Models of last passage percolation

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Summary

The thesis provides the discussion of three last passage percolation models. In particular, we focus on three aspects of probability theory: the law of large numbers, the order of the variance and large deviation estimates.

In Chapter 1, we give a brief introduction to the percolation models in general and we present some important results for this topic which are heavily used in the following proofs.

In Chapter 2, we prove a strong law of large numbers for directed last passage times in an independent but inhomogeneous exponential environment. Rates for the exponential random variables are obtained from a discretisation of a speed function that may be discontinuous on a locally finite set of discontinuity curves. The limiting shape is cast as a variational formula that maximises a certain functional over a set of weakly increasing curves.

Using this result, we present two examples that allow for partial analytical tractability and show that the shape function may not be strictly concave, and it may exhibit points of non-differentiability, flat segments, and non-uniqueness of the optimisers of the variational formula. Finally, in a specific example, we analyse further the macroscopic optimisers and uncover a phase transition for their behaviour.

In Chapter 3, we discuss the order of the variance on a lattice analogue of the Hammersley process with boundaries, for which the environment on each site has independent, Bernoulli distributed values. The last passage time is the maximum number of Bernoulli points that can be collected on a piecewise linear path, where each segment has strictly positive but finite slope.

We show that along characteristic directions the order of the variance of the last passage time is of order $N^{2/3}$ in the model with boundary. These characteristic directions are restricted in a cone starting at the origin, and along any direction outside the cone, the order of the variance changes to $O(N)$ in the boundary model and to $O(1)$ for the non-boundary model. This behavior is the result of the two flat edges of the shape function.

In Chapter 4, we prove a large deviation principle and give an expression for the rate function, for the last passage time in a Bernoulli environment. The model is exactly solvable and its invariant version satisfies a Burke-type property. Finally, we compute explicit limiting logarithmic moment generating functions for both the classical and the invariant models. The shape function of this model exhibits a flat edge in certain directions, and we also discuss the rate function and limiting log-moment generating functions in those directions.

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Chapter 1

Introduction

This thesis is concerned with the study of three different last passage percolation models of which we describe their main probabilistic features. Last passage percolation is a particular area of percolation theory which in turn is a branch of probability theory.

The mathematical study of percolation theory has now been going on for some sixty years. Its main sources of inspiration and motivation have been real world phenomena. Most of this work is concentrated on the development of models whose aim is to represent physical phenomena via simple random rules. Percolation theory tries to model microscopic physical aspects by defining a few local rules and one of the objectives is to show how a change in the microscopic environment may have a macroscopic impact. From this point of view there is a natural connection between percolation and statistical mechanics. In fact, there are several statistical mechanical models in which the phase transition can be understood as the percolation transition of a suitable (dependent) percolation model. The best known cases are that of the Ising and Potts models, in which the transition occurs at the percolation transition of the associated FK percolation (random cluster) model (for a survey see [36]).

Percolation models generally allow many natural and intuitive problems to be posed in a natural way, whereas satisfactory solutions to them often turn out far from trivial. This is of great appeal since it requires a creative development of new mathematical techniques in order to gain deeper understanding of the problem.

Next we give a general presentation of the percolation model and some of the models which are derived from it. In particular we will highlight how they are connected and which are the main problems that arose from them.

1.1 Percolation model

In the original formulation two different random mechanisms of percolation were considered: *site percolation* and *bond percolation*.

Bond and site percolation were motivated as models to describe the seemingly random structure of a porous material [18]. They are discrete models, where the discrete structure is provided by a suitably chosen graph. A graph consists of a set of vertices and a set of bonds between pairs of vertices. Each bond, also referred to as an edge, symbolizes a connection between the two vertices. The $\mathbb{Z}^d$ lattice, or the $\mathbb{Z}^d$ nearest neighbour graph, for $d \geq 2$, is the graph whose vertices are given by the points in $\mathbb{Z}^d$, and where two vertices are connected by an edge if they are at Euclidean distance one from each other.

The $\mathbb{Z}^d$ lattice is an infinite graph, and is used as an approximation of a large region. For the bond percolation to obtain a random structure from the $\mathbb{Z}^d$ lattice, we proceed as follows. Go through each edge one by one, flip a coin, and decide to keep the edge if the coin turns up heads and remove the edge if the coin turns up tails. Thus, each edge is removed independently of all other edges. For the site percolation the random environment is obtained removing vertices and incident edges instead of only edges.

The resulting structure can be viewed as a representation of a porous material where each vertex represents a cell in the material, and the edges symbolize neighboring cells having a sufficiently large passage between them (as to allow a fluid to pass, say). With this interpretation of the model, a fluid is able to flow from one cell to another if there is a sequence of edges between neighboring cells, also called a path, that connect the cells. Another way to describe a path between two points u and v of a graph is an alternating sequence of vertices and edges $u = v_0, \ell_1, v_1, \ldots, \ell_n, v_n = v$, starting and ending with a vertex, and such that the vertex v_k is the endpoint of the edges ℓ_k and ℓ_{k+1} preceding and succeeding v_k.

Studying the random structure obtained through coin tossing leads to questions concerning the existence of paths in the random structure. In particular, one may ask if the center of a large piece of porous material will be wet when immersed in the fluid? (This was the original question in Broadbent and Hammersley's work [18]) This corresponds to the question of how far a fluid injected at the center of the material will reach. Since the model is based on an infinite graph, is it possible for a fluid injected at the center (the origin of the graph) to wet infinitely many cells? That the fluid will wet another cell corresponds to the existence of a path from the origin to that cell. Cells that are connected by paths form components of interconnected cells. What can be said about the size of these components?

In fact, the answers to these questions differ depending on the coin being fair or being biased. Consider some fixed dimension $d \geq 2$, and let $p \in [0, 1]$ denote the probability that the coin tossed turns up heads. We avoid the trivial case when $d = 1$ since if $p < 1$, then only finite components remain after edges have been removed in accordance with the result of the coin tosses. If $p = 1$, the graph remains intact.

A coin is considered fair if $p = 1/2$, while if $p \neq 1/2$ the coin is biased. For values of p close to 1, an infinite connected component of cells will exist, whereas for values of p close to 0, all components will be finite. As p ranges from 0 to 1, the system undergoes what is called a *phase transition*, that is, a sudden change in the qualitative behaviour of the model. In the case of bond percolation, the phase transition occurs when the random structure goes from having no infinite connected component of cells when p is close to 0 and to having one for p close to 1. In fact, there is a critical value $p_c(d)$ strictly between 0 and 1 such that for $p < p_c(d)$, there is no infinite connected component, but for $p > p_c(d)$ an infinite connected component does exist. The existence and non-existence of infinite components should be understood to hold with probability 1, or almost surely. When an infinite component exists, there is also positive probability for a fluid injected at the origin to reach infinitely far.

Harry Kesten was the first to give a rigorous proof of the critical value in two dimensions in [68]. This work is considered a masterpiece for its probabilistic and geometrical arguments and he proved that $p_c(2) = 1/2$. To have results for higher dimension we have to wait until the 90s when Hara and Slade [58, 59] found an approximate solution for $p_c(d)$ as a function of the dimension d when $d \geq 19$.

1.2 Growth and related models

The growth models similar to the percolation models are defined on an underlying discrete structure. The typical image associated to growth models is the spread of an infection along the edges of the graphs according to some random rules and each vertex of the graph is eventually infected. In this framework the values assigned to edges could be thought of as times associated with the crossing of edges. Therefore if we define $\mathcal{T}_{\mathbf{v},\mathbf{w}} < \infty$ as the time that the infection takes to go from vertex $\mathbf{v}$ to vertex $\mathbf{w}$, then if we fix a time $t \geq 0$ the set of infected vertices at t will be

$$B_t = \{\mathbf{x} \in \mathcal{V} : \mathcal{T}_{\mathbf{v},\mathbf{x}} \leq t\}, \tag{1.2.1}$$

where $\mathcal{V}$ is the set of all vertices in $\mathbb{Z}^d$.

We focus on two growth models: first and last passage percolation and we define the general setting for them. One of the most general definition for the model set up in literature [48] is the following. Fix the dimension $d \in \mathbb{N}$ and let $p : \mathbb{Z}^d \to [0,1]$ be a random walk probability kernel: $\sum_{z\in\mathbb{Z}^d} p(z) = 1$. Assume p has a finite support $\mathcal{R} = \{z \in \mathbb{Z}^d : p(z) > 0\}$. $\mathcal{R}$ must contain at least one nonzero point, and $\mathcal{R}$ may contain 0. $\mathcal{R}$ generates the additive subgroup $\mathcal{G} = \{\sum_{z\in\mathcal{R}} a_z z : a_z \in \mathbb{Z}\}$. $\mathcal{G}$ is isomorphic to some $\mathbb{Z}^k$. Now we are ready to give some definition:

- A path $\pi_{0,n} = (v_k)_{k=0}^n$ in $\mathbb{Z}^d$ is *admissible* if its steps satisfy $z_k \equiv v_k - v_{k-1} \in \mathcal{R}$. The probability of an admissible path from a fixed initial point π_0 is $p(\pi_{0,n}) = p(z_{1,n}) = \prod_{i=1}^n p(z_i)$.

- An *environment* ω is a sample point from a Polish probability space $(\Omega, \Sigma, \mathbb{P})$ where Σ is the Borel σ-algebra of Ω. Ω comes equipped with a group $\{T_x : x \in \mathcal{G}\}$ of measurable commuting bijections that satisfy $T_{x+y} = T_x T_y$ and T_0 is the identity. $\mathbb{P}$ is $\{T_x\}_{x\in\mathcal{G}}$-invariant probability measure on (Ω, Σ). This is summerized by the statement that $(\Omega, \Sigma, \mathbb{P}, \{T_x\}_{x\in\mathcal{G}})$ is a measurable dynamical system. We assume $\mathbb{P}$ *ergodic*. As usual this means that $\mathbb{P}\{A\} = 0$ or 1 for all events $A \in \Sigma$ that satisfy $T_z^{-1}A = A$ for all $z \in \mathcal{R}$. $\mathbb{E}$ denotes expectation under $\mathbb{P}$ if not differently specified.

- A *potential* is a measurable function $V : \Omega \times \mathcal{R}^\ell \to \mathbb{R}$ for some $\ell \in \mathbb{Z}_+$, denoted by $V(\omega, z_{1,\ell})$ for an environment ω and vector of admissible steps $z_{1,\ell} = (z_1, \dots, z_\ell) \in \mathcal{R}^\ell$. The constant ℓ represents the number of steps before to reach a certain site v that the potential has to take into account. The case $\ell = 0$ corresponds to a potential $V : \Omega \to \mathbb{R}$ that is a function of ω alone. Typically $\ell = 1$.

1.2.1 First-passage percolation

The first-passage percolation (FPP) was originally introduced by Hammersley and Welsh in [57]. Typically in FPP ω is a non-negative random variable and $\mathcal{R} = \{\pm e_1, \pm e_2\}$. The infection spreads across edges according to explicit speeds. This means that for a fixed $k \in \mathbb{N}$ the random variable $\tau_{l_k} = V(T_{v_k}\omega, v_k - v_{k-1})$ is assigned to the edge l_k which links the vertex v_{k-1} with v_k. Therefore a path π is a sequence of edges $l_1, \dots, l_n$ such that each pair l_i and l_{i+1} shares an endpoint and the passage time along a path π is $F(\pi) = \sum_{l\in\pi} \tau_l$. In particular, the first-passage time $\mathcal{T}_{\mathbf{x},\mathbf{y}}$ is the minimal amount of time that the infection, following the so called *minimal path*, takes to go from $\mathbf{x}$ to $\mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$. Formally

$$\mathcal{T}_{\mathbf{x},\mathbf{y}} = \inf\{F(\pi) : \pi \text{ is a lattice path from } \mathbf{x} \text{ to } \mathbf{y}\}.$$

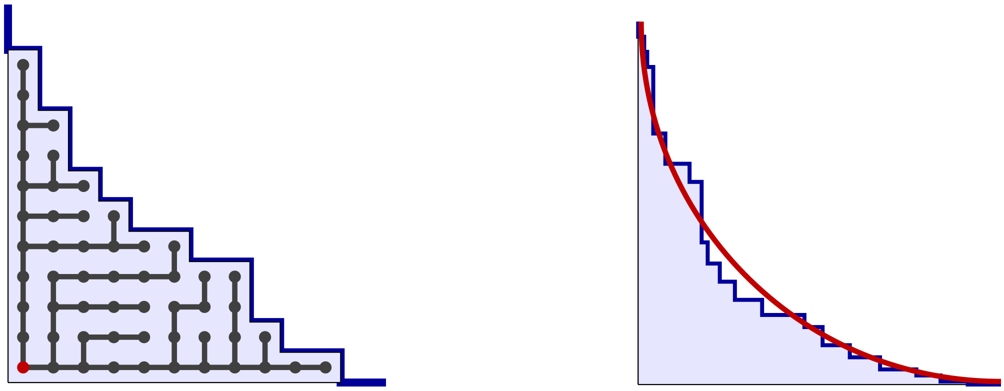


Figure 1.1: Left: A possible evolution of the corner growth model on the first quadrant of the plane. The origin is represented by the red dot. To give the random infected set B_t in (1.2.1) positive volume in $\mathbb{R}^2$, replace it with the fattened set $\tilde{B} = B_t + [-1/2, 1/2]^2$. The blue region is $\tilde{B}$. The bold dark gray edges are the paths of maximal passage time from the origin which are forced to be directed. Right: Large scale (n large) corner growth with exponentially distributed vertex weights with mean 1. The blue region represents the simulation of the scaled growing set $t^{-1}(B_t + [-1/2, 1/2]^2)$. Its boundary (the thick blue line) approximates the red limit curve $\gamma(x, y)$ with $(x, y) \in \mathbb{R}^2$, $\sqrt{x} + \sqrt{y} = 1$, as first proved by Rost in 1981 [94].

If $\mathbb{P}\{\tau_l = 0\} = 0$ then $\mathcal{T}$ is almost surely a metric on $\mathbb{Z}^d$ since it is non-negative and $\mathcal{T}_{\mathbf{x},\mathbf{y}} = 0$ only when $\mathbf{x} = \mathbf{y}$. If the edge-weight is allowed to be zero, $\mathcal{T}$ is a pseudometric. Moreover $\mathcal{T}$ satisfies the triangle inequality $\mathcal{T}_{\mathbf{x},\mathbf{y}} \leq \mathcal{T}_{\mathbf{x},\mathbf{z}} + \mathcal{T}_{\mathbf{z},\mathbf{y}}$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^d$.

In FPP, there is a shape theorem, but the limiting shape depends on the distribution of the $\{\tau_l\}$s. For a general distribution with a positive support very little is known apart from them being convex, compact, and having the symmetries of Z^d. It is expected that for most distributions, the limit shape is strictly convex, and certainly not a polygon, but strict convexity is not proved for any distribution, and there are only some two-dimensional examples of limit shapes that are not polygons. For a recent survey on FPP see [5].

1.2.2 Last passage percolation

The last-passage percolation is a modification of FPP, introduced because of its relationship with the totally asymmetric simple exclusion process (TASEP) particle system. In general the n-step point to point last passage time is defined as

$$G^V_{0,(n),x} = \max_{\pi_{0,n+\ell-1}: v_0=0, v_n=x} \sum_{k=0}^{n-1} V(T_{v_k}\omega, z_{k+1,k+\ell}),$$

where the function $V(T_{v_k}\omega, z_{k+1,k+\ell})$ defines if an admissible path can collect the weight at site v_k according to the ℓ previous steps that it made to reach that site.

Once V is specified or it is not necessary to highlight it, we omit it from the notation. Typically a path $\pi_{0,n}$ is a sequence of vertices $v_0, \ldots, v_n$ such that $||v_{i+1} - v_i||_1 = 1$ for all i, we define the random variable at vertex v_k as $\tau_{v_k} = V(T_{v_k}\omega, v_{k+1} - v_k)$ and one assigns the passage time $L(\pi) = \sum_{k=0}^{n-1} \tau_{v_k}$, as in FPP.

So far the difference between the LPP and the FPP is that the random weights are assigned to the vertices instead of the edges. But there are two main difference between them, in the LPP the passage time between two vertices is the maximal passage time of any path between them. This will generally be infinity unless a restriction to a finite set of paths is added, so that it is possible to consider only oriented paths; that is, paths such that all the coordinates of the v_is are nondecreasing ($v_i \leq v_{i+1}$). Therefore for any $\mathbf{u} \leq \mathbf{w}$, the last passage time $G_{\mathbf{u},\mathbf{w}}$ is the longest amount of time that the infection takes from $\mathbf{u}$ to $\mathbf{w}$ following the *maximal oriented path*, where $\mathbf{u}, \mathbf{w} \in \mathbb{Z}^d$. Formally

$$G_{\mathbf{u},\mathbf{w}} = \sup\{L(\pi) : \pi \text{ is a directed path from } \mathbf{u} \text{ to } \mathbf{w} \text{ with } \mathbf{u} \leq \mathbf{w}\}. \tag{1.2.2}$$

Due to directedness of the model and the fact that we are taking a maximum, G has somewhat different properties than T in FPP. One still has for $\mathbf{u} \leq \mathbf{w}$, $G_{\mathbf{u},\mathbf{w}} \geq 0$ and if $\tau_v > 0$ for all $\mathbf{v}$, then $G_{\mathbf{u},\mathbf{w}} > 0$ when $\mathbf{u} \neq \mathbf{w}$. Excluding the initial point from all our paths, we have a super-additivity property of G that corresponds to the triangle inequality in FPP:

$$G_{\mathbf{u},\mathbf{z}} \geq G_{\mathbf{u},\mathbf{w}} + G_{\mathbf{w},\mathbf{z}} \qquad \text{for } \mathbf{u} \leq \mathbf{w} \leq \mathbf{z}, \text{with } \mathbf{u}, \mathbf{w}, \mathbf{z} \in \mathbb{Z}^d.$$

By this super-additivity, the limiting shape in LPP is not convex, since the corresponding shape function g_{pp} will be super-additive. For its definition, it is necessary to use the sub-additive ergodic theorem, noting that G is super-additive. The only difficulty is to come up with conditions under which the limit is finite.

In two dimensions, it is believed that the boundary of the limit shape is the graph of a strictly concave function. In LPP, however, it is known that the limit shape is not a polygon. For a survey of LPP, see [79, 86, 103].

1.2.3 Totally asymmetric simple exclusion process (TASEP)

The most famous case of LPP is when the distribution F of the site-weights is exponential in two dimensions. In this case, there is a direct mapping from the growth of B_t in (1.2.1) to a particle system called the Totally Asymmetric Simple Exclusion Process (TASEP). TASEP is defined as follows. We imagine that at each site z of $\mathbb{Z}$ with $z \leq 0$, there sits a particle at time 0. Associated to each particle is a Poisson process, and when this process jumps, the particle attempts to move to the site directly to the right. If there is already a particle there, the move is suppressed, and the particle stays in its current location. The particle that is initially at site 0 is not restricted by particles to the right, but the other particles may sometimes be blocked by particles to their right.

TASEP is one of the most studied non-equilibrium particle systems. Its main applications include protein synthesis [75, 108] and traffic modeling [60]

The relation between TASEP and LPP with exponential weights is as follow. The procession of the first particle in TASEP is the same as the infection in LPP along the positive x-axis from 0 [65]. Indeed, the infection appears at 0 after an exponential time, just as the first particle in TASEP moves to the right. It then infects the site $(1, 0)$ after an independent exponential time, just as the same particle in TASEP moves again to the right. Generally, the infection time from $(0, 0)$ to $(n, 0)$ is achieved through the path that proceeds directly down the positive x-axis, and occurs when the first particle in TASEP reaches site $n + 1$. At the second level, the infection of site $(0, 1)$ occurs an independent exponential time after the infection appears at $(0, 0)$. This corresponds to the second particle in TASEP moving into the space left open after the first particle moves. Generally, the n-th step of the k-th particle in TASEP corresponds to the site $(n - 1, k)$ being infected from $(0, 0)$. To see this, we can derive the following relation in LPP: according to the previous notation we identify $\mathbf{u} = (0, 0)$ and $\mathbf{w} = (x_1, x_2)$ with $x_1, x_2 > 0$, one has

$$G_{(0,0),(x_1,x_2)} = \tau_{x_1,x_2} + \max\{G_{(0,0),(x_1-1,x_2)}, G_{(0,0),(x_1,x_2-1)}\}.$$

This is because the infection from $(0, 0)$ reaches (x_1, x_2) through either $(x_1 - 1, x_2)$ or $(x_1, x_2 - 1)$ (whichever is infected last), and after the one of these sites with maximal passage time from $(0, 0)$ is infected, (x_1, x_2) must wait τ_{x_1,x_2} additional time. Similarly, in TASEP, for the k-th particle to make its n-th step, it must wait an independent exponential time after both of the following events occur:

(a) the $k - 1$-st particle makes its n-th step and

(b) the k-th particle makes its $n - 1$-st step.

Another way to visualize this coupling is rotating the corner growth model by $\pi/4$ anticlockwise and the resulting shape is the so called *wedge*. Particles occupy sites of $\mathbb{Z}$, subject to the exclusion rule that does not allow for two particles to occupy the same site.

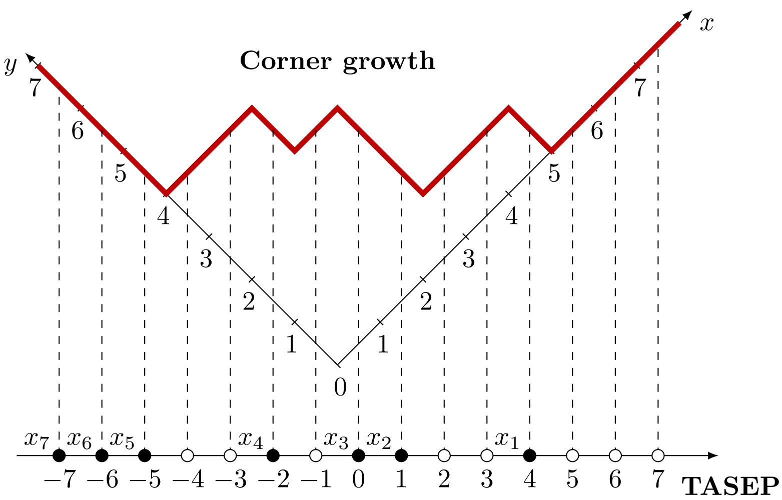


Figure 1.2: Graphical representation of the coupling between the corner growth model and TASEP. The height function h_t is represented by the red line in the plot.

In this case the connection between the corner growth and TASEP comes via the height function h_t that evolves with the particle system as time t progresses. It is a piecewise linear curve, differentiable in intervals $(x - 1/2, x + 1/2), x \in \mathbb{Z}$. For each such interval the derivative of h_t exists and it is constant 1 or -1. If the height function has a positive slope on $(x-1/2, x+1/2)$, it means that the corresponding site x on the line is not occupied by a particle at time t. Otherwise if the edge of the height function has a negative slope in $(x - 1/2, x + 1/2)$ it means that the corresponding site on the line is occupied. Particles jump to the right at random exponential times subject to the exclusion rule. With each step, the height function updates. In particular, note that the height function h_t corresponds to the level curves of the last passage time. (see Figure 1.2).

1.3 General contribution

So far we have introduced a general overview of the percolation process and its macroscopic area of study. During the thesis, as already mentioned, we will treat three different models for the last passage percolation in $\mathbb{Z}^2_+$. They differ in the distribution of the weights in each site, the admissible steps and the rule of how the maximal path collects the weights. In one of the three models the weight distribution is exponential so its connection with TASEP is straightforward as previously explained. For each model we will always derive the law of large numbers for the last passage time $G_{\mathbf{x},\mathbf{y}}$ in (1.2.2) in the homogeneous settings which means that $\{\tau_{i,j}\}_{(i,j)\in\mathbb{Z}_2^+}$ are i.i.d. under $\mathbb{P}$. Moreover, we will address three different fundamental questions (one for each model) about the last passage time:

- In Chapter 2, *Last passage percolation in an exponential environment with discontinuous rates*, the weights are exponentially distributed, $\mathcal{R} = \{e_1, e_2\}$ and we wonder what is the law of large numbers when the rates are not homogeneous but depend on the site position $(i, j) \in \mathbb{Z}_+^2$.

- In Chapter 3, *Order of the variance in the discrete Hammersley process with boundaries*, the weights are Bernoulli p distributed, $\mathcal{R} = \{e_1, e_2, e_1 + e_2\}$ and we add some proper boundary distributions in order to create an invariant model. As the title suggests we want to find the order of the last passage time fluctuations for this new model.

- In Chapter 4, *A Large deviation principle for last passage times in an asymmetric Bernoulli potential*, as in the previous chapter the weights are Bernoulli p distributed but $\mathcal{R} = \{e_1, e_2\}$ and they are collected asymmetrically in the sense that the maximal path cannot collect whenever it makes an e_2-step. Also for this model we add suitable boundary distribution for the corresponding invariant model which are different from the one in the previous model. Moreover we find an explicit formula for the right tail large deviations rate function for the model without boundaries and for the right tail large deviations logarithmic moment generating function for the model with boundaries.

We will now give an introduction to the three models under consideration in the corresponding chapters and their connections with other models. All three chapters are extracted from three different papers which are joint works with my supervisor Nicos Georgiou. The models will be formally introduced and analyzed in the following chapters.

For the three models we will treat three different topics of probability theory: the law of large numbers, fluctuations and large deviations for the last passage time. We now give a general introduction to them and how our results fit into the related literature.

1.3.1 Law of large numbers

In general finding a law of large numbers means finding a connection between a macroscopic and microscopic environment. In particular, the law of large number of the last passage time $G_{(0,0),(x,y)}$ along any direction $(x, y) \in \mathbb{R}_+^2$ in a homogeneous environment is given by

$$\lim_{n\to\infty} \frac{G_{(0,0),(\lfloor nx \rfloor, \lfloor ny \rfloor)}}{n} = g_{pp}(x, y) \quad \text{a.s} \quad \text{with } x, y \in \mathbb{R}_+^2 \text{ are fixed.}$$

$g_{pp}(x, y)$ is called the *point-to-point shape function* and its existence will be proven by Theorem 1.5.2. If the starting point is $(0, 0)$ and no confusion arises, we simply denote $G_{(0,0),(u,v)}$ with $G_{u,v}$.

Generic properties of $g_{pp}(x,y)$ have been obtained in [78], that are universal under some mild conditions on the distribution of $\tau_{i,j}$. In [15], a distributional limit to a Tracy-Widom law was proven for passage times 'near the edge', i.e. for passage times in thin rectangles of order $n \times n^a$ with $a \in (0,1)$. It is expected that several properties of the last passage models hold irrespective of the distribution of $\tau_{i,j}$; these include the fluctuation exponent of $G_{\lfloor nx \rfloor, \lfloor ny \rfloor}$, limiting laws and fluctuations of the maximal path around its macroscopic direction. As far as the law of large numbers goes, a universal approach, under only some moment assumptions on the distribution of $\tau_{i,j}$, has been developed in [48, 87, 89, 90], where the limiting shape is given in terms of variational formulas. A variational formula for the time constant in first passage percolation was proven in [73]. For two-dimensional last passage models with e_1, e_2 admissible steps the analysis and results can be sharpened; early universal results on the shape near the edge were obtained in [15, 78]. A general approach and a range of results including solutions to the variational formulas and existence of directional geodesics using invariant boundary models were developed via the use of cocycles in [49] and [50]. Similar techniques are utilized in Chapter 3 and Chapter 4, since we prove the existence of an invariant boundary model for the two models.

When the environment $\tau_{i,j} \sim \text{Exp}(1)$, the last passage model is one of the exactly solvable models of the Kardar-Parisi-Zhang (KPZ) class (see [30] for a survey). The strong law of large numbers in the exponential model is explicitly computed in [94]

$$\lim_{n\to\infty} \frac{G_{\lfloor nx \rfloor, \lfloor ny \rfloor}}{n} = \gamma(x,y) = (\sqrt{x} + \sqrt{y})^2, \quad \mathbb{P}\text{-a.s.} \tag{1.3.1}$$

The core of Chapter 2 is article [28], which is now submitted. It is concerned with directed last passage percolation on the lattice in a discontinuous environment; weights $\tau_{i,j}$ at each site (i,j) are exponentially distributed but with different rates that depend on their position. Similar arguments can be repeated when the environment comes from geometrically distributed weights, and in this case the inhomogeneity will be captured by changing the values of p, the probability of success of the geometric weight. Such models do not have the super-additivity properties that guarantee the existence of a macroscopic shape, so other techniques must be utilised to first show existence of macroscopic limits and then compute a formula for them.

Several inhomogeneous models of last passage percolation exist, each one with different ways of assigning rates (or weights in general). One way is to fix two positive sequences $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$ to assign to site (i,j) an exponential weight $\tau_{i,j}$ with rate $a_i + b_j$. Laws of large numbers for the last passage time for these models were obtained in [104] when

a_i where i.i.d. and b_j constant, and then generalised in [41]. The model enjoys several aspects of integrability, and large deviations from the shape were obtained in [42]. When admissible steps are not restricted to just e_1, e_2, [52] studies an inhomogeneous model which generalises the one introduced in [99] and obtain explicit distributional limits for fluctuations of the passage time.

Macroscopic inhomogeneities defined via the speed function (which is the inverse of the rate) have been already considered in the literature. When the speed function is continuous, [93] showed the law of large numbers for the passage times and convergence of the microscopic maximal paths to a continuous curve conditioned on uniqueness of the macroscopic maximiser.

On each site the rate of the exponentially distributed weight is completely determined by the speed function $c(\cdot,\cdot)$. When $c(x,y) = r\mathbb{1}\{x = y\} + \mathbb{1}\{x \neq y\}$ the law of large numbers was studied in [14, 101] and it was shown that for small values of r the LLN disagrees with that of the 1-homogeneous model. When the discontinuity curves of $c(x,y)$ was a locally finite set of lines of the form $\{y = x + b_i\}_{i\in\mathbb{N}}$, the law of large numbers limit was obtained in [46] and an explicit limit for the shape function was obtained in the case of the two-phase model with $c(x,y) = r_1\mathbb{1}\{x \leq y\} + r_2\mathbb{1}\{x > y\}$. In this case a flat edge was observed for the limiting shape function. A first passage (unoriented) percolation two-phase model was studied in [1], where the edge-weight distribution was different to the left and right half-planes and in certain cases proved the creation of a 'pyramid' in the limiting shape, i.e. a polygonal segment with a point of non-differentiability at the peak.

In [20] the law of large numbers for directed last passage percolation was extended when the set of discontinuity curves for $c(x,y)$ was a locally finite set of piecewise Lipschitz strictly increasing curves. A PDE approach was used, bypassing the usual techniques of TASEP particle systems, used in the earlier articles.

As previously shown there is a connection between the corner growth and TASEP which comes via the height function h_t that evolves with the particle system as time t progresses. Therefore understanding the height function in the wedge which is the level curve of the last passage time, is equivalent to studying the exclusion process for the particle system. This coupling was utilised for example in [46, 93, 101] to obtain results about hydrodynamic limits of the particle current and density, together with results for the last passage times.

Hydrodynamic limits for spatially inhomogeneous conservative systems for different versions of inhomogeneities have been extensively studied [6, 7, 23, 34, 46, 91]. An example where the discontinuity is microscopic in nature is the slow bond problem. This TASEP model was introduced in [62] and [63], in which particles jump at the same rate 1 everywhere on $\mathbb{Z}$ except at site zero where the jump happens at a slower rate than the other sites. Results regarding the hydrodynamic limits (and by extension the last passage times) were obtained in [101] and finally in [14] the full conjecture was proven that a slow bond will always affect the hydrodynamics. Recently, in [17] a totally asymmetric particle with blockage with spatial inhomogeneities was studied and limiting Tracy-Widom laws were obtained. A further improvement of the previous result has been done in [72] where they apply a different approach which allows them to extend their results to a discrete inhomogeneous space. Moreover, thanks to that approach they are able to study multitime asymptotics in the inhomogeneous exponential jump and look at a fine scaling fluctuations around a large number of particles in a small interval.

1.3.2 Fluctuations

Identifying the explicit shape function is the first step in computing fluctuations and scaling limits for last passage time quantities. When precise calculations can be performed and explicit scaling laws can be computed the model is classified as an explicitly solvable model of last passage percolation. There are only a handful of these models, and each one requires an individual treatment.

The order of the fluctuations n^{χ} is computed as that exponent χ (also called the fluctuation exponent) such that (see Figure 1.1 in the special case with exponential 1 weights) for all n large enough

$$C_1 n^{2\chi} \leq \mathrm{Var}(G_{\lfloor nx \rfloor, \lfloor ny \rfloor}) \leq C_2 n^{2\chi}. \tag{1.3.2}$$

In [8] it is proven that the fluctuations around the mean of the longest increasing subsequence (LIS) of n numbers are of order $n^{1/6}$ and the scaling limit is a Tracy-Widom distribution using a determinantal approach. The fluctuation exponent $1/3$ is often used to associate a model to the KPZ class [30, 31, 44, 55, 85], and determinental/combinatorial approaches were developed for a variety of solvable growth models in order to compute among other things explicit weak limits and formulas for Laplace transforms of last passage times and polymer partition functions. Lattice examples include the corner growth model with i.i.d. geometric weights, (admissible steps e_1, e_2) [65], the log-gamma polymer [16, 32], introduced in [102], the Brownian polymer [80, 105], the strict-weak lattice polymer [33, 81], the random walk in a beta-distributed random potential, where the zero-temperature limit is the Bernoulli-Exponential first passage percolation [11]. Particularly for percolation in Bernoulli environment see [52], where Tracy-Widom distributions where obtained for a class of models that also include the homogeneous model of [99]. The result of [65] was also used to derive explicit formulas for the distribution of the discrete Hammersley [84] with no boundaries via a particle system coupling using a mathematical physics approach.

A more probabilistic approach to estimate the order of the variance in (1.3.2), was developed in [21] and [54] where by adding Poisson distributed 'sinks' and 'sources' on the axes, they could create invariant versions of the model. For the discrete Hammersley, an invariant model with sinks and sources has been described in [12] and it was used to re-derive the law of large numbers for $G_{m,n}$.

Chapter 3 is based on article [26], which is now submitted. It studies fluctuations of a corner growth model that can be viewed as a discrete analogue of the Hammersley process [56] or an independent analogue of the longest common subsequence (LCS) problem, introduced in [25]. In particular we want to prove that the corresponding model where we add some boundary conditions belongs to the KPZ class of models for the last passage time in a particular direction. The technique that we use in this chapter to find the fluctuation order relies on finding the boundary weight distributions necessary to create an invariant boundary model. Our approach is similar to those in [9, 102, 105] where a Burke type property is first proven for the model with boundary and then exploited to obtain the order of fluctuations. The success of all those proofs is reliant on the shape function having quadratic Taylor expansion. This is the reason why we first prove the shape function for this model and then the fluctuation order.

The model under consideration in this chapter was introduced in [98] where it is studied the shape function. It is a directed corner growth model on the positive quadrant $\mathbb{Z}_+^2$. Each site v of $\mathbb{Z}_+^2$ is assigned a random weight ω_v which is Bernoulli p distributed. We have changed the notation of the random variable at each site v from τ to ω to highlight the fact that in the LCS interpretation a 1 or a 0 in a site corresponds respectively to a match or no-match between the elements of two subsequences and not to a time for an infection to reach a site. Therefore, in this case, the last passage time G corresponds to the length of longest common subsequence. The admissible steps of a potential optimal path from $(0,0)$ to (m,n) can be e_1, e_2 and e_1+e_2. In order to obtain the longest common subsequence as defined in the original problem [56] the optimal path can collect only through a diagonal step e_1+e_2 as specified by the potential

$$V(\omega, z) = \omega_{e_1+e_2}\mathbb{1}\{z = e_1+e_2\}. \tag{1.3.3}$$

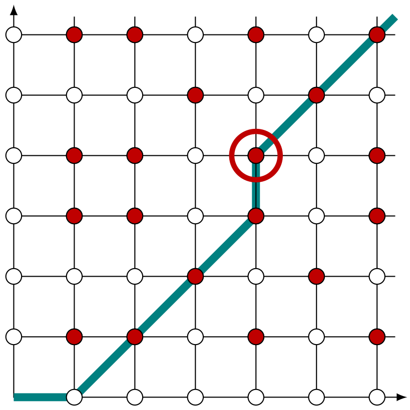


Figure 1.3: A possible representation of the maximal path (green thick line) given a fixed random environment. The red dots represent the sites where $\omega = 1$ while the white dots are when $\omega = 0$. We have circled a dot to highlight that the maximal path cannot collect that dot despite there is 1 at that site. This is due to the potential defined in (1.3.3).

At the first sight, except for the diagonal step, this model seems reminiscent of the percolation model described above. But looking at the Hammersley process more carefully it is possible to note that it has nothing to do with the percolation process since an optimal path can go through a site which has drawn a 0 without collecting anything. While in the percolation process a 0 correspond to a site or edge removal which means that a path cannot go through it.

The law of large numbers for $G^V_{(m,n)}$ was first obtained in [98] by first obtaining invariant distributions for an embedded totally asymmetric particle system. It is precisely this methodology that invites the characterization 'discrete Hammersley process' as the particle system can be viewed as a discretized version of the Aldous-Diaconis process [2] which finds the law of large numbers limit for the number of Poisson(1) points that can be collected from a strictly increasing path in $\mathbb{R}^2_+$.

The original problem is mentioned as Ulam's problem in the literature and it was about the limiting law of large numbers for the length of longest increasing subsequence of a random permutation of the first n numbers, denoted by I_n. Already in [43] it was shown that $I_n \geq \sqrt{n}$ and an elementary proof via a pigeonhole argument can be found in [56]. This gave the correct scaling and it was proven in [74, 107] that the limiting constant is 2. Then the combinatorial arguments of these papers where changed to softer probabilistic arguments in [2, 53, 97] where the full law of large numbers was obtained for a sequence of increasing Poisson points.

For the discrete Hammersley the law of large numbers for the point-to-point shape function $g_{pp}^{(p)}(s,t)$ was computed in [98] to be

$$g_{pp}^{(p)}(s,t) = \lim_{n\to\infty} \frac{G^V_{\lfloor ns\rfloor,\lfloor nt\rfloor}}{n} = \begin{cases} s, & t \geq \frac{s}{p}, \\ \frac{1}{1-p}\big(2\sqrt{pst} - p(t+s)\big), & ps \leq t < \frac{s}{p}, \\ t, & t \leq ps. \end{cases} \tag{1.3.4}$$

This is a concave, symmetric, 1-homogeneous differentiable function which is continuous up to the boundaries of $\mathbb{R}_+^2$ and it was the first completely explicit shape function for which strict concavity is not valid. In fact, the formula indicates two flat edges, for $t > s/p$ or $t < ps$.

The argument used in [98] to obtain the formula in directions of the flat edge can also be used in an identical way to obtain the law of large numbers in the same direction for the much more correlated LCS model [25]. Comparisons between the discrete Hammersley and the LCS are tantilizing. The Bernoulli environment $\eta = \{\eta_{i,j}\}$ for the LCS model is uniquely determined by two infinite random strings $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ where each digit is uniformly chosen from a k-ary alphabet (i.e. $x_i, y_j \in \{1, 2, \ldots, k\}$). Then the environment $\eta_{i,j} = \mathbb{1}\{x_i = y_j\}$ and it takes the value 1 with probability $p = 1/k$. The random variable $\mathcal{L}^{(k)}_{n,n}$ represents the longest increasing sequence of Bernoulli points in this environment, which corresponds to the longest common subsequence between the two words, of size n. The limit $c_k = \lim_{n\to\infty} n^{-1}\mathcal{L}^{(k)}_{n,n}$ is called in the literature as the the Chvatal-Sankoff constant, and it was already observed in [98] that $g_{pp}^{(1/k)}(1,1)$ of the discrete Hammersley lies between the known computational upper and lower bounds for c_k.

A formal connection between the discrete Hammersley, LCS and Hammersley models arises in the small p (large alphabet size k) limit. Sankoff and Mainville conjectured in [95] that

$$\lim_{k\to\infty} \frac{c_k}{\sqrt{k}} = 2.$$

For the discrete Hammersley model this is an immediate computation in (1.3.4) for $p = 1/k$ when we change c_k with $g_{pp}^{(1/k)}(1,1)$. For the LCS, this was proven in [71]. The value 2 is the limiting law of large numbers value for the longest increasing sequence of Poisson points in $\mathbb{R}_+^2$.

The flat edge in lattice percolation models

The discrete Hammersley is a model for which the shape function $g_{pp}(s,t)$ exhibits two flat edges, for any value of p. Flat edge in percolation is not uncommon. A flat edge for the contact process was observed in [38] and [39]. A simple explicitly solvable first passage (oriented) bond percolation model introduced in [99] allows for an exact derivation of the limiting shape function and it also exhibits a flat edge. In this model the random weight was collected only via a horizontal step, while vertical steps had a deterministic cost. For the i.i.d. oriented bond percolation where each lattice edge admits a random Bernoulli weight, a flat edge result for the shape was proved in [37] when the probability of success p is larger than some critical value and percolation occurs. This was later extended in [77] where further properties were derived. In [4] differentiability has been proven for the shape at the edge of flat edge.

These properties for oriented bond percolation can be transported to oriented site percolation and further extended to corner growth models when the environment distribution has a percolating maximum. For a general treatment to this effect, for non-exactly solvable models, see Section 3.2 in [50]. For directed percolation in a correlated environment, a shape result with flat edges can be found in [41].

Local laws of large numbers of the passage time near the flat edge of the discrete Hammersley model can be found in [45]. This work was later extended in [47], where limiting Tracy-Widom laws were obtained in special cases, using also the edge results of [15]. These 'edge results' are for the last passage time in directions that are below the critical line $(n, n/p)$ and into the concave region of g_{pp} by a mesoscopic term of n^a, $0 < a < 1$. When $a > 1/2$ the order of the fluctuations is between $O(n^{1/3})$ and $O(1)$. In the present article we further prove that in directions above the critical line (in the flat edge of g_{pp}) the variance of the passage time is bounded above by a constant that tends to 0 (see Section 3.6).

1.3.3 Large deviations

Large deviations rate functions for LPP and partition functions (for directed polymers) have been computed in several cases when the model is exactly solvable. Below G stands for a generic last passage time random variable. Define the upper (or right) and lower (or left) tail for the rate function as

$$\lim_{n\to\infty} -N^{-1}\log \mathbb{P}\{G_{Ns,Nt} \geq rN\} = J_u(r), \quad \lim_{N\to\infty} -N^{-2}\log \mathbb{P}\{G_{Ns,Nt} \leq rN\} = J_\ell(r),$$

A priori the existence of the limits is not even guaranteed, and it depends for example on the potential V and the environment ω among other things. The existence of $J_u(r)$ and $J_\ell(r)$ was proved for the exponential and geometric corner growth model in $\mathbb{Z}^2$ [65]. An earlier work where the right-tail rate function is explicitly computed appeared in [96]. Existence of the rate functions is also known in the case of the Hammersley process. Its fluctuations in the large deviations regime were studied in [35], obtaining also precise results for the upper and lower exponential tails. An explicit right-tail rate function was computed in [100], using the invariant distributions for the particle system and studying deviations for the tagged particle. In the framework of particles systems, functional large deviation principle for TASEP, which is closely connected to Exponential LPP, was obtained, for the n-speed tail in [64, 106] and for the n^2-speed tail recently in [83].

Using the invariance structure offered by Burke's property, a right-tail large deviation rate function with speed n for the partition function in the log-gamma polymer was proven [51]. Large deviations and KPZ fluctuations were computed for a random walk in a dynamic i.i.d. beta random environment in [10]. The idea of [51] was later extended for the free energy in the O'Connell-Yor polymer in [61], which is also a model with asymmetry like the one in Chapter 4 where we utilise similar techniques. Moreover for our specific model we are also able to find explicit limiting log-moment generating functions.

The approach for the existence of the right tail rate function is probabilistic in nature and utilizes super-additivity and the explicit expression is computed using probabilistic arguments. In general, the speed n^2 and the existence of lower-tail rate functions remains elusive, including for non-solvable models of last passage percolation, if one was to use only probabilistic techniques. In [69] it was shown under a boundedness condition on the environment that the n^2 speed was correct, but with no existence of the rate function results. This was for FPP. FPP and LPP have the same qualitative behavior with the role of upper and lower tails reversed, an artefact of sub-additivity vs super-additivity. Existence of the n^2 speed rate function is proven in [13] and the result is expected to extend to LPP with the same probabilistic approach. A variant of this result was earlier proved in [24] for line-to-line first passage time.

In Chapter 4 we study large deviations for the last passage time in a Bernoulli environment. All the results come from the paper [27]. The technique that we use to find the explicit formula for the log moment generating function relies on finding the invariant model. Therefore we will first add the proper boundary weights to the axes which will help proving the last passage time shape function for this model and its large deviations. Casting the model in the framework of a Burke type boundary model is part of our main contribution that is essential in computing explicit forms for the rate function and the dual for both the boundary and non-boundary model. Explicit forms of rate functions were only obtained for some of the exactly solvable models [51, 61, 65, 96, 100] and the results in this chapter add to these.

The original model was introduced in [99] as a simplified model of directed first passage percolation. In this model, the environment $\tau = \{\tau_v^{\kappa,\lambda}\}_{v\in\mathbb{Z}_+^2}$ is a collection of i.i.d. Bernoulli(p) under a background measure $\mathbb{P}$ with marginals

$$\mathbb{P}\{\tau_v = \lambda\} = p = 1 - \mathbb{P}\{\tau_v = \kappa\}, \quad \kappa > \lambda \in \mathbb{R}_+, v \in \mathbb{Z}_+^2.$$

The set of admissible paths from $(0,0)$ to $(m,n) \in \mathbb{Z}_+^2$ is denoted by $\Pi_{m,n}$ and it contains all paths of the form

$$\pi_{(0,0),(m,n)} = \{0 = v_0, v_1, \dots, v_{m+n} = (m,n)\},$$

so that $v_{i+1} - v_i \in \mathcal{R} = \{e_1, e_2\}$. We say that $\mathcal{R}$ is the set of admissible steps. The random variable under consideration is the "first passage time"

$$L^{\kappa,\lambda,p}_{(0,0),(m,n)} = \inf_{\pi\in\Pi_{m,n}} \sum_{v_i\in\pi} V(T_{v_i}\tau, v_{i+1} - v_i),$$

where T_v denotes the shift by $v \in \mathbb{Z}_+^2$ and $V : \Omega \times \mathcal{R} \to \mathbb{R}$ is the potential function given by

$$V(\tau, z) = \tau_{e_1}\mathbb{1}\{z = e_1\} + \bar{\tau}\mathbb{1}\{z = e_2\}.$$

Value $\bar{\tau}$ was constant and fixed from the beginning. The interest was to find the explicit shape function

$$\mu(s,t) = \lim_{n\to\infty} \frac{L^{\kappa,\lambda,p}_{(0,0),(\lfloor ns\rfloor,\lfloor nt\rfloor)}}{n}.$$

The model can be mapped into a last passage directed percolation by two observations. First, because the admissible paths are directed the number of vertical increments $z = e_2 \in \mathcal{R}$ are fixed for any fixed endpoint (m,n) (in fact they are n) and the cost for crossing them is deterministic $\bar{\tau}$. Thus, for simplification $\bar{\tau}$ can be set to be zero. Second, since $\lambda < \kappa$, to minimize $L^{\kappa,\lambda,p}$ one should try and take horizontal steps $e_2 \in \mathcal{R}$ when the value of the environment at the target site is λ. Define new environment

$$\omega_v = \frac{1}{\kappa - \lambda}(\kappa - \tau_v) \sim \mathrm{Ber}(p) \in \{0,1\}. \tag{1.3.5}$$

Then define the last passage time

$$G^V_{(0,0),(m,n)} = \max_{\pi_{(0,0),(m,n)}\in\Pi_{(0,0),(m,n)}} \left\{ \sum_{v_i\in\pi} V(T_{v_i}\omega, v_{i+1} - v_i) \right\}. \tag{1.3.6}$$

The value of G^V gives the number of horizontal steps through environment $\omega_v = 1$, equivalently $\tau_v = \lambda$. Each of the remaining horizontal steps contributes κ to $L^{\kappa,\lambda,p}$ and therefore we have

$$L^V_{(0,0),(m,n)} = (\lambda - \kappa)G^V_{(0,0),(m,n)} + \kappa m + \bar{\tau} n. \tag{1.3.7}$$

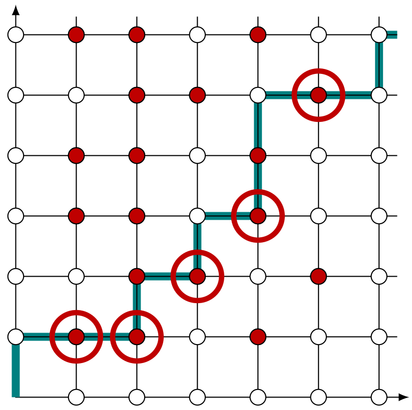


Figure 1.4: A possible representation of the maximal path (green thick line) given a fixed random environment. The red dots represent the sites where $\omega = 1$ while the white dots are when $\omega = 0$. The circled dots are the ones that the maximal path collects following the potential (1.3.8). Finally note that despite there is a column of red dots the maximal path doesn't spend a lot of time there. This is due to the fact that by (1.3.8) it cannot collect through an e_1 step. This means that the maximal path does not really see columns with high density of Bernoulli successes.

Therefore, for simplicity we study the last passage time G^V given by (1.3.6), in environment ω given by (1.3.5), under potential V given by

$$V(\omega, z) = \omega_{e_1} \mathbb{1}\{z = e_1\}. \tag{1.3.8}$$

By (1.3.7) one can translate all results to L^V.

The law of large numbers for $G_{m,n}$ was first found in [99] by first obtaining invariant distributions for an embedded totally asymmetric particle system. Most recently the LLN was reproved in [12] using an invariant boundary model with sources and sinks. This idea was utilised in the same article for the discrete version of Hammersley's process [56], introduced in [98]. The theorem states

Theorem 1.3.1 (The shape function for $G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}$ [99, 12]). *Fix p in $(0, 1)$ and $(s, t) \in \mathbb{R}_+^2$. Then we have the explicit law of large numbers limit*

$$g_{pp}(s,t) = \lim_{N\to\infty} \frac{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}}{N} = \begin{cases} \left(\sqrt{ps} + \sqrt{(1-p)t}\right)^2 - t, & t < s\frac{1-p}{p} \\ s, & t \geq s\frac{1-p}{p}. \end{cases} \tag{1.3.9}$$

This is a concave, symmetric, 1-homogeneous differentiable function which is continuous up to the boundaries of $\mathbb{R}_+^2$. Together with the shape function for the discrete Hammersley [98], are the first completely explicit shape functions for which strict concavity is not valid. In fact, the formula above indicates one flat edge, for $t > s\frac{1-p}{p}$.

This simplified Bernoulli model was studied further in [52] where Tracy-Widom distributional limits were obtained for this and a generalised inhomogeneous version where the probability of success of the Bernoulli environment changes with the first coordinate of the site. Then the LLN was used for certain estimates in proving generalised properties of the shape functions of last passage percolation in [78].

Other models for which a flat edge of the shape function exists are common, and well studied. The discrete Hammersley model discussed in [26, 45, 47, 98] and the inhomogeneous model in [41] allow for an exact derivation of the limiting shape function and they also exhibit two flat edges. Large deviations for the latter were obtained in [42]. In the present chapter, we also study the behaviour of large deviations in directions for which the shape is flat for this classical Bernoulli model.

1.4 Thesis organization

In this section we give the thesis layout and the contents of each section in each chapter.

Chapter 2

In Section 2.1 we describe the main theorems. First we state the law of large numbers limit for the passage time (2.0.5). This is Theorem 2.1.5. The limiting shape function, denoted by $\Gamma(x, y)$ comes in the form of a variational formula, where a functional is maximised over a set of suitable functions. Coninuity properties of Γ are proved in Section 2.4. The proof of the law of large numbers is in Section 2.5.

We then state results for two explicitly analysable examples. The first one is the shifted-two phase model with speed function (2.0.6); here we study properties of the shape and show analytically that there are flat edges, convexity-breaking and points of non differentiability for the shape function $\Gamma(x, y)$. The related proofs are in Section 2.2.

The other example is the corner-inhomogeneous model with a speed function (2.0.7). Under some regularity conditions on f, we are able to study properties of the maximisers of the variational formula for the shape and how their behaviour depends on f. For example, depending on f we may have points (x, y) for which the macroscopic maximiser follows the axes. For both studied examples we have cases where macroscopic maximisers are not unique. The proofs for this model can be found in Section 2.3.

Chapter 3

The chapter is structured as follows: In Section 3.1 we state our main results after describing the boundary model. In Section 3.2 we prove Burke's property for the invariant boundary model and compute the solution to the variational formula that gives the law of large numbers for the shape function of the model without boundaries. The main theorem of this paper is the order of the variance of the model with boundaries in characteristic directions. The upper bound for the order can be found in Section 3.3. The lower bound is proven in Section 3.4. For the order of the variance in off-characteristic directions see Section 3.5 and for the results for the model with no boundaries, including the order of the variance in directions in the flat edge see Section 3.6. Finally, in Section 3.7 we prove the path fluctuations in the characteristic direction, again in the model with boundaries.

Chapter 4

The chapter is organised as follows: in Section 4.1 we state our main results after describing the boundary model. In Section 4.2 we prove Burke's property for the invariant boundary model and compute the solution to the variational formula that gives the law of large numbers for the shape function of the model without boundaries. In Section 4.3 we prove a full large deviation principle (LDP) for $G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ at speed N. General properties of the rate function are also proven, including that its Legendre dual is the limiting logarithmic moment generating function (l.m.g.f.) of $G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ via Varadhan's lemma. Existence of the full LDP is a direct consequence of the existence of a right-tail rate function. In Section 4.4 we prove some important properties of the large deviations rate function which are useful to prove the main theorems of this chapter. We prove an explicit variational formula for the right-tail rate function and its Legendre dual, that we then proceed to explicitly solve and obtain a closed formula in Section 4.5. Finally, in Section 4.6 we prove an explicit expression of the limiting l.m.g.f. for the invariant boundary model.

1.5 Preliminaries

Throughout the thesis we will mention many times the theorems and lemmas that we are going to present in this section. Therefore we state and prove them for completeness and because the goal is to make the thesis self-sufficient.

1.5.1 Burke's theorem

Consider a M/M/1 queue and assume $\rho = \lambda/\mu < 1$, where μ and λ are respectively the r customer arrival and service rates, so there is an equilibrium. Let D_t denote the number of customers who have departed the queue up to time t. Let X_t denote the queue length at time t and it is a positive recurrent Markov chain for any $\mu, \lambda > 0$ with an invariant distribution.

Theorem **1.5.1** ([19]). *At equilibrium, D_t is a Poisson process with rate λ, independently of μ (so long as $\mu > \lambda$). Furthermore, X_t is independent from $(D_s, s \le t)$.*

Proof. The proof consists of a time-reversal argument. Recall that X is a birth and death chain and has an invariant distribution. So at equilibrium, X is reversible: thus for a given $T > 0$, if $\hat{X}_t = X_{T-t}$ we know that $(\hat{X}_t, 0 \le t \le T)$ has the same distribution as $(X_t, 0 \le t \le T)$. Hence $\hat{X}$ experiences a jump of size $+1$ at constant rate λ. But note that $\hat{X}$ has a jump of size $+1$ at time t if and only if a customer departs the queue at time $T - t$. Since the time reversal of a Poisson process is a Poisson process, we deduce that $(D_t, t \le T)$ is itself a Poisson process with rate λ.

Likewise, X_0 is independent from arrivals between 0 and T. Reversing the direction of time this shows that X_T is independent from departures between 0 and T. $\square$

The connection between Burke's Theorem and property comes from the queues in tandem interpretation of last passage time. The result in full generality can be found in [50]. The authors define TASEP using sequences of arrival, service and waiting times to describe the evolution of the particle system. The particle system equilibrates to what is called a fixed point [76] and the equilibrium distribution of arrival and waiting times are those of the boundary model. In the case of the exponential LPP the arrival equilibrium is Exponential(ρ) while the particle distribution is i.i.d. Bernoulli($1-\rho$). The independence and the distributions come from Burke's Theorem.

The *Burke's property* that will be mentioned in Chapter 3 and Chapter 4 is a generalization for the last passage percolation of Theorem 1.5.1. The Burke property guarantees enough analytical tractability to classify these as an exactly solvable model of the KPZ class [30]. Several well-studied models of last passage percolation and directed polymers exhibit this characteristic. There is the continuum directed polymer studied in [3], the log-gamma polymer introduced in [102], the polymer in a Brownian environment with continuous-time random walk paths, discovered in [82], subsequently worked on in [80, 81, 105], the strict-weak gamma polymer studied in [33] and [81] and the random walk in Beta-distributed random potential [11]. The exactly solvable planar polymer models with two admissible steps were recently classified in [22]. Exactly solvable models which present environment inhomogeneity are for the corner growth model [41] and for totally asymmetric particle systems associated to growth models [17, 72].

1.5.2 Subadditive ergodic theory

The subaddidive ergodic theorem was originally proved by Kingman in [70]. The improved version that we are going to present is due to Liggett. Proofs of this theorem can be found in many books of probability such as [40] and [66].

Let $\{X_{m,n} : m, n \in \mathbb{Z}_+, 0 \le m < n\}$ be a real-valued process that satisfies the following assumptions:

(i) $X_{0,n} \le X_{0,m} + X_{m,n}$ for $0 \le m < n$.

(ii) For each $k \in \mathbb{N}$, the process $\{X_{nk,(n+1)k}\}$ is stationary.

(iii) The probability distribution of the process $\{X_{m,m+j} : j \in \mathbb{N}\}$ is the same for all $m \in \mathbb{Z}_+$.

(iv) $\mathbb{E}[X_{0,1}^+] < \infty$ and for some $\gamma_0 > -\infty$, $\mathbb{E}[X_{0,n}] \ge \gamma_0 n$ for all $n \in \mathbb{N}$.

THEOREM **1.5.2.** *Under the above assumptions, there is a limit*

$$X = \lim_{n\to\infty} \frac{X_{0,n}}{n} \quad \textit{almost surely and } \mathcal{L}^1.$$

The expectation of X exists and satisfies

$$\mathbb{E}[X] = \lim_{n\to\infty} \frac{\mathbb{E}[X_{0,n}]}{n} = \liminf_{n\to\infty} \frac{\mathbb{E}[X_{0,n}]}{n}.$$

If all the stationary processes in assumption (ii) *above are ergodic, then X is constant: $\mathbb{P}\{X = \mathbb{E}[X]\} = 1$.*

Suppose that Z is a non-negative super-additive process, then the moment assumptions are not needed for almost sure convergence. Formally, let $\{Z_{m,n} : m, n \in \mathbb{Z}_+, 0 \le m < n\}$ be a process that satisfies $0 \le Z_{m,n} < \infty$, assumptions (ii) and (iii) from above, and super-additivity: $Z_{0,n} \ge Z_{0,m} + Z_{m,n}$ for $0 \le m < n$. Assume also that the processes $\{Z_{nk,(n+1)k} : n \in \mathbb{Z}_+\}$ are ergodic in addition to stationary.

Corollary 1.5.3. *There exists a constant $\gamma \in [0, \infty)$ such that $n^{-1}Z_{0,n} \to \gamma$ almost surely.*

Proof. For $K \in \mathbb{N}$, the process $Z_{m,n}^{(K)} = Z_{m,n} \wedge K(m-n)$ is super-additive, and $X_{m,n} = -Z_{m,n}^{(K)}$ satisfies all the assumptions of Theorem 1.5.2, including the ergodicity of the processes in assumption (ii). Thus there are constants $\gamma^{(K)}$ such that $n^{-1}Z_{0,n}^{(K)} \to \gamma^{(K)}$ almost surely. Since we are considering countably many $K \in \mathbb{N}$, there is a probability one event Ω_0 on which this convergence holds for all $K \in \mathbb{N}$. Let $\gamma = \sup_K \gamma(K)$. We claim that $n^{-1}Z_{0,n} \to \gamma$ on Ω_0.

Since $Z_{0,n} \ge Z_{0,n}^{(K)}$ for all K, by letting $n \to \infty$ along a suitable subsequence and then $K \nearrow \infty$ gives $\lim_{n\to\infty} n^{-1}Z_{0,n} \ge \gamma$.

If $\gamma = \infty$ this already gives the limit. Suppose $\gamma < \infty$. If $\lim_{n\to\infty} n^{-1}Z_{0,n} > \gamma$ then pick $\varepsilon > 0$ and a subsequence n_j such that $n_j^{-1}Z_{0,n_j} > \gamma + \varepsilon$ for all j. Pick $K > \gamma + \varepsilon$. Then on the one hand

$$n_j^{-1}Z_{0,n_j}^{(K)} = (n_j^{-1}Z_{0,n_j}) \wedge K > \gamma + \varepsilon \quad \text{for all } j$$

but on the other hand $n_j^{-1} Z_{0,n_j}^{(K)} \to \gamma^{(K)} \leq \gamma$. This contradiction implies that $\overline{\lim}_{n\to\infty} Z_{0,n} \leq \gamma$. $\square$

1.5.3 Borel-Cantelli lemma

Suppose that $\{A_n : n \geq 1\}$ is a sequence of events in a probability space. Then the event $A(\text{i.o}) = \{A_n \text{ occurs for infinitely many } n\}$ is given by

$$A(\text{i.o}) = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} A_n,$$

Lemma 1.5.4. *Suppose that $\{A_n : n \geq 1\}$ is a sequence of events in a probability space. If*

$$\sum_{n=1}^{\infty} \mathbb{P}\{A_n\} < \infty,$$

then $\mathbb{P}\{A(i.o)\} = 0$; only a finite number of the events occur, with probability 1.

Proof. Let $\mathbb{1}_n = \mathbb{1}\{A_n\}$ denote the indicator random variable for the event A_n, and let

$$N = \sum_{n=1}^{\infty} \mathbb{1}_n,$$

denote the total number of the events to occur. Then $\mathbb{P}\{A(\text{i.o})\} = 0$ if and only if $\mathbb{P}\{N < \infty\} = 1$. But if $\mathbb{E}[N] < \infty$, then $\mathbb{P}\{N < \infty\} = 1$ (as in the case with any random variable N), and by Tonelli's (Fubini's) theorem,

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}\{A_n\},$$

which is assumed finite, thus completing the proof. $\square$

Lemma 1.5.5. *Suppose that $\{A_n : n \geq 1\}$ is a sequence of independent events in a probability space. If*

$$\sum_{n=1}^{\infty} \mathbb{P}\{A_n\} = \infty, \tag{1.5.1}$$

then $\mathbb{P}\{A(i.o)\} = 1$.

Proof. Suppose that (1.5.1) holds, and note that if it holds then

$$\sum_{n=k}^{\infty} \mathbb{P}\{A_n\} = \infty, \quad k \geq 1. \tag{1.5.2}$$

Let $\bar{A}_n$ denote the complement of the set A_n.

$$\mathbb{P}\{A(\text{i.o})\} = \lim_{k\to\infty} \mathbb{P}\{\cup_{n=k}^{\infty} A_n\} = 1 - \lim_{k\to\infty} \mathbb{P}\{\cap_{n=k}^{\infty} \bar{A}_n\}.$$

To complete the proof we will show that

$$\mathbb{P}\{\cap_{n=k}^{\infty} \bar{A}_n\} = 0, \;\; k \geq 1.$$

By independence, and the fact that $1 - x \leq e^{-x}$, $x \geq 0$,

$$\begin{aligned}
\mathbb{P}\{\cap_{n=k}^{\infty} \bar{a}_n\} &= \prod_{n=k}^{\infty} \mathbb{P}\{\bar{A}_n\} = \prod_{n=k}^{\infty} \mathbb{P}\{\bar{A}_n\} \\
&\leq \prod_{n=k}^{\infty} e^{\mathbb{P}\{A_n\}} = e^{-\sum_{n=k}^{\infty} \mathbb{P}\{A_n\}} = 0,
\end{aligned}$$

where the last inequality is from (1.5.2). □

Chapter 2

Last passage percolation in an exponential environment with discontinuous rates

We consider a model of directed last passage growth model in two dimensions, where each lattice site (i,j) of $\mathbb{Z}_+^2$ is given a random weight $\tau_{i,j}$ according to some background measure $\mathbb{P}$.

Given lattice points $(a,b),(u,v) \in \mathbb{Z}_+^2$, $\Pi_{(a,b),(u,v)}$ is the set of lattice paths $\pi = \{(a,b) = (i_0,j_0),(i_1,j_1),\dots,(i_p,j_p) = (u,v)\}$ whose admissible steps satisfy

$$(i_\ell, j_\ell) - (i_{\ell-1}, j_{\ell-1}) \in \{(1,0),(0,1)\}. \tag{2.0.1}$$

If $(a,b) = (0,0)$ we simply denote this set by $\Pi_{u,v}$.

For $(u,v) \in \mathbb{Z}_+^2$ and $n \in \mathbb{N}$ we remind the *last passage time*

$$G_{(a,b),(u,v)} = \max_{\pi \in \Pi_{(a,b),(u,v)}} \Big\{ \sum_{(i,j)\in\pi} \tau_{i,j} \Big\}. \tag{2.0.2}$$

If $(a,b) = (0,0)$ and no confusion arises, we simply denote $G_{(0,0),(u,v)}$ with $G_{u,v}$.

In this chapter we derive the limiting constant for a sequence of scaled last passage times on the lattice. The passage times themselves are coupled through a common realization of exponential random variables. However, the rates of these random variables will be chosen according to a discrete approximation of a macroscopic function

$$c : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+.$$

Consider the lattice corner $\mathbb{Z}_+^2$. The environment $\tau = \{\tau_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ is a collection of i.i.d. exponential random variables of rate 1. For any $n \in \mathbb{N}$ we alter the rate of each of these random variables by a scalar multiplication using the macroscopic speed function $c(x, y)$. Namely, define

$$r_{i,j}^{(n)} = c\Big(\frac{i}{n}, \frac{j}{n}\Big)^{-1}, \qquad (i, j) \in \mathbb{Z}_+^2, \tag{2.0.3}$$

and define n-scaled, inhomogeneous environment by

$$\tau_{i,j}^{(n)} = r_{i,j}^{(n)} \tau_{i,j}. \tag{2.0.4}$$

The rate of the exponential random variable $\tau_{i,j}^{(n)}$ is now determined by the scalar $c\big(\frac{i}{n}, \frac{j}{n}\big)$. On each site the rate is completely determined by the speed function $c(\cdot, \cdot)$. We indicate the corresponding exponential 1 random variable as $\tau_{i,j}$.

For $(u, v) \in \mathbb{Z}_+^2$ and $n \in \mathbb{N}$ denote the last passage time

$$G_{u,v}^{(n)} = \max_{\pi \in \Pi_{u,v}} \Big\{ \sum_{(i,j)\in\pi} r_{i,j}^{(n)} \tau_{i,j}^{n} \Big\} = \max_{\pi \in \Pi_{u,v}} \Big\{ \sum_{(i,j)\in\pi} \tau_{i,j}^{(n)} \Big\}. \tag{2.0.5}$$

We impose several conditions on the function $c(x, y)$ and they are described in Section 2.1. For the moment we emphasise that for any compact set $K \subseteq \mathbb{R}_+^2$ there exist finite constant m_K and M_K such that

$$m_K \le c(x, y) \le M_K \quad \text{ for all } (x, y) \in K$$

and there are a finite number (that depends on K) of discontinuity curves of the function $c(x, y)$. These are to avoid degeneracies: If $c(x, y)$ can take the value 0 then the environment could take the value ∞ which leads to trivial passage times. If $c(x, y)$ can be infinity, that region of space will never be explored by a path. If the discontinuities have an accumulation point, then no discretization of $c(x, y)$ can capture that.

We prove a strong law of large numbers for $n^{-1} G_{\lfloor nx \rfloor, \lfloor ny \rfloor}^{(n)}$. The limiting last passage constant $\Gamma_c(x, y)$ has a variational characterization that naturally leads to a continuous version of a last passage time model (see Theorem 2.1.5). We study the variational formula and discuss properties of the shape $\Gamma_c(x, y)$ and obtain explicit minimizers in two cases of interest.

The first example is the *shifted two-phase model* with speed function

$$c_\ell(x, y) = \begin{cases} 1, & \text{if } y > x - \lambda, \\ r, & \text{if } y \le x - \lambda. \end{cases} \tag{2.0.6}$$

and the second model is the *corner-inhomogeneous* model with speed function

$$c_f(x, y) = \begin{cases} 1, & f(x) > y, \\ r, & f(x) < y, \\ 1 \wedge r, & f(x) = y. \end{cases} \tag{2.0.7}$$

Precise assumptions on f, r, λ can be found in Section 2.1.

2.0.1 Commonly used notation

$\mathbb{N}$ denotes the set of natural numbers. $\mathbb{Z}$ is the set of integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. $\mathbb{R}$ denotes the real numbers and $\mathbb{R}_+$ the non-negative reals. If a variable τ follows the exponential distribution with parameter $r > 0$ this means $\mathbb{P}\{\tau > t\} = e^{-rt}$, in other words r is the rate.

Bold-face letters (e.g. $\mathbf{v}$) indicate two dimensional vectors (e.g. $\mathbf{v} = (v_1, v_2)$). In particular letter $\mathbf{x}$ is reserved for denoting two-dimensional curves; often we write $\mathbf{x}(s) = (x_1(s), x_2(s))$ to emphasise that the curve is parametrised and seen as a function. Inequalities of vectors $\mathbf{v} \leq \mathbf{w}$ or $(v_1, v_2) \leq (w_1, w_2)$ means the inequality holds coordinate-wise. For a vector $\mathbf{v} = (v_1, v_2)$, we denote by $\lfloor \mathbf{v} \rfloor = (\lfloor v_1 \rfloor, \lfloor v_2 \rfloor)$.

Without any special mention, when we write $\|\cdot\|$ we mean $\|\cdot\|_\infty$ unless explicitly referring to a different norm. For any continuous function g we denote its modulus of continuity by ω_g and we assume

$$\|g(z_1) - g(z_2)\|_\infty \leq \omega_g(|z_1 - z_2|_\infty).$$

In the sequence we use the fact that ω_g is continuous at 0 and that $\omega_g(0) = 0$ without particular mention.

For any set $A \subseteq \mathbb{R}_2^+$, we denote the multiplication $nA = \{(nx, ny) : (x, y) \in \mathbb{R}_+^2\}$ and the floor $\lfloor nA \rfloor = \{(\lfloor nx \rfloor, \lfloor ny \rfloor) : (nx, ny) \in nA\}$. The topological interior of the set is denoted by $\mathrm{int}(A)$. For vectors $\mathbf{v}, \mathbf{w}$, $\mathbf{v} \leq \mathbf{w}$, we denote by $R(\mathbf{v}, \mathbf{w})$ the rectangle with south-west corner $\mathbf{v}$ and north-east corner $\mathbf{w}$.

Letter G is reserved for last passage times. Often we use the notation G_A to denote the last passage time in the set A, which is the maximum weight that can be collected on up-right paths that lie in the set A. If no such paths exist, $G_A = 0$.

2.1 Model and results

At this point, we state the technical conditions on $c(x, y)$ that we are imposing. There will be no special mention to these in the sequence, unless absolutely necessary. We explain why these assumptions are used after the statement of Theorem 2.1.5.

We assume the speed function $c(x, y)$ satisfies the following two assumptions:

Assumption 2.1.1 (Discontinuity curves of $c(x, y)$)**.** *Function $c(x, y)$ is discontinuous on a (potentially) countable set of curves $H_c = \{h_i\}_{i \in \mathcal{I}}$ that is locally finite in all the following properties*

1. *h_i is either a linear segment or strictly monotone.*

2. *If h_i is not a vertical line segment, it can be viewed as a graph*

$$h_i : [z_i, w_i] = \mathrm{Dom}(h_i) \to \mathbb{R},$$

3. *If h_i is strictly increasing, then*

   (a) *h_i is $C^1((z_i, w_i), \mathbb{R})$. At the boundary points z_i, w_i the derivative may take the value $\pm\infty, 0$.*

   (b) *The equation $h_i'(s) = 0$ has finitely many solutions in $[z_i, w_i]$.*

4. *If h_i is strictly decreasing, then h_i is continuous.*

The discontinuity curves $\{h_i\}_{i\in\mathcal{I}}$ separate $\mathbb{R}^2_+$ into open regions in which $c(x, y)$ is assumed continuous. The number of regions is finite in any compact set of $\mathbb{R}^2_+$. Denote the set of regions by $\mathcal{Q}$.

There are two types of points on these discontinuity curves,

1. (Interior points) These are points $\mathbf{w}$ that belong on a single discontinuity curve h_i. For any point $\mathbf{w}$ of this form, we can find an $\varepsilon > 0$ so that h_i partitions $B(\mathbf{w}, \varepsilon)$ in to three disjoint sets, $U_{\varepsilon,\mathbf{w}}$ (above h_i), $L_{\varepsilon,\mathbf{w}}$ (below h_i) and $(h_i \cap B(\mathbf{w}, \varepsilon))$.

2. (Intersection/terminal points) These are points $\mathbf{w}$ that either belong on more than one discontinuity curve or they are terminal for h_i. There are finitely many of these points in any compact set.

Assumption 2.1.2 (Further properties of $c(x, y)$)**.**

1. *$c(x, y)$ is continuous on any $Q \in \mathcal{Q}$, lower-semicontinuous on $\mathbb{R}^2_+$, that further satisfies the following stability assumption:*

   For every $i \in \mathcal{I}$ and interior point $\mathbf{w} \in h_i$, there exists $\varepsilon = \varepsilon(i, \mathbf{w}) > 0$ so that for all $\mathbf{y} \in B(\mathbf{w}, \varepsilon) \cap h_i$ there exists open set $Q_{i,\mathbf{w}} \in \{L_{\varepsilon,\mathbf{w}}, U_{\varepsilon,\mathbf{w}}\}$, so that for any sequence $\mathbf{z}_n \in Q_{i,\mathbf{w}} \cap B(\mathbf{w}, \varepsilon)$ with $\mathbf{z}_n \to \mathbf{y}$,

$$\lim_{\mathbf{z}_n \to \mathbf{y}} c(\mathbf{z}) = c(\mathbf{y}). \tag{2.1.1}$$

2. *For any compact set $K \subset \mathbb{R}^2_+$, there exist two constants $r^{(K)}_{\text{low}} > 0$ and $r^{(K)}_{\text{high}} < \infty$, so that*

$$r^{(K)}_{\text{low}} \le c(x, y) \le r^{(K)}_{\text{high}}, \quad \forall (x, y) \in K.$$

Remark 2.1.3. *Assumption 2.1.2, (1) gives by a standard compactness argument that if $c(x,y)$ is never continuous on h_i then it must be that in a strip around h_i the values of $c(x,y)$ on one of the incident regions is always smaller than the values in all other incident regions. This is consistent with assumption F3, equation (1.12) in [20]. Lower semi-continuity of $c(x,y)$ implies that the limiting value in (2.1.1) is the smallest of all possible limits on sequences that approach y. However, the assumption of [20] that $c(x,y)$ is (at least locally) Lipschitz is now removed.*

Fix an (x,y) in $\mathbb{R}_+^2$ and a speed function $c(\cdot,\cdot)$. Define the function $\Gamma_c(x,y)$ via the variational formula

$$\Gamma_c(x,y) = \sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \left\{ \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s),x_2(s))} ds \right\}, \tag{2.1.2}$$

where $\gamma(x,y) = (\sqrt{x}+\sqrt{y})^2$ is the last-passage constant in a homogeneous rate 1 environment, $\mathbf{x}(s) = (x_1(s), x_2(s))$ denotes a path in $\mathbb{R}^2$ and set

$$\mathcal{H}(x,y) = \{\mathbf{x} \in C([0,1],\mathbb{R}_+^2) : \mathbf{x} \text{ is piecewise } C^1, \mathbf{x}(0) = (0,0), \mathbf{x}(1) = (x,y),$$
$$\mathbf{x}'(s) \in \mathbb{R}_+^2 \text{ wherever the derivative is defined}\}.$$

When the speed function $c(x,y) = r$ constant, we can immediately compute

$$\begin{aligned}
\Gamma_r(x,y) &= \sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s),x_2(s))} ds = \frac{1}{r} \sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \int_0^1 \gamma(\mathbf{x}'(s))\, ds \\
&\le \frac{1}{r} \sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \gamma\Big(\int_0^1 x_1'(s)\, ds, \int_0^1 x_2'(s)\, ds \Big), \text{ by Jensen's inequality since } \gamma \text{ is concave} \\
&= \frac{1}{r}\gamma(x,y) \le \Gamma_r(x,y).
\end{aligned}$$

The last inequality follows from the fact that the straight line from 0 to (x,y) is an admissible candidate maximiser for (2.1.2). The calculation shows two things that we use freely in the sequence, namely

1. Straight lines are optimisers of (2.1.2) in homogeneous (constant) regions of $c(x,y)$. In fact, because γ is strictly concave, it is easy to show that the straight line will be the unique maximiser. We refer to this fact as 'Jensen's inequality' in the sequence.

2. $\Gamma_r(x,y)$ corresponds to the limiting shape function for last passage times in a homogeneous Exp(r) environment.

Two more properties of Γ_c can be immediately obtained:

(1) (Independence from parametrization) For any $c > 0$, $\gamma(cx, cy) = c\gamma(x, y)$ so the value of the integral

$$I(\mathbf{x}) = \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s), x_2(s))} ds \tag{2.1.3}$$

is independent of the parametrisation we choose for the curve $\mathbf{x}$.

(2) (Superadditivity) Define $\Gamma_c(x, y) := \Gamma_c((0,0),(x,y))$ and similarly define Γ_c from any starting point (a,b) to any terminal point (x,y), $(x,y) \geq (a,b)$ by

$$\Gamma_c((a,b),(x,y)) = \sup_{\mathbf{x}(\cdot)\in\mathcal{H}((a,b),(x,y))} \left\{ \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s), x_2(s))} ds \right\}, \tag{2.1.4}$$

where

$$\mathcal{H}((a,b),(x,y)) = \{\mathbf{x} \in C([0,1], \mathbb{R}^2_+) : \mathbf{x} \text{ is piecewise } C^1, \mathbf{x}(0) = (a,b), \mathbf{x}(1) = (x,y),$$
$$\mathbf{x}'(s) \in \mathbb{R}^2_+ \text{ wherever the derivative is defined}\}.$$

Then, for any $(a,b) \leq (z,w) \leq (x,y)$ we have

$$\Gamma_c((a,b),(x,y)) \geq \Gamma_c((a,b),(z,w)) + \Gamma_c((z,w),(x,y)). \tag{2.1.5}$$

In this respect, function Γ_c behaves like a 'macroscopic last passage time' and the first theorem shows that it is a continuous function.

Theorem 2.1.4 (Continuity of $\Gamma_\cdot$)**.** *Let $c(x,y)$ satisfy Assumptions 2.1.1 and 2.1.2. Fix (a,b) and $(x,y) \in \mathbb{R}^2_+$. For any $\varepsilon > 0$ there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ so that for all $\delta_1, \delta_2, \delta_3, \delta_4 \in (-\delta_0, \delta_0)$, we have*

$$|\Gamma_c((a+\delta_1, b+\delta_2),(x+\delta_3, y+\delta_4)) - \Gamma_c((a,b),(x,y))| < \varepsilon. \tag{2.1.6}$$

In the next theorem we obtain Γ_c in (2.1.2) as the law of large number of the microscopic last passage time (2.0.5).

Theorem 2.1.5. *Recall (2.0.5). Let $c(x,y)$ be a macroscopic speed function which satisfies Assumption 2.1.1, and let $(x,y) \in \mathbb{R}^2_+$. Then we have the scaling limit*

$$\lim_{n\to\infty} n^{-1} G^{(n)}_{\lfloor nx \rfloor, \lfloor ny \rfloor} = \Gamma_c(x,y) \quad \mathbb{P} - a.s. \tag{2.1.7}$$

Remark 2.1.6 (The conditions on the discontinuity curves)**.** *In [20] the discontinuity curves are assumed strictly monotone, outside of compact set. As such, when viewed as graphs of continuous functions, they are differentiable almost everywhere. This is more general than the piecewise C^1 condition in Assumption 2.1.1 3-(a). In our case we cannot relax the piecewise C^1 assumption further; in Example 2.5.4 we prove that for a certain* *speed function $c(x,y)$ the maximizing macroscopic path actually follows the discontinuity curve of $c(x,y)$ on a set of positive measure and the set of $\mathcal{H}$ contains only piecewise C^1 paths.*

We expect that under Assumptions 2.1.1 and 2.1.2 $\Gamma_c(x,y)$ is in fact a maximum and not a supremum.

We use Theorem 2.1.5 to analyse two examples.

2.1.1 The shifted two-phase model.

The first one is the shifted two-phase model. We want to study an explicit description of the limit shape function for a two-phase corner growth model with a discontinuity of the speed function along the line $y = x - \lambda$. It is a generalisation of the example provided in [46] (with $\lambda = 0$). We assume $\lambda \in \mathbb{R}_+$. For a fixed $r \in (0,1)$ we use the macroscopic speed function $c_\ell(s,t)$ on $\mathbb{R}_+^2$ defined as

$$c_\ell(x,y) = \begin{cases} 1, & \text{if } y > x - \lambda, \\ r, & \text{if } y \leq x - \lambda. \end{cases} \tag{2.1.8}$$

Subscript ℓ is to remind the reader that the small rate is *lower* than the discontinuity line, i.e. $r < 1$ in this example. Since the speed function only takes two values, the set of optimal macroscopic paths from the origin to (x,y) are piecewise linear paths.

Theorem 2.1.7. *Let $c_\ell(x,y)$ as in (2.1.8). There exist explicitly computable functions $A(r), D(r)$ (see equation (2.2.5)) and some optimal point $a^* > \lambda$ so that for any $(x,y) \in \mathbb{R}_+^2$ the limiting shape function is given by*

$$\Gamma_{c_\ell}(x,y) = \begin{cases} \gamma(x,y), & \text{if } y \geq L(x,y), \\ I(x,y), & \text{if } x - \lambda \leq y \leq L(x,y), \\ \gamma(a^*, a^* - \lambda) + r^{-1}\gamma(x - a^*, y - a^* + \lambda), & \text{if } y < x - \lambda, \end{cases}$$

where $I(x,y)$ is a linear section of $\Gamma_{c_\ell}(x,y)$, given by

$$I(x,y) = (1 + A(r))x + \Big(1 + \frac{1}{A(r)}\Big)y - D(r) = 0,$$

and $L(x,y)$ is described by the equation

$$L(x,y) = \Big(A(r)x - \frac{1}{A(r)}y\Big)^2 - 2D(r)\Big(A(r)x + \frac{1}{A(r)}y\Big) + D(r)^2 = 0.$$

2.1.2 The corner-discontinuous model.

The other example is what we call the *corner-discontinuous model*. We start with a C^2 convex decreasing function $f : [0, a_0] \longrightarrow [0, b_0]$ where $f(0) = b_0 > 0$ and $f(a_0) = 0$. Then we define the speed function

$$c_f(x,y) = \begin{cases} 1, & f(x) > y, \\ r, & f(x) < y, \\ 1 \wedge r, & f(x) = y. \end{cases} \tag{2.1.9}$$

In words, after a bounded region of rate 1 delineated by f and the coordinate axes, the rate becomes r. Computing analytically the shape function $\Gamma_{c_f}(x, y)$ is challenging; it depends on properties of the function f. When f takes the specific form

$$f(x) = (1 - \sqrt{x})^2, \quad x \in [0, 1],$$

we will explicitly identify the shape function in Example 2.3.11 and the macroscopic maximisers of (2.1.2) are straight paths from $(0, 0)$ to (x, y), despite the discontinuity.

Changing the function f, different properties of macroscopic maximisers can be obtained. From the fact that $c(x, y)$ is piecewise constant, macroscopic maximisers of (2.1.2) exist and are piecewise linear segments, one in each of the two constant regions.

For each point (x, y) in the r-region, the variational formula will be maximised by either a piecewise linear path that crosses f or by a piecewise linear path, with initial segment on one of the coordinate axes.

Definition **2.1.8** (Types of maximisers)**.** *There are two types of potential maximisers of* (2.1.2) *under speed function* (2.1.9)*:*

Type C*: We say that the maximiser is of crossing type when it crosses the function f at some optimal crossing point $(a, f(a))$, $(0 < a < a_0)$ which depends on (x, y).*

Type B*: We say that the maximiser is of boundary type, when the first linear segment of it follows one of the coordinate axes.*

Note that for $(x, y) \in (0, a_0) \times (0, f(0))$ we cannot have type B maximisers, and for (x, y) in the 1-region the maximiser must be the straight line from $(0, 0)$. Based on this definition we define

$$R_{0,f(0)} = \{(x, y) \in \mathbb{R}^2_+ : \text{maximiser of (2.1.2) is of type B and goes through } (0, f(0)\}.$$

Similarly define $R_{a_0,0}$ for which maximisers go through the horizontal axis. We would like to know when $R_{0,f(0)}$ have non-empty interior. As it turns out, this only depends on properties of the function f and the value of r.

A few definitions before stating the result. First we define a function m_2 of $a \in (0, a_0)$ by

$$m_2(a) = \frac{4}{\left(-\frac{1}{f'(a)} - 1 + D + \sqrt{\left(-\frac{1}{f'(a)} - 1 + D\right)^2 - 4\frac{1}{f'(a)}}\right)^2}, \tag{2.1.10}$$

where

$$D = D_a = r\left(1 + \sqrt{\frac{f(a)}{a}}\right)\left(\sqrt{\frac{a}{f(a)}} + \frac{1}{f'(a)}\right). \tag{2.1.11}$$

In Section 2.3 we prove that for any points $(x, y) \in \operatorname{int}(R_+^2)$ which have a candidate maximiser of type C, i.e. for any point (x, y) for which there exists at least one admissible crossing point $(a_{x,y}, f(a_{x,y}))$ with $0 < a_{x,y} < a_0$, the slope $m_2 = m_2(a_{x,y})$ of the second linear segment must satisfy the equation

$$\frac{y - f(a_{x,y})}{x - a_{x,y}} = m_2(a_{x,y}).$$

It is not necessary that for each (x, y) a unique $a_{x,y}$ will satisfy the equation above, but it will be true that $a_{x,y} < x$ and $f(a_{x,y}) < y$ (see Lemma 2.3.5).

Furthermore, we define

$$\alpha_0 = \inf\left\{s : \overline{\lim_{a\searrow 0}}\, a^s|f'(a)| = 0\right\} \quad \text{and} \quad \alpha_\infty = \sup\left\{s : \overline{\lim_{a\searrow 0}}\, a^s|f'(a)| = \infty\right\}.$$

Check that $\alpha_0 \geq \alpha_\infty$. The two values coincide when either of them is non-zero and finite. To check that the two give the same α, reason by way of contradiction; Assume that there exists a γ so that

$$\sup\left\{s : \overline{\lim_{a\searrow 0}}\, a^s|f'(a)| = \infty\right\} < \gamma < \inf\left\{s : \overline{\lim_{a\searrow 0}}\, a^s|f'(a)| = 0\right\}.$$

Then $0 < \overline{\lim}_{a\to 0}\, a^\gamma|f'(a)| < \infty$. Then for any $\varepsilon > 0$ small enough, we will have that the same condition is true for $\gamma + \varepsilon$ and that is a contradiction.

These let us define the order of growth of f' as

$$\alpha = \begin{cases} \inf\left\{s : \overline{\lim}_{a\searrow 0}\, a^s|f'(a)| = 0\right\} = \sup\left\{s : \overline{\lim}_{a\searrow 0}\, a^s|f'(a)| = \infty\right\} & \text{if } \alpha_0 \in (0, \infty) \\ \infty, & \text{if } \alpha_\infty = \infty \\ 0, & \text{if } \alpha_0 = 0. \end{cases} \tag{2.1.12}$$

When the order of growth of f' is specified to be α, we further define

$$0 \leq c_\alpha^{(-)} = \underline{\lim_{a\to 0}}\, a^\alpha|f'(a)| \leq \overline{\lim_{a\to 0}}\, a^\alpha|f'(a)| = c_\alpha^{(+)} \leq \infty. \tag{2.1.13}$$

Similarly we define

$$\beta = \begin{cases} \beta_0 = \sup\left\{s : \varliminf_{a \nearrow a_0} \dfrac{|f'(a)|}{(a_0-a)^s} = 0\right\} & \\ \quad = \inf\left\{s : \varliminf_{a \nearrow a_0} \dfrac{|f'(a)|}{(a_0-a)^s} = \infty\right\} = \beta_\infty & \text{if } \beta_0 \in (0,\infty), \\ 0, & \text{if } \beta_\infty = 0, \\ \infty, & \text{if } \beta_0 = \infty. \end{cases} \tag{2.1.14}$$

Again, at β, we similarly define $\eta_\beta^{(-)}, \eta_\beta^{(+)}$ by

$$0 \le \eta_\beta^{(-)} = \varliminf_{a \to a_0} \frac{|f'(a)|}{(a_0-a)^\beta} \le \varlimsup_{a \to a_0} \frac{|f'(a)|}{(a_0-a)^\beta} = \eta_\beta^{(+)} \le \infty. \tag{2.1.15}$$

Now we are ready to state a theorem for this model.

Theorem **2.1.9.** *Let $c_f(x,y)$ be given by (2.1.9), for some $C^2((0,a_0),(0,f(0)))$ convex function f. Assume either that $\alpha \ne 1/2$ or that $\alpha = 1/2$ and $r \notin \left[\frac{c_{1/2}^{(+)}}{c_{1/2}^{(+)}-\sqrt{f(0)}}, \frac{c_{1/2}^{(-)}}{c_{1/2}^{(-)}-\sqrt{f(0)}}\right]$. Then the following are equivalent:*

1. $\varlimsup_{a\to 0} m_2(a) = +\infty$,
2. $R_{0,f(0)} = \{0\} \times [f(0),\infty)$.

Similarly, assume either that $\beta \ne 1/2$ or that $\beta = 1/2$ and $r \notin \left[\frac{1}{1-\eta_{1/2}^{(-)}\sqrt{a_0}}, \frac{1}{1-\eta_{1/2}^{(+)}\sqrt{a_0}}\right]$. Then the following are equivalent:

1. $\varliminf_{a\to a_0} m_2(a) = 0$,
2. $R_{a_0,0} = [a_0,\infty) \times \{0\}$.

The situation when $\alpha = 1/2$ and $r \in \left[\frac{c_{1/2}^{(+)}}{c_{1/2}^{(+)}-\sqrt{f(0)}}, \frac{c_{1/2}^{(-)}}{c_{1/2}^{(-)}-\sqrt{f(0)}}\right)$ or respectively, $\beta = 1/2$ and $r \in \left[\frac{1}{1-\eta_{1/2}^{(-)}\sqrt{a_0}}, \frac{1}{1-\eta_{1/2}^{(+)}\sqrt{a_0}}\right)$, is a bit more delicate. While Theorem 2.1.9 is valid when we know the behaviour of $m_2(a)$ as a generic function of a, when $\alpha = 1/2$ we want the behaviour of $m_2(a)$ on *crossing points:*

Definition **2.1.10** (Crossing points)**.** *A point $(a, f(a))$ is a crossing point if and only if there exists $(x,y) \in \mathbb{R}_+^2$ so that a maximiser in (2.1.2) for $\Gamma_{c_f}(x,y)$ is the piecewise linear segment $(0,0) \to (a,f(a)) \to (x,y)$.*

Theorem **2.1.11.** *Assume $\alpha = 1/2, r \in \left[\frac{c_{1/2}^{(+)}}{c_{1/2}^{(+)}-\sqrt{f(0)}}, \frac{c_{1/2}^{(-)}}{c_{1/2}^{(-)}-\sqrt{f(0)}}\right)$. Then the following are equivalent:*

1. *There exists a sequence of crossing points $(a_k, f(a_k))$ so that $a_k \to 0$, $m_2(a_k) \to \infty$ and* $\displaystyle \varliminf_{a_k\to 0} a_k^{1/2}|f'(a_k)| < \frac{r\sqrt{f(0)}}{r-1}$.

2. $R_{0,f(0)} = \{0\} \times [f(0), \infty)$.

Similarly, assume that $\beta = 1/2$ *and* $r \in \left[\frac{1}{1-\eta^{(-)}_{1/2}\sqrt{a_0}}, \frac{1}{1-\eta^{(+)}_{1/2}\sqrt{a_0}}\right)$. *Then the following are equivalent:*

1. There exists a sequence of crossing points $(a_k, f(a_k))$ *so that* $a_k \to a_0$, $m_2(a_k) \to 0$ *and* $\underline{\lim}_{a_k \to a_0} a_k^{1/2}|f'(a_k)| < \dfrac{r-1}{r\sqrt{a_0}}$.

2. $R_{a_0,0} = [a_0, \infty) \times \{0\}$.

We closely look at the case for which $\alpha = 1/2$ and $c^{(-)}_{1/2} = \frac{r}{r-1}\sqrt{f(0)}$ or $\eta^{(+)}_{1/2} = \frac{r-1}{r\sqrt{a_0}}$ and show that it is a phase transition; depending on how the limits are approached it may or may not lead to non-degenerate regions for type B maximisers. We include the details that justify this statement in Section 2.3, Proposition 2.3.9.

Finally, we obtain a partition of the parameter space (α, r) where we can a priori identify whether $\overline{\lim}_{a\to 0} m_2(a) = \infty$ or $\underline{\lim}_{a\to a_0} m_2(a) = 0$ as the content of the next proposition.

Proposition 2.1.12. *Let* α, β *and* $c^{(-)}_{\alpha}, \eta^{(+)}_{\beta}$ *as defined in equations* (2.1.12), (2.1.14), (2.1.13), (2.1.15) *and let* $m_2(a)$ *be given by equation* (2.1.10). *Then, for* $(\alpha, r) \in \mathbb{R}^2_+$,

1. For $\lim_{a\to 0} f'(a) = -\infty$, *we have*

$$\overline{\lim_{a\to 0}}\, m_2(a) = \begin{cases} \dfrac{1}{(r-1)^2} & \text{if } \alpha > \dfrac{1}{2} \text{ and } r > 1, \\[2ex] \dfrac{1}{\left(r - 1 - \frac{r\sqrt{f(0)}}{c^{(-)}_{1/2}}\right)^2} & \text{if } \alpha = \dfrac{1}{2},\ c^{(-)}_{1/2} > \sqrt{f(0)},\ r > \dfrac{c^{(-)}_{1/2}}{c^{(-)}_{1/2} - \sqrt{f(0)}}, \\[2ex] +\infty & \text{otherwise.} \end{cases} \tag{2.1.16}$$

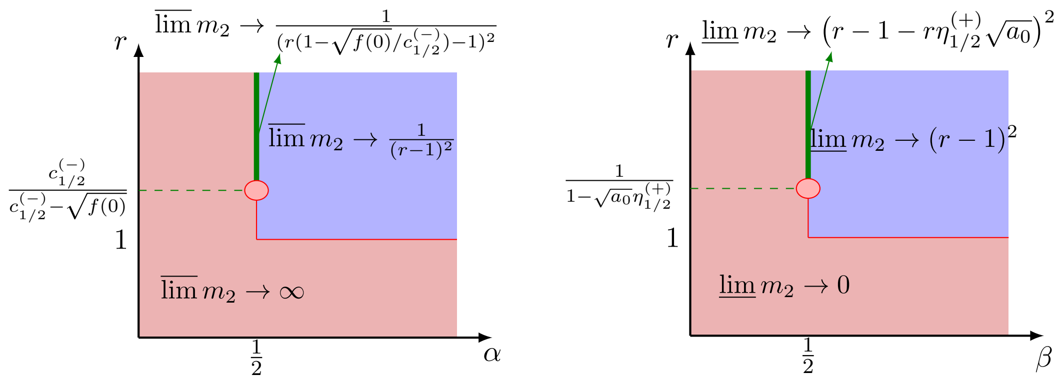
2. For $\overline{\lim}_{a\to 0} f'(a) = -c$

$$\overline{\lim_{a\to 0}}\, m_2(a) = +\infty. \tag{2.1.17}$$

By interchanging the role of the coordinates, we can obtain the corresponding results for when $a \to a_0$, *namely*

1. For $\lim_{a\to a_0} f'(a) = 0$, *we have*

$$\underline{\lim_{a\to a_0}}\, m_2(a) = \begin{cases} (r-1)^2 & \text{if } \beta > \dfrac{1}{2} \text{ and } r > 1, \\[2ex] \left(r - 1 - r\eta^{(+)}_{1/2}\sqrt{a_0}\right)^2 & \text{if } \beta = \dfrac{1}{2},\ \eta^{(+)}_{1/2} < a_0^{-1/2},\ r > \dfrac{1}{1-\eta^{(+)}_{1/2}\sqrt{a_0}}, \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{2.1.18}$$



(a) Behaviour for $\overline{\lim}\, m_2(a)$ when α and r vary, when $a \to 0$ and $f'(0) \to -\infty$, when $c^{(-)}_{1/2} > \sqrt{f(0)}$.

(b) Behaviour for $\underline{\lim}\, m_2(a)$ when β and r vary, when $a \to a_0$ and $f'(a_0) \to 0$, when $\eta^{(+)}_{1/2} < a_0^{-1/2}$.

2. *For* $\underline{\lim}_{a\to a_0} f'(a) = -c$

$$\underline{\lim}_{a\to 0}\, m_2(a) = 0. \tag{2.1.19}$$

Remark 2.1.13. *We cannot say anything meaningful about the values of* $\underline{\lim}_{a\to 0} m_2(a)$ *and* $\overline{\lim}_{a\to a_0} m_2(a)$. *These values depend on the curvature of the function* f *but it is not clear if they offer information about the maximisers. For example it is possible that the* $\underline{\lim}_{a\to 0} m_2(a) = \text{constant}$ *when* $\overline{\lim}_{a\to 0} m_2(a) = \infty$ *then we have countable a sequence of point for which two miximisers exist.*

Proposition 2.1.12 in conjunction with Theorem 2.1.9 classifies the cases for which non-trivial maximisers of type B exist when $\alpha \neq 1/2$. Theorem 2.1.11 is weaker, so without further analysis, the proposition can only guarantee trivial type B maximisers from the vertical axis when $\alpha = 1/2$ and $r \notin \left[\frac{1}{1-\eta^{(-)}_{1/2}\sqrt{a_0}}, \frac{1}{1-\eta^{(+)}_{1/2}\sqrt{a_0}}\right]$. When $\alpha = 1/2$ and $r \in \left[\frac{1}{1-\eta^{(-)}_{1/2}\sqrt{a_0}}, \frac{1}{1-\eta^{(+)}_{1/2}\sqrt{a_0}}\right)$ one needs to verify that the optimal slopes tend to $+\infty$.

We showcase the above results by performing some Monte Carlo simulations to show the maximal paths in different cases. For all simulations we considered the curve $y = f(x)$ to be

$$f(x) = (c - x^{b/k})^k,$$

and we varied the parameters b, c, k with $b < k$. See Figure 2.2.

Combining the explicit results obtained in the two examples, we can state the following theorem of counterexamples, describing situations that do not occur in the homogeneous setting.

Theorem 2.1.14. *Depending on the speed function* $c(x, y)$,

1. *$\Gamma_c(x,y)$ is not necessarily concave, and its level curves are not necessarily convex. (Γ_{c_ℓ} in Theorem 2.1.7).*

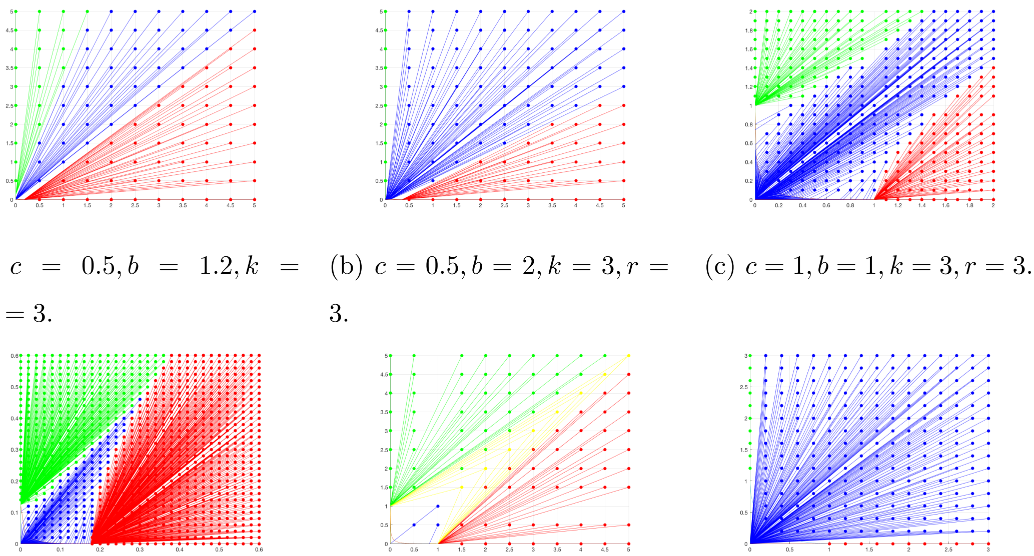
2. *$\Gamma_c(x,y)$ may exhibit flat edges. (Γ_{c_ℓ} in Theorem 2.1.7).*

3. *$\Gamma_c(x,y)$ is not necessarily differentiable on the interior of $\mathbb{R}^2_+$. (Γ_{c_ℓ} in Theorem 2.1.7).*

4. *The maximisers of (2.1.2) for some (x,y) are not necessarily unique. (See points on $L(x,y)$ in Theorem 2.1.7, Remark 2.3.2, and Fig. 2.2)*

5. *It is possible to have terminal points (x,y) for which the maximiser of (2.1.2) has an initial segment on one of the coordinate axes. (Theorem 2.1.9, Proposition 2.1.12).*

We leave the calculus details necessary for the proof of Theorem 2.1.14 to the reader.



(a) $c = 0.5, b = 1.2, k = 3, r = 3$. (b) $c = 0.5, b = 2, k = 3, r = 3$. (c) $c = 1, b = 1, k = 3, r = 3$.

(d) $c = 0.5, b = 1.2, k = 3, r = 4$. (e) $c = 1, b = 1, k = 3.5, r = 3$. (f) $c = 1, b = 1, k = 2, r = 3$.

Figure 2.2: (Colour online) Blue paths are maximisers of type C, i.e. they cross to the r-region from the interior of f. The set of all (x,y) reached by such paths may be bounded (e.g. see subfigures (D), (E)). Green and red paths are type B maximisers that follow either the y- or the x- axis respectively. Simulations suggest that when the regions $R_{0,f(0)}$ and $R_{a_0,0}$ are not degenerate they can intersect, and bound the Type C region. Finally, the target points of yellow paths are those for which the maximiser is not unique.

2.2 The shifted two-phase model

From Jensen's inequality and Theorem 2.1.5 the variational formula for the limiting last passage time can be simplified to

$$\Gamma_{c_\ell}(x,y) = \begin{cases} \sup\limits_{b_1 > a_1 \geq \lambda} \Big\{ \gamma(a_1, a_1 - \lambda) + \dfrac{1}{r}\gamma(b_1 - a_1, b_1 - a_1) \\ \qquad\qquad + \gamma(x - b_1, y - b_1 + \lambda) \Big\} \bigvee \gamma(x,y), & \text{if } y > x - \lambda, \\ \sup\limits_{a_2 \geq \lambda} \Big\{ \gamma(a_2, a_2 - \lambda) + \dfrac{1}{r}\gamma(x - a_2, y - a_2) \Big\} \bigvee \gamma(x,y), & \text{if } y = x - \lambda, \\ \sup\limits_{a_3 \geq \lambda} \Big\{ \gamma(a_3, a_3 - \lambda) + \dfrac{1}{r}\gamma(x - a_3, y - a_3 + \lambda) \Big\}, & \text{if } y < x - \lambda. \end{cases} \tag{2.2.1}$$

The top and middle expressions correspond to passage times up to (x, y) above or on the discontinuity line. If $x \geq \lambda$ then the optimal paths can either be a straight line up to (x, y) corresponding to microscopic maximal path in environment Exp(1), or a piecewise linear path which takes advantage of the smaller rate on the discontinuity line. Microscopically, the maximal path enters the region with environment Exp(r) but does not fluctuate from the discontinuity line macroscopically. It could also be that by default the maximal path is the straight line segment when $x < \lambda$ at which point the supremum takes the value $-\infty$ and only $\gamma(x, y)$ remains.

If (x, y) is below the discontinuity then it has to be that the macroscopic maximal path is piecewise linear and it crosses the line $t = s - \lambda$ at some optimal point.

In the computations that follow set

$$K(r) = \sqrt{1 + \frac{r^2}{4(1-r)}}.$$

We treat the three cases separately:

(1) **Case 1:** $y > x - \lambda$**:** Assume $x \geq \lambda$ otherwise, as we discussed the maximal path is the straight line and the shape function is $\gamma(x, y)$. We begin by explicitly computing the supremum, which after substitution of the formula for γ and some manipulation, it becomes

$$I_{c_\ell,}(x,y) = \sup_{b_1 \geq a_1} \Big\{ \Big(2 - \frac{4}{r}\Big)(a_1 - b_1) + x + y \\ + 2\big(\sqrt{a_1(a_1 - \lambda)} + \sqrt{(x - b_1)(y - b_1 + \lambda)}\big) \Big\},$$

where the parameters a_1, b_1, λ and the point (x, y) have to satisfy the constraints

$$x \geq b_1 \geq a_1 \geq \lambda, \text{ and } y \geq b_1 - \lambda.$$

The unknowns are a_1, b_1 and they are the x - coordinates of the points on the line $t = s - \lambda$ that determine the second segment of the potential piecewise linear path. Compute the first partial derivatives for a_1 and b_1 and set them equal to 0 to obtain

$$\frac{\partial I_{c_\ell}(x,y)}{\partial a_1} = 2 - \frac{4}{r} + \frac{2a_1 - \lambda}{\sqrt{a_1(a_1-\lambda)}} = 0$$
$$\frac{\partial I_{c_\ell}((x,y)}{\partial b_1} = \frac{4}{r} - 2 + \frac{2b_1 - x - y - \lambda}{\sqrt{(x-b_1)(y-b_1+\lambda)}} = 0.$$

From the first equation, imposing the condition $x \geq a_1 > 0$ to obtain the optimal entry point

$$(a_1^*, a_1^* - \lambda) = \Big(\frac{\lambda}{2}(K(r)+1), \frac{\lambda}{2}(K(r)-1)\Big). \tag{2.2.2}$$

From the second equation and the condition and $a_1 \leq b_1 \leq x$, we get

$$(b_1^*, b_1^* - \lambda) = \Big(\frac{(x+y+\lambda)+(x-y-\lambda)K(r)}{2}, \frac{(x+y-\lambda)+(x-y-\lambda)K(r)}{2}\Big) \tag{2.2.3}$$

under the constraint

$$y \leq \frac{K(r)+1}{K(r)-1}x - \frac{2K(r)}{K(r)-1}\lambda. \tag{2.2.4}$$

The constraint is equivalent to $a_1^* \leq b_1^*$. When it is not satisfied, the optimal path is the straight line. It is always true that $b_1^* < x$. Check that (a_1^*, b_1^*) gives a local maximum by computing the Hessian matrix $H(a_1, b_1)$ for which

$$\det\{H(a_1^*, b_1^*)\} = \frac{\lambda^2(x-y-\lambda)^2}{4[a_1^*(a_1^*-\lambda)(x-b_1^*)(y-b_1^*+\lambda)]^{3/2}},$$
$$\text{and } \frac{\partial^2 \Gamma_{c_\ell}(a_1^*, b_1^*)}{\partial a_1^2} = \frac{-\lambda^2}{2[a_1^*(a_1^*-\lambda)]^{3/2}}.$$

It is immediate to check that it is also a global maximum for $I_{c_\ell}(x,y)$. We substitute the values of a_1^* and b_1^* of respectively (2.2.2) and (2.2.3) into (2.2.1) to obtain the value on the trapezoidal path $I_{c_\ell}(x,y)$

$$\begin{aligned} I_{c_\ell}(x,y) = x\Big(1 + \Big(\frac{2}{r} - 1\Big)(1+K(r)) - \sqrt{K(r)^2-1}\Big) + \\ + y\Big(1 + \Big(\frac{2}{r} - 1\Big)(1-K(r)) + \sqrt{K(r)^2-1}\Big) \\ + 2\lambda\Big(\Big(1 - \frac{2}{r}\Big)K(r) + \sqrt{K(r)^2-1}\Big) \\ = (1+A(r))x + \Big(1 + \frac{1}{A(r)}\Big)y - D(r), \end{aligned}$$

where we set

$$A(r) = \frac{(1+\sqrt{1-r})^2}{r}, \quad D(r) = 4\lambda\frac{\sqrt{1-r}}{r}. \tag{2.2.5}$$

In order to find the region for which $I_{c_\ell}(x,y)$ is actually $\Gamma_{c_\ell}(x,y)$, we directly compare with $\gamma(x,y)$. The two functions give the same value on the curve

$$A(r)x + \frac{1}{A(r)}y - D(r) = 2\sqrt{xy}. \tag{2.2.6}$$

For (x,y) in the region $x-\lambda \le y \le \frac{K(r)+1}{K(r)-1}x - \frac{2K(r)}{K(r)-1}\lambda$, the left-hand side in the display above is always positive, so we can square both sides and identify the curve as

$$0 = \Big(A(r)x - \frac{1}{A(r)}y\Big)^2 - 2D(r)\Big(A(r)x + \frac{1}{A(r)}y\Big) + D(r)^2 = L(x,y),$$

where $L(x,y)$ is defined by the expression in the display above. Equation $L(x,y)=0$ defines a parabola. It has an axis of symmetry that is parallel to - and above - the line (2.2.4) and it is tangent to the discontinuity line $y = x-\lambda$ precisely at point $(a_1^*, a_1^*-\lambda)$ given by (2.2.2). Line (2.2.4) also crosses both the parabola and the discontinuity line precisely at the same point (2.2.2). Therefore,

$I_{c_\ell}(x,y) = \Gamma_{c_\ell}(x,y)$ if and only if

$$(x,y) \in \mathcal{R}_{\lambda,r} = \{(x,y) : a_1^* \le x,\ x-\lambda \le y,\ L(x,y) > 0\}. \tag{2.2.7}$$

For $(x,y)\in\mathcal{R}_{\lambda,r}$ the maximiser is the trapezoidal path with second segment on the discontinuity line of c_ℓ. For all other (x,y) with $y > x-\lambda$ the maximizing path is the straight line and $\Gamma_{c_\ell}(x,y) = \gamma(x,y)$. Points on the curve $L(x,y)=0$ have two maximizing paths.

One last remark is that if (x,y) and (z,w) both belong in $\mathcal{R}_{\lambda,r}$ then the slope of the third segments of the corresponding maximising paths are actually the same and equal to $\frac{K(r)+1}{K(r)-1}$. Therefore they are parallel to the axis of symmetry of the parabola (so they also intersect the critical parabola) and have finite macroscopic length.

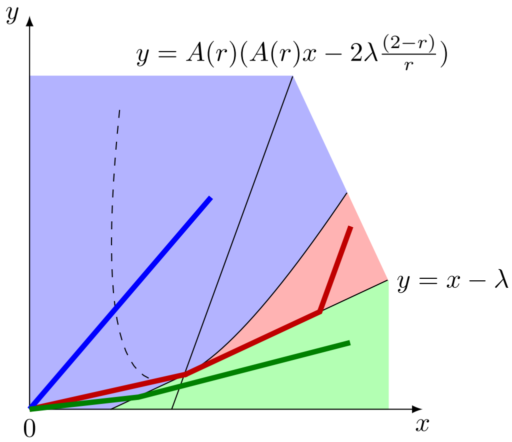
(2) **Case 2:** $y = x-\lambda$. The same steps as before (or continuity of $\Gamma_{c_\ell,}(x,y)$ as $y \searrow x-\lambda$) give

$$\begin{aligned}\Gamma_{c_\ell,}(x,y) &= (\sqrt{a_1^*} + \sqrt{a_1^*-\lambda})^2 + \frac{1}{r}(\sqrt{x-a_1^*} + \sqrt{x-a_1^*})^2 \\ &= \frac{4}{r}x + \lambda\Big(K(r) + \sqrt{K^2(r)-1} - \frac{2}{r}(1+K(r))\Big).\end{aligned}$$

When $x \ge a_1^*$, the maximiser has two linear segments; the first one goes from 0 to $(a_1^*, a_1^*-\lambda)$ and the second one follows the discontinuity line up to $(x, x-\lambda)$.

(3) **Case 3:** $y < x-\lambda$. An explicit analytical solution to the variational problem is not easily tractable. The maximisers are piecewise linear, with slopes m_1, m_2 with

$m_2 > m_1$. The optimal crossing point $(a_3^*, a_3^* - \lambda)$ on the discontinuity line always has $a_3^* < a_1^*$.



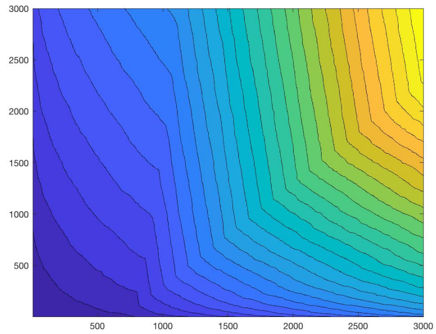


Figure 2.3: (Left) Maximal macroscopic paths for the shifted two-phase corner growth model. In the blue region we have a straight line path, in the red region we have a three piecewise linear path and in the green region we have a two piecewise linear path.
(Right) Numerical simulation of the shape function $\Gamma_{c_\ell}(x, y)$. Notice the non-convexity of the level curves, and the points of non-differentiability of the level curves, and by extension of Γ_{c_ℓ}.

Remark 2.2.1. *When the environment is homogeneous and $c(x, y) = c$, the shape function is strictly concave and in $C^2(\mathbb{R}_+^2)$. As one can see in Figure 2.3, the simulations suggest that the shape function for the shifted inhomogeneous model is no longer strictly concave or C^1 in the interior of $\mathbb{R}_+^2$. Indeed this is a straight-forward calculation because we have precise formulas for the shape function for $(x, y) \in \mathcal{R}_{\lambda,r}$ and for (x, y) for which y is above the critical parabola. We leave this calculation to the reader. The concavity-breaking does not occur in the two-phase model without shifting in [46]. The flat edge is common in both inhomogeneous models.*

2.3 The corner-discontinuous last passage percolation

It will be convenient to adopt a more general setting for the discontinuity curve f then the one described in Section 2.1. To this end, we begin from considering a C^2 function $g : \mathbb{R}_+^2 \to \mathbb{R}_+$ with the property that its level curve $g(x, y) = k$ when viewed as a function of $y = f(x)$ is strictly decreasing and twice differentiable function so that the first and second derivative never become zero, i.e.

$$\frac{df}{dx} < 0, \quad \frac{d^2 f}{dx^2} \neq 0.$$

For what follows we restrict to the case where f is convex and its second derivative strictly positive.

Since the gradient of g is always perpendicular to its level curve, for any $(a,b) \in \mathbb{R}^2_{>0}$ with $g(a,b) = h$ we have that

$$\partial_x g(a,b) \cdot \partial_y g(a,b) > 0. \tag{2.3.1}$$

Let a_0 and b_0 be defined by $g(a_0, 0) = g(0, b_0) = k$. They can also take the value infinity if g does not intersect the coordinate axes.

We define the macroscopic speed function $c_{g,k}(x,y)$ on $\mathbb{R}^2_+$ to be

$$c_{g,k}(x,y) = \begin{cases} 1, & \text{if } g(x,y) < k, \\ r, & \text{if } g(x,y) \geq k. \end{cases}$$

From Theorem 2.1.5 and the fact that macroscopic optimisers are piecewise linear in constant regions, the limiting last passage time is given by

$$\Gamma_{c_{g,k}}(x,y) = \begin{cases} \gamma(x,y), & \text{if } g(x,y) \leq k \\ \sup\limits_{a \leq x \wedge a_0,\, b \leq y \wedge b_0,\, g(a,b)=k} \Big\{ \gamma(a,b) + \dfrac{1}{r}\gamma(x-a, y-b) \Big\}, & \text{if } g(x,y) > k. \end{cases} \tag{2.3.2}$$

Except for some specific cases, the solution to the variational problem in (2.3.2) cannot be explicit but can be approximated numerically. However, this model allows for partial analysis, and despite its simplicity it demonstrates behaviour that can be rigorously shown to differ from the passage time in a homogeneous environment.

We rewrite Definition 2.1.10 using the notation introduced so far in this section.

Definition **2.3.1** (Crossing points)**.** *We say that a point (a,b) is a (g -) crossing point for point (x,y) if it belongs in the set*

$$\mathcal{S}_{x,y} = \{(a,b) : g(a,b) = k \textit{ which solve } (2.3.2) \textit{ for the given } (x,y)\}.$$

In words, (a,b) solves the optimization problem (2.3.2). The set of all crossing points is defined by

$$\mathcal{S} = \{(a,b) : g(a,b) = k \textit{ which solve } (2.3.2) \textit{ for some } (x,y)\}.$$

If $|\mathcal{S}_{x,y}| = 1$ then there is a unique piecewise linear macroscopic maximal path from the origin to (x,y) which is a maximiser of the variational formula (2.1.2), and this passes through $(a,b) \in \mathcal{S}_{x,y}$.

In the homogeneous environment ($r = 1$), maximisers of (2.1.2) are unique and are straight lines, i.e. $|\mathcal{S}_{x,y}| = 1$. Here, depending on the function g, this is no longer true, as discussed in the following remark.

Remark 2.3.2. *Depending on the function g, it is possible to have a point (x, y) that does not lead to a unique maximiser of the problem (2.3.2). Suppose you fix a point (t, t) in the r-region, and further assume that f is symmetric about the main diagonal. By carefully modulating the values of f around the main diagonal, and by appropriately lowering the value of r, one can show that the main diagonal cannot be an optimiser for Γ. Then the optimiser $\mathbf{x}$ is a concatenation of two linear segments that crosses f at some point. Because f is symmetric, the piecewise linear curve that is symmetric to $\mathbf{x}$ about the diagonal is also an optimiser. We leave the details to the reader.* □

Lemma 2.3.3. *The set of* crossing points $\mathcal{S}$ *is dense on the curve* $g(a, b) = k$.

Proof. To see this, fix an arbitrary segment on the level curve

$$\mathcal{I} = \{(a, b) : a_1 < a < a_2,\ b_1 < b < b_2,\ g(a, b) = k\}$$

and consider (x, y) so that $a_1/2 < x < a_2/2$, $b_1/2 < y < b_2/2$, $g(x, y) > k$ which is possible since the level curve is convex. The maximal path to (x, y) has to cross the curve at some point $(a_{x,y}, b_{x,y})$ with $a_1/2 < a_{x,y} < a_2/2$, $b_1/2 < b_{x,y} < b_2/2$ since it will be piecewise linear with strictly positive slope for each segment. This suffices for the proof. □

Fix a *crossing point* (a, b). Then, for some (x, y), this point solves the Lagrange multiplier problem

$$h(a, b, \lambda) = \gamma(a, b) + \frac{1}{r}\gamma(x - a, y - b) + \lambda(g(a, b) - k), \tag{2.3.3}$$
$$0 \le a \le x \wedge a_0,\ 0 \le b \le y \wedge b_0.$$

Function h has two derivatives in the interior of its domain, so we can optimize over (a, b, λ) as usual. If the local maximum is in the interior we will find it using the Lagrange multiplier method. Otherwise, we will check even the boundary value of the region. The derivatives give

$$\begin{cases} \frac{\partial h}{\partial a} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}} - \frac{1}{r}\frac{\sqrt{x-a}+\sqrt{y-b}}{\sqrt{x-a}} + \lambda\partial_a g(a, b) = 0, & \text{(2.3.4a)} \\ \frac{\partial h}{\partial b} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{b}} - \frac{1}{r}\frac{\sqrt{x-a}+\sqrt{y-b}}{\sqrt{y-b}} + \lambda\partial_b g(a, b) = 0, & \text{(2.3.4b)} \\ \frac{\partial h}{\partial \lambda} = g(a, b) - k = 0. & \text{(2.3.4c)} \end{cases}$$

Solve the first two for λ and set the two expressions equal to obtain

$$r\Big(1 + \frac{\sqrt{b}}{\sqrt{a}}\Big)\partial_a g\Big(\frac{\sqrt{a}}{\sqrt{b}} - \frac{\partial_b g}{\partial_a g}\Big) = \Big(1 + \frac{\sqrt{x-a}}{\sqrt{y-b}}\Big)\Big(\partial_a g - \partial_b g\frac{\sqrt{y-b}}{\sqrt{x-a}}\Big). \tag{2.3.5}$$

For the (x,y) for which the crossing point is the (a,b) that satisfies equation (2.3.5), the maximal path is piecewise linear with slopes

$$m_1 = \frac{b}{a} \quad \text{and} \quad m_2 = \frac{y-b}{x-a}.$$

Then equation (2.3.5) can be written as

$$\nabla g(a,b) \cdot \left(\frac{r(1+\sqrt{m_1})}{\sqrt{m_1}} - \frac{1+\sqrt{m_2}}{\sqrt{m_2}}, -r(1+\sqrt{m_1}) + (1+\sqrt{m_2}) \right) = 0. \tag{2.3.6}$$

Equation (2.3.6) has a very convenient form. It shows that if for a fixed (x,y) the crossing point (a,b) solves the Lagrange multiplier problem (2.3.3), then the same point (a,b) solves (2.3.3) for any $(x',y') = (a,b) + \lambda(x-a, y-b)$ on the line from (a,b) with slope m_2. Using the form $g(x,y) = y - f(x)$, we have that $\nabla g(a,b) = (-f'(a), 1)$. Relation (2.3.6) after some algebraic manipulations then becomes

$$\frac{r-1}{r} + \sqrt{m_1} - \frac{\sqrt{m_2}}{r} = -\frac{f'(a)}{r}\left(r - 1 + \frac{r}{\sqrt{m_1}} - \frac{1}{\sqrt{m_2}} \right). \tag{2.3.7}$$

We will use this equation later, as any crossing point away from the boundary satisfies relation (2.3.7).

The next lemma shows that if (a,b) solves (2.3.6) (or a solves (2.3.7)) does not imply that we found a global maximiser.

Lemma 2.3.4 (Maximal paths cannot cross each other)**.** *Suppose that for a point (x,y) there exist two crossing points (a_1^*, b_1^*) and (a_2^*, b_2^*) $(a_1^* > a_2^*)$ that satisfy (2.3.5), (2.3.6) subject to the constraint (2.3.4c) and in particular maximise 2.3.3. Then for $(x',y') = (a_1^*, b_1^*) + \kappa(x - a_1^*, y - b_1^*)$ we have that*

1. *If $\kappa > 1$, crossing point (a_1^*, b_1^*) is a critical point for the Lagrange multiplier problem when the terminal point is (x',y').*

2. *If $\kappa > 1$, crossing point (a_1^*, b_1^*) is not a maximiser for the Lagrange multiplier problem when the terminal point is (x',y').*

3. *If $\kappa < 1$, crossing point (a_1^*, b_1^*) is the unique maximiser for the Lagrange multiplier problem when the terminal point is (x',y').*

Proof. See Figure 2.4 for the geometric construction.

For (1) the statement follows from the fact that slope of the segment $(a_1^*, b_1^*) \to (x',y')$ is the same as that for $(a_1^*, b_1^*) \to (x,y)$. Equation (2.3.6) is automatically satisfied so (a_1^*, b_1^*) is a critical point.

For (2) we reason as follows. The path $(0,0) \to (a_2^*, b_2^*) \to (x,y) \to (x',y')$ cannot be optimal for (x',y'), because it is polygonal in the homogeneous region of rate r and the straight line (a_2^*, b_2^*) is strictly better. However it has the same weight as the path $(0,0) \to (a_1^*, b_1^*) \to (x',y')$ and therefore this path cannot be optimal for (x',y').

Part (3) follows with similar arguments. $\square$

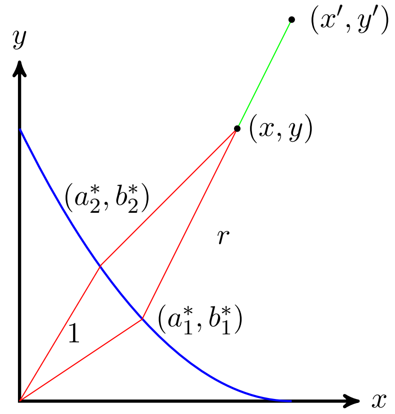


Figure 2.4: The construction described in the proof of Lemma 2.3.4.

Next, we want to verify that the maximal path will never follow a vertical or horizontal line in the r region, i.e. the slope of the second segment of a potential maximiser cannot have slope equal to zero or infinity.

Lemma 2.3.5. *Suppose that $(a,b) \in \mathcal{S}_{x,y}$. Then $a < x$ and $b < y$. In particular, any (x,y) for which the maximiser of $\Gamma_{c_g,k}(x,y)$ does not cross $(a_0,0)$ or $(0,b_0)$ has to cross at a point (a,b) that satisfies (2.3.5), (2.3.6) and the second segment has a non-zero, finite slope.*

Proof. We only show that a second segment of infinite slope is not optimal. The strictly positive slope claim follows similarly. We compare the last passage time of a path which crosses the discontinuity in the point whose x coordinate is the same of the point that it has to reach, in other words $a = x, b = f(x)$, and another path with $a = x - \varepsilon$. Under these assumptions, we have that

$$f(x-\varepsilon) = b + \delta(\varepsilon) \text{ with } \lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta(\varepsilon)} = c \in (0,\infty).$$

This is because $b + \delta(\varepsilon) = f(x-\varepsilon) = f(x) - f'(x)\varepsilon + o(\varepsilon) = b - \varepsilon f'(x) + o(\varepsilon)$ by a Taylor expansion around x and the fact that $-f'(x) \in (0,\infty)$. Then, a direct comparison between the weight of the two paths, π_1 which crosses at (x,b) and π_2 crossing at $(x-\varepsilon, f(x-\varepsilon))$ gives

$$I_{c_g,k}(\pi_2) - I_{c_g,k}(\pi_1) = (\sqrt{x-\varepsilon} + \sqrt{b+\delta(\varepsilon)})^2 + \frac{1}{r}(\sqrt{\varepsilon} + \sqrt{y-b-\delta(\varepsilon)})^2$$

$$- (\sqrt{x} + \sqrt{b})^2 - \frac{1}{r}(y-b)$$

$$= \Big(1 - \frac{1}{r}\Big)(\delta(\varepsilon) - \varepsilon) + 2\sqrt{(x-\varepsilon)(b+\delta(\varepsilon))} + \frac{2}{r}\sqrt{\varepsilon(y-b-\delta(\varepsilon))} - 2\sqrt{xb}$$

$$= \Big(1 - \frac{1}{r}\Big)(\delta(\varepsilon) - \varepsilon) - \frac{\varepsilon[b - xf'(x)]}{\sqrt{xb}} + \frac{2}{r}\sqrt{\varepsilon(y-b-\delta(\varepsilon))} + o(\varepsilon).$$

Divide through by ε and let it tend to 0 to see that the last expression is eventually positive. As such, $I_{c_{g,k}}(\pi_2)$ is a lower bound for the shape function at (x, y) and therefore the maximiser cannot be π_1. □

Lemma 2.3.6. *Let (x, y) and $(z, w) \in \mathbb{R}_+^2$ so that $(x, y) \neq \lambda(z, w)$ for any $\lambda \in \mathbb{R}$. Then*

$$\mathcal{S}_{x,y} \cap \mathcal{S}_{z,w} \in \{\varnothing, (a_0, 0), (0, b_0)\}.$$

In other words, the only possible crossing points from which more than one maximiser passes, are the axes points $(a_0, 0), (0, b_0)$.

Proof. Assume by way of contradiction that two terminal points in general position, (x, y) and (z, w) have the same crossing point (a, b) for which $0 < a < a_0$ and $0 < b < b_0$. Then the gradient of g at (a, b) is well defined. By the previous lemma, equation (2.3.6) holds for $m_1 = b/a$ and for both values of m_2,

$$m_2 = m_{x,y} = \frac{y-b}{x-a}, \quad \text{and} \quad m_2 = m_{z,w} = \frac{w-b}{z-a}.$$

For $(i, j) \in \{(x, y), (z, w)\}$ define

$$\mathbf{v}_{i,j} = \bigg(\frac{r(1+\sqrt{m_1})}{\sqrt{m_1}} - \frac{1+\sqrt{m_{i,j}}}{\sqrt{m_{i,j}}}, -r(1+\sqrt{m_1}) + (1+\sqrt{m_{i,j}})\bigg) = (v^{(1)}_{i,j}, v^{(2)}_{i,j}).$$

Vector $\mathbf{v}_{i,j}$ would be tangent to the level curve $g(x, y) = k$ at (a, b) and at such, $\mathbf{v}_{i,j} \neq 0$. The monotonicity of the level curve and the fact that (a, b) does not lie on one of the axes give that $v^{(1)}_{i,j} \cdot v^{(2)}_{i,j} \neq 0$. By planarity and (2.3.6), this and the last equation imply that there exists a $\kappa \in \mathbb{R} \setminus \{0\}$ so that $\mathbf{v}_{z,w} = \kappa \mathbf{v}_{x,y}$. The assumption that (x, y) and (z, w) are not collinear gives that $\kappa \neq \pm 1$. Assume without loss of generality that $m_{z,w} > m_{x,y}$. Then coordinate-wise,

$$v^{(1)}_{x,y} < v^{(1)}_{z,w}, \qquad v^{(2)}_{x,y} < v^{(2)}_{z,w}.$$

On the other hand, it has to be by equations (2.3.1) and (2.3.6) that the $v^{(1)}_{x,y}$ and $v^{(2)}_{x,y}$ have opposite signs, otherwise (2.3.6) would never be satisfied. Assuming $0 < v^{(1)}_{x,y}$, it has to be that $\kappa > 1$, but that would imply that $v^{(2)}_{x,y} > v^{(2)}_{z,w}$ which leads to a contradiction. Similarly, we reach a contradiction when $v^{(1)}_{x,y} < 0$. □

From Lemma 2.3.6 we know that from each crossing point except $(0, f(0))$ and $(a_0, 0)$ there is only one optimal slope that can be obtained. Remark 2.3.2 suggests that it is possible that a point could be reached by two maximal paths that both cross at the interior of f. Finally we discuss what happens when two maximal paths exists for a point (x, y), one from the axis and the other from a crossing point or both from the axes.

Proposition 2.3.7. *The following properties hold:*

1. *If a maximal path which crosses $(0, f(0))$ or $(a_0, 0)$ and a maximal path through any crossing point $(a, f(a))$ intersect, they intersect at their terminal point and that point has to belong on $\partial R_{0,f(0)}$.*

2. *If $(x, y) \in \operatorname{int}(R_{0,f(0)})$ and it also belongs on the extension of a maximiser $\mathbf{x}$ that crosses at $(a', f(a'))$, $a' \neq 0$, it has to be*

$$I(\pi_{0,(0,f(0))}) + I(\pi_{(0,f(0)),(x,y)}) > I(\pi_{0,(a',f(a'))}) + I(\pi_{(a',f(a')),(x,y)}),$$

where $\pi_{\mathbf{u},\mathbf{v}}$ is a linear segment between $\mathbf{u}$ and $\mathbf{v}$. In particular, any $(x, y) \in \operatorname{int}(R_{0,f(0)})$ has a unique maximiser that has to go through $(0, f(0))$.

3. *If $R_{0,f(0)} \cap R_{a_0,0} \neq \varnothing$ and $r > 1$, then the intersection is a segment of a (possibly degenerate) hyperbola.*

Proof of Proposition 2.3.7. We prove all the three properties one by one starting from the first.

(1) First, we show that also in this situation maximisers cannot cross. The contrary would be impossible. In fact, if it was possible to extend either maximiser, we would be able to construct a polygonal path which is not linear in a homogeneous environment, and this is not optimal with the same arguments as in Lemma 2.3.4.

$R_{0,f(0)}$ by definition is a closed, star-shaped domain. Moreover, since maximal paths cannot cross, $R_{0,f(0)}$ is simply connected. Suppose by way of contradiction that such a terminal point $(x_T, y_T) \in \operatorname{int}(R_{0,f(0)})$. Then the type C maximiser $\mathbf{x}_{0,(x_T,y_T)}$ intersects $\partial R_{0,f(0)}$ at some point (x_R, y_R). Since $R_{0,f(0)}$ is closed, (x_R, y_R) has a maximiser $\mathbf{x}_{0,(x_R,y_R)}$ that goes through $(0, f(0))$. By Lemma 2.3.4, (x_R, y_R) is also maximised by the portion of $\mathbf{x}_{0,(x_T,y_T)}$ that terminates at (x_R, y_R), and by the discussion above, (x_R, y_R) has to be a terminal point. This means that (x_T, y_T) cannot be optimised by that type C maximiser, which gives the desired contradiction.

(2) Same arguments as above imply the statement.

(3) This is a computation of the set of all points $(x, y) \in \mathbb{R}^2_+$ which take the same amount of time going through the x and y axis.

$$a_0 + \frac{1}{r}(\sqrt{x - a_0} + \sqrt{y})^2 = f(0) + \frac{1}{r}(\sqrt{x} + \sqrt{y - f(0)})^2$$
$$(a_0 - f(0))\frac{r-1}{2} = \sqrt{x(y - f(0))} - \sqrt{y(x - a_0)}.$$

Since $r > 1$, we have that $(r-1)/2 > 0$. Then, for the equality to hold, we must have $a_0 \geq f(0)$ and $y \geq xf(0)/a_0$ or $a_0 < f(0)$ and $y < xf(0)/a_0$. When either of these hold, we can square both sides and after some rearrangements we have

$$2\sqrt{xy(y - f(0))(x - a_0)} = 2xy - xf(0) - ya_0 - (a_0 - f(0))^2\frac{(r-1)^2}{4}.$$

This holds only if $y > \frac{xf(0) + (a_0 - f(0))^2 \frac{(r-1)^2}{4}}{2x - a_0}$ and it implies that both sides above are non-negative. Square both sides another time

$$0 = f(0)^2x^2 + a_0^2y^2 - 2xy\Big(a_0f(0) + (a_0 - f(0))^2\frac{(r-1)^2}{2}\Big)$$
$$+ (a_0 - f(0))^2\frac{(r-1)^2}{2}(f(0)x + a_0y) + (a_0 - f(0))^4\frac{(r-1)^4}{16},$$

which represent the equation of a hyperbola since $\big(a_0f(0) + (a_0 - f(0))^2\frac{(r-1)^2}{2}\big)^2 - a_0^2f(0)^2 > 0$. Note that if $a_0 = f(0)$, the relation that gives the boundary is $x = y$. $\square$

We have now verified that the set of crossing points is dense on the level curve (Lemma 2.3.3) and each one corresponds to a non-degenerate (Lemma 2.3.5) unique value m_2 (Lemma 2.3.6) which in turn corresponds to the slope of the second linear segment of the maximiser. Starting from equation (2.3.5), we can identify m_2.

Set

$$D = D(a, b) = r\Big(1 + \frac{\sqrt{b}}{\sqrt{a}}\Big)\Big(\frac{\sqrt{a}}{\sqrt{b}} - \frac{\partial_b g}{\partial_a g}\Big) = r\Big(1 + \sqrt{m_1}\Big)\Big(\sqrt{\frac{1}{m_1}} - \frac{\partial_b g}{\partial_a g}\Big).$$

The left-hand side in (2.3.5) becomes $\partial_a g(a, b)D$. Keep in mind that $m_2 > 0$ and solve (2.3.5) for m_2:

$$m_2 = \frac{4}{\Big(\frac{\partial_b g}{\partial_a g} - 1 + D + \sqrt{\Big(\frac{\partial_b g}{\partial_a g} - 1 + D\Big)^2 + 4\frac{\partial_b g}{\partial_a g}}\Big)^2}. \tag{2.3.8}$$

Particularly, equation (2.3.8) uniquely identifies the slope of the second segment of the optimal path for a given crossing point (a, b). Rewrite equation (2.3.8) using the fact that when $b = f(a)$, $\frac{\partial_b g}{\partial_a g}(a, f(a)) = -1/f'(a)$ to obtain equations (2.1.10) and (2.1.11).

2.3.1 Maximisers that follow the axes

We investigate whether the optimization problem (2.3.2) in the region $g(x, y) > k$ admits maximisers $(a_0, 0),\ (0, b_0)$, i.e. maximisers for which the first segment of the macroscopic maximal path follows the axes.

For $(x, y) \in [0, a_0) \times [0, f(0)) = B$ the maximal macroscopic path is obtained by the solution of (2.3.2), and it is impossible for a maximiser to follow one of the axes. For this behaviour to materialise, we consider an (x, y) outside of $[0, a_0) \times [0, f(0))$.

We are finally able to study what happens to m_2 defined in (2.3.8) if a tends to the boundary values. The idea is that if m_2 for crossing points near the y-axis (resp. x-axis) does not approach $+\infty$ (resp. 0) then it has to be that type B maximisers exist.

The behaviour of m_2 for a near 0 (resp. a_0) is the content of Proposition 2.1.12, which we prove next.

Proof of Proposition 2.1.12. We use equation (2.1.10) for the slope $m_2(a)$ and (2.1.11) for the expression $D = D_a$. We only show the case for which $a \to 0$ and leave $a \to a_0$ to the reader. Keep in mind that as $a \to 0$, $f(a)/a \to \infty$.

First we estimate the limiting behaviour of D using (2.1.11)

$$
\begin{aligned}
D_0 = \lim_{a\to 0} D &= \lim_{a\to 0} r\Big(1 + \frac{1}{f'(a)} + \frac{1}{f'(a)}\sqrt{\frac{f(a)}{a}} + \sqrt{\frac{a}{f(a)}}\Big) \\
&= r + \frac{r\sqrt{f(0)}}{\lim\limits_{a\to 0} f'(a)a^{1/2}} = \begin{cases} r, & \alpha > \frac{1}{2}, \\ r\Big(1 - \frac{\sqrt{f(0)}}{c^{(-)}_{1/2}}\Big), & \alpha = \frac{1}{2}, \\ -\infty, & \alpha < \frac{1}{2}. \end{cases}
\end{aligned}
\tag{2.3.9}
$$

(1) **Case 1: $a \to 0,\ f'(a) \to -\infty$:** Focus on the denominator in (2.1.10)

$$
\begin{aligned}
\overline{\lim_{a\to 0}} m_2(a) &= 4\,\overline{\lim_{a\to 0}}\Bigg(-\frac{1}{f'(a)} - 1 + D + \sqrt{\Big(-\frac{1}{f'(a)} - 1 + D\Big)^2 - 4\frac{1}{f'(a)}}\Bigg)^{-2} \\
&= 4\,\overline{\lim_{a\to 0}}\Bigg(-\frac{1}{f'(a)} - 1 + D + \Big|-\frac{1}{f'(a)} - 1 + D\Big| + O(\frac{1}{f'(a)})\Bigg)^{-2} \\
&= 4\,\overline{\lim_{a\to 0}}\Bigg(\Big(-\frac{1}{f'(a)} - 1 + D\Big)\Big(1 + \mathrm{sign}\Big(-\frac{1}{f'(a)} - 1 + D\Big)\Big) + O(\frac{1}{f'(a)})\Bigg)^{-2}.
\end{aligned}
\tag{2.3.10}
$$

Focus for the moment on the sign function in the last display. We have

$$
-\frac{1}{f'(a)} - 1 + D = (r-1) + \frac{r-1}{f'(a)} + r\sqrt{\frac{a}{f(a)}} + r\frac{1}{f'(a)}\sqrt{\frac{f(a)}{a}}.
$$

As $a \to 0$, the second and third term tend to 0 while the last term is negative and as $a \to 0$ the $\liminf$ of the last term is actually $D_0 - r$. Therefore, for a sufficiently small

$$\operatorname{sign}\Big(-\frac{1}{f'(a)} - 1 + D\Big) = \begin{cases} \operatorname{sign}(r-1), & \alpha > \frac{1}{2}, \\ -1, & \alpha < \frac{1}{2}. \end{cases} \tag{2.3.11}$$

We are now in a position to finish the calculation from equation (2.3.10):

(a) $r > 1, \alpha > 1/2$: From equation (2.3.11) substitute it in equation (2.3.10) to obtain

$$\overline{\lim_{a\to 0}}\, m_2(a) = \frac{1}{(r-1)^2}.$$

(b) $r < 1, \alpha > 1/2$, or $r \neq 1, \alpha < 1/2$: From equations (2.3.11), (2.3.10) we now have

$$\overline{\lim_{a\to 0}}\, m_2(a) = +\infty.$$

(c) When $\alpha = 1/2$, there are several cases to consider:

(i) $r < 1$, then $\operatorname{sign}\Big(-\frac{1}{f'(a)} - 1 + D\Big) = -1$ which implies $\overline{\lim}_{a\to 0}\, m_2(a) = +\infty$.

(ii) $r > 1$ and $c^{(+)}_{1/2} < \frac{r\sqrt{f(0)}}{r-1}$, then $\operatorname{sign}\Big(-\frac{1}{f'(a)} - 1 + D\Big) = -1$. In this case, $\overline{\lim}_{a\to 0}\, m_2(a) = +\infty$.

(iii) $r > 1$ and $c^{(-)}_{1/2} > \frac{r\sqrt{f(0)}}{r-1}$, then $\operatorname{sign}\Big(-\frac{1}{f'(a)} - 1 + D\Big) = +1$. This is the most interesting case, as it leads to yet a different possible limit. For the condition to hold it has to be that

$$c^{(-)}_{1/2} > \sqrt{f(0)} \text{ and that } r > \frac{c^{(-)}_{1/2}}{c^{(-)}_{1/2} - \sqrt{f(0)}} > 1.$$

When both these conditions are met, we have that

$$\overline{\lim_{a\to 0}}\, m_2(a) = \frac{1}{\Big(r - 1 - \frac{r\sqrt{f(0)}}{c^{(-)}_{1/2}}\Big)^2}.$$

(iv) $r > 1$ and $c^{(-)}_{1/2} < \frac{r\sqrt{f(0)}}{r-1} \leq c^{(+)}_{1/2}$, then we can find a subsequence a_k such that the $\operatorname{sign}\Big(-\frac{1}{f'(a_k)} - 1 + D\Big) = -1$ and so that $-\frac{1}{f'(a_k)} - 1 + D \to r - 1 - \frac{r\sqrt{f(0)}}{c^{(-)}_{1/2}}$. Again, $\overline{\lim}_{a\to 0}\, m_2(a) = +\infty$.

(v) $r > 1$ and $c^{(-)}_{1/2} = \frac{r\sqrt{f(0)}}{r-1}$, we cannot determine the sign function, however, we can find a subsequence a_k so that $\underline{\lim}_{a_k\to 0}(-\frac{1}{f'(a_k)} - 1 + D) = 0$ so also here $\overline{\lim}_{a\to 0}\, m_2(a) = +\infty$.

(2) **Case 2:** $a \to 0$, $f'(a) \to -c$**:** In this case, $D \to -\infty$ as $a \to 0$ so the result follows by a direct limiting argument on (2.1.10). $\square$

A close inspection of the previous proof suggests the following crucial lemma.

Lemma 2.3.8. *Suppose that* $\overline{\lim}_{a\to 0} m_2(a) = +\infty$*, and that if* $\alpha = 1/2$ *then* $r \notin \left[\frac{c^{(+)}_{1/2}}{c^{(+)}_{1/2}-\sqrt{f(0)}}, \frac{c^{(-)}_{1/2}}{c^{(-)}_{1/2}-\sqrt{f(0)}}\right]$*. Then there exists a sequence* $\{a_k\}_{k\in\mathbb{N}}$ *with distinct elements so that*

1. $\lim_{k\to\infty} a_k = 0$,
2. *Points* $(a_k, f(a_k))$ *are all crossing points,*
3. $\lim_{k\to\infty} m_2(a_k) = +\infty$.

Proof of Lemma 2.3.8. The lemma is immediately true if $r = 1$ and the environment is homogeneous.

Now assume $r \neq 1$. From Proposition 2.1.12, we know that $\overline{\lim}_{a\to 0} m_2(a) = +\infty$ when

1. $\alpha < 1/2$,
2. $\alpha > 1/2$ and $r < 1$,
3. $\alpha = 1/2$ and $r \in \left(1, \frac{c^{(-)}_{1/2}}{c^{(-)}_{1/2}-\sqrt{f(0)}}\right]$ where the interval may be potentially empty, in which case we are not concerned with this case.

These correspond to cases 1b, 1 c(i), 1c(ii), 1c(iv), 1c(v) and 2, in the proof of Proposition 2.1.12.

The assumption of the Lemma guarantees we are not in cases 1c(iv), 1c(v); For these cases $c^{(-)}_{1/2} \leq \frac{r\sqrt{f(0)}}{r-1} \leq c^{(+)}_{1/2}$ which is equivalent to

$$r \in \left[\frac{c^{(+)}_{1/2}}{c^{(+)}_{1/2} - \sqrt{f(0)}}, \frac{c^{(-)}_{1/2}}{c^{(-)}_{1/2} - \sqrt{f(0)}}\right].$$

In cases 1b, 1c(i), 1c(ii) and 2, the fact that $\overline{\lim}_{a\to 0} m_2(a) = +\infty$ is independent of which sequence of a_k we select, as long as it tends to 0. Therefore we can select a_k to be sequence that corresponds to the first coordinate of crossing points and which tends to 0, since by Lemma 2.3.3 we know they are dense on f. $\square$

Proof of Theorem 2.1.9. We only prove the theorem for $a \to 0$, as the case $a \to a_0$ is analogous.

The direction (2) $\implies$ (1) is immediate; the condition implies that all points $(x, y) \in \operatorname{int}(\mathbb{R}_+^2)$ are optimised by a type C maximiser, and by letting $x \to 0$ while keeping $y > f(0)$ fixed, the crossing points $(a_{x,y}, f(a_{x,y}))$ tend to $(0, f(0))$. This forces $m_2(a_{x,y})$ to $+\infty$.

Now for (1) $\implies$ (2). Assume that $\overline{\lim}_{a\to 0} m_2(a) = +\infty$ and assume by way of contradiction that $\operatorname{int}(R_{0,f(0)}) \neq \varnothing$.

Then we can find a sequence of points $(x_k, y_k) \in \mathbb{R}_+^2 \setminus R_{0,f(0)}$ with $(x_k, y_k) \to (0, f(0))$ so that

1. For each k, the crossing points $\{(a_k, f(a_k))\}_k$ of a maximiser that does not follow the axis are different; this is possible because the crossing points are dense on the curve.

2. The limit $\lim_{k\to\infty} m_2(a_k) = +\infty$.

This can be done by Lemma 2.3.8.

Now, by Proposition 2.3.7-(2), we have that for any point $(x, y) \in \operatorname{int}(R_{0,f(0)})$ on the line segment $\ell_k : (a_k, f(a_k)) - (x_k, y_k) - (x, y)$ the limiting passage time satisfies

$$I(\pi_{0,(0,f(0))}) + I(\pi_{(0,f(0)),(x,y)}) > I(\pi_{0,(a_k,f(a_k))}) + I(\pi_{(a_k,f(a_k)),(x,y)}).$$

For notational convenience set $\varepsilon = a_k$ and notice that the relation above stays true when we let (x, y) tend to infinity, *along the line which contains the segment* ℓ_k. We substitute the explicit values for $I(\pi)$ in the display above to obtain

$$f(0) + \frac{1}{r}(\sqrt{x} + \sqrt{y - f(0)})^2 > (\sqrt{\varepsilon} + \sqrt{f(\varepsilon)})^2 + \frac{1}{r}(\sqrt{x-\varepsilon} + \sqrt{y - f(\varepsilon)})^2. \tag{2.3.12}$$

Call $m_1(\varepsilon) = \frac{f(\varepsilon)}{\varepsilon}$, $m_2(\varepsilon) = \frac{y-f(\varepsilon)}{x-\varepsilon}$ and $m = \frac{y-f(0)}{x}$ and note that $m_2(\varepsilon) > m$. Both slopes are always finite for every $(x, y) \in (0, a_0) \times \mathbb{R}_+$. Inequality (2.3.12) is then re-written as

$$\frac{1}{r}\Big[x\Big(1+\sqrt{m}\Big)^2 - x\Big(1+\sqrt{m_2(\varepsilon)}\Big)^2\Big] > \varepsilon + f(\varepsilon) - f(0) + 2\sqrt{\varepsilon f(\varepsilon)} - \frac{\varepsilon}{r}(1+\sqrt{m_2(\varepsilon)})^2. \tag{2.3.13}$$

Since the point (x, y) belongs to the line $y = m_2(\varepsilon)(x - \varepsilon) + f(\varepsilon)$, taking $x \to \infty$ gives $m \to m_2(\varepsilon)$. We first manipulate the left-hand side of (2.3.13).

$$\begin{aligned} x\Big[\Big(1+\sqrt{m}\Big)^2 - \Big(1 + \sqrt{m_2(\varepsilon)}\Big)^2\Big] &= x\big[2(\sqrt{m} - \sqrt{m_2(\varepsilon)}) + m - m_2(\varepsilon)\big] \\ &= \frac{x(f(\varepsilon) - f(0) - \varepsilon m_2(\varepsilon)) + \varepsilon(\varepsilon m_2(\varepsilon) - f(\varepsilon) + f(0))}{x - \varepsilon}\Big[1 + \frac{2}{\sqrt{m} + \sqrt{m_2(\varepsilon)}}\Big]. \end{aligned}$$

Now take the limit $x \to \infty$ in (2.3.13). After that, and some algebraic operations, we get that the limiting version of (2.3.13) is

$$\frac{1}{r}\Big(\frac{1}{\sqrt{m_2(\varepsilon)}}+1-r\Big)\frac{f(\varepsilon)-f(0)}{\varepsilon} \geq 1-\frac{1}{r}+2\sqrt{m_1(\varepsilon)}-\frac{\sqrt{m_2(\varepsilon)}}{r}$$
$$=\sqrt{m_1(\varepsilon)}+\Big(\frac{r-1}{r}+\sqrt{m_1(\varepsilon)}-\frac{\sqrt{m_2(\varepsilon)}}{r}\Big). \quad (2.3.14)$$

This is the point where we are using the fact that $(\varepsilon, f(\varepsilon))$ is a crossing point: Utilize the relation of equation (2.3.7) to change the last parenthesis in (2.3.14) and obtain the equivalent inequality

$$\frac{1}{r}\Big(\frac{1}{\sqrt{m_2(\varepsilon)}}+1-r\Big)\frac{f(\varepsilon)-f(0)}{\varepsilon} \geq \sqrt{m_1(\varepsilon)}-\frac{f'(\varepsilon)}{r}\Big(r-1+\frac{r}{\sqrt{m_1(\varepsilon)}}-\frac{1}{\sqrt{m_2(\varepsilon)}}\Big),$$

or equivalently

$$\frac{1}{r}\Big(\frac{1}{\sqrt{m_2(\varepsilon)}}+1-r\Big)\Big(\frac{f(\varepsilon)-f(0)}{\varepsilon}-f'(\varepsilon)\Big) \geq \sqrt{m_1(\varepsilon)}-\frac{f'(\varepsilon)}{\sqrt{m_1(\varepsilon)}}. \quad (2.3.15)$$

Now, if equation (2.3.15) is violated, we automatically reach a contradiction to the assumption that $\operatorname{int}(R_{0,f(0)}) \neq \varnothing$. We will show precisely this by splitting the analysis into cases:

(1) $\lim_{\varepsilon\to 0} f'(\varepsilon) = c_0$: Then as $\varepsilon \to 0$, the left-hand side of (2.3.15) converges to 0 while the right-hand side tends to ∞. This gives the desired contradiction.

(2) $r < 1$: In this case, select an ε small enough so that $\frac{1}{\sqrt{m_2(\varepsilon)}}+1-r > 0$. The convexity and monotonicity of f imply that $\frac{f(\varepsilon)-f(0)}{\varepsilon}-f'(\varepsilon) < 0$ so the left-hand side of (2.3.15) is negative while the right-hand is strictly positive. This gives again a contradiction.

(3) $r > 1, \alpha < 1/2$: Since $\alpha < 1/2$, we have that for δ small, $\alpha+\delta < 1/2$. Then, using definition (2.1.12), for any η small, we can find ε_0 so that for all $\varepsilon < \varepsilon_0$

$$-f'(\varepsilon) < \frac{\eta}{\varepsilon^{\alpha+\delta}}.$$

Integrating the inequality from 0 to ε we get

$$f(0)-f(\varepsilon) < \frac{\eta}{1-\alpha-\delta}\varepsilon^{1-\alpha-\delta} < c\sqrt{\varepsilon}.$$

The last inequality is true for any constant c, as long as ε is small enough. We pick $c < \sqrt{\frac{f(0)}{2}}$ and reduce ε further so that $f(\varepsilon) > \frac{f(0)}{2}$. We then have for all ε small that

$$\frac{f(0)-f(\varepsilon)}{\varepsilon} < \sqrt{\frac{f(\varepsilon)}{\varepsilon}} = \sqrt{m_1(\varepsilon)}.$$

Reduce ε even more, so that $1/\sqrt{m_2(\varepsilon)} < \frac{r-1}{2}$. Then we bound

$$\frac{1}{r}\Big(\frac{1}{\sqrt{m_2(\varepsilon)}}+1-r\Big)\Big(\frac{f(\varepsilon)-f(0)}{\varepsilon}-f'(\varepsilon)\Big)$$

$$
\begin{aligned}
&= \frac{1}{r}\Big(-\frac{1}{\sqrt{m_2(\varepsilon)}} - 1 + r\Big)\Big(\frac{f(0)-f(\varepsilon)}{\varepsilon} + f'(\varepsilon)\Big) \\
&< \frac{1}{r}\Big(r - 1 - \frac{1}{\sqrt{m_2(\varepsilon)}}\Big)\Big(\frac{f(0)-f(\varepsilon)}{\varepsilon} - \frac{f'(\varepsilon)}{\sqrt{m_1(\varepsilon)}}\Big) \\
&< \frac{r-1}{r}\Big(\sqrt{m_1(\varepsilon)} - \frac{f'(\varepsilon)}{\sqrt{m_1(\varepsilon)}}\Big) \\
&< \sqrt{m_1(\varepsilon)} - \frac{f'(\varepsilon)}{\sqrt{m_1(\varepsilon)}},
\end{aligned}
$$

which is a direct violation of (2.3.15).

The remaining proof is for when $\alpha = 1/2$. In this case we have that $\overline{\lim}\, m_2(a_k) \to \infty$ for any sequence $a_k \to 0$ and $r \notin \Big[\frac{c_{1/2}^{(+)}}{c_{1/2}^{(+)} - \sqrt{f(0)}}, \frac{c_{1/2}^{(-)}}{c_{1/2}^{(-)} - \sqrt{f(0)}}\Big]$.

(4) We further impose on the subsequence of a_k that

$a_k^{1/2}|f'(a_k)| \to c_{1/2} \le c_{1/2}^{(+)} < \frac{r}{r-1}\sqrt{f(0)}$ by the assumption. Here $c_{1/2}$ can be any limit point.

For any $\delta > 0$ we can find a $K = K(\delta)$ so that for all $k > K$ we have

$$
\frac{r-1}{r}(c_{1/2} + 3\delta) < \sqrt{f(0)} - \delta < \sqrt{f(a_k)}, \quad |a_k^{1/2} f'(a_k) + c_{1/2}| < \delta.
$$

The first inequality above is true for δ sufficiently small. Then we estimate, as in case (3), that

$$
-f'(a_k) < (c_{1/2} + \delta) a_k^{-1/2}, \quad \text{for all } k > K \text{ by construction}
$$

which implies that

$$
\frac{f(0) - f(a_k)}{a_k} < 2(c_{1/2} + \delta) a_k^{-1/2}.
$$

Then use the inequalities above to bound

$$
\begin{aligned}
&\frac{1}{r}\Big(-\frac{1}{\sqrt{m_2(a_k)}} - 1 + r\Big)\Big(\frac{f(0)-f(a_k)}{a_k} + f'(a_k)\Big) \\
&\qquad < \frac{1}{r}\Big(r - 1 - \frac{1}{\sqrt{m_2(a_k)}}\Big)\Big(2(c_{1/2}+\delta)a_k^{-1/2} + f'(a_k)\Big) \\
&\qquad < \frac{1}{r}\Big(r - 1 - \frac{1}{\sqrt{m_2(a_k)}}\Big)\frac{\big(c_{1/2} + 3\delta\big)}{a_k^{1/2}} \\
&\qquad < \frac{1}{r}\Big(r - 1 - \frac{1}{\sqrt{m_2(a_k)}}\Big)\frac{r}{r-1}\frac{\sqrt{f(a_k)}}{a_k^{1/2}} \\
&\qquad < \sqrt{m_1(a_k)} - \frac{f'(a_k)}{\sqrt{m_1(a_k)}},
\end{aligned}
$$

which also contradicts (2.3.15). The last inequality follows immediately from the fact that $f' < 0$. $\square$

Proof of Theorem 2.1.11. The proof is identical to that of case (4) in the proof of Theorem 2.1.9. The reason we cannot apply the argument directly is the fact that we do not know a priori that the $\overline{\lim}_{a_k \to 0} m_2(a_k) = 0$ on a sequence of *crossing points*, since Lemma 2.3.8 does not apply here. This condition is now taken care by the assumption of Theorem 2.1.11.

To finish the proof, impose on this sequence $\{a_k\}_{k\in\mathbb{N}}$ of crossing points the extra condition that $a_k^{1/2}|f'(a_k)| \to c_{1/2} < \frac{r}{r-1}\sqrt{f(0)}$ by the assumption. Again, $c_{1/2}$ can be any limit point. Now the calculation for (4) in the proof of Theorem 2.1.9 can be repeated and it finishes the proof. $\square$

2.3.2 Phase transition at $c_{1/2}^{(-)} = \frac{r}{r-1}\sqrt{f(0)}$

Proposition 2.3.9 (Phase transition at $c_{1/2}^{(-)} = \frac{r}{r-1}\sqrt{f(0)}$). *Suppose that $c_{1/2}^{(-)} = \frac{r}{r-1}\sqrt{f(0)}$ and assume that for some $\gamma > 0$ and some $c \in \mathbb{R}$,*

$$-f'(a) = c_{1/2}^{(-)} a^{-1/2} + c a^{\gamma - 1/2}. \tag{2.3.16}$$

Then, when $\gamma < 1/4$ the equivalence of Theorem 2.1.11 is false when $c < 0$ and true when $c > 0$. When $\gamma > 1/4$, type B maximisers exist.

We first need a geometric lemma:

Lemma 2.3.10. *Assume that $R_{0,f(0)} = \{0\} \times [f(0), \infty)$ and $R_{a_0,0} = [a_0, \infty) \times \{0\}$ (i.e. they are both degenerate). Then, there exists a sequence of points (x_k, y_k) with $x_k \to \infty$ as $k \to \infty$, so that their corresponding crossing points $(\beta_k, f(\beta_k)) \to (0, f(0))$.*

Proof of Lemma 2.3.10. Suppose by way of contradiction that there exists a constant $A > 0$ so that for all $(x, y) \in \mathbb{R}^2_+$ with $x > A$, the crossing points $(a_{x,y}, f(a_{x,y}))$ satisfy $a_{x,y} > \alpha_A > 0$.

Fix an $\alpha > 0$ small and define

$$x_+ = x_+(\alpha) = \sup\{x : \exists\, y \text{ so that the crossing point } (a_{x,y}, f(a_{x,y})) \text{ satisfies } a_{x,y} \le \alpha\}. \tag{2.3.17}$$

The assumption guarantees that $x_+(\alpha)$ is bounded for α small enough, and the set for which we take the supremum is not empty, since crossing points are dense on the graph of f by Lemma 2.3.3.

For any $\delta > 0$ define the terminal point $(x_\delta, y_\delta) = (x_+ - \delta, y_\delta)$ to be such that its crossing point satisfies $a_{x_\delta, y_\delta} \le \alpha$. Then it has to be that for all points $(x_+ - \delta, y)$ with $y > y_\delta$ their corresponding crossing points has to satisfy $a_{x_\delta, y} \le \alpha$. If this is not true, then the maximal path for $(x_+ - \delta, y)$ would cross the one for (x_δ, y_δ) and this is impossible by Lemma 2.3.4.

Now there are three cases to consider:

(1) $x_+ > a_0$: In this case, consider now a point $(x_+ + \varepsilon, y_0)$, for some small $\varepsilon > 0$. Because of its x-coordinate, this point must have a crossing point with first coordinate larger than α. The maximal possible slope for its second segment is $m_{\max} = \frac{y_0}{x_+ + \varepsilon - a_0}$. Now notice that for y_0 large enough, the line $y = m_{\max}(x - a_0)$ must intersect the optimal path from 0 to (x_δ, y_δ) by planarity. In particular, the maximal paths to (x_δ, y_δ) and $(x_+ + \varepsilon, y_0)$ must intersect in the r-region, and this violates Lemma 2.3.4.

(2) $x_+ = a_0$: The same arguments as in case (1) give that the only possible crossing point for $(x_+ + \varepsilon, y_0)$ when y_0 is large enough is $(a_0, 0)$ otherwise maximal paths would intersect. This contradicts the assumption that $R_{a_0,0} = [a_0, \infty) \times \{0\}$.

(3) $x_+ < a_0$: This is the most challenging case, and we need to split it into yet two more cases.

   (a) $x_+(\alpha)$ is a maximum. Assume that $(x_+(\alpha), y_+(\alpha))$ is point with the crossing point of its maximiser less than α. Now, for any $\delta, \varepsilon > 0$, we can find $y_1 > y_+(\alpha)$ so that the point $(x_+ + \varepsilon, y_1)$ has crossing point $a_{x_+ + \varepsilon} \geq x_+ - \delta$. This is because maximal macroscopic paths cannot cross, and any point $(x_+ + \varepsilon, y_1)$ has to have a maximiser with crossing point with $a_{x_+ + \varepsilon} > \alpha$. Suppose by way of contradiction that the crossing point $a_{x_+ + \varepsilon, y_1} \leq x_+$. Keeping $\varepsilon > 0$ but raising the value of y_1, we can find a crossing point larger than $a_{x_+ + \varepsilon, y_1}$. But that would mean that maximisers cross, which cannot happen. Therefore, the crossing point $a_{x_+ + \varepsilon, y_1} > x_+$. This has to be true for all values of y_1, and it is true for all $\varepsilon > 0$.

   Now we want to understand the behaviour of the maximal paths when $\varepsilon \to 0$ as y_1 remains fixed. For each point $(x_+ + \varepsilon, y_0)$ let $(a_\varepsilon, f(a_\varepsilon))$ the corresponding crossing point. For all ε, $a_\varepsilon > x_+$ and since maximal paths cannot cross each other, $\lim_{\varepsilon \to 0} a_\varepsilon = x_+$: Then, as $\varepsilon \to 0$ and by continuity of Γ (Theorem 2.1.4), $\Gamma(x_+, y_0)$ must also be optimised by the path $0 \to (x_+, f(x_+)) \to (x_+, y_0)$. By Lemma 2.3.5 this is impossible.

   (b) $x_+(\alpha)$ is a supremum but not a maximum. Then consider terminal points of the form (x_+, y), and their crossing points $(a_{x_+,y}, f(a_{x_+,y}))$. Notice that for all y large enough we must have

$$a_{x_+,y} \in (x_+ - \delta, x_+).$$

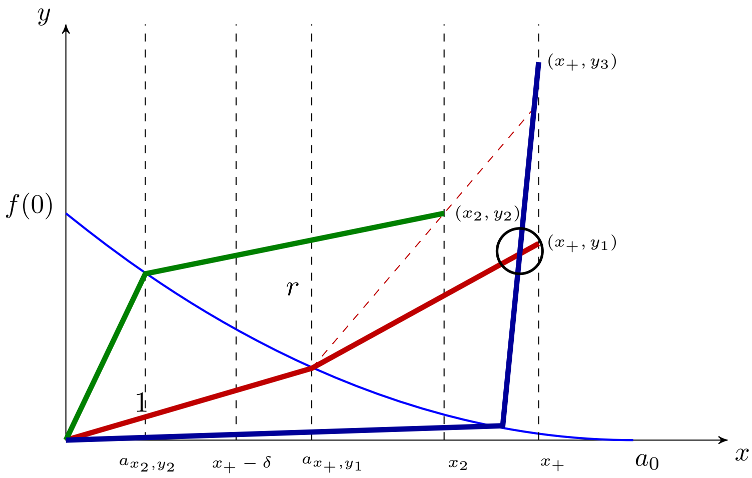


Figure 2.5: Construction in the proof of Lemma 2.3.10, part 3(b).

Set that y as y_1. Now, for all $y > y_1$, we have that $a_{x_+,y} \in (x_+ - \delta, a_{x_+,y_1})$. This is because the maximal paths cannot cross, by Lemma 2.3.4. Now consider a terminal point (x_2, y_2) so that $a_{x_+,y_1} < x_2 < x_+$, $y_2 > y_1$ and $a_{x_2,y_2} \leq \alpha$. Finally, find a $y_3 > y_2$ so that (x_+, y_3) has a crossing point with $a_{x_+,y_3} \geq x_2$. But this implies that

$$a_{x_+,y_1} < a_{x_+,y_3}, \text{ while } y_3 > y_1,$$

and in particular it means maximal paths cross. This cannot happen, so we reached a contradiction. $\square$

Proof of Proposition 2.3.9. When f' satisfies (2.3.16), we have that $c^{(-)}_{1/2} = c^{(+)}_{1/2}$. This implies that for any sequence $a_k \to 0$ we will simultaneously have

$$a_k \to 0, \quad a_k^{1/2}|f'(a_k)| \to c^{(-)}_{1/2} \text{ and } m_2(a_k) \to \infty.$$

In particular this will be true on a sequence a_k coming from crossing points.

Fix any such sequence. Integrate both sides of (2.3.16) and divide by a_k to obtain

$$\frac{f(0) - f(a_k)}{a_k} = 2c^{(-)}_{1/2} a_k^{-1/2} + \frac{c}{\gamma + 1/2} a_k^{\gamma - 1/2}. \tag{2.3.18}$$

Moreover, we have that

$$(\sqrt{f(0)} - \sqrt{f(a_k)})(\sqrt{f(0)} + \sqrt{f(a_k)}) = 2c^{(-)}_{1/2} a_k^{1/2} + \frac{c}{\gamma + 1/2} a_k^{\gamma + 1/2}.$$

Equation (2.3.16) also implies that $-a_k^{1/2} f'(a_k) = c^{(-)}_{\alpha} + c a_k^{\gamma}$. Now we are in position to estimate

$$\frac{1}{r}\Big(- \frac{1}{\sqrt{m_2(a_k)}} - 1 + r\Big)\Big(\frac{f(0) - f(a_k)}{a_k} + f'(a_k)\Big)$$

$$
\begin{aligned}
&= \frac{1}{r}\Big(r - 1 - \frac{1}{\sqrt{m_2(a_k)}}\Big)\Big(2c^{(-)}_{1/2}a_k^{-1/2} + \frac{c}{\gamma + 1/2}a_k^{\gamma - 1/2} + f'(a_k)\Big) \\
&= \frac{1}{r}\Big(r - 1 - \frac{1}{\sqrt{m_2(a_k)}}\Big)\frac{\Big(2c^{(-)}_{1/2} + \frac{c}{\gamma+1/2}a_k^{\gamma} + a_k^{1/2}f'(a_k)\Big)}{a_k^{1/2}} \\
&= \frac{1}{r}\Big(r - 1 - \frac{1}{\sqrt{m_2(a_k)}}\Big)\frac{\Big(2c^{(-)}_{1/2} + \frac{c}{\gamma+1/2}a_k^{\gamma} - c^{(-)}_{1/2} - ca_k^{\gamma}\Big)}{a_k^{1/2}} \\
&= \frac{r-1}{r}\frac{\Big(c^{(-)}_{1/2} + c\frac{1/2-\gamma}{1/2+\gamma}a_k^{\gamma}\Big)}{a_k^{1/2}} - \frac{1}{r}\frac{1}{\sqrt{m_2(a_k)}}\frac{\Big(c^{(-)}_{1/2} + c\frac{1/2-\gamma}{1/2+\gamma}a_k^{\gamma}\Big)}{a_k^{1/2}}.
\end{aligned}
$$

In the last line there are two competing terms; one is asymptotically positive and the other asymptotically negative so we must treat them separately: First the higher order positive term

$$
\begin{aligned}
\frac{r-1}{r}\frac{\Big(c^{(-)}_{\alpha} + c\frac{1/2-\gamma}{1/2+\gamma}a_k^{\gamma}\Big)}{a_k^{1/2}} &= \frac{\sqrt{f(0)}}{a_k^{1/2}} + \frac{r-1}{r}c\frac{1/2-\gamma}{1/2+\gamma}\frac{a_k^{\gamma}}{a_k^{1/2}} \\
&= \frac{\sqrt{f(a_k)}}{a_k^{1/2}} + \frac{\sqrt{f(0)} - \sqrt{f(a_k)}}{a_k^{1/2}} + c\frac{r-1}{r}\frac{1/2-\gamma}{1/2+\gamma}\frac{a_k^{\gamma}}{a_k^{1/2}} \\
&= \frac{\sqrt{f(a_k)}}{a_k^{1/2}} + \frac{2c^{(-)}_{1/2} + \frac{c}{\gamma+1/2}a_k^{\gamma}}{\sqrt{f(0)} + \sqrt{f(a_k)}} + c\frac{r-1}{r}\frac{1/2-\gamma}{1/2+\gamma}\frac{a_k^{\gamma}}{a_k^{1/2}}.
\end{aligned}
$$

Note that the term in the middle above vanishes as $a_k \to 0$. Then we work with the negative term. First we perform an asymptotic expansion on $1/\sqrt{m_2(a)}$ as a tends to 0:

$$
\frac{1}{\sqrt{m_2(a)}} = \begin{cases} \frac{1}{|c|(r-1)}a^{1/2-\gamma} + O(a^{1/2}), & \gamma \in (0, 1/2), \\ \frac{a^{1/4}}{\sqrt{c^{(-)}_{1/2}}} + O(a^{1/2}), & \gamma \in [1/2, \infty). \end{cases} \tag{2.3.19}
$$

The details for (2.3.19) can be found in the Appendix A. Using this expansion we obtain

$$
\begin{aligned}
&\frac{1}{r}\frac{1}{\sqrt{m_2(a_k)}}\frac{\Big(c^{(-)}_{1/2} + c\frac{1/2-\gamma}{1/2+\gamma}a_k^{\gamma}\Big)}{a_k^{1/2}} \\
&\qquad = \frac{1}{r}\frac{\Big(c^{(-)}_{1/2} + c\frac{1/2-\gamma}{1/2+\gamma}a_k^{\gamma}\Big)}{a_k^{1/2}} \times \begin{cases} \frac{1}{|c|(r-1)}a_k^{1/2-\gamma} + O(a_k^{1/2}), & \gamma \in (0, 1/2), \\ \frac{a_k^{1/4}}{\sqrt{c^{(-)}_{1/2}}} + O(a_k^{1/2}), & \gamma \in [1/2, \infty). \end{cases} \\
&\qquad = \begin{cases} \frac{c^{(-)}_{1/2}}{|c|r(r-1)}a_k^{-\gamma} + O(1), & \gamma \in (0, 1/2), \\ \frac{\sqrt{c^{(-)}_{1/2}}}{ra_k^{1/4}} + O(1), & \gamma \in [1/2, \infty). \end{cases}
\end{aligned}
$$

Combining the two expansions we have

$$
\frac{1}{r}\Big(-\frac{1}{\sqrt{m_2(a_k)}} - 1 + r\Big)\Big(\frac{f(0) - f(a_k)}{a_k} + f'(a_k)\Big)
$$

$$= \frac{\sqrt{f(a_k)}}{a_k^{1/2}} + c\frac{r-1}{r}\frac{1/2-\gamma}{1/2+\gamma}a_k^{\gamma-1/2} - \begin{cases} \frac{c_{1/2}^{(-)}}{|c|r(r-1)}a_k^{-\gamma} + O(1), & \gamma \in (0, 1/2), \\ \frac{\sqrt{c_{1/2}^{(-)}}}{ra_k^{1/4}} + O(1), & \gamma \in [1/2, \infty). \end{cases} \tag{2.3.20}$$

Now the phase transition reveals itself. First when $\gamma > 1/4$, the leading order terms in (2.3.20) are those in the brace; they are negative and tend to $-\infty$, so as before, (2.3.15) is violated.

Now assume $1/4 \geq \gamma$. This means $1/2 - \gamma \geq \gamma$. Then, If $c > 0$, the middle term in (2.3.20) tends to $+\infty$, and immediately gives a contradiction to (2.3.15).

If $c < 0$ with a sufficiently large modulus (if $\gamma < 1/4$ any $c < 0$ will do), we have for all a_k sufficiently small that (2.3.20) can be bounded by

$$\frac{1}{r}\Big(-\frac{1}{\sqrt{m_2(a_k)}} - 1 + r\Big)\Big(\frac{f(0)-f(a_k)}{a_k} + f'(a_k)\Big) > \frac{\sqrt{f(a_k)}}{a_k^{1/2}} + \frac{c}{2}\frac{r-1}{r}\frac{1/2-\gamma}{1/2+\gamma}a_k^{\gamma-1/2} \tag{2.3.21}$$

$$> \frac{\sqrt{f(a_k)}}{a_k^{1/2}} - \frac{f'(a_k)}{\sqrt{m_1(a_k)}} + \frac{c}{4}\frac{r-1}{r}\frac{1/2-\gamma}{1/2+\gamma}a_k^{\gamma-1/2}. \tag{2.3.22}$$

Compare (2.3.22) with equation (2.3.15). The only difference is the last term on the right-hand side, which for $c < 0$ and $\gamma < 1/4$ it is a positive term that goes to $+\infty$ as $a_k \to 0$.

Assume by way of contradiction that in this case $R_{0,f(0)}$ is degenerate. Then we can find a sequence of terminal points (x_k, y_k) with $x_k \to \infty$ (as $k \to \infty$) with corresponding crossing points $(\beta_k, f(\beta_k)) \to (0, f(0))$ by Lemma 2.3.10. Then it must be that $m_2(\beta_k) \to \infty$ and we may assume without loss of generality that $m_2(\beta_k)$ is strictly increasing.

Assume x_k is large enough so that $\frac{x_k}{x_k-\beta_k} - 1 < A\beta_k$ for some constant A. Moreover we have the relations

$$m_1(\beta_k) = \frac{f(\beta_k)}{\beta_k}, \quad m_2(\beta_k) = \frac{y_k - f(\beta_k)}{x_k - \beta_k},$$
$$m(\beta_k) = \frac{y_k - f(0)}{x_k} \text{ and } y_k = m_2(\beta_k)(x_k - \beta_k) + f(\beta_k).$$

Since we are assuming that the region $R_{0,f(0)}$ is degenerate, the weight collected on a piecewise linear path that goes through $(0, f(0))$ and then to (x_k, y_k) must be less than the weight collected on the path from the crossing point. As such, the same calculation that led to (2.3.13), now gives the inequality

$$\frac{1}{r}\frac{x_k(f(\beta_k) - f(0) - \beta_k m_2(\beta_k)) + \beta_k(\beta_k m_2(\beta_k) - f(\beta_k) + f(0))}{x_k - \beta_k}\Big[1 + \frac{2}{\sqrt{m(\beta_k)} + \sqrt{m_2(\beta_k)}}\Big] \tag{2.3.23}$$

$$< \beta_k + f(\beta_k) - f(0) + 2\sqrt{\beta_k f(\beta_k)} - \frac{\beta_k}{r}\big(1+\sqrt{m_2(\beta_k)}\big)^2.$$

In the left hand side use the bounds $1 < \frac{x_k}{x_k - \beta_k} < 1 + A\beta_k$ and $m_2(\beta_k) > m(\beta_k)$ to bound from below

$$\begin{aligned}\frac{1}{r}(f(\beta_k) - f(0) - \beta_k m_2(\beta_k))(1+A\beta_k)\Big[1 + \frac{2}{\sqrt{m(\beta_k)}+\sqrt{m_2(\beta_k)}}\Big]\\ + \frac{1}{r}\frac{\beta_k(\beta_k m_2(\beta_k) - f(\beta_k) + f(0))}{x_k - \beta_k}\Big[1 + \frac{1}{\sqrt{m_2(\beta_k)}}\Big]\\ < \beta_k + f(\beta_k) - f(0) + 2\sqrt{\beta_k f(\beta_k)} - \frac{\beta_k}{r}\big(1+\sqrt{m_2(\beta_k)}\big)^2.\end{aligned}$$

Using equation (2.3.19), we have that $\beta_k m_2(\beta_k) \to 0$, so we simplify the inequality above one more time as

$$\begin{aligned}\frac{1}{r}(f(\beta_k) - f(0) - \beta_k m_2(\beta_k))\Big[1 + \frac{2}{\sqrt{m(\beta_k)}+\sqrt{m_2(\beta_k)}}\Big] + O(\beta_k)\\ < \beta_k + f(\beta_k) - f(0) + 2\sqrt{\beta_k f(\beta_k)} - \frac{\beta_k}{r}\big(1+\sqrt{m_2(\beta_k)}\big)^2.\end{aligned} \tag{2.3.24}$$

We finally use the estimate

$$|\sqrt{m(\beta_k)} - \sqrt{m_2(\beta_k)}| \le C_x\sqrt{m_2(\beta_k)}(f(0) - f(\beta_k)) \le C_x'\beta_k^{1/2}.$$

The last inequality comes from (2.3.18). We use this for one last simplification in (2.3.24) to

$$\begin{aligned}\frac{1}{r}(f(\beta_k) - f(0) - \beta_k m_2(\beta_k))\Big[1 + \frac{1}{\sqrt{m_2(\beta_k)}}\Big] + O(\beta_k)\\ < \beta_k + f(\beta_k) - f(0) + 2\sqrt{\beta_k f(\beta_k)} - \frac{\beta_k}{r}\big(1+\sqrt{m_2(\beta_k)}\big)^2.\end{aligned}$$

With the same algebraic manipulations that led to (2.3.15), we obtain

$$\frac{1}{r}\Big(\frac{1}{\sqrt{m_2(\beta_k)}} + 1 - r\Big)\Big(\frac{f(\beta_k) - f(0)}{\beta_k} - f'(\beta_k)\Big) \le \sqrt{m_1(\beta_k)} - \frac{f'(\beta_k)}{\sqrt{m_1(\beta_k)}} + O(1). \tag{2.3.25}$$

This gives the desired contradiction, since equality (2.3.25) is precisely opposite of inequality (2.3.22). □

Example 2.3.11 (An exactly solvable corner-step model: $(g(a,b) = \sqrt{a} + \sqrt{b}, k = 1)$**.**

We have that $\partial_b g/\partial_a g = 1/\sqrt{m_1}$ and therefore $D = 0$. Then

$$m_2 = \frac{4}{\Big(\frac{\partial_b g}{\partial_a g} - 1 + \sqrt{\Big(\frac{\partial_b g}{\partial_a g}+1\Big)^2}\Big)^2} = \Big(\frac{\partial_a g}{\partial_b g}\Big)^2 = m_1.$$

Therefore, the optimal paths are straight lines and the last passage time can be explicitly computed for any (x, y). If (x, y) are such so that $\sqrt{x}+\sqrt{y} > 1$ the common optimal slope will be $m = y/x \in \mathbb{R}_+$. The crossing point is given by

$$(a^*, b^*) = \Big(\frac{x}{(\sqrt{x}+\sqrt{y})^2}, \frac{y}{(\sqrt{x}+\sqrt{y})^2}\Big), \tag{2.3.26}$$

and the last passage time shape function can be computed to be

$$\Gamma_{c_g,1}(x, y) = \begin{cases} \Big(1 - \frac{1}{r}\Big) + \frac{1}{r}(\sqrt{x}+\sqrt{y})^2, & \text{if } \sqrt{x}+\sqrt{y} > 1 \\ (\sqrt{x}+\sqrt{y})^2, & \text{if } \sqrt{x}+\sqrt{y} \le 1. \end{cases}$$

One can verify directly that going through the axes is not optimal and all maximisers have to cross the curve.

In fact, this is the unique case of a speed function with this form, for which the optimal paths are straight lines. Assume that always $m_2 = m_1 = m = b/a$. From equation (2.3.6) we have

$$0 = \nabla g(a, b) \cdot \left(\frac{1}{\sqrt{m}} + 1, -(\sqrt{m}+1)\right) = \nabla g(a, b) \cdot \left(\frac{\sqrt{a}}{\sqrt{b}} + 1, -\frac{\sqrt{b}}{\sqrt{a}} - 1\right). \tag{2.3.27}$$

Solve the differential equation (2.3.27) for a and b to conclude that there exists $c_1, c_2 \in \mathbb{R}$ such that

$$g(a, b) = c_2(\sqrt{a}+\sqrt{b})^2 + c_1.$$

Then the level curve is enforced by (2.3.4c) and is given by $\sqrt{a}+\sqrt{b} = \alpha$ for some $\alpha = \alpha(k, c_1, c_2)$ in $\mathbb{R}_+$. $\square$

2.4 Continuity properties of $\Gamma(x, y)$

Now, we want to study what happen to the difference of the macroscopic last passage time of two points that are very close to each other.

Lemma 2.4.1. *Fix $a, b, z, w > 0$ and a speed function c. Then there exists a constant $C = C(a, b, z, w, c(\cdot,\cdot)) < \infty$ such that for any $\delta > 0$ we can find sufficiently small $\delta_1, \delta_2 > 0$ so that the following two regularity conditions hold: For $0 \le a \le z$,*

$$\Gamma((a, 0), (z+\delta_1, \delta_2)) - \Gamma((a, 0), (z, 0)) \le C\sqrt{\delta}. \tag{2.4.1}$$

For $0 \le b \le w$,

$$\Gamma((0, b), (\delta_1, w+\delta_2)) - \Gamma((0, b), (0, w)) \le C\sqrt{\delta}. \tag{2.4.2}$$

Proof. The arguments will be symmetric, so we will prove only (2.4.2). Pick a δ positive.

First select $\delta_1 \in [0, 1)$, $\delta_2 \in [0, 1)$ small enough such that

1. Any discontinuity curve h_i in $[0, \delta_1] \times [0, w + \delta_2]$ is monotone and their domain is the interval $[0, \delta_1]$.

2. The intersection points of the discontinuity curves in $[0, \delta_1] \times [0, w + \delta_2]$ (if any) all lie on the y-axis.

The first one is possible since the h_i are finitely many in any compact set, and piecewise monotone functions. The second one because there only finitely many intersections points. Let H be the number of discontinuity curves in this rectangle, and enumerate them from the lowest to the highest, including the north and south straight boundaries. Decrease δ_1 further so that

$$\max_{1 \le i \le H} \{\omega_{h_i}(\delta_1)\} < \delta$$

and select an $\eta = \eta(\delta_1) > 0$ which satisfies the condition

$$\eta \le \min_{1 \le i \le H} \{\omega_{h_i}(\delta_1)\}.$$

Keep in mind that $\eta \to 0$ as $\delta_1 \to 0$. Decrease δ_1 further so that $H\eta << w$. Since $c(x, y)$ is piecewise constant, we have that in-between these discontinuity curves the rates are fixed, and on the discontinuity curve the value is the smallest of the rates in the two adjacent regions by condition (1) in Assumption 2.1.2.

From the hypotheses so far, we have that the rectangles $Q_i = [0, \delta_1] \times [h_i(0) \wedge h_i(\delta_1), h_i(0) \vee h_i(\delta_1)]$, have completely disjoint interiors for all $1 \le i \le H$ and $c(x, y)$ takes two values. In the rectangles $R_i = [0, \delta_1] \times [h_i(0) \vee h_i(\delta_1), h_{i+1}(0) \wedge h_{i+1}(\delta_1)]$, the speed function is constant. We allow the rectangles R_i, Q_i to be degenerate horizontal lines.

For any $\mathbf{x} = (x^1(s), x^2(s)) \in \mathcal{H}(\delta_1, w + \delta_2)$ set

$$I(\mathbf{x}) = \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x^1(s), x^2(s))} ds. \tag{2.4.3}$$

Let $\varepsilon > 0$ and assume that $\phi = (\phi^1, \phi^2) \in \mathcal{H}(\delta_1, w + \delta_2)$ is a path such that $\Gamma(\delta_1, w + \delta_2) - I(\phi) < \varepsilon$. It is possible to decompose ϕ into disjoint segments ϕ_j so that $\phi = \sum_{j=1}^{2H} \phi_j$ and that

1. For j even, $\phi_j \subseteq R_{j/2}$, and therefore it is a linear segment with derivative ϕ_j' in $\mathbb{R}_+^2$

2. For j odd, $\phi_j \subseteq Q_{(j+1)/2}$.

The sum $\sum_{j=1}^{2H} \phi_j$ means path concatenation.

For j odd, the total contribution of ϕ_j to $I(\phi)$ can be bounded by $\frac{1}{r_\ell}\gamma(\delta_1, \eta(\delta_1))$ where $r_\ell = \min_{(x,y)\in[0,\delta_1]\times[0,w+\delta_2]} c(x,y)$. Over all, the total contribution of the odd-indexed segments is bounded above by $4Hr_\ell^{-1}(\eta(\delta_1) \vee \delta_1)$.

For j even, the path segment is linear and the maximum contribution of any such segment is given by

$$\begin{aligned}I(\phi_j) &= \frac{1}{r_{R_j}}\gamma(\delta_1, \text{height}(R_j)) = \frac{1}{r_{R_j}}(\delta_1 + \text{height}(R_j) + 2\sqrt{\delta_1 \text{height}(R_j)})\\ &\le \frac{1}{r_{R_j}}\text{height}(R_j) + 2C_j\sqrt{\delta_1}.\end{aligned}$$

Overall, on the even-indexed segments, the total contribution to $I(\phi)$ is bounded above by $\sum_{k=1}^{H}(\frac{1}{r_{R_{2k}}}\text{height}(R_{2k}) + 2C_{2k})\sqrt{\delta_1} \le \sum_{k=1}^{H} \frac{1}{r_{R_{2k}}}\text{height}(R_{2k}) + C\sqrt{\delta_1}$.

Then,

$$\begin{aligned}\Gamma(\delta_1, w+\delta_2) - \varepsilon \le I(\phi) &\le \sum_{k=1}^{H} \frac{1}{r_{R_{2k}}}\text{height}(R_{2k}) + C\sqrt{\delta_1} + 4Hr_\ell^{-1}(\eta(\delta_1) \vee \delta_1)\\ &\le \Gamma(0, w+\delta_2) + C\sqrt{\delta_1} + 4Hr_\ell^{-1}(\eta(\delta_1) \vee \delta_1)\\ &\le \Gamma(0, w) + \frac{1}{r_\ell}\delta_2 + C\sqrt{\delta_1} + 4Hr_\ell^{-1}(\eta(\delta_1) \vee \delta_1)\\ &\le \Gamma(0, w) + C\delta_2 \vee \sqrt{\delta_1} \vee \eta(\delta_1).\end{aligned}$$

Let $\varepsilon \to 0$. □

Corollary 2.4.2. *Fix $(x, y) \in \mathbb{R}_+^2$ and a speed function c. Then there exists $C = C(x, y, c(\cdot,\cdot)) < \infty$ such that for any δ positive, there exist δ_1, δ_2 sufficiently small*

$$\Gamma(x+\delta_1, y+\delta_2) - \Gamma(x, y) < C\delta. \tag{2.4.4}$$

Proof. Let $B_{(x,y)}$ be a rectangle, where the north-east corner point is (x, y) and south-west corner is $(0, 0)$.

Let $\varepsilon > 0$ and ϕ^ε a path such that $\Gamma(x+\delta_1, y+\delta_2) - I(\phi^\varepsilon) < \varepsilon$. Moreover, let $\mathbf{u}$ be the point where ϕ^ε first intersects the north or the east boundary of $B_{(x,y)}$. Without loss of generality assume is the east boundary and so $\mathbf{u} = (x, b)$ for some $b \in [0, y]$. Then,

$$\begin{aligned}\Gamma(x+\delta_1, y+\delta_2) - \varepsilon &\le I(\phi^\varepsilon)\\ &\le \Gamma(x, b) + \Gamma((x, b), (x+\delta_1, y+\delta_2))\\ &= \Gamma(x, b) + \Gamma((x, b), (x, y)) + \Gamma((x, b), (x+\delta_1, y+\delta_2)) - \Gamma((x, b), (x, y))\\ &\le \Gamma(x, y) + \Gamma((x, b), (x+\delta_1, y+\delta_2)) - \Gamma((x, b), (x, y)).\end{aligned}$$

A rearrangement of terms gives

$$\Gamma(x+\delta_1, y+\delta_2) - \Gamma(x, y) \le \Gamma((x, b), (x+\delta_1, y+\delta_2)) - \Gamma((x, b), (x, y)) + \varepsilon$$

$$\leq C\delta + \varepsilon$$

where we used (2.4.2), albeit with a starting point of (x, b). Let $\varepsilon \to 0$ to prove the corollary. $\square$

We are now ready to prove Theorem 2.1.4.

Proof of Theorem 2.1.4. Fix an $\varepsilon > 0$ and let ζ_1, ζ_2 small enough so that by Corollary 2.4.2 we have

$$\Gamma((a, b), (x + \zeta_3, y + \zeta_4)) - \Gamma((a, b), (x, y)) < \varepsilon/4.$$

Then, keep ζ_3, ζ_4 fixed and find a ζ_1, ζ_2 small enough so that again by Corollary 2.4.2,

$$\Gamma((a - \zeta_1, b - \zeta_2), (x + \zeta_3, y + \zeta_4)) - \Gamma((a, b), (x + \zeta_3, y + \zeta_4)) < \varepsilon/4.$$

Together the inequalities above give

$$\Gamma((a - \zeta_1, b - \zeta_2), (x + \zeta_3, y + \zeta_4)) - \Gamma((a, b), (x, y)) < \varepsilon/2. \tag{2.4.5}$$

Similarly, one can approximate from the inside, and find ζ_5, ζ_6, ζ_7, ζ_8 so that

$$\Gamma((a, b), (x, y)) - \Gamma((a + \zeta_5, b + \zeta_6), (x - \zeta_7, y - \zeta_8)) < \varepsilon/2. \tag{2.4.6}$$

Let $\delta_0 = \min_{1 \leq i \leq 8}\{\zeta_i\}$. Since $\Gamma(u, v)$ decreases in the first argument and increases in the second argument the inequalities (2.4.5) and (2.4.6), together with our choice of δ_0 give

$$\Gamma((a - \delta_0, b - \delta_0), (x + \delta_0, y + \delta_0)) - \Gamma((a + \delta_0, b + \delta_0), (x - \delta_0, y - \delta_0)) < \varepsilon.$$

and that for any $\tilde{a} \in [a - \delta_0, a + \delta_0]$, $\tilde{b} \in [b - \delta_0, b + \delta_0]$, $\tilde{x} \in [x - \delta_0, x + \delta_0]$, $\tilde{y} \in [y - \delta_0, y + \delta_0]$, we have

$$\Gamma((a + \delta_0, b + \delta_0), (x - \delta_0, y - \delta_0)) \leq \Gamma((\tilde{a}, \tilde{b}), (\tilde{x}, \tilde{y})) \leq \Gamma((a - \delta_0, b - \delta_0), (x + \delta_0, y + \delta_0)).$$

The last two inequalities combined give the result. $\square$

The reason for this technical approximation is the statements in the next lemma, motivated by the following argument. In the simplest case we would like to approximate the limits of last passage times using the limiting Γ_c in rectangles where $c(x, y)$ has one discontinuity line. Unfortunately, unless the discontinuity of the speed is a line of slope 1, we cannot say at this point that the limit is $\Gamma_c(x, y)$. However, if the speed function is continuous, the fact that the limit of passage times is Γ_c in that environment is given by Theorem 3.1. in [46]. So we may approximate Γ_c with the value $\Gamma_{\tilde{c}}$ where $\tilde{c}$ will be a continuous speed function that approximates $c(s, t)$.

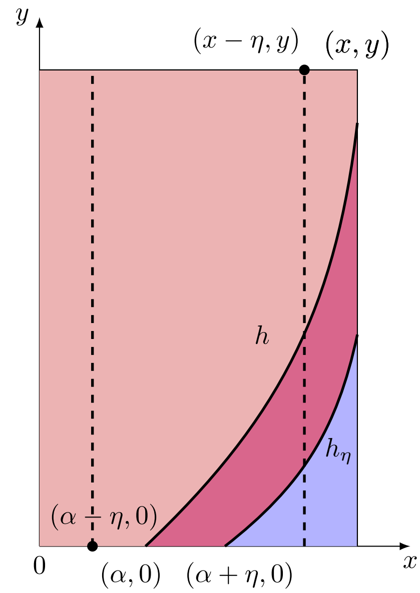


Figure 2.6: Graphical representation for the proof of Lemma 2.4.3.

Lemma 2.4.3 (Continuity of Γ in the speed function)**.** *Let $c(s,t)$ take only two values r_1, r_2 in two regions of $[a,x]\times[b,y]$ separated by a weakly monotone curve h, which satisfies Assumption 2.1.1. Then, for every $\varepsilon > 0$ there exists a $\eta_{h,\varepsilon} > 0$ so that for all $\eta < \eta_{h,\varepsilon}$ there exists a continuous speed function $c^{cont}_{\eta}(s,t) \le c(s,t)$ so that*

$$\Gamma_{c^{cont}_{\eta}}((a,b)(x,y)) - \Gamma_c((a,b),(x,y)) \le \varepsilon.$$

Proof of Lemma 2.4.3. Fix (x,y) and without loss assume that the starting point is $(a,b) = (\alpha, 0)$ for some $\alpha > 0$. We present the case when the curve h starts somewhere on $[\alpha, x]$ and exits somewhere on the east boundary $\{x\}\times[0,y]$ and the rates above the curve is $r_1 < r_2$. Symmetric arguments as the one below will work in all other cases, and are left to the reader.

For a fixed $\varepsilon > 0$ we can find an $\eta_{\varepsilon,h} > 0$ so that for all $\eta < \eta_{\varepsilon,h} > 0$ we have $|\Gamma_c((\alpha-\eta,0),(x-\eta,y)) - \Gamma_c((\alpha,0),(x,y))| < \varepsilon$. This is possible by Theorem 2.1.4. Fix any such η and define the curve h_η by the relation $h_\eta(t) = h(t+\eta)$, i.e. this correspond to shift of h by η to the right. Then, we define a speed function $c_\eta(\cdot,\cdot)$ on $[\alpha,x]\times[0,y]$

$$c_\eta(z,w) = \begin{cases} r_1, & \text{if } (z,w) \text{ is above or on the graph of } h_\eta, \\ r_2, & \text{otherwise.} \end{cases}$$

We make two observations:

1. $c(z,w) \ge c_\eta(z,w)$ for all $(z,w) \in [\alpha,x]\times[0,y]$, giving $\Gamma_{c_\eta}((\alpha,0),(x,y)) \ge \Gamma_c((\alpha,0),(x,y))$.

2. By construction

$$\Gamma_c((\alpha-\eta,0),(x-\eta,y)) = \Gamma_{c_\eta}((\alpha,0),(x,y)). \tag{2.4.7}$$

From these observations we define a new, continuous function $c_\eta^{\text{cont}}(\cdot,\cdot)$ on $[\alpha,x]\times[0,y]$ so that

$$c_\eta(z,w) \le c_\eta^{\text{cont}}(z,w) \le c(z,w), \qquad \text{for all } (z,w)\in[\alpha,x]\times[0,y].$$

This and (2.4.7) imply

$$\Gamma_{c_\eta^{\text{cont}}}((\alpha,0),(x,y)) \le \Gamma_{c_\eta}((\alpha,0),(x,y)) = \Gamma_c((\alpha-\eta,0),(x-\eta,y)) \le \Gamma_c((\alpha,0),(x,y))+\varepsilon, \tag{2.4.8}$$

which in turn yields the Lemma. □

2.5 Proof of Theorem 2.1.5

To prove Theorem 2.1.5 we need some Lemmas which help us to define some properties of the last passage time in a 2D inhomogeneous environment.

We begin by identifying the last passage time limits in simple cases of speed function, that will be used as building blocks for approximations to the general case. We first find the law of large numbers without fixing the maximal path but forcing it to stay in a homogeneous corridor. Let the speed function be

$$c(x,y) = \begin{cases} r_2 & y > x+\lambda, \\ r_1 & x-\lambda \le y \le x+\lambda, \\ r_3 & y < x-\lambda. \end{cases} \tag{2.5.1}$$

with $\lambda \in \mathbb{R}_+$.

Lemma 2.5.1 (Passage times in homogeneous corridors). *Assume $c(x,y)$ in (2.5.1) for all $(x,y)\in(0,b)\times(0,e)$. Let $(z,w)\in(0,b]\times(0,e]$ with $w\in(z-\lambda,z+\lambda)$ and let $\tilde{G}_{(\lfloor nz\rfloor,\lfloor nw\rfloor)}$ be the last passage time from $(0,0)$ to $(\lfloor nz\rfloor,\lfloor nw\rfloor)$ subject to the constraint that*

admissible paths stay in the r_1-rate region inside the strip $\lfloor nb\rfloor-\lambda \le \lfloor ne\rfloor \le \lfloor nb\rfloor+\lambda$, except possibly for a bounded number of initial and final steps.

Then

$$\lim_{n\to\infty} n^{-1}\tilde{G}_{(\lfloor nz\rfloor,\lfloor nw\rfloor)} = r_1^{-1}\gamma(z,w), \quad \mathbb{P}-a.s. \tag{2.5.2}$$

Proof. To obtain the upper bound $\lim_{n\to\infty} n^{-1}\tilde{G}_{(\lfloor nz\rfloor,\lfloor nw\rfloor)} \leq r_1^{-1}\gamma(z,w)$ ignore the path restrictions and assume that the environment in the whole region is homogeneous of constant rates r_1.

For the lower bound we use a coarse graining argument, taking into account the path restrictions. Fix an $\varepsilon \in (0,1)$ and consider the points

$$\mathscr{P}_{z,w,\varepsilon} = \{(k\lfloor \varepsilon nz\rfloor, k\lfloor \varepsilon nw\rfloor) : k = 1,2,\ldots,\lfloor \varepsilon^{-1}\rfloor\} \cup \{\lfloor nz\rfloor, \lfloor nw\rfloor)\}.$$

To bound $\tilde{G}_{(\lfloor nz\rfloor,\lfloor nw\rfloor)}$ from below, force the path to go through the partition points of $\mathscr{P}_{z,w,\varepsilon}$. By possibly reducing ε further, for each $1 \leq k \leq \lfloor \varepsilon^{-1}\rfloor$, each rectangle with lower-left and upper-right corners two consecutive points of $\mathscr{P}_{z,w,\varepsilon}$ is completely inside the region of rate r_1. For these rectangles we allow the path segments to explore space.

For $2 \leq k < \lfloor \varepsilon^{-1}\rfloor$ let $G_{R_k^n}$ be the last passage time from $((k-1)\lfloor \varepsilon nz\rfloor, (k-1)\lfloor \varepsilon nw\rfloor)$ to $(k\lfloor nz\varepsilon\rfloor, k\lfloor nw\varepsilon\rfloor)$. R_k^n refers to the rectangle that contains all the admissible paths between the two points.

Let $0 \leq \delta = \delta(\varepsilon) < \varepsilon r^{-1}\gamma(z,w)$ and assume without loss that $\delta/\varepsilon \to 0$ as $\varepsilon \to 0$. A large deviation estimate (Theorem 4.1 in [96]) gives a constant $C = C(r,z,w,\varepsilon,\delta)$ such that for k fixed

$$\mathbb{P}\{G_{R_k^n} \leq n(\varepsilon r^{-1}\gamma(z,w) - \delta)\} \leq e^{-Cn^2}. \tag{2.5.3}$$

The sequence of passage times $\{G_{R_k^n}\}_k$ are i.i.d. and as such, a Cramèr large deviation estimate and a Borel-Cantelli argument give for large n,

$$\tilde{G}_{(\lfloor nz\rfloor,\lfloor nw\rfloor)} \geq \sum_{k=1}^{\lfloor \varepsilon^{-1}\rfloor - 1} G_{R_k^n} \geq n(\lfloor \varepsilon^{-1}\rfloor - 1)(\varepsilon r^{-1}\gamma(z,w) - \delta), \quad \mathbb{P}\text{-a.s.}$$

Divide the inequality through by n and take the $\liminf$ as $n \to \infty$. After that, send $\varepsilon \to 0$ to finish the proof. $\square$

From the coarse graining argument in the previous proof, we see that when we restrict to maximal paths in a narrow (but macroscopic) homogeneous corridor we still obtain the same limiting passage time as if the environment was homogeneous throughout. This is a consequence of the mesoscopic fluctuations of the maximal paths and the strict concavity of γ. As the width ε of the corridor tends to 0, the limiting shape of the corridor is a straight line, which is the shape of the macroscopic maximal path in a homogeneous region.

Lemma 2.5.2 (Passage times in C^1 homogeneous corridors)**.** *Let $\mathbf{x}(\mathbf{s})$ be a C^1 increasing path from (a,b) to (c,d), and let $\mathcal{N}(\mathbf{x},\varepsilon)$ be a neighborhood subject to the constraint that*

$c(\mathbf{x}(\mathbf{s})) = r$ *(constant) on* $\mathcal{N}(\mathbf{x},\varepsilon)$*. Let* $G^{(n)}_{n\mathcal{N}(\mathbf{x},\varepsilon)}$ *be the passage time from* $\lfloor n(a,b)\rfloor$ *to* $\lfloor n(c,d)\rfloor$*, subject to the constraint that maximal paths never exit* $n\mathcal{N}(\mathbf{x},\varepsilon)$*. Then*

$$\varliminf_{n\to\infty} n^{-1}G^{(n)}_{n\mathcal{N}(\mathbf{x},\varepsilon)} \geq \frac{1}{r}\int_0^1 \gamma(\mathbf{x}'(s))\,ds.$$

Proof. Consider a partition of the interval $[0,1]$ $\mathcal{P} = \{0 = s_0 < s_1 < \ldots < s_N = 1\}$ fine enough so that the rectangles $R(\mathbf{x}(s_i),\mathbf{x}(s_{i+1}))$ are completely inside the neighborhood $\mathcal{N}(\mathbf{x},\varepsilon)$. Then,

$$\begin{aligned}
\varliminf_{n\to\infty} n^{-1}G^{(n)}_{n\mathcal{N}(\mathbf{x},\varepsilon)} &\geq \varliminf_{n\to\infty} n^{-1}\sum_{i=0}^{N-1} G^{(n)}_{\lfloor n\mathbf{x}(s_i)\rfloor,\lfloor n\mathbf{x}(s_{i+1})\rfloor} \geq \sum_{i=0}^{N-1}\varliminf_{n\to\infty} n^{-1}G^{(n)}_{\lfloor n\mathbf{x}(s_i)\rfloor,\lfloor n\mathbf{x}(s_{i+1})\rfloor}\\
&\geq \frac{1}{r}\sum_{i=0}^{N-1}\gamma(\mathbf{x}(s_{i+1})-\mathbf{x}(s_i)) = \frac{1}{r}\sum_{i=0}^{N-1}\gamma\Big(\frac{\mathbf{x}(s_{i+1})-\mathbf{x}(s_i)}{s_{i+1}-s_1}\Big)(s_{i+1}-s_i)\\
&= \frac{1}{r}\sum_{i=0}^{N-1}\gamma\big(\mathbf{x}'(\xi_i)\big)(s_{i+1}-s_i),\ \text{for some } \xi_i\in[s_i,s_{i+1}], \text{ by the mean value theorem.}
\end{aligned}$$

As the mesh of the partition tends to 0, the last line converges to $\frac{1}{r}\int_0^1\gamma(\mathbf{x}'(s))\,ds$, as it is a Riemann sum. This gives the result. ☐

Lemma 2.5.3 (Passage times in two-phase rectangles)**.** *Consider a* C^1 *function* $h : [0,a]\to[0,b]$ *and a macroscopic rectangle* $[0,a]\times[0,b]$ *and in which the speed function is*

$$c(x,y) = r_1\mathbb{1}_{\{y>h(x)\}} + r_2\mathbb{1}_{\{y<h(x)\}} + r_1\wedge r_2\mathbb{1}_{\{y=h(x)\}}.$$

We further assume that

1. $h([0,a]) = [0,b]$*,* h *is monotone and* $h(x)\notin\{0,b\}$*, for any* $x\in(0,a)$*.*
2. *There exists* $\eta>0$ *so that* $\min_{x\in(0,a)}|h'(x)| > \eta > 0$*.*
3. *If* h *is increasing, then we further assume that for the same* $\eta>0$ *as in (2), we have* $\sup_{x\in(0,a)}\left|h'(x)-\frac{b}{a}\right| < \eta$*. In particular, the first derivative is bounded and there exists a constant* L *so that the curve is Lipschitz-*L*.*

Assume for convenience that $r_1<r_2$*. Then, there exists a uniform constant* C_h *so that last passage time limits satisfy*

1. *For* h *increasing ,*

$$\frac{1}{r_1}\gamma(a,b) - \frac{2}{r_1}C_h\text{length}(h)\eta \leq \varliminf_{n} n^{-1}G^{(n)}_{\lfloor na\rfloor,\lfloor nb\rfloor} \leq \varlimsup_{n} n^{-1}G^{(n)}_{\lfloor na\rfloor,\lfloor nb\rfloor} \leq \frac{1}{r_1}\gamma(a,b). \tag{2.5.4}$$

Moreover,

$$\frac{1}{r_1}\gamma(a,b) - \frac{2}{r_1}C_h\mathrm{length}(h)\eta \le \Gamma(a,b) < \frac{1}{r_1}\gamma(a,b), \tag{2.5.5}$$

which in turn implies

$$\varlimsup_{n\to\infty} |n^{-1}G^{(n)}_{\lfloor na\rfloor,\lfloor nb\rfloor} - \Gamma(a,b)| \le \frac{2}{r_1}C_h\mathrm{length}(h)\eta. \tag{2.5.6}$$

2. *When h is decreasing*

$$\lim_{n\to\infty} n^{-1}G^{(n)}_{\lfloor na\rfloor,\lfloor nb\rfloor} = \Gamma(a,b). \tag{2.5.7}$$

Proof. We first treat the case of increasing h. Without loss, assume $h(0) = 0$ and $h(a) = b$. Since $r_1 < r_2$ we obtain the upper bound in (2.5.4) if we lower r_2 to r_1 and assume a homogeneous environment with constant speed function $c_{\text{low}}(x,y) = r_1$. This also gives the upper bound in (2.5.5) since $c_{\text{low}}(x,y) \le c(x,y)$.

Now for the lower bound. Let $\varepsilon > 0, \delta > 0$ sufficiently small. First consider a graph $h_\varepsilon(x) = (h(x)+\varepsilon)\wedge b$ which lies solely in the r_1 region of $c(x,y)$.

By hypothesis (1), assume ε is small enough so that the first time h_ε touches the top boundary $[0,a]\times\{b\}$, is precisely at some point $x_\varepsilon > a-\delta$. Consider a parametrisation for h, $(h^{(1)}(s), h^{(2)}(s)) : [0,1]\to\mathbb{R}^2$. Then point x_ε corresponds to some $1-s_\varepsilon \in [0,1]$.

Then define the curve $\mathbf{x}$ that goes from $(0,0)$ to $(0,h_\varepsilon(0))$ by time s_ε, then follows h_ε until it takes the value b by time 1 and then stays on the north boundary at value b for time s_ε.

Since h is rectifiable, so is h_ε, and we assume without loss that h_ε has the Lipschitz parametrization

$$\Big(h^{(1)}\Big((s-s_\varepsilon)\frac{1-s_\varepsilon}{1-2s_\varepsilon}\Big), h^{(2)}\Big((s-s_\varepsilon)\frac{1-s_\varepsilon}{1-2s_\varepsilon}\Big)+\varepsilon\Big), \quad s\in[s_\varepsilon, 1-s_\varepsilon].$$

Then we estimate

$$\begin{aligned}
\Gamma(a,b) &\ge \int_{s_\varepsilon}^{1-s_\varepsilon} \frac{\gamma(\mathbf{x}'(s))}{r_1}\,ds = \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{1-s_\varepsilon} \frac{\gamma(h^{(1)\prime}(s), h^{(2)\prime}(s))}{r_1}\,ds \\
&= \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{1-s_\varepsilon} h^{(1)\prime}(s)\frac{\gamma(1,\frac{h^{(2)\prime}(s)}{h^{(1)\prime}(s)})}{r_1}\,ds = \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{1-s_\varepsilon} h^{(1)\prime}(s)\frac{\gamma(1,h'(h^{(1)}(s)))}{r_1}\,ds \\
&= \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{h^{(1)}(1-s_\varepsilon)} \frac{\gamma(1,h'(u))}{r_1}\,du \ge \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{h^{(1)}(1-s_\varepsilon)} \frac{\gamma(1,\frac{b}{a}-\eta)}{r_1}\,du \\
&= \frac{1-s_\varepsilon}{1-2s_\varepsilon}h^{(1)}(1-s_\varepsilon)\frac{\gamma(1,\frac{b}{a}-\eta)}{r_1} \\
&\ge a\frac{\gamma(1,\frac{b}{a}-\eta)}{r_1} - \delta\frac{1-s_\varepsilon}{1-2s_\varepsilon}\frac{\gamma(1,\frac{b}{a}-\eta)}{r_1} - \frac{s_\varepsilon}{1-2s_\varepsilon}\frac{\gamma(1,\frac{b}{a}-\eta)}{r_1}.
\end{aligned} \tag{2.5.8}$$

Letting $\varepsilon \to 0$ makes the last term vanish, and by then letting $\delta \to 0$ we obtain

$$\Gamma(a,b) \geq \frac{\gamma(a, b-a\eta)}{r_1} = \frac{1}{r_1}\Big(a+b-a\eta+2\sqrt{a}\sqrt{b}\sqrt{1-\frac{a\eta}{b}}\Big). \tag{2.5.9}$$

By the mean value theorem $\eta < \min|h'(s)| < ba^{-1}$ and by item (2) in the hypothesis, one can check that

$$\sqrt{1-\frac{a\eta}{b}} \geq 1-\frac{a\eta}{b}.$$

We now estimate the γ-term in the left hand side of (2.5.9).

$$\gamma(a, b-a\eta) = a+b-a\eta+2\sqrt{a}\sqrt{b}\Big(1-\frac{a\eta}{b}\Big) = a+b-a\eta+2\sqrt{a}\sqrt{b}-2\frac{a^{3/2}\eta}{b^{1/2}} \tag{2.5.10}$$

$$\geq \gamma(a,b) - 2\eta\Big(a+\frac{a^{3/2}}{b^{1/2}}\Big). \tag{2.5.11}$$

Now the lower bound in (2.5.5). Let

$$C_h^2 > \frac{1+2\sqrt{L}}{L^3} \vee \Big(1+\frac{1+2\sqrt{L}}{\min_{x\in(0,a)} h'(x)}\Big).$$

Keep in mind that by the mean value theorem, $b/a \geq \min_{x\in(0,a)} h'(x)$ and by the choice of C_h we have

$$\frac{b}{a} \geq \min_{x\in(0,a)} h'(x) \geq \frac{1+2\sqrt{L}}{C_h^2-1}.$$

Then we can bound

$$\begin{aligned}0 &\leq a^2((C_h^2-1)b-(1+2\sqrt{L})a) < (C_h^2-1)a^2b-(1+2\sqrt{L})a^3+C_h^2b^3\\ &= (C_h^2-1)a^2b-a^3-2\sqrt{L}a^3+C_h^2b^3 < (C_h^2-1)a^2b-a^3-2a^{5/2}b^{1/2}+C_h^2b^3.\end{aligned}$$

In the last inequality above we used (3), since it implies $h(a)-h(0) = b \leq La$. An equivalent way to write the last inequality is

$$\Big(a+\frac{a^{3/2}}{b^{1/2}}\Big)^2 < C_h^2(a^2+b^2). \tag{2.5.12}$$

From (2.5.12), we conclude that $a+\frac{a^{3/2}}{b^{1/2}} < C_h\sqrt{a^2+b^2} \leq C_h\text{length}(h)$. Substitute this in (2.5.11) to finally prove the lower bound in (2.5.5).

For the lower bound in (2.5.4) consider again the function h_ε and s_ε from before and consider a partition of $[0, 1-s_\varepsilon]$, $\mathscr{P}_{s_\varepsilon,\delta} = \{x_k = k\delta(1-s_\varepsilon)\}_{0\leq k\leq\lfloor\delta^{-1}\rfloor}$, of mesh $\delta > 0$. We assume the partition is fine enough so that the rectangles $R_k = [x_k, x_{k+1}] \times [h_\varepsilon(x_k), h_\varepsilon(x_{k+1})]$ completely lie in the homogeneous region of rate r_1 and so that Riemann sum

$$\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma(h^{(1)'}(x_{k+1}), h^{(2)'}(x_{k+1}))(x_{k+1}-x_k) \geq \int_0^{1-s_\varepsilon} \frac{\gamma(h^{(1)'}(s), h^{(2)'}(s))}{r_1}\,ds - \theta_1 \tag{2.5.13}$$

for some fixed tolerance $\theta_1 > 0$. Moreover, assume the partition is fine enough so that for η_1 sufficiently small, with $0 < \eta_1 < \alpha$

$$\Big|\frac{h^{(i)}(x_{k+1}) - h^{(i)}(x_k)}{x_{k+1} - x_k} - h^{(i)\prime}(x_{k+1})\Big| < \eta_1, \quad \text{for } i = 1, 2.$$

Finally, fix a small $\theta_2 > 0$ and let n large enough so that Theorem 4.1 in [96] gives

$$\mathbb{P}\{G_{nR_k} < nr_1^{-1}\gamma(h^{(1)}(x_{k+1}) - h^{(1)}(x_k), h_\varepsilon^{(2)}(x_{k+1}) - h_\varepsilon^{(2)}(x_k)) - n\theta_2\} \le e^{-cn}.$$

By the Borel-Cantelli lemma we can then let n be large enough so that $\mathbb{P}$-a.s. for all k

$$G_{nR_k} > nr_1^{-1}\gamma(h^{(1)}(x_{k+1}) - h^{(1)}(x_k), h_\varepsilon^{(2)}(x_{k+1}) - h_\varepsilon^{(2)}(x_k)) - n\theta_2.$$

Above we denoted by G_{nR_k} the maximum weight that can be collected from oriented paths in the set nR_k.

Remind that $\omega_\gamma(\cdot)$ is the modulus of continuity of γ. By superadditivity, the passage times satisfy

$$\begin{aligned}
G^{(n)}_{\lfloor na\rfloor,\lfloor nb\rfloor} &\ge \sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} G_{nR_k} \\
&\ge n\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma(h^{(1)}(x_{k+1}) - h^{(1)}(x_k), h_\varepsilon^{(2)}(x_{k+1}) - h_\varepsilon^{(2)}(x_k)) - n\theta_2\delta^{-1} \\
&= n\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma\Big(\frac{h^{(1)}(x_{k+1}) - h^{(1)}(x_k)}{x_{k+1} - x_k}, \frac{h_\varepsilon^{(2)}(x_{k+1}) - h_\varepsilon^{(2)}(x_k)}{x_{k+1} - x_k}\Big)(x_{k+1} - x_k) - n\theta_2\delta^{-1} \\
&\ge n\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma(h^{(1)\prime}(x_{k+1}) - \eta_1, h^{(2)\prime}(x_{k+1}) - \eta_1)(x_{k+1} - x_k) - n\theta_2\delta^{-1} \\
&\ge n\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma(h^{(1)\prime}(x_{k+1}), h^{(2)\prime}(x_{k+1}))(x_{k+1} - x_k) - \frac{n}{r_1}\omega_\gamma(\eta_1) - n\theta_2\delta^{-1}, \\
&\ge n\int_0^{1-s_\varepsilon} \frac{\gamma(h'(s))}{r_1}\,ds - \frac{n}{r_1}\omega_\gamma(\eta_1) - n\theta_1 - n\theta_2\delta^{-1}, \text{ by (2.5.13).}
\end{aligned}$$

Divide through by n and take the $\underline{\lim}$ on both sides. First let $\theta_1, \theta_2 \to 0$. After that take $\eta_1 \to 0$. The final estimate comes from a repetition of computation (2.5.8) and bounds (2.5.11), (2.5.12).

When h is decreasing, the approximation argument is simpler. We briefly highlight it but leave the details to the reader. First of all, any monotone curve from $[0, a]$ to $[0, b]$ will have to cross h at a unique point $(\zeta, h(\zeta))$. Then from Jensen's inequality, the piecewise linear curve from 0 to $(\zeta, h(\zeta))$ and then to (a, b) achieves a higher value for the functional (2.1.3). So, candidate macroscopic optimisers can be restricted to piecewise linear curves, and this gives the lower bound

$$\Gamma(a,b) \le \varliminf_{n\to\infty} n^{-1} G^{(n)}_{\lfloor na \rfloor, \lfloor nb \rfloor}$$

by a coarse graining argument as for the case when h was increasing. For the upper bound, partition the curve h finely enough with a mesh $\delta > 0$. Any microscopic optimal path will have to cross the microscopic curve $[nh]$ at some point $(\lfloor n\zeta \rfloor, \lfloor n(h(\zeta)) \rfloor)$, lying between two of the partition points. For n large enough, the passage time on this path will $\mathbb{P}$-a.s , be no more than $nr_1^{-1}\gamma(\zeta, h(\zeta)) + nr_2^{-1}\gamma(a-\zeta, b-h(\zeta)) + n\varepsilon + Cn\sqrt{\delta}$ for any fixed ε. Divide by n, take the quantifiers to 0 and then take supremum over all crossing points to obtain the upper bound. $\square$

Example 2.5.4. *Consider a square with south-west corner* $(0,0)$ *and north-east corner* $(1,1)$. *This square is subdivided in two constant-rate regions by a parabola* $h(x) = x^2$ *where above the rate is* 1 *and below is* $r \in (0,1)$. *Then the set of the all potential optimisers is a concatenation of straight lines in the* 1 *region and convex segments along the discontinuity* $h(x)$.

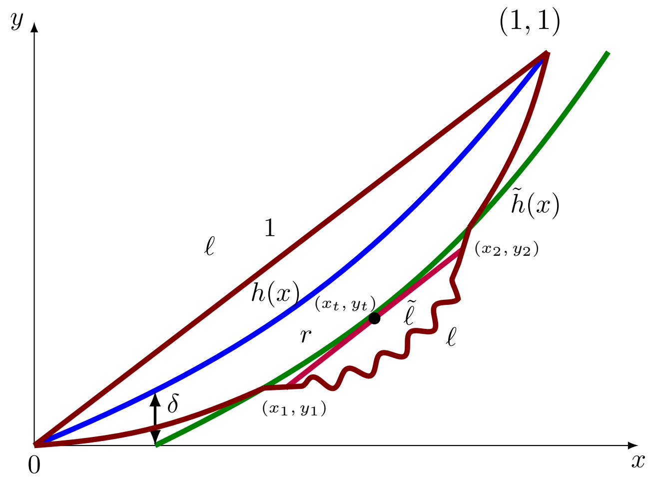


Figure 2.7: Graphical representation for the Example 2.5.4.

From Jensen's inequality and the convexity of $h(x)$ it is immediate to see that any segment of an optimiser in the rate 1 region will have to be a straight line from the entry point to the exit point of the optimiser in the region. Therefore it remains to prove the shape of the maximal path in the r region.

We first claim that for any potential optimiser $\ell \in \mathcal{H}(1,1)$, there exists a neighborhood $\mathcal{N}_\ell$ on $[0,1]$ such that for every $x \in \mathcal{N}_\ell$ a potential optimiser in $\mathcal{H}(1,1)$ takes the value $h(x)$ for $x \in \mathcal{N}_\ell$.

To see this we use a proof by contradiction: First, we show that for r small enough, any potential optimiser has to enter the r-region. If that was not the case, Jensen's inequality would give that the straight line from $(0,0)$ to $(1,1)$ is actually an optimiser and the last passage time constant would be

$$I_\ell(1,1) = \int_0^1 (\sqrt{1} + \sqrt{1})^2 dt = 4.$$

However, the C^1 curve $h(x)$ is also an admissible curve, and it achieves potential

$$I_{h(x)}(1,1) = \frac{1}{r}\int_0^1 (1 + \sqrt{2t})^2 dt = \frac{2}{r}(1 + \frac{2}{3}\sqrt{2}),$$

by the lower semicontinuity assumption on $c(x,y)$. Therefore, for $r < \frac{1}{2} + \frac{\sqrt{2}}{3}$, we have $I_\ell(1,1) < I_{h(x)}(1,1)$, so the optimiser ℓ has to enter the slow region.

Now suppose that $r > \frac{1}{2} + \frac{\sqrt{2}}{3}$ in order to complete the example. We can find points $(a, h(a))$ and $(b, h(b))$ so that ℓ enters in the r region through the point $(a, h(a))$ with $a \in [0,1)$ and stays in there without touching $h(x)$ except until $(b, h(b))$. We allow that potentially $(1,1) = (b, h(b))$. Since ℓ is continuous, it is possible to find a $\delta > 0$ so that for t in some open interval $\mathcal{N}_\ell$ we have

$$|h(t) - \ell(t)| > \delta. \tag{2.5.14}$$

To see that (2.5.14) is not respected by a potential optimiser, consider a δ shift $\tilde{h} = (h - \delta/2)^+$. Since ℓ is continuous it will cross $\tilde{h}$ at least in two points $(a_1, \tilde{h}(a_1))$ and $(b_1, \tilde{h}(b_1))$ and without loss assume $[a_1, b_1] \subseteq \mathcal{N}_\ell$. Pick any $t \in (a_1, b_1)$ and consider the tangent line at $(t, \tilde{h}(t))$ on $\tilde{h}$. By construction, this should cross ℓ in (x_1, y_1) and (x_2, y_2) (see Figure 2.7). By Jensen's inequality we know that the path $\tilde{\ell}$ which goes through ℓ up to point (x_1, y_1), straight to (x_2, y_2) and then follows ℓ. Then, $I(\tilde{\ell}) > I(\ell)$ and therefore, ℓ cannot be an optimiser. This gives the desired contradiction.

The contradiction was reached by assuming that a potential optimiser enters the slow region, but without following the discontinuity curve h. This completes the example. □

Remark 2.5.5. *In the above example, we only used the explicit form of the discontinuity h just to argue that a potential optimiser will eventually enter the slow region. If this information is known, the latter part of the proof is completely general and it uses local convexity properties of the discontinuity. In particular it just uses the fact that the discontinuity curve and the potential optimiser are continuous, piecewise C^1 and there exists a point $(t, h(t))$ for which the tangent line does not enter the fast region.* □

Remark 2.5.6. *The previous example suggests that potential optimisers cannot be more regular than the discontinuity curves.* □

Lemma 2.5.7 (Exponential concentration of passage times with continuous speed)**.** *Let $c(s,t)$ be a continuous speed function in $[0,x]\times[0,y]$. Then, for any $\theta>0$, there exists constants A and $\kappa_{\theta,c}$*

$$\mathbb{P}\{G^{(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor}\geq n\Gamma_c(x,y)+n\theta\}\leq Ae^{-\kappa_{\theta,c}n}. \tag{2.5.15}$$

Proof of Lemma 2.5.7. Fix a tolerance ε small. Its size will be determined in the proof. For a $K\in\mathbb{N}$, consider the two partitions

$$\mathscr{P}_x^{(K)}=\{\alpha_\ell=\ell xK^{-1}\}_{0\leq\ell\leq K},\ \text{and}\ \mathscr{P}_y^{(K)}=\{\beta_\ell=\ell yK^{-1}\}_{0\leq\ell\leq K}$$

of $[0,x]$ and $[0,y]$ respectively. Let $R_{i,j}$ denote the open rectangle with south-west corner (α_i,β_j). Let

$$r_{i,j}=\inf_{(s,t)\in R_{i,j}}c(s,t).$$

Define a speed function

$$c_{\text{low}}(s,t)=\sum_{(i,j)}r_{i,j}\mathbb{1}_{\{(s,t)\in R_{i,j}\}}+\sum_{(i,j)}r_{i-1,j}\wedge r_{i,j}\mathbb{1}_{\{s=\alpha_i,\beta_j<t<\beta_{j+1}\}}$$
$$+\sum_{(i,j)}r_{i,j-1}\wedge r_{i,j}\mathbb{1}_{\{\alpha_i<s<\alpha_{i+1},t=\beta_j\}}.$$

The value of $c(\alpha_i,\beta_j)$ is the minimum of the values in a neighborhood around it.

We are assuming the initial condition that $r_{i,-1}=r_{-1,j}=\infty$. In words, $c_{\text{low}}(s,t)$ is a step function with the minimum value of the neighbouring rates on the boundaries of $R_{i,j}$. Note that $c_{\text{low}}(s,t)\leq c(s,t)$. Let $\overline{R}_{i,j}$ denote the rectangle together with any of its boundaries for which it contributed the rate, using some rules to break ties, if the boundary value agrees for two rectangles.

At this point we assume that $K=K(\varepsilon)$ is large enough so that $\|c-c_{\text{low}}\|_\infty<\varepsilon$. This implies that

$$\Gamma_{c_{\text{low}}}(x,y)-\Gamma_c(x,y)\leq\varepsilon\gamma(x,y)r_{\min}^{-2},$$

where $r_{\min}$ is the smallest value of $c_{\text{low}}(x,y)$. This is because for any path $\mathbf{x}$,

$$\int_0^1\left\{\frac{\gamma(\mathbf{x}'(s))}{c_{\text{low}}(x_1(s),x_2(s))}-\frac{\gamma(\mathbf{x}'(s))}{c(x_1(s),x_2(s))}\right\}ds$$
$$=\int_0^1\frac{\gamma(\mathbf{x}'(s))(c(x_1(s),x_2(s))-c_{\text{low}}(x_1(s),x_2(s)))}{c(x_1(s),x_2(s))c_{\text{low}}(x_1(s),x_2(s))}ds\leq\varepsilon\int_0^1\frac{\gamma(\mathbf{x}'(s))}{c^2_{\text{low}}(x_1(s),x_2(s))}ds$$
$$\leq\epsilon r_{\min}^{-2}\gamma(x,y),$$

and the bound extends to the supremum over paths $\mathbf{x}$.

Pick a $L>0$ so that $L^{-1}<<K^{-1}$ and further partition each axis segment

$$\mathscr{H}_i^{(L)}=\{\alpha_i+\ell(\alpha_{i+1}-\alpha_i)L^{-1}\}_{0\leq\ell\leq L},\ \text{and}\ \mathscr{V}_j^{(L)}=\{\beta_j+\ell(\beta_{j+1}-\beta_j)L^{-1}\}_{0\leq\ell\leq L}.$$

Define

$$\mathscr{D}_{i,j} = \{\mathbf{d}^{\ell}_{i,j} = (\alpha_i + \ell(\alpha_{i+1} - \alpha_i)L^{-1}, \beta_j)\}, \qquad \mathscr{E}_{i,j} = \{\mathbf{e}^{\ell}_{i,j} = (\alpha_i, \beta_i + \ell(\beta_{i+1} - \beta_i)L^{-1})\}.$$

These completely partition all boundaries of the rectangles.

We are now ready to prove the concentration estimate. Let $G^{\text{low}}_{\lfloor nx \rfloor, \lfloor ny \rfloor}$ denote the last passage time in environment determined by c_{low}. Let $\pi_{\max}$ be the maximal path, and let π_k be the segment of the path in the k-th rectangle it visits $n\overline{R}_{i_k,j_k}$.

Now, for each k, π_k will enter and exit $n\overline{R}_{i_k,j_k}$ between two consecutive points of $n\mathscr{D}_{i_k,j_k}, n\mathscr{E}_{i_k,j_k}$. We denote by $n\mathbf{z_1}_{i_k,j_k}, n\mathbf{z_2}_{i_k,j_k}$ the consecutive points for the entrance and by $n\mathbf{z_1}_{i_{k+1},j_{k+1}}, n\mathbf{z_2}_{i_{k+1},j_{k+1}}$ for the exit.

Let $\mathbf{x}$ be a continuous, piecewise linear path from $(0,0)$ to (x,y) so that it crosses through the boundary segments $[n\mathbf{z_1}_{i_k,j_k}, n\mathbf{z_2}_{i_k,j_k}]$ at some point $\mathbf{x}_k$. Then for L small enough, we have that for some predetermined δ that

$$\Big|\frac{\gamma(\mathbf{z_2}_{i_{k+1},j_{k+2}} - \mathbf{z_1}_{i_k,j_k})}{r_{i_k,j_k}} - \frac{\gamma(\mathbf{x}_{k+1} - \mathbf{x}_k)}{r_{i_k,j_k}}\Big| < \delta.$$

We estimate

$$\begin{aligned}
&\mathbb{P}\{G^{(n)}_{\lfloor nx \rfloor, \lfloor ny \rfloor} \geq n\Gamma_c(x,y) + n\theta\} \leq \mathbb{P}\{G^{\text{low}}_{\lfloor nx \rfloor, \lfloor ny \rfloor} \geq n\Gamma_c(x,y) + n\theta\} \\
&\quad \leq \mathbb{P}\Big\{\sum_k G^{\text{low}}_{\pi_k} \geq n\Gamma_{c_{\text{low}}}(x,y) + n(\theta - \varepsilon\gamma(x,y)r^{-2}_{\min})\Big\} \\
&\quad \leq \mathbb{P}\Big\{\sum_k G^{\text{low}}_{\lfloor n\mathbf{z_1}_{i_k,j_k}\rfloor, \lfloor n\mathbf{z_2}_{i_{k+1},j_{k+1}}\rfloor} \geq n\Gamma_{c_{\text{low}}}(x,y) + n(\theta - \varepsilon\gamma(x,y)r^{-2}_{\min})\Big\} \\
&\quad \leq \mathbb{P}\Big\{\sum_k G^{\text{low}}_{\lfloor n\mathbf{z_1}_{i_k,j_k}\rfloor, \lfloor n\mathbf{z_2}_{i_{k+1},j_{k+1}}\rfloor} \geq n\sum_k \frac{\gamma(\mathbf{x}_{k+1} - \mathbf{x}_k)}{r_{i_k,j_k}} + n(\theta - \varepsilon\gamma(x,y)r^{-2}_{\min})\Big\} \\
&\quad \leq \mathbb{P}\Big\{\sum_k G^{\text{low}}_{\lfloor n\mathbf{z_1}_{i_k,j_k}\rfloor, \lfloor n\mathbf{z_2}_{i_{k+1},j_{k+1}}\rfloor} \\
&\qquad\qquad\qquad \geq n\sum_k \frac{\gamma(\mathbf{z_2}_{i_{k+1},j_{k+2}} - \mathbf{z_1}_{i_k,j_k})}{r_{i_k,j_k}} + n(\theta - \varepsilon\gamma(x,y)r^{-2}_{\min} - K^2\delta)\Big\} \\
&\quad \leq \sum_k \mathbb{P}\Big\{G^{\text{low}}_{\lfloor n\mathbf{z_1}_{i_k,j_k}\rfloor, \lfloor n\mathbf{z_2}_{i_{k+1},j_{k+1}}\rfloor} \\
&\qquad\qquad\qquad \geq n\frac{\gamma(\mathbf{z_2}_{i_{k+1},j_{k+2}} - \mathbf{z_1}_{i_k,j_k})}{r_{i_k,j_k}} + nK^{-2}(\theta - \varepsilon\gamma(x,y)r^{-2}_{\min} - K^2\delta)\Big\} \\
&\quad \leq Ae^{-\kappa_{\theta,\varepsilon}n}, \quad \text{by Theorem 4.2 in [96]}.
\end{aligned}$$

The last inequality is only true if $\theta - \varepsilon\gamma(x,y)r^{-2}_{\min} - K^2\delta > 0$ which can be achieved when ε is small enough so that $\varepsilon\gamma(x,y)r^{-2}_{\min} < \theta/3$ and then we reduce δ so that $K^2\delta = K^2(\varepsilon)\delta < \theta/3$. Theorem 4.2 in [96] is a large deviation principle which gives an exponential concentration inequality for passage times in a homogeneous environment. $\square$

The final approximation before the proof of the main theorem is the limiting time constant in any piecewise constant environment.

Proposition 2.5.8. *Let $c(s,t)$ be a piecewise constant speed function satisfying assumption 2.1.2, with a set of discontinuity curves $\{h_i\}_i$ satisfying Assumption 2.1.1. Let $\mathbf{u} = (x,y) \in \mathbb{R}_2^+$. Then the following law of large numbers holds*

$$\lim_{n\to\infty} \frac{1}{n} G^{(n)}_{\lfloor n\mathbf{u}\rfloor} = \Gamma_c(\mathbf{u}), \quad \mathbb{P} - a.s. \tag{2.5.16}$$

Proof. Fix $\mathbf{u} = (x,y) \in \mathbb{R}_+^2$ and consider any admissible path $\mathbf{x} \in \mathcal{H}(x,y)$, viewed as a curve $s \in [0,1] \mapsto \mathbf{x}(s) = (x_1(s), x_2(s))$. Recall the definition of $I(\mathbf{x})$ from (2.1.3) and remember that $\Gamma = \sup_{\mathbf{x}\in\mathcal{H}(x,y)} I(\mathbf{x})$.

Before proceeding with the technicalities, we highlight the intuition and main approximation idea. The most used technique in literature to prove this kind of limit is to find an upper and lower bound for the microscopic last passage time and then show that they tend to the same macroscopic last passage time in the limit $n \to \infty$. For the lower bound we use the superadditivity property of the microscopic last passage time, and any path acts as a lower bound. For the upper bound we have to construct a particular path which will represent an upper bound for the microscopic last passage time, while approximating the macroscopic limit after scaling its weight by n. For this, we first partition the rectangle $R_{0,(x,y)} = [0,x] \times [0,y]$ in a very specific way so the following conditions are all satisfied.

1. Isolate the finitely many points of intersection of the discontinuity curves in squares of size δ, where δ will be sufficiently small.

2. Isolate the finitely many points on strictly increasing h_i for which $h_i'(s) = 0$ or $h_i'(s)$ is not defined, in squares of size δ.

Call the collection of these squares by $\mathcal{I}_\delta = \{I_i\}_{1\le i\le Q}$. This include points of intersections with the boundary of $R_{0,(x,y)}$. It is fine if these squares overlap, as long as all these problematic points are in their interior.

Away from $\mathcal{I}_\delta$, the discontinuity curves are isolated so that for all curves we can partition each curve h_i finely enough so that for a given tolerance η,

1. Rectangles $R_{h_i(x_j),h_i(x_{j+1})}$ only contain the discontinuity curve h_i. Each rectangle now satisfies Assumption (1) of Lemma 2.5.3.

2. Assumption (3) in Lemma 2.5.3 holds for any rectangle $R_{h_i(x_j),h_i(x_{j+1})}$. Assumption (2) of Lemma 2.5.3 is automatically satisfied away from $\mathcal{I}_\delta$.

Call the collection of these rectangles that cover curve h_i by $\mathcal{J}_{h_i,\eta} = \{R_{i,j} = R_{h_i(x_j),h_i(x_{j+1})}\}_j$.

Lower Bound: Any macroscopic path $\mathbf{x}$ can be viewed as the concatenation of a finite number of segments $\mathbf{x}_j$ so that each segment belongs either in a constant rate region, or in one of the rectangles $\mathcal{I}_\delta$ or in one of the rectangles $\cup_i \mathcal{J}_{h_i,\varepsilon}$. Write

$$\mathbf{x}(s) = \sum_{k=1}^{Q} \mathbf{x}(s)\mathbb{1}\{\mathbf{x}(s) \in I_k\} + \sum_{k,\ell} \mathbf{x}(s)\mathbb{1}\{\mathbf{x}(s) \in R_{k,\ell}\} + \sum_{k=1}^{D} \mathbf{x}(s)\mathbb{1}\{\mathbf{x}(s) \in D_k\}.$$

Refine the partition further, so that if $\mathbf{x} : [s_i, s_{i+1}] \to \mathbb{R}^2 \in D_k$, then the open rectangle $R_{\mathbf{x}(s_i),\mathbf{x}(s_{i+1})} \subseteq D_k$.

Let $(x_1(s), x_2(s))$ a parametrization of the path $\mathbf{x}$. Partition the interval $[0,1]$ into $\mathcal{P} = \{0 = s_0 < s_1 < s_2 < \ldots < s_K = 1\}$ so that the path segment $\mathbf{x} : [s_i, s_{i+1}] \to \mathbb{R}^2$ belongs to exactly one I_k, $R_{k,\ell}$, or D_k. Note that $I(\mathbf{x}) = \sum_{i=0}^{K-1} \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\, ds$. The constant $K = K_{\delta,\eta}$ is the total number of different regions the path touches.

We bound each contribution separately:

(1) $\mathbf{x} : [s_i, s_{i+1}] \to \mathbb{R}^2 \in I_k$. Then, at most,

$$\int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))} < C\delta.$$

Then for all n large enough

$$\Big| \frac{G_{[\lfloor n\mathbf{x}(s_i)\rfloor, \lfloor n\mathbf{x}(s_{i+1})\rfloor]}}{n} - \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\, ds \Big| < C\delta,$$

since also passage times in these rectangles are bounded by $Cn\delta$.

(2) $\mathbf{x} : [s_i, s_{i+1}] \to \mathbb{R}^2 \in D_k$, where D_k is the homogeneous region of rate r_k. Fix a small $\theta_1 > 0$. Then for all n large enough, by the concentration estimates in [96]

$$\frac{G_{[\lfloor n\mathbf{x}(s_k)\rfloor, \lfloor n\mathbf{x}(s_{k+1})\rfloor]}}{n} > \frac{\gamma\big(\mathbf{x}(s_{k+1}) - \mathbf{x}(s_k)\big)}{r_k} - \theta_1 > \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\, ds - \theta_1.$$

(3) $\mathbf{x} : [s_i, s_{i+1}] \to \mathbb{R}^2 \in R_{k,\ell}$. Define

$$s_- = \inf\{s \in [s_i, s_{i+1}] : \mathbf{x}(s) - h_k = 0\}, \quad s_+ = \sup\{s \in [s_i, s_{i+1}] : \mathbf{x}(s) - h_k = 0\}.$$

In words, $\mathbf{x}(s_-)$ and $\mathbf{x}(s_+)$ are the points of first and last intersection of $\mathbf{x}$ with h_k in the rectangle $R_{k,\ell}$. Before $\mathbf{x}(s_-)$ and after $\mathbf{x}(s_+)$, $\mathbf{x}$ stays in a constant-rate region, in this rectangle. Between $\mathbf{x}(s_-)$ and $\mathbf{x}(s_+)$, $\mathbf{x}$ touches the discontinuity curve. This rectangle has two constant-rate regions. Denote the smallest one of those by r_{low}.

We bound in the case where the discontinuity curve in the rectangle is increasing. If it is decreasing, $s_- = s_+$, and the argument simplifies since the path $\mathbf{x}$ only intersects the discontinuity at a single point.

Let $G^{(n),\mathcal{N}(\mathbf{x},\varepsilon)}_{\lfloor n\mathbf{x}(s)\rfloor,\lfloor n\mathbf{x}(t)\rfloor}$ denote the passage time from $\lfloor n\mathbf{x}(s)\rfloor$ to $\lfloor n\mathbf{x}(t)\rfloor$, subject to the constraint that paths stay in the strip $n\mathcal{N}(\mathbf{x},\varepsilon)$. We assume ε is small enough so that the speed function stays constant on $n\mathcal{N}(\mathbf{x},\varepsilon)\cap R(\lfloor n\mathbf{x}(s)\rfloor, \lfloor n\mathbf{x}(t)\rfloor)$ except possibly at an $O(\varepsilon)$ region near the beginning and end points of the rectangle.

$$
\begin{aligned}
\frac{G^{(n)}_{\lfloor n\mathbf{x}(s_i)\rfloor,\lfloor n\mathbf{x}(s_{i+1})\rfloor}}{n} &\geq \frac{G^{(n),\mathcal{N}(\mathbf{x},\varepsilon)}_{\lfloor n\mathbf{x}(s_i)\rfloor,\lfloor n\mathbf{x}(s_-)\rfloor}}{n} + \frac{G^{(n)}_{\lfloor n\mathbf{x}(s_-)\rfloor,\lfloor n\mathbf{x}(s_+)\rfloor}}{n} + \frac{G^{(n),\mathcal{N}(\mathbf{x},\varepsilon)}_{\lfloor n\mathbf{x}(s_+)\rfloor,\lfloor n\mathbf{x}(s_{i+1})\rfloor}}{n} \\
&\geq \int_{s_i}^{s_-} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds - \theta_1 + \frac{\gamma\big(\mathbf{x}(s_-)-\mathbf{x}(s_+)\big)}{r_{\text{low}}} - C_{k,\ell}\text{length}(h_k\cap R_{k,\ell})\eta \\
&\qquad + \int_{s_+}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds - \theta_1 - O(\varepsilon) \qquad (2.5.17) \\
&\geq \int_{s_i}^{s_-} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds + \int_{s_-}^{s_+} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds + \int_{s_+}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds \\
&\qquad - 2\theta_1 - C_{k,\ell}\text{length}(h_k\cap R_{k,\ell})\eta - O(\varepsilon) \\
&= \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds - 2\theta_1 - C_{k,\ell}\text{length}(h_k\cap R_{k,\ell})\eta - O(\varepsilon). \qquad (2.5.18)
\end{aligned}
$$

Line (2.5.17) follows from Lemma 2.5.2 for some $\theta_1 > 0$ and n large enough. The line before last follows because either $c(\mathbf{x}(s_k))$ is the largest rate in $R_{i,j}$ or, if it is the smallest of the two, we use Lemma 2.5.3. The fact that these estimates hold for all large n follows from a Borel-Cantelli argument and the large deviation estimates, as seen in the proof of Lemma 2.5.1. Constants $C_{k,\ell}$ are the constants given in Lemma 2.5.3, that show up in bound (2.5.4). They are all bounded above by some constant $\tilde{C}_\delta$ (which also depends on x, y), since all points where the derivative of increasing h_i is 0 or undefined are isolated in cubes of side δ.

We are now in a position to bound, for all n large enough

$$
\begin{aligned}
G^{(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor} &\geq \sum_{i=0}^{K_{\delta,\eta}-1} G^{(n)}_{\lfloor n\mathbf{x}(s_i)\rfloor,\lfloor n\mathbf{x}(s_{i+1})\rfloor} \\
&\geq n\sum_{i=0}^{K_{\delta,\eta}-1} \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds - 3K_{\delta,\eta}n(\theta_1+O(\varepsilon)) - \tilde{C}\delta n - n\eta\tilde{C}_\delta\sum_{i=1}^{Q}\text{length}(h_i).
\end{aligned}
$$

Divide by n, and take the $\underline{\lim}$ as $n\to\infty$ to obtain

$$
\varliminf_{n\to\infty} \frac{G^{(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor}}{n} \geq I(\mathbf{x}) - 3K_{\delta,\eta}(\theta_1+O(\varepsilon)) - C\delta - C_\delta\eta - O(\varepsilon). \qquad (2.5.19)
$$

As the quantifiers go to 0, $K_{\delta,\eta}$ and C_δ blow up, so we first send θ_1 to 0 and $\varepsilon\to 0$. After that send $\eta\to 0$ and finally $\delta\to 0$ to obtain that for an arbitrary $\mathbf{x}\in\mathcal{H}(x,y)$,

$$
\varliminf_{n\to\infty} \frac{G^{(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor}}{n} \geq I(\mathbf{x}).
$$

A supremum over the class $\mathcal{H}(x,y)$ in the right hand-side of the display above gives

$$\varliminf_{n\to\infty} \frac{G^{(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor}}{n} \geq \Gamma_c(x,y). \tag{2.5.20}$$

Upper bound: For the upper bound we first partition $[0,x]\times[0,y]$ into rectangles, so that it is a refinement of the partition used for the lower bound: This way conditions (1)-(2) are satisfied and all rectangles in $\cup_i \mathcal{J}_{h_i,\eta}$ and $\mathcal{I}_\delta$ are part of this partition. Outside of the union of $\cup_i \mathcal{J}_{h_i,\eta}$ and $\mathcal{I}_\delta$, only the regions of constant rate remain. Divide each one of the constant region into rectangles, of side no longer than $\delta_1 > 0$ and assume $\delta_1 < \delta$.

Enumerate the rectangles in the two-dimensional partition by $Q_{i,j} = [x_i, x_{i+1}) \times [y_j, y_{j+1})$ and their total number is $N_{\eta,\delta,\delta_1} < \infty$.

Now, for any $n \in \mathbb{N}$ define the environment according to $c(x,y)$ and consider the maximizing path $(0,0)$ to $(\lfloor nx\rfloor, \lfloor ny\rfloor)$ which we denote by $\pi^{\max}_{0,(\lfloor nx\rfloor,\lfloor ny\rfloor)}$. The path can be written as a finite concatenation of sub-paths

$$\pi^{\max}_{0,(\lfloor nx\rfloor,\lfloor ny\rfloor)} = \sum_{(x_i,y_j)} \pi_{\lfloor nQ_{i,j}\rfloor}$$

where $\pi_{\lfloor nQ_{i,j}\rfloor}$ is the segment of the path in the rectangle $[\lfloor nx_i\rfloor, \lfloor nx_{i+1}\rfloor)\times[\lfloor ny_j\rfloor, \lfloor ny_{j+1}\rfloor)$. Some of these segments will be empty.

We partition the sides of each rectangle $Q_{i,j}$ further: Fix a $\delta_2 > 0$ and define partitions

$$\mathscr{P}_{e_1,(i,j)} = \{\mathbf{h}^{(i,j)}_k = (x_i,y_j) + k\delta_2 e_1\}_{0\leq k\leq \frac{x_{i+1}-x_i}{\delta_2}},$$

$$\mathscr{P}_{e_2,(i,j)} = \{\mathbf{v}^{(i,j)}_k = (x_i,y_j) + k\delta_2 e_2\}_{0\leq k\leq \frac{y_{i+1}-y_i}{\delta_2}}.$$

These completely define a partition of the boundaries $Q_{i,j}$. Now, the entry point of $\pi_{\lfloor nQ_{i,j}\rfloor}$ into $nQ_{i,j}$ will be between two consecutive partition points, say $\mathbf{a}^{(i,j)}_k \leq \mathbf{a}^{(i,j)}_{k+1}$ and its exit point will be between $\mathbf{b}^{(i,j)}_\ell \leq \mathbf{b}^{(i,j)}_{\ell+1}$. Note that exit point of one rectangle will be the entry point in an adjacent one, and all these points belong to some partition $\mathscr{P}_{e_k,(i,j)}$. If it so happens and the path enters (or exits) from one of the macroscopic partition points, we set $\mathbf{a}^{(i,j)}_k = \mathbf{a}^{(i,j)}_{k+1}$ (equiv. $\mathbf{b}^{(i,j)}_\ell = \mathbf{b}^{(i,j)}_{\ell+1}$).

When the environment in $Q_{i,j}$ is constant $r_{i,j}$, we have the bound

$$\begin{aligned} G^{(n)}_{\lfloor nQ_{i,j}\rfloor}(\pi) &= \sum_{v\in\pi_{\lfloor nQ_{i,j}\rfloor}} \tau^{(n)}_v \leq G^{(n)}_{n\mathbf{a}^{(i,j)}_k, n\mathbf{b}^{(i,j)}_{\ell+1}} \leq n\Big(\frac{\gamma(\mathbf{b}^{(i,j)}_{\ell+1} - \mathbf{a}^{(i,j)}_k)}{r_{i,j}} + \theta_1\Big) \\ &\leq n\Big(\frac{\gamma(\mathbf{b}^{(i,j)}_{\ell} - \mathbf{a}^{(i,j)}_{k+1})}{r_{i,j}} + C_{i,j}\omega_\gamma(\delta_2) + \theta_1\Big). \end{aligned} \tag{2.5.21}$$

The second-to-last inequality follows by Theorem 4.2 in [96], for n large enough.

When $c(s,t)$ on $Q_{i,j}$ takes two values, r_1, r_2 separated by a curve h, we bound as follows. First fix a tolerance ε and find $\delta_3 > 0$ so that we may define a continuous speed function $c_{\delta_3,h}(s,t)$ as in Lemma 2.4.3, with the property $c_{\delta_3,h}(s,t) \le c(s,t)$ and

$$\sup_{\mathbf{a}_k,\mathbf{b}_\ell} (\Gamma_{c_{\delta_3,h}}(\mathbf{a}_k,\mathbf{b}_\ell) - \Gamma_c(\mathbf{a}_k,\mathbf{b}_\ell)) < \varepsilon. \tag{2.5.22}$$

Then,

$$\begin{aligned}
G^{(n)}_{\lfloor nQ_{i,j}\rfloor}(\pi) &= \sum_{v\in\pi_{\lfloor nQ_{i,j}\rfloor}} \tau_v^{(n)} \le G^{(c_{\delta_3,h})}_{n\mathbf{a}_k^{(i,j)}, n\mathbf{b}_{\ell+1}^{(i,j)}} \\
&\le n\big(\Gamma_{c_{\delta_3,h}}(\mathbf{a}_k^{(i,j)}, \mathbf{b}_{\ell+1}^{(i,j)}) + \theta_1\big) \text{ by a Borel-Cantelli argument and Lemma 2.5.7}, \\
&\le n\big(\Gamma_{c_{\delta_3,h}}(\mathbf{a}_{k+1}^{(i,j)}, \mathbf{b}_{\ell}^{(i,j)}) + \omega_{\Gamma_c}(2\delta_2) + \theta_1\big) \text{ by Theorem 2.1.4}, && (2.5.23)\\
&\le n\big(\Gamma_{c}(\mathbf{a}_{k+1}^{(i,j)}, \mathbf{b}_{\ell}^{(i,j)}) + \varepsilon + \omega_{\Gamma_c}(2\delta_2) + \theta_1\big) \text{ by equation 2.5.22}. && (2.5.24)
\end{aligned}$$

Using the estimates (2.5.21) and (2.5.24), we have total upper bound for the passage time

$$\begin{aligned}
G^{(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor} &= \sum_{(i,j)} G^{(n)}_{\lfloor nQ_{i,j}\rfloor}(\pi) \\
&\le n\sum_{(i,j)} \Gamma_c(\mathbf{a}_{k+1}^{(i,j)}, \mathbf{b}_{\ell}^{(i,j)}) + nN_{\eta,\delta,\delta_1}(\max_{(i,j)} C_{i,j}\omega_\gamma(\delta_2) + \theta_1 + \varepsilon + \omega_{\Gamma_c}(2\delta_2)) + nC|\mathcal{I}_\delta|\delta \\
&\le n(\Gamma_c(x,y) + N_{\eta,\delta,\delta_1}(\max_{(i,j)} C_{i,j}\omega_\gamma(\delta_2) + \theta_1 + \varepsilon + \omega_{\Gamma_c}(2\delta_2)) + C|\mathcal{I}_\delta|\delta)
\end{aligned}$$

The last line follows from superadditivity of Γ. To finish the bound, divide by n and take the $\overline{\lim}_{n\to\infty}$. Then, let $\delta_2 \to 0$. This will result to finer $\mathscr{P}_{e_k,(i,j)}$ partitions, but by modulating δ_3 we can still keep estimate (2.5.22) with the same ε. Then let θ_1 and ε tend to 0. These are independent of the other quantifiers η, δ_1 and δ. Finally send $\delta \to 0$. $\square$

Proof of Theorem 2.1.5. Fix (x,y) and fix an $\epsilon > 0$. It is always possible to find piecewise strictly positive constant functions c_1 and c_2 such that $||c_1 - c_2||_\infty \le \varepsilon$ that definitely have the same discontinuity curves as the function c (but perhaps more). On $[0,x]\times[0,y]$ we can further impose $c_1(x,y) \le c(x,y) \le c_2(x,y)$, by defining each c_i on smaller rectangles.

When the weights in (2.0.3) are defined via the speed function c_i we write G^i for last passage time and Γ_{c_i} for their limits. A coupling using common exponential variables $\{\tau_{i,j}\}$ gives

$$G^{1,(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor} \ge G^{(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor} \ge G^{2,(n)}_{\lfloor nx\rfloor,\lfloor ny\rfloor}.$$

Letting $r_{\min} > 0$ denote a lower bound for $c(x,y)$ in the rectangle $[0,x]\times[0,y]$. Then we bound for any $\mathbf{x} \in \mathcal{H}$:

$$\begin{aligned}
0 &\leq \int_0^1 \left\{ \frac{\gamma(\mathbf{x}'(s))}{c_1(x_1(s), x_2(s))} - \frac{\gamma(\mathbf{x}'(s))}{c_2(x_1(s), x_2(s))} \right\} ds \\
&= \int_0^1 \frac{\gamma(\mathbf{x}'(s))(c_2(x_1(s), x_2(s)) - c_1(x_1(s), x_2(s)))}{c_1(x_1(s), x_2(s))c_2(x_1(s), x_2(s))} ds \leq \varepsilon \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c_1^2(x_1(s), x_2(s))} ds \\
&\leq \varepsilon r_{\min}^{-2} \gamma(x, y).
\end{aligned}$$

As the inequality is uniform across $\mathbf{x}$, the bound extends to the suprema

$$0 \leq \Gamma_{c_1}(x, y) - \Gamma_{c_2}(x, y) \leq C(x, y)\varepsilon.$$

From Proposition 2.5.8 we know that the Γ_{c_i} are the limits for G^i. To obtain Theorem 2.1.5, let $\varepsilon \to 0$. □

Chapter 3

Order of the variance in the discrete Hammersley process with boundaries

The model under consideration in this chapter is a directed corner growth model on the positive quadrant $\mathbb{Z}_+^2$. Each site v of $\mathbb{Z}_+^2$ is assigned a random weight ω_v. The collection $\{\omega_v\}_{v\in\mathbb{Z}_+^2}$ is the random environment and it is i.i.d. under the environment measure $\mathbb{P}$, with Bernoulli marginals

$$\mathbb{P}\{\omega_v = 1\} = p, \quad \mathbb{P}\{\omega_v = 0\} = 1 - p.$$

Throughout the chapter we exclude the values $p = 0$ or $p = 1$. One way to view the environment, is to treat site v as *present* when $\omega_v = 1$ and as *deleted* when $\omega_v = 0$. With this interpretation, the longest strictly increasing Bernoulli path up to (m, n) is a sequence of present sites

$$L_{m,n} = \{v_1 = (i_1, j_1), v_2 = (i_2, j_2), \dots, v_M = (i_M, j_M)\}$$

so that $0 < i_1 < i_2 < \dots < i_M \le m$ and $0 < j_1 < j_2 < \dots < j_M \le n$ and so that if $\{w_1, w_2, \dots, w_K\}$ is a different strictly increasing sequence of present sites, then it must be the case that $K \le M$.

In this chapter we cast the random variable $L_{m,n}$ as a last passage time as in the framework of [48]. With the previous description, a step of a potential optimal path up to (m, n) can take one of $O(m, n)$ values - any site is accessible as long as it has strictly larger coordinates from the previous site. However, any integer vector of positive coordinates can be written as a linear combination of e_1, e_2 and $e_1 + e_2$ steps. Our set of admissible steps is then restricted to $\mathcal{R} = \{e_1, e_2, e_1+e_2\}$ and an admissible path from $(0,0)$ to (m,n) is an ordered sequence of sites

$$\pi_{0,(m,n)} = \{0 = v_0, v_1, v_2, \dots, v_M = (m,n)\},$$

so that $v_{k+1} - v_k \in \mathcal{R}$. The collection of all these paths is denoted by $\Pi_{0,(m,n)}$. In order to obtain the same variable $L_{m,n}$ over this set of paths as the one from only strictly increasing steps, we need to specify the measurable potential function $V(\omega, z) : \mathbb{R}^{\mathbb{Z}_+^2} \times \mathcal{R} \to \mathbb{R}$ already defined in (1.3.3)

$$V(\omega, z) = \omega_{e_1+e_2} \mathbb{1}\{z = e_1 + e_2\}.$$

This way, the path π will collect the Bernoulli weight at site v if and only there exists a k such that $v_{k+1} = v$ and $v_k = v - e_1 - e_2$. No gain can be made through a horizontal or vertical step. Using this potential function V we define the last passage time as

$$G^V_{m,n} = \max_{\pi_{0,(m,n)} \in \Pi_{0,(m,n)}} \left\{ \sum_{v_i \in \pi} V(T_{v_i}\omega, v_{i+1} - v_i) \right\}. \tag{3.0.1}$$

Above we used T_{v_i} as the environment shift by v_i in $\mathbb{Z}_+^2$. Now one can see that $G^V_{0,(m,n)} = L_{m,n}$.

Common notation

Throughout the paper, $\mathbb{N}$ denotes the natural numbers, and $\mathbb{Z}_+$ the non-negative integers. When we write inequality between two vectors $v = (k, \ell) \le w = (m,n)$ we mean $k \le m$ and $\ell \le n$. We reserve the symbol G for last passage times. We omit from the notation the superscript V that was used to denote the dependence of potential function in (3.0.1), since for the sequence we fix V as in (1.3.3), unless otherwise mentioned. The symbol π is reserved for a generic admissible path.

3.1 The model and its invariant version

3.1.1 The invariant boundary model

The boundary model has altered distributions of weights on the two axes. The new environment there will depend on a parameter $u \in (0,1)$ that will be under our control. Each u defines different boundary distributions. At the origin we set $\omega_0 = 0$. For weights on the horizontal axis, for any $k \in \mathbb{N}$ we set $\omega_{ke_1} \sim \text{Bernoulli}(u)$, with independent marginals

$$\mathbb{P}\{\omega_{ke_1} = 1\} = u = 1 - \mathbb{P}\{\omega_{ke_1} = 0\}. \tag{3.1.1}$$

On the vertical axis, for any $k \in \mathbb{N}$, we set $\omega_{ke_2} \sim \text{Bernoulli}\Big(\frac{p(1-u)}{u+p(1-u)}\Big)$ with independent marginals

$$\mathbb{P}\{\omega_{ke_2} = 1\} = \frac{p(1-u)}{u+p(1-u)} = 1 - \mathbb{P}\{\omega_{ke_2} = 0\}. \tag{3.1.2}$$

The environment in the bulk $\{\omega_w\}_{w\in\mathbb{N}^2}$ remains unchanged with i.i.d. $\text{Ber}(p)$ marginal distributions. Denote this environment by $\omega^{(u)}$ to emphasise the different distributions on the axes that depend on u.

In summary, for any $i \geq 1, j \geq 1$, the $\omega^{(u)}$ marginals are independent under a background environment measure $\mathbb{P}$ with marginals

$$\omega^{(u)}_{i,j} \sim \begin{cases} \text{Ber}(p), & \text{if } (i,j) \in \mathbb{N}^2, \\ \text{Ber}(u), & \text{if } i \in \mathbb{N}, j = 0, \\ \text{Ber}\Big(\frac{p(1-u)}{u+p(1-u)}\Big), & \text{if } i = 0, j \in \mathbb{N}, \\ 0, & \text{if } i = 0, j = 0. \end{cases} \tag{3.1.3}$$

In this environment we slightly alter the way a path can collect weight on the boundaries. Consider any path π from 0. If the path moves horizontally before entering the bulk, then it collects the $\text{Bernoulli}(u)$ weights until it takes the first vertical step, and after that, it collects weight according to the potential function (1.3.3). If π moves vertically from 0 then it **also** collects the Bernoulli weights on the vertical axis, and after it enters the bulk, it collects according to (1.3.3).

Fix a parameter $u \in (0,1)$. Denote the last passage time from 0 to w in environment $\omega^{(u)}$ by $G^{(u)}_{0,w}$. The variational equality, using the above description, is

$$\begin{aligned} G^{(u)}_{0,w} = \max_{1\leq k\leq w\cdot e_1} \max_{z\in\{e_2, e_1+e_2\}} \Big\{ \sum_{i=1}^{k} \omega_{ie_1} + V(T_{ke_1}\omega, z) + G_{ke_1+z,w} \Big\} \\ \bigvee \max_{1\leq k\leq w\cdot e_2} \max_{z\in\{e_1, e_1+e_2\}} \Big\{ \sum_{j=1}^{k} \omega_{je_2} + V(T_{ke_2}\omega, z) + G_{z+ke_2,w} \Big\}. \end{aligned} \tag{3.1.4}$$

Our two first statements give the explicit formula for the shape function.

Theorem 3.1.1 (Law of large numbers for $G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$). *For fixed parameter $0 < u \leq 1$ and $(s,t) \in \mathbb{R}^2_+$ we have*

$$\lim_{N\to\infty} \frac{G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}{N} = su + t\,\frac{p(1-u)}{u+p(1-u)}, \quad \mathbb{P} - a.s. \tag{3.1.5}$$

Theorem **3.1.2** ([98], [12]). *Fix p in $(0,1)$ and $(s,t)\in\mathbb{R}_+^2$. Then we have the explicit law of large numbers limit*

$$\lim_{N\to\infty}\frac{G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}{N}=\inf_{0<u\le 1}\{s\mathbb{E}(\omega_{1,0}^{(u)})+t\mathbb{E}(\omega_{0,1}^{(u)})\}$$
$$=\begin{cases} s, & t\ge \frac{s}{p}\\ \frac{1}{1-p}\big(2\sqrt{pst}-p(t+s)\big), & ps\le t<\frac{s}{p}\\ t, & t\le ps.\end{cases} \tag{3.1.6}$$

The main theorems of this article verify with probabilistic techniques the variance of $G^{(u)}$ along deterministic directions. For a given boundary parameter u, there will exist a unique direction (m_u,n_u) along which the last passage time at point $N(m_u,n_u)$ time will have variance of order $O(N^{2/3})$ for large N. That is what we call the characteristic direction. The form of the characteristic direction will become apparent from the variance formula in Proposition 3.3.1; it is precisely the direction for which the higher order variance terms cancel out. As it turns out, the characteristic direction ends up being

$$(m_u(N),n_u(N))=\left(N,\left\lfloor\frac{N}{p}(p+(1-p)u)^2\right\rfloor\right). \tag{3.1.7}$$

Throughout the paper we will often compare last passage times over two different boundaries that have different characteristic directions. For this reason we explicitly denote the parameter in the subscript.

Note that as $N\to\infty$, the scaled direction converges to the macroscopic characteristic direction

$$N^{-1}(m_u(N),n_u(N))\to\Big(1,\frac{(p+(1-p)u)^2}{p}\Big), \tag{3.1.8}$$

which gives that for large enough N the endpoint $(m_u(N),n_u(N))$ is always between the two critical lines $y=\frac{x}{p}$ and $y=px$ that separate the flat edges from the strictly concave part of g_{pp}.This defines the macroscopic set of characteristic directions

$$\mathfrak{J}_p=\Big\{\Big(1,\frac{(p+(1-p)u)^2}{p}\Big):u\in(0,1)\Big\}.$$

Note that any $(s,t)\in\mathbb{R}_+^2$ for which $(1,ts^{-1})\in\mathfrak{J}_p$, the shape function g_{pp} has a strictly positive curvature at (s,t).

Theorem **3.1.3.** *Fix a parameter $u\in(0,1)$ and let (m_u,n_u) the characteristic direction corresponding to u as in* (3.1.8) *and large scale approximation, $(m_u(N),n_u(N))$ as in* (3.1.7)*. Then there exists constants C_1 and C_2 that depend on p and u so that*

$$C_1N^{2/3}\le \mathrm{Var}(G^{(u)}_{m_u(N),n_u(N)})\le C_2N^{2/3}. \tag{3.1.9}$$

In the off-characteristic direction, the process $G^{(u)}_{m(N),n(N)}$ satisfies a central limit theorem, and therefore the variance is of order N. This is due to the boundary effect, as we show that maximal paths spend a macroscopic amount of steps along a boundary, and enter the bulk at a point which creates a characteristic rectangle with the projected exit point.

Theorem 3.1.4. *Fix a $c \in \mathbb{R}$. Fix a parameter $u \in (0,1)$ and let (m_u, n_u) the characteristic direction corresponding to u as in (3.1.7). Then for $\alpha \in (2/3, 1]$,*

$$\lim_{N\to\infty} \frac{G^{(u)}_{m_u(N),n_u(N)+\lfloor cN^\alpha\rfloor} - \mathbb{E}[G^{(u)}_{m_u(N),n_u(N)+\lfloor cN^\alpha\rfloor}]}{N^{\alpha/2}} \xrightarrow{\mathcal{D}} Z \sim \mathcal{N}(0, \mathrm{Var}(\omega^{(u)}_{1,0})\mathbb{1}\{c<0\} + \mathrm{Var}(\omega^{(u)}_{0,1})\mathbb{1}\{c>0\}).$$

Remark 3.1.5. *The set $\mathfrak{J}_p$ contains only the directions $(1,t)$ for which $p < t < 1/p$. Any other directions with $t < p$ or $t > p^{-1}$ -that also correspond to the flat edge of the non-boundary model- and for an arbitrary $u \in (0,1)$, are necessarily off-characteristic directions and along those, the last passage time satisfies a central limit theorem.* □

We also have partial results for the model without boundaries. The approach does not allow to access the variance of the non-boundary model directly, but we have

Theorem 3.1.6. *Fix $x, y \in (0,\infty)$ so that $p < y/x < p^{-1}$. Then, there exist finite constants N_0 and $C = C(x,y,p)$, such that, for $b \geq C$, $N \geq N_0$ and any $0 < \alpha < 1$,*

$$\mathbb{P}\{|G_{(1,1),(\lfloor Nx\rfloor,\lfloor Ny\rfloor)} - Ng_{pp}(x,y)| \geq bN^{1/3}\} \leq Cb^{-3\alpha/2}. \tag{3.1.10}$$

In particular, for $N > N_0$, and $1 \leq r < 3\alpha/2$ we get the moment bound

$$\mathbb{E}\left[\left|\frac{G_{(1,1),(\lfloor Nx\rfloor,\lfloor Ny\rfloor)} - Ng_{pp}(x,y)}{N^{1/3}}\right|^r\right] \leq C(x,y,p,r) < \infty. \tag{3.1.11}$$

The bounds in the previous theorem work in directions where the shape function is strictly concave. In directions of flat edge we have

Theorem 3.1.7. *Fix $x, y \in (0,\infty)$ so that $p > y/x$ or $y/x > p^{-1}$. Then, there exist finite constants $c = c(x,y,p)$ and $C = C(x,y,p)$, such that*

$$\mathrm{Var}(G_{(1,1),(\lfloor Nx\rfloor,\lfloor Ny\rfloor)}) \leq CN^2e^{-cN} \to 0 \quad (N\to\infty). \tag{3.1.12}$$

For finer asymptotics on the variance and also weak limits, particularly close to the critical lines $y = px$ and $y = p^{-1}x$ we direct the reader to [45, 47].

For this particular model, the maximal path is not unique - this is because of the discrete nature of the environment distribution, so we need to enforce an a priori condition that makes our choice unique when we refer to it. Unless otherwise specified, the maximal path we select is the *right-most* one (it is also the *down-most* maximal path).

Definition **3.1.8.** *An admissible maximal path from* 0 *to* (m, n)

$$\hat{\pi}_{0,(m,n)} = \{\{(0,0) = \hat{\pi}_0, \hat{\pi}_1, \dots, \hat{\pi}_K = (m,n)\}$$

is the right-most (or down-most) maximal path if and only if it is maximal and if $\hat{\pi}_i = (v_i, w_i) \in \hat{\pi}_{0,(m,n)}$ *then the sites* $(k, \ell), v_i < k < m, 0 \leq \ell < w_i$ *cannot belong on any maximal path from 0 to* (m, n).

In words, no site underneath the right-most maximal path can belong on a different maximal path. An algorithm to construct the right-most path iteratively is given in (3.4.1).

For this right-most path $\hat{\pi}$ we define $\xi^{(u)}$ its exit point from the axes in the environment $\omega^{(u)}$. We indicate with $\xi^{(u)}_{e_1}$ the exit point from the x-axis and $\xi^{(u)}_{e_2}$ the exit point from the y-axis. If $\xi^{(u)}_{e_1} > 0$ the maximal path $\hat{\pi}$ chooses to go through the $x-$axis and $\xi^{(u)}_{e_2} = 0$ and vice versa. If $\xi^{(u)}_{e_1} = \xi^{(u)}_{e_2} = 0$ it means the maximal path directly enters into the bulk with a diagonal step. When we do not need to distinguish from which axes we exit, we just denote the generic exit point by $\xi^{(u)}$.

The exit point $\xi^{(u)}_{e_1}$ represents the exit of the maximal path from level 0. To study the fluctuations of this path around its enforced direction, define

$$v_0(j) = \min\{i \in \{0, \dots, m\} : \exists k \text{ such that } \hat{\pi}_k = (i, j)\}, \tag{3.1.13}$$

and

$$v_1(j) = \max\{i \in \{0, \dots, m\} : \exists k \text{ such that } \hat{\pi}_k = (i, j)\}. \tag{3.1.14}$$

These represent, respectively, the entry and exit point from a fixed horizontal level j of a path $\hat{\pi}$. Since our paths can take diagonal steps, it may be that $v_0(j) = v_1(j)$ for some j.

Now, we can state the theorem which shows that $N^{2/3}$ is the correct order of the magnitude of the path fluctuations. We show that the path stays in an ℓ^1 ball of radius $CN^{2/3}$ with high probability, and simultaneously, avoid balls of radius $\delta N^{2/3}$ again with high probability for δ small enough.

Theorem **3.1.9.** *Consider the last passage time in environment* $\omega^{(u)}$ *and let* $\hat{\pi}_{0,m_u(N),n_u(N)}$ *be the right-most maximal path from the origin up to* $(m_u(N), n_u(N))$ *as in* (3.1.7)*. Fix a* $0 \leq \tau < 1$*. Then, there exist constants* $C_1, C_2 < \infty$ *such that for* $N \geq 1$, $b \geq C_1$

$$\mathbb{P}\{v_0(\lfloor \tau n_u(N) \rfloor) < \tau m_u(N) - bN^{2/3} \text{ or } v_1(\lfloor \tau n_u(N) \rfloor) > \tau m_u(N) + bN^{2/3}\} \leq C_2 b^{-3}. \tag{3.1.15}$$

The same bound holds for vertical displacements.

Moreover, for a fixed $\tau \in (0,1)$ *and given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\lim_{N\to\infty} \mathbb{P}\{\exists k \textit{ such that } |\hat{\pi}_k - (\tau m_u(N), \tau n_u(N))| \le \delta N^{2/3}\} \le \varepsilon. \tag{3.1.16}$$

3.2 Burke's property and law of large numbers

To simplify the notation in what follows, set $w = (i,j) \in \mathbb{Z}_+^2$ and define the last passage time gradients by

$$I^{(u)}_{i+1,j} = G^{(u)}_{i+1,j} - G^{(u)}_{i,j} \quad \text{and} \quad J^{(u)}_{i,j+1} = G^{(u)}_{i,j+1} - G^{(u)}_{i,j}. \tag{3.2.1}$$

When there is no confusion we will drop the superscript (u) from the above. When $j = 0$ we have that $\{I^{(u)}_{i,0}\}_{i,\in\mathbb{N}}$ is a collection of i.i.d. Bernoulli(u) random variables since $I^{(u)}_{i,0} = \omega_{(i,0)}$. Similarly, for $i = 0$, $\{J^{(u)}_{0,j}\}_{j\in\mathbb{N}}$ is a collection of i.i.d. Bernoulli$\left(\frac{p(1-u)}{u+p(1-u)}\right)$ random variables.

The gradients and the passage time satisfy recursive equations. This is the content of the next lemma.

Lemma 3.2.1. *Let* $u \in (0,1)$ *and* $(i,j) \in \mathbb{N}^2$. *Then the last passage time can be recursively computed as*

$$G^{(u)}_{i,j} = \max\left\{G^{(u)}_{i,j-1},\ G^{(u)}_{i-1,j},\ G^{(u)}_{i-1,j-1} + \omega_{i,j}\right\} \tag{3.2.2}$$

Furthermore, the last passage time gradients satisfy the recursive equations

$$\begin{aligned} I^{(u)}_{i,j} &= \max\{\omega_{i,j},\ J^{(u)}_{i-1,j},\ I^{(u)}_{i,j-1}\} - J^{(u)}_{i-1,j} \\ J^{(u)}_{i,j} &= \max\{\omega_{i,j},\ J^{(u)}_{i-1,j},\ I^{(u)}_{i,j-1}\} - I^{(u)}_{i,j-1}. \end{aligned} \tag{3.2.3}$$

Proof. Equation (3.2.2) is immediate from the description of the dynamics in the boundary model and the fact that (i,j) is in the bulk. We only prove the recursive equation (3.2.3) for the J and the other one is done similarly and left to the reader. Compute

$$\begin{aligned} J^{(u)}_{i,j} &= G^{(u)}_{i,j} - G^{(u)}_{i,j-1} \\ &= \max\left\{G^{(u)}_{i,j-1},\ G^{(u)}_{i-1,j},\ G^{(u)}_{i-1,j-1} + \omega_{i,j}\right\} - G^{(u)}_{i,j-1} \qquad \text{by (3.2.2)}, \\ &= \max\left\{0, G^{(u)}_{i-1,j} - G^{(u)}_{i,j-1}, G^{(u)}_{i-1,j-1} - G^{(u)}_{i,j-1} + \omega_{i,j}\right\} \\ &= \max\left\{0, G^{(u)}_{i-1,j} - G^{(u)}_{i-1,j-1} + G^{(u)}_{i-1,j-1} - G^{(u)}_{i,j-1}, G^{(u)}_{i-1,j-1} - G^{(u)}_{i,j-1} + \omega_{i,j}\right\} \\ &= \max\left\{0, J^{(u)}_{i-1,j} - I^{(u)}_{i,j-1}, -I^{(u)}_{i,j-1} + \omega_{i,j}\right\} \\ &= \max\{\omega_{i,j},\ J^{(u)}_{i-1,j},\ I^{(u)}_{i,j-1}\} - I^{(u)}_{i,j-1}. \end{aligned}$$

□

The recursive equations are sufficient to prove a partial independence property.

Lemma 3.2.2. *Assume that $(\omega_{i,j}, I^{(u)}_{i,j-1}, J^{(u)}_{i-1,j})$ are mutually independent with marginal distributions given by*

$$\omega_{i,j} \sim Ber(p), \quad I^{(u)}_{i,j-1} \sim Ber(u), \quad J^{(u)}_{i-1,j} \sim Ber\Big(\frac{p(1-u)}{u+p(1-u)}\Big). \tag{3.2.4}$$

Then, $I^{(u)}_{i,j}, J^{(u)}_{i,j}$, computed using the recursive equations (3.2.3) are independent with marginals $Ber(u)$ and $Ber(\frac{p(1-u)}{u+p(1-u)})$ respectively.

Proof. The marginal distributions are immediate from the definitions and the independence follows when one shows

$$\begin{aligned}
&\mathbb{E}(h(I^{(u)}_{i,j})k(J^{(u)}_{i,j}))\\
&\quad= \mathbb{E}(h(\omega_{i,j} \vee J^{(u)}_{i-1,j} \vee I^{(u)}_{i,j-1} - J^{(u)}_{i-1,j})k(\omega_{i,j} \vee J^{(u)}_{i-1,j} \vee I^{(u)}_{i,j-1} - I^{(u)}_{i,j-1}))\\
&\quad= \mathbb{E}(h(I^{(u)}_{i,j-1})k(J^{(u)}_{i-1,j})).
\end{aligned}$$

for any bounded continuous functions h, k. We omit the details, as they are similar to the proof of Lemma 3.2.4 below. However, in order to prove Lemma 3.2.4, one first needs to prove Lemma 3.2.2. □

A down-right path ψ on the lattice $\mathbb{Z}^2_+$ is an ordered sequence of sites $\{v_i\}_{i\in\mathbb{Z}}$ that satisfy

$$v_i - v_{i-1} \in \{e_1, -e_2\}. \tag{3.2.5}$$

For a given down-right path ψ, define $\psi_i = v_i - v_{i-1}$ to be the i-th edge of the path and set

$$L_{\psi_i} = \begin{cases} I^{(u)}_{v_i}, & \text{if } \psi_i = e_1 \\ J^{(u)}_{v_{i-1}}, & \text{if } \psi_i = -e_2. \end{cases} \tag{3.2.6}$$

The first observation is that the random variables in the collection $\{L_{\psi_i}\}_{i\in\mathbb{Z}}$ satisfy the following:

Lemma 3.2.3. *Fix a down-right path ψ. Then the random variables $\{L_{\psi_i}\}_{i\in\mathbb{Z}}$ are mutually independent, with marginals*

$$L_{\psi_i} \sim \begin{cases} Ber(u), & \text{if } \psi_i = e_1 \\ Ber\Big(\frac{p(1-u)}{u+p(1-u)}\Big), & \text{if } \psi_i = -e_2. \end{cases}$$

Proof. The proof goes by an inductive "corner - flipping" argument: The base case is the path that follows the axes, and there the result follows immediately by the definitions of boundaries. Then we flip the corner at zero, i.e. we consider the down right path

$$\psi^{(1)} = \{\ldots, (0,2), (0,1), (1,1), (1,0), (2,0), \ldots\}.$$

Equivalently, we now consider the collection $\big\{\{J^{(u)}_{0,j}\}_{j\geq 2}, I^{(u)}_{1,1}, J^{(u)}_{1,1}, \{I^{(u)}_{i,0}\}_{i\geq 2}\big\}$. The only place where the independence or the distributions may have been violated, is for $I^{(u)}_{1,1}$, $J^{(u)}_{1,1}$. Lemma 3.2.2 shows this does not happen. As a consequence, variables on the new path satisfy the assumption of Lemma 3.2.2. We can now repetitively use Lemma 3.2.3 by flipping down-right west-south corners into north-east corners. This way, starting from the axes we can obtain any down-right path, while the distributional properties are maintained. The details are left to the reader. □

For any triplet $(\omega_{i,j}, I^{(u)}_{i,j-1}, J^{(u)}_{i-1,j})$ with $i \geq 1, j \geq 1$, we define the event

$$\mathcal{B}_{i,j} = \big\{(\omega_{i,j}, I^{(u)}_{i,j-1}, J^{(u)}_{i-1,j}) \in (1,0,0), (0,1,0), (0,0,1), (1,0,1), (1,1,0)\big\}. \qquad (3.2.7)$$

Using the gradients (3.2.3), the environment $\{\omega_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ and the events $\mathcal{B}_{i,j}$ we also define new random variables $\alpha_{i,j}$ on $\mathbb{Z}^2_+$

$$\alpha_{i-1,j-1} = \mathbb{1}\{I^{(u)}_{i,j-1} = J^{(u)}_{i-1,j} = 1\} + \beta_{i-1,j-1}\mathbb{1}\{\mathcal{B}_{i,j}\} \qquad \text{for } (i,j) \in \mathbb{N}^2. \qquad (3.2.8)$$

$\beta_{i-1,j-1}$ is a $\text{Ber}(p)$ random variable and is independent of everything else. Note that $\alpha_{i-1,j-1}$ is automatically 0 when $\omega_{i,j} = I^{(u)}_{i,j-1} = J^{(u)}_{i-1,j} = 0$ and check, with the help of Lemma 3.2.2, that $\alpha_{i-1,j-1} \overset{D}{=} \omega_{i,j}$. The following lemma gives the distribution of the triple $(I^{(u)}_{i,j}, J^{(u)}_{i,j}, \alpha_{i-1,j-1})$. It is an analogue of Burke's property for $M/M/1$ queues.

Lemma 3.2.4 (Burke's property). *Let $(\omega_{i,j}, I^{(u)}_{i,j-1}, J^{(u)}_{i-1,j})$ mutually independent Bernoulli random variables with distributions*

$$\omega_{i,j} \sim Ber(p), \quad I^{(u)}_{i,j-1} \sim Ber(u), \quad J^{(u)}_{i-1,j} \sim Ber\Big(\frac{p(1-u)}{u+p(1-u)}\Big).$$

Then the random variables $(\alpha_{i-1,j-1}, I^{(u)}_{i,j}, J^{(u)}_{i,j})$ are mutually independent with marginal distributions

$$\alpha_{i-1,j-1} \sim Ber(p), \quad I^{(u)}_{i,j} \sim Ber(u), \quad J^{(u)}_{i,j} \sim Ber\Big(\frac{p(1-u)}{u+p(1-u)}\Big).$$

Proof. Let g, h, k be bounded continuous functions. To simplify the notation slightly, set $\ell = \ell(u) = \frac{p(1-u)}{u+p(1-u)}$. In the computation below we use equations (3.2.3) without special mention.

$$\mathbb{E}(g(\alpha_{i-1,j-1})h(I^{(u)}_{i,j})k(J^{(u)}_{i,j}))$$

$$
\begin{aligned}
&= g(1)\mathbb{E}\Big(h(I^{(u)}_{i,j})k(J^{(u)}_{i,j})\mathbb{1}\{\ I^{(u)}_{i,j-1} = J^{(u)}_{i-1,j} = 1\}\Big)\\
&\qquad + g(0)\mathbb{E}\Big(h(I^{(u)}_{i,j})k(J^{(u)}_{i,j})\mathbb{1}\{\omega_{i,j} = I^{(u)}_{i,j-1} = J^{(u)}_{i-1,j} = 0\}\Big)\\
&\qquad\qquad + \mathbb{E}\Big(g(\beta_{i,j})h(I^{(u)}_{i,j})k(J^{(u)}_{i,j})\mathbb{1}\{\mathcal{B}_{i,j}\}\Big)\\
&= g(1)h(0)k(0)u\ell + g(0)h(0)k(0)(1-p)(1-u)(1-\ell)\\
&\qquad + \mathbb{E}(g(\beta_{i,j}))\mathbb{E}\Big(h(I^{(u)}_{i,j})k(J^{(u)}_{i,j})\mathbb{1}\{\mathcal{B}_{i,j}\}\Big)\\
&= h(0)k(0)(1-u)(1-\ell)(pg(1)+(1-p)g(0))\\
&\qquad + \mathbb{E}(g(\beta_{i,j}))\sum_{x\in\mathcal{B}_{i,j}}\mathbb{E}\Big(h(I^{(u)}_{i,j})k(J^{(u)}_{i,j})\mathbb{1}\{x\in\mathcal{B}_{i,j}\}\Big)\\
&= h(0)k(0)(1-u)(1-\ell)(pg(1)+(1-p)g(0))\\
&\quad + \mathbb{E}(g(\beta_{i,j}))\\
&\qquad \times\Big(h(1)k(1)p(1-u)(1-\ell) + h(0)k(1)[(1-p)(1-u)\ell + p(1-u)\ell)]\\
&\qquad\qquad + h(1)k(0)[(1-p)u(1-\ell)+pu(1-\ell)]\Big)\\
&= h(0)k(0)(1-u)(1-\ell)(pg(1)+(1-p)g(0))\\
&\quad + \mathbb{E}(g(\beta_{i,j}))\Big(h(1)k(1)u\ell + h(0)k(1)(1-u)\ell + h(1)k(0)u(1-\ell)\Big)\\
&= (pg(1)+(1-p)g(0))\mathbb{E}(h(I^{(u)}_{i,j}))\mathbb{E}(k(J^{(u)}_{i,j}))\\
&= \mathbb{E}(g(\alpha_{i-1,j-1}))\mathbb{E}(h(I^{(u)}_{i,j}))\mathbb{E}(k(J^{(u)}_{i,j})). \qquad \square
\end{aligned}
$$

The last necessary preliminary step is a corollary of Lemma 3.2.4 which generalises Lemma 3.2.3 by incorporating the random variables $\{\alpha_{i-1,j-1}\}_{i,j\geq 1}$. To this effect, for any down-right path ψ satisfying (3.2.5), define the interior sites $\mathcal{I}_\psi$ of ψ to be

$$
\mathcal{I}_\psi = \{w \in \mathbb{Z}^2_+ : \exists\, v_i \in \psi \text{ s.t. } w < v_i \text{ coordinate-wise}\}. \tag{3.2.9}
$$

Then

Corollary 3.2.5. *Fix a down-right path ψ and recall definitions (3.2.6), (3.2.9). The random variables*

$$
\{\{\alpha_w\}_{w\in\mathcal{I}_\psi}, \{L_{\psi_i}\}_{i\in\mathbb{Z}}\}
$$

are mutually independent, with marginals

$$
\alpha_w \sim Ber(p), \quad L_{\psi_i} \sim \begin{cases} Ber(u), & \text{if } \psi_i = e_1\\ Ber\Big(\frac{p(1-u)}{u+p(1-u)}\Big), & \text{if } \psi_i = -e_2. \end{cases}
$$

The proof is similar to that of Lemma 3.2.3 and equal to that of Corollary 4.2.3 in Chapter 4. Therefore we omit it here.

3.2.1 Law of large numbers for the boundary model.

Proof of Theorem 3.1.1. From equations (3.2.1) we can write

$$G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} = \sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{0,j} + \sum_{i=1}^{\lfloor Ns\rfloor} I^{(u)}_{i,\lfloor Nt\rfloor}$$

since the I, J variables are increments of the passage time. By the definition of the boundary model, the variables are i.i.d. $\text{Ber}(p(1-u)/(u+p(1-u))$. Scaled by N, the first sum converges to $t\mathbb{E}(J_{0,1})$ by the law of large numbers.

By Corollary 3.2.5 they are i.i.d. $\text{Ber}(u)$, since they belong on the down-right path that follows the vertical axes from ∞ down to $(0, \lfloor Nt\rfloor)$ and then moves horizontally. We cannot immediately appeal to the law of large numbers as the whole sequence changes with N so we first appeal to the Borel-Cantelli lemma via a large deviation estimate. Fix an $\varepsilon > 0$.

$$\begin{aligned}\mathbb{P}\Big\{N^{-1}\sum_{i=1}^{\lfloor Ns\rfloor} I^{(u)}_{i,\lfloor Nt\rfloor} \notin (u-\varepsilon, u+\varepsilon)\Big\} &= \mathbb{P}\Big\{N^{-1}\sum_{i=1}^{\lfloor Ns\rfloor} I^{(u)}_{i,0} \notin (su-\varepsilon, su+\varepsilon)\Big\}\\ &\le e^{-c(u,s,\varepsilon)N},\end{aligned}$$

for some appropriate positive constant $c(u, s, \varepsilon)$. By the Borel-Cantelli lemma we have almost sure that for each $\varepsilon > 0$ there exists a random N_ε so that for all $N > N_\varepsilon$

$$su - \varepsilon < N^{-1}\sum_{i=1}^{\lfloor Ns\rfloor} I^{(u)}_{i,\lfloor Nt\rfloor} \le su + \varepsilon.$$

Then we have

$$su + t\frac{p(1-u)}{u+p(1-u)} - \varepsilon \le \varliminf_{N\to\infty} \frac{G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}{N} \le \varlimsup_{N\to\infty} \frac{G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}{N} \le su + t\frac{p(1-u)}{u+p(1-u)} + \varepsilon.$$

Let ε tend to 0 to finish the proof. □

3.2.2 Law of large numbers for the i.i.d. model

Proof of Theorem 3.1.2. Let $g_{pp}(s,t) = \lim_{N\to\infty} N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ and denote by $g^{(u)}_{pp}(s,t) = \lim_{N\to\infty} N^{-1}G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$. Recall that $g_{pp}(s,t)$ is 1-homogeneous and concave.

The starting point is equation (3.1.4). Scaling that equation by N gives us the macroscopic variational formulation

$$\begin{aligned}&g^{(u)}_{pp}(1,1)\\ &\quad= \sup_{0\le z\le 1}\{g^{(u)}_{pp}(z,0) + g_{pp}(1-z,1)\} \bigvee \sup_{0\le z\le 1}\{g^{(u)}_{pp}(0,z) + g_{pp}(1,1-z)\}\\ &\quad= \sup_{0\le z\le 1}\{z\mathbb{E}(I^{(u)}) + g_{pp}(1-z,1)\} \bigvee \sup_{0\le z\le 1}\{z\mathbb{E}(J^{(u)}) + g_{pp}(1,1-z)\}. \qquad (3.2.10)\end{aligned}$$

We postpone the proof of (3.2.10) until the end. Assume (3.2.10) holds. Observe that since $g_{pp}(s,t)$ is symmetric then $g_{pp}(1-z,1) = g_{pp}(1,1-z)$ which we abbreviate with $g_{pp}(1-z,1) = \psi(1-z)$. Therefore

$$g^{(u)}_{pp}(1,1) = \sup_{0\le z\le 1}\{z\mathbb{E}(I^{(u)}) + \psi(1-z)\} \bigvee \sup_{0\le z\le 1}\{z\mathbb{E}(J^{(u)}) + \psi(1-z)\}. \tag{3.2.11}$$

Moreover if $u \in [\frac{\sqrt{p}}{1+\sqrt{p}}, 1]$ then $\mathbb{E}(I^{(u)}) \ge \mathbb{E}(J^{(u)})$. We restrict the parameter u to the subset $u \in [\frac{\sqrt{p}}{1+\sqrt{p}}, 1]$ of its original range $u \in (0,1]$. Then we can drop the second expression in the braces from the right-hand side of (3.2.11) because at each z-value the first expression in braces dominates. Then

$$u + \frac{p(1-u)}{u+p(1-u)} = \sup_{0\le z\le 1}\{zu + \psi(1-z)\}. \tag{3.2.12}$$

Set $x = 1-z$. x still ranges in $[0,1]$ and after a rearrangement of the terms, we obtain

$$-\frac{p(1-u)}{u+p(1-u)} = \inf_{0\le x\le 1}\{xu - \psi(x)\}. \tag{3.2.13}$$

The expression on the right-hand side is the Legendre transform of ψ, and we have that its concave dual $\psi^*(u) = -\frac{p(1-u)}{u+p(1-u)}$ with $u \in [\frac{\sqrt{p}}{1+\sqrt{p}}, 1]$. Since $\psi(x)$ is concave, the Legendre transform of ψ^* will give back ψ, i.e. $\psi^{**} = \psi$. Therefore,

$$\begin{aligned} g_{pp}(x,1) = \psi(x) = \psi^{**}(x) &= \inf_{\frac{\sqrt{p}}{1+\sqrt{p}}\le u\le 1}\{xu - \psi^*(u)\} = \inf_{\frac{\sqrt{p}}{1+\sqrt{p}}\le u\le 1}\Big\{xu + \frac{p(1-u)}{u+p(1-u)}\Big\} \\ &= \inf_{\frac{\sqrt{p}}{1+\sqrt{p}}\le u\le 1}\big\{x\mathbb{E}(I^{(u)}) + \mathbb{E}(J^{(u)})\big\}, \quad \text{for all } x \in [0,1]. \end{aligned} \tag{3.2.14}$$

Since $g_{pp}(s,t) = tg_{pp}(st^{-1},1)$, the first equality in (3.1.6) is now verified. For the second equality, we solve the variational problem (3.2.14). The derivative of the expression in the braces has a critical point $u^* \in [\frac{\sqrt{p}}{1+\sqrt{p}}, 1]$ only when $p < x < 1$. In that case, the infimum is achieved at

$$u^* = \frac{1}{1-p}\Big(\sqrt{\frac{p}{x}} - p\Big)$$

and $g_{pp}(x,1) = 1/(1-p)[2\sqrt{xp} - p(x+1)]$. Otherwise, when $x \le p$ the first derivative for $u \in [\frac{\sqrt{p}}{1+\sqrt{p}}, 1]$ is always negative and the minimum occurs at $u^* = 1$. This gives $g_{pp}(x,1) = x$. Again, extend to all (s,t) via the relation $g_{pp}(s,t) = tg_{pp}(st^{-1},1)$. This concludes the proof for the explicit shape under (3.2.10) which we now prove.

For a lower bound, fix any $z \in [0,1]$. Then

$$G^{(u)}_{N,N} \ge \sum_{i=1}^{\lfloor Nz\rfloor} I^{(u)}_{i,0} + G_{(\lfloor Nz\rfloor,1),(N,N)}.$$

Divide through by N. The left hand side converges a.s. to $g_{pp}^{(u)}(1,1)$. The first term on the right converges a.s. to $z\mathbb{E}(I^u)$. The second on the right, converges in probability to the constant $g_{pp}(1-z,1)$. In particular, we can find a subsequence N_k such that the convergence is almost sure for the second term. Taking limits on this subsequence, we conclude

$$g_{pp}^{(u)}(1,1) \geq z\mathbb{E}(I^{(u)}) + g_{pp}(1-z,1).$$

Since z is arbitrary we can take supremum over z in both sides of the equation above. The same arguments will work if we move on the vertical axis. Thus, we obtain the lower bound for (3.2.10). For the upper bound, fix $\varepsilon > 0$ and let $\{0 = q_0, \varepsilon = q_1, 2\varepsilon = q_2, \ldots, \lfloor \varepsilon^{-1} \rfloor \varepsilon, 1 = q_M\}$ a partition of $(0,1)$. We partition both axes. The maximal path that optimises $G_{N,N}^{(u)}$ has to exit between $\lfloor Nk\varepsilon \rfloor$ and $\lfloor N(k+1)\varepsilon \rfloor$ for some k. Therefore, we may write

$$\begin{aligned} G_{N,N}^{(u)} \leq \max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} &\left\{ \sum_{i=1}^{\lfloor N(k+1)\varepsilon \rfloor} I_{i,0}^{(u)} + G_{(\lfloor Nk\varepsilon \rfloor, 1),(N,N)} \right\} \\ &\bigvee \max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \left\{ \sum_{j=1}^{\lfloor N(k+1)\varepsilon \rfloor} J_{0,j}^{(u)} + G_{1,(\lfloor Nk\varepsilon \rfloor),(N,N)} \right\}. \end{aligned}$$

Divide by N. The right-hand side converges in probability to the constant

$$\begin{aligned} &\max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \{(k+1)\varepsilon u + g_{pp}(1 - \varepsilon k, 1)\} \\ &\qquad \bigvee \max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \left\{ (k+1)\varepsilon \frac{p(1-u)}{u + p(1-u)} + g_{pp}(1, 1 - \varepsilon k) \right\} \\ &= \max_{q_k} \{q_k u + g_{pp}(1 - q_k, 1)\} + \varepsilon u \\ &\qquad \bigvee \max_{q_k} \left\{ q_k \frac{p(1-u)}{u + p(1-u)} + g_{pp}(1, 1 - q_k) \right\} + \varepsilon \frac{p(1-u)}{u + p(1-u)} \\ &\leq \sup_{0 \leq z \leq 1} \{zu + g_{pp}(1-z, 1)\} + \varepsilon u \\ &\qquad \bigvee \max_{0 \leq z \leq 1} \left\{ z \frac{p(1-u)}{u + p(1-u)} + g_{pp}(1, 1-z) \right\} + \varepsilon \frac{p(1-u)}{u + p(1-u)}. \end{aligned}$$

The convergence becomes a.s. on a subsequence. The upper bound for (3.2.10) now follows by letting $\varepsilon \to 0$ in the last inequality. □

In the following sections, either when the explicit dependence on u is not important or when it is not necessary and there will be no confusion, we omit the superscripts (u) from the gradients I, J without a particular mention.

3.3 Upper bound for the variance in characteristic directions

We follow the approach of [9, 102] in order to find the order of the variance. All the difficulties and technicalities in our case arise from two facts: First that the random variables are discrete and small perturbations in the distribution do not alter the value of the random weight. Second, we have three potential steps to contest with rather than then usual two.

3.3.1 The variance formula

Let (m, n) be a generic lattice site. Eventually we will define how m, n grow to infinity using the parameter u. Define the passage time increments (labelled by compass directions) by

$$\mathcal{W} = G^{(u)}_{0,n} - G^{(u)}_{0,0}, \quad \mathcal{N} = G^{(u)}_{m,n} - G^{(u)}_{0,n}, \quad \mathcal{E} = G^{(u)}_{m,n} - G^{(u)}_{m,0}, \quad \mathcal{S} = G^{(u)}_{m,0} - G^{(u)}_{0,0}.$$

From Corollary 3.2.5 we get that each of $\mathcal{W}, \mathcal{N}, \mathcal{E}$ and $\mathcal{S}$ is a sum of i.i.d. random variables and most importantly, $\mathcal{N}$ is independent of $\mathcal{E}$ and $\mathcal{W}$ is independent of $\mathcal{S}$ by the definition of the boundary random variables. From the definitions it is immediate to show the cocycle property for the whole rectangle $[m] \times [n]$

$$\mathcal{W} + \mathcal{N} = G^{(u)}_{m,n} = \mathcal{S} + \mathcal{E}. \tag{3.3.1}$$

We can immediately attempt to evaluate the variance of $G^{(u)}_{m,n}$ using these relations, by

$$\begin{aligned} \mathrm{Var}(G^{(u)}_{m,n}) &= \mathrm{Var}(\mathcal{W} + \mathcal{N}) \\ &= \mathrm{Var}(\mathcal{W}) + \mathrm{Var}(\mathcal{N}) + 2\,\mathrm{Cov}(\mathcal{S} + \mathcal{E} - \mathcal{N}, \mathcal{N}) \\ &= \mathrm{Var}(\mathcal{W}) - \mathrm{Var}(\mathcal{N}) + 2\,\mathrm{Cov}(\mathcal{S}, \mathcal{N}), \end{aligned} \tag{3.3.2}$$

Equivalently, one may use $\mathcal{E}$ and $\mathcal{S}$ to obtain

$$\mathrm{Var}(G^{(u)}_{m,n}) = \mathrm{Var}(\mathcal{S}) - \mathrm{Var}(\mathcal{E}) + 2\,\mathrm{Cov}(\mathcal{E}, \mathcal{W}). \tag{3.3.3}$$

In the sequence, when several Bernoulli parameters will need to be considered simultaneously, will add a superscript (u) on the quantities $\mathcal{N}, \mathcal{E}, \mathcal{W}, \mathcal{S}$ to explicitly denote dependance on parameter u.

The covariance is not an object that can be computed directly, so the biggest proportion of this subsection is dedicated in finding a different way to compute the covariance that also allows for estimates and connects fluctuations of the maximal path with fluctuations of the last passage time.

In the exponential exactly solvable model there is a clear expression for the covariance term [9]. Unfortunately this does not happen here, so we must estimate the order of magnitude. This is the content of the next proposition.

Proposition 3.3.1. *Fix $0 < u \leq 1$. There exist functions $A_{\mathcal{N}(u)}$, $A_{\mathcal{E}(u)}$ such that for any $m, n \in \mathbb{N}$ we have*

$$\begin{aligned}\mathrm{Var}(G_{m,n}^{(u)}) &= n\frac{pu(1-u)}{[u+p(1-u)]^2} - mu(1-u) + 2u(1-u)A_{\mathcal{N}(u)} \\ &= mu(1-u) - n\frac{pu(1-u)}{[u+p(1-u)]^2} - 2u(1-u)A_{\mathcal{E}(u)}.\end{aligned} \tag{3.3.4}$$

The result is proved by perturbing the parameter on one of the boundaries. Throughout the proof, the endpoint (m, n) and the parameter u are fixed. Pick an $\varepsilon > 0$ and define a new parameter u_ε so that $u_\varepsilon = u + \varepsilon < 1$. The only way this is not possible is when $u = 1$. If that's the case, $G_{m,n}^{(1)} = m$ is deterministic and the variance is zero. Equation (3.3.4) remains true as the right-hand side is a multiple of $(1 - u)$.

For any fixed realization of $\omega^{(u)} = \{\omega_{i,0}^{(u)}, \omega_{0,j}^{(u)}, \omega_{i,j}^{(u)}\}$ with marginal distributions (3.1.3) we use the parameter ε to modify the weights on the south boundary only. Define new bernoulli weights ω^{u_ε} via the conditional distributions

$$\begin{aligned}&\mathbb{P}\{\omega_{i,0}^{u_\varepsilon} = 1 | \omega_{i,0}^{(u)} = 1\} = 1, \\ &\mathbb{P}\{\omega_{i,0}^{u_\varepsilon} = 1 | \omega_{i,0}^{(u)} = 0\} = \frac{\varepsilon}{1-u}, \\ &\mathbb{P}\{\omega_{i,0}^{u_\varepsilon} = 0 | \omega_{i,0}^{(u)} = 0\} = 1 - \frac{\varepsilon}{1-u},\end{aligned} \tag{3.3.5}$$

i.e. we go through the values on the south boundary, and conditioning on the environment returned a 0, we change the value to a 1 with probability $\frac{\varepsilon}{1-u}$. Then $\{\omega_{i,0}^{u_\varepsilon}\}_{1 \leq i \leq m}$ is a collection of independent $\mathrm{Ber}(u_\varepsilon)$ r.v. . It is convenient to introduce an algebraic mechanism to construct ω^{u_ε} directly. To this effect introduce a sequence of independent Bernoulli random variables $\mathscr{H}_i^{(\varepsilon)} \sim \mathrm{Ber}\big(\frac{\varepsilon}{1-u}\big)$, $1 \leq i \leq m$ that are independent of the $\omega^{(u)}$. Denote their joint distribution by μ_ε. Then construct ω^{u_ε} the following way:

$$\omega_{i,0}^{u_\varepsilon} = \mathscr{H}_i^{(\varepsilon)} \vee \omega_{i,0}^{(u)}. \tag{3.3.6}$$

Check that (3.3.6) satisfies (3.3.5). It also follows that

$$\omega_{i,0}^{u_\varepsilon} - \omega_{i,0}^{(u)} \leq \mathscr{H}_i^{(\varepsilon)}. \tag{3.3.7}$$

Under this modified environment,

$$\omega_{i,0}^{u_\varepsilon} \sim \text{Ber}(u_\varepsilon), \quad \omega_{i,j}^{(u)} \sim \text{Ber}(p), \quad \omega_{0,j}^{(u)} \sim \text{Ber}\Big(\frac{p(1-u)}{u+p(1-u)}\Big), \tag{3.3.8}$$

the passage time is denoted by G^{u_ε} and when we are referring to quantities in this model we will distinguish them with a superscript u_ε. With these definitions we have $\mathcal{S}^{u_\varepsilon} \sim \text{Bin}(m, u+\varepsilon)$, with mass function denoted by $f_{\mathcal{S}^{u_\varepsilon}}(k) = \mathbb{P}\{\mathcal{S}^{u_\varepsilon} = k\}$, $0 \le k \le m$.

Similarly, there will be instances for which we want to perturb only the weights of the vertical axis, again when the parameter will change from u to $u+\varepsilon$. In that case, we denote the modified environment by $\mathcal{W}^{u_\varepsilon}$ and it is given by

$$\omega_{i,0}^{(u)} \sim \text{Ber}(u), \quad \omega_{i,j}^{(u)} \sim \text{Ber}(p), \quad \omega_{0,j}^{u_\varepsilon} \sim \text{Ber}\Big(\frac{p(1-u-\varepsilon)}{u+\varepsilon+p(1-u-\varepsilon)}\Big), \tag{3.3.9}$$

Again, we use auxiliary i.i.d. Bernoulli variables $\{\mathscr{V}_j^{(\varepsilon)}\}_{1\le j\le n}$ with

$$\mathscr{V}_j^{(\varepsilon)} \sim \text{Ber}\Big(1 - \varepsilon\frac{1+u(1-p)}{(1-u)(p+u(1-p))+(1-p)\varepsilon}\Big),$$

where we assume that ε is sufficiently small so that the distributions are well defined. Then, the perturbed weights on the vertical axis are defined by

$$\omega_{0,j}^{u_\varepsilon} = \omega_{0,j}^{(u)} \cdot \mathscr{V}_j^{(\varepsilon)}. \tag{3.3.10}$$

Denote by ν_ε the joint distribution of $\mathscr{V}_j^{(\varepsilon)}$. It will also be convenient to couple the environments with different parameters. In that case we use common realizations of i.i.d. Uniform$[0,1]$ random variables $\eta = \{\eta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$. The Bernoulli environment in the bulk is then defined as

$$\omega_{i,j} = \mathbb{1}\{\eta_{i,j} < p\}$$

and similarly defined for the boundary values. The joint distribution for the uniforms we denote by $\mathbb{P}_\eta$.

Proposition 3.3.2. *The following bounds in terms of the right-most exit points of the maximal paths hold*

$$A_{\mathcal{N}^{(u)}} = \begin{cases} \frac{\text{Cov}(\mathcal{S}^{(u)},\mathcal{N}^{(u)})}{u(1-u)} = \lim_{\varepsilon\to 0} \dfrac{\mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}(\mathcal{N}^{u_\varepsilon} - \mathcal{N}^{(u)})}{\varepsilon}, & 0<u<1 \\ 0 & u=0,1. \end{cases} \tag{3.3.11}$$

Similarly,

$$A_{\mathcal{E}^{(u)}} = \begin{cases} \frac{\text{Cov}(\mathcal{W}^{(u)},\mathcal{E}^{(u)})}{u(1-u)} = \lim_{\varepsilon\to 0} \dfrac{\mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}(\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)})}{\varepsilon}, & 0<u<1 \\ 0 & u=0,1. \end{cases} \tag{3.3.12}$$

Proof of Proposition 3.3.1. The conditional joint distribution of $(\omega^{u_\varepsilon}_{i,0})_{1\le i\le m}$ given the value of their sum $\mathcal{S}^{u+\varepsilon}$ is independent of the parameter ε. This is because the sum of i.i.d. Bernoulli is a sufficient statistic for the parameter of the distribution. In particular this implies that $E[\mathcal{N}^{u+\varepsilon}|\mathcal{S}^{u+\varepsilon}=k] = E_{\mathbb{P}\otimes\mu_\varepsilon}[\mathcal{N}^{(u)}|\mathcal{S}^{(u)}=k]$. For clarity, we added the superscript (u) on the background measure $\mathbb{P}$ to emphasise that it is the measure on environment $\omega^{(u)}$.

Then we can compute the $\mathbb{E}(\mathcal{N}^{u+\varepsilon})$

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}(\mathcal{N}^{u_\varepsilon}-\mathcal{N}^{(u)}) &= \sum_{k=0}^{m} E[\mathcal{N}^{u_\varepsilon}|\mathcal{S}^{u_\varepsilon}=k]\mathbb{P}\otimes\mu_\varepsilon\{\mathcal{S}^{u_\varepsilon}=k\} - \mathbb{E}_{\mathbb{P}}(\mathcal{N}^{(u)})\\
&= \sum_{k=0}^{m} E[\mathcal{N}^{(u)}|\mathcal{S}^{(u)}=k]\mathbb{P}\otimes\mu_\varepsilon\{\mathcal{S}^{u_\varepsilon}=k\} - \mathbb{E}_{\mathbb{P}}(\mathcal{N}^{(u)})\\
&= \sum_{k=0}^{m} E[\mathcal{N}^{(u)}|\mathcal{S}^{(u)}=k]\big(\mathbb{P}\otimes\mu_\varepsilon\{\mathcal{S}^{u_\varepsilon}=k\} - \mathbb{P}\{\mathcal{S}^{(u)}=k\}\big) \qquad (3.3.13)
\end{aligned}
$$

To show that the limits in the statement are well defined, it suffices to compute

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\frac{\mathbb{P}\otimes\mu_\varepsilon\{\mathcal{S}^{u_\varepsilon}=k\}-\mathbb{P}\{\mathcal{S}^{(u)}=k\}}{\varepsilon}\\
&\quad=\binom{m}{k}\lim_{\varepsilon\to0}\frac{(u+\varepsilon)^k(1-u-\varepsilon)^{m-k}-u^k(1-u)^{m-k}}{\varepsilon}\\
&\quad=\binom{m}{k}\frac{d}{du}u^k(1-u)^{m-k}=\binom{m}{k}\frac{k-mu}{u(1-u)}u^k(1-u)^{m-k}.
\end{aligned}
$$

Combine this with (3.3.13) to obtain

$$
\begin{aligned}
\lim_{\varepsilon\to0}\frac{\mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}(\mathcal{N}^{u_\varepsilon}-\mathcal{N}^{(u)})}{\varepsilon} &= \frac{1}{u(1-u)}\sum_{k=0}^{m}E[\mathcal{N}^{(u)}|\mathcal{S}^{(u)}=k]k\mathbb{P}\{\mathcal{S}^{(u)}=k\}\\
&\qquad-\frac{mu}{u(1-u)}\sum_{k=0}^{m}E[\mathcal{N}^{(u)}|\mathcal{S}^{(u)}=k]\mathbb{P}\{\mathcal{S}^{(u)}=k\}\\
&= \frac{1}{u(1-u)}\Big(\mathbb{E}(\mathcal{N}^{(u)}\mathcal{S}^{(u)})-\mathbb{E}(\mathcal{N}^{(u)})\mathbb{E}(\mathcal{S}^{(u)})\Big)\\
&= \frac{1}{u(1-u)}\mathrm{Cov}(\mathcal{N}^{(u)},\mathcal{S}^{(u)}). \qquad (3.3.14)
\end{aligned}
$$

Identical symmetric arguments, prove the remaining part of the proposition. $\square$

For the rest of this proof, we prove the estimates on $A_{\mathcal{N}^{(u)}}$ by estimating the covariance in a different way.

Fix any boundary site $w=(w_1,w_2)\in\{(i,0),(0,j):1\le i\le m, 1\le j\le n\}$. The total weight in environment ω collected on the boundaries by a path that exits from the axes at w is

$$
\mathscr{S}_w = \sum_{i=1}^{w_1}\omega_{i,0}+\sum_{j=1}^{w_2}\omega_{0,j}, \qquad (3.3.15)
$$

where the empty sum takes the value 0. Let $\mathscr{S}^{(u)}$ be the above sum in environment $\omega^{(u)}$ and let $\mathscr{S}^{u_e}$ denote the same, but in environment (3.3.8).

Recall that ξ_{e_1} is the rightmost exit point of any potential maximal path from the horizontal boundary, since it is the exit point of the right-most maximal path. Similarly, if $\xi_{e_2} > 0$, ξ_{e_2} is the down-most possible exit point. When the dependence on the parameter u is important, we put superscripts (u) to denote that.

Lemma 3.3.3. *Let ξ_{e_1} be the exit point of the maximal path in environment $\omega^{(u)}$. Let $\mathcal{N}^{u_\varepsilon}$ denote the last passage increment in environment (3.3.8) of the north boundary and $\mathscr{S}^{u_\varepsilon}_{\xi_{e_1}}$ the weight collected on the horizontal axis in the same environment, but only up to the exit point of the maximal path in environment $\omega^{(u)}$. $\mathcal{N}^{(u)}$ $\mathscr{S}^{(u)}_{\xi_{e_1}}$ are the same quantities in environment $\omega^{(u)}$. Then*

$$\mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}(\mathscr{S}^{u_\varepsilon}_{\xi_{e_1}} - \mathscr{S}^{(u)}_{\xi_{e_1}}) \le \mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}(\mathcal{N}^{u_\varepsilon} - \mathcal{N}^{(u)}) \le \mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}(\mathscr{S}^{u_\varepsilon}_{\xi_{e_1}} - \mathscr{S}^{(u)}_{\xi_{e_1}}) + C(m,u,p)\varepsilon^{3/2}. \tag{3.3.16}$$

Similarly, in environments (3.3.9) and $\omega^{(u)}$,

$$\mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}(\mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} - \mathscr{S}^{(u)}_{\xi_{e_2}}) \ge \mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}(\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)}) \ge \mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}(\mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} - \mathscr{S}^{(u)}_{\xi_{e_2}}) - C(n,u,p)\varepsilon^{4/3}. \tag{3.3.17}$$

Proof. We only prove (3.3.17) as the same arguments work for (3.3.16). Modify the weights on the vertical axis and create environment $\mathcal{W}^{u_\varepsilon}$ given by (3.3.9). The first inequality in equation (3.3.17) follows by first noting that

$$\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)} \le \mathcal{W}^{u_\varepsilon} - \mathcal{W} \le 0. \tag{3.3.18}$$

The left inequality in (3.3.17) is then immediate, because the modification decreases all weights on the west boundary by (3.3.10). To see the inequality in (3.3.18), do a double induction on m, n using equations (3.2.3) and the cocycle property (3.3.1), starting from the first corner square.

The remaining proof is to establish the second inequality in (3.3.17). Consider the event $\{\xi^{u_\varepsilon} \ne \xi\}$. Since we only modify weights on the vertical axis, the exit point ξ of the original maximal path will be different from ξ^{u_ε} only if $\xi^{u_\varepsilon} = \xi^{u_\varepsilon}_{e_2}$. Moreover, since the modification actually decreases the weights, one of two things may happen:

1. $\xi^{u_\varepsilon} \ne \xi$ and $\mathscr{S}^{u_\varepsilon}_{\xi^{u_\varepsilon}_{e_2}} + G_{(1,\xi^{u_\varepsilon}_{e_2}),(m,n)} > \mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} + G_{(1,\xi_{e_2}),(m,n)}$, or
2. $\xi^{u_\varepsilon} \ne \xi$ and $\mathscr{S}^{u_\varepsilon}_{\xi^{u_\varepsilon}_{e_2}} + G_{(1,\xi^{u_\varepsilon}_{e_2}),(m,n)} = \mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} + G_{(1,\xi_{e_2}),(m,n)}$

We use these cases to define two auxiliary events:

$$\mathcal{A}_1 = \{\xi^{u_\varepsilon} \ne \xi \text{ and } \mathscr{S}^{u_\varepsilon}_{\xi^{u_\varepsilon}_{e_2}} + G_{(1,\xi^{u_\varepsilon}_{e_2}),(m,n)} > \mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} + G_{(1,\xi_{e_2}),(m,n)}\},$$

$$\mathcal{A}_2 = \{\xi^{u_\varepsilon} \neq \xi \text{ and } \mathscr{S}^{u_\varepsilon}_{\xi^{u_\varepsilon}_{e_2}} + G_{(1,\xi^{u_\varepsilon}_{e_2}),(m,n)} = \mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} + G_{(1,\xi_{e_2}),(m,n)}\}$$

and note that $\{\xi^{u_\varepsilon} \neq \xi\} = \mathcal{A}_1 \cup \mathcal{A}_2$. On $\mathcal{A}_2$ we can bound

$$\begin{aligned}\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)} = G^{u_\varepsilon}_{m,n} - G^{(u)}_{m,n} &= \mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} + G_{(1,\xi_{e_2}),(m,n)} - \mathscr{S}^{(u)}_{\xi_{e_2}} - G_{(1,\xi_{e_2}),(m,n)} \\ &= \mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} - \mathscr{S}^{(u)}_{\xi_{e_2}}.\end{aligned}$$

Then we estimate

$$\begin{aligned}\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)} &= (\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)}) \cdot \mathbb{1}\{\xi^{u_\varepsilon} = \xi\} + (\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)}) \cdot \mathbb{1}\{\xi^{u_\varepsilon} \neq \xi\} \\ &= (\mathscr{S}^{u_\varepsilon}_{\xi} - \mathscr{S}^{(u)}_{\xi}) \cdot \mathbb{1}\{\xi^{u_\varepsilon} = \xi\} + (\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)}) \cdot (\mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\mathcal{A}_2}) \qquad (3.3.19)\\ &\geq (\mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} - \mathscr{S}^{(u)}_{\xi_{e_2}}) + (\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)}) \cdot \mathbb{1}_{\mathcal{A}_1}. \qquad (3.3.20)\end{aligned}$$

The last inequality is justified in the following way: Only the weights on the vertical axis were changed-actually decreased. Therefore, if the maximal path chose to move horizontally before the modification, it would do so after and the first term in (3.3.19) must be 0. The first term may not equal zero only when the maximal path takes a vertical first step before the modification. On the event $\mathbb{1}\{\xi^{u_\varepsilon} = \xi\}$ the bound in (3.3.20) still holds.

To bound the second term of (3.3.20), we use Hölder's inequality with exponents $p = 3, q = 3/2$ to obtain

$$\mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}((\mathcal{E}^{(u)} - \mathcal{E}^{u_\varepsilon}) \cdot \mathbb{1}_{\mathcal{A}_1}) \leq \mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}((\mathcal{E}^{(u)} - \mathcal{E}^{u_\varepsilon})^3)^{1/3}(\mathbb{P}\otimes\nu_\varepsilon\{\mathcal{A}_1\})^{2/3}. \qquad (3.3.21)$$

The first expectation on the right is bounded above by $C(u,p)n$ since $\mathcal{E}^{(u)}$ is a sum of i.i.d. Bernoulli random variables that bounds above $\mathcal{E}^{(u)} - \mathcal{E}^{u_\varepsilon}$.

Now to bound the probability. Consider the equality of events

$$\begin{aligned}\mathcal{A}_1 &= \{\mathscr{S}^{u_\varepsilon}_k + G_{(1,k),(m,n)} > \mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} + G_{(1,\xi_{e_2}),(m,n)} \text{ for some } 0 \leq k \neq \xi \leq n\} \\ &= \{\mathscr{S}^{u_\varepsilon}_k - \mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} > G_{(1,\xi_{e_2}),(m,n)} - G_{(1,k),(m,n)} \text{ for some } 0 \leq k \neq \xi \leq n\} \\ &= \{\mathscr{S}^{u_\varepsilon}_k - \mathscr{S}^{u_\varepsilon}_{\xi_{e_2}} > G_{(1,\xi_{e_2}),(m,n)} - G_{(1,k),(m,n)} \geq \mathscr{S}^{(u)}_k - \mathscr{S}^{(u)}_{\xi_{e_2}} \\ &\qquad\qquad \text{for some } 0 \leq k \neq \xi \leq n\}.\end{aligned}$$

Coupling (3.3.10) implies that the events above are empty when $k > \xi_{e_2}$. Therefore, consider the case $\xi_{e_2} > k$. In that case, since ξ_{e_2} is the down-most possible exit point, the second inequality in the event above can be strict as well. Thus

$$\mathcal{A}_1 \subseteq \bigcup_{(k,i):0\leq k<i\leq n} \{\mathscr{S}^{u_\varepsilon}_k - \mathscr{S}^{u_\varepsilon}_i > G_{(1,i),(m,n)} - G_{(1,k),(m,n)} > \mathscr{S}^{(u)}_k - \mathscr{S}^{(u)}_i\}.$$

The strict inequalities in the event and the fact that these random variables are integer, we see that the difference $\mathscr{S}_k^{u_\varepsilon} - \mathscr{S}_i^{u_\varepsilon} - \mathscr{S}_k^{(u)} + \mathscr{S}_i^{(u)} \geq 2$. This way, for some k, i

$$\begin{aligned} 2 \leq \mathscr{S}_k^{u_\varepsilon} - \mathscr{S}_i^{u_\varepsilon} - \mathscr{S}_k^{(u)} + \mathscr{S}_i^{(u)} &= - \sum_{j=k+1}^{i} \omega_{0,j}^{u_\varepsilon} + \sum_{j=k+1}^{i} \omega_{0,j}^{(u)} \\ &= \sum_{j=k+1}^{i} \left(\omega_{0,j}^{(u)} - \omega_{0,j}^{u_\varepsilon}\right) \quad \text{by (3.3.10)} \\ &\leq \sum_{j=0}^{n} \left(\omega_{0,j}^{(u)} - \omega_{0,j}^{u_\varepsilon}\right) = \sum_{j=0}^{n} \omega_{0,j}^{(u)}\big(1 - \mathscr{V}_j^{(\varepsilon)}\big) = \mathscr{W}_\varepsilon. \end{aligned} \tag{3.3.22}$$

$\mathscr{W}_\varepsilon$ is defined by the last equality above and we therefore just showed $\mathcal{A}_1 \subseteq \{\mathscr{W}_\varepsilon \geq 2\}$.

The event $\{\mathscr{W}_\varepsilon \geq 2\}$ holds if at least 2 indices j satisfy with $\omega_{0,j}^{(u)}\big(1 - \mathscr{V}_j^{(\varepsilon)}\big) = 1$. By definition (3.3.22) we have that $\mathscr{W}_\varepsilon$ is binomially distributed with probability of success $C\varepsilon$ under $\mathbb{P} \otimes \nu_\varepsilon$ and therefore, in order to have at least two successes,

$$\mathbb{P} \otimes \nu_\varepsilon\{\mathscr{W}_\varepsilon \geq 2\} \leq C(n,u)\varepsilon^2. \tag{3.3.23}$$

Combine (3.3.20) and (3.3.23) to conclude

$$\mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}(\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)}) \geq \mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}(\mathscr{S}_{\xi_{e_2}}^{u_\varepsilon} - \mathscr{S}_{\xi_{e_2}}^{(u)}) - C(n,u)\varepsilon^{4/3}. \tag{3.3.24}$$

□

Lemma 3.3.4. *Let $0 < u < 1$. Then,*

$$A_{\mathcal{N}^{(u)}} \leq \frac{\mathbb{E}(\xi_{e_1}^{(u)})}{1-u}, \qquad \textit{and} \qquad A_{\mathcal{E}^{(u)}} \geq -\frac{p(1+u(1-p))}{(u+p(1-u))^2}\mathbb{E}(\xi_{e_2}^{(u)}) \tag{3.3.25}$$

Proof. Now we bound the first term. Compute

$$\begin{aligned} \mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}(\mathscr{S}_{\xi_{e_1}}^{u_\varepsilon} - \mathscr{S}_{\xi_{e_1}}^{(u)}) &= \sum_{y=1}^{m} E\Big[\mathscr{S}_y^{u_\varepsilon} - \mathscr{S}_y^{(u)}\Big|\xi_{e_1} = y\Big]\mathbb{P}\{\xi_{e_1} = y\} \\ &\leq \sum_{y=1}^{m} E\Big[\sum_{i=1}^{y} \mathscr{H}_i^{(\varepsilon)}\Big|\xi_{e_1} = y\Big]\mathbb{P}\{\xi_{e_1} = y\}, \text{ from (3.3.7)}, \\ &= \sum_{y=1}^{m} \mathbb{E}_{\mu_\varepsilon}\Big[\sum_{i=1}^{y} \mathscr{H}_i^{(\varepsilon)}\Big]\mathbb{P}\{\xi_{e_1} = y\}, \text{ since } \mathscr{H}_i,\, \omega^{(u)} \text{ independent}, \\ &= \varepsilon\frac{\mathbb{E}(\xi_{e_1})}{1-u}. \end{aligned}$$

Now substitute in (3.3.16), divide through by ε and take the limit as $\varepsilon \to 0$ to obtain

$$\lim_{\varepsilon\to 0} \frac{\mathbb{E}(\mathcal{N}^{\lambda_\varepsilon} - \mathcal{N})}{\varepsilon} \leq \frac{\mathbb{E}(\xi_{e_1}^{(u)})}{1-u}.$$

For the second bound, write

$$\mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}(\mathcal{E}^{u_\varepsilon} - \mathcal{E}^{(u)}) \geq \mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}(\mathscr{S}_{\xi_{e_2}^{(u)}}^{u_\varepsilon} - \mathscr{S}_{\xi_{e_2}^{(u)}}^{(u)}) + o(\varepsilon)$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}\Big(\sum_{j=1}^{\xi_{e_2}^{(u)}} \omega_{0,j}^{u_\varepsilon} - \omega_{0,j}^{(u)}\Big) + o(\varepsilon) = -\mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}\Big(\sum_{j=1}^{\xi_{e_2}^{(u)}} \omega_{0,j}^{(u)}(1 - \mathscr{V}_j^{(\varepsilon)})\Big) + o(\varepsilon)\\
&= -\sum_{k=1}^{n}\sum_{j=1}^{k} \mathbb{E}_{\mathbb{P}\otimes\nu_\varepsilon}\big(\omega_{0,j}^{(u)}(1 - \mathscr{V}_j^{(\varepsilon)})\mathbb{1}\{\xi_{e_2}^{(u)} = k\}\big) + o(\varepsilon)\\
&= -\sum_{k=1}^{n}\sum_{j=1}^{k} \mathbb{E}_{\mathbb{P}}\big(\omega_{0,j}^{(u)}\mathbb{1}\{\xi_{e_2}^{(u)} = k\}\big)\mathbb{E}_{\nu_\varepsilon}(1 - \mathscr{V}_j^{(\varepsilon)}) + o(\varepsilon)\\
&\geq -\sum_{k=1}^{n}\sum_{j=1}^{k} \mathbb{P}\{\xi_{e_2}^{(u)} = k\}E_{\nu_\varepsilon}(1 - \mathscr{V}_j^{(\varepsilon)}) + o(\varepsilon)\\
&= -\varepsilon\mathbb{E}_{\mathbb{P}}(\xi_{e_2}^{(u)}) \cdot \frac{1 + u(1-p)}{(1-u)(u + p(1-u)) + (1-p)\varepsilon} + o(\varepsilon).
\end{aligned}
$$

Divide both sides of the inequality by ε and let it tend to 0. □

Lemma 3.3.5. *Let $0 < r_1 < r_2 < 1$ and let $\xi^{(r_i)}$ the corresponding right-most (resp. down-most) exit points for the maximal paths in environments coupled by common uniforms η. Then*

$$
\xi_{e_1}^{(r_1)} \leq \xi_{e_1}^{(r_2)} \text{ and } \xi_{e_2}^{(r_1)} \geq \xi_{e_2}^{(r_2)}.
$$

Proof. Assume that in environment $\omega^{(r_1)}$ the maximal path exits from the vertical axis. Then, since $r_2 > r_1$ and the weights coupled through common uniforms, realization by realization $\omega_{0,j}^{(r_2)} \leq \omega_{0,j}^{(r_1)}$. Assume by way of contradiction that $\xi_{e_2}^{(r_1)} < \xi_{e_2}^{(r_2)}$. Then

$$
\begin{aligned}
G_{(1,\xi_{e_2}^{(r_1)}),(m,n)} &\geq G_{(1,\xi_{e_2}^{(r_2)}),(m,n)} + \mathscr{S}^{(r_1)}_{\xi_{e_2}^{(r_2)}} - \mathscr{S}^{(r_1)}_{\xi_{e_2}^{(r_1)}}\\
&\geq G_{(1,\xi_{e_2}^{(r_2)}),(m,n)} + \mathscr{S}^{(r_2)}_{\xi_{e_2}^{(r_2)}} - \mathscr{S}^{(r_2)}_{\xi_{e_2}^{(r_1)}},
\end{aligned}
$$

giving

$$
G_{(0,\xi_{e_2}^{(r_1)}),(m,n)} + \mathscr{S}^{(r_2)}_{\xi_{e_2}^{(r_1)}} \geq G_{(0,\xi_{e_2}^{(r_2)}),(m,n)} + \mathscr{S}^{(r_2)}_{\xi_{e_2}^{(r_2)}} = G^{(r_2)}_{m,n},
$$

which cannot be true because $\xi_{e_2}^{(r_2)}$ is the down-most exit point in $\omega^{(r_2)}$. The proof for a maximal path exiting the horizontal axis is similar. □

3.3.2 Upper bound

In this section we prove the upper bound in Theorem (3.1.3). We begin with three comparison lemmas. One is for the two functions $A_{\mathcal{N}^{(u)}}$ that appear in Proposition 3.3.1 when we vary the parameter. The other comparison is between variances in environments with different parameters.

Lemma 3.3.6. *Pick two parameters $0 < r_1 < r_2 < 1$. Then*

$$
\mathcal{A}_{\mathcal{N}^{(r_1)}} \leq \mathcal{A}_{\mathcal{N}^{(r_2)}} + m(r_2 - r_1). \tag{3.3.26}
$$

Proof of Lemma 3.3.6. Fix an $\varepsilon > 0$ small enough so that $r_1 + \varepsilon < r_2$ and $r_2 + \varepsilon < 1$. This is not a restriction as we will let ε tend to 0 at the end of the proof. We use a common realization of the Bernoulli variables $\mathscr{H}_i^{(\varepsilon)}$ and we couple the weights in the $\omega^{(r_2)}$ and $\omega^{(r_1)}$ environments using common uniforms $\eta = \{\eta_{i,j}\}$ (with law $\mathbb{P}_\eta$), independent of the $\mathscr{H}_i^{(\varepsilon)}$.

We need to bound in a different way starting from the line before (3.3.20).

$$\begin{aligned}
\mathscr{N}^{u+\varepsilon} - \mathscr{N}^{(u)} &= (\mathscr{S}_\xi^{u+\varepsilon} - \mathscr{S}_\xi^{(u)}) \cdot \mathbb{1}\{\xi^{u+\varepsilon} = \xi\} + (\mathscr{N}^{u+\varepsilon} - \mathscr{N}) \cdot (\mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\mathcal{A}_2}) \\
&= (\mathscr{S}_{\xi_{e_1}}^{u+\varepsilon} - \mathscr{S}_{\xi_{e_1}}^{(u)}) \cdot \mathbb{1}\{\xi_{e_1}^{u+\varepsilon} = \xi_{e_1}\} + (\mathscr{N}^{u+\varepsilon} - \mathscr{N}) \cdot (\mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\mathcal{A}_2}) \\
&= (\mathscr{S}_{\xi_{e_1}}^{u+\varepsilon} - \mathscr{S}_{\xi_{e_1}}^{(u)}) + (\mathscr{N}^{u+\varepsilon} - \mathscr{N} - (\mathscr{S}_{\xi_{e_1}}^{u+\varepsilon} - \mathscr{S}_{\xi_{e_1}}^{(u)})) \cdot (\mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\mathcal{A}_2}). \quad (3.3.27)
\end{aligned}$$

We first show that the second term can never be negative. Write

$$\begin{aligned}
\mathscr{N}^{u+\varepsilon} - \mathscr{N} - (\mathscr{S}_{\xi_{e_1}}^{u+\varepsilon} - \mathscr{S}_{\xi_{e_1}}^{(u)}) &= G_{m,n}^{u+\varepsilon} - G_{m,n}^{(u)} - (\mathscr{S}_{\xi_{e_1}}^{u+\varepsilon} - \mathscr{S}_{\xi_{e_1}}^{(u)}) \\
&= \mathscr{S}_{\xi_{e_1}^{u+\varepsilon}}^{u+\varepsilon} + G_{(\xi_{e_1}^{u+\varepsilon},1)(m,n)} - G_{(\xi_{e_1},1)(m,n)} - \mathscr{S}_{\xi_{e_1}}^{u+\varepsilon}.
\end{aligned}$$

On $\mathcal{A}_2$ this expression is 0. On $\mathcal{A}_1$ the sum of the first two terms is strictly larger than the sum of the last two. Then, (3.3.27) becomes

$$\mathscr{N}^{u+\varepsilon} - \mathscr{N}^{(u)} \geq \mathscr{S}_{\xi_{e_1}}^{u+\varepsilon} - \mathscr{S}_{\xi_{e_1}}^{(u)}.$$

Use this to bound the first term in the computation that follows. The second term we bound with (3.3.16).

$$\begin{aligned}
&E_{\mu_\varepsilon \otimes \mathbb{P}_\eta}(\mathscr{N}^{r_2+\varepsilon} - \mathscr{N}^{(r_2)}) - E_{\mu_\varepsilon \otimes \mathbb{P}_\eta}(\mathscr{N}^{r_1+\varepsilon} - \mathscr{N}^{(r_1)}) \\
&\quad \geq E_{\mu_\varepsilon \otimes \mathbb{P}_\eta}(\mathscr{S}_{\xi_{e_1}^{(r_2)}}^{r_2+\varepsilon} - \mathscr{S}_{\xi_{e_1}^{(r_2)}}^{(r_2)}) - E_{\mu_\varepsilon \otimes \mathbb{P}_\eta}(\mathscr{S}_{\xi_{e_1}^{(r_1)}}^{r_1+\varepsilon} - \mathscr{S}_{\xi_{e_1}^{(r_1)}}^{(r_1)}) + o(\varepsilon) \\
&\quad = E_{\mu_\varepsilon \otimes \mathbb{P}_\eta}\Big(\sum_{i=1}^{\xi_{e_1}^{(r_2)}} \mathbb{1}\{\mathscr{H}_i^{(\varepsilon)} = 1\} \mathbb{1}\{\eta_{i,0} > r_2\} \Big) \\
&\qquad\qquad - E_{\mu_\varepsilon \otimes \mathbb{P}_\eta}\Big(\sum_{i=1}^{\xi_{e_1}^{(r_1)}} \mathbb{1}\{\mathscr{H}_i^{(\varepsilon)} = 1\} \mathbb{1}\{\eta_{i,0} > r_1\} \Big) + o(\varepsilon) \\
&\quad \geq E_{\mu_\varepsilon \otimes \mathbb{P}_\eta}\Big(\sum_{i=1}^{\xi_{e_1}^{(r_1)}} \mathbb{1}\{\mathscr{H}_i^{(\varepsilon)} = 1\} \Big(\mathbb{1}\{\eta_{i,0} > r_2\} - \mathbb{1}\{\eta_{i,0} > r_1\} \Big) \Big) + o(\varepsilon) \\
&\quad \geq -m E_{\mu_\varepsilon \otimes \mathbb{P}_\eta}\Big(\mathbb{1}\{\mathscr{H}_i^{(\varepsilon)} = 1\} \Big(\mathbb{1}\{\eta_{i,0} > r_1\} - \mathbb{1}\{\eta_{i,0} > r_2\} \Big) \Big) + o(\varepsilon) \\
&\quad = -m\varepsilon(r_2 - r_1) + o(\varepsilon).
\end{aligned}$$

Divide by ε and let $\varepsilon \to 0$ to get the result. ∎

Lemma 3.3.7 (Variance comparison). *Fix $\delta_0 > 0$ and parameters u, r so that $p < p+\delta_0 < u < r < 1$. Then, there exists a constant $C = C(\delta_0, p) > 0$ so that for all admissible values of u and r we have*

$$\frac{\mathrm{Var}(G^{(u)}_{m,n})}{u(1-u)} \le \frac{\mathrm{Var}(G^{(r)}_{m,n})}{r(1-r)} + C(m+n)(r-u). \tag{3.3.28}$$

Proof. Begin from equation (3.3.4), and bound

$$\begin{aligned}
\frac{\mathrm{Var}(G^{(u)}_{m,n})}{u(1-u)} &= n\frac{p}{[u+p(1-u)]^2} - m + 2A_{\mathcal{N}^{(u)}}\\
&= n\frac{p}{[r+p(1-r)]^2} - m + 2A_{\mathcal{N}^{(u)}} + np(\frac{1}{[u+p(1-u)]^2} - \frac{1}{[r+p(1-r)]^2})\\
&\le \frac{\mathrm{Var}(G^{(r)}_{m,n})}{r(1-r)} + np(\frac{1}{[u+p(1-u)]^2} - \frac{1}{[r+p(1-r)]^2}) + 2m(r-u)\\
&\le \frac{\mathrm{Var}(G^{(r)}_{m,n})}{r(1-r)} + 2np(1-p)\frac{(r-u)}{[u+p(1-u)]^3} + 2m(r-u)\\
&\le \frac{\mathrm{Var}(G^{(r)}_{m,n})}{r(1-r)} + 2n\frac{p(1-p)}{\delta_0^3}(r-u) + 2m(r-u).
\end{aligned}$$

In the third line from the top we used Lemma 3.3.6. Set $C = 2\frac{p(1-p)}{\delta_0^3} \vee 2$ to finish the proof. $\square$

From this point onwards we proceed by a perturbative argument. We introduce the scaling parameter N that will eventually go to ∞ and the characteristic shape of the rectangle, given the boundary parameter. We will need to use the previous lemma, so we fix a $\delta_0 > 0$, so that $\delta_0 < \lambda < 1$ and we choose a parameter $u = u(N, b, v) < \lambda$ so that

$$\lambda - u = b\frac{v}{N}$$

At this point v is free but b is a constant so that $\delta_0 < \lambda < u$. The north-east endpoint of the rectangle with boundary of parameter λ is defined by $(m_\lambda(N), n_\lambda(N))$ which is the microscopic characteristic direction corresponding to λ defined in (3.1.7).

The quantities $G_{(\xi_{e_2},1),(m,n)}, \xi_{e_2}$ and $G_{m,n}$ connected to these indices are denoted by $G_{(\xi_{e_2},1),(m,n)}(N), \xi_{e_2}(N), G_{m,n}(N)$. In the proof we need to consider different boundary conditions and this will be indicated by a superscript. When the superscript u will be used, the reader should remember that this signifies changes on the boundary conditions and not the endpoint $(m_\lambda(N), n_\lambda(N))$, which will always be defined by (3.1.7) for a fixed λ.

Since the weights $\{\omega_{i,j}\}_{i,j\ge1}$ in the interior are not affected by changes in boundary conditions, the passage time $G_{(z,1),(m,n)}(N)$ will not either, for any $z < m_\lambda(N)$.

Proposition 3.3.8. *Fix $\lambda \in (0,1)$. Then, there exists a constant $K = K(\lambda, p) > 0$ so that for any $b < K$, and N sufficiently large*

$$\mathbb{P}\{\xi_{e_2}^{(\lambda)}(N) > v\} \le C\frac{N^2}{bv^3}\Big(\frac{\mathbb{E}(\xi_{e_2}^{(\lambda)})}{bv} + 1\Big), \tag{3.3.29}$$

for all $v \ge 1$.

Proof. We use an auxiliary parameter $u < \lambda$ so that

$$u = \lambda - bvN^{-1} > 0.$$

Constant b is under our control. We abbreviate $(m_\lambda(N), n_\lambda(N)) = \mathbf{t}_N(\lambda)$. Whenever we use auxiliary parameters we explicitly mention it to alert the reader that the environments are coupled through common realizations of uniform random variables η. The measure that we are using for all computations is the background measure $\mathbb{P}_\eta$ but to keep the notation simple we omit the subscript η.

Since $G^{(u)}_{\mathbf{t}_N(\lambda)}(N)$ is utilised on the maximal path,

$$\mathscr{S}^{(u)}_z + G_{(1,z),\mathbf{t}_N(\lambda)}(N) \le G^{(u)}_{\mathbf{t}_N(\lambda)}(N)$$

for all $1 \le z \le n_\lambda(N)$ and all parameters $p + \delta_0 < u < \lambda < 1$. Consequently, for integers $v \ge 0$,

$$\begin{aligned}
\mathbb{P}\{\xi_{e_2}^{(\lambda)}(N) > v\} &= \mathbb{P}\{\exists\, z > v : \mathscr{S}^{(\lambda)}_z + G_{(1,z),\mathbf{t}_N(\lambda)}(N) = G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)\}\\
&\le \mathbb{P}\{\exists\, z > v : \mathscr{S}^{(\lambda)}_z - \mathscr{S}^{(u)}_z + G^{(u)}_{\mathbf{t}_N(\lambda)}(N) \ge G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)\}\\
&= \mathbb{P}\{\exists\, z > v : \mathscr{S}^{(\lambda)}_z - \mathscr{S}^{(u)}_z + G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N) \ge 0\}\\
&\le \mathbb{P}\{\mathscr{S}^{(\lambda)}_v - \mathscr{S}^{(u)}_v + G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N) \ge 0\}.
\end{aligned} \tag{3.3.30}$$

The last line above follows from the fact that $u < \lambda$, which implies that $\mathscr{S}^{(\lambda)}_k - \mathscr{S}^{(u)}_k$ is non-positive and decreasing in k when the weights are coupled through common uniforms. The remaining part of the proof goes into bounding the last probability above. For any $\alpha \in \mathbb{R}$ we further bound

$$\mathbb{P}\{\xi_{e_2}^{(\lambda)}(N) > v\} \le \mathbb{P}\{\mathscr{S}^{(\lambda)}_v - \mathscr{S}^{(u)}_v \ge -\alpha\} \tag{3.3.31}$$

$$+ \mathbb{P}\{G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N) \ge \alpha\}. \tag{3.3.32}$$

We treat (3.3.31) and (3.3.32) separately for

$$\alpha = -\mathbb{E}[\mathscr{S}^{(\lambda)}_v - \mathscr{S}^{(u)}_v] - C_0\frac{v^2}{N} \tag{3.3.33}$$

where $C_0 > 0$. Restrictions on C_0 will be enforced in the course of the proof.

Probability (3.3.31)**:** That is a sum of i.i.d. random variables so we simply bound using Chebyshev's inequality. The variance is estimated by

$$\mathrm{Var}(\mathscr{S}_v^{(\lambda)} - \mathscr{S}_v^{(u)}) = \sum_{j=1}^{v} \mathrm{Var}\left(\omega_{0,j}^{(\lambda)} - \omega_{0,j}^{(u)}\right) \leq C_{p,\lambda} v(\lambda - u) = c_{p,\lambda} \frac{bv^2}{N}.$$

Then by Chebyshev's inequality we obtain

$$\mathbb{P}\Big\{\mathscr{S}_v^{(\lambda)} - \mathscr{S}_v^{(u)} \geq \mathbb{E}[\mathscr{S}_v^{(\lambda)} - \mathscr{S}_v^{(u)}] + C_0 \frac{v^2}{N}\Big\} \leq \frac{c_{p,\lambda}}{C_0^2} \cdot b \frac{N}{v^2}. \tag{3.3.34}$$

Probability (3.3.32)**:** Substitute in the value of α and subtract from both sides $\mathbb{E}[G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)]$. Then

$$\begin{aligned}
&\mathbb{P}\{G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N) \geq \alpha\} \\
&\quad = \mathbb{P}\{G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N) - \mathbb{E}[G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)] \\
&\qquad\qquad \geq v(\lambda - u)\frac{p}{(p+(1-p)u)(p+(1-p)\lambda)} - C_0 \frac{v^2}{N} - \mathbb{E}[G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)]\} \\
&\quad \leq \mathbb{P}\{G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N) - \mathbb{E}[G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)] \\
&\qquad\qquad \geq v(\lambda - u)C_{\lambda,p} - C_0 \frac{v^2}{N} - \mathbb{E}[G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)]\}.
\end{aligned} \tag{3.3.35}$$

where

$$C_{\lambda,p} = \frac{p}{(p+(1-p)\lambda)^2}.$$

We then estimate

$$\begin{aligned}
\mathbb{E}[G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)] &= m_\lambda(N)(u-\lambda) + n_\lambda(N)\Big(\frac{p(1-u)}{u+p(1-u)} - \frac{p(1-\lambda)}{\lambda + p(1-\lambda)}\Big) \\
&= m_\lambda(N)(u-\lambda) - n_\lambda(N)\frac{p}{(p+(1-p)u)(p+(1-p)\lambda)}(u-\lambda) \\
&\leq N\frac{1-p}{p+(1-p)u}(\lambda - u)^2 \\
&\leq \frac{D_{u,p}}{N} b^2 v^2.
\end{aligned}$$

The first inequality above comes from removing the integer parts for $n_\lambda(N)$. The constant $D_{u,p}$ is defined as

$$D_{u,p} = \frac{1-p}{p+(1-p)u}.$$

It is now straightforward to check that line (3.3.35) is non-negative when

$$b < \frac{C_{\lambda,p}}{4D_{u,p}} \quad \text{and} \quad C_0 = b\frac{C_{\lambda,p}}{2}.$$

With values of b, C_0 as are in the display above, for any c smaller than $b\,C_{\lambda,p}/4$, we have that the difference

$$G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N) - G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - \mathbb{E}[G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N) - G^{(u)}_{\mathbf{t}_N(\lambda)}(N)] \geq cv^2N^{-1} > 0.$$

Using this, we can apply Chebyshev's inequality one more time. In order, from Chebyshev's inequality, Lemma 3.3.7 and finally Proposition 3.3.1

$$\begin{aligned}
\text{Probability}(3.3.35) &\le \mathbb{P}\{|G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N) \\
&\qquad - \mathbb{E}[G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)]| \ge cv^2N^{-1}\} \\
&\le \frac{N^2}{c^2v^4}\operatorname{Var}(G^{(u)}_{\mathbf{t}_N(\lambda)}(N) - G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)) \\
&\le \frac{N^2}{c^2v^4}\Big(\operatorname{Var}(G^{(u)}_{\mathbf{t}_N(\lambda)}(N)) + \operatorname{Var}(G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N))\Big) \\
&\le 4\frac{N^2}{c^2v^4}\Big(\operatorname{Var}(G^{(\lambda)}_{\mathbf{t}_N(\lambda)}(N)) + CN(\lambda - u)\Big) \\
&\le 4\frac{N^2}{c^2v^4}|A_{\mathcal{E}^{(\lambda)}}| + Cb\frac{N^2}{c^2v^3}.
\end{aligned}$$

This together with the bound in Lemma 3.3.4 suffice for the conclusion of this proposition.

□

Proof of Theorem 3.1.3, upper bound. We first bound the expected exit point for boundary with parameter λ. In what follows, r is a parameter under our control, that will eventually go to ∞.

$$\begin{aligned}
\mathbb{E}(\xi^{(\lambda)}_{e_2}(N)) &\le rN^{2/3} + \sum_{v=rN^{2/3}}^{n_\lambda(N)} \mathbb{P}\{\xi^{(\lambda)}_{e_2}(N) > v\} \\
&\le rN^{2/3} + \sum_{v=rN^{2/3}}^{\infty} C\frac{N^2}{v^3}\Big(\frac{\mathbb{E}(\xi^{(\lambda)}_{e_2})}{v} + 1\Big) \quad \text{by}(3.3.29) \\
&\le rN^{2/3} + \frac{C\mathbb{E}(\xi^{(\lambda)}_{e_2})}{r^3} + \frac{C}{r^2}N^{2/3}.
\end{aligned}$$

Let r sufficiently large so that $C/r^3 < 1$. Then, after rearranging the terms in the inequality above, we conclude

$$\mathbb{E}(\xi^{(\lambda)}_{e_2}(N)) \le CN^{2/3}.$$

The variance bound follows from this, Lemma 3.3.4 and equation (3.3.4) when m, n satisfy (3.1.7). □

An immediate corollary of this is the following bound in probability that is obtained directly from expression (3.3.29) is

Corollary 3.3.9. *Fix $\lambda \in (0,1)$. Then, there exists a constant $K = K(\lambda, p) > 0$ so that for any $r > 0$, and N sufficiently large*

$$\mathbb{P}\{\xi^{(\lambda)}_{e_2}(N) > rN^{2/3}\} \le \frac{K}{r^3}. \tag{3.3.36}$$

3.4 Lower bound for the variance in characteristic directions

3.4.1 Down-most maximal path and Competition interface

In this section first we want to construct the down-most maximal path and a possible competition interface. Then we identify their properties and relations which will be crucial to find the lower bound for the order of fluctuations of the maximal path.

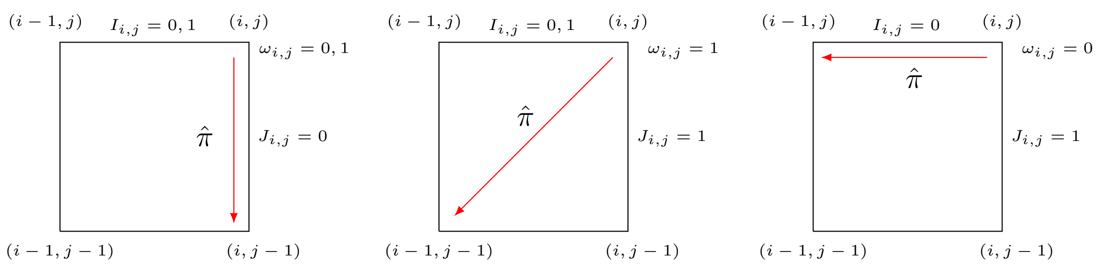
The down-most maximal path

Consider a triple $(I_{i,j}, J_{i,j}, \omega_{i,j})$ defined in (3.2.4), and keep in mind the increment definition (3.2.1). Recall that the maximal path in the interior process collects weights only with a diagonal step with probability given by ω. We define the down-most maximal path $\hat{\pi}$ starting from the target point (m, n) and going backward following the rules

$$\hat{\pi}_{k+1} = \begin{cases} \hat{\pi}_k + (0,1) & \text{if } G(\hat{\pi}_k + (0,1)) = G(\hat{\pi}_k), \\ \hat{\pi}_k + (1,0) & \text{if } G(\hat{\pi}_k + (1,-1)) < G(\hat{\pi}_k + (0,1)) \text{ and } \omega_{\hat{\pi}_k+(1,0)} = 0, \\ \hat{\pi}_k + (1,1) & \text{if } G(\hat{\pi}_k) = G(\hat{\pi}_k + (1,0)) \text{ and } \omega_{\hat{\pi}_k+(1,1)} = 1. \end{cases} \tag{3.4.1}$$

The moment that $\hat{\pi}$ hits one of the two axes (or the origin) it starts to collect from the axis, which it has hit, down to the origin.
The maximal path $\hat{\pi}$ can be formalized in the following way.

The graphical representation is in Figure 3.1.



(a) Combination of I, J and ω for a down $(-e_2)$ step.

(b) Combination of I, J and ω for a diagonal step.

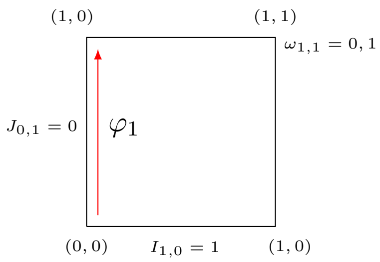
(c) Combination of I, J and ω for a left $(-e_1)$ step.

Figure 3.1: One-step backward construction for the down-most maximal path $\hat{\pi}$.

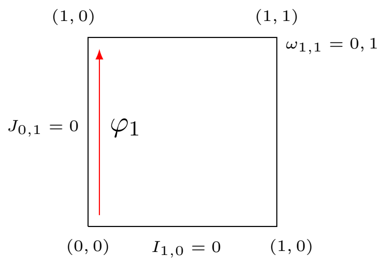
The competition interface

The competition interface is an infinite path φ which takes only the same admissible steps as the paths we optimise over. $\varphi = \{\varphi_0 = (0,0), \varphi_1, \ldots\}$ is completely determined by the values of I, J and ω. In particular, for any $k \in \mathbb{N}$,

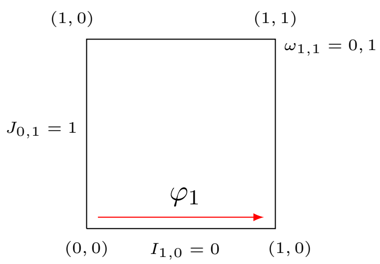
$$\varphi_{k+1} = \begin{cases} \varphi_k + (0,1) & \text{if } \begin{array}{l} G(\varphi_k + (0,1)) < G(\varphi_k + (1,0)) \text{ or} \\ G(\varphi_k + (0,1)) = G(\varphi_k + (1,0)) \text{ and } G(\varphi_k + (0,1)) = G(\varphi_k), \end{array} \\ \varphi_k + (1,0) & \text{if } G(\varphi_k + (1,0)) < G(\varphi_k + (0,1)), \\ \varphi_k + (1,1) & \text{if } G(\varphi_k + (0,1)) = G(\varphi_k + (1,0)) \text{ and } G(\varphi_k + (0,1)) > G(\varphi_k). \end{cases} \tag{3.4.2}$$



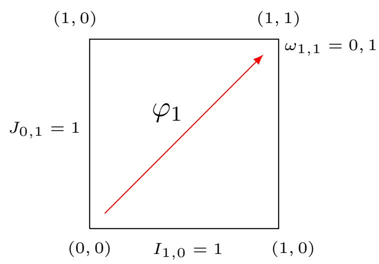
(a) Combination of I, J and ω for an up step.



(b) Combination of I, J and ω for an up step.



(c) Combination of I, J and ω for a right step.



(d) Combination of I, J and ω for a diagonal step.

Figure 3.2: Constructive admissible steps for φ_1.

In words, the path φ always chooses its step according to the smallest of the possible G-values. If they are equal, the competition interface decides to go up if the last passage time of the up and right point are equal and they are also equal to the last passage time of the starting point otherwise it takes a diagonal step.

Remark 3.4.1. *In literature the name competition interface comes from the fact that it represents the threshold interface between the points which will be reached by the maximal* *path whose first step is right or up. Since our model is discrete, and we have three (rather than two) possible steps and our maximal path is not unique, our definition of φ depends on our choice of maximal path; here we chose the down-most path as our maximal path and then we accordingly defined the competition interface, so that we exploit certain good duality properties in the sequence.* $\square$

This being said, the partition of the plain into the two competing clusters is useful in some parts of the proofs that follow, so we would like to develop it in this setting. Define

$$\mathcal{C}_{\uparrow,\nearrow} = \{v = (v_1, v_2) \in \mathbb{Z}_+^2 : \text{ there exists a maximal path from 0 to } v \\ \text{with first step } e_2 \text{ or } e_1 + e_2\}.$$

The remaining sites of $\mathbb{Z}_+^2$ are sites for which all possible maximal paths to them *have to* take a horizontal first step. We denote that cluster by $\mathcal{C}_{\rightarrow} = \mathbb{Z}_+^2 \setminus \mathcal{C}_{\uparrow,\nearrow}$.

Some immediate observations follow. First note that the vertical axis $\{(0, v_2)\}_{v_2 \in \mathbb{N}} \in \mathcal{C}_{\uparrow,\nearrow}$ while $\{(v_1, 0)\}_{v_1 \in \mathbb{N}} \in \mathcal{C}_{\rightarrow}$. We include $(0,0) \in \mathcal{C}_{\uparrow,\nearrow}$ in a vacuous way.

Then observe that if $(v_1, v_2) \in \mathcal{C}_{\uparrow,\nearrow}$ then it has to be that $(v_1, y) \in \mathcal{C}_{\uparrow,\nearrow}$ for all $y \geq v_2$. This is a consequence of planarity. Assume that for some $y > v_2$ the maximal path $\pi_{0,(v_1,y)}$ has to take a horizontal first step. Then it will intersect with the maximal path $\pi_{0,(v_1,v_2)}$ to (v_1, v_2) with a non-horizontal first step. At the point of intersection z, the two passage times are the same, so in fact there exists a maximal path to (v_1, y) with a non-horizontal first step: it is the concatenation of the $\pi_{0,(v_1,v_2)}$ up to site z and from z onwards we follow $\pi_{0,(v_1,y)}$.

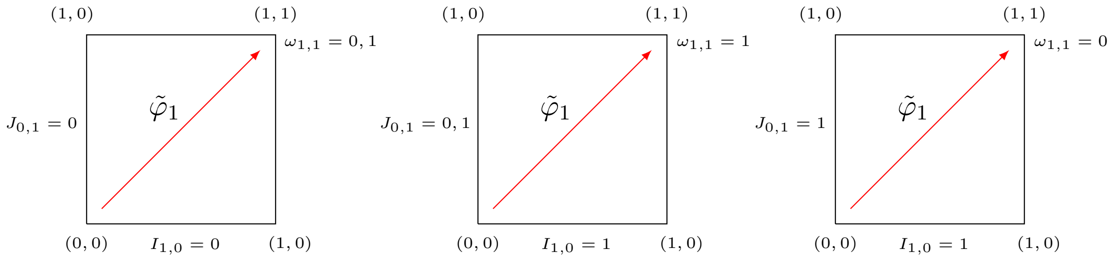
Finally, note that if $v \neq 0$ and $v \in \mathcal{C}_{\uparrow,\nearrow}$ and $v + e_1 \in \mathcal{C}_{\rightarrow}$, it must be the case that

$$I_{v+e_1} = G_{0,v+e_1} - G_{0,v} = 1.$$

Assume the contrary. Then, if the two passage times are the same, a potential maximal path to $v + e_1$ is the one that goes to v without a horizontal initial step, and after v it takes an e_1 step. This would also imply that $v + e_1 \in \mathcal{C}_{\uparrow,\nearrow}$ which is a contradiction.

These observations allow us to define a boundary between the two clusters as a piecewise linear curve $\tilde{\varphi} = \{0 = \tilde{\varphi}_0, \tilde{\varphi}_1, \ldots\}$ which takes one of the three admissible steps, $e_1, e_2, e_1 + e_2$. We first describe the first step of this curve when all of the $\{\omega, I, J\}$ are known. (see Figure 3.3).

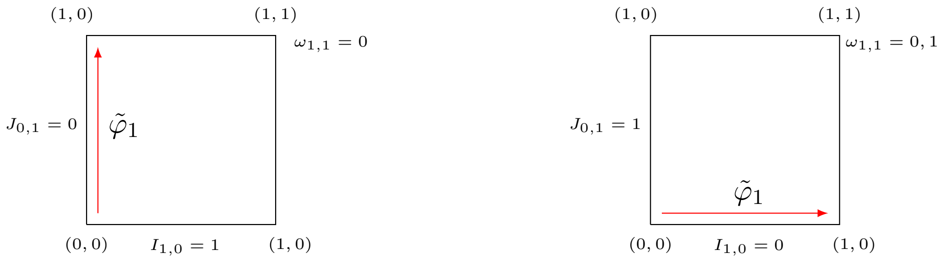
$$\tilde{\varphi}_1 = \begin{cases} (1,0), & \text{when } (\omega_{1,1}, I_{1,0}, J_{0,1}) \in \{(1,0,1),(0,0,1)\}, \\ (1,1), & \text{when } (\omega_{1,1}, I_{1,0}, J_{0,1}) \in \{(1,0,0),(0,0,0),(1,1,0),(1,1,1),(0,1,1)\}, \\ (0,1), & \text{when } (\omega_{1,1}, I_{1,0}, J_{0,1}) \in \{(0,1,0)\}. \end{cases} \tag{3.4.3}$$



(a) Combination of I, J and ω for a diagonal step.

(b) Combination of I, J and ω for a diagonal step.

(c) Combination of I, J and ω for a diagonal step.



(d) Combination of I, J and ω for an up step.

(e) Combination of I, J and ω for a right step.

Figure 3.3: Constructive admissible steps for $\tilde{\varphi}_1$. Compare with Figure 3.2 and see that the $\tilde{\varphi}$ steps are always lower or equal than the ones for φ

From this definition we see that $\tilde{\varphi}_1$ stays on the x-axis only when $I_{1,0} = 0$ and $J_{0,1} = 1$. If that is the case, repeat the steps in (3.4.3) until $\tilde{\varphi}$ increases its y-coordinate and changes level. Any time $\tilde{\varphi}$ changes level from $\ell - 1$ to ℓ, it takes horizontal steps (the number of steps could be 0) until a site (v_ℓ, ℓ) where $(v_\ell, \ell) \in \mathcal{C}_{\uparrow,\nearrow}$ but $(v_\ell + 1, \ell) \in \mathcal{C}_{\rightarrow}$. In that case, $I_{v_\ell+1,\ell} = 1$, by the second and third observations above, and $\tilde{\varphi}$ will change level, again following the steps in (3.4.3).

From the description of the evolution of $\tilde{\varphi}$, starting from (3.4.3) and evolving as we describe in the previous paragraph, the definition of the competition interface φ in (3.4.2), implies as piecewise linear curves, (as it is possible to see comparing the admissible steps in Figures 3.2 and 3.3)

$$\varphi \geq \tilde{\varphi}, \tag{3.4.4}$$

i.e. if $(x, y_1) \in \varphi$ and $(x, y_2) \in \tilde{\varphi}$ then, $y_1 \geq y_2$. Similarly, if $(x_1, y) \in \varphi$ and $(x_2, y) \in \tilde{\varphi}$ then, $x_1 \leq x_2$. Moreover, if $u \in \mathbb{Z}_+^2 \notin \tilde{\varphi}$ then it has to belong to one of the clusters; $\mathcal{C}_{\rightarrow}$ if u is below $\tilde{\varphi}$ and $\mathcal{C}_{\uparrow,\nearrow}$ otherwise. (see Figure 3.4).

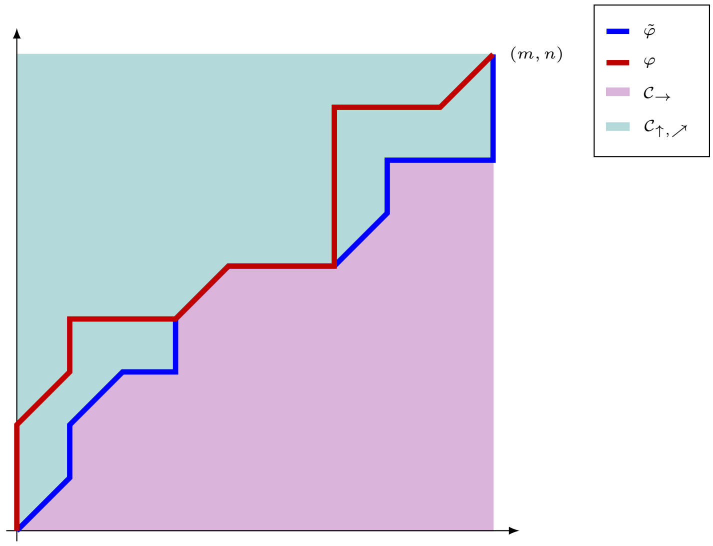


Figure 3.4: Graphical representation of $\tilde{\varphi}$ and φ. Both curves can be thought as competition interfaces. $\tilde{\varphi}$ separates competing clusters, depending on the first step of the right-most maximal path, while φ follows the smallest increment of passage times with a rule to break ties. As curves they are geometrically ordered, $\tilde{\varphi} \leq \varphi$.

The reversed process

Let (m, n) with $m, n > 0$ be the target point. Define

$$G^*_{i,j} = G_{m,n} - G_{m-i,n-j}, \qquad \text{for } 0 \leq i < m \text{ and } 0 \leq i < n. \tag{3.4.5}$$

It represents the time to reach point (i, j) starting from (m, n) for the reversed process. We also define the new edge and the bulk weights by

$$I^*_{i,j} = I_{m-i+1,n-j}, \qquad \text{when } i \geq 1, j \geq 0 \tag{3.4.6}$$

$$J^*_{i,j} = J_{m-i,n-j+1}, \qquad \text{when } i \geq 0, j \geq 1 \tag{3.4.7}$$

$$\omega^*_{i,j} = \alpha_{m-i,n-j}, \qquad \text{when } i \geq 1, j \geq 1. \tag{3.4.8}$$

Then we have the reverse identities.

Lemma 3.4.2. *Let I^* and J^* be respectively the horizontal and vertical increment for the reversed process. Then, for $0 \leq i < m$ and $0 \leq i < n$, we have*

$$I^*_{i,j} = \omega^*_{i,j} \vee I^*_{i,j-1} \vee J^*_{i-1,j} - J^*_{i-1,j} = G^*_{i,j} - G^*_{i-1,j} \tag{3.4.9}$$

$$J^*_{i,j} = \omega^*_{i,j} \vee I^*_{i,j-1} \vee J^*_{i-1,j} - I^*_{i,j-1} = G^*_{i,j} - G^*_{i,j-1}. \tag{3.4.10}$$

Proof. First note that

$$I_{m-i+1,n-j} = G_{m-i+1,n-j} - G_{m-i,n-j}$$

$$= G_{m-i+1,n-j} - G_{m,n} + G_{m,n} - G_{m-i,n-j} = G^*_{i,j} - G^*_{i-1,j}.$$

by (3.4.5). We also prove the other identity only for the $I^*_{i,j}$ and leave the proof for the second set of equations to the reader. A direct substitution to the right-hand side gives

$$\begin{aligned}
&\omega^*_{i,j} \vee I^*_{i,j-1} \vee J^*_{i-1,j} - J^*_{i-1,j}\\
&= \alpha_{m-i,n-j} \vee I_{m-i+1,n-j+1} \vee J_{m-i+1,n-j+1} - J_{m-i+1,n-j+1}\\
&= (\alpha_{m-i,n-j} - J_{m-i+1,n-j+1}) \vee (G_{m-i+1,n-j} - G_{m-i,n-j+1}) \vee 0\\
&= (\alpha_{m-i,n-j} - J_{m-i+1,n-j+1}) \vee (G_{m-i+1,n-j} - G_{m-i,n-j} + G_{m-i,n-j} - G_{m-i,n-j+1}) \vee 0\\
&= (\alpha_{m-i,n-j} - (\omega_{m-i+1,n-j+1} \vee I_{m-i+1,n-j} \vee J_{m-i,n-j+1} - I_{m-i+1,n-j}))\\
&\qquad\qquad \vee (I_{m-i+1,n-j} - J_{m-i,n-j+1}) \vee 0\\
&= I_{m-i+1,n-j} + \Big((\alpha_{m-i,n-j} - \omega_{m-i+1,n-j+1} \vee I_{m-i+1,n-j} \vee J_{m-i,n-j+1})\\
&\qquad\qquad\qquad\qquad\qquad\qquad \vee (-J_{m-i,n-j+1}) \vee (-I_{m-i+1,n-j})\Big).
\end{aligned}$$

Focus on the expression in the parenthesis. We will show that it is always 0, and therefore the lemma follows by (3.4.6). We use equations (3.2.3) and (3.2.8). If the pair $(I_{m-i+1,n-j}, J_{m-i,n-j+1}) = (1,1)$ then $\alpha_{m-i,n-j} = 1$ and the first maximum is zero. Similarly, when the triple $(\omega_{m-i+1,n-j+1}, I_{m-i+1,n-j}, J_{m-i,n-j+1}) = (0,0,0)$, $\alpha_{m-i,n-j} = 0$ and the value is zero again. When exactly one of $I_{m-i+1,n-j}, J_{m-i,n-j+1}$ is zero the overall maximum in the parenthesis is 0, irrespective of the values of $\alpha_{m-i,n-j}, \omega_{m-i+1,n-j+1}$. Finally, when $\omega_{m-i+1,n-j+1} = 1$ and both the increment variables $(I_{m-i+1,n-j}, J_{m-i,n-j+1}) = (0,0)$, the first term is either 0 or -1 and again the overall maximum is zero. $\square$

Throughout the paper quantities defined in the reversed process will be denoted by a superscript *, and they will always be equal in distribution to their original forward versions.

Competition interface for the forward process vs maximal path for the reversed process

We want to show that the competition interface defined in (3.4.2) is always below or coincides (as piecewise linear curves) with the down - most maximal path $\hat{\pi}^*$ for the reversed process. The steps of the competition interface for the forward process coincide with those of $\hat{\pi}^*$ in all cases, except when $(I_{i,j}, J_{i,j}, \omega_{i,j}) = (0,1,1)$. In that case, $\hat{\pi}^*$ will go diagonally up, while φ will move horizontally. Thus, φ is to the right and below $\hat{\pi}^*$ as curves.

Now, define

$$v(n) = \inf\{i : (i,n) = \varphi_k \text{ for some } k \geq 0\}$$
$$w(m) = \inf\{j : (m,j) = \varphi_k \text{ for some } k \geq 0\} \tag{3.4.11}$$

with the convention $\inf \varnothing = \infty$. In words, the point $(v(n), n)$ is the left-most point of the competition interface on the horizontal line $j = n$, while $(m, w(m))$ is the lowest point on the vertical line $i = m$. This observation implies

$$v(n) \geq m \implies w(m) < n \qquad \text{or} \qquad w(m) \geq n \implies v(n) < m. \tag{3.4.12}$$

Then, on the event $\{w(m) \geq n\}$, we know that $\hat{\pi}^*$ will hit the north boundary of the rectangle at a site (ℓ, n) so that

$$m - \ell = \xi^*_{e_1}(\hat{\pi}^*), \quad \ell \leq v(n).$$

Then, we have just showed that

Lemma 3.4.3. *Let φ be the competition interface constructed for the process $G^{(\lambda)}$ and $\hat{\pi}^*$ the down-most maximal path for the process $G^{*,(\lambda)}$ defined by (3.4.5) from (m,n) to $(0,0)$. Then on the event $\{v(n) \geq m\}$,*

$$m - v(n) \leq \xi^{*(\lambda)}_{e_1}(\hat{\pi}^*) \tag{3.4.13}$$

Finally, note that by reversed process definition we have

$$\xi^{*(\lambda)}_{e_1} \stackrel{\mathcal{D}}{=} \xi^{(\lambda)}_{e_1}. \tag{3.4.14}$$

3.4.2 Last passage time under different boundary conditions

In our setting the competition interface is important because it bounds the region where the boundary conditions on the axes are felt. For this reason we want to give a Lemma which describes how changes in the boundary conditions are felt by the increments in the interior part.

Lemma 3.4.4. *Given two different weights $\{\omega_{i,j}\}$ and $\{\tilde{\omega}_{i,j}\}$ which satisfy $\omega_{0,0} = \tilde{\omega}_{0,0}$, $\omega_{0,j} \geq \tilde{\omega}_{0,j}$, $\omega_{i,0} \leq \tilde{\omega}_{i,0}$ and $\omega_{i,j} = \tilde{\omega}_{i,j}$ for all $i, j \geq 1$. Then all increments satisfy $I_{i,j} \leq \tilde{I}_{i,j}$ and $J_{i,j} \geq \tilde{J}_{i,j}$.*

Proof. By following the same corner-flipping inductive proof as that of Lemma 3.2.3 one can show that the statement holds for all increments between points in $L_\psi \cup \mathcal{I}_\psi$ where L_ψ and $\mathcal{I}_\psi$ are respectively defined in (3.2.6) and (3.2.9) for those paths for which $\mathcal{I}_\psi$ is finite. The base case is when $\mathcal{I}_\psi$ is empty and the statement follows from the assumption made on the weights $\{\omega_{i,j}\}$ and $\{\tilde{\omega}_{i,j}\}$ and from the definition of the increments made in (3.2.3). ∎

Lemma 3.4.5. *We are in the settings of Lemma 3.4.4. Let $G^{\mathcal{W}=0}$ (resp.$G^{\mathcal{S}=0}$) be the last passage times of a system where we set $\tilde{\omega}_{0,j} = 0$ for all $j \geq 1$ (resp. $\omega_{i,0} = 0$) and the paths are allowed to collect weights while on the boundaries. Let $v(n)$ be given by (3.4.11).*

Then, for $v(n) < m_1 \leq m_2$,

$$\begin{aligned} G_{(1,1),(m_2,n)} - G_{(1,1),(m_1,n)} &\leq G^{\mathcal{W}=0}_{(0,0),(m_2,n)} - G^{\mathcal{W}=0}_{(0,0),(m_1,n)} \\ &= G_{(0,0),(m_2,n)} - G_{(0,0),(m_1,n)}. \end{aligned} \tag{3.4.15}$$

Alternatively, for $0 \leq m_1 \leq m_2 < v(n)$,

$$\begin{aligned} G_{(1,1),(m_2,n)} - G_{(1,1),(m_1,n)} &\geq G^{\mathcal{S}=0}_{(0,0),(m_2,n)} - G^{\mathcal{S}=0}_{(0,0),(m_1,n)} \\ &= G_{(0,0),(m_2,n)} - G_{(0,0),(m_1,n)}. \end{aligned} \tag{3.4.16}$$

Proof. We prove (3.4.16) and similar arguments prove (3.4.15). The first inequality in (3.4.16) follows from Lemma 3.4.4 in the case $\tilde{\omega}_{0,j} = \tilde{\omega}_{i,0} = 0$. The subsequent equality comes from the fact that if $v(n) \geq m_2 \geq m_1$. By (3.4.11) the target points (m_1, n) and (m_2, n) are above the competition interface φ and therefore, by (3.4.4) are strictly above $\tilde{\varphi}$. This implies that (m_1, n) and (m_2, n) belong to the cluster $\mathcal{C}_{\uparrow,\nearrow}$ and therefore we can choose the respective maximal paths to not take a horizontal first step. In turn, the maximal path does not need to go through the x-axis and hence it does not see the boundary values $\omega_{i,0}$. Thus, $G^{\mathcal{S}=0}_{(0,0),(m,n)} = G_{(0,0),(m,n)}$. □

3.4.3 Lower bound

In this section we prove the lower bound for the order of the variance. Before giving the proof we need to prove two preliminary lemmas. For the rest of this section, whenever we say maximal path, we mean the down-most maximal path.

Lemma 3.4.6. *Let $a, b > 0$ two positive numbers. Then there exist a positive integer $N_0 = N(a, b)$ and constant $C = C(a, b)$ such that for all $N > N_0$ we have*

$$\begin{aligned} \mathbb{P}\Big\{ \sup_{0 \leq z \leq aN^{2/3}} \{\mathscr{S}^{(u)}_z + G_{(z\vee 1,1),(m_u(N),n_u(N))} - G_{(1,1),(m_u(N),n_u(N))}\} \geq bN^{1/3}\Big\} \\ \leq Ca^3(b^{-3} + b^{-6}). \end{aligned} \tag{3.4.17}$$

Proof. First note that if the supremum in the probability is attained at $z = 0$ then the expression in the braces is tautologically 0 and the statement of the lemma is vacuously true. Therefore without loss of generality, we can prove the bound for the supremum when $1 \leq z \leq aN^{2/3}$.

Select and fix any parameter $0 < r < b/a$ and let N large enough. The exact dependence of r on the parameters a and b will be obtained later in the proof. Define λ by

$$\lambda = u - rN^{-1/3}. \tag{3.4.18}$$

and use it to define boundary weights on both axes using that parameter and independently of the original boundary weights with parameter u. The environment in the bulk is the same for both processes. Let $\varphi^{(\lambda)}$ be the competition interface under environment $\omega^{(\lambda)}$ and let $v^{(\lambda)}$ be as in equation (3.4.11). Restrict on the event $v^{\lambda}(n) > m$. Define the increment $\mathcal{V}^{(\lambda)}_{z-1} = G^{(\lambda)}_{(0,0),(m,n)} - G^{(\lambda)}_{(0,0),(m-z+1,n)}$. Then use Lemma 3.4.5 to obtain

$$G_{(1,1),(m,n)} - G_{(1,1),(m-z+1,n)} \geq \mathcal{V}^{(\lambda)}_{z-1}.$$

Recall that $\mathcal{V}^{(\lambda)}_{z-1}$ is a sum of i.i.d. Bernoulli(λ) variables and it is independent of $\mathscr{S}^{(u)}_z$. When (m,n) equals the characteristic direction $(m_u(N), n_u(N))$ corresponding to u,

$$\begin{aligned}
\mathbb{P}\{\sup_{1\leq z\leq aN^{2/3}} &\{\mathscr{S}^{(u)}_z + G_{(z,1),(m_u(N),n_u(N))} - G_{(1,1),(m_u(N),n_u(N))}\} \geq bN^{1/3}\} \\
&\leq \mathbb{P}\Big\{v^{(\lambda)}\Big(\Big\lfloor \frac{N}{p}\big(p+(1-p)u\big)^2\Big\rfloor\Big) \leq \lfloor N\rfloor\Big\} \\
&\qquad + \mathbb{P}\Big\{\sup_{1\leq z\leq aN^{2/3}}\{\mathscr{S}^{(u)}_z - \mathcal{V}^{(\lambda)}_{z-1}\} \geq bN^{1/3}\Big\} \\
&\leq \mathbb{P}\Big\{v^{(\lambda)}\Big(\Big\lfloor \frac{N}{p}\big(p+(1-p)u\big)^2\Big\rfloor\Big) \leq \lfloor N\rfloor\Big\} \qquad (3.4.19)\\
&\qquad + \mathbb{P}\Big\{\sup_{1\leq z\leq aN^{2/3}}\{\mathscr{S}^{(u)}_{z-1} - \mathcal{V}^{(\lambda)}_{z-1}\} \geq bN^{1/3} - 1\Big\}.
\end{aligned} \tag{3.4.20}$$

Where the first inequality is obtained by applying the law of total probability and making $\mathbb{P}\Big\{\sup_{1\leq z\leq aN^{2/3}}\{\mathscr{S}^{(u)}_z + G_{(z,1),(m_u(N),n_u(N))} - G_{(1,1),(m_u(N),n_u(N))}\} \geq bN^{1/3} | v^{(\lambda)}\Big(\Big\lfloor \frac{N}{p}\big(p+(1-p)u\big)^2\Big\rfloor\Big) \leq \lfloor N\rfloor\Big\} = \mathbb{P}\Big\{v^{(\lambda)}\Big(\Big\lfloor \frac{N}{p}\big(p+(1-p)u\big)^2\Big\rfloor\Big) > \lfloor N\rfloor\Big\} = 1$. We bound the two probabilities separately. We begin with (3.4.20). Define the martingale as $M_{z-1} = \mathscr{S}^{(u)}_{z-1} - \mathcal{V}^{(\lambda)}_{z-1} - \mathbb{E}[\mathscr{S}^{(u)}_{z-1} - \mathcal{V}^{(\lambda)}_{z-1}]$, and note that for $1 \leq z \leq aN^{2/3}$,

$$\mathbb{E}[\mathscr{S}^{(u)}_{z-1} - \mathcal{V}^{(\lambda)}_{z-1}] = (z-1)u - (z-1)\lambda \leq raN^{1/3}. \tag{3.4.21}$$

From (3.4.21) follows that

$$\mathscr{S}^{(u)}_{z-1} - \mathcal{V}^{(\lambda)}_{z-1} \leq M_{z-1} + raN^{1/3}.$$

Using this result and taking N large enough so that

$$b > ra + N^{-1/3} \tag{3.4.22}$$

we get by Doob's inequality, for any $d \geq 1$.

$$\begin{aligned}
&\mathbb{P}\{\sup_{1\leq z\leq aN^{2/3}} \{\mathscr{S}^{(u)}_{z-1} - \mathcal{V}^{(\lambda)}_{z-1}\} \geq bN^{1/3} - 1\} \\
&\qquad\qquad \leq \mathbb{P}\{\sup_{1\leq z\leq aN^{2/3}} M_{z-1} \geq N^{1/3}(b - ra - N^{-1/3})\} \\
&\qquad\qquad \leq \frac{C(d)N^{-d/3}}{(b-ra-N^{-1/3})^d}\mathbb{E}[|M_{\lfloor aN^{2/3}\rfloor}|^d] \leq \frac{C(d,u)a^{d/2}}{(b-ra-N^{-1/3})^d}. \qquad (3.4.23)
\end{aligned}$$

Then for $N \geq 4^3 b^{-3}$ the above bound is further dominated by $C(d,u)a^{d/2}(\frac{3b}{4} - ra)^{-d}$ which becomes $C(d,u)a^3b^{-6}$ once we choose

$$r = \frac{b}{4a}, \qquad (3.4.24)$$

$d = 6$, and properly re-define the constant $C(d,u)$. This concludes the bound for (3.4.20).

For (3.4.19), we rescale N as

$$N' = \Big(\frac{p+(1-p)u}{p+(1-p)\lambda}\Big)^2 N.$$

Then we write

$$\mathbb{P}\Big\{v^{(\lambda)}\Big(\Big\lfloor\frac{N'}{p}\big(p+(1-p)\lambda\big)^2\Big\rfloor\Big) < \Big\lfloor\Big(\frac{p+(1-p)\lambda}{p+(1-p)u}\Big)^2 N'\Big\rfloor\Big\}$$

Since $u > \lambda$, then

$$\Big\lfloor\Big(\frac{p+(1-p)\lambda}{p+(1-p)u}\Big)^2 N'\Big\rfloor \leq \lfloor N'\rfloor.$$

Thus, by redefining (3.1.7) and (3.4.13) with N' and λ, we have that the event $v^{(\lambda)}(\lfloor\frac{N'}{p}(p+(1-p)\lambda)^2\rfloor) < \lfloor(\frac{p+(1-p)\lambda}{p+(1-p)u})^2 N'\rfloor$ is equivalent to

$$\begin{aligned}
\xi^{*(\lambda)}_{e_1}(N') &\geq \lfloor N'\rfloor - v^{(\lambda)}\Big(\Big\lfloor\frac{N'}{p}\big(p+(1-p)\lambda\big)^2\Big\rfloor\Big) \\
&> \lfloor N'\rfloor - \Big\lfloor\Big(\frac{p+(1-p)\lambda}{p+(1-p)u}\Big)^2 N'\Big\rfloor.
\end{aligned}$$

By (3.4.14), we conclude

$$\begin{aligned}
&\mathbb{P}\Big\{v^{(\lambda)}\Big(\Big\lfloor\frac{N}{p}\big(p+(1-p)u\big)^2\Big\rfloor\Big) < \lfloor N\rfloor\Big\} \\
&\qquad\qquad = \mathbb{P}\Big\{\xi^{(\lambda)}_{e_1}(N') > \lfloor N'\rfloor - \Big\lfloor\Big(\frac{p+(1-p)\lambda}{p+(1-p)u}\Big)^2 N'\Big\rfloor\Big\}. \qquad (3.4.25)
\end{aligned}$$

Utilizing the definitions (3.4.18) and (3.4.24) of λ and r, for $N \geq N_0$ there exists a constant $C = C(u)$ such that

$$\lfloor N'\rfloor - \Big\lfloor\Big(\frac{p+(1-p)\lambda}{p+(1-p)u}\Big)^2 N'\Big\rfloor \geq CrN'^{2/3}.$$

Combining this with Corollary 3.3.9 and definition (3.4.24) of r we get the bound

$$\mathbb{P}\Big\{v^{(\lambda)}\Big(\Big\lfloor\frac{N}{p}\big(p+(1-p)u\big)^2\Big\rfloor\Big)<\lfloor N\rfloor\Big\}\leq\mathbb{P}[\xi_{e_1}^{(\lambda)}(N')>CrN'^{2/3}]$$
$$\leq Cr^{-3}\leq C(a/b)^3. \tag{3.4.26}$$

The result now follows. □

The other Lemma gives an asymptotic limit of the probability order of the exit point from the x-axis. We will discuss the exit point from the y-axis as a Corollary of this Lemma.

Lemma 3.4.7. *Let $u\in(0,1)$ and $(m_u(N),n_u(N))$ the characteristic direction. Then the exit point of a maximal path from 0 to $(m_u(N),n_u(N))$ satisfies*

$$\lim_{\delta\to 0}\varlimsup_{N\to\infty}\mathbb{P}\Big\{0\leq\xi_{e_1}^{(u)}(N)\vee\xi_{e_2}^{(u)}(N)\leq\delta N^{2/3}\Big\}=0.$$

Proof. We only show the result for $\xi_{e_1}^{(u)}(N)$. The same result for $\xi_{e_2}^{(u)}(N)$ follows by interchanging vertical and horizontal directions and the fact that both boundaries have Bernoulli variables.

First pick a parameter $\delta>0$. Recall that $\xi_{e_1}^{(u)}(N)=0$ if the down-most maximal path makes the first step diagonally or up. Also keep in mind that $\xi_{e_1}^{(u)}(N)=0$ is the right-most possible exit point, therefore all paths that exit later, have to have a smaller passage time. Then, we may bound

$$\mathbb{P}\{0\leq\xi_{e_1}^{(u)}(N)\leq\delta N^{2/3}\}\leq\mathbb{P}\{\sup_{\delta N^{2/3}<x\leq N^{2/3}}\{\mathscr{S}_x^{(u)}+G_{(x,1),(m_u(N),n_u(N))}\}$$
$$<\sup_{0\leq x\leq\delta N^{2/3}}\{\mathscr{S}_x^{(u)}+G_{(x\vee 1,1),(m_u(N),n_u(N))}\}\}.$$

Then, we subtract the term $G_{(1,1),(m(N),n(N))}$ from both sides and we bound the resulting probability from above by

$$\mathbb{P}\{\sup_{\delta N^{2/3}<x\leq N^{2/3}}\{\mathscr{S}_x^{(u)}+G_{(x,1),(m_u(N),n_u(N))}-G_{(1,1),(m_u(N),n_u(N))}\}$$
$$<\sup_{0\leq x\leq\delta N^{2/3}}\{\mathscr{S}_x^{(u)}+G_{(x\vee 1,1),(m_u(N),n_u(N))}-G_{(1,1),(m_u(N),n_u(N))}\}\}$$
$$\leq\mathbb{P}\{\sup_{\delta N^{2/3}<x\leq N^{2/3}}\{\mathscr{S}_x^{(u)}+G_{(x,1),(m_u(N),n_u(N))}-G_{(1,1),(m_u(N),n_u(N))}\}<bN^{1/3}\} \tag{3.4.27}$$
$$+\mathbb{P}\{\sup_{0\leq x\leq\delta N^{2/3}}\{\mathscr{S}_x^{(u)}+G_{(x\vee 1,1),(m_u(N),n_u(N))}-G_{(1,1),(m_u(N),n_u(N))}\}>bN^{1/3}\}. \tag{3.4.28}$$

(3.4.28) is bounded from above using Lemma 3.4.6 by $C\delta^3(b^{-3}+b^{-6})$.

To bound (3.4.27) we use similar arguments that we employed in the proof of Lemma 3.4.6. Define an auxiliary parameter λ

$$\lambda = u + rN^{-1/3}, \tag{3.4.29}$$

where conditions on r will be specified in the course of the proof. From Lemma 3.4.5 the following inequalities hold

$$\begin{aligned} G_{(x,1),(m_u(N),n_u(N))} - G_{(1,1),(m_u(N),n_u(N))} &\geq G^{(\lambda)}_{(0,0),(m_u(N)-x+1,n_u(N))} - G^{(\lambda)}_{(0,0),(m_u(N),n_u(N))} \\ &= -\mathcal{V}^{(\lambda)}_{x-1} \geq -\mathcal{V}^{(\lambda)}_x. \end{aligned}$$

whenever $v^{(\lambda)}\Big(\Big\lfloor \frac{N}{p}\big(p+(1-p)u\big)^2\Big\rfloor\Big) \leq \lfloor N \rfloor - x$. Using these, we have

$$\begin{aligned} \mathbb{P}\Big\{\sup_{\delta N^{2/3}<x\leq N^{2/3}} \{\mathscr{S}^{(u)}_x + G_{(x,1),(m_u(N),n_u(N))} - G_{(1,1),(m_u(N),n_u(N))}\} < bN^{1/3}\Big\} \\ \leq \mathbb{P}\Big\{v^{(\lambda)}\Big(\Big\lfloor \frac{N}{p}\big(p+(1-p)u\big)^2\Big\rfloor\Big) > \lfloor N \rfloor - N^{2/3}\Big\} \qquad (3.4.30) \\ + \mathbb{P}\Big\{\sup_{\delta N^{2/3}<x\leq N^{2/3}} \{\mathscr{S}^{(u)}_x - \mathcal{V}^{(\lambda)}_x\} < bN^{1/3}\Big\}. \qquad (3.4.31) \end{aligned}$$

We claim that, for $\eta > 0$ and parameter r, it is possible to fix $\delta, b > 0$ small enough so that, for some $N_0 < \infty$, the probability in (3.4.31) satisfies

$$\mathbb{P}\Big\{\sup_{\delta N^{2/3}<x\leq N^{2/3}} \{\mathscr{S}^{(u)}_x - \mathcal{V}^{(\lambda)}_x\} < bN^{1/3}\Big\} \leq \eta \qquad \text{for all } N \geq N_0. \tag{3.4.32}$$

In order to prove this, we use a scaling argument: Uniformly over $y \in [\delta, 1]$ as $N \to \infty$,

$$N^{-1/3}\mathbb{E}[\mathscr{S}^{(u)}_{\lfloor yN^{2/3}\rfloor} - \mathcal{V}^{(\lambda)}_{\lfloor yN^{2/3}\rfloor}] = N^{-1/3}(\lfloor yN^{2/3}\rfloor u - \lfloor yN^{2/3}\rfloor(u+rN^{-1/3})) \to -ry$$

and

$$\begin{aligned} N^{-2/3}\,\mathrm{Var}(\mathscr{S}^{(u)}_{\lfloor yN^{2/3}\rfloor} - \mathcal{V}^{(\lambda)}_{\lfloor yN^{2/3}\rfloor}) &= yu(1-u) + y(u+rN^{-1/3})(1-u-rN^{-1/3}) \\ &\to 4u(1-u)y = \sigma^2(u)y \end{aligned}$$

Since we are scaling the supremum of a random walk with bounded increments, the probability (3.4.32) converges as $N \to \infty$, to

$$\mathbb{P}\Big\{\sup_{\delta\leq y\leq 1}\{\sigma(u)\mathfrak{B}(y) - ry\} \leq b\Big\}$$

where $\mathfrak{B}(\cdot)$ is a standard Brownian motion. The random variable

$$\sup_{\delta\leq y\leq 1}\{\sigma(u)\mathfrak{B}(y) - ry\}$$

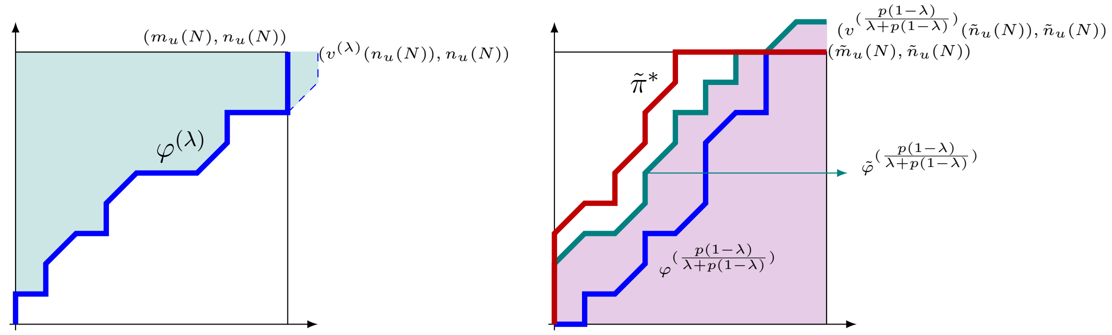


Figure 3.5: Comparison of various curves in $\omega_{i,j}$ and $\tilde{\omega}_{i,j} = \omega_{j,i}$ environments. The thickset blue curve (color online) in the left figure is the competition interface in $\omega_{i,j}$ and the reflected curve can be seen in the same color to the right. The green curve is the competition interface in $\tilde{\omega}_{i,j}$ weights which is higher than the reflected φ and the red curve is the right-most maximal paths in the reversed $\tilde{\omega}^*_{i,j}$ weights with boundaries on north and east, which is higher than both the other curves.

is positive almost surely when δ is sufficiently small. Therefore, the above probability is less than $\eta/2$ for a suitably small b. This implies (3.4.32).

Finally we bound (3.4.30). Using (3.4.12) and the transpose environment $\tilde{\omega}_{i,j} = \omega_{j,i}$ for $i, j \geq 0$ under the measure $\widetilde{\mathbb{P}}$

$$\mathbb{P}\Big\{v^{(\lambda)}\Big(\Big\lfloor \frac{N}{p}(p+(1-p)u)^2\Big\rfloor\Big) > \lfloor N - N^{2/3} - 1\rfloor\Big\}$$
$$\leq \widetilde{\mathbb{P}}\Big\{v^{\left(\frac{p(1-\lambda)}{\lambda+p(1-\lambda)}\right)}(\lfloor N - N^{2/3} - 1\rfloor) \leq \Big\lfloor \frac{N}{p}(p+(1-p)u)^2\Big\rfloor\Big\}. \tag{3.4.33}$$

Under measure $\widetilde{\mathbb{P}}$ the environment is still i.i.d. and the only change is the alternation of parameter values on the boundaries. Moreover, in the transposed environment, the new competition interface $\varphi^{(\frac{p(1-\lambda)}{\lambda+p(1-\lambda)})}$ constructed using (3.4.2), would be above (as a curve) from the transposed competition interface $\varphi^{(\lambda)}$, so it would still exit from the north boundary. (see Figure 3.5). From (3.4.29) substitute u as a function of λ,

$$\text{Probability in (3.4.33)} = \widetilde{\mathbb{P}}\Big\{v^{\left(\frac{p(1-\lambda)}{\lambda+p(1-\lambda)}\right)}(\lfloor N - N^{2/3} - 1\rfloor)$$
$$\leq \Big\lfloor \frac{N}{p}(p+(1-p)\lambda)^2 - \frac{2}{p}(p+(1-p)\lambda)(1-p)rN^{2/3} + o(N^{2/3})\Big\rfloor\Big\}.$$

Define N' as

$$N' = N - N^{2/3} - 1 \implies N = N' + N'^{2/3} + o(N'^{2/3}).$$

Replace N with N' in the probability above to obtain

$$\text{Probability in (3.4.33)} \leq \widetilde{\mathbb{P}}\Big\{v^{\left(\frac{p(1-\lambda)}{\lambda+p(1-\lambda)}\right)}(\lfloor N'\rfloor) \leq \Big\lfloor \frac{N'}{p}(p+(1-p)\lambda)^2\Big\rfloor - KN'^{2/3}\Big\},$$

where $K = p^{-1}(p+(1-p)\lambda)(2(1-p)r-(p+(1-p)\lambda))$ which is positive for r large enough. Using (3.4.13) and (3.4.14)

$$\text{Probability}(3.4.33)$$
$$\le \widetilde{\mathbb{P}}\{\xi_{e_1}^{*\left(\frac{p(1-\lambda)}{\lambda+p(1-\lambda)}\right)}(N') > KN'^{2/3}] = \widetilde{\mathbb{P}}\{\xi_{e_1}^{\left(\frac{p(1-\lambda)}{\lambda+p(1-\lambda)}\right)}(N') > KN'^{2/3}] \le CK^{-3},$$

where the last inequality follows from Corollary 3.3.9. We are now ready to prove the lemma. Start with a fixed $\eta > 0$. Then, fix an r large enough so that $CK^{-3} < \eta$ and probability (3.4.30) is controlled. This also imposes a restriction on the smallest value of N that we can take, since we must have $\lambda < 1$. Under a fixed r, we can modulate δ, b and select them small enough, so that (3.4.32) holds. Finally, make δ smaller so that $C\delta^3(b^{-3}+b^{-6}) < \eta$ and probability (3.4.28) is also controlled. Thus, unifying all these results we have

$$\mathbb{P}\{0 \le \xi_{e_1}^{(u)}(N) \le \delta N^{2/3}\} \le 2\eta. \tag{3.4.34}$$

Note that by shrinking δ while b remains fixed, (3.4.32) is reinforced. This concludes the proof of the lemma. $\square$

Proof of Theorem 3.1.3, lower bound. We first claim that

$$\mathcal{A}_{\mathcal{N}^{(u)}} \ge \mathbb{E}\Big(\xi - \sum_{i=1}^{\xi}\omega_{i,0}\Big) = \mathbb{E}\Big(\sum_{i=1}^{\xi}(1-\omega_{i,0})\Big). \tag{3.4.35}$$

Under this claim, we can write

$$\begin{aligned}
\mathcal{A}_{\mathcal{N}^{(u)}} &\ge \mathbb{E}\Big(\sum_{i=1}^{\xi_{e_1}^{(u)}}(1-\omega_{i,0})\Big)\\
&\ge \mathbb{E}\Big(\mathbb{1}\{\xi_{e_1}^{(u)}(N) \ge \delta N^{2/3}\}\sum_{i=1}^{\lfloor \delta N^{2/3}\rfloor}(1-\omega_{i,0})\Big)\\
&\ge \alpha N^{2/3}\mathbb{P}\Big\{\xi_{e_1}^{(u)}(N) \ge \delta N^{2/3}, \sum_{i=1}^{\lfloor \delta N^{2/3}\rfloor}(1-\omega_{i,0}) \ge \alpha N^{2/3}\Big\}.
\end{aligned}$$

Fix an η positive and smaller than $1/4$. Now, by making δ sufficiently small, we can make the event $\{\xi_{e_1}^{(u)}(N) \ge \delta N^{2/3}\}$ have probability larger than $1-\eta$ by Lemma 3.4.7, for N sufficiently large. With δ fixed, we can make α smaller, so that the event $\Big\{\sum_{i=1}^{\lfloor \delta N^{2/3}\rfloor}(1-\omega_{i,0}) \ge \alpha N^{2/3}\Big\}$ also has probability larger than $1-\eta$. Therefore their intersection has probability greater than $1-2\eta$.

By Proposition 3.3.1 and the fact that we are in a characteristic direction, the result follows.

It now remains to verify (3.4.35). Using the fact that

$$\mathscr{H}_{i,0}^{(\varepsilon)} \vee \omega_{i,0} - \omega_{i,0} = \mathscr{H}_{i,0}^{(\varepsilon)} - \mathscr{H}_{i,0}^{(\varepsilon)} \omega_{i,0},$$

we write using (3.3.16)

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}(\mathcal{N}^{u_\varepsilon} - \mathcal{N}^{(u)}) \geq \mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}\Big(\mathscr{S}^{u_\varepsilon}_{\xi^{(u)}_{e_1}} - \mathscr{S}^{(u)}_{\xi^{(u)}_{e_1}}\Big) &= \mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}\Big(\sum_{i=1}^{\xi^{(u)}_{e_1}} \mathscr{H}_{i,0}^{(\varepsilon)} - \mathscr{H}_{i,0}^{(\varepsilon)}\omega_{i,0}\Big)\\
&= \varepsilon\mathbb{E}(\xi^{(u)}_{e_1}) - \mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}\Big(\sum_{i=1}^{\xi^{(u)}_{e_1}} \mathscr{H}_{i,0}^{(\varepsilon)}\omega_{i,0}\Big)\\
&= \varepsilon\mathbb{E}(\xi^{(u)}_{e_1}) - \mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}\Big(\sum_{y=1}^{m_u(N)}\sum_{i=1}^{y} \mathscr{H}_{i,0}^{(\varepsilon)}\omega_{i,0}\mathbb{1}\{\xi^{(u)}_{e_1} = y\}\Big)\\
&= \varepsilon\mathbb{E}(\xi^{(u)}_{e_1}) - \mathbb{E}_{\mathbb{P}\otimes\mu_\varepsilon}\Big(\sum_{i=1}^{m_u(N)} \mathscr{H}_{i,0}^{(\varepsilon)}\omega_{i,0}\mathbb{1}\{\xi^{(u)}_{e_1} \geq i\}\Big)\\
&= \varepsilon\mathbb{E}(\xi^{(u)}_{e_1}) - \varepsilon\mathbb{E}\Big(\sum_{i=1}^{m_u(N)} \omega_{i,0}\mathbb{1}\{\xi^{(u)}_{e_1} \geq i\}\Big) = \varepsilon\mathbb{E}(\xi^{(u)}_{e_1}) - \varepsilon\mathbb{E}\Big(\sum_{i=1}^{\xi^{(u)}_{e_1}}\omega_{i,0}\Big).
\end{aligned}$$

Combine the expectations and divide by ε. Then take a limit as $\varepsilon \to 0$ to finish the proof. □

3.5 Variance in off-characteristic directions

In this section we want to deduce the central limit theorem for rectangles that do not have characteristic shape.

Proof of Theorem 3.1.4. We prove the theorem in the case $c < 0$, analogue arguments follow for $c > 0$. Set $m_u^*(N) = m_u(N) + \lfloor cN^\alpha \rfloor$. Now, the point $(m_u^*(N), n_u(N) + \lfloor cN^\alpha \rfloor)$ is in the characteristic direction. Thus

$$G^{(u)}_{m_u(N), n_u(N)+\lfloor cN^\alpha\rfloor} = G^{(u)}_{m_u^*(N), n_u(N)+\lfloor cN^\alpha\rfloor} + \sum_{i=m_u^*(N)+1}^{m_u(N)} I_{i, n_u(N)+\lfloor cN^\alpha\rfloor}.$$

Note that the second the term on the right hand side is a sum of $m_u(N) - m_u^*(N) = \lfloor cN^\alpha \rfloor$ i.i.d Bernoulli distributed with parameter u. We center by the mean of each random variable and we indicate them with a bar over the random variable. Multiply both sides by $N^{-\alpha/2}$ to obtain

$$N^{-\alpha/2}\bar{G}^{(u)}_{m_u(N), n_u(N)+\lfloor cN^\alpha\rfloor} = N^{-\alpha/2}(\bar{G}^{(u)}_{m_u^*(N), n_u(N)+\lfloor cN^\alpha\rfloor} + \sum_{i=m_u^*(N)+1}^{m_u(N)} \bar{I}_{i, n_u(N)+\lfloor cN^\alpha\rfloor}).$$

The first term on the right hand side is stochastically $O(N^{1/3-\alpha/2})$. Since $\alpha > 2/3$ this term converges to zero in probability. On the other hand the second term satisfies a CLT. □

Note that for any $\lambda \in (0,1)$, for any $\varepsilon > 0$ the endpoint $(N, pN - \varepsilon N)$ (resp. $(N, N/p + \varepsilon N)$) will always be the north-east corner of an off-characteristic rectangle no matter what the value of λ.

3.6 Variance without boundary

In this section we prove some results for the last passage time in the model without boundaries but still with fixed endpoint. We begin reminding the last passage time of the model without boundaries to reach a point in the characteristic direction (3.1.7) is $G_{(1,1),(m_u(N),n_u(N))}$ and the last passage time of the model with boundaries to reach the same point is $G^{(u)}_{(0,0),(m_\lambda(N),n_\lambda(N))}$. We want to prove another version of Lemma 3.4.6.

Lemma 3.6.1. *Fix $0 < \alpha < 1$. Then there exist a positive integer $N_0 = N(b,u)$ and constant $C = C(\alpha, u)$ such that, for all $N \geq N_0$ and $b \geq C_0$ we have*

$$\mathbb{P}\{G^{(u)}_{(0,0),(m_\lambda(N),n_\lambda(N))} - G_{(1,1),(m_u(N),n_u(N))} \geq bN^{1/3}\} \leq Cb^{-3\alpha/2}.$$

Proof. We prove only the case where the maximal path exits from the x-axis. Similar arguments hold for the maximal path exits from the y-axis and find the same bound.

Note that

$$\begin{aligned}
&\mathbb{P}\{G^{(u)}_{(0,0),(m_\lambda(N),n_\lambda(N))} - G_{(1,1),(m_u(N),n_u(N))} \geq bN^{1/3}\} \\
&\quad \leq \mathbb{P}\Big\{\sup_{1\leq z\leq aN^{2/3}} \big\{\mathscr{S}^{(u)}_z + G_{(z,1),(m_u(N),n_u(N))} - G_{(1,1),(m_u(N),n_u(N))}\big\} \geq bN^{1/3}\Big\} \qquad (3.6.1)\\
&\qquad + \mathbb{P}\Big\{\sup_{1\leq z\leq aN^{2/3}} \big\{\mathscr{S}^{(u)}_z + G_{(z,1),(m_u(N),n_u(N))}\big\} \neq G_{(1,1),(m_u(N),n_u(N))}\Big\}. \qquad (3.6.2)
\end{aligned}$$

For (3.6.2) using 3.3.9, there exists a $C = C(u)$ such that

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{1\leq z\leq aN^{2/3}} \big\{\mathscr{S}^{(u)}_z + G_{(z,1),(m_u(N),n_u(N))}\big\} \neq G_{(1,1),(m_u(N),n_u(N))}\Big\} \\
&\qquad \leq \mathbb{P}[\xi^{(u)}_{e_1}(N) \geq aN^{2/3}] \leq Ca^{-3}.
\end{aligned} \qquad (3.6.3)$$

For (3.6.1) we use the results from the proof of Lemma 3.4.6. Define

$$\lambda = u - rN^{-1/3}$$

From (3.4.20) and (3.4.23), where we choose $a = b^{\alpha/2}$, $d = 2$ and $r = b^{\alpha/2}$ we have the upper bound

$$\mathbb{P}\Big\{\sup_{1\leq z\leq aN^{2/3}} \{\mathscr{S}^{(u)}_{z-1} - \mathcal{V}^{(\lambda)}_{z-1}\} \geq bN^{1/3} - 1\Big\} \leq \frac{C(\alpha,u)b^{\alpha/2}}{(b - b^{\alpha} - N^{-1/3})^2} \qquad (3.6.4)$$

where $C(\alpha, u) > 0$ is large enough so that for $b \geq C$ (3.4.22) is satisfied and the denominator in (3.6.4) is at least $b/2$. Then we can claim that for all $b \geq C$ and $N \geq N_0 = 4^3 b^{-3}$

$$\mathbb{P}\Big\{ \sup_{1 \leq z \leq aN^{2/3}} \{\mathscr{S}_z^{(u)} + G_{(z,1),(m_u(N),n_u(N))} - G_{(1,1),(m_u(N),n_u(N))}\} \geq bN^{1/3} \Big\} \\ \leq \mathbb{P}\Big\{ v^{(\lambda)}\Big(\Big\lfloor \frac{N}{p}(p + (1-p)u)^2 \Big\rfloor\Big) \leq \lfloor N \rfloor \Big\} + Cb^{\alpha/2-2}.$$

Since $N \geq N_0$ we can use the result (3.4.26) and remembering that $r = b^{\alpha/2}$ in this case we obtain

$$\mathbb{P}\Big\{ \sup_{1 \leq z \leq aN^{2/3}} \{\mathscr{S}_z^{(u)} + G_{(z,1),(m_u(N),n_u(N))} - G_{(1,1),(m_u(N),n_u(N))}\} \geq bN^{1/3} \Big\} \\ \leq Cb^{-3\alpha/2} + Cb^{\alpha/2-2}. \tag{3.6.5}$$

Combining (3.6.5) and (3.6.3) we obtain the final result. □

All the constants which will be defined in this section depend on the values x, y and p.

Proof of Theorem 3.1.6. By Chebyshev, Theorem 3.1.3 for the upper bound, Lemma 3.6.1

$$\begin{aligned} &\mathbb{P}\{|G_{(1,1),(\lfloor Nx \rfloor,\lfloor Ny \rfloor)} - Ng_{pp}(x,y)| \geq bN^{1/3}\} \\ &\quad \leq \mathbb{P}\{|G_{(1,1),(m_u(N),n_u(N))} - G^{(u)}_{(0,0),(m_\lambda(N),n_\lambda(N))}| \geq \frac{1}{2}bN^{1/3}\} \\ &\qquad\qquad + \mathbb{P}[|G^{(u)}_{(0,0),(m_\lambda(N),n_\lambda(N))} - Ng_{pp}(x,y)| \geq \frac{1}{4}bN^{1/3}] \\ &\quad \leq Cb^{-3\alpha/2} + Cb^{-2} \leq Cb^{-3\alpha/2}. \end{aligned}$$

To get the moment bound,

$$\mathbb{E}\bigg[\bigg|\frac{G_{(1,1),(\lfloor Nx \rfloor,\lfloor Ny \rfloor)} - Ng_{pp}(x,y)}{N^{1/3}}\bigg|^r\bigg] = \int_0^\infty \mathbb{P}\bigg[\bigg|\frac{G_{(1,1),(\lfloor Nx \rfloor,\lfloor Ny \rfloor)} - Ng_{pp}(x,y)}{N^{1/3}}\bigg|^r \geq b\bigg]db.$$

At this point using (3.1.10) where b in this case is $b^{1/r}$

$$\int_0^\infty \mathbb{P}\bigg[\bigg|\frac{G_{(1,1),(\lfloor Nx \rfloor,\lfloor Ny \rfloor)} - Ng_{pp}(x,y)}{N^{1/3}}\bigg|^r \geq b\bigg]db \leq C_0 + \int_{C_0}^\infty Cb^{\frac{-3\alpha}{2r}}db < \infty,$$

which converges iff $1 \leq r < 3\alpha/2$. □

3.6.1 Variance in flat-edge directions without boundary

We only treat explicitly the case for which $y \leq px$. Since our model is symmetric, the same arguments can be repeated to prove the case $y \geq \frac{1}{p}x$.

We force macroscopic distance from the critical line, i.e. we assume that we can find $\varepsilon > 0$ so that the sequence of endpoints $(N, n(N))$ satisfy

$$\overline{\lim_{n \to \infty}} \frac{n(N)}{N} \leq p - \varepsilon. \tag{3.6.6}$$

Proof of Theorem 3.1.7. Consider the following naive strategy: We construct an approximate maximal path π for $G_{N,n(N)}$, knowing that for large $n(N) < \lfloor (p-\varepsilon/2)N \rfloor$ without using the boundaries. π enters immediately inside the bulk and moves right until it finds a weight to collect diagonally. After that this procedure repeats. For each iteration of this procedure, the horizontal length of this path increases by a random Geometric(p) length, independently of the past.

The probability that π will take more than N steps before reaching level $n(N)$ is the same as the probability that the sum of $n(N)$ independent $X_i \sim$ Geometric(p) r.v.'s exceeds the value N which is a large deviation event. In symbols

$$\mathbb{P}\{G_{N,n(N)}(\pi) < n(N)\} = \mathbb{P}\Big\{\sum_{i=1}^{n(N)} X_i > N\Big\} \leq \mathbb{P}\Big\{\sum_{i=1}^{\lfloor (p-\varepsilon/2)N \rfloor} X_i > N\Big\} \leq e^{-cN}.$$

Now, let $\mathcal{A} = \{G_{N,n(N)}(\pi) = n(N)\}$.

$$\begin{aligned}\mathrm{Var}(G_{N,n(N)}) &= \mathbb{E}(G^2_{N,n(N)}) - (\mathbb{E}(G_{N,n(N)}))^2 \\ &\leq (n(N))^2 - (\mathbb{E}(G_{N,n(N)}\mathbb{1}_{\mathcal{A}})^2 = (n(N))^2 - (n(N))^2\mathbb{P}\{\mathcal{A}\}^2 \\ &\leq (n(N))^2(1-(1-e^{-cN})^2) \leq CN^2e^{-cN} \to 0.\end{aligned}$$ □

3.7 Fluctuations of the maximal path in the boundary model

In this last section we prove the path fluctuations in the characteristic direction in the model with boundaries. The idea behind it is to study how long the maximal path spends on any horizontal (or vertical) level and find a bound for the distance between the maximal path and the line which links the starting and the ending point which corresponds to the macroscopic maximal path.

Fix a boundary parameter λ and for this section the characteristic direction in (3.1.8) $(m_\lambda(N), n_\lambda(N))$ is abbreviated by (m,n) and it is the endpoint for the maximal path. Consider two rectangles $\mathcal{R}_{(k,\ell),(m,n)} \subset \mathcal{R}_{(0,0),(m,n)}$ with $0 < k < m_\lambda(N)$ and $0 < \ell < n_\lambda(N)$. In the smaller rectangle $\mathcal{R}_{(k,\ell),(m_\lambda(N),n_\lambda(N))}$ impose boundary conditions on the south and west edges given by the distributions defined in Lemma 3.2.4.

$$I_{i,\ell} \overset{\mathcal{D}}{=} I_{i,0} \quad J_{k,j} \overset{\mathcal{D}}{=} J_{0,j} \text{ with } i \in \{k+1,\dots,m\}, j \in \{\ell+1,\dots,n\}. \tag{3.7.1}$$

Recall that (3.1.13) and (3.1.14) define respectively the i coordinate where the maximal path enters and exits from a fixed horizontal level j. Since we are interested in studying either the horizontal and vertical fluctuations we also define the j coordinate where the maximal path enters and exits from a fixed vertical level i as

$$w_0(i) = \min\{j \in \{0,\dots,n\} : \exists k \text{ such that } \pi_k = (i,j)\}, \tag{3.7.2}$$

and

$$w_1(i) = \max\{j \in \{0, \dots, n\} : \exists k \text{ such that } \pi_k = (i, j)\}. \tag{3.7.3}$$

To make our notation clearer we distinguish the exit point for the path which starts from $(0,0)$ to the one which starts from (k,ℓ) adding the superscript $(0,0)$ or (k,ℓ). We define the exit point from the south edge of the rectangle $\mathcal{R}_{(k,\ell),(m,n)}$ as

$$\xi_{e_1}^{(k,\ell)} = \max_{\pi \in \Pi_{(k,\ell),(m,n)}} \{r \geq 0 : (k+i, \ell) \in \pi \text{ for } 0 \leq i \leq r, \pi \text{ is the right-most maximal}\}. \tag{3.7.4}$$

Observe from (3.7.1) that $\xi_{e_1}^{(k,\ell)}$ and $v_1(\ell) - k$ have the same distribution, i.e.

$$\mathbb{P}\{\xi_{e_1}^{(k,\ell)} = r\} = \mathbb{P}\{v_1(\ell) = k + r\}. \tag{3.7.5}$$

Proof of Theorem 3.1.9. Note that if $\tau = 0$ (3.1.15) and (3.1.16) are already contained in (3.3.29) and (3.4.34).

For $0 < \tau < 1$ set $v = \lfloor bN^{2/3} \rfloor$ and $(k,\ell) = (\lfloor \tau m \rfloor, \lfloor \tau n \rfloor)$. We add a superscript $\mathbb{P}^{(\cdot,\cdot)}\{\cdot\}$ when we want to emphasise the target point for which we are computing the probability. Remember that the rectangle $\mathcal{R}_{(k,\ell),(m,n)}$ has boundary condition (3.7.1). By Lemma 3.2.4

$$\begin{aligned} \mathbb{P}^{(m,n)}\{v_1(\lfloor \tau n \rfloor) \geq \lfloor \tau m \rfloor + v\} &= \mathbb{P}^{(m,n)}\{\xi_{e_1}^{(k,\ell)} \geq v\}, && \text{by (3.7.5)} \\ &= \mathbb{P}^{(m-k,n-\ell)}\{\xi_{e_1}^{(0,0)} \geq v\}, && \text{by (3.2.1), (3.7.1).} \end{aligned} \tag{3.7.6}$$

Note that $(m-k, n-\ell)$ is still in the characteristic direction since $(m-k, n-\ell) = (1-\tau)(m,n)$. Therefore, from (3.7.6) and Corollary 3.3.9

$$\mathbb{P}^{(m,n)}\{v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3}\} \leq C_2 b^{-3}.$$

To prove the other part of (3.1.15) notice that

$$\mathbb{P}^{(m,n)}\{v_0(\lfloor \tau n \rfloor) < \lfloor \tau m \rfloor - v\} \leq \mathbb{P}^{(m,n)}\{w_1(\lfloor \tau m \rfloor - v) \geq \lfloor \tau n \rfloor\}. \tag{3.7.7}$$

Let $k = \lfloor \tau m \rfloor - v$ and $\ell = \lfloor \tau n \rfloor - \lfloor nv/m \rfloor$. Then, up to integer-part corrections, $k/\ell = m/n$. For a constant $C_\lambda > 0$ and N sufficiently large , $\lfloor \tau n \rfloor \geq \ell + C_\lambda b N^{2/3}$. Note that from (3.7.1) we can write

$$\mathbb{P}^{(m,n)}\{\xi_{e_2}^{(k,\ell)} = r\} = \mathbb{P}^{(m,n)}\{w_1(k) = \ell + r\}. \tag{3.7.8}$$

The vertical analogue of (3.7.6) for w_1 is

$$\begin{aligned} \mathbb{P}^{(m,n)}\{w_1(\lfloor \tau m \rfloor - v) \geq \lfloor \tau n \rfloor\} &= \mathbb{P}^{(m,n)}\{w_1(k) \geq \ell + C_\lambda b N^{2/3}\} \\ &= \mathbb{P}^{(m,n)}\{\xi_{e_2}^{(k,\ell)} \geq C_\lambda b N^{2/3}\} && \text{by (3.7.8)} \end{aligned}$$

$$= \mathbb{P}^{(m-k,n-\ell)}\{\xi_{e_2}^{(0,0)} \geq C_\lambda b N^{2/3}\} \qquad \text{by (3.2.1), (3.7.1).}$$

Combine this last result with (3.7.7) and from Corollary 3.3.9 applied to ξ_{e_2} (3.1.15) follows.

Finally, we prove (3.1.16). We want to compute

$$\mathbb{P}\{\exists\, k \text{ such that } |\hat{\pi}_k - (\tau m, \tau n)| \leq \delta N^{2/3}\}.$$

If the path $\hat{\pi}$ comes within ℓ_∞ distance $\delta N^{2/3}$ of $(\tau m, \tau n)$, then it necessarily enters through the south or west side of the rectangle $\mathcal{R}_{(k+1,\ell+1),(k+4\lfloor \delta N^{2/3}\rfloor,\ell+4\lfloor c\delta N^{2/3}\rfloor)}$ (or via a diagonal step from the south-west corner), where the point $(k,\ell) = (\lfloor \tau m\rfloor - 2\lfloor \delta N^{2/3}\rfloor, \lfloor \tau n\rfloor - 2\lfloor c\delta N^{2/3}\rfloor$ and the constant $c > m/n$ for large enough N. The constant c is there to make the rectangle of characteristic shape.

From the perspective of the rectangle $\mathcal{R}_{(k,\ell),(m,n)}$ this event is equivalent to either $0 \leq \xi_{e_1}^{(k,\ell)} \leq 4\delta N^{2/3}$ or $0 \leq \xi_{e_2}^{(k,\ell)} \leq 4c\delta N^{2/3}$. For these reasons we have

$$\begin{aligned}
&\mathbb{P}^{(m,n)}\{\exists k \text{ such that } |\hat{\pi}_k - (\tau m, \tau n)| \leq \delta N^{2/3}\} \\
&\qquad \leq \mathbb{P}^{(m,n)}\{0 < \xi_{e_1}^{(k,\ell)} \leq 4\delta N^{2/3} \text{ or } 0 < \xi_{e_2}^{(k,\ell)} \leq 4c\delta N^{2/3}\} \\
&\qquad = \mathbb{P}^{(m-k,n-l)}\{0 \leq \xi_{e_1}^{(0,0)} \leq 4\delta N^{2/3} \text{ or } 0 \leq \xi_{e_2}^{(0,0)} \leq 4c\delta N^{2/3}\}.
\end{aligned}$$

We get the result using equation (3.4.34) for both exit points. $\square$

Chapter 4

A Large deviation principle for last passage times in an asymmetric Bernoulli potential

The model under consideration in this chapter is a directed corner growth model on the positive quadrant $\mathbb{Z}_+^2$. Each site v of $\mathbb{Z}_+^2$ is assigned a random weight ω_v. The environment is the same as the one in the previous chapter. In fact, the collection $\{\omega_v\}_{v\in\mathbb{Z}_+^2}$ is i.i.d. under the environment measure $\mathbb{P}$, with Bernoulli marginals

$$\mathbb{P}\{\omega_v = 1\} = p, \quad \mathbb{P}\{\omega_v = 0\} = 1 - p.$$

Throughout the chapter we exclude the values $p = 0$ or $p = 1$. One way to view the environment, is to treat site v as *present* when $\omega_v = 1$ and as *deleted* when $\omega_v = 0$. The last passage Bernoulli path up to (m, n) is a sequence of present sites

$$L_{m,n} = \{v_1 = (i_1, j_1), v_2 = (i_2, j_2), \ldots, v_M = (i_M, j_M)\} \tag{4.0.1}$$

so that $0 < i_1 < i_2 < \ldots < i_M \le m$ and $0 < j_1 < j_2 < \ldots < j_M \le n$.

What differentiate this model from the previous one are the admissible steps and the potential. In particular, the set of admissible steps is then restricted to $\mathcal{R} = \{e_1, e_2\}$ and an admissible path from $(0, 0)$ to (m, n) is an ordered sequence of sites

$$\pi_{(0,0),(m,n)} = \{(0, 0) = v_0, v_1, v_2, \ldots, v_M = (m, n)\},$$

so that $v_{k+1} - v_k \in \mathcal{R}$. The collection of all these paths is denoted by $\Pi_{(0,0),(m,n)}$. Moreover, the admissible paths can collect the random weights only via a horizontal step and no gain can be made through a vertical step. This is specified by the measurable potential function $V(\omega, z) : \mathbb{R}^{\mathbb{Z}_+^2} \times \mathcal{R} \to \mathbb{R}$ defined in (1.3.8).

Using this potential function V we define the last passage time as

$$G^V_{(0,0),(m,n)} = \max_{\pi_{(0,0),(m,n)} \in \Pi_{(0,0),(m,n)}} \Big\{ \sum_{v_i \in \pi} V(T_{v_i}\omega, v_{i+1} - v_i) \Big\}. \tag{4.0.2}$$

Above we used T_{v_i} as the environment shift by v_i in $\mathbb{Z}^2_+$. Now that V is specified we omit it from the notation. We also omit $(0,0)$ as the starting point, when it is implied. Therefore, the last passage time (4.0.2) is simply denoted by $G_{m,n}$. If the starting point is (k,ℓ) we write $G_{(k,\ell),(m,n)}$.

4.0.1 Commonly used notation

Throughout the paper, $\mathbb{N}$ denotes the natural numbers, and $\mathbb{Z}_+$ the non-negative integers. Symbol G is always denoting a last passage time. As we already mentioned, the superscript V will be omitted as there is no confusion on the potential; in our case we always use (1.3.8). Letter π signifies a generic admissible path.

Bold-face letters (e.g. $\mathbf{v}$) indicate two-dimensional vectors (e.g. $\mathbf{w} = (w_1, w_2)$). In the rare cases where we write $\mathbf{v} \le \mathbf{w}$ we mean the inequality holds coordinate-wise.

The Legendre (convex) dual of a function $f : \mathbb{R} \to (-\infty, \infty]$ is defined $f^*(y) = \sup_{x\in\mathbb{R}}\{xy - f(x)\}$. The statement $f = f^{**}$ is used throughout the article without any special mention, and it is true if and only if f is convex and lower semicontinuous, which is why we pay particular attention into having the rate function lower-semicontinuous at the boundaries of their set that they are finite. Finally, in two occasions we need the infimal convolution of two generalised convex functions f, g, and we write

$$f \Box g(r) = \inf_{x\in\mathbb{R}}\{f(x) + g(r - x)\}.$$

The important fact is that $(f \Box g)^* = f^* + g^*$. We refer to [92] for the necessary convex analysis.

4.1 The model and its invariant model

4.1.1 The invariant boundary model

For the boundary model we alter the distribution of the weights on the two axes. The new environment there will depend on a parameter $u \in (p, 1]$ that will be under our control. Each u defines different boundary distributions. At the origin we set $\omega_0 = 0$. For weights on the horizontal axis, for any $k \in \mathbb{N}$ we set $\omega_{ke_1} \sim \text{Bernoulli}(u)$, with independent marginals

$$\mathbb{P}\{\omega_{ke_1} = 1\} = u = 1 - \mathbb{P}\{\omega_{ke_1} = 0\}. \tag{4.1.1}$$

On the vertical axis, for any $k \in \mathbb{N}$, we set $\omega_{ke_2} \sim \text{Geometric}\big(\frac{u-p}{u(1-p)}\big)$ with independent marginals

$$\mathbb{P}\{\omega_{ke_2} = \ell\} = \frac{u-p}{u(1-p)}\Big(\frac{p(1-u)}{u(1-p)}\Big)^{\ell}, \quad \ell \in \mathbb{Z}_+. \tag{4.1.2}$$

The environment in the bulk $\{\omega_w\}_{w\in\mathbb{N}^2}$ remains unchanged with i.i.d. Ber(p) marginal distributions. Denote this environment by $\omega^{(u)}$ to emphasise the different distributions on the axes that depend on u. In summary, for any $i \geq 1, j \geq 1$, the $\omega^{(u)}$ marginals are independent with marginals

$$\omega^{(u)}_{i,j} \sim \begin{cases} \text{Ber}(p), & \text{if } (i,j) \in \mathbb{N}^2, \\ \text{Ber}(u), & \text{if } i \in \mathbb{N}, j = 0, \\ \text{Geom}\Big(\frac{u-p}{u(1-p)}\Big), & \text{if } i = 0, j \in \mathbb{N}, \\ \delta_0, & \text{if } i = 0, j = 0. \end{cases} \tag{4.1.3}$$

In this environment we slightly alter the way a path can collect weight on the boundaries. Consider any path π from 0. If the path moves horizontally before entering the bulk, then it collects the Bernoulli(u) weights until it takes the first vertical step, and after that, it collects weight according to the potential function (1.3.8). If π moves vertically from 0 then it **also** collects the geometric weights on the vertical axis, and after it enters the bulk, it collects according to V. This is the only difference from the potential V of the i.i.d. model, namely while on the y-axis, the path can still collect positive weight.

Fix a parameter $u \in (p, 1]$. Denote the last passage time from 0 to w in environment $\omega^{(u)}$ by $G^{(u)}_{0,w}$. The variational equality, using the above description, is

$$G^{(u)}_{0,w} = \max_{1\leq k\leq w\cdot e_1}\Big\{\sum_{i=1}^{k}\omega_{ie_1} + G^V_{ke_1+e_2,w}\Big\}$$
$$\bigvee \max_{1\leq k\leq w\cdot e_2}\Big\{\sum_{j=1}^{k}\omega_{je_2} + \omega_{e_1+ke_2} + G^V_{e_1+ke_2,w}\Big\}. \tag{4.1.4}$$

Our first statement give the explicit formula for the shape function of the invariant model.

Theorem 4.1.1 (Law of large numbers for $G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$). *For fixed parameter $p < u \leq 1$ and $(s,t) \in \mathbb{R}^2_+$ we have*

$$g^{(u)}_{pp}(s,t) = \lim_{N\to\infty}\frac{G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}{N} = su + t\,\frac{p(1-u)}{u-p}, \quad \mathbb{P}-a.s. \tag{4.1.5}$$

It is convenient to introduce to passage times, depending on the first step of the set of paths we are optimizing over. Define

$$G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} = \max_{1\leq k\leq\lfloor Ns\rfloor}\Big\{\sum_{i=1}^{k}\omega_{i,0} + G_{(k,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}\Big\} \tag{4.1.6}$$

and

$$G^{(u),\text{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} = \max_{1\le\ell\le\lfloor Nt\rfloor}\Big\{\sum_{j=1}^{\ell}\omega_{0,j} + \omega_{1,\ell} + G_{(1,\ell),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}\Big\}. \tag{4.1.7}$$

Then, by (4.1.4)

$$G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} = G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \vee G^{(u),\text{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}. \tag{4.1.8}$$

Remind from the introduction in Chapter 1 that $g_{pp}(s,t)$ represents the shape function for the model without boundaries. Passage times (4.1.6) and (4.1.7) satisfy a law of large numbers as well, given in the next

THEOREM **4.1.2.** *Let $s,t\ge 0$, $u\in(p,1]$.*

(a) The following limit exists and is given by

$$g^{(u),hor}_{pp}(s,t) = \lim_{N\to\infty} N^{-1}G^{(u),hor}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} = \begin{cases} g^{(u)}_{pp}(s,t) & \text{if } t < s\frac{(u-p)^2}{p(1-p)},\\ g_{pp}(s,t) & \text{if } t \ge s\frac{(u-p)^2}{p(1-p)}.\end{cases} \tag{4.1.9}$$

(b) The following limit exists and is given by

$$g^{(u),ver}_{pp}(s,t) = \lim_{N\to\infty} N^{-1}G^{(u),ver}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} = \begin{cases} g^{(u)}_{pp}(s,t) & \text{if } t > s\frac{(u-p)^2}{p(1-p)},\\ g_{pp}(s,t) & \text{if } t \le s\frac{(u-p)^2}{p(1-p)}.\end{cases} \tag{4.1.10}$$

As is usual in the exactly solvable models of last passage percolation, there is the notion of a *characteristic direction.* In this case, for the model with boundaries for a given boundary parameter $u\in(p,1]$, there exists a unique direction $(m(N),n(N))$ whose scaled direction, as $N\to\infty$, converges to the macroscopic characteristic direction

$$N^{-1}(m_u(N),n_u(N)) \to \Big(1,\frac{(u-p)^2}{p(1-p)}\Big), \tag{4.1.11}$$

which gives that for large enough N the endpoint $(m(N),n(N))$ is always below the critical line $y=\frac{1-p}{p}x$ that separates the flat edge from the strictly concave part of $g_{pp}(s,t)$ in Theorem 1.3.1. Here the characteristic direction already manifested in Theorem 4.1.2 as the cutting line between feeling the boundary effect versus entering the bulk.

The full rate function is described in Theorem 4.1.4. As it is usually the case with models of last passage percolation, large deviations of the passage time above its mean are of different exponential scale than the deviations below its mean. With this in mind, in order to obtain a full LDP, one only needs the right-tail rate function. This is our starting point.

Suppose that the target point is (s,t), then, since the last passage time collects Bernoulli weights only through the right step, the last passage time definition implies that the probability

$$\mathbb{P}\{G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr\} \neq 0 \text{ if and only if } r < s. \tag{4.1.12}$$

In the particular case where s is rational, the probability above can be strictly positive for certain values of N, but otherwise it is 0.

Theorem 4.1.3. *For $((s,t),r)$ with $0 \leq r < s < \infty$ and $t \in \mathbb{R}_+$, the following $\mathbb{R}_+$-valued limit exists:*

$$-\lim_{N\to\infty} N^{-1} \log \mathbb{P}\{G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr\} = J_{s,t}(r). \tag{4.1.13}$$

$J_{s,t}(r)$, as a function of $((s,t),r)$ is a continuous convex function on the interior of the set $A = \{((s,t),r) : s \geq r \vee 0, t \in \mathbb{R}_+, r \in \mathbb{R}_+\}$. It can be uniquely extended to a finite continuous convex function on $\bar{A}$ which we denote by $J_{s,t}(r)$. Moreover, $J_{s,t}(r) > 0$ for $r > g_{pp}(s,t)$.

We show the existence of a good rate function $I_{s,t}(r)$ and list its properties; this is the content of the next theorem. We restrict $r \in [0,s]$ because $I_{s,t}(r) = \infty$ for any r outside this interval.

Theorem 4.1.4. *Let $\omega_{i,j} \sim$Bernoulli(p) with $i,j \geq 1$ and $(s,t) \in (0,\infty)^2$. Then there exists a generalised function $I_{s,t}(r)$ so that the distributions of $N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ satisfy an LDP with normalisation N and rate function $I_{s,t}(r)$. To be precise, the following bounds hold for any open set H and any closed set F in $[0,s]$:*

$$\overline{\lim_{N\to\infty}} N^{-1} \log \mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \in F\} \leq -\inf_{r\in F} I_{s,t}(r) \tag{4.1.14}$$

and

$$\underline{\lim_{N\to\infty}} N^{-1} \log \mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \in H\} \geq -\inf_{r\in H} I_{s,t}(r). \tag{4.1.15}$$

The rate function $I_{s,t}(r)$ is defined by

$$I_{s,t}(r) = \begin{cases} J_{s,t}(r), & r \in [g_{pp}(s,t), s], \\ \infty, & \textit{otherwise}. \end{cases} \tag{4.1.16}$$

Rate function $J_{s,t}(r)$ is the right-tail rate function computed in Theorem 4.1.3. In particular, on $[g_{pp}(s,t), s]$ the rate function $I_{s,t}$ is finite, strictly increasing, continuous and convex. Moreover, the unique zero of $I_{s,t}(r)$ is at $r = g_{pp}(s,t)$.

Corollary 4.1.5. *Let $\xi \in \mathbb{R}$. Then*

$$\lim_{N\to\infty} N^{-1} \log \mathbb{E}e^{\xi G_{0,(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}} = I^*_{s,t}(\xi) = \begin{cases} J^*_{s,t}(\xi) & \text{if } \xi > 0,\\ 0 & \text{if } \xi = 0,\\ \xi g_{pp}(s,t) & \text{if } \xi < 0. \end{cases} \tag{4.1.17}$$

The variational characterization of $J_{s,t}$ requires the log-moment generating functions for Bernoulli(p) random variables, given by

$$C^{(p)}_{\mathcal{B}}(\xi) = \log(1-p+pe^{\xi}), \ \ \xi \in \mathbb{R}. \tag{4.1.18}$$

and Geometric(p) random variables given by

$$C^{(p)}_{\mathcal{G}}(\xi) = \begin{cases} \log \frac{p}{1-(1-p)e^{\xi}}, & \xi < -\log(1-p)\\ \infty, & \text{otherwise}. \end{cases} \tag{4.1.19}$$

Both log-moment generating functions can be seen as the Legandre duals of the rate functions for sums of i.i.d. Bernoulli (4.1.18) and for sums of i.i.d. geometric random variables, given by

$$I^{(p)}_{\mathcal{G}}(r) = \sup_{\xi<-\log(1-p)} \big\{ r\xi - C^{(p)}_{\mathcal{G}}(\xi) \big\} = r \log \frac{r}{(1-p)(1+r)} - \log(1+r)p \qquad \text{for } r > 0. \tag{4.1.20}$$

The two theorems that give the precise forms for J and J^* follow.

Proposition 4.1.6. *Let $(s,t) \in \mathbb{R}^2_+$. Then for all $\xi \in \mathbb{R}$, the convex dual $J^*_{(s,t)}(\xi)$ is given by*

$$J^*_{s,t}(\xi) = \begin{cases} \inf_{u\in(p,1]}\{sC^{(u)}_{\mathcal{B}}(\xi) - tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(-\xi)\}, & \text{if } \xi > 0,\\ 0, & \text{if } \xi = 0,\\ \infty, & \text{if } \xi < 0. \end{cases}$$

The closed form for J^* is given in the following

Theorem 4.1.7. *Fix $p \in (0,1)$, $\xi \geq 0$ and $(s,t) \in \mathbb{R}^2_+$. Define*

$$\Delta = \Delta_{p,s,t,\xi} = p(1-p)(e^{\xi}+e^{-\xi}-2)\big[p(1-p)(s+t)^2(e^{\xi}+e^{-\xi}-2)+4st\big]. \tag{4.1.21}$$

Then,

$$J^*_{s,t}(\xi) = \begin{cases} s \log \dfrac{p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+2s+\sqrt{\Delta}}{2s(1-p(1-e^{-\xi}))} \\ \quad + t \log \dfrac{[p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+\sqrt{\Delta}](1-p(1-e^{-\xi}))}{p(1-p)(t-s)(e^{\xi}+e^{-\xi}-2)+\sqrt{\Delta}}, & \text{if } t < \frac{1-p}{p}s,\\ s\xi, & \text{if } t \geq \frac{1-p}{p}s. \end{cases} \tag{4.1.22}$$

Define the last passage time's l.m.g.f. for the boundary model

$$\Lambda^{(u)}_{(s,t)}(\xi) = \lim_{N\to\infty} N^{-1} \log \mathbb{E} e^{\xi G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}. \tag{4.1.23}$$

It will be convenient to also define the l.m.g.f. for the two passage times conditional on the first step being e_1 or e_2, $G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ and $G^{(u),\text{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ given by (4.1.6) and (4.1.7) respectively. The corresponding l.m.g.f. are

$$\Lambda^{(u),\text{hor}}_{(s,t)}(\xi) = \lim_{N\to\infty} N^{-1} \log \mathbb{E} e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}} \tag{4.1.24}$$

and

$$\Lambda^{(u),\text{ver}}_{(s,t)}(\xi) = \lim_{N\to\infty} N^{-1} \log \mathbb{E} e^{\xi G^{(u),\text{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}. \tag{4.1.25}$$

The existence of the above limits is verified in Lemma 4.6.1 below, but we state it as part of the main Theorem 4.1.8.

The existence of the two limits above then gives rise to the formula

$$\Lambda^{(u)}_{(s,t)}(\xi) = \Lambda^{(u),\text{hor}}_{(s,t)}(\xi) \vee \Lambda^{(u),\text{ver}}_{(s,t)}(\xi) \quad \text{for any } \xi > 0. \tag{4.1.26}$$

Thus, finding $\Lambda^{(u)}_{(s,t)}(\xi)$ is equivalent to finding $\Lambda^{(u),\text{hor}}_{(s,t)}(\xi), \Lambda^{(u),\text{ver}}_{(s,t)}(\xi)$, which is the content of Theorem 4.1.8 below.

Heuristically, one expects the creation of some critical direction for (s,t) that will depend on ξ, p, u; below the direction the boundary effect will be felt at the l.m.g.f. level, and otherwise the model will behave like the boundary is not present. This was also observed at the LLN level in Theorem 4.1.2. In fact this is the case.

For $\xi > 0$

$$k^{(u)}(\xi) := \Big(\frac{\partial C^{(u)}_{\mathcal{B}}(\xi)}{\partial u}\Big) / \Big(\frac{\partial C^{(u)}_{\mathcal{G}}(-\xi)}{\partial u}\Big). \tag{4.1.27}$$

The relevant conditions that create a critical line are

$$t = k^{(u)}(\xi)s, \quad \text{and} \quad t = k^{(u)}(-\xi)s, \tag{4.1.28}$$

for $\Lambda^{(u),\text{hor}}$ and $\Lambda^{(u),\text{ver}}$ respectively. Recall that $l.m.g.f$ of $G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ is given by Corollary 4.1.5, and is equal to $I^*_{s,t}(\xi) = J^*_{s,t}(\xi)$. For uniformity of notation in the section, set $\Lambda_{(s,t)}(\xi) = I^*_{s,t}(\xi)$.

Theorem **4.1.8.** *Let $s, t \geq 0$, $u \in (p,1)$ and $\xi \geq 0$.*

(a) The limit in (4.1.24) exists and is given by

$$\Lambda^{(u),\text{hor}}_{(s,t)}(\xi) = \begin{cases} sC^{(u)}_{\mathcal{B}}(\xi) - tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(-\xi) & \text{if } t < k^{(u)}(\xi)s, \\ \Lambda_{(s,t)}(\xi) & \text{if } t \geq k^{(u)}(\xi)s. \end{cases} \tag{4.1.29}$$

(b) The limit in (4.1.25) exists and is given by

$$\Lambda^{(u),\mathrm{ver}}_{(s,t)}(\xi) = \begin{cases} tC_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(\xi) - sC_{\mathcal{B}}^{(u)}(-\xi), & \text{if } \xi \in [0, \log \frac{u(1-p)}{p(1-u)}) \text{ and } t > k^{(u)}(-\xi)s, \\ \Lambda_{(s,t)}(\xi), & \text{if } \xi \in [0, \log \frac{u(1-p)}{p(1-u)}) \text{ and } t \le k^{(u)}(-\xi)s, \\ \infty, & \text{if } \xi \in [\log \frac{u(1-p)}{p(1-u)}, \infty). \end{cases} \tag{4.1.30}$$

The last theorem proves the full l.m.g.f. for the boundary model. Define

$$\ell^{(u)}(\xi) = \frac{C_{\mathcal{B}}^{(u)}(\xi) + C_{\mathcal{B}}^{(u)}(-\xi)}{C_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(\xi) + C_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(-\xi)}. \tag{4.1.31}$$

Then, the l.g.m.f. for the boundary last passage time is given by

THEOREM **4.1.9.** *Let $s, t \ge 0$ and $u \in (p, 1]$. Then the limit in (4.1.23) exists for $\xi \ge 0$ and is given by*

$$\Lambda^{(u)}_{(s,t)}(\xi) = \begin{cases} sC_{\mathcal{B}}^{(u)}(\xi) - tC_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(-\xi), & \text{if } \xi \in [0, \log \frac{u(1-p)}{p(1-u)}) \text{ and } t < \ell^{(u)}(\xi)s, \\ tC_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(\xi) - sC_{\mathcal{B}}^{(u)}(-\xi), & \text{if } \xi \in [0, \log \frac{u(1-p)}{p(1-u)}) \text{ and } t \ge \ell^{(u)}(\xi)s, \\ \infty, & \text{if } \xi \in [\log \frac{u(1-p)}{p(1-u)}, \infty). \end{cases} \tag{4.1.32}$$

4.2 Burke's property and law of large numbers

To simplify the notation in what follows, set $w = (i, j) \in \mathbb{Z}_+^2$ and define the last passage time gradients by

$$I^{(u)}_{i+1,j} = G^{(u)}_{0,(i+1,j)} - G^{(u)}_{0,(i,j)}, \quad \text{and} \quad J^{(u)}_{i,j+1} = G^{(u)}_{0,(i,j+1)} - G^{(u)}_{0,(i,j)}. \tag{4.2.1}$$

When there is no confusion we will drop the superscript (u) from the above. When $j = 0$ we have that $\{I^{(u)}_{i,0}\}_{i\in\mathbb{N}}$ is a collection of i.i.d. Bernoulli(u) random variables since $I^{(u)}_{i,0} = \omega_{(i,0)}$. Similarly, for $i = 0$, $\{J^{(u)}_{0,j}\}_{j\in\mathbb{N}}$ is a collection of i.i.d. Geometric$(\frac{u-p}{u(1-p)})$ random variables.

The gradients and the passage time satisfy recursive equations. This is the content of the next lemma.

Lemma 4.2.1. *Let $u \in (p, 1]$ and $(i, j) \in \mathbb{N}^2$. Then the last passage time can be recursively computed as*

$$G^{(u)}_{0,(i,j)} = \max\{G^{(u)}_{0,(i,j-1)},\ G^{(u)}_{0,(i-1,j)} + \omega_{i,j}\}. \tag{4.2.2}$$

Furthermore, the last passage time gradients satisfy the recursive equations

$$\begin{aligned} I^{(u)}_{i,j} &= \max\{I^{(u)}_{i,j-1} - J^{(u)}_{i-1,j},\ \omega_{i,j}\}, \\ J^{(u)}_{i,j} &= (J^{(u)}_{i-1,j} - I^{(u)}_{i,j-1} + \omega_{i,j})^+. \end{aligned} \tag{4.2.3}$$

Proof. Equation (4.2.2) is immediate from the description of the dynamics in the boundary model and the fact that (i, j) is in the bulk. We only prove the recursive equation (4.2.3) for the J and the other one is done similarly and left to the reader. Compute

$$\begin{aligned}
J^{(u)}_{i,j} &= G^{(u)}_{0,(i,j)} - G^{(u)}_{0,(i,j-1)} \\
&= \max\big\{G^{(u)}_{0,(i,j-1)}, G^{(u)}_{0,(i-1,j)} + \omega_{i,j}\big\} - G^{(u)}_{0,(i,j-1)} \qquad \text{by (4.2.2)}, \\
&= \max\big\{G^{(u)}_{0,(i,j-1)} - G^{(u)}_{0,(i,j-1)}, G^{(u)}_{0,(i-1,j)} - G^{(u)}_{0,(i,j-1)} + \omega_{i,j}\big\} \\
&= \max\big\{0, G^{(u)}_{0,(i-1,j)} - G^{(u)}_{0,(i,j-1)} + G^{(u)}_{0(i-1,j-1)} - G^{(u)}_{0(i-1,j-1)} + \omega_{i,j}\big\} \\
&= (J^{(u)}_{i-1,j} - I^{(u)}_{i,j-1} + \omega_{i,j})^{+}.
\end{aligned}$$

□

Using the gradients (4.2.3) and the environment $\{\omega_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ we also define new random variables $\alpha_{i,j}$ on $\mathbb{Z}^2_+$

$$\alpha_{i-1,j-1} = \min\{I^{(u)}_{i,j-1}, J^{(u)}_{i-1,j} + \omega_{i,j}\} \qquad \text{for } (i,j) \in \mathbb{N}^2. \tag{4.2.4}$$

Since the $I^{(u)}_{i,j}$ are Bernoulli, so are the $\alpha_{i,j}$. The following lemma gives the distribution of the triple $(I^{(u)}_{i,j}, J^{(u)}_{i,j}, \alpha_{i-1,j-1})$. It is an analogue of Burke's property for $M/M/1$ queues.

Lemma 4.2.2 (Burke's property)**.** *Let independent random variables be distributed by*

$$(I^{(u)}_{i,j-1}, J^{(u)}_{i-1,j}, \omega_{i,j}) \sim \Big(\mathit{Ber}(u), \mathit{Geom}\Big(\frac{u-p}{u(1-p)}\Big), \mathit{Ber}(p)\Big), \tag{4.2.5}$$

where we assume $u > p$. Then, for $(i, j) \in \mathbb{N}^2$, the triple obtained via equations (4.2.3), (4.2.4) is also an independent triple

$$(I^{(u)}_{i,j}, J^{(u)}_{i,j}, \alpha_{i-1,j-1}) \sim \Big(\mathit{Ber}(u), \mathit{Geom}\Big(\frac{u-p}{u(1-p)}\Big), \mathit{Ber}(p)\Big). \tag{4.2.6}$$

Proof. We omit the superscripts and indices from the I, J and we simply denote

$$\tilde{I} = \max\{I - J, \omega\}, \quad \text{and} \quad \tilde{J} = (J - I + \omega)^{+}.$$

The marginal distributions of $(\tilde{I}, \tilde{J}, \alpha)$ can be computed directly, using equations (4.2.3), (4.2.4). For example, since α only takes the values 0 or 1 it suffices to compute

$$\begin{aligned}
\mathbb{P}\{\alpha = 1\} &= \mathbb{P}\{\min\{I, J + \omega\} = 1\} = \mathbb{P}\{I = 1, J + \omega \geq 1\} \\
&= u\Big(p + (1-p)\Big(1 - \frac{u-p}{u(1-p)}\Big)\Big) = p.
\end{aligned}$$

The remaining calculations are left to the reader.

The proof of independence goes by calculating the Laplace transform of the triple $(\tilde{I}, \tilde{J}, \alpha)$. Let $x \in \mathbb{R}, z \in \mathbb{R}$ and $y > \log[p(1-u)/(u(1-p))]$. Recall that $u \in (p, 1]$. Then compute, using (4.2.3) and (4.2.4), the joint Laplace transform

$$\mathbb{E}(e^{-x\tilde{I} - y\tilde{J} - z\alpha}) = \mathbb{E}[e^{-x\max\{I-J,\omega\} - y(J-I+\omega)^{+} - z\min\{I,J+\omega\}}]$$

$$
\begin{aligned}
&= pu\mathbb{E}[e^{-x(\max\{1-J,1\})-yJ-z\min\{1,J+1\}}] + p(1-u)\mathbb{E}[e^{-x-y(J+1)^+}] \\
&\qquad\qquad + (1-p)u\mathbb{E}[e^{-x(1-J)^+-y(J-1)^+-z(1\wedge J)}] + (1-p)(1-u)\mathbb{E}[e^{-yJ}] \\
&= pu\frac{u-p}{u(1-p)}e^{-(x+z)}\sum_{j=0}^{\infty}\left(\frac{p(1-u)}{u(1-p)}\right)^j e^{-yj} \\
&\quad + p(1-u)\frac{u-p}{u(1-p)}e^{-(x+y)}\sum_{j=0}^{\infty}\left(\frac{p(1-u)}{u(1-p)}\right)^j e^{-yj} \\
&\qquad + (1-p)u\frac{u-p}{u(1-p)}\left(e^{-x}+\sum_{j=1}^{\infty}\left(\frac{p(1-u)}{u(1-p)}\right)^j e^{-y(j-1)-z}\right) \\
&\qquad\qquad + (1-p)(1-u)\sum_{j=0}^{\infty}\left(\frac{p(1-u)}{u(1-p)}\right)^j e^{-yj} \\
&= \frac{\frac{u-p}{u(1-p)}}{1-\frac{p(1-u)}{u(1-p)}e^{-y}}\left(pue^{-(x+z)}+p(1-u)e^{-(x+y)}+(1-p)(1-u)\right) \\
&\quad + (1-p)u\frac{\frac{u-p}{u(1-p)}}{1-\frac{p(1-u)}{u(1-p)}e^{-y}}\left[e^{-x}\left(1-\frac{p(1-u)}{u(1-p)}e^{-y}\right)+e^{-z}\frac{p(1-u)}{u(1-p)}\right] \\
&= \frac{\frac{u-p}{u(1-p)}}{1-\frac{p(1-u)}{u(1-p)}e^{-y}}\left(pue^{-(x+z)}+(1-p)(1-u)+(1-p)ue^{-x}+p(1-u)e^{-z}\right) \\
&= \mathbb{E}(e^{-y\tilde{J}})\mathbb{E}(e^{-x\tilde{I}})\mathbb{E}(e^{-z\alpha}) \qquad \square
\end{aligned}
$$

A down-right path ψ on the lattice $\mathbb{Z}_+^2$ is an ordered sequence of sites $\{v_i\}_{i\in\mathbb{Z}}$ that satisfy

$$v_i - v_{i-1} \in \{e_1, -e_2\}.$$

For a given down-right path ψ, define $\psi_i = v_i - v_{i-1}$ to be the i-th edge of the path and set

$$L_{\psi_i} = \begin{cases} I^{(u)}_{v_i}, & \text{if } \psi_i = e_1 \\ J^{(u)}_{v_{i-1}}, & \text{if } \psi_i = -e_2. \end{cases}$$

Also define the interior sites $\mathcal{I}_\psi$ of ψ to be

$$\mathcal{I}_\psi = \{w \in \mathbb{Z}_+^2 : \exists\, v_i \in \psi \text{ s.t. } w < v_i \text{ coordinate-wise}\}.$$

A convenient way to state Lemma 4.2.2 is the following.

Corollary 4.2.3. *Fix a down-right path ψ. Then the random variables*

$$\{\{\alpha_w\}_{w\in\mathcal{I}_\psi}, \{L_{\psi_i}\}_{i\in\mathbb{Z}}\} \tag{4.2.7}$$

are mutually independent, with marginals

$$\alpha_w \sim Ber(p), \quad L_{\psi_i} \sim \begin{cases} Ber(u), & \text{if } \psi_i = e_1 \\ Geom\Big(\frac{u-p}{u(1-p)}\Big), & \text{if } \psi_i = -e_2. \end{cases}$$

Proof. The proof is inductive. Consider the countable set of paths Ψ that connect the y-axis to the x-axis. The trivial case is when $\mathcal{I}_{\psi_0} = \emptyset$ (i.e. ψ_0 is the union of the two axes, $\psi_0 \in \Psi$) and then the statement reduces to the independence of the $\omega_{i,j}$'s on the x and y axes which is true by the definition of the environment.

Assume that for a $\psi \in \Psi$ the statement holds. We say that a lattice vertex v_{i_0} on ψ $(i,j) \in \mathbb{Z}_+^2$ is a west-south corner of ψ if

$$(v_{i_0-1}, v_{i_0}, v_{i_0+1}) = ((i, j+1), (i,j), (i+1, j)).$$

Now define a new path $\tilde{\psi}$ by replacing v_{i_0} with $\tilde{v}_{i_0} = (i+1, j+1)$ and keep all the other points intact which means that $v_i = \tilde{v}_i$ for $i \neq i_0$. In this way we have $\mathcal{I}_{\tilde{\psi}} = \mathcal{I}_\psi \cup \{(i,j)\}$.

Going from ψ to $\tilde{\psi}$ we have also a change in the set of random variables in (4.2.7). In fact

$$\{I_{i+1,j}, J_{i,j+1}\} \tag{4.2.8}$$

have been replaced by

$$\{I_{i+1,j+1}, J_{i+1,j+1}, \alpha_{i+1,j+1}\}. \tag{4.2.9}$$

By (4.2.3) and (4.2.4) the variables in (4.2.9) are determined by (4.2.8) and $\omega_{i+1,j+1}$. By construction $\omega_{i+1,j+1}$ is independent of (4.2.7) for the ψ under consideration. By construction the triple $\{I_{i+1,j}, J_{i,j+1}, \omega_{i+1,j+1}\}$ are independent random variables and by the induction assumption we have they are in turn independent of the all other variables (4.2.7). Finally Lemma 4.2.2 implies that also the triple $\{I_{i+1,j+1}, J_{i+1,j+1}, \omega_{i,j}\}$ are independent random variables with the correct marginal distribution and they are independent of all the random variables of $\tilde{\psi}$. All these observations prove that also $\tilde{\psi}$ satisfies the statement of the corollary.

Note that if we start with ψ_0, we can build a path $\psi \in \Psi$ by flipping west-south corners finitely many times. The induction argument guarantees that class Ψ satisfies the corollary.

The general statement follows also for an arbitrary down-right path ψ using the independence of finite subcollections. Consider any square $\mathcal{R} = \{i \leq 0, j \leq M\}$ large enough so that the corner (M, M) lies outside $\psi \cup \mathcal{I}_\psi$. The α and $L(\psi)$ variables associated to ψ that lie in $\mathcal{R}$ are a subset of the variables of the path $\tilde{\psi}$ that goes through the points $(0, M), (M, M)$ and $(M, 0)$. This path $\tilde{\psi}$ connects the axes so the first part of the proof applies to it. Thus the variables (4.2.7) that lie inside an arbitrarily large square are independent. $\square$

Theorem **4.2.4** (Variational formula for the LLN of the non boundary model)**.** *Fix p in $(0,1)$ and $(s,t) \in \mathbb{R}_+^2$. Then we have the explicit law of large numbers limit*

$$g_{pp}(s,t) = \inf_{p<u\leq 1}\{s\mathbb{E}(I^{(u)}) + t\mathbb{E}(J^{(u)})\} = \inf_{p<u\leq 1} g_{pp}^{(u)}(s,t). \tag{4.2.10}$$

Remark 4.2.5. *From (4.2.10) it is possible to see the characteristic direction manifesting in a different way. Without loss set $s=1$. Then the u^* that minimizes the expression above is $u^* = p + \sqrt{tp(1-p)}$ if $t < q/p$ and 1 otherwise. Assume $t < q/p$. Solve the expression for t we obtain*

$$t = \frac{(u^*-p)^2}{p(1-p)}.$$

In other words, $g_{pp}(1,t) = g_{pp}^{(u^*)}(1,t)$ and direction $(1,t)$ is characteristic according to (4.1.11) for the boundary model with parameter u^*. Note that the range of characteristic directions only covers the directions for which $g_{pp}^{(u)}(s,t)$ is strictly concave. The flat edge of g_{pp} corresponds to $u^* = 1$.

Remark 4.2.6. *Along the characteristic direction the last passage time at point $N(m,n)$ it is expected (but not proven yet) to have variance of order $O(N^{2/3})$ for large N, while in the other directions the fluctuations of $G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ to have order of magnitude $N^{1/2}$ and they are asymptotically Gaussian. Finally it is possible to prove using similar arguments as in [26] that the order of the variance in the flat edge is $o(1)$.*

From these considerations, we expect that the large deviations, for the boundary model, to be 'unusual' in the characteristic direction, while in the off-characteristic directions to be the typical decay of order e^{-N} for both tails. We can show that the right tail has deviations of order e^{-cN}, but conditional on one of the boundaries being absent. This is essentially equation (4.6.2). In Lemma 4.3.2 we give a bound on the left tail that indicates superexponential decay when we move along direction (4.1.11) for the boundary model.

Proof of Theorem 4.1.1. From equations (4.2.1) we can write the last passage time of the invariant model as

$$G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} = \sum_{j=1}^{\lfloor Ns\rfloor} I^{(u)}_{i,0} + \sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{\lfloor Ns\rfloor,j}$$

where the I,J variables are respectively the horizontal and vertical increments of the passage time. By the definition of the boundary model, the I variables are i.i.d. Ber(u). Scaled by N, the first sum converges to $s\mathbb{E}(I_{1,0})$ by the law of large numbers.

By Corollary 4.2.3 the J variables are i.i.d. Geom($\frac{u-p}{u(1-p)}$), since they belong on the down-right path that goes from $(0,\lfloor Nt\rfloor)$ horizontally to $(\lfloor Ns\rfloor, \lfloor Nt\rfloor)$ and then vertically down to $(\lfloor Ns\rfloor, 0)$. At this point we cannot immediately evoke the law of large numbers as before since the whole sequence changes with N. Therefore, we first appeal to the Borel-Cantelli lemma via a large deviation estimate. Fix an $\varepsilon > 0$.

$$\begin{aligned}
&\mathbb{P}\Big\{N^{-1}\sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{\lfloor Ns\rfloor,j} \notin \Big(\frac{p(1-u)}{u-p}-\varepsilon, \frac{p(1-u)}{u-p}+\varepsilon\Big)\Big\}\\
&\qquad\qquad\qquad\qquad = \mathbb{P}\Big\{N^{-1}\sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{0,j} \notin \Big(t\ \frac{p(1-u)}{u-p}-\varepsilon, t\ \frac{p(1-u)}{u-p}+\varepsilon\Big)\Big\}\\
&\qquad\qquad\qquad\qquad \le e^{-c(u,p,t,\varepsilon)N},
\end{aligned}$$

for some proper positive constant $c(u,p,t,\varepsilon)$. By the Borel-Cantelli lemma we have that for each $\varepsilon > 0$ there exists a random N_ε so that for all $N > N_\varepsilon$

$$t\ \frac{p(1-u)}{u-p} - \varepsilon < N^{-1}\sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{\lfloor Ns\rfloor,j} \le t\ \frac{p(1-u)}{u-p} + \varepsilon.$$

Then we have

$$su + t\ \frac{p(1-u)}{u-p} - \varepsilon \le \varliminf_{N\to\infty} \frac{G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}{N} \le \varlimsup_{N\to\infty} \frac{G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}{N} \le su + t\ \frac{p(1-u)}{u-p} + \varepsilon.$$

Let ε tend to 0 to finish the proof. $\square$

In order to prove Theorem 1.3.1 and Proposition 4.1.6 we need the following technical result. This is in the spirit of Proposition 3.10 in [61] but tailored to our particular case.

Proposition 4.2.7. *Let $I = (a,b] \subseteq \mathbb{R}$ with $a, b \in \mathbb{R}$. Let the convex functions $h, g\colon I \to \mathbb{R}$ be twice continuously differentiable with $h'(u) > 0$ and $g'(u) < 0$ for every $u \in I$. Define*

$$f_{s,t}(u) = sh(u) + tg(u) \quad \text{with } (s,t) \in \mathbb{R}^2_+.$$

Suppose that $f''_{s,t}(u) > 0$ for all $(s,t) \in \mathbb{R}^2_+$, $\lim_{u\searrow a} f_{s,t}(u) = \infty$ and $f_{s,t}(b) = c < \infty$ with $c \in \mathbb{R}$. If $\Lambda(s,t)$ is a continuous function in (s,t) with the property that for all $(s,t) \in \mathbb{R}^2_+$ and $u \in I$ the identity

$$0 = \sup_{0\le z\le s}\{\Lambda(s-z,t) - f_{s-z,t}(u)\} \vee \sup_{0\le \tilde z\le t}\{\Lambda(s,t-\tilde z) - f_{s,t-\tilde z}(u)\} \tag{4.2.11}$$

holds, then for every $t < -\frac{h'(b)}{g'(b)}s$,

$$\Lambda(s,t) = \min_{u\in I}\{f_{s,t}(u)\}.$$

Proof. Fix $(s,t) \in \mathbb{R}^2_+$ and call $\nu = \frac{t}{s}$. Observe that under the hypotheses of this proposition there exists a unique $u^*_{s,t} = \arg\min_{u\in I} f_{s,t}(u) = u^*_{1,\nu}$. This minimum point can be eventually reached at $u^*_{s,t} = b$ if $f'_{s,t}(u) \le 0$ for all $u \in I$. In particular, $u^*_{s,t}$ solves the equation

$$f'_{s,t}(u) = sh'(u) + tg'(u) = 0 \implies t = -\frac{h'(u)}{g'(u)}s. \tag{4.2.12}$$

The largest value $-\frac{h'(u)}{g'(u)}$ can take is when $u = b$. For any (s,t) above the line

$$t = -\frac{h'(b)}{g'(b)}s, \tag{4.2.13}$$

equation (4.2.12) has no solution and in fact $f'_{s,t}(u) < 0$ and $\arg\min f_{s,t}(u) = b$. For any (s,t) below this line (4.2.13) a solution to (4.2.12) exists and is giving the minimizing argument. We call the line (4.2.13) *the critical line.*

The identity in (4.2.11) implies that for all $z \in [0,s]$ and $\tilde{z} \in [0,t]$ the following inequalities hold

$$\Lambda(s-z,t) \le f_{s-z,t}(u^*_{s-z,t}), \quad \Lambda(s,t-\tilde{z}) \le f_{s,t-\tilde{z}}(u^*_{s,t-\tilde{z}}).$$

Fix a $u \in I$ and subtract, from each side of the inequalities above, $f_{s-z,t}(u)$ and $f_{s,t-\tilde{z}}(u)$ respectively, to obtain

$$\Lambda(s-z,t) - f_{s-z,t}(u) \le f_{s-z,t}(u^*_{s-z,t}) - f_{s-z,t}(u), \tag{4.2.14}$$

$$\Lambda(s,t-\tilde{z}) - f_{s,t-\tilde{z}}(u) \le f_{s,t-\tilde{z}}(u^*_{s,t-\tilde{z}}) - f_{s,t-\tilde{z}}(u). \tag{4.2.15}$$

Since the minimizer is unique we have that $f_{s-z,t}(u^*_{s-z,t}) - f_{s-z,t}(u) < 0$ unless $u = u^*_{s-z,t}$ and $f_{s,t-\tilde{z}}(u^*_{s,t-\tilde{z}}) - f_{s,t-\tilde{z}}(u) < 0$ unless $u = u^*_{s,t-\tilde{z}}$. Set $u = u^*_{s,t}$ and substitute it in (4.2.14) and (4.2.15)

$$\Lambda(s-z,t) - f_{s-z,t}(u^*_{s,t}) \le f_{s-z,t}(u^*_{s-z,t}) - f_{s-z,t}(u^*_{s,t}), \tag{4.2.16}$$

$$\Lambda(s,t-\tilde{z}) - f_{s,t-\tilde{z}}(u^*_{s,t}) \le f_{s,t-\tilde{z}}(u^*_{s,t-\tilde{z}}) - f_{s,t-\tilde{z}}(u^*_{s,t}). \tag{4.2.17}$$

Note that (4.2.11) implies that there exists a sequence $z_n \to z \in [0,s]$ or $\tilde{z}_n \to \tilde{z} \in [0,t]$ such that at least one of the following limits holds

$$\Lambda(s-z_n,t) - f_{s-z_n,t}(u^*_{s,t}) \to 0, \tag{4.2.18}$$

$$\Lambda(s,t-\tilde{z}_n) - f_{s,t-z'_n}(u^*_{s,t}) \to 0. \tag{4.2.19}$$

If $t < -\frac{h'(b)}{g'(b)}s$ then the point $(s,t-\tilde{z})$ is below the critical line for every $\tilde{z} \in [0,t]$. The point $(s-z,t)$ can be above or below the critical line according to the value of z. We analyse these two cases for the first supremum in (4.2.11). The case for the second supremum is identical to case (a) below, as for all $\tilde{z}$, the index point stays below the critical line.

(a) If $0 \le z < s + t\frac{g'(b)}{h'(b)}$, we have that both $u^*_{s,t}, u^*_{s-z,t} \in (a,b)$. In particular

$$h'(u^*_{1,\nu}) + \nu g'(u^*_{1,\nu}) = 0. \tag{4.2.20}$$

By the implicit function theorem we can take the derivative of the previous expression respect to ν and find

$$\frac{du^*_{1,\nu}}{d\nu} = -\frac{g'(u^*_{1,\nu})}{h''(u^*_{1,\nu}) + \nu g''(u^*_{1,\nu})} > 0.$$

This implies that for all $z \in (0, s + t\frac{g'(b)}{h'(b)})$ and $\tilde{z} \in (0,t)$, $u^*_{s,t-\tilde{z}} < u^*_{s,t} < u^*_{s-z,t}$.

We want to show that (4.2.18) is possible if only $z_n \to 0$ from which the result follows from continuity. The right hand side in (4.2.16) is negative and therefore, by continuity we can argue that the supremum will be attained at one of the boundary points. Thus, we have only to show that

$$\lim_{z \nearrow s + t\frac{g'(b)}{h'(b)}} f_{s-z,t}(u^*_{s-z,t}) - f_{s-z,t}(u^*_{s,t}) < 0. \tag{4.2.21}$$

For any fixed $z \in (0, s + t\frac{g'(b)}{h'(b)})$ we have that

$$f_{s-z,t}(u^*_{s-z,t}) - f_{s-z,t}(u^*_{s,t}) < 0.$$

Therefore we obtain the proof if we show that the last expression is decreasing in z. Take the derivative in z, use (4.2.20), recall that $u^*_{s,t} < u^*_{s-z,t}$ and $h(u)$ is an increasing function by hypothesis

$$\begin{aligned}
&\frac{d}{dz}\Big((s-z)h(u^*_{s-z,t}) + tg(u^*_{s-z,t}) - [(s-z)h(u^*_{s,t}) + tg(u^*_{s,t})]\Big) \\
&= -h(u^*_{s-z,t}) + \Big((s-z)h'(u^*_{s-z,t}) + tg'(u^*_{s-z,t})\Big)\frac{du^*_{s-z,t}}{dz} + h(u^*_{s,t}) \\
&= h(u^*_{s,t}) - h(u^*_{s-z,t}) < 0.
\end{aligned}$$

(b) If $s + t\frac{g'(b)}{h'(b)} \le z \le s$, we have that $u^*_{s-z,t} = b$. Note that $u^*_{s,t} < u^*_{s-z,t} = b$ in this case and therefore $f_{s-z,t}(b) - f_{s-z,t}(u^*_{s,t}) < 0$ for every $z \in [s + t\frac{g'(b)}{h'(b)}, s]$. This implies that (4.2.18) can never be true for $z \in (s + t\frac{g'(b)}{h'(b)}, s]$. But the boundary point $z = s + t\frac{g'(b)}{h'(b)}$ is also not optimal by continuity considerations and (4.2.21).

Therefore, the potential maximum happens at $z = 0$. Similarly, this will be true for $\tilde{z} = 0$ and therefore $\Lambda(s,t) = f_{s,t}(u^*_{s,t})$ as required. □

Proof of Theorems 4.2.4, 1.3.1. Let $g^{(u),\text{ver}}_{pp}(s,t) = \lim_{N\to\infty} N^{-1} G^{(u),\text{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$. Recall that $g_{pp}(s,t)$ is 1-homogeneous and concave.

If $t < \frac{1-p}{p}s$, the starting point is equation (4.1.4). Scaling that equation by N gives us the macroscopic variational formulation

$$\begin{aligned} g_{pp}^{(u)}(s,t) &= g_{pp}^{(u),\mathrm{hor}}(s,t) \bigvee g_{pp}^{(u),\mathrm{ver}}(s,t) \\ &= \sup_{0\le z\le s}\{g_{pp}^{(u)}(z,0)+g_{pp}(s-z,t)\} \bigvee \sup_{0\le \tilde z\le t}\{g_{pp}^{(u)}(0,\tilde z)+g_{pp}(s,t-\tilde z)\} \\ &= \sup_{0\le z\le s}\{z\mathbb{E}(I^{(u)})+g_{pp}(s-z,t)\} \bigvee \sup_{0\le \tilde z\le t}\{\tilde z\mathbb{E}(J^{(u)})+g_{pp}(s,t-\tilde z)\}. \end{aligned} \tag{4.2.22}$$

We postpone this bit of the proof until the end. Assume (4.2.22) holds. Subtract $g_{pp}^{(u)}(s,t)$ from either side of (4.2.22)

$$\begin{aligned} 0 = \sup_{z\in[0,s]} &\Big\{ g_{pp}(s-z,t) - \Big[(s-z)u + t\frac{p(1-u)}{u-p}\Big]\Big\} \\ &\bigvee \sup_{\tilde z\in[0,t]} \Big\{ g_{pp}(s,t-\tilde z) - \Big[(t-\tilde z)\frac{p(1-u)}{u-p} + su\Big]\Big\}. \end{aligned}$$

We use Proposition 4.2.7 by identifying as $I=(p,1]$, $\Lambda(s,t)=g_{pp}(s,t)$, $h(u)=s$, $g(u)=\frac{p(1-u)}{u-p}$ and therefore $f_{s,t}(u)=su+t\frac{p(1-u)}{u-p}$. Note that $h'(u)>0$, $g'(u)<0$ for every $u\in(p,1]$ and in particular $f''_{s,t}(u)>0$ for every $(s,t)\in\mathbb{R}_+^2$. Moreover $\lim_{u\searrow p} f_{s,t}(u)=\infty$ and $f_{s,t}(1)=s<\infty$. Therefore

$$g_{pp}(s,t) = \min_{u\in(p,1]}\Big\{su+t\frac{p(1-u)}{u-p}\Big\} = \big(\sqrt{ps}+\sqrt{(1-p)t}\big)^2 - t, \qquad \text{if } t < s\frac{1-p}{p}. \tag{4.2.23}$$

If $t \ge \frac{1-p}{p}s$, We want to find an upper and a lower bound for $G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$. The upper bound is trivial since by model definition $G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}\le\lfloor Ns\rfloor$. For the lower bound, force a macroscopic distance from the critical line, i.e. assume that it is possible to find a $\varepsilon>0$ so that the sequence of endpoints $(\lfloor Ns\rfloor, \lfloor Nt\rfloor)$ satisfy

$$\varliminf_{N\to\infty}\frac{\lfloor Nt\rfloor}{\lfloor Ns\rfloor} \ge \frac{1-p}{p}+\varepsilon. \tag{4.2.24}$$

Then consider the following strategy: construct an approximate maximal path π for $G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$, knowing that for large $\lfloor Nt\rfloor \ge (\frac{1-p}{p}+\varepsilon)\lfloor Ns\rfloor$, π starts from $(0,0)$ and moves up until it finds a weight to collect horizontally on his right. After that this procedure repeats. For each iteration of this procedure, the vertical length of this path increases by a random Geometric(p) length, independently of the past. Define $Y\sim$ Geometric(p) with range on $0,1,\ldots$. By construction, we have

$$\Big\{\sum_{i=1}^{\lfloor Ns\rfloor} Y_i > \lfloor Nt\rfloor\Big\} \supseteq \{G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} < \lfloor Ns\rfloor\}.$$

The relation on (s,t) implies that the larger event above is large deviation event, and therefore by the Borel-Cantelli lemma, $G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}=\lfloor Ns\rfloor$ almost surely. Scaling by N and letting it tend to ∞ completes the proof.

We finally prove (4.2.22). For a lower bound, fix any $z \in [0, s]$ and $\tilde{z} \in [0, t]$. Then if we move on the horizontal axis

$$G^{(u)}_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \geq \sum_{i=1}^{\lfloor Nz \rfloor} I^{(u)}_{i,0} + G_{(\lfloor Nz \rfloor, 1), (\lfloor Ns \rfloor, \lfloor Nt \rfloor)}.$$

Divide by N. Observe that the left hand side converges a.s. to $g^{(u)}_{pp}(s, t)$. While the first term on the right converges a.s. to $z\mathbb{E}(I^u)$. The second on the right, converges in probability to $g_{pp}(s - z, t)$. In particular, we can find a subsequence N_k such that the convergence is almost sure for the second term. Taking limits on this subsequence, we conclude

$$g^{(u)}_{pp}(s, t) \geq z\mathbb{E}(I^u) + g_{pp}(s - z, t).$$

Since z is arbitrary we can take supremum over z in both sides of the inequality above. The same arguments will work if we move on the vertical axis. Thus, we obtain the lower bound for (4.2.22).

For the upper bound, we partition both axes. Fix $\varepsilon, \tilde{\varepsilon} > 0$ and let $\{0 = q_0, \varepsilon = q_1, 2\varepsilon = q_2, \dots, s \left\lfloor \varepsilon^{-1} \right\rfloor \varepsilon, s = q_M\}$ a partition of $(0, s)$ and $\{0 = q_0, \tilde{\varepsilon} = q_1, 2\tilde{\varepsilon} = q_2, \dots, t \left\lfloor \tilde{\varepsilon}^{-1} \right\rfloor \tilde{\varepsilon}, t = q_{\tilde{M}}\}$ a partition of $(0, t)$. The maximal path that utilises $G^{(u)}_{N,N}$ has to exit between $\lfloor Nk\varepsilon \rfloor$ and $\lfloor N(k + 1)\varepsilon \rfloor$ for some k if it chooses to go through the x-axis and between $\left\lfloor N\tilde{k}\tilde{\varepsilon} \right\rfloor$ and $\left\lfloor N(\tilde{k} + 1)\tilde{\varepsilon} \right\rfloor$ for some $\tilde{k}$ if it goes through the y-axis. Therefore, we may write

$$\begin{aligned} G^{(u)}_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \leq \max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} &\left\{ \sum_{i=1}^{\lfloor N(k+1)\varepsilon \rfloor} I^{(u)}_{i,0} + G_{(\lfloor Nk\varepsilon \rfloor, 1), (\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \right\} \\ &\bigvee \max_{0 \leq \tilde{k} \leq \lfloor \tilde{\varepsilon}^{-1} \rfloor} \left\{ \sum_{j=1}^{\left\lfloor N(\tilde{k}+1)\tilde{\varepsilon} \right\rfloor} J^{(u)}_{0,j} + G_{(1, (\left\lfloor N\tilde{k}\tilde{\varepsilon} \right\rfloor)), (\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \right\}. \end{aligned}$$

Divide by N. The right-hand side converges in probability to the constant

$$\begin{aligned} &\max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \{(k + 1)\varepsilon u + g_{pp}(s - \varepsilon k, t)\} \\ &\qquad \bigvee \max_{0 \leq \tilde{k} \leq \lfloor \tilde{\varepsilon}^{-1} \rfloor} \left\{ (\tilde{k} + 1)\tilde{\varepsilon}\frac{p(1 - u)}{u - p} + g_{pp}(s, t - \tilde{\varepsilon}\tilde{k}) \right\} \\ &= \max_{0 \leq k \leq \lfloor \varepsilon^{-1} \rfloor} \{k\varepsilon u + g_{pp}(s - \varepsilon k, t)\} + \varepsilon u \\ &\qquad \bigvee \max_{0 \leq \tilde{k} \leq \lfloor \tilde{\varepsilon}^{-1} \rfloor} \left\{ \tilde{k}\tilde{\varepsilon}\frac{p(1 - u)}{u - p} + g_{pp}(s, t - \tilde{\varepsilon}\tilde{k}) \right\} + \tilde{\varepsilon}\frac{p(1 - u)}{u - p} \\ &= \max_{q_k} \{q_k u + g_{pp}(s - q_k, t)\} + \varepsilon u \\ &\qquad \bigvee \max_{q_{\tilde{k}}} \left\{ q_{\tilde{k}}\frac{p(1 - u)}{u - p} + g_{pp}(s, t - q_{\tilde{k}}) \right\} + \tilde{\varepsilon}\frac{p(1 - u)}{u - p} \\ &\leq \sup_{0 \leq z \leq s} \{zu + g_{pp}(s - z, t)\} + \tilde{\varepsilon}u \end{aligned}$$

$$\bigvee \max_{0\le\tilde z\le t}\left\{\tilde z\frac{p(1-u)}{u-p}+g_{pp}(s,t-\tilde z)\right\}+\tilde\varepsilon\frac{p(1-u)}{u-p}.$$

The convergence becomes a.s. on a subsequence. The upper bound for (4.2.22) now follows by letting $\varepsilon\to0$ and $\tilde\varepsilon\to0$ in the final equation. $\square$

Proof of Theorem 4.1.2. By definition (4.1.6) and (1.3.9) we have

$$\begin{aligned}
g_{pp}^{(u),\mathrm{hor}}(s,t)&=\lim_{N\to\infty}N^{-1}G^{(u),\mathrm{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}\\
&=\lim_{N\to\infty}\max_{1\le k\le\lfloor Ns\rfloor}\left\{N^{-1}\sum_{i=1}^{k}I^{(u)}_{i,0}+N^{-1}G_{(k,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}\right\}\\
&=\sup_{0\le a\le s}\{au+g_{pp}(s-a,t)\}.
\end{aligned}\tag{4.2.25}$$

The last line follows by the same coarse graining arguments as in the proof of Theorem 4.2.4.

If $t<\frac{1-p}{p}s$

$$g_{pp}^{(u),\mathrm{hor}}(s,t)=\sup_{0\le a\le s-\frac{pt}{1-p}}\{au+(\sqrt{p(s-a)}+\sqrt{(1-p)t})^2-t\}\vee\sup_{s-\frac{pt}{1-p}<a\le s}\{a(u-1)+s\}.$$

The second supremum is attained at the boundary point $s-\frac{pt}{1-p}$ since it optimizes a decreasing function of a. In the first supremum, a unique minimizing point exists and it is either a boundary point or the critical point a^* of the derivative of $f(a)=au+(\sqrt{p(s-a)}+\sqrt{(1-p)t})^2-t$, given by

$$a^*=s-\frac{p(1-p)t}{(u-p)^2}.$$

If $s-\frac{p(1-p)t}{(u-p)^2}<0$ then we have that $a^*=0$. Otherwise, we can substitute a^* into $f(a)$ and obtain

$$f(a^*)=su+\frac{p(1-u)}{u-p}t=g_{pp}^{(u)}(s,t).$$

Finally, if $t\ge\frac{1-p}{p}s$

$$g_{pp}^{(u),\mathrm{hor}}(s,t)=\sup_{0\le a\le s}\{au+s-a\}=s=g_{pp}(s,t).$$

The proof for $g_{pp}^{(u),\mathrm{ver}}(s,t)$ is similar and left to the reader. $\square$

4.3 I.i.d. Model: Full LDP

We first focus on the model without boundaries. Recall that the maximal path can collect Bernoulli weights only when it takes a step to the right.

Proof of Theorem 4.1.3. First we prove the existence of limit (4.1.13). Take $m, n \in \mathbb{N}$ and an error due to the floor function $\mathbf{x}_{m,n} \in (0,1)^2$ such that $(\lfloor (m+n)s \rfloor, \lfloor (m+n)t \rfloor) = (\lfloor ms \rfloor, \lfloor mt \rfloor) + (\lfloor ns \rfloor, \lfloor nt \rfloor) + \mathbf{x}_{m,n}$. We have

$$
\begin{aligned}
&\mathbb{P}\{G_{\lfloor (m+n)s \rfloor, \lfloor (m+n)t \rfloor} \geq (m+n)r\} \\
&\quad \geq \mathbb{P}\{G_{\lfloor ms \rfloor, \lfloor mt \rfloor} + G_{(\lfloor ms \rfloor, \lfloor mt \rfloor),(\lfloor (m+n)s \rfloor, \lfloor (m+n)t \rfloor)} \geq (m+n)r\}, \qquad \text{by superadditivity} \\
&\quad \geq \mathbb{P}\{G_{\lfloor ms \rfloor, \lfloor mt \rfloor} \geq mr\}\mathbb{P}\{G_{\lfloor ns \rfloor, \lfloor nt \rfloor} \geq nr\}\mathbb{P}\{G_{\lfloor \mathbf{x}_{m,n} \rfloor} \geq 0\}, \qquad \text{by independence.}
\end{aligned}
$$

By (4.1.12) $\mathbb{P}\{G_{\lfloor x_{m,n} \rfloor} \geq 0\} = 1$. Take logarithms in the last inequality; then by Fekete's lemma the limit

$$\lim_{N\to\infty} N^{-1} \log \mathbb{P}\{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \geq Nr\}$$

exists for any $(s,t) \in \mathbb{R}^2 \setminus \{0\}$ and $r \in [0, s]$ and in fact equals $\sup_N N^{-1} \log \mathbb{P}\{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \geq Nr\}$. The value of the limit is now denoted by $-J_{s,t}(r)$.

From the superadditivity of G we can also obtain the convexity of the limit. Pick any $\lambda \in (0,1)$ and define the triple $((s,t), r) = \lambda((s_1,t_1), r_1) + (1-\lambda)((s_2,t_2), r_2)$ with $r_1 \in [0, s_1]$ and $r_2 \in [0, s_2]$. Then

$$
\begin{aligned}
&N^{-1} \log \mathbb{P}\{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \geq Nr\} \\
&\qquad \geq \lambda(\lambda N)^{-1} \log \mathbb{P}\{G_{\lfloor N\lambda s_1 \rfloor, \lfloor N\lambda t_1 \rfloor} \geq N\lambda r_1\} \\
&\qquad\quad + (1-\lambda)((1-\lambda)N)^{-1} \log \mathbb{P}\{G_{\lfloor N(1-\lambda)s_2 \rfloor, \lfloor N(1-\lambda)t_2 \rfloor} \geq N(1-\lambda)r_2\}.
\end{aligned}
$$

Multiply both sides by -1 and invert the sign of the inequality to obtain for $N \to \infty$

$$J_{s,t}(r) \leq \lambda J_{s_1,t_1}(r_1) + (1-\lambda) J_{s_2,t_2}(r_2). \tag{4.3.1}$$

From (4.1.12) we know that J is finite and we have just proven that it is also convex. This implies that J is continuous on A and upper semicontinuous on the whole set $\bar{A}$, from Theorems 10.1 and 10.2 in [92]. Moreover, $J_{s,t}(r)$ on A can be uniquely extended to a continuous function on $\bar{A}$ by Theorem 10.3 in [92].

Finally, the law of large numbers for the last passage time implies $J_{(s,t)}(r) = 0$ for $r < g_{pp}(s,t)$ and then by continuity for $r \leq g_{pp}(s,t)$. Use the same method of proof of Proposition 3.1(b) of [29] to get the concentration inequality:

$$\mathbb{P}\{|G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} - \mathbb{E}[G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}]| \geq N\varepsilon\} \leq 2e^{-c\varepsilon^2 N} \quad \forall N \in \mathbb{N}. \tag{4.3.2}$$

This holds for a given $(s,t) \in \mathbb{R}^2_+$, and $\varepsilon > 0$. Constant $c > 0$ will depend on s, t, ε. Since $N^{-1}\mathbb{E}[G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}] \to g_{pp}(s,t)$, this implies that $J_{(s,t)}(r) > 0$ for $r > g_{pp}(s,t)$ (without excluding the value ∞). $\square$

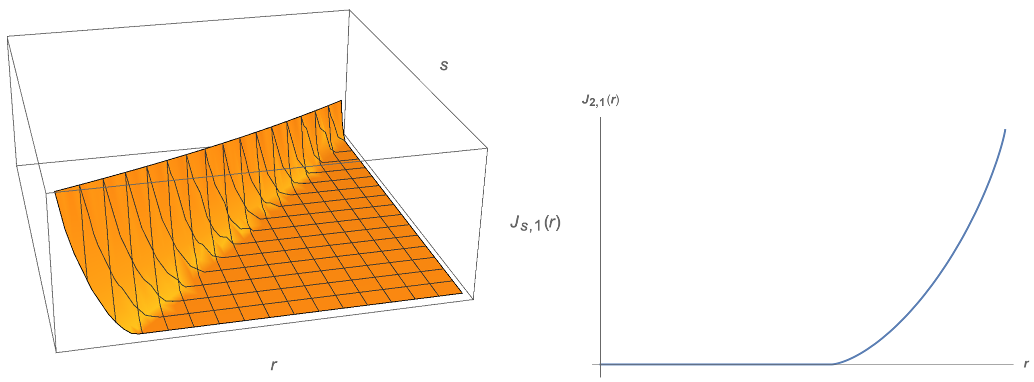


Figure 4.1: Graphical representation of the function $J_{s,t}(r)$. In both figures we used $p = 1/2$ and $t = 1$. To the left we have the lower-semicontinuous version of $J_{s,1}(r)$ as a function of (s, r). You can see that it is finite for $s \leq r$. To the right is the function $J_{2,1}(r)$.

The continuous extension up to $\bar{A}$ makes the function $J_{s,t}(r)$ lower-semicontinuous on $\mathbb{R}^3$ where it takes the value ∞ outside of $\bar{A}$.

It will be useful to also know some of the boundary values of the lower semi-continuous extension. We summarise the results in the following corollary:

Corollary 4.3.1. *The lower-semicontinuous extension of $J_{s,t}(r)$ takes the following values on ∂A*

$$J_{s,t}(r) = \begin{cases} 0, & t = 0, r \leq 0, s \in \mathbb{R}_+, \\ 0, & t \in \mathbb{R}_+, r \leq 0, s = 0, \\ sI_{\mathcal{B}}^{(p)}(r/s), & t = 0, 0 \leq r \leq s, s \in \mathbb{R}_+ \\ \lim_{r \nearrow s} J_{s,t}(r), & t, s \in \mathbb{R}_>0, r = s. \end{cases} \tag{4.3.3}$$

This follows from the fact that $J_{s,t}$ needs to be lower-semicontinuous, as it has briefly been pointed out before. Above we defined $I_{\mathcal{B}}$ to be the Cramér rate function for sums of i.i.d. $\omega_i \sim \text{Bernoulli}(p)$,

$$I_{\mathcal{B}}^{(p)}(r) = \begin{cases} -\lim_{N\to\infty} N^{-1} \log \mathbb{P}\Big\{\sum_{i=1}^{N} \omega_i \geq Nr\Big\} = r \log \frac{r}{p} + (1-r) \log \frac{1-r}{1-p}, & r \in [p, 1], \\ 0, & r < p \\ \infty, & r > 1. \end{cases} \tag{4.3.4}$$

In order to obtain a full large deviation principle, we must estimate the lower tail for the probabilities of the last passage time. As it is usual in the solvable models of last passage percolation, the speed for the lower tail is different than N. Our first lemma establishes the same fact for this model. In turn, this gives left tail bounds strong enough to imply $I_{s,t}(r) = \infty$ for $r < g_{pp}(s,t)$ for both boundary and i.i.d. model.

Lemma 4.3.2. *There exist constants $c > 0$, $C < \infty$ that depend on parameters s, t, p, u, so that for all $N \geq 1$ the following estimates hold:*

(a) For $(s,t) \in (0,\infty)^2$ and $r \in [0, g_{pp}(s,t))$

$$\mathbb{P}\{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \leq Nr\} \leq Ce^{-cN^2}.$$

(b) For $(s,t) = \alpha\Big(1, \frac{(u-p)^2}{p(1-p)}\Big)$ for some $\alpha > 0$, parallel to the characteristic direction, and $r \in [0, g^{(u)}_{pp}(s,t))$,

$$\mathbb{P}\{G^{(u)}_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \leq Nr\} \leq Ce^{-cN^2}.$$

Proof. We prove (b) but similar arguments work for (a). We bound $G^{(u)}_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}$ from below, using the superadditivity property of the last passage times.

For this reason we consider a subset of lattice paths, arranged in a collection of i.i.d. last passage time over subsets of rectangles. This block argument proof was first used in [69] and later adapted in [96] for the last passage time and in [51, 61] for the log-gamma polymer and the Brownian polymer model respectively.

Note that if (s,t) are chosen in the characteristic direction it is immediate to see that $g^{(u)}_{pp}(s,t) = g_{pp}(s,t)$.

We first show the result for $(s,t) \in \mathbb{Q}^2_+$. In order to highlight this distinction we assume that the target point is $(q_1, q_2) \in \mathbb{Q}^2_+$. Fix $0 < \varepsilon < 1/4(g^{(u)}_{pp}(q_1,q_2) - r)$. Define a new scale parameter $m \in \mathbb{N}$ large enough so that $m(q_1 \wedge q_2) \geq 1$, $mq_1, mq_2 \in \mathbb{N}_+$ and

$$\mathbb{E}[G_{mq_1, mq_2}] > m(r + 2\varepsilon). \tag{4.3.5}$$

We will use mq_1 and mq_2 to coarse-grain our environment. Let $\mathcal{R}^{k,\ell}_{a,b} = \{a, \dots, a+k-1\} \times \{b, \dots, \ell+b-1\}$ denote the $k \times \ell$ rectangle with lower left corner at (a,b). For $i, \ell \geq 0$ define pairwise disjoint $mq_1 \times mq_2$ rectangles

$$\mathcal{R}^i_\ell = \mathcal{R}^{mq_1, mq_2}_{(\ell+i)mq_1+i+1, \ell(mq_2+1)+1}.$$

The rectangles $\mathcal{R}^i_\ell$ are separated by the inter-site distance to avoid a scenario where a path goes along a common edge between two rectangles. This way, we will be able to clearly say in which one of the two rectangles the path goes through. For each i we define the diagonal union of rectangles as $\Delta_i = \bigcup_{\ell \geq 0} \mathcal{R}^i_\ell$, $i \geq 0$ and in the sequence we are considering potential paths that stay in a fixed Δ_i.

Moreover, note that the last passage times $G_{\mathbf{v}^w_{i,\ell},\mathbf{v}^e_{i,\ell}}$ in each rectangle are all identically distributed, where $\mathbf{v}^w_{i,\ell} = (\ell + i)mq_1 + 1 + i, \ell(mq_2 + 1) + 1)$ and $\mathbf{v}^e_{i,\ell} = ((i+1)mq_1 + \ell mq_1, (1+\ell)(mq_2+1))$ are respectively the south-west and north-east corners of $\mathcal{R}^i_\ell$.

Define $B, M = M(B) \in \mathbb{N}$ the maximal integers which satisfy

$$(M+1)mq_1 + Bmq_1 \le Nq_1 \qquad \text{and} \tag{4.3.6}$$

$$(1+B)(mq_2+1) \le Nq_2. \tag{4.3.7}$$

The fact that B is maximal and (4.3.7) imply that

$$B = \left\lfloor \frac{Nq_2}{mq_2+1} \right\rfloor - 1. \tag{4.3.8}$$

Substituting (4.3.8) in (4.3.6) we obtain

$$M = \left\lfloor \frac{N}{m} - \left\lfloor \frac{Nq_2}{mq_2+1} \right\rfloor \right\rfloor \text{ and hence } \left\lfloor \frac{N}{m(mq_2+1)} \right\rfloor \le M \le \left\lfloor N \frac{mq_2+2}{m(mq_2+1)} \right\rfloor. \tag{4.3.9}$$

Since m is a constant and assumed much smaller than N, we have that $B = B(N) = \mathcal{O}(N)$ and $M = M(N) = \mathcal{O}(N)$. Fix a diagonal Δ_i for $0 \le i \le M$ and define the union of rectangles in $\Delta_i \cap ([0, Nq_1] \times [0, Nq_2])$ as $\Delta^B_i = \bigcup_{0 \le \ell \le B} \mathcal{R}^i_\ell$.

Let G^Δ_i be the last passage time of all lattice paths in Δ^B_i from the lower left corner of $\mathcal{R}^i_0$ to the upper right corner of $\mathcal{R}^i_B$. G^Δ_i are i.i.d, where in particular G^Δ_0 is the sum of the B last passage times of rectangle $\mathcal{R}^0$ whose mean is controlled by (4.3.5). A standard large deviation estimate for an i.i.d sum gives the following bound

$$\begin{aligned}
\mathbb{P}\{G^{(u)}_{\lfloor Nq_1 \rfloor, \lfloor Nq_2 \rfloor} \le Nr\} &\le \mathbb{P}\{G^\Delta_i \le Nr \text{ for } 0 \le i \le M\} \\
&\le \mathbb{P}\{G^\Delta_0 \le Nr\}^M \le \mathbb{P}\Big\{ \sum_{k=0}^{B(N)} G^0_k \le Nr \Big\}^{M(N)} \\
&\le e^{-cB(N)M(N)} \le e^{-c_1 N^2}.
\end{aligned} \tag{4.3.10}$$

This completes the proof for $(s,t) \in \mathbb{Q}^2_+$.

Finally we show (4.3.10) holds also for $s, t \in \mathbb{R}_+$. We bound $G^{(u)}_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}$ using $G^{(u)}_{\lfloor Nq_1 \rfloor, \lfloor Nq_2 \rfloor}$ for some special $(q_1, q_2) \in \mathbb{Q}^2_+$ which are close enough to $(s,t) \in \mathbb{R}^2_+$. For any $(q_1, q_2) \le (s,t)$ we have that

$$\mathbb{P}\{G^{(u)}_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \le Nr\} \le \mathbb{P}\{G^{(u)}_{\lfloor Nq_1 \rfloor, \lfloor Nq_2 \rfloor} \le Nr\}, \text{ for all } r \in [0, g^{(u)}_{pp}(s,t)). \tag{4.3.11}$$

For any $\delta > 0$ find (q_1, q_2) so that $\delta > g^{(u)}_{pp}(s,t) - g^{(u)}_{pp}(q_1, q_2) > 0$. This is possible by the continuity and monotonicity of $g^{(u)}_{pp}$. We choose $\delta < \frac{g^{(u)}_{pp}(s,t) - r}{2}$ and therefore

$$r < g^{(u)}_{pp}(s,t) - 2\delta < g^{(u)}_{pp}(q_1, q_2) - \delta < g^{(u)}_{pp}(s,t),$$

for some $(q_1, q_2) \in \mathbb{Q}^2_+$ Then (4.3.11) is a left-tail large deviation anyway for $G^{(u)}_{\lfloor Nq_1 \rfloor, \lfloor Nq_2 \rfloor}$ so (4.3.10) holds. $\square$

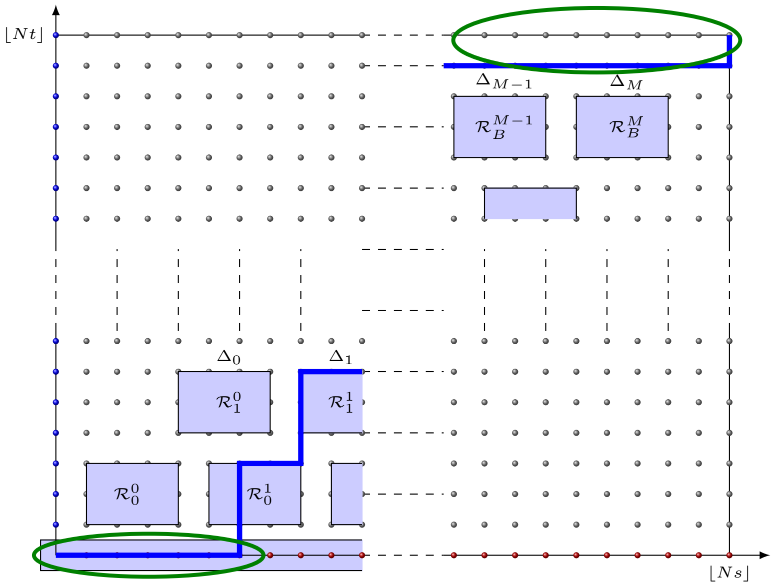


Figure 4.2: Representation of the coarse grained $\lfloor ms \rfloor \times \lfloor mt \rfloor$ rectangles and the diagonals Δ_i in the proof of Lemma 4.3.2. The blue thick line is one of the possible maximal paths. For the bound needed, we are allowed to ignore the path segments outside of the coarse-grained diagonals, particularly we may ignore the correlated segments when candidate paths traverse the south and north boundary of $[0, \lfloor Ns \rfloor] \times [0, \lfloor Nt \rfloor]$. Passage times in each Δ_i are i.i.d. and smaller than the overall passage time.

Proof of Theorem 4.1.4. This proof is a consequence of the lemmas and theorems that we have already proved. Define for $r \in \mathbb{R}$ function $I_{s,t}(r)$ by (4.1.16).

Then, the regularity properties proved for J in Theorems 4.1.3 and 4.3.1 are also valid for $I_{s,t}$. For the upper large deviation bound (4.1.14) we consider two cases:

(1) if $F \subseteq [0, g_{pp}(s,t))$, then $r^* = \max\{x : x \in F\} < g_{pp}(s,t)$ and we have

$$\mathbb{P}\{N^{-1} G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \in F\} \le \mathbb{P}\{G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \le N r^*\} \le e^{-N^2}.$$

The last inequality comes from Lemma 4.3.2. Take logarithms on both sides, divide by N, take the limit $N \to \infty$ and finally by definition (4.1.16) conclude that

$$\lim_{N\to\infty} N^{-1} \log \mathbb{P}\{N^{-1} G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \in F\} = -\infty = -\inf_{r \in F} I_{s,t}(r).$$

(2) If $F \cap [g_{pp}(s,t), s] \neq \emptyset$ then we split into two different cases:

Case 1: $F \not\ni g_{pp}(s,t)$. Then there exists an $\varepsilon > 0$ such that $(g_{pp}(s,t) - \varepsilon, g_{pp}(s,t) + \varepsilon) \subseteq F^c$. Then we bound

$$\mathbb{P}\{N^{-1} G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \in F\} \le \mathbb{P}\{N^{-1} G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \le g_{pp}(s,t) - \varepsilon\}$$

$$+\,\mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \in F\cap[g_{pp}(s,t)+\varepsilon, s]\}.$$

By the previous calculations, we already control the first addend by e^{-cN^2} therefore we focus only on the second one which will be of an exponential order of magnitude larger and control the value of the $\overline{\lim}$. Since F and $[g_{pp}(s,t)+\varepsilon, s]$ are two closed sets there exists an r^* such that $r^* = \min\{r : r \in F\cap[g_{pp}(s,t)+\varepsilon, s]\}$. It follows that

$$\mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \in F\} \le e^{-cN^2} + \mathbb{P}\{G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \ge Nr^*\}.$$

Now take the logarithm of both sides, divide by N and take the $\overline{\lim}$,

$$\begin{aligned}\overline{\lim_{N\to\infty}} N^{-1}\log\mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \in F\} &\le \overline{\lim_{N\to\infty}} N^{-1}\log(e^{-cN^2} + \mathbb{P}\{G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \ge Nr^*\})\\ &= -J_{s,t}(r^*) = -\inf_{r\in F} I_{s,t}(r).\end{aligned}$$

The last line is obtained using (4.1.13), (4.1.16) and the fact that $I_{s,t}(r)$ is a strictly increasing function.

Case 2: $F \ni g_{pp}(s,t)$. In this case, $\inf_{r\in F} I_{s,t}(r) = 0$, therefore, inequality (4.1.14) is automatically satisfied.

For the lower large deviation bound (4.1.15), we need to consider three cases according to H:

(1) If $g_{pp}(s,t) \in H$, then $\mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \in H\} \to 1$ and (4.1.15) holds as an equality.

(2) If $H \subseteq [0, g_{pp}(s,t))$, (4.1.15) holds because its right-hand side is $-\infty$.

(3) The remaining case is the one where H contains an interval $(a,b) \subset (g_{pp}(s,t), s)$. Then for any $\varepsilon > 0$ small enough, we can find a non-trivial interval $[a+\varepsilon, b-\varepsilon] \subseteq H$ and bound

$$\begin{aligned}N^{-1}\log&\mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \in H\} \ge N^{-1}\log\mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \in [a+\varepsilon, b-\varepsilon]\}\\ &= N^{-1}\log\Big(\mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \ge a+\varepsilon\} - \mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \ge b-\varepsilon\Big)\\ &\to -J_{s,t}(a+\varepsilon).\end{aligned} \tag{4.3.12}$$

Equation (4.3.12) follows after taking $\underline{\lim}$ on both sides and keeping in mind that the two terms in the logarithm have different exponential orders of magnitude.

Monotonicity and convexity $J_{s,t}$ on $[g_{pp}(s,t), s]$ implies that for some constant C, $J_{s,t}(a+\varepsilon) \le J_{s,t}(a) + C\varepsilon$. Then, (4.3.12) becomes

$$\underline{\lim_{N\to\infty}} N^{-1}\log\mathbb{P}\{N^{-1}G_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \in H\} \ge -J_{s,t}(a) - C\varepsilon$$

Let $\varepsilon \to 0$ in the last display. Then take $a = \inf H \cap (g_{pp}(s,t), s)$ to finish using

$$J_{s,t}(a) = \inf_{r \in H \cap (g_{pp}(s,t),s)} I_{s,t}(r) = \inf_{r \in H} I_{s,t}(r). \qquad \square$$

Proof of Corollary 4.1.5. Since $G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor} \le Ns$, for any $\gamma > 1$ and $\xi \in \mathbb{R}$,

$$\sup_N \left(\mathbb{E} e^{\gamma \xi \, G_{\lfloor Ns \rfloor, \lfloor Nt \rfloor}} \right)^{1/N} < \infty.$$

This bound together with Theorem 4.1.4 suffice to apply Varadhan's theorem (e.g. page 38 in [88]) which gives

$$\begin{aligned}
\lim_{N \to \infty} N^{-1} \log \mathbb{E} e^{\xi G_{0,(\lfloor Ns \rfloor, \lfloor Nt \rfloor)}} &= I^*_{s,t}(\xi) = \sup_{r \in \mathbb{R}} \{ r\xi - I_{s,t}(r) \} \\
&= \sup_{r \in [g_{pp}(s,t), s]} \{ r\xi - I_{s,t}(r) \} = \sup_{r \in [g_{pp}(s,t), s]} \{ r\xi - J_{s,t}(r) \}.
\end{aligned}$$

The first equality on the second line is because $I_{s,t}(r) = \infty$ if $r \in (-\infty, g_{pp}(s,t))$ or $r > s$ and there is no difference in excluding that interval from the supremum.

Then we can compute $I^*_{s,t}$. $I_{s,t}$ is increasing for $r \in [g_{pp}(s,t), s]$, therefore if $\xi < 0$, the supremum is always attained at $r = g_{pp}(s,t)$. Instead, when $\xi \ge 0$, $I_{s,t}(\xi)^* = J^*_{s,t}(\xi)$ since r can range over all of $\mathbb{R}$ and the last supremum will still be attained for some $r \in [g_{pp}(s,t), s]$. $\square$

4.4 Basic properties of the rate function

In this section we prove some important properties of the rate function which will be necessary later on.

Lemma 4.4.1 (Continuity in the macroscopic directions). *Let $(s,t) \in \mathbb{R}^2_{>0}$ and $\mathbf{u}_N = (s_N, t_N) \in \mathbb{Z}^2_+$ an increasing sequence such that $N^{-1}\mathbf{u}_N \to (s,t)$. Then for $r \in [0, s)$*

$$\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{ G_{s_N, t_N} \ge Nr \} = -J_{s,t}(r). \tag{4.4.1}$$

Proof. Since $\mathbf{u}_N$ and $(\lfloor Ns \rfloor, \lfloor Nt \rfloor)$ are non-decreasing in N, for each N we can find two sequences ℓ_N and m_N such that

$$\lfloor \ell_N(s,t) \rfloor \le \mathbf{u}_N \le \lfloor m_N(s,t) \rfloor \quad \text{with } N - m_N, N - \ell_N = o(N).$$

Then it is immediate that

$$G_{\lfloor \ell_N s \rfloor, \lfloor \ell_N t \rfloor} \le G_{s_N, t_N} \le G_{\lfloor m_N s \rfloor, \lfloor m_N t \rfloor},$$

which gives

$$\mathbb{P}\{ G_{\lfloor m_N s \rfloor, \lfloor m_N t \rfloor} \ge Nr \} \ge \mathbb{P}\{ G_{s_N, t_N} \ge Nr \} \ge \mathbb{P}\{ G_{\lfloor \ell_N s \rfloor, \lfloor \ell_N t \rfloor} \ge Nr \}.$$

Taking the $\overline{\lim}$ of both sides and by the continuity of the rate function we have

$$\begin{aligned}
\overline{\lim_{N\to\infty}} N^{-1}\log\mathbb{P}\{G_{s_N,t_N}\geq Nr\} &\leq \lim_{N\to\infty} N^{-1}\log\mathbb{P}\{G_{\lfloor m_N s\rfloor,\lfloor m_N t\rfloor}\geq Nr\}\\
&\leq \overline{\lim_{N\to\infty}} m_N^{-1}(\frac{m_N}{N})\log\mathbb{P}\{G_{\lfloor m_N s\rfloor,\lfloor m_N t\rfloor}\geq m_N r-(m_N-N)r\}\\
&\leq \overline{\lim_{N\to\infty}} m_N^{-1}(\frac{m_N}{N})\log\mathbb{P}\{G_{\lfloor m_N s\rfloor,\lfloor m_N t\rfloor}\geq m_N(r-\varepsilon)\}\\
&\qquad\qquad\qquad\qquad \text{for any } \varepsilon>0 \text{ and } N \text{ large enough}\\
&= -J_{s,t}(r-\varepsilon).
\end{aligned}$$

Then let $\varepsilon\to 0$ and invoke the continuity of J for the upper bound. Same arguments are valid for the lower bound, using $\underline{\lim}_{N\to\infty}$. □

From Theorem 4.1.3 we have that $J_{s,t}(r)$ can be continuously extended to the boundary of the domain $A=\{(s,t,r):J_{s,t}(r)<\infty\}$,

$$\partial A=\{s=0,t\geq 0,r\leq 0\}\cup\{t=0,s\geq r\vee 0\}\cup\{s=r,t\geq 0\}.$$

It will be convenient to understand the values of the continuation of $J_{s,t}(r)$ on ∂A.

For any $s,t>0$ and $r\leq 0$, $J_{s,t}(r)=0$. Therefore, we will have that

$$J_{s,0}(r)=J_{0,t}(r)=0,\quad r\leq 0.$$

Now for the $r>0$ case. Since we want $J_{s,0}(r)$ with $(s\geq r)$ continuous we define $J_{s,0}(r)=\lim_{h\to 0}J_{s,h}(r)$. An approximation using thin rectangles as in [51] gives that

$$J_{s,0}(r)=sI_{\mathcal{B}}(r/s)=r\log\frac{r}{sp}+\big(s-r\big)\log\frac{1-r/s}{1-p}.$$

Recall that $I_{\mathcal{B}}$ is the Cramér rate function for sums of i.i.d. $\omega_i\sim$ Bernoulli(p). This discussion is summarised in Corollary 4.3.1.

Lemma 4.4.2 (Infimal convolutions)**.** *For each N let L_N and Z_N be two independent random variables. Assume their rate functions*

$$\lambda(s)=-\lim_{N\to\infty}N^{-1}\log\mathbb{P}\{L_N\geq Ns\},\tag{4.4.2}$$

$$\phi(s)=-\lim_{N\to\infty}N^{-1}\log\mathbb{P}\{Z_N\geq Ns\}\tag{4.4.3}$$

exists and

1. *$\lambda(s)$ is finite in $(-\infty,b)$ with $b\in\bar{\mathbb{R}}$ and $\lambda(s)=\infty$ when $s>b$.*

2. *λ is continuous at all points for which is finite and lower semi-continuous on $\mathbb{R}$.*

3. $\phi(s)$ *is finite for all* $s \in \mathbb{R}$.

4. $\lambda(a_\lambda) = \phi(a_\phi) = 0$ *for some* $a_\lambda, a_\phi \in \mathbb{R}$.

Then for $r \in \mathbb{R}$

$$\lim_{N\to\infty} N^{-1} \log \mathbb{P}\{L_N + Z_N \geq Nr\} = \begin{cases} -\inf_{a_\lambda \leq s \leq b\wedge(r-a_\phi)}\{\phi(r-s)+\lambda(s)\}, & r > a_\phi + a_\lambda, \\ 0, & r \leq a_\phi + a_\lambda. \end{cases} \tag{4.4.4}$$

Proof. First observe that the infimum in (4.4.4) is obtained when s satisfies $a_\lambda \leq s \leq b \wedge (r - a_\phi)$.

The lower bound follows from the independence of the two random variables

$$\mathbb{P}\{L_N + Z_N \geq Nr\} \geq \mathbb{P}\{Z_N \geq N(r-s)\}\mathbb{P}\{L_N \geq Ns\}.$$

To upper bound for $r \leq a_\lambda + a_\phi$ is immediate.

We therefore only discuss the case $r > a_\lambda + a_\phi$. Take a finite partition $a_\lambda = q_{-1} = q_0 < \cdots < q_{m-1} = b \wedge (r - a_\phi) < q_m = q_{m+1}$.

Use a union bound and the independence of L_N, Z_N to derive

$$\begin{aligned} \mathbb{P}\{L_N + Z_N \geq Nr\} &\leq \mathbb{P}\{L_N + Z_N \geq Nr, L_N < Nq_0\} \\ &\quad + \sum_{i=0}^{m-1} \mathbb{P}\{L_N + Z_N \geq Nr, nq_i \leq L_N \leq Nq_{i+1}\} + P\{L_N \geq Nq_m\} \\ &\leq P\{Z_N \geq N(r-q_0)\} + \sum_{i=0}^{m-1} \mathbb{P}\{Z_N \geq N(r-q_{i+1})\}\mathbb{P}\{L_N \geq Nq_i\} + P\{L_N \geq Nq_m\}. \end{aligned}$$

Now take the logarithm on both sides, divide by N and finally take $N \to \infty$ to obtain

$$\begin{aligned} &\overline{\lim_{N\to\infty}} N^{-1} \log \mathbb{P}\{L_N + Z_N \geq Nr\} \\ &\qquad \leq -\min\Big\{\phi(r-q_0), \min_{0\leq i\leq m-1}\{\phi(r-q_{i+1})+\lambda(q_i)\}, \lambda(q_m)\Big\}. \end{aligned}$$

We may simplify the last inequality as

$$\mathbb{P}\{L_N + Z_N \geq Nr\} \leq -\min_{-1\leq i\leq m}\{\phi(r-q_{i+1})+\lambda(q_i)\}$$

This is because $\lambda(q_0) = 0$. Also, if $b \leq r - a_\phi$ then $\lambda(q_m) = \infty$ and it can be omitted from the minimum. If $b > r - a_\phi$ then $\phi(r - q_m) = 0$. The result then follows by the continuity of λ on $[a_\lambda, b]$ by arbitrarily refining the partition. □

4.5 I.i.d. model: Right tail rate function and log moment generating function

The main goal of this section is to prove an explicit variational formula for the rate function $J_{s,t}(r)$. That formula, while precise does not enjoy enough analytical tractability to further obtain a closed formula. However, its dual $J^*_{s,t}(\xi)$ will be explicitly computed by the end of the section.

4.5.1 Exact computations for $J_{s,t}(r)$

We first present a series of key technical lemmas, and we encourage the reader familiar with these techniques to proceed to the proof of Proposition 4.1.6.

We will use the invariance property of the model with boundaries first. Consider the last passage time in the model with boundary $G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ and we iteratively apply equation (4.2.1) to obtain

$$G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} - G^{(u)}_{0,\lfloor Nt\rfloor} = \sum_{i=1}^{\lfloor Ns\rfloor} I^{(u)}_{i,\lfloor Nt\rfloor}.$$

Focus on the left hand side. From equation (4.1.4) and (4.2.1) we can write the previous difference as

$$\begin{aligned}
\sum_{i=1}^{\lfloor Ns\rfloor} I^{(u)}_{i,\lfloor Nt\rfloor} &= G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} - G^{(u)}_{0,\lfloor Nt\rfloor}\\
&= \max_{1\le k\le\lfloor Ns\rfloor}\Big\{\sum_{i=1}^{k} I^{(u)}_{i,0} + G_{(k,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} - \sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{0,j}\Big\}\\
&\qquad \bigvee \max_{1\le k\le\lfloor Nt\rfloor}\Big\{\sum_{j=1}^{k} J^{(u)}_{0,j} + \omega_{1,k} + G_{(1,k),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} - \sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{0,j}\Big\}\\
&= \max_{1\le k\le\lfloor Ns\rfloor}\Big\{\sum_{i=1}^{k} I^{(u)}_{i,0} - \sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{0,j} + G_{(k,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}\Big\}\\
&\qquad \bigvee \max_{1\le k\le\lfloor Nt\rfloor}\Big\{-\sum_{j=k+1}^{\lfloor Nt\rfloor} J^{(u)}_{0,j} + \omega_{1,k} + G_{(1,k),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}\Big\}.
\end{aligned}$$

To compactify notation we use a convention where the y-axis is labeled by negative indices and we define

$$\text{for } k\in\mathbb{Z} \qquad \eta_k = \begin{cases} -\sum_{j=-k+1}^{\lfloor Nt\rfloor} J^{(u)}_{0,j} & k\le 0,\\ \sum_{i=1}^{k} I^{(u)}_{i,0} - \sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{0,j} & k\ge 1.\end{cases} \tag{4.5.1}$$

As such, we can say that the last passage time can be obtained on path that enters the bulk $\mathbb{N}^2$ at a point $\mathbf{v}_z$ defined by

$$\text{for } z \in \mathbb{R} \qquad \mathbf{v}(z) = \begin{cases} (1, \lfloor -z \rfloor) & z \leq -1, \\ (1,1) & -1 < z < 1, \\ (\lfloor z \rfloor, 1) & z \geq 1, \end{cases} \tag{4.5.2}$$

and the gradient can be written as

$$\sum_{i=1}^{\lfloor Ns \rfloor} I^{(u)}_{i,\lfloor Nt \rfloor} = \max_{\lfloor -Nt \rfloor \leq k \leq \lfloor Ns \rfloor, k \neq 0} \big\{ \eta_k + \omega_{\mathbf{v}(k)} \mathbb{1}\{k < 0\} + G_{\mathbf{v}(k),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \big\}.$$

Then the following inequalities are immediate:

$$\begin{aligned} &\eta_k + G_{\mathbf{v}(k),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} && (4.5.3) \\ &\quad \leq \sum_{i=1}^{\lfloor Ns \rfloor} I^{(u)}_{i,\lfloor Nt \rfloor} \\ &\quad\quad \leq \max_{\lfloor -Nt \rfloor \leq k \leq \lfloor Ns \rfloor, k \neq 0} \{ \eta_k + G_{\mathbf{v}(k),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \} + 1. && (4.5.4) \end{aligned}$$

This inequality will be crucial for our purposes. We briefly discuss the main idea.

The second line in the last display is a sum of i.i.d. Bernoulli, so it has a known large deviation rate function. A deviation for the $\sum I^{(u)}$ is controlled above and below by deviations for the expressions $\eta_k + G_{\mathbf{v}(k),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)}$. η_k itself is either a sum of i.i.d. geometric random variables or a difference of two independent sums; in either case the large deviation rate function for η_k is computable, and the only unknown will be the large deviation rate function for G (albeit in a complicated expression). The subsection is devoted to following this program and to solve for the rate function of G.

It will be crucial to understand the function defined by

$$H^{a,b}_{s,t}(r) = -\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{\eta_{\lfloor Na \rfloor} + G_{\mathbf{v}(Nb),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \geq Nr\}, \tag{4.5.5}$$

where $a, b \in [-t, s]$. We first argue why the limit exists. This fact will be a direct consequence of Lemma 4.4.2, when we show that the $\eta_{\lfloor Na \rfloor}$ and $G_{\mathbf{v}(Nb),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)}$ will have a right tail rate function.

We begin by computing the rate function for the η_k. For real $a \in [-t, s]$, and $r \in \mathbb{R}$ define

$$\kappa_a(r) = -\lim_{N \to \infty} N^{-1} \log \mathbb{P}\{\eta_{\lfloor Na \rfloor} \geq Nr\}. \tag{4.5.6}$$

From (4.5.1) we observe that if $k \leq 0$ η_k is a sum of i.i.d. geometric distributed random variables while if $k \geq 1$, η_k is the difference of two independent sums of i.i.d. random variables.

The convex dual is

$$
\begin{aligned}
\kappa_a^*(\xi) &= \sup_{r\in\mathbb{R}}\{\xi r - \kappa_a(r)\}\\
&= \begin{cases}
(t+a)\Big[\log\frac{u-p}{u(1-p)} - \log\Big(1-\frac{p(1-u)}{u(1-p)}e^{-\xi}\Big)\Big], & \text{for } \xi > \log\frac{p(1-u)}{u(1-p)}, -t\le a\le 0,\\
t\Big[\log\frac{u-p}{u(1-p)} - \log\Big(1-\frac{p(1-u)}{u(1-p)}e^{-\xi}\Big)\Big] + a\log(ue^{\xi}+1-u), & \\
& \text{for } \xi > \log\frac{p(1-u)}{u(1-p)},\ 0<a\le s,\\
\infty, & \text{otherwise.}
\end{cases}\\
&= \begin{cases}
(t+a)C_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(-\xi), & \text{for } \xi > \log\frac{p(1-u)}{u(1-p)}, -t\le a\le 0,\\
tC_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(-\xi) + aC_{\mathcal{B}}^{(u)}(\xi), & \text{for } \xi > \log\frac{p(1-u)}{u(1-p)},\ 0<a\le s,\\
\infty, & \text{otherwise.}
\end{cases}
\end{aligned}
\tag{4.5.7}
$$

The first line in (4.5.7) follows from Cramer's theorem when the random variables are geometric. The second line follows from Lemma 4.4.2 when $L_N = \sum_{i=1}^{\lfloor Na\rfloor} I_{i,0}^{(u)}$ and $Z_N = -\sum_{j=1}^{\lfloor Nt\rfloor} J_{0,j}^{(u)}$, and the fact that the dual of an infimal convolution is the sum of the corresponding duals.

Remark 4.5.1. *The condition on ξ can be stated equivalently in terms of u. In fact, if $\xi\in\mathbb{R}$ is fixed, the inequality $\xi > \log\frac{p(1-u)}{u(1-p)}$ becomes $u > \frac{pe^{-\xi}}{1-p+pe^{-\xi}}$. Moreover if $\xi > 0$, $\frac{pe^{-\xi}}{1-p+pe^{-\xi}} < p$ and so it remains $u\in(p,1]$.*

The rightmost zero $m_{\kappa,a}$ of κ_a is the law of large numbers limit

$$
m_{\kappa,a} = \lim_{N\to\infty} N^{-1}\eta_{\lfloor Na\rfloor} = \begin{cases}
-(t+a)\frac{u-p}{p(1-u)}, & -t\le a\le 0,\\
au - t\frac{u-p}{p(1-u)}, & 0<a\le s.
\end{cases}
\tag{4.5.8}
$$

Note that when viewed as functions of a, κ_a, κ_a^* and $m_{\kappa,a}$ are all continuous at $a=0$.

For the rate function of $G_{\mathbf{v}(Nb),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}$, we first introduce the equivalent macroscopic version of (4.5.2) for $a\in\mathbb{R}$, by

$$
N^{-1}\mathbf{v}(Na) \to \bar{\mathbf{v}}(a) = \begin{cases}
(0,-a), & -t\le a\le 0,\\
(a,0), & 0<a\le s.
\end{cases}
\tag{4.5.9}
$$

With this notation, the rate function of the last past passage time in the interior is

$$
J_{(s,t)-\bar{\mathbf{v}}(a)}(r) = -\lim_{N\to\infty} N^{-1}\log\mathbb{P}\{G_{\mathbf{v}(Na),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} \ge Nr\}.
\tag{4.5.10}
$$

This is because $G_{\mathbf{v}(Na),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}$ equals in distribution $G_{(0,0),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)-\mathbf{v}(Na)}$. There will be a small discrepancy between $(\lfloor Ns\rfloor, \lfloor Nt\rfloor) - \mathbf{v}(Na)$ and $\lfloor N((s,t)-\bar{\mathbf{v}}(a))\rfloor$ but Lemma 4.4.1 proves that it is negligible in the limit.

Let $m_{\kappa,a}$ and $m_{J,b}$ be the rightmost zeros respectively of κ_a (defined by (4.5.8)) and $J_{(s,t)-\bar{\mathbf{v}}(b)}$ (which equals $g_{pp}((s,t)-\bar{\mathbf{v}}(b))$. Using Lemma 4.4.2 for $(a,b)\in[-t,s]^2$, we have

$$H^{a,b}_{s,t}(r)=\begin{cases}0, & r<m_{\kappa,a}+m_{J,b},\\ \inf_{m_{\kappa,a}\le x\le r-m_{J,b}}\{\kappa_a(x)+J_{(s,t)-\bar{\mathbf{v}}(b)}(r-x)\}, & m_{\kappa,a}+m_{J,b}\le r\le s.\end{cases} \tag{4.5.11}$$

The following regularity lemma follows from the continuity properties we discussed up to this point, and the details are left to the reader.

Lemma 4.5.2. *Fix $s,t\in(0,\infty)$ and fix any compact set $K\subseteq(-\infty,s]$. Then $H^{a,b}_{s,t}(r)$ is a uniformly continuous function of $(b,r)\in[-t,s]\times K$, uniformly in $a\in[-t,s]$. In symbols*

$$\lim_{\delta\to 0}\sup_{\substack{a,b,b'\in[-t,s],r,r'\in K:\\ |b-b'|\le\delta,|r-r'|\le\delta}}|H^{a,b}_{s,t}(r)-H^{a,b'}_{s,t}(r')|=0. \tag{4.5.12}$$

When $a=b$ we simplify the notation as $H^a_{s,t}(r)=H^{a,a}_{s,t}(r)$. Observe that at this point an expression involving $J_{s,t}$ manifested on the right-hand side of (4.5.11). Our goal is to invert the relation and isolate $J_{s,t}$.

The next lemma is the continuous version of the discrete inequalities (4.5.3), (4.5.4) at the level of the rate functions.

Lemma 4.5.3. *Let $s,t\in(0,\infty)$ and $r\in[0,s]$. Then*

$$sI^{(u)}_{\mathcal{B}}(r/s)=\inf_{-t\le a\le s}H^a_{s,t}(r). \tag{4.5.13}$$

Proof. For any $a\in[-t,s]$, by (4.5.3)

$$\begin{aligned}-sI^{(u)}_{\mathcal{B}}(r/s)&=\lim_{N\to\infty}N^{-1}\log\mathbb{P}\Big\{\sum_{i=1}^{\lfloor Ns\rfloor}I^{(u)}_{i,\lfloor Nt\rfloor}\ge Nr\Big\}\\ &\ge\lim_{N\to\infty}N^{-1}\log\mathbb{P}\{\eta_{\lfloor Na\rfloor}+G_{\mathbf{v}(\lfloor Na\rfloor),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}\ge Nr\}\\ &=-H^a_{s,t}(r).\end{aligned}$$

This is true for an arbitrary a, therefore

$$sI^{(u)}_{\mathcal{B}}(r/s)\le\inf_{-t\le a\le s}H^a_{s,t}(r). \tag{4.5.14}$$

To get the lower bound we use (4.5.4) together with a coarse graining argument.

We begin describing the partition which will be helpful when we will use (4.5.4). Fix a small enough $\delta>0$ to partition the interval $[-t,s]$. In particular, define $-t=a_0<a_1<$

$\cdots < a_q = 0 < \cdots < a_m = s$ where $|a_{i+1} - a_i| < \delta$. Moreover, we fix an $\varepsilon > 0$ and we assume that N is large enough so that $N\varepsilon > 1$.

When $a_i \geq 0$, for any $k \in [\lfloor Na_i \rfloor, \lfloor Na_{i+1} \rfloor] \cap \mathbb{Z}$,

$$\mathbb{P}\{\eta_k + G_{\mathbf{v}(k),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \geq Nr\} \leq \mathbb{P}\Big\{\eta_{\lfloor Na_{i+1} \rfloor} + G_{\mathbf{v}(\lfloor Na_i \rfloor),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \geq Nr\Big\}.$$

Similarly, when $a_i < 0$ and $\lfloor Na_i \rfloor < k \leq \lfloor Na_{i+1} \rfloor$ the bound becomes

$$\mathbb{P}\{\eta_k + G_{\mathbf{v}(k),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \geq Nr\} \leq \mathbb{P}\Big\{\eta_{\lfloor Na_i \rfloor} + G_{\mathbf{v}(\lfloor Na_{i+1} \rfloor),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \geq Nr\Big\}.$$

From (4.5.4) we bound

$$\begin{aligned}
\mathbb{P}\Big\{\sum_{i=1}^{\lfloor Ns \rfloor} I_{i,\lfloor Nt \rfloor} \geq Nr\Big\} &\leq \mathbb{P}\Big\{\max_{\substack{\lfloor -Nt \rfloor \leq k \leq \lfloor Ns \rfloor, \\ k \neq 0}} \{\eta_k + G_{\mathbf{v}(k),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)}\} + 1 \geq Nr\Big\} \\
&\leq \mathbb{P}\Big\{\max_{\substack{\lfloor -Nt \rfloor \leq k \leq \lfloor Ns \rfloor, \\ k \neq 0}} \{\eta_k + G_{\mathbf{v}(k),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)}\} \geq N(r - \varepsilon)\Big\}.
\end{aligned}$$

Take logarithm on both sides and divide by N and use a union bound to obtain

$$\begin{aligned}
&N^{-1} \log \mathbb{P}\Big\{\sum_{i=1}^{\lfloor Ns \rfloor} I_{i,\lfloor Nt \rfloor} \geq Nr\Big\} \\
&\quad \leq N^{-1} \log m + \Big\{\max_{0 \leq i \leq q-1} \Big\{N^{-1} \log \mathbb{P}\{\eta_{\lfloor Na_i \rfloor} + G_{\mathbf{v}(\lfloor Na_{i+1} \rfloor),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \geq N(r - \varepsilon)\}\Big\}\Big\} \\
&\qquad\qquad \vee \Big\{\max_{q \leq i \leq m-1} \Big\{N^{-1} \log \mathbb{P}\{\{\eta_{\lfloor Na_{i+1} \rfloor} + G_{\mathbf{v}(\lfloor Na_i \rfloor),(\lfloor Ns \rfloor, \lfloor Nt \rfloor)} \geq N(r - \varepsilon)\}\Big\}.
\end{aligned}$$

Take $N \to \infty$ to get

$$\begin{aligned}
-sI_{\mathcal{B}}^{(u)}(r/s) &\leq \Big\{\max_{0 \leq i \leq q-1} \{-H_{s,t}^{a_i, a_{i+1}}(r - \varepsilon)\}\Big\} \vee \Big\{\max_{q \leq i \leq m-1} \{-H_{s,t}^{a_{i+1}, a_i}(r - \varepsilon)\}\Big\} \\
&\leq \sup_{a,b \in [-t,s]: |a-b| \leq \delta} \{-H_{s,t}^{a,b}(r - \varepsilon)\}.
\end{aligned}$$

Use Lemma 4.5.2 by letting $\delta \to 0$; this also implies $b \to a$. Then let $\varepsilon \to 0$. $\square$

The following lemma is the last technical tool we need in order to finally solve (4.5.13) for the unknown rate function J. It proves convexity and lower semi-continuity of the Legendre dual of J.

Lemma 4.5.4. *For a fixed $\xi \in \mathbb{R}_+$, the function $J^*_{s,t}(\xi)$, as a function of (s,t), is continuous and finite on $\mathbb{R}^2_+$.*

Proof. By definition $J^*_{s,t}(\xi) = \sup_{r \in \mathbb{R}}\{\xi r - J_{s,t}(r)\}$, but, since $J_{s,t}(r) = \infty$ for $r > s$, and $J_{s,t}(r) = 0$ for $r < g_{pp}(s,t)$, we can write for $\xi \geq 0$ that

$$J^*_{s,t}(\xi) = \sup_{r \in [g_{pp}(s,t), s]} \{\xi r - J_{s,t}(r)\}.$$

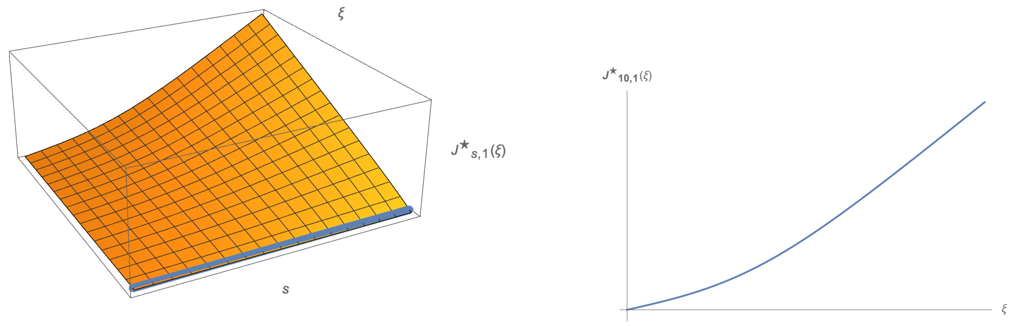


Figure 4.3: Graphical representation of the function $J^*_{s,t}(r)$. In both figures we used $p = 0.1$ and $t = 1$. To the left we have $J^*_{s,1}(\xi)$ as a function of (s, ξ) and one see the directions of convexity when s is fixed and ξ varies, and the direction of concavity ranges when ξ is fixed and s varies as described in the proof of Lemma 4.5.4. The blue line is at $s = 1/9$ which the is characteristic point for $p = 0.1$ and $t = 1$. For smaller s, $J^*_{s,1}(\xi) = s\xi$. To the right is the convex continuous function $J^*_{10,1}(\xi)$.

Then it is immediate to see that

$$J^*_{s,t}(\xi) \le \xi s, \quad \text{for all}(s,t) \in \mathbb{R}^2_+.$$

Continuity will follow once we prove that $J^*_{s,t}(\xi)$ is a concave finite function. Let $\lambda \in (0,1)$ and $(s,t) = \lambda(s_1,t_1) + (1-\lambda)(s_2,t_2)$ for some $(s_i,t_i) \in \mathbb{R}^2_+$. Recall that J is convex and lower-semicontinuous in (s,t,r) from Theorem 4.1.3. Write r as the convex combination $r = \lambda r_1 + (1-\lambda) r_2$ for some $r_1, r_2 \in \mathbb{R}$. By convexity

$$\begin{aligned}
&\inf_{r\in\mathbb{R}}\{J_{s,t}(r) - \xi r\}\\
&\quad\le \inf_{r\in\mathbb{R}}\left\{\inf_{(r_1,r_2):\lambda r_1+(1-\lambda)r_2=r}\{\lambda(J_{s_1,t_1}(r_1) - \xi r_1) + (1-\lambda)(J_{s_2,t_2}(r_2) - \xi r_2)\}\right\}\\
&\quad= \inf_{(r_1,r_2)\in\mathbb{R}^2}\{\lambda(J_{s_1,t_1}(r_1) - \xi r_1) + (1-\lambda)(J_{s_2,t_2}(r_2) - \xi r_2)\}\\
&\quad= \lambda \inf_{r_1\in\mathbb{R}]}\{J_{s_1,t_1}(r_1) - \xi r_1\} + (1-\lambda)\inf_{r_2\in\mathbb{R}}\{J_{s_2,t_2}(r_2) - \xi r_2\}\\
&\quad= -\lambda J^*_{s_1,t_1}(\xi) - (1-\lambda) J^*_{s_2,t_2}(\xi).
\end{aligned}$$

In the end we have

$$J^*_{s,t}(\xi) \ge \lambda J^*_{s_1,t_1}(\xi) + (1-\lambda) J^*_{s_2,t_2}(\xi),$$

which is enough to prove the concavity of $J^*_{s,t}(\xi)$ in (s,t). □

Now we can find a variational expression for J^*.

Proof of Proposition 4.1.6. If $\xi < 0$, by definition

$$J^*_{s,t}(\xi) = \sup_{r\in\mathbb{R}}\{r\xi - J_{s,t}(r)\} = \sup_{r<g_{pp}(s,t)}\{r\xi - J_{s,t}(r)\} \vee \sup_{r\in[g_{pp}(s,t),s]}\{r\xi - J_{s,t}(r)\} \vee \sup_{r>s}\{r\xi - J_{s,t}(r)\}.$$

Note that the first supremum is $+\infty$ since $J_{s,t}(r) = 0$ for $r < g_{pp}(s,t)$ and $\xi < 0$. Therefore $J^*_{s,t}(\xi) = \infty$ if $\xi < 0$.

If $\xi \geq 0$, equation (4.5.11) gives that $H^a_{(s,t)}$ is the infimal convolution of κ_a and $J_{(s,t)-\bar{\mathbf{v}}(a)}$ since the value of the infimum does not change when we allow r to range over all of $\mathbb{R}$. We compactify the notation by writing $H^a_{s,t}(r) = \kappa_a \Box J_{(s,t)-\bar{\mathbf{v}}(a)}(r)$. By Theorem 16.4 in [92], the addition operation is dual to the infimal convolution operation. From (4.5.13) of Lemma 4.5.3, take the Legendre dual on both sides to obtain

$$\begin{aligned} sC^{(u)}_{\mathcal{B}}(\xi) &= \sup_{-t\leq a\leq s}\Big\{\sup_{r\in\mathbb{R}}\{r\xi - (\kappa_a \Box J_{(s,t)-\bar{\mathbf{v}}(a)})(r)\}\Big\} \\ &= \sup_{-t\leq a\leq s}\big\{(\kappa_a \Box J_{(s,t)-\bar{\mathbf{v}}(a)})^*(\xi)\big\} = \sup_{-t\leq a\leq s}\big\{(\kappa^*_a(\xi) + J^*_{(s,t)-\bar{\mathbf{v}}(a)}(\xi)\big\}. \end{aligned} \tag{4.5.15}$$

From (4.5.7) we can substitute the explicit expression of $\kappa^*_a(\xi)$. Define

$$\begin{aligned} -\ell_\xi(u) &= C_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(-\xi) = \log\frac{u-p}{u(1-p) - p(1-u)e^{-\xi}}, \\ d_\xi(u) &= C^{(u)}_{\mathcal{B}}(\xi) = \log(ue^\xi + 1 - u). \end{aligned} \tag{4.5.16}$$

Use this to simplify (4.5.15) into

$$sd_\xi(u) + t\ell_\xi(u) = \sup_{0\leq a\leq t}\{a\ell_\xi(u) + J^*_{s,t-a}(\xi)\} \bigvee \sup_{0\leq a\leq s}\{ad_\xi(u) + J^*_{s-a,t}(\xi)\}.$$

Subtract $sd_\xi(u) + t\ell_\xi(u)$ to either side

$$0 = \sup_{0\leq z\leq s}\{J^*_{s-a,t}(\xi) - [(s-a)d_\xi(u) + t\ell_\xi(u)]\} \vee \sup_{0\leq \tilde{z}\leq t}\{J^*_{s,t-a}(\xi) - [sd_\xi(u) + (t-a)\ell_\xi(u)]\}.$$

Use Proposition 4.2.7 identifying as $I = (p,1]$, $\Lambda(s,t) = J^*_{s,t}(\xi)$, $h(u) = d_\xi(u)$, $g(u) = \ell_\xi(u)$ and therefore $f_{s,t}(u) = sd_\xi(u) + t\ell_\xi(u)$. The only hypothesis that is not immediately verifiable is continuity of J^* in s,t, but that is now covered by Lemma 4.5.4. Therefore, if $t < \frac{1-p}{p}s$

$$J^*_{s,t}(\xi) = \min_{u\in(p,1]}\{sd_\xi(u) + t\ell_\xi(u)\} = \min_{u\in(p,1]}\{sC^{(u)}_{\mathcal{B}}(\xi) - tC_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(-\xi)\}.$$

For $t \geq \frac{1-p}{p}s$ we reason directly: $J_{s,t}(r) = +\infty\mathbb{1}\{r > s\}$ and its convex dual will be $s\xi$ for $\xi > 0$. This is also the $\min_{u\in(p,1]}\{sC^{(u)}_{\mathcal{B}}(\xi) - tC_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(-\xi)\}$, with the minimum obtained at $u = 1$. $\square$

4.5.2 Closed formula for $J^*_{s,t}(\xi)$

Proof of Theorem 4.1.7. The aim of this proof is to find an analytical result for the infimum in Proposition 4.1.6 when $t < \frac{1-p}{p}s$. Therefore we start computing the derivatives of the two cumulant-generating function and to find the optimizing point we solve the equation

$$
\begin{aligned}
0 =& s\frac{\partial C^{(u)}_{\mathcal{B}}(\xi)}{\partial u} - t\frac{\partial C^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(-\xi)}{\partial u} = s\frac{e^{\xi}-1}{1+u(e^{\xi}-1)} \\
& - t\frac{p(p-1)(e^{-\xi}-1)}{u^2(1+p(e^{-\xi}-1)) - up[1+e^{-\xi}+p(e^{-\xi}-1)] + p^2e^{-\xi}}
\end{aligned}
$$

or equivalently, after the algebraic simplification of denominators

$$
\begin{aligned}
0 =& u^2 2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})] - up[(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+2s(1-e^{-\xi})] \\
& + (e^{-\xi}-1)p((1-p)(s+t)-s).
\end{aligned}
$$

The minimum is in fact attained to the solution to this equation (for further details see Appendix B.1). The minimizing point is

$$
u^* = \frac{p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+2sp(1-e^{-\xi})+\sqrt{\Delta}}{2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})]} \tag{4.5.17}
$$

with $\Delta = p(1-p)(e^{\xi}+e^{-\xi}-2)[(1-p)p(s+t)^2(e^{\xi}+e^{-\xi}-2)+4st]$. Then (4.1.22) follows directly by

$$
J^*_{s,t}(\xi) = sC^{(u^*)}_{\mathcal{B}}(\xi) - tsC^{(u^*)}_{\mathcal{G}}(-\xi). \qquad \square
$$

4.6 Invariant model: Limiting log-moment generating functions

Before proving the two main theorems, we begin by verifying the existence of limits (4.1.24) and (4.1.25). We begin by noting that similar arguments as in Lemma 4.5.3 give that

$$
-\lim_{N\to\infty} N^{-1}\log\mathbb{P}\{G^{(u),\mathrm{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \ge Nr\} = \inf_{a\in[0,s]}\inf_{x\in\mathbb{R}}\{aI^{(u)}_{\mathcal{B}}((r-x)/a) + J_{s-a,t}(x)\}. \tag{4.6.1}
$$

Equation (4.6.1) in particular verifies the existence of the limit in the left-hand side, and we denote it by

$$
-\lim_{N\to\infty} N^{-1}\log\mathbb{P}\{G^{(u),\mathrm{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \ge Nr\} = J^{(u),\mathrm{hor}}_{s,t}(r). \tag{4.6.2}
$$

Finally, observe that we take the Legendre transform, equation (4.6.1) becomes

$$
(J^{(u),\mathrm{hor}}_{s,t})^*(\xi) = \sup_{a\in[0,s]}\{aC^{(u)}_{\mathcal{B}}(\xi) + J^*_{s-a,t}(\xi)\}. \tag{4.6.3}
$$

Symmetric definitions and arguments give similar equations for $J^{(u),\mathrm{ver}}_{s,t}$.

Lemma 4.6.1. *Let $G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ be the last passage time given by (4.1.6), and let $(J^{(u),\text{hor}}_{s,t})^*(\xi)$ given by (4.6.3). Then for $\xi > 0$,*

$$\lim_{N\to\infty} N^{-1}\log\mathbb{E}[e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}] = (J^{(u),\text{hor}}_{s,t})^*(\xi). \tag{4.6.4}$$

Corresponding statements hold for $G^{(u),\text{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$.

Proof. Let $\xi \geq 0$. Set

$$\underline{\gamma} = \varliminf_{N\to\infty} N^{-1}\log\mathbb{E}[e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}] \qquad \text{and} \qquad \bar{\gamma} = \varlimsup_{N\to\infty} N^{-1}\log\mathbb{E}[e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}].$$

The lower bound is immediate using the exponential Chebyshev inequality

$$N^{-1}\log\mathbb{P}\{G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr\} \leq -\xi r + N^{-1}\log\mathbb{E}[e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}].$$

Letting $N\to\infty$ along a suitable subsequence gives $\underline{\gamma} \geq \xi r - J^{(u),\text{hor}}_{s,t}(r)$ for all $r\in[0,s]$. Thus $\underline{\gamma} \geq (J^{(u),\text{hor}}_{s,t})^*(\xi)$ holds.

For the upper bound we first claim that for every $r > s$

$$N^{-1}\log\mathbb{E}[e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}\mathbb{1}\{G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr\}] = -\infty. \tag{4.6.5}$$

To see this, apply Holder's inequality to the expectation in (4.6.5). For any $\alpha > 1$,

$$\begin{aligned} &N^{-1}\log\mathbb{E}[e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}\mathbb{1}\{G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr\}] \\ &\leq N^{-1}\log\{\mathbb{E}[e^{\alpha\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}]^{\alpha^{-1}}\mathbb{E}[\mathbb{1}\{G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr\}^{\frac{\alpha}{\alpha-1}}]^{\frac{\alpha-1}{\alpha}}\} \\ &= (\alpha N)^{-1}\log(\mathbb{E}[e^{\alpha\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}]) + (\alpha-1)\alpha^{-1}N^{-1}\log\mathbb{P}\{G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr\}. \end{aligned}$$

The first term is finite since $G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \leq \lfloor Ns\rfloor$ and for the same reason the second term equals $-\infty$.

To show the upper bound in (4.6.4) pick a $\delta > 0$ and partition $\mathbb{R}$ with $r_i = i\delta$, $i\in\mathbb{Z}$:

$$\begin{aligned} &N^{-1}\log\mathbb{E}[e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}] \\ &\leq N^{-1}\log\Big[\sum_{i=-m}^{m} e^{N\xi r_{i+1}}\mathbb{P}\{G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr_i\} \\ &\qquad + e^{N\xi r_{-m}} + \mathbb{E}[e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}\mathbb{1}\{G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr_m\}]\Big]. \end{aligned} \tag{4.6.6}$$

By (4.6.5), for each $M > 0$ there exists $m = m(M)$ so that for all N large enough

$$N^{-1}\log\mathbb{E}[e^{\xi G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}\mathbb{1}\{G^{(u),\text{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr_m\}] < -M.$$

Take a limit as $N\to\infty$ along any subsequence that achieves $\bar{\gamma}$ to see that (4.6.6) implies

$$\bar{\gamma} \leq \max_{-m\leq i\leq m}\{\xi r_{i+1} - J^{(u),\text{hor}}_{s,t}(r_i)\} \vee \xi r_{-m} \vee (-M)$$

$$\leq \Big(\sup_{r\in[0,s]}\{\xi r - J^{(u),\mathrm{hor}}_{s,t}(r)\} + \xi\delta\Big) \vee \xi r_{-m} \vee (-M).$$

The statement of the Lemma follows by letting $\delta \to 0$, $m \to \infty$ and $M \to \infty$.

In order to repeat the estimates for $G^{(u),\mathrm{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}$ the equivalent statement for (4.6.5) is

$$\lim_{r\to\infty}\overline{\lim_{N\to\infty}} N^{-1}\log\mathbb{E}[e^{\xi G^{(u),\mathrm{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}\mathbb{1}\{G^{(u),\mathrm{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \geq Nr\}] = -\infty.$$

We omit the remaining details, but the interested reader can find a similar calculation in [51]. $\square$

Proof of Theorem 4.1.8. The existence of limit (4.1.24) is verified by Lemma 4.6.1. Then, use in sequence equations (4.6.3) and (4.6.4) and Proposition 4.1.6 to write

$$\begin{aligned}\Lambda^{(u),\mathrm{hor}}_{(s,t)}(\xi) &= \sup_{a\in[0,s]}\{aC^{(u)}_{\mathcal{B}}(\xi) + J^{*}_{s-a,t}(\xi)\}\\ &= \sup_{a\in[0,s]}\Big\{\inf_{\theta\in(p,1]}\{a\Big(C^{(u)}_{\mathcal{B}}(\xi) - C^{(\theta)}_{\mathcal{B}}(\xi)\Big) + sC^{(\theta)}_{\mathcal{B}}(\xi) - tC_{\mathcal{G}}^{(\frac{\theta-p}{\theta(1-p)})}(-\xi)\}\Big\}.\end{aligned}$$

The sup and inf can be interchanged by a minimax theorem (e.g. [67]). The function inside the supremum is linear in a. Thus the supremum will be reached at one of the two boundary points according to the sign of the difference

$$C^{(u)}_{\mathcal{B}}(\xi) - C^{(\theta)}_{\mathcal{B}}(\xi)\begin{cases} > 0, & \text{if } \theta\in(u,1],\\ = 0, & \text{if } \theta = u,\\ < 0, & \text{if } \theta\in(p,u).\end{cases}$$

Therefore we have

$$\begin{aligned}\Lambda^{(u),\mathrm{hor}}_{(s,t)}(\xi) = \inf_{\theta\in(u,1]}\{sC^{(u)}_{\mathcal{B}}(\xi) - tC_{\mathcal{G}}^{(\frac{\theta-p}{\theta(1-p)})}(-\xi)\} \wedge \{sC^{(u)}_{\mathcal{B}}(\xi) - tC_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(-\xi)\}\\ \wedge \inf_{\theta\in(p,u)}\{sC^{(\theta)}_{\mathcal{B}}(\xi) - tC_{\mathcal{G}}^{(\frac{\theta-p}{\theta(1-p)})}(-\xi)\}.\end{aligned} \tag{4.6.7}$$

Note that, since $-C_{\mathcal{G}}^{(\frac{\theta-p}{\theta(1-p)})}(-\xi)$ is increasing in θ, the first term on the right-hand side of (4.6.7) is always greater than the second one. So, it remains to compare the second and the third term.

Call θ^* the minimizing point in $(p,1]$ for the expression $sC^{(\theta)}_{\mathcal{B}}(\xi) - tC_{\mathcal{G}}^{(\frac{\theta-p}{\theta(1-p)})}(-\xi)$ (4.5.17) in this specific case. Then, there are two possible cases:

(1) If $\theta^* \leq u$, then

$$\Lambda^{(u),\mathrm{hor}}_{(s,t)}(\xi) = \inf_{\theta\in(p,u)}\{sC^{(\theta)}_{\mathcal{B}}(\xi) - tC_{\mathcal{G}}^{(\frac{\theta-p}{\theta(1-p)})}(-\xi)\} = sC^{(\theta^*)}_{\mathcal{B}}(\xi) - tC_{\mathcal{G}}^{(\frac{\theta^*-p}{\theta^*(1-p)})}(-\xi) = \Lambda_{(s,t)}(\xi).$$

(2) If $\theta^* > u$ then

$$\Lambda^{(u),\text{hor}}_{(s,t)}(\xi) = sC^{(u)}_{\mathcal{B}}(\xi) - tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(-\xi).$$

This concludes the proof of (4.1.29). For the analogous result in the vertical case, first note that we may write

$$\Lambda_{(s,t)}(\xi) = J^*_{s,t}(\xi) = \inf_{u\in(p,1]}\{tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(\xi) - sC^{(u)}_{\mathcal{B}}(-\xi)\}. \tag{4.6.8}$$

That is possible to prove either by repeating the same computation in the subsection 4.5.1 but starting $G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} - G^{(u)}_{\lfloor Ns\rfloor,0} = \sum_{j=1}^{\lfloor Nt\rfloor} J^{(u)}_{\lfloor Ns\rfloor,j}$, or by computing (4.6.8) as in the proof of Theorem 4.1.7 and observe that it gives the same result.

Then as in the case for the horizontal boundary only,

$$\begin{aligned}
\Lambda^{(u),\text{ver}}_{(s,t)}(\xi) &= \sup_{a\in[0,t]}\{aC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(\xi) + J^*_{s,t-a}(\xi)\}\\
&= \sup_{a\in[0,t]}\Big\{\inf_{\theta\in(p,1]}\{a\Big(C^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(\xi) - C^{(\frac{\theta-p}{\theta(1-p)})}_{\mathcal{G}}(\xi)\Big) + tC^{(\frac{\theta-p}{\theta(1-p)})}_{\mathcal{G}}(\xi) - sC^{(\theta)}_{\mathcal{B}}(-\xi)\}\Big\}.
\end{aligned}$$

From this expression we see that we need to restrict $\xi \in [0, \log\frac{u(1-p)}{p(1-u)})$, otherwise $\Lambda^{(u),\text{ver}}_{(s,t)}(\xi)$ is not finite. Then, as before, for $\xi \in [0, \log\frac{u(1-p)}{p(1-u)})$

$$\begin{aligned}
&\Lambda^{(u),\text{ver}}_{(s,t)}(\xi)\\
&= \begin{cases} \inf_{\theta\in(p,1]}\{tC^{(\frac{\theta-p}{\theta(1-p)})}_{\mathcal{G}}(\xi) - sC^{(\theta)}_{\mathcal{B}}(-\xi)\} = \Lambda_{(s,t)}(\xi) & \text{if } t \le k^{(u)}(-\xi)s,\\ \inf_{\theta\in(p,u]}\{tC^{(\frac{\theta-p}{\theta(1-p)})}_{\mathcal{G}}(\xi) - sC^{(u)}_{\mathcal{B}}(-\xi)\} = tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(\xi) - sC^{(u)}_{\mathcal{B}}(-\xi) & \text{if } t > k^{(u)}(-\xi)s. \end{cases}
\end{aligned}$$

This concludes the proof of the theorem. □

Since the following proof is based on (4.1.26), we want to show first that it is true. As usual we proceed finding an upper and a lower bound for $\Lambda^{(u)}_{s,t}(\xi)$. We start from the lower bound. By (4.1.8) we have that

$$0 \le G^{(u),i}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} \le G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}, \quad \text{where } i = \{\text{hor}, \text{ver}\}.$$

Then, if $\xi > 0$

$$\mathbb{E}[e^{\xi G^{(u),i}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}] \le \mathbb{E}[e^{\xi G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}].$$

Take the logarithm of both sides, divide by N and let $N \to \infty$ to obtain the lower bound

$$\Lambda^{(u),i}_{(s,t)}(\xi) \le \Lambda^{(u)}_{(s,t)}(\xi).$$

For the upper bound, let v_1 be the first step that the maximal path makes starting from $(0,0)$ and note that

$$\mathbb{E}[e^{\xi G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}] = \mathbb{E}[e^{\xi G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}\mathbb{1}\{v_1 = e_1\}] + \mathbb{E}[e^{\xi G^{(u)}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}\mathbb{1}\{v_1 = e_2\}]$$

$$\leq \mathbb{E}[e^{\xi G^{(u),\mathrm{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}] + \mathbb{E}[e^{\xi G^{(u),\mathrm{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}] \leq 2\Big(\mathbb{E}[e^{\xi G^{(u),\mathrm{hor}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}] \vee \mathbb{E}[e^{\xi G^{(u),\mathrm{ver}}_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}}]\Big).$$

Take the logarithm of both sides, divide by N and let $N \to \infty$ to get

$$\Lambda^{(u)}_{(s,t)}(\xi) \leq \Lambda^{(u),\mathrm{hor}}_{(s,t)}(\xi) \vee \Lambda^{(u),\mathrm{ver}}_{(s,t)}(\xi).$$

Proof of Theorem 4.1.9. All the proof is based on (4.1.26). First note that by Proposition 4.1.6 and (4.6.8) we have that for any u

$$\Lambda_{(s,t)}(\xi) \leq sC^{(u)}_{\mathcal{B}}(\xi) - tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(-\xi) \quad \text{and} \quad \Lambda_{(s,t)}(\xi) \leq tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(-\xi) - sC^{(u)}_{\mathcal{B}}(-\xi). \tag{4.6.9}$$

Therefore, if $\xi \in [\log \frac{u(1-p)}{p(1-u)}, \infty)$, $\Lambda^{(u)}_{(s,t)}(\xi) = \infty$.

If $\xi \in (0, \log \frac{u(1-p)}{p(1-u)})$ we define three regions in the quadrant by

$$L = \{(s,t) : t < k^{(u)}(-\xi)s\},\ M = \{(s,t) : k^{(u)}(-\xi)s \leq t \leq k^{(u)}(\xi)s\},\ U = \mathbb{R}^2_+ \setminus (M \cup L).$$

$k^{(u)}(\xi)$ is defined by (4.1.27) and one can directly verify that $k^{(u)}(-\xi) < k^{(u)}(\xi)$. For $(s,t) \in L$, $\Lambda^{(u)}_{s,t}(\xi) = sC^{(u)}_{\mathcal{B}}(\xi) - tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(-\xi) = \Lambda^{(u),\mathrm{hor}}_{(s,t)}(\xi)$ by (4.1.26),(4.6.9), since $\Lambda^{(u),\mathrm{ver}}_{(s,t)}(\xi) = \Lambda_{(s,t)}(\xi)$. For $(s,t) \in U$ the arguments are symmetric, with $\Lambda^{(u)}_{(s,t)}(\xi) = tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(\xi) - sC^{(u)}_{\mathcal{B}}(-\xi)$.

From (4.1.26), (4.6.9) and Theorem 4.1.8, we have that

$$\Lambda^{(u)}_{(s,t)}(\xi) = \begin{cases} \Lambda^{(u),\mathrm{ver}}_{(s,t)}(\xi), & t \geq k^{(u)}(\xi)s, \\ \Lambda^{(u),\mathrm{ver}}_{(s,t)}(\xi) \vee \Lambda^{(u),\mathrm{hor}}_{(s,t)}(\xi), & k^{(u)}(\xi)s < t < k^{(u)}(\xi)s, \\ \Lambda^{(u),\mathrm{hor}}_{(s,t)}(\xi), & t \leq k^{(u)}(-\xi)s. \end{cases}$$

By (4.1.26) and Theorem 4.1.8, $\Lambda^{(u)}_{(s,t)}(\xi)$ is continuous in (s,t). From this and the fact that the middle branch above is linear in (s,t), we conclude that the slope $\ell^{(u)}(\xi)$ of the line

$$t = \ell^{(u)}(\xi)s \iff \{(s,t) \in \mathbb{R}^2_+ : \Lambda^{(u),\mathrm{ver}}_{(s,t)}(\xi) = \Lambda^{(u),\mathrm{hor}}_{(s,t)}(\xi)\}$$

satisfies $k^{(u)}(\xi) \geq \ell^{(u)}(\xi) \geq k^{(u)}(-\xi)$ and therefore

$$\Lambda^{(u)}_{(s,t)}(\xi) = \begin{cases} sC^{(u)}_{\mathcal{B}}(\xi) - tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(-\xi), & \text{if } k^{(u)}(-\xi))s \leq t \leq \ell^{(u)}(\xi)s, \\ tC^{(\frac{u-p}{u(1-p)})}_{\mathcal{G}}(\xi) - sC^{(u)}_{\mathcal{B}}(-\xi), & \text{if } \ell^{(u)}(\xi)s < t \leq k^{(u)}(\xi))s. \end{cases}$$

This gives the theorem. $\square$

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Appendix A

A.1 Approximation in 2.3.19

In this appendix section we perform all the computations step by step to get (2.3.19). From (2.3.16)

$$f(a) = f(0) + \int_0^a f'(s)ds = f(0) - 2c_{1/2}^{(-)} a^{1/2} - \frac{c}{\gamma + 1/2} a^{\gamma+1/2}, \tag{A.1.1}$$

for a small enough. Since m_2 in (2.3.8) is defined as a very complicated function of a we prefer to approximate every addend separately and then put all together.

Recall

$$\frac{1}{c_1 x^\alpha + c_2 x^\beta} = \frac{1}{c_1 x^\alpha} \frac{1}{1 + \frac{c_2}{c_1} x^{\beta-\alpha}} = \frac{1}{c_1 x^\alpha}\Big(1 - \frac{c_2}{c_1} x^{\beta-\alpha} + O(x^{2(\beta-\alpha)})\Big) \quad \alpha < \beta. \tag{A.1.2}$$

Use (A.1.2) to compute

$$\begin{aligned}\frac{1}{f'(a)} &= \frac{-a^{1/2}}{c_{1/2}^{(-)} + ca^\gamma} = -\frac{a^{1/2}}{c_{1/2}^{(-)}} \frac{1}{1 + \frac{c}{c_{1/2}^{(-)}} a^\gamma} \\ &= -\frac{a^{1/2}}{c_{1/2}^{(-)}}\Big(1 - \frac{c}{c_{1/2}^{(-)}} a^\gamma + O(a^{2\gamma})\Big) = -\frac{a^{1/2}}{c_{1/2}^{(-)}} + \frac{c}{c_{1/2}^{(-)2}} a^{\gamma+1/2} + O(a^{2\gamma+1/2}).\end{aligned} \tag{A.1.3}$$

Since $m_1(a) = f(a)/a = \frac{f(0)}{a}(1 - \frac{2r}{r-1}\frac{a^{1/2}}{\sqrt{f(0)}} - \frac{c}{\gamma+1/2}\frac{a^{\gamma+1/2}}{f(0)})$ we then have

$$\frac{m_1(a)}{f'(a)} = -\frac{(r-1)\sqrt{f(0)}}{r} a^{-1/2} + c\frac{(r-1)^2}{r^2} a^{\gamma-1/2} + 2 - \frac{2c\gamma}{(\gamma+1/2)c_{1/2}^{(-)}} a^\gamma + O(a^{2\gamma+1/2}). \tag{A.1.4}$$

By the Taylor expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2) \tag{A.1.5}$$

we obtain

$$\begin{aligned}\sqrt{m_1(a)} &= \sqrt{\frac{f(0)}{a}}\Big(1 - \frac{r}{r-1}\frac{a^{1/2}}{\sqrt{f(0)}} - \frac{c}{2(\gamma+1/2)}\frac{a^{\gamma+1/2}}{f(0)} + O(a)\Big) \\ &= \sqrt{f(0)}a^{-1/2} - \frac{r}{r-1} - \frac{c}{2(\gamma+1/2)}\frac{a^\gamma}{\sqrt{f(0)}} + O(a^{1/2})\end{aligned}$$

and using $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + O(x^2)$ we get

$$\frac{1}{\sqrt{m_1(a)}} = \sqrt{\frac{a}{f(0)}}\Big(1 + \frac{r}{r-1}\frac{a^{1/2}}{\sqrt{f(0)}} + \frac{c}{2(\gamma+1/2)f(0)}a^{\gamma+1/2} + O(a)\Big)$$

$$= \frac{a^{1/2}}{\sqrt{f(0)}} - \frac{r}{(r-1)f(0)}a - \frac{c}{2(\gamma+1/2)(f(0))^{3/2}}a^{\gamma+1} + O(a^{3/2}). \qquad \text{(A.1.6)}$$

From (2.1.11) we are able to expand $-\frac{1}{f'(a)} - 1 + D$ which after some rearrangement we can substitute (A.1.3), (A.1.4), (A.1.6) in and obtain

$$-\frac{1}{f'(a)} - 1 + D = (r-1)\Big(\frac{1}{f'(a)} + 1\Big) + r\frac{1}{\sqrt{m_1(a)}}\Big(\frac{m_1(a)}{f'(a)} + 1\Big) \qquad \text{(A.1.7)}$$

$$= (r-1) - \frac{(r-1)}{c^{(-)}_{1/2}}a^{1/2} + c\frac{(r-1)}{c^{(-)2}_{1/2}}a^{\gamma+1/2} + O(a^{2\gamma+1/2}) + \Big(\frac{ra^{1/2}}{\sqrt{f(0)}} - \frac{r^2}{(r-1)f(0)}a$$

$$- \frac{rc}{2(\gamma+1/2)(f(0))^{3/2}}a^{\gamma+1} + O(a^{3/2})\Big)\Big(- \frac{(r-1)\sqrt{f(0)}}{r}a^{-1/2} + c\frac{(r-1)^2}{r^2}a^{\gamma-1/2} + 3$$

$$- \frac{2c\gamma}{(\gamma+1/2)c^{(-)}_{1/2}}a^{\gamma} + O(a^{2\gamma+1/2})\Big)$$

$$= -\frac{3r^2}{(r-1)f(0)}a + \frac{3r^2+2r-1}{r\sqrt{f(0)}}a^{1/2} + c\frac{(r-1)^2}{r\sqrt{f(0)}}a^{\gamma} \qquad \text{(A.1.8)}$$

$$- c\Big(2 - \frac{(r-1)^2}{r^2} + \frac{\gamma-1}{\gamma+1/2}\Big)\frac{r-1}{f(0)}a^{\gamma+1/2} + c\frac{r(4\gamma-1)}{2(\gamma+1/2)f(0)^{3/2}}a^{\gamma+1} + O(a^{2\gamma+1/2}).$$

To know at which order of a we can approximate we split out analysis into two cases according to the value of γ

1. $\gamma \in (0, 1/2)$,

2. $\gamma \in [1/2, \infty)$.

If $\gamma \in (0, 1/2)$, from (A.1.8)

$$\Big(- \frac{1}{f'(a)} - 1 + D\Big)^2 = c^2\frac{(r-1)^2}{c^{(-)2}_{1/2}}a^{2\gamma} + 2c\frac{(3r^2+2r-1)}{c^{(-)2}_{1/2}}a^{\gamma+1/2} + O(a^{2\gamma+1/2}). \qquad \text{(A.1.9)}$$

Substitute (A.1.9) into the following expression

$$\sqrt{\Big(- \frac{1}{f'(a)} - 1 + D\Big)^2 - 4\frac{1}{f'(a)}} = \Big(c^2\frac{(r-1)^2}{c^{(-)2}_{1/2}}a^{2\gamma} + \frac{4}{c^{(-)}_{1/2}}a^{1/2} + c\Big(- r^2(4\gamma-1)$$

$$- 4r(\gamma+1/2) + 2(\gamma+1/2)\Big)\frac{(r-1)(3r^2+2r-1)}{r^3(\gamma+1/2)f(0)^{3/2}}a^{\gamma+1/2} + O(a^{2\gamma+1/2})\Big)^{1/2}$$

$$= c\frac{r-1}{c^{(-)}_{1/2}}a^{\gamma}\Big(1 + 4\frac{c^{(-)}_{1/2}}{c^2(r-1)^2}a^{-(2\gamma-1/2)}$$

$$+ \Big(- r^2(4\gamma-1) - 4r(\gamma+1/2) + 2(\gamma+1/2)\Big)\frac{(3r^2+2r-1)}{c(\gamma+1/2)(r-1)^4c^{(-)}_{1/2}}a^{-\gamma+1/2} + O(a^{1/2})\Big)^{1/2}.$$

and by (A.1.5) we can Taylor expand

$$
\begin{aligned}
&\sqrt{\Big(-\frac{1}{f'(a)}-1+D\Big)^2-4\frac{1}{f'(a)}}\\
&=c\frac{r-1}{c^{(-)}_{1/2}}a^{\gamma}+\frac{2}{c(r-1)}a^{-\gamma+1/2}\\
&\quad+\Big(-r^2(4\gamma-1)-4r(\gamma+1/2)+2(\gamma+1/2)\Big)\frac{(3r^2+2r-1)}{(2\gamma+1)(r-1)^3c^{(-)2}_{1/2}}a^{1/2}+O(a^{\gamma+1/2}).
\end{aligned}
$$

In the end, putting all estimates together, we approximate (2.3.8)

$$
\begin{aligned}
&\frac{1}{\sqrt{m_2(\varepsilon)}}=\frac{1}{2}\Big|-\frac{1}{f'(a)}-1+D+\sqrt{\Big(-\frac{1}{f'(a)}-1+D\Big)^2-4\frac{1}{f'(a)}}\Big|\\
&=\frac{1}{2}\Big|\frac{3r^2+2r-1}{r\sqrt{f(0)}}a^{1/2}-c\frac{(r-1)^2}{r\sqrt{f(0)}}a^{\gamma}+c\frac{(r-1)^2}{r\sqrt{f(0)}}a^{\gamma}+\frac{2}{c(r-1)}a^{-\gamma+1/2}\\
&+\Big(-r^2(4\gamma-1)-4r(\gamma+1/2)+2(\gamma+1/2)\Big)\frac{(3r^2+2r-1)}{(2\gamma+1)(r-1)^3c^{(-)2}_{1/2}}a^{1/2}+O(a^{\gamma+1/2})\Big|\\
&=\frac{1}{2}\Big|\Big((2\gamma+1)\Big(2r-1+r(r-1)\sqrt{f(0)}\Big)+r^2(4\gamma-1)\Big)\frac{(3r^2+2r-1)}{(2\gamma+1)(r-1)^3c^{(-)2}_{1/2}}a^{1/2}\\
&\qquad\qquad+\frac{2}{c(r-1)}a^{-\gamma+1/2}+O(a^{\gamma+1/2})\Big|.
\end{aligned}
\tag{A.1.10}
$$

If $\gamma\in[1/2,\infty)$, from (A.1.8)

$$
\Big(-\frac{1}{f'(a)}-1+D\Big)^2=\frac{(3r^2+2r-1)^2}{r^2f(0)}a-2c\frac{(3r^2+2r-1)}{c^{(-)2}_{1/2}}a^{\gamma+1/2}+O(a^{\gamma+1}).\tag{A.1.11}
$$

Use (A.1.11) to obtain

$$
\begin{aligned}
&\sqrt{\Big(-\frac{1}{f'(a)}-1+D\Big)^2-4\frac{1}{f'(a)}}=\Big(\frac{4}{c^{(-)}_{1/2}}a^{1/2}+\frac{(3r^2+2r-1)^2}{r^2f(0)}a\\
&+c\Big(-r^2(4\gamma-1)-4r(\gamma+1/2)+2(\gamma+1/2)\Big)\frac{(r-1)(3r^2+2r-1)}{r^3(\gamma+1/2)f(0)^{3/2}}a^{\gamma+1/2}+O(a^{\gamma+1})\Big)^{1/2}\\
&=2\sqrt{c^{(-)}_{1/2}}a^{1/4}\Big(1+\frac{(3r^2+2r-1)^2}{4r(r-1)\sqrt{f(0)}}a^{1/2}\\
&\quad+c\Big(-r^2(4\gamma-1)-4r(\gamma+1/2)+2(\gamma+1/2)\Big)\frac{3r^2+2r-1}{4r^2(\gamma+1/2)f(0)}a^{\gamma}+O(a^{\gamma+1/2})\Big)^{1/2}.
\end{aligned}
$$

By (A.1.5)

$$
\begin{aligned}
&\sqrt{\Big(-\frac{1}{f'(a)}-1+D\Big)^2-4\frac{1}{f'(a)}}=2\sqrt{c^{(-)}_{1/2}}a^{1/4}+\frac{(3r^2+2r-1)^2}{2r^{3/2}\sqrt{r-1}f(0)^{3/4}}a^{3/4}\\
&+c\Big(-r^2(4\gamma-1)-4r(\gamma+1/2)+2(\gamma+1/2)\Big)\frac{(3r^2+2r-1)\sqrt{r-1}}{2r^{5/2}(\gamma+1/2)f(0)^{5/4}}a^{\gamma+1/4}+O(a^{\gamma+3/4})\Big).
\end{aligned}
$$

Finally, combining the estimates we have

$$\begin{aligned}\frac{1}{\sqrt{m_2(\varepsilon)}} &= \frac{1}{2}\Big| -\frac{1}{f'(a)} - 1 + D + \sqrt{\Big(-\frac{1}{f'(a)} - 1 + D\Big)^2 - 4\frac{1}{f'(a)}}\Big| \\ &= \frac{1}{2}\Big|2\sqrt{c^{(-)}_{1/2}}a^{1/4} + \frac{3r^2+2r-1}{r\sqrt{f(0)}}a^{1/2} + c\frac{r-1}{c^{(-)}_{1/2}}a^{\gamma} + O(a^{3/4})\Big|. \end{aligned} \tag{A.1.12}$$

Equation (2.3.19), follows from (A.1.10) and (A.1.12). □

Appendix B

B.1 Proof of the minimum point u^* in (4.5.17)

In this appendix section we want to find the solutions to

$$\begin{aligned}0 =& u^2 2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})] - up[(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+2s(1-e^{-\xi})] \\ &+ (e^{-\xi}-1)p((1-p)(s+t)-s).\end{aligned} \tag{B.1.1}$$

And prove that the correspondent result is a minimum point for the function

$$f(u) = sC_{\mathcal{B}}^{(u)}(\xi) - tC_{\mathcal{G}}^{(\frac{u-p}{u(1-p)})}(-\xi).$$

(B.1.1) is a second degree equation in u which has two real solutions if its $\Delta \geq 0$.

$$\begin{aligned}\Delta =& e^{2\xi}(1-p)^2(s+t)^2p^2 + e^{-2\xi}(1-p)^2(s+t)^2p^2 + 6(1-p)^2(s+t)^2p^2 \\ &- 8p(1-p)st - 4e^{-\xi}[p^2(1-p)^2(s+t)^2 - p(1-p)st] - 4e^{\xi}\{p^2(1-p)^2(s+t)^2 \\ &- p(1-p)st\} \\ =& [(1-p)p(s+t)(e^{\xi}+e^{-\xi}-2)]^2 + 4p(1-p)st(e^{\xi}+e^{-\xi}-2) \\ =& p(1-p)(e^{\xi}+e^{-\xi}-2)[(1-p)p(s+t)^2(e^{\xi}+e^{-\xi}-2)+4st] \\ =& 4p(1-p)(\cosh\xi-1)[(1-p)p(s+t)^2(\cosh\xi-1)+2st] \geq 0 \qquad \forall\xi \geq 0,\end{aligned}$$

since $\cosh\xi \geq 1$. So the two optimal solutions are given by

$$u^* = \frac{p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+2sp(1-e^{-\xi}) \pm \sqrt{\Delta}}{2s(2p-1+(1-p)e^{\xi}-pe^{-\xi})}.$$

To be proper candidates, these u^* have to satisfy two features

(1) $u^* \in (p,1]$,

(2) u^* have to be two minimum points.

We begin from checking if $u^* \in (p,1]$. We analyze the two solutions separately starting from the plus one.

$$\frac{p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+2sp(1-e^{-\xi})+\sqrt{\Delta}}{2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})]} > p$$

$$\frac{p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+2sp(1-e^{-\xi})(1-p)+\sqrt{\Delta}-p2s(1-p)(e^{\xi}-1)}{2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})]}>0$$
$$\frac{p(1-p)(t-s)(e^{\xi}+e^{-\xi}-2)+\sqrt{\Delta}}{2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})]}>0$$

The numerator is always positive while the denominator is positive if $\xi > 0$. The other bound

$$\frac{p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+2sp(1-e^{-\xi})+\sqrt{\Delta}}{2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})]}\leq 1$$
$$\frac{p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+\sqrt{\Delta}-2s(1-p)(e^{\xi}-1)}{2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})]}\leq 0.$$

The denominator is always positive for $\xi > 0$, therefore the overall fraction is negative if and only if

$$p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+\sqrt{\Delta}-2s(1-p)(e^{\xi}-1)\leq 0 \tag{B.1.2}$$

If $p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2) > 2s(1-p)(e^{\xi}-1)$ the numerator is automatically positive and so the all fraction is never less than zero. Thus, we treat the case $p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2) < 2s(1-p)(e^{\xi}-1)$ for which it is useful to know the hyperbolic function equality $2e^{\xi}\cosh\xi = e^{2\xi}+1$.

$$\begin{aligned}
&p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2) < 2s(1-p)(e^{\xi}-1)\\
&p(s+t)(\cosh\xi-1) < s(e^{\xi}-1)
\end{aligned}$$

where in the last inequality we have divide both sides by $1-p$. Substitute $\cosh\xi = (e^{\xi}+e^{-\xi})/2$ and divide both sides by $(e^{\xi}-1)$

$$\begin{aligned}
&p(s+t)e^{-\xi}(e^{\xi}-1)^2 < 2s(e^{\xi}-1)\\
&p(s+t)(1-e^{-\xi}) < 2s\\
&t < \Big(\frac{2}{p(1-e^{-\xi})}-1\Big)s.
\end{aligned}$$

If the above condition is satisfied, we can isolate on one side $\sqrt{\Delta}$ in (B.1.2) and square both sides

$$\begin{aligned}
&p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+\sqrt{\Delta}-2s(1-p)(e^{\xi}-1)\leq 0\\
&4(1-p)^2p^2(s+t)^2(\cosh\xi-1)^2+8stp(1-p)(\cosh\xi-1)\\
&\qquad\leq 4p^2(1-p)^2(s+t)^2(\cosh\xi-1)^2+4s^2(1-p)^2(e^{2\xi}+1-2e^{\xi})\\
&\qquad\qquad -8sp(1-p)^2(s+t)(\cosh\xi-1)(e^{\xi}-1)\\
&8stp(\cosh\xi-1)\leq 8s^2(1-p)e^{\xi}(\cosh\xi-1)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - 8sp(1-p)(s+t)(\cosh\xi - 1)(e^{\xi}-1)\\
&stp(e^{\xi} - p(e^{\xi}-1)) \leq s^2(1-p)e^{\xi} - s^2p(1-p)(e^{\xi}-1)\\
&stp(e^{\xi}(1-p)+p) \leq s^2(1-p)(e^{\xi}(1-p)+p)\\
&t \leq \frac{(1-p)}{p}s.
\end{aligned}
$$

It is immediate to verify that $\left(\frac{2}{p(1-e^{-\xi})} - 1\right)s > \frac{(1-p)}{p}s$ for every $\xi > 0$.

Now repeat the same computations for the minus solution

$$
\begin{aligned}
&\frac{p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2) + 2sp(1-e^{-\xi}) - \sqrt{\Delta}}{2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})]} > p\\
&\frac{p(1-p)(t-s)(e^{\xi}+e^{-\xi}-2) - \sqrt{\Delta}}{2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})]} > 0
\end{aligned}
$$

The numerator in this case is always negative therefore the all fraction is positive if $\xi < 0$. Therefore we automatically know that this solution is not acceptable.

It remains to see if u^* is a minimum point when $u^* \in (p, 1]$ is satisfied. Since computing the second derivative of the two logarithm generating functions is demanding it is quicker to study the sign of their first derivative since most of the calculus has already been done.

If $\xi > 0$

$$
\begin{aligned}
&s\frac{\partial C_{\mathcal{B}}^{(u)}(\xi)}{\partial u} - t\frac{\partial C_{\mathcal{G}}^{\left(\frac{u-p}{u(1-p)}\right)}(\xi)}{\partial u} > 0\\
&s\frac{e^{\xi}-1}{1+u(e^{\xi}-1)} - t\frac{p(p-1)(e^{-\xi}-1)}{u^2(1+p(e^{-\xi}-1)) - up[1+e^{-\xi}+p(e^{-\xi}-1)] + p^2e^{-\xi}} > 0.
\end{aligned}
$$

Find the least common multiple and treat the numerator and the denominator of the resulting fraction separately and restrict the analysis to the interval $u \in (p, 1]$.

The numerator is

$$
\begin{aligned}
N(u,\xi) =& u^2s[2p-1+e^{\xi}(1-p) - pe^{-\xi}] - up\{-2((1-p)(s+t)-s) + e^{\xi}(1-p)(s+t)\\
&- e^{-\xi}[s(1+p) - t(1-p)]\} + (e^{-\xi}-1)p((1-p)(s+t)-s) > 0.
\end{aligned}
$$

This is a parabola with upward concavity if $\xi > 0$ and downward concavity if $\xi < 0$. Hence

$$
N(u,\xi)\begin{cases} \geq 0 & \text{if } \xi > 0 \text{ and } u \in [u^*, 1],\\ < 0 & \text{if } \xi > 0 \text{ and } u \in (p, u^*), \end{cases}
$$

where $u^* = \frac{p(1-p)(s+t)(e^{\xi}+e^{-\xi}-2)+2sp(1-e^{-\xi})+\sqrt{\Delta}}{2s[(1-p)(e^{\xi}-1)+p(1-e^{-\xi})]}$.

The denominator is

$$
D(u,\xi) = [1-u+ue^{\xi}][u^2(1+p(e^{-\xi}-1)) - up[1+e^{-\xi}+p(e^{-\xi}-1)] + p^2e^{-\xi}] > 0.
$$

The first factor is always positive for this reason we study the sign of the parabola in the second factor. This parabola has upward concavity for every $\xi \geq 0$ and we compute its zeros

$$\begin{aligned} u_{\pm}^{**} &= \frac{p(1+e^{-\xi}+p(e^{-\xi}-1)) \pm \sqrt{p^2(1+e^{-\xi}+p(e^{-\xi}-1))^2 - 4p^2e^{-\xi}(1+p(e^{-\xi}-1))}}{2(1+p(e^{-\xi}-1))} \\ &= \frac{p[1+e^{-\xi}+p(e^{-\xi}-1) \pm \sqrt{(1+e^{-\xi})^2 - 4e^{-\xi} + p^2(e^{-\xi}-1)^2 - 2p(e^{-\xi}-1)^2}]}{2(1+p(e^{-\xi}-1))} \\ &= \frac{p[1+e^{-\xi}+p(e^{-\xi}-1) \pm (e^{-\xi}-1)(1-p)]}{2(1+p(e^{-\xi}-1))} \end{aligned}$$

from which we obtain $u_{-}^{**} = p$, $u_{+}^{**} = \frac{pe^{-\xi}}{1-p+pe^{-\xi}}$. We already know from the previous part that $u_{+}^{**} < p$ if $\xi > 0$. Therefore we have

$$D(u,\xi) > 0 \qquad \text{if } \xi > 0 \text{ and } u \in (p,1].$$

This means than u^* is a minimum point if $\xi > 0$.