}_1 \otimes \mathcal{G}_2$ coupled to fermions and scalars. We have two non-Abelian gauge couplings α_1 and α_2, which are related to the fundamental gauge couplings via $\alpha_i = g_i^2/(4\pi)^2$. The running of gauge couplings within perturbation theory is given by

$$\begin{aligned}\beta_1 &= -B_1\,\alpha_1^2 + C_1\,\alpha_1^3 + G_1\,\alpha_1^2\,\alpha_2\,,\\ \beta_2 &= -B_2\,\alpha_2^2 + C_2\,\alpha_2^3 + G_2\,\alpha_2^2\,\alpha_1\,.\end{aligned} \tag{IV.4}$$

Here, B_i are the well known one-loop coefficients. In theories without Yukawa interactions, or where Yukawa interactions take Gaussian values, the numbers C_i and G_i are the two-loop coefficients which arise owing to the gauge loops and owing to the mixing between

fixed point		α_{gauge}	α_{Yukawa}
Gauss	G	$= 0$	$= 0$
Banks-Zaks	BZ	$\neq 0$	$= 0$
gauge-Yukawa	GY	$\neq 0$	$\neq 0$

Table 6. Conventions to denote the basic fixed points (Gaussian, Banks-Zaks, or gauge-Yukawa) of simple gauge theories weakly coupled to matter.

gauge groups, meaning $C_i \equiv C_{ii}$ (no sum), and $G_1 \equiv C_{12}$, $G_2 \equiv C_{12}$, see (IV.1). In this case, we also have that $C_i, G_i \geq 0$ as soon as $B_i < 0$.[33] For theories where Yukawa couplings take interacting fixed points the numbers C_i and G_i receive corrections due to the Yukawas, $C_i \equiv C'_{ii}$ (no sum), and $G_1 \equiv C'_{12}$, $G_2 \equiv C'_{12}$, see (IV.2). Most notably, strict positivity of C_i and G_i is then no longer guaranteed [144].

In either case, the fixed points of the combined system are determined by the vanishing of (IV.4). For a general semi-simple gauge theory with two gauge factors, one finds four different types of fixed points. The Gaussian fixed point

$$(\alpha_1^*, \alpha_2^*) = (0, 0) \tag{IV.5}$$

always exist (see Tab. **6** for our conventions). It is the UV fixed point of the theory as long as the one-loop coefficients obey $B_i > 0$. The theory may also develop partially interacting fixed points,

$$(\alpha_1^*, \alpha_2^*) = \left(0, \frac{B_2}{C_2}\right) , \tag{IV.6}$$

$$(\alpha_1^*, \alpha_2^*) = \left(\frac{B_1}{C_1}, 0\right) . \tag{IV.7}$$

Here, one of the gauge coupling is taking Gaussian values whereas the other one is interacting. The interacting fixed point is of the Banks-Zaks type [72, 110], provided Yukawa interactions are absent. This then also implies that the gauge coupling is asymptotically free. Alternatively, the interacting fixed point can be of the gauge-Yukawa type, provided that Yukawa couplings take an interacting fixed point themselves. In this case, and depending on the details of the Yukawa sector, the fixed point can be either IR or UV. Finally, we also observe fully interacting fixed points

$$(\alpha_1^*, \alpha_2^*) = \left(\frac{C_2 B_1 - B_2 G_1}{C_1 C_2 - G_1 G_2}, \frac{C_1 B_2 - B_1 G_2}{C_1 C_2 - G_1 G_2}\right) . \tag{IV.8}$$

As such, fully interacting fixed points (IV.8) can be either UV or IR, depending on the specific field content of the theory. In all cases we will additionally require that the

[33] General formal expressions of loop coefficients in the conventions used here are given in [144].

coupling	order in perturbation theory			
β_{gauge}	1	2	2	$n+1$
β_{Yukawa}	0	1	1	n
β_{scalar}	0	0	1	n
approximation	LO	NLO	NLO$'$	nNLO$'$

Table 7. Relation between approximation level and the loop order up to which couplings are retained in perturbation theory, following the terminology of [55, 58].

couplings obey

$$\begin{aligned} \alpha_1 &\geq 0\,, \\ \alpha_2 &\geq 0\,. \end{aligned} \tag{IV.9}$$

to ensure they reside in the physical regime of the theory [113].

C. Yukawa couplings

In order to proceed, we must specify the Yukawa sector. We assume three types of non-trivially charged fermions with charges under $\mathcal{G}_1$ and $\mathcal{G}_2$. Some or all of the fermions which are only charged under $\mathcal{G}_1$ ($\mathcal{G}_2$) also couple to scalar fields via Yukawa couplings α_Y (α_y), respectively. The scalars may or may not be charged under the gauge symmetries. They will have quartic self couplings which play no primary role for the fixed point analysis at weak coupling [144]. Within perturbation theory, the beta functions for the gauge and Yukawa couplings are of the form

$$\begin{aligned} \beta_1 &= -B_1\,\alpha_1^2 + C_1\,\alpha_1^3 - D_1\,\alpha_1^2\,\alpha_Y + G_1\,\alpha_1^2\,\alpha_2\,, \\ \beta_Y &= \quad E_1\,\alpha_Y^2 - F_1\,\alpha_Y\,\alpha_1\,, \\ \beta_2 &= -B_2\,\alpha_2^2 + C_2\,\alpha_2^3 - D_2\,\alpha_2^2\,\alpha_y + G_2\,\alpha_2^2\,\alpha_1\,, \\ \beta_y &= \quad E_2\,\alpha_y^2 - F_2\,\alpha_y\,\alpha_2\,. \end{aligned} \tag{IV.10}$$

The RG flow is given up to two-loop in the gauge couplings, and up to one-loop in the Yukawa couplings. We refer to this as the NLO approximation, see Tab. **7** for the terminology.

We are interested in the fixed points of the theory, defined implicitly via the vanishing of the beta functions for all couplings. The Yukawa couplings can display either a Gaussian or an interacting fixed point

$$\begin{aligned} \alpha_Y^* = 0\,, &\quad \alpha_Y^* = \frac{F_1}{E_1}\,\alpha_1^*\,, \\ \alpha_y^* = 0\,, &\quad \alpha_y^* = \frac{F_2}{E_2}\,\alpha_2^*\,. \end{aligned} \tag{IV.11}$$

Depending on whether none, one, or both of the Yukawa couplings take an interacting fixed point, the system (IV.10) reduces to (IV.4) whereby the two-loop coefficients C_i of the gauge beta functions are shifted according to

$$\begin{aligned} \alpha_Y^* \neq 0: &\quad C_1 \to C_1' = C_1 - D_1 \frac{F_1}{E_1} \leq C_1 \,, \\ \alpha_y^* \neq 0: &\quad C_2 \to C_2' = C_2 - D_2 \frac{F_2}{E_2} \leq C_2 \,. \end{aligned} \tag{IV.12}$$

Notice also that in this model the values for the mixing terms G_i do not depend on whether the corresponding Yukawa couplings vanish, or not, due to the fact that no fermions charged under both groups are involved in Yukawa interactions. Owing to the fixed point structure of the Yukawa sector (IV.11), the formal fixed points (IV.5), (IV.6), (IV.7) and (IV.8) have the multiplicity $1, 2, 2$ and 4, respectively. In total, we end up with nine qualitatively different fixed points $\mathrm{FP}_1 - \mathrm{FP}_9$, summarised in Tab. **8**: FP_1 denotes the unique Gaussian fixed point. FP_2 and FP_3 correspond to a Banks-Zaks fixed point in one of the gauge couplings, and a Gaussian in the other. They can therefore be interpreted effectively as a "product" of a Banks-Zaks with a Gaussian fixed point. Similarly, at FP_4 and FP_5, one of the Yukawa couplings remains interacting, and they can therefore effectively be viewed as the product of a gauge-Yukawa (GY) type fixed point in one gauge coupling with a Gaussian fixed point in the other. The remaining fixed points $\mathrm{FP}_6 - \mathrm{FP}_9$ are interacting in both gauge couplings. These fixed points are the only ones which are sensitive to the two-loop mixing coefficients G_1 and G_2. At FP_6, both Yukawa couplings vanish meaning that it is effectively a product of two Banks-Zaks type fixed points. At FP_7 and FP_8, only one of the Yukawa couplings vanish, implying that these are products of a gauge-Yukawa with a Banks-Zaks fixed point. Finally, at FP_9, both Yukawa couplings are non-vanishing meaning that this is effectively the product of two gauge-Yukawa fixed points.

In theories where none of the fermions carries gauge charges under both gauge groups, we have that $G_1 = 0 = G_2$. In this limit, and at the present level of approximation, the gauge sectors do not communicate with each other and the "direct product" interpretation of the fixed points as detailed above becomes exact. For the purpose of this work we will find it useful to refer to the effective "product" structure of interacting fixed points even in settings with $G_1, G_2 \neq 0$. Whether any of the fixed points is factually realised in a given theory crucially depends on the explicit values of the various loop coefficients. We defer an explicit investigation for certain "minimal models" to Sec. **III**.

D. Scalar couplings

In [144], it has been established that scalar self-interactions play no role for the primary occurrence of weakly interacting fixed points in the gauge- or gauge-Yukawa sector. On the

fixed point	gauge couplings α_1^*	α_2^*	Yukawa couplings α_Y^*	α_y^*	fixed point type
$\mathbf{FP_1}$	0	0	0	0	**G · G**
$\mathbf{FP_2}$	$\frac{B_1}{C_1}$	0	0	0	**BZ · G**
$\mathbf{FP_3}$	0	$\frac{B_2}{C_2}$	0	0	**G · BZ**
$\mathbf{FP_4}$	$\frac{B_1}{C_1'}$	0	$\frac{F_1}{E_1}\alpha_1$	0	**GY · G**
$\mathbf{FP_5}$	0	$\frac{B_2}{C_2'}$	0	$\frac{F_2}{E_2}\alpha_2$	**G · GY**
$\mathbf{FP_6}$	$\frac{C_2B_1-B_2G_1}{C_1C_2-G_1G_2}$	$\frac{C_1B_2-B_1G_2}{C_1C_2-G_1G_2}$	0	0	**BZ · BZ**
$\mathbf{FP_7}$	$\frac{C_2B_1-B_2G_1}{C_1'C_2-G_1G_2}$	$\frac{C_1'B_2-B_1G_2}{C_1'C_2-G_1G_2}$	$\frac{F_1}{E_1}\alpha_1$	0	**GY · BZ**
$\mathbf{FP_8}$	$\frac{C_2'B_1-B_2G_1}{C_1C_2'-G_1G_2}$	$\frac{C_1B_2-B_1G_2}{C_1C_2'-G_1G_2}$	0	$\frac{F_2}{E_2}\alpha_2$	**BZ · GY**
$\mathbf{FP_9}$	$\frac{C_2'B_1-B_2G_1}{C_1'C_2'-G_1G_2}$	$\frac{C_1'B_2-B_1G_2}{C_1'C_2'-G_1G_2}$	$\frac{F_1}{E_1}\alpha_1$	$\frac{F_2}{E_2}\alpha_2$	**GY · GY**

Table 8. The various types of fixed points in gauge-Yukawa theories with semi-simple gauge group $\mathcal{G}_1 \otimes \mathcal{G}_2$ and (IV.10), (IV.12). We also indicate how the nine qualitatively different fixed points can be interpreted as products of the Gaussian (G), Banks-Zaks (BZ) and gauge-Yukawa (GY) fixed points as seen from the individual gauge group factors (see main text).

other hand, for consistency, scalar couplings must nevertheless take free or interacting fixed points on their own. The necessary and sufficient conditions for this to arise have been given in [144]. Firstly, scalar couplings must take physical (real) fixed points. Secondly, the theory must display a stable ground state at the fixed point in the scalar sector. Below, we will analyse concrete models and show that both of these conditions are non-empty.

E. Universal scaling exponents

We briefly comment on the universal behaviour and scaling exponents at the interacting fixed points of Tab. **8**. Scaling exponents arise as the eigenvalues ϑ_i of the stability matrix

$$M_{ij} = \partial\beta_i/\partial\alpha_j|_* \tag{IV.13}$$

at fixed points. Negative or positive eigenvalues correspond to relevant or irrelevant couplings respectively. They imply that couplings approach the fixed point following a power-law behaviour in RG momentum scale,

$$\alpha_i(\mu) - \alpha_i^* = \sum_n c_n\, V_i^n \left(\frac{\mu}{\Lambda}\right)^{\vartheta_n} + \text{subleading}\,. \tag{IV.14}$$

Classically, we have that $\vartheta \equiv 0$. Quantum-mechanically, and at a Gaussian fixed point, eigenvalues continue to vanish and the behaviour of couplings is determined by higher order effects. Then couplings are either exactly marginal $\vartheta \equiv 0$ or marginally relevant $\vartheta \to 0^-$ or marginally irrelevant $\vartheta \to 0^+$. In a slight abuse of language we will from now on denote relevant and marginally relevant ones as $\vartheta \leq 0$, and vice versa for irrelevant ones.

Given that the scalar couplings do not feed back to the gauge-Yukawa sector at the leading non-trivial order in perturbation theory, we may neglect them for a discussion of the eigenvalue spectrum

$$\{\vartheta_i, i = 1, \cdots 4\}, \tag{IV.15}$$

related to the two gauge and Yukawa couplings. The fixed point FP_1 is Gaussian in all couplings, and the scaling of couplings are either marginally relevant or marginally irrelevant. Only if $B_i > 0$ trajectories can emanate from the Gaussian, meaning that it is a UV fixed point iff the theory is asymptotically free in both couplings. Furthermore, asymptotic freedom in the gauge couplings entails asymptotic freedom in the Yukawa couplings leading to four marginally relevant couplings with eigenvalues

$$\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \leq 0 \tag{IV.16}$$

The fixed points FP_2 and FP_3 are products of a Banks-Zaks in one gauge sector with a Gaussian fixed point in the other. Scaling exponents are then of the form

$$\vartheta_1, \vartheta_2, \vartheta_3 \leq 0 < \vartheta_4 \tag{IV.17}$$

provided the gauge sector with Gaussian fixed point is asymptotically free. For IR free gauge coupling, we instead have the pattern

$$\vartheta_1 < 0 \leq \vartheta_2, \vartheta_3, \vartheta_4 \,. \tag{IV.18}$$

At the fixed points FP_4 and FP_5, the theory is the product of a Gaussian and a gauge-Yukawa fixed point. Consequently, four possibilities arise: Provided that the theory is asymptotically safe at the gauge-Yukawa fixed point and asymptotically or infrared free at the Gaussian, scaling exponents are of the form (IV.17) or (IV.18), respectively. Conversely, if the gauge Yukawa fixed point is IR, the eigenvalue spectrum reads

$$\vartheta_1, \vartheta_2 \leq 0 \leq \vartheta_3, \vartheta_4 \tag{IV.19}$$

if the Gaussian is asymptotically free. Finally, if the Gaussian is IR free and the gauge-

Yukawa fixed point IR, all couplings are UV irrelevant and

$$0 \leq \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \,. \tag{IV.20}$$

More work is required to determine the scaling exponents at the fully interacting fixed points FP_6 – FP_9. To that end, we write the characteristic polynomial of the stability matrix as

$$\sum_{n=0}^{4} T_n \, \vartheta^n = 0 \,. \tag{IV.21}$$

The coefficients T_n are functions of the loop coefficients. Introducing $B = |B_1|$ and $B_2 = P\,B_1$, with P some free parameter, we can make a scaling analysis in the limit $B \ll 1$. Normalising the coefficient T_4 to unity, $T_4 = 1$, it then follows from the structure of the beta functions that $T_0 = \mathcal{O}(B^6), T_1 = \mathcal{O}(B^4), T_2 = \mathcal{O}(B^2)$ and $T_3 = \mathcal{O}(B)$ to leading order in B. In the limit where $B \ll 1$ we can deduce exact closed expressions for the leading order behaviour of the eigenvalues from solutions to two quadratic equations,

$$\begin{aligned} 0 &= \quad \vartheta^2 + T_3 \, \vartheta + T_2 \\ 0 &= T_2 \, \vartheta^2 + T_1 \, \vartheta + T_0 \,. \end{aligned} \tag{IV.22}$$

The general expressions are quite lengthy and shall not be given here explicitly. We note that the four eigenvalues of the four couplings at the four fully interacting fixed points FP_6 – FP_9 are the four solutions to (IV.22). Irrespective of their signs, and barring exceptional numerical cancellations, we conclude that two scaling exponents are quadratic and two are linear in B,

$$\begin{aligned} \vartheta_{1,2} &= -\frac{1}{2}\left(T_3 \pm \sqrt{T_3^2 - 4T_2}\right) = \mathcal{O}(B^2) \\ \vartheta_{3,4} &= -\frac{1}{2T_2}\left(T_1 \pm \sqrt{T_1^2 - 4T_0\,T_2}\right) = \mathcal{O}(B) \,. \end{aligned} \tag{IV.23}$$

This is reminiscent of fixed points in gauge-Yukawa theories with a simple gauge group. The main reason for the appearance of two eigenvalues of order $\mathcal{O}(B^2)$ relates to the gauge sector, where the interacting fixed point arises through the cancellation at two-loop level. Conversely, two eigenvalues of order $\mathcal{O}(B)$ relate to the Yukawa couplings, as they arise from a cancellation at one-loop level. This completes the discussion of fixed points in general weakly coupled semi-simple gauge theories.

III. MINIMAL MODELS

In this section we introduce in concrete terms a family of semi simple gauge theories whose interacting fixed points will be analysed exactly within perturbation theory in the

Veneziano limit.

A. Semi-simple gauge theory

We consider families of massless four-dimensional quantum field theories with a semi-simple gauge group

$$SU(N_{\rm C}) \times SU(N_{\rm c}) \tag{IV.24}$$

for general non-Abelian factors with $N_{\rm C} \geq 2$ and $N_{\rm c} \geq 2$. Specifically, our models contains $SU(N_{\rm C})$ gauge fields A_μ with field strength $F_{\mu\nu}$, and $SU(N_{\rm c})$ gauge fields a_μ with field strength $f_{\mu\nu}$. The gauge fields are coupled to $N_{\rm F}$ flavours of fermions Q_i, $N_{\rm f}$ flavours of fermions q_i, and N_ψ flavours of fermions ψ_i. The fermions (Q, q, ψ) transform in the fundamental representation of the first, the second, and both gauge group(s) (IV.24), respectively, as summarised in Tab. **9**. The Dirac fermions ψ are responsible for the semi-simple character of the theory and serve as messengers to communicate between gauge sectors. All fermions are Dirac to guarantee anomaly cancellation. The fermions (Q, q) additionally couple via Yukawa interactions to an $N_{\rm F} \times N_{\rm F}$ matrix scalar field H and an $N_{\rm f} \times N_{\rm f}$ matrix scalar field h, respectively. The scalars H and h are invariant under $U(N_{\rm F})_L \times U(N_{\rm F})_R$ and $U(N_{\rm f})_L \times U(N_{\rm f})_R$ global flavour rotations, respectively, and singlets under the gauge symmetry. They can be viewed as elementary mesons in that they carry the same global quantum numbers as the singlet scalar bound states $\sim \langle Q\bar{Q}\rangle$ and $\sim \langle q\bar{q}\rangle$. The fermions ψ are not furnished with Yukawa interactions.

The fundamental action is taken to be the sum of the individual Yang-Mills actions, the fermion kinetic terms, the Yukawa interactions, and the scalar kinetic and self-interaction Lagrangeans $L = L_{\rm YM} + L_F + L_Y + L_S + L_{\rm pot}$, with

$$\begin{aligned}
L_{\rm YM} &= -\frac{1}{2}{\rm Tr}\, F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}{\rm Tr}\, f^{\mu\nu}f_{\mu\nu} \\
L_F &= {\rm Tr}\left(\overline{Q}\, i\not{D}\, Q\right) + {\rm Tr}\left(\overline{q}\, i\not{D}\, q\right) + {\rm Tr}\left(\overline{\psi}\, i\not{D}\, \psi\right) \\
L_Y &= Y\, {\rm Tr}\left(\overline{Q}_L H Q_R + \overline{Q}_R H^\dagger Q_L\right) + y\, {\rm Tr}\left(\overline{q}_L h q_R + \overline{q}_R h^\dagger q_L\right) \\
L_S &= {\rm Tr}\,(\partial_\mu H^\dagger\, \partial^\mu H) + {\rm Tr}\,(\partial_\mu h^\dagger\, \partial^\mu h) \\
L_{\rm pot} &= -U\,{\rm Tr}\,(H^\dagger H)^2 - V\,({\rm Tr}\, H^\dagger H)^2 \\
&\qquad -u\,{\rm Tr}\,(h^\dagger h)^2 - v\,({\rm Tr}\, h^\dagger h)^2 - w\,{\rm Tr}\, H^\dagger H\, {\rm Tr}\, h^\dagger h\,.
\end{aligned} \tag{IV.25}$$

The trace Tr denotes the trace over both color and flavour indices, and the decomposition $Q = Q_L + Q_R$ with $Q_{L/R} = \frac{1}{2}(1 \pm \gamma_5)Q$ is understood for all fermions Q and q. Mass terms are neglected at the present stage as their effect is subleading to the main features developed below. In four dimensions, the theory is renormalisable in perturbation theory.

The theory has nine classically marginal coupling constants given by the two gauge

representation	**fermions**			**scalars**		**gauge fields**	
	Q	q	ψ	H	h	A_μ	a_μ
under $SU(N_C)$	N_C	1	N_C	1	1	N_C^2-1	1
under $SU(N_c)$	1	N_c	N_c	1	1	1	N_c^2-1
multiplicity	N_F	N_f	N_ψ	N_F^2	N_f^2	1	1

Table 9. Representation under the semi-simple gauge symmetry (IV.24) together with flavour multiplicities of all fields. Gauge (fermion) fields are either in the adjoint (fundamental) or trivial representation.

couplings, the two Yukawa couplings, and five quartic scalar couplings. We write them as

$$\begin{aligned}
&\alpha_1 = \frac{g_1^2\,N_\mathrm{C}}{(4\pi)^2}\,, \quad \alpha_2 = \frac{g_2^2\,N_\mathrm{c}}{(4\pi)^2}\,, \quad \alpha_Y = \frac{Y^2\,N_\mathrm{C}}{(4\pi)^2}\,, \quad \alpha_y = \frac{y^2\,N_\mathrm{c}}{(4\pi)^2}\,, \\
&\alpha_U = \frac{u\,N_\mathrm{F}}{(4\pi)^2}\,, \quad \alpha_V = \frac{v\,N_\mathrm{F}^2}{(4\pi)^2}\,, \quad \alpha_u = \frac{u\,N_\mathrm{f}}{(4\pi)^2}\,, \quad \alpha_v = \frac{v\,N_\mathrm{f}^2}{(4\pi)^2}\,,
\end{aligned} \tag{IV.26}$$

where we have normalized the couplings with the appropriate loop factor and powers of N_C, N_c, N_F and N_f in view of the Veneziano limit to be adopted below. Notice the additional power of N_F and N_f in the definitions of the scalar double-trace couplings. We normalise the quartic "portal" coupling as

$$\alpha_w = \frac{w\,N_\mathrm{F}N_\mathrm{f}}{(4\pi)^2}\,. \tag{IV.27}$$

It is responsible for a mixing amongst the scalar sectors starting at tree level. Below, we use the shorthand notation $\beta_i \equiv \partial_t\alpha_i$ with $i = (1, 2, Y, y, U, u, V, v, w)$ to indicate the β-functions for the couplings (IV.26). To obtain explicit expressions for these, we exploit the formal results summarised in [74, 75, 77]. The semi-simple character of the theory is switched off if the N_ψ messenger fermions (which carry charges under both gauge groups) are replaced by N_1 and N_2 Yukawa-less fermions in the fundamental of $SU(N_C)$ and $SU(N_c)$, respectively, with

$$\begin{aligned}
N_1 &= N_c\,N_\psi\,, \\
N_2 &= N_C\,N_\psi\,.
\end{aligned} \tag{IV.28}$$

If in addition $\alpha_w = 0$, the theories (IV.25) reduce to a "direct product" of simple gauge Yukawa theories with (IV.28). Also, in the limit where one of the gauge groups is switched off, $\alpha_1 \equiv 0$ (or $\alpha_2 \equiv 0$), one gauge sector and the scalars decouples straightaway, and we are left with a simple gauge theory. Finally, if $N_1 = 0 = N_2$, we recover the models of [55] in each gauge sector (displaying asymptotic safety for certain field multiplicities). Below, we will find it useful to contrast results with those from the "direct product" limit.

B. Free parameters and Veneziano limit

We now discuss the set of fundamentally free parameters of our models. On the level of the Lagrangean, the free parameters of the theory are the matter field multiplicities

$$N_{\rm C}\,,\quad N_{\rm c}\,,\quad N_{\rm F}\,,\quad N_{\rm f}\,,\quad N_\psi\,. \tag{IV.29}$$

Notice that the N_ψ fermions ψ are centrally responsible for interactions between the gauge sectors. In the limit

$$N_\psi = 0 \tag{IV.30}$$

the interaction between gauge sectors reduces to effects mediated by the portal coupling $\alpha_w \neq 0$, which are strongly loop-suppressed. In this limit, results for fixed points and running couplings fall back to those for the individual gauge sectors [55]. Results for fixed points for general N_ψ are deferred to App. **A**. Here, we will set N_ψ to a finite value,

$$N_\psi = 1\,. \tag{IV.31}$$

This leaves us with four free parameters. In order to achieve exact perturbativity, we perform a Veneziano limit [116] by sending the number of colors and the number of flavours $(N_{\rm C}, N_{\rm c}, N_{\rm F}, N_{\rm f})$ to infinity but keeping their ratios fixed. This reduces the set of free parameters of the model down to three, which we chose to be

$$R = \frac{N_{\rm c}}{N_{\rm C}}\,,\quad S = \frac{N_{\rm F}}{N_{\rm C}}\,,\quad T = \frac{N_{\rm f}}{N_{\rm c}}\,. \tag{IV.32}$$

The ratio

$$F = \frac{N_{\rm f}}{N_{\rm F}} \tag{IV.33}$$

is then no longer a free parameter, but fixed as $F = RT/S$ from (IV.32). By their very definition, the parameters (IV.32) are positive semi-definite and can take values $0 \leq F, R, S, T \leq \infty$. However, we will see below that their values are further constrained if we impose perturbativity for all couplings.

C. Perturbativity to leading order

The RG evolution of couplings is analysed within the perturbative loop expansion. To leading order (LO), the running of the gauge couplings reads $\beta_i = -B_i\,\alpha_i^2$ (no sum), with B_i the one-loop gauge coefficients for the gauge coupling α_i. In the Veneziano limit, the one-loop coefficients take the form

$$B_i = -\frac{4}{3}\epsilon_i \,. \tag{IV.34}$$

In terms of (IV.32) and in the Veneziano limit, the parameters ϵ_i are given by

$$\begin{aligned} \epsilon_1 &= S + R - \frac{11}{2}\,, \\ \epsilon_2 &= T + \frac{1}{R} - \frac{11}{2}\,. \end{aligned} \tag{IV.35}$$

We can therefore trade the free parameters (S, T) defined in (IV.32) for (ϵ_1, ϵ_2) and consider the set

$$(\epsilon_1, \epsilon_2, R) \tag{IV.36}$$

as free parameters which characterise the matter content of the theory. Under the exchange of gauge groups we have

$$(\epsilon_1, \epsilon_2, R) \to (\epsilon_2, \epsilon_1, R^{-1})\,. \tag{IV.37}$$

For fixed R, we observe that $R - \frac{11}{2} \leq \epsilon_1 < \infty$ and $1/R - \frac{11}{2} \leq \epsilon_2 < \infty$. Perturbativity in either of the gauge couplings requires that both one-loop coefficients B_i are parametrically small compared to unity. Therefore we impose

$$0 < |\epsilon_i| \ll 1\,. \tag{IV.38}$$

This requirement of exact perturbativity in both gauge sectors entails the important constraint

$$\frac{2}{11} < R < \frac{11}{2}\,. \tag{IV.39}$$

Outside of this range, no physical values for S and T can be found such that (IV.38) holds true. Inside this range, physical values are constrained within $0 \leq S, T \leq \frac{11}{2} - \frac{2}{11}$. The parameters (IV.36) have a simple interpretation. The small parameters ϵ_i control the perturbativity within each of the gauge sectors, whereas the parameter R controls the "interactions" between the two gauge sectors. It is the presence of R which makes these theories intrinsically semi-simple, rather than being the direct product of two simple gauge theories. Perturbativity is no longer required in the limit where one of the gauge sectors is switched off, and the constraint (IV.39) is relaxed into

$$\begin{aligned} 0 \leq R < \frac{11}{2} \quad &\text{if} \quad \alpha_2^* \equiv 0\,, \\ \frac{2}{11} < R < \infty \quad &\text{if} \quad \alpha_1^* \equiv 0\,. \end{aligned} \tag{IV.40}$$

The parametrisation (IV.36) is most convenient for expressing the relevant RG beta functions for all couplings.

Finally, for some of the subsequent considerations we replace the two small parameters (ϵ_1, ϵ_2) by (ϵ, P), a single small parameter ϵ proportional to ϵ_1 together with a parameter P related to the ratio between ϵ_1 and ϵ_2. Specifically, we introduce

$$\begin{aligned}\epsilon_1 &= R\,\epsilon\,,\\ \epsilon_2 &= P\,\frac{\epsilon}{R}\,.\end{aligned} \tag{IV.41}$$

which is equivalent to $P = R^2\,\epsilon_2/\epsilon_1$ together with $\epsilon = \epsilon_1/R$ and $\epsilon = P\,R\,\epsilon_2$.[34] Since R can only take finite positive values, the additional rescaling with R does not affect the relative sign between ϵ_1 and ϵ. In this manner we have traded the free parameters $(\epsilon_1, \epsilon_2, R)$ for

$$(R, P, \epsilon)\,. \tag{IV.42}$$

Notice that the parameter P can be expressed as

$$P = \frac{1 + (N_\mathrm{f} - \frac{11}{2}N_\mathrm{c})/N_\mathrm{C}}{1 + (N_\mathrm{F} - \frac{11}{2}N_\mathrm{C})/N_\mathrm{c}} \tag{IV.43}$$

in terms of the field multiplicities (IV.29). It thus may take any real value of either sign with $-\infty < P < \infty$, whereas R must take values within the range (IV.39). Moreover,

$$\epsilon = 1 + \frac{N_\mathrm{F} - \frac{11}{2}N_\mathrm{C}}{N_\mathrm{c}}\,. \tag{IV.44}$$

In this parametrisation, the ratio of fermion flavour multiplicities (IV.33) becomes

$$F = \frac{11R-2}{11-2R} + \frac{2R}{11-2R}\left(\frac{P}{R} - \frac{11R-2}{11-2R}\right)\epsilon + \mathcal{O}(\epsilon^2)\,. \tag{IV.45}$$

We also observe that the substitution

$$(R, P, \epsilon) \to \left(R^{-1}, P^{-1}, P\epsilon\right) \tag{IV.46}$$

relates to the exchange of gauge groups. The parametrisation (IV.42) is most convenient for analysing the various interacting fixed points and their scaling exponents (see below). This completes the definition of our models.

D. Anomalous dimensions

We provide results for the anomalous dimensions associated to the fermions and scalars. Furthermore, if mass terms are present, their renormalisation is induced through the RG

[34] The choice (IV.41) can be motivated by dimensional analysis of (IV.35) which shows that ϵ_1 and ϵ_2 formally scale as $\sim R$ and $\sim 1/R$ for large or small R, respectively, whereby their ratio ϵ_1/ϵ_1 scales as $\sim R^2$. The large-R behaviour is factored-out by our parametrisation.

flow of the gauge, Yukawa, and scalar couplings. Following [55], we define the scalar anomalous dimensions as $\Delta_S = 1 + \gamma_S$, where $\gamma_S \equiv \frac{1}{2} d \ln Z_S / d \ln \mu$ and $S = H, h$. Within perturbation theory, the one and two loop contributions read

$$\begin{aligned}\gamma_H &= \alpha_Y - \frac{3}{2}\left(\frac{11}{2} - \epsilon_1 - R\right)\alpha_Y^2 + \frac{5}{2}\alpha_Y\,\alpha_1 + 2\alpha_U^2 + \mathcal{O}(\alpha^3)\,,\\ \gamma_h &= \alpha_y - \frac{3}{2}\left(\frac{11}{2} + \epsilon_2 - \frac{1}{R}\right)\alpha_y^2 + \frac{5}{2}\alpha_y\,\alpha_2 + 2\alpha_u^2 + \mathcal{O}(\alpha^3)\,.\end{aligned}\tag{IV.47}$$

For the fermion anomalous dimensions $\gamma_F \equiv d \ln Z_F / d \ln \mu$ with $F = Q, q, \psi$, we find

$$\begin{aligned}\gamma_Q &= \left(\frac{11}{2} + \epsilon_1 - R\right)\alpha_Y + \xi_1\,\alpha_1 + \mathcal{O}(\alpha^2)\,,\\ \gamma_q &= \left(\frac{11}{2} + \epsilon_2 - \frac{1}{R}\right)\alpha_Y + \xi_2\,\alpha_2 + \mathcal{O}(\alpha^2)\,,\\ \gamma_\psi &= \xi_1\,\alpha_1 + \xi_2\,\alpha_2 + \mathcal{O}(\alpha^2)\,,\end{aligned}\tag{IV.48}$$

where ξ_1 and ξ_2 denote the gauge fixing parameters for the first and second gauge group respectively.

The anomalous dimension for the scalar mass terms can be derived from the composite operator $\sim M^2\,\mathrm{Tr}\,H^\dagger H$ and $\sim m^2\,\mathrm{Tr}\,h^\dagger h$. Introducing the mass anomalous dimension $\gamma_M = d \ln M^2 / d \ln \mu$, and similarly for m, one finds

$$\begin{aligned}\gamma_M &= 8\alpha_U + 4\alpha_V + 2\alpha_Y + \mathcal{O}(\alpha^2)\\ \gamma_m &= 8\alpha_u + 4\alpha_v + 2\alpha_y + \mathcal{O}(\alpha^2)\,,\end{aligned}\tag{IV.49}$$

to one-loop order. We also compute the running of the mass terms for the scalars

$$\begin{aligned}\beta_{M^2} &= \gamma_M\,M^2 + 2F\,m^2\,\alpha_w + \mathcal{O}(\alpha^2, \alpha\,m_F^2)\,,\\ \beta_{m^2} &= \gamma_m\,m^2 + 2F^{-1}\,M^2\,\alpha_w + \mathcal{O}(\alpha^2, \alpha\,m_F^2)\,,\end{aligned}\tag{IV.50}$$

where the parameter $F \equiv N_\mathrm{f}/N_\mathrm{F}$ solely depends on R to leading order in ϵ, see (IV.45). Notice that the coupling α_w induces a mixing amongst the different scalar masses already at one-loop level.

Analogously, the anomalous dimension for the fermion mass operator is defined as $\Delta_F = 3 + \gamma_{M_F}$ with $\gamma_{M_F} \equiv d \ln M_F / d \ln \mu$, and M_F stands for one of the fermion masses with $F = Q, q$ or ψ. Within perturbation theory, the one loop contributions read

$$\begin{aligned}\gamma_{M_Q} &= \alpha_Y\,\left(\frac{13}{2} + \epsilon_1 - R\right) - 3\,\alpha_1 - + \mathcal{O}(\alpha^2)\,,\\ \gamma_{M_q} &= \alpha_y\,\left(\frac{13}{2} + \epsilon_2 - \frac{1}{R}\right) - 3\,\alpha_2 + \mathcal{O}(\alpha^2)\\ \gamma_{M_\psi} &= -3\,(\alpha_1 + \alpha_2) + \mathcal{O}(\alpha^2)\,.\end{aligned}\tag{IV.51}$$

For the fermion masses we have the running

$$\begin{aligned}\beta_{m_Q} &= \gamma_{M_Q}\, m_Q\,,\\ \beta_{m_q} &= \gamma_{M_q}\, m_q\,,\\ \beta_{m_\psi} &= \gamma_{M_\psi}\, m_\psi\,.\end{aligned} \tag{IV.52}$$

We note that γ_{M_ψ} is manifestly negative. For γ_{M_Q} and γ_{M_q} we observe that the gauge and Yukawa contributions arise with manifestly opposite signs in the parameter regime (IV.38), (IV.39). Hence either of these may take either sign, depending on whether the gauge or Yukawa contributions dominate.

E. Running couplings beyond the leading order

We now go beyond the leading order in perturbation theory and provide the complete, minimal set of RG equations which display exact and weakly interacting fixed points. To that end, we must retain terms up to two loop order in the gauge coupling, or else an interacting fixed point cannot arise. At the same time, in order to explore the feasibility of asymptotically safe UV fixed points we must retain the Yukawa couplings [144], minimally at the leading non-trivial order which is one loop. Following [55] we refer to this level of approximation in the gauge-Yukawa sector as next-to-leading order (NLO). In the presence of scalar fields, we also must retain the quartic scalar couplings at their leading non-trivial order. We refer to this approximation of the gauge-Yukawa-scalar sector as NLO$'$ [58], see Tab. **7**. This is the minimal order in perturbation theory at which a fully interacting fixed point can be determined in all couplings with canonically vanishing mass dimension.

In general, the RG flow for the gauge and Yukawa couplings at NLO$'$ is strictly independent of the scalar couplings owing to the fact that scalar loops only arise starting from the two loop order in the Yukawa sector, and at three (four) loop order in the gauge sector, if the scalars are charged (uncharged). Furthermore, the scalar sector at NLO$'$ depends on the Yukawa couplings, but not on the gauge couplings owing to the fact that the scalars are uncharged. Consequently, we observe a partial decoupling of the gauge-Yukawa sector $(\alpha_1, \alpha_2, \alpha_Y, \alpha_y)$ and the scalar sector $(\alpha_U, \alpha_V, \alpha_u, \alpha_v, \alpha_w)$. This structure will be exploited systematically below to identify all interacting fixed points.

We begin with the gauge-Yukawa sector where we find the coupled beta functions (IV.10) which are characterised by ten loop coefficients C_i, D_i, E_i, F_i and G_i $(i = 1, 2)$, together with the coefficients B_i given in (IV.34) or, equivalently, the perturbative control parameters (IV.35). The one-loop coefficients arise in the Yukawa sector and take the values

$$\begin{aligned} E_1 &= 13 + 2\,(\epsilon_1 - R)\;, \qquad & F_1 &= 6\,, \\ E_2 &= 13 + 2\left(\epsilon_2 - \frac{1}{R}\right)\;, \qquad & F_2 &= 6\,. \end{aligned} \tag{IV.53}$$

At the two-loop level we have six coefficients related to the gauge, Yukawa, and mixing contribution, which are found to be

$$\begin{aligned} C_1 &= 25 + \frac{26}{3}\epsilon_1\,, \quad & D_1 &= 2\left(\epsilon_1 - R + \frac{11}{2}\right)^2\;, \quad & G_1 &= 2R \\ C_2 &= 25 + \frac{26}{3}\epsilon_2\,, \quad & D_2 &= 2\left(\epsilon_2 - \frac{1}{R} + \frac{11}{2}\right)^2\;, \quad & G_2 &= \frac{2}{R} \end{aligned} \tag{IV.54}$$

A few comments are in order. Firstly, the loop coefficients $D_i, E_i, F_i, G_i > 0$ as they must for any quantum field theory. Additionally we confirm that $C_i > 0$ [144], provided the parameters ϵ_i are in the perturbative regime (IV.38). Secondly, provided that $R = 0$ in the expressions for ϵ_1, E_1 and G_1, and $1/R = 0$ in those for ϵ_2, E_2 and G_2, the system (IV.10) falls back onto a direct product of simple gauge-Yukawa theories, each of the type discussed in [55]. Notice that this limit cannot be achieved parametrically in R. The reason for this is the presence of N_ψ fermions which are charged under both gauge groups. They contribute with reciprocal multiplicity $R \leftrightarrow 1/R$ to the Yukawa-induced loop terms D_i and E_i as well as to the mixing terms G_i. Exact decoupling of the gauge sectors then becomes visible only in the parametric limit where $N_\psi \to 0$ whereby all terms involving R or $1/R$ drop out. Finally, we note that the exchange of gauge groups $\mathcal{G}_1 \leftrightarrow \mathcal{G}_2$ corresponds to $R \leftrightarrow 1/R$ and $S \leftrightarrow T$, implying $\epsilon_1 \leftrightarrow \epsilon_2$ and $P \leftrightarrow 1/P$, respectively. Evidently, at the symmetric point $R = 1$ and $\epsilon_1 = \epsilon_2$ (or $P = 1$) we have exact exchange symmetry between gauge group factors.

Inserting (IV.53), (IV.54) and (IV.34) into the general expression (IV.10), we obtain the perturbative RG flow for the gauge-Yukawa system at NLO accuracy

$$\begin{aligned} \beta_1 &= \frac{4}{3}\epsilon_1\,\alpha_1^2 + \left(25 + \frac{26}{3}\epsilon_1\right)\alpha_1^3 - 2\left(\epsilon_1 - R + \frac{11}{2}\right)^2 \alpha_1^2\,\alpha_Y + 2R\,\alpha_1^2\,\alpha_2\,, \\ \beta_2 &= \frac{4}{3}\epsilon_2\,\alpha_2^2 + \left(25 + \frac{26}{3}\epsilon_2\right)\alpha_2^3 - 2\left(\epsilon_2 - \frac{1}{R} + \frac{11}{2}\right)^2 \alpha_2^2\,\alpha_y + \frac{2}{R}\,\alpha_2^2\,\alpha_1\,, \\ \beta_Y &= \big[13 + 2\,(\epsilon_1 - R)\,\big]\alpha_Y^2\; - 6\,\alpha_Y\,\alpha_1\,, \\ \beta_y &= \big[13 + 2\left(\epsilon_2 - \tfrac{1}{R}\right)\big]\,\alpha_y^2 - 6\,\alpha_y\,\alpha_2\,. \end{aligned} \tag{IV.55}$$

We observe that the running of Yukawa couplings at one loop is determined by the fermion mass anomalous dimension (IV.51),

$$\begin{aligned} \beta_Y &= 2\,\gamma_{M_Q}\,\alpha_Y\,, \\ \beta_y &= 2\,\gamma_{M_q}\,\alpha_y\,. \end{aligned} \tag{IV.56}$$

The result for the mass anomalous dimensions (IV.51) can also be derived diagrammatically from the flow of the Yukawa vertices (IV.55), thus offering an independent confirmation for the link (IV.56).

Next, we turn to the scalar sector and the running of quartic couplings to leading order in perturbation theory, which is one loop. At NLO$'$ accuracy, we have (IV.55) together with the beta functions for the quartic scalar couplings which are found to be

$$\begin{aligned}
\beta_U &= -[11 + 2(\epsilon_1 - R)]\alpha_Y^2 + 4\alpha_U(\alpha_Y + 2\alpha_U)\,, \\
\beta_V &= \ \ 12\alpha_U^2 + 4\alpha_V(\alpha_V + 4\alpha_U + \alpha_Y) + \alpha_w^2\,, \\
\beta_u &= -[11 + 2(\epsilon_2 - \tfrac{1}{R})]\alpha_y^2 + 4\alpha_u(\alpha_y + 2\alpha_u)\,, \\
\beta_v &= \ \ 12\alpha_u^2 + 4\alpha_v(\alpha_v + 4\alpha_u + \alpha_y) + \alpha_w^2\,, \\
\beta_w &= \ \ \alpha_w\left[8(\alpha_U + \alpha_u) + 4(\alpha_V + \alpha_v) + 2(\alpha_Y + \alpha_y)\right] .
\end{aligned} \tag{IV.57}$$

Their structure is worth a few remarks: Firstly, in the Veneziano limit, β_w contains no term quadratic in the coupling α_w as the coefficient is of the order $\mathcal{O}(N_\mathrm{F}^{-1}N_\mathrm{f}^{-1})$ and suppressed by inverse powers in flavour multiplicities. Secondly, we notice that β_w comes out proportional to α_w. Consequently, α_w is a technically natural coupling according to the rationale of [123], unlike all the other quartic interactions, implying that

$$\alpha_w^* = 0 \tag{IV.58}$$

constitutes an exact fixed point of the theory. Comparison with (IV.49) shows that the proportionality factor is the sum of the scalar mass anomalous dimensions, $\beta_w = \alpha_w(\gamma_M + \gamma_m)$. The quartic coupling α_w would be promoted to a free parameter characterising a line of fixed points with exactly marginal scaling provided that its beta function vanishes identically at one loop. This would require the vanishing of the sum of scalar anomalous mass dimensions at the fixed point,

$$\gamma_M^* + \gamma_m^* = 0\,. \tag{IV.59}$$

Below, however, we will establish that such scenarios are incompatible with vacuum stability (see Sect. **V**). Moreover, at interacting fixed points we invariably find that

$$\gamma_M^* + \gamma_m^* > 0 \tag{IV.60}$$

as a consequence of vacuum stability. This implies that α_w constitutes an infrared free coupling at any interacting fixed point with a stable ground state. For the purpose of the present study, we therefore limit ourselves to fixed points with (IV.58). We then observe that the running of the remaining scalar couplings is solely fuelled by the Yukawa couplings. Furthermore, the scalar subsectors associated to the different gauge groups are disentangled in our approximation.[35] Interestingly, the beta functions for (α_U, α_V) and (α_u, α_v) are related by the substitution $R \leftrightarrow 1/R$ and $\epsilon_1 \leftrightarrow \epsilon_2$. Moreover, the double trace scalar couplings do not couple back into any of the other couplings and their fixed points are entirely dictated by the corresponding single trace scalar and the Yukawa coupling [55]. This structure allows for a straightforward systematic analysis of all weakly coupled fixed points of the theory to which we turn next.

IV. INTERACTING FIXED POINTS

In this section, we present our results for exact fixed points in the Veneziano limit, corresponding to interacting conformal field theories, and the universal scaling exponents in their vicinity.

A. Parameter space

In Tab. **10** we state our results for the gauge and Yukawa couplings to leading order in (IV.38) at all fixed points, following the nomenclature of Tab. **8**. Expressions are given as functions of the parameters (P, R, ϵ),

$$\begin{aligned} P &= \frac{1 + (N_\mathrm{f} - \frac{11}{2}N_\mathrm{c})/N_\mathrm{C}}{1 + (N_\mathrm{F} - \frac{11}{2}N_\mathrm{C})/N_\mathrm{c}} \\ R &= \frac{N_\mathrm{c}}{N_\mathrm{C}} \\ \mathrm{sgn}\,\epsilon &= \mathrm{sgn}\left(N_\mathrm{c} + N_\mathrm{F} - \tfrac{11}{2}N_\mathrm{C}\right) \,, \end{aligned} \tag{IV.61}$$

which only depend on the matter and gauge field multiplicities (IV.29), and $N_\psi = 1$. Results for general N_ψ are given in App. **A**. We also observe (IV.39), unless stated otherwise. Constraints on the parameters (R, P, ϵ) and other information is summarised Figs. **6**, **7**, **8**, **9** and **10** and in Tabs. **11** **12**, **13** for the various fixed points. Below, certain characteristic values $\frac{2}{11} < R_1 < R_2 < R_3 < R_4 < \frac{11}{2}$ for the parameter R are of particular interest, namely

$$\begin{aligned} R_1 &= \frac{343 - 3\sqrt{9361}}{100} \approx 0.53 \,, \\ R_2 &= \frac{43 - 9\sqrt{5}}{38} \approx 0.60 \,, \\ R_3 &= \frac{43 + 9\sqrt{5}}{38} \approx 1.66 \,, \\ R_4 &= \frac{343 + 3\sqrt{9361}}{100} \approx 1.90 \,. \end{aligned} \tag{IV.62}$$

[35] The degeneracy is lifted as soon as the quartic coupling $\alpha_w \neq 0$, see (IV.25), (IV.26).

Their origin is explained in App. **B**. After these preliminaries we are in a position to analyse the fixed point spectra.

B. Partially and fully interacting fixed points

Gauge theories with (IV.55), (IV.57) can have two types of interacting fixed points: partially interacting ones where one gauge coupling takes the Gaussian fixed point (FP_2,FP_3,FP_4,FP_5), and fully interacting ones where both gauge sectors remain interacting (FP_6,FP_7,FP_8,FP_9), see Tab. **8**. At partially interacting fixed points, one gauge sector decouples and the semi-simple theory with (IV.55), (IV.57) effectively reduces to a simple gauge theory. Simple gauge theories have three possible types of perturbative fixed points: the Gaussian (G), the Banks-Zaks (BZ), and gauge-Yukawa (GY) fixed points for each independent linear combination of the Yukawa couplings [144]. In our setting, at FP_2 and FP_4 we have that $\alpha_2^* \equiv 0$, and the theory reduces to a simple gauge theory with

$$\begin{aligned}
\beta_1 &= \frac{4}{3}\epsilon_1\,\alpha_1^2 + \left(25 + \frac{26}{3}\epsilon_1\right)\alpha_1^3 - 2\left(\epsilon_1 - R + \frac{11}{2}\right)^2 \alpha_1^2\,\alpha_Y \\
\beta_Y &= \big[13 + 2\,(\epsilon_1 - R)\big]\alpha_Y^2 \; - 6\,\alpha_Y\,\alpha_1 \\
\beta_U &= -[11 + 2(\epsilon_1 - R)]\alpha_Y^2 + 4\alpha_U(\alpha_Y + 2\alpha_U)\,, \\
\beta_V &= 12\alpha_U^2 + 4\alpha_V(\alpha_V + 4\alpha_U + \alpha_Y)\,,
\end{aligned} \tag{IV.63}$$

#	gauge couplings	Yukawa couplings	type
$\mathbf{FP_1}$	$\alpha_1^* = 0\,, \quad \alpha_2^* = 0\,,$	$\alpha_Y^* = 0\,, \quad \alpha_y^* = 0\,,$	**G · G**
$\mathbf{FP_2}$	$\alpha_1^* = -\frac{4}{75} R\epsilon\,, \quad \alpha_2^* = 0\,,$	$\alpha_Y^* = 0\,, \quad \alpha_y^* = 0\,,$	**BZ · G**
$\mathbf{FP_3}$	$\alpha_1^* = 0\,, \quad \alpha_2^* = -\frac{4}{75}\frac{P\epsilon}{R}\,,$	$\alpha_Y^* = 0\,, \quad \alpha_y^* = 0\,,$	**G · BZ**
$\mathbf{FP_4}$	$\alpha_1^* = \frac{2}{3}\frac{(13-2R)R\epsilon}{(2R-1)(3R-19)}\,, \quad \alpha_2^* = 0\,,$	$\alpha_Y^* = \frac{4R\epsilon}{(2R-1)(3R-19)}\,, \quad \alpha_y^* = 0\,,$	**GY · G**
$\mathbf{FP_5}$	$\alpha_1^* = 0\,, \quad \alpha_2^* = \frac{2}{3}\frac{(13-2/R)}{(2/R-1)(3/R-19)}\frac{P\epsilon}{R}\,,$	$\alpha_Y^* = 0\,, \quad \alpha_y^* = \frac{4\,P\epsilon/R}{(2/R-1)(3/R-19)}\,,$	**G · GY**
$\mathbf{FP_6}$	$\alpha_1^* = \frac{-4(25-2P/R)}{1863}R\epsilon\,, \quad \alpha_2^* = \frac{-4(25-2R/P)}{1863}\frac{P\epsilon}{R}$	$\alpha_Y^* = 0\,, \quad \alpha_y^* = 0\,,$	**BZ · BZ**
$\mathbf{FP_7}$	$\alpha_1^* = \frac{2}{9}\frac{(13-2R)(25-2P/R)}{50R^2-343R+167}R\epsilon$ $\alpha_2^* = \frac{4}{9}\frac{(13-2R)R/P+(2R-1)(19-3R)}{50R^2-343R+167}\frac{P\epsilon}{R}$	$\alpha_Y^* = \frac{4}{3}\frac{25-2P/R}{50R^2-343R+167}R\epsilon$ $\alpha_y^* = 0$	**GY · BZ**
$\mathbf{FP_8}$	$\alpha_1^* = \frac{4}{9}\frac{(13-2/R)P/R+(2/R-1)(19-3/R)}{50/R^2-343/R+167}R\epsilon$ $\alpha_2^* = \frac{2}{9}\frac{(13-2/R)(25-2R/P)}{50/R^2-343/R+167}\frac{P\epsilon}{R}$	$\alpha_Y^* = 0$ $\alpha_y^* = \frac{4}{3}\frac{25-2R/P}{50/R^2-343/R+167}\frac{P\epsilon}{R}$	**BZ · GY**
$\mathbf{FP_9}$	$\alpha_1^* = \frac{2}{9}\frac{(13-2R)[(13-2/R)P/R+(\frac{2}{R}-1)(3/R-19)]}{(19R^2-43R+19)(2/R^2-13/R+2)}R\epsilon$ $\alpha_2^* = \frac{2}{9}\frac{(13-2/R)[(13-2R)R/P+(2R-1)(3R-19)]}{(19/R^2-43/R+19)(2R^2-13R+2)}\frac{P\epsilon}{R}$	$\alpha_Y^* = \frac{6\,\alpha_1^*}{13-2R}$ $\alpha_y^* = \frac{6\,\alpha_2^*}{13-2/R}$	**GY · GY**

Table 10. Gauge and Yukawa couplings at interacting fixed points following Tab. **8** to the leading order in ϵ and in terms of (R, P, ϵ). Valid domains for (ϵ, P, R) in (IV.61) are detailed in Tab. **12**, **13**.

at NLO$'$ accuracy, where the parameter R with

$$0 \leq R = \frac{N_1}{N_\mathrm{C}} < \frac{11}{2} \tag{IV.64}$$

measures the number of Yukawa-less Dirac fermions N_1 in the fundamental representation in units of N_C. Notice that N_1 is related to N_ψ via (IV.28) in the theories (IV.25). On the other hand, N_1 can be viewed as an independent parameter (counting the Yukawa-less fermions in the fundamental representation of the gauge group) if one were to switch off the semi-simple character of the theory. For $R = 0$ the theory (IV.63) reduces to the one investigated in [55]. The lower bound on R (IV.39) is relaxed in (IV.64), because the requirement of perturbativity for an interacting fixed point in the other gauge sector has become redundant. We observe the R-independent Banks-Zaks (BZ) fixed point $\alpha_1^* = \frac{4}{75}\epsilon_1$ which is, invariably, IR. To leading order in ϵ_1 we also find a gauge-Yukawa (GY) fixed point

$$\begin{aligned}
\alpha_g^* &= \frac{26-4R}{57-9R}\,\frac{\epsilon_1}{1-2R} \\
\alpha_Y^* &= \frac{4}{19-3R}\,\frac{\epsilon_1}{1-2R} \\
\alpha_U^* &= \frac{\sqrt{23-4R}-1}{19-3R}\,\frac{\epsilon_1}{1-2R} \\
\alpha_V^* &= \frac{-2\sqrt{23-4R}+\sqrt{20-4R+6\sqrt{23-4R}}}{19-3R}\,\frac{\epsilon_1}{1-2R}\,.
\end{aligned} \tag{IV.65}$$

For $\epsilon_1 > 0$, the GY fixed point is UV and physical as long as $0 \leq R < \frac{1}{2}$. It can be interpreted as a "deformation" of the UV fixed point analysed in [55] owing to the presence of charged Yukawa-less fermions. Once $R > \frac{1}{2}$, however, the fixed point is physical iff $\epsilon_1 < 0$ where it becomes an IR fixed point. This new regime is entirely due to the Yukawa-less fermions and does not arise in the model of [55]. This pattern can also be read off from the scaling exponents, which, at the gauge Yukawa fixed point and to the leading non-trivial order in ϵ_1, are given by

$$\begin{aligned}
\vartheta_g &= -\frac{104}{171}\,\frac{1-\frac{2}{13}R}{1-\frac{3}{19}R}\,\frac{\epsilon_1^2}{1-2R} \\
\vartheta_y &= \frac{4}{19}\,\frac{1}{1-\frac{3}{19}R}\,\frac{\epsilon_1}{1-2R} \\
\vartheta_u &= \frac{16\sqrt{23-4R}}{19-3R}\,\frac{\epsilon_1}{1-2R} \\
\vartheta_v &= \frac{8\sqrt{20+6\sqrt{23-4R}-4R}}{19-3R}\,\frac{\epsilon_1}{1-2R}\,.
\end{aligned} \tag{IV.66}$$

For $\epsilon_1 > 0$ and $R < \frac{1}{2}$ asymptotic safety is guaranteed with $\vartheta_g < 0 < \vartheta_y, \vartheta_u, \vartheta_v$, showing that the UV fixed point has one relevant direction. The scaling exponents reduce to those in [55] for $R = 0$. Conversely, for $\epsilon_1 < 0$ and $R > \frac{1}{2}$ the theory is asymptotically free and the interacting fixed point is fully IR attractive with $0 < \vartheta_g, \vartheta_y, \vartheta_u, \vartheta_v$. Results straightforwardly translate to the partially interacting fixed points FP_3 and FP_5 where $\alpha_1^* \equiv 0$. The explicit β-functions in the other gauge sector are found from (IV.63) – (IV.66) via the replacements $\epsilon_1 \leftrightarrow \epsilon_2$ and $R \leftrightarrow 1/R$, leading to

$$\begin{aligned}
\beta_2 =&\quad \frac{4}{3}\epsilon_2\,\alpha_2^2 + \left(25 + \frac{26}{3}\epsilon_2\right)\alpha_2^3 - 2\left(\epsilon_2 - \frac{1}{R} + \frac{11}{2}\right)^2 \alpha_2^2\,\alpha_y \\
\beta_y =&\quad \left[13 + 2\left(\epsilon_2 - \frac{1}{R}\right)\right]\alpha_y^2 - 6\,\alpha_y\,\alpha_2 \\
\beta_u =& -\left[11 + 2(\epsilon_2 - \frac{1}{R})\right]\alpha_y^2 + 4\alpha_u(\alpha_y + 2\alpha_u)\,, \\
\beta_v =&\quad 12\alpha_u^2 + 4\alpha_v(\alpha_v + 4\alpha_u + \alpha_y)\,.
\end{aligned} \tag{IV.67}$$

Evidently, the coordinates of the fully interacting gauge-Yukawa fixed point and the corresponding universal scaling exponents of (IV.67) are given by (IV.65), (IV.66) after obvious replacements. Moreover, in (IV.67) the parameter R with

$$0 \le \frac{1}{R} = \frac{N_2}{N_\mathrm{c}} < \frac{11}{2} \tag{IV.68}$$

measures the number of Yukawa-less Dirac fermions N_2 in the fundamental representation in units of N_c, see (IV.28). The only direct communication between the different gauge sectors in (IV.25) is through the off-diagonal two-loop gauge contributions G_i. Were it not for the fermions ψ which are charged under both gauge groups, the theory (IV.25) with (IV.55), (IV.57) would be the "direct product" of the simple model (IV.63), (IV.64) with its counterpart (IV.67), (IV.68). In this limit we will find nine "direct product" fixed points with scaling exponents from each pairing of the possibilities (G, BZ, GY) in each sector.

Below, we contrast findings for the full semi-simple setting (IV.55), (IV.57) with those from the "direct product" limit in order to pin-point effects which uniquely arise from the semi-simple character of the theories (IV.25).

At any of the partially interacting fixed points, the semi-simple character of the theory becomes visible in the non-interacting sector. In fact, contributions from the ψ fermions modify the effective one-loop coefficient $B_i \to B_i'$ according to

$$\begin{aligned}
\alpha_1^* = 0: &\quad B_1 \to B_1' = B_1 + G_1\,\alpha_2^* \\
\alpha_2^* = 0: &\quad B_2 \to B_2' = B_2 + G_2\,\alpha_1^*\,.
\end{aligned} \tag{IV.69}$$

No such effects can materialize in a "direct product" limit. Moreover, these contributions always arise with a positive coefficient ($B' > B$) and are absent if $N_\psi = 0$ (where $G_i = 0$). For $N_\psi \neq 0$, asymptotic freedom can thereby be changed into infrared freedom, but not the other way around. This result is due to the fact that the Yukawa couplings are tied to individual gauge groups, and so by this structure we cannot have any Yukawa contributions

#	B' coefficient
FP$_2$	$B_2' = -\frac{4}{3}\left(1 - \frac{2}{25}R/P\right)\frac{P\epsilon}{R}$
FP$_3$	$B_1' = -\frac{4}{3}\left(1 - \frac{2}{25}P/R\right)R\epsilon$
FP$_4$	$B_2' = -\frac{4}{3}(1 - X(R)/P)\frac{P\epsilon}{R}$
FP$_5$	$B_1' = -\frac{4}{3}\left(1 - P/\widetilde{X}(R)\right)R\epsilon$

Table 11. Shown are the effective one-loop coefficients B' for the non-interacting gauge coupling at FP$_2$, FP$_3$, FP$_4$ and FP$_5$, and their dependence on model parameters. $B' > 0$ corresponds to asymptotic freedom. Notice that B' changes sign across the boundaries $P = 2R/25, 25R/2, X(R)$, and $\widetilde{X}(R)$, respectively, with X and $\widetilde{X}$ given in (IV.**B**.1).

to B'. In principle, the opposite effect can equally arise: it would require Yukawa couplings which contribute to both gauge coupling β-functions, and would therefore have to involve at least one field which is charged under both gauge groups [144]. Tab. **11** shows the effective one loop coefficients at partially interacting fixed points as a function of field multiplicities.

C. Gauss with Banks-Zaks

Next, we discuss all fixed points one-by-one, and determine the valid parameter regimes (R, P, ϵ) for each of them. We recall that $N_\psi = 1$ in our models. Whenever appropriate, we also compare results with the "direct product" limit, whereby the diagonal contributions from the Yukawa-less ψ-fermions are retained but their off-diagonal contributions to the other gauge sectors suppressed (see Sect. **IV** B). This comparison allows us to quantify the effect related to the semi-simple nature of the models (IV.25).

For convenience and better visibility, we scale the axes in Figs. **6** **7**, **8**, **9** and **10** as

$$X \to \frac{X}{1 + |X|} \quad \text{where} \quad X = P \text{ or } R\,, \tag{IV.70}$$

and within their respective domains of validity $R \in (\frac{2}{11}, \frac{11}{2})$ and $P \in (-\infty, \infty)$. The rescaling permits easy display of the entire range of parameters.

Fig. **6** shows the results for FP$_2$ (BZ$\cdot$G, upper) and FP$_3$ (G$\cdot$BZ, lower panel), and parameter ranges are given in Tab. **12**. We observe that the Banks-Zaks fixed point always requires an asymptotically free gauge sector. Hence, FP$_2$ exists for any R as long as $\epsilon < 0$. Provided that $P\epsilon < 0$, the other gauge sector either remain asymptotically free (region 1) or becomes infrared free (region 2). On the other hand, if $P\epsilon > 0$, the other gauge sector is invariably infrared free. This is a consequence of (IV.69) which states that the interacting gauge sector can turn asymptotic freedom of the non-interacting gauge sector into infrared freedom (region 2), but not the other way around. The existence of the parameter region 2 is thus entirely due to the semi-simple character of the theory which cannot arise from a "direct product".

The Banks-Zaks fixed point is invariably attractive in the gauge coupling, and repulsive in the Yukawa coupling. The eigenvalue spectrum in the gauge-Yukawa sector is therefore of the form (IV.17) or (IV.18), depending on whether the free gauge sector is asymptotically free or infrared free, see Tab. **12**.

Under the exchange of gauge groups we have $(R, P, \epsilon) \leftrightarrow \left(R^{-1}, P^{-1}, P\epsilon\right)$, see (IV.46). On the level of Fig. **6** this corresponds to a simple rotation by 180 degree around the symmetric points $(R, P) = (1, 1)$ (for $P > 0$) and $(R, P) = (1, -1)$ (for $P < 0$), owing to the rescaling of parameters. Consequently, the results for FP_3 can be deduced from those at FP_2 by a simple rotation, see Fig. **6**. More generally, this exchange symmetry relates the partially interacting fixed points $\mathrm{FP}_2 \leftrightarrow \mathrm{FP}_3$ (Fig. **6**), $\mathrm{FP}_4 \leftrightarrow \mathrm{FP}_5$ (Fig. **7**), and the fully interacting fixed points $\mathrm{FP}_7 \leftrightarrow \mathrm{FP}_8$ (Fig. **9**). The exchange symmetry is manifest at

#	parameter range: sign ϵ	R	P	eigenvalue spectrum	info
FP$_1$	$\pm$	$\left(\frac{2}{11}, \frac{11}{2}\right)$	$(-\infty, +\infty)$	(IV.16), (IV.19), or (IV.20)	Gaussian
FP$_2$					Fig. **6** (upper panel)
	$-$	$\left(\frac{2}{11}, \frac{11}{2}\right)$	$\left(\frac{2}{25}R, +\infty\right)$	$\vartheta_{1,2,3} \leq 0 < \vartheta_4$	region 1
	$-$	$\left(\frac{2}{11}, \frac{11}{2}\right)$	$\left(0, \frac{2}{25}R\right)$	$\vartheta_1 < 0 \leq \vartheta_{2,3,4}$	region 2
	$-$	$\left(\frac{2}{11}, \frac{11}{2}\right)$	$(-\infty, 0)$	$\vartheta_1 < 0 \leq \vartheta_{2,3,4}$	region 3
FP$_3$					Fig. **6** (lower panel)
	$-$	$\left(\frac{2}{11}, \frac{11}{2}\right)$	$\left(0, \frac{25}{2}R\right)$	$\vartheta_{1,2,3} \leq 0 < \vartheta_4$	region 1
	$-$	$\left(\frac{2}{11}, \frac{11}{2}\right)$	$\left(\frac{25}{2}R, +\infty\right)$	$\vartheta_1 < 0 \leq \vartheta_{2,3,4}$	region 2
	$+$	$\left(\frac{2}{11}, \frac{11}{2}\right)$	$(-\infty, 0)$	$\vartheta_1 < 0 \leq \vartheta_{2,3,4}$	region 3
FP$_4$					Fig. **7** (upper panel)
	$-$	$\left(\frac{1}{2}, \frac{11}{2}\right)$	$(-\infty, X(R))$	$0 \leq \vartheta_{1,2,3,4}$	region 1 & 3
	$-$	$\left(\frac{1}{2}, \frac{11}{2}\right)$	$(X(R), +\infty)$	$\vartheta_{1,2} \leq 0 < \vartheta_{3,4}$	region 2
	$+$	$\left(\frac{2}{11}, \frac{1}{2}\right)$	$(X(R), +\infty)$	$\vartheta_1 < 0 \leq \vartheta_{2,3,4}$	region 4 & 6
	$+$	$\left(\frac{2}{11}, \frac{1}{2}\right)$	$(-\infty, X(R))$	$\vartheta_{1,2,3} \leq 0 < \vartheta_4$	region 5
FP$_5$					Fig. **7** (lower panel)
	$-$	$\left(\frac{2}{11}, 2\right)$	$\left(\widetilde{X}(R), +\infty\right)$	$0 \leq \vartheta_{1,2,3,4}$	region 1
	$-$	$\left(\frac{2}{11}, 2\right)$	$\left(0, \widetilde{X}(R)\right)$	$\vartheta_{1,2} \leq 0 < \vartheta_{3,4}$	region 2
	$+$	$\left(\frac{2}{11}, 2\right)$	$\left(-\infty, 0\right)$	$0 \leq \vartheta_{1,2,3,4}$	region 3
	$-$	$\left(2, \frac{11}{2}\right)$	$\left(-\infty, \widetilde{X}(R)\right)$	$\vartheta_1 < 0 \leq \vartheta_{2,3,4}$	region 4
	$-$	$\left(2, \frac{11}{2}\right)$	$\left(\widetilde{X}(R), 0\right)$	$\vartheta_{1,2,3} \leq 0 < \vartheta_4$	region 5
	$+$	$\left(2, \frac{11}{2}\right)$	$(0, +\infty)$	$\vartheta_1 < 0 \leq \vartheta_{2,3,4}$	region 6

Table 12. Parameter regions where the partially interacting fixed points $\mathrm{FP}_1 - \mathrm{FP}_5$ exist, along with regions of relevancy for eigenvalues and effective one-loop terms, where applicable. The boundary functions $X(R)$ and $\widetilde{X}(R)$ are given in (IV.**B**.1). The coefficient B' for the gauge coupling at the Gaussian fixed point is given in Tab. **11**.

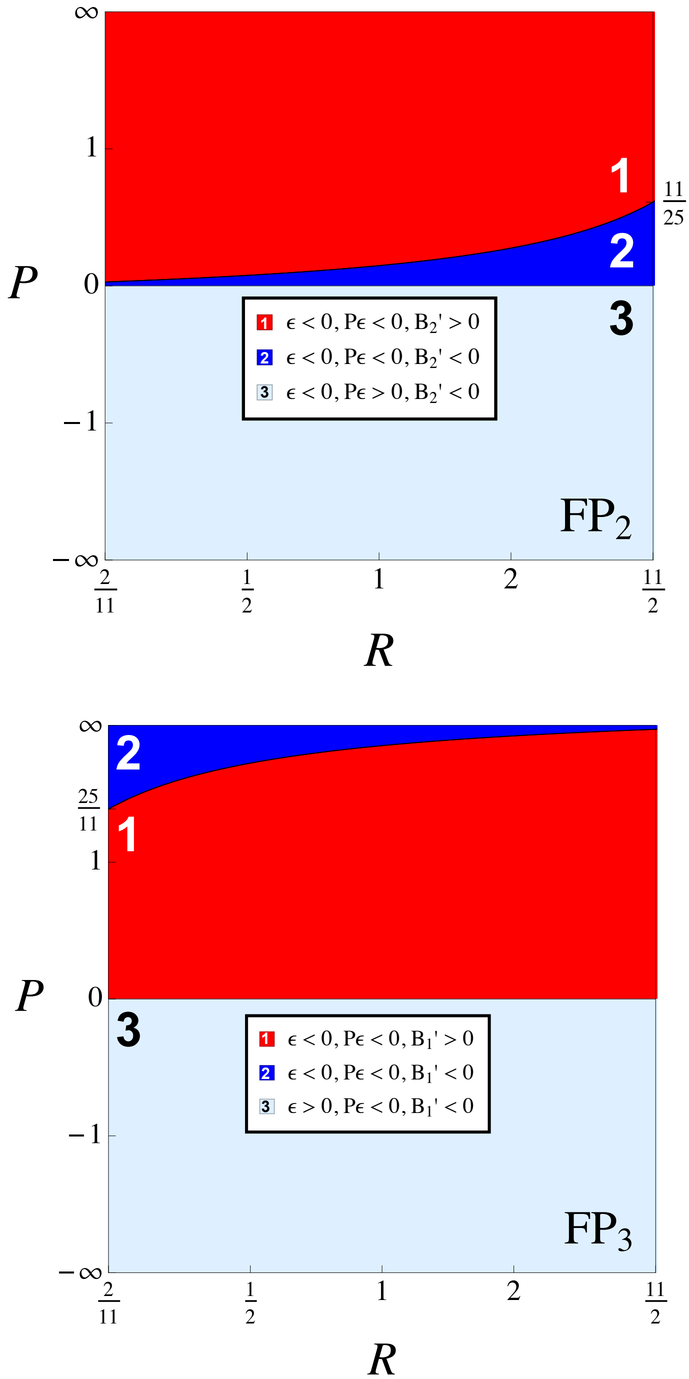

Figure 6. The phase space of parameters (IV.61) for the partially interacting fixed points FP_2 (upper panel) and FP_3 (lower panel) where one of the two gauge sectors remains interacting and all Yukawa couplings vanish. The inset indicates the different parameter regions and conditions for existence, including whether the non-interacting gauge sector is asymptotically free ($B' > 0$) or infrared free ($B' < 0$), see Tab. **11**,**12**.

the fully interacting fixed points FP_6 (Fig. **8**) and FP_9 (Fig. **10**).

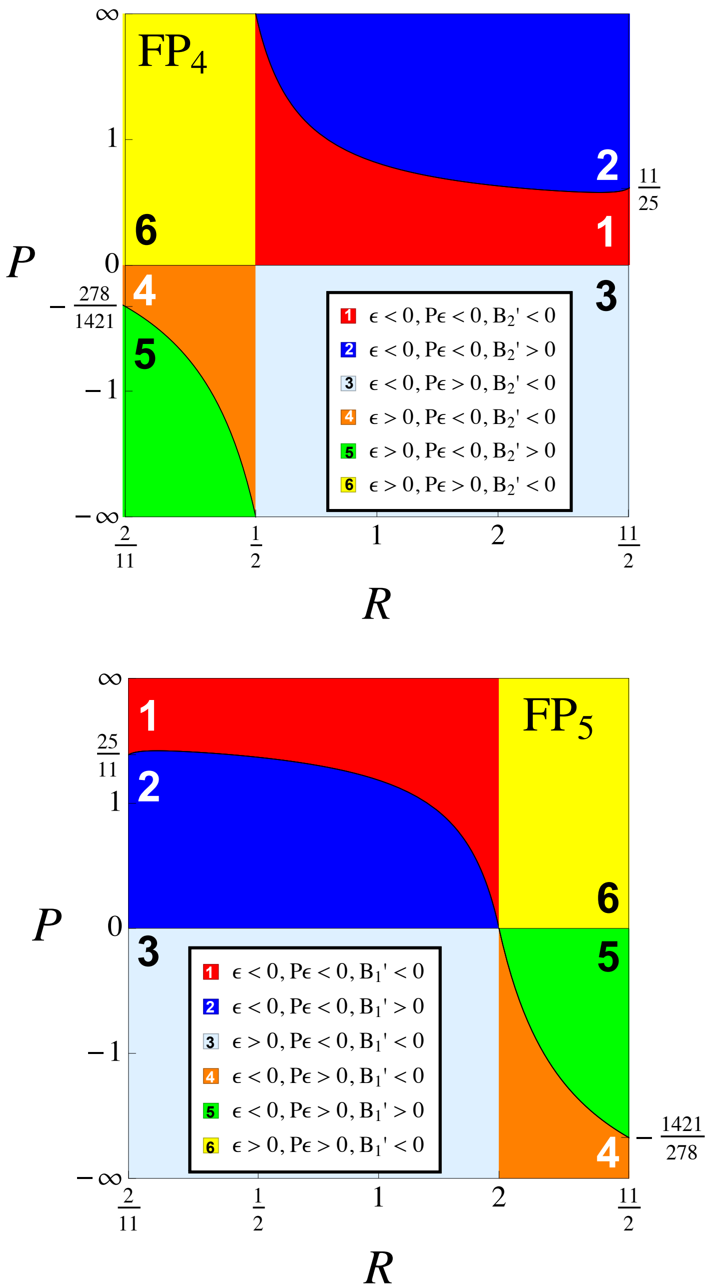

Figure 7. The phase space of parameters for the partially interacting fixed points FP_4 and FP_5, where the gauge and Yukawa coupling in one gauge sector take interacting fixed points while those of the other sector remain trivial. The insets indicate the different parameter regions and conditions for existence, and whether the non-interacting gauge sector is asymptotically free ($B' > 0$) or infrared free ($B' < 0$), see Tab. **11**,**12**.

D. Gauss with Gauge-Yukawa

In Fig. **7** we show the domains of existence for FP_4 ($\mathrm{GY}\cdot\mathrm{G}$, upper) and FP_5 ($\mathrm{G}\cdot\mathrm{GY}$, lower panel). We observe that the fixed point exists for any parameter choice though its features vary with matter multiplicities. Specifically, for FP_4, six qualitatively different parameter regions are found. If the interacting gauge coupling is asymptotically free

($\epsilon < 0$) and provided that $P\epsilon < 0$, the other gauge sector either remains asymptotically free (region 2) or becomes infrared free (region 1), whereas for $P\epsilon > 0$ the other gauge sector invariably remains infrared free (region 3). Conversely, if the interacting gauge coupling is infrared free ($\epsilon > 0$) and provided that $P\epsilon < 0$, the other gauge sector either remains asymptotically free (region 5) or becomes infrared free (region 4), whereas for $P\epsilon > 0$ the other gauge sector invariably remains infrared free (region 6). Moreover, as explained in Tab. **11**, the interacting gauge sector can turn asymptotic freedom of the non-interacting gauge sector into infrared freedom (region 1 and 4). The eigenvalue spectrum in the gauge-Yukawa sector has therefore no relevant eigendirection (IV.20) in region 1 and 3, one relevant eigendirection (IV.18) in region 4 and 6, two relevant eigendirections (IV.19) in region 2, and three relevant eigendirections (IV.17) in region 5, see Tab. **12**.

We make the following observations. Firstly, we note that FP_4 in region 1 and 3 corresponds to a fully attractive IR fixed point with all RG trajectories terminating in it. The fixed point then acts as an infrared "sink" for massless trajectories and all canonically marginal couplings of the theory. Once scalar masses are switched on, RG flows may run away from the hypercritical surface of exactly massless theories, leading to massive phases with or without spontaneous breaking of symmetry. The quantum phase transition at FP_4 in region 1 and 3 is of the second order. Notice that in the "direct product" limit only models with $P\epsilon > 0 > \epsilon$ and $R > \frac{1}{2}$ (analogous to region 3) would lead to a fully infrared attractive "sink". Hence, the availability of region 1 is an entirely new effect, solely due to the ψ fermions and the semi-simple nature of our models. We conclude that the semi-simple structure opens up new types of fixed points which cannot be achieved through a product structure. In region 2, we find that FP_4 has two relevant eigendirections as it would in "direct product" settings.

Secondly, in regions 4 and 6, FP_4 shows a single relevant eigendirection. In the "direct product" limit, only models with $P, \epsilon > 0$ and $R < \frac{1}{2}$ (analogous to region 6) would lead to a single relevant direction. Again, the availability of region 4 is a novel feature, and solely due to the ψ fermions and thus a consequence of the semi-simple nature of the model.

In the parameter region 5 the fixed point shows the largest number of UV relevant directions as it would without the ψ fermions. Moreover, in this parameter regime the Gaussian fixed point has only two relevant directions ($\epsilon > 0, P\epsilon < 0$). Therefore FP_4 in region 5 qualifies as an asymptotically safe UV fixed point. On the other hand, in region 2,4 and 6, it takes the role of a cross-over fixed point. Results for FP_5 (Fig. **7**, lower panel) follow from those for FP_4 via (IV.46), and the distinct regions stated for FP_5 relate to the same physics as those for FP_4.

E. Banks-Zaks with Banks-Zaks

Next, we turn to fully interacting fixed points where both gauge couplings are non-vanishing, see Tab. **13**. In general, the eigenvalue spectrum is determined through (IV.22) with solutions (IV.23), with ϵ taking the role of the parameter B. In the "direct product" limit, fully interacting fixed points reduce to direct products from each pairing of the possibilities (BZ, GY) in each of the simple gauge sectors. For $N_\psi \neq 0$, the fermions ψ introduce a direct mixing between the gauge groups and we may then expect to find something close to a product structure, potentially modified by new effects parametrised via R in fixed points not involving Gaussian factors.

The first such fixed point is FP_6 (BZ·BZ), where each gauge sector achieves a Banks-Zaks fixed point. Yukawa couplings play no role, see Fig. **8**. The fixed point invariably requires $\epsilon < 0$ and $P\epsilon < 0$ and entails an eigenvalue spectrum with two relevant directions of order $\mathcal{O}(\epsilon^2)$, and two irrelevant directions of order $\mathcal{O}(\epsilon)$ associated to the Yukawas,

$$\vartheta_1, \vartheta_2 < 0 < \vartheta_3, \vartheta_4 \,. \tag{IV.71}$$

The quartics are marginally irrelevant. The Gaussian is necessarily the UV fixed point in these settings which makes FP_6 a cross-over fixed point. The accessible parameter region, shown in Fig. **8**, is invariant under the exchange of gauge groups (IV.46). The "direct product" limit has qualitatively the same spectrum (IV.71). The main effect due to the semi-simple character of the theory relates to the exclusion of certain parameter regions (white regions). We conclude that the semi-simple nature of the theory leads to parameter restrictions without otherwise changing the overall appearance of the fixed point.

#	parameter range			eigenvalue spectrum	info
	sign ϵ	R	P		
FP$_6$	$-$	$\left(\frac{2}{11}, \frac{11}{2}\right)$	$\left(\frac{2}{25}R, \frac{25}{2}R\right)$	$\vartheta_{1,2} < 0 < \vartheta_{3,4}$	Fig. **8**
FP$_7$					Fig. **9** (upper panel)
	$-$	$\left(\frac{2}{11}, \frac{1}{2}\right)$	$\left(\frac{25}{2}R, +\infty\right)$	$\vartheta_{1,2} < 0 < \vartheta_{3,4}$	region 1
	$-$	$\left(\frac{1}{2}, R_1\right)$	$\left(\frac{25}{2}R, X(R)\right)$	$\vartheta_{1,2} < 0 < \vartheta_{3,4}$	region 1
	$-$	$\left(R_1, \frac{11}{2}\right)$	$\left(X(R), \frac{25}{2}R\right)$	$\vartheta_1 < 0 < \vartheta_{2,3,4}$	region 2
	$+$	$\left(\frac{2}{11}, \frac{1}{2}\right)$	$\left(-\infty, X(R)\right)$	$\vartheta_{1,2} < 0 < \vartheta_{3,4}$	region 3
FP$_8$					Fig. **9** (lower panel)
	$-$	$\left(R_4, \frac{11}{2}\right)$	$\left(\widetilde{X}(R), \frac{2}{25}R\right)$	$\vartheta_{1,2} < 0 < \vartheta_{3,4}$	region 1 & 3
	$-$	$\left(\frac{2}{11}, R_4\right)$	$\left(\frac{2}{25}R, \widetilde{X}(R)\right)$	$\vartheta_1 < 0 < \vartheta_{2,3,4}$	region 2
FP$_9$					Fig. **10**
	$-$	$\left(\frac{2}{11}, \frac{1}{2}\right)$	$\left(\widetilde{X}(R), +\infty\right)$	$\vartheta_1 < 0 < \vartheta_{2,3,4}$	region 1
	$-$	$\left(\frac{1}{2}, R_2\right)$	$\left(\widetilde{X}(R), X(R)\right)$	$\vartheta_1 < 0 < \vartheta_{2,3,4}$	region 1
	$-$	$\left(R_3, \frac{11}{2}\right)$	$\left(\widetilde{X}(R), X(R)\right)$	$\vartheta_1 < 0 < \vartheta_{2,3,4}$	region 1 & 4
	$-$	$\left(R_2, R_3\right)$	$\left(X(R), \widetilde{X}(R)\right)$	$0 < \vartheta_{1,2,3,4}$	region 2
	$+$	$\left(\frac{2}{11}, \frac{1}{2}\right)$	$\left(-\infty, X(R)\right)$	$\vartheta_1 < 0 < \vartheta_{2,3,4}$	region 3

Table 13. Parameter regions where the fully interacting fixed points $\mathrm{FP}_6 - \mathrm{FP}_9$ exist, along with the eigenvalue spectrum for the various parameter regions.

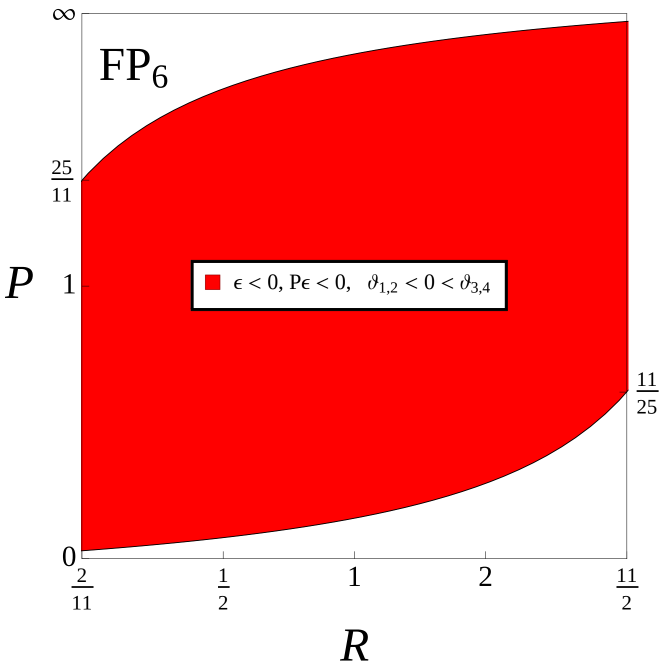

Figure 8. The phase space of parameters for the interacting fixed point FP_6 (red) where both gauge sectors take interacting and physical fixed points while all Yukawa couplings vanish. The eigenvalue spectrum at the fixed point always displays exactly two relevant eigenvalues of $O(\epsilon)$ and two irrelevant eigenvalues of order $O(\epsilon^2)$, see Tab. **13**. Note that this fixed point invariably requires asymptotic freedom for both gauge sectors (see main text).

F. Banks-Zaks with Gauge-Yukawa

At the interacting fixed points FP_7 ($\mathrm{BZ}\cdot\mathrm{GY}$, upper panel), and FP_8 ($\mathrm{GY}\cdot\mathrm{BZ}$, lower panel), we have that both gauge and one of the Yukawa couplings are non-trivial. Our results for the condition of existence and the eigenvalue spectra are displayed in Fig. **9**. By definition, this type of fixed point requires that either $\epsilon < 0$ or $P\epsilon < 0$, or both, meaning that at least one of the gauge sectors is asymptotically free. In Fig. **9**, this relates to three different parameter regions (see inset for the colour coding). In region 1 and 2, the theory is asymptotically free in both gauge sectors, whereas in region 3 the theory is asymptotically free in only one gauge sector. We observe that large regions of parameter space are excluded. Valid parameter regions are further distinguished by their eigenvalue spectrum which either takes the form (IV.19) or (IV.18), meaning that minimally one and maximally two eigenoperators constructed out of the gauge kinetic terms and the Yukawa interactions are UV relevant, $\vartheta_1 < 0 < \vartheta_3, \vartheta_4$. The sign of ϑ_2 depends on the matter field multiplicities. In region 1 and 3, and for either of FP_7 and FP_8, we find that

$$\vartheta_1, \vartheta_2 < 0 < \vartheta_3, \vartheta_4 \,. \tag{IV.72}$$

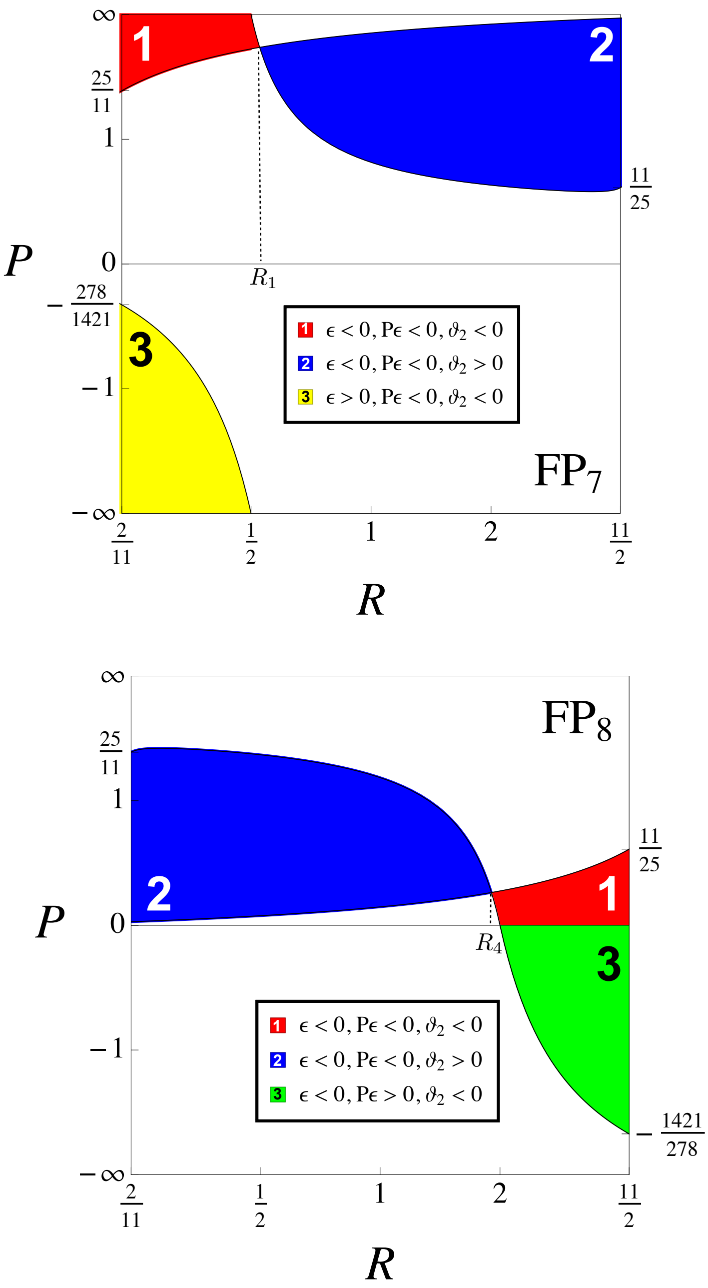

Figure 9. The phase space of parameters for the fixed points FP_7 and FP_8 where two gauge and one of the Yukawa couplings take interacting and physical fixed points, while the other Yukawa coupling remains trivial. The inset indicates the signs for ϵ and $P\epsilon$, together with the sign for the eigenvalue ϑ_2, Tab. **13** (see main text).

In region 2, conversely, we have

$$\vartheta_1 < 0 < \vartheta_2, \vartheta_3, \vartheta_4 \,. \tag{IV.73}$$

Hence, at FP_7 and in the regime where both gauge sectors are asymptotically free ($P > 0 > \epsilon$), two types of valid fixed points are found. For sufficiently low $R < R_1$ (IV.62), and large P, the fixed point has two relevant directions (region 1). Increasing $R > R_1$ at fixed P may lead to a second type of IR fixed point with a single relevant direction (region 2). On the other hand, in the regime $\epsilon > 0 > P$ only one type of fixed point exists with two relevant directions (region 3). It is worth comparing these results with the "direct product" limit. For $P > 0 > \epsilon$ the latter leads to the eigenvalue spectrum (IV.73), as found in region 2. Also, for $\epsilon > 0 > P$ the "direct product" fixed point has the eigenvalue spectrum (IV.72), which is qualitatively in accord with findings in region 3. We conclude that the semi-simple nature of the interactions plays a minor quantitative role in region 2 and 3. On the other hand, in region 1 where $P > 0 > \epsilon$, the semi-simple nature of the theory leads to an important qualitative modification: an eigenvalue spectrum with two relevant directions at FP_7 cannot be achieved through a direct product setting; rather, it necessarily requires matter fields charged under both gauge groups. We conclude that the semi-simple nature of interactions play a key qualitative role in region 1. Analogous results hold true for FP_8 after the substitutions (IV.46) and the replacement $R_1 \to R_4 = 1/R_1$, see (IV.62).

G. Gauge-Yukawa with Gauge-Yukawa

At the fully interacting fixed point FP_9 ($\mathrm{GY} \cdot \mathrm{GY}$), we have that both gauge and both Yukawa couplings are non-trivial. We find that the eigenvalue spectrum in the gauge-Yukawa sector reads either (IV.18) or (IV.20), meaning that at least three of the four eigenoperators constructed out of the gauge and fermion fields are strictly irrelevant, $0 < \vartheta_2, \vartheta_3, \vartheta_4$. The sign of the eigenvalue ϑ_1 depends on the matter field multiplicities of the model.

Our results for the condition of existence and the eigenvalue spectrum are stated in Fig. **10**. We observe four qualitatively different parameter regions (see inset for the colour coding). For $P > 0 > \epsilon$, the theory is asymptotically free in both gauge sectors and we find two types of valid parameter regions, depending on whether R takes values below R_2 or above R_3 (region 1), or in between (region 2); see (IV.62). Moreover, in region 2, we find that the fixed point is strictly IR attractive in all couplings, owing to

$$0 < \vartheta_1 \,, \vartheta_2, \vartheta_3, \vartheta_4 \,. \tag{IV.74}$$

Hence, the fixed point FP_9 in region 2 corresponds to a fully attractive IR fixed point acting as an infrared "sink" for massless trajectories and all canonically marginal couplings of the theory. Ultimately it describes a second order quantum phase transition between a symmetric and a symmetry broken phase, characterised by the vacuum expectation value of the scalar field. Qualitatively, the same type of result is achieved in the "direct product" limit. Hence, the main effect due to semi-simple interactions is to have generated

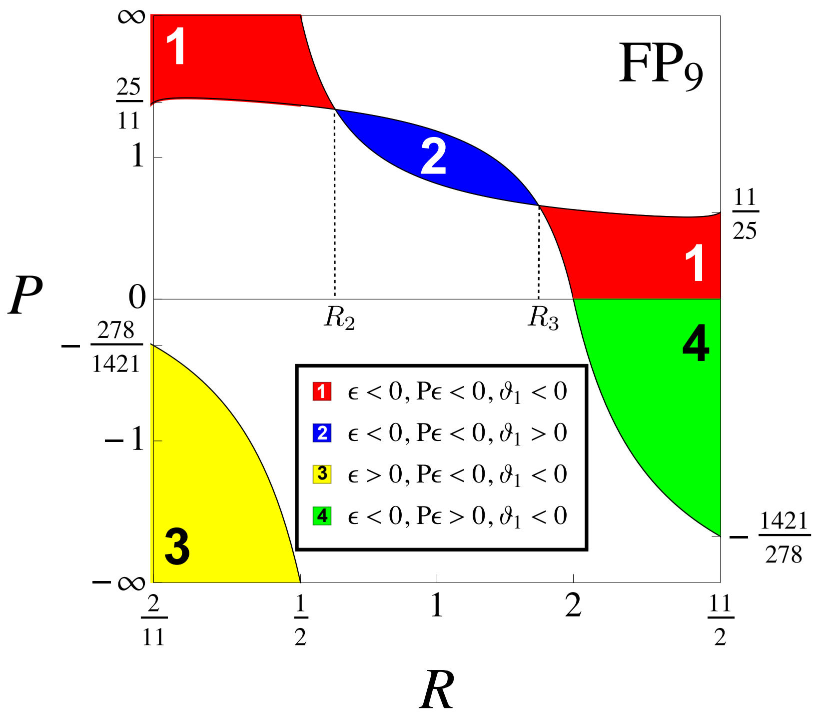

Figure 10. The phase space of parameters for the fully interacting fixed point FP$_9$ where all gauge and all Yukawa couplings are non-trivial. The coloured regions relate to the portions of parameter space where the fully interacting fixed point is physical. The inset provides additional information including the sign for the eigenvalue ϑ_4 (see main text).

a boundary in parameter space. In region 1 we find

$$\vartheta_1 < 0 < \vartheta_2, \vartheta_3, \vartheta_4 \,. \tag{IV.75}$$

This type of eigenvalue spectrum cannot be achieved without semi-simple interactions mediated by the ψ fields and is therefore a novel feature, entirely due to the semi-simple nature of the theory. In this regime, FP$_9$ corresponds to a cross-over fixed point (the Gaussian is the UV fixed point) with a single unstable direction where trajectories escape either towards a weakly coupled IR fixed point, or towards a regime of strong coupling with (chiral) symmetry breaking, confinement, or infrared conformality.

For $\epsilon > 0 > P$ or $P\epsilon > 0 > \epsilon$, the theory is asymptotically free in one and infrared free in the other gauge sector. Valid fixed points then correspond to region 3 or region 4, respectively. In either of these cases, the eigenvalue spectrum shows a single relevant direction, (IV.75). This agrees qualitatively with the eigenvalue spectrum in the "direct product" limit. We conclude, once more, that the main impact of the ψ fields relates to the boundaries in parameter space which restrict the fixed point's domain of availability.

Finally, for $\epsilon, P > 0$, the theory is infrared free in both gauge sectors. We observe that no such interacting fixed point arises, irrespective of matter multiplicities. Interestingly though, such fixed points do exist in the "direct product" limit with spectrum

$$\vartheta_1, \vartheta_2 < 0 < \vartheta_3, \vartheta_4 \,. \tag{IV.76}$$

The reason for their non-existence in our models is the presence of the ψ fermions. The requirement of perturbativity in both gauge couplings then leads to limitations on the parameter R which cannot be satisfied at FP_9 with eigenvalue spectrum (IV.76). This result provides us with an example where the semi-simple nature of the theory "disables" a fixed point. This completes the overview of interacting fixed points in the gauge-Yukawa sector and their key properties.

V. SCALAR FIXED POINTS AND VACUUM STABILITY

In this section, we analyse the scalar sector and establish conditions for stability of the quantum vacuum. We also provide results for all scalar couplings at all interacting fixed points, Tab. **14**.

A. Yukawa and scalar nullclines

Following [144], we begin by exploiting the results (IV.53) to express the Yukawa nullclines in terms of the gauge couplings and the parameter R. To leading order in the small expansion parameters (IV.38), and using (IV.53) together with (IV.10), the non-trivial Yukawa nullclines $\beta_Y = 0$ and $\beta_y = 0$ take the explicit form

$$\frac{\alpha_Y}{\alpha_1} = \frac{6}{13 - 2R}\,, \quad \frac{\alpha_y}{\alpha_2} = \frac{6}{13 - 2/R}\,. \tag{IV.77}$$

For fixed gauge couplings, we observe that the Yukawa couplings are enhanced over their values in the absence of the fermions ψ. The relevance of nullcline solutions (IV.77) is as follows. By their very definition, the Yukawa couplings no longer run with the RG scale when taking the values (IV.77). If at the same time the gauge couplings take fixed points on their own, the nullcline relations then provide us with the correct fixed point values for the Yukawa couplings. Evidently, (IV.77) together with (IV.39) guarantees that the Yukawa fixed points are physical as long as the gauge fixed points are. Note also that the slope of the nullcline remains positive and finite for all R within the domain (IV.39). Hence strict perturbativity in the Yukawa couplings follows from strict perturbativity in the gauge couplings, in accord with the general discussion in [144] based on dimensional analysis.

Next we turn to the scalar nullclines. Since the beta functions for the two scalar sectors decouple at this order, we may analyse their nullclines individually.[36] All results for the

[36] This simplification solely arises provided the mixing coupling α_w takes its exact Gaussian fixed point (IV.58). For non-trivial α_w the nullclines take more general forms.

subsystem (α_U, α_V) can straightforwardly be translated to the subsystem (α_u, α_v) by substituting $R \leftrightarrow 1/R$, also using (IV.37). Furthermore, since the scalars are uncharged, their one loop beta functions are independent of the gauge coupling. Dimensional analysis then shows that all non-trivial scalar nullclines are proportional to the corresponding Yukawa coupling [144]. The scalar nullclines represent exact fixed points of the theory provided the Yukawa couplings take interacting fixed points. Perturbativity of scalar couplings at an interacting fixed point then follows from the perturbativity of Yukawa couplings which, in turn, follows from perturbativity in the gauge couplings.

Specifically, the nullclines for the single trace scalar couplings are found from (IV.57) by resolving $\beta_U = 0$ for α_U. We find two solutions

$$\frac{\alpha_{U\pm}}{\alpha_Y} = \frac{1}{4}\left(-1 \pm \sqrt{23 - 4R}\right) . \tag{IV.78}$$

Note that the double trace coupling does not couple back into the running of the single trace coupling. Within the parameter range (IV.39) we observe that $\alpha_{U+} > 0 > \alpha_{U-}$. Next, we consider the nullclines for the double-trace quartic coupling α_V. Inserting α_{U+} into $\beta_V = 0$, we find a pair of nullclines given by

$$\frac{\alpha_{V\pm}}{\alpha_Y} = \frac{1}{4}\left(-2\sqrt{23 - 4R} \pm \sqrt{20 - 4R + 6\sqrt{23 - 4R}}\right) . \tag{IV.79}$$

Both nullclines take real values for all R within the range (IV.39), and we end up with $\alpha_{U+} \geq 0$ together with $0 > \alpha_{V+} > \alpha_{V-}$. Analogously, inserting α_{U-} into $\beta_V = 0$, we find a second pair of nullclines given by

$$\frac{\alpha_{V2\pm}}{\alpha_Y} = \frac{1}{4}\left(2\sqrt{23 - 4R} \pm \sqrt{20 - 4R - 6\sqrt{23 - 4R}}\right) . \tag{IV.80}$$

In this case, however, the result (IV.80) comes out complex within the parameter range (IV.39), meaning that even if α_Y^* takes a real positive fixed point the corresponding scalar fixed point is invariably unphysical.

The replacement $R \to 1/R$ in (IV.78) and (IV.79), (IV.80) allows us to obtain explicit expressions for the nullclines for $\alpha_{u\pm}/\alpha_y$ and $\alpha_{v\pm}/\alpha_y$. The real solutions are given by

$$\frac{\alpha_{u\pm}}{\alpha_y} = \frac{1}{4}\left(-1 \pm \sqrt{23 - 4/R}\right) \tag{IV.81}$$

with $\alpha_{u+} \geq 0 > \alpha_{u-}$. The solution α_{u+} leads to real nullclines for the double-trace coupling α_v given by

$$\frac{\alpha_{v\pm}}{\alpha_y} = \frac{1}{4}\left(-2\sqrt{23 - 4/R} \pm \sqrt{20 - 4/R + 6\sqrt{23 - 4/R}}\right) , \tag{IV.82}$$

and we end up with $\alpha_{u+} \geq 0$ together with $0 > \alpha_{v+} > \alpha_{v-}$. On the other hand, the solution α_{u-} does not lead to real solutions for $\alpha_{v\pm}$. This completes the overview of

Yukawa and scalar nullcline solutions.

B. Stability of the vacuum

We are now in a position to reach firm conclusions concerning the stability of the ground state at interacting fixed points. The reason for this is that this information is encoded in the scalar nullclines. The explicit form of the fixed point in the gauge-Yukawa sector is not needed. To that end, we recall the stability analysis for potentials of the form

$$W \propto \alpha_U \,\mathrm{Tr}\,(H^\dagger H)^2 + \alpha_V/N_\mathrm{F}\,(\mathrm{Tr}\,H^\dagger H)^2\,, \tag{IV.83}$$

In the limit where $\alpha_w = 0$ the scalar field potential in our models (IV.25) are given by (IV.83) together with its counterpart $(H, \alpha_U, \alpha_V) \leftrightarrow (h, \alpha_u, \alpha_v)$. For potentials of the form (IV.83), the general conditions for vacuum stability read [58, 162]

$$\begin{array}{llll} a) & \alpha_U^* > 0 & \text{and} & \alpha_U^* + \alpha_V^* > 0 \\ b) & \alpha_U^* < 0 & \text{and} & \alpha_U^* + \alpha_V^*/N_\mathrm{F} > 0 \end{array} \tag{IV.84}$$

and similarly for $(\alpha_U, \alpha_V) \leftrightarrow (\alpha_u, \alpha_v)$. In the Veneziano limit, case b) effectively becomes void and cannot be satisfied for any α_U^*, irrespective of the sign of α_V^*. Inserting the fixed points into (IV.84) we find

$$\begin{aligned} \alpha_{U+}^* + \alpha_{V+}^* &= \frac{\alpha_Y^*}{4}\left(+\sqrt{20 - 4R + 6\sqrt{23-4R}} - \sqrt{23-4R} - 1\right) \geq 0\,, \\ \alpha_{U+}^* + \alpha_{V-}^* &= \frac{\alpha_Y^*}{4}\left(-\sqrt{20 - 4R + 6\sqrt{23-4R}} - \sqrt{23-4R} - 1\right) \leq -\alpha_Y^*\,, \end{aligned} \tag{IV.85}$$

Stability of the quantum vacuum is evidently achieved at the fixed point $(\alpha_{U+}^*, \alpha_{V+}^*)$ following case a) and irrespective of the value for the Yukawa fixed point as long as $\alpha_Y^* > 0$. The potential (IV.83) becomes exactly flat at the fixed point iff $R = \frac{11}{2}$. In this case, higher order or radiative corrections must be taken into consideration to guarantee stability in the presence of flat directions. Stability is not achieved at the fixed point $(\alpha_{U+}^*, \alpha_{V-}^*)$, for any R. Turning to the second scalar sector, we find

$$\begin{aligned} \alpha_{u+}^* + \alpha_{v+}^* &= \frac{\alpha_y^*}{4}\left(+\sqrt{20 - 4/R + 6\sqrt{23-4/R}} - \sqrt{23-4/R} - 1\right) \geq 0\,, \\ \alpha_{u+}^* + \alpha_{v-}^* &= \frac{\alpha_y^*}{4}\left(-\sqrt{20 - 4/R + 6\sqrt{23-4/R}} - \sqrt{23-4/R} - 1\right) \leq -\alpha_y^*\,, \end{aligned} \tag{IV.86}$$

where the bounds refer to R varying within the range (IV.39). This part of the potential becomes exactly flat at the fixed point iff $R = \frac{2}{11}$. The result establishes vacuum stability at the fixed point $(\alpha_{u+}^*, \alpha_{v+}^*)$. We also confirm that the fixed point $(\alpha_{u+}^*, \alpha_{v-}^*)$ does not lead to a stable ground state. We conclude that vacuum stability is guaranteed at the interacting fixed points $(\alpha_{U+}^*, \alpha_{V+}^*)$ and $(\alpha_{u+}^*, \alpha_{v+}^*)$, together with $\alpha_w = 0$, irrespective of the fixed points in the gauge Yukawa sector, as long as the later is physical. Out of the a priori 2^3 different fixed point candidates in the scalar sector at one loop (half of which lead to real fixed points) the additional requirement of vacuum stability has identified a *unique* viable solution. In this light, vacuum stability dictates that the anomalous dimensions (IV.49) are strictly positive at interacting fixed points, (IV.60), with

$$\begin{aligned} \gamma_M^* &= \alpha_Y^* \sqrt{20 - 4R + 6\sqrt{23 - 4R}} > 0\,, \\ \gamma_m^* &= \alpha_y^* \sqrt{20 - 4/R + 6\sqrt{23 - 4/R}} > 0\,, \end{aligned} \tag{IV.87}$$

and provided that (IV.39) is observed.

C. Portal coupling

Now we clarify whether the stability of the vacuum is affected by the presence of the "portal" coupling $\alpha_w \neq 0$ which induces a mixing between the scalar sectors. In this case the scalar potential is given by $W = -L_{\text{pot}}$ in (IV.25),

$$\begin{aligned} W =& U\,\mathrm{Tr}(H^\dagger H)^2 + V\,(\mathrm{Tr} H^\dagger H)^2 + u\,\mathrm{Tr}(h^\dagger h)^2 + v\,(\mathrm{Tr} h^\dagger h)^2 \\ &+ w\,\mathrm{Tr}(H^\dagger H)\mathrm{Tr}(h^\dagger h)\,, \end{aligned} \tag{IV.88}$$

where H and h are $N_\text{F} \times N_\text{F}$ and $N_\text{f} \times N_\text{f}$ matrices, respectively. Following the reasoning of [58, 162], we observe that the potential has a global $U(N_\text{F})_L \otimes U(N_\text{F})_R \otimes U(N_\text{f})_L \otimes U(N_\text{f})_R$ symmetry which allows us to bring each of the fields into a real diagonal configuration, $H = \mathrm{diag}(\Phi_1, \Phi_2, \dots)$ and $h = \mathrm{diag}(\phi_1, \phi_2, \dots)$. As the potential is homogeneous in either field, $W(c\,H, c\,h) = c^4 W(H, h)$, it suffices to guarantee positivity on a fixed surface $\sum_i \Phi_i^2 = 1 = \sum_j \phi_j^2$ which is implemented using Lagrange multipliers Λ and λ. From

$$\begin{aligned} \frac{\partial W}{\partial \Phi_i} &= 2\Phi_i (2U\Phi_i^2 + 2V + w - 2\Lambda)\,, \\ \frac{\partial W}{\partial \phi_i} &= 2\phi_i (2u\phi_i^2 + 2v + w - 2\lambda)\,, \end{aligned} \tag{IV.89}$$

it follows that extremal field configurations are those where all non-zero fields take equal values. If we have M non-zero Φ fields and m non-zero ϕ fields, the extremal field values are $\Phi_i^2 = 0$ or $\Phi_i^2 = \frac{1}{M}$ alongside with $\phi_i^2 = 0$ or $\phi_i^2 = \frac{1}{m}$. Three non-trivial cases arise. If $m = 0$ the extremal potential is $W_e = U/M + V$. Likewise if $M = 0$ we have $W_e = u/m + v$. Lastly, if both $m, M \neq 0$, we have $W_e = U/M + V + u/m + v + w$. The values of M, m for which these extremal potentials are minima depend on the signs of the couplings U, u, leaving us with the four possible cases $U, u > 0$, $u > 0 > U$, $U > 0 > u$, and $0 > U, u$. We thus obtain four distinct sets of conditions for vacuum stability which we summarise

#	quartic scalar couplings
$\mathbf{FP_{1-3}}$	$\alpha_U^* = 0\,, \quad \alpha_V^* = 0\,, \quad \alpha_u^* = 0\,, \quad \alpha_v^* = 0\,,$
$\mathbf{FP_4}$	$\alpha_U^* = \frac{4F_1(R)R\epsilon}{(2R-1)(3R-19)}\,, \quad \alpha_V^* = \frac{4F_2(R)R\epsilon}{(2R-1)(3R-19)}\,, \quad \alpha_u^* = 0\,, \quad \alpha_v^* = 0\,,$
$\mathbf{FP_5}$	$\alpha_U^* = 0\,, \quad \alpha_V^* = 0\,, \quad \alpha_u^* = \frac{4\,F_1(1/R)P\epsilon/R}{(2/R-1)(3/R-19)}\,, \quad \alpha_v^* = \frac{4\,F_2(1/R)P\epsilon/R}{(2/R-1)(3/R-19)}\,,$
$\mathbf{FP_6}$	$\alpha_U^* = 0\,, \quad \alpha_V^* = 0\,, \quad \alpha_u^* = 0\,, \quad \alpha_v^* = 0\,,$
$\mathbf{FP_7}$	$\alpha_U^* = \frac{4}{3}\frac{(25-2P/R)F_1(R)}{50R^2-343R+167}R\epsilon\,, \quad \alpha_V^* = \frac{4}{3}\frac{(25-2P/R)F_2(R)}{50R^2-343R+167}R\epsilon\,, \quad \alpha_u^* = 0\,, \quad \alpha_v^* = 0\,,$
$\mathbf{FP_8}$	$\alpha_U^* = 0\,, \quad \alpha_V^* = 0\,, \quad \alpha_u^* = \frac{4}{3}\frac{(25-2R/P)F_1(1/R)}{50/R^2-343/R+167}\frac{P\epsilon}{R}\,, \quad \alpha_v^* = \frac{4}{3}\frac{(25-2R/P)F_2(1/R)}{50/R^2-343/R+167}\frac{P\epsilon}{R}\,,$
$\mathbf{FP_9}$	$\alpha_U^* = \frac{4}{3}\frac{[(13-2/R)P/R+(2/R-1)(3/R-19)]F_1(R)}{(19R^2-43R+19)(2/R^2-13/R+2)}R\epsilon,\ \alpha_u^* = \frac{4}{3}\frac{[(13-2R)R/P+(2R-1)(3R-19)]F_1(1/R)}{(19/R^2-43/R+19)(2R^2-13R+2)}\frac{P\epsilon}{R},$ $\alpha_V^* = \frac{4}{3}\frac{[(13-2/R)P/R+(2/R-1)(3/R-19)]F_2(R)}{(19R^2-43R+19)(2/R^2-13/R+2)}R\epsilon,\ \alpha_v^* = \frac{4}{3}\frac{[(13-2R)R/P+(2R-1)(3R-19)]F_2(1/R)}{(19/R^2-43/R+19)(2R^2-13R+2)}\frac{P\epsilon}{R}.$

Table 14. Quartic scalar couplings at all weakly interacting fixed points to leading order in ϵ following Tab. **8** using the auxiliary functions (IV.92). Same conventions as in Tab. **10**. Within the admissible parameter ranges (Tab. **12**, **13**) we observe vacuum stability.

as follows:

$$
\begin{aligned}
&a)\ \ \alpha_u, \alpha_U \geq 0\,, \quad \alpha_U + \alpha_V \geq 0\,,\ \ \alpha_u + \alpha_v \geq 0\,,\ \ F\left(\alpha_U + \alpha_V\right) + \frac{\alpha_u + \alpha_v}{F} + \alpha_w \geq 0\,,\\
&b)\ \ \alpha_u > 0 > \alpha_U\,, \alpha_U + \frac{\alpha_V}{N_\mathrm{F}} \geq 0\,,\ \ \alpha_u + \alpha_v \geq 0\,,\ \ \alpha_U + \frac{\alpha_V}{N_\mathrm{F}} + \frac{\alpha_u + \alpha_v}{F\,N_\mathrm{f}} + \frac{\alpha_w}{N_\mathrm{f}} \geq 0\,,\\
&c)\ \ \alpha_U > 0 > \alpha_u\,, \alpha_U + \alpha_V \geq 0\,,\ \ \alpha_u + \frac{\alpha_v}{N_\mathrm{f}} \geq 0\,,\ \ \alpha_u + \frac{\alpha_v}{N_\mathrm{f}} + F\,\frac{\alpha_U + \alpha_V}{N_\mathrm{F}} + \frac{\alpha_w}{N_\mathrm{F}} \geq 0\,,\\
&d)\ \ 0 \geq \alpha_u, \alpha_U\,, \quad \alpha_U + \frac{\alpha_V}{N_\mathrm{F}} \geq 0\,,\ \alpha_u + \frac{\alpha_v}{N_\mathrm{f}} \geq 0\,,\ \alpha_u + \frac{\alpha_v}{N_\mathrm{f}} + F\left(\alpha_U + \frac{\alpha_V}{N_\mathrm{F}}\right) + \frac{\alpha_w}{N_\mathrm{F}} \geq 0\,.
\end{aligned}
\tag{IV.90}
$$

Notice that we have rescaled the couplings as in (IV.26) and (IV.27) to make contact with the notation used in this paper. The parameter $F \equiv N_\mathrm{f}/N_\mathrm{F} > 0$ can be expressed in terms of the parameter R to leading order in $\epsilon \ll 1$, see (IV.45).

We make the following observations. In all four cases, the additional condition owing to the mixing coupling (IV.27) takes the form of a lower bound for α_w. Furthermore, α_w is allowed to be negative without destroying the stability of the potential, provided it does not become too negative. We also note that none of the three cases b), c) or d) in (IV.90) can have consistent solutions in the Veneziano limit where $N_\mathrm{F}, N_\mathrm{f} \to \infty$. This uniquely leaves the case a) as the only possibility for vacuum stability in the parameter regions considered here. These solutions neatly fall back onto the solutions discussed previously in the limit $\alpha_w \to 0$. As long as the auxiliary condition

$$
\alpha_w \geq -[F\left(\alpha_U + \alpha_V\right) + F^{-1}\left(\alpha_u + \alpha_v\right)] \tag{IV.91}
$$

is satisfied, we can safely conclude that a non-vanishing $\alpha_w \neq 0$ does not spoil vacuum stability, not even for negative portal coupling α_w.

D. Unique scalar fixed points

In Tab. **14**, we summarise our results for the quartic scalar couplings at all weakly interacting fixed points to leading order in ϵ following Tab. **8**, using (IV.61). We also introduce the auxiliary functions

$$
\begin{aligned}
F_1(x) &= \tfrac{1}{4}\left(\sqrt{23-4x}-1\right) \,, \\
F_2(x) &= \tfrac{1}{4}\left(\sqrt{20-4x+6\sqrt{23-4x}}-2\sqrt{23-4x}\right)
\end{aligned}
\tag{IV.92}
$$

which originate from the scalar nullclines. The main result is that vacuum stability together with a physical fixed point in the gauge-Yukawa sector singles out a *unique* fixed point in the scalar sector. The scalar fixed points do not offer further parameter constraints other than those already stated in Tabs. **12** and **13**. Within the admissible parameter ranges we invariably find that the scalar couplings are either strictly irrelevant (at interacting fixed points) or marginally irrelevant (at the Gaussian fixed point).

VI. ULTRAVIOLET COMPLETIONS

In this section, we discuss interacting fixed points and the weak coupling phase structure of minimal models (IV.25) in dependence on matter field multiplicities. Differences from the viewpoint of their high- and low-energy behaviour are highlighted.

A. Classification

In Figs. **11**, **12**, **13** and **14** we summarise results for the qualitatively different types of quantum field theories with Lagrangean (IV.25) in view of their fixed point structure at weak coupling, together with their behaviour in the deep UV and IR. Theories differ primarily through their matter multiplicities (IV.32), which translate to the parameters (P, R) and the sign of ϵ, (IV.61). As such, the "phase space" shown in Fig. **11** arises as the overlay of Figs. **6**, **7**, **8**, **9** and **10**. Distinctive parameter regions are separated from each other by the seven characteristic curves $P = 0\,, X\,, \widetilde{X}\,, Y$ or $\widetilde{Y}$ and $R = \frac{1}{2}$ or 2. The functions $X(R)\,, \widetilde{X}(R)\,, Y(R)$ and $\widetilde{Y}(R)$ are given explicitly in (IV.**B**.1). Overall, this leads to the 22 distinct regions shown in Fig. **11** and denoted by capital letters. Together with the sign of ϵ this leaves us with 44 different cases. Some of these are redundant and related under the exchange of gauge groups, see (IV.46). In fact, for $P > 0$ and for either sign of ϵ, we find nine fundamentally independent cases corresponding to the parameter regions

$$
\mathrm{A}\,, \mathrm{B}\,, \mathrm{C}\,, \mathrm{D}\,, \mathrm{E}\,, \mathrm{F}\,, \mathrm{G}\,, \mathrm{H}\,, \mathrm{I} \tag{IV.93}
$$

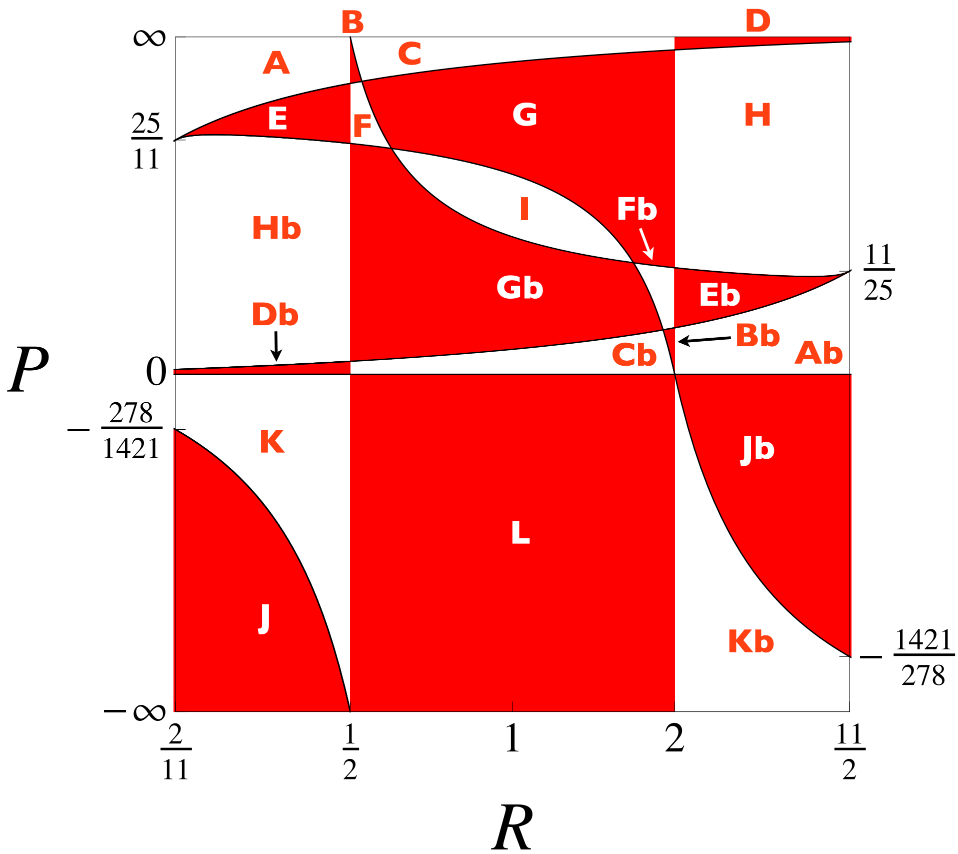

Figure 11. The "phase space" of quantum field theories with fundamental action (IV.25) expressed as a function of field multiplicities and written in terms of (P, R), see (IV.61). The 22 different parameter regions are indicated by roman letters. Theories with parameters in region X are dual to those in region Xb under the exchange of gauge groups following the map (IV.46). Further details on fixed points and their eigenvalue spectra per parameter region are summarised in Figs. **12**, **13** and **14**.

given in Fig. **11**. Theories with parameters in the regime

$$\text{Ab}\,,\ \text{Bb}\,,\ \text{Cb}\,,\ \text{Db}\,,\ \text{Eb}\,,\ \text{Fb}\,,\ \text{Gb}\,,\ \text{Hb}\,, \tag{IV.94}$$

are "dual" to those in (IV.93) under the exchange of gauge groups (X $\leftrightarrow$ Xb) and for the same sign of ϵ, except for the theories within (I, ϵ), which are "selfdual" and mapped onto themselves under (IV.46). For $P < 0$ we find five parameter regions for either sign of ϵ,

$$\text{J}\,,\ \text{K}\,,\ \text{L}\,,\ \text{Kb}\,,\ \text{Jb}\,. \tag{IV.95}$$

For these, the manifest "duality" under exchange of gauge groups involves a change of sign for ϵ with (X, $\epsilon < 0$) being dual to (Xb, $-\epsilon > 0$) except for the parameter region L which is selfdual. In total, we end up with $2 \times 9 + 5 = 23$ fundamentally distinct scenarios underneath the $2 \times (9 + 8 + 5) = 44$ cases tabulated in Figs. **12**, **13** and **14** and discussed more extensively below.

A comment on the nomenclature: in each row of Figs. **12**, **13** and **14**, we indicate the parameter region (P, R) as in Fig. **11** together with the sign of ϵ (if required), followed by the set of fixed points. For each of these, the (marginally) relevant and irrelevant eigenvalues in the gauge-Yukawa sector are indicated by a $-$ and $+$ sign. For the Gaussian fixed point FP_1, the signs relate pairwise to the $SU(N_\text{C})$ and $SU(N_\text{c})$ gauge sector, respectively; for all other fixed points eigenvalues are sorted by magnitude. Red shaded slots indicate eigenvalue spectra which uniquely arise due to the semi-simple character of the theory. The column " UV" states the UV fixed point, differentiating between complete asymptotic freedom (AF), asymptotic safety (AS), asymptotic freedom in one sector without asymptotic safety in the other (pAF), asymptotic safety in one sector without asymptotic freedom in the other (pAS), or none of the above. The column "IR" states the fully attractive IR fixed point (provided it exists), distinguishing the cases where none (0), one (Y) or (y), or both (Yy) Yukawa couplings are non-trivial at the fixed point; a hyphen indicates that the IR regime is strongly coupled.

B. Asymptotic freedom

We discuss main features of the different quantum field theories (IV.25) starting with those where each gauge sector is asymptotically free from the outset ($P > 0 > \epsilon$), corresponding to the cases $1 - 17$ in Fig. **12**. The Gaussian fixed point FP_1 is always the UV fixed point. Any other weakly interacting fixed point displays a lower number of relevant directions. All weakly interacting fixed points can be reached from the Gaussian. Another point in common is that all theories are completely asymptotically free meaning that– besides the gauge and the Yukawa couplings – all quartics reach the Gaussian UV fixed point.

Differences arise as to the set of interacting fixed points, summarised in Fig. **12**. Overall, theories display between three and eight distinct weakly interacting fixed points. The partial Banks-Zaks fixed points (FP_2, FP_3) are invariably present in all 17 cases. This is a consequence of a general theorem established in [144], which states that the two loop gauge coefficient is strictly positive for any gauge theory in the limit where the one-loop coefficient vanishes. This guarantees the existence of a partial Banks-Zaks fixed point in either gauge sector. At least one of the partial gauge-Yukawa fixed points (FP_4, FP_5) also arises in all cases. Moreover, the fully interacting Banks-Zaks (FP_6) as well as the fully interacting gauge-Yukawa fixed points (FP_7, FP_8, FP_9) are present in many, though not all, cases. All nine distinct fixed points are available in the "most symmetric" parameter region I (case 9).

It is noteworthy that many theories display a fully IR attractive "sink", invariably given by an IR gauge-Yukawa fixed point in one (FP_4, FP_5) or both gauge sectors (FP_9). In Fig. **11**, this happens for matter field multiplicities in the regions A, B, C, E, F, G, I and their duals (cases 1, 2, 3, 5, 6, 7, 9, 11, 12, 13, 15, 16 and 17 of Fig. **12**).

case	region	complete asymptotic freedom (eps < 0, P > 0)									UV	IR			
		G.G	BZ.G	G.BZ	GY.G	G.GY	BZ.BZ	GY.BZ	BZ.GY	GY.GY		0	Y	y	Yy
1	**A**	1----	2---+	3-+++		5++++		7--++		9-+++	**1**	**3**	**3**	**5**	**5**
2	**B**	1----	2---+	3-+++	4++++	5++++		7--++		9-+++	**1**	**3**	**4**	**5**	**4,5**
3	**C**	1----	2---+	3-+++	4--++	5++++					**1**	**3**	**3**	**5**	**5**
4	**D**	1----	2---+	3-+++	4--++						**1**	**3**	**3**	**-**	**-**
5	**E**	1----	2---+	3---+		5++++	6--++			9-+++	**1**	**6**	**-**	**5**	**5**
6	**F**	1----	2---+	3---+	4++++	5++++	6--++			9-+++	**1**	**6**	**4**	**5**	**4,5**
7	**G**	1----	2---+	3---+	4--++	5++++	6--++	7-+++			**1**	**6**	**7**	**5**	**5**
8	**H**	1----	2---+	3---+	4--++		6--++	7-+++			**1**	**6**	**7**	**-**	**-**
9	**I**	1----	2---+	3---+	4--++	5--++	6--++	7-+++	8-+++	9++++	**1**	**6**	**7**	**8**	**9**
10	**Hb**	1----	2---+	3---+		5--++	6--++		8-+++		**1**	**6**	**-**	**8**	**-**
11	**Gb**	1----	2---+	3---+	4++++	5--++	6--++		8-+++		**1**	**6**	**4**	**8**	**4**
12	**Fb**	1----	2---+	3---+	4++++	5++++	6--++			9-+++	**1**	**6**	**4**	**5**	**4,5**
13	**Eb**	1----	2---+	3---+	4++++		6--++			9-+++	**1**	**6**	**4**	**-**	**4**
14	**Db**	1----	2-+++	3---+		5--++					**1**	**2**	**-**	**2**	**-**
15	**Cb**	1----	2-+++	3---+	4++++	5--++					**1**	**2**	**4**	**4**	**4**
16	**Bb**	1----	2-+++	3---+	4++++	5++++			8--++	9-+++	**1**	**2**	**4**	**5**	**4,5**
17	**Ab**	1----	2-+++	3---+	4++++				8--++	9-+++	**1**	**2**	**2**	**4**	**4**

Figure 12. Shown are the fixed points and eigenvalue spectra of quantum field theories with Lagrangean (IV.25) for the 17 parameter regions with $\epsilon < 0$ and $P > 0$ in Fig. **11**. Scalar selfinteractions are irrelevant at fixed points. All cases display complete asymptotic freedom in the UV. Red shaded slots indicate eigenvalue spectra which arise due to the semi-simple character of the theory. In the deep IR, various types of interacting conformal fixed points are achieved depending on whether both, one, or none of the Yukawa couplings Y and y vanish (from left to right). Regimes with "strong coupling only" in the IR are indicated by a hyphen.

At FP_9, the fully IR attractive fixed point is largely a consequence of IR attractive fixed points in each gauge sector individually. This is not altered qualitatively by the semi-simple nature of the model. As such, a fully IR attractive fixed point FP_9 also arises in the "direct product" limit where the ψ fermions are removed.

At FP_4 and FP_5, in contrast, the IR sink is a direct consequence of the semi simple nature of the theory in that it would be strictly absent as soon as the messenger fermions ψ are removed. Most importantly, the IR gauge Yukawa fixed point in one gauge sector changes the sign of the effective one loop coefficient in the other, mediated via the ψ fermions. This secondary effect means that one gauge sector becomes IR free dynamically, rather than remaining UV free. Overall, the fixed point becomes IR attractive in all canonically marginal couplings (including the quartic couplings). In most cases the IR sink is unique except in parameter regions B and F (case 2, 6, 12 and 16) where we find two competing and inequivalent IR sinks (FP_4 versus FP_5).

Provided that one or both Yukawa couplings take Gaussian values, other fixed points may take over the role of IR "sinks". In these settings, one or both of the elementary "meson" fields remain free for all scales and decouple from the outset. Specifically, the IR sink is given by FP_6 provided that $y = 0 = Y$ (cases 5 – 13); by FP_2 or FP_7 provided that $y = 0$ (cases 14 or 7 – 9, respectively); and by FP_3 or FP_8 provided that $Y = 0$ (cases 4 or 9 – 11). We note that FP_6, FP_7 and FP_8 are natural IR sinks, with or without ψ fermions,

case	region	eps	asymptotic safety and effective theories (P < 0)								UV	IR			
			G.G	BZ.G	G.BZ	GY.G	G.GY	GY.BZ	BZ.GY	GY.GY		0	Y	y	Yy
18	**J**	**-**	1--++	2-+++							(1 pAF)	**2**	**-**	**2**	**-**
19	**K**	**-**	1--++	2-+++							(1 pAF)	**2**	**-**	**2**	**-**
20	**L**	**-**	1--++	2-+++		4++++					(1 pAF)	**2**	**4**	**2**	**4**
21	**Kb**	**-**	1--++	2-+++		4++++	5-+++				(1 pAF, 5 pAS)	**2**	**4**	**2**	**4**
22	**Jb**	**-**	1--++	2-+++		4++++	5---+		8--++	9-+++	**5 AS**	**2**	**4**	**2**	**4**
23	**J**	**+**	1++--		3-+++	4---+	5++++	7--++		9-+++	**4 AS**	**3**	**3**	**5**	**5**
24	**K**	**+**	1++--		3-+++	4-+++	5++++				(1 pAF, 4 pAS)	**3**	**3**	**5**	**5**
25	**L**	**+**	1++--		3-+++		5++++				(1 pAF)	**3**	**3**	**5**	**5**
26	**Kb**	**+**	1++--		3-+++						(1 pAF)	**3**	**3**	**-**	**-**
27	**Jb**	**+**	1++--		3-+++						(1 pAF)	**3**	**3**	**-**	**-**

Figure 13. Same as Fig. **12**, covering the 10 parameter regions with $P < 0$ of Fig. **11**. Notice that FP_6 is absent throughout. Exact asymptotic safety (AS) is realised in the cases 22 and 23. Red shaded slots indicate eigenvalue spectra which arise due to the semi-simple character of the theory. For the cases $18-21$ and $24-27$, partial asymptotic freedom (pAF) or partial asymptotic safety (pAS) is observed whereby one gauge sector decouples entirely at all scales. The latter theories are only UV complete in one of the two gauge sectors and must be viewed as effective rather than fundamental.

provided that all Yukawa couplings of those fermions which interact with the Banks-Zaks fixed point(s) vanish. On the other hand, the result that FP_2 and FP_3 may become IR sinks is a strict consequence of the ψ fermions and would not arise otherwise. Once more, one of the gauge sectors becomes IR free owing to the BZ fixed point in the other, an effect which is mediated via the ψ fermions. In the presence of non-trivial Yukawa couplings, no fully IR stable fixed point arises for theories with field multiplicities in the parameter regions D and H (case 4, 8, 10 and 14). Generically, trajectories will then run towards strong coupling with *e.g.* confinement or strongly-coupled IR conformality. Analogous conclusions hold true in settings with fully attractive IR fixed points provided their basins of attraction do not include the Gaussian.

Finally, another interesting feature which is entirely due to the semi simple nature of the theory are models where FP_9 has a single relevant direction (cases 1, 2, 5, 6, 12, 13, 16 and 17). Whenever this arises, the theory also always displays a fully IR attractive fixed point (FP_4, FP_5, or both).

C. Asymptotic safety

We now turn to quantum field theories with (IV.25) where asymptotic safety is realised. Asymptotic safety relates to settings where some or all couplings take non-zero values in the UV [144]. A prerequisite for this is the absence of asymptotic freedom in at least one of the gauge sectors. We find two such examples provided $P < 0$ (cases 22 and 23 in Fig. **13**), corresponding precisely to settings where one gauge sector is QCD-like whereas the other is QED-like. For these theories, we furthermore find that all other interacting fixed points are also present, except those of the Banks-Zaks type involving the QED-like

case	region	effective theories (eps > 0, P > 0)			UV	IR
		G.G	GY.G	G.GY		
28	**A**	1++++	4-+++		(4 pAS)	**1**
29	**B**	1++++			none	**1**
30	**C**	1++++			none	**1**
31	**D**	1++++		5-+++	(5 pAS)	**1**
32	**E**	1++++	4-+++		(4 pAS)	**1**
33	**F**	1++++			none	**1**
34	**G**	1++++			none	**1**
35	**H**	1++++		5-+++	(5 pAS)	**1**
36	**I**	1++++			none	**1**
37	**Hb**	1++++	4-+++		(4 pAS)	**1**
38	**Gb**	1++++			none	**1**
39	**Fb**	1++++			none	**1**
40	**Eb**	1++++		5-+++	(5 pAS)	**1**
41	**Db**	1++++	4-+++		(4 pAS)	**1**
42	**Cb**	1++++			none	**1**
43	**Bb**	1++++			none	**1**
44	**Ab**	1++++		5-+++	(5 pAS)	**1**

Figure 14. Same as Figs **12** and **13**, covering the 17 parameter regions where $\epsilon > 0$ and $P > 0$ in Fig. **11**. Asymptotic freedom is absent in both gauge sectors implying that FP_2, FP_3, FP_6, FP_7 and FP_8 cannot arise. Partial asymptotic safety (in one gauge sector) is observed in case $28, 31, 32, 35, 37, 40, 41$ and 44, whereby the other gauge sector remains free at all scales (pAS). All models must be viewed as effective rather than fundamental theories. All theories become trivial in the IR.

gauge sector. More specifically, in case 22 the role of the asymptotically safe UV fixed point is now taken by FP_5. The UV critical surface is three-dimensional, in distinction to asymptotically free settings where it is four-dimensional. This reduction, ultimately a consequence of an interacting fixed point in one of the Yukawa couplings, leads to enhanced predictivity of the theory. The Gaussian necessarily becomes a cross-over fixed point with both attractive and repulsive directions, similar to the interacting FP_8. Also, FP_2 and FP_9 are realised with a one-dimensional critical surface. The fully IR attractive FP_4 – the counterpart of the UV fixed point FP_5 – takes the role of an IR "sink". In the low energy limit, the theory displays free $SU(N_c)$ "gluons" in one gauge sector and weakly interacting $SU(N_C)$ "gluons" in the other. Moreover, the spectrum includes both free and weakly interacting mesons related to the former and the latter sectors, as well as free and weakly interacting fermions. Qualitatively, a similar result arises in the "direct product" limit, showing that the semi-simple nature of (IV.25) is not crucial for this scenario.

A noteworthy feature of semi-simple theories with asymptotic safety is that they connect an interacting UV fixed point with an interacting IR fixed point. Hence, our models offer examples of quantum field theories with exact UV and IR conformality, strictly controlled by perturbation theory for all scales. In the massless limit, the phase diagram has trajectories connecting the interacting UV fixed point with the interacting IR fixed point. Some trajectories may escape towards the regime of strong coupling where the theory is expected to display confinement, possibly infrared conformality. The same picture arises in case 23 after exchange of gauge groups.

No asymptotically safe fixed point arises if both gauge sectors are IR free ($P, \epsilon > 0$). This result is in marked contrast to findings in the "direct product" limit where models with an interacting UV fixed points exist – simply because it exists for the simple gauge factors (IV.63) and (IV.67), given suitable matter field multiplicities. We conclude that it is precisely the semi-simple nature of the specific set of theories (IV.25) which disallows asymptotic safety for settings with $P, \epsilon > 0$, see (IV.61).

D. Effective field theories

We now turn to quantum field theories with (IV.25) which are not UV complete semi-simple gauge theories and, as such, must be seen as effective field theories. We find three different types of these. Firstly, we find models with partial asymptotic freedom (pAF), where one gauge sector remains asymptotically free whereas the other stays infrared free. These models always realise a Banks-Zaks fixed point (as they must), and some also realise an IR gauge-Yukawa fixed point. When viewed as a fundamental theory, the IR free sector decouples exactly, for all RG scales, and the theory becomes a simple asymptotically free gauge theory (which is UV complete). The IR-free sector can be interacting when viewed as an effective theory, very much like the $U(1)_Y$ sector of the Standard Model. This setting requires $P < 0$ and is realised in cases 18 – 21 and 23 – 27.

Secondly, we find models with partial asymptotic safety (pAS), where one gauge sector becomes asymptotically safe whereas the other remains free at all scales. All such models display a UV gauge-Yukawa fixed point. When viewed as a fundamental theory, these semi-simple gauge theories in fact reduce to a simple asymptotically safe gauge theory (which is UV complete). The IR-free sector can be interacting when viewed as a non-UV complete effective theory. This setting mostly requires $P, \epsilon > 0$ and is realised in cases 28, 31, 32, 35, 37, 40, 41 and 44. Curiously, pAS is also realised in cases 21 and 24 where $P < 0$ alongside pAF in the other gauge sector — such models have two disconnected UV scenarios, where we can choose to have either asymptotic freedom in one sector, or asymptotic safety in the other, in each case with the remaining sector decoupling at all scales. Once more, if both gauge sectors are interacting these models must be viewed as (non-UV complete) effective theories.

Finally, we find models with none of the above. In these settings (cases 29, 30, 33, 34, 36, 38, 39, 42 and 43), both gauge sectors are IR free and no other weakly coupled fixed points are realised, leaving us with no perturbative UV completion. In the cases 28 – 44, the Gaussian acts as in IR "sink" for RG trajectories. Along these, the long-distance behaviour is trivial, characterised by free massless non-Abelian gauge fields, quarks, and elementary mesons.

In summary, the semi-simple gauge Yukawa theories (IV.25) have a well-defined UV

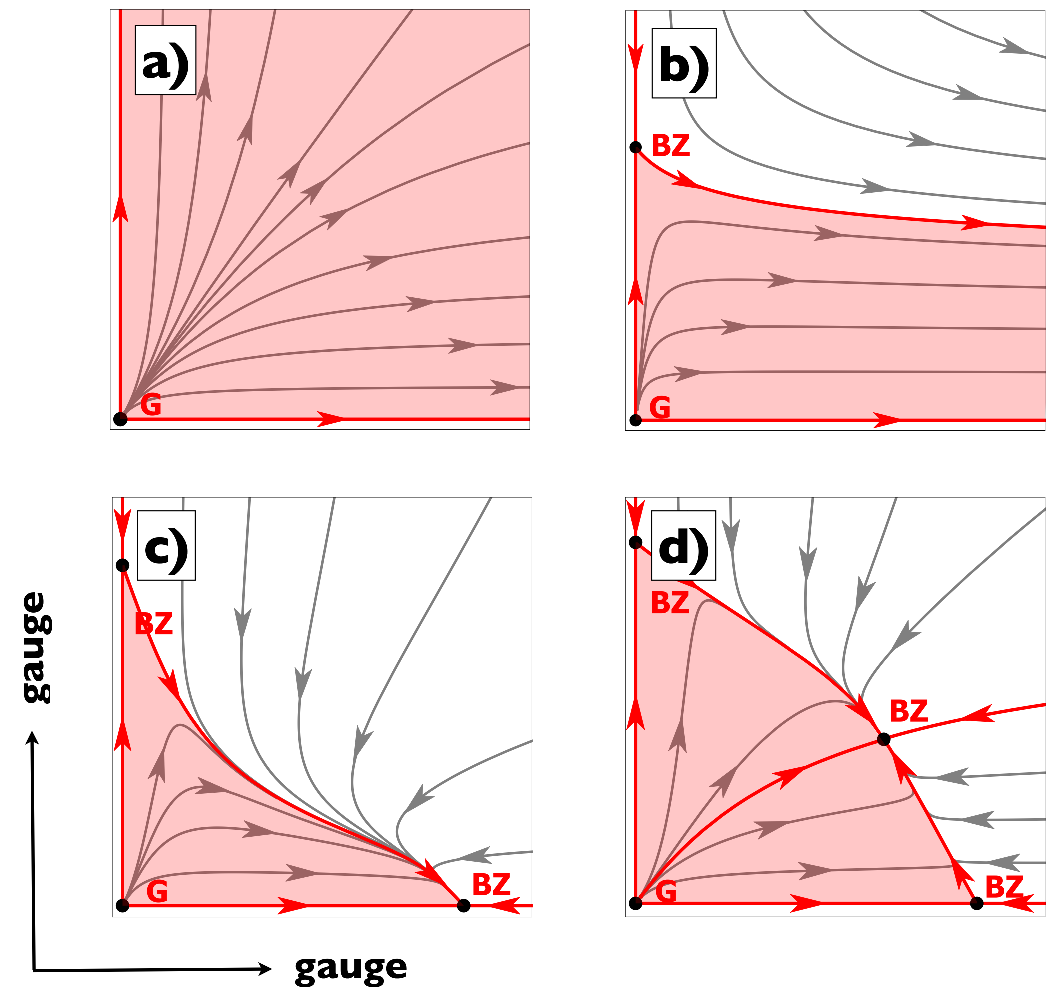

Figure 15. Phase diagrams of asymptotically free semi-simple gauge theories (two gauge groups) coupled to matter without Yukawas, covering *a*) asymptotic freedom and the Gaussian (G) without interacting fixed points and trajectories running towards strong coupling and confinement, *b*) the same, with an additional Banks-Zaks (BZ) fixed point, *c*) two BZ fixed points, one of which turned into an IR sink for all trajectories, or *d*) three BZ fixed points, the fully interacting one now becoming the IR sink. Axes show the running gauge couplings, fixed points (black) are connected by separatrices (red), and red-shaded areas cover all UV free trajectories with arrows pointing from the UV to the IR.

limit with either asymptotic freedom or asymptotic safety in 9+1=10 cases out of the 23 fundamentally distinct parameter settings covered in Fig. **11**. The remaining 4+9=13 parameter settings do not offer a well-defined UV limit at weak coupling. This completes the classification of the models with (IV.25).

VII. PHASE DIAGRAMS OF GAUGE THEORIES

In this section, we discuss the phase diagrams of UV complete theories of the type (IV.25), particularly in view of theories with asymptotic freedom or asymptotic safety.

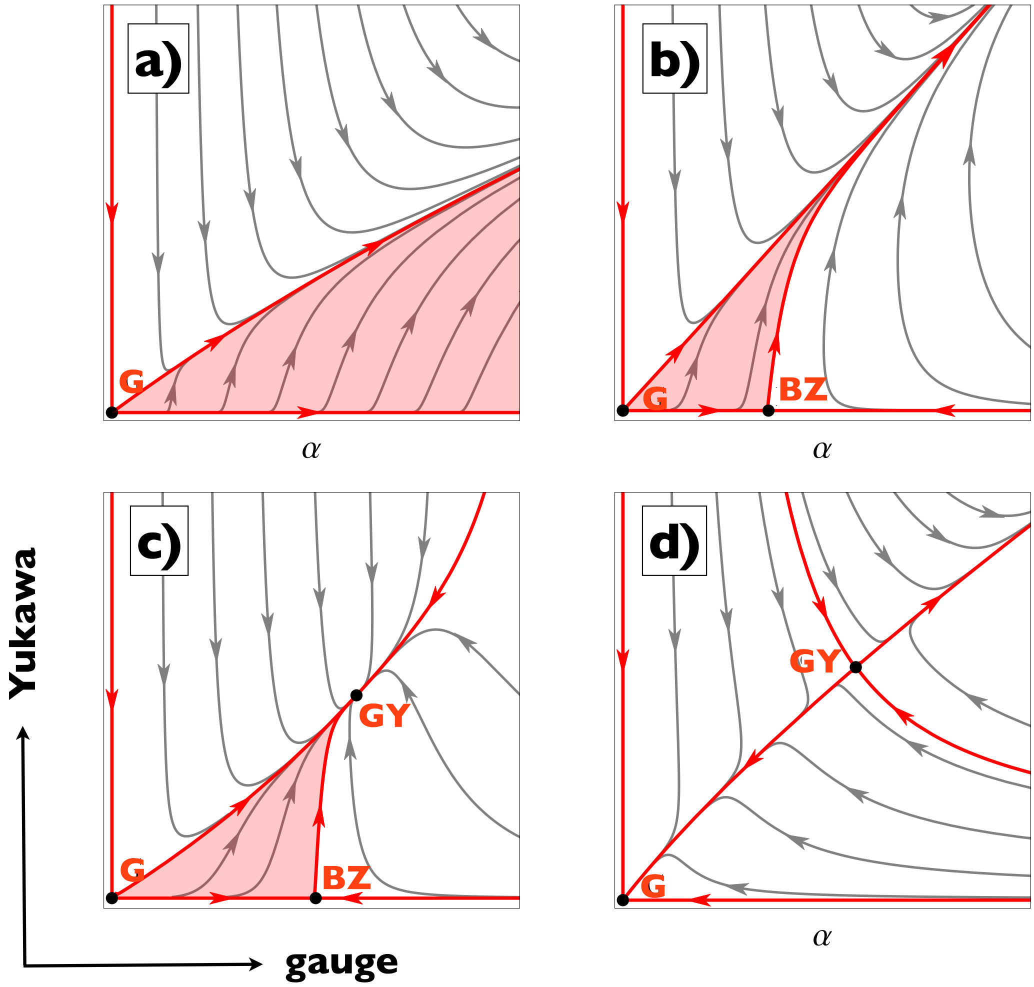

Figure 16. Phase diagrams of UV complete and weakly interacting simple gauge theories coupled to matter with a single Yukawa coupling, covering *a*) asymptotic freedom with the Gaussian UV fixed point and no other weakly interacting fixed point, *b*) asymptotic freedom with a Banks-Zaks (BZ) fixed point, *c*) asymptotic freedom with a Banks-Zaks and an IR gauge-Yukawa (GY) fixed point, and *d*) asymptotic safety with an UV gauge-Yukawa fixed point. Axes display the running gauge and Yukawa couplings, fixed points (black) are connected by separatrices (red), and red-shaded areas cover all UV free trajectories with arrows pointing from the UV to the IR [144, 159]. Examples are given by (IV.63), (IV.67) (see main text).

A. Semi-simple gauge theories without Yukawas

We begin with settings where Yukawa couplings are switched off. In these cases, interacting fixed points can only arise for asymptotically free gauge sectors, and fixed points are of the Banks-Zaks type or products thereof [144, 159]. Qualitatively different cases realised amongst the theories (IV.25) are summarised in Fig. **15** for semi-simple gauge theories with two gauge groups $G_1 \times G_2$. Results generalise to more gauge groups in an obvious manner.

Specifically, Fig. **15***a*) shows theories with asymptotic freedom but without any BZ fixed points. UV free trajectories emanate out of the Gaussian fixed point and invariably

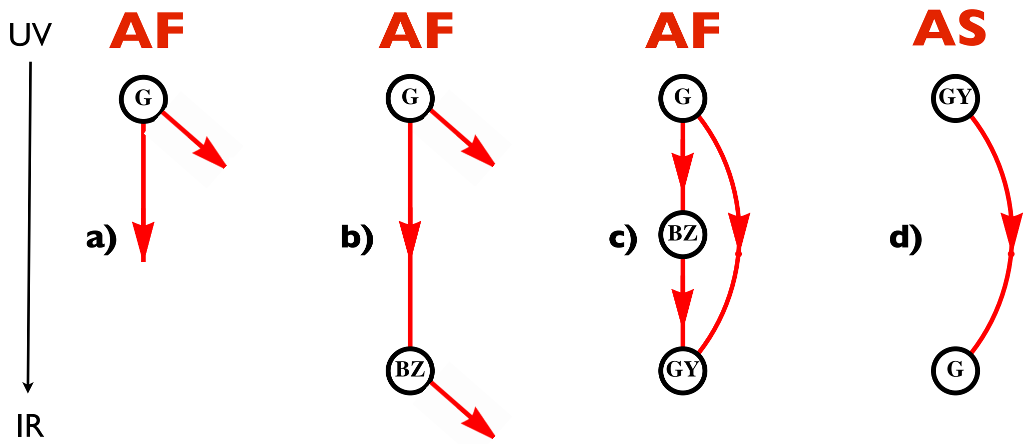

Figure 17. "Primitives" for phase diagrams of simple gauge-Yukawa theories with asymptotic freedom (AF) or asymptotic safety (AS), corresponding to the different setting shown in Fig. **16**. Arrows point from the UV to the IR and connect the different fixed points. Open arrows point towards strong coupling in the IR. The number of outgoing red arrows gives the dimensionality of the UV critical surface. The separate UV safe trajectory towards strong coupling in case d) is not indicated. Yukawa-induced IR unstable directions in a, b) or gauge Yukawa fixed points in c, d) are absent as soon as Yukawa interactions are switched off from the outset.

escape towards strong coupling where the theory is expected to display confinement, or IR conformality. Similarly, Fig. **15**b) shows theories with asymptotic freedom and a BZ fixed point in one of the gauge sectors. The other gauge coupling remains an IR relevant perturbation even at the BZ. Therefore UV free trajectories will again escape towards strong coupling in the IR.

Fig. **15**c) shows asymptotic freedom with a BZ fixed point in both gauge sectors individually. Here, and much unlike Fig. **15**b), one of the BZ fixed points has turned into an exact IR "sink", and both BZ fixed points are connected by a separatrix. As we have already noticed in Sect. **VI** B, the presence of an interacting fixed point in one gauge sector can turn the other gauge sector from UV free to IR free. This new type of phenomenon has become possible owing to the ψ fermions and is once again due to the semi-simple nature of the theory. Therefore, all UV free trajectories invariably are attracted into the IR sink. In the deep IR, the theory approaches a conformal fixed point with massless and unconfined free and weakly coupled gluons and quarks. Regimes of strong coupling cannot be reached.

Fig. **15**d) shows asymptotic freedom with a (partial) BZ fixed point in either gauge sector individually, as well as a fully interacting BZ fixed point. Most notably, all UV free trajectories are attracted by the later, which acts as an IR sink. No trajectories can escape towards strong coupling. The long distance physics is characterised by an interacting conformal field theory with massless weakly coupled gauge fields and fermions. Here, and unlike in Fig. **15**c), all fields remain weakly coupled in the IR.

In the scenarios of Fig. **15**a) and b) UV free trajectories run towards strong coupling and confinement in the IR, in one or both gauge sectors. In contrast, the scenarios in Fig. **15**c)

and d) show that all UV free trajectories are attracted by an IR-stable conformal fixed point. These theories remain unconfined and perturbative at all scales. All four scenarios in Fig. **15** are realised for our template of semi-simple gauge theories with Lagrangean (IV.25). Explicit examples are given for models without Yukawa couplings ($Y = 0 = y$) and for field multiplicities in the parameter regions a) $\epsilon_1, \epsilon_2 < -75/26$, b) $\epsilon_1 < -75/26$ and $-75/26 < \epsilon_2 < 0$, or ($\epsilon_1 \leftrightarrow \epsilon_2$), c) the cases 1 – 4 and 14 – 17 of Fig. **12**, and d) the cases 5 – 13 of Fig. **12**.

B. Simple gauge theories with Yukawas

We continue the discussion of phase diagrams with simple gauge theories with gauge group G and a single Yukawa coupling. Four distinct cases can arise [144, 159], summarised in Fig. **16**. For asymptotically free settings, the theory either shows a) only the Gaussian UV fixed point, b) the Gaussian together with the Banks-Zaks, or c) the Gaussian together with the Banks-Zaks and an IR gauge-Yukawa fixed point. Simple gauge theories can also become asymptotically safe, in which case d) a UV gauge-Yukawa fixed point arises. Trajectories are directed towards the IR. The red-shaded areas indicate the set of UV complete trajectories emanating out of the UV fixed point. We genuinely observe a two-dimensional area of trajectories for asymptotically free settings, which is reduced to a one-dimensional set in the asymptotically safe scenario. The IR regime is characterised by either strong interactions and confinement such as in Fig. **16**a, b, d), or by an interacting conformal field theory with weakly coupled gluons and fermions alongside free or interacting scalar mesons —corresponding to the BZ fixed points in Fig. **16**b) and c), or the IR GY fixed point in Fig. **16**c), respectively—, or by Gaussian scaling, Fig. **16**d).

All four scenarios in Fig. **16** are realised for simple gauge theories with (IV.63) corresponding to the parameter regions a) $\epsilon_1 < -75/26$, b) $-75/26 < \epsilon_1 < 0$ and $R > \frac{1}{2}$, c) $-75/26 < \epsilon_1 < 0$ and $R < \frac{1}{2}$, or d) $\epsilon_1 > 0$ and $R < \frac{1}{2}$, respectively, with R additionally bounded by (IV.64).

An economic way to display phase diagrams for semi-simple theories with or without Yukawas is achieved by introducing a schematic diagrammatic language, see Fig. **17**. Each of the four basic phase diagrams in Fig. **16** are represented by a "primitive" diagram, Fig. **17**, where full dots indicate (free or interacting) fixed points, red arrows indicate the outgoing trajectories, and RG flows schematically run "top-down" from the UV to the IR. Also, at each fixed point the number of outgoing arrows indicates the dimensionality of the fixed point's "UV critical surface". Fixed points are connected by separatrices. We use straight lines to indicate separatrices involving the BZ fixed point, curved lines to indicate separatrices connecting GY fixed points with the Gaussian, and open-ended lines to denote RG trajectories running towards strong coupling without reaching any weakly coupled fixed points.

Specifically, in case a), a two-dimensional array of RG flows are running out of the

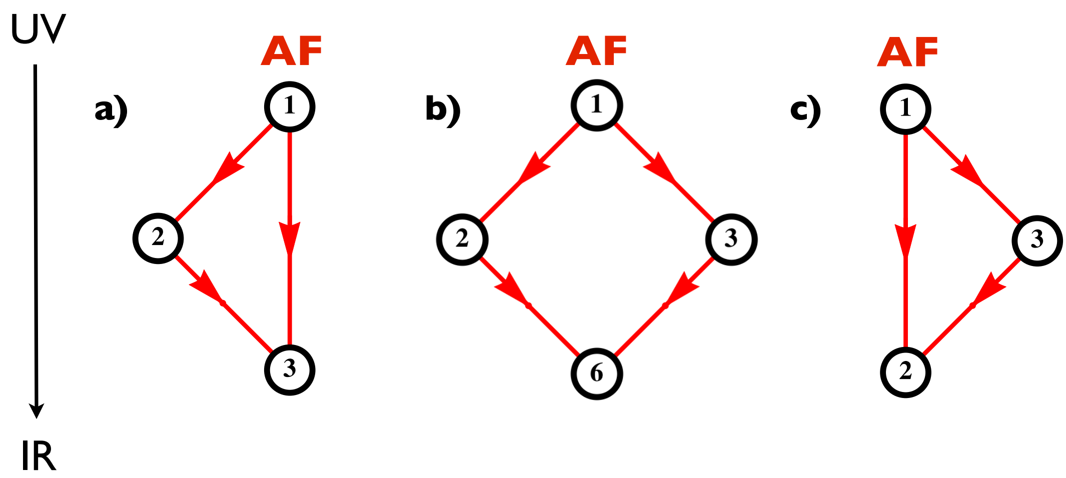

Figure 18. Schematic phase diagram for asymptotically free semi-simple gauge theories (IV.25) with Banks-Zaks type fixed points without Yukawas and exact IR conformality. Field multiplicities correspond to the cases *a*) 1 – 4, *b*) 5 – 13, and *c*) 14 – 17 of Fig. **12**, respectively, with scalars decoupled. RG flows point from the UV to the IR (top to bottom). At each fixed point, the dimensionality of the UV critical surface is given by the number of outgoing red arrows. All UV free trajectories terminate at FP_2, FP_6 and FP_3, respectively, which act as fully attractive IR "sinks". The topology of the phase diagram *b*) is the "square" of Fig. **17***b*), representing Fig. **15***d*). The phase diagrams *a*) and *c*), representing Fig. **15***c*), cannot be constructed from the primitives in Fig. **17**.

Gaussian UV fixed point towards strong coupling, with no weakly interacting fixed points. In case *b*), we additionally observe a Banks-Zaks fixed point. It is connected with the Gaussian by a separatrix shown in red. Arrows invariably point towards the IR. Yukawa couplings act as an unstable direction at both fixed points. In case *c*), we additionally observe a gauge-Yukawa fixed point besides the Gaussian and the BZ. All three fixed points are connected by separatrices. Note that two lines emanate from the Gaussian, reflecting that the UV critical surface is two dimensional. The GY fixed point arises as an IR sink, which attracts all UV-free trajectories emanating out of the Gaussian. In case *d*), the model is asymptotically safe and the GY fixed point has become the interacting UV fixed point. A Banks-Zaks fixed point can no longer arise [144]. The theory has a one-dimensional UV critical surface connecting the GY fixed point with the IR Gaussian fixed point via a separatrix. A second UV safe trajectory which leaves the GY fixed point towards strong coupling is not depicted. Finally, we note that the Yukawa-induced IR unstable directions in *a*) and *b*) or gauge Yukawa fixed points in *c*) and *d*) are absent as soon as Yukawa interactions are switched off from the outset.

C. Semi-simple gauge theories with asymptotic freedom

We consider phase diagrams for semi-simple theories (IV.25) with complete asymptotic freedom, exemplified by all models in Fig. **12**. When Yukawa couplings are absent, the meson-like scalar degrees of freedom remain free at all scales and decouple from the theory.

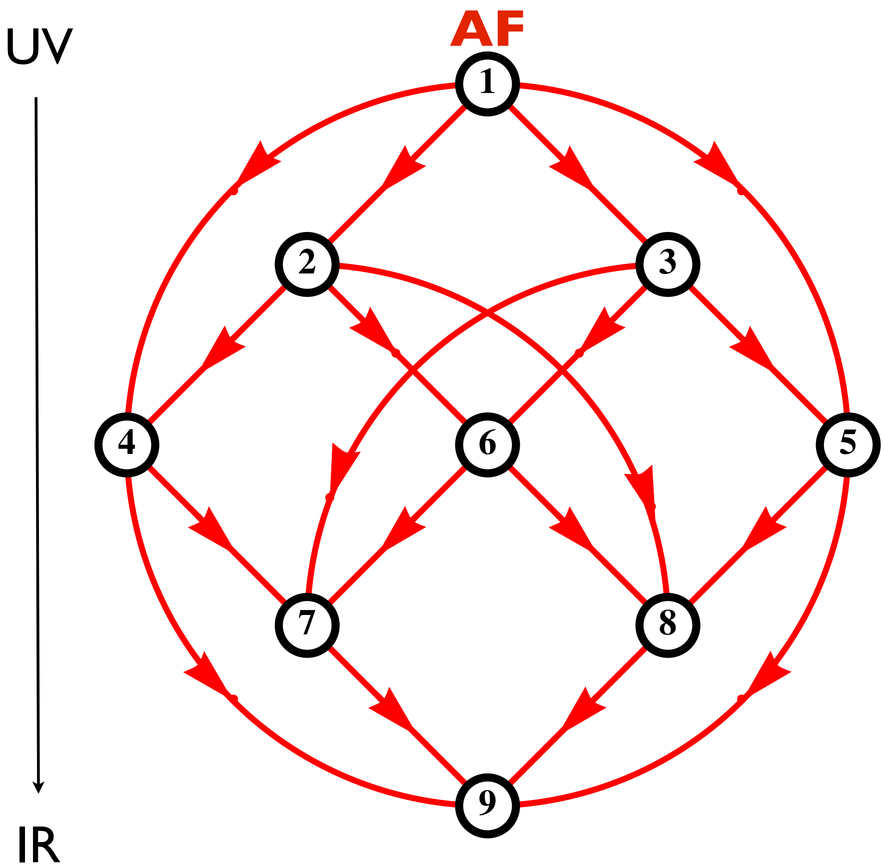

Figure 19. Asymptotic freedom and schematic phase diagram for semi simple gauge-Yukawa theories with field multiplicities as in case 9 of Fig. **12**. RG flows point from the UV to the IR (top to bottom). Besides the Gaussian UV fixed point (FP_1), the theory displays all eight weakly interacting fixed points, see Tab. **8**. At each fixed point, the dimensionality of the UV critical surface is given by the number of outgoing red arrows. FP_9 is fully attractive and acts as an IR "sink". The topology of the phase diagram is the "square" of Figs. **16**,**17***c*); see main text.

In the regime with asymptotic freedom solely Banks-Zaks fixed points can arise in the IR. Fig. **15***d*) and Fig. **18***b*) shows settings where all Banks-Zaks fixed points are present, corresponding to the cases 5 – 13 of Fig. **12**. RG flows point from the UV to the IR (top to bottom) and connect the Gaussian UV fixed point (FP_1) with either of the partially (FP_2 and FP_3) and the fully interacting (FP_6) Banks-Zaks fixed points. The latter is fully attractive and acts as an IR sink. The topology of the phase diagram is the "square" of Figs. **16**,**17***b*). In the deep IR the theory is unconfined yet weakly interacting, and the elementary gauge fields A, a and fermions Q, q and ψ appear as massless particles at the IR conformal fixed point. The phase diagrams in Figs. **18***a*) and *c*) cannot be constructed out of the simple primitives, Fig. **17**. The reason for this is that the eigenvalue spectrum at one of the fixed points deviates from the "direct product" spectrum due to interactions.

Next we include Yukawa interactions. We have already concluded from Fig. **12** that the eigenvalue spectrum in the cases 8, 9 and 10 agrees qualitatively, for all fixed points, with

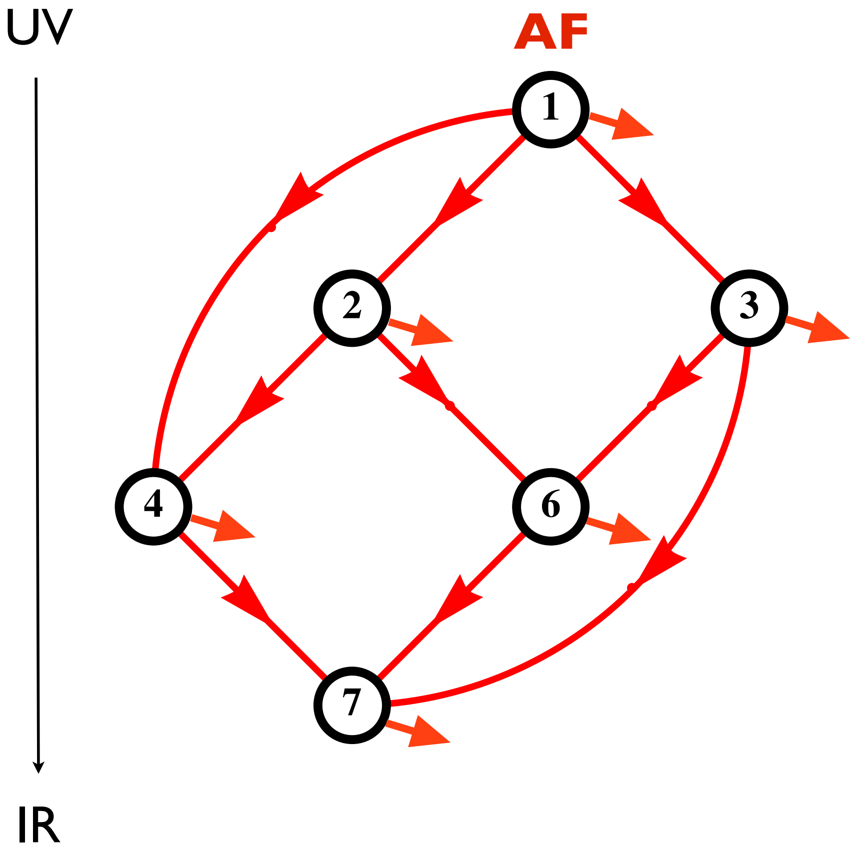

Figure 20. Asymptotic freedom and schematic phase diagrams for semi-simple gauge-Yukawa theories with field multiplicities as in case 8 of Fig. **12**. Flows point from the UV to the IR (top to bottom). The theories display five weakly interacting fixed points besides the Gaussian UV fixed point (FP_1). The unavailability of FP_5, FP_8 and FP_9 implies that some trajectories escape towards strong coupling (short arrows), and none of the fixed points acts as a complete IR attractor. The topology of the phase diagram is the "direct product" of Fig. **16**, **17***c*) with Fig. **16**, **17***b*); see main text. The IR unstable direction is removed provided that the Yukawa coupling $y \equiv 0$, in which case the singlet mesons h decouple.

the eigenvalue spectrum in the corresponding "direct product" limit. In these settings, we may then use the primitives in Fig. **17** to find the semi-simple phase diagrams. We consider the case where the parameters (IV.61) take values within the range I of Fig. **11** and for $\epsilon < 0$, corresponding to case 9 of Fig. **12**. This family of theories includes the "symmetric" setup $(R, P) = (1, 1)$ where symmetry under the exchange of gauge groups is manifest. The UV fixed point is given by the Gaussian (FP_1), and the UV critical surface at the Gaussian is four-dimensional, owing to the marginal UV relevancy of the two gauge and the two Yukawa couplings. All scalar couplings are irrelevant in the UV and can be expressed in terms of the gauge and the Yukawa couplings along UV-free trajectories. Moreover, each gauge sector displays the Banks-Zaks and a gauge-Yukawa fixed point individually, and all nine fixed points are realised in the full theory.

Since the sign pattern of the eigenvalue spectra at all fixed points is equivalent to the "direct product" limit, the topology of the semi-simple phase diagram is the "square" of Fig. **17***c*) – shown in Fig. **19**. Fixed points are connected by separatrices (red lines), and arrows always point towards the IR. From top to bottom, the fixed points FP_1 $(\mathrm{FP}_{2,3})$ $[\mathrm{FP}_{4,5,6}]$ $(\mathrm{FP}_{7,8})$ and FP_9 have a 4 (3) [2] (1) and 0-dimensional UV critical surface, respectively, corresponding to the number of outgoing red arrows. FP_9 acts as an IR attractor for all trajectories within its basin of attraction. Consequently, the elementary quarks and gluons are not confined and the theory corresponds to a conformal field theory of weakly interacting massless gluons, fermions and mesons in the deep IR. For certain fine-tuned settings, the IR limit would, instead, correspond to one of the other interacting fixed points $\mathrm{FP}_2 - \mathrm{FP}_8$, relating to different conformal field theories. Also, while all other fixed points can be reached from the Gaussian FP_1 (whose UV critical surface has the largest dimensionality), it is not true in general that a fixed point with a smaller UV critical dimension can be reached from a fixed point with a larger one. Fixed points are also not connected "horizontally".

As a further example we consider a less symmetrical setting given by models with (IV.61) in the parameter range H (or Hb) of Fig. **11**, and for $\epsilon < 0$. In these theories, only one of the two gauge sectors can achieve a gauge-Yukawa fixed point. Consequently, six different types of fixed points are realised. The sign pattern of the eigenvalue spectrum (cases 8 or 10, Fig. **12**) ensures that the topology of the semi-simple phase diagram obtains as the direct product of Fig. **17***b*) with Fig. **17***c*), shown in Fig. **20**. From top to bottom, the fixed points FP_1 $(\mathrm{FP}_{2,3})$ $[\mathrm{FP}_{4,6}]$ and FP_7 have a 4 (3) [2] and 1-dimensional UV critical surface, respectively. Fixed points are connected by separatrices. The absence of FP_5, FP_8 and FP_9 implies that some trajectories escape towards strong coupling, indicated by short arrows, from each of the fixed points. The unstable direction relates to the Yukawa coupling y in (IV.25). Provided it is switched off, FP_7 would become the fully attractive IR "sink". In this case, the elementary mesons h are spectators and remain free at all scales. Also, the elementary quarks and gluons remain unconfined. In the deep IR, the theory corresponds to a conformal field theory of weakly interacting massless gluons A, fermions Q, ψ and mesons H, together with free and massless gluons a, fermions q and mesons h, see Tab. **9**. For certain fine-tuned settings, the IR limit would, instead, correspond to one of the other interacting fixed points $\mathrm{FP}_2 - \mathrm{FP}_8$, relating to different conformal field theories.

The phase diagrams of asymptotically free theories in the cases $1-7$ and $11-17$ of Fig. **12** cannot be constructed out of the simple primitives, Fig. **17**. The reason for this is that their eigenvalue spectrum at some of the interacting fixed points deviates from the "direct product" spectrum. Once again this effect is due to the semi-simple nature of the theory. A more detailed study of these cases is left for future work.

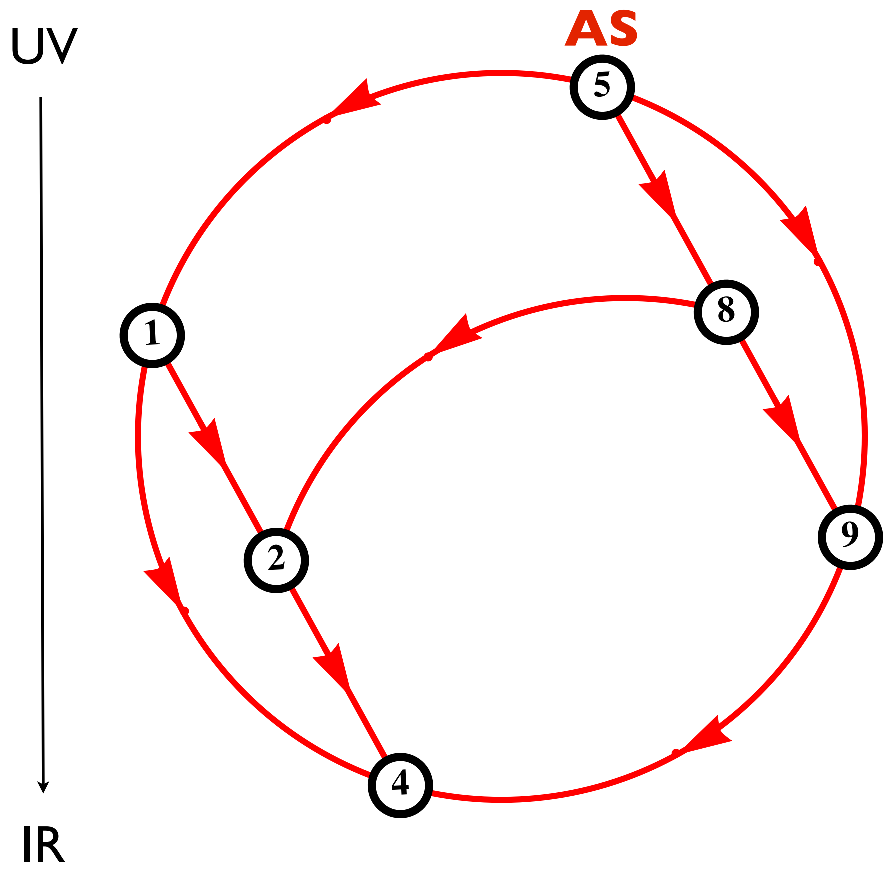

Figure 21. Asymptotic safety and schematic phase diagram of semi simple gauge-Yukawa theories with field multiplicities as in case 22 of Fig. **13**). Besides the partially interacting UV fixed point (FP_5), the theory displays five weakly interacting fixed points. The Gaussian (FP_1) takes the role of a crossover fixed point and FP_4 takes the role of an IR sink. The topology of the phase diagram is the "direct product" of Fig. **16**, **17***c*) with Fig. **16**, **17***d*); see main text.

D. Semi-simple gauge theories with asymptotic safety

We finally turn to the phase diagram of semi-simple gauge theories with exact asymptotic safety. From Figs. **13** and **14** we conclude that asymptotic safety arises through a partially interacting UV fixed point where one gauge sector is interacting whereas the other gauge sector is free. This is achieved for matter field multiplicities (IV.61) taking values within the range J or Jb of Fig. **11**, corresponding to cases 22 or 23 of Fig. **13**. Once more, the eigenvalue spectra at all fixed points are equivalent to the ones in the direct product limit, implying that the phase diagram arises as the direct product of the corresponding "simple factors" Fig. **16**, **17***c*) and Fig. **16**, **17***d*).

Fig. **21** shows the schematic phase diagram for case 22, where the asymptotically safe UV fixed point FP_5 is of the $\mathrm{G}\cdot\mathrm{GY}$ type (see Tab. **8** and **10**). Unlike the cases with asymptotic freedom, here, the UV hypercritical surface is three rather than four dimensional. The reason for this is that one of the Yukawa couplings is taking an interacting UV fixed point. At each fixed point, the number of outgoing directions indicate the dimensionality of the fixed point's critical hypersurface. From top to bottom, the fixed points FP_5 ($\mathrm{FP}_{1,8}$) [$\mathrm{FP}_{2,9}$] and FP_4 have a 3 (2) [1] and 0-dimensional UV critical surface. UV finite trajectories connect FP_5 via intermediate cross-over fixed points with the fully IR attractive fixed point FP_4, which is of the $\mathrm{GY}\cdot\mathrm{G}$ type. At weak coupling, all UV-IR connecting trajectories proceed either via the Gaussian FP_1 ($\mathrm{G}\cdot\mathrm{G}$) and FP_2 ($\mathrm{BZ}\cdot\mathrm{G}$), or via FP_8 ($\mathrm{BZ}\cdot\mathrm{GY}$) and FP_2 or FP_9 ($\mathrm{GY}\cdot\mathrm{GY}$). The Gaussian fixed point is IR free in one of the gauge couplings meaning that is necessarily arises as a cross-over fixed point. There are no trajectories connecting the fixed points FP_1 with FP_9 because the sole relevant direction at the latter is an irrelevant direction at the former. FP_4 acts as an IR "sink" for RG trajectories. While all other fixed points can be reached from the interacting UV fixed point FP_5 (whose UV critical surface has the largest dimensionality), it is not true in general that a fixed point with a smaller UV critical dimension can be reached from a fixed point with a larger one (e.g. FP_9 cannot be reached from FP_1). Fixed points are also not connected "horizontally".

An intriguing novelty of our models with asymptotic safety is that both the deep UV and the deep IR limits are characterised by weakly interacting conformal field theories. For example, in the deep UV the theories of case 22 correspond to conformal field theories of weakly interacting massless gluons a, fermions q, ψ and mesons h, together with free and massless gluons A, fermions Q and mesons H. Along the UV – IR transition, the fields (A, Q, H) and (a, q, h) effectively "interchange" their roles, ultimately approaching conformal field theories of weakly interacting massless gluons A, fermions Q, ψ, and mesons H, together with free and massless gluons a, fermions q and mesons h in the IR. Hence, one may say that IR conformality in the $SU(N_\mathrm{c})$ gauge sector arises from UV conformality in the $SU(N_\mathrm{C})$ gauge sector through a "see-saw" mechanism transmitted via the ψ fermions, *i.e.* the only fields which are interacting at all scales including the UV and the IR limits. For certain fine-tuned settings, the IR limit would, instead, correspond to one of the other interacting fixed points FP_1, FP_2, FP_8 or FP_9, relating to different conformal field theories. Also, for certain UV parameters, theories may escape towards strong coupling in the IR.

E. Mass deformations and phase transitions

In the vicinity of fixed points phase transitions between different phases arise once mass terms are switched on. At weak coupling mass anomalous dimensions are perturbatively small (Sec. **III** D). The running of scalar or fermion mass terms, once switched on, will then be dominated by their canonical mass dimensions – modulo small quantum corrections. Consequently, mass terms add additional relevant directions at all fixed points (e.g. Figs. **19** – **21**). Each of the eight interacting UV fixed points relates to a quantum phase transition between phases with and without spontaneous breaking of symmetry where the vacuum expectation value of the scalar fields serves as an order parameter. In particular, fixed points which act as IR sinks for the canonically marginal interactions (such as FP_9 in Fig. **19** and FP_4 in Fig. **21**) develop new unstable directions driven by the mass. Scalar fields may or may not develop vacuum expectation values leading to symmetric and symmetry broken phases, respectively. Also, fermions may acquire masses spontaneously. Thereby a variety of different phases may arise, connected by first and higher order quantum phase transitions. Close to interacting fixed points, phase transitions are continuous and, in some cases, of the Wilson-Fisher type with a single relevant parameter. We leave a more detailed investigation of phase transitions for a future study.

VIII. DISCUSSION

In this section, we address further aspects of interacting fixed points covering universality and operator ordering, triviality bounds, perturbativity in and beyond the Veneziano limit, conformal symmetry, and conformal windows.

A. Gap, universality, and operator ordering

At partially or fully interacting fixed points, the degeneracy of the nine classically marginal couplings (IV.26), (IV.27) is partly or fully lifted. We have computed scaling exponents to the leading non-trivial order in ϵ. Interacting fixed points have non-trivial exponents of order $\sim \epsilon$, except if a gauge coupling is involved in which case one of the exponents is parametrically smaller $\sim \epsilon^2$. Hence, the eigenvalue spectrum opens up $\sim \epsilon$ because eigenvalues of order ϵ are invariably present at any of the interacting fixed points. It is convenient to denote the difference between the smallest negative eigenvalue and the smallest positive eigenvalue as the "gap" in the eigenvalue spectrum, which serves as an indicator for interaction strength [44, 112]. Simple $SU(N)$ gauge theories in the Veneziano limit such as (IV.63) display a gap of order $\sim \epsilon$ (ϵ^2) at the Banks-Zaks or the UV gauge Yukawa (IR gauge Yukawa) fixed point, respectively [55]. In semi-simple theories, and depending on the specifics of the fixed point, we again find that the gap is either of order ϵ or of order ϵ^2. (The gap trivially vanishes if one of the gauge sectors is asymptotically free and takes Gaussian values.) The gap still depends on the remaining free parameters (P, R).

Also, all results for fixed points and scaling exponents are universal and independent of the RG scheme, although we have used a specific scheme (MS bar) throughout. This is obviously correct for dimensionless couplings at one loop where divergences are logarithmic. We have checked that it also holds at two loop level both for the gauge sectors, and for the Yukawa contributions to the running of the gauge coupling(s) [55]. The field strengths and the Yukawa couplings are marginally relevant operators at asymptotically free Gaussian UV fixed points (case 1 – 17 of Fig. **12**). At asymptotically safe UV fixed points, one of the field strengths becomes relevant and the corresponding Yukawa coupling irrelevant (case 22, 23 of Fig. **13**). There is no UV fixed point where both gauge sectors remain interacting. The scalar selfinteractions are (marginally) irrelevant at any fixed point.

B. Elementary gauge fields and scalars

Triviality bounds relate to perturbative UV Landau poles of infrared free interactions. They limit the predictivity of theories to a maximal UV extension [131]. For theories with action (IV.25), perturbative UV Landau poles can arise for gauge couplings in the absence of asymptotic freedom or asymptotic safety. Examples for this are given in cases 18 – 21 and 24 – 27 of Fig. **13** where one gauge sector is IR free, as well as in cases 28 – 44 of Fig. **14** where both gauge sectors are IR free. In these cases the theories can at best be treated as effective rather than fundamental (see Sect. **VI** D). Conversely, triviality in gauge sectors is trivially avoided in settings with asymptotic freedom (such as in cases 1 – 17), and non-trivially in settings with asymptotic safety (case 22 and 23). In the latter cases, the loss of asymptotic freedom is compensated through an interacting fixed point in the Yukawa and scalar couplings, which enabled a fixed point for the gauge coupling [55]. We stress that scalar fields and Yukawa interactions play a key role. Without them, triviality of any QED-like gauge theories cannot be avoided [144, 159].

Triviality also relates to the difficulty of defining elementary self-interacting scalar quantum fields in four dimensions [163–165]. It is interesting to notice that the quartic scalar couplings always take a unique physical fixed points as soon as the gauge and Yukawa coupling take weakly coupled fixed points. Hence, in theories with (IV.25) scalar fields can be viewed as elementary and triviality is evaded in all settings with asymptotic freedom and asymptotic safety. In either case gauge fields play an important role, albeit for different reasons [55]. For gauge interactions with asymptotic freedom, the running of gauge couplings dictates the running for Yukawa and scalar couplings, and conditions for complete asymptotic freedom have been derived [124] which ensure that gauge theories coupled to matter reach the free UV fixed point [125]. For theories with asymptotic safety, scalars are required to help generate a combined fixed point in the gauge, Yukawa, and quartic scalar couplings. This leads invariably to a “reduction of couplings” and enhanced predictivity over models with asymptotic freedom through a reduced UV critical surface.

C. Veneziano limit and beyond

Our findings, throughout, rely on the existence of exact small parameters $\epsilon_1 \ll 1$ and $\epsilon_2 \ll 1$ (IV.35) [or $\epsilon \ll 1$ see (IV.41)] in the Veneziano limit, which relate to the gauge one loop coefficients. Consequently, an iterative solution of perturbative beta functions becomes exact and interacting fixed points arise as exact power series in the small parameters. More specifically, the leading non-trivial approximation which is NLO$'$ (Tab. **7**) retains the gauge beta functions up to two loop, and the Yukawa and scalar beta functions up to one loop. The parametric smallness of the gauge one-loop coefficients allows an exact cancellation of one and two loop terms implying that interacting fixed points for the gauge couplings must be of the order of the one loop coefficient $\sim \epsilon$. The Yukawa nullclines at one loop imply that Yukawa couplings are necessarily proportional to the gauge couplings, and the scalar nullcline impose that scalar couplings are proportional to the Yukawas (see Sect. **V** A); hence either of these come out $\sim \epsilon$. Higher order loop approximations nNLO$'$ starting with $n = 2$ then correspond to retaining $n + 1$ loops in the gauge, and n loops in the Yukawa and scalar beta functions respectively, see Tab. **7**. Hence, solving the beta functions for interacting fixed points order-by-order in perturbation theory ($n \to n + 1$) we have that

$$\alpha_i^* = \alpha_i^*\Big|_{n\mathrm{NLO}'} + \mathcal{O}(\epsilon^{n+1}) \tag{IV.96}$$

for all couplings (IV.26), (IV.27) and all fixed points, with corrections from the $(n+1)$NLO$'$ level being at least one power in ϵ smaller than those from the preceding level. We conclude that the expressions for the interacting fixed points $\alpha_i^*|_{n\mathrm{NLO}'}$ are accurate polynomials in ϵ up to including terms of order ϵ^n, for all n.

Beyond the Veneziano limit, the parametrically small control parameter ϵ is no longer available. Instead, ϵ will take finite, possibly large, values dictated by the (finite) field multiplicities. Still, for sufficiently large matter field multiplicities, ϵ remains sufficiently small and perturbativity remains in reach [56]. It is then conceivable that the fixed points found in the Veneziano limit persist even for finite N.[37] At finite N, however, we stress that the nNLO$'$ approximations and (IV.96) are no longer exact order-by-order. It then becomes important to check numerical convergence of higher loop approximations, including non-perturbative resummations. In this context it would be particularly useful to know the radius of convergence of beta functions (in ϵ) in the Veneziano limit. A finite radius of convergence has been established rigorously in certain large-N_F limits of gauge theories without Yukawa interactions [166, 167] which makes it conceivable that the radius of convergence might be finite here as well.[38] If so, this would offer additional indications for the existence of interacting fixed points beyond the Veneziano limit.

[37] An example for a conformal window with asymptotic safety is given in [62] for the model introduced in [55].

[38] Results for resummed beta functions of large-N gauge theories with Yukawa couplings are presently not available.

D. Conformal symmetry and conformal windows

By their very definition, the gauge-Yukawa theories investigated here are scale-invariant at (interacting) fixed points. Conditions under which scale invariance entails exact conformal invariance have been discussed by Polchinski [147] (see also [141]). Applied to the theories (IV.25) at weak coupling, it implies that exact conformal invariance is realised at all interacting fixed points discovered here. It would then be interesting to find the full conformally invariant effective action beyond the classically marginal invariants retained in (IV.25). First steps into these directions have been reported in [60]. Moreover, for a quantum theory to be compatible with unitarity, scaling dimension of (primary) scalar fields must be larger than unity. This is confirmed for all fixed points by using the results of Sect. **III** D for the anomalous dimensions of fields and composite scalar operators, together with the results for fixed points at NLO$'$ accuracy (Tab. **12** and **13**). We conclude that the residual interactions are compatible with unitarity.

Away from the Veneziano limit, findings for the various interacting conformal fixed points persist once ϵ is finite. One may then think of keeping the parameters in the gauge sectors $(N_\mathrm{C}, N_\mathrm{c})$ fixed and finite while varying the matter field content $(N_\mathrm{F}, N_\mathrm{f}, N_\psi)$. Then, the domain of existence for each of the interacting fixed points (Tab. **12**, **13**) turns into a “conformal window” as a function of the matter field multiplicities. The fixed point ceases to exist outside the conformal window. The conformal window for asymptotic safety with a simple $SU(N)$ gauge factor has been determined in [62]. Boundaries of conformal windows can be estimated within perturbation theory though more accurate results invariably require non-perturbative tools.[39]

IX. SUMMARY

We have used perturbation theory and large-N techniques for a rigorous and comprehensive investigation of weakly interacting fixed points of gauge theories coupled to fermionic and scalar matter. For concrete families of simple and semi-simple gauge theories with action (IV.25) and following the classification of fixed points put forward in [144, 159], we have discovered a large variety of exact high- and low-energy fixed points (Tab. **8**, **12**, **13**). These include partially interacting ones (Tab. **12**) where one gauge sector remains free, and fully interacting ones (Tab. **13**) where both gauge sectors are interacting. We have determined the domains of existence for all of them (Fig. **6**– **10**). Interestingly, we also find that the requirement of vacuum stability always singles out a unique viable fixed point in the scalar sector.

As a function of field multiplicities, the phase space of distinct quantum field theories (Fig. **11**) includes models with asymptotic safety and asymptotic freedom, and effective

[39] See [168] for lattice studies of conformal windows in QCD with fermionic matter (Banks-Zaks fixed points).

theories without UV completion (Figs. **12**, **13** and **14**). In the IR, theories display either strong coupling and confinement, or weakly coupled fixed points where the elementary gauge fields and fermions are unconfined and appear as massless particles. Many features are a consequence of the semi-simple nature and would not arise in simple (or "direct products" of simple) gauge theories. Highlights include massless semi-simple gauge-matter theories where one gauge sector can be both UV free and IR free owing to a fixed point in the other, Fig. **15** *c*), and theories with inequivalent scaling limits in the IR. Semi-simple effects are particularly pronounced for asymptotically free theories where they enhance the diversity of different IR scaling regimes (Fig. **12**).

Another central outcome of our study is the first explicit "proof of existence" for asymptotic safety in semi-simple quantum field theories with elementary gauge fields, scalars and fermions. It establishes the important result that asymptotic safety is not limited to simple gauge factors [55], fully in line with general theorems and structural results [144]. Our findings, together with their supersymmetric counterparts in [146], make it conceivable that semi-simple theories display interacting UV fixed points even beyond the Veneziano limit, thus further paving the way for asymptotic safety beyond the Standard Model [56]. The stability of the vacuum (Sect. **V**) in all models studied here suggests that the near-criticality of the Standard Model Higgs [160, 161] can very well expand into full criticality at an interacting UV fixed point [56].

In addition, we have investigated phase diagrams for simple and semi-simple gauge theories with and without Yukawa interactions, continuing an analysis initiated in [144, 159]. We find that transitions from the UV to the IR can proceed from free or interacting fixed points to confinement and strong coupling. We also find transitions from free to interacting (Figs. **15**, **16**, **17**, **18**, **19**, **20**) or from interacting to other interacting conformal fixed points (Fig. **21**). In the latter cases, theories display a variety of exact "IR sinks", meaning free or interacting IR conformal fixed points which are fully attractive in all classically marginal interactions. Once more, many new features have come to light beyond those observed in simple gauge theories [144, 159].

Our study used minimal models with a low number of Yukawa and gauge couplings. Already at this basic level, an intriguing diversity of fixed points and scaling regimes has emerged, with many novel characteristics both at high and low energies. We believe that these findings warrant more extensive studies in view of rigorous results [144, 146], extensions towards strong coupling [60], and its exciting potential for physics beyond the Standard Model [56].

Appendix A: General expressions for fixed points

Most results in the main text relate to the choice $N_\psi = 1$. For completeness, we summarize fixed point results for general N_ψ species of fermions in the fundamental of both gauge groups $SU(N_C)$ and $SU(N_c)$. We observe that N_ψ is restricted within the range

$$0 \le N_\psi \le \frac{11}{2} \,. \tag{IV.A.1}$$

Outside of this range, exact perturbativity is lost. Substituting N_ψ into the RG coefficients and solving for fixed points, we find the following expressions at the partially interacting Banks-Zaks fixed points FP_2 and FP_3,

$$\mathbf{FP_2}: \quad \alpha_1 = -\frac{4}{75} R\epsilon \tag{IV.A.2}$$

$$\mathbf{FP_3}: \quad \alpha_2 = -\frac{4}{75}\,\frac{P\epsilon}{R} \tag{IV.A.3}$$

At the partially interacting fixed points FP_4 and FP_5 we have

$$\mathbf{FP_4}: \quad \begin{cases} \alpha_1 = \dfrac{2}{3}\dfrac{13 - 2N_\psi R}{(2N_\psi R - 1)(3N_\psi R - 19)} R\epsilon \\ \alpha_Y = \dfrac{4}{(2N_\psi R - 1)(3N_\psi R - 19)} R\epsilon \end{cases} \tag{IV.A.4}$$

$$\mathbf{FP_5}: \quad \begin{cases} \alpha_2 = \dfrac{2}{3}\dfrac{13 - 2N_\psi/R}{(2N_\psi/R - 1)(3N_\psi/R - 19)} \dfrac{P\epsilon}{R} \\ \alpha_y = \dfrac{4}{(2N_\psi/R - 1)(3N_\psi/R - 19)} \dfrac{P\epsilon}{R} \end{cases} \tag{IV.A.5}$$

For the Banks-Zaks times Banks-Zaks-type fixed point FP_6 we find

$$\mathbf{FP_6}: \quad \begin{cases} \alpha_1 = -\dfrac{4}{3}\left(\dfrac{25 - 2N_\psi P/R}{625 - 4N_\psi^2}\right) R\epsilon \\ \alpha_2 = -\dfrac{4}{3}\left(\dfrac{25 - 2N_\psi R/P}{625 - 4N_\psi^2}\right) \dfrac{P\epsilon}{R} \end{cases} \tag{IV.A.6}$$

For the interacting fixed points FP_7 and FP_8 we find

$$\mathbf{FP_7}: \quad \begin{cases} \alpha_1 = \dfrac{2}{3}\left(\dfrac{(13 - 2N_\psi R)(25 - 2N_\psi P/R)}{150 N_\psi^2 R^2 - (4N_\psi^2 + 1025)N_\psi R + 26N_\psi^2 + 475}\right) R\epsilon \\ \alpha_2 = -\dfrac{4}{3}\left(\dfrac{(13 - 2N_\psi R)N_\psi R/P + (2N_\psi R - 1)(3N_\psi R - 19)}{150 N_\psi^2 R^2 - (4N_\psi^2 + 1025)N_\psi R + 26N_\psi^2 + 475}\right) \dfrac{P\epsilon}{R} \\ \alpha_Y = \dfrac{4(25 - 2N_\psi P/R)}{150 N_\psi^2 R^2 - (4N_\psi^2 + 1025)N_\psi R + 26N_\psi^2 + 475} R\epsilon \end{cases} \tag{IV.A.7}$$

$$\mathbf{FP_8}: \quad \begin{cases} \alpha_1 = -\dfrac{4}{3}\left(\dfrac{(13-2N_\psi/R)N_\psi P/R + (2N_\psi/R-1)(3N_\psi/R-19)}{150N_\psi^2/R^2 - (4N_\psi^2+1025)N_\psi/R + 26N_\psi^2+475}\right) R\epsilon \\ \alpha_2 = \dfrac{2}{3}\left(\dfrac{(13-2N_\psi/R)(25-2N_\psi R/P)}{150N_\psi^2/R^2 - (4N_\psi^2+1025)N_\psi/R + 26N_\psi^2+475}\right) \dfrac{P\epsilon}{R} \\ \alpha_y = \dfrac{4(25-2N_\psi R/P)}{150N_\psi^2/R^2 - (4N_\psi^2+1025)N_\psi/R + 26N_\psi^2+475}\,\dfrac{P\epsilon}{R} \end{cases} \tag{IV.A.8}$$

Finally, at the fully interacting fixed point FP_9 we have

$$\mathbf{FP_9}: \begin{cases} \alpha_1 = \dfrac{2}{3}\dfrac{(13-2N_\psi R)\left[(13-2N_\psi R)N_\psi P/R + (2N_\psi/R-1)(3N_\psi/R-19)\right] R\epsilon}{114N_\psi^2(R^2+1/R^2) + (32N_\psi^4+1512N_\psi^2+361) - (220N_\psi^2+779)(R+1/R)} \\ \alpha_2 = \dfrac{2}{3}\dfrac{(13-2N_\psi/R)\left[(13-2N_\psi/R)N_\psi R/P + (2N_\psi R-1)(3N_\psi R-19)\right] P\epsilon/R}{114N_\psi^2(R^2+1/R^2) + (32N_\psi^4+1512N_\psi^2+361) - (220N_\psi^2+779)(R+1/R)} \\ \alpha_Y = \dfrac{4\left[(13-2N_\psi R)N_\psi P/R + (2N_\psi/R-1)(3N_\psi/R-19)\right] R\epsilon}{114N_\psi^2(R^2+1/R^2) + (32N_\psi^4+1512N_\psi^2+361) - (220N_\psi^2+779)(R+1/R)} \\ \alpha_y = \dfrac{4\left[(13-2N_\psi/R)N_\psi R/P + (2N_\psi R-1)(3N_\psi R-19)\right] P\epsilon/R}{114N_\psi^2(R^2+1/R^2) + (32N_\psi^4+1512N_\psi^2+361) - (220N_\psi^2+779)(R+1/R)}\,. \end{cases} \tag{IV.A.9}$$

All expressions reduce to those given in the main body in the limit $N_\psi = 1$. We note that the parameter range in which fixed points exist changes both qualitatively and quantitatively when varying N_ψ within the range (IV.**A**.1). Moreover, we also observe that the characteristic boundaries in parameter space depend on N_ψ, indicating that domains of existence and eigenvalue spectra depend on N_ψ. It is straightforward, if tedious, to investigate regions of validity and scaling exponents for the general case, and to find the analogues of Figs. **6**, **7**, **8**, **9** and **10** and of Tabs. **11** **12**, **13** for general N_ψ.

Appendix B: Boundaries

We find that the existence and relevancy of fixed points in the parameter space (P, R), see (IV.61), is controlled by characteristic curves $P = X(R), Y(R), \widetilde{X}(R)$ or $\widetilde{Y}(R)$ with the functions

$$\begin{aligned} X(R) &= \frac{(2R-13)R}{(2R-1)(3R-19)}\,, \\ Y(R) &= \frac{25}{2}R\,, \\ \widetilde{X}(R) &= \frac{(2/R-1)(3/R-19)}{(2/R-13)/R}\,, \\ \widetilde{Y}(R) &= \frac{2}{25}R\,. \end{aligned} \tag{IV.B.1}$$

These appear as boundaries of the "phase space" of parameters (R, P) characterising valid fixed points. Note that the functions $(X, \widetilde{X})$ and $(Y, \widetilde{Y})$ in (IV.**B**.1) are "dual" to each other,

$$X(R)\cdot\widetilde{X}(R^{-1}) = 1 = Y(R)\cdot\widetilde{Y}(R^{-1})\,. \qquad \text{(IV.}\mathbf{B}\text{.2)}$$

A further set of boundaries is given by the straight lines $R = R_{\rm low}$ or $R_{\rm high}$, with

$$\begin{aligned} R_{\rm low} &= \frac{1}{2} \\ R_{\rm high} &= 2\,. \end{aligned} \qquad \text{(IV.}\mathbf{B}\text{.3)}$$

The boundaries $P = X(R), Y(R), \widetilde{X}(R)$ or $\widetilde{Y}(R)$ with (IV.**B**.1) together with (IV.**B**.3) delimit the qualitatively different quantum field theories in the “phase space” shown in Fig. **11**.

Certain characteristic values for the parameter R arise in its domain of validity $\frac{2}{11} < R < \frac{11}{2}$ at points where the boundaries (IV.**B**.1) cross. We find four of these $R_{1,\cdots,4}$ with

$$\frac{2}{11} < R_{\rm low} < R_1 < R_2 < 1 < R_3 < R_4 < R_{\rm high} < \frac{11}{2}\,, \qquad \text{(IV.}\mathbf{B}\text{.4)}$$

with R_1 and R_2 arising from

$$\begin{aligned} X(R_1) &= Y(R_1)\,, \\ X(R_2) &= \widetilde{X}(R_2) \end{aligned} \qquad \text{(IV.}\mathbf{B}\text{.5)}$$

together with $R_3 = 1/R_2$ and $R_4 = 1/R_1$. Quantitatively we have (IV.62) for $R_{1,\cdots,4}$ as stated in the main text. The expressions (IV.**B**.1), (IV.**B**.3) for the boundaries are modified once $N_\psi \neq 1$.

Part V

Asymptotic safety guaranteed in supersymmetry

Andrew D. Bond[1] and Daniel F. Litim[1]

[1]Department of Physics and Astronomy, U Sussex, Brighton, BN1 9QH, U.K.

We explain how asymptotic safety arises in four-dimensional supersymmetric gauge theories. We provide asymptotically safe supersymmetric gauge theories together with their superconformal fixed points, R-charges, phase diagrams, and UV-IR connecting trajectories. Strict perturbative control is achieved in a Veneziano limit. Consistency with unitarity and the a-theorem is established. We find that supersymmetry enhances the predictivity of asymptotically safe theories.

Introduction.— The discovery of asymptotic freedom for non-abelian gauge theories in 1973 has initiated a new era in particle physics [16, 17]. Asymptotic freedom explains why certain types of quantum field theories such as the strong and weak sector of the Standard Model, can be truly fundamental and predictive up to highest energies. It implies that interactions are switched off asymptotically, and theories become free. Asymptotic freedom constitutes a cornerstone in the Standard Model of particle physics, and continues to play an important role in the search for models beyond.

The discovery of exact asymptotic safety for non-abelian gauge theories with matter [55, 144, 145] has raised substantial interest. Asymptotic safety explains how theories can be fundamental, predictive, and *interacting* at highest energies [12]. Initially put forward as a scenario to quantize gravity [15, 38, 40, 44], asymptotic safety also arises in many other theories [29, 32, 148, 169]. In particle physics, asymptotic safety offers intriguing new directions to ultraviolet (UV) complete the Standard Model beyond the confines of asymptotic freedom [56, 111, 170].

In this Letter, we investigate whether asymptotic safety can be achieved in supersymmetric gauge theories. In the language of the renormalisation group, asymptotic safety corresponds to an interacting UV fixed point for the running couplings [12]. Supersymmetry modifies fixed points and the evolution of couplings because it links bosonic with fermionic degrees of freedom [96, 97, 144]. Additional constraints arise as bounds on the superconformal R-charges [109] from both unitarity [171] and the a-theorem [104, 105, 107, 172]. Hence, our task consists of finding supersymmetric gauge theories without asymptotic freedom, but with viable interacting UV fixed points, and in accord with all constraints.

One arena in which we may hope to find reliable answers is that of perturbation theory. For sufficiently small couplings [116], the loop expansion and weakly interacting fixed points are trustworthy [144]. In this spirit, we obtain fixed points, phase diagrams, superconformal R-charges, and UV-IR connecting trajectories for supersymmetric gauge theories in a controlled setting. Previously, this philosophy has been used successfully for proofs of asymptotic safety in non-supersymmetric simple [55] and semi-simple [145] gauge theories.

The model.— We consider a family of massless supersymmetric Yang-Mills theories in four space-time dimensions with product gauge group $SU(N_1)\otimes SU(N_2)$, coupled to chiral superfields (ψ, χ, Ψ, Q) with flavour multiplicities $(N_F, N_F, 1, N_Q)$. The main novelty is the use of a semi-simple gauge group as otherwise asymptotic safety cannot arise at weak coupling [97, 144]. For each superfield we introduce a left- and right-handed copy with gauge charges as in Tab. **15** to ensure the absence of gauge anomalies. Also, viable models with asymptotic safety must have Yukawa couplings [144]. Therefore, we allow for superpotentials of the form

$$W = y\,\mathrm{Tr}\big[\psi_L\,\Psi_L\,\chi_L + \psi_R\,\Psi_R\,\chi_R\big]\,, \tag{V.1}$$

where the trace sums over flavour and gauge indices. The superfields Q are not furnished with Yukawa interactions. The theory has a global $SU(N_F)_L \otimes SU(N_F)_R \otimes SU(N_Q)_L \otimes SU(N_Q)_R$ flavour and a $U(1)_R$ symmetry. Moreover, the theory is renormalisable in perturbation theory and characterised by two gauge couplings g_1 and g_2 and the Yukawa coupling y, which we write as

$$\alpha_1 = \frac{N_1\, g_1^2}{(4\pi)^2}\,, \quad \alpha_2 = \frac{N_2\, g_2^2}{(4\pi)^2}\,, \quad \alpha_y = \frac{N_1\, y^2}{(4\pi)^2}\,. \tag{V.2}$$

Sending field multiplicities (N_1, N_2, N_F, N_Q) to infinity while keeping their ratios fixed reduces the number of free parameters down to three, which we choose to be

$$\begin{aligned} R &= \frac{N_2}{N_1}\,, \quad P = \frac{N_1}{N_2}\frac{N_Q + N_1 + N_F - 3N_2}{N_F + N_2 - 3N_1}\,, \\ \epsilon &= \frac{N_F + N_2 - 3N_1}{N_1}\,. \end{aligned} \tag{V.3}$$

In the large-N limit [116] the model parameters (R, P, ϵ) are continuous. We can always arrange to find (V.3) with

$$1 < R < 3\,, \quad P = \text{finite}\,, \quad 0 < |\epsilon| \ll 1\,. \tag{V.4}$$

The smallness of ϵ ensures perturbative control in both gauge sectors [144, 145], which is the regime of interest for the rest of this work (the general case is discussed elsewhere [173]). This completes the definition of our models.

Superconformal fixed points.— The running of couplings is controlled by the beta functions $\beta_i = d\alpha_i/d\ln\mu$, with μ denoting the RG momentum scale. To find accurate fixed points, we must minimally retain terms up to two loop in the gauge and one loop in the Yukawa beta functions [144]. Using the results of [74, 174] and suppressing subleading terms in ϵ, we find

$$\begin{aligned} \beta_1 &= 2\alpha_1^2\big[\,\epsilon + 6\alpha_1 + 2R\alpha_2 - 4R(3-R)\alpha_y\big]\,, \\ \beta_2 &= 2\alpha_2^2\Big[P\epsilon + 6\alpha_2 + \frac{2}{R}\alpha_1 - \frac{4}{R}(3-R)\alpha_y\Big]\,, \\ \beta_y &= 4\alpha_y\big[2\alpha_y - \alpha_1 - \alpha_2\big]\,. \end{aligned} \tag{V.5}$$

Chiral superfields	ψ_L	ψ_R	Ψ_L	Ψ_R	χ_L	χ_R	Q_L	Q_R
$SU(N_1)$	$\overline{\Box}$	$\Box$	$\Box$	$\overline{\Box}$	1	1	1	1
$SU(N_2)$	1	1	$\Box$	$\overline{\Box}$	$\overline{\Box}$	$\Box$	$\overline{\Box}$	$\Box$

Table 15. Chiral superfields and their gauge charges.

FP	**G**	**$\mathbf{BZ_1}$**	**$\mathbf{BZ_2}$**	**$\mathbf{GY_1}$**	**$\mathbf{GY_2}$**	**$\mathbf{BZ_{12}}$**	**$\mathbf{GY_{12}}$**
$\boldsymbol{\alpha_1^*}$	0	$-\frac{\epsilon}{6}$	0	$\frac{-\epsilon}{2(3-3R+R^2)}$	0	$\frac{PR-3}{16}\epsilon$	$\frac{3-4R-2PR^2+PR^3}{(R-1)(9-8R+3R^2)}\frac{\epsilon}{2}$
$\boldsymbol{\alpha_2^*}$	0	0	$-\frac{P\epsilon}{6}$	0	$\frac{-PR}{4R-3}\frac{\epsilon}{2}$	$\frac{1-3PR}{16R}\epsilon$	$\frac{R-2-3PR+3PR^2-PR^3}{(R-1)(9-8R+3R^2)}\frac{\epsilon}{2}$
$\boldsymbol{\alpha_y^*}$	0	0	0	$\frac{1}{2}\alpha_1^*$	$\frac{1}{2}\alpha_2^*$	0	$\frac{1}{2}(\alpha_1^* + \alpha_2^*)$

Table 16. The Gaussian (G) and all Banks-Zaks (BZ) and gauge-Yukawa (GY) fixed points to leading order in ϵ.

Anomalous dimensions of the superfields are given by

$$\begin{aligned}
\gamma_\Psi &= (3-R)\alpha_y - \alpha_1 - \alpha_2\,,\\
\gamma_\psi &= R\,\alpha_y - \alpha_1\,,\\
\gamma_\chi &= \alpha_y - \alpha_2\,,\\
\gamma_Q &= -\alpha_2\,,
\end{aligned} \tag{V.6}$$

up to corrections of order $\mathcal{O}(\epsilon\,\alpha, \alpha^2)$. The simultaneous vanishing of (V.5) implies fixed points and scale invariance. Besides the free Gaussian (G), the model has weakly coupled fixed points α^* of order ϵ. These are either of the Banks-Zaks (BZ) or gauge-Yukawa (GY) type, depending on whether the Yukawa coupling is free or interacting [144]. We find partially interacting Banks-Zaks ($\mathrm{BZ}_1, \mathrm{BZ}_2$) and gauge-Yukawa ($\mathrm{GY}_1, \mathrm{GY}_2$) fixed points, and fully interacting ones ($\mathrm{BZ}_{12}, \mathrm{GY}_{12}$), all summarised in Tab. **16**. Results are exact to the leading order in ϵ, with higher loop orders only correcting subleading terms. We also note that (V.5), (V.6), and fixed points, are universal and RG scheme independent at weak coupling [55, 144].

At superconformal fixed points, our models display a global and anomaly-free $U(1)_R$ symmetry. In terms of the superfield anomalous dimensions (V.6), the R-charges (not to be confused with the parameter R) read

$$R_i = 2\,(1+\gamma_i^*)\,/3\,. \tag{V.7}$$

Non-perturbative expressions for the R-charges are found using the method of a-maximisation [109]. For small couplings, findings agree with (V.6), (V.7) and deviate mildly from Gaussian values, in accord with unitarity [171].

Asymptotic freedom of (V.5) is guaranteed for $P > 0 > \epsilon$. Then, all three couplings (V.2) are marginally relevant at the Gaussian UV fixed point. The set of asymptotically free trajectories is characterised by three free parameters, the initial values $0 < \delta\alpha_i(\Lambda) \ll 1$ at the high scale Λ. Some or all interacting fixed points of Tab. **16** arise within specific parameter ranges (V.3) and take the role of IR fixed points. Trajectories run either towards a regime with strong coupling and confinement, or terminate at a superconformal IR fixed point. By and large, this is very similar to the generic behaviour of asymptotically free non-supersymmetric gauge theories [145].

Asymptotic safety.— Next, we turn to regimes (V.3) where asymptotic freedom is lost, starting with

$$P < 0 < \epsilon\,. \tag{V.8}$$

Clearly, the Gaussian has ceased to be the UV fixed point for the full theory and one might wonder whether its role is taken over by one of the interacting fixed points in Tab. **16**. Available candidates in the regime (V.8) are BZ_2, GY_2, and GY_{12}. At the partially interacting BZ_2, only the Yukawa term (V.1) is a relevant perturbation. The theory becomes interacting in α_2 and α_y, yet α_1 remains switched off at all scales. From the eigenvalue spectrum we learn that GY_{12}, once it exists, is IR attractive in all couplings. Hence, neither the Gaussian, nor BZ_2, nor GY_{12} qualify as UV fixed points. A new effect occurs at GY_2. While α_2 and α_y are irrelevant in its vicinity [144], the relevancy of α_1 now depends on the magnitude of α_2^* and α_y^* at GY_2. We find

$$\begin{aligned}\beta_1\big|_{\mathrm{GY}_2} &= -B_{1,\mathrm{eff}}\,\alpha_1^2 + \mathcal{O}(\alpha_1^3)\,,\\ B_{1,\mathrm{eff}} &= -2\epsilon + 2\epsilon\,P/Q_1\,,\end{aligned} \tag{V.9}$$

with $Q_1(R) = (4R-3)/(R^3-2R^2)$. The first term in $B_{1,\mathrm{eff}}$ is the conventional one loop coefficient. It is negative in the regime (V.8) and documents the irrelevancy of α_1 at the Gaussian. The second term is sourced through the fixed point GY_2. Most notably, the

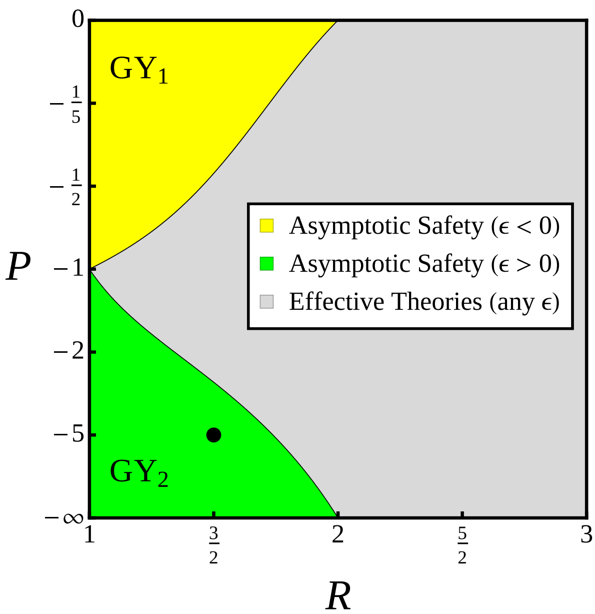

Figure 22. Phase space for asymptotic safety, showing the parameter regions (V.10) and (V.11). Models in the gray-shaded area are UV incomplete. P-axis is scaled as $P/(1-P)$ for better display. The full dot indicates the example in Figs. **23**, **24**.

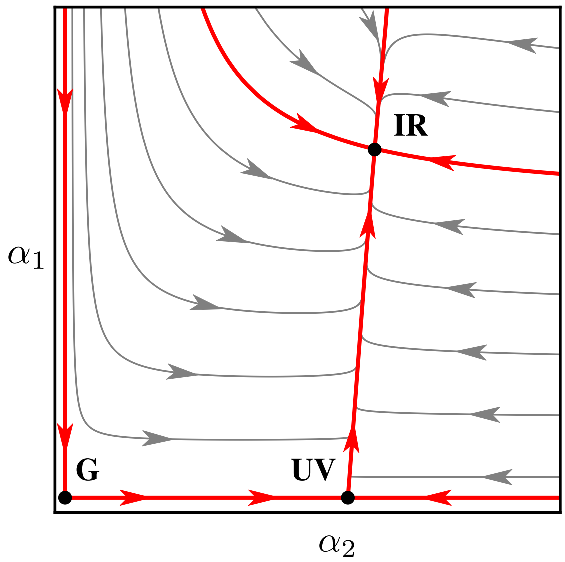

Figure 23. Phase diagram with asymptotic safety for supersymmetry ($P = -5, R = \frac{3}{2}, \epsilon = \frac{1}{1000}$; Fig. **22**) projected onto $\alpha_y = \frac{\alpha_1+\alpha_2}{2}$. Trajectories are pointing towards the IR. Notice that α_1 is destabilised and asymptotic freedom is absent. Dots show the Gaussian, the UV and the IR fixed points. Also shown are separatrices (red) and sample trajectories (gray).

sign of $B_{1,\text{eff}}$ is positive provided that

$$P < Q_1 < 0\,, \quad 1 < R < 2\,, \quad \epsilon > 0\,, \tag{V.10}$$

thereby turning α_1 into a relevant coupling. We emphasize that the Yukawa term (V.1) is crucial to achieve $B_{1,\text{eff}} > 0$; without it, the required change of sign would be impossible [144]. In other words, while α_1 is IR free close to the Gaussian or BZ_2 fixed points, it has become UV free close to the GY_2 fixed point. It is precisely for this reason that the gauge-Yukawa fixed point GY_2 takes the role of an asymptotically safe UV fixed point with one marginally relevant and two irrelevant directions.

The same mechanism is operative once $P, \epsilon < 0$, where α_1 and α_2 have interchanged their roles. Near GY_1, the effective one-loop coefficient for α_2 reads $B_{2,\text{eff}} = 2(Q_2 - P)\,\epsilon$, with $Q_2 = (R-2)/(R^3 - 3R^2 + 3R)$. Consequently, α_2 becomes a relevant coupling for

$$Q_2 < P < 0\,, \quad 1 < R < 2\,, \quad \epsilon < 0\,, \tag{V.11}$$

thereby promoting GY_1 to an UV fixed point. As soon as both gauge sectors are destabilised ($P, \epsilon > 0$), no fixed point other than the IR attractive Gaussian can arise. Theories are UV incomplete and must be viewed as effective. Fig.**22** summarises our results once $P < 0$, also indicating the parameter regions (V.10) and (V.11) with exact asymptotic safety.

From the UV to the IR.— At either of the superconformal UV fixed points, the elementary “quarks” and “gluons” are unconfined and appear as interacting (free) massless particles in one (the other) gauge sector. The free gauge sector acts as a marginally relevant perturbation which drives the theory away from the UV fixed point. The corresponding phase diagram in the regime (V.10) is shown in Fig.**23**. It confirms that GY_2, unlike the Gaussian, is the unique UV fixed point. Close to the UV fixed point, the critical surface of asymptotically safe trajectories running out of it is given by

$$\begin{aligned} \alpha_1(\mu) &= \frac{\delta\alpha_1(\Lambda)}{1 + B_{1,\mathrm{eff}}\,\delta\alpha_1(\Lambda)\,\ln(\mu/\Lambda)}\,, \\ \alpha_2(\mu) &= \alpha_2^* + \frac{2-R}{4R-3}\,\alpha_1(\mu)\,, \\ \alpha_y(\mu) &= \alpha_y^* + \frac{3R-1}{8R-6}\,\alpha_1(\mu)\,. \end{aligned} \tag{V.12}$$

We emphasize that the theory has only one free parameter $\delta\alpha_1(\Lambda) \ll 1$ related to the relevant gauge coupling at the high scale Λ. Both α_2 and α_y have become irrelevant couplings and are strictly determined by α_1. (Similar expressions are found for the regime (V.11).) Dimensional transmutation leads to the RG invariant mass scale

$$\mu_{\mathrm{tr}} = \Lambda \exp\big[-B_{1,\mathrm{eff}}\,\delta\alpha_1(\Lambda)\big]^{-1}\,, \tag{V.13}$$

which is independent of the high scale. It characterises the scale where couplings stop being controlled by the UV fixed point. For RG scales $\mu \ll \mu_{\mathrm{tr}}$, we observe a cross-over into another superconformal fixed point (GY_{12}) governing the IR. There, the elementary quarks and gluons of either gauge sector remain unconfined and appear as interacting massless particles, different from those observed in the UV. Fig. **24** exemplifies the running of couplings from the UV to the IR.

The UV fixed point persists in the presence of mass terms for the chiral superfields. Once masses are switched on, with or without soft supersymmetry-breaking ones such as those for the "gluinos", they lead to decoupling [175] and low-energy modifications of the RG flow (V.5). Then, UV safe trajectories may terminate in regimes with strong coupling and confinement in the IR, with or without softly broken supersymmetry.

Asymptotic safety and the a-theorem.— We are now in a position to establish consistency with a more formal aspect of the renormalisation group known as the a-theorem [104, 105, 107, 172]. It states that the central charge $a = \frac{3}{32}\left[2d_G + \sum_i (1-R_i)(1-3(1-R_i)^2)\right]$ [107], must be a decreasing function along RG trajectories in any $4d$ quantum field theory (d_G denotes the dimension of the gauge groups and i runs over all chiral superfields). Using (V.6), (V.7), and Tab. **16**, we find

$$\Delta a \equiv a_{\mathrm{UV}} - a_{\mathrm{IR}} > 0 \tag{V.14}$$

on any of the UV-IR connecting trajectories in the parameter ranges (V.10), (V.11) shown in Fig. **22**. Had the IR limit been the Gaussian, validity of the a-theorem implies strong coupling and large R-charges in the UV, at least for some of the fields [97, 107]. In our

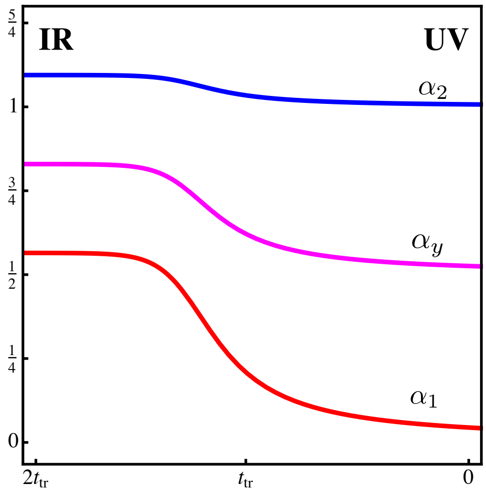

Figure 24. The running couplings $\alpha_i(t)$ in units of RG time $t = \ln(\mu/\Lambda)$ along the separatrix from the UV to the IR fixed point. Parameters as in Fig. **23**. All couplings in units of $\alpha^*_{2,\rm UV}$ with $t_{\rm tr} = \ln(\mu_{\rm tr}/\Lambda)$ and Λ the high scale, see (V.13).

models, this implication is circumvented because the IR is not free. In fact, there is not a single trajectory flowing from the UV fixed point to the Gaussian, Fig. **23**, which again is in accord with the a-theorem ($a_{\rm UV} - a_{\rm G} < 0$).

Discussion.— In supersymmetry, and for superpotentials of the form (V.1) including mass terms, the scalar potential is always a sum of squares of absolute values [176]. Hence, the stability of the quantum vacuum is automatic. Also, a fixed point for the gauge and Yukawa couplings implies a fixed point for the scalar potential. Without supersymmetry, physicality of scalar fixed points and vacuum stability do not come by default [144] and must be checked case by case [58, 145].

Also, without supersymmetry, at least one Yukawa coupling is required to help generate an interacting UV fixed point [144]. Invariably, this reduces the number of fundamentally free parameters in the UV by at least one, thereby enhancing the predictive power [55]. In supersymmetry, asymptotic safety at weak coupling cannot arise with only a single gauge factor [97, 144]. Then, as we have seen in (V.12), at least one of the Yukawa couplings together with at least one of multiple gauge couplings must be non-trivial in the UV, thereby reducing the number of free parameters by two. We conclude that supersymmetry additionally enhances the predictive power of asymptotic safety.

We have shown that asymptotic safety is operative in supersymmetric gauge theories. Yukawa couplings continue to play a distinctive role at weak coupling, as they do for asymptotic safety without supersymmetry [144]. Explicit examples with superpotential (V.1) and matter content as in Tab. **15** are provided, including the phase space (Fig. **22**) and phase diagram (Fig. **23**). Results are consistent with unitarity and the a-theorem. Our construction makes it clear that asymptotic safety exists in supersymmetry beyond the models discussed here. It is interesting to include more gauge groups, expand Yukawa sectors, switch on mass terms, and explore the potential for asymptotically safe supersymmetric model building.

Part VI

Conclusions

In the papers comprising this work we have explored several features of weakly coupled fixed points in four-dimensional local quantum field theories containing field content consisting of scalars, vector bosons, and spin-$\frac{1}{2}$ fermions. We established the mechanism by which such fixed points may arise — crucially containing gauge and Yukawa couplings, thus necessitating particles of each spin. This can be realised either by having fully interacting fixed points, or fixed points interacting in a single gauge sector, for which asymptotic freedom in another sector is generated via the fixed point. Additionally we have constructed new examples of theories which exhibit exactly controllable perturbative asymptotic safety in settings where asymptotic freedom for the full theory is no longer possible. In particular this extends the class of known theories with this property to include semisimple theories, as well as theories which are supersymmetric, which have the advantage of automatically satisfying constraints from the scalar sector, which provide non-trivial barriers to non-supersymmetric theories.

Firstly focusing on gauge theories in Part II , we found that the generation of perturbative fixed points is driven by the gauge coupling. Fixed points are found to be one of three primary types, or to be product-like copies of these for each gauge sector in the semisimple case. As well as the non-interacting Gaussian fixed point, the other possibilities are of Banks-Zaks type, being interacting in the gauge but not the Yukawa couplings, or of gauge-Yukawa type, where additionally some Yukawa couplings take interacting values. It was established that fixed points of the Banks-Zaks type cannot arise in a setting where asymptotic freedom is lost, and therefore serve as the ultimate ultraviolet limit of theories. In contrast, gauge-Yukawa fixed points can be ultraviolet fixed points for the theory, provided certain algebraic conditions are satisfied. Scalar quartics do not affect the gauge-Yukawa sector at leading order, but provide non-trivial constraints on whether fixed points correspond to physically sensible theories.

We then examined in Part III generic theories of only fermions and scalars, where gauge fields are no longer present. We firstly generalised the arguments of Coleman and Gross [124], that such theories may not be asymptotically free, to allow for the case of Weyl, and not only Dirac, fermions, which required the use of a stricter inequality on certain combinations of Yukawa matrices. We then demonstrated how the very same inequalities that disallow asymptotic freedom of such theories in fact also preclude the possibility of having a weakly coupled fixed point generated only by Yukawa and quartic couplings. Combined with the results for gauge theories, this fully categorises the possible types of fixed points for all sets of couplings in general four-dimensional perturbatively renormalisable local quantum field theories. In particular, this establishes that perturbative asymptotic safety necessitates gauge fields and Yukawa couplings, and therefore requires, scalars, fermions and gauge bosons.

In Part IV we systematically examined a specific class of semisimple gauge-Yukawa models. By studying it in a generalised Veneziano limit we retained full perturbative control over the fixed points. This model was in some sense minimally semisimple, obtained by squaring a simple gauge-Yukawa theory which has an ultraviolet fixed point for certain parameter values, but adding a single messenger fermion charged under both of the simple gauge factors of the theory. By comparing with the exact product theory which does not have this additional fermion, we could see explicitly where the semisimple nature of the theory directly affected the physical existence and relevance of the various possible fixed points. We analysed the full parameter space of the theory, finding many distinct regions with different phase structures. These include asymptotically free regions with up to eight interacting infrared or saddle-type fixed points, and regions where no ultraviolet completion is possible. Additionally, there are regions where asymptotic freedom is not retained overall at the Gaussian fixed point, but the theory can attain asymptotic safety by developing an interacting gauge-Yukawa fixed point in one of the gauge sectors.

Finally in Part V we constructed a semisimple theory which additionally has $\mathcal{N} = 1$ supersymmetry. Crucially, in such a model for suitable choices of parameter, despite the fact that the Gaussian fixed point is not fully ultraviolet, the theory can have an ultraviolet completion through an interacting gauge-Yukawa fixed point. In such a setting the only fully semisimple trajectory with a perturbative ultraviolet limit is the one leading to the interacting ultraviolet fixed point. Here one of the gauge couplings, despite not being asymptotically free near the non-interacting theory, becomes effectively asymptotically free only in the vicinity of an interacting theory in the other gauge sector. This established that perturbative asymptotic safety is in fact compatible with supersymmetry.

Of course, there are still many exciting avenues for future work to explore. So far, the known landscape of theories which permit perturbative asymptotic safety is relatively small, and it would be interesting to understand this further by continuing to attempt to discover new examples. If many further such theories are found, it would be interesting to understand to what degree they are similar to currently known examples, and which features they share. On the other hand, if additional examples prove to be relatively few in number, it should also help focus understanding precisely what it is about the structure of existing examples that permit ultraviolet fixed points to arise.

Additionally, pushing the understanding of current asymptotically safe theories would be of tremendous benefit. In particular, while perturbation theory is a very powerful tool, it would be interesting to gain insight into what happens when theories move away from weakly coupled regimes. Tools such as the functional renormalisation group, and lattice field theory, may allow separate investigation into these theories, which allow understanding of nonperturbative regimes. The existence of a supersymmetric example furthermore opens the door to investigation using techniques enabled by the higher degree of symmetry for such theories.

Lastly, continuing to look at more advanced and realistic models is of great interest, both directly for standard model extensions, and for theories additionally including gravity. This is of course deeply inter-related to other directions. Understanding the landscape of asymptotically safe theories, and exactly which features allow the generation of ultraviolet fixed points, will open up new possibilities for constructing further testable models which are compatible with experimental constraints. Similarly understanding non-perturbative aspects of theories will allow greater understanding of more realistic models which may not be under strict perturbative control, as well as proving useful for theories, such as those coupled to gravity, where strong coupling dynamics takes hold.

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