ON DECOMPOSITIONS OF FINITE PROJECTIVE PLANES AND THEIR APPLICATIONS

By

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Abstract

Let PG$(2, q)$ be the projective plane over the field $\mathbb{F}_q$. Singer [19] notes that PG$(2, q)$ has a cyclic group of order $q^2 + q + 1$ that permutes the points of PG$(2, q)$ in a single cycle. A k-arc is a set of k points no three of which are collinear. A k-arc is called complete if it is not contained in a $(k + 1)$-arc of PG$(2, q)$.

By taking the orbits of points under a proper subgroup of a single cycle, one can decompose the projective plane PG$(2, q^k)$ into disjoint copies of subplanes isomorphic to PG$(2, q)$ if and only if k is not divisible by three. Moreover, by taking the orbits of points under a proper subgroup, one can decompose the projective plane PG$(2, q^2)$ into disjoint copies of complete $(q^2 - q + 1)$-arcs. In this thesis, our main problem is to classify (up to isomorphism) the different types of decompositions of PG$(2, q^2)$ for $q = 3, 4, 5, 7$ into particular subgeometries, primarily subplanes and arcs. By using the computer program GAP, we provide a detailed mathematical investigation into particular subgeometries of PG$(2, q^2)$ for $q = 3, 4, 5, 7$, namely subplanes and arcs. We further illustrate some of the connections between these subgeometry decompositions and other areas of combinatorial interest; in particular, we explain the relationship between coding theory and projective spaces and describe the links with Hermitian unital. Furthermore, projective codes are obtained by taking the disjoint union of such subgeometries.

Dedication

To my beloved

Mother

Father

Wife

Sons

Daughters

Brothers

Sisters

Declaration

I hereby declare and confirm that the work presented in this thesis is entirely my own unless otherwise stated, and has been not submitted for examination, in whole or in part, to this or any other university.

Signature:

Date:

Acknowledgment

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Contents

Chapter 1

Introduction

In the first section of this chapter, we give an overview of the results in this thesis. The remainder of this chapter will introduce the necessary concepts from group theory, finite field theory and design theory that are needed for a full understanding of later chapters. The content of this chapter is based on the following standard references: [1],[3], [5], [6], [7],[8], [9], [10], [12],[13],[14] [15], [16], [17] ,[19] and [20].

1.1 Introduction

A projective plane of order q, denoted by $\mathrm{PG}(2, q)$, consists of a set of $q^2 + q + 1$ points and a set of $q^2 + q + 1$ lines, where each line contains exactly $q + 1$ points and two distinct points lie on exactly one line. It follows from the definition that each point is contained in exactly $q + 1$ lines and two distinct lines have exactly one common point. A k-arc $\mathcal{K}$ in the projective plane $\mathrm{PG}(2, q)$ is a set of k points no three of which are collinear. A k-arc is called complete if it is not contained in a $(k + 1)$-arc of $\mathrm{PG}(2, q)$.

By taking the orbits of points under a proper subgroup, one can decompose the projective plane $\mathrm{PG}(2, q^k)$ into disjoint copies of subplanes isomorphic to $\mathrm{PG}(2, q)$ if and only if k is not divisible by three [10, Corollary 4.31]. The idea of decomposing $\mathrm{PG}(2, q^2)$ into disjoint subplanes is essentially due to Bruck [6]. His decomposition is obtained by using Singer's [19] result that every finite projective plane is cyclic. Throughout this thesis, the decompositions that arise as a collection of orbits under an appropriate subgroup of the Singer group will refer to as classical. Yff [20] reports that there are other types of decompositions in cases $q = 3, 4, 5, 7$. An illustration in case of $\mathrm{PG}(2, 3^2)$ is given but, for $q = 4, 5, 7$, no proof or list provided. He gave an example of a non-classical subplane decomposition of $\mathrm{PG}(2, 3^2)$. Additionally, Mathon and Hamilton [17] show using a computer search that there are other types of decompositions in the cases $q = 3, 4, 5$. They approved Yff's result that there is only one non-classical subplane decomposition of $\mathrm{PG}(2, 3^2)$ and found two non-classical subplane decompositions of $\mathrm{PG}(2, 4^2)$ and found two non-classical subplane decompositions of $PG(2, 5^2)$.

Similarly, Boros and Szönyi [3, Theorem 1.1] show that by taking the orbits of points under a proper subgroup, one can decompose the projective plane $PG(2, q^2)$ into disjoint copies of complete $(q^2 - q + 1)$-arcs. The idea of decomposing $PG(2, q^2)$ into disjoint arcs is essentially due to Kestenband [13, 14] where he studied the arcs as one of the possible categories of the intersection of two Hermitian curves in $PG(2, q^2)$. The cyclic arcs were then investigated in details by Boros and Szönyi [3], Fisher, Hirschfeld, and Thas [7].

In this thesis, our main problem is to classify (up to isomorphism) the different types of decompositions of $PG(2, q^2)$ for $q = 3, 4, 5, 7$ into particular subgeometries, primarily subplanes and arcs. By using the technique described in Section 2.4 together with the computer program GAP [8], decompositions of $PG(2, q^2)$ for $q = 3, 4, 5, 7$ into subplanes and arcs are examined. We provide a detailed mathematical investigation into these class of subgeometries. In this dissertation, some applications are also discussed; in particular, the geometrical objects considered in this work give specific linear codes. By taking the union of i such subgeometries, a projective code, which is a linear code, is obtained for all i. Furthermore, we demonstrate the relationship between coding theory and projective spaces and describe the links with Hermitian unital.

The structure of the thesis is as follows.

The next three sections of this chapter introduce a collection of mathematical foundations for subsequent chapters. The purpose is to introduce the basic concepts and results without presenting proofs. In Section 1.2, we review and provide the basic background and definitions from group theory. Additionally, Section 1.3 presents a brief introduction of finite fields which play an important role in this thesis. Moreover, the fourth section of chapter one addresses some elementary definitions and properties of combinatorial designs and their relationship to projective planes.

Chapter 2 collects the necessary ideas and notations in the first section making the general background of this thesis which includes definitions and properties of projective spaces. In Section 2.2, we present cyclic projectivity that illustrates the structure of projective spaces over finite fields. Also, the third section of chapter two provides the definitions and main results in decompositions of $PG(2, q^2)$ into particular subgeometries, primarily subplanes and arcs. Furthermore, Section 2.4 is devoted to describing fully an adapted version of the algorithm that is introduced in [9] and [16] which is used for finding the decompositions.

Chapter 3 discusses the different types of decompositions of the projective plane $PG(2, 3^2)$ into disjoint subplanes of order three and disjoint 7-arcs. The first section of chapter three presents two non-isomorphic decompositions of $PG(2, 3^2)$ into disjoint subplanes of order three where the first partition is the classical one, and the other is Yff's result [20]. Also, in section 3.2, it is shown that in $\mathrm{PG}(2,3^2)$ there are, up to isomorphism, two decompositions into disjoint 7-arcs where the first partition is the classical one, and the other decomposition is new.

Chapter 4 discusses the different types of decompositions of the projective plane $\mathrm{PG}(2,4^2)$ into disjoint subplanes of order two, disjoint subplanes of order four and disjoint 13-arcs. In Section 4.1, we show four non-isomorphic decompositions of $\mathrm{PG}(2,4^2)$ into disjoint subplanes of order two where the first partition is the classical one, and the other three decompositions are new. Moreover, the second section of chapter four points that, in $\mathrm{PG}(2,4^2)$, there are, up to isomorphism, three types of decompositions into disjoint subplanes of order four where the first partition is the classical one, and the other two decompositions are presented by Mathon and Hamilton in [17]. Additionally, Section 4.3 provides three non-isomorphic decompositions of $\mathrm{PG}(2,4^2)$ into disjoint 13-arcs where the first partition is the classical one, and the other two decompositions are new.

Chapter 5 presents the different types of decompositions of the projective plane $\mathrm{PG}(2,5^2)$ into disjoint subplanes of order five and disjoint 21-arcs. The first section of chapter five shows three non-isomorphic decompositions of $\mathrm{PG}(2,5^2)$ into disjoint subplanes of order five where the first partition is the classical one, and the other two decompositions are given by Mathon and Hamilton in [17]. Also, Section 5.2 points that, in $\mathrm{PG}(2,5^2)$, there are, up to isomorphism, three decompositions into disjoint 21-arcs where the first partition is the classical one, and the other two decompositions are new.

Chapter 6 introduces the different types of decompositions of the projective plane $\mathrm{PG}(2,7^2)$ into disjoint subplanes of order seven and disjoint 43-arcs. In Section 6.1, we show four non-isomorphic decompositions of $\mathrm{PG}(2,7^2)$ into disjoint subplanes of order seven where the first partition is the classical one, and the other three decompositions are new. Moreover, the second section of chapter six shows that, in $\mathrm{PG}(2,7^2)$, there are, up to isomorphism, three decompositions into disjoint 43-arcs where the first partition is the classical one, and the other two decompositions are new.

In Chapter 7, we explain the relationship between coding theory and projective spaces and describe the links with Hermitian unital. Also, projective codes are obtained by taking the union of i such subgeometries.

1.2 Group theory

In this section, we review and provide the basic background from group theory that is needed for this thesis. The main reference used in this section is [12].

Definition 1.1 (Group [12, Definition 1.1]). A group is a non-empty set G together with a binary operation $\times$ on G such that

(i) if a and b belong to G then $a \times b$ is also in G (closure);

(ii) $a \times (b \times c) = (a \times b) \times c$ for all $a, b, c \in G$ (associativity);

(iii) there is an element $e \in G$ such that $a \times e = e \times a = a$ for all $a \in G$ (identity);

(iv) if $a \in G$, then there is an element $a^{-1} \in G$ such that $a \times a^{-1} = a^{-1} \times a = e$ (inverse).

Definition 1.2 (Finite group [12, Definition 1.2]). A group G is called finite if the underlying set contains a finite number of elements. The order of G is the number of elements in its set and the order of $g \in G$ is the smallest positive integer m such that $g^m = e$.

Definition 1.3 (Abelian group [12, Definition 1.3]). A group G is called abelian if the binary operation is commutative; that is,

$$a \times b = b \times a, \quad \text{for all } a, b \in G.$$

Definition 1.4 (Subgroup [12, Definition 4.1]). A subgroup H of a group G is a non-empty subset of G that forms a group under the binary operation of G.

Let S and T be finite sets that are closed under some binary operations; probably groups or fields.

Definition 1.5 (Homomorphism [12, Definition 8.1]). A map $\phi : S \longrightarrow T$ is called a homomorphism of S into T if and only if

$$(ab)\phi = (a)\phi(b)\phi, \quad \text{for all } a, b \in S.$$

Definition 1.6 (Isomorphism [12, Definition 8.8]). A homomorphism $\phi : S \longrightarrow T$ is called an isomorphism of S into T if and only if ϕ is a bijection. Two sets S and T are called isomorphic if and only if there exists an isomorphism between them; such a relationship is denoted by $S \cong T$.

Definition 1.7 (Automorphism [12, Definition 8.10]). An isomorphism of S into itself is called an automorphism of S.

The automorphism group of S, denoted by $Aut(S)$, is the set of all automorphisms of S together with the operation of composition of maps. Consequently, for $\phi_1, \phi_2 \in Aut(S)$,

$$(s)\phi_1\phi_2 = ((s)\phi_1)\phi_2, \quad \text{for all } s \in S.$$

The identity of $Aut(S)$ is e, where $(s)e = s$, for all $s \in S$ and for every $\phi \in Aut(S)$ there exists an $\phi^{-1} \in Aut(S)$ such that

$$((s)\phi)\phi^{-1} = ((s)\phi^{-1})\phi = s, \quad \text{for all } s \in S.$$

Let G be a finite group that contains an element g of order m. The subgroup C_m is generated by a single element of G; that is,

$$C_m = \langle g \rangle = \{1, g, \dots, g^{m-1}\}.$$

Multiplication is determined by the rule

$$g^a g^b = g^{(a+b)} \mod m, \quad \text{for all } 0 \leq a, b \leq m-1.$$

The identity is 1 and

$$(g^a)^{-1} = g^{m-a}, \quad \text{for all } 1 \leq a \leq m-1.$$

The group G is called the cyclic group of order m if and only if $G = C_m$. The cyclic group of order m is isomorphic to the additive group of integers modulo m; that is, $C_m \cong \mathbb{Z}_m$, see [12, Example 1.6].

Definition 1.8 (Direct product [12, Example 1.10]). The direct product of the set of groups $G_1, G_2, \dots, G_n$, denoted by

$$G = G_1 \times G_2 \times \cdots \times G_n,$$

is the set

$$\{(g_1, g_2, \dots, g_n) \mid g_i \in G_i\}.$$

The set G together with the composition law

$$(g_1, g_2, \dots, g_n)(h_1, h_2, \dots, h_n) = (g_1h_1, g_2h_2, \dots, g_nh_n)$$

form a group with identity element

$$(e_{G_1}, e_{G_2}, \dots, e_{G_n})$$

and inverse

$$(g_1, g_2, \dots, g_n)^{-1} = (g_1^{-1}, g_2^{-1}, \dots, g_n^{-1}).$$

Proposition 1.1 ([12, Proposition 13.5]). Let $H_1, H_2, \dots, H_n$ be subgroups of the group G such that

(i) $h_i h_j = h_j h_i$, where $h_i \in H_i, h_j \in H_j$ and $i \neq j$;

(ii) $G = H_1 H_2 \cdots H_n$;

(iii) $H_i \cap H_1 H_2 \cdots H_{i-1} H_{i+1} \cdots H_n = e$.

Then G is isomorphic to $H_1 \times H_2 \times \cdots \times H_n$.

The order of the direct product is given by

$$|H_1 \times H_2 \times \cdots \times H_n| = \prod_{i=1}^{n} |H_i|.$$

Definition 1.9 (Normal subgroup [12, Definition 7.3]). A subgroup H of G is called normal if and only if $gH = Hg$ for any $g \in G$.

Definition 1.10 (Semidirect product [12, Definition 19.1]). A group G with subgroups N and H such that

(i) N is normal subgroup of G,

(ii) $G = NH$,

(iii) $N \cap H = e$,

is called a semidirect product of N by H, denoted by $N \rtimes H$. The order of G is $|N| \times |H|$.

Definition 1.11 (Group action [12, Definition 10.1]). Let G be a group with identity e and S be a non-empty set. A (right) action of G on S is a function $\phi : S \times G \longrightarrow S$ such that

(i) $\phi(x, e) = x \quad$ for all $x \in S$,

(ii) $\phi(\phi(x, g_1), g_2) = \phi(x, g_1 g_2) \quad$ for all $g_1, g_2 \in G$ and $x \in S$.

We shall generally write xg for $\phi(x, g)$, except where this leads to ambiguities, or where other notation is more convenient.

Definition 1.12 (Orbit and Stabiliser [12, Definitions 10.8 and 10.14]). Let G be a group acting on a set S with an action $(s, g) \mapsto sg$ for $g \in G$ and $s \in S$. Let x be an element of S.

(i) The orbit of an element x of S is the set of elements in S which are the images of x under elements of G. It is denoted by $\mathrm{Orb}_G(x)$.

$$\mathrm{Orb}_G(x) = \{xg \mid g \in G\}.$$

(ii) The stabiliser of an element x of S is the set of elements of G which send x to itself. It is denoted by $\mathrm{Stab}_G(x)$.

$$\mathrm{Stab}_G(x) = \{g \in G \mid xg = x\}.$$

(iii) The group G is said to act transitively on S if, given any two elements x, y in S, there exists g in G such that $y = xg$. In that case, there is only one orbit. The action is regular if it is transitive and $\mathrm{Stab}_G(x) = \{e\}$ for every x in S.

Remark 1.1 ([12, Proposition 10.9]). The orbit of x is a subset of S, whereas the stabiliser of x is a subgroup of G.

Theorem 1.1 (Orbit-Stabilizer Theorem [12, Theorem 10.16]). Let G be a finite group acting on a set S, and let $x \in S$. Then

$$|G| = |\mathrm{Orb}_G(x)||\mathrm{Stab}_G(x)|.$$

1.3 Finite fields

Our objective of this section is to introduce necessary notations and results of finite fields. We will recall and summarise them without providing proofs. The main reference used in this section is [10].

Definition 1.13 (Field [10, Section 1.1]). A field is a set F closed under two operations $+, \times$ such that

(i) $(F, +)$ is an abelian group with identity 0;

(ii) $(F\backslash\{0\}, \times)$ is an abelian group with identity 1;

(iii) for all $x, y, z \in F$,

$$x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz.$$

Definition 1.14 (Finite field [10, Section 1.1]). A finite field is a field with only a finite number of elements. The smallest positive integer p, and hence a prime, such that

$$px = \underbrace{x + x + \cdots + x}_{p \text{ times}} = 0 \quad \text{for all } x \in F$$

is called the characteristic.

A finite field, with p elements such that p is a prime, denoted by $\mathbb{F}_p$ consists of the residue classes of the integers modulo p under the natural addition and multiplication, see [10, Section 1.1].

Definition 1.15 (Construction of $\mathbb{F}_{p^h}$ [10, Section 1.1]). Given an irreducible polynomial $f(x)$ of degree h over $\mathbb{F}_p$, we define the field $\mathbb{F}_q$, where $q = p^h$, for some prime p, and some positive integer h, as

$$\begin{aligned} \mathbb{F}_q &= \mathbb{F}_p[x]/\langle f(x)\rangle \\ &= \{a_0 + a_1 x + \cdots + a_{h-1}x^{h-1} \mid a_i \in \mathbb{F}_p; f(x) = 0\}. \end{aligned}$$

Definition 1.16 (Automorphism of $\mathbb{F}_q$ [10, Section 1.2]). An automorphism σ of $\mathbb{F}_q$ is a permutation of the elements of $\mathbb{F}_q$ such that

(i) $(x + y)\sigma = x\sigma + y\sigma$,

(ii) $(xy)\sigma = (x\sigma)(y\sigma)$,

for all x, y in $\mathbb{F}_q$.

Theorem 1.2 (Properties of finite fields [10, Sections 1.1 and 1.2]). *Let $q = p^h$ for some prime p and some positive integer h. The field $\mathbb{F}_q$ has the following properties.*

(i) $(x + y)^p = x^p + y^p$ *for all* $x, y \in \mathbb{F}_q$.

(ii) $x^q = x$ *for all* $x \in \mathbb{F}_q$.

(iii) *There exists* $\mu \in \mathbb{F}_q$ *such that*

$$\mathbb{F}_q = \{0, 1, \mu, \ldots, \mu^{q-2} \mid \mu^{q-1} = 1\}.$$

Such a μ *is called a primitive element of* $\mathbb{F}_q$.

(iv) *Under multiplication,* $\mathbb{F}_q\backslash\{0\}$ *is a cyclic group of order* $q - 1$:

$$\mathbb{F}_q\backslash\{0\} \cong \mathbb{Z}_{q-1}.$$

(v) *The additive structure of* $\mathbb{F}_q$ *is given by the group isomorphism*

$$\mathbb{F}_q \cong \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{h \text{ factors}}.$$

(vi) *If* F_1, F_2 *are finite fields such that* $F_1 \subset F_2$*, then* $|F_1|$ *divides* $|F_2|$.

(vii) The automorphism group of $\mathbb{F}_q$ is

$$Aut(\mathbb{F}_q) = \{1, \varphi, \ldots, \varphi^{h-1}\} \cong \mathbb{Z}_h,$$

where $\varphi(x) = x^p$.

Definition 1.17 (Companion matrix [10, Section 2.4]). Let $f(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_0$ be a polynomial of degree n over $\mathbb{F}_q$. Then its companion matrix $C(f)$ is the $n \times n$ matrix given by

$$C(f) = \begin{pmatrix} 0 & & & \\ \vdots & & \mathrm{I}_{n-1} & \\ 0 & & & \\ a_0 & a_1 & \cdots & a_{n-1} \end{pmatrix},$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. $f(x)$ is the characteristic polynomial of the matrix $C(f)$.

Definition 1.18 (Primitive and subprimitive polynomials [10, Section 1.6]). Let $f(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_0$ be an irreducible polynomial of degree n over $\mathbb{F}_q$. Then

(i) $f(x)$ is called primitive in $\mathbb{F}_q$ if the smallest positive integer m such that $f(x)$ divides $x^m - 1$ is equal to $q^n - 1$.

(ii) $f(x)$ is called subprimitive in $\mathbb{F}_q$ if the smallest positive integer m such that $f(x)$ divides $x^m - c$ for some $c \in \mathbb{F}_q \backslash \{0\}$ is equal to $q^{n-1} + \cdots + q + 1$.

1.4 Design theory

In this section, we discuss some basic definitions and properties of combinatorial designs and their relationship to projective planes. The main references used in this section are [1], [5] and [15].

Definition 1.19 (t-design [1, Definition 1.1.1]). A t-(v, k, λ) design, or in short a t-design, is a pair (P, B) where P is a set of v elements, called points, and B is a collection of distinct subsets of P of size k, called blocks, such that each subset of points of size t is contained in exactly λ blocks.

Definition 1.20 (Symmetric design [1, Section 1.1]). A 2-(v, k, λ) design is called a symmetric design if the number of blocks is equal to the number of points.

Definition 1.21 (Difference set [1, Definition 2.1.1]). A (v, k, λ) difference set modulo v or, a cyclic (v, k, λ) difference set is a set $D = \{d_1, \ldots, d_k\}$ of distinct elements of $\mathbb{Z}_v$ such that each non-zero d in $\mathbb{Z}_v$ can be expressed in the form $d = d_i - d_j$ in precisely λ ways.

Difference sets that seem to have been first investigated by Kirkman [15] are a useful tool in the construction of symmetric designs. He considered the case $\lambda = 1$ and noticed that they form $(q^2 + q + 1, q + 1, 1)$ designs for $q = 2, 3, 4, 5$. The generalisation of the cyclic difference set concept to more general groups is due to Bruck [5].

Lemma 1.1 ([1, Lemma 2.1.1]). If D is a difference set in the group G, then the parameters (v, k, λ) satisfy the relation

$$k(k - 1) = \lambda(v - 1).$$

Proof. (Proof from [1, Lemma 2.1.1]). There are $k(k - 1)$ differences arising from the k elements of the difference set, and they represent each of the $v - 1$ non-zero integers modulo v, λ times each. ∎

Remark 1.2 ([1, Theorem 2.1.5]). The set-theoretic complement of D in the group G is also a difference set in G, properly called the complementary difference set of D and has parameters $(v, v - k, v - 2k + \lambda)$.

Definition 1.22 (Translate [1, Definitions 2.1.2 and 2.5.1]). Let G be a group. For any element $g \in G$, the set

$$Dg := \{dg \mid d \in D\}$$

is called the translate of D and it is also a difference set with the same parameters as D. Two difference sets D_1 and D_2 in G are equivalent if there is some automorphism of G mapping D_1 onto a translate of D_2, and such an automorphism is called a multiplier.

Definition 1.23 (Finite projective plane [1, Definition 1.1.2]). A $(q^2 + q + 1, q + 1, 1)$ design, which is symmetric, is called a finite projective plane of order q.

In summary, this chapter gathers some necessary concepts, background and basic definitions from group theory, finite field theory and design theory that we need throughout this thesis. For example, Orbit-Stabilizer Theorem is used in Chapter 2 to prove some corollaries, difference sets are used to represent the points and lines of the projective plane in Section 2.2, and an associated companion matrix of the subprimitive polynomial is employed in Section 2.2 to describe the cyclic projectivity.

Chapter 2

Galois geometries

In this chapter, formal definitions and properties of projective spaces are given in the first section. Section 2.2 deals with cyclic projectivity that illustrates the structure of projective spaces over finite fields. In the third section, we provide the definitions and main results in decompositions of $\mathrm{PG}(2, q^2)$ into particular subgeometries such as subplanes and arcs. The fourth section is devoted to describing the procedure that is used for finding the decompositions of $\mathrm{PG}(2, q^2)$.

This chapter is mainly based on the following references: [2],[3], [6],[7], [8], [9], [10], [13], [14], [16], [17], [19] and [20].

2.1 Projective spaces and general properties

Let $\mathrm{V} = \mathrm{V}(n+1, q)$ be an $(n+1)$-dimensional vector space over the finite field $\mathbb{F}_q$, where $q = p^h$. Then any two vectors $X = (x_0, \ldots, x_n)$ and $Y = (y_0, \ldots, y_n)$ of $\mathrm{V}\backslash\{0\}$ are called equivalent if $Y = tX$, for some $t \in \mathbb{F}_q\backslash\{0\}$. The equivalence class of a vector $X = (x_0, \ldots, x_n)$ of $\mathrm{V}\backslash\{0\}$ is the subset of $\mathrm{V}\backslash\{0\}$ given by $\{tX = (tx_0, \ldots, tx_n) \mid t \in \mathbb{F}_q\backslash\{0\}\}$.

Definition 2.1 (Finite projective space [10, Section 2.1.1]). The n-dimensional projective space over the finite field $\mathbb{F}_q$, denoted by $\mathrm{PG}(n, q)$, is the set consisting of the equivalence classes of vectors of the $(n+1)$-dimensional vector space $\mathrm{V}(n+1, q)$.

The elements of $\mathrm{PG}(n, q)$ are called points and the point, denoted by $P(X)$, is the equivalence class of the vector X. Since $P(tX) = P(X)$, for all $t \in \mathbb{F}_q\backslash\{0\}$, if the first non-zero component of tX is a one, $tX = (0, \ldots, 0, 1, \ldots)$, then for consistency this vector is used to represent the point $P(X)$. The number of points in $\mathrm{PG}(n, q)$ is $\theta(n, q) = \frac{q^{n+1}-1}{q-1}$.

Definition 2.2 (Subspace [10, Section 2.1.1]). The subspaces of the projective space $\mathrm{PG}(n, q)$ are defined as follows:

(i) the (-1)-dimensional subspace of $\mathrm{PG}(n, q)$ is the empty set;

(ii) the points or 0-dimensional subspaces of $\mathrm{PG}(n, q)$ are the 1-dimensional subspaces of V;

(iii) the lines or 1-dimensional subspaces of $\mathrm{PG}(n, q)$ are the 2-dimensional subspaces of V;

(iv) the planes or 2-dimensional subspaces of $\mathrm{PG}(n, q)$ are the 3-dimensional subspaces of V;

(v) the hyperplanes or $(n-1)$-dimensional subspaces of $\mathrm{PG}(n, q)$ are the n-dimensional subspaces of V.

Definition 2.3 (Incidence structure [10, Section 2.3]). An incidence structure $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ consists of a set of objects $\mathcal{P}$ called points, and a second set of objects $\mathcal{B}$ called blocks together with a subset $\mathcal{I}$ of $\mathcal{P} \times \mathcal{B}$ called incidence relation such that if $P \in \mathcal{P}$, $l \in \mathcal{B}$ and $P \in l$, then P is incident with l, or l is incident with P.

A k-dimensional subspace or k-space of $\mathrm{PG}(n, q)$ is denoted by $\prod_k$. A k-dimensional subspace of $\mathrm{PG}(n, q)$ is incident with a $(k + j)$-dimensional subspace of $\mathrm{PG}(n, q)$ if the corresponding $(k + 1)$-dimensional subspace of V is contained in the corresponding $(k + j + 1)$-dimensional subspace of V.

The points $P(X_0), \ldots, P(X_m)$ are linearly independent if and only if the vectors $X_0, \ldots, X_m$ are linearly independent; that is, no vector can be expressed as a linear combination of the others. Hence, the point $P(X_j)$ is incident with the line through the points $P(X_0)$ and $P(X_1)$ if and only if the vector X_j is a linear combination of the vectors X_0 and X_1. In addition, the points $P(X_0), \ldots, P(X_m)$ are collinear if they are all incident with the same line.

The 2-dimensional subspace of V containing the vectors X and Y is

$$\{aX + bY \mid a, b \in \mathbb{F}_q\}$$

and

$$P(aX + bY) = \begin{cases} P(X + ba^{-1}Y) & \text{if} \quad a \neq 0, \\ P(Y) & \text{if} \quad a = 0. \end{cases}$$

Therefore, the line through the points $P(X)$ and $P(Y)$ in $\mathrm{PG}(n, q)$ is the set of points

$$\{P(X + cY) \cup P(Y) \mid c \in \mathbb{F}_q\}.$$

Remark 2.1 The 2-dimensional projective space $\mathrm{PG}(2, q)$ is called the projective plane. It contains $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Every point of $\mathrm{PG}(2, q)$ is incident with $q + 1$ lines and every line in $\mathrm{PG}(2, q)$ is incident with $q + 1$ points. The points of $\mathrm{PG}(2, q)$ have the form $P(0, 0, 1)$, $P(0, 1, x_2)$ or $P(1, x_1, x_2)$, where $x_1, x_2 \in \mathbb{F}_q$.

Theorem 2.1 ([2, Theorem 1.5.3]). Let S be a k-dimensional subspace of the finite projective space $\mathrm{PG}(n, q)$, where $1 \leq k \leq n$. Then the following facts hold:

(i) the number of points of S is

$$q^k + q^{k-1} + \cdots + q + 1;$$

(ii) if s is a point of S, then in S, the number of lines through the point s is

$$q^{k-1} + \cdots + q + 1;$$

(iii) the number of lines in S is

$$\frac{(q^k + q^{k-1} + \cdots + q + 1)(q^{k-1} + \cdots + q + 1)}{q + 1}.$$

Proof. (Proof from [2, Theorem 1.5.3]).

(i) A $(k + 1)$-dimensional subspace of the vector space V, given by

$$\{a_0X_0 + a_1X_1 + \cdots + a_kX_k \mid a_i \in \mathbb{F}_q\},$$

contains $q^{k+1} - 1$ non-zero vectors; hence, there are

$$\frac{q^{k+1} - 1}{q - 1} = q^k + q^{k-1} + \cdots + q + 1$$

points of S.

(ii) Let s be a point that is incident with S, then there are $q^k + \cdots + q$ additional points that are incident with S. Since each line in S that is incident with P is incident with $q + 1$ points of S, in S there are

$$\frac{q^k + \cdots + q}{q} = q^{k-1} + \cdots + q + 1$$

lines through the point s.

(iii) The subspace S consists of $q^k + \cdots + q + 1$ points and every point of S is incident with $q^{k-1} + \cdots + q + 1$ lines in S, but every line in S is incident with $q + 1$ points of S. Hence, the number of lines in S is

$$\frac{(q^k + q^{k-1} + \cdots + q + 1)(q^{k-1} + \cdots + q + 1)}{q + 1}.$$

∎

Definition 2.4 (Collineation [10, Section 2.1.1]). If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two projective spaces PG(n, q), then a collineation of PG(n, q) is a bijection $\mathcal{T} : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ that preserves incidence structures.

Definition 2.5 (Projectivity [10, Section 2.1.1]). A collineation $\mathcal{T} : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is called a projectivity of PG(n, q) if it is induced by a linear map of V$(n + 1, q)$. That is, $\mathcal{T}$ maps $P(X)$, a point of $\mathcal{S}_1$, onto $P(Y)$, a point of $\mathcal{S}_2$, if and only if $tY = Xg$, where $t \in \mathbb{F}_q\backslash\{0\}$ and g is a non-singular matrix. Such a projectivity is denoted by $(\mathcal{T}, g)$ or g if the map $\mathcal{T}$ is not stated; $(\mathcal{T}, g) = (\mathcal{T}, ag)$, for all $a \in \mathbb{F}_q\backslash\{0\}$.

Let $x = P(X)$ be a point of PG(n, q). Then the result of the right action of a matrix g on x, denoted by x^g, is the point $P(Xg)$. For a projectivity $(\mathcal{T}, g)$ of PG(n, q), if $\prod_{k_1} \subset \prod_{k_2}$, then $\prod_{k_1} g \subset \prod_{k_2} g$, since incidence structures are preserved.

Example 2.1 Let $\mathcal{T}$: PG$(2, q) \longrightarrow$ PG$(2, q)$ be a projectivity. Then $\mathcal{T}$ maps the line

$$L(X, Y) = \{P(Z) \mid Z \in \{X + aY \mid a \in \mathbb{F}_q\} \cup Y\}$$

to the line

$$L(Xg, Yg) = \{P(Z) \mid Z \in \{Xg + aYg \mid a \in \mathbb{F}_q\} \cup Yg\}.$$

Definition 2.6 (Frame [10, Section 2.1.4]). A set of $n + 2$ points of PG(n, q) is called a frame if and only if the vectors representing $n+1$ of these points form a basis of V$(n+1, q)$; that is, the vectors are linearly independent and every vector in V$(n + 1, q)$ is a linear combination of this set.

Therefore, in PG(n, q), no hyperplane is incident with more than n points of a frame. In PG$(2, q)$, no line is incident with more than 2 points of a frame. Let the set of $n + 1$ vectors $X_0, \ldots, X_n$ form a basis of V$(n+1, q)$. Then the set of $n+2$ points $P(X_0), \ldots, P(X_n)$, $P(X_0 + \cdots + X_n)$ is a frame of PG(n, q). The standard frame is

$$\begin{aligned} e_0 &= P(1, 0, \ldots, 0), \\ &\vdots \\ e_n &= P(0, \ldots, 0, 1), \\ e_{n+1} &= P(1, \ldots, 1, 1). \end{aligned}$$

Theorem 2.2 ([2, Theorem 3.6.2]). For $(\mathcal{T}, g)$ a projectivity and $\mathcal{P}$ a projective space PG(n, q), if the projectivity $\mathcal{T} : \mathcal{P} \longrightarrow \mathcal{P}$ maps every point of a frame to itself, then $\mathcal{T}$ is the identity mapping $\mathcal{I}$ or $g = I$. This projectivity is denoted by $(\mathcal{I}, I)$.

Proof. (Proof from [2, Theorem 3.6.2]). Let $P(X_i)$ be a point of P. Then $X_i g = a_i X_i$, where $a_i \in \mathbb{F}_q\backslash\{0\}$, for all $i = 0, \ldots, n$; thus,

$$(X_0 + \cdots + X_n)g = a_0 X_0 + \cdots + a_n X_n.$$

However, it is correspondingly true that

$$(X_0 + \cdots + X_n)g = a(X_0 + \cdots + X_n) = aX_0 + \cdots + aX_n.$$

Therefore, $g = aI$. ∎

Corollary 2.1 ([2, Corollary 3.6.3]). Let the $n+2$ points $P(X_0), \ldots, P(X_n)$, $P(X_{n+1} = X_0 + \cdots + X_n)$ be a frame of $\mathrm{PG}(n,q)$. Then there is precisely one projectivity of $\mathrm{PG}(n,q)$ such that $\mathcal{T}(e_i) = P(X_i)$.

Proof. (Proof from [2, Corollary 3.6.3]). The linear map $\mathcal{T}$ that is defined by $\mathcal{T}(e_i) = P(X_i)$, for $i = 0, \ldots, n$, similarly maps e_{n+1} onto $P(X_{n+1})$. If there are two such projectivities $\mathcal{T}_1$ and $\mathcal{T}_2$, then $\mathcal{T}_1 {\mathcal{T}_2}^{-1}$ is the identity. Hence, $\mathcal{T}_1 = \mathcal{T}_2$. This follows immediately from the previous theorem. ∎

Definition 2.7 ([10, Section 2.4]).

(i) The general linear group $\mathrm{GL}(n+1,q)$ is the group of bijective linear transformations of V with respect to the operation of composition of maps.

(ii) The projective general linear group $\mathrm{PGL}(n+1,q)$ is the group of projectivities of $\mathrm{PG}(n,q)$ with respect to the operation of composition of maps.

(iii) The collineation group $\mathrm{P\Gamma L}(n+1,q)$ is the group of collineations of $\mathrm{PG}(n,q)$ with respect to the operation of composition of maps.

Theorem 2.3 ([10, Theorem 2.8]). The cardinality of the groups $\mathrm{GL}(n+1,q)$, $\mathrm{PGL}(n+1,q)$ and $\mathrm{P\Gamma L}(n+1,q)$ are

(i) $|\mathrm{GL}(n+1,q)| = (q^{n+1}-1)(q^{n+1}-q)\cdots(q^{n+1}-q^n)$,

(ii) $|\mathrm{PGL}(n+1,q)| = |\mathrm{GL}(n+1,q)|/(q-1)$,

(iii) $|\mathrm{P\Gamma L}(n+1,q)| = h|\mathrm{PGL}(n+1,q)|$,

where $q = p^h$, for some prime p, and some positive integer h.

Proof. (Proof from [10, Theorem 2.8]).

(i) Let g be an $(n+1)\times(n+1)$ non-singular matrix with elements chosen from $\mathbb{F}_q$. Then as no row can be zero or a linear combination of previous rows, the rows may be selected in $q^{n+1}-1, q^{n+1}-q, \ldots, q^{n+1}-q^n$ ways.

(ii) Since $(\mathcal{T}, g) = (\mathcal{T}, ag)$, for all $a \in \mathbb{F}_q \backslash \{0\}$, every projectivity is given by $q-1$ matrices.

(iii) See the fundamental theorem of projective geometry.

∎

Definition 2.8 (k-arc [10, Section 3.3]). A (k, r)-arc $\mathcal{K}$ in the projective plane $\mathrm{PG}(2, q)$ is a set of k points of $\mathrm{PG}(2, q)$ such that no $r+1$ points are collinear and some r points are collinear. If $r = 2$, then $\mathcal{K}$ is called a k-arc.

A line l in $\mathrm{PG}(2, q)$ is called an i-secant of a (k, r)-arc $\mathcal{K}$ if and only if l is incident with i points of $\mathcal{K}$. A 0-secant is called an external line, an i-secant is called an internal line, for $1 \leq i \leq q+1$. Also, a line of $\mathrm{PG}(2, q)$ is a tangent to the k-arc if they have one or two points in common.

Definition 2.9 (Complete arc [10, Section 8.1]). A (k, r)-arc $\mathcal{K}$ in the projective plane $\mathrm{PG}(2, q)$ is complete if it is not contained in a $(k+1, r)$-arc.

Two (k, r)-arcs $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathrm{PG}(2, q)$ are said to be projectively equivalent if there exists a projectivity $(\mathcal{T}, g) \in \mathrm{PGL}(3, q)$ that maps $\mathcal{K}_1$ onto $\mathcal{K}_2$; that is, $X_i g = tY_i$, for all points $P(X_i) \in \mathcal{K}_1$ and $P(Y_i) \in \mathcal{K}_2$, where $i = 0, \ldots, k-1$.

Theorem 2.4 (Fundamental Theorem of Projective Geometry [10, Section 2.1.2]).

(i) If $\mathcal{T}' : S \longrightarrow S$ is a collineation, then $\mathcal{T}' = \sigma\mathcal{T}$, where σ is an automorphic collineation on $\mathrm{PG}(n, q)$ and $\mathcal{T}$ is a projectivity. If $q = p^h$, then

$$Aut(\mathbb{F}_{p^h}) = \{1, \varphi, \varphi^2, \ldots, \varphi^{h-1}\};$$

hence, for every projectivity there exist h collineations. If $P(X') = P(X)\mathcal{T}'$, then for every $1 \leq m \leq h$, there exists $t \in \mathbb{F}_q \backslash \{0\}$ and g a non-singular matrix such that

$$tX' = X^{p^m} g,$$

where $X^{p^m} = (x_0^{p^m}, \ldots, x_n^{p^m})$.

(ii) If $\{P_0, \ldots, P_{n+1}\}$ and $\{P'_0, \ldots, P'_{n+1}\}$ are $(n+2)$-arcs of $\mathrm{PG}(2, q)$, then there exists a unique projectivity $\mathcal{T}$ such that $P'_i = P_i\mathcal{T}$, for all $0 \leq i \leq n+1$.

By the Fundamental Theorem of Projective Geometry, any four points no three collinear can be mapped projectively to any other four points, no three collinear.

Definition 2.10 (The Principle of Duality [10, Section 2.1.3]). For any projective space $\mathrm{PG}(n, q)$, there is a dual projective space $\mathrm{PG}(n, q)'$. The points and hyperplanes of $\mathrm{PG}(n, q)'$ are the hyperplanes and points of $\mathrm{PG}(n, q)$, respectively.

If there is a theorem that is true in $\mathrm{PG}(n, q)$, then there exists an equivalent theorem that is true in $\mathrm{PG}(n, q)'$. The theorem in $\mathrm{PG}(n, q)'$ is obtained by substituting points with hyperplanes and hyperplanes with points in $\mathrm{PG}(n, q)$. However, since any i linearly independent points define an $(i-1)$-dimensional projective space and i linearly independent hyperplanes intersect in an $(n-i)$-dimensional projective space, one needs to adjust the language used to explain the incidence structures of the theorem in $\mathrm{PG}(n, q)'$ from that used to define the incidence structures of the theorem in $\mathrm{PG}(n, q)$. For illustration, in $\mathrm{PG}(2, q)$ two points describe a line, and two lines intersect on a point.

2.2 Cyclic projectivities

Definition 2.11 (Cyclic projectivity [10, Section 4.2]). A projectivity $\mathcal{T}$ is called a cyclic projectivity if it permutes the $\theta(n)$ points of $\mathrm{PG}(n, q)$ in a single cycle; also, it is called a Singer cycle and the group it generates a Singer group.

Theorem 2.5 (Hirschfeld [10, Theorem 4.2]). A projectivity $\mathcal{T}$ of $\mathrm{PG}(n, q)$ is cyclic if and only if the characteristic polynomial of an associated matrix is subprimitive.

Corollary 2.2 ([10, Corollary 4.3]). By the principle of duality, a cyclic projectivity permutes the hyperplanes of $\mathrm{PG}(n, q)$ in a single cycle.

For given polynomial $f(x) = x^3 - a_2x^2 - a_1x - a_0$ over a finite field $\mathbb{F}_q$, the companion matrix $\mathrm{C}(f)$ of $f(x)$ is defined by the following:

$$T = \mathrm{C}(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{pmatrix},$$

where $f(x)$ is the characteristic polynomial of the matrix. If $f(x)$ is a subprimitive polynomial over $\mathbb{F}_q$, then $\mathrm{C}(f)$ is a cyclic projectivity over $\mathrm{PG}(2, q)$. This is an example of a cyclic projectivity. let μ be a root of $f(x)$; then

$$\mu^3 = a_2\mu^2 + a_1\mu + a_0.$$

Therefore, for $i = 0, 1, \ldots, q^2 + q$, there exist $z_0^{(i)}, z_1^{(i)}, z_2^{(i)} \in \mathbb{F}_q$ such that

$$\begin{aligned} \mu^i &= z_2^{(i)}\mu^2 + z_1^{(i)}\mu + z_0^{(i)}, \\ \mu^{i+1} &= z_2^{(i+1)}\mu^2 + z_1^{(i+1)}\mu + z_0^{(i+1)}. \end{aligned}$$

Thus, each point $P(z_0^{(i)}, z_1^{(i)}, z_2^{(i)})$ in $\mathrm{PG}(2, q)$ can be generated as follows:

$$(z_0^{(i+1)}, z_1^{(i+1)}, z_2^{(i+1)}) = (z_0^{(i)}, z_1^{(i)}, z_2^{(i)})T;$$

here $z_0^{(i)}, z_1^{(i)}, z_2^{(i)}$ are the coordinates of the point P in PG$(2, q)$. Now, let $P(i) = P(z_0^{(i)}, z_1^{(i)}, z_2^{(i)})$; then

$$P(i+1) = P(i)T.$$

In general, $P(j) = P(i)T^{j-i}$ for $0 \leq i < j \leq q^2 + q$. Let $P(0) = P(1, 0, 0)$; then $P(i) = P(0)T^i$. Since the order of the projectivity $\mathcal{T}$ is $q^2 + q + 1$, so

$$\mathrm{PG}(2, q) = \{P(0)T^i \mid i = 0, 1, \ldots, q^2 + q\}.$$

Hence, the existence of cyclic projectivities produces correspondence between the points of PG(n, q) and the elements of $\mathbb{F}_{q^{n+1}}$, see [10, Section 4.2]. The idea of the correspondence between $\mathbb{F}_{q^{n+1}} \backslash \{0\}$ as a cyclic group and PG(n, q) is due to Singer [19]. In PG$(2, q)$, the existence of a cyclic projectivity provides a representation of the points and lines. By the duality, $\mathcal{T}$ acts cyclically on the lines of PG$(2, q)$ as it acts cyclically on the points of PG$(2, q)$. Therefore, an array with $q + 1$ rows and $q^2 + q + 1$ columns can always represent the plane, where each element of the array is a point, and each column consists of the points on a line. Consequently, a regular array can demonstrate the plane; that is, each row is a cyclic permutation of the preceding row.

Suppose that $P(i)$ with indices $i = d_2, d_3, \ldots, d_q$ are the points collinear with $P(0)$ and $P(1)$. Write d_0 for 0 and d_1 for 1, and consider the array A

$$\begin{array}{ccccc}
d_0 & d_0 + 1 & d_0 + 2 & \cdots & d_0 + q^2 + q \\
d_1 & d_1 + 1 & d_1 + 2 & \cdots & d_1 + q^2 + q \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
d_q & d_q + 1 & d_q + 2 & \cdots & d_q + q^2 + q.
\end{array}$$

Let A′ be the previous array A with each element x reduced modulo $q^2 + q + 1$, so that $0 \leq x \leq q^2 + q$.

Theorem 2.6 ([10, Theorem 4.8]). A′ is a regular array and represents the points and lines of PG$(2, q)$.

Proof. (Proof from [10, Theorem 4.8]). The first row of A′ represents all the points $P(0), P(1), \ldots, P(q^2 + q)$ of PG$(2, q)$. Also, the first column demonstrates the points of a line. Each successive column is the transformation by $\mathcal{T}$ of the previous one as $\mathcal{T}$ acts cyclically on the lines of the plane. Hence, every column represents the points of a line. Moreover, each row is a cyclic permutation of the previous one by construction; therefore A′ is regular. ∎

Corollary 2.3 ([10, Corollary 4.9]). The integers $d_0, d_1, \ldots, d_q$ form a perfect difference set; that is, the $q^2 + q$ integers $d_i - d_j$ with $i \neq j$ are all distinct modulo $q^2 + q + 1$.

Proof. (Proof from [10, Corollary 4.9]). Let B be the array that is produced by the columns of A′ with $d_0 = 0$. The elements could be selected modulo $q^2 + q + 1$ as follows:

$$\begin{array}{ccccc} d_0 - d_0 & d_0 - d_1 & d_0 - d_2 & \cdots & d_0 - d_q \\ d_1 - d_0 & d_1 - d_1 & d_1 - d_2 & \cdots & d_1 - d_q \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ d_q - d_0 & d_q - d_1 & d_q - d_2 & \cdots & d_q - d_q. \end{array}$$

The lines of PG$(2, q)$ through $P(0)$ are represented by the columns of B. Also, every point of the plane apart from $P(0)$ lies on just one of these lines. Therefore, the $q^2 + q$ integers in B aside from those on the main diagonal are distinct modulo $q^2 + q + 1$ and consist of all the differences $d_i - d_j$ with $i \neq j$. Hence any column of A′ forms a perfect difference set. ∎

2.3 Subgeometry decompositions

An additional application of cyclic projectivities is that it explains the structure of projective spaces over finite fields when the order of the field is not prime. Since $\mathbb{F}_q$ is a subfield of $\mathbb{F}_{q^k}$, where k is a positive integer, thus with a fixed coordinate system, PG(n, q^k) inherently contains PG(n, q). Therefore, any PG(n, q) contained within PG(n, q^k) is a subgeometry of PG(n, q^k). Let S be the set that denotes such subgeometries, and let $s(n, q, q^k)$ be the size of S.

Lemma 2.1 ([10, Lemma 4.20]). The number of subgeometries PG(n, q) of PG(n, q^k) is given by

$$s(n, q, q^k) = \frac{|\mathrm{PGL}(n + 1, q^k)|}{|\mathrm{PGL}(n + 1, q)|}.$$

Proof. (Proof from [10, Lemma 4.20]). A PG(n, q) is determined by $n+2$ points such that no $n+1$ of which lie in a hyperplane. Consequently, PGL$(n+1, q^k)$ acts transitively on the set of subgeometries PG(n, q) each of which has a stabiliser isomorphic to PGL$(n + 1, q)$. Therefore,

$$s(n, q, q^k) = \frac{|\mathrm{PGL}(n + 1, q^k)|}{|\mathrm{PGL}(n + 1, q)|}.$$

∎

Corollary 2.4 ([10, Corollary 4.23]). In the projective plane of square order PG$(2, q^2)$,

$$s(2, q, q^2) = q^3(q^3 + 1)(q^2 + 1).$$

Proof. From Lemma 2.1, we have

$$
\begin{aligned}
s(2,q,q^2) &= \frac{|\mathrm{PGL}(3,q^2)|}{|\mathrm{PGL}(3,q)|} \\
&= \frac{(q^6-1)(q^6-q^2)(q^6-q^4)/(q^2-1)}{(q^3-1)(q^3-q)(q^3-q^2)/(q-1)} \quad \text{by Theorem 2.3} \\
&= \frac{(q^6-1)(q^6-q^2)(q^6-q^4)(q-1)}{(q^3-1)(q^3-q)(q^3-q^2)(q^2-1)} \\
&= \frac{[(q^3-1)(q^3+1)][q^2(q^2-1)(q^2+1)][q^4(q^2-1)](q-1)}{(q^3-1)[q(q^2-1)][q^2(q-1)](q^2-1)} \\
&= q^3(q^3+1)(q^2+1).
\end{aligned}
$$

∎

Corollary 2.5 ([10, Corollary 4.21]). If $\mathrm{PG}(n,q)$ is embedded in $\mathrm{PG}(n,q^k)$ and $\mathrm{PG}(n,q^k)$ is embedded in $\mathrm{PG}(n,q^{kl})$, then

$$s(n,q,q^{kl}) = s(n,q,q^k)s(n,q^k,q^{kl}).$$

Proof. From Lemma 2.1, we have

$$s(n,q,q^k) = \frac{|\mathrm{PGL}(n+1,q^k)|}{|\mathrm{PGL}(n+1,q)|}, \quad \text{and} \quad s(n,q^k,q^{kl}) = \frac{|\mathrm{PGL}(n+1,q^{kl})|}{|\mathrm{PGL}(n+1,q^k)|}.$$

Thus,

$$
\begin{aligned}
s(n,q,q^k)s(n,q^k,q^{kl}) &= \frac{|\mathrm{PGL}(n+1,q^k)|}{|\mathrm{PGL}(n+1,q)|}\frac{|\mathrm{PGL}(n+1,q^{kl})|}{|\mathrm{PGL}(n+1,q^k)|} \\
&= \frac{|\mathrm{PGL}(n+1,q^{kl})|}{|\mathrm{PGL}(n+1,q)|} \\
&= s(n,q,q^{kl}).
\end{aligned}
$$

∎

The first corollary is a special case of Lemma 2.1 when $n = k = 2$ and the later corollary states that the number $s(n,q,q^k)$ is multiplicative. For example,

$$
\begin{aligned}
s(2,2,4) &= 360; \\
s(2,4,16) &= 70720; \\
s(2,2,16) &= 25459200 = 360 \times 70720 = s(2,2,4)s(2,4,16).
\end{aligned}
$$

Theorem 2.7 ([10, Corollary 4.25]). The number of points in PG(n, q) divides the number of points in PG(n, q^k) if and only if k and $n + 1$ are relatively prime; that is, they share no common positive divisors besides one .

Proof. (Proof from [10, Lemma 4.24 and Corollary 4.25]). The number of points in PG(n, q) is $\theta(n, q) = \frac{q^{n+1}-1}{q-1}$ and the number of points in PG(n, q^k) is $\theta(n, q^k) = \frac{q^{k(n+1)}-1}{q^k-1}$. So

$$\begin{aligned} N &= \frac{\theta(n, q^k)}{\theta(n, q)} \\ &= \frac{\frac{q^{k(n+1)}-1}{q^k-1}}{\frac{q^{n+1}-1}{q-1}} \\ &= \frac{(q^{k(n+1)} - 1)(q - 1)}{(q^k - 1)(q^{n+1} - 1)} \\ &= \frac{q^{(n+1)(k-1)} + q^{(n+1)(k-2)} + \cdots + q^{(n+1)} + 1}{q^{k-1} + q^{k-2} + \cdots + q + 1}. \end{aligned}$$

Let l, m be positive integers with $kl \leq m$. Then

$$q^m - q^{m-kl} = q^{m-kl}(q^{kl} - 1),$$

which is divisible by $q^k - 1$ and consequently by $q^{k-1} + q^{k-2} + \cdots + q + 1$. Hence

$$\frac{q^m}{q^{k-1} + q^{k-2} + \cdots + q + 1} - \frac{q^{m-kl}}{q^{k-1} + q^{k-2} + \cdots + q + 1}$$

is an integer.

Let d be the greatest common divisor of k and $n + 1$; then the set of residues of $(n+1)(k-1), (n+1)(k-2), \ldots, (n+1), 0$ modulo k is, in some order, $k-d, k-2d, \ldots, d, 0$, each of which appears d times. Hence, in the expression for N, if $q^{(n+1)i}$ is replaced by q^j where $(n + 1)i \equiv j \mod (k)$ and $0 \leq j < k$, then N is an integer if and only if M is, where

$$M = \frac{d(q^{k-d} + q^{k-2d} + \cdots + q^d + 1)}{q^{k-1} + q^{k-2} + \cdots + q + 1}.$$

If $d = 1$, then $M = 1$ and so N is an integer. If $d > 1$, then $M < 1$ and as a result N is not an integer. ■

Theorem 2.8 (Hirschfeld [10, Theorem 4.28]). If $\mathcal{S}$ is a projectivity of PG(n, q^k) which acts as a cyclic projectivity on some PG(n, q) contained within PG(n, q^k) and k and $n + 1$ are relatively prime, then

(i) there exists a cyclic projectivity $\mathcal{T}$ of PG(n, q^k) such that $\mathcal{T}^a = \mathcal{S}$; where

$$a = \frac{(q^{k(n+1)} - 1)(q - 1)}{(q^k - 1)(q^{n+1} - 1)};$$

(ii) every orbit of $\mathcal{S}$ is a subgeometry $\mathrm{PG}(n, q)$;

(iii) if $\mathcal{T}_0$ is any cyclic projectivity of $\mathrm{PG}(n, q^k)$, then the orbits of $\mathcal{T}_0^a$ are subgeometries $\mathrm{PG}(n, q)$.

The previous two theorems state that a decomposition of $\mathrm{PG}(n, q^k)$ into subgeometries $\mathrm{PG}(n, q)$ exists if and only if the number of points in $\mathrm{PG}(n, q)$ divides the number of points in $\mathrm{PG}(n, q^k)$ if and only if k and $n + 1$ are relatively prime.

Corollary 2.6 ([10, Corollary 4.31]). The projective plane $\mathrm{PG}(2, q^k)$ can be decomposed into disjoint copies of subplanes $\mathrm{PG}(2, q)$ if and only if k is not divisible by three.

Corollary 2.7 ([10, Corollary 4.32]). The projective plane of square order $\mathrm{PG}(2, q^2)$ can be decomposed into $q^2 - q + 1$ disjoint copies of subplanes $\mathrm{PG}(2, q)$.

Proof. From Theorem 2.8, every orbit of the projectivity $\mathcal{S}$ of $\mathrm{PG}(2, q^2)$ is a subplane $\mathrm{PG}(2, q)$. By the Orbit-Stabilizer Theorem, the number of orbits is equal to the number of points in $\mathrm{PG}(2, q^2)$ divided by the number of points in $\mathrm{PG}(2, q)$; that is, $(q^4 + q^2 + 1)/(q^2 + q + 1) = q^2 - q + 1$. Hence, $\mathrm{PG}(2, q^2)$ can be decomposed into $q^2 - q + 1$ disjoint subplanes $\mathrm{PG}(2, q)$. ∎

Corollary 2.8 The projective plane $\mathrm{PG}(2, q^4)$ can be decomposed into $(q^2 - q + 1)(q^4 - q^2 + 1)$ disjoint copies of subplanes $\mathrm{PG}(2, q)$.

Proof. From Corollary 2.6, the projective plane $\mathrm{PG}(2, q^4)$ can be decomposed into $[(q^2)^2 - (q^2) + 1 = q^4 - q^2 + 1]$ disjoint subplanes $\mathrm{PG}(2, q^2)$ and every $\mathrm{PG}(2, q^2)$ can be decomposed into $q^2 - q + 1$ disjoint subplanes $\mathrm{PG}(2, q)$. Therefore, $\mathrm{PG}(2, q^4)$ can be decomposed into $(q^2 - q + 1)(q^4 - q^2 + 1)$ disjoint subplanes $\mathrm{PG}(2, q)$. ∎

For instance, the plane $\mathrm{PG}(2, 2^2)$ can be decomposed into 3 disjoint copies of subplanes $\mathrm{PG}(2, 2)$. The plane $\mathrm{PG}(2, 2^4)$ can be decomposed into 39 disjoint copies of subplanes $\mathrm{PG}(2, 2)$.

The idea of decomposing $\mathrm{PG}(2, q^2)$ into disjoint subplanes is essentially due to Bruck [6]. His decomposition is obtained by using Singer's [19] result that every finite projective plane is cyclic. Throughout this thesis, the decompositions that arise as a collection of orbits under an appropriate subgroup of the Singer group will refer to as classical. Yff [20] reports that there are other types of decompositions in cases $q = 3, 4, 5, 7$. An illustration in case of $\mathrm{PG}(2, 3^2)$ is given but, for $q = 4, 5, 7$, no proof or list provided. He gave an example of a non-classical subplane decomposition of $\mathrm{PG}(2, 3^2)$. Additionally, Mathon and Hamilton [17] show using a computer search that there are other types of decompositions in the cases $q = 3, 4, 5$. They approved Yff's result that there is only one non-classical subplane decomposition of $\mathrm{PG}(2,3^2)$ and found two non-classical subplane decompositions of $\mathrm{PG}(2,4^2)$ and found two non-classical subplane decompositions of $\mathrm{PG}(2,5^2)$.

Now a further connection with an observation to an attractive class of substructures of the projective plane, called arcs, is to be investigated. Let us represent $\mathrm{PG}(2,q^2)$ by considering $\mathbb{F}_{q^6}$ as a vector space over $\mathbb{F}_{q^2}$. Accordingly, the points are represented by the elements of $\mathbb{F}_{q^6}\backslash\{0\}$, and $u, v \in \mathbb{F}_{q^6}$ correspond to the same point if and only if $u/v \in \mathbb{F}_{q^2}$. The point of $\mathrm{PG}(2,q^2)$, denoted by (u), represents $u \in \mathbb{F}_{q^6}$. The lines in this representation are defined by an equation $\mathrm{Tr}(ax) = 0$, where $a \neq 0$ is fixed and Tr stands for the trace function from $\mathbb{F}_{q^6}$ to $\mathbb{F}_{q^2}$.

If μ is a primitive element of $\mathbb{F}_{q^6}$, then the group G, where

$$G = \{\varphi : \mathbb{F}_{q^6} \to \mathbb{F}_{q^6} \mid \varphi(x) = x\mu^{i(q^2+q+1)}, \quad \forall i = 0, 1, \ldots, q^2 - q\},$$

and the projective plane of square order $\mathrm{PG}(2,q^2)$ can be cyclically decomposed in two different ways:

The subsets

$$\mathcal{B}_s = \{\mu^{i(q^2-q+1)+s} \mid i = 0, 1, \ldots, q^2 + q\}, \quad \text{for all } s = 0, 1, \ldots, q^2 - q,$$

form a cyclic decomposition of $\mathrm{PG}(2,q^2)$ into disjoint subplanes. Furthermore, a line l of $\mathrm{PG}(2,q^2)$ belongs to one and only one subplane $\mathcal{B}_s$. Also, we say that the line l belongs to $\mathcal{B}_s$ if $|l \cap \mathcal{B}_s| \geq 2$.

The other natural cyclic decomposition of $\mathrm{PG}(2,q^2)$ is connected to arcs. The subsets

$$\mathcal{K}_r = \{\mu^{i(q^2+q+1)+r} \mid i = 0, 1, \ldots, q^2 - q\}, \quad \text{for all } r = 0, 1, \ldots, q^2 + q,$$

form a cyclic decomposition of $\mathrm{PG}(2,q^2)$ into disjoint arcs as shown below.

Lemma 2.2 ([3, Lemma 1.2]). $|\mathcal{K}_0 \cap \mathcal{B}_s| = 1, \quad$ for all $s = 0, 1, \ldots, q^2 - q$.

Proof. (Proof from [3, Lemma 1.2]). Let $\mu^x, \mu^y \in \mathcal{K}_0 \cap \mathcal{B}_s$. From the definitions of $\mathcal{K}_0$ and $\mathcal{B}_s$, it follows that $q^2 - q + 1$ and $q^2 + q + 1$ both divide $x - y$, and as they are relatively prime, $q^4 + q^2 + 1$ is a divisor of $x - y$. Consequently, μ^x and μ^y represent the same point of $\mathrm{PG}(2,q^2)$. Since the number of distinct subplanes of form $\mathcal{B}_s = |\mathcal{K}_0| = q^2 - q + 1$, the lemma immediately follows. ∎

Lemma 2.3 ([3, Lemma 1.3]). Let l be an arbitrary line of $\mathcal{B}_s$ not containing $x \in \mathcal{K}_0 \cap \mathcal{B}_s$, for every $s = 0, 1, \ldots, q^2 - q$. Then $|l \cap \mathcal{K}_0|$ is even.

Proof. (Proof from [3, Lemma 1.3]). Let us prove the lemma for $s = 0$. The collineation $\mathcal{S} : x \longrightarrow x^{q^3}$, such that $\mathcal{S}^2$ is the identity mapping $\mathcal{I}$, fixes $\mathcal{B}_0$ pointwise and $\mathcal{K}_0$ setwise because $\mu^{i(q^2+q+1)}\mathcal{S} = \mu^{(q^2-q+1-i)(q^2+q+1)}$. Therefore, $\mathcal{K}_0\backslash\{\mu^0\}$ is divided into disjoint pairs under the action of $\mathcal{S}$. Since l is a line of $\mathcal{B}_0$, $l\mathcal{S} = l$, so

$$(l \cap \mathcal{K}_0)\mathcal{S} = (l\mathcal{S}) \cap (\mathcal{K}_0\mathcal{S}) = (l \cap \mathcal{K}_0).$$

Also, since $\mu^0 \notin l$, the intersection $l \cap \mathcal{K}_0$ contains both members of certain pairs of $\mathcal{K}_0$. Consequently, the lemma follows by applying the collineation $\mathcal{T}^{s(q^2-q+1)}$ that maps $\mathcal{B}_0$ into $\mathcal{B}_s$ and fixes $\mathcal{K}_0$ setwise. ■

Theorem 2.9 (Boros and Szönyi [3, Theorem 1.1]). For $q > 3$, each orbit $\mathcal{K}_r$ is a complete k-arc in $\mathrm{PG}(2, q^2)$, with $k = q^2 - q + 1$.

Proof. (Proof from [3, Theorem 1.1]). $\mathcal{K}_r$ can be obtained from $\mathcal{K}_0$ by applying the r-th power of the cyclic collineation $\mathcal{T}$. Thus, it is sufficient to prove that $\mathcal{K}_0$ is a complete arc. Initially, we show that $\mathcal{K}_0$ is an arc. From the previous lemma, it follows that among the lines of the subplane $\mathcal{B}_s$ only those $q + 1$ lines could be tangents to $\mathcal{K}_0$ which are incident to $x \in \mathcal{K}_0 \cap \mathcal{B}_s$. Consequently, $\mathcal{K}_0$ has at most $(q + 1)(q^2 - q + 1)$ tangents. On the other hand, $|\mathcal{K}_0| = q^2 - q + 1$ and so at least $q^2 + 1 - (q^2 - q) = q + 1$ tangents pass through every point of $\mathcal{K}_0$. Therefore, $\mathcal{K}_0$ has at least $(q+1)(q^2 - q + 1)$ tangent lines. Hence, the number of tangents is exactly $(q + 1)(q^2 - q + 1)$, and the tangents of $\mathcal{K}_0$ at $x \in \mathcal{K}_0 \cap \mathcal{B}_s$ are the lines of $\mathcal{B}_s$ containing x. Now, let l be a line of $\mathcal{B}_t$, with $t \neq s$, through $x \in \mathcal{K}_0 \cap \mathcal{B}_s$. From $x \in l$ and $x \notin \mathcal{B}_t$, it follows that l is not a tangent to $\mathcal{K}_0$, hence by the previous lemma, $|l \cap \mathcal{K}_0| \geq 2$. Since the number of such lines is $q^2 - q$, $|l \cap \mathcal{K}_0| = 2$ for every non-tangent line through x. Therefore, $\mathcal{K}_0$ is actually an arc.

Next, we show that $\mathcal{K}_0$ is complete. Assume that there is a point $x \in \mathcal{K}_r$ with $r \neq 0$ for which $\mathcal{K}_0 \cup \{x\}$ is also an arc. The collineation group $G = \{\varphi(x) = x\mu^{i(q^2+q+1)}, \quad \forall i = 0, 1, \ldots, q^2 - q\}$ leaves $\mathcal{K}_0$ invariant and acts transitively on the points of $\mathcal{K}_r$, therefore for every $y \in \mathcal{K}_r$ where $\mathcal{K}_0 \cup \{y\}$ will be an arc, too. Consequently, for every $x \in \mathcal{K}_0$ and $y \in \mathcal{K}_r$ the line l joining them is a tangent to $\mathcal{K}_0$. Since $\mathcal{K}_r$ is also an arc, $|l \cap \mathcal{K}_0| \leq 2$. From this it follows that at least $\frac{q^2-q+1}{2}$ tangents pass through a point x of $\mathcal{K}_0$. However, this is a contradiction for $q \geq 4$ since $\frac{q^2-q+1}{2} > q + 1$ if $q \geq 4$. ■

Remark 2.2 In $\mathrm{PG}(2, 2^2)$, $\mathcal{K}_0 = \{\mu^0, \mu^7, \mu^{14}\}$ cannot be complete but for $\mathrm{PG}(2, 3^2)$, calculation shows that $\mathcal{K}_0 = \{\mu^0, \mu^{13}, \mu^{26}, \mu^{39}, \mu^{52}, \mu^{65}, \mu^{78}\}$ is a complete arc with a group $\mathbb{Z}_m$ acting on it, see [10, Section 14.7].

Corollary 2.9 For $q > 2$, the projective plane of square order $\mathrm{PG}(2, q^2)$ can be decomposed into $q^2 + q + 1$ disjoint copies of complete arcs of size $q^2 - q + 1$.

Proof. From Theorem 2.9, every orbit $\mathcal{K}_r$ is a complete k-arc in $\mathrm{PG}(2, q^2)$, with $k = q^2 - q + 1$. By the Orbit-Stabilizer Theorem, the number of orbits is equal to the number of points in $\mathrm{PG}(2, q^2)$ divided by the number of points in $\mathcal{K}_r$; that is, $(q^4+q^2+1)/(q^2-q+1) = q^2+q+1$. Hence, $\mathrm{PG}(2, q^2)$ can be decomposed into q^2+q+1 disjoint copies of complete arcs $\mathcal{K}_r$. ■

The idea of decomposing $\mathrm{PG}(2, q^2)$ into disjoint arcs is essentially due to Kestenband [13, 14] where he studied the arcs as one of the possible categories of the intersection of two Hermitian curves in $\mathrm{PG}(2, q^2)$. The cyclic arcs were then investigated in details by Boros and Szönyi [3], Fisher, Hirschfeld, and Thas [7].

Example 2.2 ([10, Example 4.37]). In the projective plane of order four $\mathrm{PG}(2, 2^2)$, the number of points is $\theta(2, 4) = (4^3 - 1)/(4 - 1) = 21$. For the ground field, we have $\mathbb{F}_4 = \{0, 1, \mu, \mu^2\}$ with $\mu^2+\mu+1 = 0$. To construct $\mathbb{F}_{4^3} = \mathbb{F}_4[x]/\langle f(x)\rangle$, $f(x) = x^3+x^2+x+\mu$ is chosen. Because $f(x)$ has no zeroes in $\mathbb{F}_4$, it is irreducible in $\mathbb{F}_4[x]$. We then have $x^3 = x^2 + x^1 + \mu x^0$. In general, the vectors x^2, x^1, x^0 form a basis for the 3-dimensional vector space $\mathbb{F}_{4^3}$. We still need to check that x is a generator for $\mathbb{F}_{4^3}\backslash\{0\}$. To achieve this, the other powers of x are expressed as linear combinations of the basis vectors. Note that $x^0 = 1$ is actually in the field $\mathbb{F}_4$. One calculation is shown in detail as follows:

$$\begin{aligned} x^4 &= x^3 + x^2 + \mu x^1 \\ &= x^2 + x^1 + \mu x^0 + x^2 + \mu x^1 \\ &= \mu^2 x^1 + \mu x^0. \end{aligned}$$

Likewise, we find that

$$x^5 = \mu^2 x^2 + \mu x^1, x^6 = x^2 + \mu^2 x^1 + x^0, x^7 = \mu x^2 + \mu x^0, \ldots.$$

Checking the exponents that are multiples of 21, we find what we anticipated: $x^{21} = \mu, x^{42} = \mu^2$, and $x^{63} = 1$ are all elements of the base field $\mathbb{F}_4$. We can accordingly select $x^0, x^1, \ldots, x^{20}$ as generators of the 21 points of $\mathrm{PG}(2, 2^2)$, and we mark the point represented by x^s by s. The points are then symbolised by $0, 1, \ldots, 20$. As $x^{21} = \mu x^0$ determines the same projective point as x^0, we see that $\tau(s) = s + 1$, with addition modulo 21, defines a regular action on the 21 points of $\mathrm{PG}(2, 2^2)$(see details in Appendix A). Consequently, the projective plane $\mathrm{PG}(2, 2^2)$ can be demonstrated by a regular array of 5 rows and 21 columns, where each element of the array is a point and each column consists of the points on a line.

PG$(2, 2^2)$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3
14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13
16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The projective plane PG$(2, 2^2)$ consists of three disjoint subplanes of order two. A subplane PG$(2, 2)$ in PG$(2, 2^2)$ can be selected by choosing multiples of three. Therefore, there are the following subplanes of order two PG$(2, 2)$ in PG$(2, 2^2)$.

Subplane $\mathcal{B}_0$

0	3	6	9	12	15	18
3	6	9	12	15	18	0
15	18	0	3	6	9	12

Subplane $\mathcal{B}_1$

1	4	7	10	13	16	19
4	7	10	13	16	19	1
16	19	1	4	7	10	13

Subplane $\mathcal{B}_2$

2	5	8	11	14	17	20
5	8	11	14	17	20	2
17	20	2	5	8	11	14

If the subplane $\mathcal{B}_0$ is fixed, then the subplanes $\mathcal{B}_1$ and $\mathcal{B}_2$ can be chosen in a variety of ways. The following are all the possible selection of $\mathcal{B}_1$ and $\mathcal{B}_2$ with fixed $\mathcal{B}_0$. There are two arrays one for selecting the subplane $\mathcal{B}_1$ and the other for choosing the subplane $\mathcal{B}_2$ organised such that the first row of each array complements the decomposition or the second row of each array completes the decomposition and so on.

Subplane $\mathcal{B}_1$

1	4	7	10	13	16	19
14	19	4	1	13	2	17
17	1	7	4	16	5	20
20	4	10	7	19	8	2
2	7	13	10	1	11	5
5	10	16	13	4	14	8
8	13	19	16	7	17	11
11	16	1	19	10	20	14

Subplane $\mathcal{B}_2$

2	5	8	11	14	17	20
5	16	10	11	20	7	8
8	19	13	14	2	10	11
11	1	16	17	5	13	14
14	4	19	20	8	16	17
17	7	1	2	11	19	20
20	10	4	5	14	1	2
2	13	7	8	17	4	5

Furthermore, the projective plane PG$(2, 2^2)$ consists of seven disjoint 3-arcs. A 3-arc in PG$(2, 2^2)$ can be selected by choosing multiples of seven. Therefore, from the regular array of PG$(2, 2^2)$, there are the following 3-arcs such that each row represents the points of one of the 7 disjoint 3-arcs in PG$(2, 2^2)$.

0	7	14
3	10	17
6	13	20
9	16	2
12	19	5
15	1	8
18	4	11

If the 3-arc $\mathcal{K}_0 = \{0, 7, 14\}$ is fixed, then the other 3-arcs $\mathcal{K}_i$ with $i = 1, \ldots, 6$ can be chosen in a variety of ways. The following are some of the other decompositions of $\mathrm{PG}(2, 2^2)$ into seven disjoint 3-arcs with fixed $\mathcal{K}_0$.

Partition 1			Partition 2			Partition 3		
0	7	14	0	7	14	0	7	14
8	16	18	15	2	4	1	9	11
15	20	2	1	6	9	8	13	16
3	1	19	10	8	5	17	15	12
12	4	13	19	11	20	5	18	6
11	5	9	18	12	16	4	19	2
10	17	6	17	3	13	3	10	20

Remark 2.3 Let μ be a generator for a cyclic group of order $q^4 + q^2 + 1$ in $\mathrm{PG}(2, q^2)$. Then, by Corollary 2.7, the orbits of the group generated by μ^{q^2-q+1} are a collection of $q^2 - q + 1$ disjoint subplanes $\mathcal{B}_i$ with $i = 0, \ldots, q^2 - q$. Also, by Corollary 2.9, the orbits of the group generated by μ^{q^2+q+1} are a collection $\mathcal{K}_i$ with $i = 0, \ldots, q^2 + q$, of $q^2 + q + 1$ disjoint $(q^2 - q + 1)$-arcs. An array with $q^2 - q + 1$ rows and $q^2 + q + 1$ columns can always represent the points of $\mathrm{PG}(2, q^2)$, where each element of the array is a point, and each row consists of the points of a subplane of order q, and each column consists of the points of a $(q^2 - q + 1)$-arc.

For example, an array with 3 rows and 7 columns represents the projective plane $\mathrm{PG}(2, 2^2)$, where each element of the array is a point, and each column represents a 3-arc in $\mathrm{PG}(2, 2^2)$ and every row represents a subplane $\mathrm{PG}(2, 2)$ in $\mathrm{PG}(2, 2^2)$.

	PG(2, 2)						
3-arc	0	3	6	9	12	15	18
	7	10	13	16	19	1	4
	14	17	20	2	5	8	11

2.4 The algorithm

In this section, we describe fully an adapted version of the algorithm that is introduced in [16] by Mathon and then by Mathon and Hamilton in [17] which is used for finding the decompositions of projective planes. The content of this section is based on the references [8], [9], [16] and [17].

The procedure for constructing all decompositions of projective planes up to isomorphism is as follows. A decomposition of the points P of $\mathrm{PG}(2, q^2)$ into disjoint copies of subplanes can be translated as a spread of subplanes from a collection C of subsets of P that form subplanes. This idea is due to Mathon and Hamilton [17]. Similarly, a decomposition of the points P of $\mathrm{PG}(2, q^2)$ into disjoint $(q^2 - q + 1)$-arcs can be interpreted as a spread of $(q^2 - q + 1)$-arcs from a collection C of subsets of P that form $(q^2 - q + 1)$-arcs. Therefore, our basic algorithm for searching for spreads is a standard backtrack, see [9] and [16], where we successively try to supplement prescribed subgeometries, possibly subplanes or arcs, to a partial spread until we either get to a full spread or have no specified subgeometries remaining that are disjoint from our partial spread, in which case we backtrack. A spread S is constructed one prescribed subgeometry at a time. If S_i is a partial spread of i disjoint subgeometries and $P_i \subset P$ is the set of points not covered by S_i a new subgeometry c must lie in the set $C_i \in C$ of subgeometries disjoint from all members of S_i. For S_i to have a completion, every point in P_i must belong to some subgeometry in C_i. This is a strong condition which allows an early detection of bad partial spreads. Briefly, a recursive procedure is formulated to describe the algorithm.

Algorithm 1

procedure SPREAD((i, P_i, C_i, S_i))
  Begin
    If $P_i = \emptyset$ **then** record S_i as a solution;
    else
      Begin
        Find a point $p \in P_i$ incident with the minimum number of prescribed subgeometries $C_i(p)$ in C_i
        for all $c \in C_i(p)$ **do**
          Begin
            Find the set $P(c)$ of points in P_i incident with c;
            Find the set $C(c)$ of prescribed subgeometries in C_i intersecting c;
            $P_{i+1} := P_i \backslash P(c)$;
            $C_{i+1} := C_i \backslash C(c)$;
            $S_{i+1} := S_i \cup \{c\}$;
            Spread$(i + 1, P_{i+1}, C_{i+1}, S_{i+1})$;
          End
        end for
      End
  End
end procedure

The initial conditions Spread$(0, P_0, C_0, S_0)$ of the algorithm are as follows:

P_0 is the set points P of PG$(2, q^2)$ where the points of PG$(2, q^2)$ written in vector forms are represented in terms of distinct integers modulo $q^4 + q^2 + 1$ by using a projectivity $\mathcal{T}$ as in Section 2.2. C_0 is the subset $\mathcal{B}_0 = \{i(q^2 - q + 1) \mid i = 0, 1, \ldots, q^2 + q\}$ if we want to decompose PG$(2, q^2)$ into disjoint subplanes or $\mathcal{K}_0 = \{i(q^2 + q + 1) \mid i = 0, 1, \ldots, q^2 - q\}$ if we want to decompose PG$(2, q^2)$ into disjoint arcs. S_0 is usually set to the empty set as a starting point.

To increase the effectiveness of this technique, we conduct an isomorph rejection stage in advance of backtracking to establish some starters as follows covering the search space. Using the computer program GAP [8], we launch our initial data structures including the prescribed subgeometry C_0 and the collineation group of PG$(2, q^2)$ which is the projective semi-linear group PΓL$(3, q^2)$. Furthermore, GAP is operated to find the generators of the stabiliser of the prescribed subgeometry. Then, the orbits O_i of specified subgeometries disjoint from the subgeometry C_0 are computed. For each orbit O_i with $0 \leq i \leq n$, a representative r_i is subsequently nominated. Correspondingly, the prescribed subgeometries disjoint from the representative r_i and C_0 are then initiated. Subsequently, the search space $(C_0, \emptyset)$ is divided into the search spaces with starters $(\{C_0, r_i\}, \bigcup_{k=i+1}^{n} O_k)$. Each of these pairs of the prescribed subgeometries is accordingly used as a starter in the backtracking algorithm. We rehearse this technique on the assumption that one of the starters has a preliminary partial spread with a nontrivial stabiliser or turns out to be inconvenient. The problem of isomorphism regarding decompositions is solved using GAP after they are determined. The practicality of executing searches for finding different types of decompositions into disjoint copies of prescribed subgeometries, particularly subplanes and arcs, of PG$(2, q^2)$ with $q = 3, 4, 5, 7$ is conceivable using the above approach. The following four chapters discuss the different types of decompositions discovered.

Backtracking is a useful technique for solving algorithmic problems. Moreover, the technique is manageable to implement and straightforward to code, and the accuracy is accepted. However, the overall runtime of the backtracking algorithm usually is slow, and the procedure needs a large amount of memory space for storing various stages in the stack for a big problem; also the method has exponential running time. For example, in case of decomposition into disjoint subplanes, the isomorph rejection stage of the search took a minute in PG$(2, 3^2)$ and took a few days for PG$(2, 7^2)$. Furthermore, the backtracking stage took a few minutes for PG$(2, 3^2)$, and more than four months for PG$(2, 7^2)$ due to a large number of starters where many of them turn out to be trivial.

In summary, from Section 2.2, the points of PG$(2, q^2)$ written in vector forms can be represented in terms of distinct integers modulo q^4+q^2+1 by using a projectivity $\mathcal{T}$. Also, Corollary 2.6 states that by taking the orbits of points under a proper subgroup of a Singer cycle, one can decompose the projective plane $\mathrm{PG}(2, q^k)$ into disjoint copies of subplanes isomorphic to $\mathrm{PG}(2, q)$ if and only if k is not divisible by three. Moreover, Theorem 2.9 shows that by taking the orbits of points under a proper subgroup, one can decompose the projective plane $\mathrm{PG}(2, q^2)$ into disjoint copies of complete $(q^2 - q + 1)$-arcs. By using the technique described in Section 2.4, the following four chapters are devoted to discussing the different types of decompositions of $\mathrm{PG}(2, q^2)$ for $q = 3, 4, 5, 7$ into subplanes and arcs.

Chapter 3

The plane of order nine

From Corollary 2.7 and Corollary 2.9, the projective plane $\mathrm{PG}(2, 3^2)$ can be decomposed into disjoint subplanes isomorphic to $\mathrm{PG}(2, 3)$ and 7-arcs. In this chapter, we discuss the different types of decompositions of $\mathrm{PG}(2, 3^2)$ into disjoint subplanes of order three and disjoint 7-arcs found by using the procedure described in Section 2.4.

The projective plane of order nine $\mathrm{PG}(2, 3^2)$ has 91 points, 91 lines, 10 points on each line and 10 lines passing through each point. Let μ be a primitive element of the finite field $\mathbb{F}_9$ including minimum polynomial $f(x) = x^2 - x - 1$ over the finite field $\mathbb{F}_3$. Accordingly, for the ground field, we have

$$\mathbb{F}_9 = \{0, 1, \mu, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6, \mu^7 \mid \mu^2 - \mu - 1 = 0\}.$$

To construct $\mathbb{F}_{9^3} = \mathbb{F}_9[x]/\langle g(x)\rangle$, we choose $g(x) = x^3 - x^2 - x - \mu$. Since $g(x)$ has no zeros in $\mathbb{F}_9$, it is irreducible in $\mathbb{F}_9[x]$. We then have the projectivity $\mathcal{T} = \mathrm{M(T)}$ given by

$$\mathrm{T} = \mathrm{C}(x^3 - x^2 - x - \mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mu & 1 & 1 \end{pmatrix}$$

is cyclic on $\mathrm{PG}(2, 3^2)$. Then $P(i) = P(0)\mathcal{T}^i$ for all $i = 0, 1, \ldots, 90$. Thus, the points of $\mathrm{PG}(2, 3^2)$ written in vector forms can be represented in terms of distinct integers modulo 91 as follows:

$$0 = P(0) = (1, 0, 0), \ldots, i = P(i), \ldots \quad \text{for all } i = 0, 1, \ldots, 90$$

(see details in Appendix B). They form the following difference set:

$$\mathrm{D} = \{1, 2, 13, 16, 26, 49, 58, 66, 86, 88\}.$$

Therefore, a regular array giving the lines of $\mathrm{PG}(2, 3^2)$ is as follows:

$$\begin{array}{cccccccccc}
1 & 2 & 13 & 16 & 26 & 49 & 58 & 66 & 86 & 88 \\
2 & 3 & 14 & 17 & 27 & 50 & 59 & 67 & 87 & 89 \\
3 & 4 & 15 & 18 & 28 & 51 & 60 & 68 & 88 & 90 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 12 & 15 & 25 & 48 & 57 & 65 & 85 & 87\,.
\end{array}$$

Each row represents one of the 91 lines of $PG(2, 3^2)$.

3.1 Decomposition into subplanes of order three

A subplane of order three in $PG(2, 3^2)$ is a set of 13 points that has 13 lines, 4 points on each line and 4 lines passing through each point. From Corollary 2.7, the projective plane $PG(2, 3^2)$ consists of 7 disjoint subplanes of order three that are isomorphic to $PG(2, 3)$. By using the procedure described in Section 2.4, we identified two non-isomorphic decompositions of the projective plane $PG(2, 3^2)$ into seven disjoint subplanes of order three. The first decomposition has a projective collineation stabiliser of size 546. Additionally, the second decomposition has a projective collineation stabiliser of size 21 and is isomorphic to $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

The first decomposition is given by 7 sets $\mathcal{B}_i$ for $i = 1, \ldots, 7$ each set consists of 13 points of one of the 7 disjoint subplanes of order three in $PG(2, 3^2)$ as follows:

$$\mathcal{B}_1 = \{0, 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84\};$$

$$\mathcal{B}_2 = \{13, 20, 27, 34, 41, 48, 55, 62, 69, 76, 83, 90, 6\};$$

$$\mathcal{B}_3 = \{26, 33, 40, 47, 54, 61, 68, 75, 82, 89, 5, 12, 19\};$$

$$\mathcal{B}_4 = \{39, 46, 53, 60, 67, 74, 81, 88, 4, 11, 18, 25, 32\};$$

$$\mathcal{B}_5 = \{52, 59, 66, 73, 80, 87, 3, 10, 17, 24, 31, 38, 45\};$$

$$\mathcal{B}_6 = \{65, 72, 79, 86, 2, 9, 16, 23, 30, 37, 44, 51, 58\};$$

$$\mathcal{B}_7 = \{78, 85, 1, 8, 15, 22, 29, 36, 43, 50, 57, 64, 71\}.$$

The second decomposition is given by 7 sets $\mathcal{B}_i$ for $i = 1, \ldots, 7$ each set consists of 13 points of one of the 7 disjoint subplanes of order three in $PG(2, 3^2)$ as follows:

$$\mathcal{B}_1 = \{0, 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84\};$$

$$\mathcal{B}_2 = \{86, 8, 22, 75, 32, 4, 9, 12, 33, 89, 19, 6, 69\};$$

$$\mathcal{B}_3 = \{46, 39, 62, 41, 10, 13, 64, 83, 51, 2, 26, 48, 16\};$$

$\mathcal{B}_4 = \{50, 90, 80, 73, 59, 3, 15, 27, 18, 40, 61, 45, 58\}$;

$\mathcal{B}_5 = \{85, 17, 30, 54, 53, 29, 65, 76, 74, 60, 1, 25, 47\}$;

$\mathcal{B}_6 = \{67, 24, 38, 68, 23, 66, 5, 81, 57, 82, 71, 72, 36\}$;

$\mathcal{B}_7 = \{43, 78, 55, 11, 87, 20, 88, 52, 37, 34, 31, 79, 44\}$.

3.2 Decomposition into 7-arcs

A 7-arc in PG$(2, 3^2)$ is a set of 7 points, no three of which are collinear. From Corollary 2.9, the projective plane PG$(2, 3^2)$ consists of 13 disjoint 7-arcs. By using the procedure described in Section 2.4, we identified two non-isomorphic decompositions of the projective plane PG$(2, 3^2)$ into thirteen disjoint 7-arcs. The first decomposition has a projective collineation stabiliser of size 546. Additionally, the second decomposition has a projective collineation stabiliser of size 39 and is isomorphic to $\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$.

The first decomposition is given by 13 sets $\mathcal{K}_i$ for $i = 1, \ldots, 13$ each set consists of 7 points of one of the 13 disjoint 7-arcs in PG$(2, 3^2)$ as follows:

$\mathcal{K}_1 = \{0, 13, 26, 39, 52, 65, 78\}$;

$\mathcal{K}_2 = \{7, 20, 33, 46, 59, 72, 85\}$;

$\mathcal{K}_3 = \{14, 27, 40, 53, 66, 79, 1\}$;

$\mathcal{K}_4 = \{21, 34, 47, 60, 73, 86, 8\}$;

$\mathcal{K}_5 = \{28, 41, 54, 67, 80, 2, 15\}$;

$\mathcal{K}_6 = \{35, 48, 61, 74, 87, 9, 22\}$;

$\mathcal{K}_7 = \{42, 55, 68, 81, 3, 16, 29\}$;

$\mathcal{K}_8 = \{49, 62, 75, 88, 10, 23, 36\}$;

$\mathcal{K}_9 = \{56, 69, 82, 4, 17, 30, 43\}$;

$\mathcal{K}_{10} = \{63, 76, 89, 11, 24, 37, 50\}$;

$\mathcal{K}_{11} = \{70, 83, 5, 18, 31, 44, 57\}$;

$\mathcal{K}_{12} = \{77, 90, 12, 25, 38, 51, 64\}$;

$\mathcal{K}_{13} = \{84, 6, 19, 32, 45, 58, 71\}$.

The second decomposition is given by 13 sets $\mathcal{K}_i$ for $i = 1, \ldots, 13$ each set consists of

7 points of one of the 13 disjoint 7-arcs in PG$(2, 3^2)$ as follows:

$$\mathcal{K}_1 = \{0, 13, 26, 39, 52, 65, 78\};$$

$$\mathcal{K}_2 = \{29, 68, 15, 33, 80, 85, 54\};$$

$$\mathcal{K}_3 = \{21, 34, 22, 82, 38, 43, 61\};$$

$$\mathcal{K}_4 = \{89, 50, 62, 35, 10, 48, 49\};$$

$$\mathcal{K}_5 = \{55, 42, 86, 1, 17, 40, 60\};$$

$$\mathcal{K}_6 = \{58, 32, 88, 28, 45, 41, 23\};$$

$$\mathcal{K}_7 = \{90, 77, 18, 47, 73, 8, 44\};$$

$$\mathcal{K}_8 = \{37, 11, 16, 75, 66, 36, 81\};$$

$$\mathcal{K}_9 = \{7, 20, 51, 6, 3, 84, 25\};$$

$$\mathcal{K}_{10} = \{19, 71, 79, 5, 87, 57, 53\};$$

$$\mathcal{K}_{11} = \{56, 69, 46, 67, 24, 2, 72\};$$

$$\mathcal{K}_{12} = \{64, 12, 9, 30, 59, 4, 74\};$$

$$\mathcal{K}_{13} = \{63, 76, 27, 83, 31, 70, 14\}.$$

In summary, two non-isomorphic decompositions of PG$(2, 3^2)$ into disjoint subplanes of order three are found in Section 3.1 where the first partition is the classical one, and the other decomposition is Yff's result [20]. Section 3.2 shows that, in PG$(2, 3^2)$, there are, up to isomorphism, two decompositions into disjoint 7-arcs where the first partition is the classical one, and the other decomposition is new.

Chapter 4

The plane of order sixteen

The projective plane $PG(2, 4^2)$ can be decomposed into disjoint subplanes isomorphic to $PG(2, 4)$ by Corollary 2.7 and $PG(2, 2)$ by Corollary 2.8, and disjoint 13-arcs by Corollary 2.9. In this chapter, we discuss the different types of decompositions of $PG(2, 4^2)$ into disjoint subplanes of order two, disjoint subplanes of order four and disjoint 13-arcs found by using the procedure described in Section 2.4.

The projective plane of order sixteen $PG(2, 4^2)$ has 273 points, 273 lines, 17 points on each line and 17 lines passing through each point. Let μ be a primitive element of the finite field $\mathbb{F}_{16}$ including minimum polynomial $f(x) = x^4 + x + 1$ over the finite field $\mathbb{F}_2$. Accordingly, for the ground field, we have

$$\mathbb{F}_{16} = \{\, 0, 1, \mu, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6, \mu^7, \mu^8, \mu^9, \mu^{10}, \mu^{11}, \mu^{12}, \mu^{13}, \mu^{14} \mid 2 = \mu^4 + \mu + 1 = 0 \,\}.$$

To construct $\mathbb{F}_{16^3} = \mathbb{F}_{16}[x]/\langle g(x)\rangle$, we choose $g(x) = x^3 + x^2 + x + \mu$. Since $g(x)$ has no zeros in $\mathbb{F}_{16}$, it is irreducible in $\mathbb{F}_{16}[x]$. We then have the projectivity $\mathcal{T} = \mathrm{M(T)}$ given by

$$\mathrm{T} = \mathrm{C}(x^3 + x^2 + x + \mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mu & 1 & 1 \end{pmatrix}$$

is cyclic on $PG(2, 4^2)$. Then $P(i) = P(0)\mathcal{T}^i$ for all $i = 0, 1, \ldots, 272$. Thus, the points of $PG(2, 4^2)$ written in vector forms can be represented in terms of distinct integers modulo 273 as follows:

$$0 = P(0) = (1, 0, 0), \ldots, i = P(i), \ldots \quad \text{for all } i = 0, 1, \ldots, 272$$

(see details in Appendix C).

4.1 Decomposition into subplanes of order two

A subplane of order two in $PG(2, 4^2)$ is a set of 7 points that has 7 lines, 3 points on each line and 3 lines passing through each point. From Corollary 2.8, the projective plane $PG(2, 4^2)$ consists of 39 disjoint subplanes of order two that are isomorphic to $PG(2, 2)$. By using the procedure described in Section 2.4, we identified four non-isomorphic decompositions of the projective plane $PG(2, 4^2)$ into 39 disjoint subplanes of order two. The first decomposition has a projective collineation stabiliser of size 3276. Additionally, the second decomposition has a projective collineation stabiliser of size 117 and is isomorphic to $\mathbb{Z}_{39} \times \mathbb{Z}_3$. Also, the third decomposition has a projective collineation stabiliser of size 39 and is isomorphic to $\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$. Moreover, the fourth decomposition has a projective collineation stabiliser of size 12 and is isomorphic to $\mathbb{Z}_{12}$.

The first decomposition is given by 39 sets $\mathcal{B}_i$ for $i = 1, \ldots, 39$ each set consists of 7 points of one of the 39 disjoint subplanes of order two in $PG(2, 4^2)$ as follows:

$$\mathcal{B}_1 = \{0, 39, 78, 117, 156, 195, 234\};$$

$$\mathcal{B}_2 = \{7, 46, 85, 124, 163, 202, 241\};$$

$$\mathcal{B}_3 = \{14, 53, 92, 131, 170, 209, 248\};$$

$$\mathcal{B}_4 = \{21, 60, 99, 138, 177, 216, 255\};$$

$$\mathcal{B}_5 = \{28, 67, 106, 145, 184, 223, 262\};$$

$$\mathcal{B}_6 = \{35, 74, 113, 152, 191, 230, 269\};$$

$$\mathcal{B}_7 = \{42, 81, 120, 159, 198, 237, 3\};$$

$$\mathcal{B}_8 = \{49, 88, 127, 166, 205, 244, 10\};$$

$$\mathcal{B}_9 = \{56, 95, 134, 173, 212, 251, 17\};$$

$$\mathcal{B}_{10} = \{63, 102, 141, 180, 219, 258, 24\};$$

$$\mathcal{B}_{11} = \{70, 109, 148, 187, 226, 265, 31\};$$

$$\mathcal{B}_{12} = \{77, 116, 155, 194, 233, 272, 38\};$$

$$\mathcal{B}_{13} = \{84, 123, 162, 201, 240, 6, 45\};$$

$$\mathcal{B}_{14} = \{91, 130, 169, 208, 247, 13, 52\};$$

$$\mathcal{B}_{15} = \{98, 137, 176, 215, 254, 20, 59\};$$

$$\mathcal{B}_{16} = \{105, 144, 183, 222, 261, 27, 66\};$$

$$\mathcal{B}_{17} = \{112, 151, 190, 229, 268, 34, 73\};$$

$\mathcal{B}_{18} = \{119, 158, 197, 236, 2, 41, 80\}$;

$\mathcal{B}_{19} = \{126, 165, 204, 243, 9, 48, 87\}$;

$\mathcal{B}_{20} = \{133, 172, 211, 250, 16, 55, 94\}$;

$\mathcal{B}_{21} = \{140, 179, 218, 257, 23, 62, 101\}$;

$\mathcal{B}_{22} = \{147, 186, 225, 264, 30, 69, 108\}$;

$\mathcal{B}_{23} = \{154, 193, 232, 271, 37, 76, 115\}$;

$\mathcal{B}_{24} = \{161, 200, 239, 5, 44, 83, 122\}$;

$\mathcal{B}_{25} = \{168, 207, 246, 12, 51, 90, 129\}$;

$\mathcal{B}_{26} = \{175, 214, 253, 19, 58, 97, 136\}$;

$\mathcal{B}_{27} = \{182, 221, 260, 26, 65, 104, 143\}$;

$\mathcal{B}_{28} = \{189, 228, 267, 33, 72, 111, 150\}$;

$\mathcal{B}_{29} = \{196, 235, 1, 40, 79, 118, 157\}$;

$\mathcal{B}_{30} = \{203, 242, 8, 47, 86, 125, 164\}$;

$\mathcal{B}_{31} = \{210, 249, 15, 54, 93, 132, 171\}$;

$\mathcal{B}_{32} = \{217, 256, 22, 61, 100, 139, 178\}$;

$\mathcal{B}_{33} = \{224, 263, 29, 68, 107, 146, 185\}$;

$\mathcal{B}_{34} = \{231, 270, 36, 75, 114, 153, 192\}$;

$\mathcal{B}_{35} = \{238, 4, 43, 82, 121, 160, 199\}$;

$\mathcal{B}_{36} = \{245, 11, 50, 89, 128, 167, 206\}$;

$\mathcal{B}_{37} = \{252, 18, 57, 96, 135, 174, 213\}$;

$\mathcal{B}_{38} = \{259, 25, 64, 103, 142, 181, 220\}$;

$\mathcal{B}_{39} = \{266, 32, 71, 110, 149, 188, 227\}$.

The second decomposition is given by 39 sets $\mathcal{B}_i$ for $i = 1, \ldots, 39$ each set consists of 7 points of one of the 39 disjoint subplanes of order two in $\mathrm{PG}(2, 4^2)$ as follows:

$\mathcal{B}_1 = \{0, 39, 78, 117, 156, 195, 234\}$;

$\mathcal{B}_2 = \{13, 26, 91, 130, 169, 65, 143\}$;

$\mathcal{B}_3 = \{253, 150, 162, 94, 48, 213, 147\}$;

$\mathcal{B}_4 = \{186, 245, 154, 64, 248, 226, 100\}$;

$\mathcal{B}_5 = \{176, 106, 109, 206, 166, 92, 105\}$;

$\mathcal{B}_6 = \{74, 215, 8, 228, 223, 103, 262\}$;

$\mathcal{B}_7 = \{217, 28, 172, 218, 266, 240, 5\}$;

$\mathcal{B}_8 = \{20, 197, 56, 67, 15, 90, 135\}$;

$\mathcal{B}_9 = \{183, 268, 160, 211, 31, 23, 132\}$;

$\mathcal{B}_{10} = \{29, 203, 159, 216, 44, 229, 151\}$;

$\mathcal{B}_{11} = \{205, 125, 61, 163, 54, 270, 204\}$;

$\mathcal{B}_{12} = \{264, 149, 57, 77, 145, 45, 11\}$;

$\mathcal{B}_{13} = \{236, 113, 47, 124, 46, 139, 246\}$;

$\mathcal{B}_{14} = \{24, 227, 209, 250, 38, 72, 96\}$;

$\mathcal{B}_{15} = \{167, 114, 180, 76, 50, 126, 40\}$;

$\mathcal{B}_{16} = \{107, 3, 82, 81, 233, 102, 30\}$;

$\mathcal{B}_{17} = \{196, 202, 269, 174, 219, 58, 7\}$;

$\mathcal{B}_{18} = \{243, 259, 99, 267, 232, 69, 71\}$;

$\mathcal{B}_{19} = \{194, 120, 121, 238, 175, 201, 75\}$;

$\mathcal{B}_{20} = \{265, 255, 184, 27, 170, 239, 230\}$;

$\mathcal{B}_{21} = \{9, 138, 86, 212, 25, 53, 214\}$;

$\mathcal{B}_{22} = \{171, 200, 220, 66, 14, 198, 134\}$;

$\mathcal{B}_{23} = \{16, 225, 181, 199, 19, 251, 108\}$;

$\mathcal{B}_{24} = \{35, 111, 89, 146, 140, 256, 189\}$;

$\mathcal{B}_{25} = \{51, 142, 118, 168, 97, 115, 112\}$;

$\mathcal{B}_{26} = \{158, 254, 2, 148, 153, 178, 55\}$;

$\mathcal{B}_{27} = \{185, 34, 43, 141, 237, 4, 187\}$;

$\mathcal{B}_{28} = \{18, 88, 17, 110, 188, 165, 119\}$;

$\mathcal{B}_{29} = \{177, 190, 161, 116, 133, 179, 98\}$;

$\mathcal{B}_{30} = \{95, 129, 263, 123, 249, 137, 191\}$;

$\mathcal{B}_{31} = \{ 87, 85, 192, 63, 42, 10, 70 \}$;

$\mathcal{B}_{32} = \{ 60, 22, 173, 21, 144, 12, 271 \}$;

$\mathcal{B}_{33} = \{ 36, 222, 32, 155, 193, 164, 210 \}$;

$\mathcal{B}_{34} = \{ 252, 152, 79, 1, 244, 131, 157 \}$;

$\mathcal{B}_{35} = \{ 83, 235, 272, 68, 136, 207, 80 \}$;

$\mathcal{B}_{36} = \{ 52, 182, 104, 260, 247, 208, 221 \}$;

$\mathcal{B}_{37} = \{ 41, 241, 224, 73, 261, 6, 257 \}$;

$\mathcal{B}_{38} = \{ 122, 242, 37, 258, 33, 49, 128 \}$;

$\mathcal{B}_{39} = \{ 62, 84, 101, 231, 93, 59, 127 \}$.

The third decomposition is given by 39 sets $\mathcal{B}_i$ for $i = 1, \ldots, 39$ each set consists of 7 points of one of the 39 disjoint subplanes of order two in $\mathrm{PG}(2, 4^2)$ as follows:

$\mathcal{B}_1 = \{ 0, 39, 78, 117, 156, 195, 234 \}$;

$\mathcal{B}_2 = \{ 219, 258, 239, 122, 113, 119, 118 \}$;

$\mathcal{B}_3 = \{ 80, 41, 253, 69, 18, 210, 148 \}$;

$\mathcal{B}_4 = \{ 176, 216, 242, 74, 164, 138, 217 \}$;

$\mathcal{B}_5 = \{ 263, 126, 152, 222, 259, 240, 109 \}$;

$\mathcal{B}_6 = \{ 187, 16, 115, 9, 43, 215, 20 \}$;

$\mathcal{B}_7 = \{ 40, 133, 214, 197, 88, 268, 243 \}$;

$\mathcal{B}_8 = \{ 212, 213, 211, 194, 147, 238, 121 \}$;

$\mathcal{B}_9 = \{ 161, 155, 128, 15, 125, 267, 68 \}$;

$\mathcal{B}_{10} = \{ 129, 101, 178, 174, 132, 203, 111 \}$;

$\mathcal{B}_{11} = \{ 34, 98, 23, 186, 83, 35, 11 \}$;

$\mathcal{B}_{12} = \{ 81, 165, 144, 12, 170, 248, 93 \}$;

$\mathcal{B}_{13} = \{ 244, 255, 62, 114, 141, 97, 140 \}$;

$\mathcal{B}_{14} = \{ 13, 130, 143, 247, 260, 208, 182 \}$;

$\mathcal{B}_{15} = \{ 266, 107, 77, 160, 179, 226, 261 \}$;

$\mathcal{B}_{16} = \{ 87, 241, 231, 2, 28, 116, 6 \}$;

$\mathcal{B}_{17} = \{196, 224, 229, 145, 67, 36, 57\}$;

$\mathcal{B}_{18} = \{4, 90, 137, 24, 51, 58, 59\}$;

$\mathcal{B}_{19} = \{180, 89, 94, 53, 232, 139, 246\}$;

$\mathcal{B}_{20} = \{256, 7, 251, 84, 183, 82, 188\}$;

$\mathcal{B}_{21} = \{22, 225, 79, 227, 32, 66, 27\}$;

$\mathcal{B}_{22} = \{48, 159, 181, 202, 166, 269, 110\}$;

$\mathcal{B}_{23} = \{198, 154, 123, 200, 146, 17, 201\}$;

$\mathcal{B}_{24} = \{108, 136, 76, 204, 157, 150, 55\}$;

$\mathcal{B}_{25} = \{49, 168, 14, 223, 149, 177, 257\}$;

$\mathcal{B}_{26} = \{54, 86, 271, 270, 106, 19, 46\}$;

$\mathcal{B}_{27} = \{52, 65, 104, 169, 91, 104, 26\}$;

$\mathcal{B}_{28} = \{163, 189, 167, 61, 175, 167, 199\}$;

$\mathcal{B}_{29} = \{228, 33, 185, 236, 21, 185, 230\}$;

$\mathcal{B}_{30} = \{8, 218, 172, 44, 173, 172, 42\}$;

$\mathcal{B}_{31} = \{162, 37, 60, 158, 70, 60, 249\}$;

$\mathcal{B}_{32} = \{127, 209, 95, 63, 193, 95, 120\}$;

$\mathcal{B}_{33} = \{272, 5, 134, 245, 112, 134, 207\}$;

$\mathcal{B}_{34} = \{151, 29, 233, 38, 99, 233, 184\}$;

$\mathcal{B}_{35} = \{124, 171, 131, 153, 102, 131, 64\}$;

$\mathcal{B}_{36} = \{47, 190, 30, 250, 56, 30, 31\}$;

$\mathcal{B}_{37} = \{142, 73, 205, 262, 3, 205, 100\}$;

$\mathcal{B}_{38} = \{235, 96, 92, 206, 45, 92, 265\}$;

$\mathcal{B}_{39} = \{254, 75, 50, 1, 237, 50, 135\}$.

The fourth decomposition is given by 39 sets $\mathcal{B}_i$ for $i = 1, \ldots, 39$ each set consists of 7 points of one of the 39 disjoint subplanes of order two in $\mathrm{PG}(2, 4^2)$ as follows:

$\mathcal{B}_1 = \{0, 39, 78, 117, 156, 195, 234\}$;

$\mathcal{B}_2 = \{79, 196, 270, 17, 7, 258, 168\}$;

$\mathcal{B}_3 = \{ 97, 194, 119, 48, 124, 3, 49 \}$;

$\mathcal{B}_4 = \{ 148, 101, 158, 133, 36, 263, 47 \}$;

$\mathcal{B}_5 = \{ 150, 176, 142, 67, 128, 151, 228 \}$;

$\mathcal{B}_6 = \{ 64, 162, 50, 211, 27, 186, 185 \}$;

$\mathcal{B}_7 = \{ 6, 70, 1, 25, 96, 225, 69 \}$;

$\mathcal{B}_8 = \{ 118, 82, 123, 2, 120, 51, 170 \}$;

$\mathcal{B}_9 = \{ 26, 104, 130, 143, 169, 182, 247 \}$;

$\mathcal{B}_{10} = \{ 201, 9, 257, 155, 262, 222, 272 \}$;

$\mathcal{B}_{11} = \{ 19, 10, 237, 122, 8, 121, 214 \}$;

$\mathcal{B}_{12} = \{ 146, 116, 248, 191, 164, 23, 167 \}$;

$\mathcal{B}_{13} = \{ 75, 38, 240, 83, 187, 183, 105 \}$;

$\mathcal{B}_{14} = \{ 137, 163, 197, 226, 166, 141, 89 \}$;

$\mathcal{B}_{15} = \{ 210, 37, 213, 198, 152, 209, 251 \}$;

$\mathcal{B}_{16} = \{ 28, 171, 160, 59, 54, 239, 93 \}$;

$\mathcal{B}_{17} = \{ 229, 267, 165, 42, 135, 35, 236 \}$;

$\mathcal{B}_{18} = \{ 58, 81, 40, 181, 249, 200, 115 \}$;

$\mathcal{B}_{19} = \{ 33, 5, 100, 180, 154, 111, 144 \}$;

$\mathcal{B}_{20} = \{ 86, 131, 15, 98, 63, 61, 175 \}$;

$\mathcal{B}_{21} = \{ 56, 30, 12, 43, 129, 212, 45 \}$;

$\mathcal{B}_{22} = \{ 85, 252, 106, 250, 244, 159, 20 \}$;

$\mathcal{B}_{23} = \{ 21, 220, 265, 245, 68, 202, 18 \}$;

$\mathcal{B}_{24} = \{ 126, 125, 4, 227, 80, 102, 22 \}$;

$\mathcal{B}_{25} = \{ 66, 172, 103, 259, 24, 184, 41 \}$;

$\mathcal{B}_{26} = \{ 188, 62, 205, 177, 92, 231, 88 \}$;

$\mathcal{B}_{27} = \{ 13, 91, 208, 221, 260, 52, 65 \}$;

$\mathcal{B}_{28} = \{ 44, 84, 76, 254, 57, 266, 204 \}$;

$\mathcal{B}_{29} = \{ 207, 178, 132, 46, 71, 112, 109 \}$;

$\mathcal{B}_{30} = \{127, 14, 34, 99, 60, 264, 255\}$;

$\mathcal{B}_{31} = \{149, 95, 217, 253, 243, 161, 77\}$;

$\mathcal{B}_{32} = \{230, 271, 107, 218, 206, 11, 134\}$;

$\mathcal{B}_{33} = \{173, 74, 224, 90, 32, 203, 179\}$;

$\mathcal{B}_{34} = \{157, 113, 31, 242, 29, 190, 238\}$;

$\mathcal{B}_{35} = \{87, 139, 246, 215, 219, 193, 216\}$;

$\mathcal{B}_{36} = \{108, 16, 192, 138, 269, 147, 223\}$;

$\mathcal{B}_{37} = \{241, 199, 189, 233, 114, 94, 174\}$;

$\mathcal{B}_{38} = \{268, 140, 256, 53, 235, 136, 261\}$;

$\mathcal{B}_{39} = \{110, 145, 73, 72, 153, 55, 232\}$.

4.2 Decomposition into subplanes of order four

A subplane of order four in $PG(2, 4^2)$ is a set of 21 points that has 21 lines, 5 points on each line and 5 lines passing through each point. From Corollary 2.7, the projective plane $PG(2, 4^2)$ consists of 13 disjoint subplanes of order four that are isomorphic to $PG(2, 4)$. By using the procedure described in Section 2.4, we identified three non-isomorphic decompositions of the projective $PG(2, 4^2)$ into 13 disjoint subplanes of order four. The first decomposition has a projective collineation stabiliser of size 3276. Additionally, the second decomposition has a projective collineation stabiliser of size 39 and is isomorphic to $\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$. Also, the third decomposition has a projective collineation stabiliser of size 12 and is isomorphic to $\mathbb{Z}_{12}$.

The first decomposition is given by 13 sets $\mathcal{B}_i$ for $i = 1, \ldots, 13$ each set consists of 21 points of one of the 13 disjoint subplanes of order four in $PG(2, 4^2)$ as follows:

$\mathcal{B}_1 = \{0, 13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, 182, 195, 208, 221, 234, 247, 260\}$;

$\mathcal{B}_2 = \{8, 21, 34, 47, 60, 73, 86, 99, 112, 125, 138, 151, 164, 177, 190, 203, 216, 229, 242, 255, 268\}$;

$\mathcal{B}_3 = \{3, 16, 29, 42, 55, 68, 81, 94, 107, 120, 133, 146, 159, 172, 185, 198, 211, 224, 237, 250, 263\}$;

$\mathcal{B}_4 = \{11, 24, 37, 50, 63, 76, 89, 102, 115, 128, 141, 154, 167, 180, 193, 206, 219, 232, 245, 258, 271\}$;

$\mathcal{B}_5 = \{6, 19, 32, 45, 58, 71, 84, 97, 110, 123, 136, 149, 162, 175, 188, 201, 214, 227, 240, 253, 266\}$;

$\mathcal{B}_6 = \{1, 14, 27, 40, 53, 66, 79, 92, 105, 118, 131, 144, 157, 170, 183, 196, 209, 222, 235, 248, 261\}$;

$\mathcal{B}_7 = \{9, 22, 35, 48, 61, 74, 87, 100, 113, 126, 139, 152, 165, 178, 191, 204, 217, 230, 243, 256, 269\}$;

$\mathcal{B}_8 = \{4, 17, 30, 43, 56, 69, 82, 95, 108, 121, 134, 147, 160, 173, 186, 199, 212, 225, 238, 251, 264\}$;

$\mathcal{B}_9 = \{12, 25, 38, 51, 64, 77, 90, 103, 116, 129, 142, 155, 168, 181, 194, 207, 220, 233, 246, 259, 272\}$;

$\mathcal{B}_{10} = \{7, 20, 33, 46, 59, 72, 85, 98, 111, 124, 137, 150, 163, 176, 189, 202, 215, 228, 241, 254, 267\}$;

$\mathcal{B}_{11} = \{2, 15, 28, 41, 54, 67, 80, 93, 106, 119, 132, 145, 158, 171, 184, 197, 210, 223, 236, 249, 262\}$;

$\mathcal{B}_{12} = \{10, 23, 36, 49, 62, 75, 88, 101, 114, 127, 140, 153, 166, 179, 192, 205, 218, 231, 244, 257, 270\}$;

$\mathcal{B}_{13} = \{5, 18, 31, 44, 57, 70, 83, 96, 109, 122, 135, 148, 161, 174, 187, 200, 213, 226, 239, 252, 265\}$.

The second decomposition is given by 13 sets $\mathcal{B}_i$ for $i = 1, \ldots, 13$ each set consists of 21 points of one of the 13 disjoint subplanes of order four in $\mathrm{PG}(2, 4^2)$ as follows:

$\mathcal{B}_1 = \{0, 13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, 182, 195, 208, 221, 234, 247, 260\}$;

$\mathcal{B}_2 = \{48, 64, 92, 94, 100, 105, 106, 109, 147, 150, 154, 162, 166, 176, 186, 206, 213, 226, 245, 248, 253\}$;

$\mathcal{B}_3 = \{5, 8, 15, 20, 28, 56, 67, 74, 90, 103, 135, 172, 197, 215, 217, 218, 223, 228, 240, 262, 266\}$;

$\mathcal{B}_4 = \{23, 29, 31, 44, 54, 61, 125, 132, 151, 159, 160, 163, 183, 203, 204, 205, 211, 216, 229, 268, 270\}$;

$\mathcal{B}_5 = \{11, 24, 38, 45, 46, 47, 57, 72, 77, 96, 113, 124, 139, 145, 149, 209, 227, 236, 246, 250, 264\}$;

$\mathcal{B}_6 = \{3, 7, 30, 40, 50, 58, 76, 81, 82, 102, 107, 114, 126, 167, 174, 180, 196, 202, 219, 233, 269\}$;

$\mathcal{B}_7 = \{27, 69, 71, 75, 99, 120, 121, 170, 175, 184, 194, 201, 230, 232, 238, 239, 243, 255, 259, 265, 267\}$;

$\mathcal{B}_8 = \{9, 14, 16, 19, 25, 53, 66, 86, 108, 134, 138, 171, 181, 198, 199, 200, 212, 214, 220, 225, 251\}$;

$\mathcal{B}_9 = \{2, 35, 51, 55, 89, 97, 111, 112, 115, 118, 140, 142, 146, 148, 153, 158, 168, 178, 189, 254, 256\}$;

$\mathcal{B}_{10} = \{4, 17, 18, 34, 43, 88, 98, 110, 116, 119, 133, 141, 161, 165, 177, 179, 185, 187, 188, 190, 237\}$;

$\mathcal{B}_{11} = \{10, 12, 21, 22, 42, 60, 63, 70, 85, 87, 95, 123, 129, 137, 144, 173, 191, 192, 249, 263, 271\}$;

$\mathcal{B}_{12} = \{1, 32, 36, 68, 79, 80, 83, 131, 136, 152, 155, 157, 164, 193, 207, 210, 222, 235, 244, 252, 272\}$;

$\mathcal{B}_{13} = \{6, 33, 37, 41, 49, 59, 62, 73, 84, 93, 101, 122, 127, 128, 224, 231, 241, 242, 257, 258, 261\}$.

The third decomposition is given by 13 sets $\mathcal{B}_i$ for $i = 1, \ldots, 13$ each set consists of 21 points of one of the 13 disjoint subplanes of order four in $\mathrm{PG}(2, 4^2)$ as follows:

$\mathcal{B}_1 = \{0, 13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, 182, 195, 208, 221, 234, 247, 260\}$;

$\mathcal{B}_2 = \{61, 77, 105, 107, 113, 118, 119, 122, 160, 163, 167, 175, 179, 189, 199, 219, 226, 239, 258, 261, 266\}$;

$\mathcal{B}_3 = \{2, 6, 18, 21, 28, 33, 41, 69, 80, 87, 103, 116, 148, 185, 210, 228, 230, 231, 236, 241, 253\}$;

$\mathcal{B}_4 = \{8, 10, 36, 42, 44, 57, 67, 74, 138, 145, 164, 172, 173, 176, 196, 216, 217, 218, 224, 229, 242\}$;

$\mathcal{B}_5 = \{4, 24, 37, 51, 58, 59, 60, 70, 85, 90, 109, 126, 137, 152, 158, 162, 222, 240, 249, 259, 263\}$;

$\mathcal{B}_6 = \{9, 16, 20, 43, 53, 63, 71, 89, 94, 95, 115, 120, 127, 139, 180, 187, 193, 209, 215, 232, 246\}$;

$\mathcal{B}_7 = \{5, 7, 40, 82, 84, 88, 112, 133, 134, 183, 188, 197, 207, 214, 243, 245, 251, 252, 256, 268, 272\}$;

$\mathcal{B}_8 = \{22, 27, 29, 32, 38, 66, 79, 99, 121, 147, 151, 184, 194, 211, 212, 213, 225, 227, 233, 238, 264\}$;

$\mathcal{B}_9 = \{15, 48, 64, 68, 102, 110, 124, 125, 128, 131, 153, 155, 159, 161, 166, 171, 181, 191, 202, 267, 269\}$;

$\mathcal{B}_{10} = \{17, 30, 31, 47, 56, 101, 111, 123, 129, 132, 146, 154, 174, 178, 190, 192, 198, 200, 201, 203, 250\}$;

$\mathcal{B}_{11} = \{3, 11, 23, 25, 34, 35, 55, 73, 76, 83, 98, 100, 108, 136, 142, 150, 157, 186, 204, 205, 262\}$;

$\mathcal{B}_{12} = \{12, 14, 45, 49, 81, 92, 93, 96, 144, 149, 165, 168, 170, 177, 206, 220, 223, 235, 248, 257, 265\}$;

$\mathcal{B}_{13} = \{1, 19, 46, 50, 54, 62, 72, 75, 86, 97, 106, 114, 135, 140, 141, 237, 244, 254, 255, 270, 271\}$.

4.3 Decomposition into 13-arcs

A 13-arc in PG$(2, 4^2)$ is a set of 13 points, no three of which are collinear. From Corollary 2.9, the projective plane PG$(2, 4^2)$ consists of 21 disjoint 13-arcs. By using the procedure described in Section 2.4, we identified three non-isomorphic decompositions of the projective plane PG$(2, 4^2)$ into 21 disjoint 13-arcs. The first decomposition has a projective collineation stabiliser of size 3276. Additionally, the second decomposition has a projective collineation stabiliser of size 84 and is isomorphic to $\mathbb{Z}_{21} \rtimes \mathbb{Z}_4$. Moreover, the third decomposition has a projective collineation stabiliser of size 18 and is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_3$.

The first decomposition is given by 21 sets $\mathcal{K}_i$ for $i = 1, \ldots, 21$ each set consists of 13 points of one of the 21 disjoint 13-arcs in PG$(2, 4^2)$ as follows:

$\mathcal{K}_1 = \{0, 21, 42, 63, 84, 105, 126, 147, 168, 189, 210, 231, 252\}$;

$\mathcal{K}_2 = \{13, 34, 55, 76, 97, 118, 139, 160, 181, 202, 223, 244, 265\}$;

$\mathcal{K}_3 = \{5, 26, 47, 68, 89, 110, 131, 152, 173, 194, 215, 236, 257\}$;

$\mathcal{K}_4 = \{18, 39, 60, 81, 102, 123, 144, 165, 186, 207, 228, 249, 270\}$;

$\mathcal{K}_5 = \{10, 31, 52, 73, 94, 115, 136, 157, 178, 199, 220, 241, 262\}$;

$\mathcal{K}_6 = \{2, 23, 44, 65, 86, 107, 128, 149, 170, 191, 212, 233, 254\}$;

$\mathcal{K}_7 = \{15, 36, 57, 78, 99, 120, 141, 162, 183, 204, 225, 246, 267\}$;

$\mathcal{K}_8 = \{7, 28, 49, 70, 91, 112, 133, 154, 175, 196, 217, 238, 259\}$;

$\mathcal{K}_9 = \{20, 41, 62, 83, 104, 125, 146, 167, 188, 209, 230, 251, 272\}$;

$\mathcal{K}_{10} = \{12, 33, 54, 75, 96, 117, 138, 159, 180, 201, 222, 243, 264\}$;

$\mathcal{K}_{11} = \{4, 25, 46, 67, 88, 109, 130, 151, 172, 193, 214, 235, 256\}$;

$\mathcal{K}_{12} = \{17, 38, 59, 80, 101, 122, 143, 164, 185, 206, 227, 248, 269\}$;

$\mathcal{K}_{13} = \{9, 30, 51, 72, 93, 114, 135, 156, 177, 198, 219, 240, 261\}$;

$\mathcal{K}_{14} = \{1, 22, 43, 64, 85, 106, 127, 148, 169, 190, 211, 232, 253\}$;

$\mathcal{K}_{15} = \{14, 35, 56, 77, 98, 119, 140, 161, 182, 203, 224, 245, 266\}$;

$\mathcal{K}_{16} = \{6, 27, 48, 69, 90, 111, 132, 153, 174, 195, 216, 237, 258\}$;

$\mathcal{K}_{17} = \{19, 40, 61, 82, 103, 124, 145, 166, 187, 208, 229, 250, 271\}$;

$\mathcal{K}_{18} = \{11, 32, 53, 74, 95, 116, 137, 158, 179, 200, 221, 242, 263\}$;

$\mathcal{K}_{19} = \{3, 24, 45, 66, 87, 108, 129, 150, 171, 192, 213, 234, 255\}$;

$\mathcal{K}_{20} = \{16, 37, 58, 79, 100, 121, 142, 163, 184, 205, 226, 247, 268\}$;

$\mathcal{K}_{21} = \{8, 29, 50, 71, 92, 113, 134, 155, 176, 197, 218, 239, 260\}$.

The second decomposition is given by 21 sets $\mathcal{K}_i$ for $i = 1, \ldots, 21$ each set consists of 13 points of one of the 21 disjoint 13-arcs in $\mathrm{PG}(2, 4^2)$ as follows:

$\mathcal{K}_1 = \{0, 21, 42, 63, 84, 105, 126, 147, 168, 189, 210, 231, 252\}$;

$\mathcal{K}_2 = \{9, 11, 53, 64, 69, 70, 82, 102, 125, 136, 151, 201, 205\}$;

$\mathcal{K}_3 = \{1, 61, 66, 119, 139, 149, 159, 164, 175, 184, 196, 220, 223\}$;

$\mathcal{K}_4 = \{5, 35, 41, 57, 79, 96, 150, 162, 178, 193, 248, 256, 257\}$;

$\mathcal{K}_5 = \{12, 14, 27, 45, 51, 65, 115, 138, 153, 158, 165, 203, 269\}$;

$\mathcal{K}_6 = \{17, 23, 31, 77, 83, 99, 107, 110, 145, 194, 230, 253, 259\}$;

$\mathcal{K}_7 = \{16, 62, 74, 128, 140, 143, 176, 195, 207, 229, 236, 268, 272\}$;

$\mathcal{K}_8 = \{20, 46, 71, 86, 87, 117, 130, 166, 183, 186, 202, 245, 270\}$;

$\mathcal{K}_9 = \{6, 22, 52, 78, 122, 123, 152, 181, 219, 247, 250, 258, 261\}$;

$\mathcal{K}_{10} = \{15, 33, 39, 81, 92, 100, 132, 173, 174, 221, 227, 234, 264\}$;

$\mathcal{K}_{11} = \{18, 80, 106, 114, 134, 141, 180, 188, 190, 206, 211, 213, 249\}$;

$\mathcal{K}_{12} = \{4, 30, 43, 91, 109, 116, 146, 148, 157, 187, 215, 217, 235\}$;

$\mathcal{K}_{13} = \{44, 73, 88, 113, 121, 135, 142, 155, 212, 226, 238, 251, 266\}$;

$\mathcal{K}_{14} = \{19, 26, 37, 50, 98, 124, 144, 208, 240, 246, 255, 263, 271\}$;

$\mathcal{K}_{15} = \{47, 55, 75, 97, 101, 103, 111, 112, 163, 167, 172, 216, 241\}$;

$\mathcal{K}_{16} = \{2, 10, 32, 38, 48, 67, 76, 95, 108, 171, 199, 224, 254\}$;

$\mathcal{K}_{17} = \{13, 24, 36, 54, 56, 58, 68, 72, 104, 137, 192, 228, 233\}$;

$\mathcal{K}_{18} = \{3, 28, 40, 49, 60, 89, 90, 129, 133, 169, 170, 225, 265\}$;

$\mathcal{K}_{19} = \{7, 8, 59, 93, 156, 191, 197, 198, 200, 204, 232, 262, 267\}$;

$\mathcal{K}_{20} = \{29, 85, 120, 131, 160, 161, 177, 182, 214, 222, 237, 239, 244\}$;

$\mathcal{K}_{21} = \{25, 34, 94, 118, 127, 154, 179, 185, 209, 218, 242, 243, 260\}$.

The third decomposition is given by 21 sets $\mathcal{K}_i$ for $i = 1, \dots, 21$ each set consists of 13 points of one of the 21 disjoint 13-arcs in $\mathrm{PG}(2, 4^2)$ as follows:

$\mathcal{K}_1 = \{0, 21, 42, 63, 84, 105, 126, 147, 168, 189, 210, 231, 252\}$;

$\mathcal{K}_2 = \{6, 11, 20, 48, 53, 56, 82, 92, 151, 175, 184, 243, 270\}$;

$\mathcal{K}_3 = \{1, 13, 60, 66, 107, 205, 211, 223, 234, 250, 256, 257, 258\}$;

$\mathcal{K}_4 = \{4, 18, 33, 45, 59, 123, 128, 150, 164, 165, 221, 265, 272\}$;

$\mathcal{K}_5 = \{30, 39, 99, 166, 178, 194, 202, 206, 213, 226, 244, 262, 269\}$;

$\mathcal{K}_6 = \{16, 29, 78, 91, 114, 118, 124, 135, 138, 176, 215, 230, 261\}$;

$\mathcal{K}_7 = \{19, 43, 77, 94, 100, 117, 142, 155, 172, 201, 227, 245, 268\}$;

$\mathcal{K}_8 = \{47, 64, 76, 113, 141, 180, 181, 186, 220, 229, 247, 255, 271\}$;

$\mathcal{K}_9 = \{32, 35, 55, 68, 71, 81, 112, 119, 173, 187, 217, 219, 263\}$;

$\mathcal{K}_{10} = \{2, 49, 58, 67, 73, 115, 116, 122, 134, 162, 167, 249, 266\}$;

$\mathcal{K}_{11} = \{65, 72, 98, 132, 137, 145, 157, 170, 171, 232, 235, 238, 246\}$;

$\mathcal{K}_{12} = \{37, 40, 83, 88, 97, 129, 143, 161, 163, 190, 198, 233, 251\}$;

$\mathcal{K}_{13} = \{3, 38, 50, 74, 87, 108, 109, 111, 144, 156, 179, 182, 267\}$;

$\mathcal{K}_{14} = \{8, 25, 36, 44, 75, 86, 95, 102, 131, 152, 160, 216, 228\}$;

$\mathcal{K}_{15} = \{10, 15, 52, 54, 70, 89, 90, 139, 209, 214, 218, 224, 240\}$;

$\mathcal{K}_{16} = \{7, 24, 28, 61, 69, 103, 104, 125, 174, 188, 193, 197, 242\}$;

$\mathcal{K}_{17} = \{85, 96, 106, 120, 148, 149, 159, 169, 199, 200, 203, 225, 248\}$;

$\mathcal{K}_{18} = \{5, 27, 41, 146, 153, 154, 177, 185, 191, 192, 204, 212, 259\}$;

$\mathcal{K}_{19} = \{12, 23, 26, 31, 57, 121, 133, 140, 158, 196, 237, 239, 260\}$;

$\mathcal{K}_{20} = \{17, 34, 46, 51, 62, 79, 93, 101, 127, 208, 236, 253, 264\}$;

$\mathcal{K}_{21} = \{9, 14, 22, 80, 110, 130, 136, 183, 195, 207, 222, 241, 254\}$.

In summary, four non-isomorphic decompositions of $\mathrm{PG}(2, 4^2)$ into disjoint subplanes of order two are found in Section 4.1 where the first partition is the classical one, and the other three decompositions are new. The second section shows that, in $\mathrm{PG}(2, 4^2)$, there are, up to isomorphism, three types of decompositions into disjoint subplanes of order four where the first partition is the classical one, and the other two decompositions are given by Mathon and Hamilton in [17]. Section 4.3 presents three non-isomorphic decompositions of $\mathrm{PG}(2, 4^2)$ into disjoint 13-arcs where the first partition is the classical one, and the other two decompositions are new.

Chapter 5

The plane of order twenty-five

From Corollary 2.7 and Corollary 2.9, the projective plane $\mathrm{PG}(2,5^2)$ can be decomposed into disjoint subplanes isomorphic to $\mathrm{PG}(2,5)$ and 21-arcs. In this chapter, we discuss the different types of decompositions of $\mathrm{PG}(2,5^2)$ into disjoint subplanes of order five and disjoint 21-arcs found by using the procedure described in Section 2.4.

The projective plane of order twenty-five $\mathrm{PG}(2,5^2)$ has 651 points, 651 lines, 26 points on each line and 26 lines passing through each point. Let μ be a primitive element of the finite field $\mathbb{F}_{25}$ including minimum polynomial $f(x) = x^2 + x + 2$ over the finite field $\mathbb{F}_5$. Accordingly, for the ground field, we have

$$\begin{aligned}\mathbb{F}_{25} = \{\, &0, 1, \mu, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6, \mu^7, \mu^8, \mu^9, \mu^{10}, \mu^{11}, \mu^{12}, \mu^{13}, \mu^{14}, \mu^{15}, \mu^{16}, \mu^{17}, \mu^{18},\\ &\mu^{19}, \mu^{20}, \mu^{21}, \mu^{22}, \mu^{23} \mid \mu^2 + \mu + 2 = 0\,\}.\end{aligned}$$

To construct $\mathbb{F}_{25^3} = \mathbb{F}_{25}[x]/\langle g(x)\rangle$, we choose $g(x) = x^3 - x^2 - x - \mu^8$. Since $g(x)$ has no zeros in $\mathbb{F}_{25}$, it is irreducible in $\mathbb{F}_{25}[x]$. We then have the projectivity $\mathcal{T} = \mathrm{M(T)}$ given by

$$\mathrm{T} = \mathrm{C}(x^3 - x^2 - x - \mu^8) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mu^8 & 1 & 1 \end{pmatrix}$$

is cyclic on $\mathrm{PG}(2,5^2)$. Then $P(i) = P(0)\mathcal{T}^i$ for all $i = 0, 1, \ldots, 650$. Thus, the points of $\mathrm{PG}(2,5^2)$ written in vector forms can be represented in terms of distinct integers modulo 651 as follows:

$$0 = P(0) = (1,0,0), \ldots, i = P(i), \ldots \quad \text{for all } i = 0, 1, \ldots, 650$$

(see details in Appendix D).

5.1 Decomposition into subplanes of order five

A subplane of order five in $PG(2, 5^2)$ is a set of 31 points that has 31 lines, 6 points on each line and 6 lines passing through each point. From Corollary 2.7, the projective plane $PG(2, 5^2)$ consists of 21 disjoint subplanes of order five that are isomorphic to $PG(2, 5)$. By using the procedure described in Section 2.4, we identified three non-isomorphic decompositions of the projective plane $PG(2, 5^2)$ into 21 disjoint subplanes of order five. The first decomposition has a projective collineation stabiliser of size 3906. Additionally, the second decomposition has a projective collineation stabiliser of size 63 and is isomorphic to $\mathbb{Z}_{21} \times \mathbb{Z}_3$. Also, the third decomposition has a projective collineation stabiliser of size 18 and is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_3$.

The first decomposition is given by 21 sets $\mathcal{B}_i$ for $i = 1, \ldots, 21$ each set consists of 31 points of one of the 21 disjoint subplanes of order five in $PG(2, 5^2)$ as follows:

$$\mathcal{B}_1 = \{0, 21, 42, 63, 84, 105, 126, 147, 168, 189, 210, 231, 252, 273, 294, 315, 336, 357, 378, 399, 420, 441, 462, 483, 504, 525, 546, 567, 588, 609, 630\};$$

$$\mathcal{B}_2 = \{10, 31, 52, 73, 94, 115, 136, 157, 178, 199, 220, 241, 262, 283, 304, 325, 346, 367, 388, 409, 430, 451, 472, 493, 514, 535, 556, 577, 598, 619, 640\};$$

$$\mathcal{B}_3 = \{20, 41, 62, 83, 104, 125, 146, 167, 188, 209, 230, 251, 272, 293, 314, 335, 356, 377, 398, 419, 440, 461, 482, 503, 524, 545, 566, 587, 608, 629, 650\};$$

$$\mathcal{B}_4 = \{9, 30, 51, 72, 93, 114, 135, 156, 177, 198, 219, 240, 261, 282, 303, 324, 345, 366, 387, 408, 429, 450, 471, 492, 513, 534, 555, 576, 597, 618, 639\};$$

$$\mathcal{B}_5 = \{19, 40, 61, 82, 103, 124, 145, 166, 187, 208, 229, 250, 271, 292, 313, 334, 355, 376, 397, 418, 439, 460, 481, 502, 523, 544, 565, 586, 607, 628, 649\};$$

$$\mathcal{B}_6 = \{8, 29, 50, 71, 92, 113, 134, 155, 176, 197, 218, 239, 260, 281, 302, 323, 344, 365, 386, 407, 428, 449, 470, 491, 512, 533, 554, 575, 596, 617, 638\};$$

$$\mathcal{B}_7 = \{18, 39, 60, 81, 102, 123, 144, 165, 186, 207, 228, 249, 270, 291, 312, 333, 354, 375, 396, 417, 438, 459, 480, 501, 522, 543, 564, 585, 606, 627, 648\};$$

$\mathcal{B}_8 = \{7, 28, 49, 70, 91, 112, 133, 154, 175, 196, 217, 238, 259, 280, 301, 322, 343, 364, 385, 406, 427, 448, 469, 490, 511, 532, 553, 574, 595, 616, 637\}$;

$\mathcal{B}_9 = \{17, 38, 59, 80, 101, 122, 143, 164, 185, 206, 227, 248, 269, 290, 311, 332, 353, 374, 395, 416, 437, 458, 479, 500, 521, 542, 563, 584, 605, 626, 647\}$;

$\mathcal{B}_{10} = \{6, 27, 48, 69, 90, 111, 132, 153, 174, 195, 216, 237, 258, 279, 300, 321, 342, 363, 384, 405, 426, 447, 468, 489, 510, 531, 552, 573, 594, 615, 636\}$;

$\mathcal{B}_{11} = \{16, 37, 58, 79, 100, 121, 142, 163, 184, 205, 226, 247, 268, 289, 310, 331, 352, 373, 394, 415, 436, 457, 478, 499, 520, 541, 562, 583, 604, 625, 646\}$;

$\mathcal{B}_{12} = \{5, 26, 47, 68, 89, 110, 131, 152, 173, 194, 215, 236, 257, 278, 299, 320, 341, 362, 383, 404, 425, 446, 467, 488, 509, 530, 551, 572, 593, 614, 635\}$;

$\mathcal{B}_{13} = \{15, 36, 57, 78, 99, 120, 141, 162, 183, 204, 225, 246, 267, 288, 309, 330, 351, 372, 393, 414, 435, 456, 477, 498, 519, 540, 561, 582, 603, 624, 645\}$;

$\mathcal{B}_{14} = \{4, 25, 46, 67, 88, 109, 130, 151, 172, 193, 214, 235, 256, 277, 298, 319, 340, 361, 382, 403, 424, 445, 466, 487, 508, 529, 550, 571, 592, 613, 634\}$;

$\mathcal{B}_{15} = \{14, 35, 56, 77, 98, 119, 140, 161, 182, 203, 224, 245, 266, 287, 308, 329, 350, 371, 392, 413, 434, 455, 476, 497, 518, 539, 560, 581, 602, 623, 644\}$;

$\mathcal{B}_{16} = \{3, 24, 45, 66, 87, 108, 129, 150, 171, 192, 213, 234, 255, 276, 297, 318, 339, 360, 381, 402, 423, 444, 465, 486, 507, 528, 549, 570, 591, 612, 633\}$;

$\mathcal{B}_{17} = \{13, 34, 55, 76, 97, 118, 139, 160, 181, 202, 223, 244, 265, 286, 307, 328, 349, 370, 391, 412, 433, 454, 475, 496, 517, 538, 559, 580, 601, 622, 643\}$;

$\mathcal{B}_{18} = \{2, 23, 44, 65, 86, 107, 128, 149, 170, 191, 212, 233, 254, 275, 296, 317, 338, 359, 380, 401, 422, 443, 464, 485, 506, 527, 548, 569, 590, 611, 632\}$;

$\mathcal{B}_{19} = \{12, 33, 54, 75, 96, 117, 138, 159, 180, 201, 222, 243, 264, 285, 306, 327, 348, 369, 390, 411, 432, 453, 474, 495, 516, 537, 558, 579, 600, 621, 642\}$;

$\mathcal{B}_{20}$ = { 1, 22, 43, 64, 85, 106, 127, 148, 169, 190, 211, 232, 253, 274, 295, 316, 337, 358, 379, 400, 421, 442, 463, 484, 505, 526, 547, 568, 589, 610, 631 };

$\mathcal{B}_{21}$ = { 11, 32, 53, 74, 95, 116, 137, 158, 179, 200, 221, 242, 263, 284, 305, 326, 347, 368, 389, 410, 431, 452, 473, 494, 515, 536, 557, 578, 599, 620, 641 }.

The second decomposition is given by 21 sets $\mathcal{B}_i$ for $i = 1, \ldots, 21$ each set consists of 31 points of one of the 21 disjoint subplanes of order five in PG$(2, 5^2)$ as follows:

$\mathcal{B}_1$ = { 0, 21, 42, 63, 84, 105, 126, 147, 168, 189, 210, 231, 252, 273, 294, 315, 336, 357, 378, 399, 420, 441, 462, 483, 504, 525, 546, 567, 588, 609, 630 };

$\mathcal{B}_2$ = { 15, 22, 121, 125, 139, 156, 188, 197, 217, 227, 249, 253, 257, 271, 296, 321, 400, 447, 449, 466, 473, 477, 491, 507, 550, 558, 591, 593, 623, 634, 643 };

$\mathcal{B}_3$ = { 35, 55, 106, 119, 120, 129, 179, 186, 194, 203, 288, 363, 373, 401, 408, 410, 424, 435, 436, 438, 471, 474, 480, 493, 544, 547, 548, 557, 563, 613, 622 };

$\mathcal{B}_4$ = { 11, 53, 67, 75, 83, 102, 111, 115, 127, 145, 154, 155, 182, 199, 234, 235, 244, 263, 276, 289, 360, 372, 380, 382, 430, 446, 464, 509, 577, 579, 644 };

$\mathcal{B}_5$ = { 2, 20, 31, 37, 47, 70, 73, 144, 150, 239, 274, 283, 284, 287, 319, 339, 349, 356, 366, 391, 407, 440, 448, 455, 475, 484, 500, 501, 602, 624, 642 };

$\mathcal{B}_6$ = { 48, 72, 103, 108, 110, 112, 143, 149, 164, 190, 225, 228, 233, 236, 272, 303, 304, 314, 355, 402, 413, 416, 431, 444, 454, 531, 539, 571, 610, 612, 641 };

$\mathcal{B}_7$ = { 10, 40, 45, 76, 116, 140, 172, 180, 185, 196, 212, 223, 243, 327, 365, 411, 414, 426, 451, 457, 458, 460, 517, 545, 554, 559, 570, 597, 615, 617, 625 };

$\mathcal{B}_8$ = { 6, 33, 34, 41, 71, 136, 160, 177, 270, 275, 297, 302, 317, 352, 354, 361, 362, 379, 384, 390, 392, 421, 442, 485, 526, 530, 541, 542, 543, 606, 632 };

$\mathcal{B}_9$ = { 12, 13, 18, 32, 46, 57, 60, 90, 159, 169, 238, 291, 322, 324, 377, 412, 418, 443, 453, 495, 536, 540, 549, 565, 569, 574, 578, 581, 583, 589, 614 };

$\mathcal{B}_{10}$ = { 1, 9, 16, 17, 28, 29, 52, 61, 87, 95, 96, 141, 176, 211, 254, 259, 266, 269, 326, 353, 422, 498, 511, 537, 556, 584, 592, 598, 621, 637, 639 };

$\mathcal{B}_{11} = \{ 43, 85, 89, 91, 132, 146, 158, 161, 165, 202, 206, 207, 208, 216, 232, 295, 308, 323, 332, 346, 368, 374, 376, 397, 406, 432, 445, 468, 552, 600, 605 \}$;

$\mathcal{B}_{12} = \{ 4, 8, 14, 58, 101, 133, 142, 148, 153, 175, 178, 200, 215, 242, 312, 359, 388, 389, 394, 405, 429, 437, 452, 465, 478, 561, 573, 587, 611, 631, 638 \}$;

$\mathcal{B}_{13} = \{ 79, 124, 131, 137, 138, 166, 170, 171, 219, 220, 224, 262, 280, 305, 318, 328, 393, 409, 419, 467, 472, 479, 499, 508, 521, 551, 580, 627, 629, 636, 648 \}$;

$\mathcal{B}_{14} = \{ 27, 51, 62, 69, 77, 78, 86, 152, 157, 261, 286, 309, 325, 337, 338, 340, 348, 381, 386, 428, 433, 461, 503, 562, 572, 595, 607, 616, 626, 645, 650 \}$;

$\mathcal{B}_{15} = \{ 25, 44, 80, 93, 135, 174, 191, 218, 230, 256, 264, 282, 290, 311, 330, 331, 345, 375, 398, 417, 439, 494, 496, 516, 533, 555, 560, 566, 604, 635, 646 \}$;

$\mathcal{B}_{16} = \{ 19, 30, 50, 64, 66, 68, 82, 88, 99, 100, 162, 222, 245, 277, 279, 292, 313, 342, 343, 369, 385, 404, 476, 481, 489, 510, 513, 514, 522, 523, 594 \}$;

$\mathcal{B}_{17} = \{ 26, 38, 39, 92, 104, 109, 134, 173, 181, 205, 221, 246, 247, 250, 265, 268, 281, 293, 306, 310, 344, 347, 396, 487, 490, 497, 568, 575, 582, 599, 633 \}$;

$\mathcal{B}_{18} = \{ 3, 5, 7, 23, 56, 74, 81, 97, 123, 193, 204, 209, 237, 241, 320, 329, 333, 334, 341, 351, 364, 367, 383, 387, 505, 512, 529, 564, 576, 590, 619 \}$;

$\mathcal{B}_{19} = \{ 49, 59, 94, 98, 107, 114, 118, 151, 163, 167, 183, 192, 201, 213, 267, 278, 298, 300, 301, 335, 358, 370, 463, 470, 482, 486, 492, 506, 585, 596, 608 \}$;

$\mathcal{B}_{20} = \{ 117, 128, 130, 198, 214, 226, 229, 240, 251, 255, 258, 285, 307, 371, 395, 415, 423, 425, 450, 456, 459, 502, 515, 518, 527, 553, 601, 618, 620, 647, 649 \}$;

$\mathcal{B}_{21} = \{ 24, 36, 54, 65, 113, 122, 184, 187, 195, 248, 260, 299, 316, 350, 403, 427, 434, 469, 488, 519, 520, 524, 528, 532, 534, 535, 538, 586, 603, 628, 640 \}$.

The third decomposition is given by 21 sets $\mathcal{B}_i$ for $i = 1, \ldots, 21$ each set consists of

31 points of one of the 21 disjoint subplanes of order five in $PG(2, 5^2)$ as follows:

$\mathcal{B}_1$ = { 0, 21, 42, 63, 84, 105, 126, 147, 168, 189, 210, 231, 252, 273, 294, 315, 336, 357, 378, 399, 420, 441, 462, 483, 504, 525, 546, 567, 588, 609, 630 };

$\mathcal{B}_2$ = { 5, 26, 32, 33, 43, 82, 91, 114, 122, 125, 185, 198, 207, 230, 243, 259, 267, 280, 297, 335, 343, 370, 402, 415, 440, 449, 531, 560, 610, 623, 649 };

$\mathcal{B}_3$ = { 38, 52, 66, 72, 116, 128, 129, 175, 183, 200, 204, 214, 232, 260, 321, 344, 384, 408, 412, 437, 438, 442, 457, 488, 502, 513, 517, 586, 605, 618, 626 };

$\mathcal{B}_4$ = { 15, 113, 130, 136, 145, 151, 153, 170, 174, 181, 197, 220, 237, 244, 247, 249, 250, 275, 293, 306, 310, 333, 355, 389, 406, 434, 460, 521, 530, 599, 636 };

$\mathcal{B}_5$ = { 2, 57, 69, 71, 79, 100, 101, 102, 163, 186, 217, 301, 320, 348, 350, 371, 377, 383, 407, 409, 453, 461, 505, 512, 562, 576, 590, 591, 613, 617, 619 };

$\mathcal{B}_6$ = { 1, 9, 11, 55, 60, 70, 93, 117, 152, 154, 173, 180, 182, 209, 236, 254, 300, 313, 396, 498, 501, 507, 528, 547, 561, 563, 606, 635, 646, 647, 650 };

$\mathcal{B}_7$ = { 28, 31, 53, 68, 111, 158, 162, 195, 226, 282, 316, 330, 339, 351, 364, 379, 397, 414, 416, 432, 445, 464, 469, 481, 492, 500, 508, 574, 575, 622, 643 };

$\mathcal{B}_8$ = { 8, 14, 19, 37, 48, 59, 73, 77, 92, 109, 121, 164, 178, 229, 246, 269, 272, 277, 298, 299, 361, 374, 380, 401, 454, 473, 536, 564, 615, 631, 642 };

$\mathcal{B}_9$ = { 81, 110, 137, 171, 190, 193, 194, 212, 234, 258, 281, 314, 329, 363, 419, 468, 477, 491, 493, 497, 518, 532, 541, 555, 556, 558, 573, 579, 581, 625, 648 };

$\mathcal{B}_{10}$ = { 3, 24, 47, 49, 51, 83, 87, 106, 127, 146, 157, 184, 227, 270, 286, 288, 289, 292, 347, 349, 372, 390, 394, 413, 424, 480, 486, 499, 597, 614, 616 };

$\mathcal{B}_{11}$ = { 44, 107, 118, 139, 141, 202, 225, 235, 238, 257, 263, 284, 311, 319, 334, 362, 366, 367, 395, 398, 426, 429, 439, 448, 467, 554, 570, 572, 589, 601, 644 };

$\mathcal{B}_{12}$ = { 30, 34, 50, 88, 95, 99, 103, 143, 150, 166, 172, 222, 283, 285, 304, 352, 359, 435, 455, 510, 527, 539, 540, 543, 548, 584, 594, 596, 608, 611, 645 };

$\mathcal{B}_{13} = \{39, 45, 46, 54, 75, 115, 132, 138, 240, 265, 308, 342, 365, 382, 392, 418, 421, 425, 428, 436, 443, 484, 487, 511, 520, 524, 537, 545, 585, 592, 612\}$;

$\mathcal{B}_{14} = \{7, 23, 29, 56, 58, 76, 85, 156, 160, 161, 177, 206, 253, 266, 268, 274, 337, 354, 356, 417, 465, 509, 549, 553, 578, 593, 602, 603, 621, 628, 641\}$;

$\mathcal{B}_{15} = \{10, 13, 18, 36, 108, 120, 131, 169, 201, 213, 239, 242, 248, 318, 322, 341, 353, 385, 405, 410, 423, 431, 485, 494, 544, 550, 565, 569, 580, 598, 639\}$;

$\mathcal{B}_{16} = \{12, 17, 67, 80, 96, 119, 155, 176, 188, 223, 228, 251, 262, 309, 326, 328, 331, 338, 376, 400, 422, 430, 433, 447, 451, 514, 519, 538, 552, 634, 637\}$;

$\mathcal{B}_{17} = \{20, 41, 86, 104, 144, 165, 191, 196, 203, 211, 216, 256, 276, 279, 287, 295, 296, 332, 346, 373, 404, 466, 471, 478, 503, 516, 534, 557, 600, 632, 638\}$;

$\mathcal{B}_{18} = \{4, 35, 64, 112, 133, 135, 142, 148, 159, 205, 208, 264, 271, 303, 324, 360, 368, 369, 387, 391, 444, 446, 452, 474, 476, 542, 551, 577, 582, 587, 627\}$;

$\mathcal{B}_{19} = \{22, 40, 61, 78, 90, 124, 149, 167, 192, 215, 219, 221, 224, 278, 305, 358, 388, 458, 463, 470, 472, 475, 479, 523, 529, 533, 559, 568, 595, 624, 633\}$;

$\mathcal{B}_{20} = \{6, 16, 25, 27, 62, 74, 97, 98, 134, 179, 233, 241, 290, 302, 307, 317, 323, 325, 327, 381, 386, 393, 450, 456, 459, 515, 522, 571, 607, 620, 629\}$;

$\mathcal{B}_{21} = \{65, 89, 94, 123, 140, 187, 199, 218, 245, 255, 261, 291, 312, 340, 345, 375, 403, 411, 427, 482, 489, 490, 495, 496, 506, 526, 535, 566, 583, 604, 640\}$.

5.2 Decomposition into 21-arcs

A 21-arc in $\mathrm{PG}(2, 5^2)$ is a set of 21 points, no three of which are collinear. From Corollary 2.9, the projective plane $\mathrm{PG}(2, 5^2)$ consists of 31 disjoint 21-arcs. By using the procedure described in Section 2.4, we identified three non-isomorphic decompositions of the projective plane $\mathrm{PG}(2, 5^2)$ into 31 disjoint 21-arcs. The first decomposition has a projective collineation stabiliser of size 3906. Additionally, the second decomposition has a projective collineation stabiliser of size 93 and is isomorphic to $\mathbb{Z}_{31} \rtimes \mathbb{Z}_3$. Also, the third decomposition has a projective collineation stabiliser of size 18 and is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_3$.

The first decomposition is given by 31 sets $\mathcal{K}_i$ for $i = 1, \ldots, 31$ each set consists of 21 points of one of the 31 disjoint 21-arcs in $PG(2,5^2)$ as follows:

$$\mathcal{K}_1 = \{0, 31, 62, 93, 124, 155, 186, 217, 248, 279, 310, 341, 372, 403, 434, 465, 496, 527, 558, 589, 620\};$$

$$\mathcal{K}_2 = \{21, 52, 83, 114, 145, 176, 207, 238, 269, 300, 331, 362, 393, 424, 455, 486, 517, 548, 579, 610, 641\};$$

$$\mathcal{K}_3 = \{11, 42, 73, 104, 135, 166, 197, 228, 259, 290, 321, 352, 383, 414, 445, 476, 507, 538, 569, 600, 631\};$$

$$\mathcal{K}_4 = \{1, 32, 63, 94, 125, 156, 187, 218, 249, 280, 311, 342, 373, 404, 435, 466, 497, 528, 559, 590, 621\};$$

$$\mathcal{K}_5 = \{22, 53, 84, 115, 146, 177, 208, 239, 270, 301, 332, 363, 394, 425, 456, 487, 518, 549, 580, 611, 642\};$$

$$\mathcal{K}_6 = \{12, 43, 74, 105, 136, 167, 198, 229, 260, 291, 322, 353, 384, 415, 446, 477, 508, 539, 570, 601, 632\};$$

$$\mathcal{K}_7 = \{2, 33, 64, 95, 126, 157, 188, 219, 250, 281, 312, 343, 374, 405, 436, 467, 498, 529, 560, 591, 622\};$$

$$\mathcal{K}_8 = \{23, 54, 85, 116, 147, 178, 209, 240, 271, 302, 333, 364, 395, 426, 457, 488, 519, 550, 581, 612, 643\};$$

$$\mathcal{K}_9 = \{13, 44, 75, 106, 137, 168, 199, 230, 261, 292, 323, 354, 385, 416, 447, 478, 509, 540, 571, 602, 633\};$$

$$\mathcal{K}_{10} = \{3, 34, 65, 96, 127, 158, 189, 220, 251, 282, 313, 344, 375, 406, 437, 468, 499, 530, 561, 592, 623\};$$

$$\mathcal{K}_{11} = \{24, 55, 86, 117, 148, 179, 210, 241, 272, 303, 334, 365, 396, 427, 458, 489, 520, 551, 582, 613, 644\};$$

$$\mathcal{K}_{12} = \{14, 45, 76, 107, 138, 169, 200, 231, 262, 293, 324, 355, 386, 417, 448, 479, 510, 541, 572, 603, 634\};$$

$$\mathcal{K}_{13} = \{4, 35, 66, 97, 128, 159, 190, 221, 252, 283, 314, 345, 376, 407, 438, 469, 500, 531, 562, 593, 624\};$$

$$\mathcal{K}_{14} = \{25, 56, 87, 118, 149, 180, 211, 242, 273, 304, 335, 366, 397, 428, 459, 490, 521, 552, 583, 614, 645\};$$

$$\mathcal{K}_{15} = \{15, 46, 77, 108, 139, 170, 201, 232, 263, 294, 325, 356, 387, 418, 449, 480, 511, 542, 573, 604, 635\};$$

$$\mathcal{K}_{16} = \{5, 36, 67, 98, 129, 160, 191, 222, 253, 284, 315, 346, 377, 408, 439, 470, 501, 532, 563, 594, 625\};$$

$\mathcal{K}_{17} = \{26, 57, 88, 119, 150, 181, 212, 243, 274, 305, 336, 367, 398, 429, 460, 491, 522, 553, 584, 615, 646\};$

$\mathcal{K}_{18} = \{16, 47, 78, 109, 140, 171, 202, 233, 264, 295, 326, 357, 388, 419, 450, 481, 512, 543, 574, 605, 636\};$

$\mathcal{K}_{19} = \{6, 37, 68, 99, 130, 161, 192, 223, 254, 285, 316, 347, 378, 409, 440, 471, 502, 533, 564, 595, 626\};$

$\mathcal{K}_{20} = \{27, 58, 89, 120, 151, 182, 213, 244, 275, 306, 337, 368, 399, 430, 461, 492, 523, 554, 585, 616, 647\};$

$\mathcal{K}_{21} = \{17, 48, 79, 110, 141, 172, 203, 234, 265, 296, 327, 358, 389, 420, 451, 482, 513, 544, 575, 606, 637\};$

$\mathcal{K}_{22} = \{7, 38, 69, 100, 131, 162, 193, 224, 255, 286, 317, 348, 379, 410, 441, 472, 503, 534, 565, 596, 627\};$

$\mathcal{K}_{23} = \{28, 59, 90, 121, 152, 183, 214, 245, 276, 307, 338, 369, 400, 431, 462, 493, 524, 555, 586, 617, 648\};$

$\mathcal{K}_{24} = \{18, 49, 80, 111, 142, 173, 204, 235, 266, 297, 328, 359, 390, 421, 452, 483, 514, 545, 576, 607, 638\};$

$\mathcal{K}_{25} = \{8, 39, 70, 101, 132, 163, 194, 225, 256, 287, 318, 349, 380, 411, 442, 473, 504, 535, 566, 597, 628\};$

$\mathcal{K}_{26} = \{29, 60, 91, 122, 153, 184, 215, 246, 277, 308, 339, 370, 401, 432, 463, 494, 525, 556, 587, 618, 649\};$

$\mathcal{K}_{27} = \{19, 50, 81, 112, 143, 174, 205, 236, 267, 298, 329, 360, 391, 422, 453, 484, 515, 546, 577, 608, 639\};$

$\mathcal{K}_{28} = \{9, 40, 71, 102, 133, 164, 195, 226, 257, 288, 319, 350, 381, 412, 443, 474, 505, 536, 567, 598, 629\};$

$\mathcal{K}_{29} = \{30, 61, 92, 123, 154, 185, 216, 247, 278, 309, 340, 371, 402, 433, 464, 495, 526, 557, 588, 619, 650\};$

$\mathcal{K}_{30} = \{20, 51, 82, 113, 144, 175, 206, 237, 268, 299, 330, 361, 392, 423, 454, 485, 516, 547, 578, 609, 640\};$

$\mathcal{K}_{31} = \{10, 41, 72, 103, 134, 165, 196, 227, 258, 289, 320, 351, 382, 413, 444, 475, 506, 537, 568, 599, 630\}.$

The second decomposition is given by 31 sets $\mathcal{K}_i$ for $i = 1, \ldots, 31$ each set consists of 21 points of one of the 31 disjoint 21-arcs in $\mathrm{PG}(2, 5^2)$ as follows:

$\mathcal{K}_1 = \{0, 31, 62, 93, 124, 155, 186, 217, 248, 279, 310, 341, 372, 403, 434, 465, 496, 527, 558, 589, 620\};$

$\mathcal{K}_2 = \{\,52, 83, 99, 113, 172, 184, 187, 200, 224, 238, 297, 315, 345, 363, 375, 416, 530, 537, 547, 548, 559\,\};$

$\mathcal{K}_3 = \{\,104, 120, 134, 193, 205, 208, 221, 241, 245, 318, 336, 366, 384, 396, 437, 443, 469, 551, 568, 579, 580\,\};$

$\mathcal{K}_4 = \{\,125, 141, 176, 214, 226, 229, 242, 262, 266, 339, 357, 387, 405, 417, 458, 464, 490, 572, 600, 601, 610\,\};$

$\mathcal{K}_5 = \{\,162, 197, 235, 247, 250, 263, 283, 287, 360, 378, 408, 426, 438, 479, 485, 511, 524, 593, 621, 622, 631\,\};$

$\mathcal{K}_6 = \{\,1, 183, 218, 256, 268, 271, 284, 304, 308, 381, 399, 429, 447, 459, 500, 506, 532, 545, 614, 642, 643\,\};$

$\mathcal{K}_7 = \{\,12, 13, 22, 204, 239, 277, 289, 292, 305, 325, 329, 402, 420, 450, 468, 480, 521, 553, 566, 569, 635\,\};$

$\mathcal{K}_8 = \{\,5, 33, 34, 43, 225, 260, 298, 313, 326, 331, 346, 350, 423, 441, 471, 489, 501, 542, 574, 587, 590\,\};$

$\mathcal{K}_9 = \{\,26, 54, 55, 64, 246, 281, 319, 334, 347, 352, 367, 371, 444, 462, 492, 510, 522, 563, 595, 608, 611\,\};$

$\mathcal{K}_{10} = \{\,47, 75, 76, 85, 267, 302, 340, 355, 368, 373, 388, 392, 483, 486, 513, 531, 543, 584, 616, 629, 632\,\};$

$\mathcal{K}_{11} = \{\,2, 68, 96, 97, 106, 288, 323, 361, 376, 389, 394, 409, 413, 504, 507, 534, 552, 564, 605, 637, 650\,\};$

$\mathcal{K}_{12} = \{\,7, 20, 23, 89, 117, 118, 127, 309, 344, 382, 397, 410, 415, 430, 455, 525, 528, 555, 573, 585, 626\,\};$

$\mathcal{K}_{13} = \{\,28, 41, 44, 110, 138, 139, 148, 330, 365, 418, 424, 431, 436, 451, 476, 546, 549, 576, 594, 606, 647\,\};$

$\mathcal{K}_{14} = \{\,17, 49, 65, 131, 146, 159, 160, 169, 351, 386, 439, 445, 452, 457, 472, 497, 567, 570, 597, 615, 627\,\};$

$\mathcal{K}_{15} = \{\,38, 70, 86, 152, 167, 180, 181, 190, 393, 407, 460, 466, 473, 478, 493, 518, 588, 591, 618, 636, 648\,\};$

$\mathcal{K}_{16} = \{\,6, 18, 59, 91, 107, 173, 188, 201, 202, 211, 414, 428, 481, 487, 494, 499, 514, 539, 609, 612, 639\,\};$

$\mathcal{K}_{17} = \{\,9, 27, 39, 80, 112, 128, 194, 209, 222, 223, 232, 435, 449, 502, 508, 515, 520, 535, 560, 630, 633\,\};$

$\mathcal{K}_{18} = \{\,3, 21, 30, 48, 60, 101, 133, 149, 215, 230, 243, 244, 253, 456, 470, 523, 529, 536, 541, 556, 581\,\};$

$\mathcal{K}_{19} = \{24, 42, 51, 69, 81, 122, 154, 170, 236, 251, 264, 265, 274, 477, 491, 544, 550, 557, 562, 577, 602\}$;

$\mathcal{K}_{20} = \{45, 63, 72, 90, 102, 143, 175, 191, 257, 272, 285, 286, 295, 498, 512, 565, 571, 578, 583, 598, 623\}$;

$\mathcal{K}_{21} = \{66, 84, 111, 114, 123, 164, 196, 212, 278, 293, 306, 307, 316, 519, 533, 586, 592, 599, 604, 619, 644\}$;

$\mathcal{K}_{22} = \{14, 87, 105, 132, 135, 144, 185, 233, 259, 299, 314, 327, 328, 337, 540, 554, 607, 613, 625, 640, 641\}$;

$\mathcal{K}_{23} = \{10, 11, 35, 108, 126, 153, 156, 165, 206, 254, 280, 320, 335, 348, 349, 358, 561, 575, 628, 634, 646\}$;

$\mathcal{K}_{24} = \{4, 16, 32, 56, 73, 129, 147, 174, 177, 207, 227, 275, 301, 356, 362, 369, 370, 379, 582, 596, 649\}$;

$\mathcal{K}_{25} = \{19, 25, 37, 53, 77, 94, 150, 168, 195, 198, 228, 269, 296, 322, 377, 383, 390, 391, 400, 603, 617\}$;

$\mathcal{K}_{26} = \{40, 46, 58, 74, 98, 115, 171, 189, 216, 219, 249, 290, 317, 343, 398, 404, 411, 412, 421, 624, 638\}$;

$\mathcal{K}_{27} = \{8, 61, 67, 79, 95, 119, 136, 192, 210, 237, 240, 270, 311, 338, 364, 419, 425, 432, 433, 442, 645\}$;

$\mathcal{K}_{28} = \{15, 29, 82, 88, 100, 116, 140, 157, 213, 231, 258, 261, 291, 332, 359, 385, 440, 446, 453, 454, 463\}$;

$\mathcal{K}_{29} = \{36, 50, 103, 109, 121, 137, 161, 178, 234, 252, 282, 300, 312, 353, 380, 406, 461, 467, 474, 475, 484\}$;

$\mathcal{K}_{30} = \{57, 71, 130, 142, 145, 158, 182, 199, 255, 273, 303, 321, 333, 374, 401, 427, 482, 488, 495, 505, 517\}$;

$\mathcal{K}_{31} = \{78, 92, 151, 163, 166, 179, 203, 220, 276, 294, 324, 342, 354, 395, 422, 448, 503, 509, 516, 526, 538\}$.

The third decomposition is given by 31 sets $\mathcal{K}_i$ for $i = 1, \ldots, 31$ each set consists of 21 points of one of the 31 disjoint 21-arcs in PG$(2, 5^2)$ as follows:

$\mathcal{K}_1 = \{0, 31, 62, 93, 124, 155, 186, 217, 248, 279, 310, 341, 372, 403, 434, 465, 496, 527, 558, 589, 620\}$;

$\mathcal{K}_2 = \{11, 27, 30, 68, 106, 116, 147, 167, 201, 230, 243, 270, 360, 387, 402, 455, 459, 466, 538, 542, 644\}$;

$\mathcal{K}_3 = \{9, 87, 173, 187, 264, 272, 274, 284, 285, 306, 311, 373, 413, 414, 444, 446, 509, 552, 585, 609, 624\}$;

$\mathcal{K}_4 = \{1, 49, 53, 63, 66, 77, 88, 151, 157, 162, 200, 215, 227, 265, 275, 333, 337, 382, 572, 573, 618\}$;

$\mathcal{K}_5 = \{22, 39, 44, 108, 114, 169, 176, 234, 250, 293, 326, 328, 358, 367, 384, 431, 432, 456, 470, 491, 650\}$;

$\mathcal{K}_6 = \{32, 38, 42, 103, 117, 135, 211, 222, 338, 349, 357, 421, 471, 485, 488, 497, 519, 521, 604, 607, 615\}$;

$\mathcal{K}_7 = \{43, 65, 97, 118, 190, 198, 245, 257, 268, 294, 323, 330, 346, 351, 418, 440, 464, 520, 583, 613, 622\}$;

$\mathcal{K}_8 = \{54, 69, 89, 107, 148, 242, 249, 253, 353, 363, 379, 461, 475, 492, 525, 540, 564, 606, 614, 627, 634\}$;

$\mathcal{K}_9 = \{13, 56, 57, 74, 75, 110, 112, 181, 209, 262, 305, 364, 425, 451, 458, 472, 534, 551, 567, 568, 602\}$;

$\mathcal{K}_{10} = \{2, 61, 67, 109, 125, 127, 143, 223, 226, 231, 235, 244, 263, 301, 386, 441, 483, 522, 539, 576, 645\}$;

$\mathcal{K}_{11} = \{6, 26, 59, 83, 161, 241, 252, 283, 287, 309, 316, 371, 381, 407, 474, 500, 515, 535, 591, 594, 643\}$;

$\mathcal{K}_{12} = \{41, 101, 120, 136, 142, 182, 292, 295, 368, 401, 447, 450, 487, 506, 560, 569, 597, 629, 633, 639, 641\}$;

$\mathcal{K}_{13} = \{12, 34, 36, 51, 78, 92, 134, 138, 149, 213, 232, 239, 278, 329, 409, 442, 463, 477, 489, 526, 605\}$;

$\mathcal{K}_{14} = \{5, 15, 23, 48, 55, 94, 104, 133, 172, 224, 238, 258, 286, 315, 331, 345, 544, 555, 587, 608, 616\}$;

$\mathcal{K}_{15} = \{28, 47, 50, 129, 164, 191, 193, 205, 251, 266, 281, 355, 366, 374, 443, 476, 479, 493, 510, 565, 611\}$;

$\mathcal{K}_{16} = \{17, 45, 111, 132, 163, 197, 199, 212, 229, 237, 302, 339, 392, 394, 429, 430, 481, 494, 505, 543, 632\}$;

$\mathcal{K}_{17} = \{18, 24, 60, 91, 100, 156, 183, 240, 273, 298, 299, 318, 352, 495, 502, 571, 575, 580, 628, 638, 647\}$;

$\mathcal{K}_{18} = \{79, 85, 146, 175, 179, 195, 207, 282, 359, 417, 420, 435, 490, 503, 508, 518, 529, 533, 577, 596, 649\}$;

$\mathcal{K}_{19} = \{19, 21, 96, 121, 122, 144, 158, 228, 260, 261, 393, 416, 530, 556, 582, 584, 599, 617, 621, 623, 631\}$;

$\mathcal{K}_{20} = \{105, 123, 128, 165, 168, 170, 216, 220, 246, 256, 290, 308, 336, 343, 380, 499, 513, 550, 579, 603, 637\}$;

$\mathcal{K}_{21} = \{ 20, 82, 90, 102, 119, 126, 130, 141, 152, 160, 171, 184, 203, 259, 289, 313, 325, 405, 457, 478, 566 \};$

$\mathcal{K}_{22} = \{ 14, 29, 73, 84, 95, 159, 221, 276, 280, 291, 322, 378, 385, 389, 408, 424, 501, 523, 548, 574, 619 \};$

$\mathcal{K}_{23} = \{ 3, 35, 40, 58, 70, 131, 137, 154, 236, 370, 396, 397, 453, 484, 528, 557, 563, 595, 600, 640, 648 \};$

$\mathcal{K}_{24} = \{ 10, 37, 72, 76, 145, 150, 192, 206, 255, 269, 314, 319, 365, 426, 428, 436, 467, 498, 517, 524, 561 \};$

$\mathcal{K}_{25} = \{ 4, 194, 225, 271, 307, 332, 335, 354, 356, 388, 404, 411, 422, 427, 452, 482, 507, 516, 586, 612, 642 \};$

$\mathcal{K}_{26} = \{ 25, 64, 113, 166, 177, 178, 233, 277, 297, 342, 375, 377, 415, 419, 433, 480, 504, 536, 554, 630, 636 \};$

$\mathcal{K}_{27} = \{ 33, 153, 180, 196, 204, 208, 254, 324, 327, 344, 361, 376, 399, 412, 438, 449, 454, 537, 546, 547, 625 \};$

$\mathcal{K}_{28} = \{ 16, 99, 139, 185, 202, 210, 300, 304, 334, 348, 369, 410, 437, 468, 469, 570, 581, 590, 601, 626, 646 \};$

$\mathcal{K}_{29} = \{ 52, 71, 81, 86, 140, 296, 312, 317, 340, 350, 362, 390, 395, 398, 460, 511, 514, 532, 553, 592, 610 \};$

$\mathcal{K}_{30} = \{ 46, 174, 188, 189, 214, 247, 267, 288, 383, 400, 406, 423, 439, 445, 448, 462, 473, 541, 545, 593, 598 \};$

$\mathcal{K}_{31} = \{ 7, 8, 80, 98, 115, 218, 219, 303, 320, 321, 347, 391, 486, 512, 531, 549, 559, 562, 578, 588, 635 \}.$

In summary, three non-isomorphic decompositions of $\mathrm{PG}(2, 5^2)$ into disjoint subplanes of order five are found in Section 5.1 where the first partition is the classical one, and the other two decompositions are given by Mathon and Hamilton in [17]. Section 5.2 shows that, in $\mathrm{PG}(2, 5^2)$, there are, up to isomorphism, three decompositions into disjoint 21-arcs where the first partition is the classical one, and the other two decompositions are new.

Chapter 6

The plane of order forty-nine

From Corollary 2.7 and Corollary 2.9, the projective plane $\mathrm{PG}(2,7^2)$ can be decomposed into disjoint subplanes isomorphic to $\mathrm{PG}(2,7)$ and 43-arcs. In this chapter, we discuss the different types of decompositions of $\mathrm{PG}(2,7^2)$ into disjoint subplanes of order seven and disjoint 43-arcs found by using the procedure described in Section 2.4.

The projective plane of order forty-nine $\mathrm{PG}(2,7^2)$ has 2451 points, 2451 lines, 50 points on each line and 50 lines passing through each point. Let μ be a primitive element of the finite field $\mathbb{F}_{49}$ including minimum polynomial $f(x) = x^2 - 2x - 2$ over the finite field $\mathbb{F}_7$. Accordingly, for the ground field, we have

$$\mathbb{F}_{49} \quad = \quad \{0, 1, \mu, \mu^2, \ldots, \mu^{47} \mid \mu^2 - 2\mu - 2 = 0\}.$$

To construct $\mathbb{F}_{49^3} = \mathbb{F}_{49}[x]/\langle g(x)\rangle$, we choose $g(x) = x^3 - x^2 - x - \mu^{13}$. Since $g(x)$ has no zeros in $\mathbb{F}_{49}$, it is irreducible in $\mathbb{F}_{49}[x]$. We then have the projectivity $\mathcal{T} = \mathrm{M}(\mathrm{T})$ given by

$$\mathrm{T} = \mathrm{C}(x^3 - x^2 - x - \mu^{13}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mu^{13} & 1 & 1 \end{pmatrix}$$

is cyclic on $\mathrm{PG}(2,7^2)$. Then $P(i) = P(0)\mathcal{T}^i$ for all $i = 0, 1, \ldots, 2450$. Thus, the points of $\mathrm{PG}(2,7^2)$ written in vector forms can be represented in terms of distinct integers modulo 2451 as follows:

$$0 = P(0) = (1,0,0), \ldots, i = P(i), \ldots \quad \text{for all } i = 0, 1, \ldots, 2450$$

(see details in Appendix E).

6.1 Decomposition into subplanes of order seven

A subplane of order seven in $PG(2, 7^2)$ is a set of 57 points that has 57 lines, 8 points on each line and 8 lines passing through each point. From Corollary 2.7, the projective plane $PG(2, 7^2)$ consists of 43 disjoint subplanes of order seven that are isomorphic to $PG(2, 7)$. By using the procedure described in Section 2.4, we identified four non-isomorphic decompositions of the projective plane $PG(2, 7^2)$ into 43 disjoint subplanes of order seven. The first decomposition has a projective collineation stabiliser of size 14706. Additionally, the second decomposition has a projective collineation stabiliser of size 129 and is isomorphic to $\mathbb{Z}_{43} \rtimes \mathbb{Z}_3$. Also, the third decomposition has a projective collineation stabiliser of size 18 and is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_3$. Moreover, the fourth decomposition has a projective collineation stabiliser of size 6 and is isomorphic to $\mathbb{Z}_6$.

The first decomposition is given by 43 sets $\mathcal{B}_i$ for $i = 1, \ldots, 43$ each set consists of 57 points of one of the 43 disjoint subplanes of order seven in $PG(2, 7^2)$ as follows:

$$\begin{aligned}\mathcal{B}_1 = \{ & 0, 43, 86, 129, 172, 215, 258, 301, 344, 387, 430, 473, 516, 559, 602, 645,\\ & 688, 731, 774, 817, 860, 903, 946, 989, 1032, 1075, 1118, 1161, 1204,\\ & 1247, 1290, 1333, 1376, 1419, 1462, 1505, 1548, 1591, 1634, 1677,\\ & 1720, 1763, 1806, 1849, 1892, 1935, 1978, 2021, 2064, 2107, 2150,\\ & 2193, 2236, 2279, 2322, 2365, 2408 \};\end{aligned}$$

$$\begin{aligned}\mathcal{B}_2 = \{ & 14, 57, 100, 143, 186, 229, 272, 315, 358, 401, 444, 487, 530, 573, 616,\\ & 659, 702, 745, 788, 831, 874, 917, 960, 1003, 1046, 1089, 1132, 1175,\\ & 1218, 1261, 1304, 1347, 1390, 1433, 1476, 1519, 1562, 1605, 1648,\\ & 1691, 1734, 1777, 1820, 1863, 1906, 1949, 1992, 2035, 2078, 2121,\\ & 2164, 2207, 2250, 2293, 2336, 2379, 2422 \};\end{aligned}$$

$$\begin{aligned}\mathcal{B}_3 = \{ & 28, 71, 114, 157, 200, 243, 286, 329, 372, 415, 458, 501, 544, 587, 630,\\ & 673, 716, 759, 802, 845, 888, 931, 974, 1017, 1060, 1103, 1146, 1189,\\ & 1232, 1275, 1318, 1361, 1404, 1447, 1490, 1533, 1576, 1619, 1662,\\ & 1705, 1748, 1791, 1834, 1877, 1920, 1963, 2006, 2049, 2092, 2135,\\ & 2178, 2221, 2264, 2307, 2350, 2393, 2436 \};\end{aligned}$$

$$\begin{aligned}\mathcal{B}_4 = \{ & 42, 85, 128, 171, 214, 257, 300, 343, 386, 429, 472, 515, 558, 601, 644,\\ & 687, 730, 773, 816, 859, 902, 945, 988, 1031, 1074, 1117, 1160, 1203,\\ & 1246, 1289, 1332, 1375, 1418, 1461, 1504, 1547, 1590, 1633, 1676,\\ & 1719, 1762, 1805, 1848, 1891, 1934, 1977, 2020, 2063, 2106, 2149,\\ & 2192, 2235, 2278, 2321, 2364, 2407, 2450 \};\end{aligned}$$

$\mathcal{B}_5 = \{13, 56, 99, 142, 185, 228, 271, 314, 357, 400, 443, 486, 529, 572, 615, 658, 701, 744, 787, 830, 873, 916, 959, 1002, 1045, 1088, 1131, 1174, 1217, 1260, 1303, 1346, 1389, 1432, 1475, 1518, 1561, 1604, 1647, 1690, 1733, 1776, 1819, 1862, 1905, 1948, 1991, 2034, 2077, 2120, 2163, 2206, 2249, 2292, 2335, 2378, 2421\}$;

$\mathcal{B}_6 = \{27, 70, 113, 156, 199, 242, 285, 328, 371, 414, 457, 500, 543, 586, 629, 672, 715, 758, 801, 844, 887, 930, 973, 1016, 1059, 1102, 1145, 1188, 1231, 1274, 1317, 1360, 1403, 1446, 1489, 1532, 1575, 1618, 1661, 1704, 1747, 1790, 1833, 1876, 1919, 1962, 2005, 2048, 2091, 2134, 2177, 2220, 2263, 2306, 2349, 2392, 2435\}$;

$\mathcal{B}_7 = \{41, 84, 127, 170, 213, 256, 299, 342, 385, 428, 471, 514, 557, 600, 643, 686, 729, 772, 815, 858, 901, 944, 987, 1030, 1073, 1116, 1159, 1202, 1245, 1288, 1331, 1374, 1417, 1460, 1503, 1546, 1589, 1632, 1675, 1718, 1761, 1804, 1847, 1890, 1933, 1976, 2019, 2062, 2105, 2148, 2191, 2234, 2277, 2320, 2363, 2406, 2449\}$;

$\mathcal{B}_8 = \{12, 55, 98, 141, 184, 227, 270, 313, 356, 399, 442, 485, 528, 571, 614, 657, 700, 743, 786, 829, 872, 915, 958, 1001, 1044, 1087, 1130, 1173, 1216, 1259, 1302, 1345, 1388, 1431, 1474, 1517, 1560, 1603, 1646, 1689, 1732, 1775, 1818, 1861, 1904, 1947, 1990, 2033, 2076, 2119, 2162, 2205, 2248, 2291, 2334, 2377, 2420\}$;

$\mathcal{B}_9 = \{26, 69, 112, 155, 198, 241, 284, 327, 370, 413, 456, 499, 542, 585, 628, 671, 714, 757, 800, 843, 886, 929, 972, 1015, 1058, 1101, 1144, 1187, 1230, 1273, 1316, 1359, 1402, 1445, 1488, 1531, 1574, 1617, 1660, 1703, 1746, 1789, 1832, 1875, 1918, 1961, 2004, 2047, 2090, 2133, 2176, 2219, 2262, 2305, 2348, 2391, 2434\}$;

$\mathcal{B}_{10} = \{40, 83, 126, 169, 212, 255, 298, 341, 384, 427, 470, 513, 556, 599, 642, 685, 728, 771, 814, 857, 900, 943, 986, 1029, 1072, 1115, 1158, 1201, 1244, 1287, 1330, 1373, 1416, 1459, 1502, 1545, 1588, 1631, 1674, 1717, 1760, 1803, 1846, 1889, 1932, 1975, 2018, 2061, 2104, 2147, 2190, 2233, 2276, 2319, 2362, 2405, 2448\}$;

$\mathcal{B}_{11} = \{11, 54, 97, 140, 183, 226, 269, 312, 355, 398, 441, 484, 527, 570, 613, 656, 699, 742, 785, 828, 871, 914, 957, 1000, 1043, 1086, 1129, 1172, 1215, 1258, 1301, 1344, 1387, 1430, 1473, 1516, 1559, 1602, 1645, 1688, 1731, 1774, 1817, 1860, 1903, 1946, 1989, 2032, 2075, 2118, 2161, 2204, 2247, 2290, 2333, 2376, 2419\}$;

$\mathcal{B}_{12} = \{ 25, 68, 111, 154, 197, 240, 283, 326, 369, 412, 455, 498, 541, 584, 627, 670, 713, 756, 799, 842, 885, 928, 971, 1014, 1057, 1100, 1143, 1186, 1229, 1272, 1315, 1358, 1401, 1444, 1487, 1530, 1573, 1616, 1659, 1702, 1745, 1788, 1831, 1874, 1917, 1960, 2003, 2046, 2089, 2132, 2175, 2218, 2261, 2304, 2347, 2390, 2433 \}$;

$\mathcal{B}_{13} = \{ 39, 82, 125, 168, 211, 254, 297, 340, 383, 426, 469, 512, 555, 598, 641, 684, 727, 770, 813, 856, 899, 942, 985, 1028, 1071, 1114, 1157, 1200, 1243, 1286, 1329, 1372, 1415, 1458, 1501, 1544, 1587, 1630, 1673, 1716, 1759, 1802, 1845, 1888, 1931, 1974, 2017, 2060, 2103, 2146, 2189, 2232, 2275, 2318, 2361, 2404, 2447 \}$;

$\mathcal{B}_{14} = \{ 10, 53, 96, 139, 182, 225, 268, 311, 354, 397, 440, 483, 526, 569, 612, 655, 698, 741, 784, 827, 870, 913, 956, 999, 1042, 1085, 1128, 1171, 1214, 1257, 1300, 1343, 1386, 1429, 1472, 1515, 1558, 1601, 1644, 1687, 1730, 1773, 1816, 1859, 1902, 1945, 1988, 2031, 2074, 2117, 2160, 2203, 2246, 2289, 2332, 2375, 2418 \}$;

$\mathcal{B}_{15} = \{ 24, 67, 110, 153, 196, 239, 282, 325, 368, 411, 454, 497, 540, 583, 626, 669, 712, 755, 798, 841, 884, 927, 970, 1013, 1056, 1099, 1142, 1185, 1228, 1271, 1314, 1357, 1400, 1443, 1486, 1529, 1572, 1615, 1658, 1701, 1744, 1787, 1830, 1873, 1916, 1959, 2002, 2045, 2088, 2131, 2174, 2217, 2260, 2303, 2346, 2389, 2432 \}$;

$\mathcal{B}_{16} = \{ 38, 81, 124, 167, 210, 253, 296, 339, 382, 425, 468, 511, 554, 597, 640, 683, 726, 769, 812, 855, 898, 941, 984, 1027, 1070, 1113, 1156, 1199, 1242, 1285, 1328, 1371, 1414, 1457, 1500, 1543, 1586, 1629, 1672, 1715, 1758, 1801, 1844, 1887, 1930, 1973, 2016, 2059, 2102, 2145, 2188, 2231, 2274, 2317, 2360, 2403, 2446 \}$;

$\mathcal{B}_{17} = \{ 9, 52, 95, 138, 181, 224, 267, 310, 353, 396, 439, 482, 525, 568, 611, 654, 697, 740, 783, 826, 869, 912, 955, 998, 1041, 1084, 1127, 1170, 1213, 1256, 1299, 1342, 1385, 1428, 1471, 1514, 1557, 1600, 1643, 1686, 1729, 1772, 1815, 1858, 1901, 1944, 1987, 2030, 2073, 2116, 2159, 2202, 2245, 2288, 2331, 2374, 2417 \}$;

$\mathcal{B}_{18} = \{ 23, 66, 109, 152, 195, 238, 281, 324, 367, 410, 453, 496, 539, 582, 625, 668, 711, 754, 797, 840, 883, 926, 969, 1012, 1055, 1098, 1141, 1184, 1227, 1270, 1313, 1356, 1399, 1442, 1485, 1528, 1571, 1614, 1657, 1700, 1743, 1786, 1829, 1872, 1915, 1958, 2001, 2044, 2087, 2130, 2173, 2216, 2259, 2302, 2345, 2388, 2431 \}$;

$\mathcal{B}_{19} = \{$ 37, 80, 123, 166, 209, 252, 295, 338, 381, 424, 467, 510, 553, 596, 639, 682, 725, 768, 811, 854, 897, 940, 983, 1026, 1069, 1112, 1155, 1198, 1241, 1284, 1327, 1370, 1413, 1456, 1499, 1542, 1585, 1628, 1671, 1714, 1757, 1800, 1843, 1886, 1929, 1972, 2015, 2058, 2101, 2144, 2187, 2230, 2273, 2316, 2359, 2402, 2445 $\}$;

$\mathcal{B}_{20} = \{$ 8, 51, 94, 137, 180, 223, 266, 309, 352, 395, 438, 481, 524, 567, 610, 653, 696, 739, 782, 825, 868, 911, 954, 997, 1040, 1083, 1126, 1169, 1212, 1255, 1298, 1341, 1384, 1427, 1470, 1513, 1556, 1599, 1642, 1685, 1728, 1771, 1814, 1857, 1900, 1943, 1986, 2029, 2072, 2115, 2158, 2201, 2244, 2287, 2330, 2373, 2416 $\}$;

$\mathcal{B}_{21} = \{$ 22, 65, 108, 151, 194, 237, 280, 323, 366, 409, 452, 495, 538, 581, 624, 667, 710, 753, 796, 839, 882, 925, 968, 1011, 1054, 1097, 1140, 1183, 1226, 1269, 1312, 1355, 1398, 1441, 1484, 1527, 1570, 1613, 1656, 1699, 1742, 1785, 1828, 1871, 1914, 1957, 2000, 2043, 2086, 2129, 2172, 2215, 2258, 2301, 2344, 2387, 2430 $\}$;

$\mathcal{B}_{22} = \{$ 36, 79, 122, 165, 208, 251, 294, 337, 380, 423, 466, 509, 552, 595, 638, 681, 724, 767, 810, 853, 896, 939, 982, 1025, 1068, 1111, 1154, 1197, 1240, 1283, 1326, 1369, 1412, 1455, 1498, 1541, 1584, 1627, 1670, 1713, 1756, 1799, 1842, 1885, 1928, 1971, 2014, 2057, 2100, 2143, 2186, 2229, 2272, 2315, 2358, 2401, 2444 $\}$;

$\mathcal{B}_{23} = \{$ 7, 50, 93, 136, 179, 222, 265, 308, 351, 394, 437, 480, 523, 566, 609, 652, 695, 738, 781, 824, 867, 910, 953, 996, 1039, 1082, 1125, 1168, 1211, 1254, 1297, 1340, 1383, 1426, 1469, 1512, 1555, 1598, 1641, 1684, 1727, 1770, 1813, 1856, 1899, 1942, 1985, 2028, 2071, 2114, 2157, 2200, 2243, 2286, 2329, 2372, 2415 $\}$;

$\mathcal{B}_{24} = \{$ 21, 64, 107, 150, 193, 236, 279, 322, 365, 408, 451, 494, 537, 580, 623, 666, 709, 752, 795, 838, 881, 924, 967, 1010, 1053, 1096, 1139, 1182, 1225, 1268, 1311, 1354, 1397, 1440, 1483, 1526, 1569, 1612, 1655, 1698, 1741, 1784, 1827, 1870, 1913, 1956, 1999, 2042, 2085, 2128, 2171, 2214, 2257, 2300, 2343, 2386, 2429 $\}$;

$\mathcal{B}_{25} = \{$ 35, 78, 121, 164, 207, 250, 293, 336, 379, 422, 465, 508, 551, 594, 637, 680, 723, 766, 809, 852, 895, 938, 981, 1024, 1067, 1110, 1153, 1196, 1239, 1282, 1325, 1368, 1411, 1454, 1497, 1540, 1583, 1626, 1669, 1712, 1755, 1798, 1841, 1884, 1927, 1970, 2013, 2056, 2099, 2142, 2185, 2228, 2271, 2314, 2357, 2400, 2443 $\}$;

$\mathcal{B}_{26} = \{ 6, 49, 92, 135, 178, 221, 264, 307, 350, 393, 436, 479, 522, 565, 608, 651, 694, 737, 780, 823, 866, 909, 952, 995, 1038, 1081, 1124, 1167, 1210, 1253, 1296, 1339, 1382, 1425, 1468, 1511, 1554, 1597, 1640, 1683, 1726, 1769, 1812, 1855, 1898, 1941, 1984, 2027, 2070, 2113, 2156, 2199, 2242, 2285, 2328, 2371, 2414 \}$;

$\mathcal{B}_{27} = \{ 20, 63, 106, 149, 192, 235, 278, 321, 364, 407, 450, 493, 536, 579, 622, 665, 708, 751, 794, 837, 880, 923, 966, 1009, 1052, 1095, 1138, 1181, 1224, 1267, 1310, 1353, 1396, 1439, 1482, 1525, 1568, 1611, 1654, 1697, 1740, 1783, 1826, 1869, 1912, 1955, 1998, 2041, 2084, 2127, 2170, 2213, 2256, 2299, 2342, 2385, 2428 \}$;

$\mathcal{B}_{28} = \{ 34, 77, 120, 163, 206, 249, 292, 335, 378, 421, 464, 507, 550, 593, 636, 679, 722, 765, 808, 851, 894, 937, 980, 1023, 1066, 1109, 1152, 1195, 1238, 1281, 1324, 1367, 1410, 1453, 1496, 1539, 1582, 1625, 1668, 1711, 1754, 1797, 1840, 1883, 1926, 1969, 2012, 2055, 2098, 2141, 2184, 2227, 2270, 2313, 2356, 2399, 2442 \}$;

$\mathcal{B}_{29} = \{ 5, 48, 91, 134, 177, 220, 263, 306, 349, 392, 435, 478, 521, 564, 607, 650, 693, 736, 779, 822, 865, 908, 951, 994, 1037, 1080, 1123, 1166, 1209, 1252, 1295, 1338, 1381, 1424, 1467, 1510, 1553, 1596, 1639, 1682, 1725, 1768, 1811, 1854, 1897, 1940, 1983, 2026, 2069, 2112, 2155, 2198, 2241, 2284, 2327, 2370, 2413 \}$;

$\mathcal{B}_{30} = \{ 19, 62, 105, 148, 191, 234, 277, 320, 363, 406, 449, 492, 535, 578, 621, 664, 707, 750, 793, 836, 879, 922, 965, 1008, 1051, 1094, 1137, 1180, 1223, 1266, 1309, 1352, 1395, 1438, 1481, 1524, 1567, 1610, 1653, 1696, 1739, 1782, 1825, 1868, 1911, 1954, 1997, 2040, 2083, 2126, 2169, 2212, 2255, 2298, 2341, 2384, 2427 \}$;

$\mathcal{B}_{31} = \{ 33, 76, 119, 162, 205, 248, 291, 334, 377, 420, 463, 506, 549, 592, 635, 678, 721, 764, 807, 850, 893, 936, 979, 1022, 1065, 1108, 1151, 1194, 1237, 1280, 1323, 1366, 1409, 1452, 1495, 1538, 1581, 1624, 1667, 1710, 1753, 1796, 1839, 1882, 1925, 1968, 2011, 2054, 2097, 2140, 2183, 2226, 2269, 2312, 2355, 2398, 2441 \}$;

$\mathcal{B}_{32} = \{ 4, 47, 90, 133, 176, 219, 262, 305, 348, 391, 434, 477, 520, 563, 606, 649, 692, 735, 778, 821, 864, 907, 950, 993, 1036, 1079, 1122, 1165, 1208, 1251, 1294, 1337, 1380, 1423, 1466, 1509, 1552, 1595, 1638, 1681, 1724, 1767, 1810, 1853, 1896, 1939, 1982, 2025, 2068, 2111, 2154, 2197, 2240, 2283, 2326, 2369, 2412 \}$;

$\mathcal{B}_{33} = \{ 18, 61, 104, 147, 190, 233, 276, 319, 362, 405, 448, 491, 534, 577, 620, 663, 706, 749, 792, 835, 878, 921, 964, 1007, 1050, 1093, 1136, 1179, 1222, 1265, 1308, 1351, 1394, 1437, 1480, 1523, 1566, 1609, 1652, 1695, 1738, 1781, 1824, 1867, 1910, 1953, 1996, 2039, 2082, 2125, 2168, 2211, 2254, 2297, 2340, 2383, 2426 \}$;

$\mathcal{B}_{34} = \{ 32, 75, 118, 161, 204, 247, 290, 333, 376, 419, 462, 505, 548, 591, 634, 677, 720, 763, 806, 849, 892, 935, 978, 1021, 1064, 1107, 1150, 1193, 1236, 1279, 1322, 1365, 1408, 1451, 1494, 1537, 1580, 1623, 1666, 1709, 1752, 1795, 1838, 1881, 1924, 1967, 2010, 2053, 2096, 2139, 2182, 2225, 2268, 2311, 2354, 2397, 2440 \}$;

$\mathcal{B}_{35} = \{ 3, 46, 89, 132, 175, 218, 261, 304, 347, 390, 433, 476, 519, 562, 605, 648, 691, 734, 777, 820, 863, 906, 949, 992, 1035, 1078, 1121, 1164, 1207, 1250, 1293, 1336, 1379, 1422, 1465, 1508, 1551, 1594, 1637, 1680, 1723, 1766, 1809, 1852, 1895, 1938, 1981, 2024, 2067, 2110, 2153, 2196, 2239, 2282, 2325, 2368, 2411 \}$;

$\mathcal{B}_{36} = \{ 17, 60, 103, 146, 189, 232, 275, 318, 361, 404, 447, 490, 533, 576, 619, 662, 705, 748, 791, 834, 877, 920, 963, 1006, 1049, 1092, 1135, 1178, 1221, 1264, 1307, 1350, 1393, 1436, 1479, 1522, 1565, 1608, 1651, 1694, 1737, 1780, 1823, 1866, 1909, 1952, 1995, 2038, 2081, 2124, 2167, 2210, 2253, 2296, 2339, 2382, 2425 \}$;

$\mathcal{B}_{37} = \{ 31, 74, 117, 160, 203, 246, 289, 332, 375, 418, 461, 504, 547, 590, 633, 676, 719, 762, 805, 848, 891, 934, 977, 1020, 1063, 1106, 1149, 1192, 1235, 1278, 1321, 1364, 1407, 1450, 1493, 1536, 1579, 1622, 1665, 1708, 1751, 1794, 1837, 1880, 1923, 1966, 2009, 2052, 2095, 2138, 2181, 2224, 2267, 2310, 2353, 2396, 2439 \}$;

$\mathcal{B}_{38} = \{ 2, 45, 88, 131, 174, 217, 260, 303, 346, 389, 432, 475, 518, 561, 604, 647, 690, 733, 776, 819, 862, 905, 948, 991, 1034, 1077, 1120, 1163, 1206, 1249, 1292, 1335, 1378, 1421, 1464, 1507, 1550, 1593, 1636, 1679, 1722, 1765, 1808, 1851, 1894, 1937, 1980, 2023, 2066, 2109, 2152, 2195, 2238, 2281, 2324, 2367, 2410 \}$;

$\mathcal{B}_{39} = \{ 16, 59, 102, 145, 188, 231, 274, 317, 360, 403, 446, 489, 532, 575, 618, 661, 704, 747, 790, 833, 876, 919, 962, 1005, 1048, 1091, 1134, 1177, 1220, 1263, 1306, 1349, 1392, 1435, 1478, 1521, 1564, 1607, 1650, 1693, 1736, 1779, 1822, 1865, 1908, 1951, 1994, 2037, 2080, 2123, 2166, 2209, 2252, 2295, 2338, 2381, 2424 \}$;

$\mathcal{B}_{40} = \{30, 73, 116, 159, 202, 245, 288, 331, 374, 417, 460, 503, 546, 589, 632, 675, 718, 761, 804, 847, 890, 933, 976, 1019, 1062, 1105, 1148, 1191, 1234, 1277, 1320, 1363, 1406, 1449, 1492, 1535, 1578, 1621, 1664, 1707, 1750, 1793, 1836, 1879, 1922, 1965, 2008, 2051, 2094, 2137, 2180, 2223, 2266, 2309, 2352, 2395, 2438\}$;

$\mathcal{B}_{41} = \{1, 44, 87, 130, 173, 216, 259, 302, 345, 388, 431, 474, 517, 560, 603, 646, 689, 732, 775, 818, 861, 904, 947, 990, 1033, 1076, 1119, 1162, 1205, 1248, 1291, 1334, 1377, 1420, 1463, 1506, 1549, 1592, 1635, 1678, 1721, 1764, 1807, 1850, 1893, 1936, 1979, 2022, 2065, 2108, 2151, 2194, 2237, 2280, 2323, 2366, 2409\}$;

$\mathcal{B}_{42} = \{15, 58, 101, 144, 187, 230, 273, 316, 359, 402, 445, 488, 531, 574, 617, 660, 703, 746, 789, 832, 875, 918, 961, 1004, 1047, 1090, 1133, 1176, 1219, 1262, 1305, 1348, 1391, 1434, 1477, 1520, 1563, 1606, 1649, 1692, 1735, 1778, 1821, 1864, 1907, 1950, 1993, 2036, 2079, 2122, 2165, 2208, 2251, 2294, 2337, 2380, 2423\}$;

$\mathcal{B}_{43} = \{29, 72, 115, 158, 201, 244, 287, 330, 373, 416, 459, 502, 545, 588, 631, 674, 717, 760, 803, 846, 889, 932, 975, 1018, 1061, 1104, 1147, 1190, 1233, 1276, 1319, 1362, 1405, 1448, 1491, 1534, 1577, 1620, 1663, 1706, 1749, 1792, 1835, 1878, 1921, 1964, 2007, 2050, 2093, 2136, 2179, 2222, 2265, 2308, 2351, 2394, 2437\}$.

The second decomposition is given by 43 sets $\mathcal{B}_i$ for $i = 1, \ldots, 43$ each set consists of 57 points of one of the 43 disjoint subplanes of order seven in PG$(2, 7^2)$ as follows:

$\mathcal{B}_1 = \{0, 43, 86, 129, 172, 215, 258, 301, 344, 387, 430, 473, 516, 559, 602, 645, 688, 731, 774, 817, 860, 903, 946, 989, 1032, 1075, 1118, 1161, 1204, 1247, 1290, 1333, 1376, 1419, 1462, 1505, 1548, 1591, 1634, 1677, 1720, 1763, 1806, 1849, 1892, 1935, 1978, 2021, 2064, 2107, 2150, 2193, 2236, 2279, 2322, 2365, 2408\}$;

$\mathcal{B}_2 = \{7, 20, 50, 62, 93, 117, 226, 315, 397, 440, 483, 501, 669, 705, 712, 716, 869, 879, 905, 909, 912, 942, 955, 971, 1130, 1176, 1421, 1464, 1537, 1540, 1554, 1584, 1596, 1597, 1630, 1639, 1640, 1683, 1701, 1802, 1846, 1851, 1855, 2068, 2098, 2105, 2186, 2198, 2239, 2249, 2282, 2292, 2317, 2325, 2391, 2401, 2419\}$;

$\mathcal{B}_3 = \{42, 49, 103, 124, 146, 189, 190, 235, 238, 281, 324, 457, 461, 497, 500, 526, 540, 583, 627, 698, 722, 746, 776, 795, 833, 850, 885, 922, 952, 1001, 1048, 1119, 1125, 1171, 1181, 1480, 1543, 1577, 1586, 1594, 1667, 1727, 1746, 1920, 1942, 1994, 2104, 2159, 2232, 2278, 2284, 2289, 2314, 2321, 2362, 2364, 2376\};$

$\mathcal{B}_4 = \{21, 183, 204, 207, 278, 441, 584, 623, 626, 660, 798, 823, 838, 890, 908, 919, 937, 993, 1008, 1120, 1128, 1147, 1174, 1217, 1246, 1260, 1271, 1365, 1371, 1385, 1389, 1414, 1428, 1457, 1459, 1528, 1585, 1605, 1617, 1628, 1671, 1710, 1804, 1810, 1826, 1847, 1883, 1909, 2102, 2146, 2180, 2310, 2353, 2389, 2395, 2396, 2417\};$

$\mathcal{B}_5 = \{35, 64, 149, 250, 261, 424, 495, 518, 561, 604, 647, 700, 743, 754, 759, 784, 797, 812, 842, 980, 1042, 1061, 1076, 1100, 1123, 1167, 1175, 1228, 1263, 1291, 1341, 1422, 1566, 1598, 1618, 1672, 1713, 1786, 1796, 1805, 1833, 1845, 1848, 1853, 1876, 1884, 1891, 1919, 1971, 2048, 2069, 2083, 2145, 2237, 2266, 2287, 2383\};$

$\mathcal{B}_6 = \{85, 159, 180, 290, 329, 393, 420, 493, 494, 514, 576, 617, 654, 661, 682, 701, 704, 729, 741, 744, 747, 753, 772, 787, 790, 793, 815, 832, 851, 884, 944, 995, 1089, 1107, 1142, 1162, 1166, 1210, 1254, 1316, 1359, 1512, 1514, 1670, 1737, 1778, 1888, 1927, 1939, 1987, 2101, 2103, 2144, 2258, 2298, 2361, 2426\};$

$\mathcal{B}_7 = \{23, 61, 150, 243, 321, 332, 347, 408, 436, 438, 536, 563, 635, 703, 841, 883, 982, 999, 1018, 1029, 1037, 1072, 1097, 1150, 1164, 1195, 1211, 1233, 1257, 1356, 1360, 1403, 1432, 1446, 1493, 1506, 1581, 1758, 1857, 1861, 1900, 1928, 1944, 2008, 2016, 2051, 2076, 2094, 2106, 2119, 2137, 2162, 2170, 2179, 2291, 2341, 2357\};$

$\mathcal{B}_8 = \{26, 107, 241, 252, 256, 284, 299, 323, 327, 342, 366, 421, 456, 523, 662, 678, 778, 866, 904, 947, 990, 991, 1002, 1023, 1033, 1046, 1047, 1099, 1104, 1136, 1207, 1236, 1238, 1261, 1313, 1355, 1357, 1370, 1381, 1469, 1479, 1559, 1600, 1602, 1715, 1724, 1858, 2013, 2020, 2091, 2152, 2207, 2212, 2213, 2230, 2271, 2272\};$

$\mathcal{B}_9 = \{8, 40, 79, 105, 169, 192, 200, 209, 365, 415, 418, 445, 464, 488, 522, 531, 549, 574, 607, 780, 821, 916, 969, 987, 1126, 1132, 1134, 1209, 1227, 1235, 1293, 1351, 1361, 1424, 1448, 1498, 1522, 1541, 1557, 1603, 1646, 1661, 1689, 1813, 1870, 1970, 1973, 1986, 2190, 2218, 2231, 2261, 2295, 2297, 2311, 2405, 2448\};$

$\mathcal{B}_{10} = \{$ 30, 75, 106, 123, 148, 269, 286, 452, 479, 484, 527, 557, 570, 650, 687, 692, 699, 709, 846, 889, 893, 894, 925, 926, 932, 975, 1021, 1028, 1039, 1071, 1077, 1082, 1191, 1278, 1282, 1394, 1476, 1575, 1608, 1782, 1815, 1894, 1977, 2043, 2071, 2073, 2192, 2219, 2228, 2262, 2299, 2305, 2316, 2334, 2368, 2384, 2416 $\}$;

$\mathcal{B}_{11} = \{$ 87, 118, 119, 121, 132, 232, 471, 499, 550, 630, 721, 737, 767, 808, 873, 874, 882, 917, 928, 936, 960, 967, 1003, 1036, 1078, 1080, 1088, 1143, 1184, 1186, 1220, 1229, 1265, 1358, 1364, 1377, 1442, 1627, 1825, 1904, 1932, 1937, 1975, 2018, 2034, 2037, 2063, 2072, 2100, 2242, 2256, 2273, 2330, 2375, 2386, 2418, 2429 $\}$;

$\mathcal{B}_{12} = \{$ 11, 28, 39, 54, 63, 69, 71, 80, 97, 114, 116, 157, 162, 168, 175, 212, 242, 262, 308, 338, 379, 447, 507, 538, 673, 736, 796, 864, 918, 933, 959, 994, 1079, 1081, 1168, 1253, 1292, 1393, 1532, 1609, 1641, 1747, 1798, 1818, 1841, 2023, 2028, 2029, 2057, 2080, 2189, 2255, 2404, 2407, 2439, 2440, 2447 $\}$;

$\mathcal{B}_{13} = \{$ 46, 77, 94, 203, 305, 307, 333, 350, 407, 428, 490, 537, 579, 592, 643, 670, 693, 713, 742, 756, 765, 835, 924, 976, 1084, 1085, 1149, 1213, 1256, 1299, 1300, 1303, 1319, 1330, 1339, 1343, 1384, 1386, 1427, 1435, 1437, 1463, 1500, 1515, 1589, 1652, 1655, 1665, 1911, 1934, 1989, 2166, 2169, 2233, 2280, 2434, 2450 $\}$;

$\mathcal{B}_{14} = \{$ 55, 98, 161, 197, 248, 263, 331, 336, 376, 453, 485, 506, 582, 625, 665, 668, 735, 830, 862, 878, 949, 951, 992, 1004, 1035, 1062, 1067, 1109, 1185, 1281, 1342, 1346, 1400, 1401, 1507, 1550, 1593, 1615, 1636, 1721, 1764, 1808, 1821, 1860, 1868, 1931, 1962, 1974, 1990, 2010, 2017, 2084, 2127, 2301, 2323, 2324, 2344 $\}$;

$\mathcal{B}_{15} = \{$ 16, 38, 68, 155, 205, 211, 218, 335, 378, 391, 433, 649, 671, 711, 714, 827, 858, 870, 913, 1005, 1025, 1068, 1101, 1262, 1264, 1295, 1296, 1305, 1307, 1308, 1350, 1367, 1438, 1458, 1467, 1489, 1650, 1678, 1693, 1736, 1779, 1800, 1864, 1929, 1951, 1972, 2005, 2015, 2027, 2053, 2059, 2067, 2126, 2222, 2274, 2360, 2409 $\}$;

$\mathcal{B}_{16} = \{$ 19, 32, 191, 348, 354, 384, 385, 427, 463, 521, 533, 546, 564, 566, 589, 609, 622, 632, 675, 748, 814, 847, 901, 927, 964, 970, 1013, 1020, 1063, 1106, 1219, 1222, 1329, 1353, 1363, 1553, 1556, 1558, 1663, 1685, 1706, 1728, 1771, 1901, 1950, 2058, 2110, 2116, 2202, 2205, 2226, 2250, 2265, 2319, 2331, 2352, 2382 $\}$;

$\mathcal{B}_{17} = \{22, 89, 151, 194, 225, 237, 259, 277, 289, 370, 398, 454, 504, 606, 807, 914, 921, 957, 979, 1248, 1270, 1324, 1344, 1404, 1485, 1571, 1620, 1658, 1679, 1691, 1696, 1700, 1722, 1732, 1734, 1739, 1765, 1801, 1814, 1844, 1887, 1893, 1913, 1936, 1956, 1979, 2022, 2055, 2138, 2139, 2194, 2248, 2293, 2351, 2369, 2397, 2425\}$;

$\mathcal{B}_{18} = \{3, 33, 76, 81, 83, 166, 280, 325, 346, 381, 404, 413, 467, 491, 596, 730, 789, 845, 888, 948, 950, 1057, 1065, 1122, 1163, 1179, 1197, 1325, 1326, 1328, 1368, 1369, 1412, 1434, 1447, 1477, 1510, 1520, 1563, 1573, 1616, 1643, 1648, 1686, 1729, 1735, 1822, 1865, 1908, 2003, 2096, 2181, 2196, 2241, 2251, 2348, 2379\}$;

$\mathcal{B}_{19} = \{60, 126, 137, 160, 171, 176, 201, 214, 219, 223, 234, 318, 352, 383, 389, 426, 489, 613, 718, 738, 761, 773, 802, 804, 813, 867, 910, 953, 965, 1012, 1055, 1091, 1098, 1170, 1190, 1199, 1208, 1306, 1455, 1523, 1533, 1666, 1695, 1835, 1878, 1921, 1964, 2061, 2077, 2136, 2153, 2285, 2318, 2328, 2354, 2373, 2374\}$;

$\mathcal{B}_{20} = \{4, 47, 90, 92, 128, 202, 229, 247, 285, 298, 330, 419, 539, 656, 749, 792, 859, 996, 1022, 1041, 1054, 1140, 1214, 1218, 1269, 1272, 1287, 1454, 1518, 1551, 1561, 1654, 1723, 1730, 1743, 1753, 1773, 1781, 1829, 1836, 1863, 1906, 1949, 1992, 2065, 2085, 2108, 2120, 2151, 2160, 2164, 2214, 2257, 2300, 2326, 2438, 2441\}$;

$\mathcal{B}_{21} = \{82, 133, 304, 328, 358, 372, 410, 448, 476, 505, 577, 620, 639, 663, 725, 732, 828, 1017, 1060, 1103, 1146, 1165, 1314, 1315, 1318, 1334, 1475, 1497, 1549, 1606, 1626, 1632, 1649, 1669, 1692, 1712, 1817, 1830, 1834, 1866, 1869, 1873, 1886, 1896, 1924, 1967, 2038, 2177, 2206, 2220, 2229, 2260, 2315, 2343, 2356, 2414, 2444\}$;

$\mathcal{B}_{22} = \{6, 18, 25, 91, 120, 125, 135, 178, 221, 253, 273, 296, 343, 386, 395, 414, 429, 472, 481, 528, 562, 619, 644, 683, 706, 791, 875, 1073, 1090, 1116, 1160, 1429, 1487, 1622, 1623, 1642, 1675, 1738, 1740, 1752, 1755, 1757, 1770, 1794, 1795, 1838, 1856, 1963, 1981, 1985, 2007, 2024, 2050, 2093, 2134, 2149, 2188\}$;

$\mathcal{B}_{23} = \{2, 12, 56, 95, 108, 138, 264, 297, 306, 375, 462, 525, 547, 571, 666, 674, 752, 877, 881, 1030, 1045, 1144, 1193, 1266, 1276, 1286, 1289, 1312, 1398, 1472, 1491, 1513, 1529, 1680, 1690, 1733, 1761, 1776, 1783, 1809, 1819, 1852, 1881, 1895, 1912, 1955, 1991, 1998, 2030, 2035, 2078, 2121, 2281, 2296, 2339, 2420, 2442\}$;

$\mathcal{B}_{24} = \{$ 27, 34, 36, 78, 122, 145, 163, 164, 177, 182, 185, 199, 206, 249, 293, 519, 585, 628, 633, 657, 702, 715, 857, 1034, 1187, 1189, 1206, 1232, 1250, 1275, 1283, 1304, 1399, 1481, 1519, 1572, 1644, 1704, 1915, 1938, 1958, 1995, 2041, 2052, 2054, 2095, 2124, 2167, 2210, 2345, 2349, 2377, 2392, 2403, 2424, 2430, 2435 $\}$;

$\mathcal{B}_{25} = \{$ 24, 67, 220, 260, 282, 292, 295, 303, 341, 349, 392, 435, 458, 515, 542, 558, 600, 601, 811, 824, 834, 844, 854, 900, 1038, 1124, 1154, 1177, 1241, 1245, 1273, 1288, 1320, 1331, 1349, 1352, 1374, 1387, 1492, 1546, 1565, 1568, 1611, 1744, 1751, 1880, 1923, 1966, 2062, 2079, 2114, 2197, 2245, 2253, 2269, 2307, 2446 $\}$;

$\mathcal{B}_{26} = \{$ 29, 51, 73, 88, 131, 141, 167, 174, 184, 216, 227, 245, 255, 270, 319, 364, 388, 403, 442, 446, 450, 478, 567, 590, 610, 695, 871, 898, 941, 958, 986, 997, 1010, 1156, 1321, 1395, 1430, 1524, 1567, 1610, 1614, 1633, 1862, 1905, 1947, 1948, 2009, 2046, 2140, 2171, 2191, 2235, 2270, 2288, 2313, 2410, 2412 $\}$;

$\mathcal{B}_{27} = \{$ 5, 9, 48, 57, 70, 102, 112, 113, 156, 217, 231, 274, 317, 422, 510, 529, 534, 740, 757, 783, 785, 800, 843, 856, 886, 968, 998, 1058, 1066, 1087, 1131, 1152, 1249, 1420, 1484, 1494, 1516, 1527, 1530, 1574, 1583, 1592, 1653, 1657, 1750, 1793, 1914, 1980, 2042, 2089, 2183, 2208, 2283, 2312, 2355, 2380, 2398 $\}$;

$\mathcal{B}_{28} = \{$ 17, 74, 109, 140, 152, 158, 208, 251, 266, 340, 360, 367, 405, 470, 513, 532, 553, 556, 599, 638, 646, 689, 771, 819, 839, 899, 961, 1133, 1180, 1188, 1223, 1252, 1338, 1382, 1392, 1417, 1444, 1449, 1460, 1503, 1578, 1621, 1662, 1664, 1703, 1709, 1790, 2123, 2143, 2175, 2203, 2267, 2276, 2359, 2402, 2411, 2445 $\}$;

$\mathcal{B}_{29} = \{$ 84, 179, 186, 187, 230, 254, 275, 288, 309, 313, 345, 355, 356, 399, 474, 475, 517, 560, 686, 708, 733, 799, 962, 985, 988, 1000, 1019, 1043, 1086, 1129, 1183, 1301, 1362, 1405, 1416, 1456, 1499, 1534, 1580, 1684, 1687, 1707, 1714, 1766, 1879, 1968, 2011, 2014, 2115, 2158, 2200, 2201, 2224, 2243, 2246, 2303, 2427 $\}$;

$\mathcal{B}_{30} = \{$ 15, 31, 58, 101, 432, 580, 581, 588, 603, 618, 624, 631, 667, 726, 764, 775, 876, 929, 972, 1014, 1015, 1096, 1139, 1212, 1226, 1255, 1390, 1410, 1431, 1433, 1470, 1526, 1555, 1562, 1601, 1635, 1637, 1659, 1702, 1745, 1787, 1788, 1791, 1946, 1960, 2033, 2060, 2081, 2111, 2154, 2309, 2332, 2335, 2337, 2346, 2358, 2366 $\}$;

$\mathcal{B}_{31}$ = { 144, 154, 198, 210, 233, 276, 287, 316, 416, 451, 459, 469, 502, 508, 512, 543, 544, 551, 555, 568, 575, 587, 598, 616, 642, 659, 685, 690, 728, 769, 770, 907, 938, 956, 1031, 1074, 1117, 1205, 1279, 1322, 1542, 1756, 1799, 1837, 1842, 1899, 1907, 1952, 1965, 2047, 2129, 2172, 2223, 2275, 2387, 2432, 2443 };

$\mathcal{B}_{32}$ = { 13, 45, 52, 142, 320, 363, 444, 487, 530, 545, 611, 717, 853, 855, 861, 880, 896, 923, 966, 1009, 1056, 1111, 1172, 1215, 1258, 1297, 1298, 1336, 1340, 1354, 1379, 1383, 1397, 1408, 1415, 1440, 1451, 1468, 1511, 1564, 1607, 1708, 1760, 1816, 1859, 1890, 1898, 1902, 1922, 1945, 2117, 2221, 2264, 2308, 2378, 2388, 2421 };

$\mathcal{B}_{33}$ = { 188, 193, 236, 279, 311, 394, 402, 437, 460, 503, 573, 586, 629, 652, 672, 697, 745, 766, 786, 794, 801, 809, 818, 837, 852, 1108, 1151, 1224, 1284, 1465, 1508, 1547, 1582, 1590, 1625, 1651, 1668, 1694, 1711, 1831, 1872, 1874, 1916, 1917, 1930, 1959, 2002, 2045, 2049, 2092, 2135, 2215, 2217, 2290, 2336, 2367, 2431 };

$\mathcal{B}_{34}$ = { 59, 66, 339, 359, 382, 411, 425, 443, 468, 486, 498, 640, 641, 684, 727, 768, 803, 939, 1040, 1251, 1294, 1327, 1375, 1402, 1407, 1418, 1450, 1461, 1490, 1496, 1535, 1539, 1726, 1741, 1768, 1769, 1772, 1780, 1784, 1807, 1811, 1812, 1823, 1850, 1854, 1897, 1926, 1933, 1976, 1999, 2019, 2056, 2099, 2142, 2148, 2178, 2350 };

$\mathcal{B}_{35}$ = { 181, 224, 267, 271, 310, 314, 357, 431, 480, 482, 492, 524, 535, 565, 578, 608, 621, 651, 760, 816, 829, 831, 872, 915, 1044, 1052, 1083, 1095, 1102, 1115, 1138, 1141, 1145, 1153, 1196, 1285, 1348, 1391, 1406, 1411, 1413, 1504, 1536, 1579, 1676, 1682, 1725, 1759, 1824, 1867, 1957, 1988, 2031, 2066, 2074, 2109, 2112 };

$\mathcal{B}_{36}$ = { 37, 41, 302, 400, 406, 449, 548, 572, 591, 655, 681, 788, 836, 930, 973, 1016, 1127, 1169, 1280, 1323, 1366, 1409, 1441, 1445, 1488, 1531, 1645, 1660, 1749, 1754, 1792, 1797, 1827, 1840, 2000, 2001, 2044, 2087, 2088, 2113, 2130, 2131, 2156, 2163, 2174, 2195, 2209, 2238, 2252, 2302, 2342, 2371, 2385, 2423, 2428, 2436, 2449 };

$\mathcal{B}_{37}$ = { 165, 222, 371, 373, 496, 511, 554, 593, 597, 605, 636, 648, 679, 724, 755, 897, 940, 983, 1026, 1059, 1105, 1148, 1158, 1194, 1198, 1201, 1231, 1237, 1239, 1244, 1345, 1373, 1388, 1423, 1439, 1466, 1482, 1509, 1552, 1624, 1697, 1762, 1777, 1820, 1940, 1983, 2026, 2086, 2244, 2277, 2294, 2304, 2320, 2338, 2363, 2381, 2393 };

$\mathcal{B}_{38}$ = { 1, 44, 127, 244, 265, 353, 396, 401, 439, 653, 658, 664, 696, 707, 710, 739, 750, 779, 782, 810, 822, 865, 920, 931, 954, 963, 974, 1114, 1173, 1216, 1234, 1259, 1277, 1337, 1380, 1569, 1629, 1688, 1718, 1719, 1731, 1767, 1774, 1843, 1903, 1961, 1996, 2004, 2039, 2082, 2125, 2141, 2157, 2184, 2199, 2399, 2406 };

$\mathcal{B}_{39}$ = { 10, 111, 130, 134, 173, 257, 272, 300, 351, 614, 615, 676, 719, 720, 763, 806, 849, 981, 1053, 1302, 1317, 1452, 1471, 1474, 1495, 1525, 1538, 1570, 1599, 1612, 1613, 1656, 1674, 1699, 1717, 1789, 1832, 1871, 1875, 1885, 1910, 1953, 1954, 1993, 1997, 2006, 2036, 2040, 2173, 2216, 2259, 2327, 2340, 2347, 2370, 2390, 2433 };

$\mathcal{B}_{40}$ = { 65, 110, 213, 291, 294, 334, 337, 377, 380, 423, 465, 595, 634, 677, 777, 820, 840, 863, 906, 1024, 1051, 1064, 1069, 1094, 1112, 1155, 1157, 1200, 1230, 1243, 1274, 1309, 1332, 1335, 1372, 1378, 1426, 1502, 1545, 1588, 1595, 1604, 1638, 1647, 1681, 1698, 1877, 1918, 1941, 2090, 2122, 2165, 2204, 2227, 2247, 2394, 2437 };

$\mathcal{B}_{41}$ = { 14, 53, 72, 96, 115, 136, 139, 170, 228, 268, 322, 412, 434, 455, 477, 520, 691, 734, 825, 868, 911, 943, 984, 1092, 1110, 1121, 1135, 1178, 1203, 1221, 1240, 1267, 1425, 1478, 1521, 1560, 1631, 1803, 1839, 1882, 1984, 2032, 2075, 2097, 2118, 2168, 2187, 2211, 2254, 2263, 2286, 2306, 2329, 2372, 2413, 2415, 2422 };

$\mathcal{B}_{42}$ = { 99, 100, 143, 153, 196, 239, 240, 283, 326, 368, 374, 417, 751, 781, 826, 848, 887, 891, 892, 934, 935, 977, 978, 1006, 1007, 1049, 1050, 1093, 1137, 1159, 1182, 1202, 1225, 1268, 1310, 1311, 1436, 1473, 1483, 1517, 1576, 1619, 1673, 1716, 1742, 1785, 1828, 1943, 1969, 1982, 2025, 2070, 2161, 2176, 2185, 2240, 2333 };

$\mathcal{B}_{43}$ = { 104, 147, 195, 246, 312, 361, 362, 369, 390, 409, 466, 509, 541, 552, 569, 594, 612, 637, 680, 694, 723, 758, 762, 805, 895, 902, 945, 1011, 1027, 1070, 1113, 1192, 1242, 1347, 1396, 1443, 1453, 1486, 1501, 1544, 1587, 1705, 1748, 1775, 1889, 1925, 2012, 2128, 2132, 2133, 2147, 2155, 2182, 2225, 2234, 2268, 2400 }.

The third decomposition is given by 43 sets $\mathcal{B}_i$ for $i = 1, \ldots, 43$ each set consists of 57 points of one of the 43 disjoint subplanes of order seven in PG$(2, 7^2)$ as follows:

$\mathcal{B}_1$ = { 0, 43, 86, 129, 172, 215, 258, 301, 344, 387, 430, 473, 516, 559, 602, 645, 688, 731, 774, 817, 860, 903, 946, 989, 1032, 1075, 1118, 1161, 1204, 1247, 1290, 1333, 1376, 1419, 1462, 1505, 1548, 1591, 1634, 1677, 1720, 1763, 1806, 1849, 1892, 1935, 1978, 2021, 2064, 2107, 2150, 2193, 2236, 2279, 2322, 2365, 2408 };

$\mathcal{B}_2 = \{$ 10, 12, 24, 59, 95, 102, 147, 247, 357, 360, 422, 459, 551, 578, 593, 604, 608, 757, 931, 954, 986, 1107, 1188, 1190, 1208, 1214, 1222, 1226, 1251, 1274, 1298, 1301, 1302, 1383, 1463, 1470, 1531, 1562, 1579, 1609, 1664, 1676, 1683, 1702, 1728, 1752, 1770, 1885, 1946, 1965, 2026, 2048, 2087, 2189, 2273, 2285, 2432 $\}$;

$\mathcal{B}_3 = \{$ 110, 154, 170, 194, 257, 279, 298, 304, 333, 470, 487, 512, 572, 628, 666, 747, 750, 779, 855, 861, 896, 898, 944, 983, 1004, 1016, 1085, 1164, 1218, 1244, 1279, 1314, 1322, 1361, 1366, 1382, 1406, 1426, 1449, 1480, 1511, 1596, 1781, 1805, 1809, 1824, 1848, 1871, 1915, 1918, 1960, 2017, 2238, 2287, 2385, 2387, 2397 $\}$;

$\mathcal{B}_4 = \{$ 3, 60, 78, 302, 322, 341, 345, 372, 402, 437, 505, 548, 595, 619, 636, 687, 701, 708, 716, 739, 740, 744, 753, 770, 811, 828, 833, 837, 913, 958, 984, 1000, 1035, 1111, 1185, 1231, 1336, 1369, 1379, 1415, 1509, 1517, 1538, 1604, 1631, 1774, 1859, 1954, 2092, 2094, 2124, 2142, 2188, 2291, 2360, 2363, 2381 $\}$;

$\mathcal{B}_5 = \{$ 42, 109, 127, 136, 291, 311, 318, 375, 456, 562, 567, 605, 651, 652, 803, 822, 826, 871, 910, 990, 1013, 1038, 1176, 1259, 1277, 1287, 1350, 1360, 1403, 1410, 1418, 1487, 1539, 1574, 1656, 1659, 1681, 1701, 1784, 1880, 1928, 1959, 2030, 2034, 2059, 2082, 2117, 2133, 2149, 2161, 2180, 2185, 2202, 2263, 2294, 2337, 2433 $\}$;

$\mathcal{B}_6 = \{$ 47, 88, 124, 169, 173, 195, 242, 243, 244, 256, 287, 297, 299, 314, 364, 369, 380, 382, 411, 434, 469, 531, 694, 806, 831, 877, 920, 1034, 1045, 1074, 1076, 1106, 1159, 1196, 1243, 1389, 1399, 1456, 1469, 1474, 1483, 1484, 1530, 1571, 1577, 1725, 1875, 1887, 1901, 1997, 1999, 2130, 2217, 2254, 2257, 2267, 2428 $\}$;

$\mathcal{B}_7 = \{$ 139, 142, 208, 224, 231, 272, 295, 334, 340, 361, 365, 449, 467, 617, 633, 640, 676, 679, 681, 699, 710, 863, 895, 932, 973, 978, 1007, 1026, 1066, 1177, 1212, 1270, 1285, 1411, 1540, 1588, 1589, 1598, 1603, 1605, 1644, 1646, 1651, 1669, 1704, 1729, 1816, 1924, 2020, 2055, 2090, 2156, 2165, 2372, 2417, 2421, 2436 $\}$;

$\mathcal{B}_8 = \{$ 49, 51, 73, 115, 166, 178, 222, 223, 239, 390, 477, 482, 485, 494, 600, 629, 759, 782, 801, 865, 916, 960, 1018, 1098, 1127, 1178, 1237, 1252, 1257, 1268, 1335, 1358, 1407, 1482, 1568, 1578, 1687, 1744, 1762, 1786, 1803, 1877, 1907, 1916, 1981, 2044, 2118, 2129, 2182, 2200, 2219, 2228, 2241, 2262, 2320, 2371, 2429 $\}$;

$\mathcal{B}_9 = \{$ 85, 114, 143, 168, 326, 330, 339, 474, 496, 658, 680, 682, 737, 853, 934, 940, 965, 968, 1046, 1050, 1096, 1101, 1113, 1140, 1203, 1216, 1357, 1380, 1420, 1478, 1569, 1575, 1697, 1699, 1724, 1758, 1787, 1826, 1841, 1869, 1932, 1947, 1955, 1975, 1976, 2040, 2066, 2106, 2148, 2184, 2270, 2296, 2308, 2333, 2357, 2373, 2427 $\}$;

$\mathcal{B}_{10} = \{$ 5, 11, 15, 54, 63, 77, 90, 99, 112, 120, 181, 187, 200, 206, 210, 350, 374, 377, 394, 423, 438, 471, 508, 541, 696, 764, 814, 866, 872, 955, 961, 981, 1002, 1011, 1044, 1317, 1432, 1587, 1600, 1640, 1748, 1753, 1832, 1843, 1953, 2143, 2174, 2209, 2225, 2231, 2297, 2315, 2370, 2386, 2399, 2401, 2416 $\}$;

$\mathcal{B}_{11} = \{$ 52, 134, 192, 213, 263, 306, 324, 392, 397, 416, 520, 588, 597, 670, 677, 713, 717, 758, 765, 882, 907, 969, 1021, 1094, 1105, 1180, 1355, 1356, 1362, 1468, 1488, 1545, 1599, 1613, 1660, 1661, 1684, 1741, 1775, 1798, 1818, 1856, 1896, 1900, 1941, 1942, 1977, 2000, 2001, 2073, 2091, 2127, 2215, 2276, 2293, 2362, 2415 $\}$;

$\mathcal{B}_{12} = \{$ 1, 18, 65, 79, 252, 307, 332, 337, 355, 366, 378, 439, 441, 444, 497, 557, 580, 616, 724, 734, 745, 752, 794, 838, 873, 894, 897, 939, 951, 987, 1009, 1078, 1093, 1153, 1201, 1267, 1309, 1311, 1373, 1390, 1438, 1442, 1447, 1481, 1560, 1567, 1671, 1679, 1872, 1882, 1931, 1968, 1974, 2057, 2075, 2316, 2434 $\}$;

$\mathcal{B}_{13} = \{$ 8, 133, 164, 265, 283, 338, 480, 495, 544, 653, 723, 768, 773, 827, 870, 909, 942, 991, 1014, 1049, 1080, 1100, 1193, 1240, 1371, 1393, 1496, 1500, 1532, 1541, 1711, 1731, 1764, 1822, 1903, 1904, 1969, 1993, 2025, 2049, 2071, 2084, 2097, 2111, 2113, 2126, 2176, 2221, 2226, 2259, 2268, 2269, 2329, 2334, 2350, 2355, 2443 $\}$;

$\mathcal{B}_{14} = \{$ 2, 69, 74, 84, 94, 111, 233, 235, 335, 463, 499, 524, 706, 718, 781, 805, 923, 925, 927, 967, 970, 1082, 1124, 1134, 1149, 1155, 1199, 1225, 1250, 1342, 1354, 1375, 1439, 1544, 1547, 1570, 1612, 1620, 1670, 1682, 1827, 1889, 1897, 1973, 2120, 2155, 2172, 2199, 2240, 2255, 2275, 2282, 2289, 2361, 2390, 2394, 2412 $\}$;

$\mathcal{B}_{15} = \{$ 30, 145, 146, 212, 271, 294, 328, 362, 379, 406, 443, 450, 521, 534, 592, 606, 621, 637, 697, 711, 733, 766, 880, 911, 912, 937, 1024, 1037, 1041, 1200, 1211, 1239, 1372, 1404, 1408, 1431, 1441, 1525, 1565, 1593, 1616, 1625, 1648, 1779, 1801, 1808, 1844, 1855, 1881, 2065, 2100, 2128, 2141, 2169, 2233, 2389, 2422 $\}$;

$\mathcal{B}_{16}$ = { 25, 50, 66, 96, 98, 107, 165, 268, 281, 286, 410, 426, 506, 650, 675, 730, 735, 802, 812, 876, 930, 1102, 1123, 1152, 1179, 1181, 1194, 1377, 1409, 1465, 1492, 1576, 1582, 1606, 1641, 1643, 1675, 1686, 1696, 1709, 1726, 1765, 1811, 1828, 1906, 1951, 1984, 2047, 2058, 2072, 2212, 2227, 2239, 2286, 2327, 2353, 2419 };

$\mathcal{B}_{17}$ = { 33, 103, 128, 196, 368, 388, 535, 537, 538, 549, 552, 561, 571, 627, 649, 847, 902, 918, 1012, 1051, 1052, 1055, 1060, 1071, 1182, 1286, 1308, 1310, 1337, 1338, 1413, 1476, 1537, 1552, 1628, 1757, 1760, 1766, 1768, 1773, 1780, 1814, 1825, 1883, 1908, 1970, 2007, 2063, 2077, 2157, 2223, 2277, 2298, 2354, 2364, 2413, 2449 };

$\mathcal{B}_{18}$ = { 4, 32, 64, 162, 176, 182, 280, 285, 363, 386, 479, 492, 594, 630, 635, 692, 704, 712, 742, 804, 959, 1053, 1070, 1130, 1137, 1169, 1173, 1187, 1242, 1260, 1319, 1323, 1345, 1405, 1454, 1475, 1507, 1513, 1536, 1594, 1665, 1754, 1839, 1873, 1888, 1891, 1910, 2012, 2035, 2069, 2081, 2125, 2290, 2310, 2326, 2380, 2418 };

$\mathcal{B}_{19}$ = { 20, 89, 282, 296, 305, 465, 498, 518, 634, 749, 778, 784, 787, 836, 849, 900, 1151, 1189, 1191, 1253, 1265, 1280, 1299, 1401, 1433, 1455, 1466, 1584, 1586, 1632, 1636, 1722, 1733, 1746, 1789, 1837, 1863, 1909, 1919, 1925, 1927, 1940, 1961, 1982, 2051, 2086, 2088, 2131, 2134, 2151, 2195, 2256, 2324, 2384, 2407, 2430, 2445 };

$\mathcal{B}_{20}$ = { 7, 16, 45, 87, 104, 138, 211, 404, 428, 436, 507, 511, 515, 528, 554, 579, 587, 661, 664, 668, 691, 721, 722, 810, 815, 878, 884, 906, 947, 996, 1017, 1030, 1125, 1154, 1156, 1294, 1303, 1351, 1423, 1425, 1428, 1446, 1459, 1502, 1624, 1674, 1692, 1713, 1759, 1815, 1846, 1865, 2039, 2201, 2338, 2392, 2440 };

$\mathcal{B}_{21}$ = { 21, 46, 160, 188, 230, 342, 343, 351, 353, 396, 488, 655, 667, 693, 705, 751, 761, 763, 797, 864, 908, 998, 1006, 1144, 1184, 1207, 1221, 1234, 1254, 1281, 1288, 1363, 1397, 1452, 1549, 1558, 1721, 1767, 1862, 1867, 1958, 1962, 1996, 2008, 2015, 2028, 2032, 2053, 2093, 2095, 2162, 2204, 2213, 2321, 2343, 2376, 2377 };

$\mathcal{B}_{22}$ = { 55, 70, 80, 130, 150, 184, 201, 240, 259, 303, 327, 367, 412, 461, 464, 542, 584, 591, 631, 720, 755, 762, 809, 820, 854, 857, 858, 889, 904, 924, 977, 1090, 1217, 1437, 1453, 1467, 1522, 1535, 1595, 1658, 1689, 1690, 1698, 1755, 1771, 1842, 1868, 1884, 2083, 2108, 2110, 2121, 2144, 2173, 2216, 2340, 2441 };

$\mathcal{B}_{23} = \{40, 81, 161, 191, 205, 229, 237, 250, 255, 264, 336, 393, 420, 431, 445, 446, 454, 610, 648, 659, 738, 777, 800, 1064, 1069, 1110, 1112, 1120, 1135, 1213, 1275, 1278, 1347, 1387, 1417, 1421, 1491, 1501, 1629, 1639, 1649, 1673, 1714, 1736, 1769, 1845, 1876, 2013, 2122, 2168, 2186, 2205, 2249, 2251, 2305, 2341, 2349\};$

$\mathcal{B}_{24} = \{72, 218, 313, 348, 370, 457, 501, 513, 522, 563, 569, 570, 582, 601, 632, 678, 741, 772, 785, 825, 846, 868, 891, 892, 963, 1092, 1095, 1119, 1121, 1210, 1236, 1245, 1262, 1263, 1296, 1328, 1412, 1471, 1519, 1527, 1727, 1793, 1834, 1937, 2018, 2022, 2041, 2046, 2050, 2070, 2085, 2112, 2170, 2264, 2314, 2374, 2423\};$

$\mathcal{B}_{25} = \{62, 83, 97, 100, 141, 159, 246, 251, 292, 373, 421, 533, 536, 577, 622, 623, 660, 689, 719, 821, 823, 840, 848, 856, 893, 921, 929, 953, 1043, 1136, 1160, 1170, 1229, 1293, 1396, 1436, 1443, 1444, 1497, 1563, 1590, 1663, 1742, 1743, 1785, 1797, 1804, 1929, 1948, 1979, 2078, 2080, 2098, 2119, 2278, 2299, 2336\};$

$\mathcal{B}_{26} = \{6, 35, 39, 56, 149, 156, 254, 284, 289, 401, 414, 466, 478, 486, 509, 539, 550, 607, 613, 639, 642, 647, 700, 756, 850, 883, 976, 1036, 1061, 1174, 1192, 1232, 1238, 1344, 1378, 1394, 1490, 1506, 1515, 1520, 1554, 1559, 1597, 1608, 1685, 1691, 1705, 1964, 1983, 2096, 2187, 2203, 2243, 2284, 2300, 2311, 2400\};$

$\mathcal{B}_{27} = \{118, 119, 151, 395, 558, 564, 707, 715, 736, 790, 816, 845, 851, 948, 952, 993, 995, 1008, 1047, 1089, 1139, 1145, 1171, 1172, 1272, 1316, 1321, 1364, 1386, 1424, 1430, 1503, 1516, 1521, 1601, 1615, 1653, 1694, 1712, 1761, 1782, 1783, 1833, 1886, 1921, 1943, 1992, 2003, 2006, 2153, 2218, 2230, 2303, 2323, 2351, 2368, 2448\};$

$\mathcal{B}_{28} = \{34, 38, 41, 82, 148, 171, 175, 221, 232, 262, 278, 321, 399, 409, 417, 527, 657, 726, 729, 813, 819, 874, 917, 992, 1028, 1029, 1031, 1091, 1146, 1271, 1312, 1326, 1332, 1448, 1450, 1464, 1495, 1551, 1566, 1642, 1739, 1796, 1831, 1851, 1864, 1905, 1911, 1949, 1980, 1986, 2023, 2062, 2089, 2137, 2163, 2175, 2379\};$

$\mathcal{B}_{29} = \{28, 36, 71, 76, 108, 113, 220, 228, 248, 290, 371, 452, 472, 490, 568, 641, 654, 746, 835, 867, 899, 936, 962, 980, 985, 1015, 1023, 1033, 1103, 1186, 1273, 1307, 1324, 1388, 1429, 1479, 1518, 1533, 1638, 1866, 1939, 1994, 1998, 2029, 2076, 2109, 2123, 2145, 2147, 2160, 2166, 2265, 2331, 2375, 2378, 2438, 2439\};$

$\mathcal{B}_{30} = \{ 23, 61, 93, 219, 241, 249, 347, 391, 429, 483, 574, 586, 673, 728, 769, 859, 879, 887, 890, 1019, 1062, 1063, 1079, 1084, 1147, 1166, 1209, 1235, 1283, 1330, 1334, 1370, 1427, 1460, 1510, 1529, 1550, 1622, 1700, 1718, 1819, 1987, 1988, 1989, 2004, 2010, 2079, 2177, 2211, 2214, 2246, 2252, 2260, 2309, 2332, 2388, 2447 \};$

$\mathcal{B}_{31} = \{ 27, 29, 197, 234, 315, 356, 435, 453, 583, 596, 611, 614, 662, 683, 792, 793, 795, 824, 869, 875, 935, 964, 971, 975, 979, 1073, 1126, 1148, 1175, 1205, 1258, 1282, 1284, 1343, 1395, 1626, 1667, 1717, 1776, 1802, 1860, 1894, 1933, 2067, 2068, 2206, 2234, 2237, 2281, 2344, 2346, 2366, 2403, 2405, 2409, 2414, 2450 \};$

$\mathcal{B}_{32} = \{ 44, 57, 68, 117, 135, 190, 207, 226, 238, 329, 352, 418, 484, 525, 698, 829, 905, 949, 972, 1003, 1022, 1040, 1068, 1122, 1230, 1276, 1346, 1374, 1457, 1458, 1486, 1581, 1607, 1610, 1740, 1778, 1800, 1810, 1917, 1920, 1950, 1967, 2037, 2043, 2139, 2171, 2183, 2194, 2232, 2242, 2245, 2247, 2382, 2420, 2424, 2431, 2435 \};$

$\mathcal{B}_{33} = \{ 157, 167, 270, 354, 455, 503, 585, 609, 669, 685, 727, 767, 776, 798, 841, 885, 933, 943, 1048, 1128, 1143, 1165, 1167, 1246, 1264, 1269, 1292, 1304, 1320, 1327, 1329, 1331, 1365, 1445, 1556, 1583, 1585, 1614, 1662, 1695, 1735, 1912, 1957, 2005, 2024, 2036, 2138, 2179, 2196, 2224, 2292, 2345, 2383, 2398, 2402, 2426, 2442 \};$

$\mathcal{B}_{34} = \{ 9, 22, 140, 163, 177, 204, 216, 225, 227, 308, 319, 346, 419, 432, 458, 500, 510, 620, 656, 709, 839, 886, 901, 919, 988, 1020, 1083, 1150, 1158, 1215, 1228, 1241, 1341, 1422, 1435, 1473, 1543, 1611, 1672, 1707, 1715, 1716, 1738, 1850, 1934, 1956, 2074, 2136, 2146, 2192, 2280, 2339, 2395, 2404, 2425, 2437, 2444 \};$

$\mathcal{B}_{35} = \{ 14, 75, 121, 261, 266, 325, 331, 349, 358, 462, 476, 489, 532, 575, 599, 603, 612, 663, 695, 799, 830, 843, 997, 1042, 1088, 1197, 1291, 1300, 1315, 1352, 1368, 1385, 1391, 1494, 1514, 1557, 1655, 1688, 1737, 1795, 1821, 1847, 1874, 1985, 2014, 2102, 2105, 2132, 2164, 2181, 2208, 2235, 2248, 2258, 2266, 2313, 2335 \};$

$\mathcal{B}_{36} = \{ 17, 48, 125, 132, 144, 158, 389, 413, 519, 543, 556, 576, 671, 684, 743, 791, 832, 881, 926, 982, 1001, 1067, 1081, 1129, 1131, 1142, 1162, 1183, 1305, 1318, 1400, 1489, 1493, 1524, 1555, 1619, 1627, 1652, 1678, 1747, 1792, 1836, 1838, 1852, 1914, 1922, 1944, 1963, 1971, 2042, 2060, 2115, 2140, 2222, 2317, 2328, 2347 \};$

$\mathcal{B}_{37}$ = { 126, 186, 293, 376, 384, 385, 407, 447, 451, 523, 545, 547, 581, 638, 643, 644, 703, 732, 818, 834, 945, 956, 1086, 1157, 1168, 1219, 1223, 1233, 1289, 1313, 1359, 1472, 1512, 1617, 1618, 1654, 1706, 1788, 1790, 1820, 1829, 1890, 1895, 2016, 2027, 2031, 2152, 2159, 2167, 2198, 2250, 2274, 2283, 2312, 2358, 2359, 2391 };

$\mathcal{B}_{38}$ = { 53, 67, 123, 179, 185, 203, 209, 273, 359, 405, 408, 433, 440, 448, 514, 560, 573, 590, 598, 686, 725, 754, 786, 914, 922, 974, 1056, 1072, 1104, 1109, 1206, 1261, 1339, 1353, 1384, 1528, 1564, 1572, 1732, 1734, 1745, 1756, 1791, 1858, 1870, 1899, 1923, 1990, 1991, 2011, 2104, 2116, 2158, 2295, 2356, 2369, 2411 };

$\mathcal{B}_{39}$ = { 91, 101, 131, 153, 155, 183, 199, 275, 300, 415, 424, 460, 481, 491, 540, 555, 624, 665, 674, 748, 760, 844, 862, 888, 941, 1117, 1132, 1227, 1249, 1295, 1297, 1349, 1402, 1440, 1485, 1526, 1573, 1580, 1602, 1623, 1630, 1650, 1693, 1710, 1861, 1930, 1945, 2033, 2052, 2101, 2154, 2178, 2271, 2301, 2330, 2348, 2446 };

$\mathcal{B}_{40}$ = { 13, 19, 26, 31, 180, 193, 198, 214, 245, 260, 274, 276, 381, 383, 493, 502, 504, 526, 546, 589, 702, 780, 783, 788, 796, 842, 852, 938, 1005, 1025, 1027, 1115, 1266, 1367, 1392, 1398, 1414, 1504, 1546, 1637, 1647, 1666, 1680, 1772, 1799, 1807, 1853, 1857, 1902, 1938, 2045, 2061, 2191, 2210, 2261, 2288, 2319 };

$\mathcal{B}_{41}$ = { 92, 106, 122, 137, 152, 288, 323, 398, 400, 403, 442, 468, 530, 566, 626, 672, 690, 714, 807, 966, 1077, 1087, 1097, 1141, 1163, 1195, 1256, 1340, 1348, 1451, 1553, 1561, 1592, 1621, 1645, 1657, 1723, 1730, 1812, 1879, 1893, 1898, 1936, 1952, 1966, 1995, 2002, 2054, 2056, 2103, 2190, 2253, 2306, 2307, 2325, 2352, 2393 };

$\mathcal{B}_{42}$ = { 37, 174, 189, 236, 269, 277, 310, 320, 425, 427, 475, 517, 553, 565, 625, 775, 808, 950, 999, 1054, 1057, 1058, 1059, 1133, 1138, 1202, 1220, 1224, 1248, 1306, 1381, 1434, 1477, 1498, 1499, 1508, 1633, 1635, 1668, 1703, 1708, 1719, 1749, 1750, 1751, 1830, 1913, 2009, 2019, 2038, 2114, 2135, 2197, 2220, 2272, 2302, 2304 };

$\mathcal{B}_{43}$ = { 58, 105, 116, 202, 217, 253, 267, 309, 312, 316, 317, 529, 615, 618, 646, 771, 789, 915, 928, 957, 994, 1010, 1039, 1065, 1099, 1108, 1114, 1116, 1198, 1255, 1325, 1416, 1461, 1523, 1534, 1542, 1777, 1794, 1813, 1817, 1823, 1835, 1840, 1854, 1878, 1926, 1972, 2099, 2207, 2229, 2244, 2318, 2342, 2367, 2396, 2406, 2410 }.

The fourth decomposition is given by 43 sets $\mathcal{B}_i$ for $i = 1, \ldots, 43$ each set consists of 57 points of one of the 43 disjoint subplanes of order seven in PG$(2, 7^2)$ as follows:

$\mathcal{B}_1 = \{$ 0, 43, 86, 129, 172, 215, 258, 301, 344, 387, 430, 473, 516, 559, 602, 645, 688, 731, 774, 817, 860, 903, 946, 989, 1032, 1075, 1118, 1161, 1204, 1247, 1290, 1333, 1376, 1419, 1462, 1505, 1548, 1591, 1634, 1677, 1720, 1763, 1806, 1849, 1892, 1935, 1978, 2021, 2064, 2107, 2150, 2193, 2236, 2279, 2322, 2365, 2408 $\}$;

$\mathcal{B}_2 = \{$ 2, 35, 74, 94, 108, 145, 182, 219, 234, 265, 416, 476, 538, 600, 630, 659, 706, 708, 735, 757, 887, 901, 927, 985, 991, 997, 1033, 1063, 1078, 1091, 1106, 1143, 1174, 1216, 1283, 1292, 1329, 1364, 1507, 1508, 1519, 1601, 1613, 1651, 1710, 1725, 1747, 1880, 1989, 2121, 2127, 2222, 2259, 2340, 2379, 2398, 2428 $\}$;

$\mathcal{B}_3 = \{$ 10, 30, 84, 150, 197, 225, 282, 315, 337, 378, 444, 449, 457, 643, 667, 679, 733, 791, 792, 824, 839, 1006, 1022, 1031, 1064, 1155, 1275, 1308, 1393, 1407, 1410, 1435, 1492, 1533, 1540, 1593, 1612, 1665, 1801, 1823, 1832, 1853, 1881, 1914, 1947, 2023, 2199, 2234, 2248, 2250, 2264, 2336, 2362, 2404, 2407, 2415, 2438 $\}$;

$\mathcal{B}_4 = \{$ 32, 49, 52, 69, 112, 325, 331, 413, 440, 441, 498, 542, 547, 601, 638, 656, 859, 865, 909, 911, 1019, 1024, 1074, 1077, 1156, 1165, 1199, 1237, 1300, 1303, 1311, 1334, 1387, 1427, 1431, 1458, 1490, 1525, 1545, 1590, 1643, 1804, 1842, 1876, 1889, 1920, 1967, 1990, 2013, 2014, 2049, 2066, 2166, 2171, 2194, 2324, 2329 $\}$;

$\mathcal{B}_5 = \{$ 16, 89, 126, 155, 179, 235, 255, 264, 354, 540, 700, 723, 741, 772, 816, 875, 998, 1012, 1037, 1041, 1100, 1206, 1220, 1225, 1246, 1305, 1315, 1375, 1380, 1400, 1555, 1583, 1678, 1735, 1759, 1826, 1828, 1872, 1948, 1962, 1984, 2024, 2040, 2046, 2047, 2075, 2206, 2209, 2227, 2233, 2238, 2276, 2341, 2344, 2366, 2388, 2421 $\}$;

$\mathcal{B}_6 = \{$ 62, 98, 116, 117, 125, 140, 156, 187, 283, 292, 355, 367, 377, 404, 410, 429, 451, 552, 629, 637, 655, 678, 856, 858, 905, 937, 1068, 1105, 1219, 1276, 1284, 1310, 1349, 1363, 1414, 1526, 1672, 1706, 1707, 1760, 1907, 1953, 1973, 2041, 2119, 2136, 2158, 2163, 2212, 2214, 2274, 2319, 2339, 2361, 2381, 2413, 2425 $\}$;

$\mathcal{B}_7 = \{$ 1, 75, 160, 245, 284, 371, 398, 412, 478, 520, 549, 588, 603, 609, 624, 631, 677, 768, 776, 799, 850, 912, 938, 1014, 1040, 1048, 1071, 1123, 1166, 1277, 1304, 1463, 1472, 1503, 1514, 1564, 1626, 1714, 1734, 1757, 1776, 1779, 1815, 1879, 1891, 1906, 1961, 1976, 2012, 2113, 2116, 2164, 2181, 2203, 2205, 2308, 2369 $\}$;

$\mathcal{B}_8 = \{$ 95, 132, 135, 144, 173, 281, 370, 389, 399, 458, 460, 491, 524, 616, 675, 692, 734, 775, 787, 869, 872, 888, 919, 1004, 1057, 1060, 1093, 1257, 1318, 1439, 1485, 1513, 1572, 1582, 1584, 1592, 1653, 1674, 1718, 1799, 1811, 1875, 1898, 1929, 1956, 1957, 2060, 2298, 2303, 2309, 2317, 2395, 2401, 2418, 2435, 2441, 2448 $\}$;

$\mathcal{B}_9 = \{$ 83, 184, 214, 238, 246, 250, 316, 381, 386, 447, 464, 527, 535, 545, 614, 620, 635, 644, 726, 819, 956, 1015, 1125, 1133, 1205, 1224, 1291, 1331, 1340, 1357, 1368, 1441, 1446, 1488, 1537, 1588, 1628, 1666, 1685, 1690, 1768, 1807, 1829, 1912, 1915, 1942, 2141, 2159, 2165, 2204, 2221, 2235, 2266, 2268, 2318, 2360, 2446 $\}$;

$\mathcal{B}_{10} = \{$ 9, 21, 24, 115, 137, 151, 163, 165, 203, 227, 236, 319, 342, 414, 492, 568, 573, 613, 725, 728, 778, 800, 811, 832, 838, 962, 1056, 1131, 1162, 1186, 1201, 1214, 1230, 1323, 1348, 1370, 1528, 1534, 1561, 1585, 1607, 1676, 1723, 1733, 1736, 1795, 1926, 1991, 2118, 2202, 2231, 2246, 2288, 2290, 2327, 2328, 2349 $\}$;

$\mathcal{B}_{11} = \{$ 133, 143, 199, 326, 349, 424, 452, 481, 503, 513, 522, 532, 567, 572, 632, 703, 762, 825, 843, 862, 898, 943, 1051, 1054, 1126, 1233, 1245, 1258, 1272, 1280, 1314, 1326, 1341, 1351, 1424, 1466, 1494, 1571, 1654, 1657, 1689, 1731, 1749, 1761, 1790, 1852, 1884, 1930, 2022, 2038, 2073, 2099, 2157, 2178, 2220, 2346, 2392 $\}$;

$\mathcal{B}_{12} = \{$ 17, 82, 141, 148, 180, 226, 393, 467, 553, 556, 608, 657, 686, 740, 867, 1005, 1043, 1053, 1104, 1116, 1231, 1261, 1288, 1398, 1421, 1423, 1442, 1473, 1484, 1504, 1524, 1546, 1563, 1579, 1587, 1742, 1748, 1772, 1777, 1794, 1839, 1850, 1917, 1971, 1979, 2039, 2043, 2095, 2112, 2167, 2188, 2229, 2258, 2299, 2305, 2356, 2390 $\}$;

$\mathcal{B}_{13} = \{$ 91, 109, 122, 127, 208, 223, 242, 303, 309, 338, 400, 415, 442, 465, 466, 483, 550, 577, 727, 767, 785, 885, 931, 936, 982, 1086, 1097, 1132, 1141, 1200, 1265, 1273, 1383, 1391, 1429, 1474, 1512, 1520, 1531, 1554, 1597, 1618, 1738, 1923, 1964, 1982, 2010, 2030, 2033, 2083, 2132, 2215, 2293, 2312, 2352, 2385, 2412 $\}$;

$\mathcal{B}_{14}$ = { 37, 96, 104, 131, 147, 198, 257, 279, 308, 408, 420, 446, 462, 493, 514, 523, 534, 583, 636, 736, 763, 939, 952, 1058, 1059, 1079, 1140, 1171, 1244, 1248, 1266, 1344, 1399, 1425, 1430, 1444, 1447, 1456, 1529, 1589, 1652, 1745, 1792, 1833, 1921, 1938, 1992, 2050, 2067, 2074, 2090, 2114, 2146, 2200, 2272, 2411, 2440 };

$\mathcal{B}_{15}$ = { 46, 196, 254, 274, 275, 278, 295, 358, 376, 485, 519, 555, 729, 751, 769, 771, 773, 789, 820, 822, 953, 966, 1010, 1042, 1096, 1146, 1150, 1195, 1271, 1457, 1549, 1604, 1655, 1709, 1774, 1802, 1803, 1813, 1820, 1861, 1870, 1911, 1949, 2003, 2004, 2054, 2078, 2089, 2142, 2173, 2244, 2253, 2271, 2315, 2382, 2403, 2406 };

$\mathcal{B}_{16}$ = { 26, 31, 66, 102, 248, 433, 472, 508, 611, 651, 695, 698, 780, 813, 834, 899, 974, 980, 1008, 1067, 1069, 1070, 1101, 1103, 1142, 1190, 1222, 1232, 1252, 1294, 1302, 1621, 1633, 1668, 1693, 1699, 1717, 1746, 1766, 1789, 1856, 1885, 1902, 1963, 1981, 1987, 2009, 2076, 2138, 2145, 2197, 2224, 2245, 2263, 2300, 2333, 2433 };

$\mathcal{B}_{17}$ = { 20, 106, 217, 241, 271, 300, 346, 369, 423, 432, 517, 529, 541, 558, 589, 590, 690, 748, 752, 798, 814, 1159, 1218, 1243, 1250, 1289, 1413, 1426, 1449, 1459, 1468, 1502, 1544, 1547, 1614, 1712, 1796, 1819, 1854, 1918, 1932, 2002, 2016, 2027, 2042, 2068, 2081, 2120, 2160, 2190, 2207, 2247, 2296, 2397, 2427, 2431, 2447 };

$\mathcal{B}_{18}$ = { 3, 11, 27, 39, 58, 139, 164, 185, 190, 221, 230, 269, 320, 322, 328, 345, 361, 372, 397, 425, 443, 455, 489, 504, 510, 575, 578, 717, 722, 764, 794, 808, 930, 951, 1169, 1249, 1327, 1343, 1353, 1404, 1454, 1565, 1631, 1647, 1816, 1837, 1858, 1905, 1936, 1939, 2032, 2055, 2070, 2260, 2359, 2405, 2416 };

$\mathcal{B}_{19}$ = { 45, 50, 61, 113, 239, 240, 266, 267, 318, 497, 683, 730, 745, 754, 844, 848, 882, 907, 908, 928, 949, 994, 1083, 1102, 1108, 1124, 1137, 1163, 1227, 1365, 1366, 1374, 1386, 1392, 1396, 1477, 1630, 1675, 1716, 1778, 1836, 1916, 1922, 1998, 2086, 2088, 2100, 2126, 2149, 2161, 2185, 2241, 2277, 2306, 2314, 2337, 2374 };

$\mathcal{B}_{20}$ = { 15, 47, 188, 224, 243, 270, 287, 288, 348, 450, 463, 511, 525, 526, 571, 612, 663, 694, 732, 810, 818, 823, 854, 965, 979, 992, 1009, 1045, 1113, 1173, 1183, 1202, 1251, 1264, 1306, 1371, 1422, 1460, 1501, 1509, 1515, 1641, 1743, 1867, 1878, 2057, 2077, 2101, 2162, 2179, 2191, 2210, 2350, 2364, 2372, 2400, 2449 };

$\mathcal{B}_{21} = \{$ 142, 149, 202, 273, 285, 312, 351, 353, 356, 359, 537, 564, 574, 591, 606, 610, 626, 639, 681, 710, 879, 886, 921, 955, 1020, 1087, 1090, 1122, 1130, 1152, 1187, 1260, 1268, 1307, 1345, 1500, 1535, 1594, 1615, 1627, 1635, 1704, 1711, 1737, 1753, 1824, 1838, 1857, 1966, 2020, 2044, 2082, 2087, 2320, 2332, 2345, 2358 $\}$;

$\mathcal{B}_{22} = \{$ 12, 38, 59, 72, 130, 174, 222, 259, 323, 324, 360, 395, 431, 434, 459, 475, 548, 584, 599, 674, 760, 801, 806, 831, 851, 868, 916, 971, 972, 975, 983, 1168, 1179, 1212, 1241, 1338, 1453, 1491, 1493, 1527, 1542, 1632, 1658, 1667, 1695, 1698, 1702, 1703, 1739, 1895, 1896, 1897, 1899, 1909, 2096, 2105, 2432 $\}$;

$\mathcal{B}_{23} = \{$ 18, 73, 76, 81, 100, 123, 251, 317, 487, 512, 528, 605, 618, 621, 685, 702, 739, 788, 855, 863, 890, 917, 993, 1001, 1003, 1007, 1110, 1129, 1158, 1312, 1384, 1405, 1406, 1416, 1440, 1498, 1569, 1575, 1639, 1671, 1713, 1781, 1785, 1810, 1845, 1848, 2048, 2122, 2129, 2148, 2152, 2153, 2232, 2251, 2331, 2354, 2436 $\}$;

$\mathcal{B}_{24} = \{$ 4, 36, 71, 157, 302, 313, 422, 471, 495, 505, 509, 604, 716, 744, 852, 853, 920, 944, 981, 1039, 1044, 1062, 1128, 1144, 1178, 1193, 1213, 1215, 1226, 1254, 1298, 1409, 1420, 1433, 1551, 1562, 1566, 1617, 1656, 1688, 1700, 1705, 1762, 1788, 1859, 1946, 1965, 2092, 2182, 2237, 2243, 2295, 2301, 2307, 2330, 2383, 2393 $\}$;

$\mathcal{B}_{25} = \{$ 42, 154, 290, 380, 391, 394, 484, 560, 562, 582, 587, 596, 598, 658, 676, 709, 747, 796, 836, 857, 861, 929, 934, 961, 995, 1025, 1084, 1139, 1191, 1228, 1279, 1321, 1322, 1381, 1382, 1403, 1418, 1552, 1596, 1650, 1660, 1670, 1694, 1719, 1812, 1818, 1866, 1874, 1934, 1941, 1959, 2079, 2106, 2291, 2294, 2347, 2409 $\}$;

$\mathcal{B}_{26} = \{$ 29, 87, 101, 105, 121, 244, 299, 321, 347, 352, 382, 402, 453, 456, 546, 566, 615, 642, 684, 830, 833, 876, 877, 964, 1002, 1011, 1138, 1157, 1211, 1256, 1262, 1267, 1317, 1336, 1342, 1362, 1373, 1379, 1389, 1397, 1415, 1450, 1464, 1622, 1624, 1741, 1770, 1945, 1950, 1980, 1993, 2001, 2053, 2094, 2102, 2125, 2434 $\}$;

$\mathcal{B}_{27} = \{$ 53, 57, 107, 169, 213, 272, 311, 502, 625, 633, 662, 666, 669, 711, 746, 803, 809, 849, 897, 990, 1023, 1081, 1135, 1153, 1172, 1176, 1269, 1274, 1390, 1489, 1609, 1646, 1661, 1663, 1701, 1764, 1767, 1800, 1840, 1893, 1900, 1903, 1913, 1924, 1944, 1969, 2028, 2031, 2045, 2109, 2110, 2123, 2139, 2180, 2223, 2281, 2394 $\}$;

$\mathcal{B}_{28}$ = { 40, 78, 111, 124, 153, 170, 333, 363, 366, 385, 407, 411, 418, 448, 468, 530, 531, 544, 557, 653, 699, 821, 891, 906, 924, 940, 1076, 1147, 1149, 1209, 1313, 1325, 1377, 1434, 1470, 1543, 1556, 1560, 1568, 1577, 1662, 1683, 1691, 1769, 1831, 1877, 1970, 1972, 2026, 2155, 2196, 2252, 2262, 2273, 2422, 2424, 2444 };

$\mathcal{B}_{29}$ = { 22, 136, 204, 209, 310, 336, 479, 570, 617, 641, 696, 750, 793, 845, 918, 932, 988, 1027, 1035, 1038, 1049, 1095, 1109, 1148, 1151, 1175, 1203, 1221, 1285, 1320, 1354, 1358, 1361, 1372, 1385, 1394, 1480, 1517, 1541, 1570, 1576, 1605, 1623, 1728, 1732, 1808, 1835, 1904, 1975, 1985, 2135, 2169, 2176, 2270, 2387, 2417, 2426 };

$\mathcal{B}_{30}$ = { 5, 44, 54, 55, 60, 99, 118, 168, 171, 194, 216, 261, 268, 294, 329, 364, 388, 506, 634, 650, 687, 759, 766, 805, 881, 896, 902, 960, 977, 1018, 1082, 1107, 1111, 1127, 1197, 1229, 1295, 1296, 1319, 1350, 1461, 1483, 1538, 1581, 1680, 1786, 1797, 1817, 1951, 1988, 2052, 2115, 2133, 2256, 2325, 2335, 2348 };

$\mathcal{B}_{31}$ = { 19, 85, 114, 167, 175, 189, 260, 293, 339, 357, 368, 437, 496, 507, 543, 576, 581, 622, 648, 649, 652, 671, 682, 691, 713, 738, 742, 758, 847, 895, 1046, 1066, 1164, 1180, 1287, 1347, 1369, 1518, 1616, 1636, 1681, 1726, 1825, 1855, 1983, 1995, 2015, 2034, 2036, 2061, 2208, 2286, 2292, 2378, 2380, 2386, 2414 };

$\mathcal{B}_{32}$ = { 64, 67, 77, 92, 158, 162, 181, 200, 327, 384, 403, 426, 438, 470, 490, 500, 586, 693, 707, 712, 755, 807, 910, 958, 963, 1016, 1052, 1181, 1182, 1242, 1301, 1401, 1432, 1559, 1679, 1740, 1751, 1756, 1771, 1784, 1798, 1843, 1890, 1968, 1974, 1996, 1999, 2085, 2130, 2170, 2177, 2254, 2391, 2396, 2429, 2437, 2445 };

$\mathcal{B}_{33}$ = { 14, 186, 211, 263, 305, 307, 421, 594, 719, 720, 786, 846, 870, 914, 941, 950, 978, 986, 1000, 1026, 1047, 1073, 1196, 1297, 1355, 1411, 1495, 1497, 1574, 1586, 1599, 1629, 1722, 1750, 1754, 1773, 1822, 1841, 1847, 1883, 1927, 1933, 1960, 1977, 2058, 2091, 2104, 2111, 2189, 2201, 2218, 2257, 2275, 2302, 2363, 2410, 2442 };

$\mathcal{B}_{34}$ = { 6, 51, 65, 79, 177, 183, 193, 201, 233, 350, 390, 436, 607, 646, 718, 743, 770, 782, 783, 829, 873, 880, 970, 987, 1028, 1085, 1188, 1198, 1259, 1378, 1402, 1428, 1438, 1448, 1471, 1522, 1523, 1558, 1573, 1619, 1638, 1659, 1669, 1791, 1814, 1863, 1865, 1873, 1940, 2017, 2069, 2084, 2098, 2156, 2267, 2371, 2420 };

$\mathcal{B}_{35}$ = { 152, 229, 247, 280, 296, 335, 383, 469, 486, 585, 654, 664, 761, 797, 840, 842, 900, 913, 945, 968, 1017, 1036, 1115, 1117, 1119, 1184, 1185, 1240, 1278, 1352, 1359, 1408, 1445, 1482, 1521, 1532, 1598, 1673, 1697, 1758, 1782, 1783, 1844, 1997, 2056, 2065, 2071, 2117, 2211, 2213, 2226, 2284, 2285, 2287, 2334, 2375, 2384 };

$\mathcal{B}_{36}$ = { 7, 13, 23, 34, 80, 119, 138, 178, 298, 304, 334, 343, 417, 494, 499, 521, 565, 569, 628, 660, 721, 753, 777, 781, 935, 967, 1013, 1072, 1145, 1253, 1263, 1339, 1452, 1455, 1465, 1499, 1600, 1606, 1611, 1620, 1645, 1686, 1730, 1834, 1851, 1937, 2059, 2103, 2108, 2140, 2143, 2144, 2184, 2187, 2217, 2282, 2399 };

$\mathcal{B}_{37}$ = { 41, 134, 146, 159, 212, 220, 262, 341, 379, 477, 579, 619, 640, 665, 672, 828, 864, 893, 969, 996, 999, 1055, 1061, 1092, 1160, 1255, 1395, 1486, 1595, 1602, 1648, 1649, 1664, 1684, 1696, 1780, 1830, 1887, 1901, 1928, 1943, 1955, 1958, 1994, 2007, 2080, 2128, 2147, 2186, 2283, 2304, 2313, 2321, 2342, 2357, 2370, 2430 };

$\mathcal{B}_{38}$ = { 33, 48, 56, 93, 206, 237, 253, 297, 375, 405, 409, 461, 482, 563, 580, 593, 802, 835, 841, 866, 871, 976, 1050, 1089, 1094, 1099, 1114, 1154, 1210, 1217, 1235, 1270, 1281, 1309, 1367, 1388, 1437, 1469, 1487, 1496, 1506, 1536, 1729, 1765, 1805, 1925, 2019, 2035, 2168, 2183, 2239, 2261, 2316, 2326, 2353, 2419, 2423 };

$\mathcal{B}_{39}$ = { 103, 128, 161, 166, 176, 192, 286, 365, 373, 392, 428, 454, 536, 561, 627, 701, 714, 715, 779, 795, 874, 892, 904, 915, 942, 1034, 1098, 1120, 1136, 1189, 1207, 1223, 1335, 1412, 1467, 1530, 1553, 1580, 1644, 1682, 1715, 1793, 1868, 1869, 1882, 1894, 1908, 2018, 2029, 2097, 2131, 2195, 2225, 2323, 2343, 2402, 2443 };

$\mathcal{B}_{40}$ = { 28, 88, 97, 110, 120, 191, 205, 207, 218, 277, 330, 340, 401, 406, 427, 445, 480, 515, 551, 554, 673, 689, 704, 749, 804, 894, 1121, 1170, 1177, 1238, 1316, 1328, 1356, 1360, 1443, 1475, 1478, 1557, 1567, 1637, 1721, 1744, 1755, 1864, 1888, 1954, 2011, 2037, 2062, 2063, 2228, 2240, 2289, 2310, 2311, 2338, 2373 };

$\mathcal{B}_{41}$ = { 90, 231, 249, 252, 256, 291, 306, 362, 374, 396, 518, 533, 539, 647, 670, 737, 756, 784, 878, 883, 926, 933, 954, 957, 959, 984, 1029, 1065, 1080, 1167, 1194, 1234, 1236, 1330, 1436, 1479, 1511, 1550, 1687, 1724, 1752, 1827, 1846, 1862, 1952, 2006, 2025, 2051, 2134, 2151, 2154, 2174, 2265, 2355, 2368, 2389, 2450 };

$\mathcal{B}_{42}$ = { 8, 25, 63, 70, 232, 276, 289, 314, 435, 501, 597, 661, 680, 790, 812, 826, 837, 884, 889, 923, 947, 1030, 1192, 1208, 1239, 1282, 1293, 1299, 1324, 1332, 1337, 1346, 1481, 1516, 1578, 1603, 1692, 1708, 1727, 1775, 1787, 1809, 1860, 1871, 1886, 1919, 1931, 2000, 2172, 2192, 2198, 2216, 2230, 2255, 2280, 2297, 2376 };

$\mathcal{B}_{43}$ = { 68, 195, 210, 228, 332, 419, 439, 474, 488, 592, 595, 623, 668, 697, 705, 724, 765, 815, 827, 922, 925, 948, 973, 1021, 1088, 1112, 1134, 1286, 1417, 1451, 1476, 1510, 1539, 1608, 1610, 1625, 1640, 1642, 1821, 1910, 1986, 2005, 2008, 2072, 2093, 2124, 2137, 2175, 2219, 2242, 2249, 2269, 2278, 2351, 2367, 2377, 2439 }.

6.2 Decomposition into 43-arcs

A 43-arc in PG$(2, 7^2)$ is a set of 43 points, no three of which are collinear. From Corollary 2.9, the projective plane PG$(2, 7^2)$ consists of 57 disjoint 43-arcs. By using the procedure described in Section 2.4, we identified three non-isomorphic decompositions of the projective plane PG$(2, 7^2)$ into 57 disjoint 43-arcs of PG$(2, 7^2)$. The first decomposition has a projective collineation stabiliser of size 14706. Additionally, the second decomposition has a projective collineation stabiliser of size 171 and is isomorphic to $\mathbb{Z}_{57} \times \mathbb{Z}_3$. Also, the third decomposition has a projective collineation stabiliser of size 18 and is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_3$.

The first decomposition is given by 57 sets $\mathcal{K}_i$ for $i = 1, \ldots, 57$ each set consists of 43 points of one of the 57 disjoint 43-arcs in PG$(2, 7^2)$ as follows:

$\mathcal{K}_1$ = { 0, 57, 114, 171, 228, 285, 342, 399, 456, 513, 570, 627, 684, 741, 798, 855, 912, 969, 1026, 1083, 1140, 1197, 1254, 1311, 1368, 1425, 1482, 1539, 1596, 1653, 1710, 1767, 1824, 1881, 1938, 1995, 2052, 2109, 2166, 2223, 2280, 2337, 2394 };

$\mathcal{K}_2$ = { 43, 100, 157, 214, 271, 328, 385, 442, 499, 556, 613, 670, 727, 784, 841, 898, 955, 1012, 1069, 1126, 1183, 1240, 1297, 1354, 1411, 1468, 1525, 1582, 1639, 1696, 1753, 1810, 1867, 1924, 1981, 2038, 2095, 2152, 2209, 2266, 2323, 2380, 2437 };

$\mathcal{K}_3$ = { 29, 86, 143, 200, 257, 314, 371, 428, 485, 542, 599, 656, 713, 770, 827, 884, 941, 998, 1055, 1112, 1169, 1226, 1283, 1340, 1397, 1454, 1511, 1568, 1625, 1682, 1739, 1796, 1853, 1910, 1967, 2024, 2081, 2138, 2195, 2252, 2309, 2366, 2423 };

$\mathcal{K}_4$ = { 15, 72, 129, 186, 243, 300, 357, 414, 471, 528, 585, 642, 699, 756, 813, 870, 927, 984, 1041, 1098, 1155, 1212, 1269, 1326, 1383, 1440, 1497, 1554, 1611, 1668, 1725, 1782, 1839, 1896, 1953, 2010, 2067, 2124, 2181, 2238, 2295, 2352, 2409 };

$\mathcal{K}_5$ = { 1, 58, 115, 172, 229, 286, 343, 400, 457, 514, 571, 628, 685, 742, 799, 856, 913, 970, 1027, 1084, 1141, 1198, 1255, 1312, 1369, 1426, 1483, 1540, 1597, 1654, 1711, 1768, 1825, 1882, 1939, 1996, 2053, 2110, 2167, 2224, 2281, 2338, 2395 };

$\mathcal{K}_6$ = { 44, 101, 158, 215, 272, 329, 386, 443, 500, 557, 614, 671, 728, 785, 842, 899, 956, 1013, 1070, 1127, 1184, 1241, 1298, 1355, 1412, 1469, 1526, 1583, 1640, 1697, 1754, 1811, 1868, 1925, 1982, 2039, 2096, 2153, 2210, 2267, 2324, 2381, 2438 };

$\mathcal{K}_7$ = { 30, 87, 144, 201, 258, 315, 372, 429, 486, 543, 600, 657, 714, 771, 828, 885, 942, 999, 1056, 1113, 1170, 1227, 1284, 1341, 1398, 1455, 1512, 1569, 1626, 1683, 1740, 1797, 1854, 1911, 1968, 2025, 2082, 2139, 2196, 2253, 2310, 2367, 2424 };

$\mathcal{K}_8$ = { 16, 73, 130, 187, 244, 301, 358, 415, 472, 529, 586, 643, 700, 757, 814, 871, 928, 985, 1042, 1099, 1156, 1213, 1270, 1327, 1384, 1441, 1498, 1555, 1612, 1669, 1726, 1783, 1840, 1897, 1954, 2011, 2068, 2125, 2182, 2239, 2296, 2353, 2410 };

$\mathcal{K}_9$ = { 2, 59, 116, 173, 230, 287, 344, 401, 458, 515, 572, 629, 686, 743, 800, 857, 914, 971, 1028, 1085, 1142, 1199, 1256, 1313, 1370, 1427, 1484, 1541, 1598, 1655, 1712, 1769, 1826, 1883, 1940, 1997, 2054, 2111, 2168, 2225, 2282, 2339, 2396 };

$\mathcal{K}_{10}$ = { 45, 102, 159, 216, 273, 330, 387, 444, 501, 558, 615, 672, 729, 786, 843, 900, 957, 1014, 1071, 1128, 1185, 1242, 1299, 1356, 1413, 1470, 1527, 1584, 1641, 1698, 1755, 1812, 1869, 1926, 1983, 2040, 2097, 2154, 2211, 2268, 2325, 2382, 2439 };

$\mathcal{K}_{11}$ = { 31, 88, 145, 202, 259, 316, 373, 430, 487, 544, 601, 658, 715, 772, 829, 886, 943, 1000, 1057, 1114, 1171, 1228, 1285, 1342, 1399, 1456, 1513, 1570, 1627, 1684, 1741, 1798, 1855, 1912, 1969, 2026, 2083, 2140, 2197, 2254, 2311, 2368, 2425 };

$\mathcal{K}_{12}$ = { 17, 74, 131, 188, 245, 302, 359, 416, 473, 530, 587, 644, 701, 758, 815, 872, 929, 986, 1043, 1100, 1157, 1214, 1271, 1328, 1385, 1442, 1499, 1556, 1613, 1670, 1727, 1784, 1841, 1898, 1955, 2012, 2069, 2126, 2183, 2240, 2297, 2354, 2411 };

$\mathcal{K}_{13} = \{\, 3, 60, 117, 174, 231, 288, 345, 402, 459, 516, 573, 630, 687, 744, 801, 858, 915, 972, 1029, 1086, 1143, 1200, 1257, 1314, 1371, 1428, 1485, 1542, 1599, 1656, 1713, 1770, 1827, 1884, 1941, 1998, 2055, 2112, 2169, 2226, 2283, 2340, 2397 \,\};$

$\mathcal{K}_{14} = \{\, 46, 103, 160, 217, 274, 331, 388, 445, 502, 559, 616, 673, 730, 787, 844, 901, 958, 1015, 1072, 1129, 1186, 1243, 1300, 1357, 1414, 1471, 1528, 1585, 1642, 1699, 1756, 1813, 1870, 1927, 1984, 2041, 2098, 2155, 2212, 2269, 2326, 2383, 2440 \,\};$

$\mathcal{K}_{15} = \{\, 32, 89, 146, 203, 260, 317, 374, 431, 488, 545, 602, 659, 716, 773, 830, 887, 944, 1001, 1058, 1115, 1172, 1229, 1286, 1343, 1400, 1457, 1514, 1571, 1628, 1685, 1742, 1799, 1856, 1913, 1970, 2027, 2084, 2141, 2198, 2255, 2312, 2369, 2426 \,\};$

$\mathcal{K}_{16} = \{\, 18, 75, 132, 189, 246, 303, 360, 417, 474, 531, 588, 645, 702, 759, 816, 873, 930, 987, 1044, 1101, 1158, 1215, 1272, 1329, 1386, 1443, 1500, 1557, 1614, 1671, 1728, 1785, 1842, 1899, 1956, 2013, 2070, 2127, 2184, 2241, 2298, 2355, 2412 \,\};$

$\mathcal{K}_{17} = \{\, 4, 61, 118, 175, 232, 289, 346, 403, 460, 517, 574, 631, 688, 745, 802, 859, 916, 973, 1030, 1087, 1144, 1201, 1258, 1315, 1372, 1429, 1486, 1543, 1600, 1657, 1714, 1771, 1828, 1885, 1942, 1999, 2056, 2113, 2170, 2227, 2284, 2341, 2398 \,\};$

$\mathcal{K}_{18} = \{\, 47, 104, 161, 218, 275, 332, 389, 446, 503, 560, 617, 674, 731, 788, 845, 902, 959, 1016, 1073, 1130, 1187, 1244, 1301, 1358, 1415, 1472, 1529, 1586, 1643, 1700, 1757, 1814, 1871, 1928, 1985, 2042, 2099, 2156, 2213, 2270, 2327, 2384, 2441 \,\};$

$\mathcal{K}_{19} = \{\, 33, 90, 147, 204, 261, 318, 375, 432, 489, 546, 603, 660, 717, 774, 831, 888, 945, 1002, 1059, 1116, 1173, 1230, 1287, 1344, 1401, 1458, 1515, 1572, 1629, 1686, 1743, 1800, 1857, 1914, 1971, 2028, 2085, 2142, 2199, 2256, 2313, 2370, 2427 \,\};$

$\mathcal{K}_{20} = \{\, 19, 76, 133, 190, 247, 304, 361, 418, 475, 532, 589, 646, 703, 760, 817, 874, 931, 988, 1045, 1102, 1159, 1216, 1273, 1330, 1387, 1444, 1501, 1558, 1615, 1672, 1729, 1786, 1843, 1900, 1957, 2014, 2071, 2128, 2185, 2242, 2299, 2356, 2413 \,\};$

$\mathcal{K}_{21} = \{\, 5, 62, 119, 176, 233, 290, 347, 404, 461, 518, 575, 632, 689, 746, 803, 860, 917, 974, 1031, 1088, 1145, 1202, 1259, 1316, 1373, 1430, 1487, 1544, 1601, 1658, 1715, 1772, 1829, 1886, 1943, 2000, 2057, 2114, 2171, 2228, 2285, 2342, 2399 \,\};$

$\mathcal{K}_{22} = \{ 48, 105, 162, 219, 276, 333, 390, 447, 504, 561, 618, 675, 732, 789, 846, 903, 960, 1017, 1074, 1131, 1188, 1245, 1302, 1359, 1416, 1473, 1530, 1587, 1644, 1701, 1758, 1815, 1872, 1929, 1986, 2043, 2100, 2157, 2214, 2271, 2328, 2385, 2442 \};$

$\mathcal{K}_{23} = \{ 34, 91, 148, 205, 262, 319, 376, 433, 490, 547, 604, 661, 718, 775, 832, 889, 946, 1003, 1060, 1117, 1174, 1231, 1288, 1345, 1402, 1459, 1516, 1573, 1630, 1687, 1744, 1801, 1858, 1915, 1972, 2029, 2086, 2143, 2200, 2257, 2314, 2371, 2428 \};$

$\mathcal{K}_{24} = \{ 20, 77, 134, 191, 248, 305, 362, 419, 476, 533, 590, 647, 704, 761, 818, 875, 932, 989, 1046, 1103, 1160, 1217, 1274, 1331, 1388, 1445, 1502, 1559, 1616, 1673, 1730, 1787, 1844, 1901, 1958, 2015, 2072, 2129, 2186, 2243, 2300, 2357, 2414 \};$

$\mathcal{K}_{25} = \{ 6, 63, 120, 177, 234, 291, 348, 405, 462, 519, 576, 633, 690, 747, 804, 861, 918, 975, 1032, 1089, 1146, 1203, 1260, 1317, 1374, 1431, 1488, 1545, 1602, 1659, 1716, 1773, 1830, 1887, 1944, 2001, 2058, 2115, 2172, 2229, 2286, 2343, 2400 \};$

$\mathcal{K}_{26} = \{ 49, 106, 163, 220, 277, 334, 391, 448, 505, 562, 619, 676, 733, 790, 847, 904, 961, 1018, 1075, 1132, 1189, 1246, 1303, 1360, 1417, 1474, 1531, 1588, 1645, 1702, 1759, 1816, 1873, 1930, 1987, 2044, 2101, 2158, 2215, 2272, 2329, 2386, 2443 \};$

$\mathcal{K}_{27} = \{ 35, 92, 149, 206, 263, 320, 377, 434, 491, 548, 605, 662, 719, 776, 833, 890, 947, 1004, 1061, 1118, 1175, 1232, 1289, 1346, 1403, 1460, 1517, 1574, 1631, 1688, 1745, 1802, 1859, 1916, 1973, 2030, 2087, 2144, 2201, 2258, 2315, 2372, 2429 \};$

$\mathcal{K}_{28} = \{ 21, 78, 135, 192, 249, 306, 363, 420, 477, 534, 591, 648, 705, 762, 819, 876, 933, 990, 1047, 1104, 1161, 1218, 1275, 1332, 1389, 1446, 1503, 1560, 1617, 1674, 1731, 1788, 1845, 1902, 1959, 2016, 2073, 2130, 2187, 2244, 2301, 2358, 2415 \};$

$\mathcal{K}_{29} = \{ 7, 64, 121, 178, 235, 292, 349, 406, 463, 520, 577, 634, 691, 748, 805, 862, 919, 976, 1033, 1090, 1147, 1204, 1261, 1318, 1375, 1432, 1489, 1546, 1603, 1660, 1717, 1774, 1831, 1888, 1945, 2002, 2059, 2116, 2173, 2230, 2287, 2344, 2401 \};$

$\mathcal{K}_{30} = \{ 50, 107, 164, 221, 278, 335, 392, 449, 506, 563, 620, 677, 734, 791, 848, 905, 962, 1019, 1076, 1133, 1190, 1247, 1304, 1361, 1418, 1475, 1532, 1589, 1646, 1703, 1760, 1817, 1874, 1931, 1988, 2045, 2102, 2159, 2216, 2273, 2330, 2387, 2444 \};$

$\mathcal{K}_{31} = \{ 36, 93, 150, 207, 264, 321, 378, 435, 492, 549, 606, 663, 720, 777, 834, 891, 948, 1005, 1062, 1119, 1176, 1233, 1290, 1347, 1404, 1461, 1518, 1575, 1632, 1689, 1746, 1803, 1860, 1917, 1974, 2031, 2088, 2145, 2202, 2259, 2316, 2373, 2430 \}$;

$\mathcal{K}_{32} = \{ 22, 79, 136, 193, 250, 307, 364, 421, 478, 535, 592, 649, 706, 763, 820, 877, 934, 991, 1048, 1105, 1162, 1219, 1276, 1333, 1390, 1447, 1504, 1561, 1618, 1675, 1732, 1789, 1846, 1903, 1960, 2017, 2074, 2131, 2188, 2245, 2302, 2359, 2416 \}$;

$\mathcal{K}_{33} = \{ 8, 65, 122, 179, 236, 293, 350, 407, 464, 521, 578, 635, 692, 749, 806, 863, 920, 977, 1034, 1091, 1148, 1205, 1262, 1319, 1376, 1433, 1490, 1547, 1604, 1661, 1718, 1775, 1832, 1889, 1946, 2003, 2060, 2117, 2174, 2231, 2288, 2345, 2402 \}$;

$\mathcal{K}_{34} = \{ 51, 108, 165, 222, 279, 336, 393, 450, 507, 564, 621, 678, 735, 792, 849, 906, 963, 1020, 1077, 1134, 1191, 1248, 1305, 1362, 1419, 1476, 1533, 1590, 1647, 1704, 1761, 1818, 1875, 1932, 1989, 2046, 2103, 2160, 2217, 2274, 2331, 2388, 2445 \}$;

$\mathcal{K}_{35} = \{ 37, 94, 151, 208, 265, 322, 379, 436, 493, 550, 607, 664, 721, 778, 835, 892, 949, 1006, 1063, 1120, 1177, 1234, 1291, 1348, 1405, 1462, 1519, 1576, 1633, 1690, 1747, 1804, 1861, 1918, 1975, 2032, 2089, 2146, 2203, 2260, 2317, 2374, 2431 \}$;

$\mathcal{K}_{36} = \{ 23, 80, 137, 194, 251, 308, 365, 422, 479, 536, 593, 650, 707, 764, 821, 878, 935, 992, 1049, 1106, 1163, 1220, 1277, 1334, 1391, 1448, 1505, 1562, 1619, 1676, 1733, 1790, 1847, 1904, 1961, 2018, 2075, 2132, 2189, 2246, 2303, 2360, 2417 \}$;

$\mathcal{K}_{37} = \{ 9, 66, 123, 180, 237, 294, 351, 408, 465, 522, 579, 636, 693, 750, 807, 864, 921, 978, 1035, 1092, 1149, 1206, 1263, 1320, 1377, 1434, 1491, 1548, 1605, 1662, 1719, 1776, 1833, 1890, 1947, 2004, 2061, 2118, 2175, 2232, 2289, 2346, 2403 \}$;

$\mathcal{K}_{38} = \{ 52, 109, 166, 223, 280, 337, 394, 451, 508, 565, 622, 679, 736, 793, 850, 907, 964, 1021, 1078, 1135, 1192, 1249, 1306, 1363, 1420, 1477, 1534, 1591, 1648, 1705, 1762, 1819, 1876, 1933, 1990, 2047, 2104, 2161, 2218, 2275, 2332, 2389, 2446 \}$;

$\mathcal{K}_{39} = \{ 38, 95, 152, 209, 266, 323, 380, 437, 494, 551, 608, 665, 722, 779, 836, 893, 950, 1007, 1064, 1121, 1178, 1235, 1292, 1349, 1406, 1463, 1520, 1577, 1634, 1691, 1748, 1805, 1862, 1919, 1976, 2033, 2090, 2147, 2204, 2261, 2318, 2375, 2432 \}$;

$$\mathcal{K}_{40} = \{ 24, 81, 138, 195, 252, 309, 366, 423, 480, 537, 594, 651, 708, 765, 822, 879, 936, 993, 1050, 1107, 1164, 1221, 1278, 1335, 1392, 1449, 1506, 1563, 1620, 1677, 1734, 1791, 1848, 1905, 1962, 2019, 2076, 2133, 2190, 2247, 2304, 2361, 2418 \};$$

$$\mathcal{K}_{41} = \{ 10, 67, 124, 181, 238, 295, 352, 409, 466, 523, 580, 637, 694, 751, 808, 865, 922, 979, 1036, 1093, 1150, 1207, 1264, 1321, 1378, 1435, 1492, 1549, 1606, 1663, 1720, 1777, 1834, 1891, 1948, 2005, 2062, 2119, 2176, 2233, 2290, 2347, 2404 \};$$

$$\mathcal{K}_{42} = \{ 53, 110, 167, 224, 281, 338, 395, 452, 509, 566, 623, 680, 737, 794, 851, 908, 965, 1022, 1079, 1136, 1193, 1250, 1307, 1364, 1421, 1478, 1535, 1592, 1649, 1706, 1763, 1820, 1877, 1934, 1991, 2048, 2105, 2162, 2219, 2276, 2333, 2390, 2447 \};$$

$$\mathcal{K}_{43} = \{ 39, 96, 153, 210, 267, 324, 381, 438, 495, 552, 609, 666, 723, 780, 837, 894, 951, 1008, 1065, 1122, 1179, 1236, 1293, 1350, 1407, 1464, 1521, 1578, 1635, 1692, 1749, 1806, 1863, 1920, 1977, 2034, 2091, 2148, 2205, 2262, 2319, 2376, 2433 \};$$

$$\mathcal{K}_{44} = \{ 25, 82, 139, 196, 253, 310, 367, 424, 481, 538, 595, 652, 709, 766, 823, 880, 937, 994, 1051, 1108, 1165, 1222, 1279, 1336, 1393, 1450, 1507, 1564, 1621, 1678, 1735, 1792, 1849, 1906, 1963, 2020, 2077, 2134, 2191, 2248, 2305, 2362, 2419 \};$$

$$\mathcal{K}_{45} = \{ 11, 68, 125, 182, 239, 296, 353, 410, 467, 524, 581, 638, 695, 752, 809, 866, 923, 980, 1037, 1094, 1151, 1208, 1265, 1322, 1379, 1436, 1493, 1550, 1607, 1664, 1721, 1778, 1835, 1892, 1949, 2006, 2063, 2120, 2177, 2234, 2291, 2348, 2405 \};$$

$$\mathcal{K}_{46} = \{ 54, 111, 168, 225, 282, 339, 396, 453, 510, 567, 624, 681, 738, 795, 852, 909, 966, 1023, 1080, 1137, 1194, 1251, 1308, 1365, 1422, 1479, 1536, 1593, 1650, 1707, 1764, 1821, 1878, 1935, 1992, 2049, 2106, 2163, 2220, 2277, 2334, 2391, 2448 \};$$

$$\mathcal{K}_{47} = \{ 40, 97, 154, 211, 268, 325, 382, 439, 496, 553, 610, 667, 724, 781, 838, 895, 952, 1009, 1066, 1123, 1180, 1237, 1294, 1351, 1408, 1465, 1522, 1579, 1636, 1693, 1750, 1807, 1864, 1921, 1978, 2035, 2092, 2149, 2206, 2263, 2320, 2377, 2434 \};$$

$$\mathcal{K}_{48} = \{ 26, 83, 140, 197, 254, 311, 368, 425, 482, 539, 596, 653, 710, 767, 824, 881, 938, 995, 1052, 1109, 1166, 1223, 1280, 1337, 1394, 1451, 1508, 1565, 1622, 1679, 1736, 1793, 1850, 1907, 1964, 2021, 2078, 2135, 2192, 2249, 2306, 2363, 2420 \};$$

$\mathcal{K}_{49} = \{\,12, 69, 126, 183, 240, 297, 354, 411, 468, 525, 582, 639, 696, 753, 810, 867, 924, 981, 1038, 1095, 1152, 1209, 1266, 1323, 1380, 1437, 1494, 1551, 1608, 1665, 1722, 1779, 1836, 1893, 1950, 2007, 2064, 2121, 2178, 2235, 2292, 2349, 2406\,\};$

$\mathcal{K}_{50} = \{\,55, 112, 169, 226, 283, 340, 397, 454, 511, 568, 625, 682, 739, 796, 853, 910, 967, 1024, 1081, 1138, 1195, 1252, 1309, 1366, 1423, 1480, 1537, 1594, 1651, 1708, 1765, 1822, 1879, 1936, 1993, 2050, 2107, 2164, 2221, 2278, 2335, 2392, 2449\,\};$

$\mathcal{K}_{51} = \{\,41, 98, 155, 212, 269, 326, 383, 440, 497, 554, 611, 668, 725, 782, 839, 896, 953, 1010, 1067, 1124, 1181, 1238, 1295, 1352, 1409, 1466, 1523, 1580, 1637, 1694, 1751, 1808, 1865, 1922, 1979, 2036, 2093, 2150, 2207, 2264, 2321, 2378, 2435\,\};$

$\mathcal{K}_{52} = \{\,27, 84, 141, 198, 255, 312, 369, 426, 483, 540, 597, 654, 711, 768, 825, 882, 939, 996, 1053, 1110, 1167, 1224, 1281, 1338, 1395, 1452, 1509, 1566, 1623, 1680, 1737, 1794, 1851, 1908, 1965, 2022, 2079, 2136, 2193, 2250, 2307, 2364, 2421\,\};$

$\mathcal{K}_{53} = \{\,13, 70, 127, 184, 241, 298, 355, 412, 469, 526, 583, 640, 697, 754, 811, 868, 925, 982, 1039, 1096, 1153, 1210, 1267, 1324, 1381, 1438, 1495, 1552, 1609, 1666, 1723, 1780, 1837, 1894, 1951, 2008, 2065, 2122, 2179, 2236, 2293, 2350, 2407\,\};$

$\mathcal{K}_{54} = \{\,56, 113, 170, 227, 284, 341, 398, 455, 512, 569, 626, 683, 740, 797, 854, 911, 968, 1025, 1082, 1139, 1196, 1253, 1310, 1367, 1424, 1481, 1538, 1595, 1652, 1709, 1766, 1823, 1880, 1937, 1994, 2051, 2108, 2165, 2222, 2279, 2336, 2393, 2450\,\};$

$\mathcal{K}_{55} = \{\,42, 99, 156, 213, 270, 327, 384, 441, 498, 555, 612, 669, 726, 783, 840, 897, 954, 1011, 1068, 1125, 1182, 1239, 1296, 1353, 1410, 1467, 1524, 1581, 1638, 1695, 1752, 1809, 1866, 1923, 1980, 2037, 2094, 2151, 2208, 2265, 2322, 2379, 2436\,\};$

$\mathcal{K}_{56} = \{\,28, 85, 142, 199, 256, 313, 370, 427, 484, 541, 598, 655, 712, 769, 826, 883, 940, 997, 1054, 1111, 1168, 1225, 1282, 1339, 1396, 1453, 1510, 1567, 1624, 1681, 1738, 1795, 1852, 1909, 1966, 2023, 2080, 2137, 2194, 2251, 2308, 2365, 2422\,\};$

$\mathcal{K}_{57} = \{\,14, 71, 128, 185, 242, 299, 356, 413, 470, 527, 584, 641, 698, 755, 812, 869, 926, 983, 1040, 1097, 1154, 1211, 1268, 1325, 1382, 1439, 1496, 1553, 1610, 1667, 1724, 1781, 1838, 1895, 1952, 2009, 2066, 2123, 2180, 2237, 2294, 2351, 2408\,\}.$

The second decomposition is given by 57 sets $\mathcal{K}_i$ for $i = 1, \dots, 57$ each set consists of 43 points of one of the 57 disjoint 43-arcs in $\mathrm{PG}(2, 7^2)$ as follows:

$\mathcal{K}_1 = \{$ 0, 57, 114, 171, 228, 285, 342, 399, 456, 513, 570, 627, 684, 741, 798, 855, 912, 969, 1026, 1083, 1140, 1197, 1254, 1311, 1368, 1425, 1482, 1539, 1596, 1653, 1710, 1767, 1824, 1881, 1938, 1995, 2052, 2109, 2166, 2223, 2280, 2337, 2394 $\}$;

$\mathcal{K}_2 = \{$ 19, 41, 98, 151, 182, 221, 236, 259, 271, 461, 476, 576, 608, 661, 666, 723, 792, 795, 803, 1010, 1021, 1024, 1060, 1065, 1108, 1117, 1125, 1389, 1398, 1426, 1522, 1579, 1612, 1785, 1828, 1840, 1876, 1890, 1923, 1932, 2137, 2184, 2400 $\}$;

$\mathcal{K}_3 = \{$ 26, 175, 196, 251, 301, 492, 560, 574, 678, 746, 805, 835, 914, 940, 996, 1053, 1142, 1157, 1184, 1208, 1274, 1340, 1363, 1416, 1479, 1520, 1673, 1730, 1738, 1758, 1759, 1781, 1865, 1965, 2092, 2146, 2248, 2286, 2305, 2361, 2375, 2386, 2420 $\}$;

$\mathcal{K}_4 = \{$ 54, 102, 174, 296, 325, 330, 369, 469, 539, 548, 584, 633, 642, 822, 935, 1032, 1048, 1089, 1175, 1358, 1382, 1434, 1516, 1572, 1645, 1646, 1677, 1701, 1753, 1757, 1858, 1862, 1948, 1951, 2008, 2025, 2032, 2054, 2246, 2292, 2348, 2349, 2405 $\}$;

$\mathcal{K}_5 = \{$ 20, 192, 318, 371, 391, 409, 411, 448, 521, 556, 819, 907, 928, 968, 978, 1025, 1111, 1164, 1194, 1223, 1242, 1251, 1264, 1307, 1324, 1351, 1354, 1444, 1593, 1765, 1801, 1822, 1855, 1917, 1937, 2083, 2118, 2129, 2132, 2224, 2370, 2440, 2443 $\}$;

$\mathcal{K}_6 = \{$ 22, 40, 78, 97, 155, 190, 230, 235, 302, 341, 439, 451, 530, 544, 623, 751, 864, 963, 1008, 1095, 1232, 1296, 1353, 1400, 1406, 1488, 1591, 1625, 1631, 1688, 1830, 1946, 1982, 2059, 2124, 2171, 2219, 2242, 2249, 2265, 2335, 2414, 2423 $\}$;

$\mathcal{K}_7 = \{$ 31, 110, 139, 212, 324, 337, 372, 434, 449, 518, 568, 669, 675, 708, 759, 847, 876, 917, 927, 938, 950, 995, 1076, 1110, 1167, 1168, 1189, 1255, 1275, 1332, 1337, 1442, 1698, 1725, 1770, 1869, 1920, 1945, 1998, 2080, 2090, 2121, 2317 $\}$;

$\mathcal{K}_8 = \{$ 58, 239, 252, 292, 295, 335, 459, 484, 691, 720, 748, 755, 775, 810, 836, 856, 867, 929, 1013, 1051, 1132, 1171, 1209, 1347, 1392, 1409, 1473, 1585, 1594, 1600, 1680, 1724, 1726, 1813, 1872, 1906, 2065, 2087, 2122, 2262, 2388, 2417, 2429 $\}$;

$\mathcal{K}_9 = \{ 93, 128, 138, 184, 274, 283, 332, 340, 545, 606, 614, 682, 773, 911, 979, 1028, 1082, 1133, 1149, 1151, 1276, 1291, 1388, 1393, 1532, 1541, 1573, 1589, 1696, 1731, 1854, 1903, 1972, 2006, 2069, 2073, 2105, 2267, 2290, 2333, 2364, 2372, 2435 \};$

$\mathcal{K}_{10} = \{ 100, 132, 134, 216, 314, 338, 382, 622, 657, 729, 747, 749, 757, 852, 868, 939, 1086, 1124, 1147, 1162, 1181, 1344, 1376, 1391, 1410, 1455, 1467, 1489, 1493, 1523, 1649, 1652, 1685, 1803, 1806, 1860, 1980, 2016, 2034, 2176, 2186, 2355, 2448 \};$

$\mathcal{K}_{11} = \{ 27, 39, 59, 106, 123, 339, 358, 635, 725, 745, 901, 937, 973, 1044, 1121, 1165, 1169, 1191, 1216, 1225, 1280, 1282, 1294, 1350, 1404, 1432, 1535, 1545, 1818, 1894, 1944, 1960, 1969, 1988, 2011, 2067, 2079, 2136, 2149, 2330, 2336, 2359, 2416 \};$

$\mathcal{K}_{12} = \{ 94, 150, 213, 416, 423, 471, 481, 583, 596, 640, 721, 812, 869, 904, 926, 982, 1037, 1077, 1152, 1198, 1236, 1265, 1303, 1313, 1346, 1433, 1536, 1621, 1636, 1773, 1795, 1832, 1885, 1889, 1928, 1943, 2047, 2111, 2156, 2192, 2259, 2273, 2312 \};$

$\mathcal{K}_{13} = \{ 10, 28, 83, 92, 95, 111, 194, 364, 367, 447, 468, 489, 501, 581, 617, 625, 706, 717, 821, 841, 898, 956, 990, 1000, 1139, 1239, 1299, 1439, 1671, 1721, 1875, 1884, 2051, 2053, 2064, 2108, 2144, 2274, 2331, 2334, 2397, 2431, 2450 \};$

$\mathcal{K}_{14} = \{ 61, 80, 179, 327, 361, 406, 467, 547, 567, 589, 601, 649, 778, 1054, 1177, 1180, 1234, 1377, 1408, 1411, 1466, 1537, 1629, 1699, 1788, 1797, 1811, 1820, 1848, 1878, 1898, 1910, 1967, 2004, 2123, 2187, 2212, 2254, 2288, 2379, 2392, 2406, 2436 \};$

$\mathcal{K}_{15} = \{ 29, 122, 142, 201, 279, 621, 659, 698, 700, 701, 724, 782, 850, 878, 915, 1088, 1160, 1172, 1204, 1212, 1244, 1247, 1263, 1269, 1277, 1316, 1334, 1435, 1477, 1499, 1501, 1521, 1603, 1764, 1864, 1879, 2078, 2095, 2194, 2251, 2306, 2378, 2421 \};$

$\mathcal{K}_{16} = \{ 2, 9, 18, 66, 269, 311, 343, 400, 516, 519, 531, 564, 610, 652, 670, 713, 730, 736, 794, 882, 925, 1349, 1518, 1555, 1622, 1633, 1737, 1782, 1838, 1895, 1950, 1975, 1976, 2043, 2045, 2061, 2169, 2178, 2228, 2237, 2256, 2271, 2444 \};$

$\mathcal{K}_{17} = \{ 15, 101, 158, 183, 241, 264, 331, 495, 651, 732, 752, 857, 859, 913, 972, 1224, 1235, 1278, 1372, 1423, 1494, 1508, 1534, 1641, 1651, 1733, 1739, 1837, 1867, 1918, 1933, 1984, 1990, 2074, 2075, 2128, 2141, 2143, 2316, 2329, 2371, 2393, 2418 \};$

$\mathcal{K}_{18}$ = { 38, 45, 104, 238, 270, 305, 404, 421, 465, 485, 533, 537, 671, 727, 787, 807, 820, 832, 854, 865, 942, 943, 944, 967, 999, 1041, 1118, 1380, 1500, 1547, 1597, 1709, 1742, 2005, 2018, 2033, 2042, 2049, 2077, 2107, 2179, 2264, 2432 };

$\mathcal{K}_{19}$ = { 85, 124, 172, 224, 280, 293, 303, 497, 603, 750, 762, 768, 853, 895, 1101, 1116, 1182, 1196, 1335, 1355, 1359, 1364, 1445, 1512, 1734, 1791, 1796, 1798, 1808, 1841, 1989, 2037, 2048, 2115, 2177, 2198, 2238, 2296, 2341, 2343, 2345, 2353, 2389 };

$\mathcal{K}_{20}$ = { 178, 223, 245, 312, 336, 415, 437, 475, 665, 735, 779, 804, 858, 892, 922, 988, 997, 1023, 1073, 1098, 1144, 1321, 1369, 1378, 1452, 1459, 1474, 1524, 1557, 1598, 1655, 1694, 1702, 1723, 1749, 1793, 1852, 1934, 1942, 2180, 2196, 2376, 2381 };

$\mathcal{K}_{21}$ = { 81, 163, 198, 229, 263, 281, 491, 641, 673, 674, 680, 722, 737, 740, 771, 786, 843, 880, 992, 1019, 1099, 1127, 1202, 1267, 1339, 1496, 1507, 1635, 1665, 1690, 1713, 1715, 2072, 2085, 2089, 2168, 2190, 2209, 2258, 2320, 2327, 2398, 2407 };

$\mathcal{K}_{22}$ = { 24, 88, 135, 145, 219, 284, 381, 401, 438, 613, 766, 767, 827, 888, 980, 983, 1068, 1130, 1138, 1153, 1290, 1356, 1403, 1424, 1431, 1468, 1528, 1607, 1686, 1777, 1790, 1902, 1957, 1968, 2084, 2119, 2125, 2157, 2175, 2189, 2358, 2433, 2438 };

$\mathcal{K}_{23}$ = { 86, 129, 170, 233, 319, 453, 488, 522, 612, 639, 658, 760, 790, 828, 866, 871, 883, 921, 955, 986, 1043, 1047, 1136, 1318, 1427, 1443, 1478, 1558, 1564, 1648, 1682, 1752, 1774, 1846, 1963, 1979, 2002, 2020, 2040, 2062, 2142, 2257, 2277 };

$\mathcal{K}_{24}$ = { 3, 60, 112, 164, 173, 392, 394, 419, 427, 463, 483, 498, 527, 615, 837, 932, 984, 1123, 1179, 1188, 1319, 1342, 1421, 1487, 1556, 1567, 1664, 1711, 1750, 1839, 2014, 2024, 2030, 2063, 2120, 2147, 2159, 2163, 2182, 2284, 2285, 2298, 2446 };

$\mathcal{K}_{25}$ = { 71, 115, 197, 290, 291, 310, 313, 435, 494, 499, 510, 573, 663, 688, 789, 893, 909, 953, 966, 970, 1009, 1016, 1122, 1211, 1288, 1304, 1326, 1438, 1449, 1469, 1484, 1674, 1761, 1812, 2021, 2229, 2240, 2269, 2326, 2368, 2419, 2430, 2442 };

$\mathcal{K}_{26}$ = { 50, 68, 69, 91, 195, 253, 388, 410, 455, 457, 535, 664, 672, 728, 815, 889, 991, 1036, 1090, 1114, 1183, 1219, 1229, 1245, 1257, 1286, 1568, 1575, 1588, 1807, 1871, 1893, 1901, 1907, 1908, 2028, 2101, 2206, 2247, 2263, 2310, 2324, 2408 };

$\mathcal{K}_{27} = \{8, 215, 254, 321, 353, 377, 395, 412, 432, 464, 588, 645, 653, 692, 704, 710, 714, 764, 783, 870, 890, 1072, 1213, 1260, 1345, 1365, 1366, 1502, 1569, 1707, 1779, 1799, 1834, 1900, 1931, 1993, 2012, 2131, 2233, 2252, 2283, 2380, 2425\}$;

$\mathcal{K}_{28} = \{1, 21, 32, 63, 96, 126, 176, 211, 304, 383, 450, 472, 515, 525, 534, 553, 582, 598, 619, 861, 872, 877, 906, 998, 1029, 1056, 1134, 1158, 1193, 1252, 1429, 1486, 1554, 1592, 1604, 1650, 1880, 1947, 1958, 2068, 2098, 2202, 2207\}$;

$\mathcal{K}_{29} = \{37, 55, 161, 218, 220, 242, 300, 307, 350, 496, 685, 711, 844, 894, 1031, 1058, 1109, 1131, 1293, 1317, 1338, 1370, 1420, 1451, 1678, 1804, 1842, 1859, 1873, 1888, 1999, 2046, 2050, 2093, 2154, 2208, 2210, 2225, 2260, 2340, 2401, 2410, 2449\}$;

$\mathcal{K}_{30} = \{25, 43, 137, 214, 232, 278, 417, 424, 440, 443, 558, 597, 600, 662, 676, 731, 791, 839, 896, 949, 981, 1173, 1230, 1231, 1323, 1394, 1413, 1456, 1460, 1503, 1666, 1712, 1778, 1882, 1904, 1919, 1986, 2023, 2155, 2231, 2255, 2293, 2360\}$;

$\mathcal{K}_{31} = \{125, 222, 306, 454, 524, 549, 624, 636, 683, 715, 796, 900, 919, 1045, 1049, 1062, 1092, 1102, 1104, 1112, 1115, 1221, 1258, 1266, 1283, 1284, 1310, 1320, 1395, 1415, 1552, 1614, 1640, 1667, 1743, 1794, 1844, 1851, 1916, 1954, 2086, 2311, 2354\}$;

$\mathcal{K}_{32} = \{4, 30, 109, 187, 202, 248, 298, 355, 414, 474, 520, 565, 569, 585, 739, 758, 793, 808, 1011, 1105, 1203, 1256, 1295, 1371, 1422, 1560, 1576, 1613, 1615, 1618, 1637, 1638, 1639, 1760, 1772, 1821, 1829, 1921, 2173, 2232, 2350, 2427, 2441\}$;

$\mathcal{K}_{33} = \{141, 167, 257, 268, 276, 333, 368, 444, 562, 563, 578, 630, 667, 826, 897, 976, 1002, 1064, 1091, 1100, 1186, 1187, 1292, 1315, 1361, 1367, 1384, 1453, 1580, 1676, 1705, 1766, 1823, 1940, 1996, 2031, 2058, 2152, 2167, 2226, 2250, 2299, 2342\}$;

$\mathcal{K}_{34} = \{47, 84, 117, 136, 146, 240, 366, 376, 408, 433, 445, 470, 575, 594, 686, 712, 754, 770, 834, 846, 874, 1020, 1096, 1097, 1128, 1166, 1417, 1533, 1565, 1632, 1695, 1754, 1802, 1817, 1925, 2010, 2041, 2094, 2151, 2197, 2199, 2204, 2319\}$;

$\mathcal{K}_{35} = \{72, 113, 144, 181, 206, 315, 356, 430, 436, 442, 644, 946, 989, 1001, 1038, 1039, 1046, 1059, 1069, 1141, 1390, 1437, 1446, 1450, 1506, 1510, 1530, 1601, 1856, 1909, 1956, 1966, 2060, 2071, 2088, 2164, 2188, 2222, 2325, 2332, 2374, 2383, 2404\}$;

$\mathcal{K}_{36}$ = { 23, 51, 56, 99, 105, 193, 328, 346, 347, 348, 360, 380, 405, 441, 511, 536, 554, 862, 863, 899, 910, 931, 1093, 1154, 1215, 1328, 1343, 1498, 1511, 1550, 1553, 1559, 1610, 1627, 1630, 1654, 1661, 1669, 1763, 1991, 2112, 2138, 2301 };

$\mathcal{K}_{37}$ = { 74, 205, 256, 345, 384, 396, 452, 702, 800, 813, 918, 971, 1201, 1206, 1302, 1305, 1352, 1397, 1405, 1412, 1440, 1458, 1464, 1509, 1525, 1529, 1582, 1658, 1692, 1703, 1717, 1719, 1740, 1751, 1927, 2056, 2103, 2117, 2133, 2158, 2215, 2243, 2365 };

$\mathcal{K}_{38}$ = { 13, 90, 107, 108, 143, 153, 159, 160, 162, 199, 200, 208, 255, 277, 571, 595, 643, 679, 699, 788, 818, 823, 887, 947, 1004, 1006, 1192, 1330, 1402, 1505, 1542, 1543, 1602, 1693, 1720, 1974, 2009, 2253, 2307, 2351, 2384, 2395, 2439 };

$\mathcal{K}_{39}$ = { 76, 79, 140, 177, 237, 246, 262, 275, 308, 329, 378, 591, 648, 695, 811, 923, 945, 961, 1079, 1106, 1156, 1178, 1210, 1357, 1462, 1481, 1517, 1571, 1691, 1735, 1741, 1800, 1896, 1912, 1953, 1981, 2015, 2057, 2153, 2281, 2294, 2396, 2413 };

$\mathcal{K}_{40}$ = { 7, 12, 64, 188, 227, 317, 359, 458, 480, 543, 561, 599, 689, 780, 817, 875, 930, 934, 993, 1027, 1084, 1087, 1161, 1170, 1287, 1374, 1491, 1562, 1584, 1672, 1809, 1835, 1987, 1992, 2003, 2039, 2161, 2201, 2220, 2266, 2279, 2289, 2323 };

$\mathcal{K}_{41}$ = { 67, 73, 156, 209, 234, 294, 407, 429, 555, 577, 580, 681, 733, 776, 785, 838, 842, 848, 884, 886, 905, 941, 964, 1066, 1126, 1159, 1544, 1624, 1644, 1656, 1762, 1771, 1827, 1845, 1897, 1936, 2097, 2127, 2270, 2352, 2362, 2399, 2445 };

$\mathcal{K}_{42}$ = { 11, 48, 186, 243, 326, 370, 420, 503, 517, 831, 849, 903, 1119, 1176, 1199, 1261, 1268, 1348, 1399, 1418, 1463, 1471, 1483, 1626, 1642, 1683, 1736, 1744, 1783, 1784, 1826, 1877, 1886, 1962, 1964, 1973, 1978, 2044, 2195, 2218, 2261, 2390, 2422 };

$\mathcal{K}_{43}$ = { 14, 148, 266, 387, 390, 425, 431, 529, 586, 602, 703, 765, 801, 833, 1003, 1014, 1085, 1240, 1241, 1281, 1308, 1448, 1497, 1515, 1549, 1563, 1578, 1608, 1617, 1675, 1689, 1700, 1732, 1768, 1825, 1836, 2066, 2091, 2102, 2203, 2287, 2297, 2357 };

$\mathcal{K}_{44}$ = { 154, 180, 261, 273, 297, 316, 352, 477, 490, 616, 628, 677, 709, 761, 908, 1033, 1081, 1148, 1150, 1205, 1222, 1309, 1314, 1387, 1430, 1526, 1583, 1616, 1716, 2019, 2038, 2082, 2140, 2236, 2241, 2245, 2282, 2303, 2309, 2339, 2373, 2412, 2437 };

$\mathcal{K}_{45} = \{$ 120, 133, 386, 478, 509, 587, 609, 631, 656, 802, 809, 948, 975, 985, 1035, 1052, 1055, 1075, 1103, 1129, 1214, 1253, 1322, 1470, 1527, 1590, 1606, 1611, 1647, 1718, 1874, 1922, 2026, 2027, 2104, 2114, 2150, 2162, 2221, 2344, 2367, 2385, 2424 $\}$;

$\mathcal{K}_{46} = \{$ 34, 89, 152, 168, 210, 231, 287, 334, 357, 426, 462, 473, 493, 508, 550, 611, 618, 654, 734, 742, 772, 829, 860, 924, 965, 1227, 1270, 1327, 1386, 1504, 1581, 1814, 1863, 1930, 1985, 2055, 2106, 2145, 2160, 2191, 2315, 2402, 2409 $\}$;

$\mathcal{K}_{47} = \{$ 46, 130, 250, 375, 393, 397, 500, 546, 572, 593, 629, 647, 650, 718, 719, 952, 977, 994, 1005, 1007, 1063, 1381, 1396, 1465, 1540, 1566, 1574, 1609, 1670, 1727, 1792, 1810, 1815, 1870, 1899, 1905, 1926, 1939, 1941, 1959, 2227, 2272, 2291 $\}$;

$\mathcal{K}_{48} = \{$ 6, 70, 247, 289, 309, 363, 398, 418, 428, 626, 687, 744, 830, 959, 1040, 1061, 1094, 1137, 1163, 1218, 1228, 1238, 1298, 1331, 1341, 1375, 1419, 1457, 1492, 1561, 1697, 1747, 1756, 1776, 1833, 1850, 1883, 1892, 1915, 2174, 2211, 2268, 2363 $\}$;

$\mathcal{K}_{49} = \{$ 77, 87, 165, 189, 244, 322, 502, 559, 634, 690, 705, 716, 726, 738, 816, 851, 954, 974, 1012, 1034, 1071, 1145, 1249, 1306, 1472, 1513, 1538, 1570, 1722, 1816, 1847, 1971, 2126, 2139, 2181, 2193, 2216, 2230, 2235, 2346, 2347, 2391, 2434 $\}$;

$\mathcal{K}_{50} = \{$ 17, 33, 36, 49, 75, 121, 373, 422, 541, 552, 579, 605, 620, 697, 769, 881, 1030, 1042, 1143, 1195, 1237, 1312, 1441, 1475, 1476, 1519, 1657, 1706, 1708, 1748, 1805, 1819, 1843, 1866, 1887, 1913, 1929, 1970, 1997, 2035, 2081, 2100, 2411 $\}$;

$\mathcal{K}_{51} = \{$ 52, 119, 185, 249, 344, 379, 389, 413, 482, 592, 743, 753, 784, 891, 916, 958, 962, 1050, 1070, 1107, 1185, 1246, 1271, 1272, 1289, 1329, 1383, 1407, 1428, 1485, 1605, 1619, 1681, 1746, 2029, 2036, 2134, 2135, 2165, 2200, 2213, 2318, 2356 $\}$;

$\mathcal{K}_{52} = \{$ 35, 42, 65, 157, 191, 226, 288, 351, 354, 362, 385, 479, 487, 514, 538, 540, 551, 632, 637, 694, 774, 933, 1074, 1078, 1233, 1243, 1300, 1325, 1401, 1447, 1620, 1853, 1911, 1983, 2007, 2099, 2113, 2116, 2170, 2234, 2302, 2369, 2382 $\}$;

$\mathcal{K}_{53} = \{$ 53, 103, 203, 260, 272, 349, 506, 507, 638, 646, 660, 814, 845, 902, 920, 1067, 1080, 1155, 1174, 1207, 1226, 1273, 1285, 1297, 1385, 1454, 1480, 1495, 1577, 1587, 1662, 1684, 1728, 1775, 1789, 2001, 2130, 2172, 2275, 2278, 2304, 2308, 2447 $\}$;

$\mathcal{K}_{54}$ = { 116, 118, 131, 286, 323, 402, 512, 528, 532, 566, 668, 756, 777, 879, 1113, 1135, 1190, 1262, 1360, 1362, 1373, 1514, 1531, 1546, 1687, 1787, 1857, 1861, 1891, 1914, 1935, 1952, 2000, 2013, 2022, 2070, 2148, 2183, 2205, 2217, 2321, 2366, 2377 };

$\mathcal{K}_{55}$ = { 16, 44, 149, 217, 265, 267, 320, 403, 460, 466, 504, 526, 781, 824, 825, 840, 873, 1120, 1146, 1200, 1250, 1259, 1279, 1301, 1333, 1336, 1461, 1599, 1623, 1634, 1668, 1704, 1729, 1755, 1786, 1924, 1961, 2076, 2110, 2300, 2328, 2338, 2387 };

$\mathcal{K}_{56}$ = { 5, 62, 127, 207, 365, 505, 542, 604, 607, 655, 693, 696, 799, 806, 885, 936, 951, 1015, 1057, 1220, 1379, 1414, 1436, 1586, 1628, 1643, 1659, 1663, 1679, 1714, 1769, 1780, 1849, 1868, 1994, 2017, 2096, 2214, 2244, 2276, 2313, 2415, 2428 };

$\mathcal{K}_{57}$ = { 82, 147, 166, 169, 204, 225, 258, 282, 299, 374, 446, 486, 523, 557, 590, 707, 763, 797, 957, 960, 987, 1017, 1018, 1022, 1217, 1248, 1490, 1548, 1551, 1595, 1660, 1745, 1831, 1949, 1955, 1977, 2185, 2239, 2295, 2314, 2322, 2403, 2426 }.

The third decomposition is given by 57 sets $\mathcal{K}_i$ for $i = 1, \ldots, 57$ each set consists of 43 points of one of the 57 disjoint 43-arcs in $\mathrm{PG}(2, 7^2)$ as follows:

$\mathcal{K}_1$ = { 0, 57, 114, 171, 228, 285, 342, 399, 456, 513, 570, 627, 684, 741, 798, 855, 912, 969, 1026, 1083, 1140, 1197, 1254, 1311, 1368, 1425, 1482, 1539, 1596, 1653, 1710, 1767, 1824, 1881, 1938, 1995, 2052, 2109, 2166, 2223, 2280, 2337, 2394 };

$\mathcal{K}_2$ = { 1, 31, 207, 216, 312, 348, 393, 405, 447, 476, 477, 523, 528, 542, 566, 599, 690, 697, 732, 821, 827, 895, 928, 998, 1024, 1029, 1089, 1145, 1211, 1327, 1373, 1632, 1709, 1791, 1797, 2018, 2032, 2115, 2139, 2152, 2282, 2339, 2399 };

$\mathcal{K}_3$ = { 49, 129, 230, 300, 448, 547, 594, 629, 700, 750, 763, 770, 832, 844, 903, 929, 1042, 1267, 1378, 1470, 1474, 1527, 1620, 1684, 1695, 1747, 1793, 1865, 1971, 2004, 2015, 2037, 2100, 2110, 2132, 2158, 2188, 2215, 2286, 2295, 2328, 2344, 2352 };

$\mathcal{K}_4$ = { 2, 72, 131, 181, 223, 263, 284, 344, 369, 597, 734, 737, 772, 829, 864, 947, 1003, 1004, 1160, 1207, 1289, 1374, 1445, 1477, 1483, 1510, 1645, 1647, 1661, 1665, 1688, 1711, 1843, 1847, 1852, 1868, 1869, 2200, 2207, 2255, 2268, 2397, 2428 };

$\mathcal{K}_5 = \{$ 6, 119, 258, 337, 362, 434, 438, 519, 550, 579, 590, 591, 612, 648, 711, 733, 933, 1005, 1138, 1154, 1171, 1241, 1324, 1332, 1469, 1496, 1552, 1570, 1599, 1614, 1616, 1641, 1654, 1670, 1727, 1849, 1976, 1983, 1991, 2103, 2184, 2212, 2358 $\}$;

$\mathcal{K}_6 = \{$ 4, 7, 17, 52, 64, 78, 341, 349, 427, 569, 574, 650, 687, 744, 748, 943, 975, 1006, 1036, 1074, 1172, 1278, 1285, 1412, 1414, 1458, 1468, 1561, 1612, 1666, 1692, 1774, 1816, 1872, 1949, 1973, 1979, 2069, 2082, 2172, 2310, 2329, 2400 $\}$;

$\mathcal{K}_7 = \{$ 71, 93, 148, 164, 334, 374, 483, 531, 551, 669, 683, 731, 823, 848, 865, 897, 905, 914, 948, 985, 1034, 1086, 1106, 1118, 1174, 1187, 1190, 1213, 1330, 1349, 1485, 1531, 1589, 1617, 1757, 1759, 1833, 1841, 1945, 1972, 2183, 2253, 2429 $\}$;

$\mathcal{K}_8 = \{$ 50, 102, 134, 224, 287, 318, 320, 335, 435, 515, 548, 649, 651, 658, 761, 762, 858, 886, 901, 923, 1011, 1119, 1156, 1176, 1200, 1300, 1306, 1372, 1438, 1443, 1461, 1504, 1574, 1743, 1787, 1860, 1993, 2057, 2073, 2199, 2210, 2289, 2304 $\}$;

$\mathcal{K}_9 = \{$ 104, 120, 122, 178, 190, 192, 238, 260, 321, 383, 455, 458, 561, 600, 606, 673, 826, 977, 991, 1015, 1146, 1218, 1489, 1556, 1615, 1675, 1732, 1762, 1784, 1805, 1855, 1891, 1926, 1953, 2017, 2071, 2154, 2220, 2284, 2312, 2376, 2415, 2416 $\}$;

$\mathcal{K}_{10} = \{$ 302, 347, 464, 505, 564, 573, 765, 783, 801, 877, 917, 1016, 1019, 1066, 1084, 1107, 1116, 1126, 1148, 1162, 1205, 1228, 1250, 1338, 1419, 1456, 1490, 1507, 1671, 1800, 1831, 1889, 1930, 1954, 1963, 2048, 2059, 2070, 2084, 2097, 2261, 2342, 2426 $\}$;

$\mathcal{K}_{11} = \{$ 61, 322, 350, 352, 388, 422, 449, 460, 484, 546, 598, 616, 694, 719, 846, 906, 951, 988, 1060, 1079, 1256, 1280, 1290, 1302, 1399, 1563, 1607, 1672, 1745, 1749, 1770, 1785, 1876, 1918, 1932, 1964, 2028, 2063, 2126, 2195, 2293, 2331, 2354 $\}$;

$\mathcal{K}_{12} = \{$ 43, 53, 55, 82, 128, 151, 175, 233, 242, 363, 418, 444, 453, 720, 780, 835, 842, 880, 909, 946, 1143, 1150, 1152, 1234, 1237, 1244, 1253, 1310, 1371, 1402, 1497, 1643, 1673, 1715, 1799, 1836, 1846, 1900, 2027, 2162, 2260, 2297, 2375 $\}$;

$\mathcal{K}_{13} = \{$ 100, 176, 182, 204, 219, 221, 251, 509, 545, 625, 628, 666, 896, 935, 958, 980, 1032, 1112, 1158, 1186, 1188, 1276, 1309, 1314, 1319, 1358, 1386, 1559, 1581, 1638, 1677, 1683, 1750, 1863, 1890, 1969, 2020, 2205, 2206, 2225, 2230, 2256, 2362 $\}$;

$\mathcal{K}_{14}$ = { 11, 79, 146, 197, 210, 240, 249, 281, 319, 386, 391, 414, 416, 492, 586, 653, 749, 785, 918, 994, 1129, 1134, 1264, 1288, 1348, 1363, 1396, 1413, 1434, 1453, 1479, 1761, 1851, 2046, 2118, 2120, 2128, 2150, 2227, 2248, 2355, 2369, 2370 };

$\mathcal{K}_{15}$ = { 37, 69, 89, 237, 277, 325, 355, 404, 525, 623, 654, 664, 715, 907, 932, 949, 959, 966, 1040, 1048, 1094, 1108, 1191, 1227, 1308, 1448, 1591, 1619, 1626, 1631, 1728, 1792, 1862, 1884, 1901, 1956, 1996, 2003, 2116, 2164, 2242, 2348, 2433 };

$\mathcal{K}_{16}$ = { 41, 94, 118, 130, 137, 180, 254, 291, 324, 346, 351, 506, 518, 521, 529, 554, 891, 1012, 1033, 1049, 1164, 1194, 1208, 1223, 1260, 1275, 1359, 1406, 1517, 1564, 1609, 1613, 1699, 1754, 1823, 1892, 1899, 2019, 2090, 2161, 2263, 2391, 2445 };

$\mathcal{K}_{17}$ = { 40, 54, 126, 144, 194, 209, 289, 619, 626, 630, 659, 691, 708, 710, 738, 751, 777, 963, 996, 1021, 1095, 1125, 1131, 1161, 1301, 1320, 1392, 1447, 1459, 1473, 1509, 1529, 1606, 1664, 1874, 1883, 1906, 2081, 2101, 2104, 2151, 2232, 2275 };

$\mathcal{K}_{18}$ = { 13, 18, 90, 110, 152, 295, 323, 445, 510, 568, 577, 582, 793, 890, 920, 938, 940, 954, 965, 979, 1071, 1231, 1333, 1351, 1377, 1383, 1472, 1498, 1520, 1532, 1573, 1705, 1859, 1951, 1962, 1966, 2006, 2035, 2050, 2072, 2145, 2313, 2317 };

$\mathcal{K}_{19}$ = { 33, 250, 338, 373, 440, 482, 514, 534, 584, 769, 800, 806, 810, 822, 850, 852, 859, 968, 1292, 1362, 1409, 1418, 1431, 1481, 1503, 1549, 1605, 1610, 1803, 1810, 1844, 1856, 1905, 1933, 2076, 2105, 2185, 2186, 2204, 2249, 2262, 2413, 2417 };

$\mathcal{K}_{20}$ = { 39, 86, 215, 265, 272, 283, 354, 395, 409, 413, 428, 461, 556, 689, 790, 919, 1078, 1149, 1203, 1215, 1248, 1271, 1296, 1393, 1395, 1492, 1506, 1545, 1582, 1663, 1689, 1806, 1809, 2042, 2121, 2149, 2174, 2356, 2359, 2364, 2401, 2432, 2439 };

$\mathcal{K}_{21}$ = { 96, 165, 200, 288, 365, 385, 562, 596, 603, 622, 632, 722, 791, 851, 868, 904, 937, 1056, 1093, 1111, 1113, 1124, 1279, 1416, 1493, 1577, 1594, 1621, 1624, 1625, 1644, 1734, 1741, 1760, 1832, 1839, 2030, 2056, 2181, 2219, 2245, 2278, 2350 };

$\mathcal{K}_{22}$ = { 16, 24, 103, 188, 339, 452, 468, 475, 552, 668, 755, 788, 879, 927, 970, 1037, 1062, 1268, 1312, 1321, 1367, 1379, 1454, 1462, 1516, 1519, 1548, 1587, 1674, 1678, 1718, 1721, 1744, 1802, 1920, 1944, 2031, 2079, 2144, 2274, 2336, 2410, 2435 };

$\mathcal{K}_{23}$ = { 58, 136, 153, 213, 225, 466, 502, 567, 583, 680, 728, 756, 939, 1051, 1090, 1097, 1177, 1202, 1240, 1303, 1446, 1514, 1562, 1618, 1623, 1636, 1656, 1687, 1712, 1769, 1778, 1788, 2062, 2129, 2134, 2160, 2165, 2319, 2330, 2333, 2334, 2357, 2392 };

$\mathcal{K}_{24}$ = { 28, 42, 83, 92, 127, 150, 294, 381, 624, 782, 862, 873, 900, 957, 1063, 1069, 1091, 1151, 1180, 1204, 1421, 1450, 1471, 1488, 1522, 1569, 1579, 1716, 1735, 1864, 1894, 1913, 1931, 2034, 2051, 2060, 2157, 2234, 2267, 2276, 2300, 2309, 2338 };

$\mathcal{K}_{25}$ = { 26, 51, 202, 311, 431, 472, 657, 706, 836, 847, 911, 1075, 1080, 1122, 1163, 1165, 1272, 1435, 1533, 1550, 1595, 1611, 1659, 1703, 1714, 1737, 1835, 1921, 1977, 2045, 2078, 2096, 2123, 2138, 2187, 2202, 2218, 2257, 2308, 2379, 2406, 2420, 2448 };

$\mathcal{K}_{26}$ = { 138, 179, 183, 253, 266, 307, 368, 396, 419, 436, 446, 486, 559, 681, 742, 794, 1018, 1020, 1100, 1239, 1265, 1294, 1405, 1440, 1603, 1879, 1923, 1950, 1952, 1974, 2011, 2087, 2095, 2113, 2135, 2194, 2241, 2292, 2315, 2316, 2346, 2349, 2377 };

$\mathcal{K}_{27}$ = { 111, 269, 353, 421, 495, 500, 536, 571, 608, 621, 631, 661, 713, 753, 764, 824, 845, 1054, 1169, 1313, 1423, 1505, 1693, 1731, 1738, 1765, 1819, 1822, 1838, 1840, 1924, 1959, 1970, 1978, 2007, 2107, 2176, 2192, 2229, 2231, 2277, 2374, 2444 };

$\mathcal{K}_{28}$ = { 15, 23, 140, 155, 212, 222, 226, 264, 329, 425, 471, 497, 501, 696, 698, 745, 922, 972, 1123, 1142, 1147, 1224, 1274, 1329, 1341, 1375, 1382, 1408, 1480, 1588, 1650, 1719, 1814, 1857, 1896, 1947, 1975, 2064, 2077, 2264, 2318, 2345, 2361 };

$\mathcal{K}_{29}$ = { 12, 169, 244, 267, 297, 330, 336, 375, 377, 496, 633, 670, 768, 787, 803, 820, 1027, 1067, 1110, 1167, 1170, 1192, 1206, 1336, 1354, 1420, 1537, 1576, 1597, 1729, 1928, 1948, 1988, 2021, 2053, 2075, 2170, 2177, 2179, 2216, 2254, 2306, 2321 };

$\mathcal{K}_{30}$ = { 5, 21, 36, 73, 247, 293, 489, 620, 679, 701, 807, 825, 902, 1023, 1196, 1307, 1322, 1355, 1394, 1436, 1676, 1690, 1691, 1723, 1739, 1758, 1811, 1893, 1919, 1936, 1986, 1997, 2022, 2065, 2083, 2122, 2178, 2198, 2221, 2259, 2378, 2423, 2443 };

$\mathcal{K}_{31}$ = { 168, 201, 280, 282, 326, 361, 364, 408, 526, 530, 592, 611, 735, 740, 882, 960, 974, 993, 1008, 1041, 1082, 1139, 1178, 1185, 1221, 1232, 1266, 1366, 1524, 1557, 1651, 1725, 1780, 1790, 1826, 1858, 1888, 1914, 2086, 2169, 2196, 2320, 2449 };

$\mathcal{K}_{32} = \{$ 20, 65, 139, 172, 248, 480, 487, 522, 580, 723, 727, 739, 767, 779, 797, 808, 839, 856, 876, 892, 1014, 1068, 1088, 1099, 1277, 1281, 1339, 1410, 1465, 1467, 1555, 1566, 1786, 1837, 1882, 1929, 1984, 1994, 1998, 2089, 2288, 2332, 2398 $\}$;

$\mathcal{K}_{33} = \{$ 9, 62, 67, 70, 81, 167, 252, 279, 292, 316, 380, 401, 507, 614, 634, 854, 857, 883, 944, 983, 992, 1047, 1130, 1225, 1282, 1385, 1580, 1640, 1794, 1796, 1813, 1830, 2029, 2043, 2058, 2189, 2236, 2305, 2322, 2363, 2381, 2390, 2446 $\}$;

$\mathcal{K}_{34} = \{$ 30, 45, 84, 191, 298, 303, 402, 463, 494, 527, 712, 771, 773, 869, 921, 926, 955, 1017, 1070, 1098, 1133, 1182, 1189, 1214, 1389, 1417, 1455, 1475, 1538, 1633, 1861, 1877, 2040, 2091, 2119, 2133, 2142, 2143, 2175, 2235, 2290, 2360, 2405 $\}$;

$\mathcal{K}_{35} = \{$ 132, 206, 217, 229, 231, 305, 356, 358, 394, 424, 499, 805, 841, 898, 997, 1039, 1092, 1144, 1183, 1222, 1251, 1304, 1342, 1384, 1404, 1478, 1578, 1648, 1649, 1708, 1720, 1733, 1746, 1766, 1798, 1866, 1911, 1961, 2001, 2014, 2024, 2125, 2247 $\}$;

$\mathcal{K}_{36} = \{$ 46, 56, 97, 245, 290, 328, 465, 524, 565, 635, 641, 686, 693, 709, 784, 795, 814, 893, 973, 1179, 1326, 1360, 1427, 1568, 1575, 1628, 1646, 1681, 1834, 1910, 1960, 1967, 2094, 2111, 2137, 2147, 2240, 2271, 2301, 2302, 2327, 2396, 2438 $\}$;

$\mathcal{K}_{37} = \{$ 116, 149, 166, 195, 384, 457, 613, 615, 809, 828, 870, 888, 987, 995, 1064, 1077, 1193, 1212, 1245, 1257, 1269, 1299, 1325, 1397, 1433, 1449, 1457, 1584, 1713, 1717, 1775, 1781, 1821, 1853, 1909, 1917, 2092, 2131, 2148, 2156, 2325, 2371, 2389 $\}$;

$\mathcal{K}_{38} = \{$ 22, 107, 199, 343, 366, 400, 437, 470, 533, 539, 601, 667, 699, 717, 760, 804, 816, 867, 894, 908, 1022, 1065, 1072, 1155, 1297, 1345, 1476, 1528, 1553, 1604, 1627, 1682, 1753, 1755, 1828, 1875, 2106, 2182, 2217, 2233, 2244, 2368, 2431 $\}$;

$\mathcal{K}_{39} = \{$ 25, 32, 74, 80, 101, 158, 163, 196, 278, 286, 340, 371, 378, 411, 442, 488, 560, 589, 609, 778, 815, 952, 984, 1057, 1120, 1466, 1484, 1487, 1499, 1525, 1565, 1592, 1657, 1660, 1946, 1958, 2190, 2208, 2281, 2294, 2343, 2366, 2404 $\}$;

$\mathcal{K}_{40} = \{$ 44, 115, 125, 143, 296, 389, 516, 549, 672, 792, 878, 934, 942, 999, 1181, 1220, 1263, 1293, 1316, 1317, 1403, 1437, 1511, 1542, 1572, 1668, 1702, 1763, 1777, 1842, 1850, 1955, 2023, 2039, 2124, 2191, 2266, 2283, 2298, 2335, 2388, 2421, 2442 $\}$;

$\mathcal{K}_{41}$ = { 91, 214, 232, 239, 331, 407, 429, 491, 676, 688, 725, 736, 799, 813, 884, 885, 910, 925, 1103, 1105, 1136, 1255, 1291, 1357, 1401, 1422, 1444, 1464, 1535, 1593, 1667, 1722, 2061, 2098, 2136, 2146, 2252, 2291, 2296, 2324, 2353, 2403, 2425 };

$\mathcal{K}_{42}$ = { 135, 186, 268, 317, 439, 462, 557, 578, 595, 640, 704, 714, 837, 849, 872, 1013, 1058, 1137, 1195, 1235, 1236, 1273, 1283, 1323, 1391, 1415, 1534, 1536, 1554, 1598, 1639, 1655, 1680, 1742, 1751, 1783, 1871, 1916, 2171, 2239, 2246, 2393, 2395 };

$\mathcal{K}_{43}$ = { 29, 63, 123, 184, 241, 255, 257, 270, 367, 382, 417, 432, 537, 585, 675, 685, 695, 786, 796, 843, 978, 1085, 1101, 1128, 1166, 1337, 1560, 1768, 1854, 1885, 2000, 2033, 2036, 2068, 2108, 2127, 2153, 2203, 2272, 2340, 2419, 2427, 2434 };

$\mathcal{K}_{44}$ = { 68, 85, 88, 124, 145, 273, 304, 310, 430, 443, 451, 467, 504, 517, 540, 543, 617, 757, 775, 881, 976, 1038, 1061, 1210, 1258, 1335, 1370, 1526, 1558, 1637, 1652, 1782, 1873, 1942, 1992, 2010, 2047, 2155, 2167, 2197, 2299, 2303, 2436 };

$\mathcal{K}_{45}$ = { 19, 59, 95, 117, 141, 227, 410, 511, 558, 575, 604, 644, 662, 674, 705, 838, 913, 950, 986, 1031, 1035, 1043, 1115, 1141, 1284, 1328, 1365, 1388, 1495, 1629, 1700, 1764, 1817, 1897, 1925, 1934, 1980, 2026, 2180, 2251, 2285, 2367, 2441 };

$\mathcal{K}_{46}$ = { 3, 60, 121, 187, 211, 261, 306, 315, 345, 473, 474, 508, 512, 555, 610, 639, 671, 746, 747, 789, 899, 1002, 1010, 1096, 1199, 1217, 1270, 1352, 1501, 1530, 1635, 1686, 1698, 1707, 1736, 1795, 1848, 1895, 2088, 2112, 2258, 2311, 2383 };

$\mathcal{K}_{47}$ = { 106, 112, 113, 133, 160, 193, 370, 387, 553, 572, 576, 758, 766, 840, 874, 889, 962, 1000, 1007, 1109, 1209, 1216, 1315, 1343, 1439, 1508, 1512, 1608, 1630, 1669, 1701, 1740, 1867, 1902, 1927, 1940, 1941, 1965, 2173, 2228, 2269, 2307, 2326 };

$\mathcal{K}_{48}$ = { 14, 47, 309, 415, 441, 469, 485, 563, 643, 655, 682, 702, 811, 860, 866, 887, 953, 971, 981, 1081, 1229, 1233, 1286, 1318, 1346, 1380, 1398, 1411, 1451, 1502, 1571, 1601, 1756, 1801, 1898, 1943, 2002, 2117, 2140, 2211, 2226, 2243, 2386 };

$\mathcal{K}_{49}$ = { 105, 108, 159, 246, 256, 259, 299, 301, 392, 454, 587, 588, 645, 716, 729, 834, 853, 861, 915, 930, 990, 1059, 1175, 1350, 1513, 1523, 1543, 1590, 1602, 1789, 1815, 1908, 1922, 1937, 2041, 2159, 2163, 2168, 2238, 2279, 2347, 2414, 2437 };

$\mathcal{K}_{50} = \{ 27, 75, 98, 156, 205, 271, 275, 327, 406, 412, 426, 481, 493, 663, 730, 819, 931, 961, 1001, 1009, 1055, 1087, 1121, 1157, 1262, 1295, 1356, 1369, 1390, 1429, 1486, 1694, 1706, 1771, 1912, 1915, 2054, 2055, 2224, 2265, 2351, 2411, 2430 \}$;

$\mathcal{K}_{51} = \{ 77, 142, 161, 174, 198, 218, 235, 357, 359, 360, 423, 503, 535, 581, 593, 818, 863, 916, 982, 1053, 1127, 1132, 1219, 1242, 1247, 1287, 1340, 1381, 1407, 1428, 1518, 1658, 1704, 1807, 1870, 1880, 1907, 2080, 2141, 2201, 2250, 2270, 2422 \}$;

$\mathcal{K}_{52} = \{ 10, 154, 162, 177, 208, 379, 398, 538, 544, 607, 642, 692, 754, 830, 831, 1076, 1153, 1159, 1173, 1201, 1230, 1243, 1305, 1500, 1515, 1697, 1724, 1776, 1779, 1827, 1845, 1878, 1887, 1939, 1968, 2008, 2066, 2099, 2237, 2323, 2372, 2440, 2450 \}$;

$\mathcal{K}_{53} = \{ 8, 76, 170, 189, 203, 220, 262, 450, 479, 638, 660, 703, 707, 724, 726, 871, 941, 1025, 1045, 1052, 1102, 1246, 1249, 1252, 1261, 1347, 1387, 1400, 1442, 1452, 1642, 1696, 1773, 1820, 1886, 1987, 2016, 2038, 2074, 2222, 2341, 2384, 2418 \}$;

$\mathcal{K}_{54} = \{ 35, 87, 99, 173, 243, 308, 314, 390, 459, 498, 532, 541, 602, 656, 759, 776, 875, 924, 945, 1030, 1073, 1334, 1353, 1430, 1432, 1460, 1521, 1547, 1622, 1662, 1748, 1772, 1804, 1829, 1903, 1935, 1982, 2005, 2013, 2093, 2102, 2114, 2407 \}$;

$\mathcal{K}_{55} = \{ 66, 185, 234, 274, 276, 313, 333, 372, 397, 403, 420, 490, 618, 652, 781, 817, 1050, 1104, 1135, 1168, 1259, 1364, 1376, 1424, 1426, 1441, 1491, 1494, 1600, 1812, 1825, 1904, 1989, 2067, 2085, 2130, 2213, 2273, 2365, 2373, 2385, 2409, 2412 \}$;

$\mathcal{K}_{56} = \{ 34, 157, 236, 332, 376, 433, 478, 520, 605, 636, 637, 665, 718, 743, 752, 774, 802, 812, 964, 967, 1114, 1117, 1184, 1198, 1238, 1540, 1544, 1546, 1583, 1586, 1634, 1679, 1685, 1726, 1752, 1818, 1957, 1990, 2009, 2044, 2214, 2287, 2408 \}$;

$\mathcal{K}_{57} = \{ 38, 48, 109, 147, 646, 647, 677, 678, 721, 833, 936, 956, 989, 1028, 1044, 1046, 1226, 1298, 1331, 1344, 1361, 1463, 1541, 1551, 1567, 1585, 1730, 1808, 1981, 1985, 1999, 2012, 2025, 2049, 2193, 2209, 2314, 2380, 2382, 2387, 2402, 2424, 2447 \}$.

In summary, four non-isomorphic decompositions of $\mathrm{PG}(2, 7^2)$ into disjoint subplanes of order seven are found in Section 6.1 where the first partition is the classical one, and the other three decompositions are new. Section 6.2 shows that, in $\mathrm{PG}(2, 7^2)$, there are, up to isomorphism, three decompositions into disjoint 43-arcs where the first partition is the classical one, and the other two decompositions are new.

Chapter 7

Applications

7.1 Coding theory

In this chapter, we explain the relationship between coding theory and projective spaces and describe the links with Hermitian unital. Also, by taking the disjoint union of such subgeometries, a projective code, which is a linear code, is obtained. The content of this chapter is based on the following standard references: [4], [7], [10], [11] and [18].

The geometrical objects considered in this thesis can be seen as linear codes defined over a finite field. Let $\mathbb{F}_q$ be the finite field of order q and let $\mathrm{V} := \mathbb{F}_q^n$ be the n-dimensional vector space of the n-tuples over $\mathbb{F}_q$.

Definition 7.1 (Linear code [11, Section 1.2]). A linear code C is a k-dimensional subspace of V and it is called an $[n, k]$ code over $\mathbb{F}_q$ or an $[n, k]_q$ code. From this we see that $|C| = q^k$.

Definition 7.2 (Hamming distance [11, Section 1.4]). The Hamming distance between two codewords $c, c_0 \in \mathbb{F}_q^n$, denoted by $d(c, c_0)$, is the number of positions in which $c_i \neq c_{0i}$, for $c = (c_1, \ldots, c_n)$ and $c_0 = (c_{01}, \ldots, c_{0n})$.

Definition 7.3 (Minimum distance [11, Section 1.4]). The minimum distance d of a linear code C is the smallest number of positions in which two different elements of C differ,

$$d = \min\{d(c, c_0) \mid c, c_0 \in C, c \neq c_0\}.$$

A linear $[n, k]$ code or $[n, k]_q$ code having minimum distance d is denoted by $[n, k, d]$ code or $[n, k, d]_q$ code.

Definition 7.4 (Weight of codeword [11, Section 1.4]). The weight $w(c)$ of a codeword c is the number of the non–zero components of c; that is, $w(c) = d(c, 0)$.

For a linear code C, the minimum distance and minimum weight are the same for linear codes [11, Section 1.4].

$$\min\{w(c) \mid c \in C\backslash\{0\}\} = \min\{d(c, c_0) \mid c, c_0 \in C, c \neq c_0\}.$$

One of the main reasons why the weight of the codewords of a linear code is measured is that a code C with minimum distance d can correct up to t errors, where $2t + 1 \leq d$.

Definition 7.5 (Generator matrix [11, Section 1.2]). A generator matrix G of an $[n, k]$ code C is a $k \times n$ matrix whose rows form a basis for C.

Remark 7.1 ([11, Section 1.2]). We can describe each k-dimensional subspace of $\mathrm{V} := \mathbb{F}_q^n$ using a generator matrix.

Definition 7.6 (Dual code [11, Section 1.3]). The dual code $C^\perp$ of a q-ary linear code C of length m is the set of all vectors orthogonal to all the codewords of C, hence

$$C^\perp = \{v \in \mathrm{V}(m, q) \mid (v, c) = 0, \forall c \in C\}.$$

Definition 7.7 (Parity-check matrix [11, Sections 1.2 and 1.3]). A parity-check matrix H for an $[n, k]$ code C is an $(n - k) \times n$ matrix which is a generator matrix for $C^\perp$.

Definition 7.8 (Projective code [10, Section 2.14.2]). A linear $[n, k, d]$ code C over $\mathbb{F}_q$ with $k \geq 2$ and generator matrix G is called projective, if the n columns of G, construed as homogenous coordinates, form distinct points in the projective space $\mathrm{PG}(k - 1, q)$.

Definition 7.9 (MDS code [11, Section 2.4]). If C is an $[n, k, n - k + 1]$ linear code; that is, $d = n - k + 1$, then C is called a maximum distance separable code or an MDS code, for short.

Over finite fields, projective geometries and linear codes connect to each other by their underlying vector spaces; that is, in Galois geometry to investigating the column space and in coding theory to analysing the row space of a generator matrix of a code. In particular, linear $[n, k, n - k + 1]$ MDS codes, with $k \geq 3$, and n-arcs in $\mathrm{PG}(k - 1, q)$ are equivalent objects, see [10, Section 2.14.2]. For example, a complete 7-arc in the projective plane $\mathrm{PG}(2, 3^2)$ is a set of 7 points, no three of which are collinear. Also, a $[7, 3, 5]_9$ MDS linear code is equivalent to a 7-arc in $\mathrm{PG}(2, 3^2)$.

Definition 7.10 (Grassmann graph [18, Section 3.2]). The Grassmann graph $\mathrm{Gr}_q(n, k)$ is defined as follows:

(i) vertices are the k-dimensional subspaces of the vector space $\mathbb{F}_q^n$;

(ii) two vertices are adjacent if the associated subspaces intersect in a $(k-1)$-dimensional subspace.

Being pairwise disjoint means that the components are far away on the Grassmannian. This can be made accurate utilising the distance function on the graph $\mathcal{G}$. For two subspaces $x, y \in \mathcal{G}$, let $d(x, y)$ be the length of the shortest path from x to y. This forms a metric on the graph $\mathcal{G}$. Two k-subspaces $x, y \in \mathrm{Gr}_q(n, k)$ are disjoint if and only if $d(x, y) = k$. Additionally, if two subspaces intersect only in the zero vector, then the corresponding subspaces of projective space do not intersect, see [18, Chapter 3].

From Corollary 2.7, we know that the projective $\mathrm{PG}(2, q^2)$ can be decomposed into pairwise disjoint subplanes with cardinality $q^2 + q + 1$. Therefore, by taking the union of i such subplanes, their points form the generator matrix of a projective $[i(q^2 + q + 1), 3, i(q^2 + q) - q]$ code over $\mathbb{F}_{q^2}$ for all $i = 2, \ldots, q^2 - q + 1$, see [4, Section 4.2.8]. Similarly, from Corollary 2.8, we know that the projective plane $\mathrm{PG}(2, q^4)$ can be decomposed into pairwise disjoint subplanes with cardinality $q^2 + q + 1$. Consequently, by taking the union of i such subplanes, their points form the generator matrix of a projective $[i(q^2 + q + 1), 3, i(q^2 + q) - q]$ code over $\mathbb{F}_{q^4}$ for all $i = 2, \ldots, (q^8 + q^4 + 1)/(q^2 + q + 1)$. Furthermore, from Corollary 2.9, we know that the projective plane $\mathrm{PG}(2, q^2)$ can be decomposed into pairwise disjoint $(q^2 - q + 1)$-arcs. Hence, by taking the union of i such subsets, their points form the generator matrix of a projective $[i(q^2 - q + 1), 3, i(q^2 - q)]$ code over $\mathbb{F}_{q^2}$ for all $i = 2, \ldots, q^2 + q + 1$. In particular, when $i = (q + 1)$, points form the generator matrix of a projective $[q^3 + 1, 3, q^3 - q]$ code over $\mathbb{F}_{q^2}$.

In the projective plane $\mathrm{PG}(2, 3^2)$, there exist 7 disjoint subplanes of order three. By taking the union of i such subplanes, a projective $[i(3^2 + 3 + 1), 3, i(3^2 + 3) - 3]_9$ code is obtained for all $i = 2, \ldots, 7$. $[n, k, d]$ codes for using union of i disjoint subplanes in $\mathrm{PG}(2, 3^2)$ are given in the following table.

i	$[n, k, d]_q$
2	$[26, 3, 21]_9$
3	$[39, 3, 33]_9$
4	$[52, 3, 45]_9$
5	$[65, 3, 57]_9$
6	$[78, 3, 69]_9$
7	$[91, 3, 81]_9$

In the projective plane $PG(2, 4^2)$, there exist 39 disjoint subplanes of order two. By taking the union of i disjoint subplanes, a projective $[i(2^2+2+1), 3, i(2^2+2)-2]_{16}$ code is obtained for all $i = 2, \ldots, 39$. $[n, k, d]$ codes for using union of i disjoint subplanes in $PG(2, 4^2)$ are given in the following table.

i	$[n,k,d]_q$	i	$[n,k,d]_q$
2	$[14,3,10]_{16}$	21	$[147,3,124]_{16}$
3	$[21,3,16]_{16}$	22	$[154,3,130]_{16}$
4	$[28,3,22]_{16}$	23	$[161,3,136]_{16}$
5	$[35,3,28]_{16}$	24	$[168,3,142]_{16}$
6	$[42,3,34]_{16}$	25	$[175,3,148]_{16}$
7	$[49,3,40]_{16}$	26	$[182,3,154]_{16}$
8	$[56,3,46]_{16}$	27	$[189,3,160]_{16}$
9	$[63,3,52]_{16}$	28	$[196,3,166]_{16}$
10	$[70,3,58]_{16}$	29	$[203,3,172]_{16}$
11	$[77,3,64]_{16}$	30	$[210,3,178]_{16}$
12	$[84,3,70]_{16}$	31	$[217,3,184]_{16}$
13	$[91,3,76]_{16}$	32	$[224,3,190]_{16}$
14	$[98,3,82]_{16}$	33	$[231,3,196]_{16}$
15	$[105,3,88]_{16}$	34	$[238,3,202]_{16}$
16	$[112,3,94]_{16}$	35	$[245,3,208]_{16}$
17	$[119,3,100]_{16}$	36	$[252,3,214]_{16}$
18	$[126,3,106]_{16}$	37	$[259,3,220]_{16}$
19	$[133,3,112]_{16}$	38	$[266,3,226]_{16}$
20	$[140,3,118]_{16}$	39	$[273,3,232]_{16}$

In the projective plane $PG(2, 4^2)$, there exist 13 disjoint subplanes of order four. By taking the union of i such subplanes, a projective $[i(4^2+4+1), 3, i(4^2+4)-4]_{16}$ code is obtained for all $i = 2, \ldots, 13$. $[n, k, d]$ codes for using union of i disjoint subplanes in $PG(2, 4^2)$ are given in the following table.

i	$[n,k,d]_q$	i	$[n,k,d]_q$
2	$[42,3,36]_{16}$	8	$[168,3,156]_{16}$
3	$[63,3,56]_{16}$	9	$[189,3,176]_{16}$
4	$[84,3,76]_{16}$	10	$[210,3,196]_{16}$
5	$[105,3,96]_{16}$	11	$[231,3,216]_{16}$
6	$[126,3,116]_{16}$	12	$[252,3,236]_{16}$
7	$[147,3,136]_{16}$	13	$[273,3,256]_{16}$

In the projective plane $PG(2,5^2)$, there exist 21 disjoint subplanes of order five. By taking the union of i such subplanes, a projective $[i(5^2+5+1),3,i(5^2+5)-5]_{25}$ code is obtained for all $i=2,\dots,21$. $[n,k,d]$ codes for using union of i disjoint subplanes in $PG(2,5^2)$ are given in the following table.

i	$[n,k,d]_q$	i	$[n,k,d]_q$
2	$[62,3,55]_{25}$	12	$[372,3,355]_{25}$
3	$[93,3,85]_{25}$	13	$[403,3,385]_{25}$
4	$[124,3,115]_{25}$	14	$[434,3,415]_{25}$
5	$[155,3,145]_{25}$	15	$[465,3,445]_{25}$
6	$[186,3,175]_{25}$	16	$[496,3,475]_{25}$
7	$[217,3,205]_{25}$	17	$[527,3,505]_{25}$
8	$[248,3,235]_{25}$	18	$[558,3,535]_{25}$
9	$[279,3,265]_{25}$	19	$[589,3,565]_{25}$
10	$[310,3,295]_{25}$	20	$[620,3,595]_{25}$
11	$[341,3,325]_{25}$	21	$[651,3,625]_{25}$

In the projective plane $PG(2,7^2)$, there exist 43 disjoint subplanes of order seven. By taking the union of i such subplanes, a projective $[i(7^2+7+1),3,i(7^2+7)-7]_{49}$ code is obtained for all $i=2,\dots,43$. $[n,k,d]$ codes for using union of i disjoint subplanes in $PG(2,7^2)$ are given in the following table.

i	$[n,k,d]_q$	i	$[n,k,d]_q$
2	$[114,3,105]_{49}$	23	$[1311,3,1281]_{49}$
3	$[171,3,161]_{49}$	24	$[1368,3,1337]_{49}$
4	$[228,3,217]_{49}$	25	$[1425,3,1393]_{49}$
5	$[285,3,273]_{49}$	26	$[1482,3,1449]_{49}$
6	$[342,3,329]_{49}$	27	$[1539,3,1505]_{49}$
7	$[399,3,385]_{49}$	28	$[1596,3,1561]_{49}$
8	$[456,3,441]_{49}$	29	$[1653,3,1617]_{49}$
9	$[513,3,497]_{49}$	30	$[1710,3,1673]_{49}$
10	$[570,3,553]_{49}$	31	$[1767,3,1729]_{49}$
11	$[627,3,609]_{49}$	32	$[1824,3,1785]_{49}$
12	$[684,3,665]_{49}$	33	$[1881,3,1841]_{49}$
13	$[741,3,721]_{49}$	34	$[1938,3,1897]_{49}$
14	$[798,3,777]_{49}$	35	$[1995,3,1953]_{49}$
15	$[855,3,833]_{49}$	36	$[2052,3,2009]_{49}$
16	$[912,3,889]_{49}$	37	$[2109,3,2065]_{49}$
17	$[969,3,495]_{49}$	38	$[2166,3,2121]_{49}$
18	$[1026,3,1001]_{49}$	39	$[2223,3,2177]_{49}$
19	$[1083,3,1057]_{49}$	40	$[2280,3,2233]_{49}$
20	$[1140,3,1113]_{49}$	41	$[2337,3,2289]_{49}$
21	$[1197,3,1169]_{49}$	42	$[2394,3,2345]_{49}$
22	$[1254,3,1225]_{49}$	43	$[2451,3,2401]_{49}$

In the projective plane PG$(2, 3^2)$, there exist 13 disjoint 7-arcs. By taking the union of i such arcs, a projective $[i(3^2-3+1), 3, i(3^2-3)]_9$ code is obtained for all $i = 2, \ldots, 13$. $[n, k, d]$ codes for using union of i disjoint 7-arcs in PG$(2, 3^2)$ are given in the following table.

i	$[n, k, d]_q$	i	$[n, k, d]_q$
2	$[14, 3, 12]_9$	8	$[56, 3, 48]_9$
3	$[21, 3, 18]_9$	9	$[63, 3, 54]_9$
4	$[28, 3, 24]_9$	10	$[70, 3, 60]_9$
5	$[35, 3, 30]_9$	11	$[77, 3, 66]_9$
6	$[42, 3, 36]_9$	12	$[84, 3, 72]_9$
7	$[49, 3, 42]_9$	13	$[91, 3, 78]_9$

In the projective plane PG$(2, 4^2)$, there exist 21 disjoint 13-arcs. By taking the union of i such arcs, a projective $[i(4^2-4+1), 3, i(4^2-4)]_{16}$ code is obtained for all $i = 2, \ldots, 21$. $[n, k, d]$ codes for using union of i disjoint 13-arcs in PG$(2, 4^2)$ are given in the following table.

i	$[n, k, d]_q$	i	$[n, k, d]_q$
2	$[26, 3, 24]_{16}$	12	$[156, 3, 144]_{16}$
3	$[39, 3, 36]_{16}$	13	$[169, 3, 156]_{16}$
4	$[52, 3, 48]_{16}$	14	$[182, 3, 168]_{16}$
5	$[65, 3, 60]_{16}$	15	$[195, 3, 180]_{16}$
6	$[78, 3, 72]_{16}$	16	$[208, 3, 192]_{16}$
7	$[91, 3, 84]_{16}$	17	$[221, 3, 204]_{16}$
8	$[104, 3, 96]_{16}$	18	$[234, 3, 216]_{16}$
9	$[117, 3, 108]_{16}$	19	$[247, 3, 228]_{16}$
10	$130, 3, 120]_{16}$	20	$[260, 3, 240]_{16}$
11	$[143, 3, 132]_{16}$	21	$[273, 3, 252]_{16}$

In the projective plane $PG(2,5^2)$, there exist 31 disjoint 21-arcs. By taking the union of i such arcs, a projective $[i(5^2-5+1),3,i(5^2-5)]_{25}$ code is obtained for all $i=2,\ldots,31$. $[n,k,d]$ codes for using union of i disjoint 21-arcs in $PG(2,5^2)$ are given in the following table.

i	$[n,k,d]_q$	i	$[n,k,d]_q$
2	$[42,3,40]_{25}$	17	$[357,3,340]_{25}$
3	$[63,3,60]_{25}$	18	$[378,3,360]_{25}$
4	$[84,3,80]_{25}$	19	$[399,3,380]_{25}$
5	$[105,3,100]_{25}$	20	$[420,3,400]_{25}$
6	$[126,3,120]_{25}$	21	$[441,3,420]_{25}$
7	$[147,3,140]_{25}$	22	$[462,3,440]_{25}$
8	$[168,3,160]_{25}$	23	$[483,3,460]_{25}$
9	$[189,3,180]_{25}$	24	$[504,3,480]_{25}$
10	$[210,3,200]_{25}$	25	$[525,3,500]_{25}$
11	$[231,3,220]_{25}$	26	$[546,3,520]_{25}$
12	$[252,3,240]_{25}$	27	$[567,3,540]_{25}$
13	$[273,3,260]_{25}$	28	$[588,3,560]_{25}$
14	$[294,3,280]_{25}$	29	$[609,3,580]_{25}$
15	$[315,3,300]_{25}$	30	$[630,3,600]_{25}$
16	$[336,3,320]_{25}$	31	$[651,3,620]_{25}$

In the projective plane PG$(2, 7^2)$, there exist 57 disjoint 43-arcs. By taking the union of i such arcs, a projective $[i(7^2-7+1), 3, i(7^2-7)]_{49}$ code is obtained for all $i = 2, \ldots, 57$. $[n, k, d]$ codes for using union of i disjoint 43-arcs in PG$(2, 7^2)$ are given in the following table.

i	$[n,k,d]_q$	i	$[n,k,d]_q$
2	$[86,3,84]_{49}$	30	$[1290,3,1260]_{49}$
3	$[129,3,126]_{49}$	31	$[1333,3,1302]_{49}$
4	$[172,3,168]_{49}$	32	$[1376,3,1344]_{49}$
5	$[215,3,210]_{49}$	33	$[1419,3,1386]_{49}$
6	$[258,3,252]_{49}$	34	$[1462,3,1428]_{49}$
7	$[301,3,294]_{49}$	35	$[1505,3,1470]_{49}$
8	$[344,3,336]_{49}$	36	$[1548,3,1512]_{49}$
9	$[387,3,378]_{49}$	37	$[1591,3,1554]_{49}$
10	$[430,3,420]_{49}$	38	$[1634,3,1596]_{49}$
11	$[473,3,462]_{49}$	39	$[1677,3,1638]_{49}$
12	$[516,3,504]_{49}$	40	$[1720,3,1680]_{49}$
13	$[559,3,546]_{49}$	41	$[1763,3,1722]_{49}$
14	$[602,3,588]_{49}$	42	$[1806,3,1764]_{49}$
15	$[645,3,630]_{49}$	43	$[1849,3,1806]_{49}$
16	$[688,3,672]_{49}$	44	$[1892,3,1848]_{49}$
17	$[731,3,714]_{49}$	45	$[1935,3,1890]_{49}$
18	$[774,3,756]_{49}$	46	$[1978,3,1932]_{49}$
19	$[817,3,798]_{49}$	47	$[2021,3,1974]_{49}$
20	$[860,3,840]_{49}$	48	$[2064,3,2016]_{49}$
21	$[903,3,882]_{49}$	49	$[2107,3,2058]_{49}$
22	$[946,3,924]_{49}$	50	$[2150,3,2100]_{49}$
23	$[989,3,966]_{49}$	51	$[2193,3,2142]_{49}$
24	$[1032,3,1008]_{49}$	52	$[2236,3,2184]_{49}$
25	$[1075,3,1050]_{49}$	53	$[2279,3,2226]_{49}$
26	$[1118,3,1092]_{49}$	54	$[2322,3,2268]_{49}$
27	$[1161,3,1134]_{49}$	55	$[2365,3,2310]_{49}$
28	$[1204,3,1176]_{49}$	56	$[2408,3,2352]_{49}$
29	$[1247,3,1218]_{49}$	57	$[2451,3,2394]_{49}$

7.2 Embedded unitals

Definition 7.11 (Unital [10, Section 12.3]). A unital is a 2-$(n^3 + 1, n + 1, 1)$ design for some integer $n \geq 3$; that is, a geometry having $n^3 + 1$ points, with $n + 1$ points on each line such that any two distinct points are on exactly one line.

Let q be a prime power, and let $\mathrm{PG}(2, q)$ be the projective plane over $\mathbb{F}_q$. A Hermitian unital $\mathcal{U}$ has the property that any line of $\mathrm{PG}(2, q)$ meets $\mathcal{U}$ in either 1 or $q + 1$ points, and these lines are called tangents and secants, respectively. Considering the intersections of $\mathcal{U}$ and its secants as blocks, the Hermitian unital in $\mathrm{PG}(2, q^2)$ is a 2-$(q^3 + 1, q + 1, 1)$ design. Let $\mathrm{PG}(2, q^2)$ be represented using homogeneous coordinates. Then the points (X, Y, Z) for which

$$X^{q+1} + Y^{q+1} + Z^{q+1} = 0$$

form a unital. This unital is a Hermitian curve and these points form the generator matrix of a projective $[q^3+1, 3, q^3-q]$ code over $\mathbb{F}_{q^2}$. Moreover, Fisher, Hirschfeld, and Thas [7] show that for every prime power q the Hermitian curve admits a complete (q^2-q+1)-arc. In fact, the Hermitian curve is the disjoint union of $q + 1$ complete arcs of size $q^2 - q + 1$.

For example, in the projective plane $\mathrm{PG}(2, 4^2)$, a unital is a 2-$(65, 5, 1)$ design; that is, a geometry having 65 points, such that its intersection with any line contains either 1 or 5 points. The Hermitian unital consists of all points $(X, Y, Z) \in \mathrm{PG}(2, 4^2)$ with

$$X^5 + Y^5 + Z^5 = 0.$$

These points form the generator matrix of a projective $[65, 3, 60]_{16}$ code. Furthermore, the Hermitian curve can be obtained by choosing five complete arcs of size 13 from some of the decomposition of $\mathrm{PG}(2, 4^2)$ found in Section 4.3.

In summary, projective geometries connect to linear codes by their underlying vector spaces. Section 7.1 shows that we can construct a code by taking the disjoint union of i such subplanes or arcs. In the second section, we obtain a Hermitian curve by taking the union of $q + 1$ complete arcs of size $q^2 - q + 1$.

7.3 Conclusion

The work presented in this dissertation opens many doors for further study. Several of these possibilities are addressed as follows.

Due to the highly parallelizable nature of our search, it is likely possible to make use of the existing search procedures to find different types of decomposition into disjoint copies of given substructures, subplanes or arcs, of a projective plane of square order

$\mathrm{PG}(2, q^2)$ with $q \geq 8$. For example, one problem is to find different types of decompositions into disjoint subplanes of order eight in $\mathrm{PG}(2, 8^2)$. The other is to find different types of decompositions into disjoint arcs of size 57 in $\mathrm{PG}(2, 8^2)$. In general, the existing search techniques are likely possible to be implemented to complete a similar investigation of partitioning the points of $\mathrm{PG}(n, q^k)$ into subgeometries $\mathrm{PG}(n, q)$ or even other types of substructures. For instance, as the Hermitian unital of a projective plane $\mathrm{PG}(2, q^2)$ consists of disjoint $(q^2 - q + 1)$-arcs, it can be seen as a spread of $(q^2 - q + 1)$-arcs from a collection C. Therefore, a problem would be to investigate such decompositions.

Appendices

Appendix A

Points of $\mathrm{PG}(2, 2^2)$

The table of the points of $\mathrm{PG}(2, 2^2)$ written in numeral and vector forms.

The points of $\mathrm{PG}(2, 2^2)$ generated by $[(0, 1, 0), (0, 0, 1), (\mu, 1, 1)]$

0	$(1, 0, 0)$	1	$(0, 1, 0)$	2	$(0, 0, 1)$	3	$(\mu, 1, 1)$	4	$(\mu^2, 1, 0)$
5	$(0, \mu^2, 1)$	6	$(1, \mu^2, 1)$	7	$(1, 0, 1)$	8	$(\mu, 0, 1)$	9	$(\mu, \mu^2, 1)$
10	$(1, \mu, 1)$	11	$(\mu^2, 0, 1)$	12	$(\mu, \mu, 1)$	13	$(\mu^2, 1, 1)$	14	$(1, 1, 0)$
15	$(0, 1, 1)$	16	$(\mu, 1, 0)$	17	$(0, \mu, 1)$	18	$(\mu^2, \mu, 1)$	19	$(\mu^2, \mu^2, 1)$
20	$(1, 1, 1)$								

Appendix B

Points of PG$(2, 3^2)$

The table of the points of PG$(2, 3^2)$ written in numeral and vector forms.

The points of PG$(2, 3^2)$ generated by $[(0, 1, 0), (0, 0, 1), (\mu, 1, 1)]$

0	$(1,0,0)$	1	$(0,1,0)$	2	$(0,0,1)$	3	$(\mu,1,1)$	4	$(\mu^5,\mu^6,1)$
5	$(\mu^4,\mu^6,1)$	6	$(\mu^4,0,1)$	7	$(\mu,0,1)$	8	$(\mu,\mu^2,1)$	9	$(\mu^2,\mu^3,1)$
10	$(\mu^3,\mu,1)$	11	$(\mu^7,\mu^4,1)$	12	$(1,1,0)$	13	$(0,1,1)$	14	$(\mu^5,\mu^4,1)$
15	$(\mu^6,1,0)$	16	$(0,\mu^6,1)$	17	$(\mu^4,\mu^3,1)$	18	$(\mu^3,0,1)$	19	$(\mu,\mu^6,1)$
20	$(\mu^4,\mu^5,1)$	21	$(\mu^6,0,1)$	22	$(\mu,\mu^5,1)$	23	$(\mu^6,\mu^7,1)$	24	$(1,\mu^4,1)$
25	$(\mu^5,1,0)$	26	$(0,\mu^5,1)$	27	$(\mu^6,\mu^5,1)$	28	$(\mu^6,\mu^2,1)$	29	$(\mu^2,\mu^6,1)$
30	$(\mu^4,\mu^2,1)$	31	$(\mu^2,0,1)$	32	$(\mu,\mu^7,1)$	33	$(1,\mu,1)$	34	$(\mu^7,\mu^2,1)$
35	$(\mu^2,\mu^2,1)$	36	$(\mu^2,1,1)$	37	$(\mu^5,\mu^3,1)$	38	$(\mu^3,\mu^5,1)$	39	$(\mu^6,\mu^3,1)$
40	$(\mu^3,\mu^7,1)$	41	$(1,\mu^5,1)$	42	$(\mu^6,\mu,1)$	43	$(\mu^7,\mu^3,1)$	44	$(\mu^3,\mu^3,1)$
45	$(\mu^3,1,1)$	46	$(\mu^5,\mu^2,1)$	47	$(\mu^2,\mu^4,1)$	48	$(\mu^2,1,0)$	49	$(0,\mu^2,1)$
50	$(\mu^2,\mu,1)$	51	$(\mu^7,\mu^5,1)$	52	$(\mu^6,\mu^6,1)$	53	$(\mu^4,1,1)$	54	$(\mu^5,0,1)$
55	$(\mu,\mu^3,1)$	56	$(\mu^3,\mu^4,1)$	57	$(\mu^3,1,0)$	58	$(0,\mu^3,1)$	59	$(\mu^3,\mu^2,1)$
60	$(\mu^2,\mu^7,1)$	61	$(1,\mu^6,1)$	62	$(\mu^4,\mu^7,1)$	63	$(1,0,1)$	64	$(\mu,\mu^4,1)$
65	$(\mu^7,1,0)$	66	$(0,\mu^7,1)$	67	$(1,\mu^7,1)$	68	$(1,\mu^3,1)$	69	$(\mu^3,\mu^6,1)$
70	$(\mu^4,\mu,1)$	71	$(\mu^7,0,1)$	72	$(\mu,\mu,1)$	73	$(\mu^7,1,1)$	74	$(\mu^5,\mu^5,1)$
75	$(\mu^6,1,1)$	76	$(\mu^5,\mu,1)$	77	$(\mu^7,\mu,1)$	78	$(\mu^7,\mu^7,1)$	79	$(1,1,1)$
80	$(\mu^5,1,1)$	81	$(\mu^5,\mu^7,1)$	82	$(1,\mu^2,1)$	83	$(\mu^2,\mu^5,1)$	84	$(\mu^6,\mu^4,1)$
85	$(\mu^4,1,0)$	86	$(0,\mu^4,1)$	87	$(\mu,1,0)$	88	$(0,\mu,1)$	89	$(\mu^7,\mu^6,1)$
90	$(\mu^4,\mu^4,1)$								

Appendix C

Points of $\mathrm{PG}(2, 4^2)$

The table of the points of $\mathrm{PG}(2, 4^2)$ written in numeral and vector forms.

The points of $\mathrm{PG}(2, 4^2)$ generated by $[(0, 1, 0), (0, 0, 1), (\mu, 1, 1)]$

0	$(1,0,0)$	1	$(0,1,0)$	2	$(0,0,1)$	3	$(\mu,1,1)$	4	$(\mu^{12},1,0)$
5	$(0,\mu^{12},1)$	6	$(\mu^{5},\mu^{4},1)$	7	$(1,\mu^{9},1)$	8	$(\mu^{9},0,1)$	9	$(\mu,\mu^{7},1)$
10	$(\mu^{7},\mu^{10},1)$	11	$(\mu^{11},\mu^{4},1)$	12	$(1,\mu^{11},1)$	13	$(\mu^{4},0,1)$	14	$(\mu,\mu,1)$
15	$(\mu^{12},1,1)$	16	$(\mu^{5},1,0)$	17	$(0,\mu^{5},1)$	18	$(\mu^{6},\mu^{5},1)$	19	$(\mu^{6},\mu^{3},1)$
20	$(\mu^{2},\mu^{14},1)$	21	$(\mu^{13},\mu^{5},1)$	22	$(\mu^{6},\mu^{11},1)$	23	$(\mu^{4},\mu,1)$	24	$(\mu^{12},\mu^{12},1)$
25	$(\mu^{5},1,1)$	26	$(\mu^{6},1,0)$	27	$(0,\mu^{6},1)$	28	$(\mu^{3},\mu^{2},1)$	29	$(\mu^{8},\mu^{6},1)$
30	$(\mu^{3},\mu^{4},1)$	31	$(1,\mu^{13},1)$	32	$(\mu^{10},0,1)$	33	$(\mu,\mu^{5},1)$	34	$(\mu^{6},\mu^{9},1)$
35	$(\mu^{9},\mu^{6},1)$	36	$(\mu^{3},\mu^{9},1)$	37	$(\mu^{9},\mu^{7},1)$	38	$(\mu^{7},\mu^{13},1)$	39	$(\mu^{10},\mu^{3},1)$
40	$(\mu^{2},\mu^{6},1)$	41	$(\mu^{3},\mu^{10},1)$	42	$(\mu^{11},\mu^{9},1)$	43	$(\mu^{9},\mu^{5},1)$	44	$(\mu^{6},\mu^{12},1)$
45	$(\mu^{5},\mu^{2},1)$	46	$(\mu^{8},\mu^{2},1)$	47	$(\mu^{8},\mu^{9},1)$	48	$(\mu^{9},\mu^{10},1)$	49	$(\mu^{11},\mu^{2},1)$
50	$(\mu^{8},\mu^{4},1)$	51	$(1,\mu,1)$	52	$(\mu^{12},0,1)$	53	$(\mu,\mu^{11},1)$	54	$(\mu^{4},\mu^{7},1)$
55	$(\mu^{7},\mu^{7},1)$	56	$(\mu^{7},1,1)$	57	$(\mu^{7},1,0)$	58	$(0,\mu^{7},1)$	59	$(\mu^{7},\mu^{6},1)$
60	$(\mu^{3},\mu^{11},1)$	61	$(\mu^{4},\mu^{2},1)$	62	$(\mu^{8},\mu^{8},1)$	63	$(\mu^{14},1,1)$	64	$(\mu^{13},1,0)$
65	$(0,\mu^{13},1)$	66	$(\mu^{10},\mu^{9},1)$	67	$(\mu^{9},\mu^{13},1)$	68	$(\mu^{10},\mu,1)$	69	$(\mu^{12},\mu,1)$
70	$(\mu^{12},\mu^{7},1)$	71	$(\mu^{7},\mu^{2},1)$	72	$(\mu^{8},\mu,1)$	73	$(\mu^{12},\mu^{13},1)$	74	$(\mu^{10},\mu^{5},1)$
75	$(\mu^{6},\mu^{10},1)$	76	$(\mu^{11},\mu^{8},1)$	77	$(\mu^{14},\mu^{10},1)$	78	$(\mu^{11},\mu^{13},1)$	79	$(\mu^{10},\mu^{6},1)$
80	$(\mu^{3},\mu^{7},1)$	81	$(\mu^{7},\mu^{5},1)$	82	$(\mu^{6},\mu^{14},1)$	83	$(\mu^{13},\mu^{10},1)$	84	$(\mu^{11},\mu,1)$
85	$(\mu^{12},\mu^{8},1)$	86	$(\mu^{14},\mu^{9},1)$	87	$(\mu^{9},\mu^{11},1)$	88	$(\mu^{4},\mu^{10},1)$	89	$(\mu^{11},\mu^{11},1)$
90	$(\mu^{4},1,1)$	91	$(1,1,0)$	92	$(0,1,1)$	93	$(\mu,1,0)$	94	$(0,\mu,1)$
95	$(\mu^{12},\mu^{11},1)$	96	$(\mu^{4},\mu^{14},1)$	97	$(\mu^{13},\mu^{13},1)$	98	$(\mu^{10},1,1)$	99	$(\mu^{11},1,0)$
100	$(0,\mu^{11},1)$	101	$(\mu^{4},\mu^{3},1)$	102	$(\mu^{2},\mu^{2},1)$	103	$(\mu^{8},1,1)$	104	$(\mu^{14},1,0)$
105	$(0,\mu^{14},1)$	106	$(\mu^{13},\mu^{12},1)$	107	$(\mu^{5},\mu^{10},1)$	108	$(\mu^{11},\mu^{5},1)$	109	$(\mu^{6},\mu^{2},1)$
110	$(\mu^{8},\mu^{5},1)$	111	$(\mu^{6},\mu^{7},1)$	112	$(\mu^{7},\mu^{4},1)$	113	$(1,\mu^{8},1)$	114	$(\mu^{14},0,1)$
115	$(\mu,\mu^{3},1)$	116	$(\mu^{2},\mu^{5},1)$	117	$(\mu^{6},\mu^{13},1)$	118	$(\mu^{10},\mu^{7},1)$	119	$(\mu^{7},\mu^{11},1)$
120	$(\mu^{4},\mu^{12},1)$	121	$(\mu^{5},\mu^{5},1)$	122	$(\mu^{6},1,1)$	123	$(\mu^{3},1,0)$	124	$(0,\mu^{3},1)$
125	$(\mu^{2},\mu,1)$	126	$(\mu^{12},\mu^{4},1)$	127	$(1,\mu^{10},1)$	128	$(\mu^{11},0,1)$	129	$(\mu,\mu^{12},1)$
130	$(\mu^{5},\mu^{8},1)$	131	$(\mu^{14},\mu^{8},1)$	132	$(\mu^{14},\mu,1)$	133	$(\mu^{12},\mu^{14},1)$	134	$(\mu^{13},\mu^{8},1)$

Continued on next page

Table C.1 – continued from previous page (The points of $\mathrm{PG}(2,4^2)$**)**

135	$(\mu^{14},\mu^4,1)$	136	$(1,\mu^2,1)$	137	$(\mu^8,0,1)$	138	$(\mu,\mu^2,1)$	139	$(\mu^8,\mu^{11},1)$
140	$(\mu^4,\mu^5,1)$	141	$(\mu^6,\mu^6,1)$	142	$(\mu^3,1,1)$	143	$(\mu^2,1,0)$	144	$(0,\mu^2,1)$
145	$(\mu^8,\mu^7,1)$	146	$(\mu^7,\mu^8,1)$	147	$(\mu^{14},\mu^7,1)$	148	$(\mu^7,\mu^9,1)$	149	$(\mu^9,\mu^2,1)$
150	$(\mu^8,\mu^{14},1)$	151	$(\mu^{13},\mu^{14},1)$	152	$(\mu^{13},\mu^3,1)$	153	$(\mu^2,\mu^7,1)$	154	$(\mu^7,\mu^{14},1)$
155	$(\mu^{13},\mu^6,1)$	156	$(\mu^3,\mu^8,1)$	157	$(\mu^{14},\mu^{12},1)$	158	$(\mu^5,\mu^7,1)$	159	$(\mu^7,\mu,1)$
160	$(\mu^{12},\mu^5,1)$	161	$(\mu^6,\mu,1)$	162	$(\mu^{12},\mu^9,1)$	163	$(\mu^9,\mu^4,1)$	164	$(1,\mu^6,1)$
165	$(\mu^3,0,1)$	166	$(\mu,\mu^{14},1)$	167	$(\mu^{13},\mu,1)$	168	$(\mu^{12},\mu^2,1)$	169	$(\mu^8,\mu^3,1)$
170	$(\mu^2,\mu^3,1)$	171	$(\mu^2,\mu^9,1)$	172	$(\mu^9,\mu,1)$	173	$(\mu^{12},\mu^3,1)$	174	$(\mu^2,\mu^{12},1)$
175	$(\mu^5,\mu^{12},1)$	176	$(\mu^5,\mu^{14},1)$	177	$(\mu^{13},\mu^7,1)$	178	$(\mu^7,\mu^{12},1)$	179	$(\mu^5,\mu^{13},1)$
180	$(\mu^{10},\mu^4,1)$	181	$(1,\mu^4,1)$	182	$(1,0,1)$	183	$(\mu,0,1)$	184	$(\mu,\mu^4,1)$
185	$(1,\mu^3,1)$	186	$(\mu^2,0,1)$	187	$(\mu,\mu^8,1)$	188	$(\mu^{14},\mu^2,1)$	189	$(\mu^8,\mu^{10},1)$
190	$(\mu^{11},\mu^{12},1)$	191	$(\mu^5,\mu,1)$	192	$(\mu^{12},\mu^6,1)$	193	$(\mu^3,\mu^{13},1)$	194	$(\mu^{10},\mu^8,1)$
195	$(\mu^{14},\mu^3,1)$	196	$(\mu^2,\mu^4,1)$	197	$(1,\mu^7,1)$	198	$(\mu^7,0,1)$	199	$(\mu,\mu^9,1)$
200	$(\mu^9,\mu^{12},1)$	201	$(\mu^5,\mu^{11},1)$	202	$(\mu^4,\mu^{13},1)$	203	$(\mu^{10},\mu^{10},1)$	204	$(\mu^{11},1,1)$
205	$(\mu^4,1,0)$	206	$(0,\mu^4,1)$	207	$(1,\mu^{14},1)$	208	$(\mu^{13},0,1)$	209	$(\mu,\mu^6,1)$
210	$(\mu^3,\mu^6,1)$	211	$(\mu^3,\mu,1)$	212	$(\mu^{12},\mu^{10},1)$	213	$(\mu^{11},\mu^6,1)$	214	$(\mu^3,\mu^{14},1)$
215	$(\mu^{13},\mu^{11},1)$	216	$(\mu^4,\mu^9,1)$	217	$(\mu^9,\mu^9,1)$	218	$(\mu^9,1,1)$	219	$(\mu^9,1,0)$
220	$(0,\mu^9,1)$	221	$(\mu^9,\mu^8,1)$	222	$(\mu^{14},\mu^5,1)$	223	$(\mu^6,\mu^8,1)$	224	$(\mu^{14},\mu^{11},1)$
225	$(\mu^4,\mu^6,1)$	226	$(\mu^3,\mu^3,1)$	227	$(\mu^2,1,1)$	228	$(\mu^8,1,0)$	229	$(0,\mu^8,1)$
230	$(\mu^{14},\mu^{13},1)$	231	$(\mu^{10},\mu^{12},1)$	232	$(\mu^5,\mu^9,1)$	233	$(\mu^9,\mu^3,1)$	234	$(\mu^2,\mu^8,1)$
235	$(\mu^{14},\mu^6,1)$	236	$(\mu^3,\mu^5,1)$	237	$(\mu^6,\mu^4,1)$	238	$(1,\mu^{12},1)$	239	$(\mu^5,0,1)$
240	$(\mu,\mu^{10},1)$	241	$(\mu^{11},\mu^{14},1)$	242	$(\mu^{13},\mu^9,1)$	243	$(\mu^9,\mu^{14},1)$	244	$(\mu^{13},\mu^4,1)$
245	$(1,\mu^5,1)$	246	$(\mu^6,0,1)$	247	$(\mu,\mu^{13},1)$	248	$(\mu^{10},\mu^{13},1)$	249	$(\mu^{10},\mu^{14},1)$
250	$(\mu^{13},\mu^2,1)$	251	$(\mu^8,\mu^{13},1)$	252	$(\mu^{10},\mu^{11},1)$	253	$(\mu^4,\mu^8,1)$	254	$(\mu^{14},\mu^{14},1)$
255	$(\mu^{13},1,1)$	256	$(\mu^{10},1,0)$	257	$(0,\mu^{10},1)$	258	$(\mu^{11},\mu^{10},1)$	259	$(\mu^{11},\mu^7,1)$
260	$(\mu^7,\mu^3,1)$	261	$(\mu^2,\mu^{10},1)$	262	$(\mu^{11},\mu^3,1)$	263	$(\mu^2,\mu^{13},1)$	264	$(\mu^{10},\mu^2,1)$
265	$(\mu^8,\mu^{12},1)$	266	$(\mu^5,\mu^6,1)$	267	$(\mu^3,\mu^{12},1)$	268	$(\mu^5,\mu^3,1)$	269	$(\mu^2,\mu^{11},1)$
270	$(\mu^4,\mu^{11},1)$	271	$(\mu^4,\mu^4,1)$	272	$(1,1,1)$				

Appendix D

Points of PG$(2, 5^2)$

The table of the points of PG$(2, 5^2)$ written in numeral and vector forms.

The points of PG$(2, 5^2)$ generated by $[(0, 1, 0), (0, 0, 1), (\mu^8, 1, 1)]$

0	$(1,0,0)$	1	$(0,1,0)$	2	$(0,0,1)$	3	$(\mu^8,1,1)$	4	$(\mu^2,\mu^{22},1)$
5	$(\mu^{17},\mu^2,1)$	6	$(\mu^{15},\mu^8,1)$	7	$(\mu^4,\mu^{22},1)$	8	$(\mu^{17},\mu^8,1)$	9	$(\mu^4,\mu^{21},1)$
10	$(\mu,\mu^{16},1)$	11	$(\mu^{12},\mu^2,1)$	12	$(\mu^{15},0,1)$	13	$(\mu^8,\mu^2,1)$	14	$(\mu^{15},\mu^{11},1)$
15	$(\mu^{16},\mu^{10},1)$	16	$(\mu^{19},\mu^7,1)$	17	$(1,\mu,1)$	18	$(\mu^{10},\mu^8,1)$	19	$(\mu^4,\mu^9,1)$
20	$(\mu^{21},\mu^{12},1)$	21	$(\mu,1,0)$	22	$(0,\mu,1)$	23	$(\mu^{10},\mu^2,1)$	24	$(\mu^{15},\mu^{20},1)$
25	$(\mu^{13},\mu^7,1)$	26	$(1,\mu^{21},1)$	27	$(\mu,\mu^{23},1)$	28	$(\mu^{11},\mu,1)$	29	$(\mu^{10},\mu^{18},1)$
30	$(\mu^{20},\mu,1)$	31	$(\mu^{10},\mu^{21},1)$	32	$(\mu,\mu^6,1)$	33	$(\mu^{14},\mu^4,1)$	34	$(\mu^9,\mu^4,1)$
35	$(\mu^9,\mu^{12},1)$	36	$(\mu^{21},1,0)$	37	$(0,\mu^{21},1)$	38	$(\mu,\mu^{17},1)$	39	$(\mu^7,\mu^{21},1)$
40	$(\mu,\mu,1)$	41	$(\mu^{10},1,1)$	42	$(\mu^2,\mu^7,1)$	43	$(1,\mu^9,1)$	44	$(\mu^{21},\mu^{19},1)$
45	$(\mu^{23},\mu^{22},1)$	46	$(\mu^{17},\mu^6,1)$	47	$(\mu^{14},\mu^7,1)$	48	$(1,\mu^{19},1)$	49	$(\mu^{23},\mu^{21},1)$
50	$(\mu,\mu^{14},1)$	51	$(\mu^5,\mu^{19},1)$	52	$(\mu^{23},\mu^5,1)$	53	$(\mu^{18},\mu^7,1)$	54	$(1,\mu^4,1)$
55	$(\mu^9,\mu^7,1)$	56	$(1,\mu^3,1)$	57	$(\mu^{22},\mu^{20},1)$	58	$(\mu^{13},\mu^{20},1)$	59	$(\mu^{13},\mu^{10},1)$
60	$(\mu^{19},\mu^{16},1)$	61	$(\mu^{12},\mu^{13},1)$	62	$(\mu^3,0,1)$	63	$(\mu^8,\mu^{10},1)$	64	$(\mu^{19},\mu^{15},1)$
65	$(\mu^6,\mu^7,1)$	66	$(1,\mu^{10},1)$	67	$(\mu^{19},\mu^{17},1)$	68	$(\mu^7,\mu^8,1)$	69	$(\mu^4,\mu^4,1)$
70	$(\mu^9,1,1)$	71	$(\mu^2,\mu^5,1)$	72	$(\mu^{18},\mu^3,1)$	73	$(\mu^{22},\mu^2,1)$	74	$(\mu^{15},\mu^{22},1)$
75	$(\mu^{17},\mu^{11},1)$	76	$(\mu^{16},\mu^9,1)$	77	$(\mu^{21},\mu^9,1)$	78	$(\mu^{21},\mu^{20},1)$	79	$(\mu^{13},\mu^{12},1)$
80	$(\mu^3,1,0)$	81	$(0,\mu^3,1)$	82	$(\mu^{22},\mu^{14},1)$	83	$(\mu^5,\mu^{12},1)$	84	$(\mu^{18},1,0)$
85	$(0,\mu^{18},1)$	86	$(\mu^{20},\mu^{12},1)$	87	$(\mu^{13},1,0)$	88	$(0,\mu^{13},1)$	89	$(\mu^3,\mu^{19},1)$
90	$(\mu^{23},\mu,1)$	91	$(\mu^{10},\mu^{23},1)$	92	$(\mu^{11},\mu^{16},1)$	93	$(\mu^{12},\mu^{20},1)$	94	$(\mu^{13},0,1)$
95	$(\mu^8,\mu^5,1)$	96	$(\mu^{18},\mu^{14},1)$	97	$(\mu^5,\mu^9,1)$	98	$(\mu^{21},\mu^3,1)$	99	$(\mu^{22},\mu^{21},1)$
100	$(\mu,\mu^8,1)$	101	$(\mu^4,\mu^{18},1)$	102	$(\mu^{20},\mu^{11},1)$	103	$(\mu^{16},\mu^3,1)$	104	$(\mu^{22},\mu^{10},1)$
105	$(\mu^{19},\mu^2,1)$	106	$(\mu^{15},\mu^{16},1)$	107	$(\mu^{12},\mu^6,1)$	108	$(\mu^{14},0,1)$	109	$(\mu^8,\mu^3,1)$
110	$(\mu^{22},\mu^{18},1)$	111	$(\mu^{20},\mu^3,1)$	112	$(\mu^{22},\mu^9,1)$	113	$(\mu^{21},\mu^4,1)$	114	$(\mu^9,\mu^8,1)$
115	$(\mu^4,\mu^7,1)$	116	$(1,\mu^{15},1)$	117	$(\mu^6,\mu^4,1)$	118	$(\mu^9,\mu^{19},1)$	119	$(\mu^{23},\mu^2,1)$
120	$(\mu^{15},\mu^4,1)$	121	$(\mu^9,\mu^3,1)$	122	$(\mu^{22},\mu,1)$	123	$(\mu^{10},\mu^{17},1)$	124	$(\mu^7,\mu^{12},1)$
125	$(1,1,0)$	126	$(0,1,1)$	127	$(\mu^2,\mu^{18},1)$	128	$(\mu^{20},\mu^5,1)$	129	$(\mu^{18},\mu^5,1)$
130	$(\mu^{18},\mu^{22},1)$	131	$(\mu^{17},\mu^{21},1)$	132	$(\mu,\mu^{18},1)$	133	$(\mu^{20},\mu^{10},1)$	134	$(\mu^{19},\mu^6,1)$

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Table D.1 – continued from previous page (The points of $\mathrm{PG}(2,5^2)$**)**

135	$(\mu^{14},\mu^{15},1)$	136	$(\mu^{6},\mu,1)$	137	$(\mu^{10},\mu^{20},1)$	138	$(\mu^{13},\mu^{18},1)$	139	$(\mu^{20},\mu^{17},1)$
140	$(\mu^{7},\mu^{18},1)$	141	$(\mu^{20},\mu^{20},1)$	142	$(\mu^{13},1,1)$	143	$(\mu^{2},\mu^{23},1)$	144	$(\mu^{11},\mu^{20},1)$
145	$(\mu^{13},\mu^{21},1)$	146	$(\mu,\mu^{22},1)$	147	$(\mu^{17},\mu^{7},1)$	148	$(1,\mu^{17},1)$	149	$(\mu^{7},\mu^{5},1)$
150	$(\mu^{18},\mu^{18},1)$	151	$(\mu^{20},1,1)$	152	$(\mu^{2},\mu^{13},1)$	153	$(\mu^{3},\mu^{12},1)$	154	$(\mu^{22},1,0)$
155	$(0,\mu^{22},1)$	156	$(\mu^{17},\mu^{9},1)$	157	$(\mu^{21},\mu^{14},1)$	158	$(\mu^{5},\mu^{4},1)$	159	$(\mu^{9},\mu^{15},1)$
160	$(\mu^{6},\mu^{9},1)$	161	$(\mu^{21},\mu^{7},1)$	162	$(1,\mu^{23},1)$	163	$(\mu^{11},\mu^{9},1)$	164	$(\mu^{21},\mu^{5},1)$
165	$(\mu^{18},\mu^{17},1)$	166	$(\mu^{7},\mu^{11},1)$	167	$(\mu^{16},\mu^{16},1)$	168	$(\mu^{12},1,1)$	169	$(\mu^{2},0,1)$
170	$(\mu^{8},\mu^{17},1)$	171	$(\mu^{7},\mu^{3},1)$	172	$(\mu^{22},\mu^{22},1)$	173	$(\mu^{17},1,1)$	174	$(\mu^{2},\mu^{19},1)$
175	$(\mu^{23},\mu^{8},1)$	176	$(\mu^{4},\mu^{17},1)$	177	$(\mu^{7},\mu^{22},1)$	178	$(\mu^{17},\mu^{17},1)$	179	$(\mu^{7},1,1)$
180	$(\mu^{2},\mu^{2},1)$	181	$(\mu^{15},1,1)$	182	$(\mu^{2},\mu^{20},1)$	183	$(\mu^{13},\mu^{22},1)$	184	$(\mu^{17},\mu^{14},1)$
185	$(\mu^{5},\mu^{22},1)$	186	$(\mu^{17},\mu^{23},1)$	187	$(\mu^{11},\mu^{4},1)$	188	$(\mu^{9},\mu^{17},1)$	189	$(\mu^{7},\mu^{10},1)$
190	$(\mu^{19},\mu^{19},1)$	191	$(\mu^{23},1,1)$	192	$(\mu^{2},\mu^{15},1)$	193	$(\mu^{6},\mu^{15},1)$	194	$(\mu^{6},\mu^{16},1)$
195	$(\mu^{12},\mu^{22},1)$	196	$(\mu^{17},0,1)$	197	$(\mu^{8},\mu,1)$	198	$(\mu^{10},\mu^{6},1)$	199	$(\mu^{14},\mu^{19},1)$
200	$(\mu^{23},\mu^{18},1)$	201	$(\mu^{20},\mu^{9},1)$	202	$(\mu^{21},\mu^{8},1)$	203	$(\mu^{4},\mu^{3},1)$	204	$(\mu^{22},\mu^{13},1)$
205	$(\mu^{3},\mu^{10},1)$	206	$(\mu^{19},\mu^{21},1)$	207	$(\mu,\mu^{2},1)$	208	$(\mu^{15},\mu^{5},1)$	209	$(\mu^{18},\mu^{12},1)$
210	$(\mu^{20},1,0)$	211	$(0,\mu^{20},1)$	212	$(\mu^{13},\mu^{5},1)$	213	$(\mu^{18},\mu^{15},1)$	214	$(\mu^{6},\mu^{10},1)$
215	$(\mu^{19},\mu^{5},1)$	216	$(\mu^{18},\mu^{19},1)$	217	$(\mu^{23},\mu^{3},1)$	218	$(\mu^{22},\mu^{11},1)$	219	$(\mu^{16},\mu^{23},1)$
220	$(\mu^{11},\mu^{23},1)$	221	$(\mu^{11},\mu^{19},1)$	222	$(\mu^{23},\mu^{7},1)$	223	$(1,\mu^{13},1)$	224	$(\mu^{3},\mu,1)$
225	$(\mu^{10},\mu^{12},1)$	226	$(\mu^{19},1,0)$	227	$(0,\mu^{19},1)$	228	$(\mu^{23},\mu^{15},1)$	229	$(\mu^{6},\mu^{19},1)$
230	$(\mu^{23},\mu^{9},1)$	231	$(\mu^{21},\mu^{10},1)$	232	$(\mu^{19},\mu^{18},1)$	233	$(\mu^{20},\mu^{21},1)$	234	$(\mu,\mu^{12},1)$
235	$(\mu^{10},1,0)$	236	$(0,\mu^{10},1)$	237	$(\mu^{19},\mu^{11},1)$	238	$(\mu^{16},\mu^{17},1)$	239	$(\mu^{7},\mu^{19},1)$
240	$(\mu^{23},\mu^{23},1)$	241	$(\mu^{11},1,1)$	242	$(\mu^{2},\mu^{10},1)$	243	$(\mu^{19},\mu^{4},1)$	244	$(\mu^{9},\mu^{10},1)$
245	$(\mu^{19},\mu^{22},1)$	246	$(\mu^{17},\mu^{18},1)$	247	$(\mu^{20},\mu^{13},1)$	248	$(\mu^{3},\mu^{14},1)$	249	$(\mu^{5},\mu^{7},1)$
250	$(1,\mu^{6},1)$	251	$(\mu^{14},\mu^{12},1)$	252	$(\mu^{5},1,0)$	253	$(0,\mu^{5},1)$	254	$(\mu^{18},\mu^{10},1)$
255	$(\mu^{19},\mu^{23},1)$	256	$(\mu^{11},\mu^{12},1)$	257	$(\mu^{16},1,0)$	258	$(0,\mu^{16},1)$	259	$(\mu^{12},\mu^{4},1)$
260	$(\mu^{9},0,1)$	261	$(\mu^{8},\mu^{11},1)$	262	$(\mu^{16},\mu^{12},1)$	263	$(\mu^{12},1,0)$	264	$(0,\mu^{12},1)$
265	$(\mu^{8},1,0)$	266	$(0,\mu^{8},1)$	267	$(\mu^{4},\mu^{20},1)$	268	$(\mu^{13},\mu^{4},1)$	269	$(\mu^{9},\mu^{6},1)$
270	$(\mu^{14},\mu^{17},1)$	271	$(\mu^{7},\mu^{2},1)$	272	$(\mu^{15},\mu^{15},1)$	273	$(\mu^{6},1,1)$	274	$(\mu^{2},\mu^{12},1)$
275	$(\mu^{15},1,0)$	276	$(0,\mu^{15},1)$	277	$(\mu^{6},\mu^{22},1)$	278	$(\mu^{17},\mu^{3},1)$	279	$(\mu^{22},\mu^{15},1)$
280	$(\mu^{6},\mu^{13},1)$	281	$(\mu^{3},\mu^{13},1)$	282	$(\mu^{3},\mu^{5},1)$	283	$(\mu^{18},\mu^{20},1)$	284	$(\mu^{13},\mu^{17},1)$
285	$(\mu^{7},\mu^{4},1)$	286	$(\mu^{9},\mu^{9},1)$	287	$(\mu^{21},1,1)$	288	$(\mu^{2},\mu,1)$	289	$(\mu^{10},\mu^{19},1)$
290	$(\mu^{23},\mu^{4},1)$	291	$(\mu^{9},\mu^{22},1)$	292	$(\mu^{17},\mu^{20},1)$	293	$(\mu^{13},\mu^{6},1)$	294	$(\mu^{14},\mu^{11},1)$
295	$(\mu^{16},\mu^{11},1)$	296	$(\mu^{16},\mu^{4},1)$	297	$(\mu^{9},\mu^{21},1)$	298	$(\mu,\mu^{4},1)$	299	$(\mu^{9},\mu^{23},1)$
300	$(\mu^{11},\mu^{14},1)$	301	$(\mu^{5},\mu^{13},1)$	302	$(\mu^{3},\mu^{9},1)$	303	$(\mu^{21},\mu^{23},1)$	304	$(\mu^{11},\mu^{10},1)$
305	$(\mu^{19},\mu^{3},1)$	306	$(\mu^{22},\mu^{23},1)$	307	$(\mu^{11},\mu^{18},1)$	308	$(\mu^{20},\mu^{4},1)$	309	$(\mu^{9},\mu^{20},1)$
310	$(\mu^{13},\mu^{16},1)$	311	$(\mu^{12},\mu^{9},1)$	312	$(\mu^{21},0,1)$	313	$(\mu^{8},\mu^{7},1)$	314	$(1,\mu^{20},1)$
315	$(\mu^{13},\mu^{11},1)$	316	$(\mu^{16},\mu^{13},1)$	317	$(\mu^{3},\mu^{15},1)$	318	$(\mu^{6},\mu^{8},1)$	319	$(\mu^{4},\mu^{14},1)$
320	$(\mu^{5},\mu^{20},1)$	321	$(\mu^{13},\mu^{19},1)$	322	$(\mu^{23},\mu^{20},1)$	323	$(\mu^{13},\mu^{2},1)$	324	$(\mu^{15},\mu^{12},1)$
325	$(\mu^{6},1,0)$	326	$(0,\mu^{6},1)$	327	$(\mu^{14},\mu^{6},1)$	328	$(\mu^{14},\mu^{9},1)$	329	$(\mu^{21},\mu^{16},1)$
330	$(\mu^{12},\mu^{11},1)$	331	$(\mu^{16},0,1)$	332	$(\mu^{8},\mu^{20},1)$	333	$(\mu^{13},\mu^{9},1)$	334	$(\mu^{21},\mu^{18},1)$
335	$(\mu^{20},\mu^{19},1)$	336	$(\mu^{23},\mu^{10},1)$	337	$(\mu^{19},\mu^{8},1)$	338	$(\mu^{4},\mu^{5},1)$	339	$(\mu^{18},\mu^{9},1)$
340	$(\mu^{21},\mu,1)$	341	$(\mu^{10},\mu^{9},1)$	342	$(\mu^{21},\mu^{2},1)$	343	$(\mu^{15},\mu^{14},1)$	344	$(\mu^{5},\mu^{23},1)$
345	$(\mu^{11},\mu^{17},1)$	346	$(\mu^{7},\mu^{15},1)$	347	$(\mu^{6},\mu^{6},1)$	348	$(\mu^{14},1,1)$	349	$(\mu^{2},\mu^{21},1)$
350	$(\mu,\mu^{10},1)$	351	$(\mu^{19},\mu^{9},1)$	352	$(\mu^{21},\mu^{22},1)$	353	$(\mu^{17},\mu^{16},1)$	354	$(\mu^{12},\mu^{5},1)$
355	$(\mu^{18},0,1)$	356	$(\mu^{8},\mu^{12},1)$	357	$(\mu^{4},1,0)$	358	$(0,\mu^{4},1)$	359	$(\mu^{9},\mu,1)$

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Table D.1 – continued from previous page (The points of $\mathrm{PG}(2,5^2)$**)**

360	$(\mu^{10},\mu^{13},1)$	361	$(\mu^{3},\mu^{8},1)$	362	$(\mu^{4},\mu^{6},1)$	363	$(\mu^{14},\mu^{5},1)$	364	$(\mu^{18},\mu^{13},1)$
365	$(\mu^{3},\mu^{7},1)$	366	$(1,\mu^{2},1)$	367	$(\mu^{15},\mu^{13},1)$	368	$(\mu^{3},\mu^{21},1)$	369	$(\mu,\mu^{3},1)$
370	$(\mu^{22},\mu^{12},1)$	371	$(\mu^{17},1,0)$	372	$(0,\mu^{17},1)$	373	$(\mu^{7},\mu^{23},1)$	374	$(\mu^{11},\mu^{11},1)$
375	$(\mu^{16},1,1)$	376	$(\mu^{2},\mu^{14},1)$	377	$(\mu^{5},\mu^{14},1)$	378	$(\mu^{5},\mu^{11},1)$	379	$(\mu^{16},\mu^{22},1)$
380	$(\mu^{17},\mu^{5},1)$	381	$(\mu^{18},\mu^{11},1)$	382	$(\mu^{16},\mu^{20},1)$	383	$(\mu^{13},\mu,1)$	384	$(\mu^{10},\mu^{7},1)$
385	$(1,\mu^{5},1)$	386	$(\mu^{18},\mu^{16},1)$	387	$(\mu^{12},\mu^{16},1)$	388	$(\mu^{12},0,1)$	389	$(\mu^{8},0,1)$
390	$(\mu^{8},\mu^{4},1)$	391	$(\mu^{9},\mu^{5},1)$	392	$(\mu^{18},\mu^{21},1)$	393	$(\mu,\mu^{5},1)$	394	$(\mu^{18},\mu^{8},1)$
395	$(\mu^{4},\mu^{8},1)$	396	$(\mu^{4},\mu^{19},1)$	397	$(\mu^{23},\mu^{14},1)$	398	$(\mu^{5},\mu^{18},1)$	399	$(\mu^{20},\mu^{2},1)$
400	$(\mu^{15},\mu^{2},1)$	401	$(\mu^{15},\mu^{9},1)$	402	$(\mu^{21},\mu^{15},1)$	403	$(\mu^{6},\mu^{5},1)$	404	$(\mu^{18},\mu^{4},1)$
405	$(\mu^{9},\mu^{13},1)$	406	$(\mu^{3},\mu^{6},1)$	407	$(\mu^{14},\mu^{16},1)$	408	$(\mu^{12},\mu^{7},1)$	409	$(1,0,1)$
410	$(\mu^{8},\mu^{6},1)$	411	$(\mu^{14},\mu^{10},1)$	412	$(\mu^{19},\mu^{14},1)$	413	$(\mu^{5},\mu^{6},1)$	414	$(\mu^{14},\mu^{20},1)$
415	$(\mu^{13},\mu^{8},1)$	416	$(\mu^{4},\mu,1)$	417	$(\mu^{10},\mu,1)$	418	$(\mu^{10},\mu^{15},1)$	419	$(\mu^{6},\mu^{11},1)$
420	$(\mu^{16},\mu^{2},1)$	421	$(\mu^{15},\mu^{3},1)$	422	$(\mu^{22},\mu^{16},1)$	423	$(\mu^{12},\mu^{19},1)$	424	$(\mu^{23},0,1)$
425	$(\mu^{8},\mu^{21},1)$	426	$(\mu,\mu^{21},1)$	427	$(\mu,\mu^{15},1)$	428	$(\mu^{6},\mu^{20},1)$	429	$(\mu^{13},\mu^{23},1)$
430	$(\mu^{11},\mu^{8},1)$	431	$(\mu^{4},\mu^{12},1)$	432	$(\mu^{9},1,0)$	433	$(0,\mu^{9},1)$	434	$(\mu^{21},\mu^{13},1)$
435	$(\mu^{3},\mu^{2},1)$	436	$(\mu^{15},\mu^{17},1)$	437	$(\mu^{7},\mu,1)$	438	$(\mu^{10},\mu^{10},1)$	439	$(\mu^{19},1,1)$
440	$(\mu^{2},\mu^{3},1)$	441	$(\mu^{22},\mu^{7},1)$	442	$(1,\mu^{7},1)$	443	$(1,\mu^{22},1)$	444	$(\mu^{17},\mu^{15},1)$
445	$(\mu^{6},\mu^{23},1)$	446	$(\mu^{11},\mu^{21},1)$	447	$(\mu,\mu^{9},1)$	448	$(\mu^{21},\mu^{11},1)$	449	$(\mu^{16},\mu^{15},1)$
450	$(\mu^{6},\mu^{18},1)$	451	$(\mu^{20},\mu^{6},1)$	452	$(\mu^{14},\mu,1)$	453	$(\mu^{10},\mu^{5},1)$	454	$(\mu^{18},\mu^{23},1)$
455	$(\mu^{11},\mu^{15},1)$	456	$(\mu^{6},\mu^{14},1)$	457	$(\mu^{5},\mu^{15},1)$	458	$(\mu^{6},\mu^{12},1)$	459	$(\mu^{14},1,0)$
460	$(0,\mu^{14},1)$	461	$(\mu^{5},\mu^{21},1)$	462	$(\mu,\mu^{7},1)$	463	$(1,\mu^{14},1)$	464	$(\mu^{5},\mu^{3},1)$
465	$(\mu^{22},\mu^{4},1)$	466	$(\mu^{9},\mu^{16},1)$	467	$(\mu^{12},\mu^{15},1)$	468	$(\mu^{6},0,1)$	469	$(\mu^{8},\mu^{18},1)$
470	$(\mu^{20},\mu^{16},1)$	471	$(\mu^{12},\mu^{23},1)$	472	$(\mu^{11},0,1)$	473	$(\mu^{8},\mu^{16},1)$	474	$(\mu^{12},\mu^{8},1)$
475	$(\mu^{4},0,1)$	476	$(\mu^{8},\mu^{23},1)$	477	$(\mu^{11},\mu^{7},1)$	478	$(1,\mu^{8},1)$	479	$(\mu^{4},\mu^{2},1)$
480	$(\mu^{15},\mu^{6},1)$	481	$(\mu^{14},\mu^{8},1)$	482	$(\mu^{4},\mu^{23},1)$	483	$(\mu^{11},\mu^{2},1)$	484	$(\mu^{15},\mu^{23},1)$
485	$(\mu^{11},\mu^{5},1)$	486	$(\mu^{18},\mu^{2},1)$	487	$(\mu^{15},\mu^{19},1)$	488	$(\mu^{23},\mu^{17},1)$	489	$(\mu^{7},\mu^{20},1)$
490	$(\mu^{13},\mu^{13},1)$	491	$(\mu^{3},1,1)$	492	$(\mu^{2},\mu^{4},1)$	493	$(\mu^{9},\mu^{18},1)$	494	$(\mu^{20},\mu^{23},1)$
495	$(\mu^{11},\mu^{22},1)$	496	$(\mu^{17},\mu,1)$	497	$(\mu^{10},\mu^{3},1)$	498	$(\mu^{22},\mu^{3},1)$	499	$(\mu^{22},\mu^{5},1)$
500	$(\mu^{18},\mu,1)$	501	$(\mu^{10},\mu^{14},1)$	502	$(\mu^{5},\mu^{10},1)$	503	$(\mu^{19},\mu,1)$	504	$(\mu^{10},\mu^{11},1)$
505	$(\mu^{16},\mu^{21},1)$	506	$(\mu,\mu^{13},1)$	507	$(\mu^{3},\mu^{17},1)$	508	$(\mu^{7},\mu^{9},1)$	509	$(\mu^{21},\mu^{21},1)$
510	$(\mu,1,1)$	511	$(\mu^{2},\mu^{16},1)$	512	$(\mu^{12},\mu^{21},1)$	513	$(\mu,0,1)$	514	$(\mu^{8},\mu^{22},1)$
515	$(\mu^{17},\mu^{13},1)$	516	$(\mu^{3},\mu^{20},1)$	517	$(\mu^{13},\mu^{15},1)$	518	$(\mu^{6},\mu^{3},1)$	519	$(\mu^{22},\mu^{8},1)$
520	$(\mu^{4},\mu^{11},1)$	521	$(\mu^{16},\mu^{7},1)$	522	$(1,\mu^{12},1)$	523	$(\mu^{2},1,0)$	524	$(0,\mu^{2},1)$
525	$(\mu^{15},\mu^{7},1)$	526	$(1,\mu^{18},1)$	527	$(\mu^{20},\mu^{18},1)$	528	$(\mu^{20},\mu^{7},1)$	529	$(1,\mu^{11},1)$
530	$(\mu^{16},\mu^{14},1)$	531	$(\mu^{5},\mu^{17},1)$	532	$(\mu^{7},\mu^{13},1)$	533	$(\mu^{3},\mu^{3},1)$	534	$(\mu^{22},1,1)$
535	$(\mu^{2},\mu^{9},1)$	536	$(\mu^{21},\mu^{6},1)$	537	$(\mu^{14},\mu^{13},1)$	538	$(\mu^{3},\mu^{22},1)$	539	$(\mu^{17},\mu^{19},1)$
540	$(\mu^{23},\mu^{16},1)$	541	$(\mu^{12},\mu,1)$	542	$(\mu^{10},0,1)$	543	$(\mu^{8},\mu^{13},1)$	544	$(\mu^{3},\mu^{23},1)$
545	$(\mu^{11},\mu^{13},1)$	546	$(\mu^{3},\mu^{11},1)$	547	$(\mu^{16},\mu^{18},1)$	548	$(\mu^{20},\mu^{8},1)$	549	$(\mu^{4},\mu^{15},1)$
550	$(\mu^{6},\mu^{21},1)$	551	$(\mu,\mu^{11},1)$	552	$(\mu^{16},\mu^{6},1)$	553	$(\mu^{14},\mu^{2},1)$	554	$(\mu^{15},\mu^{10},1)$
555	$(\mu^{19},\mu^{13},1)$	556	$(\mu^{3},\mu^{4},1)$	557	$(\mu^{9},\mu^{11},1)$	558	$(\mu^{16},\mu^{19},1)$	559	$(\mu^{23},\mu^{11},1)$
560	$(\mu^{16},\mu^{5},1)$	561	$(\mu^{18},\mu^{6},1)$	562	$(\mu^{14},\mu^{18},1)$	563	$(\mu^{20},\mu^{15},1)$	564	$(\mu^{6},\mu^{17},1)$
565	$(\mu^{7},\mu^{17},1)$	566	$(\mu^{7},\mu^{7},1)$	567	$(1,1,1)$	568	$(\mu^{2},1,1)$	569	$(\mu^{2},\mu^{11},1)$
570	$(\mu^{16},\mu,1)$	571	$(\mu^{10},\mu^{22},1)$	572	$(\mu^{17},\mu^{22},1)$	573	$(\mu^{17},\mu^{10},1)$	574	$(\mu^{19},\mu^{12},1)$
575	$(\mu^{23},1,0)$	576	$(0,\mu^{23},1)$	577	$(\mu^{11},\mu^{3},1)$	578	$(\mu^{22},\mu^{6},1)$	579	$(\mu^{14},\mu^{21},1)$
580	$(\mu,\mu^{20},1)$	581	$(\mu^{13},\mu^{3},1)$	582	$(\mu^{22},\mu^{19},1)$	583	$(\mu^{23},\mu^{6},1)$	584	$(\mu^{14},\mu^{3},1)$

Continued on next page

Table D.1 – continued from previous page (The points of PG$(2,5^2)$)

585	$(\mu^{22},\mu^{17},1)$	586	$(\mu^{7},\mu^{14},1)$	587	$(\mu^{5},\mu^{5},1)$	588	$(\mu^{18},1,1)$	589	$(\mu^{2},\mu^{6},1)$
590	$(\mu^{14},\mu^{23},1)$	591	$(\mu^{11},\mu^{6},1)$	592	$(\mu^{14},\mu^{22},1)$	593	$(\mu^{17},\mu^{12},1)$	594	$(\mu^{7},1,0)$
595	$(0,\mu^{7},1)$	596	$(1,\mu^{16},1)$	597	$(\mu^{12},\mu^{10},1)$	598	$(\mu^{19},0,1)$	599	$(\mu^{8},\mu^{9},1)$
600	$(\mu^{21},\mu^{17},1)$	601	$(\mu^{7},\mu^{6},1)$	602	$(\mu^{14},\mu^{14},1)$	603	$(\mu^{5},1,1)$	604	$(\mu^{2},\mu^{8},1)$
605	$(\mu^{4},\mu^{13},1)$	606	$(\mu^{3},\mu^{18},1)$	607	$(\mu^{20},\mu^{22},1)$	608	$(\mu^{17},\mu^{4},1)$	609	$(\mu^{9},\mu^{2},1)$
610	$(\mu^{15},\mu^{18},1)$	611	$(\mu^{20},\mu^{14},1)$	612	$(\mu^{5},\mu^{16},1)$	613	$(\mu^{12},\mu^{18},1)$	614	$(\mu^{20},0,1)$
615	$(\mu^{8},\mu^{19},1)$	616	$(\mu^{23},\mu^{19},1)$	617	$(\mu^{23},\mu^{12},1)$	618	$(\mu^{11},1,0)$	619	$(0,\mu^{11},1)$
620	$(\mu^{16},\mu^{8},1)$	621	$(\mu^{4},\mu^{16},1)$	622	$(\mu^{12},\mu^{3},1)$	623	$(\mu^{22},0,1)$	624	$(\mu^{8},\mu^{15},1)$
625	$(\mu^{6},\mu^{2},1)$	626	$(\mu^{15},\mu,1)$	627	$(\mu^{10},\mu^{4},1)$	628	$(\mu^{9},\mu^{14},1)$	629	$(\mu^{5},\mu^{8},1)$
630	$(\mu^{4},\mu^{10},1)$	631	$(\mu^{19},\mu^{10},1)$	632	$(\mu^{19},\mu^{20},1)$	633	$(\mu^{13},\mu^{14},1)$	634	$(\mu^{5},\mu^{2},1)$
635	$(\mu^{15},\mu^{21},1)$	636	$(\mu,\mu^{19},1)$	637	$(\mu^{23},\mu^{13},1)$	638	$(\mu^{3},\mu^{16},1)$	639	$(\mu^{12},\mu^{14},1)$
640	$(\mu^{5},0,1)$	641	$(\mu^{8},\mu^{14},1)$	642	$(\mu^{5},\mu,1)$	643	$(\mu^{10},\mu^{16},1)$	644	$(\mu^{12},\mu^{17},1)$
645	$(\mu^{7},0,1)$	646	$(\mu^{8},\mu^{8},1)$	647	$(\mu^{4},1,1)$	648	$(\mu^{2},\mu^{17},1)$	649	$(\mu^{7},\mu^{16},1)$
650	$(\mu^{12},\mu^{12},1)$								

Appendix E

Points of PG$(2, 7^2)$

The table of the points of PG$(2, 7^2)$ written in numeral and vector forms.

The points of PG$(2, 7^2)$ generated by $[(0, 1, 0), (0, 0, 1), (\mu^{13}, 1, 1)]$

0	$(1,0,0)$	1	$(0,1,0)$	2	$(0,0,1)$	3	$(\mu^{13},1,1)$	4	$(\mu^{45},\mu^{23},1)$
5	$(\mu^{31},\mu^{12},1)$	6	$(\mu^{47},\mu^{35},1)$	7	$(\mu^{35},\mu^{26},1)$	8	$(\mu,\mu^{14},1)$	9	$(\mu^{32},\mu^{24},1)$
10	$(\mu^{21},1,0)$	11	$(0,\mu^{21},1)$	12	$(\mu^{34},\mu^{21},1)$	13	$(\mu^{34},\mu^{36},1)$	14	$(\mu^{11},\mu^{13},1)$
15	$(\mu^{22},\mu^{40},1)$	16	$(\mu^{37},\mu^{10},1)$	17	$(\mu^{15},\mu^{22},1)$	18	$(\mu^{27},\mu^{3},1)$	19	$(\mu^{16},\mu^{9},1)$
20	$(\mu^{42},\mu^{37},1)$	21	$(\mu^{41},\mu^{21},1)$	22	$(\mu^{34},\mu,1)$	23	$(\mu^{8},\mu^{10},1)$	24	$(\mu^{15},\mu^{34},1)$
25	$(\mu^{46},\mu^{22},1)$	26	$(\mu^{27},\mu^{23},1)$	27	$(\mu^{31},\mu^{24},1)$	28	$(\mu^{12},1,0)$	29	$(0,\mu^{12},1)$
30	$(\mu^{47},\mu^{34},1)$	31	$(\mu^{46},\mu^{37},1)$	32	$(\mu^{41},\mu^{37},1)$	33	$(\mu^{41},\mu^{8},1)$	34	$(\mu^{29},\mu^{44},1)$
35	$(1,\mu^{12},1)$	36	$(\mu^{47},\mu^{2},1)$	37	$(\mu^{2},\mu^{41},1)$	38	$(\mu^{33},\mu^{31},1)$	39	$(\mu^{12},\mu^{21},1)$
40	$(\mu^{34},\mu^{35},1)$	41	$(\mu^{35},\mu^{37},1)$	42	$(\mu^{41},\mu^{6},1)$	43	$(\mu^{14},\mu^{29},1)$	44	$(\mu^{36},\mu^{4},1)$
45	$(\mu^{44},\mu^{33},1)$	46	$(\mu^{39},\mu^{39},1)$	47	$(\mu^{3},1,1)$	48	$(\mu^{45},\mu^{29},1)$	49	$(\mu^{36},\mu^{17},1)$
50	$(\mu^{43},\mu^{32},1)$	51	$(\mu^{21},\mu^{41},1)$	52	$(\mu^{33},\mu^{47},1)$	53	$(\mu^{9},\mu^{18},1)$	54	$(\mu^{40},\mu^{46},1)$
55	$(\mu^{4},\mu^{15},1)$	56	$(\mu^{24},\mu^{28},1)$	57	$(\mu^{38},0,1)$	58	$(\mu^{13},\mu^{36},1)$	59	$(\mu^{11},\mu^{37},1)$
60	$(\mu^{41},\mu^{11},1)$	61	$(\mu^{30},\mu^{45},1)$	62	$(\mu^{19},\mu^{9},1)$	63	$(\mu^{42},\mu^{25},1)$	64	$(\mu^{6},\mu^{34},1)$
65	$(\mu^{46},\mu^{32},1)$	66	$(\mu^{21},\mu^{17},1)$	67	$(\mu^{43},\mu^{9},1)$	68	$(\mu^{42},\mu^{14},1)$	69	$(\mu^{32},\mu^{12},1)$
70	$(\mu^{47},\mu^{26},1)$	71	$(\mu,\mu^{40},1)$	72	$(\mu^{37},\mu^{29},1)$	73	$(\mu^{36},\mu^{43},1)$	74	$(\mu^{28},\mu^{17},1)$
75	$(\mu^{43},\mu^{5},1)$	76	$(\mu^{23},\mu^{43},1)$	77	$(\mu^{28},\mu^{45},1)$	78	$(\mu^{19},\mu^{29},1)$	79	$(\mu^{36},\mu^{19},1)$
80	$(\mu^{17},\mu^{6},1)$	81	$(\mu^{14},\mu^{19},1)$	82	$(\mu^{17},\mu^{33},1)$	83	$(\mu^{39},\mu^{44},1)$	84	$(1,\mu^{45},1)$
85	$(\mu^{19},\mu^{22},1)$	86	$(\mu^{27},\mu^{10},1)$	87	$(\mu^{15},\mu^{8},1)$	88	$(\mu^{29},\mu^{5},1)$	89	$(\mu^{23},\mu^{35},1)$
90	$(\mu^{35},\mu^{4},1)$	91	$(\mu^{44},\mu^{9},1)$	92	$(\mu^{42},\mu^{42},1)$	93	$(\mu^{20},1,1)$	94	$(\mu^{45},\mu^{27},1)$
95	$(\mu^{7},\mu^{36},1)$	96	$(\mu^{11},\mu^{33},1)$	97	$(\mu^{39},\mu^{9},1)$	98	$(\mu^{42},\mu^{39},1)$	99	$(\mu^{3},\mu^{31},1)$
100	$(\mu^{12},\mu^{44},1)$	101	$(1,\mu,1)$	102	$(\mu^{8},\mu^{11},1)$	103	$(\mu^{30},\mu,1)$	104	$(\mu^{8},\mu^{46},1)$
105	$(\mu^{4},\mu^{23},1)$	106	$(\mu^{31},\mu^{35},1)$	107	$(\mu^{35},\mu^{23},1)$	108	$(\mu^{31},\mu^{44},1)$	109	$(1,\mu^{36},1)$
110	$(\mu^{11},\mu^{14},1)$	111	$(\mu^{32},\mu^{2},1)$	112	$(\mu^{2},\mu^{29},1)$	113	$(\mu^{36},\mu^{34},1)$	114	$(\mu^{46},\mu^{35},1)$
115	$(\mu^{35},\mu^{31},1)$	116	$(\mu^{12},\mu^{25},1)$	117	$(\mu^{6},\mu^{7},1)$	118	$(\mu^{26},\mu^{12},1)$	119	$(\mu^{47},\mu^{46},1)$
120	$(\mu^{4},\mu^{43},1)$	121	$(\mu^{28},\mu^{32},1)$	122	$(\mu^{21},\mu^{31},1)$	123	$(\mu^{12},\mu^{26},1)$	124	$(\mu,\mu^{2},1)$
125	$(\mu^{2},\mu^{42},1)$	126	$(\mu^{20},\mu^{18},1)$	127	$(\mu^{40},\mu^{22},1)$	128	$(\mu^{27},\mu^{38},1)$	129	$(\mu^{25},\mu^{18},1)$
130	$(\mu^{40},\mu^{34},1)$	131	$(\mu^{46},\mu^{9},1)$	132	$(\mu^{42},\mu^{38},1)$	133	$(\mu^{25},\mu^{5},1)$	134	$(\mu^{23},\mu^{17},1)$

Continued on next page

Table E.1 – continued from previous page (The points of $\mathrm{PG}(2, 7^2)$)

135	$(\mu^{43}, \mu^{12}, 1)$	136	$(\mu^{47}, \mu^{19}, 1)$	137	$(\mu^{17}, \mu^{8}, 1)$	138	$(\mu^{29}, \mu^{34}, 1)$	139	$(\mu^{46}, \mu^{10}, 1)$
140	$(\mu^{15}, \mu^{11}, 1)$	141	$(\mu^{30}, \mu^{6}, 1)$	142	$(\mu^{14}, \mu^{4}, 1)$	143	$(\mu^{44}, \mu^{12}, 1)$	144	$(\mu^{47}, \mu^{47}, 1)$
145	$(\mu^{9}, 1, 1)$	146	$(\mu^{45}, \mu^{3}, 1)$	147	$(\mu^{16}, \mu^{45}, 1)$	148	$(\mu^{19}, \mu^{14}, 1)$	149	$(\mu^{32}, \mu^{15}, 1)$
150	$(\mu^{24}, \mu^{3}, 1)$	151	$(\mu^{16}, 0, 1)$	152	$(\mu^{13}, \mu^{8}, 1)$	153	$(\mu^{29}, \mu^{7}, 1)$	154	$(\mu^{26}, \mu^{38}, 1)$
155	$(\mu^{25}, \mu^{24}, 1)$	156	$(\mu^{6}, 1, 0)$	157	$(0, \mu^{6}, 1)$	158	$(\mu^{14}, \mu, 1)$	159	$(\mu^{8}, \mu^{24}, 1)$
160	$(\mu^{29}, 1, 0)$	161	$(0, \mu^{29}, 1)$	162	$(\mu^{36}, \mu^{23}, 1)$	163	$(\mu^{31}, \mu^{20}, 1)$	164	$(\mu^{18}, \mu^{6}, 1)$
165	$(\mu^{14}, \mu^{22}, 1)$	166	$(\mu^{27}, \mu^{43}, 1)$	167	$(\mu^{28}, \mu^{21}, 1)$	168	$(\mu^{34}, \mu^{44}, 1)$	169	$(1, \mu^{2}, 1)$
170	$(\mu^{2}, \mu^{5}, 1)$	171	$(\mu^{23}, \mu^{21}, 1)$	172	$(\mu^{34}, \mu^{3}, 1)$	173	$(\mu^{16}, \mu^{18}, 1)$	174	$(\mu^{40}, \mu^{35}, 1)$
175	$(\mu^{35}, \mu^{46}, 1)$	176	$(\mu^{4}, \mu^{17}, 1)$	177	$(\mu^{43}, \mu^{47}, 1)$	178	$(\mu^{9}, \mu^{29}, 1)$	179	$(\mu^{36}, \mu^{42}, 1)$
180	$(\mu^{20}, \mu^{9}, 1)$	181	$(\mu^{42}, \mu^{24}, 1)$	182	$(\mu^{20}, 1, 0)$	183	$(0, \mu^{20}, 1)$	184	$(\mu^{18}, \mu^{5}, 1)$
185	$(\mu^{23}, \mu^{31}, 1)$	186	$(\mu^{12}, \mu^{29}, 1)$	187	$(\mu^{36}, \mu^{37}, 1)$	188	$(\mu^{41}, \mu^{30}, 1)$	189	$(\mu^{10}, \mu^{25}, 1)$
190	$(\mu^{6}, \mu^{39}, 1)$	191	$(\mu^{3}, \mu^{37}, 1)$	192	$(\mu^{42}, \mu^{25}, 1)$	193	$(\mu^{6}, \mu^{21}, 1)$	194	$(\mu^{34}, \mu^{20}, 1)$
195	$(\mu^{18}, \mu^{20}, 1)$	196	$(\mu^{18}, \mu^{26}, 1)$	197	$(\mu, \mu^{9}, 1)$	198	$(\mu^{42}, \mu^{34}, 1)$	199	$(\mu^{46}, \mu^{26}, 1)$
200	$(\mu, \mu^{45}, 1)$	201	$(\mu^{19}, \mu^{11}, 1)$	202	$(\mu^{30}, \mu^{13}, 1)$	203	$(\mu^{22}, \mu^{12}, 1)$	204	$(\mu^{47}, \mu^{20}, 1)$
205	$(\mu^{18}, \mu^{9}, 1)$	206	$(\mu^{42}, \mu^{2}, 1)$	207	$(\mu^{2}, \mu^{30}, 1)$	208	$(\mu^{10}, \mu^{8}, 1)$	209	$(\mu^{29}, \mu^{14}, 1)$
210	$(\mu^{32}, \mu^{44}, 1)$	211	$(1, \mu^{27}, 1)$	212	$(\mu^{7}, \mu^{10}, 1)$	213	$(\mu^{15}, \mu^{37}, 1)$	214	$(\mu^{41}, \mu^{17}, 1)$
215	$(\mu^{43}, \mu^{10}, 1)$	216	$(\mu^{15}, \mu^{35}, 1)$	217	$(\mu^{35}, \mu^{11}, 1)$	218	$(\mu^{30}, \mu^{43}, 1)$	219	$(\mu^{28}, \mu^{18}, 1)$
220	$(\mu^{40}, \mu^{2}, 1)$	221	$(\mu^{2}, \mu^{13}, 1)$	222	$(\mu^{22}, \mu^{20}, 1)$	223	$(\mu^{18}, \mu^{39}, 1)$	224	$(\mu^{3}, \mu^{11}, 1)$
225	$(\mu^{30}, \mu^{14}, 1)$	226	$(\mu^{32}, \mu^{22}, 1)$	227	$(\mu^{27}, \mu^{6}, 1)$	228	$(\mu^{14}, \mu^{7}, 1)$	229	$(\mu^{26}, \mu^{42}, 1)$
230	$(\mu^{20}, \mu^{19}, 1)$	231	$(\mu^{17}, \mu^{47}, 1)$	232	$(\mu^{9}, \mu^{14}, 1)$	233	$(\mu^{32}, \mu^{38}, 1)$	234	$(\mu^{25}, \mu^{4}, 1)$
235	$(\mu^{44}, \mu^{38}, 1)$	236	$(\mu^{25}, \mu^{25}, 1)$	237	$(\mu^{6}, 1, 1)$	238	$(\mu^{45}, \mu^{31}, 1)$	239	$(\mu^{12}, \mu^{41}, 1)$
240	$(\mu^{33}, \mu^{34}, 1)$	241	$(\mu^{46}, \mu^{7}, 1)$	242	$(\mu^{26}, \mu^{22}, 1)$	243	$(\mu^{27}, \mu^{26}, 1)$	244	$(\mu, \mu^{42}, 1)$
245	$(\mu^{20}, \mu^{12}, 1)$	246	$(\mu^{47}, \mu^{29}, 1)$	247	$(\mu^{36}, \mu^{27}, 1)$	248	$(\mu^{7}, \mu^{44}, 1)$	249	$(1, \mu^{22}, 1)$
250	$(\mu^{27}, \mu^{30}, 1)$	251	$(\mu^{10}, \mu^{3}, 1)$	252	$(\mu^{16}, \mu, 1)$	253	$(\mu^{8}, \mu^{3}, 1)$	254	$(\mu^{16}, \mu^{35}, 1)$
255	$(\mu^{35}, \mu^{30}, 1)$	256	$(\mu^{10}, \mu^{23}, 1)$	257	$(\mu^{31}, \mu^{16}, 1)$	258	$(\mu^{5}, \mu^{41}, 1)$	259	$(\mu^{33}, \mu^{10}, 1)$
260	$(\mu^{15}, \mu^{24}, 1)$	261	$(\mu^{24}, 1, 0)$	262	$0, \mu^{24}, 1)$	263	$(\mu^{13}, 1, 0)$	264	$(0, \mu^{13}, 1)$
265	$(\mu^{22}, \mu^{9}, 1)$	266	$(\mu^{42}, \mu^{15}, 1)$	267	$(\mu^{24}, \mu^{4}, 1)$	268	$(\mu^{44}, 0, 1)$	269	$(\mu^{13}, \mu^{13}, 1)$
270	$(\mu^{22}, 1, 1)$	271	$(\mu^{45}, \mu^{18}, 1)$	272	$(\mu^{40}, \mu^{21}, 1)$	273	$(\mu^{34}, \mu^{45}, 1)$	274	$(\mu^{19}, \mu^{21}, 1)$
275	$(\mu^{34}, \mu^{17}, 1)$	276	$(\mu^{43}, \mu^{45}, 1)$	277	$(\mu^{19}, \mu^{39}, 1)$	278	$(\mu^{3}, \mu^{34}, 1)$	279	$(\mu^{46}, \mu^{30}, 1)$
280	$(\mu^{10}, \mu^{6}, 1)$	281	$(\mu^{14}, \mu^{47}, 1)$	282	$(\mu^{9}, \mu^{25}, 1)$	283	$(\mu^{6}, \mu^{12}, 1)$	284	$(\mu^{47}, \mu^{33}, 1)$
285	$(\mu^{39}, \mu^{30}, 1)$	286	$(\mu^{10}, \mu^{7}, 1)$	287	$(\mu^{26}, \mu^{11}, 1)$	288	$(\mu^{30}, \mu^{29}, 1)$	289	$(\mu^{36}, \mu^{26}, 1)$
290	$(\mu, \mu^{38}, 1)$	291	$(\mu^{25}, \mu^{17}, 1)$	292	$(\mu^{43}, \mu^{37}, 1)$	293	$(\mu^{41}, \mu^{13}, 1)$	294	$(\mu^{22}, \mu^{37}, 1)$
295	$(\mu^{41}, \mu^{14}, 1)$	296	$(\mu^{32}, \mu^{47}, 1)$	297	$(\mu^{9}, \mu^{36}, 1)$	298	$(\mu^{11}, \mu^{17}, 1)$	299	$(\mu^{43}, \mu^{13}, 1)$
300	$(\mu^{22}, \mu^{42}, 1)$	301	$(\mu^{20}, \mu^{41}, 1)$	302	$(\mu^{33}, \mu^{15}, 1)$	303	$(\mu^{24}, \mu^{33}, 1)$	304	$(\mu^{39}, 0, 1)$
305	$(\mu^{13}, \mu^{10}, 1)$	306	$(\mu^{15}, \mu^{41}, 1)$	307	$(\mu^{33}, \mu^{9}, 1)$	308	$(\mu^{42}, \mu^{3}, 1)$	309	$(\mu^{16}, \mu^{44}, 1)$
310	$(1, \mu^{43}, 1)$	311	$(\mu^{28}, \mu^{31}, 1)$	312	$(\mu^{12}, \mu^{22}, 1)$	313	$(\mu^{27}, \mu^{28}, 1)$	314	$(\mu^{38}, \mu^{31}, 1)$
315	$(\mu^{12}, \mu^{35}, 1)$	316	$(\mu^{35}, \mu^{36}, 1)$	317	$(\mu^{11}, \mu^{24}, 1)$	318	$(\mu^{30}, 1, 0)$	319	$(0, \mu^{30}, 1)$
320	$(\mu^{10}, \mu^{45}, 1)$	321	$(\mu^{19}, \mu^{4}, 1)$	322	$(\mu^{44}, \mu^{27}, 1)$	323	$(\mu^{7}, \mu^{7}, 1)$	324	$(\mu^{26}, 1, 1)$
325	$(\mu^{45}, \mu^{44}, 1)$	326	$(1, \mu^{29}, 1)$	327	$(\mu^{36}, \mu^{39}, 1)$	328	$(\mu^{3}, \mu^{40}, 1)$	329	$(\mu^{37}, \mu^{21}, 1)$
330	$(\mu^{34}, \mu^{41}, 1)$	331	$(\mu^{33}, \mu^{35}, 1)$	332	$(\mu^{35}, \mu^{44}, 1)$	333	$(1, \mu^{13}, 1)$	334	$(\mu^{22}, \mu^{25}, 1)$
335	$(\mu^{6}, \mu^{27}, 1)$	336	$(\mu^{7}, \mu^{41}, 1)$	337	$(\mu^{33}, \mu^{7}, 1)$	338	$(\mu^{26}, \mu^{35}, 1)$	339	$(\mu^{35}, \mu^{34}, 1)$
340	$(\mu^{46}, \mu^{11}, 1)$	341	$(\mu^{30}, \mu^{26}, 1)$	342	$(\mu, \mu^{39}, 1)$	343	$(\mu^{3}, \mu^{43}, 1)$	344	$(\mu^{28}, \mu^{12}, 1)$
345	$(\mu^{47}, \mu^{9}, 1)$	346	$(\mu^{42}, \mu^{33}, 1)$	347	$(\mu^{39}, \mu^{19}, 1)$	348	$(\mu^{17}, \mu^{14}, 1)$	349	$(\mu^{32}, \mu^{37}, 1)$
350	$(\mu^{41}, \mu^{20}, 1)$	351	$(\mu^{18}, \mu^{33}, 1)$	352	$(\mu^{39}, \mu^{47}, 1)$	353	$(\mu^{9}, \mu^{6}, 1)$	354	$(\mu^{14}, \mu^{20}, 1)$
355	$(\mu^{18}, \mu^{34}, 1)$	356	$(\mu^{46}, \mu^{6}, 1)$	357	$(\mu^{14}, \mu^{10}, 1)$	358	$(\mu^{15}, \mu^{31}, 1)$	359	$(\mu^{12}, \mu^{36}, 1)$

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Table E.1 – continued from previous page (The points of PG$(2,7^2)$)

360	$(\mu^{11},\mu^{12},1)$	361	$(\mu^{47},\mu^{17},1)$	362	$(\mu^{43},\mu^{34},1)$	363	$(\mu^{46},\mu^{18},1)$	364	$(\mu^{40},\mu^{36},1)$
365	$(\mu^{11},\mu^{22},1)$	366	$(\mu^{27},\mu^{45},1)$	367	$(\mu^{19},\mu^{12},1)$	368	$(\mu^{47},\mu^{30},1)$	369	$(\mu^{10},\mu,1)$
370	$(\mu^{8},\mu^{41},1)$	371	$(\mu^{33},\mu^{4},1)$	372	$(\mu^{44},\mu^{5},1)$	373	$(\mu^{23},\mu^{23},1)$	374	$(\mu^{31},1,1)$
375	$(\mu^{45},\mu^{33},1)$	376	$(\mu^{39},\mu^{20},1)$	377	$(\mu^{18},\mu^{15},1)$	378	$(\mu^{24},\mu^{32},1)$	379	$(\mu^{21},0,1)$
380	$(\mu^{13},\mu^{27},1)$	381	$(\mu^{7},\mu^{33},1)$	382	$(\mu^{39},\mu^{13},1)$	383	$(\mu^{22},\mu^{19},1)$	384	$(\mu^{17},\mu^{38},1)$
385	$(\mu^{25},\mu^{30},1)$	386	$(\mu^{10},\mu^{4},1)$	387	$(\mu^{44},\mu^{29},1)$	388	$(\mu^{36},\mu^{36},1)$	389	$(\mu^{11},1,1)$
390	$(\mu^{45},\mu^{15},1)$	391	$(\mu^{24},\mu^{5},1)$	392	$(\mu^{23},0,1)$	393	$(\mu^{13},\mu^{30},1)$	394	$(\mu^{10},\mu^{36},1)$
395	$(\mu^{11},\mu^{44},1)$	396	$(1,\mu^{18},1)$	397	$(\mu^{40},\mu^{43},1)$	398	$(\mu^{28},\mu^{39},1)$	399	$(\mu^{3},\mu^{13},1)$
400	$(\mu^{22},\mu^{6},1)$	401	$(\mu^{14},\mu^{35},1)$	402	$(\mu^{35},\mu^{3},1)$	403	$(\mu^{16},\mu^{29},1)$	404	$(\mu^{36},\mu^{31},1)$
405	$(\mu^{12},\mu,1)$	406	$(\mu^{8},\mu^{9},1)$	407	$(\mu^{42},\mu^{13},1)$	408	$(\mu^{22},\mu^{2},1)$	409	$(\mu^{2},\mu^{23},1)$
410	$(\mu^{31},\mu^{29},1)$	411	$(\mu^{36},\mu^{24},1)$	412	$(\mu^{11},1,0)$	413	$(0,\mu^{11},1)$	414	$(\mu^{30},\mu^{17},1)$
415	$(\mu^{43},\mu^{33},1)$	416	$(\mu^{39},\mu^{11},1)$	417	$(\mu^{30},\mu^{27},1)$	418	$(\mu^{7},\mu^{45},1)$	419	$(\mu^{19},\mu^{41},1)$
420	$(\mu^{33},\mu^{16},1)$	421	$(\mu^{5},\mu^{14},1)$	422	$(\mu^{32},\mu^{9},1)$	423	$(\mu^{42},\mu^{21},1)$	424	$(\mu^{34},\mu^{14},1)$
425	$(\mu^{32},\mu^{34},1)$	426	$(\mu^{46},\mu^{25},1)$	427	$(\mu^{6},\mu^{2},1)$	428	$(\mu^{2},\mu^{36},1)$	429	$(\mu^{11},\mu^{9},1)$
430	$(\mu^{42},\mu^{12},1)$	431	$(\mu^{47},\mu^{27},1)$	432	$(\mu^{7},\mu^{46},1)$	433	$(\mu^{4},\mu^{26},1)$	434	$(\mu,\mu^{5},1)$
435	$(\mu^{23},\mu^{15},1)$	436	$(\mu^{24},\mu^{41},1)$	437	$(\mu^{33},0,1)$	438	$(\mu^{13},\mu^{22},1)$	439	$(\mu^{27},\mu^{5},1)$
440	$(\mu^{23},\mu^{16},1)$	441	$(\mu^{5},\mu^{22},1)$	442	$(\mu^{27},\mu^{4},1)$	443	$(\mu^{44},\mu^{37},1)$	444	$(\mu^{41},\mu^{41},1)$
445	$(\mu^{33},1,1)$	446	$(\mu^{45},\mu^{6},1)$	447	$(\mu^{14},\mu^{43},1)$	448	$(\mu^{28},\mu^{44},1)$	449	$(1,\mu^{10},1)$
450	$(\mu^{15},\mu^{18},1)$	451	$(\mu^{40},\mu^{16},1)$	452	$(\mu^{5},\mu^{16},1)$	453	$(\mu^{5},\mu^{30},1)$	454	$(\mu^{10},\mu^{35},1)$
455	$(\mu^{35},\mu^{20},1)$	456	$(\mu^{18},\mu^{31},1)$	457	$(\mu^{12},\mu^{20},1)$	458	$(\mu^{18},\mu^{19},1)$	459	$(\mu^{17},\mu^{25},1)$
460	$(\mu^{6},\mu^{11},1)$	461	$(\mu^{30},\mu^{16},1)$	462	$(\mu^{5},\mu^{43},1)$	463	$(\mu^{28},\mu^{5},1)$	464	$(\mu^{23},\mu^{33},1)$
465	$(\mu^{39},\mu^{8},1)$	466	$(\mu^{29},\mu^{26},1)$	467	$(\mu,\mu^{13},1)$	468	$(\mu^{22},\mu^{14},1)$	469	$(\mu^{32},\mu^{5},1)$
470	$(\mu^{23},\mu^{2},1)$	471	$(\mu^{2},\mu^{19},1)$	472	$(\mu^{17},\mu^{15},1)$	473	$(\mu^{24},\mu^{29},1)$	474	$(\mu^{36},0,1)$
475	$(\mu^{13},\mu^{2},1)$	476	$(\mu^{2},\mu^{28},1)$	477	$(\mu^{38},\mu^{36},1)$	478	$(\mu^{11},\mu^{34},1)$	479	$(\mu^{46},\mu^{16},1)$
480	$(\mu^{5},\mu,1)$	481	$(\mu^{8},\mu^{33},1)$	482	$(\mu^{39},\mu^{10},1)$	483	$(\mu^{15},\mu^{12},1)$	484	$(\mu^{47},\mu^{23},1)$
485	$(\mu^{31},\mu^{22},1)$	486	$(\mu^{27},\mu^{15},1)$	487	$(\mu^{24},\mu^{17},1)$	488	$(\mu^{43},0,1)$	489	$(\mu^{13},\mu^{33},1)$
490	$(\mu^{39},\mu^{17},1)$	491	$(\mu^{43},\mu^{40},1)$	492	$(\mu^{37},\mu^{9},1)$	493	$(\mu^{42},\mu,1)$	494	$(\mu^{8},\mu^{36},1)$
495	$(\mu^{11},\mu^{30},1)$	496	$(\mu^{10},\mu^{28},1)$	497	$(\mu^{38},\mu^{23},1)$	498	$(\mu^{31},\mu^{6},1)$	499	$(\mu^{14},\mu^{2},1)$
500	$(\mu^{2},\mu^{18},1)$	501	$(\mu^{40},\mu^{38},1)$	502	$(\mu^{25},\mu^{36},1)$	503	$(\mu^{11},\mu^{5},1)$	504	$(\mu^{23},\mu^{41},1)$
505	$(\mu^{33},\mu^{2},1)$	506	$(\mu^{2},\mu^{11},1)$	507	$(\mu^{30},\mu^{28},1)$	508	$(\mu^{38},\mu^{28},1)$	509	$(\mu^{38},\mu^{13},1)$
510	$(\mu^{22},\mu^{45},1)$	511	$(\mu^{19},\mu^{40},1)$	512	$(\mu^{37},\mu^{20},1)$	513	$(\mu^{18},\mu^{25},1)$	514	$(\mu^{6},\mu^{14},1)$
515	$(\mu^{32},\mu^{18},1)$	516	$(\mu^{40},\mu^{19},1)$	517	$(\mu^{17},\mu^{28},1)$	518	$(\mu^{38},\mu^{43},1)$	519	$(\mu^{28},\mu^{3},1)$
520	$(\mu^{16},\mu^{26},1)$	521	$(\mu,\mu^{44},1)$	522	$(1,\mu^{40},1)$	523	$(\mu^{37},\mu^{40},1)$	524	$(\mu^{37},\mu^{44},1)$
525	$(1,\mu^{7},1)$	526	$(\mu^{26},\mu^{29},1)$	527	$(\mu^{36},\mu^{35},1)$	528	$(\mu^{35},\mu^{24},1)$	529	$(\mu^{35},1,0)$
530	$(0,\mu^{35},1)$	531	$(\mu^{35},\mu^{22},1)$	532	$(\mu^{27},\mu^{40},1)$	533	$(\mu^{37},\mu^{30},1)$	534	$(\mu^{10},\mu^{17},1)$
535	$(\mu^{43},\mu^{28},1)$	536	$(\mu^{38},\mu^{10},1)$	537	$(\mu^{15},\mu^{38},1)$	538	$(\mu^{25},\mu,1)$	539	$(\mu^{8},\mu^{2},1)$
540	$(\mu^{2},\mu^{21},1)$	541	$(\mu^{34},\mu^{32},1)$	542	$(\mu^{21},\mu^{23},1)$	543	$(\mu^{31},\mu^{45},1)$	544	$(\mu^{19},\mu^{7},1)$
545	$(\mu^{26},\mu^{9},1)$	546	$(\mu^{42},\mu^{41},1)$	547	$(\mu^{33},\mu^{13},1)$	548	$(\mu^{22},\mu^{31},1)$	549	$(\mu^{12},\mu^{33},1)$
550	$(\mu^{39},\mu^{40},1)$	551	$(\mu^{37},\mu^{34},1)$	552	$(\mu^{46},\mu^{5},1)$	553	$(\mu^{23},\mu^{19},1)$	554	$(\mu^{17},\mu^{34},1)$
555	$(\mu^{46},\mu^{3},1)$	556	$(\mu^{16},\mu^{12},1)$	557	$(\mu^{47},\mu^{42},1)$	558	$(\mu^{20},\mu^{11},1)$	559	$(\mu^{30},\mu^{12},1)$
560	$(\mu^{47},\mu^{37},1)$	561	$(\mu^{41},\mu^{32},1)$	562	$(\mu^{21},\mu^{36},1)$	563	$(\mu^{11},\mu^{25},1)$	564	$(\mu^{6},\mu^{24},1)$
565	$(\mu^{14},1,0)$	566	$(0,\mu^{14},1)$	567	$(\mu^{32},\mu^{19},1)$	568	$(\mu^{17},\mu^{44},1)$	569	$(1,\mu^{5},1)$
570	$(\mu^{23},\mu^{26},1)$	571	$(\mu,\mu^{18},1)$	572	$(\mu^{40},\mu^{32},1)$	573	$(\mu^{21},\mu^{32},1)$	574	$(\mu^{21},\mu^{35},1)$
575	$(\mu^{35},\mu,1)$	576	$(\mu^{8},\mu^{21},1)$	577	$(\mu^{34},\mu^{5},1)$	578	$(\mu^{23},\mu^{25},1)$	579	$(\mu^{6},\mu^{23},1)$
580	$(\mu^{31},\mu^{17},1)$	581	$(\mu^{43},\mu^{31},1)$	582	$(\mu^{12},\mu^{32},1)$	583	$(\mu^{21},\mu^{22},1)$	584	$(\mu^{27},\mu^{41},1)$

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Table E.1 – continued from previous page (The points of PG$(2,7^2)$)

585	$(\mu^{33},\mu^{26},1)$	586	$(\mu,\mu^{10},1)$	587	$(\mu^{15},\mu^{7},1)$	588	$(\mu^{26},\mu^{2},1)$	589	$(\mu^{2},\mu,1)$
590	$(\mu^{8},\mu^{6},1)$	591	$(\mu^{14},\mu^{33},1)$	592	$(\mu^{39},\mu^{7},1)$	593	$(\mu^{26},\mu^{23},1)$	594	$(\mu^{31},\mu^{30},1)$
595	$(\mu^{10},\mu^{46},1)$	596	$(\mu^{4},\mu^{37},1)$	597	$(\mu^{41},\mu^{45},1)$	598	$(\mu^{19},\mu^{34},1)$	599	$(\mu^{46},\mu^{29},1)$
600	$(\mu^{36},\mu^{32},1)$	601	$(\mu^{21},\mu^{10},1)$	602	$(\mu^{15},\mu^{29},1)$	603	$(\mu^{36},\mu^{12},1)$	604	$(\mu^{47},\mu^{36},1)$
605	$(\mu^{11},\mu^{2},1)$	606	$(\mu^{2},\mu^{20},1)$	607	$(\mu^{18},\mu^{16},1)$	608	$(\mu^{5},\mu^{13},1)$	609	$(\mu^{22},\mu^{47},1)$
610	$(\mu^{9},\mu^{30},1)$	611	$(\mu^{10},\mu^{16},1)$	612	$(\mu^{5},\mu^{38},1)$	613	$(\mu^{25},\mu^{2},1)$	614	$(\mu^{2},\mu^{44},1)$
615	$(1,\mu^{46},1)$	616	$(\mu^{4},\mu^{7},1)$	617	$(\mu^{26},\mu^{30},1)$	618	$(\mu^{10},\mu^{9},1)$	619	$(\mu^{42},\mu^{27},1)$
620	$(\mu^{7},\mu^{35},1)$	621	$(\mu^{35},\mu^{9},1)$	622	$(\mu^{42},\mu^{7},1)$	623	$(\mu^{26},\mu^{6},1)$	624	$(\mu^{14},\mu^{13},1)$
625	$(\mu^{22},\mu^{38},1)$	626	$(\mu^{25},\mu^{46},1)$	627	$(\mu^{4},\mu^{46},1)$	628	$(\mu^{4},\mu^{8},1)$	629	$(\mu^{29},\mu^{33},1)$
630	$(\mu^{39},\mu^{3},1)$	631	$(\mu^{16},\mu^{13},1)$	632	$(\mu^{22},\mu^{17},1)$	633	$(\mu^{43},\mu^{16},1)$	634	$(\mu^{5},\mu^{25},1)$
635	$(\mu^{6},\mu^{31},1)$	636	$(\mu^{12},\mu^{46},1)$	637	$(\mu^{4},\mu^{5},1)$	638	$(\mu^{23},\mu^{27},1)$	639	$(\mu^{7},\mu^{24},1)$
640	$(\mu^{26},1,0)$	641	$(0,\mu^{26},1)$	642	$(\mu,\mu^{36},1)$	643	$(\mu^{11},\mu^{3},1)$	644	$(\mu^{16},\mu^{34},1)$
645	$(\mu^{46},\mu^{41},1)$	646	$(\mu^{33},\mu^{29},1)$	647	$(\mu^{36},\mu^{45},1)$	648	$(\mu^{19},\mu^{8},1)$	649	$(\mu^{29},\mu^{12},1)$
650	$(\mu^{47},\mu^{11},1)$	651	$(\mu^{30},\mu^{21},1)$	652	$(\mu^{34},\mu^{24},1)$	653	$(\mu^{46},1,0)$	654	$(0,\mu^{46},1)$
655	$(\mu^{4},\mu^{39},1)$	656	$(\mu^{3},\mu^{7},1)$	657	$(\mu^{26},\mu^{10},1)$	658	$(\mu^{15},\mu^{14},1)$	659	$(\mu^{32},\mu^{8},1)$
660	$(\mu^{29},\mu^{8},1)$	661	$(\mu^{29},\mu^{41},1)$	662	$(\mu^{33},\mu^{45},1)$	663	$(\mu^{19},\mu^{28},1)$	664	$(\mu^{38},\mu^{21},1)$
665	$(\mu^{34},\mu^{9},1)$	666	$(\mu^{42},\mu^{44},1)$	667	$(1,\mu^{28},1)$	668	$(\mu^{38},\mu^{41},1)$	669	$(\mu^{33},\mu^{8},1)$
670	$(\mu^{29},\mu^{38},1)$	671	$(\mu^{25},\mu^{37},1)$	672	$(\mu^{41},\mu^{35},1)$	673	$(\mu^{35},\mu^{2},1)$	674	$(\mu^{2},\mu^{15},1)$
675	$(\mu^{24},\mu^{22},1)$	676	$(\mu^{27},0,1)$	677	$(\mu^{13},\mu^{6},1)$	678	$(\mu^{14},\mu^{40},1)$	679	$(\mu^{37},\mu^{5},1)$
680	$(\mu^{23},\mu^{30},1)$	681	$(\mu^{10},\mu^{27},1)$	682	$(\mu^{7},\mu^{40},1)$	683	$(\mu^{37},\mu^{11},1)$	684	$(\mu^{30},\mu^{37},1)$
685	$(\mu^{41},\mu^{31},1)$	686	$(\mu^{12},\mu^{27},1)$	687	$(\mu^{7},\mu^{8},1)$	688	$(\mu^{29},\mu^{3},1)$	689	$(\mu^{16},\mu^{28},1)$
690	$(\mu^{38},\mu^{33},1)$	691	$(\mu^{39},\mu^{14},1)$	692	$(\mu^{32},\mu^{29},1)$	693	$(\mu^{36},\mu^{15},1)$	694	$(\mu^{24},\mu^{13},1)$
695	$(\mu^{22},0,1)$	696	$(\mu^{13},\mu^{34},1)$	697	$(\mu^{46},\mu^{24},1)$	698	$(\mu^{4},1,0)$	699	$(0,\mu^{4},1)$
700	$(\mu^{44},\mu^{31},1)$	701	$(\mu^{12},\mu^{12},1)$	702	$(\mu^{47},1,1)$	703	$(\mu^{45},\mu^{36},1)$	704	$(\mu^{11},\mu^{40},1)$
705	$(\mu^{37},\mu^{7},1)$	706	$(\mu^{26},\mu^{33},1)$	707	$(\mu^{39},\mu^{38},1)$	708	$(\mu^{25},\mu^{22},1)$	709	$(\mu^{27},\mu^{21},1)$
710	$(\mu^{34},\mu^{27},1)$	711	$(\mu^{7},\mu^{9},1)$	712	$(\mu^{42},\mu^{16},1)$	713	$(\mu^{5},\mu^{33},1)$	714	$(\mu^{39},\mu^{16},1)$
715	$(\mu^{5},\mu^{2},1)$	716	$(\mu^{2},\mu^{27},1)$	717	$(\mu^{7},\mu^{5},1)$	718	$(\mu^{23},\mu^{45},1)$	719	$(\mu^{19},\mu^{36},1)$
720	$(\mu^{11},\mu^{42},1)$	721	$(\mu^{20},\mu^{38},1)$	722	$(\mu^{25},\mu^{7},1)$	723	$(\mu^{26},\mu^{20},1)$	724	$(\mu^{18},\mu^{17},1)$
725	$(\mu^{43},\mu^{3},1)$	726	$(\mu^{16},\mu^{36},1)$	727	$(\mu^{11},\mu^{6},1)$	728	$(\mu^{14},\mu^{32},1)$	729	$(\mu^{21},\mu^{37},1)$
730	$(\mu^{41},\mu^{7},1)$	731	$(\mu^{26},\mu^{41},1)$	732	$(\mu^{33},\mu^{32},1)$	733	$(\mu^{21},\mu^{30},1)$	734	$(\mu^{10},\mu^{24},1)$
735	$(\mu^{15},1,0)$	736	$(0,\mu^{15},1)$	737	$(\mu^{24},\mu^{11},1)$	738	$(\mu^{30},0,1)$	739	$(\mu^{13},\mu^{3},1)$
740	$(\mu^{16},\mu^{42},1)$	741	$(\mu^{20},\mu^{15},1)$	742	$(\mu^{24},\mu^{6},1)$	743	$(\mu^{14},0,1)$	744	$(\mu^{13},\mu^{29},1)$
745	$(\mu^{36},\mu^{14},1)$	746	$(\mu^{32},\mu^{21},1)$	747	$(\mu^{34},\mu^{13},1)$	748	$(\mu^{22},\mu^{24},1)$	749	$(\mu^{27},1,0)$
750	$(0,\mu^{27},1)$	751	$(\mu^{7},\mu^{42},1)$	752	$(\mu^{20},\mu^{42},1)$	753	$(\mu^{20},\mu^{2},1)$	754	$(\mu^{2},\mu^{32},1)$
755	$(\mu^{21},\mu^{19},1)$	756	$(\mu^{17},\mu^{31},1)$	757	$(\mu^{12},\mu^{17},1)$	758	$(\mu^{43},\mu^{44},1)$	759	$(1,\mu^{20},1)$
760	$(\mu^{18},\mu^{21},1)$	761	$(\mu^{34},\mu^{42},1)$	762	$(\mu^{20},\mu^{22},1)$	763	$(\mu^{27},\mu^{9},1)$	764	$(\mu^{42},\mu^{35},1)$
765	$(\mu^{35},\mu^{15},1)$	766	$(\mu^{24},\mu^{37},1)$	767	$(\mu^{41},0,1)$	768	$(\mu^{13},\mu^{28},1)$	769	$(\mu^{38},\mu^{16},1)$
770	$(\mu^{5},\mu^{28},1)$	771	$(\mu^{38},\mu^{15},1)$	772	$(\mu^{24},\mu^{47},1)$	773	$(\mu^{9},0,1)$	774	$(\mu^{13},\mu^{19},1)$
775	$(\mu^{17},\mu^{43},1)$	776	$(\mu^{28},\mu^{33},1)$	777	$(\mu^{39},\mu,1)$	778	$(\mu^{8},\mu^{5},1)$	779	$(\mu^{23},\mu^{42},1)$
780	$(\mu^{20},\mu^{37},1)$	781	$(\mu^{41},\mu^{23},1)$	782	$(\mu^{31},\mu^{46},1)$	783	$(\mu^{4},\mu^{40},1)$	784	$(\mu^{37},\mu^{41},1)$
785	$(\mu^{33},\mu^{40},1)$	786	$(\mu^{37},\mu^{46},1)$	787	$(\mu^{4},\mu^{11},1)$	788	$(\mu^{30},\mu^{34},1)$	789	$(\mu^{46},\mu^{36},1)$
790	$(\mu^{11},\mu^{7},1)$	791	$(\mu^{26},\mu^{44},1)$	792	$(1,\mu^{47},1)$	793	$(\mu^{9},\mu^{12},1)$	794	$(\mu^{47},\mu^{5},1)$
795	$(\mu^{23},\mu^{14},1)$	796	$(\mu^{32},\mu,1)$	797	$(\mu^{8},\mu^{35},1)$	798	$(\mu^{35},\mu^{6},1)$	799	$(\mu^{14},\mu^{27},1)$
800	$(\mu^{7},\mu^{23},1)$	801	$(\mu^{31},\mu^{5},1)$	802	$(\mu^{23},\mu^{11},1)$	803	$(\mu^{30},\mu^{47},1)$	804	$(\mu^{9},\mu^{47},1)$
805	$(\mu^{9},\mu^{15},1)$	806	$(\mu^{24},\mu^{30},1)$	807	$(\mu^{10},0,1)$	808	$(\mu^{13},\mu^{46},1)$	809	$(\mu^{4},\mu^{30},1)$

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Table E.1 – continued from previous page (The points of $\mathrm{PG}(2,7^2)$)

810	$(\mu^{10},\mu^{14},1)$	811	$(\mu^{32},\mu^{17},1)$	812	$(\mu^{43},\mu^{22},1)$	813	$(\mu^{27},\mu^{47},1)$	814	$(\mu^{9},\mu^{2},1)$
815	$(\mu^{2},\mu^{8},1)$	816	$(\mu^{29},\mu^{27},1)$	817	$(\mu^{7},\mu^{19},1)$	818	$(\mu^{17},\mu^{39},1)$	819	$(\mu^{3},\mu^{8},1)$
820	$(\mu^{29},\mu^{13},1)$	821	$(\mu^{22},\mu^{34},1)$	822	$(\mu^{46},\mu^{19},1)$	823	$(\mu^{17},\mu^{13},1)$	824	$(\mu^{22},\mu^{27},1)$
825	$(\mu^{7},\mu^{28},1)$	826	$(\mu^{38},\mu^{12},1)$	827	$(\mu^{47},\mu^{22},1)$	828	$(\mu^{27},\mu^{18},1)$	829	$(\mu^{40},\mu^{33},1)$
830	$(\mu^{39},\mu^{2},1)$	831	$(\mu^{2},\mu^{47},1)$	832	$(\mu^{9},\mu^{7},1)$	833	$(\mu^{26},\mu^{32},1)$	834	$(\mu^{21},\mu^{20},1)$
835	$(\mu^{18},\mu^{32},1)$	836	$(\mu^{21},\mu^{29},1)$	837	$(\mu^{36},\mu^{2},1)$	838	$(\mu^{2},\mu^{39},1)$	839	$(\mu^{3},\mu,1)$
840	$(\mu^{8},\mu^{40},1)$	841	$(\mu^{37},\mu^{8},1)$	842	$(\mu^{29},\mu^{36},1)$	843	$(\mu^{11},\mu^{23},1)$	844	$(\mu^{31},\mu,1)$
845	$(\mu^{8},\mu^{44},1)$	846	$(1,\mu^{19},1)$	847	$(\mu^{17},\mu^{20},1)$	848	$(\mu^{18},\mu^{23},1)$	849	$(\mu^{31},\mu^{39},1)$
850	$(\mu^{3},\mu^{39},1)$	851	$(\mu^{3},\mu^{35},1)$	852	$(\mu^{35},\mu^{19},1)$	853	$(\mu^{17},\mu^{30},1)$	854	$(\mu^{10},\mu^{15},1)$
855	$(\mu^{24},\mu^{9},1)$	856	$(\mu^{42},0,1)$	857	$(\mu^{13},\mu^{41},1)$	858	$(\mu^{33},\mu^{11},1)$	859	$(\mu^{30},\mu^{39},1)$
860	$(\mu^{3},\mu^{41},1)$	861	$(\mu^{33},\mu^{17},1)$	862	$(\mu^{43},\mu^{4},1)$	863	$(\mu^{44},\mu^{16},1)$	864	$(\mu^{5},\mu^{5},1)$
865	$(\mu^{23},1,1)$	866	$(\mu^{45},\mu^{14},1)$	867	$(\mu^{32},\mu^{13},1)$	868	$(\mu^{22},\mu,1)$	869	$(\mu^{8},\mu^{29},1)$
870	$(\mu^{36},\mu^{7},1)$	871	$(\mu^{26},\mu^{15},1)$	872	$(\mu^{24},\mu^{23},1)$	873	$(\mu^{31},0,1)$	874	$(\mu^{13},\mu,1)$
875	$(\mu^{8},\mu^{34},1)$	876	$(\mu^{46},\mu^{17},1)$	877	$(\mu^{43},\mu^{39},1)$	878	$(\mu^{3},\mu^{23},1)$	879	$(\mu^{31},\mu^{15},1)$
880	$(\mu^{24},\mu^{12},1)$	881	$(\mu^{47},0,1)$	882	$(\mu^{13},\mu^{4},1)$	883	$(\mu^{44},\mu^{22},1)$	884	$(\mu^{27},\mu^{27},1)$
885	$(\mu^{7},1,1)$	886	$(\mu^{45},\mu^{19},1)$	887	$(\mu^{17},\mu^{46},1)$	888	$(\mu^{4},\mu^{9},1)$	889	$(\mu^{42},\mu^{46},1)$
890	$(\mu^{4},\mu^{32},1)$	891	$(\mu^{21},\mu^{25},1)$	892	$(\mu^{6},\mu^{20},1)$	893	$(\mu^{18},\mu^{4},1)$	894	$(\mu^{44},\mu^{4},1)$
895	$(\mu^{44},\mu^{44},1)$	896	$(1,1,1)$	897	$(\mu^{45},1,1)$	898	$(\mu^{45},\mu^{26},1)$	899	$(\mu,\mu^{30},1)$
900	$(\mu^{10},\mu^{2},1)$	901	$(\mu^{2},\mu^{35},1)$	902	$(\mu^{35},\mu^{33},1)$	903	$(\mu^{39},\mu^{4},1)$	904	$(\mu^{44},\mu^{41},1)$
905	$(\mu^{33},\mu^{33},1)$	906	$(\mu^{39},1,1)$	907	$(\mu^{45},\mu^{42},1)$	908	$(\mu^{20},\mu,1)$	909	$(\mu^{8},\mu^{38},1)$
910	$(\mu^{25},\mu^{44},1)$	911	$(1,\mu^{42},1)$	912	$(\mu^{20},\mu^{23},1)$	913	$(\mu^{31},\mu^{13},1)$	914	$(\mu^{22},\mu^{10},1)$
915	$(\mu^{15},\mu^{36},1)$	916	$(\mu^{11},\mu^{35},1)$	917	$(\mu^{35},\mu^{5},1)$	918	$(\mu^{23},\mu^{36},1)$	919	$(\mu^{11},\mu^{28},1)$
920	$(\mu^{38},\mu^{8},1)$	921	$(\mu^{29},\mu^{4},1)$	922	$(\mu^{44},\mu^{8},1)$	923	$(\mu^{29},\mu^{29},1)$	924	$(\mu^{36},1,1)$
925	$(\mu^{45},\mu^{34},1)$	926	$(\mu^{46},\mu^{27},1)$	927	$(\mu^{7},\mu^{3},1)$	928	$(\mu^{16},\mu^{38},1)$	929	$(\mu^{25},\mu^{20},1)$
930	$(\mu^{18},\mu^{12},1)$	931	$(\mu^{47},\mu^{7},1)$	932	$(\mu^{26},\mu^{17},1)$	933	$(\mu^{43},\mu^{42},1)$	934	$(\mu^{20},\mu^{40},1)$
935	$(\mu^{37},\mu^{19},1)$	936	$(\mu^{17},\mu^{24},1)$	937	$(\mu^{43},1,0)$	938	$(0,\mu^{43},1)$	939	$(\mu^{28},\mu^{15},1)$
940	$(\mu^{24},\mu^{34},1)$	941	$(\mu^{46},0,1)$	942	$(\mu^{13},\mu^{9},1)$	943	$(\mu^{42},\mu^{20},1)$	944	$(\mu^{18},\mu^{46},1)$
945	$(\mu^{4},\mu^{12},1)$	946	$(\mu^{47},\mu^{3},1)$	947	$(\mu^{16},\mu^{7},1)$	948	$(\mu^{26},\mu^{21},1)$	949	$(\mu^{34},\mu^{33},1)$
950	$(\mu^{39},\mu^{41},1)$	951	$(\mu^{33},\mu^{30},1)$	952	$(\mu^{10},\mu^{19},1)$	953	$(\mu^{17},\mu^{2},1)$	954	$(\mu^{2},\mu^{7},1)$
955	$(\mu^{26},\mu^{24},1)$	956	$(\mu,1,0)$	957	$(0,\mu,1)$	958	$(\mu^{8},\mu^{43},1)$	959	$(\mu^{28},\mu^{47},1)$
960	$(\mu^{9},\mu^{19},1)$	961	$(\mu^{17},\mu^{23},1)$	962	$(\mu^{31},\mu^{36},1)$	963	$(\mu^{11},\mu^{47},1)$	964	$(\mu^{9},\mu^{27},1)$
965	$(\mu^{7},\mu^{13},1)$	966	$(\mu^{22},\mu^{44},1)$	967	$(1,\mu^{21},1)$	968	$(\mu^{34},\mu^{37},1)$	969	$(\mu^{41},\mu^{43},1)$
970	$(\mu^{28},\mu^{43},1)$	971	$(\mu^{28},\mu^{38},1)$	972	$(\mu^{25},\mu^{35},1)$	973	$(\mu^{35},\mu^{29},1)$	974	$(\mu^{36},\mu,1)$
975	$(\mu^{8},\mu^{45},1)$	976	$(\mu^{19},\mu^{38},1)$	977	$(\mu^{25},\mu^{8},1)$	978	$(\mu^{29},\mu^{23},1)$	979	$(\mu^{31},\mu^{43},1)$
980	$(\mu^{28},\mu^{16},1)$	981	$(\mu^{5},\mu^{15},1)$	982	$(\mu^{24},\mu,1)$	983	$(\mu^{8},0,1)$	984	$(\mu^{13},\mu^{32},1)$
985	$(\mu^{21},\mu^{47},1)$	986	$(\mu^{9},\mu^{23},1)$	987	$(\mu^{31},\mu^{37},1)$	988	$(\mu^{41},\mu^{29},1)$	989	$(\mu^{36},\mu^{3},1)$
990	$(\mu^{16},\mu^{5},1)$	991	$(\mu^{23},\mu^{18},1)$	992	$(\mu^{40},\mu^{9},1)$	993	$(\mu^{42},\mu^{5},1)$	994	$(\mu^{23},\mu^{3},1)$
995	$(\mu^{16},\mu^{33},1)$	996	$(\mu^{39},\mu^{34},1)$	997	$(\mu^{46},\mu^{43},1)$	998	$(\mu^{28},\mu^{24},1)$	999	$(\mu^{38},1,0)$
1000	$(0,\mu^{38},1)$	1001	$(\mu^{25},\mu^{12},1)$	1002	$(\mu^{47},\mu^{41},1)$	1003	$(\mu^{33},\mu^{24},1)$	1004	$(\mu^{39},1,0)$
1005	$(0,\mu^{39},1)$	1006	$(\mu^{3},\mu^{38},1)$	1007	$(\mu^{25},\mu^{9},1)$	1008	$(\mu^{42},\mu^{36},1)$	1009	$(\mu^{11},\mu^{39},1)$
1010	$(\mu^{3},\mu^{21},1)$	1011	$(\mu^{34},\mu^{18},1)$	1012	$(\mu^{40},\mu^{42},1)$	1013	$(\mu^{20},\mu^{31},1)$	1014	$(\mu^{12},\mu^{42},1)$
1015	$(\mu^{20},\mu^{21},1)$	1016	$(\mu^{34},\mu^{16},1)$	1017	$(\mu^{5},\mu^{7},1)$	1018	$(\mu^{26},\mu^{3},1)$	1019	$(\mu^{16},\mu^{15},1)$
1020	$(\mu^{24},\mu^{19},1)$	1021	$(\mu^{17},0,1)$	1022	$(\mu^{13},\mu^{18},1)$	1023	$(\mu^{40},\mu^{18},1)$	1024	$(\mu^{40},\mu^{3},1)$
1025	$(\mu^{16},\mu^{27},1)$	1026	$(\mu^{7},\mu^{2},1)$	1027	$(\mu^{2},\mu^{24},1)$	1028	$(\mu^{2},1,0)$	1029	$(0,\mu^{2},1)$
1030	$(\mu^{2},\mu^{37},1)$	1031	$(\mu^{41},\mu^{39},1)$	1032	$(\mu^{3},\mu^{18},1)$	1033	$(\mu^{40},\mu^{24},1)$	1034	$(\mu^{37},1,0)$

Continued on next page

Table E.1 – continued from previous page (The points of $\mathrm{PG}(2, 7^2)$)

1035	$(0, \mu^{37}, 1)$	1036	$(\mu^{41}, \mu^{28}, 1)$	1037	$(\mu^{38}, \mu^{5}, 1)$	1038	$(\mu^{23}, \mu^{46}, 1)$	1039	$(\mu^{4}, \mu^{21}, 1)$
1040	$(\mu^{34}, \mu^{38}, 1)$	1041	$(\mu^{25}, \mu^{27}, 1)$	1042	$(\mu^{7}, \mu, 1)$	1043	$(\mu^{8}, \mu^{30}, 1)$	1044	$(\mu^{10}, \mu^{29}, 1)$
1045	$(\mu^{36}, \mu^{21}, 1)$	1046	$(\mu^{34}, \mu^{23}, 1)$	1047	$(\mu^{31}, \mu^{33}, 1)$	1048	$(\mu^{39}, \mu^{27}, 1)$	1049	$(\mu^{7}, \mu^{4}, 1)$
1050	$(\mu^{44}, \mu^{18}, 1)$	1051	$(\mu^{40}, \mu^{40}, 1)$	1052	$(\mu^{37}, 1, 1)$	1053	$(\mu^{45}, \mu^{4}, 1)$	1054	$(\mu^{44}, \mu^{25}, 1)$
1055	$(\mu^{6}, \mu^{6}, 1)$	1056	$(\mu^{14}, 1, 1)$	1057	$(\mu^{45}, \mu^{13}, 1)$	1058	$(\mu^{22}, \mu^{3}, 1)$	1059	$(\mu^{16}, \mu^{37}, 1)$
1060	$(\mu^{41}, \mu^{36}, 1)$	1061	$(\mu^{11}, \mu^{26}, 1)$	1062	$(\mu, \mu^{19}, 1)$	1063	$(\mu^{17}, \mu^{9}, 1)$	1064	$(\mu^{42}, \mu^{47}, 1)$
1065	$(\mu^{9}, \mu^{37}, 1)$	1066	$(\mu^{41}, \mu^{47}, 1)$	1067	$(\mu^{9}, \mu^{24}, 1)$	1068	$(\mu^{42}, 1, 0)$	1069	$(0, \mu^{42}, 1)$
1070	$(\mu^{20}, \mu^{7}, 1)$	1071	$(\mu^{26}, \mu^{8}, 1)$	1072	$(\mu^{29}, \mu^{28}, 1)$	1073	$(\mu^{38}, \mu^{2}, 1)$	1074	$(\mu^{2}, \mu^{25}, 1)$
1075	$(\mu^{6}, \mu^{4}, 1)$	1076	$(\mu^{44}, \mu^{30}, 1)$	1077	$(\mu^{10}, \mu^{10}, 1)$	1078	$(\mu^{15}, 1, 1)$	1079	$(\mu^{45}, \mu^{21}, 1)$
1080	$(\mu^{34}, \mu^{15}, 1)$	1081	$(\mu^{24}, \mu^{26}, 1)$	1082	$(\mu, 0, 1)$	1083	$(\mu^{13}, \mu^{5}, 1)$	1084	$(\mu^{23}, \mu, 1)$
1085	$(\mu^{8}, \mu^{25}, 1)$	1086	$(\mu^{6}, \mu^{25}, 1)$	1087	$(\mu^{6}, \mu^{40}, 1)$	1088	$(\mu^{37}, \mu^{23}, 1)$	1089	$(\mu^{31}, \mu^{38}, 1)$
1090	$(\mu^{25}, \mu^{13}, 1)$	1091	$(\mu^{22}, \mu^{16}, 1)$	1092	$(\mu^{5}, \mu^{26}, 1)$	1093	$(\mu, \mu^{26}, 1)$	1094	$(\mu, \mu^{41}, 1)$
1095	$(\mu^{33}, \mu^{25}, 1)$	1096	$(\mu^{6}, \mu^{15}, 1)$	1097	$(\mu^{24}, \mu^{10}, 1)$	1098	$(\mu^{15}, 0, 1)$	1099	$(\mu^{13}, \mu^{37}, 1)$
1100	$(\mu^{41}, \mu^{19}, 1)$	1101	$(\mu^{17}, \mu^{32}, 1)$	1102	$(\mu^{21}, \mu^{26}, 1)$	1103	$(\mu, \mu^{15}, 1)$	1104	$(\mu^{24}, \mu^{16}, 1)$
1105	$(\mu^{5}, 0, 1)$	1106	$(\mu^{13}, \mu^{38}, 1)$	1107	$(\mu^{25}, \mu^{3}, 1)$	1108	$(\mu^{16}, \mu^{10}, 1)$	1109	$(\mu^{15}, \mu^{10}, 1)$
1110	$(\mu^{15}, \mu^{39}, 1)$	1111	$(\mu^{3}, \mu^{27}, 1)$	1112	$(\mu^{7}, \mu^{39}, 1)$	1113	$(\mu^{3}, \mu^{25}, 1)$	1114	$(\mu^{6}, \mu^{38}, 1)$
1115	$(\mu^{25}, \mu^{11}, 1)$	1116	$(\mu^{30}, \mu^{24}, 1)$	1117	$(\mu^{10}, 1, 0)$	1118	$(0, \mu^{10}, 1)$	1119	$(\mu^{15}, \mu^{2}, 1)$
1120	$(\mu^{2}, \mu^{26}, 1)$	1121	$(\mu, \mu^{47}, 1)$	1122	$(\mu^{9}, \mu, 1)$	1123	$(\mu^{8}, \mu^{14}, 1)$	1124	$(\mu^{32}, \mu^{3}, 1)$
1125	$(\mu^{16}, \mu^{43}, 1)$	1126	$(\mu^{28}, \mu^{23}, 1)$	1127	$(\mu^{31}, \mu^{41}, 1)$	1128	$(\mu^{33}, \mu^{21}, 1)$	1129	$(\mu^{34}, \mu^{43}, 1)$
1130	$(\mu^{28}, \mu^{30}, 1)$	1131	$(\mu^{10}, \mu^{20}, 1)$	1132	$(\mu^{18}, \mu^{3}, 1)$	1133	$(\mu^{16}, \mu^{24}, 1)$	1134	$(\mu^{5}, 1, 0)$
1135	$(0, \mu^{5}, 1)$	1136	$(\mu^{23}, \mu^{10}, 1)$	1137	$(\mu^{15}, \mu^{32}, 1)$	1138	$(\mu^{21}, \mu^{45}, 1)$	1139	$(\mu^{19}, \mu^{33}, 1)$
1140	$(\mu^{39}, \mu^{22}, 1)$	1141	$(\mu^{27}, \mu^{24}, 1)$	1142	$(\mu^{7}, 1, 0)$	1143	$(0, \mu^{7}, 1)$	1144	$(\mu^{26}, \mu^{13}, 1)$
1145	$(\mu^{22}, \mu^{21}, 1)$	1146	$(\mu^{34}, \mu^{7}, 1)$	1147	$(\mu^{26}, \mu^{28}, 1)$	1148	$(\mu^{38}, \mu^{37}, 1)$	1149	$(\mu^{41}, \mu^{16}, 1)$
1150	$(\mu^{5}, \mu^{20}, 1)$	1151	$(\mu^{18}, \mu^{43}, 1)$	1152	$(\mu^{28}, \mu^{36}, 1)$	1153	$(\mu^{11}, \mu^{21}, 1)$	1154	$(\mu^{34}, \mu^{4}, 1)$
1155	$(\mu^{44}, \mu^{46}, 1)$	1156	$(\mu^{4}, \mu^{4}, 1)$	1157	$(\mu^{44}, 1, 1)$	1158	$(\mu^{45}, \mu^{45}, 1)$	1159	$(\mu^{19}, 1, 1)$
1160	$(\mu^{45}, \mu^{28}, 1)$	1161	$(\mu^{38}, \mu^{19}, 1)$	1162	$(\mu^{17}, \mu^{40}, 1)$	1163	$(\mu^{37}, \mu^{42}, 1)$	1164	$(\mu^{20}, \mu^{27}, 1)$
1165	$(\mu^{7}, \mu^{37}, 1)$	1166	$(\mu^{41}, \mu^{15}, 1)$	1167	$(\mu^{24}, \mu^{39}, 1)$	1168	$(\mu^{3}, 0, 1)$	1169	$(\mu^{13}, \mu^{45}, 1)$
1170	$(\mu^{19}, \mu^{45}, 1)$	1171	$(\mu^{19}, \mu^{2}, 1)$	1172	$(\mu^{2}, \mu^{33}, 1)$	1173	$(\mu^{39}, \mu^{37}, 1)$	1174	$(\mu^{41}, \mu^{38}, 1)$
1175	$(\mu^{25}, \mu^{40}, 1)$	1176	$(\mu^{37}, \mu^{31}, 1)$	1177	$(\mu^{12}, \mu^{19}, 1)$	1178	$(\mu^{17}, \mu^{18}, 1)$	1179	$(\mu^{40}, \mu^{45}, 1)$
1180	$(\mu^{19}, \mu^{30}, 1)$	1181	$(\mu^{10}, \mu^{41}, 1)$	1182	$(\mu^{33}, \mu^{18}, 1)$	1183	$(\mu^{40}, \mu, 1)$	1184	$(\mu^{8}, \mu^{19}, 1)$
1185	$(\mu^{17}, \mu^{36}, 1)$	1186	$(\mu^{11}, \mu^{16}, 1)$	1187	$(\mu^{5}, \mu^{23}, 1)$	1188	$(\mu^{31}, \mu^{8}, 1)$	1189	$(\mu^{29}, \mu^{17}, 1)$
1190	$(\mu^{43}, \mu^{7}, 1)$	1191	$(\mu^{26}, \mu^{46}, 1)$	1192	$(\mu^{4}, \mu^{3}, 1)$	1193	$(\mu^{16}, \mu^{20}, 1)$	1194	$(\mu^{18}, \mu^{13}, 1)$
1195	$(\mu^{22}, \mu^{30}, 1)$	1196	$(\mu^{10}, \mu^{31}, 1)$	1197	$(\mu^{12}, \mu^{45}, 1)$	1198	$(\mu^{19}, \mu^{20}, 1)$	1199	$(\mu^{18}, \mu, 1)$
1200	$(\mu^{8}, \mu^{16}, 1)$	1201	$(\mu^{5}, \mu^{24}, 1)$	1202	$(\mu^{23}, 1, 0)$	1203	$(0, \mu^{23}, 1)$	1204	$(\mu^{31}, \mu^{18}, 1)$
1205	$(\mu^{40}, \mu^{28}, 1)$	1206	$(\mu^{38}, \mu, 1)$	1207	$(\mu^{8}, \mu^{31}, 1)$	1208	$(\mu^{12}, \mu^{31}, 1)$	1209	$(\mu^{12}, \mu^{13}, 1)$
1210	$(\mu^{22}, \mu^{23}, 1)$	1211	$(\mu^{31}, \mu^{4}, 1)$	1212	$(\mu^{44}, \mu^{32}, 1)$	1213	$(\mu^{21}, \mu^{21}, 1)$	1214	$(\mu^{34}, 1, 1)$
1215	$(\mu^{45}, \mu^{47}, 1)$	1216	$(\mu^{9}, \mu^{38}, 1)$	1217	$(\mu^{25}, \mu^{31}, 1)$	1218	$(\mu^{12}, \mu^{6}, 1)$	1219	$(\mu^{14}, \mu^{15}, 1)$
1220	$(\mu^{24}, \mu^{40}, 1)$	1221	$(\mu^{37}, 0, 1)$	1222	$(\mu^{13}, \mu^{20}, 1)$	1223	$(\mu^{18}, \mu^{44}, 1)$	1224	$(1, \mu^{8}, 1)$
1225	$(\mu^{29}, \mu^{32}, 1)$	1226	$(\mu^{21}, \mu^{33}, 1)$	1227	$(\mu^{39}, \mu^{5}, 1)$	1228	$(\mu^{23}, \mu^{20}, 1)$	1229	$(\mu^{18}, \mu^{35}, 1)$
1230	$(\mu^{35}, \mu^{43}, 1)$	1231	$(\mu^{28}, \mu^{41}, 1)$	1232	$(\mu^{33}, \mu^{43}, 1)$	1233	$(\mu^{28}, \mu^{37}, 1)$	1234	$(\mu^{41}, \mu^{3}, 1)$
1235	$(\mu^{16}, \mu^{31}, 1)$	1236	$(\mu^{12}, \mu^{7}, 1)$	1237	$(\mu^{26}, \mu^{27}, 1)$	1238	$(\mu^{7}, \mu^{6}, 1)$	1239	$(\mu^{14}, \mu^{36}, 1)$
1240	$(\mu^{11}, \mu^{27}, 1)$	1241	$(\mu^{7}, \mu^{25}, 1)$	1242	$(\mu^{6}, \mu^{28}, 1)$	1243	$(\mu^{38}, \mu^{24}, 1)$	1244	$(\mu^{25}, 1, 0)$
1245	$(0, \mu^{25}, 1)$	1246	$(\mu^{6}, \mu^{41}, 1)$	1247	$(\mu^{33}, \mu^{19}, 1)$	1248	$(\mu^{17}, \mu^{26}, 1)$	1249	$(\mu, \mu^{6}, 1)$
1250	$(\mu^{14}, \mu^{6}, 1)$	1251	$(\mu^{14}, \mu^{30}, 1)$	1252	$(\mu^{10}, \mu^{26}, 1)$	1253	$(\mu, \mu^{34}, 1)$	1254	$(\mu^{46}, \mu^{38}, 1)$
1255	$(\mu^{25}, \mu^{21}, 1)$	1256	$(\mu^{34}, \mu^{28}, 1)$	1257	$(\mu^{38}, \mu^{40}, 1)$	1258	$(\mu^{37}, \mu^{12}, 1)$	1259	$(\mu^{47}, \mu^{6}, 1)$

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Table E.1 – continued from previous page (The points of $\mathrm{PG}(2,7^2)$**)**

1260	$(\mu^{14},\mu^{5},1)$	1261	$(\mu^{23},\mu^{39},1)$	1262	$(\mu^{3},\mu^{20},1)$	1263	$(\mu^{18},\mu^{2},1)$	1264	$(\mu^{2},\mu^{10},1)$
1265	$(\mu^{15},\mu^{13},1)$	1266	$(\mu^{22},\mu^{46},1)$	1267	$(\mu^{4},\mu^{25},1)$	1268	$(\mu^{6},\mu^{10},1)$	1269	$(\mu^{15},\mu,1)$
1270	$(\mu^{8},\mu^{32},1)$	1271	$(\mu^{21},\mu^{40},1)$	1272	$(\mu^{37},\mu^{3},1)$	1273	$(\mu^{16},\mu^{23},1)$	1274	$(\mu^{31},\mu^{26},1)$
1275	$(\mu,\mu^{37},1)$	1276	$(\mu^{41},\mu^{33},1)$	1277	$(\mu^{39},\mu^{6},1)$	1278	$(\mu^{14},\mu^{11},1)$	1279	$(\mu^{30},\mu^{46},1)$
1280	$(\mu^{4},\mu^{42},1)$	1281	$(\mu^{20},\mu^{24},1)$	1282	$(\mu^{18},1,0)$	1283	$(0,\mu^{18},1)$	1284	$(\mu^{40},\mu^{27},1)$
1285	$(\mu^{7},\mu^{18},1)$	1286	$(\mu^{40},\mu^{14},1)$	1287	$(\mu^{32},\mu^{43},1)$	1288	$(\mu^{28},\mu^{7},1)$	1289	$(\mu^{26},\mu^{36},1)$
1290	$(\mu^{11},\mu^{10},1)$	1291	$(\mu^{15},\mu^{33},1)$	1292	$(\mu^{39},\mu^{15},1)$	1293	$(\mu^{24},\mu^{21},1)$	1294	$(\mu^{34},0,1)$
1295	$(\mu^{13},\mu^{15},1)$	1296	$(\mu^{24},\mu^{2},1)$	1297	$(\mu^{2},0,1)$	1298	$(\mu^{13},\mu^{11},1)$	1299	$(\mu^{30},\mu^{8},1)$
1300	$(\mu^{29},\mu^{19},1)$	1301	$(\mu^{17},\mu^{29},1)$	1302	$(\mu^{36},\mu^{41},1)$	1303	$(\mu^{33},\mu^{22},1)$	1304	$(\mu^{27},\mu^{36},1)$
1305	$(\mu^{11},\mu^{4},1)$	1306	$(\mu^{44},\mu^{14},1)$	1307	$(\mu^{32},\mu^{32},1)$	1308	$(\mu^{21},1,1)$	1309	$(\mu^{45},\mu^{11},1)$
1310	$(\mu^{30},\mu^{11},1)$	1311	$(\mu^{30},\mu^{20},1)$	1312	$(\mu^{18},\mu^{8},1)$	1313	$(\mu^{29},\mu^{37},1)$	1314	$(\mu^{41},\mu^{5},1)$
1315	$(\mu^{23},\mu^{38},1)$	1316	$(\mu^{25},\mu^{42},1)$	1317	$(\mu^{20},\mu^{14},1)$	1318	$(\mu^{32},\mu^{14},1)$	1319	$(\mu^{32},\mu^{11},1)$
1320	$(\mu^{30},\mu^{9},1)$	1321	$(\mu^{42},\mu^{32},1)$	1322	$(\mu^{21},\mu,1)$	1323	$(\mu^{8},\mu^{22},1)$	1324	$(\mu^{27},\mu^{46},1)$
1325	$(\mu^{4},\mu^{45},1)$	1326	$(\mu^{19},\mu^{23},1)$	1327	$(\mu^{31},\mu^{14},1)$	1328	$(\mu^{32},\mu^{20},1)$	1329	$(\mu^{18},\mu^{45},1)$
1330	$(\mu^{19},\mu^{27},1)$	1331	$(\mu^{7},\mu^{38},1)$	1332	$(\mu^{25},\mu^{47},1)$	1333	$(\mu^{9},\mu^{3},1)$	1334	$(\mu^{16},\mu^{22},1)$
1335	$(\mu^{27},\mu^{22},1)$	1336	$(\mu^{27},\mu^{20},1)$	1337	$(\mu^{18},\mu^{11},1)$	1338	$(\mu^{30},\mu^{38},1)$	1339	$(\mu^{25},\mu^{15},1)$
1340	$(\mu^{24},\mu^{18},1)$	1341	$(\mu^{40},0,1)$	1342	$(\mu^{13},\mu^{24},1)$	1343	$(\mu^{22},1,0)$	1344	$(0,\mu^{22},1)$
1345	$(\mu^{27},\mu^{14},1)$	1346	$(\mu^{32},\mu^{25},1)$	1347	$(\mu^{6},\mu^{33},1)$	1348	$(\mu^{39},\mu^{25},1)$	1349	$(\mu^{6},\mu^{3},1)$
1350	$(\mu^{16},\mu^{2},1)$	1351	$(\mu^{2},\mu^{45},1)$	1352	$(\mu^{19},\mu^{17},1)$	1353	$(\mu^{43},\mu^{26},1)$	1354	$(\mu,\mu^{21},1)$
1355	$(\mu^{34},\mu^{26},1)$	1356	$(\mu,\mu^{3},1)$	1357	$(\mu^{16},\mu^{8},1)$	1358	$(\mu^{29},\mu^{24},1)$	1359	$(\mu^{36},1,0)$
1360	$(0,\mu^{36},1)$	1361	$(\mu^{11},\mu^{46},1)$	1362	$(\mu^{4},\mu^{22},1)$	1363	$(\mu^{27},\mu^{31},1)$	1364	$(\mu^{12},\mu^{5},1)$
1365	$(\mu^{23},\mu^{24},1)$	1366	$(\mu^{31},1,0)$	1367	$(0,\mu^{31},1)$	1368	$(\mu^{12},\mu^{47},1)$	1369	$(\mu^{9},\mu^{10},1)$
1370	$(\mu^{15},\mu^{21},1)$	1371	$(\mu^{34},\mu^{10},1)$	1372	$(\mu^{15},\mu^{17},1)$	1373	$(\mu^{43},\mu^{19},1)$	1374	$(\mu^{17},\mu^{37},1)$
1375	$(\mu^{41},\mu^{46},1)$	1376	$(\mu^{4},\mu^{19},1)$	1377	$(\mu^{17},\mu^{21},1)$	1378	$(\mu^{34},\mu^{39},1)$	1379	$(\mu^{3},\mu^{5},1)$
1380	$(\mu^{23},\mu^{7},1)$	1381	$(\mu^{26},\mu^{43},1)$	1382	$(\mu^{28},\mu^{27},1)$	1383	$(\mu^{7},\mu^{17},1)$	1384	$(\mu^{43},\mu^{17},1)$
1385	$(\mu^{43},\mu^{15},1)$	1386	$(\mu^{24},\mu^{44},1)$	1387	$(1,0,1)$	1388	$(\mu^{13},\mu^{16},1)$	1389	$(\mu^{5},\mu^{31},1)$
1390	$(\mu^{12},\mu^{37},1)$	1391	$(\mu^{41},\mu^{42},1)$	1392	$(\mu^{20},\mu^{35},1)$	1393	$(\mu^{35},\mu^{17},1)$	1394	$(\mu^{43},\mu^{8},1)$
1395	$(\mu^{29},\mu,1)$	1396	$(\mu^{8},\mu^{20},1)$	1397	$(\mu^{18},\mu^{37},1)$	1398	$(\mu^{41},\mu,1)$	1399	$(\mu^{8},\mu^{23},1)$
1400	$(\mu^{31},\mu^{2},1)$	1401	$(\mu^{2},\mu^{38},1)$	1402	$(\mu^{25},\mu^{23},1)$	1403	$(\mu^{31},\mu^{25},1)$	1404	$(\mu^{6},\mu^{42},1)$
1405	$(\mu^{20},\mu^{6},1)$	1406	$(\mu^{14},\mu^{44},1)$	1407	$(1,\mu^{16},1)$	1408	$(\mu^{5},\mu^{8},1)$	1409	$(\mu^{29},\mu^{6},1)$
1410	$(\mu^{14},\mu^{26},1)$	1411	$(\mu,\mu^{17},1)$	1412	$(\mu^{43},\mu^{35},1)$	1413	$(\mu^{35},\mu^{7},1)$	1414	$(\mu^{26},\mu^{39},1)$
1415	$(\mu^{3},\mu^{2},1)$	1416	$(\mu^{2},\mu^{34},1)$	1417	$(\mu^{46},\mu^{44},1)$	1418	$(1,\mu^{44},1)$	1419	$(1,\mu^{3},1)$
1420	$(\mu^{16},\mu^{19},1)$	1421	$(\mu^{17},\mu^{12},1)$	1422	$(\mu^{47},\mu^{4},1)$	1423	$(\mu^{44},\mu^{35},1)$	1424	$(\mu^{35},\mu^{35},1)$
1425	$(\mu^{35},1,1)$	1426	$(\mu^{45},\mu^{10},1)$	1427	$(\mu^{15},\mu^{44},1)$	1428	$(1,\mu^{24},1)$	1429	$(\mu^{45},1,0)$
1430	$(0,\mu^{45},1)$	1431	$(\mu^{19},\mu^{6},1)$	1432	$(\mu^{14},\mu^{45},1)$	1433	$(\mu^{19},\mu^{35},1)$	1434	$(\mu^{35},\mu^{18},1)$
1435	$(\mu^{40},\mu^{5},1)$	1436	$(\mu^{23},\mu^{34},1)$	1437	$(\mu^{46},\mu^{15},1)$	1438	$(\mu^{24},\mu^{20},1)$	1439	$(\mu^{18},0,1)$
1440	$(\mu^{13},\mu^{21},1)$	1441	$(\mu^{34},\mu^{12},1)$	1442	$(\mu^{47},\mu,1)$	1443	$(\mu^{8},\mu^{47},1)$	1444	$(\mu^{9},\mu^{28},1)$
1445	$(\mu^{38},\mu^{44},1)$	1446	$(1,\mu^{23},1)$	1447	$(\mu^{31},\mu^{34},1)$	1448	$(\mu^{46},\mu^{34},1)$	1449	$(\mu^{46},\mu^{42},1)$
1450	$(\mu^{20},\mu^{16},1)$	1451	$(\mu^{5},\mu^{35},1)$	1452	$(\mu^{35},\mu^{12},1)$	1453	$(\mu^{47},\mu^{12},1)$	1454	$(\mu^{47},\mu^{38},1)$
1455	$(\mu^{25},\mu^{16},1)$	1456	$(\mu^{5},\mu^{47},1)$	1457	$(\mu^{9},\mu^{34},1)$	1458	$(\mu^{46},\mu^{4},1)$	1459	$(\mu^{44},\mu^{40},1)$
1460	$(\mu^{37},\mu^{37},1)$	1461	$(\mu^{41},1,1)$	1462	$(\mu^{45},\mu^{12},1)$	1463	$(\mu^{47},\mu^{28},1)$	1464	$(\mu^{38},\mu^{29},1)$
1465	$(\mu^{36},\mu^{11},1)$	1466	$(\mu^{30},\mu^{19},1)$	1467	$(\mu^{17},\mu^{7},1)$	1468	$(\mu^{26},\mu^{31},1)$	1469	$(\mu^{12},\mu^{11},1)$
1470	$(\mu^{30},\mu^{31},1)$	1471	$(\mu^{12},\mu^{2},1)$	1472	$(\mu^{2},\mu^{3},1)$	1473	$(\mu^{16},\mu^{14},1)$	1474	$(\mu^{32},\mu^{27},1)$
1475	$(\mu^{7},\mu^{34},1)$	1476	$(\mu^{46},\mu^{20},1)$	1477	$(\mu^{18},\mu^{14},1)$	1478	$(\mu^{32},\mu^{40},1)$	1479	$(\mu^{37},\mu^{16},1)$
1480	$(\mu^{5},\mu^{12},1)$	1481	$(\mu^{47},\mu^{24},1)$	1482	$(\mu^{9},1,0)$	1483	$(0,\mu^{9},1)$	1484	$(\mu^{42},\mu^{29},1)$

Continued on next page

Table E.1 – continued from previous page (The points of $\mathrm{PG}(2,7^2)$**)**

1485	$(\mu^{36},\mu^{16},1)$	1486	$(\mu^{5},\mu^{42},1)$	1487	$(\mu^{20},\mu^{45},1)$	1488	$(\mu^{19},\mu,1)$	1489	$(\mu^{8},\mu^{39},1)$
1490	$(\mu^{3},\mu^{22},1)$	1491	$(\mu^{27},\mu^{11},1)$	1492	$(\mu^{30},\mu^{23},1)$	1493	$(\mu^{31},\mu^{21},1)$	1494	$(\mu^{34},\mu^{22},1)$
1495	$(\mu^{27},\mu^{29},1)$	1496	$(\mu^{36},\mu^{29},1)$	1497	$(\mu^{36},\mu^{25},1)$	1498	$(\mu^{6},\mu^{43},1)$	1499	$(\mu^{28},\mu^{14},1)$
1500	$(\mu^{32},\mu^{42},1)$	1501	$(\mu^{20},\mu^{47},1)$	1502	$(\mu^{9},\mu^{39},1)$	1503	$(\mu^{3},\mu^{9},1)$	1504	$(\mu^{42},\mu^{26},1)$
1505	$(\mu,\mu^{29},1)$	1506	$(\mu^{36},\mu^{28},1)$	1507	$(\mu^{38},\mu^{27},1)$	1508	$(\mu^{7},\mu^{30},1)$	1509	$(\mu^{10},\mu^{32},1)$
1510	$(\mu^{21},\mu^{6},1)$	1511	$(\mu^{14},\mu^{28},1)$	1512	$(\mu^{38},\mu^{6},1)$	1513	$(\mu^{14},\mu^{37},1)$	1514	$(\mu^{41},\mu^{9},1)$
1515	$(\mu^{42},\mu^{9},1)$	1516	$(\mu^{42},\mu^{22},1)$	1517	$(\mu^{27},\mu^{7},1)$	1518	$(\mu^{26},\mu^{19},1)$	1519	$(\mu^{17},\mu^{16},1)$
1520	$(\mu^{5},\mu^{10},1)$	1521	$(\mu^{15},\mu^{40},1)$	1522	$(\mu^{37},\mu^{13},1)$	1523	$(\mu^{22},\mu^{29},1)$	1524	$(\mu^{36},\mu^{9},1)$
1525	$(\mu^{42},\mu^{31},1)$	1526	$(\mu^{12},\mu^{40},1)$	1527	$(\mu^{37},\mu^{38},1)$	1528	$(\mu^{25},\mu^{32},1)$	1529	$(\mu^{21},\mu^{15},1)$
1530	$(\mu^{24},\mu^{38},1)$	1531	$(\mu^{25},0,1)$	1532	$(\mu^{13},\mu^{7},1)$	1533	$(\mu^{26},\mu^{4},1)$	1534	$(\mu^{44},\mu^{43},1)$
1535	$(\mu^{28},\mu^{28},1)$	1536	$(\mu^{38},1,1)$	1537	$(\mu^{45},\mu^{20},1)$	1538	$(\mu^{18},\mu^{47},1)$	1539	$(\mu^{9},\mu^{17},1)$
1540	$(\mu^{43},\mu,1)$	1541	$(\mu^{8},\mu^{28},1)$	1542	$(\mu^{38},\mu^{9},1)$	1543	$(\mu^{42},\mu^{17},1)$	1544	$(\mu^{43},\mu^{23},1)$
1545	$(\mu^{31},\mu^{3},1)$	1546	$(\mu^{16},\mu^{4},1)$	1547	$(\mu^{44},\mu^{39},1)$	1548	$(\mu^{3},\mu^{3},1)$	1549	$(\mu^{16},1,1)$
1550	$(\mu^{45},\mu^{40},1)$	1551	$(\mu^{37},\mu^{18},1)$	1552	$(\mu^{40},\mu^{47},1)$	1553	$(\mu^{9},\mu^{20},1)$	1554	$(\mu^{18},\mu^{24},1)$
1555	$(\mu^{40},1,0)$	1556	$(0,\mu^{40},1)$	1557	$(\mu^{37},\mu^{24},1)$	1558	$(\mu^{41},1,0)$	1559	$(0,\mu^{41},1)$
1560	$(\mu^{33},\mu^{20},1)$	1561	$(\mu^{18},\mu^{27},1)$	1562	$(\mu^{7},\mu^{15},1)$	1563	$(\mu^{24},\mu^{46},1)$	1564	$(\mu^{4},0,1)$
1565	$(\mu^{13},\mu^{17},1)$	1566	$(\mu^{43},\mu^{21},1)$	1567	$(\mu^{34},\mu^{6},1)$	1568	$(\mu^{14},\mu^{16},1)$	1569	$(\mu^{5},\mu^{21},1)$
1570	$(\mu^{34},\mu^{11},1)$	1571	$(\mu^{30},\mu^{32},1)$	1572	$(\mu^{21},\mu^{11},1)$	1573	$(\mu^{30},\mu^{44},1)$	1574	$(1,\mu^{38},1)$
1575	$(\mu^{25},\mu^{28},1)$	1576	$(\mu^{38},\mu^{32},1)$	1577	$(\mu^{21},\mu^{44},1)$	1578	$(1,\mu^{14},1)$	1579	$(\mu^{32},\mu^{35},1)$
1580	$(\mu^{35},\mu^{14},1)$	1581	$(\mu^{32},\mu^{45},1)$	1582	$(\mu^{19},\mu^{46},1)$	1583	$(\mu^{4},\mu^{35},1)$	1584	$(\mu^{35},\mu^{39},1)$
1585	$(\mu^{3},\mu^{16},1)$	1586	$(\mu^{5},\mu^{37},1)$	1587	$(\mu^{41},\mu^{18},1)$	1588	$(\mu^{40},\mu^{7},1)$	1589	$(\mu^{26},\mu^{37},1)$
1590	$(\mu^{41},\mu^{40},1)$	1591	$(\mu^{37},\mu^{4},1)$	1592	$(\mu^{44},\mu^{3},1)$	1593	$(\mu^{16},\mu^{16},1)$	1594	$(\mu^{5},1,1)$
1595	$(\mu^{45},\mu^{22},1)$	1596	$(\mu^{27},\mu^{8},1)$	1597	$(\mu^{29},\mu^{22},1)$	1598	$(\mu^{27},\mu^{39},1)$	1599	$(\mu^{3},\mu^{44},1)$
1600	$(1,\mu^{32},1)$	1601	$(\mu^{21},\mu^{24},1)$	1602	$(\mu^{34},1,0)$	1603	$(0,\mu^{34},1)$	1604	$(\mu^{46},\mu^{33},1)$
1605	$(\mu^{39},\mu^{35},1)$	1606	$(\mu^{35},\mu^{32},1)$	1607	$(\mu^{21},\mu^{34},1)$	1608	$(\mu^{46},\mu^{12},1)$	1609	$(\mu^{47},\mu^{43},1)$
1610	$(\mu^{28},\mu^{19},1)$	1611	$(\mu^{17},\mu^{27},1)$	1612	$(\mu^{7},\mu^{12},1)$	1613	$(\mu^{47},\mu^{21},1)$	1614	$(\mu^{34},\mu^{25},1)$
1615	$(\mu^{6},\mu^{8},1)$	1616	$(\mu^{29},\mu^{15},1)$	1617	$(\mu^{24},\mu^{36},1)$	1618	$(\mu^{11},0,1)$	1619	$(\mu^{13},\mu^{31},1)$
1620	$(\mu^{12},\mu^{38},1)$	1621	$(\mu^{25},\mu^{26},1)$	1622	$(\mu,\mu^{43},1)$	1623	$(\mu^{28},\mu^{20},1)$	1624	$(\mu^{18},\mu^{28},1)$
1625	$(\mu^{38},\mu^{46},1)$	1626	$(\mu^{4},\mu^{27},1)$	1627	$(\mu^{7},\mu^{11},1)$	1628	$(\mu^{30},\mu^{4},1)$	1629	$(\mu^{44},\mu^{34},1)$
1630	$(\mu^{46},\mu^{46},1)$	1631	$(\mu^{4},1,1)$	1632	$(\mu^{45},\mu,1)$	1633	$(\mu^{8},\mu^{37},1)$	1634	$(\mu^{41},\mu^{12},1)$
1635	$(\mu^{47},\mu^{14},1)$	1636	$(\mu^{32},\mu^{23},1)$	1637	$(\mu^{31},\mu^{10},1)$	1638	$(\mu^{15},\mu^{3},1)$	1639	$(\mu^{16},\mu^{40},1)$
1640	$(\mu^{37},\mu^{32},1)$	1641	$(\mu^{21},\mu^{28},1)$	1642	$(\mu^{38},\mu^{4},1)$	1643	$(\mu^{44},\mu^{19},1)$	1644	$(\mu^{17},\mu^{17},1)$
1645	$(\mu^{43},1,1)$	1646	$(\mu^{45},\mu^{17},1)$	1647	$(\mu^{43},\mu^{24},1)$	1648	$(\mu^{28},1,0)$	1649	$(0,\mu^{28},1)$
1650	$(\mu^{38},\mu^{25},1)$	1651	$(\mu^{6},\mu^{29},1)$	1652	$(\mu^{36},\mu^{22},1)$	1653	$(\mu^{27},\mu^{16},1)$	1654	$(\mu^{5},\mu^{46},1)$
1655	$(\mu^{4},\mu^{29},1)$	1656	$(\mu^{36},\mu^{40},1)$	1657	$(\mu^{37},\mu^{26},1)$	1658	$(\mu,\mu^{8},1)$	1659	$(\mu^{29},\mu^{21},1)$
1660	$(\mu^{34},\mu^{46},1)$	1661	$(\mu^{4},\mu^{6},1)$	1662	$(\mu^{14},\mu^{18},1)$	1663	$(\mu^{40},\mu^{8},1)$	1664	$(\mu^{29},\mu^{40},1)$
1665	$(\mu^{37},\mu,1)$	1666	$(\mu^{8},\mu^{15},1)$	1667	$(\mu^{24},\mu^{43},1)$	1668	$(\mu^{28},0,1)$	1669	$(\mu^{13},\mu^{23},1)$
1670	$(\mu^{31},\mu^{9},1)$	1671	$(\mu^{42},\mu^{30},1)$	1672	$(\mu^{10},\mu^{38},1)$	1673	$(\mu^{25},\mu^{10},1)$	1674	$(\mu^{15},\mu^{9},1)$
1675	$(\mu^{42},\mu^{18},1)$	1676	$(\mu^{40},\mu^{20},1)$	1677	$(\mu^{18},\mu^{29},1)$	1678	$(\mu^{36},\mu^{44},1)$	1679	$(1,\mu^{37},1)$
1680	$(\mu^{41},\mu^{44},1)$	1681	$(1,\mu^{15},1)$	1682	$(\mu^{24},\mu^{27},1)$	1683	$(\mu^{7},0,1)$	1684	$(\mu^{13},\mu^{35},1)$
1685	$(\mu^{35},\mu^{13},1)$	1686	$(\mu^{22},\mu^{35},1)$	1687	$(\mu^{35},\mu^{8},1)$	1688	$(\mu^{29},\mu^{42},1)$	1689	$(\mu^{20},\mu^{32},1)$
1690	$(\mu^{21},\mu^{3},1)$	1691	$(\mu^{16},\mu^{30},1)$	1692	$(\mu^{10},\mu^{5},1)$	1693	$(\mu^{23},\mu^{8},1)$	1694	$(\mu^{29},\mu^{46},1)$
1695	$(\mu^{4},\mu^{16},1)$	1696	$(\mu^{5},\mu^{9},1)$	1697	$(\mu^{42},\mu^{19},1)$	1698	$(\mu^{17},\mu^{45},1)$	1699	$(\mu^{19},\mu^{24},1)$
1700	$(\mu^{17},1,0)$	1701	$(0,\mu^{17},1)$	1702	$(\mu^{43},\mu^{30},1)$	1703	$(\mu^{10},\mu^{30},1)$	1704	$(\mu^{10},\mu^{43},1)$
1705	$(\mu^{28},\mu^{13},1)$	1706	$(\mu^{22},\mu^{32},1)$	1707	$(\mu^{21},\mu^{42},1)$	1708	$(\mu^{20},\mu^{34},1)$	1709	$(\mu^{46},\mu^{28},1)$

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Table E.1 – continued from previous page (The points of $\mathrm{PG}(2,7^2)$**)**

1710	$(\mu^{38},\mu^{34},1)$	1711	$(\mu^{46},\mu^{21},1)$	1712	$(\mu^{34},\mu^{30},1)$	1713	$(\mu^{10},\mu^{12},1)$	1714	$(\mu^{47},\mu^{32},1)$
1715	$(\mu^{21},\mu^{12},1)$	1716	$(\mu^{47},\mu^{13},1)$	1717	$(\mu^{22},\mu^{13},1)$	1718	$(\mu^{22},\mu^{43},1)$	1719	$(\mu^{28},\mu,1)$
1720	$(\mu^{8},\mu^{18},1)$	1721	$(\mu^{40},\mu^{11},1)$	1722	$(\mu^{30},\mu^{41},1)$	1723	$(\mu^{33},\mu^{23},1)$	1724	$(\mu^{31},\mu^{40},1)$
1725	$(\mu^{37},\mu^{25},1)$	1726	$(\mu^{6},\mu^{13},1)$	1727	$(\mu^{22},\mu^{8},1)$	1728	$(\mu^{29},\mu^{2},1)$	1729	$(\mu^{2},\mu^{14},1)$
1730	$(\mu^{32},\mu^{30},1)$	1731	$(\mu^{10},\mu^{37},1)$	1732	$(\mu^{41},\mu^{26},1)$	1733	$(\mu,\mu^{16},1)$	1734	$(\mu^{5},\mu^{45},1)$
1735	$(\mu^{19},\mu^{44},1)$	1736	$(1,\mu^{31},1)$	1737	$(\mu^{12},\mu^{15},1)$	1738	$(\mu^{24},\mu^{25},1)$	1739	$(\mu^{6},0,1)$
1740	$(\mu^{13},\mu^{47},1)$	1741	$(\mu^{9},\mu^{35},1)$	1742	$(\mu^{35},\mu^{41},1)$	1743	$(\mu^{33},\mu^{46},1)$	1744	$(\mu^{4},\mu^{13},1)$
1745	$(\mu^{22},\mu^{26},1)$	1746	$(\mu,\mu^{22},1)$	1747	$(\mu^{27},\mu^{19},1)$	1748	$(\mu^{17},\mu^{10},1)$	1749	$(\mu^{15},\mu^{20},1)$
1750	$(\mu^{18},\mu^{42},1)$	1751	$(\mu^{20},\mu^{28},1)$	1752	$(\mu^{38},\mu^{20},1)$	1753	$(\mu^{18},\mu^{41},1)$	1754	$(\mu^{33},\mu^{41},1)$
1755	$(\mu^{33},\mu^{42},1)$	1756	$(\mu^{20},\mu^{29},1)$	1757	$(\mu^{36},\mu^{18},1)$	1758	$(\mu^{40},\mu^{29},1)$	1759	$(\mu^{36},\mu^{47},1)$
1760	$(\mu^{9},\mu^{46},1)$	1761	$(\mu^{4},\mu^{10},1)$	1762	$(\mu^{15},\mu^{19},1)$	1763	$(\mu^{17},\mu^{41},1)$	1764	$(\mu^{33},\mu^{38},1)$
1765	$(\mu^{25},\mu^{34},1)$	1766	$(\mu^{46},\mu^{40},1)$	1767	$(\mu^{37},\mu^{33},1)$	1768	$(\mu^{39},\mu^{46},1)$	1769	$(\mu^{4},\mu,1)$
1770	$(\mu^{8},\mu^{12},1)$	1771	$(\mu^{47},\mu^{18},1)$	1772	$(\mu^{40},\mu^{31},1)$	1773	$(\mu^{12},\mu^{23},1)$	1774	$(\mu^{31},\mu^{32},1)$
1775	$(\mu^{21},\mu^{9},1)$	1776	$(\mu^{42},\mu^{8},1)$	1777	$(\mu^{29},\mu^{9},1)$	1778	$(\mu^{42},\mu^{6},1)$	1779	$(\mu^{14},\mu^{42},1)$
1780	$(\mu^{20},\mu^{36},1)$	1781	$(\mu^{11},\mu^{41},1)$	1782	$(\mu^{33},\mu^{3},1)$	1783	$(\mu^{16},\mu^{25},1)$	1784	$(\mu^{6},\mu,1)$
1785	$(\mu^{8},\mu^{42},1)$	1786	$(\mu^{20},\mu^{39},1)$	1787	$(\mu^{3},\mu^{33},1)$	1788	$(\mu^{39},\mu^{23},1)$	1789	$(\mu^{31},\mu^{28},1)$
1790	$(\mu^{38},\mu^{26},1)$	1791	$(\mu,\mu^{24},1)$	1792	$(\mu^{8},1,0)$	1793	$(0,\mu^{8},1)$	1794	$(\mu^{29},\mu^{16},1)$
1795	$(\mu^{5},\mu^{17},1)$	1796	$(\mu^{43},\mu^{20},1)$	1797	$(\mu^{18},\mu^{38},1)$	1798	$(\mu^{25},\mu^{33},1)$	1799	$(\mu^{39},\mu^{33},1)$
1800	$(\mu^{39},\mu^{36},1)$	1801	$(\mu^{11},\mu^{8},1)$	1802	$(\mu^{29},\mu^{47},1)$	1803	$(\mu^{9},\mu^{21},1)$	1804	$(\mu^{34},\mu^{40},1)$
1805	$(\mu^{37},\mu^{39},1)$	1806	$(\mu^{3},\mu^{10},1)$	1807	$(\mu^{15},\mu^{47},1)$	1808	$(\mu^{9},\mu^{33},1)$	1809	$(\mu^{39},\mu^{45},1)$
1810	$(\mu^{19},\mu^{16},1)$	1811	$(\mu^{5},\mu^{36},1)$	1812	$(\mu^{11},\mu^{36},1)$	1813	$(\mu^{11},\mu^{29},1)$	1814	$(\mu^{36},\mu^{6},1)$
1815	$(\mu^{14},\mu^{3},1)$	1816	$(\mu^{16},\mu^{32},1)$	1817	$(\mu^{21},\mu^{16},1)$	1818	$(\mu^{5},\mu^{19},1)$	1819	$(\mu^{17},\mu^{42},1)$
1820	$(\mu^{20},\mu^{25},1)$	1821	$(\mu^{6},\mu^{36},1)$	1822	$(\mu^{11},\mu^{45},1)$	1823	$(\mu^{19},\mu^{37},1)$	1824	$(\mu^{41},\mu^{24},1)$
1825	$(\mu^{33},1,0)$	1826	$(0,\mu^{33},1)$	1827	$(\mu^{39},\mu^{26},1)$	1828	$(\mu,\mu^{46},1)$	1829	$(\mu^{4},\mu^{44},1)$
1830	$(1,\mu^{4},1)$	1831	$(\mu^{44},\mu^{47},1)$	1832	$(\mu^{9},\mu^{9},1)$	1833	$(\mu^{42},1,1)$	1834	$(\mu^{45},\mu^{25},1)$
1835	$(\mu^{6},\mu^{35},1)$	1836	$(\mu^{35},\mu^{21},1)$	1837	$(\mu^{34},\mu^{47},1)$	1838	$(\mu^{9},\mu^{11},1)$	1839	$(\mu^{30},\mu^{36},1)$
1840	$(\mu^{11},\mu,1)$	1841	$(\mu^{8},\mu^{26},1)$	1842	$(\mu,\mu^{20},1)$	1843	$(\mu^{18},\mu^{10},1)$	1844	$(\mu^{15},\mu^{23},1)$
1845	$(\mu^{31},\mu^{7},1)$	1846	$(\mu^{26},\mu^{14},1)$	1847	$(\mu^{32},\mu^{31},1)$	1848	$(\mu^{12},\mu^{39},1)$	1849	$(\mu^{3},\mu^{4},1)$
1850	$(\mu^{44},\mu^{28},1)$	1851	$(\mu^{38},\mu^{38},1)$	1852	$(\mu^{25},1,1)$	1853	$(\mu^{45},\mu^{39},1)$	1854	$(\mu^{3},\mu^{32},1)$
1855	$(\mu^{21},\mu^{5},1)$	1856	$(\mu^{23},\mu^{37},1)$	1857	$(\mu^{41},\mu^{10},1)$	1858	$(\mu^{15},\mu^{30},1)$	1859	$(\mu^{10},\mu^{34},1)$
1860	$(\mu^{46},\mu^{31},1)$	1861	$(\mu^{12},\mu^{8},1)$	1862	$(\mu^{29},\mu^{30},1)$	1863	$(\mu^{10},\mu^{22},1)$	1864	$(\mu^{27},\mu^{12},1)$
1865	$(\mu^{47},\mu^{40},1)$	1866	$(\mu^{37},\mu^{28},1)$	1867	$(\mu^{38},\mu^{45},1)$	1868	$(\mu^{19},\mu^{42},1)$	1869	$(\mu^{20},\mu^{3},1)$
1870	$(\mu^{16},\mu^{46},1)$	1871	$(\mu^{4},\mu^{47},1)$	1872	$(\mu^{9},\mu^{13},1)$	1873	$(\mu^{22},\mu^{28},1)$	1874	$(\mu^{38},\mu^{11},1)$
1875	$(\mu^{30},\mu^{5},1)$	1876	$(\mu^{23},\mu^{13},1)$	1877	$(\mu^{22},\mu^{39},1)$	1878	$(\mu^{3},\mu^{24},1)$	1879	$(\mu^{16},1,0)$
1880	$(0,\mu^{16},1)$	1881	$(\mu^{5},\mu^{40},1)$	1882	$(\mu^{37},\mu^{14},1)$	1883	$(\mu^{32},\mu^{39},1)$	1884	$(\mu^{3},\mu^{30},1)$
1885	$(\mu^{10},\mu^{42},1)$	1886	$(\mu^{20},\mu^{5},1)$	1887	$(\mu^{23},\mu^{5},1)$	1888	$(\mu^{23},\mu^{40},1)$	1889	$(\mu^{37},\mu^{6},1)$
1890	$(\mu^{14},\mu^{21},1)$	1891	$(\mu^{34},\mu^{2},1)$	1892	$(\mu^{2},\mu^{4},1)$	1893	$(\mu^{44},\mu^{42},1)$	1894	$(\mu^{20},\mu^{20},1)$
1895	$(\mu^{18},1,1)$	1896	$(\mu^{45},\mu^{5},1)$	1897	$(\mu^{23},\mu^{4},1)$	1898	$(\mu^{44},\mu^{13},1)$	1899	$(\mu^{22},\mu^{22},1)$
1900	$(\mu^{27},1,1)$	1901	$(\mu^{45},\mu^{38},1)$	1902	$(\mu^{25},\mu^{6},1)$	1903	$(\mu^{14},\mu^{8},1)$	1904	$(\mu^{29},\mu^{45},1)$
1905	$(\mu^{19},\mu^{31},1)$	1906	$(\mu^{12},\mu^{43},1)$	1907	$(\mu^{28},\mu^{29},1)$	1908	$(\mu^{36},\mu^{46},1)$	1909	$(\mu^{4},\mu^{41},1)$
1910	$(\mu^{33},\mu^{37},1)$	1911	$(\mu^{41},\mu^{2},1)$	1912	$(\mu^{2},\mu^{17},1)$	1913	$(\mu^{43},\mu^{41},1)$	1914	$(\mu^{33},\mu^{5},1)$
1915	$(\mu^{23},\mu^{32},1)$	1916	$(\mu^{21},\mu^{38},1)$	1917	$(\mu^{25},\mu^{39},1)$	1918	$(\mu^{3},\mu^{45},1)$	1919	$(\mu^{19},\mu^{3},1)$
1920	$(\mu^{16},\mu^{47},1)$	1921	$(\mu^{9},\mu^{4},1)$	1922	$(\mu^{44},\mu^{2},1)$	1923	$(\mu^{2},\mu^{2},1)$	1924	$(\mu^{2},1,1)$
1925	$(\mu^{45},\mu^{43},1)$	1926	$(\mu^{28},\mu^{9},1)$	1927	$(\mu^{42},\mu^{4},1)$	1928	$(\mu^{44},\mu^{24},1)$	1929	$(1,1,0)$
1930	$(0,1,1)$	1931	$(\mu^{45},\mu^{32},1)$	1932	$(\mu^{21},\mu^{2},1)$	1933	$(\mu^{2},\mu^{16},1)$	1934	$(\mu^{5},\mu^{3},1)$

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Table E.1 – continued from previous page (The points of $\mathrm{PG}(2,7^2)$**)**

1935	$(\mu^{16},\mu^{41},1)$	1936	$(\mu^{33},\mu^{28},1)$	1937	$(\mu^{38},\mu^{47},1)$	1938	$(\mu^{9},\mu^{32},1)$	1939	$(\mu^{21},\mu^{27},1)$
1940	$(\mu^{7},\mu^{21},1)$	1941	$(\mu^{34},\mu^{8},1)$	1942	$(\mu^{29},\mu^{31},1)$	1943	$(\mu^{12},\mu^{24},1)$	1944	$(\mu^{47},1,0)$
1945	$(0,\mu^{47},1)$	1946	$(\mu^{9},\mu^{44},1)$	1947	$(1,\mu^{6},1)$	1948	$(\mu^{14},\mu^{17},1)$	1949	$(\mu^{43},\mu^{11},1)$
1950	$(\mu^{30},\mu^{2},1)$	1951	$(\mu^{2},\mu^{40},1)$	1952	$(\mu^{37},\mu^{35},1)$	1953	$(\mu^{35},\mu^{42},1)$	1954	$(\mu^{20},\mu^{33},1)$
1955	$(\mu^{39},\mu^{21},1)$	1956	$(\mu^{34},\mu^{31},1)$	1957	$(\mu^{12},\mu^{14},1)$	1958	$(\mu^{32},\mu^{33},1)$	1959	$(\mu^{39},\mu^{18},1)$
1960	$(\mu^{40},\mu^{37},1)$	1961	$(\mu^{41},\mu^{4},1)$	1962	$(\mu^{44},\mu^{11},1)$	1963	$(\mu^{30},\mu^{30},1)$	1964	$(\mu^{10},1,1)$
1965	$(\mu^{45},\mu^{30},1)$	1966	$(\mu^{10},\mu^{39},1)$	1967	$(\mu^{3},\mu^{36},1)$	1968	$(\mu^{11},\mu^{43},1)$	1969	$(\mu^{28},\mu^{46},1)$
1970	$(\mu^{4},\mu^{14},1)$	1971	$(\mu^{32},\mu^{36},1)$	1972	$(\mu^{11},\mu^{38},1)$	1973	$(\mu^{25},\mu^{43},1)$	1974	$(\mu^{28},\mu^{22},1)$
1975	$(\mu^{27},\mu^{37},1)$	1976	$(\mu^{41},\mu^{34},1)$	1977	$(\mu^{46},\mu^{13},1)$	1978	$(\mu^{22},\mu^{18},1)$	1979	$(\mu^{40},\mu^{13},1)$
1980	$(\mu^{22},\mu^{33},1)$	1981	$(\mu^{39},\mu^{12},1)$	1982	$(\mu^{47},\mu^{44},1)$	1983	$(1,\mu^{39},1)$	1984	$(\mu^{3},\mu^{6},1)$
1985	$(\mu^{14},\mu^{46},1)$	1986	$(\mu^{4},\mu^{20},1)$	1987	$(\mu^{18},\mu^{22},1)$	1988	$(\mu^{27},\mu^{35},1)$	1989	$(\mu^{35},\mu^{28},1)$
1990	$(\mu^{38},\mu^{3},1)$	1991	$(\mu^{16},\mu^{39},1)$	1992	$(\mu^{3},\mu^{46},1)$	1993	$(\mu^{4},\mu^{36},1)$	1994	$(\mu^{11},\mu^{15},1)$
1995	$(\mu^{24},\mu^{42},1)$	1996	$(\mu^{20},0,1)$	1997	$(\mu^{13},\mu^{43},1)$	1998	$(\mu^{28},\mu^{6},1)$	1999	$(\mu^{14},\mu^{24},1)$
2000	$(\mu^{32},1,0)$	2001	$(0,\mu^{32},1)$	2002	$(\mu^{21},\mu^{8},1)$	2003	$(\mu^{29},\mu^{43},1)$	2004	$(\mu^{28},\mu^{40},1)$
2005	$(\mu^{37},\mu^{47},1)$	2006	$(\mu^{9},\mu^{16},1)$	2007	$(\mu^{5},\mu^{11},1)$	2008	$(\mu^{30},\mu^{7},1)$	2009	$(\mu^{26},\mu^{16},1)$
2010	$(\mu^{5},\mu^{4},1)$	2011	$(\mu^{44},\mu^{21},1)$	2012	$(\mu^{34},\mu^{34},1)$	2013	$(\mu^{46},1,1)$	2014	$(\mu^{45},\mu^{41},1)$
2015	$(\mu^{33},\mu^{14},1)$	2016	$(\mu^{32},\mu^{41},1)$	2017	$(\mu^{33},\mu^{12},1)$	2018	$(\mu^{47},\mu^{8},1)$	2019	$(\mu^{29},\mu^{20},1)$
2020	$(\mu^{18},\mu^{30},1)$	2021	$(\mu^{10},\mu^{18},1)$	2022	$(\mu^{40},\mu^{25},1)$	2023	$(\mu^{6},\mu^{17},1)$	2024	$(\mu^{43},\mu^{29},1)$
2025	$(\mu^{36},\mu^{8},1)$	2026	$(\mu^{29},\mu^{18},1)$	2027	$(\mu^{40},\mu^{4},1)$	2028	$(\mu^{44},\mu^{7},1)$	2029	$(\mu^{26},\mu^{26},1)$
2030	$(\mu,1,1)$	2031	$(\mu^{45},\mu^{37},1)$	2032	$(\mu^{41},\mu^{22},1)$	2033	$(\mu^{27},\mu^{42},1)$	2034	$(\mu^{20},\mu^{13},1)$
2035	$(\mu^{22},\mu^{4},1)$	2036	$(\mu^{44},\mu^{17},1)$	2037	$(\mu^{43},\mu^{43},1)$	2038	$(\mu^{28},1,1)$	2039	$(\mu^{45},\mu^{7},1)$
2040	$(\mu^{26},\mu^{7},1)$	2041	$(\mu^{26},\mu^{25},1)$	2042	$(\mu^{6},\mu^{5},1)$	2043	$(\mu^{23},\mu^{9},1)$	2044	$(\mu^{42},\mu^{11},1)$
2045	$(\mu^{30},\mu^{10},1)$	2046	$(\mu^{15},\mu^{5},1)$	2047	$(\mu^{23},\mu^{47},1)$	2048	$(\mu^{9},\mu^{26},1)$	2049	$(\mu,\mu^{7},1)$
2050	$(\mu^{26},\mu^{18},1)$	2051	$(\mu^{40},\mu^{39},1)$	2052	$(\mu^{3},\mu^{14},1)$	2053	$(\mu^{32},\mu^{16},1)$	2054	$(\mu^{5},\mu^{32},1)$
2055	$(\mu^{21},\mu^{46},1)$	2056	$(\mu^{4},\mu^{18},1)$	2057	$(\mu^{40},\mu^{44},1)$	2058	$(1,\mu^{11},1)$	2059	$(\mu^{30},\mu^{33},1)$
2060	$(\mu^{39},\mu^{29},1)$	2061	$(\mu^{36},\mu^{33},1)$	2062	$(\mu^{39},\mu^{28},1)$	2063	$(\mu^{38},\mu^{35},1)$	2064	$(\mu^{35},\mu^{10},1)$
2065	$(\mu^{15},\mu^{28},1)$	2066	$(\mu^{38},\mu^{14},1)$	2067	$(\mu^{32},\mu^{7},1)$	2068	$(\mu^{26},\mu^{5},1)$	2069	$(\mu^{23},\mu^{22},1)$
2070	$(\mu^{27},\mu^{44},1)$	2071	$(1,\mu^{41},1)$	2072	$(\mu^{33},\mu^{36},1)$	2073	$(\mu^{11},\mu^{20},1)$	2074	$(\mu^{18},\mu^{36},1)$
2075	$(\mu^{11},\mu^{19},1)$	2076	$(\mu^{17},\mu^{35},1)$	2077	$(\mu^{35},\mu^{40},1)$	2078	$(\mu^{37},\mu^{2},1)$	2079	$(\mu^{2},\mu^{9},1)$
2080	$(\mu^{42},\mu^{40},1)$	2081	$(\mu^{37},\mu^{17},1)$	2082	$(\mu^{43},\mu^{2},1)$	2083	$(\mu^{2},\mu^{22},1)$	2084	$(\mu^{27},\mu^{25},1)$
2085	$(\mu^{6},\mu^{47},1)$	2086	$(\mu^{9},\mu^{43},1)$	2087	$(\mu^{28},\mu^{34},1)$	2088	$(\mu^{46},\mu^{8},1)$	2089	$(\mu^{29},\mu^{25},1)$
2090	$(\mu^{6},\mu^{18},1)$	2091	$(\mu^{40},\mu^{26},1)$	2092	$(\mu,\mu^{12},1)$	2093	$(\mu^{47},\mu^{39},1)$	2094	$(\mu^{3},\mu^{42},1)$
2095	$(\mu^{20},\mu^{4},1)$	2096	$(\mu^{44},\mu^{26},1)$	2097	$(\mu,\mu,1)$	2098	$(\mu^{8},1,1)$	2099	$(\mu^{45},\mu^{16},1)$
2100	$(\mu^{5},\mu^{34},1)$	2101	$(\mu^{46},\mu^{23},1)$	2102	$(\mu^{31},\mu^{27},1)$	2103	$(\mu^{7},\mu^{43},1)$	2104	$(\mu^{28},\mu^{2},1)$
2105	$(\mu^{2},\mu^{12},1)$	2106	$(\mu^{47},\mu^{45},1)$	2107	$(\mu^{19},\mu^{10},1)$	2108	$(\mu^{15},\mu^{46},1)$	2109	$(\mu^{4},\mu^{28},1)$
2110	$(\mu^{38},\mu^{42},1)$	2111	$(\mu^{20},\mu^{43},1)$	2112	$(\mu^{28},\mu^{10},1)$	2113	$(\mu^{15},\mu^{25},1)$	2114	$(\mu^{6},\mu^{30},1)$
2115	$(\mu^{10},\mu^{44},1)$	2116	$(1,\mu^{33},1)$	2117	$(\mu^{39},\mu^{42},1)$	2118	$(\mu^{20},\mu^{17},1)$	2119	$(\mu^{43},\mu^{25},1)$
2120	$(\mu^{6},\mu^{26},1)$	2121	$(\mu,\mu^{35},1)$	2122	$(\mu^{35},\mu^{27},1)$	2123	$(\mu^{7},\mu^{20},1)$	2124	$(\mu^{18},\mu^{40},1)$
2125	$(\mu^{37},\mu^{45},1)$	2126	$(\mu^{19},\mu^{26},1)$	2127	$(\mu,\mu^{32},1)$	2128	$(\mu^{21},\mu^{13},1)$	2129	$(\mu^{22},\mu^{36},1)$
2130	$(\mu^{11},\mu^{32},1)$	2131	$(\mu^{21},\mu^{39},1)$	2132	$(\mu^{3},\mu^{17},1)$	2133	$(\mu^{43},\mu^{27},1)$	2134	$(\mu^{7},\mu^{27},1)$
2135	$(\mu^{7},\mu^{29},1)$	2136	$(\mu^{36},\mu^{10},1)$	2137	$(\mu^{15},\mu^{4},1)$	2138	$(\mu^{44},\mu^{20},1)$	2139	$(\mu^{18},\mu^{18},1)$
2140	$(\mu^{40},1,1)$	2141	$(\mu^{45},\mu^{8},1)$	2142	$(\mu^{29},\mu^{10},1)$	2143	$(\mu^{15},\mu^{27},1)$	2144	$(\mu^{7},\mu^{31},1)$
2145	$(\mu^{12},\mu^{34},1)$	2146	$(\mu^{46},\mu^{47},1)$	2147	$(\mu^{9},\mu^{5},1)$	2148	$(\mu^{23},\mu^{29},1)$	2149	$(\mu^{36},\mu^{5},1)$
2150	$(\mu^{23},\mu^{12},1)$	2151	$(\mu^{47},\mu^{16},1)$	2152	$(\mu^{5},\mu^{44},1)$	2153	$(1,\mu^{25},1)$	2154	$(\mu^{6},\mu^{9},1)$
2155	$(\mu^{42},\mu^{28},1)$	2156	$(\mu^{38},\mu^{18},1)$	2157	$(\mu^{40},\mu^{15},1)$	2158	$(\mu^{24},\mu^{35},1)$	2159	$(\mu^{35},0,1)$

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Table E.1 – continued from previous page (The points of $\mathrm{PG}(2,7^2)$**)**

2160	$(\mu^{13},\mu^{26},1)$	2161	$(\mu,\mu^{27},1)$	2162	$(\mu^{7},\mu^{47},1)$	2163	$(\mu^{9},\mu^{31},1)$	2164	$(\mu^{12},\mu^{18},1)$
2165	$(\mu^{40},\mu^{41},1)$	2166	$(\mu^{33},\mu^{44},1)$	2167	$(1,\mu^{9},1)$	2168	$(\mu^{42},\mu^{45},1)$	2169	$(\mu^{19},\mu^{47},1)$
2170	$(\mu^{9},\mu^{40},1)$	2171	$(\mu^{37},\mu^{43},1)$	2172	$(\mu^{28},\mu^{35},1)$	2173	$(\mu^{35},\mu^{45},1)$	2174	$(\mu^{19},\mu^{32},1)$
2175	$(\mu^{21},\mu^{4},1)$	2176	$(\mu^{44},\mu^{10},1)$	2177	$(\mu^{15},\mu^{15},1)$	2178	$(\mu^{24},1,1)$	2179	$(\mu^{45},0,1)$
2180	$(\mu^{13},\mu^{42},1)$	2181	$(\mu^{20},\mu^{46},1)$	2182	$(\mu^{4},\mu^{34},1)$	2183	$(\mu^{46},\mu^{2},1)$	2184	$(\mu^{2},\mu^{46},1)$
2185	$(\mu^{4},\mu^{2},1)$	2186	$(\mu^{2},\mu^{6},1)$	2187	$(\mu^{14},\mu^{12},1)$	2188	$(\mu^{47},\mu^{15},1)$	2189	$(\mu^{24},\mu^{15},1)$
2190	$(\mu^{24},0,1)$	2191	$(\mu^{13},0,1)$	2192	$(\mu^{13},\mu^{39},1)$	2193	$(\mu^{3},\mu^{29},1)$	2194	$(\mu^{36},\mu^{20},1)$
2195	$(\mu^{18},\mu^{7},1)$	2196	$(\mu^{26},\mu^{34},1)$	2197	$(\mu^{46},\mu^{45},1)$	2198	$(\mu^{19},\mu^{15},1)$	2199	$(\mu^{24},\mu^{7},1)$
2200	$(\mu^{26},0,1)$	2201	$(\mu^{13},\mu^{12},1)$	2202	$(\mu^{47},\mu^{25},1)$	2203	$(\mu^{6},\mu^{45},1)$	2204	$(\mu^{19},\mu^{5},1)$
2205	$(\mu^{23},\mu^{6},1)$	2206	$(\mu^{14},\mu^{31},1)$	2207	$(\mu^{12},\mu^{28},1)$	2208	$(\mu^{38},\mu^{39},1)$	2209	$(\mu^{3},\mu^{26},1)$
2210	$(\mu,\mu^{33},1)$	2211	$(\mu^{39},\mu^{31},1)$	2212	$(\mu^{12},\mu^{9},1)$	2213	$(\mu^{42},\mu^{43},1)$	2214	$(\mu^{28},\mu^{8},1)$
2215	$(\mu^{29},\mu^{39},1)$	2216	$(\mu^{3},\mu^{15},1)$	2217	$(\mu^{24},\mu^{8},1)$	2218	$(\mu^{29},0,1)$	2219	$(\mu^{13},\mu^{25},1)$
2220	$(\mu^{6},\mu^{32},1)$	2221	$(\mu^{21},\mu^{7},1)$	2222	$(\mu^{26},\mu^{40},1)$	2223	$(\mu^{37},\mu^{36},1)$	2224	$(\mu^{11},\mu^{18},1)$
2225	$(\mu^{40},\mu^{10},1)$	2226	$(\mu^{15},\mu^{26},1)$	2227	$(\mu,\mu^{25},1)$	2228	$(\mu^{6},\mu^{46},1)$	2229	$(\mu^{4},\mu^{38},1)$
2230	$(\mu^{25},\mu^{29},1)$	2231	$(\mu^{36},\mu^{30},1)$	2232	$(\mu^{10},\mu^{47},1)$	2233	$(\mu^{9},\mu^{42},1)$	2234	$(\mu^{20},\mu^{26},1)$
2235	$(\mu,\mu^{31},1)$	2236	$(\mu^{12},\mu^{4},1)$	2237	$(\mu^{44},\mu^{45},1)$	2238	$(\mu^{19},\mu^{19},1)$	2239	$(\mu^{17},1,1)$
2240	$(\mu^{45},\mu^{2},1)$	2241	$(\mu^{2},\mu^{31},1)$	2242	$(\mu^{12},\mu^{10},1)$	2243	$(\mu^{15},\mu^{16},1)$	2244	$(\mu^{5},\mu^{29},1)$
2245	$(\mu^{36},\mu^{13},1)$	2246	$(\mu^{22},\mu^{11},1)$	2247	$(\mu^{30},\mu^{3},1)$	2248	$(\mu^{16},\mu^{6},1)$	2249	$(\mu^{14},\mu^{9},1)$
2250	$(\mu^{42},\mu^{10},1)$	2251	$(\mu^{15},\mu^{43},1)$	2252	$(\mu^{28},\mu^{4},1)$	2253	$(\mu^{44},\mu^{6},1)$	2254	$(\mu^{14},\mu^{14},1)$
2255	$(\mu^{32},1,1)$	2256	$(\mu^{45},\mu^{24},1)$	2257	$(\mu^{19},1,0)$	2258	$(0,\mu^{19},1)$	2259	$(\mu^{17},\mu^{4},1)$
2260	$(\mu^{44},\mu,1)$	2261	$(\mu^{8},\mu^{8},1)$	2262	$(\mu^{29},1,1)$	2263	$(\mu^{45},\mu^{9},1)$	2264	$(\mu^{42},\mu^{23},1)$
2265	$(\mu^{31},\mu^{11},1)$	2266	$(\mu^{30},\mu^{18},1)$	2267	$(\mu^{40},\mu^{30},1)$	2268	$(\mu^{10},\mu^{21},1)$	2269	$(\mu^{34},\mu^{19},1)$
2270	$(\mu^{17},\mu^{19},1)$	2271	$(\mu^{17},\mu^{22},1)$	2272	$(\mu^{27},\mu^{32},1)$	2273	$(\mu^{21},\mu^{14},1)$	2274	$(\mu^{32},\mu^{46},1)$
2275	$(\mu^{4},\mu^{31},1)$	2276	$(\mu^{12},\mu^{16},1)$	2277	$(\mu^{5},\mu^{6},1)$	2278	$(\mu^{14},\mu^{39},1)$	2279	$(\mu^{3},\mu^{19},1)$
2280	$(\mu^{17},\mu,1)$	2281	$(\mu^{8},\mu^{13},1)$	2282	$(\mu^{22},\mu^{41},1)$	2283	$(\mu^{33},\mu^{6},1)$	2284	$(\mu^{14},\mu^{23},1)$
2285	$(\mu^{31},\mu^{47},1)$	2286	$(\mu^{9},\mu^{45},1)$	2287	$(\mu^{19},\mu^{25},1)$	2288	$(\mu^{6},\mu^{37},1)$	2289	$(\mu^{41},\mu^{27},1)$
2290	$(\mu^{7},\mu^{22},1)$	2291	$(\mu^{27},\mu,1)$	2292	$(\mu^{8},\mu,1)$	2293	$(\mu^{8},\mu^{27},1)$	2294	$(\mu^{7},\mu^{26},1)$
2295	$(\mu,\mu^{23},1)$	2296	$(\mu^{31},\mu^{23},1)$	2297	$(\mu^{31},\mu^{19},1)$	2298	$(\mu^{17},\mu^{5},1)$	2299	$(\mu^{23},\mu^{28},1)$
2300	$(\mu^{38},\mu^{7},1)$	2301	$(\mu^{26},\mu,1)$	2302	$(\mu^{8},\mu^{7},1)$	2303	$(\mu^{26},\mu^{45},1)$	2304	$(\mu^{19},\mu^{18},1)$
2305	$(\mu^{40},\mu^{23},1)$	2306	$(\mu^{31},\mu^{42},1)$	2307	$(\mu^{20},\mu^{8},1)$	2308	$(\mu^{29},\mu^{11},1)$	2309	$(\mu^{30},\mu^{42},1)$
2310	$(\mu^{20},\mu^{10},1)$	2311	$(\mu^{15},\mu^{45},1)$	2312	$(\mu^{19},\mu^{43},1)$	2313	$(\mu^{28},\mu^{11},1)$	2314	$(\mu^{30},\mu^{40},1)$
2315	$(\mu^{37},\mu^{27},1)$	2316	$(\mu^{7},\mu^{14},1)$	2317	$(\mu^{32},\mu^{6},1)$	2318	$(\mu^{14},\mu^{41},1)$	2319	$(\mu^{33},\mu,1)$
2320	$(\mu^{8},\mu^{17},1)$	2321	$(\mu^{43},\mu^{14},1)$	2322	$(\mu^{32},\mu^{4},1)$	2323	$(\mu^{44},\mu^{23},1)$	2324	$(\mu^{31},\mu^{31},1)$
2325	$(\mu^{12},1,1)$	2326	$(\mu^{45},\mu^{46},1)$	2327	$(\mu^{4},\mu^{33},1)$	2328	$(\mu^{39},\mu^{43},1)$	2329	$(\mu^{28},\mu^{25},1)$
2330	$(\mu^{6},\mu^{16},1)$	2331	$(\mu^{5},\mu^{39},1)$	2332	$(\mu^{3},\mu^{28},1)$	2333	$(\mu^{38},\mu^{22},1)$	2334	$(\mu^{27},\mu^{2},1)$
2335	$(\mu^{2},\mu^{43},1)$	2336	$(\mu^{28},\mu^{26},1)$	2337	$(\mu,\mu^{11},1)$	2338	$(\mu^{30},\mu^{22},1)$	2339	$(\mu^{27},\mu^{17},1)$
2340	$(\mu^{43},\mu^{36},1)$	2341	$(\mu^{11},\mu^{31},1)$	2342	$(\mu^{12},\mu^{30},1)$	2343	$(\mu^{10},\mu^{11},1)$	2344	$(\mu^{30},\mu^{15},1)$
2345	$(\mu^{24},\mu^{14},1)$	2346	$(\mu^{32},0,1)$	2347	$(\mu^{13},\mu^{40},1)$	2348	$(\mu^{37},\mu^{15},1)$	2349	$(\mu^{24},\mu^{31},1)$
2350	$(\mu^{12},0,1)$	2351	$(\mu^{13},\mu^{14},1)$	2352	$(\mu^{32},\mu^{10},1)$	2353	$(\mu^{15},\mu^{42},1)$	2354	$(\mu^{20},\mu^{44},1)$
2355	$(1,\mu^{30},1)$	2356	$(\mu^{10},\mu^{13},1)$	2357	$(\mu^{22},\mu^{7},1)$	2358	$(\mu^{26},\mu^{47},1)$	2359	$(\mu^{9},\mu^{8},1)$
2360	$(\mu^{29},\mu^{35},1)$	2361	$(\mu^{35},\mu^{47},1)$	2362	$(\mu^{9},\mu^{22},1)$	2363	$(\mu^{27},\mu^{33},1)$	2364	$(\mu^{39},\mu^{32},1)$
2365	$(\mu^{21},\mu^{18},1)$	2366	$(\mu^{40},\mu^{6},1)$	2367	$(\mu^{14},\mu^{25},1)$	2368	$(\mu^{6},\mu^{22},1)$	2369	$(\mu^{27},\mu^{13},1)$
2370	$(\mu^{22},\mu^{15},1)$	2371	$(\mu^{24},\mu^{45},1)$	2372	$(\mu^{19},0,1)$	2373	$(\mu^{13},\mu^{44},1)$	2374	$(1,\mu^{26},1)$
2375	$(\mu,\mu^{4},1)$	2376	$(\mu^{44},\mu^{36},1)$	2377	$(\mu^{11},\mu^{11},1)$	2378	$(\mu^{30},1,1)$	2379	$(\mu^{45},\mu^{35},1)$
2380	$(\mu^{35},\mu^{16},1)$	2381	$(\mu^{5},\mu^{18},1)$	2382	$(\mu^{40},\mu^{17},1)$	2383	$(\mu^{43},\mu^{6},1)$	2384	$(\mu^{14},\mu^{34},1)$

Continued on next page

Table E.1 – continued from previous page (The points of PG$(2, 7^2)$)

2385	$(\mu^{46}, \mu^{14}, 1)$	2386	$(\mu^{32}, \mu^{28}, 1)$	2387	$(\mu^{38}, \mu^{17}, 1)$	2388	$(\mu^{43}, \mu^{18}, 1)$	2389	$(\mu^{40}, \mu^{12}, 1)$
2390	$(\mu^{47}, \mu^{10}, 1)$	2391	$(\mu^{15}, \mu^{6}, 1)$	2392	$(\mu^{14}, \mu^{38}, 1)$	2393	$(\mu^{25}, \mu^{41}, 1)$	2394	$(\mu^{33}, \mu^{27}, 1)$
2395	$(\mu^{7}, \mu^{16}, 1)$	2396	$(\mu^{5}, \mu^{27}, 1)$	2397	$(\mu^{7}, \mu^{32}, 1)$	2398	$(\mu^{21}, \mu^{43}, 1)$	2399	$(\mu^{28}, \mu^{42}, 1)$
2400	$(\mu^{20}, \mu^{30}, 1)$	2401	$(\mu^{10}, \mu^{40}, 1)$	2402	$(\mu^{37}, \mu^{22}, 1)$	2403	$(\mu^{27}, \mu^{34}, 1)$	2404	$(\mu^{46}, \mu^{39}, 1)$
2405	$(\mu^{3}, \mu^{47}, 1)$	2406	$(\mu^{9}, \mu^{41}, 1)$	2407	$(\mu^{33}, \mu^{39}, 1)$	2408	$(\mu^{3}, \mu^{12}, 1)$	2409	$(\mu^{47}, \mu^{31}, 1)$
2410	$(\mu^{12}, \mu^{3}, 1)$	2411	$(\mu^{16}, \mu^{17}, 1)$	2412	$(\mu^{43}, \mu^{38}, 1)$	2413	$(\mu^{25}, \mu^{45}, 1)$	2414	$(\mu^{19}, \mu^{13}, 1)$
2415	$(\mu^{22}, \mu^{5}, 1)$	2416	$(\mu^{23}, \mu^{44}, 1)$	2417	$(1, \mu^{17}, 1)$	2418	$(\mu^{43}, \mu^{46}, 1)$	2419	$(\mu^{4}, \mu^{24}, 1)$
2420	$(\mu^{44}, 1, 0)$	2421	$(0, \mu^{44}, 1)$	2422	$(1, \mu^{35}, 1)$	2423	$(\mu^{35}, \mu^{38}, 1)$	2424	$(\mu^{25}, \mu^{38}, 1)$
2425	$(\mu^{25}, \mu^{19}, 1)$	2426	$(\mu^{17}, \mu^{11}, 1)$	2427	$(\mu^{30}, \mu^{35}, 1)$	2428	$(\mu^{35}, \mu^{25}, 1)$	2429	$(\mu^{6}, \mu^{19}, 1)$
2430	$(\mu^{17}, \mu^{3}, 1)$	2431	$(\mu^{16}, \mu^{21}, 1)$	2432	$(\mu^{34}, \mu^{29}, 1)$	2433	$(\mu^{36}, \mu^{38}, 1)$	2434	$(\mu^{25}, \mu^{14}, 1)$
2435	$(\mu^{32}, \mu^{26}, 1)$	2436	$(\mu, \mu^{28}, 1)$	2437	$(\mu^{38}, \mu^{30}, 1)$	2438	$(\mu^{10}, \mu^{33}, 1)$	2439	$(\mu^{39}, \mu^{24}, 1)$
2440	$(\mu^{3}, 1, 0)$	2441	$(0, \mu^{3}, 1)$	2442	$(\mu^{16}, \mu^{3}, 1)$	2443	$(\mu^{16}, \mu^{11}, 1)$	2444	$(\mu^{30}, \mu^{25}, 1)$
2445	$(\mu^{6}, \mu^{44}, 1)$	2446	$(1, \mu^{34}, 1)$	2447	$(\mu^{46}, \mu, 1)$	2448	$(\mu^{8}, \mu^{4}, 1)$	2449	$(\mu^{44}, \mu^{15}, 1)$
2450	$(\mu^{24}, \mu^{24}, 1)$								

Notation

G	an arbitrary group G
$\|S\|$	number of elements in the set S
$X \cap Y$	the intersection of X and Y
$X \cup Y$	the union of X and Y
$X \backslash Y$	the set of elements of X not in Y
$i \neq j$	i is not equal to j
$\emptyset$	the empty set
$s \in S$	x is an element of S
$G \cong H$	the groups G and H are isomorphic
$G_1 \times G_2$	a direct product of the groups G_1 and G_2
$H \rtimes N$	a semi-direct product of H and N
$\theta(n), \theta(n, q)$	$\frac{q^{n+1}-1}{q-1}$
$\mathbb{Z}_m$	cyclic group of order m
F	an arbitrary field
$F\backslash\{0\}$	the set of elements of F without 0
$\mathbb{F}_p$	a finite field with p prime elements
$\mathbb{F}_q$	a finite field with $q = p^h$ elements
Aut(S)	the automorphism group of S
t-(v, k, λ) design	a design with parameters t, v, k, λ
$V = V(n + 1, q)$	an $(n + 1)$-dimensional vector space over the finite field $\mathbb{F}_q$
$PG(n, q)$	an n-dimensional projective space over the finite field $\mathbb{F}_q$
$\prod_k$	a k-dimensional subspace of $PG(n, q)$
$P(X), P(x_0, \ldots, x_n)$	point of $PG(n, q)$ with vector $X = (x_0, \ldots, x_n)$
$\mathcal{T}, (\mathcal{T}, g)$	a projectivity
M(T)	a projectivity with matrix T
$C(f)$	companion matrix of polynomial f
$GL(n + 1, q)$	group of non-singular linear transformations of $V(n + 1, q)$
$PGL(n + 1, q)$	group of projectivities of $PG(n, q)$
$P\Gamma L(n + 1, q)$	group of collineations of $PG(n, q)$

(k, r)-arc, $\mathcal{K}$	a set of k points of $PG(2, q)$ with at most r collinear points
k-arc	a (k, r)-arc with $r = 2$
$\mathcal{B}$	a subplane of $PG(2, q)$
n	length of a code
k	dimension of a code
$d(x, y)$	distance between codewords x and y
$w(x)$	weight of x
$d = d(C)$	minimum distance of the code C
$[n, k, d]$ code	a code with parameters n, k, d
$C^{\perp}$	code dual to C

Bibliography

[1] I. Anderson. *Combinatorial Designs and Tournaments*, Oxford University Press, 1997.

[2] A. Beutelspacher and U. Rosenbaum. *Projective geometry: from foundations to applications*, Cambridge University Press, Cambridge, 1998.

[3] E. Boros and T. Szönyi. *On the sharpness of a theorem of B. Segre*, Combinatorica, volume 6, pages 261-268, 1986.

[4] A. E. Brouwer. *Bounds on the size of linear codes*, In Vera S. Pless and W. Cary Huffman, editors, Handbook of Coding Theory, volume 1, pages 295-461, Elsevier Science, 1998.

[5] R. H. Bruck. *Difference sets in a finite group*, Transactions of the American Mathematical Society, volume 78, pages 464-481, 1955.

[6] R. H. Bruck. *Quadratic extensions of cyclic planes*, Proceedings of Symposia in Applied Mathematics, volume 10, pages 15-44, 1960.

[7] J. C. Fisher, J. W. P. Hirschfeld, and J. A. Thas. *Complete arcs in planes of square order*, Annals of Discrete Mathematics, volume 30, pages 243-250, 1986.

[8] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.9.1*; 2018, (https://www.gap-system.org).

[9] P. B. Gibbons. *Computational methods in design theory*, In The CRC Handbook of Combinatorial Designs, C.J. Colbourn, J.H. Dinitz (eds.), CRC Press, Boca Raton, pages 718-740, 1996.

[10] J. W. P. Hirschfeld. *Projective Geometries Over Finite Fields*, Second Edition, Oxford University Press, Oxford, 1998.

[11] W. C. Huffman and V. Pless. *Fundamentals of Error-Correcting Codes*, Cambridge University Press, 2003.

[12] J. F. Humphreys. *A Course in Group Theory*, Oxford University Press, Oxford, 1996.

[13] B. C. Kestenband. *Projective geometries that are disjoint unions of caps*, Canadian Journal of Mathematics, volume 32, pages 1299-1305, 1980.

[14] B. C. Kestenband. *Unital intersections in finite projective planes*, Geometriae Dedicata, volume 11, pages 107-117, 1981.

[15] T. P. Kirkman. *On the perfect r-partitions of* $r^2 - r + 1$, Transactions of the Historic Society of Lancashire and Cheshire, volume 9, pages 127-142, 1857.

[16] R. Mathon. *Searching for spreads and packings*, In S.S. Magliveras, J.W.P. Hirschfeld and M.J. de Resmini, editors, Geometry, Combinatorial Designs and Related Structures. London Mathematical Society, Lecture Notes Series 245, pages 161-176. Cambridge University Press, 1997.

[17] R. Mathon and N. Hamilton. *Baer partitions of small order projective planes*, The Journal of Combinatorial Mathematics and Combinatorial Computing, volume 29, pages 87-94, 1999.

[18] M. Pankov. *Geometry of semilinear embeddings: Relations to graphs and codes*, World Scientific, 2015.

[19] J. Singer. *A theorem in finite projective geometry and some applications to number theory*, Transactions of the American Mathematical Society, volume 43, pages 377-385, 1938.

[20] P. Yff. *On subplane partitions of a finite projective plane*, Journal of Combinatorial Theory, volume 22, pages 118-122, 1977.