UNIVERSITY OF SUSSEX

**School of Mathematics and Physical Sciences**

**Department of Mathematics**

# Rigidity for perimeter inequalities under symmetrisation

**Matteo Perugini**

Supervisor

**Filippo Cagnetti**

Submitted for the degree of Doctor of Philosophy

University of Sussex

9th April 2019

# Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

Signature:

Matteo Perugini

# ABSTRACT PHD THESIS

MATTEO PERUGINI

My PhD thesis contains a couple of results I obtained under the supervision of my advisor Filippo Cagnetti, during the past three years of my studies. In particular, I present two results about rigidity of perimeter inequality under symmetrization techniques. The first result, presented in Chapter 3, provides the characterization of rigidity for equality cases for the perimeter inequality under spherical symmetrization; whereas in Chapter 4 I will study the rigidity of equality cases for the Steiner's inequality for the anisotropic perimeter.

*"The mathematical sciences particularly exhibit order symmetry and limitations; and these are the greatest forms of the beautiful."*

Aristotle

# Acknowledgements

Vorrei ringraziare alcune persone che hanno inevitabilmente segnato questi ultimi 3 anni e mezzo della mia vita. In ordine di apparizione: il mio mentore e advisor Filippo Cagnetti, per avermi insegnato a fare Matematica e aver sempre cercato di spingermi fuori dalla mia "comfort zone", aiutandomi così a superare alcuni limiti che mi sarei altrimenti continuato a portare appresso; Brighton e la sua multi-cultura che mi ha fatto capire molte cose della vita che forse avrei perso restandomene in Italia. Vorrei ringraziare anche alcuni amici, compagni di merende e bagordi, nonché bellissime avventure. Tra questi Antoine, Davide, Peter, Silvio, Farzad, Victor, Yacine, Chiara, Patrick e Hilary.

Per concludere, vorrei ringraziere la mia famiglia e i miei amici che ho lasciato in Italia o in giro per il mondo. A loro devo molto, tra cui il sostegno morale e psicologico durante i lunghi periodi passati lontano da casa, dimostrandomi ancora una volta che non è la distanza ad uccidere un rapporto ma bensì la mancanza di volontà a coltivarlo.

# Contents

# Chapter 1

# Introduction

The aim of this thesis is the study of perimeter inequalities under symmetrisation. In particular, we are interested in the understanding of rigidity, that is, the situation in which the only extremals of the inequality are symmetric sets.

We start by studying rigidity for the perimeter inequality under spherical symmetrisation. This is the subject of Chapter 3. After that, in Chapter 4 we consider the rigidity problem for Steiner's inequality for the anisotropic perimeter. These results are collected in [12] and [35], respectively.

## 1.1 State of the art

Perimeter inequalities under symmetrisation have been studied by many authors, see for instance [30, 31] and the references therein. The study of rigidity for such inequalities can have important applications and can lead, for instance, to show that minimisers of variational problems (or solutions of PDEs) are symmetric.

Indeed for instance, Ennio De Giorgi in his proof of the Isoperimetric Inequality, using Steiner's inequality (see (1.1.2)) showed that the minimum for the Isoperimetric problem is a convex set. After De Giorgi, an important contribution in the understanding of rigidity for Steiner's inequality was given by Chlebík, Cianchi, and Fusco. In the seminal paper [14], the authors give sufficient conditions for rigidity. After that, this result was extended to the case of higher codimensions in [3], where a quantitative version of Steiner's inequality is also given. Finally, necessary and sufficient conditions for rigidity (in codimension 1) are given in [11], in the case where the distribution function is a Special Function of Bounded Variation with locally finite jump set. In the Gaussian setting, where the analogous of Steiner's inequality is given by Ehrhard's inequality (see [17, Section 4.1]), necessary and sufficient conditions for rigidity are given in [10], by making use of the notion of essential connectedness.

### 1.1.1 Basic notions on sets of finite perimeter

For every $r > 0$ and $x \in \mathbb{R}^n$, we denote by $B(x,r)$ the open ball of $\mathbb{R}^n$ with radius $r$ centred at $x$. In the special case $x = 0$, we set $B(r) := B(0,r)$. Let $n, k \in \mathbb{N}$, and $\delta > 0$. The $k$-dimensional Hausdorff measure of step $\delta$ of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^k_\delta(E) := \inf_{\mathcal{F}} \sum_{F \in \mathcal{F}} \omega_k \frac{\mathrm{diam}(F)}{2}^k,$$

where $\mathcal{F}$ is a countable covering of $E$ by sets $F \subset \mathbb{R}^n$ such that $\mathrm{diam}(F) < \delta$, and $\omega_k = \mathcal{L}^k(B(1))$ (where $B(1)$ is the unitary open ball in $\mathbb{R}^k$). The *$k$-dimensional Hausdorff measure* of $E$ is then

$$\mathcal{H}^k(E) := \sup_{\delta > 0} \mathcal{H}^k_\delta(E) = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E).$$

Let $E \subset \mathbb{R}^n$ be a measurable set, and let $t \in [0,1]$. We denote by $E^{(t)}$ the set of points of density $t$ of $E$, given by

$$E^{(t)} := \left\{ x \in \mathbb{R}^n : \lim_{\rho \to 0^+} \frac{\mathcal{H}^n(E \cap B(x,\rho))}{\omega_n \rho^n} = t \right\}.$$

The essential boundary of $E$ is then defined as

$$\partial^{\mathrm{e}} E := E \setminus (E^{(1)} \cup E^{(0)}).$$

Moreover, if $A \subset \mathbb{R}^n$ is any Borel set, we define the perimeter of $E$ relative to $A$ as the extended real number given by

$$P(E; A) := \mathcal{H}^{n-1}(\partial^{\mathrm{e}} E \cap A) \in [0, \infty].$$

We then define the perimeter of $E$ as $P(E) := P(E; \mathbb{R}^n)$. When $E$ is a set with smooth boundary, it turns out that $\partial^{\mathrm{e}} E = \partial E$, and the perimeter of $E$ agrees with the usual notion of $(n-1)$-surface dimensional measure of $\partial E$. If $P(E) < \infty$, it is possible to define the reduced boundary $\partial^* E$ of $E$. This has the property that $\partial^* E \subset \partial^{\mathrm{e}} E$, $\mathcal{H}^{n-1}(\partial^{\mathrm{e}} E \backslash \partial^* E) = 0$, and is such that for every $x \in \partial^* E$ there exists the measure theoretic outer unit normal $\nu^E(x)$ to $E$ at $x$ (see Section 2).

### 1.1.2 Steiner's inequality

Let us now recall how Steiner symmetrisation is defined. We decompose $\mathbb{R}^n$, $n \geq 2$, as the Cartesian product $\mathbb{R}^{n-1} \times \mathbb{R}$, denoting by $\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\mathbf{q} : \mathbb{R}^n \to \mathbb{R}$ the "horizontal" and "vertical" projections respectively, so that $x = (\mathbf{p}x, \mathbf{q}x)$, $\mathbf{p}x = (x_1, \ldots, x_{n-1})$, and $\mathbf{q}x = x_n$ for every $x \in \mathbb{R}^n$. Given a function $v : \mathbb{R}^{n-1} \to [0, \infty]$, we say that a set $E \subset \mathbb{R}^n$ is $v$-*distributed* if, denoting by $E_z$ its vertical section with respect to $z \in \mathbb{R}^{n-1}$, that is

$$E_z := \{t \in \mathbb{R} : (z,t) \in E\}, \quad z \in \mathbb{R}^{n-1},$$

we have that

$$v(z) = \mathcal{H}^1(E_z), \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } z \in \mathbb{R}^{n-1}.$$

Among all $v$-distributed sets, we denote by $F[v]$ the only one that is symmetric by reflection with respect to $\{\mathbf{q}x = 0\}$, and whose vertical sections are segments, that is

$$F[v] := \left\{ x \in \mathbb{R}^n : |\mathbf{q}x| < \frac{v(\mathbf{p}x)}{2} \right\}. \tag{1.1.1}$$

If $E$ is a $v$-distributed set, we define the Steiner symmetral $E^s$ of $E$ as $E^s := F[v]$. Note that, $F[v]$ is a Lebesgue measurable set, as shown in [21, Theorem 2.3]. Furthermore, by Fubini Theorem, Steiner symmetrisation preserves the volume. That is, if $E$ is a $v$-distributed set of finite volume, then $\mathcal{H}^n(E) = \mathcal{H}^n(F[v])$. A very important fact is that Steiner symmetrisation acts monotonically on the perimeter. More precisely, *Steiner's inequality* holds true: if $E$ is a $v$-distributed set then

$$P(E; G \times \mathbb{R}) \geq P(F[v]; G \times \mathbb{R}) \quad \text{for every Borel set } G \subset \mathbb{R}^{n-1}. \tag{1.1.2}$$

The next two results give the minimal regularity assumptions needed to study inequality (1.1.2) (see [14, Lemma 3.1] and [11, Proposition 3.2] respectively).

**Lemma 1.1.1.** (Chlebík, Cianchi and Fusco) *Let $E$ be a $v$-distributed set of finite perimeter in $\mathbb{R}^n$, for some measurable function $v : \mathbb{R}^{n-1} \to [0, \infty]$. Then, one and only one of the following two possibilities is satisfied:*

*i) $v(x') = \infty$ for $\mathcal{H}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$ and $F[v]$ is $\mathcal{H}^n$-equivalent to $\mathbb{R}^n$;*

*ii) $v(x') < \infty$ for $\mathcal{H}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, $\mathcal{H}^n(F[v]) < \infty$, and $v \in BV(\mathbb{R}^{n-1})$,*

*where $BV(\mathbb{R}^{n-1})$ denotes the space of functions of bounded variation in $\mathbb{R}^{n-1}$ (see Section 2).*

**Lemma 1.1.2.** *Let $v : \mathbb{R}^{n-1} \to [0, \infty)$ be measurable. Then, we have $0 < \mathcal{H}^n(F[v]) < \infty$ and $P(F[v]) < \infty$ if and only if*

$$v \in BV(\mathbb{R}^{n-1}), \quad \textit{and} \quad \mathcal{H}^{n-1}(\{v > 0\}) < \infty. \tag{1.1.3}$$

### 1.1.3 Rigidity for Steiner's inequality

Given $v$ as in (1.1.3) we set:

$$\mathcal{M}(v) = \{E \subset \mathbb{R}^n : E \text{ is } v\text{-distributed and } P(E) = P(F[v])\} . \tag{1.1.4}$$

We say that *rigidity* holds true for Steiner's inequality if the only elements of $\mathcal{M}(v)$ are ($\mathcal{H}^n$-equivalent to) vertical translations of $F[v]$, namely:

$$E \in \mathcal{M}(v) \quad \Longleftrightarrow \quad \mathcal{H}^n(E\Delta(F[v] + te_n)) = 0 \quad \text{ for some } t \in \mathbb{R}, \tag{RS}$$

where $\Delta$ stands for the symmetric difference between sets, and $e_1, \ldots, e_n$ are the elements of the canonical basis of $\mathbb{R}^n$.

A natural step in order to understand when (RS) holds true, is to study the set $\mathcal{M}(v)$. The characterization of equality cases in (1.1.2) was first addressed by Ennio De Giorgi in [19], where he showed that any set $E \in \mathcal{M}(v)$ is such that

$$E_z \text{ is } \mathcal{H}^1\text{-equivalent to a segment,} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } z \in \mathbb{R}^{n-1}, \tag{1.1.5}$$

(see also [32, Theorem 14.4]). After that, further information about $\mathcal{M}(v)$ was given by by Chlebík, Cianchi and Fusco (see [14, Theorem 1.1]). The study of equality cases in Steiner's inequality was then resumed by Cagnetti, Colombo, De Philippis and Maggi in [11], where the authors give a complete characterization of elements of $\mathcal{M}(v)$ (see Theorem 1.1.4 below). In order to explain their result, let us observe that any $v$-distributed set $E$ satisfying (1.1.5) is uniquely determined by the barycenter function $b_E : \mathbb{R}^{n-1} \to \mathbb{R}$, defined as:

$$b_E(z) = \begin{cases} \frac{1}{v(z)} \int_{E_z} t d\mathcal{H}^1(t) & \text{if } 0 < v(z) < \infty \\ 0, & \text{otherwise.} \end{cases} \tag{1.1.6}$$

In general, $b_E$ may fail to be a $BV$, or even an $L^1_{loc}$ function, even if $E$ is a set of finite perimeter (see [11, Remark 3.5]). The optimal regularity for $b_E$, when $E$ satisfies (1.1.5), is given by the following result (see [11, Theorem 1.7]).

**Theorem 1.1.3.** *Let $v$ be as in (1.1.3), and let $E$ be a $v$-distributed set of finite perimeter satisfying (1.1.5). Then,*

$$b_\delta = 1_{\{v>\delta\}} \, b_E \in GBV(\mathbb{R}^{n-1}),$$

*for every $\delta > 0$ such that $\{v > \delta\}$ is a set of finite perimeter. Moreover, $b_E$ is approximately differentiable $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$, and for every Borel set $G \subset \{v^\vee > 0\}$ the following* *coarea formula holds:*

$$\int_{\mathbb{R}} \mathcal{H}^{n-2}(G \cap \partial^e\{b_E > t\})dt = \int_G |\nabla b_E| d\mathcal{H}^{n-1} + \int_{G\cap S_{b_E}} [b_E] d\mathcal{H}^{n-2} + |D^c b_E|^+(G), \tag{1.1.7}$$

*where $|D^c b_E|^+$ is the Borel measure on $\mathbb{R}^{n-1}$ defined by*

$$|D^c b_E|^+(G) := \lim_{\rho\to 0^+} |D^c b_\delta|(G) = \sup_{\delta>0} |D^c b_\delta|(G), \qquad \forall\, G \subset \mathbb{R}^{n-1}.$$

Here $GBV$ is the space of functions of generalized bounded variation, $v^\vee$ and $v^\wedge$ are the approximate limsup and approximate liminf of $v$ respectively, $[b_E] := b_E^\vee - b_E^\wedge$ is the jump of $b_E$, and $D^c b_\delta$ is the Cantor part of the distributional derivative $Db_\delta$ of $b_\delta$ (for more details see Chapter 2). Starting from this result, the authors were able to establish a formula for the perimeter of $E$ in terms of $v$ and $b_E$ (see [11, Corollary 3.3]). With this formula at hands, as shown in the next result (see [11, Theorem 1.9]), they managed to fully characterize the equality cases in Steiner's perimeter inequality. Below, we set $\tau_M(s) := \max\{-M, \min\{M, s\}\}$ for every $s \in \mathbb{R}$.

**Theorem 1.1.4.** *Let $v$ be as in (1.1.3), and let $E$ be a $v$-distributed set of finite perimeter. Then, $E \in \mathcal{M}(v)$ if and only if*

$$E_z \text{ is } \mathcal{H}^1\text{-equivalent to a segment}, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } z \in \mathbb{R}^{n-1}, \tag{1.1.8}$$

$$\nabla b_E(z) = 0, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } z \in \mathbb{R}^{n-1}, \tag{1.1.9}$$

$$2[b_E] \le [v], \qquad \mathcal{H}^{n-2}\text{-a.e. on } \{v^\wedge > 0\}, \tag{1.1.10}$$

$$D^c\left(\tau_M(b_\delta)\right)(G) = \int_{G\cap\{v>\delta\}^{(1)}\cap\{|b_E|<M\}^{(1)}} f\, d(D^c v), \tag{1.1.11}$$

*for every bounded Borel set $G \subset \mathbb{R}^{n-1}$ and $M > 0$, and for $\mathcal{H}^1$-a.e. $\delta > 0$, where $f : \mathbb{R}^{n-1} \to [-1/2, 1/2]$ is a Borel function. In particular, if $E \in \mathcal{M}(v)$ then*

$$2|D^c b_E|^+(G) \le |D^c v|(G), \quad \text{for every Borel set } G \subset \mathbb{R}^{n-1}, \tag{1.1.12}$$

*and, if $K$ is a concentration set for $D^c v$ and $G$ is a Borel subset of $\{v^\wedge > 0\}$, then*

$$\int_{\mathbb{R}} \mathcal{H}^{n-2}(G \cap \partial^e\{b_E > t\})dt = \int_{G\cap S_{b_E}\cap S_v} [b_E] d\mathcal{H}^{n-2} + |D^c b_E|^+(G \cap K). \tag{1.1.13}$$

Theorem 1.1.3 and Theorem 1.1.4 play a key role in the study of rigidity. Indeed, (RS) holds true if and only if the following condition is satisfied:

$$E \in \mathcal{M}(v) \quad \Longleftrightarrow \quad b_E \text{ is } \mathcal{H}^{n-1}\text{-a.e. constant on } \{v > 0\}. \tag{1.1.14}$$

Based on the previous results, the authors proved several rigidity results, depending of the regularity assumptions on $v$ (see [11, Theorems 1.11-1-30]). In particular, a complete characterization of rigidity is given when $v$ is a special function of bounded variation with locally finite jump set (see [11, Theorem 1.29]).

## 1.2 Rigidity for the perimeter inequality under spherical symmetrisation

The spherical symmetrisation is a useful tool to understand the symmetry properties of solutions of certain PDEs and variational problems, when the radial symmetry has been ruled out. This turns out to be helpful also because some well established techniques, as for instance the moving plane method [38, 26], rely on convexity properties of the domain which fail, for example, when one deals with annuli. Indeed, in many applications minimisers of variational problems and solutions of PDEs turn out to be *foliated Schwarz symmetric*. Roughly speaking, a function $u : \mathbb{R}^n \to \mathbb{R}$ is foliated Schwarz symmetric if one can find a direction $p \in \mathbb{S}^{n-1}$ (here $\mathbb{S}^{n-1} := \partial B(1)$) such that $u$ only depends on $|x|$ and on the polar angle $\alpha = \arccos(\hat{x} \cdot p)$ (here $\hat{x} := x/|x|$, $|x|$ is the modulus of $x$ and $\hat{x} \cdot p$ is the scalar product between $\hat{x}$ and $p$), and $u$ is non increasing with respect to $\alpha$. We direct the interested reader to [4, 5, 6, 40] and the references therein for more information.

### 1.2.1 Spherical Symmetrisation

To the best of our knowledge, the spherical symmetrisation was first introduced by Pólya in [36], in the case $n = 2$ and in the smooth setting. Let $n \geq 2$. Given a set $E \subset \mathbb{R}^n$ and $r > 0$, we define the *spherical slice* $E_r$ of $E$ with respect to $\partial B(r)$ as

$$E_r := E \cap \partial B(r) = \{x \in \partial B(r) : x \in E\}.$$

Let $v : (0, \infty) \to [0, \infty)$ be a measurable function. We say that $E$ is *spherically $v$-distributed* if

$$v(r) = \mathcal{H}^{n-1}(E_r), \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in (0, \infty). \tag{1.2.1}$$

Note that, in order $v$ to be an admissible distribution, one needs

$$v(r) \leq \mathcal{H}^{n-1}(\partial B(r)) = n\omega_n r^{n-1} \qquad \text{for } \mathcal{H}^1\text{-a.e. } r > 0. \tag{1.2.2}$$

For every $x, y \in \mathbb{S}^{n-1}$, the *geodesic distance* between $x$ and $y$ is given by

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(x, y) := \arccos(x \cdot y).$$

Let $r > 0$, $p \in \mathbb{S}^{n-1}$, and $\beta \in [0, \pi]$ be fixed. The *open geodesic ball* (or *spherical cap*) of centre $rp$ and radius $\beta$ is the set

$$\mathbf{B}_\beta(rp) := \{x \in \partial B(r) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, p) < \beta\}.$$

The $(n-1)$-dimensional Hausdorff measure of $\mathbf{B}_\beta(rp)$ can be explicitly calculated, and is given by

$$\mathcal{H}^{n-1}(\mathbf{B}_\beta(rp)) = (n-1)\omega_{n-1}r^{n-1}\int_0^\beta (\sin\tau)^{n-2}\,d\tau.$$

The expression above shows that the function $\beta \mapsto \mathcal{H}^{n-1}(\mathbf{B}_\beta(rp))$ is strictly increasing from $[0, \pi]$ to $[0, n\omega_n r^{n-1}]$. Therefore, if $v : (0, \infty) \to [0, \infty)$ is a measurable function satisfying (1.2.2), and $E \subset \mathbb{R}^n$ is a spherically $v$-distributed set, there exists only one measurable function $\alpha_v : (0, \infty) \to [0, \pi]$ satisfying

$$v(r) = \mathcal{H}^{n-1}(\mathbf{B}_{\alpha_v(r)}(re_1)) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in (0, \infty). \tag{1.2.3}$$

Among all the spherically $v$-distributed sets of $\mathbb{R}^n$, we denote by $F_v$ the one whose spherical slices are open geodesic balls centred at the positive $e_1$ axis., i.e.

$$F_v := \{x \in \mathbb{R}^n \setminus \{0\} : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) < \alpha_v(|x|)\},$$

see Figure 1.2.1. Given any spherically $v$-distributed set $E$, we refer to $F_v$ as the *spherical symmetral* of $E$. As mentioned for the Steiner symmetrisation (see [21, Theorem 2.3]), also for the spherical symmetrisation it can be proved that $F_v$ is a Lebesgue measurable set.

### 1.2.2 Perimeter inequality under spherical symmetrisation

If $x \in \partial^* E$, it will be convenient to decompose $\nu^E(x)$ as

$$\nu^E(x) = \nu^E_\perp(x) + \nu^E_\parallel(x),$$

where $\nu^E_\perp(x) := (\nu^E(x)\cdot\hat{x})\hat{x}$ and $\nu^E_\parallel(x)$ are the radial and tangential component of $\nu^E(x)$ along $\partial B(|x|)$, respectively. We will also use the diffeomorphism $\Phi : (0, \infty) \times \mathbb{S}^{n-1} \to \mathbb{R}^n_0$ defined as

$$\Phi(r, \omega) := r\omega \qquad \text{for every } (r, \omega) \in (0, \infty) \times \mathbb{S}^{n-1},$$

where $\mathbb{R}^n_0 := \mathbb{R}^n \setminus \{0\}$. Our first result shows that the spherical symmetrisation decreases the perimeter, and gives some necessary conditions for equality cases. In our analysis we require the set $F_v$ (or, equivalently, any spherically $v$-distributed set) to have finite volume. This is not restrictive. Indeed, if $F_v$ has finite perimeter but infinite volume,

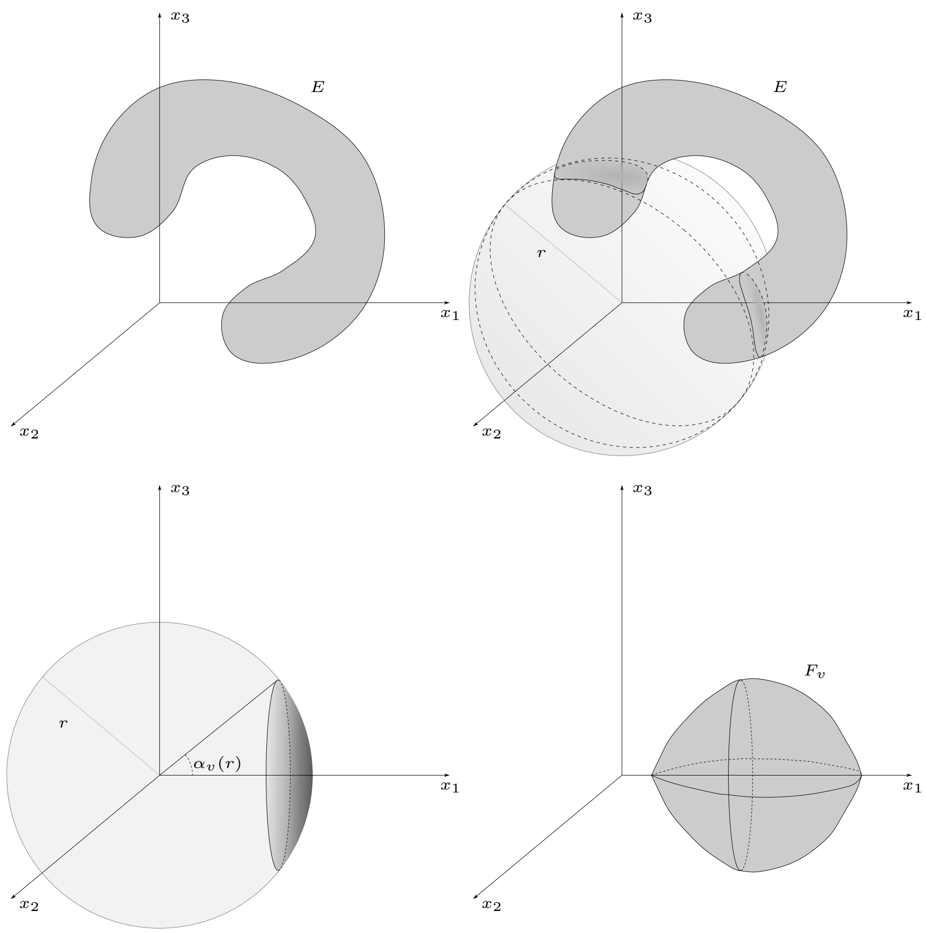


Figure 1.2.1: A pictorial idea of the spherical symmetral $F_v$ of a spherically $v$-distributed set $E$, in the case $n = 3$.

we can consider the complement $\mathbb{R}^n \setminus F_v$ which, by the relative isoperimetric inequality, has finite volume. This change corresponds to considering the complementary distribution function $r \mapsto \omega_n r^n - v(r)$, and the spherical symmetrisation with respect to the axis $-e_1$.

**Theorem 1.2.1.** *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying (1.2.2), and let $E \subset \mathbb{R}^n$ be a spherically $v$-distributed set of finite perimeter and finite volume. Then, $v \in BV(0,\infty)$. Moreover, $F_v$ is a set of finite perimeter and*

$$P(F_v; \Phi(B \times \mathbb{S}^{n-1})) \le P(E; \Phi(B \times \mathbb{S}^{n-1})), \tag{1.2.4}$$

*for every Borel set $B \subset (0,\infty)$.*

*Finally, if $P(E) = P(F_v)$, then for $\mathcal{H}^1$-a.e. $r \in \{0 < \alpha_v < \pi\}$:*

*(a) $E_r$ is $\mathcal{H}^{n-1}$-equivalent to a spherical cap and $\mathcal{H}^{n-2}(\partial^*(E_r)\Delta(\partial^* E)_r) = 0$;*

*(b) the functions $x \mapsto \nu^E(x) \cdot \hat{x}$ and $x \mapsto |\nu^E_{\|}|(x)$ are constant $\mathcal{H}^{n-2}$-a.e. in $(\partial^* E)_r$.*

The result above shows that the perimeter inequality holds on a local level, provided one considers sets of the type $\Phi(B \times \mathbb{S}^{n-1})$, with $B \subset (0, \infty)$ Borel. Inequality (1.2.4) is very well known in the literature. In the special case $n = 2$, a short proof was given by Pólya [36]. In the general $n$-dimensional case with $B = (0, \infty)$ a sketch of the proof is given in [34, Theorem 6.2] (see also [33]). As mentioned by Morgan and Pratelli in [34], certain parts of the proof of (1.2.4) follow the general lines of analogous results in the context of Steiner symmetrisation (see, for instance, [14, Lemma 3.4], [3, Theorem 1.1]). There are, however, non trivial technical difficulties that arise when one deals with the spherical case. For this reason, we give a detailed proof of Theorem 1.2.1. The tools we develop to show this result will also be useful in the study of rigidity.

We start by introducing radial and tangential components of a Radon measure, see Section 3.1.1. Since we are dealing with a symmetrisation of codimension $n-1$, we need to pay attention to some delicate effects that are not usually observed when the codimension is 1 (as, for instance, in [14]). Indeed, a crucial role is played by the measure $\lambda_E$ given by:

$$\lambda_E(B) := \int_{\partial^* E \cap \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\parallel}^E = 0\}} \hat{x} \cdot \nu^E(x) \, d\mathcal{H}^{n-1}(x), \tag{1.2.5}$$

for every Borel set $B \subset (0, \infty)$. When $n = 2$, it turns out that $\lambda_E$ is singular with respect to the Lebesgue measure in $(0, \infty)$. However, for $n > 2$ it may happen that $\lambda_E$ contains a non trivial absolutely continuous part, see Remark 3.1.9. This requires some extra care while proving inequality (1.2.4). A similar phenomenon has already been observed in [3], in the study of the Steiner symmetrisation of codimension higher than 1. Higher codimension effects play an important role also in the study of rigidity, as explained below.

### 1.2.3 Rigidity in the spherical setting

Given $v : (0, \infty) \to [0, \infty)$ measurable, satisfying (1.2.2), and such that $F_v$ is a set of finite perimeter and finite volume, we define $\mathcal{N}(v)$ as the class of extremals of (1.2.4):

$$\mathcal{N}(v) := \{E \subset \mathbb{R}^n : E \text{ is spherically } v\text{- distributed and } P(E) = P(F_v)\}.$$

Note that, by definition of $F_v$, and by the invariance of the perimeter under rigid transformations, every time we apply a rotation to $F_v$ we obtain a set that belongs to $\mathcal{N}(v)$, i.e.:

$$\mathcal{N}(v) \supset \{E \subset \mathbb{R}^n : \mathcal{H}^n(E \Delta (R\, F_v)) = 0 \quad \text{for some } R \in SO(n)\},$$

where $SO(n)$ is the set of rotations in $\mathbb{R}^n$. We would like to understand when also the opposite inclusion is satisfied, that is, when the class of extremals of (1.2.4) is just given by rotated copies of $F_v$. We will say that *rigidity* holds true for inequality (1.2.4) if

$$\mathcal{N}(v) = \{E \subset \mathbb{R}^n : \mathcal{H}^n(E\Delta(R\,F_v)) = 0 \text{ for some } R \in SO(n)\}. \tag{$\mathcal{R}$}$$

In order to explain which conditions we should expect in order ($\mathcal{R}$) to be true, let us first give some examples.

Figure 1.2.2 shows a set $E \in \mathcal{N}(v)$ that cannot be obtained by applying a single rotation to $F_v$. This is due to the fact that the set $\{0 < \alpha_v < \pi\}$ is disconnected by

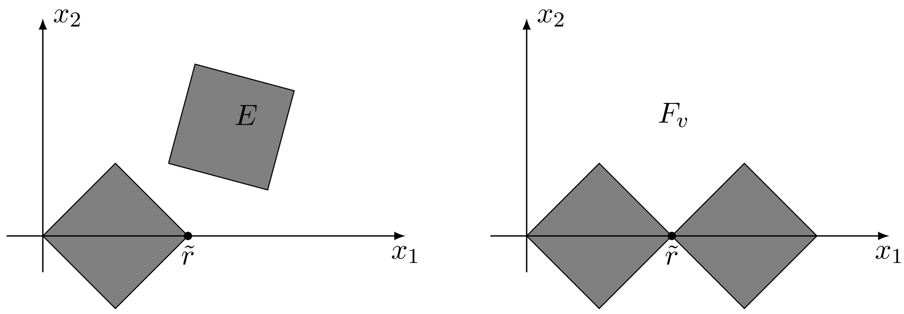


Figure 1.2.2: Rigidity ($\mathcal{R}$) fails, since the set $\{0 < \alpha_v < \pi\}$ is disconnected.

a point $\tilde{r}$ satisfying $\alpha_v(\tilde{r}) = 0$. Similar counterexamples can be provided also by using points belonging to the set $\{\alpha_v = \pi\}$. One possibility to avoid such a situation could be to request the set $\{0 < \alpha_v < \pi\}$ to be an interval. However, as Figure 1.2.3 shows, this condition depends on the representative chosen for $\alpha_v$, while the perimeters of the sets $E$ and $F_v$ don't. Indeed, in the previous example one can modify $\alpha_v$ just at the point $\tilde{r}$, in such a way that $\{0 < \alpha_v < \pi\}$ becomes an interval. Nevertheless, rigidity still fails.

To formulate a condition which is independent on the chosen representative, we consider the approximate liminf and the approximate limsup of $\alpha_v$, which we denote by $\alpha_v^\wedge$ and $\alpha_v^\vee$, respectively (see Section 2). These two functions are defined *at every point* $r \in (0,\infty)$ and satisfy $\alpha_v^\wedge \leq \alpha_v^\vee$. In addition, they do not depend on the representative chosen for $\alpha_v$, and $\alpha_v^\wedge = \alpha_v^\vee = \alpha_v$ $\mathcal{H}^1$-a.e. in $(0,\infty)$. The condition that we will impose is then the following:

$$\{0 < \alpha_v^\wedge \leq \alpha_v^\vee < \pi\} \text{ is an interval.} \tag{1.2.6}$$

One can check that in the example given in Figure 1.2.3 this condition fails, since $\alpha_v^\wedge(\tilde{r}) = \alpha_v^\vee(\tilde{r}) = 0$.

Let us show that, even imposing (1.2.6), rigidity can still be violated. In the example given in Figure 1.2.4, there is some radius $\overline{r} \in \{0 < \alpha_v^\wedge \leq \alpha_v^\vee < \pi\}$ such that the boundary of $F_v$ contains a non trivial subset of $\partial B(\overline{r})$. In this way, it is possible to rotate a proper

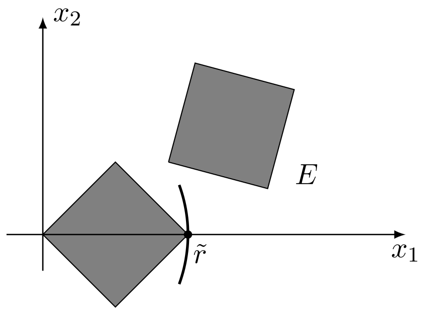


Figure 1.2.3: Modifying the function $\alpha_v$ given in Figure 1.2.2 at the point $\tilde{r}$, we can make sure that $\{0 < \alpha_v < \pi\}$ is an interval. However, rigitidy still fails.

subset of $F_v$ around the origin, without affecting the perimeter. Note that at each point of the set $\partial^* F_v \cap \partial B(\overline{r})$ the exterior normal $\nu^{F_v}$ is parallel to the radial direction. To rule out the situation described in Figure 1.2.4, we will impose the following condition:

$$\mathcal{H}^{n-1}(\{x \in \partial^* F_v : \nu_{\parallel}^{F_v}(x) = 0 \text{ and } |x| \in \{0 < \alpha_v^{\wedge} \leq \alpha_v^{\vee} < \pi\}) = 0. \tag{1.2.7}$$

Note that, from Theorem 1.2.1 and identity (1.2.3), it follows that in general we only have $\alpha_v \in BV_{\text{loc}}(0, \infty)$. However, it turns out that (1.2.7) is equivalent to ask that $\alpha_v \in W^{1,1}_{\text{loc}}(0, \infty)$, see Proposition 3.3.3.

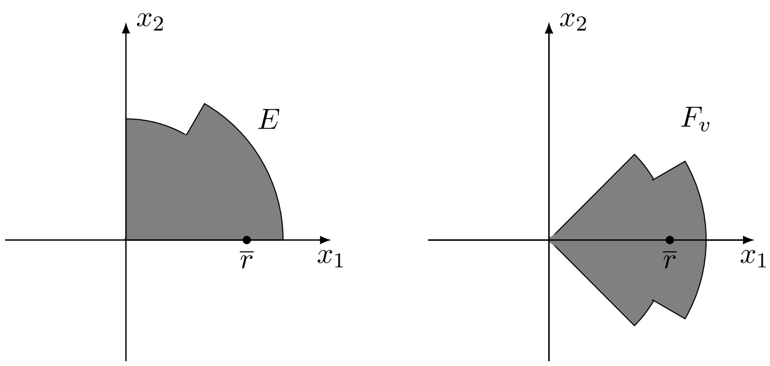


Figure 1.2.4: An example in which rigidity fails. In this case, the tangential part of $\partial^* F_v$ gives a non trivial contribution to $P(F_v)$. This allows to slide a proper subset of $F_v$ around the origin, without modifying the perimeter.

Our main result shows that the two conditions above give a complete characterisation of rigidity for inequality (1.2.4) (below, $\mathring{\mathcal{I}}$ stands for the interior of the set $\mathcal{I}$).

**Theorem 1.2.2.** *Let $v : (0, \infty) \to [0, \infty)$ be a measurable function satisfying (1.2.2) such that $F_v$ is a set of finite perimeter and finite volume, and let $\alpha_v$ be defined by (1.2.3). Then, the following two statements are equivalent:*

*(i)* $(\mathcal{R})$ *holds true;*

*(ii)* $\{0 < \alpha_v^\wedge \le \alpha_v^\vee < \pi\}$ *is a (possibly unbounded) interval* $\mathcal{I}$*, and* $\alpha_v \in W^{1,1}_{\mathrm{loc}}(\mathring{\mathcal{I}})$*.*

Let us point out that, although similar results in the context of Steiner and Ehrhard's inequalities already appeared in [11, 10], the proof of Theorem 1.2.2 cannot simply use previous ideas, especially in the implication (i) $\Longrightarrow$ (ii). We cannot rely, as in [11], on a general formula for the perimeter of sets $E$ satisfying equality in (1.2.4). Instead, we exhibit explicit counterexamples to rigidity, whenever one of the assumptions in (ii) fails. This requires a careful analysis of the transformations that one can apply to the set $F_v$, without modifying its perimeter. This turns out to be non trivial, especially if one assumes $\alpha_v$ to have a non zero Cantor part (see Proposition 3.5.4).

Also the proof of the implication (ii) $\Longrightarrow$ (i) presents some difficulties. In the context of Steiner symmetrisation, the analogous of (ii) $\Longrightarrow$ (i) has been proved in [14, Theorem 1.3] and [3, Theorem 1.2], for codimension 1 and generic codimension, respectively. In the spherical setting, this implication has already been stated in [34, Theorem 6.2], but a rigorous proof of this fact turns out to be more delicate than one would expect, and relies in the following result.

**Lemma 1.2.3.** *Let* $v : (0,\infty) \to [0,\infty)$ *be a measurable function satisfying* (1.2.2) *such that* $F_v$ *is a set of finite perimeter and finite volume. Let* $E \subset \mathbb{R}^n$ *be a spherically* $v$*-distributed set, and let* $I \subset (0,+\infty)$ *be a Borel set. Assume that*

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{\parallel}^{E}(x) = 0\right\}\right) = 0. \tag{1.2.8}$$

*Then,*

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F_v \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{\parallel}^{F_v}(x) = 0\right\}\right) = 0. \tag{1.2.9}$$

*Viceversa, let* (1.2.9) *be satisfied, and suppose that* $P(E;\Phi(I\times\mathbb{S}^{n-1})) = P(F_v;\Phi(I\times\mathbb{S}^{n-1}))$*. Then,* (1.2.8) *holds true.*

A direct proof of Lemma 1.2.3 does not seem to be obvious, due to the fact that, as pointed out above, the measure $\lambda_E$ defined in (1.2.5) can have an absolutely continuous part when $n > 2$. In the context of Steiner symmetrisation of higher codimension, the analogous of Lemma 1.2.3 (see [3, Proposition 3.6]) is proved using the fact that the result holds true in codimension 1, see [14, Proposition 4.2]. For this reason, we consider the following (codimension 1) circular symmetrisation, which was introduced by Pólya in the case $n = 3$ [36].

### 1.2.4 Circular symmetrisation

Let us choose an ordered pair of orthogonal directions in $\mathbb{R}^n$, which we will assume to be $(e_1, e_2)$. In the following, for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we will write $x = (x_{12}, x')$, where $x_{12} = (x_1, x_2) \in \mathbb{R}^2$ and $x' = (x_3, \ldots, x_n) \in \mathbb{R}^{n-2}$. When $x_{12} \neq 0$, we will write $\hat{x}_{12} := x_{12}/|x_{12}|$. For each given $z' \in \mathbb{R}^{n-2}$, we denote by $\Pi_{z'}$ the bi-dimensional plane defined by

$$\Pi_{z'} := \{x = (x_{12}, x') \in \mathbb{R}^2 \times \mathbb{R}^{n-2} : x' = z'\}.$$

Given a set $E \subset \mathbb{R}^n$ and $(r, z') \in (0, \infty) \times \mathbb{R}^{n-2}$, we define the *circular slice* $E_{(r,z')}$ of $E$ with respect to $\partial B\left((0, x'), r\right) \cap \Pi_{z'}$ as

$$E_{(r,z')} := \{x \in E : x' = z' \text{ and } x_1^2 + x_2^2 = r^2\}.$$

Let $\ell : (0, \infty) \times \mathbb{R}^{n-2} \to [0, \infty)$ be a measurable function. We say that $E$ is *circularly $\ell$-distributed* if

$$\ell(r, x') = \mathcal{H}^1(E_{(r,z')}), \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } (r, x') \in (0, \infty) \times \mathbb{R}^{n-2}.$$

If $\ell$ is a circular distribution, then we have

$$\ell(r, x') \leq \mathcal{H}^1\left(\partial B\left((0, x'), r\right) \cap \Pi_{z'}\right) = 2\pi r \tag{1.2.10}$$

for $\mathcal{H}^{n-1}$-a.e. $(r, x') \in (0, \infty) \times \mathbb{R}^{n-2}$. Among all the sets in $\mathbb{R}^n$ that are circularly $\ell$-distributed, we denote by $F^\ell$ the one whose circular slices are open circumference arcs centred at the positive $e_1$ axis. That is, we set

$$F^\ell := \left\{(x_{12}, x') \in \mathbb{R}^n \setminus \{x_{12} = 0\} : \operatorname{dist}_{\mathbb{S}^1}(\hat{x}_{12}, e_1) < \frac{1}{2r}\ell(r, x')\right\}.$$

In the following, we introduce the diffeomorphism $\Phi_{12} : (0, \infty) \times \mathbb{R}^{n-2} \times \mathbb{S}^1 \to \mathbb{R}^n \setminus \{x_{12} = 0\}$ given by

$$\Phi_{12}(r, x', \omega) := (r\omega, x') \qquad \text{for every } (r, x', \omega) \in (0, \infty) \times \mathbb{R}^{n-2} \times \mathbb{S}^1.$$

Moreover, for every $x \in \partial^* E$ we write $\nu^E(x) = (\nu^E_{12}(x), \nu^E_{x'}(x))$, where $\nu^E_{12}(x) = (\nu^E_1(x), \nu^E_2(x))$ and $\nu^E_{x'}(x) = (\nu^E_3(x), \ldots, \nu^E_n(x))$. Then, we further decompose $\nu^E_{12}(x)$ as

$$\nu^E_{12}(x) = \nu^E_{12\perp}(x) + \nu^E_{12\parallel}(x),$$

where $\nu^E_{12\perp}(x) := (\nu^E(x) \cdot \hat{x}_{12})\hat{x}_{12}$ and $\nu^E_{12\parallel}(x) := \nu^E_{12}(x) - \nu^E_{12\perp}(x)$. We can now state the analogous of Theorem 1.2.1.

**Theorem 1.2.4.** *Let $\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying (1.2.10), and let $E \subset \mathbb{R}^n$ be a circularly $\ell$-distributed set of finite perimeter and finite volume. Then, $\ell \in BV_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^{n-2})$. Moreover, $F^\ell$ is a set of finite perimeter and*

$$P(F^\ell; \Phi_{12}(B \times \mathbb{S}^1)) \leq P(E; \Phi_{12}(B \times \mathbb{S}^1)), \tag{1.2.11}$$

*for every Borel set $B \subset (0,\infty) \times \mathbb{R}^{n-2}$.*

*Finally, if $P(E) = P(F^\ell)$, then for $\mathcal{H}^{n-1}$-a.e. $(r,x') \in (0,\infty) \times \mathbb{R}^{n-2}$:*

*(a) $E_{(r,z')}$ is $\mathcal{H}^1$-equivalent to a circumference arc and $\partial^*(E_{(r,z')}) = (\partial^* E)_{(r,z')}$;*

*(b) the functions $x \mapsto \nu^E(x) \cdot \hat{x}_{12}$ and $x \mapsto |\nu^E_{12\|}|(x)$ are constant in $(\partial^* E)_{(r,z')}$.*

Let us mention that, in the smooth setting and in the case $n = 3$, inequality (1.2.11) was proved by Pólya. We can now state the analogous of Lemma 1.2.3.

**Lemma 1.2.5.** *Let $\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying (1.2.10) such that $F^\ell$ is a set of finite perimeter and finite volume. Let $E \subset \mathbb{R}^n$ be a circularly $\ell$-distributed set, and let $I \subset (0,\infty) \times \mathbb{R}^{n-2}$ be a Borel set. Assume that*

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E \cap \Phi(I \times \mathbb{S}^1) : \nu^E_{12\|}(x) = 0\right\}\right) = 0. \tag{1.2.12}$$

*Then,*

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F^\ell \cap \Phi(I \times \mathbb{S}^1) : \nu^{F^\ell}_{12\|}(x) = 0\right\}\right) = 0. \tag{1.2.13}$$

*Viceversa, let (1.2.13) be satisfied, and suppose that $P(E; \Phi(I \times \mathbb{S}^1)) = P(F^\ell; \Phi(I \times \mathbb{S}^1))$. Then, (1.2.12) holds true.*

Once Lemma 1.2.5 is established, we can show Lemma 1.2.3 through a slicing argument. Finally, the proof of (ii) $\Longrightarrow$ (i) is concluded by showing that, if $E$ satisfies equality in (1.2.4), the function associating to every $r \in (0,\infty)$ the center of $E_r$ (see (3.4.1)) is $W^{1,1}_{\mathrm{loc}}$ and, ultimately, constant (see Section 3.4).

## 1.3 Rigidity for the anisotropic perimeter inequality under Steiner symmetrisation

The second problem we address concerns the Steiner inequality for the anisotropic perimeter (see Chapter 4).

### 1.3.1 Anisotropic perimeter

Let us start by recalling some basic notions. A function $\phi : \mathbb{R}^n \to [0, \infty)$ is said to be *1-homogeneous* if

$$\phi(x) = |x|\phi\left(\frac{x}{|x|}\right) \qquad \forall\, x \in \mathbb{R}^n_0. \tag{1.3.1}$$

If $\phi$ is 1-homogeneous, then we say that it is *coercive* if there exists $c > 0$ such that

$$\phi(x) \geq c|x| \qquad \forall\, x \in \mathbb{R}^n. \tag{1.3.2}$$

In the following, we will assume that

$$K \subset \mathbb{R}^n \textit{ is open, bounded, convex and contains the origin.} \tag{1.3.3}$$

Given $K$ as in (1.3.3), one can define a one-homogeneous, convex and coercive function $\phi_K : \mathbb{R}^n \to [0, \infty)$ in this way:

$$\phi_K(x) := \sup\left\{x \cdot y :\ y \in K\right\}. \tag{1.3.4}$$

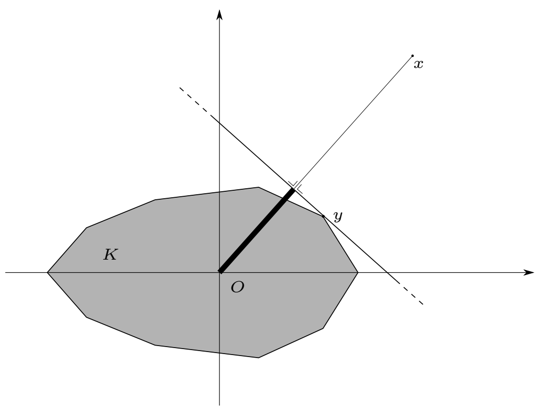


Figure 1.3.1: Note that $y$ is the point such that we have $\phi_K(x) = x \cdot y$. The length of the segment in bolt equals $\phi_K\left(\frac{x}{|x|}\right)$. Therefore, the line passing through $y$ orthogonal to the vector $x$ represents the hyperplane $\left\{y \in \mathbb{R}^n :\ y \cdot \frac{x}{|x|} = \phi_K\left(\frac{x}{|x|}\right)\right\}$.

By homogeneity, convexity of $\phi_K$ is equivalent to *subadditivity* (see for instance [32, Remark 20.2]), namely

$$\phi_K(x_1 + x_2) \leq \phi_K(x_1) + \phi_K(x_2), \qquad \forall\, x_1, x_2 \in \mathbb{R}^n. \tag{1.3.5}$$

Let us notice that there is a one to one correspondence between open, bounded and convex sets $K$ containing the origin and one-homogeneous, convex and coercive functions $\phi : \mathbb{R}^n \to$

$[0,\infty)$. Indeed, given a one-homogeneous, convex and coercive function $\phi : \mathbb{R}^n \to [0,\infty)$, then the set

$$K = \bigcap_{\omega \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : x \cdot \omega < \phi(\omega)\}\,, \tag{1.3.6}$$

satisfies (1.3.3), and is such that

$$\phi(x) = \sup\{x \cdot y : y \in K\} = \phi_K(x),$$

where $\phi_K$ is given by (1.3.4). Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and let $G \subset \mathbb{R}^n$ be a Borel set. Then, we observe that, the relative perimeter of $E$ with respect to $G$ can be written as

$$P(E;G) = \int_{\partial^* E \cap G} |\nu^E(x)| d\mathcal{H}^{n-1}(x).$$

Analogously, given $K \subset \mathbb{R}^n$ as in (1.3.3), we define the relative *anisotropic perimeter* of $E$ with respect to $G$ as

$$P_K(E;G) = \int_{\partial^* E \cap G} \phi_K(\nu^E(x)) d\mathcal{H}^{n-1}(x).$$

We define the anisotropic perimeter (with respect to $K$) $P_K(E)$ of $E$ as $P_K(E;\mathbb{R}^n)$. Observe that in the special case $\phi_K(x) = |x|$, this notion of perimeter agrees with the one above of Euclidean perimeter. Note that, in general, $\phi_K$ is not a norm, unless $\phi_K(x) = \phi_K(-x)$ for every $x \in \mathbb{R}^n$.

In the applications, the anisotropic perimeter can be used to describe the surface tension in the study of equilibrium configurations of solid crystals with sufficiently small grains [29, 43, 45], and represents the basic model for surface energies in phase transitions [27]. These applications motivate the study of the the *Wulff problem* (or *anisotropic isoperimetric problem*):

$$\inf\left\{\int_{\partial^* E} \phi_K(\nu^E(x)) d\mathcal{H}^{n-1}(x) : E \subset \mathbb{R}^n,\, \mathcal{H}^n(E) = \mathcal{H}^n(K)\right\}. \tag{1.3.7}$$

This name comes from the russian crystallographer Wulff, who was the first one to study (1.3.7) and who first conjectured that $K$ is the unique (modulo translations and scalings) minimizer of (1.3.7) (see [45]). Indeed the anisotropic perimeter inequality holds true:

$$P_K(K) \le P_K(E) \quad \textit{for every } E \subset \mathbb{R}^n \textit{ with } \mathcal{H}^n(E) = \mathcal{H}^n(K), \tag{1.3.8}$$

with equality if and only if $\mathcal{H}^n(K \Delta (E + x)) = 0$ for some $x \in \mathbb{R}^n$. The proof of the uniqueness was then given by Taylor (see [43]) and later, with a different method, by Fonseca and Müller (see [23]). We usually refer to $K$ as the *Wulff shape* for the surface tension $\phi_K$.

### 1.3.2 Steiner's inequality for the anisotropic perimeter

Note that the analogous of inequality (1.1.2) for the anisotropic perimeter in general fails. Indeed, choose $K$ as in (1.3.3) such that

$$\inf_{x \in \mathbb{R}^n} \mathcal{H}^n(K\Delta(K^s + x)) > 0,$$

where $K^s$ denotes the Steiner symmetral of $K$. Then, by (1.3.8), we have that

$$P_K(K) < P_K(K^s).$$

Let us give a simple example of the above inequality in dimension 2. Let $K$ and $K^s$ be as in Figure 1.3.2. Then, on can see that

$$P_K(K) = 8 < 10 = P_K(K^s),$$

see Figure 4.3.1. The above considerations show that, for an inequality as in (1.1.2) to hold true in the anisotropic setting, one should at least consider the perimeter $P_{K^s}$ with respect to the Steiner symmetral $K^s$ of $K$. Our first result gives the Steiner's inequality for the anisotropic perimeter. Let us mention that this result was already proved by Cianchi and Fusco in [16, Theorem 2.8].

**Theorem 1.3.1.** *Let $K \subset \mathbb{R}^n$ be as (1.3.3), let $K^s$ be its Steiner symmetral, and let $v$ as in (1.1.3). Then, for every $E \subset \mathbb{R}^n$ $v$-distributed we have*

$$P_{K^s}(E; G \times \mathbb{R}) \geq P_{K^s}(F[v]; G \times \mathbb{R}) \quad \textit{for every Borel set } G \subset \mathbb{R}^{n-1}. \tag{AS}$$

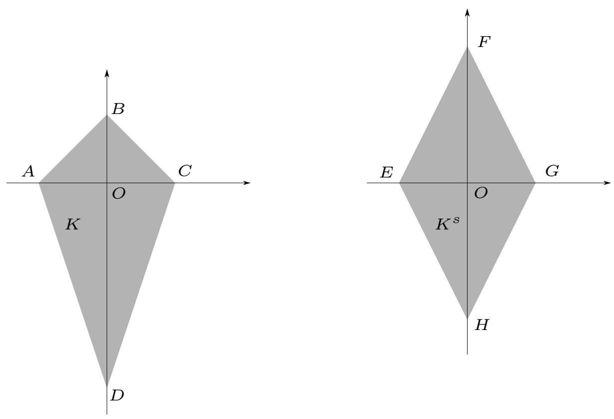


Figure 1.3.2: An example in which $P_K(K) < P_K(K^s)$. The coordinates of the vertices are $A = (-1, 0)$, $B = (0, 1)$, $C = (1, 0)$, $D = (0, -3)$, $E = (-1, 0)$, $F = (0, 2)$, $G = (1, 0)$, $H = (0, -2)$.

### 1.3.3 Rigidity for the Steiner's inequality for the anisotropic perimeter

Given $v$ as in (1.1.3), and $K \subset \mathbb{R}^n$ satisfying (1.3.3) we denote by

$$\mathcal{M}_{K^s}(v) := \{E \subset \mathbb{R}^n : E \text{ is } v\text{-distributed and } P_{K^s}(E) = P_{K^s}(F[v])\}\,, \tag{1.3.9}$$

the family of sets achieving equality in (AS). In this context, we say that *rigidity* holds true for (AS) if the only elements of $\mathcal{M}_{K^s}(v)$ are vertical translations of $F[v]$, namely

$$E \in \mathcal{M}_{K^s}(v) \quad \Longleftrightarrow \quad \mathcal{H}^n(E\Delta(F[v] + te_n)) = 0 \textit{ for some } t \in \mathbb{R}. \tag{RSA}$$

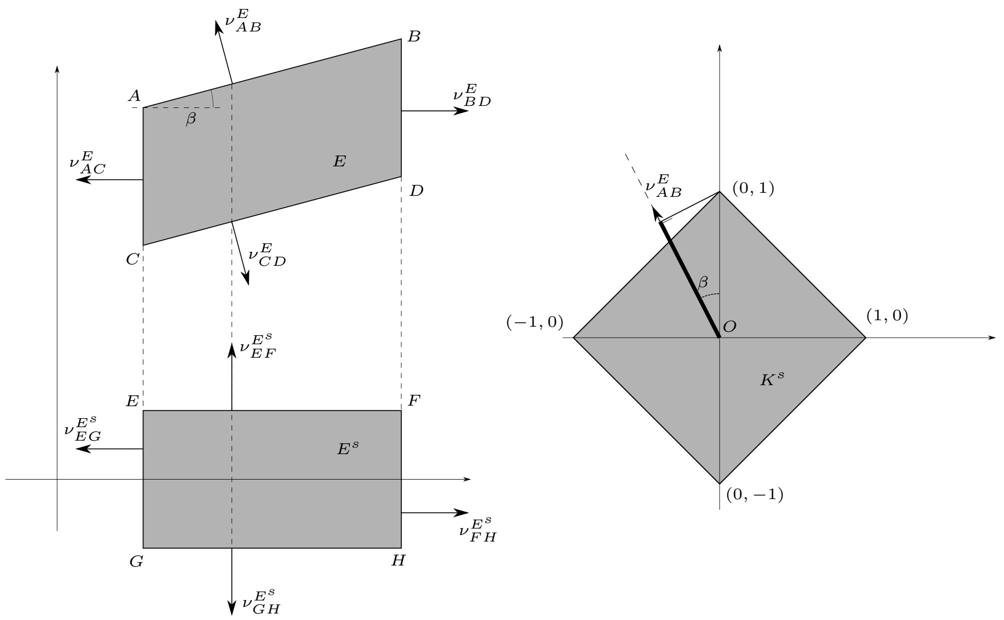


Figure 1.3.3: Suppose that $0 < \beta \leq \pi/4$. By definition of $\phi_{K^s}$, one can check that the length of the segment in bolt equals $\phi_{K^s}(\nu^E_{AB}) = \phi_{K^s}(\nu^E_{CD}) = \cos(\beta)$. As a consequence, we have $P_{K^s}(E) = P_{K^s}(E^s)$, even if $b'_E = \tan\beta \neq 0$.

As done for the study of (RS), let us first characterize the cases of equality (AS). We start by observing that the characterization of equality cases given in Theorem 1.1.4 fails when we deal with the anisotropic perimeter. In particular, let us show with an example in dimension 2, that condition (1.1.9) fails to be necessary. Let $K^s, E$, and $E^s$ be as in Figure 1.3.3. Observe that, although $b'_E = \tan(\beta) \neq 0$ we have $P_{K^s}(E) = P_{K^s}(E^s)$, if $0 < \beta \leq \pi/4$. Indeed, in this case

$$\begin{aligned} P_{K^s}(E) &= \phi_{K^s}(\nu^E_{AB})\mathcal{H}^1(AB) + \phi_{K^s}(\nu^E_{CD})\mathcal{H}^1(CD) + \phi_{K^s}(\nu^E_{AC})h + \phi_{K^s}(\nu^E_{BD})h \\ &= 2h + 2\cos(\beta)\mathcal{H}^1(AB) = 2h + 2\cos(\beta)\frac{l}{\cos(\beta)} = 2h + 2l = P_{K^s}(E^s). \end{aligned}$$

Interestingly, if $\pi/4 < \beta < \pi/2$ one can see that $P_K(E) > P_K(E^s)$.

We will see that this simple example carries some important features of the general case. In order to characterize $\mathcal{M}_{K^s}(v)$ we start by proving a formula that allows to calculate $P_K(E)$ in terms of $b_E$ and $v$ whenever $E$ is a $v$-distributed set satisfying (1.1.5) (see Corollary 4.4.11). After that, we need to carefully study under which conditions equality holds true in (1.3.5), see Proposition 4.1.22.

Before stating our results, let us give some definitions. If $K \subset \mathbb{R}^n$ is as in (1.3.3), we define the gauge function $\phi_K^* : \mathbb{R}^n \to [0,\infty)$ as

$$\phi_K^*(x) := \sup\{x \cdot y : \phi_K(y) < 1\}. \tag{1.3.10}$$

It turns out that $\phi_K^*$ is one-homogeneous, convex and coercive on $\mathbb{R}^n$ (see Proposition 4.1.4). Let now $x_0 \in \partial K$ and let $\partial\phi_K^*(x_0)$ denote the sub-differential of $\phi_K^*$ at $x_0$ (see Definition 4.1.8). We define the *positive cone* generated by $\partial\phi_K^*(x_0)$, as

$$C_K^*(x_0) := \{\lambda y : y \in \partial\phi_K^*(x_0)\, , \lambda \geq 0\}\, , \tag{1.3.11}$$

see Figure 1.3.4. Let us also mention that, if $\mu$ is an $\mathbb{R}^n$-valued Radon measure $\mathbb{R}^{n-1}$, we denote by $|\mu|_K$ the anisotropic total variation of $\mu$ (with respect to $K$), see Definition 4.1.11.

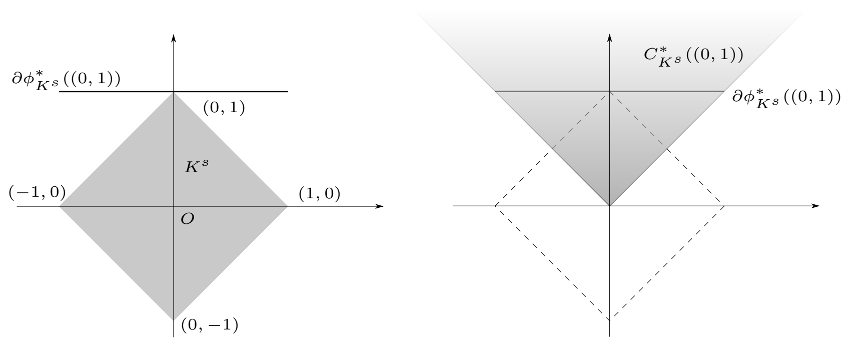


Figure 1.3.4: On the left $K^s$ and a pictorial idea of the sub-differential $\partial\phi^*_{K^s}((0,1))$ and of $C^*_{K^s}((0,1))$.

Next result is the anisotropic version of Theorem 1.1.4.

**Theorem 1.3.2.** *Let $v$ be as in (1.1.3), let $K \subset \mathbb{R}^n$ satisfy (1.3.3), and let $E$ be a $v$-distributed set of finite perimeter. Then, $E \in \mathcal{M}_{K^s}(v)$ if and only if*

*i) for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^{n-1}$ we have that $E_x$ is $\mathcal{H}^1$-equivalent to a segment;*

*ii) for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$ there exists $z(x) \in \partial K^s$ s.t.*

$$\left\{ \left( -\frac{1}{2}\nabla v(x) + t\nabla b_E(x), 1 \right) : t \in [-1,1] \right\} \subset C^*_{K^s}(z(x)); \tag{1.3.12}$$

*iii) for $\mathcal{H}^{n-2}$-a.e. $x \in \{v^\wedge > 0\}$ we have that*

$$[b_E](x) \leq \frac{[v](x)}{2}; \tag{1.3.13}$$

*iv) There exists a Borel function $g : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that*

$$D^c(\tau_M b_\delta)(G) = \int_{G\cap\{v>\delta\}^{(1)}\cap\{|b_E|<M\}^{(1)}} g(x) d|(D^c v/2, 0)|_{K^s}(x),$$

*for every Borel set $G \subset \mathbb{R}^{n-1}$, every $M > 0$, and $\mathcal{H}^1$-a.e. $\delta > 0$. Moreover, $g$ satisfies the following property: for $|D^c v|$-a.e. $x \in \{v^\wedge > 0\}$ there exists $z(x) \in \partial K$ s.t.*

$$\{h(x) + tg(x) : t \in [-1,1]\} \subset C^*_{K^s}(z(x)), \tag{1.3.14}$$

*where*

$$h(x) := \frac{-dD^c v/2}{d\,|(D^c v/2, 0)|_{K^s}}(x), \tag{1.3.15}$$

*is defined as the derivative of $-D^c v/2$ with respect to the anisotropic total variation $|(D^c v/2, 0)|_{K^s}$ in the sense of Radon measures.*

**Remark 1.3.3.** *Let us mention that the above result extend a previous one obtained by Cianchi and Fusco (see [16, Theorem 2.9]).*

In Figure 1.3.5 we give a pictorial idea of condition (1.3.12) for the example of Figure 1.3.3.

An important consequence of Theorem 1.3.2, is the following.

**Proposition 1.3.4.** *Let $v$ be as in (1.1.3) and let $K \subset \mathbb{R}^n$ satisfy (1.3.3). Then,*

$$\mathcal{M}(v) \subset \mathcal{M}_{K^s}(v).$$

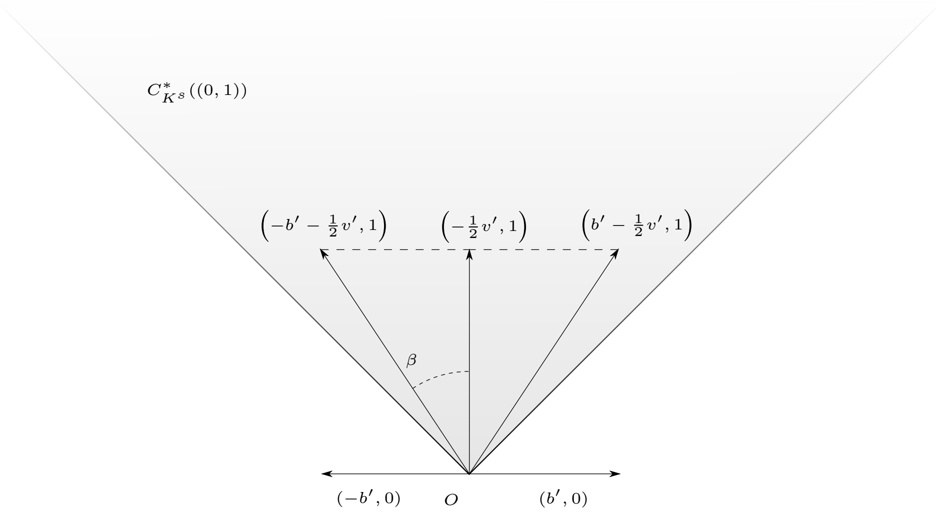


Figure 1.3.5: A pictorial idea of condition (1.3.12), for the example given in Figure 1.3.3. As long as $0 \le \beta \le \pi/4$, we have that $E \in \mathcal{M}_{K^s}(v)$. Note that since $v$ is constant, then $v' = 0$.

Therefore, to study the rigidity problem in the anisotropic setting, it is crucial to understand when the opposite inclusion $\mathcal{M}_{K^s}(v) \subset \mathcal{M}(v)$ holds true. To this aim, given $K \subset \mathbb{R}^n$ as in (1.3.3) and $y \in \mathbb{R}^n$, we set

$$\mathcal{Z}_K(y) := \{z \in \partial K : y \in C^*_K(z)\}. \tag{1.3.16}$$

Note that $\emptyset \neq \mathcal{Z}_K(y) = \mathcal{Z}_K(\lambda y)$ for ever $y \in \mathbb{R}^n$ and for every $\lambda > 0$ (see for instance relation (4.1.24) in Lemma 4.1.24). The following two conditions will play an important role in the understanding of rigidity.

**R1**: $\forall\, y \in \mathbb{R}^n$, for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$, and $\forall\, z \in \mathcal{Z}_{K^s}\left(\left(-\frac{1}{2}\nabla v(x), 1\right)\right)$,

$$\left(-\frac{1}{2}\nabla v(x), 1\right) \pm y \in C^*_{K^s}(z) \quad \Longrightarrow \quad y = \lambda\left(-\frac{1}{2}\nabla v(x), 1\right), \quad \textit{for some } \lambda \in [-1,1].$$

**R2**: $\forall\, y \in \mathbb{R}^n$, for $|D^c v|$-a.e. $x \in \{v^\wedge > 0\}$, and $\forall\, z \in \mathcal{Z}_{K^s}(h(x))$,

$$h(x) \pm y \in C^*_{K^s}(z) \quad \Longrightarrow \quad y = \lambda h(x) \qquad \textit{for some } \lambda \in [-1,1],$$

where $h$ has been defined in (1.3.15). Next result shows the importance of condition **R1** and **R2**.

**Theorem 1.3.5.** *Let $v$ be as in (1.1.3) and let $K \subset \mathbb{R}^n$ be as in (1.3.3). In addition, let us assume that **R1** and **R2** hold true. Then, $\mathcal{M}_{K^s}(v) \subset \mathcal{M}(v)$. As an immediate consequence, (RS) and (RSA) are equivalent.*

**Remark 1.3.6.** *The above result can be seen as a generalization of [16, Theorem 2.10].*

To check whether conditions **R1**, **R2** hold true might be difficult in general. Thus, in the last section of Chapter 4, we prove a result that provides necessary and sufficient conditions for **R1** and **R2** to hold true (see Proposition 4.6.1 and also Lemma 4.6.3). As a consequence, we have the following results.

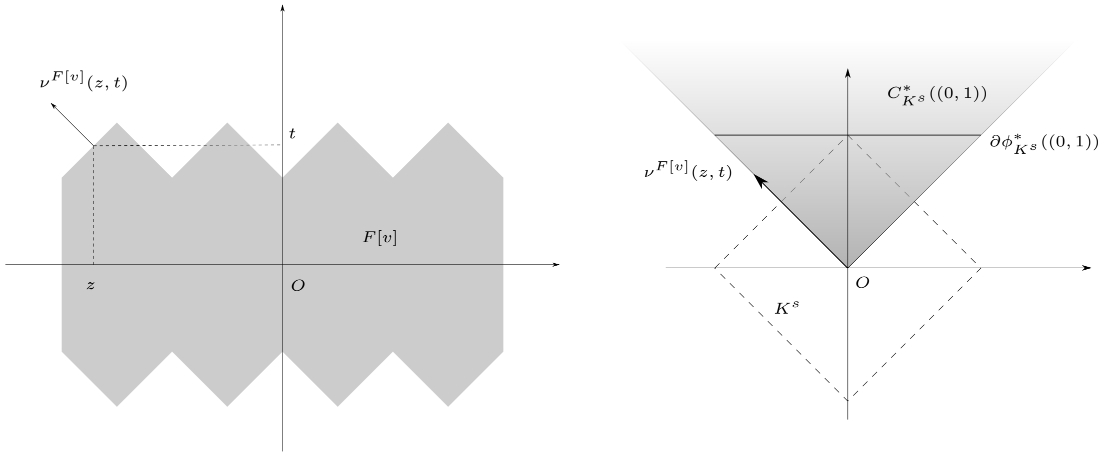


Figure 1.3.6: A pictorial idea of a situation where $v$ and $K^s$ (the same one used in Figure 1.3.4) satisfy the assumptions of Corollary 1.3.7.

**Corollary 1.3.7.** *Let $v$ be as in (1.1.3) and let $K \subset \mathbb{R}^n$ be as in (1.3.3). Moreover, assume that for $\mathcal{H}^{n-1}$-a.e. $z \in \{v > 0\}$, and for $|D^c v|$-a.e. $z \in \{v^\wedge > 0\}$ there exists $x \in \partial^* K^s$ such that $\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right) = \nu^{K^s}(x)$. Then, conditions **R1**, **R2** hold true.*

A pictorial idea of the assumptions of the above Corollary can be found in Figure 1.3.6.

**Corollary 1.3.8.** *Let $v$ be as in (1.1.3) and let $K \subset \mathbb{R}^n$ be as in (1.3.3). In addition, assume that $K^s$ has $C^1$ boundary. Then, conditions **R1**, **R2** hold true.*

Let us notice that, given any $K \subset \mathbb{R}^n$ that satisfies (1.3.3), Corollary 1.3.7, and in particular Lemma 4.6.3 ensure the existence of $v$ defined as in (1.1.3), such that $\mathcal{M}_{K^s}(v) \subset \mathcal{M}(v)$ (see Remark 4.6.4). It would be actually interesting checking whether conditions **R1** and **R2** are also necessary in order to get $\mathcal{M}_{K^s}(v) \subset \mathcal{M}(v)$. This seems quite a delicate problem, and for this reason it could be an interesting topic for some possible future discussions.

# Chapter 2

# Basic notions of Geometric Measure Theory

In this chapter we introduce some tools from Geometric Measure Theory. The interested reader can find more details in the monographs [2, 25, 32, 39]. Note that part of the notations we will use, has been already presented across the Introduction. For the seek of simplicity, we briefly restate it in the next lines, in such a way that the reader can easily access to them. For $n \in \mathbb{N}$, we denote with $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$, i.e.

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\},$$

and we set $\mathbb{R}^n_0 := \mathbb{R}^n \setminus \{0\}$. For every $x \in \mathbb{R}^n_0$, we write $\hat{x} := x/|x|$ for the radial versor of $x$. We denote by $e_1, \ldots, e_n$ the canonical basis in $\mathbb{R}^n$, and for every $x, y \in \mathbb{R}^n$, $x \cdot y$ stands for the standard scalar product in $\mathbb{R}^n$ between $x$ and $y$. For every $r > 0$ and $x \in \mathbb{R}^n$, we denote by $B(x, r)$ the open ball of $\mathbb{R}^n$ with radius $r$ centred at $x$. In the special case $x = 0$, we set $B(r) := B(0, r)$. For every $x, y \in \mathbb{R}^n$, $x \cdot y$ stands for the standard scalar product in $\mathbb{R}^n$ between $x$ and $y$. We denote the $(n-1)$-dimensional ball in $\mathbb{R}^{n-1}$ of center $z \in \mathbb{R}^{n-1}$ and radius $r > 0$ as

$$D_{z,r} = \Big\{\eta \in \mathbb{R}^{n-1} : \, |\eta - z| < r\Big\}.$$

For $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$, we will denote by $H^+_{x,\nu}$ and $H^-_{x,\nu}$ the closed half-spaces whose boundaries are orthogonal to $\nu$:

$$H^+_{x,\nu} := \Big\{y \in \mathbb{R}^n : (y - x) \cdot \nu \geq 0\Big\}, \quad H^-_{x,\nu} := \Big\{y \in \mathbb{R}^n : (y - x) \cdot \nu \leq 0\Big\}. \tag{2.0.1}$$

If $1 \leq k \leq n$, we denote by $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure in $\mathbb{R}^n$. If $\{E_h\}_{h \in \mathbb{N}}$ is a sequence of Lebesgue measurable sets in $\mathbb{R}^n$ with finite volume, and $E \subset \mathbb{R}^n$ is also measurable with finite volume, we say that $\{E_h\}_{h\in\mathbb{N}}$ converges to $E$ as $h\to\infty$, and write $E_h \to E$, if $\mathcal{H}^n(E_h \Delta E) \to 0$ as $h\to\infty$. In the following, we will denote by $\chi_E$ the characteristic function of a measurable set $E \subset \mathbb{R}^n$.

### 2.0.1 Density points

Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set and let $x \in \mathbb{R}^n$. The upper and lower $n$-dimensional densities of $E$ at $x$ are defined as

$$\theta^*(E,x) := \limsup_{r\to 0^+} \frac{\mathcal{H}^n(E\cap B(x,r))}{\omega_n\, r^n}\,, \qquad \theta_*(E,x) := \liminf_{r\to 0^+} \frac{\mathcal{H}^n(E\cap B(x,r))}{\omega_n\, r^n}\,,$$

respectively. It turns out that $x \mapsto \theta^*(E,x)$ and $x\mapsto \theta_*(E,x)$ are Borel functions that agree $\mathcal{H}^n$-a.e. on $\mathbb{R}^n$. Therefore, the $n$-dimensional density of $E$ at $x$

$$\theta(E,x) := \lim_{r\to 0^+} \frac{\mathcal{H}^n(E\cap B(x,r))}{\omega_n\, r^n}\,,$$

is defined for $\mathcal{H}^n$-a.e. $x\in\mathbb{R}^n$, and $x\mapsto\theta(E,x)$ is a Borel function on $\mathbb{R}^n$. Given $t\in[0,1]$, we set

$$E^{(t)} := \{x \in \mathbb{R}^n : \theta(E,x) = t\}.$$

By the Lebesgue differentiation theorem, the pair $\{E^{(0)}, E^{(1)}\}$ is a partition of $\mathbb{R}^n$, up to a $\mathcal{H}^n$-negligible set. The set $\partial^{\mathrm{e}} E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$ is called the *essential boundary* of $E$.

### 2.0.2 Rectifiable sets

Let $1 \le k \le n$, $k\in\mathbb{N}$. If $A, B \subset \mathbb{R}^n$ are Borel sets we say that $A \subset_{\mathcal{H}^k} B$ if $\mathcal{H}^k(B\setminus A) = 0$, and $A =_{\mathcal{H}^k} B$ if $\mathcal{H}^k(A\Delta B) = 0$, where $\Delta$ denotes the symmetric difference of sets. Let $M\subset\mathbb{R}^n$ be a Borel set. We say that $M$ is *countably $\mathcal{H}^k$-rectifiable* if there exist Lipschitz functions $f_h : \mathbb{R}^k \to \mathbb{R}^n$ ($h\in\mathbb{N}$) such that $M \subset_{\mathcal{H}^k} \bigcup_{h\in\mathbb{N}} f_h(\mathbb{R}^k)$. Moreover, we say that $M$ is *locally $\mathcal{H}^k$-rectifiable* if is *countably $\mathcal{H}^k$-rectifiable* and $\mathcal{H}^k(M\cap K) < \infty$ for every compact set $K\subset\mathbb{R}^n$, or, equivalently, if $\mathcal{H}^k \llcorner M$ is a Radon measure on $\mathbb{R}^n$. Given a $\mathbb{R}^m$-valued Radon measure $\mu$ on $\mathbb{R}^n$, we define its *total variation* $|\mu|$ as

$$|\mu|(\Omega) = \sup\left\{ \int_{\mathbb{R}^n} \varphi(x)\cdot d\mu(x) :\ \varphi \in C_c^\infty(\Omega;\mathbb{R}^m),\ |\varphi|\le 1\right\}, \quad \forall\,\Omega\subset\mathbb{R}^n \textit{ open.} \tag{2.0.2}$$

If we consider a generic Borel set $B\subset\mathbb{R}^n$ then

$$|\mu|(B) = \inf\{|\mu|(\Omega) :\ B\subset\Omega,\ \Omega\subset\mathbb{R}^n \text{ open set}\}\,.$$

Let $\mu$ be a Radon measure on $\mathbb{R}^n$, let $1 \leq p < \infty$ and $m \geq 1$ with $m \in \mathbb{N}$. The vector space $L^p(\mathbb{R}^n, \mu; \mathbb{R}^m)$ is defined as

$$L^p(\mathbb{R}^n, \mu; \mathbb{R}^m) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^m : \ f \ \textit{is } \mu\textit{-measurable}, \ \int_{\mathbb{R}^n} |f|^p d\mu < \infty \right\},$$

equipped with the norm

$$\|f\|_{L^p(\mathbb{R}^n, \mu; \mathbb{R}^m)} = \left( \int_{\mathbb{R}^n} |f|^p d\mu \right)^{\frac{1}{p}}.$$

If $p = \infty$ then $L^\infty(\mathbb{R}^n, \mu; \mathbb{R}^m)$ is defined as

$$L^\infty(\mathbb{R}^n, \mu; \mathbb{R}^m) = \{ f : \mathbb{R}^n \to \mathbb{R}^m : \ f \ \textit{is } \mu\textit{-measurable}, \ \mathrm{supess}_{\mathbb{R}^n} f < \infty \},$$

where

$$\mathrm{supess}_{\mathbb{R}^n} f := \inf \{ c > 0 : \ \mu(\{|f| > c\}) = 0 \}.$$

We equip this space with the norm

$$\|f\|_{L^\infty(\mathbb{R}^n, \mu; \mathbb{R}^m)} = \mathrm{supess}_{\mathbb{R}^n} f.$$

We say that $f \in L^p_{loc}(\mathbb{R}^n, \mu; \mathbb{R}^m)$, $1 \leq p \leq \infty$ if $f \in L^p(C, \mu; \mathbb{R}^m)$ for every compact set $C \subset \mathbb{R}^n$.

**Remark 2.0.1.** *Let $\mu$ be a Radon measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mathbb{R}^n, \mu; \mathbb{R}^m)$ with $m \geq 1$, $m \in \mathbb{N}$. Then, we define a $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$ by setting*

$$f\mu(B) = \int_B f(x)\, d\mu(x) \quad \forall \, \textit{Borel set } B \subset \mathbb{R}^n.$$

*Its total variation is then defined as*

$$|f\mu|(B) = \int_B |f(x)| d\mu(x) \quad \forall \, \textit{Borel set } B \subset \mathbb{R}^n.$$

*For more details see [32, Example 4.6, Remark 4.8].*

A Lebesgue measurable set $E \subset \mathbb{R}^n$ is said of *locally finite perimeter* in $\mathbb{R}^n$ if there exists a $\mathbb{R}^n$-valued Radon measure $\mu_E$, called the *Gauss–Green measure* of $E$, such that

$$\int_E \nabla \varphi(x)\, dx = \int_{\mathbb{R}^n} \varphi(x)\, d\mu_E(x), \qquad \forall \varphi \in C^1_c(\mathbb{R}^n).$$

The relative perimeter of $E$ in $A \subset \mathbb{R}^n$ is then defined by setting $P(E; A) := |\mu_E|(A)$ for any Borel set $A \subset \mathbb{R}^n$. The perimeter of $E$ is then defined as $P(E) := P(E; \mathbb{R}^n)$. If $P(E) < \infty$, we say that $E$ is a set of *finite perimeter* in $\mathbb{R}^n$. The *reduced boundary* of $E$ is the set $\partial^* E$ of those $x \in \mathbb{R}^n$ such that

$$\nu^E(x) = \frac{d\mu_E}{d|\mu_E|}(x) = \lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \quad \text{exists and belongs to } \mathbb{S}^{n-1},$$

Where $\frac{d\mu_E}{d|\mu_E|}$ indicates the derivative of $\mu_E$ with respect its total variation $|\mu_E|$ in the sense of Radon measure. The Borel function $\nu^E : \partial^* E \to \mathbb{S}^{n-1}$ is called the *measure-theoretic outer unit normal* to $E$. If $E$ is a set of locally finite perimeter, it is possible to show that $\partial^* E$ is a locally $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^n$ [32, Corollary 16.1], with $\mu_E = \nu^E \, \mathcal{H}^{n-1} \llcorner \partial^* E$, and

$$\int_E \nabla\varphi(x)\, dx = \int_{\partial^* E} \varphi(x)\, \nu^E(x)\, d\mathcal{H}^{n-1}(x)\,, \qquad \forall \varphi \in C^1_c(\mathbb{R}^n)\,,$$

where $C^1_c(\mathbb{R}^n)$ denotes the class of $C^1$ functions in $\mathbb{R}^n$ with compact support. Thus, $P(E; A) = \mathcal{H}^{n-1}(A \cap \partial^* E)$ for every Borel set $A \subset \mathbb{R}^n$. If $E$ is a set of locally finite perimeter, it turns out that

$$\partial^* E \subset E^{(1/2)} \subset \partial^{\mathrm{e}} E\,.$$

Moreover, *Federer's theorem* holds true (see [2, Theorem 3.61] and [32, Theorem 16.2]):

$$\mathcal{H}^{n-1}(\partial^{\mathrm{e}} E \setminus \partial^* E) = 0\,,$$

thus implying that the essential boundary $\partial^{\mathrm{e}} E$ of $E$ is locally $\mathcal{H}^{n-1}$-rectifiable in $\mathbb{R}^n$.

### 2.0.3 General facts about measurable functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function. We define the *approximate upper limit* $f^\vee(x)$ and the *approximate lower limit* $f^\wedge(x)$ of $f$ at $x \in \mathbb{R}^n$ as

$$f^\vee(x) = \inf\left\{ t \in \mathbb{R} : x \in \{f > t\}^{(0)} \right\}, \tag{2.0.3}$$

$$f^\wedge(x) = \sup\left\{ t \in \mathbb{R} : x \in \{f < t\}^{(0)} \right\}. \tag{2.0.4}$$

We observe that $f^\vee$ and $f^\wedge$ are Borel functions that are defined at *every* point of $\mathbb{R}^n$, with values in $\mathbb{R} \cup \{\pm\infty\}$. Moreover, if $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are measurable functions satisfying $f_1 = f_2$ $\mathcal{H}^n$-a.e. on $\mathbb{R}^n$, then $f_1^\vee = f_2^\vee$ and $f_1^\wedge = f_2^\wedge$ *everywhere* on $\mathbb{R}^n$. We define the *approximate discontinuity* set $S_f$ of $f$ as

$$S_f := \{f^\wedge < f^\vee\}.$$

Note that, by the above considerations, it follows that $\mathcal{H}^n(S_f) = 0$. Although $f^\wedge$ and $f^\vee$ may take infinite values on $S_f$, the difference $f^\vee(x) - f^\wedge(x)$ is well defined in $\mathbb{R} \cup \{\pm\infty\}$ for every $x \in S_f$. Then, we can define the *approximate jump* $[f]$ of $f$ as the Borel function $[f] : \mathbb{R}^n \to [0, \infty]$ given by

$$[f](x) := \begin{cases} f^\vee(x) - f^\wedge(x)\,, & \text{if } x \in S_f\,, \\ 0\,, & \text{if } x \in \mathbb{R}^n \setminus S_f\,. \end{cases}$$

The *approximate average* of $f$ is the Borel function

$$\tilde{f}(x) = \begin{cases} \frac{f^\vee(x)+f^\wedge(x)}{2}, & \text{if } x \in \mathbb{R}^n \setminus \{f^\wedge = -\infty, f^\vee = +\infty\}, \\ 0, & \text{if } x \in \{f^\wedge = -\infty, f^\vee = +\infty\}. \end{cases}$$

It also holds the following limit relation

$$\tilde{f}(x) = \lim_{M\to\infty} \widetilde{\tau_M f}(x) = \lim_{M\to\infty} \frac{\tau_M(f^\vee) + \tau_M(f^\wedge)}{2}, \quad \forall x \in \mathbb{R}^n, \tag{2.0.5}$$

that we want to be true for every Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$, where, here and in the rest of the work,

$$\tau_M(s) = \max\{-M, \min\{M, s\}\}, \quad s \in \mathbb{R} \cup \{\pm\infty\}. \tag{2.0.6}$$

By definition, $\tau_M$ is equivalently defined as

$$\tau_M(s) = \begin{cases} M & s > M \\ s & -M \le s \le M \\ -M & s < -M \end{cases}$$

and the following properties can be easily proved

$$\tau_M(s_2) \ge \tau_M(s_1) \qquad \forall\, s_2 \ge s_1, \textit{ provided } M > 0. \tag{2.0.7}$$

$$\tau_{M_2}(s) \ge \tau_{M_1}(s) \qquad \forall\, M_2 \ge M_1 \ge 0, \textit{ provided } s \ge 0. \tag{2.0.8}$$

$$\tau_{M_2}(s) \le \tau_{M_1}(s) \qquad \forall\, M_2 \ge M_1 \ge 0, \textit{ provided } s \le 0. \tag{2.0.9}$$

$$(\tau_{M_2} - \tau_{M_1})(s_2) \ge (\tau_{M_2} - \tau_{M_1})(s_1) \qquad \forall\, s_2 \ge s_1, \textit{ provided } M_2 \ge M_1 \ge 0. \tag{2.0.10}$$

$$\tau_{M_2}(s_2) - \tau_{M_2}(s_1) \ge \tau_{M_1}(s_2) - \tau_{M_1}(s_1) \qquad \forall\, M_2 \ge M_1 \ge 0, \textit{ provided } s_2 \ge s_1. \tag{2.0.11}$$

The validity of the limit relation (2.0.5) can be easily checked noticing that

$$\tau_M(f)^\wedge = \tau_M(f^\wedge), \quad \tau_M(f)^\vee = \tau_M(f^\vee), \quad \widetilde{\tau_M(f)}(x) = \frac{\tau_M(f^\vee) + \tau_M(f^\wedge)}{2}, \quad \forall x \in \mathbb{R}^n.$$

Using these above definitions, the validity of the following properties can be easily deduced. For every Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$ and for every $t \in \mathbb{R}$ we have that

$$\{|f|^\vee < t\} = \{-t < f^\wedge\} \cap \{f^\vee < t\}, \tag{2.0.12}$$

$$\{f^\vee < t\} \subset \{f < t\}^{(1)} \subset \{f^\vee \le t\}, \tag{2.0.13}$$

$$\{f^\wedge > t\} \subset \{f > t\}^{(1)} \subset \{f^\wedge \ge t\}. \tag{2.0.14}$$

Furthermore, if $f, g : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable functions and $f = g$ $\mathcal{H}^n$-a.e. on a Borel set $E$, then

$$f^\vee(x) = g^\vee(x), \quad f^\wedge(x) = g^\wedge(x), \quad [f](x) = [g](x), \quad \forall x \in E^{(1)}. \tag{2.0.15}$$

Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set. We say that $t \in \mathbb{R} \cup \{\pm\infty\}$ is the approximate limit of $f$ at $x$ with respect to $A$, and write $t = \mathrm{aplim}(f, A, x)$, if

$$\theta\Big(\{|f-t| > \varepsilon\} \cap A; x\Big) = 0\,, \qquad \forall \varepsilon > 0\,, \qquad (t \in \mathbb{R})\,, \tag{2.0.16}$$

$$\theta\Big(\{f < M\} \cap A; x\Big) = 0\,, \qquad \forall M > 0\,, \qquad (t = +\infty)\,, \tag{2.0.17}$$

$$\theta\Big(\{f > -M\} \cap A; x\Big) = 0\,, \qquad \forall M > 0\,, \qquad (t = -\infty)\,. \tag{2.0.18}$$

We say that $x \in S_f$ is a *jump point* of $f$ if there exists $\nu \in \mathbb{S}^{n-1}$ such that

$$f^\vee(x) = \mathrm{aplim}(f, H^+_{x,\nu}, x) > f^\wedge(x) = \mathrm{aplim}(f, H^-_{x,\nu}, x)\,.$$

If this is the case, we say that $\nu_f(x) := \nu$ is the approximate jump direction of $f$ at $x$. If we denote by $J_f$ the set of approximate jump points of $f$, we have that $J_f \subset S_f$ and $\nu_f : J_f \to \mathbb{S}^{n-1}$ is a Borel function.

Consider $f : \mathbb{R}^n \to \mathbb{R}$ Lebesgue measurable, then we say that $f$ is *approximately differentiable* at $x \in S_f^c$ provided $f^\wedge(x) = f^\vee(x) \in \mathbb{R}$ if there exists $\xi \in \mathbb{R}^n$ such that

$$\mathrm{aplim}(g, \mathbb{R}^n, x) = 0,$$

where $g(y) = (f(y) - \tilde{f}(x) - \xi \cdot (y-x))/|y-x|$ for $y \in \mathbb{R}^n \setminus \{x\}$. If this is the case, then $\xi$ is uniquely determined, we set $\xi = \nabla f(x)$, and call $\nabla f(x)$ the *approximate differential* of $f$ at $x$. The localization property (2.0.15) holds true also for the approximate differentials, namely if $g, f : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable functions, $f = g$ $\mathcal{H}^n$-a.e. on a Borel set $E$, and $f$ is approximately differentiable $\mathcal{H}^n$-a.e. on $E$, then so it is $g$ $\mathcal{H}^n$-a.e. on $E$ with

$$\nabla f(x) = \nabla g(x), \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in E. \tag{2.0.19}$$

### 2.0.4 Functions of bounded variation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function, and let $\Omega \subset \mathbb{R}^n$ be open. We define the *total variation of $f$ in* $\Omega$ as

$$|Df|(\Omega) = \sup\Big\{ \int_\Omega f(x)\,\mathrm{div}\,T(x)\,dx : T \in C^1_c(\Omega; \mathbb{R}^n)\,, |T| \le 1 \Big\}\,,$$

where $C^1_c(\Omega; \mathbb{R}^n)$ is the set of $C^1$ functions from $\Omega$ to $\mathbb{R}^n$ with compact support. We also denote by $C_c(\Omega; \mathbb{R}^n)$ the class of all continuous functions from $\Omega$ to $\mathbb{R}^n$. Analogously, for any $k \in \mathbb{N}$, the class of $k$ times continuously differentiable functions from $\Omega$ to $\mathbb{R}^n$ is denoted by $C^k_c(\Omega; \mathbb{R}^n)$. We say that $f$ belongs to the space of functions of bounded variations, $f \in BV(\Omega)$, if $|Df|(\Omega) < \infty$ and $f \in L^1(\Omega)$. Moreover, we say that $f \in BV_{\mathrm{loc}}(\Omega)$ if $f \in BV(\Omega')$ for every open set $\Omega'$ compactly contained in $\Omega$. Therefore, if $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$

the distributional derivative $Df$ of $f$ is an $\mathbb{R}^n$-valued Radon measure. In particular, $E$ is a set of locally finite perimeter if and only if $\chi_E \in BV_{\text{loc}}(\mathbb{R}^n)$. If $f \in BV_{loc}(\mathbb{R}^n)$, one can write the Radon–Nykodim decomposition of $Df$ with respect to $\mathcal{H}^n$ as $Df = D^a f + D^s f$, where $D^s f$ and $\mathcal{H}^n$ are mutually singular, and where $D^a f \ll \mathcal{H}^n$. We denote the density of $D^a f$ with respect to $\mathcal{H}^n$ by $\nabla f$, so that $\nabla f \in L^1(\Omega; \mathbb{R}^n)$ with $D^a f = \nabla f \, d\mathcal{H}^n$. Moreover, for a.e. $x \in \mathbb{R}^n$, $\nabla f(x)$ is the approximate differential of $f$ at $x$. If $f \in BV_{\text{loc}}(\mathbb{R}^n)$, then $S_f$ is countably $\mathcal{H}^{n-1}$-rectifiable. Moreover, we have $\mathcal{H}^{n-1}(S_f \setminus J_f) = 0$, $[f] \in L^1_{loc}(\mathcal{H}^{n-1} \llcorner J_f)$, and the $\mathbb{R}^n$-valued Radon measure $D^j f$ defined as

$$D^j f = [f] \, \nu_f \, d\mathcal{H}^{n-1} \llcorner J_f \,,$$

is called the *jump part of* $Df$. If we set $D^c f = D^s f - D^j f$, we have that $Df = D^a f + D^j f + D^c f$. The $\mathbb{R}^n$-valued Radon measure $D^c f$ is called the *Cantorian part* of $Df$, and it is such that $|D^c f|(M) = 0$ for every $M \subset \mathbb{R}^n$ which is $\sigma$-finite with respect to $\mathcal{H}^{n-1}$. In the special case $n = 1$, if $(a, b) \subset \mathbb{R}$ is an open (possibly unbounded) interval, *every* $f \in BV((a, b))$ can be written as

$$f = f^a + f^j + f^c, \tag{2.0.20}$$

where $f \in W^{1,1}(\Omega)$, $f^j$ is a jump function (i.e. $Df = D^j f$) and $f^c$ is a Cantor function (i.e. $Df = D^c f$), see [2, Corollary 3.33]. Moreover, if $f^j = 0$ (or, more in general, if $f$ is a *good representative*, see [2, Theorem 3.28]), the total variation of $Df$ can be obtained as

$$|Df|(a, b) = \sup \left\{ \sum_{i=1}^{N} |f(x_{i+1}) - f(x_i)| : a < x_1 < x_2 < \ldots < x_N < b \right\}, \tag{2.0.21}$$

where the supremum runs over all $N \in \mathbb{N}$, and all the possible partitions of $(a, b)$ with $a < x_1 < x_2 < \ldots < x_N < b$. In the one dimensional setting, we will often write $f'$ instead of $\nabla f$. Let us recall some useful properties we will need on the next sections (see [11, Lemma 2.2, Lemma 2.3] for further details).

**Lemma 2.0.2.** *If $v \in BV(\mathbb{R}^n)$, then $|D^c v|(\{v^\wedge = 0\}) = 0$. In particular, if $f = g$ $\mathcal{H}^n$-a.e. on a Borel set $E \subset \mathbb{R}^n$, then $D^c f \llcorner E^{(1)} = D^c g \llcorner E^{(1)}$.*

**Lemma 2.0.3.** *If $f, g \in BV(\mathbb{R}^n)$, $E$ is a set of finite perimeter and $f = 1_E g$, then*

$$\nabla f = 1_E \nabla g, \qquad \mathcal{H}^n\text{- a.e. on } \mathbb{R}^n, \tag{2.0.22}$$

$$D^c f = D^c g \llcorner E^{(1)}, \tag{2.0.23}$$

$$S_f \cap E^{(1)} = S_g \cap E^{(1)}. \tag{2.0.24}$$

A Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$, it's called of *generalized bounded variation* on $\mathbb{R}^n$, shortly $f \in GBV(\mathbb{R}^n)$ if and only if $\tau_M(u) \in BV_{loc}(\mathbb{R}^{n-1})$ for every $M > 0$ (where $\tau_M(s)$ has been defined in the previous subsection). It is interesting to notice that the structure theory of BV-functions holds true for GBV-functions too. Indeed, given $f \in GBV(\mathbb{R}^n)$, then, (see [2, Theorem 4.34]) $\{f > t\}$ is a set of finite perimeter too for $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$, $f$ is approximately differentiable $\mathcal{H}^n$-a.e. on $\mathbb{R}^n$, $S_f$ is countably $\mathcal{H}^{n-1}$-rectifiable and $\mathcal{H}^{n-1}$-equivalent to $J_f$ and the usual coarea formula takes the form

$$\int_{\mathbb{R}} P(\{f > t\}; G)dt = \int_G |\nabla f| d\mathcal{H}^n + \int_{G \cap S_f} [f] d\mathcal{H}^{n-1} + |D^c f|(G),$$

for every Borel set $G \subset \mathbb{R}^n$, where $|D^c f|$ denotes the Borel measure on $\mathbb{R}^n$ defined as

$$|D^c f|(G) = \lim_{M \to +\infty} |D^c(\tau_M(f))|(G) = \sup_{M > 0} |D^c(\tau_M)(f)|(G), \tag{2.0.25}$$

whenever $G$ is a Borel set in $\mathbb{R}^n$.

# Chapter 3

# Rigidity of equality cases for the spherical perimeter inequality

## 3.1 Setting of the problem and preliminary results

In this section we give the notation for the chapter, and we introduce some results that will be extensively used later. For every $x, y \in \mathbb{S}^{n-1}$, the *geodesic distance* between $x$ and $y$ is given by

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(x, y) := \arccos(x \cdot y).$$

We recall that the geodesic distance satisfies the triangle inequality:

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(x, y) \leq \mathrm{dist}_{\mathbb{S}^{n-1}}(x, z) + \mathrm{dist}_{\mathbb{S}^{n-1}}(z, y) \qquad \text{for every } x, y, z \in \mathbb{S}^{n-1}.$$

Let $r > 0$, $p \in \mathbb{S}^{n-1}$ and $\beta \in [0, \pi]$ be fixed. The *open geodesic ball* of centre $rp$ and radius $\beta$ is the set

$$\mathbf{B}_\beta(rp) := \{x \in \partial B(r) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, p) < \beta\}.$$

Note in the extreme cases $\beta = 0$ and $\beta = \pi$ we have $\mathbf{B}_0(rp) = \emptyset$ and $\mathbf{B}_\pi(rp) = \partial B(r) \setminus \{-rp\}$, respectively. Accordingly, the *geodesic sphere* of centre $rp$ and radius $\beta$ is the boundary of $\mathbf{B}_\beta(rp)$, which is given by

$$\mathbf{S}_\beta(rp) := \{x \in \partial B(r) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, p) = \beta\}.$$

The $(n-1)$-dimensional Hausdorff measure of a geodesic ball and the $(n-2)$-dimensional Hausdorff measure of a geodesic sphere are given by

$$\mathcal{H}^{n-1}(\mathbf{B}_\beta(rp)) = (n-1)\omega_{n-1} r^{n-1} \int_0^\beta (\sin\tau)^{n-2}\, d\tau, \tag{3.1.1}$$

$$\mathcal{H}^{n-2}(\mathbf{S}_\beta(rp)) = (n-1)\omega_{n-1} r^{n-2} (\sin\beta)^{n-2}. \tag{3.1.2}$$

Let $E \subset \mathbb{R}^n$ be a measurable set. For every $r > 0$, we define the *spherical slice of radius $r$ of $E$* as the set

$$E_r := E \cap \partial B(r) = \{x \in \partial B(r) : x \in E\}.$$

Let $v : (0, \infty) \to [0, \infty)$ be a Lebesgue measurable function, and let $E \subset \mathbb{R}^n$ be a measurable set in $\mathbb{R}^n$. We say that $E$ is spherically $v$-distributed if

$$v(r) = \mathcal{H}^{n-1}(E_r), \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in (0, \infty).$$

If $E$ is spherically $v$-distributed, we can define the function

$$\xi_v(r) := \frac{v(r)}{r^{n-1}} = \frac{\mathcal{H}^{n-1}(E_r)}{r^{n-1}}, \qquad \text{for every } r \in (0, \infty). \tag{3.1.3}$$

Note that $\mathcal{H}^{n-1}(\mathbf{B}_\pi) = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n$, so that

$$0 \leq \xi_v(r) \leq n\omega_n, \qquad \text{for every } r \in (0, \infty). \tag{3.1.4}$$

From (3.1.1), it follows that the function $\mathcal{F} : [0, \pi] \to [0, n\omega_n]$ given by

$$\mathcal{F}(\beta) := \mathcal{H}^{n-1}(\mathbf{B}_\beta(e_1)) \text{ is strictly increasing and smoothly invertible in } (0, n\omega_n). \tag{3.1.5}$$

Therefore, if $v : (0, \infty) \to [0, \infty)$ is measurable, thanks to (3.1.4), there exists a unique function $\alpha_v : (0, \infty) \to [0, \pi]$ such that

$$\xi_v(r) = \mathcal{H}^{n-1}(\mathbf{B}_{\alpha_v(r)}(e_1)) \qquad \text{for every } r \in (0, \infty). \tag{3.1.6}$$

Among all the spherically $v$-distributed sets of $\mathbb{R}^n$, we denote by $F_v$ the one whose spherical slices are open all geodesic balls centred in the at the positive $e_1$ axis., i.e.

$$F_v := \{x \in \mathbb{R}^n_0 : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) < \alpha_v(|x|)\}, \tag{3.1.7}$$

where $\alpha_v$ is defined by (3.1.3) and (3.1.6), and $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$. Next proposition is a special case of the Coarea formula (see [2, Theorem 2.93]).

**Proposition 3.1.1.** *Let $E$ be a set of finite perimeter in $\mathbb{R}^n$ and let $g : \mathbb{R}^n \to [0, \infty]$ be a Borel function. Then,*

$$\int_{\partial^* E} g(x) |\nu_\parallel^E(x)| d\mathcal{H}^{n-1}(x) = \int_0^\infty dr \int_{(\partial^* E)_r} g(x)\, d\mathcal{H}^{n-2}(x).$$

*Proof.* The result follows by applying [2, Remark 2.94] with $N = n - 1$, $M = n$, $k = 1$, and $f(x) = |x|$. □

We will also need the following result (see [2, Lemma 2.35]).

**Lemma 3.1.2.** *Let $B \subset \mathbb{R}^n$ be a Borel set and let $\varphi_h, \varphi : B \to \mathbb{R}$, $h \in \mathbb{N}$ be summable Borel functions such that $|\varphi_h| \le |\varphi|$ for every $h$. Then*

$$\int_B \sup_h \varphi_h dx = \sup_H \left\{ \sum_{h \in H} \int_{A_h} \varphi_h dx \right\},$$

*where the supremum ranges over all finite sets $H \subset \mathbb{N}$ and all finite partitions $A_h$, $h \in H$ of $B$ in Borel sets.*

### 3.1.1 Normal and tangential components of functions and measures

For every $\varphi \in C_c(\mathbb{R}^n_0; \mathbb{R}^n)$, we decompose $\varphi$ as $\varphi = \varphi_\perp + \varphi_\|$, where

$$\varphi_\perp(x) := (\varphi(x) \cdot \hat{x})\, \hat{x} \qquad \text{and} \qquad \varphi_\|(x) := \varphi(x) - \varphi_\perp(x)$$

are the radial and tangential components of $\varphi$, respectively. If $\varphi \in C^1_c(\mathbb{R}^n_0; \mathbb{R}^n)$, $\mathrm{div}_\| \varphi(x)$ stands for the tangential divergence of $\varphi$ at $x$ along the sphere $\partial B(|x|)$:

$$\mathrm{div}_\| \varphi(x) := \mathrm{div} \varphi(x) - (\nabla \varphi(x) \hat{x}) \cdot \hat{x}. \tag{3.1.8}$$

The following lemma gives some useful identities that will be needed later.

**Lemma 3.1.3.** *Let $\varphi \in C^1_c(\mathbb{R}^n_0; \mathbb{R}^n)$. Then, for every $x \in \mathbb{R}^n_0$ one has*

$$\mathrm{div} \varphi_\perp(x) = (\nabla \varphi(x) \hat{x}) \cdot \hat{x} + (\varphi(x) \cdot \hat{x})\, \frac{n-1}{|x|}, \tag{3.1.9}$$

$$\mathrm{div} \varphi_\|(x) = \mathrm{div}_\| \varphi_\|(x). \tag{3.1.10}$$

**Remark 3.1.4.** *Let $\varphi \in C^1_c(\mathbb{R}^n_0; \mathbb{R}^n)$. Recalling that $\varphi = \varphi_\perp + \varphi_\|$, combining (3.1.9) and (3.1.10) it follows that*

$$\mathrm{div} \varphi(x) = (\nabla \varphi(x) \hat{x}) \cdot \hat{x} + (\varphi(x) \cdot \hat{x})\, \frac{n-1}{|x|} + \mathrm{div}_\| \varphi_\|(x) \qquad \forall\, x \in \mathbb{R}^n_0.$$

*Proof.* First of all, note that

$$\nabla \left(\varphi(x) \cdot \hat{x}\right) = \left(\nabla \varphi(x)\right)^T \hat{x} + \frac{1}{|x|} \varphi_\|(x). \tag{3.1.11}$$

Indeed,

$$\nabla \left(\varphi(x) \cdot \hat{x}\right) = \left(\nabla \varphi(x)\right)^T \hat{x} + \frac{I - \hat{x} \otimes \hat{x}}{|x|} \varphi(x) = \left(\nabla \varphi(x)\right)^T \hat{x} + \frac{1}{|x|} \varphi_\|(x),$$

where $I$ represents the identity in $\mathbb{R}^n$, and $\hat{x} \otimes \hat{x}$ is the usual tensor product of $\hat{x}$ with itself (so that $I - \hat{x} \otimes \hat{x}$ is the orthogonal projection on the tangent plane to $\mathbb{S}^{n-1}$ at $\hat{x}$).

Thanks to (3.1.11), we have

$$\begin{aligned}\mathrm{div}\varphi_\perp(x) &= \mathrm{div}\left((\varphi(x)\cdot\hat{x})\hat{x}\right) = \nabla\left(\varphi(x)\cdot\hat{x}\right)\cdot\hat{x} + (\varphi(x)\cdot\hat{x})\,\mathrm{div}\hat{x}\\ &= \left[(\nabla\varphi(x))^T\hat{x} + \frac{1}{|x|}\varphi_\parallel(x)\right]\cdot\hat{x} + (\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|}\\ &= (\nabla\varphi(x)\hat{x})\cdot\hat{x} + (\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|},\end{aligned}$$

which proves (3.1.9). Note now that, by definition (3.1.8), it follows that

$$\mathrm{div}\varphi(x) = \mathrm{div}_\parallel\varphi(x) + (\nabla\varphi(x)\hat{x})\cdot\hat{x}. \tag{3.1.12}$$

On the other hand, from (3.1.9)

$$\begin{aligned}\mathrm{div}\varphi(x) &= \mathrm{div}\varphi_\parallel(x) + \mathrm{div}\varphi_\perp(x)\\ &= \mathrm{div}\varphi_\parallel(x) + (\nabla\varphi(x)\hat{x})\cdot\hat{x} + (\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|}.\end{aligned}$$

Comparing last identity with (3.1.12) we obtain that for every $\varphi\in C^1_c(\mathbb{R}^n_0;\mathbb{R}^n)$

$$\mathrm{div}_\parallel\varphi(x) = \mathrm{div}\varphi_\parallel(x) + (\varphi(x)\cdot\hat{x})\,\frac{n-1}{|x|}.$$

Applying the last identity to the function $\varphi_\parallel$ we obtain (3.1.10). □

If $\mu$ is an $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n_0$, we will write $\mu=\mu_\perp+\mu_\parallel$, where $\mu_\perp$ and $\mu_\parallel$ are the $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^n_0$ such that

$$\int_{\mathbb{R}^n_0}\varphi\cdot d\mu_\perp = \int_{\mathbb{R}^n_0}\varphi_\perp\cdot d\mu, \qquad \text{and} \qquad \int_{\mathbb{R}^n_0}\varphi\cdot d\mu_\parallel = \int_{\mathbb{R}^n_0}\varphi_\parallel\cdot d\mu,$$

for every $\varphi\in C_c(\mathbb{R}^n_0;\mathbb{R}^n)$. Note that $\mu_\perp$ and $\mu_\parallel$ are well defined by Riesz Theorem (see, for instance, [2, Theorem 1.54]). In the special case $\mu=Df$, with $f\in BV_{\mathrm{loc}}(\mathbb{R}^n_0)$, we will shorten the notation writing $D_\parallel f$ and $D_\perp f$ in place of $(Df)_\parallel$ and $(Df)_\perp$, respectively. In particular, if $f=\chi_E$ and $E\subset\mathbb{R}^n$ is a set of finite perimeter, by De Giorgi structure theorem we have

$$D_\perp\chi_E = \nu^E_\perp d\mathcal{H}^{n-1}\llcorner\partial^*E \qquad \text{and} \qquad D_\parallel\chi_E = \nu^E_\parallel d\mathcal{H}^{n-1}\llcorner\partial^*E. \tag{3.1.13}$$

Next lemma gives some useful identities concerning the radial and tangential components of the gradient of a $BV_{\mathrm{loc}}$ function.

**Lemma 3.1.5.** *Let $f\in BV_{\mathrm{loc}}(\mathbb{R}^n_0)$. Then,*

$$\int_{\mathbb{R}^n_0}\varphi(x)\cdot dD_\parallel f = -\int_{\mathbb{R}^n_0} f(x)\,\mathrm{div}_\parallel\varphi_\parallel(x)\,dx, \tag{3.1.14}$$

$$\int_{\mathbb{R}^n_0}\varphi(x)\cdot dD_\perp f = -\int_{\mathbb{R}^n_0} f(x)\,(\nabla\varphi(x)\,\hat{x})\cdot\hat{x}\,dx - \int_{\mathbb{R}^n_0} f(x)\frac{n-1}{|x|}\,(\varphi(x)\cdot\hat{x})\,dx, \tag{3.1.15}$$

*for every $\varphi\in C^1_c(\mathbb{R}^n_0;\mathbb{R}^n)$.*

*Proof.* Let $\varphi \in C^1_c(\mathbb{R}^n_0;\mathbb{R}^n)$. By definition of $D_\parallel f$ and thanks to (3.1.10) we have

$$\begin{aligned}\int_{\mathbb{R}^n_0} \varphi(x)\cdot dD_\parallel f &= \int_{\mathbb{R}^n_0} \varphi_\parallel(x)\cdot dDf \\ &= -\int_{\mathbb{R}^n_0} \mathrm{div}\varphi_\parallel(x) f(x)\,dx = -\int_{\mathbb{R}^n_0} \mathrm{div}_\parallel \varphi_\parallel(x) f(x)\,dx,\end{aligned}$$

and this shows (3.1.14). Similarly, by definition of $D_\perp f$

$$\int_{\mathbb{R}^n_0} \varphi(x)\cdot dD_\perp f = \int_{\mathbb{R}^n_0} \varphi_\perp(x)\cdot dDf = -\int_{\mathbb{R}^n_0} \mathrm{div}\varphi_\perp(x) f(x)\,dx.$$

Thanks to (3.1.9), identity (3.1.15) follows. □

An immediate consequence of identity (3.1.14) is the following.

**Corollary 3.1.6.** *Let $f \in BV_{\mathrm{loc}}(\mathbb{R}^n_0)$ and let $\Omega \subset\subset \mathbb{R}^n_0$ be open and bounded. Then,*

$$\left|D_\parallel f\right|(\Omega) = \sup\left\{\int_{\mathbb{R}^n} f(x)\,\mathrm{div}_\parallel\varphi_\parallel(x)dx :\ \varphi \in C^1_c(\Omega;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1\right\}.$$

### 3.1.2 Sets of finite perimeter on $\mathbb{S}^{n-1}$

We will follow here the notation of [7]. For more details, we direct the interested reader to [39].

The notion of set of finite perimeter can also be given in a natural way for subsets of the sphere $\mathbb{S}^{n-1}$ (and, more in general, if $r > 0$, for $\partial B(r)$). Let $A \subset \mathbb{S}^{n-1}$ be an $\mathcal{H}^{n-1}$-measurable set. We will say that $A$ is a set of finite perimeter if there exists an $(n-2)$-currents $T \in \mathcal{R}_{n-2}(\mathbb{R}^n)$ with $\mathrm{supp}\,T \subset \mathbb{S}^{n-1}$ and

$$T = \partial[\![A]\!],$$

with the property that

$$\mathbf{M}_U(T) = \mathbf{M}(\partial[\![A]\!] \llcorner U) < \infty,$$

for every $U \subset\subset \mathbb{R}^n$. Denoting by $\mu_T$ the total variation measure of $T = \partial[\![A]\!]$, by the Riesz representation theorem it follows that there exists a $\mu_T$-measurable function $\nu : \mathbb{S}^{n-1} \to T_x\mathbb{S}^{n-1}$ such that $|\nu(x)| = 1$ for $\mu_T$-a.e. $x$ and

$$\int_A \mathrm{div}_\parallel\varphi(x)\,d\mathcal{H}^{n-1}(x) = \int_{\mathbb{S}^{n-1}} \varphi(x)\cdot\nu(x)\,d\mu_T(x),$$

for every smooth vector field with $\varphi = \varphi_\parallel$. If $A \subset \mathbb{S}^{n-1}$ is a set of finite perimeter on the sphere, the reduced boundary $\partial^* A$ is the set of points $x \in \mathbb{S}^{n-1}$ such that the limit

$$\nu^A(x) := \lim_{\rho\to 0} \frac{1}{\mu_T(B(x,\rho))}\int_{B(x,\rho)} \nu(x)\,d\mu_T$$

exists, $\nu^A(x) \in T_x\mathbb{S}^{n-1}$, and $\nu^A(x) = 1$. The De Giorgi structure theorem holds true also for sets of finite perimeter on the sphere. In particular, $\partial^* A$ is countably $(n-2)$-rectifiable, $\mu_T = \mathcal{H}^{n-2} \llcorner \partial^* A$, and

$$\int_A \mathrm{div}_{\|}\varphi(x)\, d\mathcal{H}^{n-1}(x) = \int_{\partial^* A} \varphi(x) \cdot \nu^A(x)\, d\mathcal{H}^{n-2}(x), \tag{3.1.16}$$

for every smooth vector field with $\varphi = \varphi_{\|}$. The isoperimetric inequality on the sphere states that, if $\beta \in (0,\pi)$ and $A \subset \mathbb{S}^{n-1}$ is a set of finite perimeter on $\mathbb{S}^{n-1}$ with $\mathcal{H}^{n-1}(A) = \mathcal{H}^{n-1}(\mathbf{B}_\beta(e_1))$, then (see [41])

$$\mathcal{H}^{n-2}(\partial^* \mathbf{B}_\beta(e_1)) \leq \mathcal{H}^{n-2}(\partial^* A). \tag{3.1.17}$$

The next theorem is a version of a result by Vol'pert (see [44]).

**Theorem 3.1.7.** *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying (1.2.2), and let $E \subset \mathbb{R}^n$ be a spherically $v$-distributed set of finite perimeter and finite volume. Then, there exists a Borel set $G_E \subset \{\alpha_v > 0\}$ with $\mathcal{H}^1(\{\alpha_v > 0\} \setminus G_E) = 0$, such that*

*(i) for every $r \in G_E$:*

*(ia) $E_r$ is a set of finite perimeter in $\partial B(r)$;*

*(ib) $\mathcal{H}^{n-2}(\partial^*(E_r)\Delta(\partial^* E)_r) = 0$;*

*(ii) for every $r \in G_E \cap \{0 < \alpha_v < \pi\}$:*

*(iia) $|\nu^E_{\|}(r\omega)| > 0$,*

*(iib) $\nu^E_{\|}(r\omega) = \nu^{E_r}(r\omega)|\nu^E_{\|}(r\omega)|$,*

*for $\mathcal{H}^{n-2}$-a.e. $\omega \in \mathbb{S}^{n-1}$ such that $r\omega \in \partial^*(E_r) \cap (\partial^* E)_r$.*

*Proof.* The result follows applying [39, Theorem 28.5] with $f(x) = |x|$, and recalling the definition of slicing of a current (see [39, Definition 28.4]). □

We now make some important remarks about Theorem 3.1.7.

**Remark 3.1.8.** *Thanks to property (ib), we have*

$$\partial^*(E_r) =_{\mathcal{H}^{n-2}} (\partial^* E)_r \qquad \textit{for every } r \in G_E.$$

*Therefore, whenever $r \in G_E$ we will often write $\partial^* E_r$ instead of $\partial^*(E_r)$ or $(\partial^* E)_r$, without any risk of ambiguity. Moreover, for every $r \in G_E$ we will also use the notation*

$$p_E(r) := \mathcal{H}^{n-2}(\partial^* E_r).$$

**Remark 3.1.9.** *In dimension $n = 2$, the theorem above implies that, if $r \in G_E \cap \{0 < \theta < \pi\}$, then $\partial^*(E_r) = (\partial^* E)_r$ and*

$$|\nu_\parallel^E(r\omega)| > 0 \quad \textbf{for every } \omega \in \mathbb{S}^1 \text{ such that } r\omega \in (\partial^* E)_r. \tag{3.1.18}$$

*Let now $\lambda_E$ be the measure defined in (1.2.5):*

$$\lambda_E(B) = \int_{\partial^* E \cap \Phi(B \times \mathbb{S}^1) \cap \{\nu_\parallel^E = 0\}} \hat{x} \cdot \nu^E(x)\, d\mathcal{H}^1(x) \quad \text{for every Borel set } B \subset (0,\infty).$$

*If $B \subset G_E$, then by (3.1.18)*

$$|\lambda_E(B)| \le \mathcal{H}^1(\partial^* E \cap \Phi(G_E \times \mathbb{S}^1) \cap \{\nu_\parallel^E = 0\}) = 0,$$

*so that $\lambda_E(B) = 0$. As a consequence, $\lambda_E$ is singular with respect to the Lebesgue measure in $(0,\infty)$. If $n > 2$ this conclusion is in general false (unless one chooses $E = F_v$, see Remark 3.1.10 below), and it may happen that $\lambda_E$ has a non trivial absolutely continuous part.*

**Remark 3.1.10.** *If $n \ge 2$, but we consider the special case $E = F_v$, Theorem 3.1.7 gives much more information than the one we can obtain for a generic set of finite perimeter. Indeed, let $R \in SO(n)$ be any rotation that keeps fixed the $e_1$ axis. By definition of $F_v$, and thanks to [32, Exercise 15.10], we have that if $x \in \partial^* F_v$, then $Rx \in \partial^* F_v$ and*

$$\nu_\parallel^{F_v}(Rx) = R\,\nu_\parallel^{F_v}(x) \qquad \text{and} \qquad \nu_\perp^{F_v}(Rx) = R\,\nu_\perp^{F_v}(x).$$

*Therefore, applying Theorem 3.1.7 to $F_v$ we infer that*

*(j) for every $r \in G_{F_v}$:*

*(ja) $(F_v)_r$ is a spherical cap;*

*(jb) $\partial^*(F_v)_r = (\partial^* F_v)_r$;*

*(jj) for every $r \in G_{F_v} \cap \{0 < \alpha_v < \pi\}$:*

*(jja) $|\nu_\parallel^{F_v}(r\omega)| > 0$,*

*(jjb) $\nu_\parallel^{F_v}(r\omega) = \nu^{(F_v)_r}(r\omega)|\nu_\parallel^{F_v}(r\omega)|$,*

**for every** $\omega \in \mathbb{S}^{n-1}$ *such that* $r\omega \in \cap(\partial^* F_v)_r \cap \partial^*(F_v)_r$.

*Therefore,*

$$\mathcal{H}^1(B_0) = 0, \tag{3.1.19}$$

*where*

$$B_0 := \left\{ r \in (0,+\infty) : \exists\, \omega \in \mathbb{S}^{n-1} \text{ such that } r\omega \in \partial^* F_v \text{ and } \nu_{\parallel}^{F_v}(r\omega) = 0 \right\}.$$

*Moreover, repeating the argument used in Remark 3.1.9 one obtains that*

$$\mathcal{H}^{n-1}(\partial^* F_v \cap \Phi(G_{F_v} \times \mathbb{S}^{n-1}) \cap \{\nu_{\parallel}^{F_v} = 0\}) = 0.$$

*Thus, the measure $\lambda_{F_v}$ defined in (1.2.5) is purely singular with respect to the Lebesgue measure in $(0,\infty)$.*

## 3.2 Properties of $v$ and $\xi_v$

In this section we discuss several properties of the functions $v$ and $\xi_v$. We start by showing that, if $E \subset \mathbb{R}^n$ is a set of finite perimeter and volume, then $v \in BV(0,\infty)$. Next lemma gives one of the implications of Theorem 1.2.1.

**Lemma 3.2.1.** *Let $v$ be as in Theorem 1.2.1, and let $E \subset \mathbb{R}^n$ be a spherically $v$-distributed set of finite perimeter and finite volume. Then, $v \in BV(0,\infty)$. Moreover, $\xi_v \in BV_{\mathrm{loc}}(0,\infty)$ and*

$$\int_0^\infty \psi(r) r^{n-1} dD\xi_v(r) = \int_{\mathbb{R}^n_0} \psi(|x|)\, \hat{x} \cdot dD_\perp \chi_E(x), \tag{3.2.1}$$

*for every bounded Borel function $\psi : (0,\infty) \to \mathbb{R}$. As a consequence,*

$$r^{n-1}|D\xi_v|(B) \le |D_\perp \chi_E|(\Phi(B \times \mathbb{S}^{n-1})), \tag{3.2.2}$$

*for every Borel set $B \subset (0,\infty)$. In particular, $r^{n-1}D\xi_v$ is a bounded Radon measure on $(0,\infty)$.*

*Proof.* We divide the proof into steps.

**Step 1:** We show that $v \in BV(0,\infty)$. First of all, note that $v \in L^1(0,\infty)$, since

$$\|v\|_{L^1(0,\infty)} = \int_0^\infty v(r)\, dr = \int_0^\infty dr \int_{\partial B(r)} \chi_E(x)\, d\mathcal{H}^{n-1}(x) = \mathcal{H}^n(E) < \infty.$$

Let now $\psi \in C^1_c(0,\infty)$ with $|\psi| \le 1$. Applying formula (3.1.9) to the radial function $\psi(|x|)\hat{x}$, we obtain that for every $x \in \mathbb{R}^n_0$

$$\begin{aligned}
\operatorname{div}(\psi(|x|)\hat{x}) &= [\nabla(\psi(|x|)\hat{x})\,\hat{x}] \cdot \hat{x} + [\psi(|x|)\hat{x} \cdot \hat{x}]\,\frac{n-1}{|x|} \\
&= \left[\left(\psi'(|x|)\hat{x} \otimes \hat{x} + \psi(|x|)\frac{I - \hat{x} \otimes \hat{x}}{|x|}\right)\hat{x}\right] \cdot \hat{x} + \psi(|x|)\frac{n-1}{|x|} \\
&= \psi'(|x|) + \psi(|x|)\frac{n-1}{|x|}.
\end{aligned} \tag{3.2.3}$$

Thus,

$$\int_{\mathbb{R}^n}\left[\psi'(|x|)+\psi(|x|)\frac{n-1}{|x|}\right]\chi_E(x)\,dx=\int_{\mathbb{R}^n}\operatorname{div}\left(\psi(|x|)\,\hat{x}\right)\chi_E(x)\,dx$$
$$=-\int_{\mathbb{R}^n}\psi(|x|)\,\hat{x}\cdot dD\chi_E(x)=-\int_{\mathbb{R}^n}\psi(|x|)\,\hat{x}\cdot dD_\perp\chi_E(x),$$

so that

$$\begin{aligned}&\int_{\mathbb{R}^n}\psi'(|x|)\chi_E(x)\,dx\\&=-\int_{\mathbb{R}^n}\psi(|x|)\frac{n-1}{|x|}\chi_E(x)\,dx-\int_{\mathbb{R}^n}\psi(|x|)\,\hat{x}\cdot dD_\perp\chi_E(x).\end{aligned}\tag{3.2.4}$$

By Coarea formula, the integral in the left hand side can be written as

$$\int_{\mathbb{R}^n}\psi'(|x|)\chi_E(x)\,dx=\int_0^\infty dr\,\psi'(r)\int_{\partial B(r)}\chi_E(x)\,d\mathcal{H}^{n-1}(x)=\int_0^\infty\psi'(r)v(r)\,dr.\tag{3.2.5}$$

Combining (3.2.4) and (3.2.5) we find that

$$\begin{aligned}&\int_0^\infty\psi(r)\,dDv(r)\\&=\int_{\mathbb{R}^n}\psi(|x|)\frac{n-1}{|x|}\chi_E(x)\,dx+\int_{\mathbb{R}^n}\psi(|x|)\,\hat{x}\cdot dD_\perp\chi_E(x).\qquad(3.2.6)\\&\le\int_{B(1)}\psi(|x|)\frac{n-1}{|x|}\chi_E(x)\,dx+\int_{\mathbb{R}^n\setminus B(1)}\psi(|x|)\frac{n-1}{|x|}\chi_E(x)\,dx+P(E)\\&\le n(n-1)\omega_n\int_0^1\rho^{n-2}\,d\rho+(n-1)|E|+P(E)\\&=n\omega_n+(n-1)|E|+P(E)<\infty.\end{aligned}$$

Taking the supremum over $\psi$ we obtain that

$$|Dv|(0,\infty)<\infty,$$

so that $v\in BV(0,\infty)$.

**Step 2:** We conclude the proof. Since the function $r\mapsto 1/(r^{n-1})$ is smooth and locally bounded in $(0,\infty)$, we also have that $\xi_v(r)\in BV_{\mathrm{loc}}(0,\infty)$. Moreover, recalling that $v(r)=r^{n-1}\xi_v(r)$, by the chain rule in $BV$ (see [2, Example 3.97])

$$Dv=(n-1)r^{n-2}\xi_v(r)\,dr+r^{n-1}D\xi_v=(n-1)\frac{v(r)}{r}dr+r^{n-1}D\xi_v.\tag{3.2.7}$$

Let now $\psi\in C^1_c(0,\infty)$. From the previous identity it follows that

$$\int_0^\infty\psi(r)\,dDv(r)=\int_0^\infty\psi(r)\frac{n-1}{r}v(r)\,dr+\int_0^\infty\psi(r)r^{n-1}dD\xi_v(r)$$
$$=\int_0^\infty\psi(r)\frac{n-1}{r}\mathcal{H}^{n-1}(\partial B(r)\cap E)\,dr+\int_0^\infty\psi(r)r^{n-1}dD\xi_v(r)$$
$$=\int_{\mathbb{R}^n}\psi(|x|)\frac{n-1}{|x|}\chi_E(x)\,dx+\int_0^\infty\psi(r)r^{n-1}dD\xi_v(r).$$

Combining the previous identity and (3.2.6),

$$\int_0^\infty \psi(r) r^{n-1} dD\xi_v(r) = \int_{\mathbb{R}^n} \psi(|x|)\, \hat{x} \cdot dD_\perp \chi_E, \qquad \text{for every } \psi \in C_c^1(0\,\infty).$$

By approximation, the identity above is true also when $\psi$ is a bounded Borel function, and this gives (3.2.1).

If $B \subset (0, \infty)$ is open, thanks to (3.2.1) we have that for every $\psi \in C_c(B)$ with $|\psi| \leq 1$

$$\int_B \psi(r) r^{n-1} dD\xi_v(r) = \int_{\Phi(B \times \mathbb{S}^{n-1})} \psi(|x|)\, \hat{x} \cdot dD_\perp \chi_E \leq |D_\perp \chi_E|(\Phi(B \times \mathbb{S}^{n-1})).$$

Taking the supremum over all such $\psi$ gives

$$r^{n-1}|D\xi_v|(B) \leq |D_\perp \chi_E|(\Phi(B \times \mathbb{S}^{n-1})) \qquad \text{for every open set } B \subset (0, \infty).$$

By approximation, the inequality above holds true for every Borel set, and this shows inequality (3.2.2). $\square$

Next lemma gives an important property of the measure $r^{n-1}D\xi_v$.

**Lemma 3.2.2.** *Let $v$ be as in Theorem 1.2.1, and let $E \subset \mathbb{R}^n$ be a spherically $v$-distributed set of finite perimeter and finite volume. Then*

$$\begin{aligned}(r^{n-1}D\xi_v)(B) = &\int_{\partial^* E \cap \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}} \hat{x} \cdot \nu^E(x)\, d\mathcal{H}^{n-1}(x) \\ &+ \int_B dr \int_{(\partial^* E)_r \cap \{\nu_\parallel^E \neq 0\}} \frac{\hat{x} \cdot \nu^E(x)}{|\nu_\parallel^E(x)|} d\mathcal{H}^{n-2}(x).\end{aligned} \tag{3.2.8}$$

*for every Borel set $B \subset (0, +\infty)$.*

*Moreover, $r^{n-1}D\xi_v \llcorner G_{F_v} = r^{n-1}\xi_v' dr$ and for $\mathcal{H}^1$-a.e. $r \in G_{F_v} \cap \{0 < \alpha_v < \pi\}$*

$$r^{n-1}\xi_v'(r) = \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(re_1)) \frac{\hat{x} \cdot \nu_\perp^{F_v}(x)}{|\nu_\parallel^{F_v}(x)|}, \qquad \text{for every } x \in \mathbf{S}_{\alpha_v(r)}(re_1).$$

*Proof.* Let $B \subset (0, +\infty)$ be a Borel set. Then, choosing $\psi = \chi_B$ in (3.2.1), and recalling (3.1.13),

$$\begin{aligned}(r^{n-1}D\xi_v)(B) &= \int_0^{+\infty} \chi_B(r) r^{n-1} dD\xi_v(r) \\ &= \int_{\Phi(B \times \mathbb{S}^{n-1})} \hat{x} \cdot dD_\perp \chi_E(x) = \int_{\partial^* E \cap \Phi(B \times \mathbb{S}^{n-1})} \hat{x} \cdot \nu^E(x)\, d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial^* E \cap \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}} \hat{x} \cdot \nu^E(x)\, d\mathcal{H}^{n-1}(x) + \int_{\partial^* E \cap \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E \neq 0\}} \hat{x} \cdot \nu^E(x)\, d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial^* E \cap \Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_\parallel^E = 0\}} \hat{x} \cdot \nu^E(x)\, d\mathcal{H}^{n-1}(x) + \int_B dr \int_{(\partial^* E)_r \cap \{\nu_\parallel^E \neq 0\}} \frac{\hat{x} \cdot \nu^E(x)}{|\nu_\parallel^E(x)|} d\mathcal{H}^{n-2}(x),\end{aligned}$$

where in the last equality we have used the Coarea formula.

Let us now prove the second part of the statement. If one chooses $E = F_v$, thanks to Remark 3.1.10 we have

$$\begin{aligned} r^{n-1} D\xi_v \llcorner G_{F_v} &= \left( \int_{(\partial^* F_v)_r \cap \{\nu_{\|}^{F_v} \neq 0\}} \frac{\hat{x} \cdot \nu^{F_v}(x)}{|\nu_{\|}^{F_v}(x)|} d\mathcal{H}^{n-2}(x) \right) dr \llcorner G_{F_v} \\ &= \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(re_1)) \frac{\hat{x} \cdot \nu_{\perp}^{F_v}(x)}{|\nu_{\|}^{F_v}(x)|}. \end{aligned}$$

In particular,

$$r^{n-1} D\xi_v \llcorner G_{F_v} = r^{n-1} \xi_v'(r)\, dr \llcorner G_{F_v}.$$

Moreover, since $\xi_v'(r) = 0$ $\mathcal{H}^1$-a.e. in $\{\alpha = 0\} \cup \{\alpha = \pi\}$, we obtain that for $\mathcal{H}^1$-a.e. $r \in (0, \infty)$

$$r^{n-1} \xi'(r) = \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(re_1)) \frac{\hat{x} \cdot \nu_{\perp}^{F_v}(x)}{|\nu_{\|}^{F_v}(x)|}, \qquad \text{for every } x \in \mathbf{S}_{\alpha_v(r)}(re_1).$$

□

We now prove an auxiliary inequality that will be useful later.

**Proposition 3.2.3.** *Let $v$ be as in Theorem 1.2.1, and suppose that there exists a spherically $v$-distributed set $E \subset \mathbb{R}^n$ of finite perimeter and finite volume. Then, $F_v$ is a set of finite perimeter in $\mathbb{R}^n$. Moreover, for every Borel set $B \subset (0, +\infty)$*

$$P(F_v; \Phi(B \times \mathbb{S}^{n-1})) \leq r^{n-1} |D\xi_v| (B) + \left| D_{\|} \chi_{F_v} \right| (\Phi(B \times \mathbb{S}^{n-1})). \tag{3.2.9}$$

*Proof.* The proof is based on the arguments of [14, Lemma 3.5] and [3, Lemma 3.3]. Thanks to Lemma 3.2.1, $v \in BV(0, \infty)$. Let $\{v_j\}_{j \in \mathbb{N}} \subset C_c^1(0, \infty)$ be a sequence of non-negative functions such that $v_j \to v$ $\mathcal{H}^1$-a.e. in $(0, \infty)$ and $|Dv_j| \overset{*}{\rightharpoonup} |Dv|$. For every $j \in \mathbb{N}$, we denote by $F_{v_j} \subset \mathbb{R}^n$ the set defined by (3.1.7), with $v_j$ in place of $v$. Let now $\Omega \subset (0, \infty)$ be open, and let $\varphi \in C_c^1(\Phi(\Omega \times \mathbb{S}^{n-1}); \mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(\Omega \times \mathbb{S}^{n-1}); \mathbb{R}^n)} \leq 1$. Thanks to Remark 3.1.4, we have

$$\begin{aligned} \int_{\Phi(\Omega \times \mathbb{S}^{n-1})} \chi_{F_{v_j}}(x) \operatorname{div} \varphi(x) dx &= \int_{\Phi(\Omega \times \mathbb{S}^{n-1})} \chi_{F_{v_j}}(x) \operatorname{div}_{\|} \varphi_{\|}(x) dx \\ + \int_{\Phi(\Omega \times \mathbb{S}^{n-1})} \chi_{F_{v_j}}(x) \, (\nabla \varphi(x) \, \hat{x}) \cdot \hat{x} \, dx &+ \int_{\Phi(\Omega \times \mathbb{S}^{n-1})} \chi_{F_{v_j}}(x) \, \frac{n-1}{|x|} \, (\varphi(x) \cdot \hat{x}) \, dx. \end{aligned} \tag{3.2.10}$$

In the following, it will be convenient to introduce the function $V_j : (0, \infty) \to \mathbb{R}$ given by

$$V_j(r) := \int_{\mathbf{B}_{\alpha_{v_j}(r)}(re_1)} \varphi(x) \cdot \hat{x} \, d\mathcal{H}^{n-1}(x) = r^{n-1} \int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega) \cdot \omega \, d\mathcal{H}^{n-1}(\omega),$$

where $\alpha_{v_j} : (0, r) \to [0, \pi]$ is defined by (3.1.6), with $v_j$ in place of $v$. We divide the proof into several steps.

**Step 1:** We show that $V_j$ is Lipschitz continuous with compact support. Indeed,

$$\operatorname{supp} V_j \subset \Lambda(\operatorname{supp} \varphi) := \{r \in (0, +\infty) : (\operatorname{supp} \varphi) \cap \partial B(r) \neq \emptyset\}.$$

Moreover, for every $r_1, r_2 \in (0, \infty)$,

$$\begin{aligned}
|V_j(r_1) - V_j(r_2)| &\leq \int_{\mathbf{B}_{\alpha_{v_j}(r_1)}(e_1)} |r_1^{n-1}\varphi(r_1\omega)\cdot\omega - r_2^{n-1}\varphi(r_2\omega)\cdot\omega|\, d\mathcal{H}^{n-1}(\omega) \\
&+ r_2^{n-1}\left|\int_{\mathbf{B}_{\alpha_{v_j}(r_1)}(e_1)} \varphi(r_2\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega) - \int_{\mathbf{B}_{\alpha_{v_j}(r_2)}(e_1)} \varphi(r_2\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega)\right| \\
&\leq c|r_1 - r_2| + r_2^{n-1}\int_{\mathbf{B}_{\alpha_{v_j}(\widetilde{r}_1)}(e_1)\setminus\mathbf{B}_{\alpha_{v_j}(\widetilde{r}_2)}(e_1)} |\varphi(r_2\omega)\cdot\omega|\, d\mathcal{H}^{n-1}(\omega) \\
&\leq c|r_1 - r_2| + r_2^{n-1}|\xi_{v_j}(r_1) - \xi_{v_j}(r_2)| \leq c|r_1 - r_2|,
\end{aligned}$$

where we used the fact that $\xi_{v_j}$ is compactly supported in $(0, \infty)$ (since $v_j$ is), and $\widetilde{r}_1$ and $\widetilde{r}_2$ are such that $\alpha_{v_j}(\widetilde{r}_1) = \max\{\alpha_{v_j}(r_1), \alpha_{v_j}(r_2)\}$ and $\alpha_{v_j}(\widetilde{r}_2) := \min\{\alpha_{v_j}(r_1), \alpha_{v_j}(r_2)\}$.

**Step 2:** We show that $\alpha_{v_j}$ is $\mathcal{H}^1$-a.e. differentiable and that

$$\begin{aligned}
V_j'(r) = (n-1)r^{n-2} &\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega) \\
&+ r^{n-1}\left(\alpha_{v_j}'(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right) \\
&+ r^{n-1}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} (\nabla\varphi(r\omega)\,\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega),
\end{aligned} \tag{3.2.11}$$

for $\mathcal{H}^1$-a.e. $r > 0$. Let us set $A_j := \{0 < \alpha_{v_j} < \pi\}$. Since $v_j \in C_c^1(0, \infty)$, from (3.1.5) it follows that $\alpha_{v_j} \in C^1(A_j)$. Moreover, for every $r \in A_j$

$$\begin{aligned}
V_j'(r) &= \frac{d}{dr}\left(r^{n-1}\int_0^{\alpha_{v_j}(r)} d\beta \int_{\mathbf{S}_\beta(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right) \\
&= (n-1)r^{n-2}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega) + r^{n-1}\left(\alpha_{v_j}'(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right) \\
&\quad + r^{n-1}\int_0^{\alpha_{v_j}(r)} d\beta \int_{\mathbf{S}_\beta(e_1)} (\nabla\varphi(r\omega)\,\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega) \\
&= (n-1)r^{n-2}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega) + r^{n-1}\left(\alpha_{v_j}'(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\right) \\
&\quad + r^{n-1}\int_{\mathbf{B}_{\alpha_j(r)}(e_1)} (\nabla\varphi(r\omega)\,\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega).
\end{aligned}$$

This shows (3.2.11) whenever $r \in A_j$. Note now that

$$\begin{aligned}
&V_j(r) = 0 && \text{for every } r \in \operatorname{Int}(\{\alpha_{v_j} = 0\}), \\
&V_j(r) = r^{n-1}\int_{\mathbb{S}^{n-1}} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega) && \text{for every } r \in \operatorname{Int}(\{\alpha_{v_j} = \pi\}),
\end{aligned}$$

where $\mathrm{Int}(\cdot)$ stands for the interior of a set. Since $\alpha'_{v_j}(r) = 0$ for every $r \in \mathrm{Int}(\{\alpha_{v_j} = 0\}) \cup \mathrm{Int}(\{\alpha_{v_j} = \pi\})$, using the identities above one can see that (3.2.11) holds true for $\mathcal{H}^1$-a.e. $r > 0$.

**Step 3:** We show that

$$\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\, (\nabla\varphi(x)\,\hat{x})\cdot\hat{x}\,dx + \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\frac{n-1}{|x|}\,(\varphi(x)\cdot\hat{x})\,dx$$
$$= -\int_\Omega dr\, r^{n-1}\Big(\alpha'_{v_j}(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\,d\mathcal{H}^{n-2}(\omega)\Big).$$

Integrating (3.2.11), thanks to the classical divergence theorem applied in $\Omega$, and recalling that $V_j$ has compact support, we obtain

$$\begin{aligned}
0 &= (n-1)\int_\Omega dr\,r^{n-2}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\,d\mathcal{H}^{n-1}(\omega)\\
&\quad+\int_\Omega dr\,r^{n-1}\Big(\alpha'_{v_j}(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)}\varphi(r\omega)\cdot\omega\,d\mathcal{H}^{n-2}(\omega)\Big)\\
&\quad+\int_\Omega dr\,r^{n-1}\int_{\mathbf{B}_{\alpha_{v_j}(r)}(e_1)}(\nabla\varphi(r\omega)\,\omega)\cdot\omega\,d\mathcal{H}^{n-1}(\omega)\\
&=\int_{\Phi(\Omega\times\mathbb{S}^{n-1})}\chi_{F_{v_j}}(x)\,\frac{n-1}{|x|}\,(\varphi(x)\cdot\hat{x})\,dx\\
&\quad+\int_\Omega dr\,r^{n-1}\Big(\alpha'_{v_j}(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)}\varphi(r\omega)\cdot\omega\,d\mathcal{H}^{n-2}(\omega)\Big)\\
&\quad+\int_{\Phi(\Omega\times\mathbb{S}^{n-1})}\chi_{F_{v_j}}(x)\,(\nabla\varphi(x)\,\hat{x})\cdot\hat{x}\,dx,
\end{aligned}$$

which gives the claim.

**Step 4:** we prove that

$$\int_{\Phi(\Omega\times\mathbb{S}^{n-1})}\chi_{F_{v_j}}(x)\,\mathrm{div}\varphi(x)dx \le \Big|r^{n-1}D\xi_{v_j}\Big|(\Lambda(\mathrm{supp}\,\varphi)) + \int_\Omega \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j}(r)})dr, \tag{3.2.12}$$

where $\Lambda(\mathrm{supp}\,\varphi)\subset(0,\infty)$ is the compact set defined in Step 1. Thanks to Step 3, (3.2.10) can be written as

$$\begin{aligned}
\int_{\Phi(\Omega\times\mathbb{S}^{n-1})}\chi_{F_{v_j}}(x)\,\mathrm{div}\varphi(x)\,dx &= \int_{\Phi(\Omega\times\mathbb{S}^{n-1})}\chi_{F_{v_j}}(x)\,\mathrm{div}_\parallel\varphi_\parallel(x)\,dx\\
&\quad-\int_\Omega dr\,r^{n-1}\Big(\alpha'_{v_j}(r)\int_{\mathbf{S}_{\alpha_j(r)}(e_1)}\varphi(r\omega)\cdot\omega\,d\mathcal{H}^{n-2}(\omega)\Big).
\end{aligned}\tag{3.2.13}$$

We now estimate the right hand side of the expression above. Thanks to (3.1.6) and arguing as in Step 2 we have that

$$\xi'_{v_j}(r) = \alpha'_{v_j}(r)\mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j}(r)}(e_1)) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r\in(0,\infty).$$

Therefore,

$$\begin{aligned}
&-\int_{\Omega} dr\, r^{n-1}\Big(\alpha'_{v_j}(r)\int_{\mathbf{S}_{\alpha_{v_j}(r)}(e_1)} \varphi(r\omega)\cdot\omega\, d\mathcal{H}^{n-2}(\omega)\Big)\\
&\quad\le \int_{\Lambda(\operatorname{supp}\varphi)} r^{n-1}\left|\alpha'_{v_j}(r)\right| \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j}(r)}(e_1))dr \qquad (3.2.14)\\
&\quad= \int_{\Lambda(\operatorname{supp}\varphi)} r^{n-1}\left|\xi'_{v_j}(r)\right| dr = \left|r^{n-1}D\xi_{v_j}\right|(\Lambda(\operatorname{supp}\varphi)).
\end{aligned}$$

Let us now focus on the second integral in the right hand side of (3.2.13). Applying the divergence theorem (3.1.16) with $A = \mathbf{B}_{\alpha_{v_j}(r)}(re_1)$, and denoting by $\nu_*(x)$ the exterior unit normal to $\mathbf{S}_{\alpha_{v_j}(r)}(re_1)$, we have

$$\begin{aligned}
&\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\operatorname{div}_{\|}\varphi_{\|}(x)\,dx = \int_{\Omega} dr \int_{\mathbf{B}_{\alpha_{v_j}(r)}(re_1)} \operatorname{div}_{\|}\varphi_{\|}(x)\,d\mathcal{H}^{n-1}(x)\\
&= \int_{\Omega} dr \int_{\mathbf{S}_{\alpha_{v_j}(r)}(re_1)} \varphi_{\|}(x)\cdot\nu_*(x)d\mathcal{H}^{n-2}(x) \le \int_{\Omega} dr\, \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j}(r)}(re_1)). \qquad (3.2.15)
\end{aligned}$$

Combining (3.2.13), (3.2.14), and (3.2.15), we obtain (3.2.12).

**Step 5:** We show that $F_v$ is a set of finite perimeter. Note that $\chi_{F_{v_j}} \to \chi_{F_v}$ $\mathcal{H}^n$-a.e. in $\mathbb{R}^n$, and $\alpha_{v_j} \to \alpha$ $\mathcal{H}^1$-a.e. in $(0,\infty)$. Note also that, from our choice of the sequence $\{v_j\}_{j\in\mathbb{N}}$ and thanks to (3.2.7), it follows that

$$r^{n-1}|D\xi_{v_j}| \overset{*}{\rightharpoonup} r^{n-1}|D\xi_v| \qquad \text{as } j\to\infty.$$

Therefore, taking the limsup as $j\to\infty$ in (3.2.12), and using the fact that $\Lambda(\operatorname{supp}\varphi)$ is compact,

$$\begin{aligned}
&\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_v}(x)\operatorname{div}\varphi(x)dx = \limsup_{j\to\infty}\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\operatorname{div}\varphi(x)dx\\
&\le \limsup_{j\to\infty}\left|r^{n-1}D\xi_{v_j}\right|(\Lambda(\operatorname{supp}\varphi)) + \limsup_{j\to\infty}\int_{\Omega}\mathcal{H}^{n-2}(\mathbf{S}_{\alpha_{v_j}(r)}(re_1))\,dr\\
&\le \left|r^{n-1}D\xi_v\right|(\Lambda(\operatorname{supp}\varphi)) + \int_{\Omega}\mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(re_1))\,dr \le \left|r^{n-1}D\xi_v\right|(\Omega) + \int_{\Omega}\mathcal{H}^{n-2}(\partial^* E_r)\,dr\\
&\le \left|r^{n-1}D\xi_v\right|(\Omega) + P(E;\Phi(\Omega\times\mathbb{S}^{n-1})),
\end{aligned}$$

where we also used the isoperimetric inequality in the sphere (see (3.1.17)) and the Coarea formula. Taking the supremum of the above inequality over all functions $\varphi \in C^1_c(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)} \le 1$, we obtain

$$P(F_v;\Phi(\Omega\times\mathbb{S}^{n-1})) \le \left|r^{n-1}D\xi_v\right|(\Omega) + P(E;\Phi(\Omega\times\mathbb{S}^{n-1})).$$

Thanks to (3.2.2) we have

$$P(F_v;\Phi(\Omega\times\mathbb{S}^{n-1})) \le 2P(E;P(F_v;\Phi(\Omega\times\mathbb{S}^{n-1}))) < \infty,$$

since $E$ is a set of finite perimeter by assumption. Since $\Omega$ was arbitrary, this shows that $F_v$ is a set of locally finite perimeter.

**Step 6:** We conclude. Let $\Omega \subset (0,\infty)$ be open, and let $\varphi \in C^1_c(\Phi(\Omega \times \mathbb{S}^{n-1}); \mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)} \leq 1$. Combining (3.2.10), Step 3, and (3.2.14), we have that for every $j \in \mathbb{N}$

$$\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\mathrm{div}\,\varphi(x)dx \leq \left|r^{n-1}D\xi_{v_j}\right|(\Lambda(\mathrm{supp}\,\varphi)) + \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_{v_j}}(x)\,\mathrm{div}_{\|}\varphi_{\|}(x)\,dx.$$

Taking the limsup as $j \to \infty$ and thanks to Corollary 3.1.6,

$$\begin{aligned}\int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_v}(x)\,\mathrm{div}\,\varphi(x)dx &\leq \left|r^{n-1}D\xi_v\right|(\Lambda(\mathrm{supp}\,\varphi)) + \int_{\Phi(\Omega\times\mathbb{S}^{n-1})} \chi_{F_v}(x)\,\mathrm{div}_{\|}\varphi_{\|}(x)\,dx \\ &\leq \left|r^{n-1}D\xi_v\right|(\Lambda(\mathrm{supp}\,\varphi)) + |D_{\|}\chi_{F_v}|(\Phi(\Omega\times\mathbb{S}^{n-1})),\end{aligned}$$

where we also used the fact that $\Lambda(\mathrm{supp}\,\varphi)$ is compact.

Taking the supremum over all $\varphi \in C^1_c(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\Phi(\Omega\times\mathbb{S}^{n-1});\mathbb{R}^n)} \leq 1$,

$$P(F_v; \Phi(\Omega\times\mathbb{S}^{n-1})) \leq \left|r^{n-1}D\xi_v\right|(\Omega) + |D_{\|}\chi_{F_v}|(\Phi(\Omega\times\mathbb{S}^{n-1})), \tag{3.2.16}$$

which shows (3.2.9) when $B$ is an open set. Let now $B \subset (0,\infty)$ be a Borel set. From (3.2.16) it follows that

$$P(F_v; \Phi(B\times\mathbb{S}^{n-1})) \leq \left|r^{n-1}D\xi_v\right|(\Omega) + P(E; \Phi(\Omega\times\mathbb{S}^{n-1})),$$

for any open set $\Omega \subset (0,\infty)$ with $B \subset \Omega$. Taking the infimum of the above inequality over all open sets $\Omega \subset (0,\infty)$ with $B \subset \Omega$, we obtain inequality (3.2.9) when $B$ is a Borel set. $\square$

## 3.3 Proof of Theorem 1.2.1

In this section we prove Theorem 1.2.1, and state some important auxiliary results.

*Proof of Theorem 1.2.1.* We will adapt the arguments of the proof of [3, Theorem 1.1]. Let $G_{F_v}$ be the set associated with $F_v$ given by Theorem 3.1.7. We start by proving (1.2.4). We will first prove the inequality when $B \subset (0,\infty) \setminus G_{F_v}$, and then in the case $B \subset G_{F_v}$. The case of a general Borel set $B \subset (0,\infty)$ then follows by decomposing $B$ as $B = (B \setminus G_{F_v}) \cup (B \cap G_{F_v})$.

**Step 1:** We prove inequality (1.2.4) when $B \subset (0,\infty) \setminus G_{F_v}$. First observe that, thanks to Proposition 3.1.1 and (3.1.13),

$$\left|D_{\|}\chi_{F_v}\right|(\Phi(B\times\mathbb{S}^{n-1})) = \int_{\partial^* F_v\cap\Phi(B\times\mathbb{S}^{n-1})} |\nu_{\|}^{F_v}(x)|d\mathcal{H}^{n-1}(x) = \int_B \mathcal{H}^{n-2}((\partial^* F_v)_r)dr$$
$$= \int_{B\cap\{0<\alpha_v\}} \mathcal{H}^{n-2}((\partial^* F_v)_r)dr = \int_{B\cap(\{0<\alpha_v\}\setminus G_{F_v})} \mathcal{H}^{n-2}((\partial^* F_v)_r)dr = 0, \tag{3.3.1}$$

where we used the fact that $B \subset (0,\infty)\setminus G_{F_v}$ and $\mathcal{H}^1(\{0<\alpha_v\}\setminus G_{F_v}) = 0$. Therefore, thanks to Proposition 3.2.3

$$P(F_v;\Phi(B\times\mathbb{S}^{n-1})) \le r^{n-1}\left|D\xi_v\right|(B) + \left|D_{\|}\chi_{F_v}\right|(\Phi(B\times\mathbb{S}^{n-1}))$$
$$= r^{n-1}\left|D\xi_v\right|(B) \le P(E;\Phi(B\times\mathbb{S}^{n-1})), \tag{3.3.2}$$

where in the last inequality we used (3.2.2).

**Step 2:** We prove inequality (1.2.4) when $B \subset G_{F_v}$. We divide this part of the proof into further substeps.

**Step 2a:** we prove that

$$P(E;\Phi(B\times\mathbb{S}^{n-1})) \ge P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\|}^E = 0\}) + \int_B \sqrt{p_E^2(r)+g^2(r)}dr, \tag{3.3.3}$$

where $g:(0,\infty)\to\mathbb{R}$ and $p_E:(0,\infty)\to[0,\infty)$ are defined as

$$g(r) := \int_{\partial^* E\cap\partial B(r)} \frac{\hat{x}\cdot\nu^E(x)}{|\nu_{\|}^E(x)|}d\mathcal{H}^{n-2}(x) \quad\text{ and }\quad p_E(r) := \mathcal{H}^{n-2}(\partial^* E\cap\partial B(r)),$$

for $\mathcal{H}^1$-a.e. $r\in(0,\infty)$, respectively. We have

$$\begin{aligned}
&P(E;\Phi(B\times\mathbb{S}^{n-1}))\\
&= P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\|}^E = 0\}) + P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\|}^E \ne 0\})\\
&= P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\|}^E = 0\}) + \int_{\partial^* E\cap\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\|}^E\ne 0\}} d\mathcal{H}^{n-1}(x)\\
&= P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\|}^E = 0\}) + \int_B dr\int_{\partial^* E\cap\partial B(r)} \frac{1}{|\nu_{\|}^E(x)|}d\mathcal{H}^{n-2}(x)\\
&= P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\|}^E = 0\}) + \int_B dr\int_{\partial^* E\cap\partial B(r)} \sqrt{1+\left(\frac{\hat{x}\cdot\nu^E(x)}{|\nu_{\|}^E(x)|}\right)^2}d\mathcal{H}^{n-2}(x),
\end{aligned}$$

where in the last equality we used the fact that

$$1 = |\nu_{\perp}^E|^2 + |\nu_{\|}^E|^2 = (\hat{x}\cdot\nu_{\perp}^E)^2 + |\nu_{\|}^E|^2.$$

Defining the function $f:\mathbb{R}\to[0,\infty)$ as

$$f(t) := \sqrt{1+t^2},$$

we obtain

$$
\begin{aligned}
&P(E;\Phi(B\times\mathbb{S}^{n-1}))\\
&= P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\parallel}^{E}=0\}) + \int_B dr\int_{\partial^* E\cap\partial B(r)} f\left(\frac{\hat{x}\cdot\nu_{\perp}^{E}(x)}{|\nu_{\parallel}^{E}(x)|}\right)\,d\mathcal{H}^{n-2}(x).
\end{aligned}
$$

Observing that $f$ is strictly convex, (3.3.3) follows applying Jensen's inequality.

**Step 2b:** We show that

$$
\begin{aligned}
&\int_B \sqrt{p_E^2(r)+(r^{n-1}\xi_v'(r))^2}\,dr\\
&\qquad\le P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\parallel}^{E}=0\}) + \int_B\sqrt{p_E^2(r)+g^2(r)}\,dr.
\end{aligned}
\tag{3.3.4}
$$

Let $\{A_h\}_{h\in H}$ be a finite partition of Borel sets of $B$. Note that, for each $h\in\mathbb{N}$, we have $A_h\subset B\subset G_{F_v}$. Therefore, thanks to Lemma 3.2.2, for every $h\in\mathbb{N}$ we have $r^{n-1}D\xi_v \llcorner A_h = r^{n-1}\xi_v' dr \llcorner A_h$ and

$$
\begin{aligned}
&\int_{A_h} w_h r^{n-1}\xi_v'(r)\,dr = \int_{A_h} w_h r^{n-1} dD\xi_v(r)\\
&= \int_{\partial^* E\cap\Phi(A_h\times\mathbb{S}^{n-1})\cap\{\nu_{\parallel}^{E}=0\}} w_h\,\hat{x}\cdot\nu^E(x)\,d\mathcal{H}^{n-1}(x)\\
&\quad+\int_{A_h} dr\int_{(\partial^* E)_r\cap\{\nu_{\parallel}^{E}\neq 0\}} w_h\frac{\hat{x}\cdot\nu^E(x)}{|\nu_{\parallel}^{E}(x)|}d\mathcal{H}^{n-2}(x)\\
&= \int_{\partial^* E\cap\Phi(A_h\times\mathbb{S}^{n-1})\cap\{\nu_{\parallel}^{E}=0\}} w_h\,\hat{x}\cdot\nu^E(x)\,d\mathcal{H}^{n-1}(x) + \int_{A_h} w_h\,g(r)\,dr.
\end{aligned}
\tag{3.3.5}
$$

We will now use the fact that, by duality, we can write

$$
\sqrt{1+t^2} = \sup_h\left\{w_h t + \sqrt{1-w_h^2}\right\}\qquad\text{for every } t\in\mathbb{R},
\tag{3.3.6}
$$

where $\{w_h\}_h$ is a countable dense set in $(-1,1)$. Then, thanks to (3.3.5)

$$
\begin{aligned}
&\sum_{h\in H}\int_{A_h}\left(w_h r^{n-1}\xi_v'(r) + p_E(r)\sqrt{1-w_h^2}\right)dr\\
&= \sum_{h\in H}\int_{\partial^* E\cap\Phi(A_h\times\mathbb{S}^{n-1})\cap\{\nu_{\parallel}^{E}=0\}} w_h\,\hat{x}\cdot\nu^E(x) d\mathcal{H}^{n-1}(x)\\
&\quad+\sum_{h\in H}\int_{A_h}\left(w_h\,g(r)+p_E(r)\sqrt{1-w_h^2}\right)dr\\
&\le \sum_{h\in H}\int_{\partial^* E\cap\Phi(A_h\times\mathbb{S}^{n-1})\cap\{\nu_{\parallel}^{E}=0\}} |\hat{x}\cdot\nu^E(x)| d\mathcal{H}^{n-1}(x)\\
&\quad+\sum_{h\in H}\int_{A_h} p_E(r)\left(w_h\frac{g(r)}{p_E(r)}+\sqrt{1-w_h^2}\right)dr\\
&\le \sum_{h\in H}\left(P(E;\Phi(A_h\times\mathbb{S}^{n-1})\cap\{\nu_{\parallel}^{E}=0\})\right) + \int_{A_h} p_E(r)\sqrt{1+\frac{g^2(r)}{p_E^2(r)}}dr\\
&= P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\parallel}^{E}=0\}) + \int_B\sqrt{p_E^2(r)+g^2(r)}dr,
\end{aligned}
$$

where we applied identity (3.3.6) with $t = g(r)/p_E(r)$, and we also used the fact that $p_E(r) = 0$ for $\mathcal{H}^1$-a.e. $r \notin \{0 < \alpha_v < \pi\}$, thanks to Volper't theorem. Applying Lemma 3.1.2 to the functions

$$\varphi_h(r) = p_E(r)\left(w_h \frac{r^{n-1}\xi_v'(r)}{p_E(r)} + \sqrt{1-w_h^2}\right),$$

we obtain (3.3.4).

**Step 2c:** We conclude the proof of Step 2. In the special case $E = F_v$, thanks to Vol'pert Theorem and Lemma 3.2.2 we have

$$\begin{aligned}
&P(F_v;\Phi(B\times\mathbb{S}^{n-1})) = \mathcal{H}^{n-1}(\partial^* F_v \cap \Phi(B\times\mathbb{S}^{n-1}))\\
&= \int_{B\cap\{0<\alpha_v<\pi\}}\int_{\partial^*(F_v)_r} \frac{1}{|\nu_{\parallel}^{F_v}(x)|} d\mathcal{H}^{n-2}(x)dr\\
&= \int_{B\cap\{0<\alpha_v<\pi\}}\int_{\partial^*(F_v)_r} \sqrt{1+\left(\frac{\nu_{\perp}^{F_v}(x)}{|\nu_{\parallel}^{F_v}(x)|}\right)^2} d\mathcal{H}^{n-2}(x)dr\\
&= \int_{B\cap\{0<\alpha_v<\pi\}} \sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}dr.
\end{aligned} \tag{3.3.7}$$

Using the isoperimetric inequality (3.1.17) together with (3.3.4) and (3.3.3) we then have,

$$\begin{aligned}
P(F_v;\Phi(B\times\mathbb{S}^{n-1})) &\le \int_{B\cap\{0<\alpha_v<\pi\}} \sqrt{p_E^2(r) + (r^{n-1}\xi_v'(r))^2}dr\\
&\le P(E;\Phi(B\times\mathbb{S}^{n-1})\cap\{\nu_{\parallel}^E = 0\}) + \int_B \sqrt{p_E^2(r)+g^2(r)}dr\\
&\le P(E;\Phi(B\times\mathbb{S}^{n-1})),
\end{aligned}$$

from which we conclude.

**Step 3:** We conclude the proof of the theorem. Suppose $P(E) = P(F_v)$. Then, in particular, all the inequalities in Step 2 hold true as equalities. At the end of Step 2c) we used the fact that, by the isoperimetric inequality (3.1.17), we have

$$p_{F_v}(r) \le p_E(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in \{0<\alpha_v<\pi\}.$$

If the above becomes an equality, this means that for $\mathcal{H}^1$-a.e. $r \in \{0<\alpha_v<\pi\}$ the slice $E_r$ is a spherical cap. Finally, the fact that for $\mathcal{H}^1$-a.e. $r \in \{0<\alpha_v<\pi\}$ we have

$$\mathcal{H}^{n-2}(\partial^*(E_r)\Delta(\partial^* E)_r) = 0$$

follows from Vol'pert Theorem 3.1.7, and this shows (a).

Let us now prove (b). If $P(E) = P(F_v)$, the Jensen's inequality at the end of Step 2b, for the strictly convex function

$$f(t) := \sqrt{1+t^2},$$

becomes an equality. This implies that for $\mathcal{H}^1$-a.e. $r \in \{0 < \alpha_v < \pi\}$ the function

$$x \longmapsto \frac{\hat{x} \cdot \nu^E(x)}{|\nu^E_{\parallel}(x)|}$$

is $\mathcal{H}^{n-2}$-a.e. constant in $\partial^* E_r$. Since, for $\mathcal{H}^{n-2}$-a.e. $x \in \partial^* E_r$, we have

$$1 = |\nu^E_{\parallel}(x)|^2 + (\hat{x} \cdot \nu^E(x))^2,$$

this implies that

$$x \longmapsto \frac{(\hat{x} \cdot \nu^E(x))^2}{|\nu^E_{\parallel}(x)|^2} = 1 - \frac{1}{|\nu^E_{\parallel}(x)|^2}$$

is $\mathcal{H}^{n-2}$-a.e. constant in $\partial^* E_r$. Therefore, the two functions

$$x \longmapsto \nu^E(x) \cdot \hat{x} \qquad \text{and} \qquad x \longmapsto |\nu^E_{\parallel}|(x)$$

are constant $\mathcal{H}^{n-2}$-a.e. in $(\partial^* E)_r$.

$\square$

The previous result allows us to prove a useful proposition (see also [3, Proposition 3.4]).

**Proposition 3.3.1.** *Let $v : (0, \infty) \to [0, \infty)$ be a measurable function satisfying (1.2.2) such that $F_v$ is a set of finite perimeter and finite volume, let $E$ be a spherically $v$-distributed set of finite perimeter, and let $f : (0, \infty) \to [0, \infty]$ be a Borel function. Then,*

$$\begin{aligned} &\int_{\partial^* E} f(|x|)\, d\mathcal{H}^{n-1}(x) \\ &\geq \int_0^\infty f(r) \sqrt{p_E^2(r) + (r^{n-1} \xi_v'(r))^2}\, dr + \int_0^\infty f(r) r^{n-1} d|D^s \xi_v|(r). \end{aligned} \tag{3.3.8}$$

*Moreover, in the special case $E = F_v$, equality holds true.*

*Proof.* To prove the proposition it is enough to consider the case in which $f = \chi_B$, with $B \subset (0, \infty)$ Borel set.

First, suppose $B \subset (0, \infty) \setminus G_{F_v}$. Thanks to Lemma 3.2.2, in this case we have $\xi_v' = 0$ in $B$ and $|r^{n-1} D\xi_v|(B) = |r^{n-1} D^s \xi_v|(B)$. Then, from (3.2.2) it follows that

$$\begin{aligned} &\int_{\partial^* E} \chi_B(|x|)\, d\mathcal{H}^{n-1}(x) = P(E; \Phi(B \times \mathbb{S}^{n-1})) \geq |D_\perp \chi_E|(\Phi(B \times \mathbb{S}^{n-1})) \\ &\geq |r^{n-1} D\xi_v|(B) = |r^{n-1} D^s \xi_v|(B) = \int_0^\infty \chi_B(r) r^{n-1} d|D^s \xi_v|(r) \\ &= \int_0^\infty \chi_B(r) \sqrt{p_E^2(r) + (r^{n-1} \xi_v'(r))^2}\, dr + \int_0^\infty \chi_B(r) r^{n-1} d|D^s \xi_v|(r), \end{aligned}$$

where we also used the fact that $p_E = 0$ $\mathcal{H}^1$-a.e. in $B$, since

$$\mathcal{H}^n(E \cap \Phi(B \times \mathcal{S}^{n-1})) \leq \int_{v=0} dr \int_{E_r} d\mathcal{H}^{n-1}(x) = \int_{\{v=0\}} v(r)\, dr = 0.$$

Let us now assume $B \subset G_{F_v}$. In this case, by Lemma 3.2.2 we have $|r^{n-1}D^s\xi_v|(B) = 0$. Then, thanks to (3.3.3) and (3.3.4) we obtain

$$
\begin{aligned}
&\int_{\partial^* E} \chi_B(|x|)\, d\mathcal{H}^{n-1}(x) = P(E;\Phi(B \times \mathbb{S}^{n-1}))\\
&\geq P(E;\Phi(B \times \mathbb{S}^{n-1}) \cap \{\nu_{\parallel}^E = 0\}) + \int_B \sqrt{p_E^2(r) + g^2(r)}dr\\
&\geq \int_B \sqrt{p_E^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr\\
&= \int_0^\infty \chi_B(r)\sqrt{p_E^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_0^\infty \chi_B(r) r^{n-1} d|D^s\xi_v|(r),
\end{aligned}
$$

so that (3.3.8) follows.

Consider now the case $E = F_v$. If $B \subset G_{F_v}$, recalling again that by Lemma 3.2.2 we have $|r^{n-1}D^s\xi_v|(B) = 0$, thanks to (3.3.7) we obtain

$$
\begin{aligned}
&\int_{\partial^* F_v} \chi_B(|x|)\, d\mathcal{H}^{n-1}(x) = P(F_v;\Phi(B \times \mathbb{S}^{n-1})) = \int_B \sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr\\
&= \int_0^\infty \chi_B(r)\sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_0^\infty \chi_B(r) r^{n-1} d|D^s\xi_v|(r).
\end{aligned}
$$

If, instead, $B \subset (0,\infty) \setminus G_{F_v}$, then $\xi_v' = 0$ in $B$ and $|r^{n-1}D\xi_v|(B) = |r^{n-1}D^s\xi_v|(B)$. Therefore, thanks to (3.3.2),

$$
\begin{aligned}
&\int_{\partial^* F_v} \chi_B(|x|)\, d\mathcal{H}^{n-1}(x) = P(F_v;\Phi(B \times \mathbb{S}^{n-1})) \leq r^{n-1}\,|D\xi_v|\,(B) = |r^{n-1}D^s\xi_v|(B)\\
&= \int_0^\infty \chi_B(r)\sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_0^\infty \chi_B(r) r^{n-1} d|D^s\xi_v|(r).
\end{aligned}
$$

$\square$

An important consequence of the above proposition is a formula for the perimeter of $F_v$.

**Corollary 3.3.2.** *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying (1.2.2) such that $F_v$ is a set of finite perimeter and finite volume. Then*

$$
P(F_v;\Phi(B \times \mathbb{S}^{n-1})) = \int_B \sqrt{p_{F_v}^2(r) + (r^{n-1}\xi_v'(r))^2}\, dr + \int_B r^{n-1} d|D^s\xi_v|(r). \tag{3.3.9}
$$

We conclude this section with two important results, that will be used later.

**Proposition 3.3.3.** *Let $v : (0,\infty) \to [0,\infty)$ be a measurable function satisfying (1.2.2) such that $F_v$ is a set of finite perimeter and finite volume, and let $I \subset (0,+\infty)$ be an open set. Then the following three statements are equivalent:*

*(i)* $\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F_v \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{\parallel}^{F_v}(x) = 0\right\}\right) = 0$;

*(ii)* $\xi_v \in W^{1,1}_{\mathrm{loc}}(I)$;

*(iii)* $P(F_v;\Phi(B\times\mathbb{S}^{n-1}))=0$ *for every Borel set* $B\subset I$*, such that* $\mathcal{H}^1(B)=0$.

**Remark 3.3.4.** *Note that the equivalence* (iii) $\iff$ (i) *holds true also if* $I$ *is a Borel set. To show this, we only need to prove that* (i) $\implies$ (iii)*, since the opposite implication is given by repeating Step 3 of the proof of Proposition 3.3.3. Suppose* (i) *is satisfied. Then from* (3.2.8) *we have* $r^{n-1}D\xi_v\llcorner I=r^{n-1}\xi_v'\llcorner I$*. Therefore, thanks to* (3.3.9)

$$P(F_v;\Phi(B\times\mathbb{S}^{n-1}))=\int_B\sqrt{p_{F_v}^2(r)+(r^{n-1}\xi_v'(r))^2}\,dr\qquad\textit{for every Borel set } B\subset I,$$

*which implies (iii).*

*Proof.* We divide the proof into three steps.

**Step 1:** (i) $\implies$ (ii). Recall that, by Lemma 3.2.1, $\xi_v\in BV_{\rm loc}(I)$. If (i) is satisfied, from (3.2.8) we have $r^{n-1}D\xi_v\llcorner I=r^{n-1}\xi_v'\llcorner I$, which implies (ii).

**Step 2:** (ii) $\implies$ (iii). This implication follows from formula (3.3.9).

**Step 3:** (iii) $\implies$ (i) (note that we will not use the fact that $I$ is open). Assume (iii) holds true. Then,

$$\mathcal{H}^{n-1}\left(\left\{x\in\partial^*F_v\cap\Phi(I\times\mathbb{S}^{n-1}):\nu_\parallel^{\partial^*F_v}(x)=0\right\}\right)\le P(\partial^*F_v;\Phi((B_0\cap I)\times\mathbb{S}^{n-1}))=0,$$

where we used the fact that $\mathcal{H}^1(B_0)=0$, thanks to (3.1.19). □

We can now prove Lemma 1.2.3. In the proof, we will rely on Theorem 1.2.4 and Lemma 1.2.5, that we will prove in Section 3.6.

*Proof of Lemma 1.2.3.* We divide the proof into steps.

**Step 1:** We show that (1.2.8) $\implies$ (1.2.9). Suppose (1.2.8) is satisfied. Then, from (3.2.8) we have $r^{n-1}D\xi_v\llcorner I=r^{n-1}\xi_v'\llcorner I$. Thanks to (3.3.9), this implies that

$$P(F_v;\Phi(B\times\mathbb{S}^{n-1}))=\int_B\sqrt{p_{F_v}^2(r)+(r^{n-1}\xi_v'(r))^2}\,dr.\qquad\text{for every Borel set } B\subset I.$$

In particular, condition (iii) of Proposition 3.3.3 is satisfied. Then, (1.2.9) follows from Remark 3.3.4.

**Step 2:** We show that if $P(E;\Phi(I\times\mathbb{S}^{n-1}))=P(F_v;\Phi(I\times\mathbb{S}^{n-1}))$, then (1.2.9) implies (1.2.8). To this aim, we first prove an auxiliary result.

**Step 2a:** We show that if $\overline{F}\subset\mathbb{R}^n$ is a set of finite perimeter such that $(\overline{F})_r$ is a spherical cap for $\mathcal{H}^1$-a.e. $r>0$, and

$$\mathcal{H}^{n-1}\left(\left\{x\in\partial^*\overline{F}\cap\Phi(I\times\mathbb{S}^{n-1}):\nu_\parallel^{\overline{F}}(x)=0\right\}\right)=0,\tag{3.3.10}$$

then $\mathcal{H}^{n-1}(B^j)=0$ for every $j=2,\dots,n$, where

$$B^j := \left\{ x \in \partial^* \overline{F} \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{1j\|}^{\overline{F}}(x) = 0 \right\}.$$

Here, the vector $\nu_{1j\|}^{\overline{F}}$ is defined in the following way. Let $j \in \{2,\dots,n\}$, and let $\nu_{1j}^{\overline{F}}$ be the orthogonal projection of $\nu^{\overline{F}}$ on the bi-dimensional plane generated by $e_1$ and $e_j$. In this plane, we consider the following orthonormal basis $\{\widehat{x}_{1j}, \widetilde{x}_{1j}\}$:

$$\widehat{x}_{1j} = \frac{1}{\sqrt{x_1^2+x_j^2}}(x_1, \overbrace{0,\dots,0}^{j-2 \text{ times}}, x_j, \overbrace{0,\dots,0}^{n-j \text{ times}}),$$

and

$$\widetilde{x}_{1j} = \frac{1}{\sqrt{x_1^2+x_j^2}}(-x_j, \overbrace{0,\dots,0}^{j-2 \text{ times}}, x_1, \overbrace{0,\dots,0}^{n-j \text{ times}}),$$

where $\widehat{x}_{1j}$ is directed along the radial direction, and $\widetilde{x}_{1j}$ is parallel to the tangential direction. To show the claim, first of all note that, by Vol'pert Theorem 3.1.7, for $\mathcal{H}^1$-a.e. $r>0$ we have

$$(B^j)_r = \left\{ x \in \partial^* \overline{F}_r \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{\|}^{\overline{F}_r}(x) \cdot \widetilde{x}_{1j} = 0 \right\}.$$

up to an $\mathcal{H}^{n-2}$-negligible set. Since $(B^j)_r$ is a spherical cap, we have $\mathcal{H}^{n-2}((B^j)_r)=0$. Then, thanks to (3.3.10),

$$\begin{aligned}\mathcal{H}^{n-1}(B^j) &= \mathcal{H}^{n-1}\left( B^j \cap \left\{ x \in \partial^* \overline{F} \cap \Phi(I \times \mathbb{S}^{n-1}) : \nu_{\|}^{\overline{F}}(x) \neq 0 \right\} \right) \\ &= \int_I dr \int_{\partial^* \overline{F}_r \cap (B^j)_r} \chi_{\{\nu_{\|}^{\overline{F}} \neq 0\}}(x) \frac{1}{|\nu_{\|}^{\overline{F}}(x)|}\, d\mathcal{H}^{n-2}(x) = 0.\end{aligned}$$

**Step 2b:** We conclude. Let $E^1 := E$, and let $E^2$ be set obtained by applying to $E$ the circular symmetrisation with respect to $(e_1, e_2)$. Then, for $j=3,\dots,n$, we define iteratively the set $E^j$ as the circular symmetral of $E^{j-1}$ with respect to $(e_1,e_j)$. Note that, since $\mathcal{H}^1$-a.e. spherical section of $E$ is a spherical cap, we have $E^n = F_v$. Therefore, thanks to the perimeter inequality (1.2.11) under circular symmetrisation, we have

$$P(F_v; \Phi(I \times \mathbb{S}^{n-1})) = P(E^{n-1}; \Phi(I \times \mathbb{S}^{n-1})) = \ldots = P(E; \Phi(I \times \mathbb{S}^{n-1})).$$

Moreover, for $j=3,\dots,n$, we define $I_j := \Phi(I \times \mathbb{S}^{n-1}) \cap \{x_j = 0\} \cap \{x_1 > 0\}$. It is not difficult to check that

$$\Phi(I \times \mathbb{S}^{n-1}) = \Phi_{1j}(I_j \times \mathbb{S}^1) \quad \textit{for } j = 3,\dots,n.$$

Then, applying Lemma 1.2.5 to $F_v$ and $E^{n-1}$, we obtain that

$$\mathcal{H}^{n-1}\left( \left\{ x \in \partial^* E^{n-1} \cap \Phi_{1\,n-1}(I_{n-1} \times \mathbb{S}^1) : \nu_{1(n-1)\|}^{E^{n-1}}(x) = 0 \right\} \right) = 0,$$

which, in turns, implies

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E^{n-1} \cap \Phi_{1\,n-1}(I_{n-1} \times \mathbb{S}^1) : \nu_{\parallel}^{E^{n-1}}(x) = 0\right\}\right) = 0.$$

Applying iteratively this argument to $E^{n-2}, \dots, E$, we conclude. $\square$

## 3.4 Proof of Theorem 1.2.2: (ii) $\implies$ (i)

Before giving the proof of the implication (ii) $\implies$ (i) of Theorem 1.2.2, it will be convenient to introduce some useful notation. Let $v$ and $\mathcal{I} = \{0 < \alpha_v^\wedge \leq \alpha_v^\vee < \pi\}$ be as in the statement of Theorem 1.2.2. By assumption, $\mathcal{I}$ is an interval and $\alpha_v \in W^{1,1}_{\mathrm{loc}}(I)$ where, to ease the notation, we set $I := \mathring{\mathcal{I}}$. Let now $E$ be a spherically $v$-distributed set of finite perimeter. We define the *average direction of* $E$ as the map $d_E : I \to \mathbb{S}^{n-1}$ given by

$$d_E(r) := \begin{cases} \dfrac{1}{\omega_{n-1}(\sin\alpha_v(r))^{n-1} r^{n-1}} \displaystyle\int_{E_r} \hat{x}\, d\mathcal{H}^{n-1}(x), & \text{if } r \in I \cap G_E, \\ e_1 & \text{otherwise in } I, \end{cases} \tag{3.4.1}$$

where $G_E \subset (0, \infty)$ is the set given by Theorem 3.1.7. To ease our calculations, it will also be convenient to introduce the *barycentre function* $b_E : I \to \mathbb{R}^n$ *of* $E$ as

$$b_E(r) := \begin{cases} \dfrac{1}{r^{n-1}} \displaystyle\int_{E_r} \hat{x}\, d\mathcal{H}^{n-1}(x), & \text{if } r \in I \cap G_E, \\ e_1 & \text{otherwise in } I. \end{cases}$$

The importance of the functions $d_E$ and $b_E$ is given by the following lemma.

**Lemma 3.4.1.** *Let $v$ be as in Theorem 1.2.2, let $I \subset (0, \infty)$ be an open interval, and let $E$ be a spherically $v$-distributed set of finite perimeter such that $E_r$ is $\mathcal{H}^{n-1}$-equivalent to a spherical cap for $\mathcal{H}^1$-a.e. $r \in I$. Then,*

$$E \cap \Phi(I \times \mathbb{S}^{n-1}) =_{\mathcal{H}^n} \{x \in \Phi(I \times \mathbb{S}^{n-1}) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, d_E(|x|)) < \alpha_v(|x|)\}.$$

*Moreover,*

$$b_E(r) = \omega_{n-1}(\sin\alpha_v(r))^{n-1} d_E(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I. \tag{3.4.2}$$

*Proof.* Let us immediately observe that (3.4.2) follows by construction of $d_E$ and $b_E$. By assumption, for $\mathcal{H}^1$-a.e. $r \in I$, there exists $\omega(r) \in \mathbb{S}^{n-1}$ such that $E_r = \mathbf{B}_{\alpha_v(r)}(r\omega(r))$. We are left to show that

$$\omega(r) = d_E(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I. \tag{3.4.3}$$

Note that for $\mathcal{H}^1$-a.e. $r \in I$ we have $E_r = \mathbf{B}_{\alpha_v(r)}(r\omega(r))$ and $\partial^* E_r = \mathbf{S}_{\alpha_v(r)}(r\omega(r))$. Therefore, for $\mathcal{H}^1$-a.e. $r \in I$

$$\int_{E_r} \hat{x}\, d\mathcal{H}^{n-1}(x) = \int_0^{\alpha_v(r)} d\beta \int_{\mathbf{S}_\beta(r\omega(r))} x\, d\mathcal{H}^{n-2}(x), \tag{3.4.4}$$

where using the symmetry of the geodesic sphere

$$\int_{\mathbf{S}_\beta(r\omega(r))} x\,d\mathcal{H}^{n-2}(x) = \omega(r)\int_{\mathbf{S}_\beta(r\omega(r))}(x\cdot\omega(r))\,d\mathcal{H}^{n-2}(x) \tag{3.4.5}$$
$$= \omega(r) r\cos(\beta)\mathcal{H}^{n-2}(\mathbf{S}_\beta(r\omega(r))).$$

Recalling the definition of $d_E$, identity (3.4.3) follows.

□

**Remark 3.4.2.** *Let us point out that here we are using the term barycentre in a slightly imprecise way. Indeed, for a given $r \in I \cap G_E$, the geometric barycentre of $E_r$ is given by*

$$\frac{1}{\mathcal{H}^{n-1}(E_r)}\int_{E_r} x\,d\mathcal{H}^{n-1}(x) = \frac{1}{\xi_v(r)r^{n-1}}\int_{E_r} x\,d\mathcal{H}^{n-1}(x)$$
$$= \frac{r}{\xi_v(r)}\frac{1}{r^{n-1}}\int_{E_r}\hat{x}\,d\mathcal{H}^{n-1}(x) = \frac{r}{\xi_v(r)}b_E(r).$$

*Nevertheless, we will still keep this terminology, since $b_E$ turns out to be very useful for our analysis.*

We are now ready to prove the implication (ii) $\Longrightarrow$ (i) of Theorem 1.2.2.

*Proof of Theorem 1.2.2: (ii) $\Longrightarrow$ (i).* Suppose (ii) is satisfied, and let $E \in \mathcal{N}(v)$. We are going to show that there exists a rotation $R \in SO(n)$ such that $\mathcal{H}^n(E\Delta(RF_v)) = 0$. We now divide the proof into steps.

**Step 1:** First of all, we observe that

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E \cap \Phi(I\times\mathbb{S}^{n-1}) : \nu_{\parallel}^E(x) = 0\right\}\right) = 0.$$

Indeed, since $\alpha_v \in W^{1,1}_{\mathrm{loc}}(I)$, thanks to Proposition 3.3.3 we have

$$\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F_v \cap \Phi(I\times\mathbb{S}^{n-1}) : \nu_{\parallel}^{F_v}(x) = 0\right\}\right) = 0.$$

Since $E \in \mathcal{N}(v)$, applying Lemma 1.2.3 the claim follows.

**Step 2:** We show that $b_E \in W^{1,1}_{\mathrm{loc}}(I;\mathbb{R}^n)$ and

$$b_E'(r) = \frac{1}{r^n}\int_{(\partial^* E)_r \cap \{\nu_{\parallel}^E \neq 0\}} x\frac{\hat{x}\cdot\nu^E(x)}{|\nu_{\parallel}^E(x)|}\,d\mathcal{H}^{n-2}(x). \tag{3.4.6}$$

Indeed, let $\psi \in C^1_c(I)$ be arbitrary, and let $i \in \{1,\dots,n\}$. By definition of $b_E$

$$\int_I (b_E)_i(r)\psi'(r)dr = \int_I\int_{E\cap\partial B(r)}\frac{1}{r^{n-1}}\frac{x_i}{|x|}d\mathcal{H}^{n-1}(x)\psi'(r)dr$$
$$= \int_{\Phi(I\times\mathbb{S}^{n-1})}\frac{x_i}{|x|^n}\psi'(|x|)\chi_E(x)\,dx.$$

Note now that

$$\operatorname{div}\left(\frac{x_i}{|x|^n}\psi(|x|)\hat{x}\right) = \frac{x_i}{|x|^n}\psi'(|x|).$$

Indeed, recalling (3.2.3),

$$\begin{aligned}\operatorname{div}\left(\frac{x_i}{|x|^n}\psi(|x|)\hat{x}\right) &= \psi(|x|)\nabla\left(\frac{x_i}{|x|^n}\right)\cdot\hat{x} + \frac{x_i}{|x|^n}\operatorname{div}(\psi(|x|)\hat{x})\\ &= \psi(|x|)\left(\frac{e_i}{|x|^n} - \frac{n\,x_i}{|x|^{n+1}}\hat{x}\right)\cdot\hat{x} + \frac{x_i}{|x|^n}\left(\psi'(|x|) + \psi(|x|)\frac{n-1}{|x|}\right) = \frac{x_i}{|x|^n}\psi'(|x|).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_I (b_E)_i(r)\psi'(r)dr &= \int_{\Phi(I\times\mathbb{S}^{n-1})}\operatorname{div}\left(\frac{x_i}{|x|^n}\psi(|x|)\hat{x}\right)\chi_E(x)\,dx\\ &= -\int_{\Phi(I\times\mathbb{S}^{n-1})}\frac{x_i}{|x|^n}\psi(|x|)\hat{x}\cdot dD\chi_E(x)\\ &= \int_{\partial^* E\cap\Phi(I\times\mathbb{S}^{n-1})}\frac{x_i}{|x|^n}\psi(|x|)\,\hat{x}\cdot\nu^E(x)d\mathcal{H}^{n-1}(x).\end{aligned}$$

Thanks to Step 1 we then obtain

$$\begin{aligned}\int_I (b_E)_i(r)\psi'(r)dr &= \int_{\partial^* E\cap\{\nu^E_{\parallel}\neq 0\}\cap\Phi(I\times\mathbb{S}^{n-1})}\frac{x_i}{|x|^n}\psi(|x|)\,\hat{x}\cdot\nu^E(x)d\mathcal{H}^{n-1}(x)\\ &= \int_I \psi(r)\frac{1}{r^n}\left[\int_{(\partial^* E)_r\cap\{\nu^E_{\parallel}\neq 0\}} x_i\frac{\hat{x}\cdot\nu^E(x)}{|\nu^E_{\parallel}(x)|}\,d\mathcal{H}^{n-2}(x)\right]dr,\end{aligned}$$

so that (3.4.6) follows.

**Step 3:** We show that

$$b'_E(r) = (n-1)\alpha'_v(r)\,\frac{\cos\alpha_v(r)}{\sin\alpha_v(r)}\,b_E(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r\in I. \tag{3.4.7}$$

Since $E\in\mathcal{N}(v)$, from Theorem 1.2.4 we know that for $\mathcal{H}^1$-a.e. $r\in I$ the spherical slice $E_r$ is a spherical cap. Then, thanks to Lemma 3.4.1

$$E_r = \mathbf{B}_{\alpha_v(r)}(rd_E(r)) \quad \text{and} \quad (\partial^* E)_r = \mathbf{S}_{\alpha_v(r)}(rd_E(r)) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r\in I.$$

Still thanks to Theorem 1.2.4, we know that for $\mathcal{H}^1$-a.e. $r\in I$ the functions $x\mapsto\nu^E(x)\cdot\hat{x}$ and $x\mapsto|\nu^E_{\parallel}|(x)$ are constant $\mathcal{H}^{n-2}$-a.e. in $(\partial^* E)_r$, say

$$\nu^E(x)\cdot\hat{x} = a(r) \quad \text{and} \quad |\nu^E_{\parallel}|(x) = c(r), \qquad \text{for } \mathcal{H}^1\text{-a.e. } r\in I,$$

for some measurable functions $a : I\to(-1,1)$ and $c : I\to(0,1]$. Therefore, recalling the definition of $d_E$ together with (3.4.4)-(3.4.5) we obtain

$$\begin{aligned} b_E'(r) &= \frac{1}{r^n} \int_{(\partial^* E)_r \cap \{\nu_\parallel^E \neq 0\}} x \frac{\hat{x} \cdot \nu^E(x)}{|\nu_\parallel^E(x)|} \, d\mathcal{H}^{n-2}(x) \\ &= \frac{1}{r^n} \frac{a(r)}{c(r)} \int_{\mathbf{S}_{\alpha_v(r)}(r d_E(r))} x \, d\mathcal{H}^{n-2}(x) \\ &= \frac{1}{r^n} \frac{a(r)}{c(r)} r \cos(\alpha_v(r)) \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(r d_E(r))) d_E(r) \\ &= \frac{1}{r^{n-1}} \frac{a(r)}{c(r)} \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(r d_E(r))) \cos(\alpha_v(r)) d_E(r). \end{aligned} \tag{3.4.8}$$

Note now that from Step 1 and (3.2.8) it follows that for $\mathcal{H}^1$-a.e. $r \in I$

$$\begin{aligned} r^{n-1} \xi_v'(r) &= \int_{(\partial^* E)_r \cap \{\nu_\parallel^E \neq 0\}} \frac{\hat{x} \cdot \nu^E(x)}{|\nu_\parallel^E(x)|} d\mathcal{H}^{n-2}(x) \\ &= \frac{a(r)}{c(r)} \mathcal{H}^{n-2}(\mathbf{S}_{\alpha_v(r)}(r d_E(r))). \end{aligned}$$

Plugging last identity into (3.4.8) and using (3.4.2), we obtain

$$\begin{aligned} b_E'(r) &= \xi_v'(r) \cos(\alpha_v(r)) d_E(r) = \xi_v'(r) \cos(\alpha_v(r)) \frac{b_E(r)}{\omega_{n-1} (\sin \alpha_v(r))^{n-1}} \\ &= (n-1) \alpha_v'(r) \, \frac{\cos \alpha_v(r)}{\sin \alpha_v(r)} \, b_E(r), \end{aligned}$$

where we used the fact that, thanks to (3.1.1) and (3.1.3),

$$\xi_v'(r) = (n-1) \omega_{n-1} (\sin \alpha_v(r))^{n-2} \alpha_v'(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I.$$

**Step 4:** We conclude. First of all, note that from From (3.4.2) and Step 2 it follows that $d_E \in W^{1,1}_{\mathrm{loc}}(I; \mathbb{S}^{n-1})$. Then, thanks to Step 3, for $\mathcal{H}^1$-a.e. $r \in I$

$$\begin{aligned} \omega_{n-1} d_E'(r) &= \frac{d}{dr} \left[ \frac{b_E(r)}{(\sin \alpha_v(r))^{n-1}} \right] = \frac{b_E'(r)}{(\sin \alpha_v(r))^{n-1}} + b_E(r) \frac{d}{dr} \left[ \frac{1}{(\sin \alpha_v(r))^{n-1}} \right] \\ &= (n-1) \alpha_v'(r) \, \frac{\cos \alpha_v(r)}{(\sin \alpha_v(r))^n} \, b_E(r) + b_E(r) \left[ - \frac{n-1}{(\sin \alpha_v(r))^n} (\cos \alpha_v(r)) \alpha_v'(r) \right] = 0, \end{aligned}$$

for $\mathcal{H}^1$-a.e. $r \in I$. This shows that $d_E$ is $\mathcal{H}^1$-a.e. constant in $I$. Therefore, $E \cap \Phi(I \times \mathbb{S}^{n-1})$ can be obtained by applying a rotation to $F_v \cap \Phi(I \times \mathbb{S}^{n-1})$. $\square$

## 3.5 Proof of Theorem 1.2.2: (i) $\Longrightarrow$ (ii)

We start by showing that the fact that $\{0 < \alpha^\wedge \leq \alpha^\vee < \pi\}$ is an interval is a necessary condition for rigidity.

**Proposition 3.5.1.** *Let $v : (0, \infty) \to [0, \infty)$ be a measurable function satisfying (1.2.2), such that $F_v$ is a set of finite perimeter and finite volume, and let $\alpha_v$ be defined by (1.2.3).*

*Suppose that the set* $\{0 < \alpha^\wedge \le \alpha^\vee < \pi\}$ *is not an interval. That is, suppose that there exists* $\overline{r} \in \{\alpha^\wedge = 0\} \cup \{\alpha^\vee = \pi\}$ *such that*

$$(0, \overline{r}) \cap \{0 < \alpha^\wedge \le \alpha^\vee < \pi\} \neq \emptyset \qquad \text{and} \qquad (\overline{r}, \infty) \cap \{0 < \alpha^\wedge \le \alpha^\vee < \pi\} \neq \emptyset.$$

*Then, rigidity fails. More precisely, setting* $E_1 := F_v \cap B(\overline{r})$ *and* $E_2 := F_v \setminus B(\overline{r})$, *we have*

$$E_1 \cup (RE_2) \in \mathcal{N}(v) \qquad \text{for every } R \in SO(n).$$

Before giving the proof of Proposition 3.5.1 we need the following lemma.

**Lemma 3.5.2.** *Let* $v : (0, \infty) \to [0, \infty)$ *be a measurable function satisfying* (1.2.2)*, such that* $F_v$ *is a set of finite perimeter and finite volume. Let* $\alpha_v$ *be defined by* (1.2.3)*, and let* $\overline{r} > 0$*. Then,*

$$(\partial^* F_v)_{\overline{r}} =_{\mathcal{H}^{n-1}} \mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1) \setminus \mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1).$$

*Proof.* We divide the proof in two steps.

**Step 1:** We show that

$$(\partial^* F_v)_{\overline{r}} \subset \overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)} \setminus \mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1).$$

To this aim, it will be enough to show that

$$\alpha_v^\wedge(\overline{r}) \le \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \le \alpha_v^\vee(\overline{r}) \qquad \text{for every } x \in (\partial^* F_v)_{\overline{r}}. \tag{3.5.1}$$

Let us first prove that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \le \alpha_v^\vee(\overline{r}) \qquad \text{for every } x \in (\partial^* F_v)_{\overline{r}} \tag{3.5.2}$$

Note that (3.5.2) is trivial if $\alpha_v^\vee(\overline{r}) = \pi$. For this reason, we will assume $\alpha_v^\vee(\overline{r}) < \pi$. Note now that (3.5.2) follows if we prove that

$$x \in \partial B(\overline{r}) \quad \text{and} \quad \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) > \alpha_v^\vee(\overline{r}) \quad \Longrightarrow \quad x \in F_v^{(0)}. \tag{3.5.3}$$

Let now $x \in \partial B(\overline{r})$, and suppose that there exists $\delta > 0$ such that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) = \alpha_v^\vee(\overline{r}) + \delta.$$

Let now $\overline{\rho} > 0$ be so small that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \hat{x}) < \frac{\delta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}).$$

By triangle inequality for the geodesic distance we have, in particular, that

$$\alpha_v^\vee(\overline{r}) + \delta = \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \le \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, \hat{y}) + \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) < \frac{\delta}{2} + \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1),$$

so that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) > \alpha_v^\vee(\overline{r}) + \frac{\delta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}). \tag{3.5.4}$$

Thanks to the inequality above, by definition of $F_v$ we have

$$F_v \cap B(x, \overline{\rho}) \subset \left\{ y \in \mathbb{R}^n : \alpha_v^\vee(\overline{r}) + \frac{\delta}{2} < \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) < \alpha_v(|y|) \right\} \cap B(x, \overline{\rho}).$$

Therefore, for every $\rho \in (0, \overline{\rho})$

$$\begin{aligned}
\mathcal{H}^n(F_v \cap B(x,\rho)) &= \int_{\overline{r}-\rho}^{\overline{r}+\rho} \mathcal{H}^{n-1}(F_v \cap B(x,\rho) \cap \partial B(r))\, dr \\
&\leq \int_{\overline{r}-\rho}^{\overline{r}+\rho} \chi_{\{\alpha_v > \alpha_v^\vee(\overline{r}) + \delta/2\}}(r) \mathcal{H}^{n-1}(F_v \cap B(x,\rho) \cap \partial B(r))\, dr \\
&= \int_{(\overline{r}-\rho, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^\vee(\overline{r}) + \delta/2\}} \mathcal{H}^{n-1}(F_v \cap B(x,\rho) \cap \partial B(r))\, dr.
\end{aligned}$$

Note now that, for $\rho$ small enough, there exists $C = C(\overline{r}) > 0$ such that

$$B(x, \rho) \cap \partial B(r) \subset \mathbf{B}_{C\rho}(r\hat{x}) \qquad \text{for every } r \in (\overline{r} - \rho, \overline{r} + \rho).$$

Therefore,

$$\begin{aligned}
\mathcal{H}^n(F_v \cap B(x,\rho)) &\leq \int_{(\overline{r}-\rho, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^\vee(\overline{r}) + \delta/2\}} \mathcal{H}^{n-1}(\mathbf{B}_{C\rho}(r\hat{x}))\, dr \\
&= (n-1)\omega_{n-1} \int_{(\overline{r}-\rho, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^\vee(\overline{r}) + \delta/2\}} r^{n-1} \int_0^{C\rho} (\sin\tau)^{n-2}\, d\tau\, dr \\
&\leq (n-1)\omega_{n-1} \int_{(\overline{r}-\rho, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^\vee(\overline{r}) + \delta/2\}} r^{n-1} \int_0^{C\rho} \tau^{n-2}\, d\tau\, dr \\
&= \omega_{n-1} C^{n-1} (\overline{r} + \overline{\rho})^{n-1} \rho^{n-1} \mathcal{H}^1((\overline{r}-\rho, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^\vee(\overline{r}) + \delta/2\}).
\end{aligned}$$

Thus, recalling the definition of $\alpha_v^\vee(\overline{r})$,

$$\begin{aligned}
&\lim_{\rho \to 0^+} \frac{\mathcal{H}^n(F_v \cap B(x,\rho))}{\omega_n \rho^n} \\
&\leq \frac{\omega_{n-1} C^{n-1}}{\omega_n} (\overline{r} + \overline{\rho})^{n-1} \lim_{\rho \to 0^+} \frac{\mathcal{H}^1((\overline{r}-\rho, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^\vee(\overline{r}) + \delta/2\})}{\rho} = 0,
\end{aligned}$$

which gives (3.5.3) and, in turn, (3.5.2). By similar arguments, one can prove that

$$x \in \partial B(\overline{r}) \quad \text{and} \quad \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) < \alpha_v^\wedge(\overline{r}) \quad \Longrightarrow \quad x \in F_v^{(1)},$$

which implies that

$$\alpha_v^\wedge(\overline{r}) \leq \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \qquad \text{for every } x \in (\partial^* F_v)_{\overline{r}}.$$

The above inequality, together with (3.5.2), shows (3.5.1).

**Step 2:** We conclude. Thanks to Corollary 3.3.2,

$$\begin{aligned}
&\mathcal{H}^{n-1}((\partial^* F_v)_{\overline{r}}) = \mathcal{H}^{n-1}(\partial^* F_v \cap \partial B(\overline{r})) = P(F_v; \partial B(\overline{r})) = \overline{r}^{n-1}(\xi_v^\vee(\overline{r}) - \xi_v^\wedge(\overline{r})) \\
&= v^\vee(\overline{r}) - v^\wedge(\overline{r}) = \mathcal{H}^{n-1}(\overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)}) - \mathcal{H}^{n-1}(\mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1)) \\
&= \mathcal{H}^{n-1}\left(\overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)} \setminus \mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1)\right)
\end{aligned}$$

Since, by Step 1,

$$(\partial^* F_v)_{\overline{r}} \subset \overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)} \setminus \mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1),$$

we have

$$(\partial^* F_v)_{\overline{r}} =_{\mathcal{H}^{n-1}} \overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)} \setminus \mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1) =_{\mathcal{H}^{n-1}} \mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1) \setminus \mathbf{B}_{\alpha_v^\wedge(\overline{r})}(\overline{r}e_1).$$

□

We can now give the proof of Proposition 3.5.1.

*Proof of Proposition 3.5.1.* Note that, since $B(\overline{r})$ is open and $E \cap B(\overline{r}) = F_v \cap B(\overline{r})$, we have

$$E^{(t)} \cap B(\overline{r}) = (E \cap B(\overline{r}))^{(t)} = (F_v \cap B(\overline{r}))^{(t)} = F_v^{(t)} \cap B(\overline{r}) \qquad \text{for every } t \in [0,1].$$

From this, it follows that

$$\partial^* E \cap B(\overline{r}) = \partial^* F_v \cap B(\overline{r}). \tag{3.5.5}$$

Similarly, we obtain

$$\partial^* E \setminus \overline{B(\overline{r})} = \partial^*(RF_v) \setminus \overline{B(\overline{r})} = (R\,\partial^* F_v) \setminus (R\overline{B(\overline{r})}) = R(\partial^* F_v \setminus \overline{B(\overline{r})}). \tag{3.5.6}$$

Thus, thanks to (3.5.5) and (3.5.6)

$$\begin{aligned}
P(E) &= \mathcal{H}^{n-1}(\partial^* E \cap B(\overline{r}) + \mathcal{H}^{n-1}(\partial^* E \cap \partial B(\overline{r}) + \mathcal{H}^{n-1}(\partial^* E \setminus \overline{B(\overline{r})}) \\
&= \mathcal{H}^{n-1}(\partial^* F_v \cap B(\overline{r}) + \mathcal{H}^{n-1}(\partial^* E \cap \partial B(\overline{r}) + \mathcal{H}^{n-1}\left(R(\partial^* F_v \setminus \overline{B(\overline{r})})\right) \\
&= \mathcal{H}^{n-1}(\partial^* F_v \cap B(\overline{r}) + \mathcal{H}^{n-1}(\partial^* E \cap \partial B(\overline{r}) + \mathcal{H}^{n-1}(\partial^* F_v \setminus \overline{B(\overline{r})}).
\end{aligned}$$

Therefore, in order to conclude the proof we only need to show that

$$\mathcal{H}^{n-1}(\partial^* E \cap B(\overline{r})) = \mathcal{H}^{n-1}(\partial^* F_v \cap B(\overline{r})). \tag{3.5.7}$$

Without any loss of generality, we will assume that

$$\alpha_v^\vee(\overline{r}) = \operatorname{aplim}(f, (0, \overline{r}), \overline{r}), \qquad 0 = \alpha_v^\wedge(\overline{r}) = \operatorname{aplim}(f, (\overline{r}, \infty), \overline{r}). \tag{3.5.8}$$

Let now $E_1, E_2$, and $R$ be as in the statement. We divide the proof of (3.5.7) into steps.

**Step 1:** We show that

$$(\partial^* E)_{\overline{r}} \subset \overline{\mathbf{B}_{\alpha_v^\vee(\overline{r})}(\overline{r}e_1)} \cup \{R(\overline{r}e_1)\}.$$

To this aim, it will be enough to prove that

$$\operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) \leq \alpha_v^\vee(\overline{r}) \qquad \text{for every } x \in (\partial^* E)_{\overline{r}}. \tag{3.5.9}$$

If $\alpha_v^\vee(\overline{r}) = \pi$ inequality (3.5.9) is obvious, so we will assume that $\alpha_v^\vee(\overline{r}) < \pi$.

**Step 1a:** We show that

$$x \in \partial B(\overline{r}) \quad \text{and} \quad \operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) > \alpha_v^\vee(\overline{r}) \quad \Longrightarrow \quad x \in E_1^{(0)}.$$

Indeed, let $x \in \partial B(\overline{r})$, and suppose that there exists $\delta > 0$ such that

$$\operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) = \alpha_v^\vee(\overline{r}) + \delta.$$

By repeating the argument used to show (3.5.4), we can choose $\overline{\rho} > 0$ so small that

$$\operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) > \alpha_v^\vee(\overline{r}) + \frac{\delta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}).$$

By definition of $E_1$, we then have

$$\begin{aligned} &E_1 \cap B(x, \overline{\rho}) = F_v \cap B(\overline{r}) \cap B(x, \overline{\rho}) \\ &\subset \left\{ y \in \mathbb{R}^n : |y| < \overline{r} \text{ and } \alpha_v^\vee(\overline{r}) + \frac{\delta}{2} < \operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) < \alpha_v(|y|) \right\} \cap B(x, \overline{\rho}). \end{aligned}$$

Therefore, for every $\rho \in (0, \overline{\rho})$, by repeating the calculations done in Step 1 of Lemma 3.5.2, we obtain

$$\begin{aligned} &\lim_{\rho \to 0^+} \frac{1}{\omega_n \rho^n} \mathcal{H}^n(E_1 \cap B(x, \rho)) \\ &= \lim_{\rho \to 0^+} \frac{1}{\omega_n \rho^n} \int_{\overline{r}-\rho}^{\overline{r}} \mathcal{H}^{n-1}(F_v \cap B(x, \rho) \cap \partial B(r))\, dr \\ &\leq \frac{\omega_{n-1} C^{n-1}}{\omega_n} (\overline{r} + \overline{\rho})^{n-1} \lim_{\rho \to 0^+} \frac{\mathcal{H}^1((\overline{r} - \rho, \overline{r}) \cap \{\alpha_v > \alpha_v^\vee(\overline{r}) + \delta/2\})}{\rho} = 0, \end{aligned}$$

where we used (3.5.8).

**Step 1b:** We show that

$$\partial B(\overline{r}) \setminus \{R(\overline{r}e_1)\} \subset (RE_2)^{(0)}.$$

Indeed, let $x \in \partial B(\overline{r})$, and suppose that $\eta := \operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{x}, Re_1) > 0$. We are going to prove that $x \in (RE_2)^{(0)}$. By repeating the argument used to show (3.5.4), we can choose $\overline{\rho} > 0$ so small that

$$\operatorname{dist}_{\mathbb{S}^{n-1}}(\hat{y}, Re_1) > \frac{\eta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}).$$

Then,

$$\begin{aligned}&(RE_2)\cap B(x,\overline{\rho}) = \Big(R(F_v\setminus\overline{B(\overline{r})})\Big)\cap B(x,\overline{\rho})\\&\subset_{\mathcal{H}^n}\left\{y\in\mathbb{R}^n : |y|>\overline{r}\text{ and }\frac{\eta}{2}<\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y},Re_1)<\alpha_v(|y|)\right\}\cap B(x,\overline{\rho}).\end{aligned}$$

For $\rho$ small enough, there exists $C=C(\overline{r})>0$ such that

$$B(x,\rho)\cap\partial B(r)\subset\mathbf{B}_{C\rho}(r\hat{x})\qquad\text{for every } r\in(\overline{r}-\rho,\overline{r}+\rho).$$

Therefore, for every $\rho\in(0,\overline{\rho})$,

$$\begin{aligned}\mathcal{H}^n((RE_2)\cap B(x,\rho)) &\le \int_{(\overline{r},\overline{r}+\rho)\cap\{\alpha_v>\eta/2\}}\mathcal{H}^{n-1}(\mathbf{B}_{C\rho}(r\hat{x}))\,dr\\&=(n-1)\omega_{n-1}\int_{(\overline{r},\overline{r}+\rho)\cap\{\alpha_v>\eta/2\}}r^{n-1}\int_0^{C\rho}(\sin\tau)^{n-2}\,d\tau\,dr\\&=\omega_{n-1}C^{n-1}(\overline{r}+\overline{\rho})^{n-1}\rho^{n-1}\mathcal{H}^1((\overline{r},\overline{r}+\rho)\cap\{\alpha_v>\eta/2\}).\end{aligned}$$

From this, thanks to (3.5.8), we obtain

$$\begin{aligned}&\lim_{\rho\to0^+}\frac{\mathcal{H}^n((RE_2)\cap B(x,\rho))}{\omega_n\rho^n}\\&\le\frac{\omega_{n-1}C^{n-1}}{\omega_n}(\overline{r}+\overline{\rho})^{n-1}\lim_{\rho\to0^+}\frac{\mathcal{H}^1((\overline{r},\overline{r}+\rho)\cap\{\alpha_v>\eta/2\})}{\rho}=0.\end{aligned}$$

**Step 1c:** We conclude the proof of Step 1. By definition of $E$, from Step 1a and Step 1b it follows that

$$\{x\in\partial B(\overline{r}) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},e_1)>\alpha_v^{\vee}(\overline{r})\}\setminus\{Re_1\}\subset E_1^{(0)}\cap(RE_2)^{(0)}=E^{(0)}.$$

Therefore,

$$\begin{aligned}(\partial^*E)_r&\subset\partial B(\overline{r})\setminus\left(\{x\in\partial B(\overline{r}) : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},e_1)>\alpha_v^{\vee}(\overline{r})\}\setminus\{Re_1\}\right)\\&=\overline{\mathbf{B}_{\alpha_v^{\vee}(\overline{r})}(\overline{r}e_1)}\cup\{Re_1\}.\end{aligned}$$

**Step 2:** We show (3.5.7), concluding the proof. Thanks to Step 1 and Lemma 3.5.2 we have

$$\begin{aligned}P(E;\partial B(\overline{r}))&=\mathcal{H}^{n-1}(\partial^*E\cap\partial B(\overline{r}))=\mathcal{H}^{n-1}((\partial^*E)_{\overline{r}})\le\mathcal{H}^{n-1}\left(\mathbf{B}_{\alpha_v^{\vee}(\overline{r})}(\overline{r}e_1)\right)\\&=\mathcal{H}^{n-1}(\partial^*F_v\cap\partial B(\overline{r}))=P(F_v;\partial B(\overline{r}))\le P(E;\partial B(\overline{r})),\end{aligned}$$

where we also used (1.2.4) with $B=\{\overline{r}\}$. □

We now show that, if the jump part $D^j\alpha_v$ of $D\alpha_v$ is non zero, rigidity fails.

**Proposition 3.5.3.** *Let $v:(0,\infty)\to[0,\infty)$ be a measurable function satisfying (1.2.2) such that $F_v$ is a set of finite perimeter and finite volume, and let $\alpha_v$ be defined by (1.2.3). Suppose that $\alpha_v$ has a jump at some point $\overline{r}>0$. Then, rigidity fails. More precisely, setting $E_1 := F_v\cap B(\overline{r})$ and $E_2 := F_v\setminus B(\overline{r})$, we have*

$$E_1\cup(RE_2)\in\mathcal{N}(v),$$

*for every $R\in SO(n)$ such that*

$$0<\mathrm{dist}_{\mathbb{S}^{n-1}}(Re_1,e_1)<\lambda(\alpha_v^{\vee}(\overline{r})-\alpha_v^{\wedge}(\overline{r}))\qquad \textit{for some } \lambda\in(0,1). \tag{3.5.10}$$

*Proof.* Let $R\in SO(n)$, $\lambda\in(0,1)$, and $E\in\mathbb{R}^n$ be as in the statement, and set $\omega:=Re_1$. Arguing as in the proof of Proposition 3.5.1 we have:

$$P(E)=\mathcal{H}^{n-1}(\partial^*F_v\cap B(\overline{r}))+\mathcal{H}^{n-1}(\partial^*E\cap\partial B(\overline{r}))+\mathcal{H}^{n-1}(\partial^*F_v\setminus\overline{B(\overline{r})}).$$

Therefore, in order to conclude the proof we only need to show that

$$\mathcal{H}^{n-1}(\partial^*E\cap\partial B(\overline{r}))=\mathcal{H}^{n-1}(\partial^*F_v\cap\partial B(\overline{r})). \tag{3.5.11}$$

Without any loss of generality, we will assume that

$$\alpha_v^{\vee}(\overline{r})=\mathrm{aplim}(f,(0,\overline{r}),\overline{r})\,,\qquad \alpha_v^{\wedge}(\overline{r})=\mathrm{aplim}(f,(\overline{r},\infty),\overline{r})\,. \tag{3.5.12}$$

We now proceed by steps.

**Step 1:** We show that

$$(\partial^*E)_{\overline{r}}\subset\overline{\mathbf{B}_{\alpha_v^{\vee}(\overline{r})}(\overline{r}e_1)}\setminus\mathbf{B}_{\alpha_v^{\wedge}(\overline{r})}(\overline{r}\omega). \tag{3.5.13}$$

To show (3.5.13), it is enough to prove that for every $x\in(\partial^*E)_{\overline{r}}$ we have

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},e_1)\le\alpha_v^{\vee}(\overline{r})\qquad\text{for every } x\in(\partial^*E)_{\overline{r}}, \tag{3.5.14}$$

and

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},\omega)\ge\alpha_v^{\wedge}(\overline{r})\qquad\text{for every } x\in(\partial^*E)_{\overline{r}}. \tag{3.5.15}$$

We will only show (3.5.14), since (3.5.15) can be obtained in a similar way. Note that (3.5.14) is automatically satisfied if $\alpha_v^{\vee}(\overline{r})=\pi$, so we will assume $\alpha_v^{\vee}(\overline{r})<\pi$.

By arguing as in Step 1a of the proof of Proposition 3.5.1 we obtain

$$x\in\partial B(\overline{r})\quad\text{and}\quad\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},e_1)>\alpha_v^{\vee}(\overline{r})\quad\Longrightarrow\quad x\in E_1^{(0)}. \tag{3.5.16}$$

Let us now prove that

$$x\in\partial B(\overline{r})\quad\text{and}\quad\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},e_1)>\alpha_v^{\vee}(\overline{r})\quad\Longrightarrow\quad x\in(RE_2)^{(0)}. \tag{3.5.17}$$

Let $x \in \partial B(\overline{r})$, and suppose that there exists $\delta > 0$ such that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, e_1) = \alpha_v^{\vee}(\overline{r}) + \delta.$$

Thanks to the argument we used to show (3.5.4), we can choose $\overline{\rho} > 0$ so small that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) > \alpha_v^{\vee}(\overline{r}) + \frac{\delta}{2} \qquad \text{for every } y \in B(x, \overline{\rho}).$$

Therefore, for every $y \in B(x, \overline{\rho})$ we have

$$\begin{aligned} \alpha_v^{\vee}(\overline{r}) + \frac{\delta}{2} &< \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, e_1) \leq \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \omega) + \mathrm{dist}_{\mathbb{S}^{n-1}}(\omega, e_1) \\ &< \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \omega) + \lambda(\alpha_v^{\vee}(\overline{r}) - \alpha_v^{\wedge}(\overline{r})). \end{aligned}$$

Since $\overline{r}$ is a jump point for $\alpha_v$, we have $\alpha_v^{\vee}(\overline{r}) > \alpha_v^{\wedge}(\overline{r})$, and the above inequality implies that

$$\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \overline{\omega}) > (1-\lambda)\alpha_v^{\vee}(\overline{r}) + \lambda\alpha_v^{\wedge}(\overline{r}) + \frac{\delta}{2} > (1-\lambda)\alpha_v^{\wedge}(\overline{r}) + \lambda\alpha_v^{\wedge}(\overline{r}) + \frac{\delta}{2} = \alpha_v^{\wedge}(\overline{r}) + \frac{\delta}{2},$$

for every $\in B(x, \overline{\rho})$. Then, by definition of $E_2$,

$$\begin{aligned} &(RE_2) \cap B(x, \overline{\rho}) = \Big( R(F_v \setminus \overline{B(\overline{r})}) \Big) \cap B(x, \overline{\rho}) \\ &\subset_{\mathcal{H}^n} \Big\{ y \in \mathbb{R}^n : |y| > \overline{r} \text{ and } \alpha_v^{\wedge}(\overline{r}) + \frac{\delta}{2} < \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{y}, \omega) < \alpha_v(|y|) \Big\} \cap B(x, \overline{\rho}). \end{aligned}$$

As already observed in the previous proofs, for $\rho$ small enough there exists $C = C(\overline{r}) > 0$ such that

$$B(x, \rho) \cap \partial B(r) \subset \mathbf{B}_{C\rho}(r\hat{x}) \qquad \text{for every } r \in (\overline{r} - \rho, \overline{r} + \rho).$$

Therefore, for every $\rho \in (0, \overline{\rho})$ sufficiently small

$$\begin{aligned} \mathcal{H}^n((RE_2) \cap B(x, \rho)) &\leq \int_{(\overline{r}, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^{\wedge}(\overline{r}) + \delta/2\}} \mathcal{H}^{n-1}(\mathbf{B}_{C\rho}(r\hat{x}))\, dr \\ &= (n-1)\omega_{n-1} \int_{(\overline{r}, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^{\wedge}(\overline{r}) + \delta/2\}} r^{n-1} \int_0^{C\rho} (\sin\tau)^{n-2}\, d\tau\, dr \\ &= \omega_{n-1} C^{n-1} (\overline{r} + \overline{\rho})^{n-1} \rho^{n-1} \mathcal{H}^1((\overline{r}, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^{\wedge}(\overline{r}) + \delta/2\}). \end{aligned}$$

From this, thanks to (3.5.12), we obtain

$$\begin{aligned} &\lim_{\rho \to 0^+} \frac{\mathcal{H}^n((RE_2) \cap B(x, \rho))}{\omega_n \rho^n} \\ &\leq \frac{\omega_{n-1} C^{n-1}}{\omega_n} (\overline{r} + \overline{\rho})^{n-1} \lim_{\rho \to 0^+} \frac{\mathcal{H}^1((\overline{r}, \overline{r}+\rho) \cap \{\alpha_v > \alpha_v^{\wedge}(\overline{r}) + \delta/2\})}{\rho} = 0, \end{aligned}$$

which shows (3.5.17). This, together with (3.5.16), implies (3.5.14). As already mentioned, (3.5.15) can be proved in a similar way, and therefore (3.5.13) follows.

**Step 2:** We conclude. From (3.5.10) it follows that

$$\mathbf{B}_{\alpha_v^\wedge(\bar r)}(\bar r\omega) \subset \mathbf{B}_{\alpha_v^\vee(\bar r)}(\bar r e_1).$$

Therefore, thanks to (3.5.13) and Lemma 3.5.2

$$\begin{aligned}P(E;\partial B(\bar r)) &= \mathcal{H}^{n-1}(\partial^* E\cap \partial B(\bar r)) = \mathcal{H}^{n-1}((\partial^* E)_{\bar r}) \le \mathcal{H}^{n-1}\left(\mathbf{B}_{\alpha_v^\vee(\bar r)}(\bar r e_1)\setminus \mathbf{B}_{\alpha_v^\wedge(\bar r)}(\bar r\omega)\right)\\ &= v^\vee(\bar r) - v^\wedge(\bar r) = P(F_v;\partial B(\bar r)) \le P(E;\partial B(\bar r)),\end{aligned}$$

where we also used (1.2.4) with $B=\{\bar r\}$. Then, (3.5.11) follows from the last chain of inequalities. $\square$

We conclude this section showing that, if $D^c\alpha_v \neq 0$, rigidity fails.

**Proposition 3.5.4.** *Let $v:(0,\infty)\to[0,\infty)$ be a measurable function satisfying (1.2.2) such that $F_v$ is a set of finite perimeter and finite volume, and let $\alpha_v$ be defined by (1.2.3). Suppose that $D^c\alpha_v\neq 0$. Then, rigidity fails.*

*Proof.* We are going to construct a spherically $v$-distributed set $E\in\mathcal{N}(v)$ that cannot be obtained by applying a single rotation to $F_v$ (see (3.5.20) below).

First of all, let us note that it is not restrictive to assume that $\alpha_v$ is purely Cantorian. Indeed, by (2.0.20) one can decompose $\alpha_v$ into

$$\alpha_v = \alpha_v^a + \alpha_v^j + \alpha_v^c, \tag{3.5.18}$$

where $\alpha_v^a \in W^{1,1}_{\mathrm{loc}}(0,\infty)$, $\alpha_v^j$ is a purely jump function, and $\alpha_v^c$ is purely Cantorian. Thanks to (3.5.18), in the general case when $\alpha_v\neq\alpha_v^c$, the proof can be repeated by applying our argument just to the Cantorian part $\alpha_v^c$ of $\alpha_v$. Therefore, from now on we will assume that

$$D\alpha_v = D^c\alpha_v.$$

Thanks to Proposition 3.5.1, we can also assume that $\{0<\alpha^\wedge\le\alpha^\vee<\pi\}$ is an interval (otherwise there is nothing to prove, since rigidity fails). Moreover, since $\alpha_v$ is continuous, there exist $a,b>0$, with $a<b$, such that $I:=(a,b)\subset\subset\{0<\alpha^\wedge\le\alpha^\vee<\pi\}$ and

$$0<\alpha_v(r)<\pi \qquad \text{for every } r\in I. \tag{3.5.19}$$

Since $D^c\alpha_v\neq 0$, it is not restrictive to assume $|D^c\alpha_v|(I)>0$. For each $\gamma\in(-\pi,\pi)$, we define $R_\gamma \in SO(n)$ in the following way:

$$R_\gamma \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \cos\gamma - x_2 \sin\gamma \\ x_1 \sin\gamma + x_2 \cos\gamma \\ x_3 \\ \vdots \\ x_n \end{pmatrix}.$$

That is, $R_\gamma$ is a counterclockwise rotation of the angle $\gamma$ in the plane $(x_1, x_2)$. Let now fix $\lambda \in (0,1)$, and define $\beta : (0,\infty) \to (-\pi, \pi)$ as

$$\beta(r) := \begin{cases} 0 & \text{if } r \in (0,a), \\ \lambda(\alpha_v(r) - \alpha_v(a)) & \text{if } r \in [a,b], \\ \lambda(\alpha_v(b) - \alpha_v(a)) & \text{if } r \in (b,\infty). \end{cases}$$

We set

$$E := \{x \in \mathbb{R}^n : \mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x}, R_{\beta(|x|)} e_1) < \alpha_v^\vee(|x|)\}. \tag{3.5.20}$$

Clearly, $E$ cannot be obtained by applying a single rotation to $F_v$. Let us show that $E \in \mathcal{N}(v)$, so that rigidity fails. We proceed by steps.

**Step 1:** We construct a sequence of functions $v^k : I \to [0,\infty)$ satisfying the following properties:

(a) $\lim_{k\to\infty} \alpha_{v^k}(r) = \alpha_v(r)$ for $\mathcal{H}^1$-a.e. $r \in I$;

(b) $D\xi_{v^k} = D^j \xi_{v^k}$ for every $k \in \mathbb{N}$;

(c) $\lim_{k\to\infty} P(F_{v^k}; \Phi(I \times \mathbb{S}^{n-1})) = P(F_v; \Phi(I \times \mathbb{S}^{n-1}))$.

First of all note that, by (3.1.5) and by the chain rule in BV (see, [2, Theorem 3.96]), it follows that $\xi_v$ is purely Cantorian, where $\xi_v$ is given by (3.1.3). Moreover, from (2.0.21) and from the fact that $\xi_v$ is continuous, we have

$$|D\xi_v|(I) = \sup\left\{ \sum_{i=1}^{N-1} |\xi_v(r_{i+1}) - \xi_v(r_i)| : a < r_1 < r_2 < \ldots < r_N < b \right\},$$

where the supremum runs over $N \in \mathbb{N}$ and over all $r_1, \ldots, r_N$ with $a < r_1 < r_2 < \ldots < r_N < b$. Therefore, for every $k \in \mathbb{N}$ there exist $N_k \in \mathbb{N}$ and $r_1^k, \ldots, r_N^k$ with $a < r_1^k < r_2^k < \ldots < r_N^k < b$ such that

$$|D\xi_v|(I) \le \sum_{i=1}^{N_k - 1} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)| + \frac{1}{k}$$

and

$$|r_{i+1}^k - r_i^k| < \frac{1}{k} \qquad \text{for every } i = 1, \dots, N_k - 1.$$

Without any loss of generality, we can assume that the partitions are increasing in $k$. That is, we will assume that

$$\{r_1^k, \dots, r_{N_k}^k\} \subset \{r_1^{k+1}, \dots, r_{N_{k+1}}^{k+1}\} \quad \text{for every } k \in \mathbb{N}.$$

Define now, for every $k \in \mathbb{N}$,

$$\xi_v^k(r) := \sum_{i=0}^{N_k} \xi_v(r_i^k)\chi_{[r_i^k, r_{i+1}^k)}(r), \tag{3.5.21}$$

where we set $r_0^k := a$ and $r_{N_k+1}^k := b$. Let us now set

$$v^k(r) := \xi_v^k(r)/r^{n-1} \qquad \text{for every } r \in I \text{ and for every } k \in \mathbb{N},$$

and note that, by definition, $\xi_v^k = \xi_{v^k}$. Since $\xi_v$ is continuous, we have that

$$\lim_{k\to\infty} \xi_v^k(r) = \xi_v(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I. \tag{3.5.22}$$

Recalling (3.1.5) and (3.1.6), last relation implies property (a). Moreover, from (3.5.21) we have (b).

Let us now show (c). Thanks to (3.5.19) and (3.5.22), we have

$$\lim_{k\to\infty} p_{F_v^k}(r) = p_{F_v}(r) \qquad \text{for } \mathcal{H}^1\text{-a.e. } r \in I. \tag{3.5.23}$$

Moreover,

$$\begin{aligned} |D\xi_v^k|(I) &= \sum_{i=0}^{N_k} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)| \qquad (3.5.24) \\ &= |\xi_v(r_1^k) - \xi_v(a)| + |\xi_v(b) - \xi_v(r_{N_k}^k)| + \sum_{i=1}^{N_k-1} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)|. \end{aligned}$$

Since

$$|D\xi_v|(I) - \frac{1}{k} \le \sum_{i=1}^{N_k-1} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)| \le |D\xi_v|(I),$$

using (3.5.24) and the fact that $\xi_v$ is continuous we obtain

$$|D\xi_v|(I) = \lim_{k\to\infty} \sum_{i=1}^{N_k-1} |\xi_v(r_{i+1}^k) - \xi_v(r_i^k)| = \lim_{k\to\infty} |D\xi_v^k|(I). \tag{3.5.25}$$

Thanks to [2, Theorem 3.23], up to subsequences $\xi_v^k$ weakly* converges in $BV(I)$ to $\xi_v$. Since, in addition, (3.5.25) holds true, we can apply [2, Proposition 1.80] to the sequence of measures $\{|D\xi_v^k|\}_{k\in\mathbb{N}}$. Therefore, recalling that $D\xi_v^k = D^s\xi_v^k$ and $D\xi_v = D^s\xi_v$, we have

$$\lim_{k\to\infty} \int_I r^n d|D^s\xi_v^k|(r) = \lim_{k\to\infty} \int_I r^n d|D\xi_v^k|(r) = \int_I r^n d|D\xi_v|(r) = \int_I r^n d|D^s\xi_v|(r).$$

Then, from Corollary 3.3.2

$$\lim_{k\to\infty} P(F_{v^k};\Phi(I\times\mathbb{S}^{n-1})) = \lim_{k\to\infty}\left(\int_I p_{F_{v^k}}(r)\,dr + \int_I r^{n-1} d|D^s\xi_v^k|(r)\right)$$
$$= \left(\int_I p_{F_v}(r)\,dr + \int_I r^{n-1} d|D^s\xi_v|(r)\right) = P(F_v;\Phi(I\times\mathbb{S}^{n-1})),$$

where we also used (3.5.23).

**Step 2:** For each $k\in\mathbb{N}$, we construct a spherically $v^k$-distributed set $E^k$ such that

$$P(E^k;\Phi(I\times\mathbb{S}^{n-1})) = P(F_{v^k};\Phi(I\times\mathbb{S}^{n-1})).$$

From (3.1.5) and (3.1.6) it follows that $\alpha_{v^k} = \mathcal{F}^{-1}(\xi_v^k)\in BV(I)$, and

$$\alpha_{v^k}(r) = \sum_{i=0}^{N_k} \alpha_v(r_i^k)\chi_{[r_i^k, r_{i+1}^k)}(r). \tag{3.5.26}$$

Therefore, for each $k\in\mathbb{N}$ we have that $D\alpha_{v^k} = D^j\alpha_{v^k}$, and the jump set of $\alpha_{v^k}$ is a finite set. More precisely,

$$D\alpha_{v^k} = \sum_{i=1}^{N_k} (\alpha_v(r_i^k) - \alpha_v(r_{i-1}^k))\delta_{r_i^k},$$

where $\delta_r$ denotes the Dirac delta measure concentrated at $r$. Let $\lambda\in(0,1)$ be fixed, and define the set $E_1^k\subset\Phi(I\times\mathbb{S}^{n-1})$ as

$$E_1^k := \left[F_{v^k}\cap(B(r_1^k)\setminus\overline{B(a)})\right]\cup\left[R_{\lambda(\alpha_v(r_1^k)-\alpha_v(a))}(F_{v^k}\cap(B(b)\setminus B(r_1^k)))\right].$$

Thanks to Proposition 3.5.3, we have that

$$P(E_1^k;\Phi(I\times\mathbb{S}^{n-1})) = P(F_{v^k};\Phi(I\times\mathbb{S}^{n-1})).$$

Define now $E_2^k\subset\Phi(I\times\mathbb{S}^{n-1})$ as

$$E_2^k := (E_1^k\cap B(r_2^k))\cup\left[R_{\lambda(\alpha_v(r_2^k)-\alpha_v(r_1^k))}(E_1^k\setminus B(r_2^k))\right].$$

Applying again Proposition 3.5.3, we have

$$P(E_2^k;\Phi(I\times\mathbb{S}^{n-1})) = P(E_1^k;\Phi(I\times\mathbb{S}^{n-1})) = P(F_{v^k};\Phi(I\times\mathbb{S}^{n-1})).$$

Note that, since $R_\gamma$ is associative with respect to $\gamma$ (that is, we have $R_{\gamma_1}R_{\gamma_2} = R_{\gamma_1+\gamma_1}$), we can write $E_2^k$ as

$$E_2^k = \left[F_{v^k}\cap(B(r_1^k)\setminus\overline{B(a)})\right]\cup\left[R_{\lambda(\alpha_v(r_1^k)-\alpha_v(a))}(F_{v^k}\cap(B(r_2^k)\setminus B(r_1^k)))\right]$$
$$\cup\left[R_{\lambda(\alpha_v(r_2^k)-\alpha_v(a))}(F_{v^k}\cap(B(b)\setminus B(r_2^k)))\right].$$

Iterating this procedure $N_k$ times, we obtain that

$$P(E^k;\Phi(I\times\mathbb{S}^{n-1}))=P(F_{v^k};\Phi(I\times\mathbb{S}^{n-1})),$$

where

$$E_k:=E^k_{N_k}=\{x\in\Phi(I\times\mathbb{S}^{n-1}):\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},R_{\lambda(\alpha_{v^k}(|x|)-\alpha_{v^k}(a))}e_1)<\alpha_{v^k}(|x|)\}. \tag{3.5.27}$$

**Step 3:** We show that $E^k\longrightarrow\widehat{E}$ in $\Phi(I\times\mathbb{S}^{n-1})$, for some spherically $v$-distributed set $\widehat{E}$ such that

$$P(\widehat{E};\Phi(I\times\mathbb{S}^{n-1}))=P(F_v;\Phi(I\times\mathbb{S}^{n-1})).$$

From (3.5.26) and (3.5.22) it follows that

$$\lim_{k\to\infty}\alpha_{v^k}(r)=\alpha_v(r)\qquad\text{for }\mathcal{H}^1\text{-a.e. }r\in I.$$

Therefore, from (3.5.27) we have $E^k\longrightarrow\widehat{E}$ ( in $(\Phi(I\times\mathbb{S}^{n-1}))$), where $\widehat{E}$ is the spherically $v$-distributed set in $\Phi(I\times\mathbb{S}^{n-1})$ given by

$$\widehat{E}:=\{x\in\Phi(I\times\mathbb{S}^{n-1}):\mathrm{dist}_{\mathbb{S}^{n-1}}(\hat{x},R_{\lambda(\alpha_v(|x|)-\alpha_v(a))}e_1)<\alpha_v(|x|)\}. \tag{3.5.28}$$

Then, by the lower semicontinuity of the perimeter with respect to the $L^1$ convergence (see, for instance, [32, Proposition 12.15]):

$$\begin{aligned}&P(\widehat{E};\Phi(I\times\mathbb{S}^{n-1}))\le\lim_{k\to\infty}P(E^k;\Phi(I\times\mathbb{S}^{n-1}))\\&\lim_{k\to\infty}P(F_{v^k};\Phi(I\times\mathbb{S}^{n-1}))=P(F_v;\Phi(I\times\mathbb{S}^{n-1}))\\&\le P(\widehat{E};\Phi(I\times\mathbb{S}^{n-1})),\end{aligned}$$

where we also used (1.2.4).

**Step 4:** We conclude. Let $E$ be given by (3.5.20). Then, $E$ is spherically $v$-distributed and satisfies

$$E=_{\mathcal{H}^n}(F_v\cap(B(a)))\cup\left[\widehat{E}\cap(B(b)\setminus B(a))\right]\cup\left[R_{\lambda(\alpha_v(b)-\alpha_v(a))}(F_v\setminus(B(b)))\right],$$

where $\widehat{E}$ is defined in (3.5.28). By repeating the arguments used in the proof of Proposition 3.5.1, and using the fact that $\Phi(I\times\mathbb{S}^{n-1})=B(b)\setminus\overline{B(a)}$, one can see that

$$\begin{aligned}P(E)&=P(E;B(a))+P(E;\partial B(a))+P(E;B(b)\setminus\overline{B(a)})\\&+P(E;\partial B(b))+P(E;\mathbb{R}^n\setminus\overline{B(b)})\\&=P(F_v;B(a))+P(E;\partial B(a))+P(\widehat{E};B(b)\setminus\overline{B(a)})\\&+P(E;\partial B(b))+P(F_v;\mathbb{R}^n\setminus\overline{B(b)})\\&=P(F_v;B(a))+P(E;\partial B(a))+P(F_v;B(b)\setminus\overline{B(a)})\\&+P(E;\partial B(b))+P(F_v;\mathbb{R}^n\setminus\overline{B(b)}),\end{aligned}$$

where we also used Step 3 and the invariance of the perimeter under rotations. Since $\alpha_v$ is continuous, an argument similar to the one used to prove (3.5.13) shows that

$$P(E;\partial B(a)) = P(E;\partial B(b)) = 0.$$

Therefore,

$$P(E) = P(F_v;B(a)) + P(F_v;B(b)\setminus\overline{B(a)}) + P(F_v;\mathbb{R}^n\setminus\overline{B(b)}) = P(F_v).$$

□

We can now give the proof of the implication (i) $\Longrightarrow$ (ii) of Theorem 1.2.2.

*Proof of Theorem 1.2.2: (i) $\Longrightarrow$ (ii).* To show the implication, it suffices to combine Proposition 3.5.1, Proposition 3.5.3, and Proposition 3.5.4. □

## 3.6 Circular symmetrisation

In this section we sketch the proofs Theorem 1.2.4 and Lemma 1.2.5. We will only give here the important details, since in most cases they follow the lines of the proofs of Section 3.1, Section 3.2, and Section 3.3.

We start with some notation which, together with that one already given in the Introduction, will be extensively used in this section. Let $(r,x') \in (0,\infty)\times\mathbb{R}^{n-2}$, $\beta\in[0,\pi]$, and let $p\in\mathbb{S}^1$. The circular arc of centre $(rp,x')$ and radius $\beta$ is the set

$$\mathcal{B}_\beta(rp,x') := \{x \in \partial B((0,x'),r)\cap\Pi_{x'} : \mathrm{dist}_{\mathbb{S}^1}(\hat{x}_{12},rp) < \beta\},$$

If $\ell : (0,\infty)\times\mathbb{R}^{n-2}\to[0,\infty)$ is a measurable function satisfying (1.2.10), we define $\alpha^\ell : (0,\infty)\times\mathbb{R}^{n-2}\to[0,\pi]$ and $\xi^\ell : (0,\infty)\times\mathbb{R}^{n-2}\to[0,2\pi]$ as

$$\alpha^\ell := \frac{1}{2r}\ell(r,x') \qquad \text{and} \qquad \xi^\ell(r,x') = \frac{1}{r}\ell(r,x') = 2\alpha^\ell(r,x').$$

Note that in this case the relation between $\alpha^\ell$ and $\xi^\ell$ is linear. If $\mu$ is an $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n\setminus\{x_{12}=0\}$, we will write $\mu=\mu_{12\perp}+\mu_{12\|}$, where $\mu_{12\perp}$ and $12\mu_{\|}$ are the $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^n\setminus\{x_{12}=0\}$ such that

$$\int_{\mathbb{R}^n\setminus\{x_{12}=0\}} \varphi\cdot d\mu_{12\perp} = \int_{\mathbb{R}^n\setminus\{x_{12}=0\}} \varphi_{12\perp}\cdot d\mu,$$

and

$$\int_{\mathbb{R}^n\setminus\{x_{12}=0\}} \varphi\cdot d\mu_{12\|} = \int_{\mathbb{R}^n\setminus\{x_{12}=0\}} \varphi_{12\|}\cdot d\mu,$$

for every $\varphi\in C_c(\mathbb{R}^n\setminus\{x_{12}=0\};\mathbb{R}^n)$. The next two results are the analogous of Proposition 3.1.1 and Vol'pert Theorem 3.1.7, respectively.

**Proposition 3.6.1.** *Let $E$ be a set of finite perimeter in $\mathbb{R}^n$ and let $g : \mathbb{R}^n \to [0,\infty]$ be a Borel function. Then,*

$$\int_{\partial^* E} g(x)|\nu^E_{12\|}(x)|d\mathcal{H}^{n-1}(x) = \int_{(0,\infty)\times\mathbb{R}^{n-2}} dr\, dx' \int_{(\partial^* E)_{(r,x')}} g(x)\, d\mathcal{H}^0(x).$$

*Proof.* In this case, the result follows applying [2, Remark 2.94] with $N = n-1$, $M = n$, $k = n-1$, and $f(x) = (|x_{12}|, x')$. $\square$

**Theorem 3.6.2.** *Let $\ell : (0,\infty)\times\mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying (1.2.10), and let $E \subset \mathbb{R}^n$ be an circularly $\ell$-distributed set of finite perimeter and finite volume. Then, there exists a Borel set $G^\ell_E \subset \{\alpha^\ell > 0\}$ with $\mathcal{H}^{n-1}(\{\alpha^\ell > 0\}\setminus G^\ell_E) = 0$, such that*

*(i) for every $(r,x') \in G^\ell_E$:*

*(ia) $E_{(r,x')}$ is a set of finite perimeter in $\partial B_r(0,x')\cap\Pi_{x'}$;*

*(ib) $\partial^*(E_{(r,x')}) = (\partial^* E)_{(r,x')}$;*

*(ii) for every $(r,x') \in G^\ell_E \cap \{0 < \alpha^\ell < \pi\}$:*

*(iia) $|\nu^E_{12\|}(r\omega,x')| > 0$;*

*(iib) $\nu^E_{12\|}(r\omega,x') = \nu^{E_{(r,x')}}(r\omega,x')|\nu^E_{12\|}(r\omega,x')|$,*

*for **every** $\omega\in\mathbb{S}^1$ such that $(r\omega,x') \in \partial^*(E_{(r,x')}) = (\partial^* E)_{(r,x')}$.*

*Proof.* The statement follows applying the results of [25, Section 2.5], where slicing of codimension higher than one for currents are defined. $\square$

**Remark 3.6.3.** *Note that, if $(r,x') \in G^\ell_E$, conditions (iia) and (iib) are satisfied for **every** $\omega \in \mathbb{S}^1$ such that $(r\omega,x') \in \partial^*(E_{(r,x')}) = (\partial^* E)_{(r,x')}$. This is due to the fact that the circular symmetrisation has codimension 1. Such property failed, in general, for the spherical symmetrisation (see Remark 3.1.9).*

**Remark 3.6.4.** *An argument similar to that one used in Remark 3.1.9 shows that*

$$\mathcal{H}^{n-1}(\partial^* E \cap \Phi_{12}(G^\ell_E\times\mathbb{S}^1)\cap\{\nu^E_{12\|} = 0\}) = 0.$$

*As a consequence, the measure $\lambda^\ell_E$ defined as:*

$$\lambda^\ell_E(B) := \int_{\partial^* E\cap\Phi_{12}(B\times\mathbb{S}^1)\cap\{\nu^E_{12\|}=0\}} \hat{x}_{12}\cdot\nu^E(x)\, d\mathcal{H}^1(x),$$

*for every Borel set* $B \subset (0,\infty) \times \mathbb{R}^{n-2}$*, is singular with respect to the Lebesgue measure in* $(0,\infty) \times \mathbb{R}^{n-2}$.

We are now ready to state the analogous of Lemma 3.2.1.

**Lemma 3.6.5.** *Let* $\ell : (0,\infty)\times\mathbb{R}^{n-2} \to [0,\infty)$ *be a measurable function satisfying* (1.2.10)*, and let* $E \subset \mathbb{R}^n$ *be an circularly* $\ell$*-distributed set of finite perimeter and finite volume. Then,* $\ell \in BV_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^{n-2})$*. Moreover,* $\xi^\ell \in BV_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^{n-2})$ *and*

$$\int_{(0,\infty)\times\mathbb{R}^{n-2}} \psi(r,x')\, r\, dD_r\xi^\ell(r,x') = \int_{\mathbb{R}^n\setminus\{x_{12}=0\}} \psi(|x_{12}|,x')\, \hat{x}_{12} \cdot dD_{12\perp}\chi_E(x),$$

*for every bounded Borel function* $\psi : (0,\infty) \times \mathbb{R}^{n-2} \to \mathbb{R}$*, where* $D_r\xi^\ell$ *denotes the* $r$*-component of the* $\mathbb{R}^{n-1}$*-valued Radon measure* $D\xi^\ell$*. As a consequence,*

$$|rD_r\xi^\ell|(B) \le |D_{12\perp}\chi_E|(\Phi_{12}(B \times \mathbb{S}^1)),$$

*for every Borel set* $B \subset (0,\infty) \times \mathbb{R}^{n-2}$*. In particular,* $rD_r\xi^\ell$ *is a bounded Radon measure on* $(0,\infty) \times \mathbb{R}^{n-2}$*. Finally,*

$$D_{x'}\ell(B) = \int_{\partial^* E\cap\Phi_{12}(B\times\mathbb{S}^1)} \nu^E_{x'}(x)\, d\mathcal{H}^{n-1}(x),$$

*for every Borel set* $B \subset (0,\infty) \times \mathbb{R}^{n-2}$.

**Remark 3.6.6.** *Unlike what happened when we were considering the spherical symmetrisation, now the function* $\ell$ *might fail to be in* $BV((0,\infty) \times \mathbb{R}^{n-2})$*. Indeed, in Step 1 of the proof of Lemma 3.2.1 we used the fact that for* $r$ *bounded we are in a bounded set. This is not true in the context of circular symmetrisation.*

Next lemma, which is related to Lemma 3.2.2, will show the advantage of considering a symmetrisation of codimension 1.

**Lemma 3.6.7.** *Let* $\ell : (0,\infty)\times\mathbb{R}^{n-2} \to [0,\infty)$ *be a measurable function satisfying* (1.2.10)*, and let* $E \subset \mathbb{R}^n$ *be an circularly* $\ell$*-distributed set of finite perimeter and finite volume. Then*

$$\begin{aligned}(r\, dD_r\xi^\ell)(B) = &\int_{\partial^* E\cap\Phi_{12}(B\times\mathbb{S}^1)\cap\{\nu^E_{12\|}=0\}} \hat{x}_{12} \cdot \nu^E(x)\, d\mathcal{H}^{n-1}(x)\\ &+ \int_B dr\, dx' \int_{(\partial^* E)_{(r,x')}\cap\{\nu^E_{12\|}\neq 0\}} \frac{\hat{x}_{12} \cdot \nu^E(x)}{|\nu^E_{12\|}(x)|} d\mathcal{H}^0(x).\end{aligned}$$

*for every Borel set* $B \subset (0,\infty) \times \mathbb{R}^{n-2}$*. Moreover,*

$$r(\xi^\ell)'(r,x') = \int_{(\partial^* E)_{(r,x')}\cap\{\nu^E_{12\|}\neq 0\}} \frac{\hat{x}_{12} \cdot \nu^E(x)}{|\nu^E_{12\|}(x)|} d\mathcal{H}^0(x),$$

*for $\mathcal{H}^{n-1}$-a.e. $(r,x') \in (0,\infty)\times\mathbb{R}^{n-2}$, where $(\xi^\ell)'$ denotes the approximate differential of $\xi^\ell$ with respect to $r$. Similarly,*

$$D_{x'}\ell(B) = \int_{\partial^* E\cap\Phi_{12}(B\times\mathbb{S}^1)\cap\{\nu^E_{12\|}=0\}} \nu^E_{x'}(x)\,d\mathcal{H}^{n-1}(x) + \int_B dr\,dx' \int_{(\partial^* E)_{(r,x')}\cap\{\nu^E_{12\|}\neq 0\}} \frac{\nu^E_{x'}(x)}{|\nu^E_{12\|}(x)|} d\mathcal{H}^0(x).$$

*for every Borel set $B\subset(0,\infty)\times\mathbb{R}^{n-2}$, and*

$$\nabla_{x'}\ell(r,x') = \int_{(\partial^* E)_{(r,x')}\cap\{\nu^E_{12\|}\neq 0\}} \frac{\nu^E_{x'}(x)}{|\nu^E_{12\|}(x)|} d\mathcal{H}^0(x),$$

*for $\mathcal{H}^{n-1}$-a.e. $(r,x')\in(0,\infty)\times\mathbb{R}^{n-2}$, where $\nabla_{x'}\ell$ denotes the approximate gradient of $\ell$ with respect to $x'$.*

The following two results should be compared to Proposition 3.2.3 and Proposition 3.3.1, respectively.

**Proposition 3.6.8.** *Let $\ell : (0,\infty)\times\mathbb{R}^{n-2}\to[0,\infty)$ be a measurable function satisfying (1.2.10), and suppose that there exists an circularly $\ell$-distributed set $E\subset\mathbb{R}^n$ be of finite perimeter and finite volume. Then, $F^\ell$ is a set of finite perimeter in $\mathbb{R}^n$. Moreover, for every Borel set $B\subset(0,+\infty)\times\mathbb{R}^{n-2}$*

$$P(F^\ell;\Phi_{12}(B\times\mathbb{S}^1)) \le |D_{x'}\ell|(B) + r|D_r\xi^\ell|(B) + \left|D_{12\|}\chi_{F_v}\right|(\Phi_{12}(B\times\mathbb{S}^1)).$$

**Proposition 3.6.9.** *Let $\ell : (0,\infty)\times\mathbb{R}^{n-2}\to[0,\infty)$ be a measurable function satisfying (1.2.10) such that $F^\ell$ is a set of finite perimeter and finite volume, let $E\subset\mathbb{R}^n$ be an circularly $\ell$-distributed set of finite perimeter, and let $f : (0,\infty)\times\mathbb{R}^{n-2}\to[0,\infty]$ be a Borel function. Then,*

$$\begin{aligned}&\int_{\partial^* E} f(|x_{12}|,x')\,d\mathcal{H}^{n-1}(x)\\ &\ge \int_{(0,\infty)\times\mathbb{R}^{n-2}} f(r,x')\sqrt{p_E^2(r,x') + (rD_r\xi^\ell(r,x'))^2 + |\nabla_{x'}\ell(r,x')|^2}\,dr\,dx'\\ &+\int_{(0,\infty)\times\mathbb{R}^{n-2}} f(r,x')\,r\,d|D_r^s\xi^\ell|(r,x') + \int_{(0,\infty)\times\mathbb{R}^{n-2}} f(r,x')d|D^s_{x'}\ell|(r,x').\end{aligned}$$

*Moreover, in the special case $E = F^\ell$, equality holds true.*

A straightforward consequence of the previous result is the following formula for the perimeter of $F^\ell$.

**Corollary 3.6.10.** *Let $\ell : (0,\infty)\times\mathbb{R}^{n-2}\to[0,\infty)$ be a measurable function satisfying (1.2.10) such that $F^\ell$ is a set of finite perimeter and finite volume. Then*

$$\begin{aligned}&P(F^\ell;\Phi_{12}(B\times\mathbb{S}^1))\\ &= \int_B \sqrt{p_E^2(r,x') + (rD_r\xi^\ell(r,x'))^2 + |\nabla_{x'}\ell(r,x')|^2}\,dr\,dx' + |rD_r^s\xi^\ell|(B) + |D^s_{x'}\ell|(B).\end{aligned}$$

Next lemma relies on the fact that the circular symmetrisation has codimension 1. The proof can be obtained by repeating the argument used in the proof of [14, Lemma 4.1].

**Lemma 3.6.11.** *Let $\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying (1.2.10), let $E \subset \mathbb{R}^n$ be an circularly $\ell$-distributed set of finite perimeter and finite volume, and let $A \subset (0,+\infty) \times \mathbb{R}^{n-2}$ be a Borel set. Then,*

$$\mathcal{H}^{n-1}\Big(\{x \in \partial^* E : \nu^E_{12\|}(x) = 0\} \cap \Phi_{12}(A \times \mathbb{S}^1)\Big) = 0.$$

*if and only if*

$$P(E; \Phi_{12}(B \times \mathbb{S}^1)) = 0 \qquad \text{for every Borel set } B \subset A \text{ with } \mathcal{H}^{n-1}(B) = 0.$$

The previous propositions can be proved with the same arguments used to show Proposition 3.3.3.

**Proposition 3.6.12.** *Let $\ell : (0,\infty) \times \mathbb{R}^{n-2} \to [0,\infty)$ be a measurable function satisfying (1.2.10) such that $F^\ell$ is a set of finite perimeter and finite volume, and let $\Omega \subset (0,+\infty) \times \mathbb{R}^{n-2}$ be an open set. Then the following three statements are equivalent:*

*(i)* $\mathcal{H}^{n-1}\left(\left\{x \in \partial^* F^\ell \cap \Phi_{12}(\Omega \times \mathbb{S}^1) : \nu^{\partial^* F^\ell}_{12\|}(x) = 0\right\}\right) = 0;$

*(ii)* $\xi^\ell \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *and* $\ell \in W^{1,1}_{\mathrm{loc}}(\Omega)$*;*

*(iii)* $P(F^\ell; \Phi_{12}(B \times \mathbb{S}^1)) = 0$ *for every Borel set $B \subset \Omega$, such that $\mathcal{H}^{n-1}(B) = 0$.*

Once all the results above are established, Lemma 1.2.5 can be shown by using the same arguments as in the proof of [14, Proposition 4.2].

# Chapter 4

# Rigidity of equality cases for the Steiner anisotropic perimeter inequality

In this chapter, we will present in detail all the results obtained about rigidity for the Steiner's inequality for the anisotropic perimeter.

## 4.1 Setting of the problems and preliminary results

We recall in here, few results that will be useful later on for the proof of (AS) (for more details see [14, Section 2 and 3]). Let us start with a version of a result by Vol'pert (see [14, Theorem G]).

**Theorem 4.1.1.** *Let $v \in BV(\mathbb{R}^{n-1})$ such that $\mathcal{H}^{n-1}(\{v > 0\}) < \infty$. Let $E \subset \mathbb{R}^n$ be a $v$-distributed set of finite perimeter. Then, we have for $\mathcal{L}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$,*

$$E_z \text{ has finite perimeter in } \mathbb{R}; \tag{4.1.1}$$

$$(\partial^e E)_z = (\partial^* E)_z = \partial^*(E_z) = \partial^e(E_z); \tag{4.1.2}$$

$$\boldsymbol{q}(\nu^E(z,t)) \neq 0 \text{ for every } t \text{ such that } (z,t) \in \partial^* E; \tag{4.1.3}$$

*In particular, there exists a Borel set $G_E \subseteq \{v > 0\}$ such that $\mathcal{L}^{n-1}(\{v > 0\} \setminus G_E) = 0$ and (4.1.1)-(4.1.3) are satisfied for every $z \in G_E$.*

The next result is a version of the *Coarea formula* for rectifiable sets (see [14, Theorem F]).

**Theorem 4.1.2.** *Let $E$ be a set of finite perimeter in $\mathbb{R}^n$ and let $g : \mathbb{R}^n \to [0, +\infty]$ be any Borel function. Then,*

$$\int_{\partial^* E} g(x)|\boldsymbol{q}(\nu^E(x))|d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} dz \int_{(\partial^* E)_z} g(z,y)d\mathcal{H}^0(y). \tag{4.1.4}$$

Lastly, next result is a version of [14, Lemma 3.2].

**Lemma 4.1.3.** *Let $v \in BV(\mathbb{R}^{n-1})$ such that $\mathcal{H}^{n-1}(\{v > 0\}) < \infty$. Let $E \subset \mathbb{R}^n$ be a $v$-distributed set of finite perimeter. Then, for $\mathcal{L}^{n-1}$-a.e. $z \in \{v > 0\}$*

$$\frac{\partial v}{\partial x_i}(z) = -\int_{(\partial^* E)_z} \frac{\nu_i^E(z,y)}{|\boldsymbol{q}(\nu^E(z,y))|}\, d\mathcal{H}^0(y), \qquad i = 1, \dots, n-1,$$

*In particular by (4.1.2) and the above relation, we get for $\mathcal{L}^{n-1}$-a.e. $z \in \{v > 0\}$*

$$\frac{1}{2}\frac{\partial v}{\partial x_i}(z) = -\frac{\nu_i^{F[v]}(z,y)}{|\boldsymbol{q}(\nu^{F[x]}(z,y))|}\, d\mathcal{H}^0(y), \qquad i = 1, \dots, n-1,\ y \in (\partial^* F[z])_z.$$

### 4.1.1 Properties of the surface tension $\phi_K$

Let us start recalling some basic facts about the surface tension $\phi_K$. First of all, let us sum up some known properties of the gauge function in the following result, that can be easily deduced from [32, Proposition 20.10].

**Proposition 4.1.4.** *Consider $K \subset \mathbb{R}^n$ as in (1.3.3). Consider $\phi_K, \phi_K^* : \mathbb{R}^n \to [0, \infty)$ the corresponding surface tension and gauge function defined in (1.3.4), (1.3.10) respectively. Then the following properties hold true.*

*i) The function $\phi_K^*$ is one-homogeneous, convex and coercive on $\mathbb{R}^n$ and there exist positive constants $c$ and $C$ such that*

$$c|x| \le \phi_K(x) \le C|x|, \quad \forall x \in \mathbb{R}^n,$$
$$\frac{|x|}{C} \le \phi_K^*(x) \le \frac{|x|}{c}, \quad \forall x \in \mathbb{R}^n.$$

*ii) The so called* Fenchel inequality *holds true i.e.*

$$x \cdot y \le \phi_K^*(x)\phi_K(y), \quad \forall x, y \in \mathbb{R}^n. \tag{4.1.5}$$

*iii) The gauge function $\phi_K^*$ provides a new characterization for the Wulff shape $K$ i.e.*

$$K = \{x \in \mathbb{R}^n : \phi_K^*(x) < 1\},$$

*from which we can immediately derive that*

$$\phi_K(x) = \sup\{x \cdot y : \phi_K^*(x) < 1\},$$
$$\phi_K(x) = (\phi_K^*)^*(x).$$

*iv) If $x \in \partial^* K$ and $y \in \mathbb{S}^{n-1}$, then equality holds in (4.1.5) if and only if $y = \nu^K(x)$; in particular*

$$P_K(K) = n|K|. \tag{4.1.6}$$

**Remark 4.1.5.** *By (i) of Proposition 4.1.4 we have that $E$ is a set of locally finite perimeter if and only if $E$ is a set of locally finite anisotropic perimeter i.e. $P_K(E; C) < \infty$ for every $C \subset \mathbb{R}^n$ compact set.*

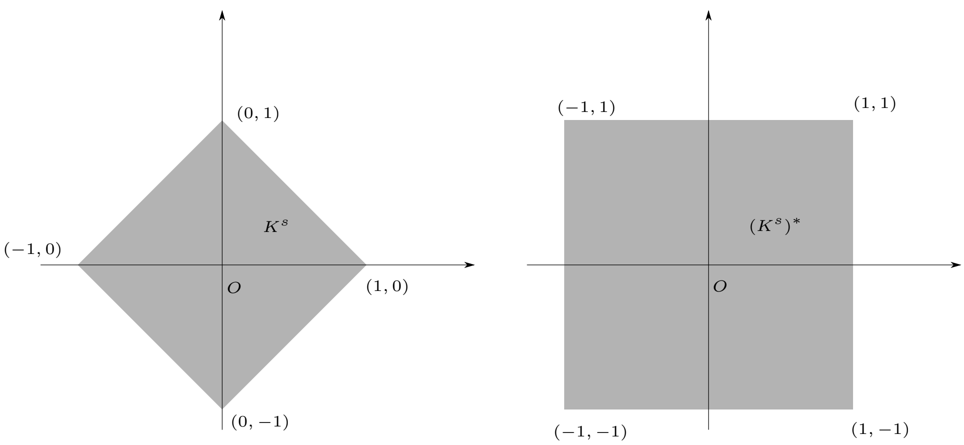


Figure 4.1.1: A two dimensional example of $K^s$ and its dual $(K^s)^*$.

**Remark 4.1.6.** *Thanks to iii) of the above proposition we have*

$$K^* = \{x \in \mathbb{R}^n : \phi_K(x) < 1\},$$

*from which together with (1.3.10) gives*

$$\phi_K^*(x) = \sup\{x \cdot y : y \in K^*\} \quad \forall\, x \in \mathbb{R}^n.$$

*For a pictorial idea of $K$ and $K^*$ see for instance Figure 4.1.1. Furthermore, observe that*

$$\phi_K(x) = 1 \quad \forall\, x \in \partial K^*, \tag{4.1.7}$$
$$\phi_K^*(x) = 1 \quad \forall\, x \in \partial K. \tag{4.1.8}$$

**Remark 4.1.7.** *Let us consider $K \subset \mathbb{R}^n$ as in (1.3.3). According to Proposition 4.1.4, iii) another way to define the Wulff shape $K$ is*

$$K := \boldsymbol{p}\left(\Sigma_{\phi_K^*} \cap \{x_{n+1} = 1\}\right),$$

*where $\Sigma_{\phi_K^*}$ is the epigraph of $\phi_K^*$ in $\mathbb{R}^{n+1}$ and $\boldsymbol{p} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ corresponds to the horizontal projection. By the one-homogeneity of $\phi_K$ we get that*

$$\phi_K(tx) = t|x|\phi_K\left(\frac{tx}{t|x|}\right) = t\phi_K(x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \forall t > 0. \tag{4.1.9}$$

*By (4.1.9), we get for every constant $\lambda > 0$ that*

$$\lambda K := \boldsymbol{p}\left(\Sigma_{\phi_K^*} \cap \{x_{n+1} = \lambda\}\right).$$

*Another thing we would like to observe is that given $x, y \in \mathbb{R}^n$ with $x \in \lambda K$ and $y \in (\lambda K)^c$, (for some $\lambda > 0$) then $\phi_K^*(x) < \phi_K^*(y)$. Naturally, these considerations hold true for $K^*$ and $\phi_K$ too.*

**Definition 4.1.8** (Sub-differential)**.** *Let $\varphi : \mathbb{R}^n \to [0, \infty]$ be a convex function. Let us fix $x_0 \in \mathbb{R}^n$ and consider all vectors $y_0 \in \mathbb{R}^n$ such that*

$$\varphi(z) \geq \phi(x_0) + y_0 \cdot (z - x_0) \quad \forall z \in \mathbb{R}^n. \tag{4.1.10}$$

*The set of all vectors $y_0$ satisfying the above property is called* sub-differential *of $\varphi$ at $x_0$ and we indicate it by $\partial\varphi(x_0)$.*

Keeping in mind Definition 1.3.11 we have the following Remarks.

**Remark 4.1.9.** *For every $x_0 \in \mathbb{R}^n$, the sub-differential $\partial\phi(x_0)$ is a closed and convex set of $\mathbb{R}^n$ (see [37] chapter 5). From this, it can be proved that, given $x \in \partial K$, also $C_K^*(x)$ is a convex set of $\mathbb{R}^n$, where $C_K^*(x)$ is defined as in (1.3.11).*

**Remark 4.1.10.** *Let $\phi : \mathbb{R}^n \to [0, \infty]$ be a convex function. It is a well known result about convex functions that, $\phi$ is differentiable in $x_0 \in \mathbb{R}^n$ if and only if $\partial\phi(x_0)$ consists of only one element. In that situation, we call $\nabla\phi(x_0)$ is the only element in the sub-differential $\partial\phi(x_0)$.*

**Definition 4.1.11.** *Fix an integer $m \geq 1$ and let $K \subset \mathbb{R}^n$ be as in (1.3.3). Given a $\mathbb{R}^n$-valued Radon measure $\mu$ on $\mathbb{R}^m$ and a generic Borel set $F \subset \mathbb{R}^m$, we define the $\phi_K$-anisotropic total variation of $\mu$ on $F$ as*

$$|\mu|_K(F) = \int_F \phi_K\left(\frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x).$$

**Remark 4.1.12.** *By condition $i)$ in Proposition 4.1.4 we have that*

$$|\mu|_K(F) = \int_F \phi_K\left(\frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) \le C\int_F d|\mu|(x) = C|\mu|(F).$$

*Analogously,*

$$|\mu|(F) = \int_F d|\mu|(x) \le \frac{1}{c}\int_F \phi_K\left(\frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) = \frac{1}{c}|\mu|_K(F).$$

*Thus, $|\mu|_K \ll |\mu|$ and $|\mu| \ll |\mu|_K$.*

**Remark 4.1.13.** *Given $f \in GBV(\mathbb{R}^{n-1})$, motivated by (2.0.25), for every Borel set $G \subset \mathbb{R}^{n-1}$ we define*

$$|(D^c f, 0)|_K(G) = \lim_{M\to+\infty} |(D^c(\tau_M(f), 0)|_K(G) = \sup_{M>0} |(D^c(\tau_M)(f), 0)|(G). \tag{4.1.11}$$

**Lemma 4.1.14.** *Fix an integer $m \ge 1$ and let $K \subset \mathbb{R}^n$ be as in (1.3.3). Let $(\mu_h)_{h\in\mathbb{N}}$ and $\mu$ be $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^m$. Let us assume that*

*i)* $\mu_h \overset{*}{\rightharpoonup} \mu$,

*ii)* $|\mu_h|(\mathbb{R}^m) \to |\mu|(\mathbb{R}^m)$ *and* $|\mu|(\mathbb{R}^m) < \infty$.

*Then,*

$$|\mu_h|_K \overset{*}{\rightharpoonup} |\mu|_K. \tag{4.1.12}$$

*Proof.* Let us first observe that, thanks to [32, Proposition 4.30] we immediately get

$$|\mu_h| \overset{*}{\rightharpoonup} |\mu|. \tag{4.1.13}$$

Moreover, given $f \in C_c^0(\mathbb{R}^m)$, if we consider the Radon measures defined as $f\mu_h$ and $f\mu$ $\forall\, h \in \mathbb{N}$, then

$$f\mu_h \overset{*}{\rightharpoonup} f\mu, \tag{4.1.14}$$

$$|f\mu_h|(\mathbb{R}^m) \to |f\mu|(\mathbb{R}^m), \quad |f\mu| < \infty. \tag{4.1.15}$$

Indeed, $\forall\, g \in C_c^0(\mathbb{R}^m;\mathbb{R}^n)$, noticing that $gf \in C_c^0(\mathbb{R}^m;\mathbb{R}^n)$ and having in mind assumption $i)$ we get

$$\lim_{h\to\infty}\int_{\mathbb{R}^m} g(x)f(x)\cdot d\mu_h(x) = \int_{\mathbb{R}^m} g(x)f(x)\cdot d\mu(x).$$

This proves (4.1.14). Whereas, thanks to Remark 2.0.1 and having in mind assumption $ii)$ we get

$$\lim_{h\to\infty}\int_{\mathbb{R}^m} |f(x)| d|\mu_h|(x) = \int_{\mathbb{R}^m} |f(x)| d|\mu|(x) < \|f\|_{L^\infty(\mathbb{R}^m)}|\mu|(\mathbb{R}^m) < \infty.$$

This proves (4.1.15). In order to prove relation (4.1.12), by definition we have to prove that

$$\lim_{h\to\infty}\int_{\mathbb{R}^m}\varphi(x)\phi_K\left(\frac{d\mu_h}{d|\mu_h|}(x)\right)d|\mu_h|(x)=\int_{\mathbb{R}^m}\varphi(x)\phi_K\left(\frac{d\mu}{d|\mu|}(x)\right)d|\mu|(x),\quad\forall\,\varphi\in C_c^0(\mathbb{R}^m).\tag{4.1.16}$$

Let us fix $\varphi\in C_c^0(\mathbb{R}^m)$ and let us write $\varphi=\varphi^+-\varphi^-$ with $\varphi^+,\varphi^-\geq 0$, so that

$$\begin{aligned}\int_{\mathbb{R}^m}\varphi(x)\phi_K\left(\frac{d\mu_h}{d|\mu_h|}(x)\right)d|\mu_h|(x)=&\int_{\mathbb{R}^m}\varphi^+(x)\phi_K\left(\frac{d\mu_h}{d|\mu_h|}(x)\right)d|\mu_h|(x)\\&-\int_{\mathbb{R}^m}\varphi^-(x)\phi_K\left(\frac{d\mu_h}{d|\mu_h|}(x)\right)d|\mu_h|(x)=I-II.\end{aligned}\tag{4.1.17}$$

Thanks to relations (4.1.14) and (4.1.15) with first $f=\varphi^+$ and then $f=\varphi^-$ and thanks to Reshetniak result [32, Proposition 20.12] we get that

$$\lim_{h\to\infty}I=\int_{\mathbb{R}^m}\varphi^+(x)\phi_K\left(\frac{d\mu}{d|\mu|}(x)\right)d|\mu|(x),\quad\lim_{h\to\infty}II=\int_{\mathbb{R}^m}\varphi^-(x)\phi_K\left(\frac{d\mu}{d|\mu|}(x)\right)d|\mu|(x).\tag{4.1.18}$$

Thus, thanks to (4.1.18), passing to the limit as $h\to\infty$ in both sides of relation (4.1.17) we prove (4.1.12). This concludes the proof. □

The following Lemma is the anisotropic version of [2, Definition 1.4 (b)].

**Lemma 4.1.15.** *Fix an integer $m\geq 1$ and let $K\subset\mathbb{R}^n$ be as in (1.3.3). Given a $\mathbb{R}^n$-valued Radon measure $\mu$ on $\mathbb{R}^m$ we have*

$$|\mu|_K(G)=\sup\left\{\sum_{h\in\mathbb{N}}\phi_K(\mu(G_h)):\ (G_h)_{h\in\mathbb{N}}\ \textit{pairwise disjoint},\ \bigcup_h G_h=G\right\},\quad\forall\,G\subset\mathbb{R}^m\ \textit{Borel},\tag{4.1.19}$$

*where $G_h$ are bounded Borel sets.*

*Proof.* Thanks to Jensen Inequality and 1-homogeneity of $\phi_K$ we get

$$\phi_K\left(\mu(G_h)\right)=\phi_K\left(\int_{G_h}\frac{d\mu}{d|\mu|}(x)d|\mu|(x)\right)\leq|\mu|_K(G_h),$$

so using that $G_h\cap G_k=\emptyset\ \forall h\neq k$

$$|\mu|_K(G)=|\mu|_K\left(\cup_h G_h\right)=\sum_{h\in\mathbb{N}}|\mu|_K(G_h)\geq\sum_{h\in\mathbb{N}}\phi_K(\mu(G_h)).$$

Taking the sup on the right hand side we proved that $|\mu|_K(G)$ is greater or equal than the right hand side of relation (4.1.19). We are then left to prove that

$$|\mu|_K(G)\leq\sup\left\{\sum_{h\in\mathbb{N}}\phi_K(\mu(G_h)):\ (G_h)_{h\in\mathbb{N}}\ \textit{pairwise disjoint},\ \bigcup_h G_h=G\right\},$$

Let $G \subset \mathbb{R}^n$ be a bounded Borel set. Let us consider the function

$$f(x) = \frac{d\mu}{d|\mu|}(x) \in L^\infty(\mathbb{R}^m, |\mu|; \mathbb{R}^n).$$

For each $i \in \{1, \ldots, n\}$ we also have

$$f_i(x) = \frac{d\mu_i}{d|\mu|}(x) \in L^1_{loc}(\mathbb{R}^m, |\mu|),$$

where $\mu = (\mu_1, \ldots, \mu_n)$. Consider $\forall\, i \in$ a sequence of step functions $\{f_{i,h}\}_{h\in\mathbb{N}}$ such that

$$\|f_{h,i} - f_i\|_{L^\infty(\mathbb{R}^m,|\mu|)} \to 0 \qquad as\ h \to \infty.$$

As a consequence, we have $\|f_h - f\|_{L^\infty(\mathbb{R}^m,|\mu|;\mathbb{R}^n)} \to 0$ as $h \to \infty$. Fix $\epsilon > 0$, then there exists $h(\epsilon) > 0$ such that

$$\|f_h - f\|_{L^\infty(\mathbb{R}^m,|\mu|;\mathbb{R}^n)} < \epsilon \qquad \forall\, h > h(\epsilon).$$

Since for each $i \in \{1, \ldots, n\}$ the function $f_{h,i}$ is simple, there exists $n(h) \in \mathbb{N}$ and a finite pairwise disjoint partition $\{G^h_k\}_{k=1,\ldots,n(h)}$ of $G$ such that $f_h$ is constant $|\mu|$-a.e. in $G^h_k$, $\forall\, k \in \{1, \ldots, n(h)\}$, namely $\exists\, a_{h,k} \in \mathbb{R}^n$ s.t. $f_h(x) = a_{h,k}$ for $|\mu|$-a.e. $x \in G^h_k$, $\forall\, k \in \{1, \ldots, n(h)\}$. Let $\epsilon > 0$ and let, then thanks to the one-homogeneity and subadditivity we get

$$\begin{aligned}
\int_G \phi_K\left(f_h(x)\right) d|\mu|(x) &= \sum_{k=1}^{n(h)} \int_{G^h_k} \phi_K\left(f_h(x)\right) d|\mu|(x) = \sum_{k=1}^{n(h)} \phi_K\left(a_{h,k}\right)|\mu|(G^h_k) \\
&= \sum_{k=1}^{n(h)} \phi_K\left(a_{h,k}|\mu|(G^h_k)\right) = \sum_{k=1}^{n(h)} \phi_K\left(\int_{G^h_k} f_h(x)\, d|\mu|(x)\right) \\
&= \sum_{k=1}^{n(h)} \phi_K\left(\int_{G^h_k} f(x)\, d|\mu|(x) + \int_{G^h_k} (f_h(x) - f(x))\, d|\mu|(x)\right) \\
&\leq \sum_{k=1}^{n(h)} \phi_K\left(\int_{G^h_k} f(x)\, d|\mu|(x)\right) + \sum_{k=1}^{n(h)} \phi_K\left(\int_{G^h_k} (f_h(x) - f(x))\, d|\mu|(x)\right) \\
&= \sum_{k=1}^{n(h)} \phi_K\left(\mu(G^h_k)\right) + \sum_{k=1}^{n(h)} \left|\int_{G^h_k} (f_h - f)\, d|\mu|\right| \phi_K\left(\frac{\int_{G^h_k} (f_h - f)\, d|\mu|}{\left|\int_{G^h_k} (f_h - f)\, d|\mu|\right|}\right) \\
&\leq \sum_{k=1}^{n(h)} \phi_K\left(\mu(G^h_k)\right) + C\sum_{k=1}^{n(h)} \int_{G^h_k} |f_h(x) - f(x)|\; d|\mu|(x) \\
&\leq \sum_{k=1}^{n(h)} \phi_K\left(\mu(G^h_k)\right) + \epsilon\, C\sum_{k=1}^{n(h)} |\mu|(G^h_k) \\
&= \sum_{k=1}^{n(h)} \phi_K\left(\mu(G^h_k)\right) + \epsilon\, C|\mu|(G) \qquad \forall\, h > h(\epsilon),
\end{aligned}$$

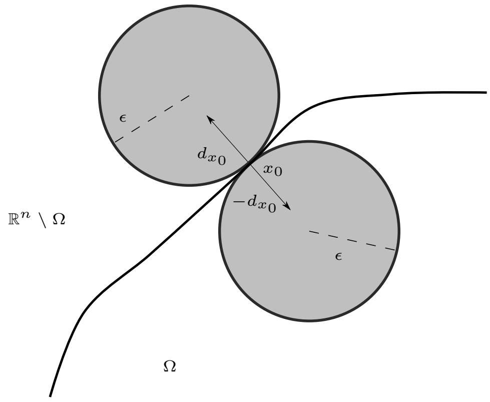


Figure 4.1.2: A pictorial idea of the $\epsilon$- ball property.

where $C := \sup_{\omega \in \mathbb{S}^{n-1}} \phi_K(\omega)$. So we proved that $\forall\, \epsilon > 0\ \exists\, h(\epsilon) > 0$, $n(h) \in \mathbb{N}$ and $\{G_k^h\}_{k=1,\dots,n(h)}$ s.t. $\forall\, h > h(\epsilon)$ the following holds

$$\begin{aligned}\int_G \phi_K\left(f_h(x)\right)\, d|\mu|(x) &\leq \sum_{k=1}^{n(h)} \phi_K\left(\mu(G_k^h)\right) + \epsilon\, C|\mu|(G)\\ &\leq \sup\left\{\sum_{h\in\mathbb{N}} \phi_K(\mu(G_h)) : (G_h)_{h\in\mathbb{N}} \textit{ pairwise disjoint, } \bigcup_h G_h = G\right\}\\ &+ \epsilon\, C|\mu|(G).\end{aligned}$$

Taking the limit as $h \to +\infty$ in the left hand side, by Lebesgue dominated theorem we get

$$\begin{aligned}|\mu|_K(G) \leq \sup&\left\{\sum_{h\in\mathbb{N}} \phi_K(\mu(G_h)) : (G_h)_{h\in\mathbb{N}} \textit{ pairwise disjoint, } \bigcup_h G_h = G\right\}\\ &+ \epsilon\, C|\mu|(G).\end{aligned}$$

By the arbitrariness of $\epsilon > 0$ we conclude for $G$ bounded. Thanks to standard considerations we can extend the result also for $G$ unbounded. $\square$

**Definition 4.1.16** (Hausdorff distance)**.** *Let $A, B \subset \mathbb{R}^n$. We define the* Hausdorff distance *between $A$ and $B$ as*

$$\mathrm{dist}_H(A, B) := \max\left\{\sup_{x\in A} d(x, B); \sup_{x\in B} d(x, A)\right\},$$

*where $d(\cdot, A)$ denotes the Euclidean distance from $A$.*

**Definition 4.1.17** ($\epsilon$-ball property)**.** *Let $\epsilon > 0$. We say that an open bounded set $\Omega \subset \mathbb{R}^n$ satisfies the $\epsilon$-ball property if for any point $x \in \partial\Omega\ \exists$ a unit vector $d_x \in \mathbb{S}^{n-1}$ s.t.*

$$B(x - \epsilon d_x, \epsilon) \subset \Omega,$$

$$B(x + \epsilon d_x, \epsilon) \subset \mathbb{R}^n \setminus \overline{\Omega}.$$

Roughly speaking, a set satisfies the $\epsilon$-ball property if it is possible to roll two tangent balls, one in the interior and the other one in the exterior part of $\Omega$ (see for instance figure 4.1.2).

**Definition 4.1.18.** *Let $\mathcal{S} \subset \mathbb{R}^n$ be non-empty. We say that $\mathcal{S}$ is a $C^{1,1}$ hypersurface if for every point $x \in \mathcal{S}$, there exists an open neighbourhood $D$ of $x$, an open set $\Omega$ of $\mathbb{R}^{n-1}$, and a continuously differentiable bijection $\varphi : E \to D \cap \mathcal{S}$ with $\varphi$ and its gradient $\nabla\varphi$ both Lipschitz continuous, and $\mathcal{J}\varphi > 0$ on $E$.*

Given $K \subset \mathbb{R}^n$ as in (1.3.3), we will now prove few more properties about the surface tension $\phi_K$. In particular, the main result we present is Proposition 4.1.22 that gives a characterization of the cases of additivity for the function $\phi_K$.

**Lemma 4.1.19.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3), and let $y_1, y_2 \in \mathbb{R}^n$. Then, the following are equivalent:*

*(i)* $\phi_K(y_1) + \phi_K(y_2) = \phi_K(y_1 + y_2)$;

*(ii)* $\exists\, \bar{z} \in \partial K$ *s.t.* $\phi_K(y_1) = y_1 \cdot \bar{z}$ *and* $\phi_K(y_2) = y_2 \cdot \bar{z}$.

*Proof.* Assume *(ii)* is satisfied. Then,

$$\phi_K(y_1 + y_2) = \max_{z \in \partial K}[(y_1 + y_2) \cdot z] \geq \bar{z} \cdot (y_1 + y_2) = \phi_K(y_1) + \phi_K(y_2),$$

which gives *(i)*. Let now *(i)* be satisfied and suppose, by contradiction, that

$$\nexists\, \bar{z} \quad \text{such that} \quad \phi_K(y_1) = y_1 \cdot \bar{z} \quad \text{and} \quad \phi_K(y_2) = y_2 \cdot \bar{z}. \tag{4.1.20}$$

Let $z_1, z_2, z_3 \in \partial K$ be such that $\phi_K(y_1) = y_1 \cdot z_1$ and $\phi_K(y_2) = y_2 \cdot z_2$, and

$$\phi_K(y_1 + y_2) = (y_1 + y_2) \cdot z_3.$$

Then,

$$y_1 \cdot z_3 \leq y_1 \cdot z_1 \qquad \text{and} \qquad y_2 \cdot z_3 \leq y_2 \cdot z_2.$$

Note that, in particular, from (4.1.20) we have that at least one of the above inequalities is strict. Thus,

$$\phi_K(y_1 + y_2) < \phi_K(y_1) + \phi_K(y_2),$$

which is impossible. $\square$

**Lemma 4.1.20.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3) and consider $\phi_K$ the associated surface tension. Let $y_0 \in \mathbb{R}^n$ and let $x_0 \in \partial K$. Then,*

$$\phi_K(y_0) = y_0 \cdot x_0 \quad \Longleftrightarrow \quad \frac{y_0}{\phi_K(y_0)} \in \partial\phi_K^*(x_0),$$

*where, we recall, $\partial\phi_K^*(x_0)$ is the sub differential of $\phi_K^*(x_0)$.*

*Proof.* We divide the proof into two steps, each for every implications.

**Step 1** Suppose

$$\frac{y_0}{\phi_K(y_0)} \in \partial\phi_K^*(x_0).$$

Then, since by (4.1.7) we have $\phi_K^*(x_0) = 1$, we deduce that for every $z \in \mathbb{R}^n$

$$\phi_K^*(z) \geq \phi_K^*(x_0) + \frac{y_0}{\phi_K(y_0)} \cdot (z - x_0) = 1 + \frac{y_0}{\phi_K(y_0)} \cdot (z - x_0).$$

In particular, if $z \in \partial K$ we have $\phi_K^*(z) = 1$, and therefore

$$1 \geq 1 + \frac{y_0}{\phi_K(y_0)} \cdot (z - x_0), \qquad \text{for every } z \in \partial K,$$

so that $y_0 \cdot x_0 \geq y_0 \cdot z$ for every $z \in \partial K$. Thus, $\phi_K(y_0) = y_0 \cdot x_0$.

**Step 2** Assume that $\phi_K(y_0) = y_0 \cdot x_0$. Then, by the Fenchel inequality, for every $z \in \mathbb{R}^n$ we have

$$\phi_K(y_0)\phi_K^*(z) \geq y_0 \cdot z \quad \Longleftrightarrow \quad \phi_K^*(z) \geq \frac{y_0 \cdot z}{y_0 \cdot x_0} \quad \Longleftrightarrow \quad \phi_K^*(z) \geq 1 + \frac{y_0 \cdot (z - x_0)}{y_0 \cdot x_0}.$$

Recalling that $\phi_K^*(x_0) = 1$, we conclude. ☐

**Remark 4.1.21.** *Let us observe that, given $y_0 \in \mathbb{R}^n$ and $x_0 \in \partial K$ then*

$$\phi_K(y_0) = y_0 \cdot x_0 \quad \Longleftrightarrow \quad y_0 \in C_K^*(x_0),$$

*where $C_K^*(x_0)$ has been defined in 1.3.11. Indeed, by the Lemma above and Definition 1.3.11, we immediately derive that if $\phi_K(y_0) = y_0 \cdot x_0$ then $y_0/\phi_K(y_0) \in \partial\phi_K^*(x_0)$ that implies $y_0 \in C_K^*(x_0)$. Whereas, if $y_0 \in C_K^*(x_0)$ then there exists $\lambda = \lambda(y_0) > 0$ such that $\lambda y_0 \in \partial\phi_K^*(x_0)$ i.e.*

$$\phi_K^*(z) \geq 1 + \lambda y_0 \cdot (z - x_0) \qquad \forall\, z \in \mathbb{R}^n.$$

*In particular, if we choose $z \in \partial K$ we get*

$$\lambda y_0 \cdot x_0 \geq \lambda y_0 \cdot z \qquad \forall\, z \in \partial K,$$

*that implies $\phi_K(y_0) = y_0 \cdot x_0$.*

As a direct consequence of Lemmas 4.1.19 and 4.1.20 we get the following proposition.

**Proposition 4.1.22.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3), and let $y_1, y_2 \in \mathbb{R}^n$. Then, the following are equivalent:*

*(i)* $\phi_K(y_1) + \phi_K(y_2) = \phi_K(y_1 + y_2)$;

*(ii)* $\exists\, \bar{z} \in \partial K$ *s.t.* $\phi_K(y_1) = y_1 \cdot \bar{z}$ *and* $\phi_K(y_2) = y_2 \cdot \bar{z}$,

*(iii)* $\exists \bar{z} \in \partial K$ *s.t.* $\frac{y_1}{\phi_K(y_1)}, \frac{y_2}{\phi_K(y_2)} \in \partial\phi_K^*(\bar{z})$.

**Remark 4.1.23.** *By Definition 1.3.11 condition $(iii)$ in the above Proposition is equivalent to say that*

$$\exists \bar{z} \in \partial K \qquad s.t. \quad y_1, y_2 \in C_K^*(\bar{z}). \tag{4.1.21}$$

*As noticed in Remark 4.1.9, $C_K^*(\bar{z})$ is a convex set and so condition (4.1.21) is equivalent to say that*

$$\exists \bar{z} \in \partial K \qquad s.t. \quad \{\lambda y_1 + (1-\lambda) y_2 :\, \lambda \in [0,1]\} \subset C_K^*(\bar{z}). \tag{4.1.22}$$

**Lemma 4.1.24.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3) and consider $\phi_K$ the associated surface tension. Let $x_0 \in \partial K$ then,*

$$\phi_K(y) = 1 \qquad \forall\, y \in \partial\phi_K^*(x_0). \tag{4.1.23}$$

*Moreover,*

$$\bigcup_{x \in \partial K} \partial\phi_K^*(x) = \partial K^*. \tag{4.1.24}$$

*Proof.* We divide the proof in two steps.

**Step 1** In this first part we prove (4.1.23). Let $y \in \partial\phi_K^*(x_0)$. By definition of sub-differential, we have that

$$\phi_K^*(z) \geq 1 + y \cdot (z - x_0) \qquad \forall\, z \in \mathbb{R}^n.$$

So, choosing $z = 0$ we get that $y \cdot x_0 \geq 1$. Observe that $y \in \partial\phi_K^*(x_0)$ implies $y \in C_K^*(x_0)$ so that, by the above Remark is equivalent to say $\phi_K(y) = y \cdot x_0$. So, $\phi_K(y) = y \cdot x_0 \geq 1$. At the same time, the fact that $\phi_K(y) = y \cdot x_0$ is equivalent to say that $y/\phi_K(y) \in \partial\phi_K^*(x_0)$. By the convexity property of the sub-differential of a convex function (see Remark 4.1.9), we have $\lambda y \in \partial\phi_K^*(x_0)$ for every $\lambda \in [1/\phi_K(y), 1]$, namely

$$\phi_K^*(z) \geq 1 + \lambda y \cdot (z - x_0) \qquad \forall\, z \in \mathbb{R}^n, \forall\, \lambda \in [1/\phi_K(y), 1].$$

Note that choosing $z = 0$ we get $\lambda \geq 1/\phi_K(y)$, while choosing $z = 2x_0$ we get, thanks to 1-homogeneity of $\phi_K^*$, that $\lambda \leq 1/\phi_K(y)$. Thus, we deduce that $1/\phi_K(y) = 1$. This concludes the proof of the first step.

**Step 2** In the last step we prove (4.1.24). Thanks to step 1 and Remark 4.1.6 we have that

$$\bigcup_{x\in\partial K} \partial\phi_K^*(x) \subseteq \partial K^*.$$

We are left to prove the other inclusion. Let $y \in \partial K^*$. By properties of convex sets there exists $\nu(y) \in \mathbb{S}^{n-1}$ such that $K^* \subset H^-_{y,\nu(y)}$ (see relations (2.0.1). So, $\forall\, z \in H^-_{y,\nu(y)}$ , and in particular $\forall\, z \in K^*$ we have

$$z \cdot \nu(y) \leq y \cdot \nu(y),$$

that implies, recalling Remark 4.1.6 that $\phi_K^*(\nu(y)) = \nu(y) \cdot y$. Thus, thanks to Lemma 4.1.20, recalling that $\phi_K(y) = 1$ we get

$$\begin{aligned}\phi_K^*(\nu(y)) = \nu(y)\cdot y \quad &\Leftrightarrow \quad \phi_K^*\left(\frac{\nu(y)}{\phi_K^*(\nu(y))}\right) = \frac{\nu(y)}{\phi_K^*(\nu(y))}\cdot y \quad \Leftrightarrow \quad 1 = \frac{\nu(y)}{\phi_K^*(\nu(y))}\cdot y \\ &\Leftrightarrow \quad \phi_K(y) = \frac{\nu(y)}{\phi_K^*(\nu(y))}\cdot y \quad \Leftrightarrow \quad y \in \partial\phi_K^*\left(\frac{\nu(y)}{\phi_K^*(\nu(y))}\right).\end{aligned}$$

Since $\nu(y)/\phi_K^*(\nu(y)) \in \partial K$ we conclude. $\square$

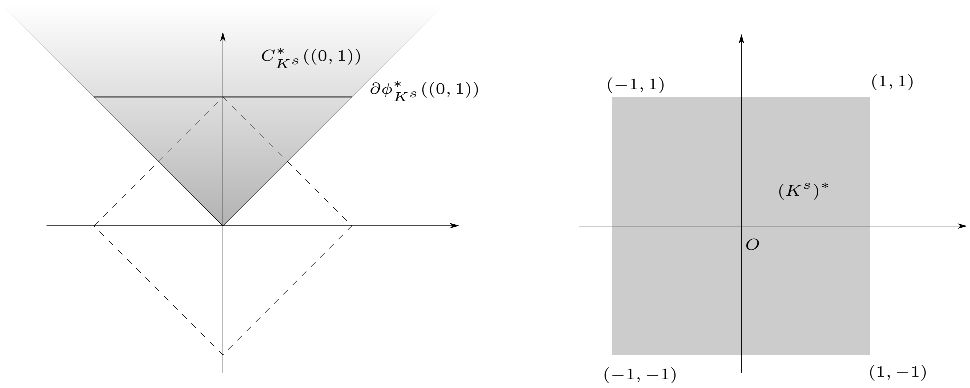


Figure 4.1.3: A pictorial idea of condition (4.1.24) with respect to the Wulff shape $K^s$ presented in Figure 4.1.1. Indeed, according to Lemma 4.1.24 and (4.1.28), we see that $\partial\phi^*_{K^s}((0,1))$ is a convex subset of the boundary of $(K^s)^*$. The fact that $\partial\phi^*_{K^s}((0,1))$ actually contains the point $(0,1)$ is just a consequence of the specific Wulff shape considered in the example.

**Corollary 4.1.25.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3) and consider $\phi_K$ the associated surface tension. Assume in addition that $\phi_K \in C^1(\mathbb{R}^n_0)$. Then,*

$$\phi_K(x) = \nabla\phi_K(x)\cdot x \quad \textit{and} \quad \phi_K^*(\nabla\phi_K(x)) = 1 \qquad \forall\, x \in \mathbb{R}^n_0. \tag{4.1.25}$$

*Proof.* Firstly, let us observe it is a well known fact that the first relation in (4.1.25) holds true for every positive and 1-homogeneous function. So, we are left to prove the second relation in (4.1.25). Let $x \in \partial K^*$. As we observed in the above Lemma, by properties of convex sets there exists $\nu(x) \in \mathbb{S}^{n-1}$ such that $K^* \subset H^-_{x,\nu(x)}$ and $\phi^*_K(\nu(x)) = \nu(x) \cdot x$. By Lemma 4.1.20, having in mind Remark 4.1.10 we have that

$$\phi^*_K(\nu(x)) = \nu(x) \cdot x \quad \iff \quad \frac{\nu(x)}{\phi^*_K(\nu(x))} = \nabla \phi_K(x). \tag{4.1.26}$$

By the 1-homogeneity of $\phi_K$ it follows that

$$\nabla \phi_K(\lambda x) = \nabla \phi_K(x) \qquad \forall\, \lambda > 0,\, \forall\, x \in \mathbb{R}^n_0, \tag{4.1.27}$$

therefore $\phi^*_K(\nabla \phi_K(x)) = 1$ for all $x \in \mathbb{R}^n_0$. This concludes the proof. $\square$

**Remark 4.1.26.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3), and consider $x \in \partial K$. Note that, thanks the above results we can deduce the following equivalent characterization for the subdifferential $\partial\phi^*_K(x)$, namely*

$$\partial\phi^*_K(x) = \left\{ y \in \partial K^* : y \cdot \frac{x}{|x|} = \phi^*_K\left(\frac{x}{|x|}\right) \right\}. \tag{4.1.28}$$

*Indeed, thanks to Lemma 4.1.24 we know that $\partial\phi^*_K(x) \subset \partial K^*$ so that $\phi_K(y) = 1$. Whereas, thanks to Lemma 4.1.20 we have that $y \in \partial\phi^*_K(x)$ is equivalent to say that $1 = \phi^*_K(x)\phi_K(y) = y \cdot x$, from which, we get $y \cdot \frac{x}{|x|} = \phi^*_K\left(\frac{x}{|x|}\right)$.*

The following two results will be used for the proof of Lemma 4.6.3.

**Lemma 4.1.27.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3). Let $x_1, x_2 \in \partial K$ and $\bar{y} \in \partial K^*$ be such that $\bar{y} \in \partial\phi^*_K(x_1) \cap \partial\phi^*_K(x_2)$. Let us now assume that there exist $y_1, y_2 \in \partial\phi^*_K(x_2)$, with $y_1 \neq \bar{y} \neq y_2$, such that $\bar{y} = (1-\bar{\lambda})y_1 + \bar{\lambda} y_2$ for some $\bar{\lambda} \in (0,1)$. Then,*

$$(1-\lambda)y_1 + \lambda y_2 \in \partial\phi^*_K(x_1) \quad \forall\, \lambda \in [0,1]. \tag{4.1.29}$$

*Proof.* Let us suppose by contradiction that there exists $\tilde{\lambda} \in [0, \bar{\lambda}]$ such that $\tilde{y} = (1-\tilde{\lambda})y_1 + \tilde{\lambda} y_2 \notin \partial\phi^*_K(x_1)$. By the Fenchel inequality (4.1.5) and (4.1.28) we get

$$\tilde{y} \cdot \frac{x_1}{|x_1|} < \bar{y} \cdot \frac{x_1}{|x_1|} = \phi^*_K\left(\frac{x_1}{|x_1|}\right). \tag{4.1.30}$$

Recall that, by (1.3.6) applied to $K^*$ we have that

$$\overline{K^*} = \bigcap_{\omega \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot \omega \le \phi^*_K(\omega) \right\}.$$

By relation (4.1.30) we have that the continuous linear function

$$\varphi(\lambda) := ((1-\lambda)y_1 + \lambda y_2) \cdot \frac{x_1}{|x_1|} > \phi^*_K\left(\frac{x_1}{|x_1|}\right)$$

for every $\lambda \in (\bar{\lambda}, 1]$, but this is impossible since

$$\{(1-\lambda)y_1 + \lambda y_2 : \lambda \in [0,1]\} \subset \partial\phi_K^*(x_2) \subset \overline{K^*}.$$

This concludes the proof. $\square$

**Corollary 4.1.28.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3). Let $x \in \partial K$ be such that the subdifferential of $\phi_K^*$ in $x$ has only one point, namely $\partial\phi_K^*(x) = \{y\}$. Then, $\forall\, z \in \mathcal{Z}_K(y)$, where $\mathcal{Z}_K(y)$ is defined in (1.3.16), and for every $y_1, y_2 \in C_K^*(z)$, if $\exists\, \lambda \in [0,1]$ s.t. $y = (1-\lambda)y_1 + \lambda y_2$, then $y_1 = \lambda_1 y$, $y_2 = \lambda_2 y$ for some $\lambda_1, \lambda_2 > 0$.*

*Proof.* So, let us fix $z \in \mathcal{Z}_K(y)$ and $y_1, y_2 \in C_K^*(z)$ and let us assume that $y = (1-\lambda)y_1 + \lambda y_2$, for some $\lambda \in [0,1]$. By the convexity of $C_K^*(z)$ together with Lemma 4.1.20 and Remark 4.1.21 we get that

$$\frac{(1-\lambda)y_1 + \lambda y_2}{\phi_K((1-\lambda)y_1 + \lambda y_2)} \in \partial\phi_K^*(z) \quad \forall\, \lambda \in [0,1].$$

Therefore, thanks to Lemma 4.1.27, we have that

$$\frac{(1-\lambda)y_1 + \lambda y_2}{\phi_K((1-\lambda)y_1 + \lambda y_2)} \in \partial\phi_K^*(x) \quad \forall\, \lambda \in [0,1],$$

but this is possible if and only if $\frac{y_1}{\phi_K(y_1)}, \frac{y_2}{\phi_K(y_2)} = y$. This concludes the proof. $\square$

We know introduce a technical result that will be used later on for the proof of the Steiner's inequality for the anisotropic perimeter.

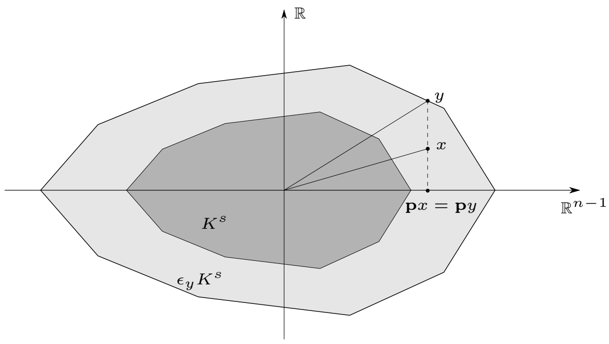


Figure 4.1.4: A pictorial idea for Lemma 4.1.29.

**Lemma 4.1.29.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3) and let us consider $K^s$, its Steiner symmetral. Then, for any two points $x, y \in \mathbb{R}^n$ such that $|x| < |y|$, $\mathbf{p}x = \mathbf{p}y$ the following inequalities hold true*

$$\phi_{K^s}^*(x) \le \phi_{K^s}^*(y),$$

$$\phi_{K^s}(x) \le \phi_{K^s}(y).$$

*Proof.* We divide the proof in two steps.

**Step 1** Let us prove the first relation. Suppose by contradiction that

$$\phi^*_{K^s}(x) > \phi^*_{K^s}(y) \tag{4.1.31}$$

and consider the constant $\epsilon_y > 0$ s.t. $y \in \partial(\epsilon_y K^s)$. By (4.1.31) we get

$$x \in \left(\overline{\epsilon_y K^s}\right)^c.$$

By the symmetry of $\epsilon_y K^s$ with respect to $\{x_n = 0\}$ we know that

$$y = (\mathbf{p}y, \mathbf{q}y) \in \partial\epsilon_y K^s,\ y_- = (\mathbf{p}y, -\mathbf{q}y) \in \partial(\epsilon_y K^s),$$

while both $x = (\mathbf{p}x, \mathbf{q}x)$ and $x_- = (\mathbf{p}x, -\mathbf{q}x)$ are in $\left(\overline{\epsilon_y K^s}\right)^c$. We found two points $y$ and $y_-$ contained in $\overline{\epsilon_y K^s}$ whom segment that links them is not totally contained in $\overline{\epsilon_y K^s}$. This is a contradiction to the convexity of $\overline{\epsilon_y K^s}$ and so we conclude that

$$\phi^*_{K^s}(x) \leq \phi^*_{K^s}(y).$$

**Step 2** In order to conclude the proof we want to apply the above argument to $(K^s)^*$. It is sufficient to prove that if $K^s$ is symmetric with respect $\{x_n = 0\}$ then $(K^s)^*$ has the same symmetric property. If $K^s$ is symmetric then, by relation (1.3.4) follows that $\phi_K(\mathbf{p}x, \mathbf{q}x) = \phi_K(\mathbf{p}x, -\mathbf{q}x)$ for every $x \in \mathbb{R}^n$. Thanks to this relation, and together with the fact that $\phi^*_K := \{x \in \mathbb{R}^n : \phi_K(x) < 1\}$ we immediately get that $K^s$ is symmetric with respect $\{x_n = 0\}$. This concludes the proof. □

We conclude this section recalling few more definitions and a couple of results very well known in convex analysis. Such tools, will play a key role in the understanding of (RSA).

**Definition 4.1.30.** *Let $C \subset \mathbb{R}^n$ be a convex set. We say that $x \in C$ is an* extreme point *of $C$ if and only if there is no way to express $x$ as a convex combination $(1-\lambda)y + \lambda z$ such that $y, z \in C$ and $0 < \lambda < 1$, except by taking $y = z = x$.*

**Definition 4.1.31.** *Let $C \subset \mathbb{R}^n$ be a convex set. We say that $x \in C$ is an* exposed point *of $C$ if and only if there exists an hyperplane of the form $H_{x,\nu}$, with $\nu \in \mathbb{S}^{n-1}$, such that $C \subset H^-_{x,\nu}$ and $\overline{C} \cap H_{x,\nu} = \{x\}$.*

**Remark 4.1.32.** *If $C \subset \mathbb{R}^n$ is a closed convex set, then by [37, Theorem 18.6], the set of exposed points of $C$ is dense in the set of extreme points of $C$, namely, every extreme point is the limit of a sequence of exposed points.*

Let us now recall an useful result about the characterization of the exposed points of a closed convex set (see for instance [37, Corollary 25.1.3]).

**Lemma 4.1.33.** *Let $C \subset \mathbb{R}^n$ be a non empty, closed, convex set, and let $g : \mathbb{R}^n \to [0, \infty)$ be any 1-homogeneous, convex function, such that*

$$C = \{z \in \mathbb{R}^n : \, z \cdot y \leq g(y) \quad \forall \, y \in \mathbb{R}^n\}.$$

*Then, $z \in C$ is an exposed point of $C$ if and only if there exists a point $y \in \mathbb{R}^n$ such that $g$ is differentiable at $y$ and $\nabla g(x) = z$.*

## 4.2 Characterization of the anisotropic total variation

In this section we will study some properties of the anisotropic total variation (see Definition 4.1.11), proving also a characterization Theorem (see 4.2.1). This result will be useful to obtain a formula for the anisotropic perimeter of the subgraph and epigraph of a function of bounded variation. Such characterization result is already known in literature but we decided to give a proof for the seek of completeness since we couldn't find a precise reference. The main result is the following.

**Theorem 4.2.1.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3). Let $\mu$ be a $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^m$, $m \geq 1$, $m \in \mathbb{N}$. Then, we have*

$$|\mu|_K(\Omega) = \sup \left\{ \int_{\Omega} \varphi(x) \cdot d\mu(x) : \, \varphi \in C^1_c(\Omega; \mathbb{R}^n), \, \phi^*_K(\varphi) \leq 1 \right\} \quad \forall \, \Omega \subset \mathbb{R}^m \text{ open.}$$

In order to prove Theorem 4.2.1 we need some intermediate results.

**Lemma 4.2.2.** *Let $\{K_h\}_{h \in \mathbb{N}} \subset \mathbb{R}^n$, $K \subset \mathbb{R}^n$ be such that $K_h, K$ are as in (1.3.3) $\forall \, h \in \mathbb{N}$. Assume moreover that*

- *i) the sequence $(K_h)_{h \in \mathbb{N}}$ is either of the form $K_h \subset K_{h+1} \subset K$, or $K \subset K_{h+1} \subset K_h$, $\forall h \in \mathbb{N}$,*
- *ii) $\lim_{h \to +\infty} \mathrm{dist}_H(K_h, K) = 0$.*

*Then, the sequence $\{\phi_{K_h}\}$ converges uniformly to $\phi_K$ in $\mathbb{S}^{n-1}$.*

*Proof.* Without loss of generality we can consider the case when $K_h \subset K_{h+1} \subset K \; \forall h \in \mathbb{N}$. For every $x \in \mathbb{S}^{n-1}$ and $h \in \mathbb{N}$, let $y(x) \in \partial K$ and $y_h(x) \in \partial K_h$ be such that $\phi_K(x) = y(x) \cdot x$ and $\phi_{K_h}(x) = y_h(x) \cdot x$, respectively. Then, since $K_h \subset K$,

$$\sup_{x \in \mathbb{S}^{n-1}} |\phi_K(x) - \phi_{K_h}(x)| = \sup_{x \in \mathbb{S}^{n-1}} \left[ x \cdot (y(x) - y_h(x)) \right].$$

Note now that, by definition of $y_h$, we have $-x \cdot y_h(x) \leq -x \cdot \bar{y} \ \ \forall \bar{y} \in \partial K_h$. In particular, choosing $\bar{y} = z(x) \in \partial K_h$ such that $|y(x) - z(x)| = \mathrm{dist}(y(x), \partial K_h)$, we have

$$\sup_{x \in \mathbb{S}^{n-1}} |\phi_K(x) - \phi_{K_h}(x)| \leq \sup_{x \in \mathbb{S}^{n-1}} [x \cdot (y(x) - z(x))] \leq \mathrm{dist}(y(x), \partial K_h) = \mathrm{dist}_H(K, K_h),$$

where in the last equality we used the fact that $K_h \subset K$. Passing to the limit as $h \to +\infty$ we conclude. □

**Lemma 4.2.3.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3). Then there exists a sequence $\{K_h\}_{h \in \mathbb{N}} \subset \mathbb{R}^n$ with $K_h$ as in (1.3.3) for every $h \in \mathbb{N}$, such that*

*i)* $K_h$ *is* $C^{1,1}$, $\forall\, h \in \mathbb{N}$;

*ii)* $K \subset \cdots \subset K_{h+1} \subset K_h \quad \forall h \in \mathbb{N}$;

*iii)* $\lim_{h \to +\infty} \mathrm{dist}_H(K_h, K) = 0$.

*Proof.* We divide the proof in few steps. Take any $\epsilon > 0$ and let $K_\epsilon = \bigcup_{x \in K} B(x, \epsilon)$ denote the $\epsilon$-neighbourhood of $K$.

**Step 1** In this Step we want to prove that $K_\epsilon$ is convex, open, bounded and it contains the origin. By construction, we need just to prove that it is convex. Consider two generic points $x_1, x_2 \in K_\epsilon$, let us show that

$$\lambda x_1 + (1 - \lambda) x_2 \in K^* \quad \forall \lambda \in [0, 1].$$

Observe that, since $x_1, x_2 \in K_\epsilon$ there exist $c_1, c_2 \in K$ such that $|x_1 - c_1| < \epsilon$ and $|x_2 - c_2| < \epsilon$. Thus,

$$\begin{aligned} \lambda x_1 + (1 - \lambda) x_2 &= \lambda[c_1 + (x_1 - c_1)] + (1 - \lambda)[c_2 + (x_2 - c_2)] \\ &= \lambda c_1 + (1 - \lambda) c_2 + \lambda(x_1 - c_1) + (1 - \lambda)(x_2 - c_2). \end{aligned}$$

Since $\lambda c_1 + (1 - \lambda) c_2 \in K$ and $|\lambda(x_1 - c_1) + (1 - \lambda)(x_2 - c_2)| < \epsilon$ we conclude the proof of step 1.

**Step 2** In this step we are going to prove that $K_\epsilon$ satisfies the $\epsilon$-ball property. This is true by construction. Indeed, since $K_\epsilon$ is as in (1.3.3), we can associate to it the function $\phi_{K_\epsilon}$. So, having in mind (1.3.6) we know that for every $y \in \partial K_\epsilon$ there exists $\nu \in \mathbb{S}^{n-1}$ and an hyperplane $H_{\phi_{K_\epsilon}(\nu)} = \{z \in \mathbb{R}^n : \ z \cdot \nu = \phi_{K_\epsilon}(\nu)\}$ such that $y \in H_{\phi_{K_\epsilon}(\nu)}$ and $K_\epsilon$ lies on one side of $H_{\phi_{K_\epsilon}(\nu)}$ (this is because $K_\epsilon$ is a convex set). So, we can construct on the exterior of $K_\epsilon$ a ball of whatever radius tangent to the hyperplane $H_{\phi_{K_\epsilon}(\nu)}$ in the point $y$. Let us now consider $z \in K_\epsilon$ such that $|z - y| = \epsilon$ in particular, $z \in \partial K$. By construction we have that $B(z, \epsilon) \subset K_\epsilon$ and this concludes the proof of step 2.

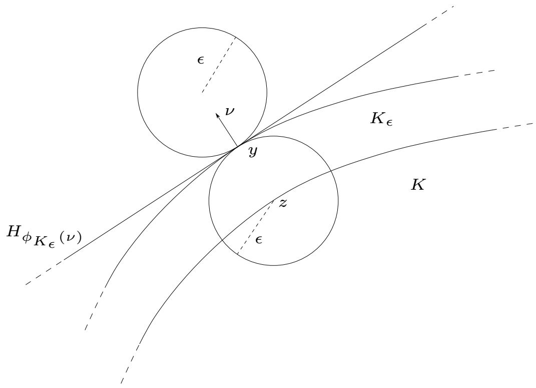


Figure 4.2.1: A pictorial idea for the proof of Lemma 4.2.3.

**Step 3** We have to prove that $\partial K_\epsilon$ is an hypersurface $C^{1,1}$ regular. This result is a straight forward consequence of [18, Theorem 1.8].

**Step 4** We are left to prove that $\mathrm{dist}_H(\bigcup_{x\in K} B(x,\epsilon), K) \le \epsilon$. By definition of Hausdorff distance we have that

$$\begin{aligned}\mathrm{dist}_H\left(K_\epsilon, K\right) &= \max\left\{\sup_{y\in K_\epsilon} d(y,K); \sup_{y\in K} d(y,K_\epsilon)\right\}\\ &= \max\left\{\epsilon; 0\right\}.\end{aligned}$$

To conclude the proof of the Lemma let us observe the following. Let us fix a decreasing sequence of positive real numbers $(\epsilon_h)_{h\in\mathbb{N}}$. We can construct the sequence $(K_h)_{h\in\mathbb{N}}$ where $K_h = K_{\epsilon_h}$ is the $\epsilon_h$-neighbourhood of $K$ $\forall\, h \in \mathbb{N}$. By all previous steps, the sequence $(K_h)_{h\in\mathbb{N}}$ satisfies $i), ii)$ and $iii)$ of the Lemma and this concludes the proof. □

**Proposition 4.2.4.** *Let $K$ be as in (1.3.3) and let $K^*$ be its dual. Consider $(K_h^*)_{h\in\mathbb{N}}$ a sequence as in (1.3.3), such that either $K_h^* \subset K_{h+1}^* \subset K^*$ or $K^* \subset K_{h+1}^* \subset K_h^*$, $\forall h \in \mathbb{N}$. Then, denoting with $K_h = (K_h^*)^*$ we have*

$$\lim_{h\to+\infty} \mathrm{dist}_H(K_h^*, K^*) = 0 \quad \textit{if and only if} \quad \lim_{h\to+\infty} \mathrm{dist}_H(K_h, K) = 0.$$

*Proof.* Let us assume that $\lim_{h\to+\infty} \mathrm{dist}_H(K_h^*, K^*) = 0$ and, without loss of generality, that $K^* \subset K_{h+1}^* \subset K_h^*$, $\forall h \in \mathbb{N}$. We can apply immediately Lemma 4.2.2 to the sequence $\{K_h^*\}_{h\in\mathbb{N}}$ to obtain that $\phi_{K_h^*}$ uniformly converges to $\phi_{K^*}$. Consider the following quantity

$$\mathrm{dist}_H(K_h, K) = \max\left\{\sup_{x\in K_h} d(x,K); \sup_{x\in K} d(x,K_h)\right\}.$$

Now, by the way the $K_h^*$ are constructed, and having in mind $iii)$ of Proposition 4.1.4, we have

$$K_h \subset K_{h+1} \subset \cdots \subset K \quad \forall h \in \mathbb{N}.$$

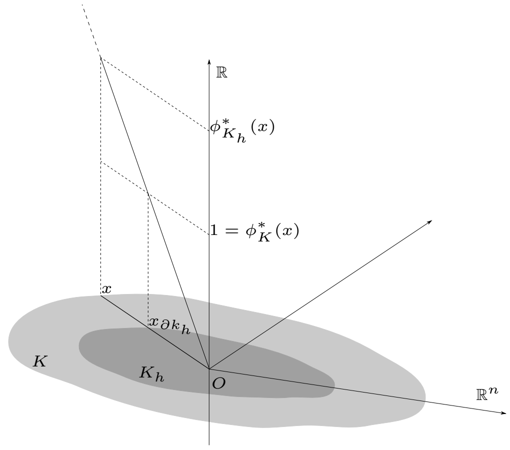


Figure 4.2.2: A pictorial idea for the proof of Proposition 4.2.4.

This fact immediately tells us that

$$\sup_{x\in K_h} d(x,K) = 0.$$

Let us focus our attention now on $\sup_{x\in K} d(x,K_h)$, thus

$$\sup_{x\in K} d(x,K_h) = \sup_{x\in\partial K} d(x,K_h) = \max_{x\in\partial K} d(x,K_h) \le \max_{x\in\partial K} |x - x_{K_h}|,$$

where $x_{K_h} = \{tx : t>0\}\cap\partial K_h$. By observing that $\phi^*_{K_h}(x) = \frac{|x|}{|x_{K_h}|}\phi^*_{K_h}(x_{K_h}) = \frac{|x|}{|x_{K_h}|}$, and since $|x| - |x_{K_h}| = |x - x_{K_h}|$, we get

$$|x - x_{K_h}|\frac{1}{|x_{K_h}|} = \left(\phi^*_{K_h}(x) - \phi^*_K(x)\right).$$

Thus,

$$\lim_{h\to+\infty} |x - x_{K_h}| = \lim_{h\to+\infty} |x_{K_h}|\left(\phi^*_{K_h}(x) - \phi^*_K(x)\right) = 0 \quad \forall x\in\partial K$$

thanks to the uniform convergence of $\phi^*_{K_h}$ to $\phi^*_K$. This shows that $\{K_h\}\subset\mathbb{R}^n$ converges in Hausdorff distance to $K$. Since $(K^*)^* = K$, $(K^*_h)^* = K_h$ the proof is complete. $\square$

We can now prove Theorem 4.2.1.

*Proof.* For the seek of clarity we decided to divide the proof in several steps.

**Step 1** Assume $\Omega\subset\mathbb{R}^n$ to be an open, bounded set. We start proving

$$\int_\Omega \phi_K\left(\frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) \ge \sup\left\{\int_\Omega \varphi(x)\cdot d\mu(x) : \varphi\in C^1_c(\Omega;\mathbb{R}^n),\ \phi^*_K(\varphi)\le 1\right\}.$$

Let us observe that by definition of $\phi_K$ we have

$$\begin{aligned}|\mu|_K(\Omega) &= \int_\Omega \phi_K\left(\frac{d\mu}{d|\mu|}(x)\right), d|\mu|(x) = \int_\Omega\left(\sup_{y\in\partial K} y\cdot\frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x)\\ &\ge \int_\Omega \varphi(x)\cdot\frac{d\mu}{d|\mu|}(x)\, d|\mu|(x),\end{aligned}$$

where $\varphi \in C^1_c(\Omega;\mathbb{R}^n)$, $\phi^*_K(\varphi) \le 1$. Passing to the sup on the right hand side we conclude the first step.

**Step 2** We want to prove the reverse inequality, namely

$$|\mu|_K(\Omega) \le \sup\left\{\int_\Omega \varphi(x)\cdot d\mu(x) : \varphi \in C^1_c(\Omega;\mathbb{R}^n),\, \phi^*_K(\varphi)\le 1\right\},$$

In order to do so, we consider at first the case when $\phi_K$ is in addition $C^1(\mathbb{R}^n_0)$. Recalling relations (4.1.26), we have

$$|\mu|_K(\Omega) = \int_\Omega \phi_K\left(\frac{d\mu}{|d\mu|}(x)\right) d|\mu|(x) = \int_\Omega \nabla\phi_K\left(\frac{d\mu}{|d\mu|}(x)\right)\cdot\frac{d\mu}{|d\mu|}(x)\, d|\mu|(x).$$

Since $\nabla\phi_K \in C^0(\mathbb{R}^n_0)$, the composition $\nabla\phi_K\left(\frac{d\mu}{|d\mu|}(x)\right)$ is well defined moreover,

$$\nabla\phi_K\left(\frac{d\mu}{|d\mu|}(\cdot)\right) \in L^1_{loc}(\Omega,|\mu|;\mathbb{R}^n),$$

with $\phi^*_K\left(\nabla\phi_K\left(\frac{d\mu}{|d\mu|}(x)\right)\right) = 1$ for $|\mu|$-a.e. $x\in\Omega$. Recall that

$$\phi^*_K\left(\nabla\phi_K\left(\frac{d\mu}{|d\mu|}(x)\right)\right) = 1 \quad \text{implies} \quad \nabla\phi_K\left(\frac{d\mu}{|d\mu|}(x)\right) \in \partial K, \quad \textit{for } |\mu|\textit{-a.e. } x\in\Omega.$$

that means $\nabla\phi_K\left(\frac{d\mu}{|d\mu|}(x)\right) \in L^\infty(\Omega,|\mu|;\mathbb{R}^n)$. By the fact that $\Omega$ is a bounded set we have that

$$\nabla\phi_K\left(\frac{d\mu}{|d\mu|}(\cdot)\right) \in L^p(\Omega,|\mu|;\mathbb{R}^n) \quad \forall p\ge 1.$$

Let us call $f := \nabla\phi_K\left(\frac{d\mu}{|d\mu|}\right)$. By [2, Remark 1.46] there exist a sequence $(g_h)_h \in C^0_c(\Omega;\mathbb{R}^n)$ such that $g_h \to f$ in $L^1(\Omega,|\mu|;\mathbb{R}^n)$. Since every function in $C^0_c$ can be uniformly approximated by functions in $C^1_c$ we can suppose without loss of generality that the sequence $(g_h)_h \in C^1_c(\Omega;\mathbb{R}^n)$. Now we consider the sequence $(\tilde g)_h \in C^0_c(\Omega;\mathbb{R}^n)$ defined as

$$\tilde g_h(x) := \frac{g_h(x)}{\phi^*_K(g_h(x)) + 1/h} \quad \forall\, h\in\mathbb{N}.$$

By construction, up to a subsequence, we have that $\tilde g_h \to f$ $|\mu|$-a.e. on $\Omega$ and, thanks to the term $1/h$ in the denominator, $\tilde g_h(x) \in \mathring{K}$, so that $\phi^*_K(\tilde g_h(x)) < 1$ for every $h\in\mathbb{N}$ and for $|\mu|$-a.e. $x\in\Omega$. By the continuity of the functions $\tilde g_h$, for every $h\in\mathbb{N}$ there exists $\lambda = \lambda(h) > 0$ such that $0 < \lambda(h) < 1$ and $\tilde g_h(x) \in \lambda(h)K$ for every $x\in\Omega$. Again, using the fact that $C^1_c(\Omega;\mathbb{R}^n)$ is dense in $C^0_c(\Omega;\mathbb{R}^n)$ we can proceed as follow: let $(\epsilon_h)_{h\in\mathbb{N}}$ be such that $\epsilon_h > 0$ for every $h\in\mathbb{N}$ and $\epsilon_h \to 0$ for $h\to\infty$. For every $h\in\mathbb{N}$ let $f_h \in C^1_c(\Omega;\mathbb{R}^n)$ be such that

$$\sup_{x\in\Omega} |f_h(x) - \tilde g_h(x)| < \epsilon_h.$$

Since $\mathrm{dist}(\partial(\lambda(h)K);\partial K) > 0$ for every $h \in \mathbb{N}$ , choosing $\epsilon_h$ small enough we get that $\forall\, h \in \mathbb{N}$ $f_h(x) \in K$ for every $x \in \Omega$ . Thus, by the Lebesgue dominated convergence theorem

$$\begin{aligned}|\mu|_K(\Omega) &= \int_\Omega \phi_K\left(\frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) = \int_\Omega \lim_{h\to\infty} f_h(x)\cdot \frac{d\mu}{d|\mu|}(x)d|\mu|(x)\\ &= \lim_{h\to\infty}\int_\Omega f_h(x)\cdot\frac{d\mu}{d|\mu|}(x)d|\mu|(x) \le \sup_{h\in\mathbb{N}}\int_\Omega f_h(x)\cdot\frac{d\mu}{d|\mu|}(x)d|\mu|(x)\\ &\le \sup_{\substack{\varphi\in C^1_c(\Omega;\mathbb{R}^n),\\ \phi^*_K(\varphi)\le 1}}\int_\Omega \varphi(x)\cdot\frac{d\mu}{d|\mu|}(x)d|\mu|(x).\end{aligned}$$

This concludes step 2.

**Step 3** We want now to prove the statement for a generic $\phi_K$. Thus, thanks to Lemma (4.2.3) consider $\{K_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^n$ a sequence as in (1.3.3) with $K_h \subset K_{h+1} \subset \cdots \subset K$ and such that the sequence satisfies the assumptions of Lemma 4.2.2. Using Proposition 4.2.4 we can immediately deduce that $\phi_{K_h}$ uniformly converges to $\phi_K$. Therefore, applying step 2 we get

$$\begin{aligned}|\mu|_{K_h}(\Omega) &= \int_\Omega \phi_{K_h}\left(\frac{d\mu}{d|\mu|}\right) d|\mu|(x) = \sup_{\substack{\varphi\in C^1_c(\Omega;\mathbb{R}^n),\\ \phi^*_{K_h}(\varphi)\le 1}}\int_\Omega \varphi(x)\cdot\frac{d\mu}{d|\mu|}(x)d|\mu|(x)\\ &\le \sup_{\substack{\varphi\in C^1_c(\Omega;\mathbb{R}^n),\\ \phi^*_K(\varphi)\le 1}}\int_\Omega \varphi(x)\cdot\frac{d\mu}{d|\mu|}(x)d|\mu|(x),\end{aligned}$$

where we used the fact that $\phi^*_K(\varphi) \le 1$ as a consequence of $\phi^*_{K_h}(\varphi) \le 1$ and of $K_h \subset K$. Now, thanks to the uniform convergence of the functions $\phi_{K_h}$ to $\phi_K$ we get

$$\begin{aligned}|\mu|_K(\Omega) &= \int_\Omega \phi_K\left(\frac{d\mu}{d|\mu|}\right) d|\mu|(x) = \lim_{h\to+\infty}\int_\Omega \phi_{K_h}\left(\frac{d\mu}{d|\mu|}\right) d|\mu|(x)\\ &\le \sup_{\substack{\varphi\in C^1_c(\Omega;\mathbb{R}^n),\\ \phi^*_K(\varphi)\le 1}}\int_\Omega \varphi(x)\cdot\frac{d\mu}{d|\mu|}(x)d|\mu|(x).\end{aligned}$$

This concludes the proof in the case $\Omega$ open and bounded. From standard considerations about outer measures, the extension of this result for unbounded open set follows. □

The following result is the anisotropic version of [11, Lemma 3.7].

**Lemma 4.2.5.** *If $\nu$ and $\mu$ are $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^m$, then*

$$2|\mu|_K(G) \le |\mu+\nu|_K(G) + |\mu-\nu|_K(G) \tag{4.2.1}$$

*for every Borel set $G \subset \mathbb{R}^m$.*

*Proof.* Fix a generic partition of $G$ made by bounded Borel sets $\{G_i\}_{i\in\mathbb{N}}$, by subadditivity we have

$$\begin{aligned}\phi_K\left(2\mu(G_i)\right) &= \phi_K\left(\mu(G_i)+\nu(G_i)+\mu(G_i)-\nu(G_i)\right)\\ &\le \phi_K\left((\mu+\nu)(G_i)\right)+\phi_K\left((\mu-\nu)(G_i)\right).\end{aligned}$$

Thus,

$$\sum_{i\in\mathbb{N}}\phi_K\left(2\mu(G_i)\right)\le\sum_{i\in\mathbb{N}}\left[\phi_K\left((\mu+\nu)(G_i)\right)+\phi_K\left((\mu-\nu)(G_i)\right)\right].$$

Then thanks to Lemma 4.1.15 and passing to the sup in both sides we get

$$\begin{aligned}|2\mu|_K(G) &\le \sup_{\{G_i\}}\sum_{i\in\mathbb{N}}\left[\phi_K\left((\mu+\nu)(G_i)\right)+\phi_K\left((\mu-\nu)(G_i)\right)\right]\\ &\le \sup_{\{G_i\}}\sum_{i\in\mathbb{N}}\phi_K\left((\mu+\nu)(G_i)\right)+\sup_{\{G_k\}}\sum_{k\in\mathbb{N}}\phi_K\left((\mu-\nu)(G_k)\right)\\ &= |\mu+\nu|_K(G)+|\mu-\nu|_K(G).\end{aligned}$$

This concludes the proof. $\square$

**Remark 4.2.6.** *Let $\mu_1,\mu_2$ be $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^m$. Let us observe that, by (4.2.1) with $\mu=\mu_1+\mu_2$ and $\nu=\mu_1-\mu_2$ we obtain*

$$|\mu_1+\mu_2|_K\le|\mu_1|_K+|\mu_2|_K. \tag{4.2.2}$$

*On the other hand, let $\nu_1,\nu_2$ be $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^m$. Then, by the above relation with $\mu_1=\nu_1+\nu_2$ and $\mu_2=-\nu_2$ we get*

$$|\nu_1+\nu_2|_K\ge|\nu_1|_K-|-\nu_2|_K. \tag{4.2.3}$$

**Remark 4.2.7.** *In this Remark we discuss the equality case for relation (4.2.1). Let us assume that*

$$2|\mu|_K(G)=|\mu+\nu|_K(G)+|\mu-\nu|_K(G)\qquad\forall\,\text{Borel set } G\subset\mathbb{R}^m. \tag{4.2.4}$$

*We immediately observe that if $|\mu|_K(G)=0$ then $|\mu+\nu|_K(G)=|\mu-\nu|_K(G)=|\nu|_K(G)=0$, so that*

$$|\nu|_K\ll|\mu|_K.$$

*Thanks to Radon-Nykodym Theorem we know that $\exists\,g,h\in L^1_{loc}(\mathbb{R}^m,|\mu|_K;\mathbb{R}^n)$ s.t.*

$$\nu=g|\mu|_K\quad\textit{and}\quad\mu=h|\mu|_K,$$

*thus,*

$$\mu \pm \nu = (h \pm g)|\mu|_K.$$

*Observing that*

$$|\mu \pm \nu|_K(G) = \int_G \phi_K \left( \frac{d(\mu \pm \nu)}{d|\mu \pm \nu|}(x) \right) d|\mu \pm \nu|(x) = \int_G \phi_K \left( \frac{(h \pm g)(x)}{|h \pm g|(x)} \right) |h \pm g|(x)\, d|\mu|_K(x),$$

*we can now rewrite (4.2.4) in the following way*

$$\int_G 2\phi_K\left(h(x)\right)\, d|\mu|_K(x) = \int_G \phi_K\left((h+g)(x)\right)\, d|\mu|_K(x) + \int_G \phi_K\left((h-g)(x)\right)\, d|\mu|_K(x).$$

*By 1-homogeneity we have*

$$\int_G \phi_K\left(2h(x)\right) - \phi_K\left((h+g)(x)\right) - \phi_K\left((h-g)(x)\right)\, d|\mu|_K(x) = 0 \quad \forall\, G \subset \mathbb{R}^m \text{ Borel.}$$

*By subadditivity we get*

$$\phi_K\left(2h(x)\right) - \phi_K\left((h+g)(x)\right) - \phi_K\left((h-g)(x)\right) \le 0 \quad |\mu|_K\text{-}a.e.\, x \in \mathbb{R}^m,$$

*thus,*

$$\phi_K(2h(x)) = \phi_K\left((h+g)(x)\right) + \phi_K\left((h-g)(x)\right) \quad |\mu|_K\text{-}a.e.\, x \in \mathbb{R}^m. \tag{4.2.5}$$

*Thus condition (4.2.4) is equivalent to (4.2.5) that is equivalent to say, thanks to Proposition 4.1.22, Remark 4.1.23 and relation (4.1.22) with $y_1 = h+g$ and $y_2 = h-g$, that for $|\mu|_K$-a.e. $x \in \mathbb{R}^{n-1}$ $\exists\, z(x) \in \partial K$ s.t.*

$$\{h(x) + tg(x) : \, t \in [-1,1]\} \subset C_K^*(z(x)). \tag{4.2.6}$$

## 4.3 The Steiner's inequality for the anisotropic perimeter

In this section we prove (AS), i.e. that the Steiner's inequality for the anisotropic perimeter holds true whenever we consider $E \subset \mathbb{R}^n$ any set of finite perimeter and a Wulff shape $K^s$ defined as in (1.3.3) that is symmetric with respect the hyperplane $\{x_n = 0\}$. The strategy we will use, follows the ideas presented in [14]. Let $E \subset \mathbb{R}^n$ be any set of finite perimeter, consider $B \subset \mathbb{R}^{n-1}$ any Borel set and let $G_E$ and $G_{E^s}$ be the two sets given by Theorem 4.1.1.

Let us start providing the details of the simple example shown in the Introduction (see Figure 4.3.1), where $P_K(K) < P_K(K^s)$. Simple calculations show that

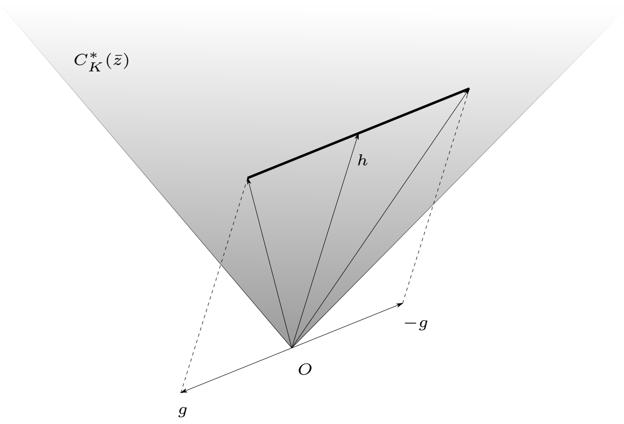


Figure 4.2.3: In this picture we give a 2-dimensional representation of condition (4.2.6) where $h \in C^*_K(\bar{z})$ and $\bar{z}$ is a fixed point in the boundary of the Wulff shape $K$.

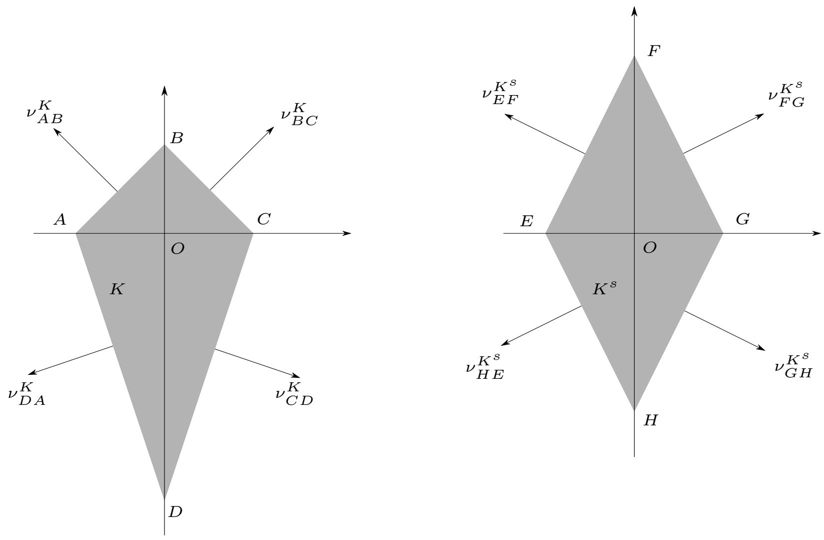


Figure 4.3.1: An example in which $P_K(K) < P_K(K^s)$. The coordinates of the vertices are $A = (-1, 0)$, $B = (0, 1)$, $C = (1, 0)$, $D = (0, -3)$, $E = (-1, 0)$, $F = (0, 2)$, $G = (1, 0)$, $H = (0, -2)$.

$$\mathcal{H}^1(AB) = \mathcal{H}^1(BC) = \sqrt{2}, \quad \mathcal{H}^1(CD) = \mathcal{H}^1(DA) = \sqrt{10},$$
$$\mathcal{H}^1(EF) = \mathcal{H}^1(FG) = \mathcal{H}^1(GH) = \mathcal{H}^1(HE) = \sqrt{5},$$

where by $\mathcal{H}^1(AB)$ for instance, we mean the length of the segment $AB$.

$$\nu^K_{AB} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad \nu^K_{BC} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \quad \nu^K_{CD} = \left(\frac{3\sqrt{10}}{10}, -\frac{\sqrt{10}}{10}\right), \quad \nu^K_{DA} = \left(-\frac{3\sqrt{10}}{10}, -\frac{\sqrt{10}}{10}\right).$$
$$\nu^{K^s}_{GH} = \left(\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right), \quad \nu^{K^s}_{HE} = \left(-\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right) \quad \nu^{K^s}_{EF} = \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right) \quad \nu^{K^s}_{FG} = \left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$$

Moreover, using relation (1.3.4) we get

$$\phi_K(\nu^K_{AB}) = \phi_K(\nu^K_{BC}) = \frac{\sqrt{2}}{2}, \quad \phi_K(\nu^K_{CD}) = \phi_K(\nu^K_{DA}) = \frac{3\sqrt{10}}{10},$$

$$\phi_K(\nu_{GH}^{K^s}) = \phi_K(\nu_{HE}^{K^s}) = \frac{3\sqrt{5}}{5}, \quad \phi_K(\nu_{EF}^{K^s}) = \phi_K(\nu_{FG}^{K^s}) = \frac{2\sqrt{5}}{5}.$$

Therefore,

$$\begin{aligned} P_K(K) &= \int_{\partial^* K} \phi_K(\nu^K(x))d\mathcal{H}^1(x) = \sqrt{2}\left(\phi_K(\nu_{AB}^K) + \phi_K(\nu_{BC}^K)\right) + \sqrt{10}\left(\phi_K(\nu_{CD}^K) + \phi_K(\nu_{DA}^K)\right) \\ &= 8, \end{aligned}$$

$$\begin{aligned} P_K(K^s) &= \int_{\partial^* K^s} \phi_K(\nu^{K^s}(x))d\mathcal{H}^1(x) = \sqrt{5}\left(\phi_K(\nu_{EF}^{K^s}) + \phi_K(\nu_{FG}^{K^s}) + \phi_K(\nu_{GH}^{K^s}) + \phi_K(\nu_{HE}^{K^s})\right) \\ &= 10. \end{aligned}$$

This shows that $P_K(K^s) > P_K(K)$ and so (AS) fails to be true.

**Remark 4.3.1.** *Let us observe that since $K^s$ is symmetric with respect to $\{x_n = 0\}$, then $\forall x \in \mathbb{R}^n$ we have that $\phi_K(\boldsymbol{p}x, \boldsymbol{q}x) = \phi_K(\boldsymbol{p}x, -\boldsymbol{q}x)$.*

First, we need the following intermediate result (see for instance [16, Lemma 5.3]).

**Lemma 4.3.2** (Auxiliary anisotropic perimeter inequality)**.** *Let $v$ as in (1.1.3) and let $K \subset \mathbb{R}^n$ be as in (1.3.3). Then, for every $E \subset \mathbb{R}^n$ $v$-distributed set we have*

$$P_{K^s}(F[v]; B \times \mathbb{R}) \leq P_{K^s}(E; B \times \mathbb{R}) + |\boldsymbol{q}(D1_{F[v]})|(B \times \mathbb{R}) \tag{4.3.1}$$

*for every Borel set $B \subset \mathbb{R}^{n-1}$, where*

$$|\boldsymbol{q}(D1_{F[v]})|(B \times \mathbb{R}) = \int_{\partial^* F[v] \cap (B \times \mathbb{R})} |\boldsymbol{q}(\nu^{F[v]}(x))| d\mathcal{H}^{n-1}(x).$$

*Proof.* The argument used in this proof follows the ideas of [14, Lemma 3.5]. Let $\{v_j\}_{j\in\mathbb{N}} \subset C_c^1(\mathbb{R}^{n-1})$ be a sequence of non negative functions such that $v_j \to v$ $L^{n-1}$-a.e. in $\mathbb{R}^{n-1}$, $\nabla v_j \overset{*}{\rightharpoonup} Dv$ and $|\nabla v_j|(\mathbb{R}^{n-1}) \to |Dv|(\mathbb{R}^{n-1})$. Moreover, let us denote by $F[v_j]$ the set $v_j$-distributed constructed as explained in (1.1.1). Fix any open set $\Omega \subset \mathbb{R}^{n-1}$ and let $f = (f_1, f_2, \ldots, f_n) \in C_c^1(\Omega \times \mathbb{R}, \mathbb{R}^n)$. Then, thanks to the divergence theorem and standard differentiation results we get

$$\begin{aligned} \int_{\Omega\times\mathbb{R}} 1_{F[v_j]} \operatorname{div} f dx &= \int_\Omega dz \int_{-v_j(z)/2}^{v_j(z)/2} \sum_{i=1}^{n-1} \frac{\partial f_i}{\partial x_i} dy + \int_{\Omega\times\mathbb{R}} 1_{F[v_j]} \frac{\partial f_n}{\partial x_n} dx \\ &= -\frac{1}{2} \int_{\mathbf{p}(\operatorname{supp} f)} \sum_{i=1}^{n-1} \left[ f_i\left(z, \frac{v_j(z)}{2}\right) + f_i\left(z, -\frac{v_j(z)}{2}\right)\right] \frac{\partial v_j}{\partial x_i} dz \\ &+ \int_{\Omega\times\mathbb{R}} 1_{F[v_j]} \frac{\partial f_n}{\partial x_n} dx. \end{aligned}$$

Thus, calling $g_i(x) := \frac{1}{2}\left(f_i\left(\mathbf{p}x, \frac{v_j(\mathbf{p}x)}{2}\right) + f_i\left(\mathbf{p}x, \frac{-v_j(\mathbf{p}x)}{2}\right)\right)$ for $i = 1, \dots, n-1$ and using the Fenchel inequality (see ii) in Proposition 4.1.4) we get

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}} 1_{F[v_j]} \mathrm{div} f dx &\leq \int_{\mathbf{p}(\mathrm{supp} f)} \phi^*\left(g_1(z), \dots, g_{n-1}(z), 0\right) \phi(-\nabla v_j(z), 0) dz + \int_{\Omega\times\mathbb{R}} 1_{F[v_j]} \frac{\partial f_n}{\partial x_n} dx \\
&\leq \int_{\mathbf{p}(\mathrm{supp} f)} \frac{1}{2}\phi^*_{K^s}\left(f_1\left(z, \frac{v_j(z)}{2}\right), \dots, f_{n-1}\left(z, \frac{v_j(z)}{2}\right), 0\right) \phi_{K^s}(-\nabla v_j(z), 0) dz \\
&+ \int_{\mathbf{p}(\mathrm{supp} f)} \frac{1}{2}\phi^*_{K^s}\left(f_1\left(z, \frac{-v_j(z)}{2}\right), \dots, f_{n-1}\left(z, \frac{-v_j(z)}{2}\right), 0\right) \phi_{K^s}(-\nabla v_j(z), 0) dz \\
&+ \int_{\Omega\times\mathbb{R}} 1_{F[v_j]} \frac{\partial f_n}{\partial x_n} dx.
\end{aligned}$$

If now we consider $\phi^*_{K^s}(f) \leq 1$, thanks also to the symmetric properties of the Wulff shape, we deduce that

$$\int_{\Omega\times\mathbb{R}} 1_{F[v_j]} \mathrm{div} f dx \leq \int_{\mathbf{p}(\mathrm{supp} f)} \phi_{K^s}(-\nabla v_j(x'), 0) dx' + \int_{\Omega\times\mathbb{R}} 1_{F[v_j]} \frac{\partial f_n}{\partial x_n} dx.$$

Now let us observe that, by Lemma 4.1.14 applied to $\mu_h = (-\nabla v_j, 0)$ and $\mu = (-Dv, 0)$ we get that $|(-\nabla v_j, 0)|_K \overset{*}{\rightharpoonup} |(-Dv, 0)|_K$. Thus, since $1_{F[v_j]} \to 1_{F[v]}$ $\mathcal{L}^n$-a.e. and $\mathbf{p}(\mathrm{supp}(f))$ is a compact subset of $\Omega$, we can take the lim sup in both side of the above inequality as $j$ goes to infinity and, recalling [32, Proposition 4.26] we get

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}} 1_{F[v]} \mathrm{div} f dx &\leq \int_{\mathbf{p}(\mathrm{supp} f)} \phi_{K^s}\left(-\frac{dDv}{d|Dv|}(x'), 0\right) d|Dv|(x') + \int_{\Omega\times\mathbb{R}} 1_{F[v]} \frac{\partial f_n}{\partial x_n} dx \\
&\leq \int_{\Omega} \phi_{K^s}\left(-\frac{dDv}{d|Dv|}(x'), 0\right) d|Dv|(x') + |\mathbf{q}(D1_{F[v]})|(\Omega\times\mathbb{R}).
\end{aligned}$$

The last inequality holds whenever $\Omega \subset \mathbb{R}^n$ is an open set and hence, we deduce that it holds true also for any Borel set. Finally, using the characterization of the anisotropic total variation (see Theorem 4.2.1) and [14, Lemma 3.1], we deduce that

$$\int_B \phi_{K^s}\left(-\frac{dDv}{d|Dv|}(x'), 0\right) d|Dv|(x') \leq P_{K^s}(E; B\times\mathbb{R}) \qquad \forall\, B \subset \mathbb{R}^{n-1} \textit{ Borel},$$

and this concludes the proof. □

We can now prove (AS).

*Proof of Theorem 1.3.1.* We divide the proof in two steps.

**Step 1** Let us consider first $B \subset \left(G_{F[v]} \cap G_E\right)$, where $G_{F[v]}$ and $G_E$ are sets given by

Theorem 4.1.1 for $F[v]$ and $E$ respectively. Then, by equation (4.1.4) we get

$$\begin{aligned} P_{K^s}(F[v];(B\times\mathbb{R}) &= \int_{\partial^* F[v]\cap(B\times\mathbb{R})} \phi_{K^s}(\nu^{F[v]}(x))d\mathcal{H}^{n-1}(x) \\ &= \int_B dz \int_{(\partial^* F[v])_z} \frac{\phi_{K^s}(\nu^{F[v]}(z,y))}{|\mathbf{q}(\nu^{F[v]}(z,y))|} d\mathcal{H}^0(y) \\ &= \int_B dz \int_{(\partial^* F[v])_z} \phi_{K^s}\left(\frac{\nu^{F[v]}(z,y))}{|\mathbf{q}(\nu^{F[v]}(z,y))|}\right) d\mathcal{H}^0(y), \qquad (4.3.2) \\ & \qquad (4.3.3) \end{aligned}$$

where the last equality holds true thanks to the one-homogeneity of $\phi_{K^s}$. Thanks to Lemma 4.1.3 we observe that,

$$\begin{aligned} \frac{\nu^{F[v]}(z,y))}{|\mathbf{q}(\nu^{F[v]}(z,y))|} &= \left(\frac{\nu_1^{F[v]}(z,y))}{|\mathbf{q}(\nu^{F[v]}(z,y))|}\dots,\frac{\nu_{n-1}^{F[v]}(z,y))}{|\mathbf{q}(\nu^{F[v]}(z,y))|},\frac{\mathbf{q}(\nu^{F[v]}(z,y))}{|\mathbf{q}(\nu^{F[v]}(z,y))|}\right) \\ &= \left(-\frac{1}{2}\frac{\partial v(z)}{\partial x_1},\dots,-\frac{1}{2}\frac{\partial v(z)}{\partial x_{n-1}},\frac{\mathbf{q}(\nu^{F[v]}(z,y))}{|\mathbf{q}(\nu^{F[v]}(z,y))|}\right), \quad (z,y)\in\partial^* F[v]. \quad (4.3.4) \end{aligned}$$

Thanks to Theorem 4.1.1 we have that $(\partial^* E)_z = \partial^* E_z$. Calling $N(z) = \mathcal{H}^0((\partial^* E)_z)$, we know that thanks to the isoperimetric inequality in $\mathbb{R}$, $N(z)\geq 2$ for $\mathcal{H}^{n-1}$ a.e. $z\in\{v>0\}$. Thus,

$$\begin{aligned} &P_{K^s}(F[v];B\times\mathbb{R}) = \int_B dz \int_{(\partial^* F[v])_z} \phi_{K^s}\left(\frac{\nu^{F[v]}(z,y))}{|\mathbf{q}(\nu^{F[v]}(z,y))|}\right) d\mathcal{H}^0(y) \\ &= \int_B dz \int_{(\partial^* F[v])_z} \phi_{K^s}\left(-\frac{1}{2}\frac{\partial v(z)}{\partial x_1},\dots,-\frac{1}{2}\frac{\partial v(z)}{\partial x_{n-1}},1\right) d\mathcal{H}^0(y) \\ &= \int_B 2\phi_{K^s}\left(-\frac{1}{2}\frac{\partial v(z)}{\partial x_1},\dots,-\frac{1}{2}\frac{\partial v(z)}{\partial x_{n-1}},1\right) dz = \int_B \phi_{K^s}\left(-\frac{\partial v(z)}{\partial x_1},\dots,-\frac{\partial v(z)}{\partial x_{n-1}},2\right) dz \\ &= \int_B \phi_{K^s}\left(\int_{\partial^* E_z}\frac{\nu_1^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|}d\mathcal{H}^0,\dots,\int_{\partial^* E_z}\frac{\nu_{n-1}^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|}d\mathcal{H}^0, 2\right) dz \qquad (4.3.5) \\ &= \int_B N(z)\phi_{K^s}\left(\fint_{\partial^* E_z}\frac{\nu_1^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|}d\mathcal{H}^0,\dots,\fint_{\partial^* E_z}\frac{\nu_{n-1}^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|}d\mathcal{H}^0,\fint_{\partial^* E_z}\frac{2}{N(z)}d\mathcal{H}^0\right) dz \\ &\leq \int_B dz \int_{\partial^* E_z} \phi_{K^s}\left(\frac{\nu_1^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|},\dots,\frac{\nu_{n-1}^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|},\frac{2}{N(z)}\right) d\mathcal{H}^0(y) \\ &\leq \int_B dz \int_{\partial^* E_z} \phi_{K^s}\left(\frac{\nu_1^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|},\dots,\frac{\nu_{n-1}^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|},1\right) d\mathcal{H}^0(y) \\ &= \int_B dz \int_{\partial^* E_z} \phi_{K^s}\left(\frac{\nu_1^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|},\dots,\frac{\nu_{n-1}^E(z,y))}{|\mathbf{q}(\nu^E(z,y))|},\frac{\mathbf{q}(\nu^E(z,y)))}{|\mathbf{q}(\nu^E(z,y))|}\right) d\mathcal{H}^0(y) \\ &= P_{K^s}(E;B\times\mathbb{R}), \end{aligned}$$

where in the first line we used (4.3.2), in the second line we used (4.3.4), from line 5 to line 6 we used Jensen inequality, from line 6 to line 7 we used Lemma 4.1.29 and from line 7 to line 8 we used the symmetric properties of $K^s$. This finishes the proof of the first step.

**Step 2** We consider $B \subset \left(\mathbb{R}^{n-1} \setminus \left(G_{F[v]} \cap G_E\right)\right)$. Using Coarea formula we get,

$$
\begin{aligned}
|\mathbf{q}(D1_{F[v]})|(B \times \mathbb{R}) &= \int_{\partial^* F[v] \cap (B \times \mathbb{R})} |\mathbf{q}(\nu^{F[v]}(x))| d\mathcal{H}^{n-1}(x) = \int_B \mathcal{H}^0((\partial^* F[v])_z) dz \\
&= \int_{B \cap \{v>0\}} \mathcal{H}^0((\partial^* F[v])_z) dz + \int_{B \setminus \{v>0\}} \mathcal{H}^0((\partial^* F[v])_z) dz = 0
\end{aligned}
$$

where for the last equality we used that $\mathcal{L}^{n-1}(B) = 0$ together with $\mathcal{H}^0((\partial^* F[v])_z) = 0$ for all $z \in B \setminus \{v > 0\}$. Putting together this result with the auxiliary anisotropic perimeter inequality (4.3.1) we obtain that

$$
P_{K^s}(F[v]; B \times \mathbb{R}) \leq P_{K^s}(E; B \times \mathbb{R}).
$$

This concludes the second step. The proof of (1.3.1) follows on splitting $B$ into $B \cap \left(G_{F[v]} \cap G_E\right)$ and $B \setminus \left(G_{F[v]} \cap G_E\right)$ and using step 1 and step 2 respectively. $\square$

## 4.4 A formula for the anisotropic perimeter

Through all this section, given $u \in BV_{loc}(\mathbb{R}^{n-1})$ we consider $\eta := (Du, -\mathcal{L}^{n-1})$ a $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^{n-1}$.

**Theorem 4.4.1.** *Let $K \subset \mathbb{R}^n$ as in (1.3.3) and let $u \in BV_{loc}(\mathbb{R}^{n-1})$, then*

$$
|\eta|_K(B) = |D1_{\Sigma^u}|_K(B \times \mathbb{R}) \quad \forall B \subset \mathbb{R}^{n-1} \text{ Borel.}
$$

*Proof.* Thanks to Theorem 4.2.1, the identity follows from a careful inspection of the proof of [25, Theorem 1 (Section 1.5)]. It is important to notice that in the present situation one should replace condition $|\varphi| \leq 1$ with $\phi^*_{K^s}(\varphi) \leq 1$ with $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$. $\square$

We recall now an important result concerning how to determine $\nu^{\Sigma^u}$ i.e. the outer normal to the reduced boundary of the subgraph of the function $u$. Recall that thanks to Radon-Nykodym Theorem we have

$$
Du = D^a u + D^j u + D^c u.
$$

With a little abuse of notation let us call $D^{ac}u = D^a u + D^c u$, so that

$$
D^c u = D^{ac} u \llcorner Z_u
$$

where,

$$
Z_u = \left\{ x \in \Omega : \frac{d|D^{ac}u|}{dL^{n-1}}(x) = +\infty \right\}.
$$

**Theorem 4.4.2.** *Let $u \in BV(\Omega)$ with $\Omega \subset \mathbb{R}^{n-1}$ open and bounded, then*

*i) for $|\eta|$-a.e. $x \in \Omega \setminus J_u$ we have*

$$\frac{d\eta}{d|\eta|}(x) = -\nu^{\Sigma^u}(x, u(x)),$$

*ii) for $|\eta|$-a.e. $x \in J_u$ we have*

$$\frac{d\eta}{d|\eta|}(x) = \left(\frac{dD^j u}{d|D^j u|}(x), 0\right) = (\nu_u(x), 0) = -\nu^{\Sigma^u}(x, y) \quad \forall\, y\, s.t.\, (x, y) \in \partial^* \Sigma^u,$$

*iii) for $|\eta|$-a.e. $x \in \Big((\Omega \setminus J_u) \cap \Big\{x \in \Omega : \boldsymbol{q}\nu^{\Sigma^u}(x, u^\vee(x)) = 0\Big\}\Big)$ we have*

$$\frac{d\eta}{d|\eta|}(x) = \left(\frac{dD^c u}{d|D^c u|}(x), 0\right).$$

*Proof.* Statement $(i)$ is proved in $(i)$ of [25, Theorem 4, section 4.5]. Statement $(ii)$ follows by combining $(ii)$ of [25, Theorem 4, section 4.5] with $(ii)$ of [25, Theorem 3, section 4.5]. We will give a proof of point $iii)$. Let $x \in \Omega$ and consider $\rho > 0$, then

$$\begin{aligned}
|\eta|(D_{x,\rho}) &= \sup_{\substack{|f| \le 1 \\ f \in C_c^0(D_{x,\rho}, \mathbb{R}^n)}} \int_{D_{x,\rho}} f(y) \cdot d\eta(y) \\
&= \sup_{\substack{|f| \le 1 \\ f \in C_c^0(D_{x,\rho}, \mathbb{R}^n)}} \left(\int_{D_{x,\rho}} (f_1(y), \dots, f_{n-1}(y)) \cdot dDu(y) - \int_{D_{x,\rho}} f_n(y) dy\right) \\
&\le \sup_{\substack{|f| \le 1 \\ f \in C_c^0(D_{x,\rho}, \mathbb{R}^n)}} \int_{D_{x,\rho}} (f_1(y), \dots, f_{n-1}(y)) \cdot dDu(y) + \sup_{\substack{|f| \le 1 \\ f \in C_c^0(D_{x,\rho}, \mathbb{R}^n)}} \int_{D_{x,\rho}} f_n(y) dy \\
&= |Du|(D_{x,\rho}) + \mathcal{L}^{n-1}(D_{x,\rho}).
\end{aligned}$$

At the same time we get

$$\begin{aligned}
|\eta|(D_{x,\rho}) &= \sup_{\substack{|f| \le 1 \\ f \in C_c^0(D_{x,\rho}, \mathbb{R}^n)}} \int_{D_{x,\rho}} f(y) \cdot d\eta(y) \\
&\ge \int_{D_{x,\rho}} (f_1(y), \dots, f_{n-1}(y)) \cdot dDu(y) \\
&\ge |Du|(D_{x,\rho}),
\end{aligned}$$

where the last inequality is obtained passing to the sup in the right hand side. Putting together these two inequalities we get

$$|Du|(D_{x,\rho}) \le |\eta|(D_{x,\rho}) \le |Du|(D_{x,\rho}) + \mathcal{L}^{n-1}(D_{x,\rho}). \tag{4.4.1}$$

Let now $x \in Z$ and let $\rho > 0$. Then,

$$\frac{\eta(D_{x,\rho})}{|\eta|(D_{x,\rho})} = \frac{\eta(D_{x,\rho})}{|Du|(D_{x,\rho})} \frac{|Du|(D_{x,\rho})}{|\eta|(D_{x,\rho})}.$$

Since

$$\lim_{\rho\to 0^+} \frac{\eta(D_{x,\rho})}{|Du|(D_{x,\rho})} = \left(\frac{dD^c u}{d|D^c u|}(x), 0\right),$$

we are left to prove that

$$\lim_{\rho\to 0^+} \frac{|Du|(D_{x,\rho})}{|\eta|(D_{x,\rho})} = 1. \tag{4.4.2}$$

Thanks to (4.4.1) we have

$$\frac{|Du|(D_{x,\rho})}{|Du|(D_{x,\rho}) + |D_{x,\rho}|} \leq \frac{|Du|(D_{x,\rho})}{|\eta|(D_{x,\rho})} \leq \frac{|Du|(D_{x,\rho})}{|Du|(D_{x,\rho})} = 1. \tag{4.4.3}$$

Recall that $x \in Z$, so that

$$\lim_{\rho\to 0^+} \frac{|D_{x,\rho}|}{|Du|(D_{x,\rho})} = 0.$$

Thus, we can calculate the following limit for the left hand side of (4.4.3)

$$\lim_{\rho\to 0^+} \frac{|Du|(D_{x,\rho})}{|Du|(D_{x,\rho}) + |D_{x,\rho}|} = \lim_{\rho\to 0^+} \frac{1}{1 + \frac{|D_{x,\rho}|}{|Du|(D_{x,\rho})}} = 1.$$

By the above calculation and relation (4.4.3) we proved (4.4.2) and so we conclude the proof. $\square$

**Proposition 4.4.3.** *Let $u \in BV_{loc}(\mathbb{R}^{n-1})$ and let $K \subset \mathbb{R}^n$ be as in (1.3.3). Then, for every Borel set $B \subset \mathbb{R}^{n-1}$ we have*

$$\begin{aligned} P_K(\Sigma^u; B \times \mathbb{R}) = &\int_{B\setminus(J_u\cup Z_u)} \phi_K(-\nabla u(x), 1)dx \\ &+ \int_{B\cap J_u} [u](x)\phi_K\left(-\frac{dD^j u}{d|D^j u|}(x), 0\right) d\mathcal{H}^{n-2}(x) \\ &+ \int_{B\cap Z_u} \phi_K\left(-\frac{dD^c u}{d|D^c u|}(x), 0\right) d|D^c u|(x), \end{aligned} \tag{4.4.4}$$

*where $Z_u$ has been defined at the beginning of this Section.*

*Proof.* Let us consider a generic Borel set $B \subset \mathbb{R}^{n-1}$. Then, thanks to the De Giorgi structure Theorem and Theorem 4.4.1 we get

$$\begin{aligned} P_K(\Sigma^u; B \times \mathbb{R}) &= \int_{\partial^*\Sigma^u\cap(B\times\mathbb{R})} \phi_K(\nu^{\Sigma^u}(x))d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial^*\Sigma^u\cap(B\times\mathbb{R})} \phi_K\left(-\frac{dD1_{\Sigma^u}}{d|D1_{\Sigma^u}|}(x)\right) d|D1_{\Sigma^u}|(x) \\ &= \int_B \phi_K\left(-\frac{d\eta}{d|\eta}(x)\right) d|\eta|(x). \end{aligned}$$

Let us split the last integral in the following way

$$\int_B \phi_K\left(-\frac{d\eta}{d|\eta}(x)\right) d|\eta|(x) = \int_{B\setminus(J_u\cup Z)} \phi_K\left(-\frac{d\eta}{d|\eta}(x)\right) d|\eta|(x) \tag{4.4.5}$$

$$+ \int_{B\cap J_u} \phi_K\left(-\frac{d\eta}{d|\eta}(x)\right) d|\eta|(x) \tag{4.4.6}$$

$$+ \int_{B\cap Z} \phi_K\left(-\frac{d\eta}{d|\eta}(x)\right) d|\eta|(x). \tag{4.4.7}$$

About the first integral on the right hand side we observe that

$$\eta \llcorner \mathbb{R}^{n-1} \setminus (J_u \cup Z_u) = (D^a u, -\mathcal{L}^{n-1}) \llcorner \mathbb{R}^{n-1} = (\nabla u, -\mathcal{L}^{n-1}) \llcorner \mathbb{R}^{n-1}.$$

Therefore, recalling Remark 2.0.1 we have

$$\eta(B) = \int_B (\nabla u, -1)dx \quad \textit{and} \quad |\eta|(B) = \int_B \sqrt{|\nabla u|^2 + 1}dx \quad \forall \textit{ Borel set } B \subset \mathbb{R}^{n-1} \setminus (J_u \cup Z).$$

Thus,

$$\int_{B\setminus(J_u\cup Z)} \phi_K\left(-\frac{d\eta}{d|\eta|}(x)\right) d|\eta|(x) = \int_{B\setminus(J_u\cup Z)} \phi_K\left(\frac{(-\nabla u(x), 1)}{\sqrt{|\nabla u|^2+1}}\right) \sqrt{|\nabla u|^2+1}\, dx$$

$$= \int_{B\setminus(J_u\cup Z)} \phi_K(-\nabla u(x), 1)\, dx. \tag{4.4.8}$$

Let us observe now that, thanks to $(ii)$ of Theorem 4.4.2

$$\eta \llcorner J_u = (D^j u, -\mathcal{L}^{n-1}) \llcorner J_u = (D^j u, 0) \llcorner J_u.$$

Thus,

$$|\eta|(B) = |D^j u|(B) \quad \forall \textit{ Borel set } B \subset J_u.$$

Then,

$$\int_{B\cap J_u} \phi_K\left(-\frac{d\eta}{d|\eta|}(x)\right) d|\eta|(x) = \int_{B\cap J_u} \phi_K\left(-\frac{dD^j u}{d|D^j u|}(x), 0\right) d|D^j u|(x)$$

$$= \int_{B\cap J_u} \phi_K\left(-\frac{dD^j u}{d|D^j u|}(x), 0\right) [u](x) d\mathcal{H}^{n-2}(x). \tag{4.4.9}$$

A similar argument holds for the integral over $B \cap Z_u$, so that

$$\int_{B\cap Z_u} \phi_K\left(-\frac{d\eta}{d|\eta|}(x)\right) d|\eta|(x) = \int_{B\cap Z_u} \phi_K\left(-\frac{dD^c u}{d|D^c u|}(x), 0\right) d|D^c u|(x). \tag{4.4.10}$$

Combining equations (4.4.5), (4.4.8), (4.4.9) and (4.4.10) we conclude. □

**Remark 4.4.4.** *We can also use the notation of the anisotropic total variation to obtain a more compact formula for the perimeter,*

$$P_K(\Sigma^u; B \times \mathbb{R}) = \int_B \phi_K(-\nabla u(x), 1)dx + |(-D^j u, 0)|_K(B) + |(-D^c u, 0)|_K(B).$$

**Remark 4.4.5.** *Note that, since* $\Sigma_u = \mathbb{R}^n \setminus \Sigma^u$*, we have* $\partial^*\Sigma_u = \partial^*\Sigma^u$ *and* $\nu^{\Sigma_u}(x) = -\nu^{\Sigma^u}(x)$ *for* $\mathcal{H}^{n-1}$*-a.e.* $x \in \partial^*\Sigma_u$*, and so*

$$P_K(\Sigma_u; B \times \mathbb{R}) = \int_B \phi_K(\nabla u(x), -1)dx + |(D^j u, 0)|_K(B) + |(D^c u, 0)|_K(B)$$

*for every Borel set* $B \subset \mathbb{R}^{n-1}$*. Note that, although* $\Sigma_u = \mathbb{R}^n \setminus \Sigma^u$*, since* $\phi_K$ *is not a norm, it might be that* $P_K(\Sigma_u; B \times \mathbb{R}) \neq P_K(\Sigma^u; B \times \mathbb{R})$*. Indeed, let us consider the following example.*

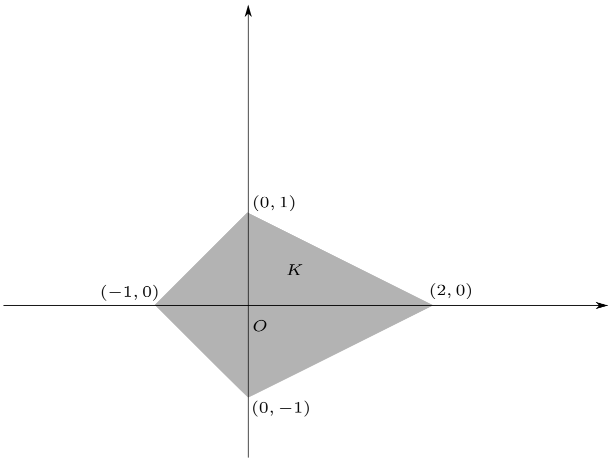


Figure 4.4.1: Since this Wulff shape is not symmetric with respect to the origin we can construct examples where $P_K(\Sigma_u; B \times \mathbb{R}) \neq P_K(\Sigma^u; B \times \mathbb{R})$.

*Let us consider* $K \subset \mathbb{R}^2$ *as shown in the figure above and let* $\phi_K$ *be its surface tension defined as*

$$\phi_K(x) := \begin{cases} \max\{|\boldsymbol{p}x|, |\boldsymbol{q}x|\} & \text{if } \boldsymbol{p}x < 0 \\ \max\{2|\boldsymbol{p}x|, |\boldsymbol{q}x|\} & \text{if } \boldsymbol{p}x \geq 0. \end{cases}$$

*Let us consider as* $u \in BV_{loc}(\mathbb{R})$ *the following function*

$$u(x) := \begin{cases} 2 & \text{if } x > 0 \\ 1 & \text{if } x < 0. \end{cases}$$

*Then, fixing* $B = (-1, 1)$ *we have*

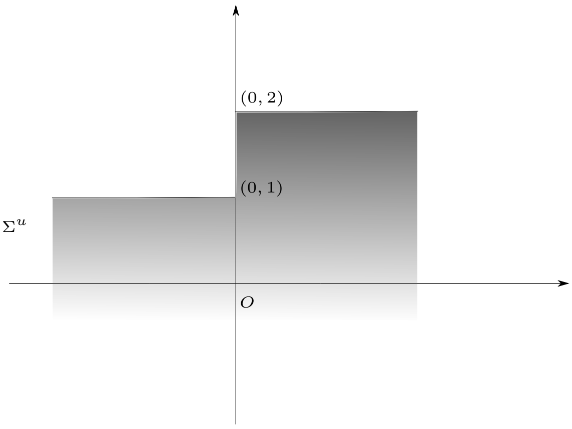


Figure 4.4.2: A pictorial idea of $\Sigma^u$.

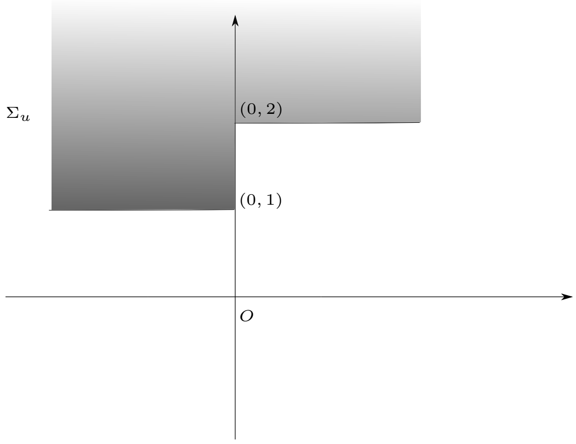


Figure 4.4.3: A pictorial idea of $\Sigma_u$.

$$P_K(\Sigma_u; B\times\mathbb{R}) := 1\phi_K((0,-1)) + 1\phi_K((1,0)) + 1\phi_K((0,-1)) = 4,$$

$$P_K(\Sigma^u; B\times\mathbb{R}) := 1\phi_K((0,1)) + 1\phi_K((-1,0)) + 1\phi_K((0,1)) = 3.$$

**Lemma 4.4.6.** *Let $K\subset\mathbb{R}^n$ be as in (1.3.3). If $u_1,u_2\in BV_{loc}(\mathbb{R}^{n-1})$ with $u_1\le u_2$ and $E=\Sigma_{u_1}\cap\Sigma^{u_2}$ has finite volume, then $E$ is a set of locally finite perimeter in $\mathbb{R}^n$ and for every Borel set $B\subset\mathbb{R}^{n-1}$*

$$\begin{aligned}P_K(E;B\times\mathbb{R}) = &\int_{B\cap\{\widetilde{u_1}<\widetilde{u_2}\}}\phi_K(\nabla u_1(x),-1)dx + \int_{B\cap\{\widetilde{u_1}<\widetilde{u_2}\}}\phi_K(-\nabla u_2(x),1)dx \\ &+\int_{B\cap J_{u_1}}\phi_K\left(\nu_{u_1}(z),0\right)\left(\min(u_1^\vee(z),u_2^\wedge(z))-u_1^\wedge(z)\right)d\mathcal{H}^{n-2}(z)\\ &+\int_{B\cap J_{u_2}}\phi_K\left(\nu_{u_2}(z),0\right)\left(u_2^\vee(z)-\max(u_2^\wedge(z),u_1^\vee(z))\right)d\mathcal{H}^{n-2}(z)\\ &+|(D^cu_1,0)|_K(B\cap\{\widetilde{u_1}<\widetilde{u_2}\})+|(-D^cu_2,0)|_K(B\cap\{\widetilde{u_1}<\widetilde{u_2}\})\end{aligned}\tag{4.4.11}$$

*Proof.* We will follow the strategy of [11, Theorem 3.1]. By [32, Theorem 16.3], if $F_1$, $F_2$ are sets of locally finite perimeter in $\mathbb{R}^n$, then

$$\partial^*(F_1\cap F_2) =_{\mathcal{H}^{n-1}} \left(F_1^{(1)}\cap\partial^*F_2\right)\cup\left(F_2^{(1)}\cap\partial^*F_1\right)\cup\left(\partial^*F_1\cap\partial^*F_2\cap\{\nu^{F_1}=\nu^{F_2}\}\right).\tag{4.4.12}$$

Moreover, in the particular case of $F_1\subset F_2$, then $\nu^{F_1}=\nu^{F_2}$ $\mathcal{H}^{n-1}$-a.e. on $\partial^*F_1\cap\partial^*F_2$. Let us observe that $u_1\le u_2$ implies $\Sigma_{u_2}\subset\Sigma_{u_1}$ and that $\Sigma^{u_2}=\mathbb{R}^n\setminus\Sigma_{u_2}$ implying $\mu_{\Sigma_{u_2}}=-\mu_{\Sigma^{u_2}}$. We thus find

$$\nu^{\Sigma_{u_1}} = -\nu^{\Sigma^{u_2}},\quad \mathcal{H}^{n-1}\text{-a.e. on }\partial^*\Sigma_{u_1}\cap\partial^*\Sigma^{u_2}.\tag{4.4.13}$$

By, (4.4.12) and (4.4.13), since $E=\Sigma_{u_1}\cap\Sigma^{u_2}$ we find

$$\partial^*E =_{\mathcal{H}^{n-1}} \left(\partial^*\Sigma_{u_1}\cap(\Sigma^{u_2})^{(1)}\right)\cup\left(\partial^*\Sigma^{u_2}\cap(\Sigma_{u_1})^{(1)}\right).$$

Thanks to [25, Section 4.1.5] we know that $\Sigma_{u_1}$ and $\Sigma^{u_2}$ are sets of locally finite perimeter in $\mathbb{R}^n$ with

$$\partial^*\Sigma_{u_1}^{(1)} \cap (S_{u_1}^c \times \mathbb{R}) =_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : \tilde{u}_1(\mathbf{p}x) = \mathbf{q}x\}, \tag{4.4.14}$$

$$\partial^*\Sigma_{u_1}^{(1)} \cap (S_{u_1} \times \mathbb{R}) =_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : u_1^\wedge(\mathbf{p}x) < \mathbf{q}x < u_1^\vee(\mathbf{p}x)\}, \tag{4.4.15}$$

$$\Sigma_{u_1}^{(1)} \cap (S_{u_1}^c \times \mathbb{R}) =_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : \tilde{u}_1(\mathbf{p}x) < \mathbf{q}x\}, \tag{4.4.16}$$

$$\Sigma_{u_1}^{(1)} \cap (S_{u_1} \times \mathbb{R}) =_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : u_1^\vee(\mathbf{p}x) < \mathbf{q}x\}, \tag{4.4.17}$$

$$(\Sigma^{u_2})^{(1)} \cap (S_{u_1}^c \times \mathbb{R}) =_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : \tilde{u}_2(\mathbf{p}x) > \mathbf{q}x\}, \tag{4.4.18}$$

$$(\Sigma^{u_2})^{(1)} \cap (S_{u_1} \times \mathbb{R}) =_{\mathcal{H}^{n-1}} \{x \in \mathbb{R}^n : u_2^\wedge(\mathbf{p}x) > \mathbf{q}x\}. \tag{4.4.19}$$

We now focus on the set $\partial^*\Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)}$. Observe that,

$$\begin{aligned} P_K\left(\Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap (B \times \mathbb{R})\right) &= P_K\left(\Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1}^c \cap J_{u_2}^c) \times \mathbb{R}]\right) \\ &+ P_K\left(\Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1} \cap J_{u_2}^c) \times \mathbb{R}]\right) \\ &+ P_K\left(\Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1} \cap J_{u_2}) \times \mathbb{R}]\right) \\ &+ P_K\left(\Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1}^c \cap J_{u_2}) \times \mathbb{R}]\right). \end{aligned}$$

Applying (4.4.14) to $u_1$ and (4.4.18) to $u_2$ we find

$$\left(\partial^*\Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)}\right) \cap ((J_{u_1}^c \cap J_{u_2}^c) \times \mathbb{R}) =_{\mathcal{H}^{n-1}} \{(z, \tilde{u}_1(z)) : z \in (J_{u_1}^c \cap J_{u_2}^c),\, \tilde{u}_1(z) < \tilde{u}_2(z)\}. \tag{4.4.20}$$

Applying (4.4.15) to $u_1$ and (4.4.18) to $u_2$ we obtain

$$\begin{aligned} &\left(\partial^*\Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)}\right) \cap ((J_{u_1} \cap J_{u_2}^c) \times \mathbb{R}) \\ &=_{\mathcal{H}^{n-1}} \{(z,t) : z \in (J_{u_1} \cap J_{u_2}^c),\, u_1^\wedge(z) < t < \min(u_1^\vee(z), \widetilde{u}_2(z))\}. \end{aligned} \tag{4.4.21}$$

Combining (4.4.15) to $u_1$ and (4.4.19) to $u_2$ we obtain

$$\begin{aligned} &\left(\partial^*\Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)}\right) \cap ((J_{u_1} \cap J_{u_2}) \times \mathbb{R}) \\ &=_{\mathcal{H}^{n-1}} \{(z,t) : z \in (J_{u_1} \cap J_{u_2}),\, u_1^\wedge(z) < t < \min(u_1^\vee(z), u_2^\wedge(z))\}. \end{aligned} \tag{4.4.22}$$

Finally, applying (4.4.14) to $u_1$ and (4.4.19) to $u_2$ we get

$$\begin{aligned} &\left(\partial^*\Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)}\right) \cap ((J_{u_1}^c \cap J_{u_2}) \times \mathbb{R}) \\ &=_{\mathcal{H}^{n-1}} \{(z, \widetilde{u}_1(z)) : z \in (J_{u_1}^c \cap J_{u_2}),\, \widetilde{u}_1(z) < u_2^\wedge(z)\}. \end{aligned} \tag{4.4.23}$$

Thus, thanks to Remark 4.4.5 and (4.4.20) we get

$$\begin{aligned} P_K\left(\Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1}^c \cap J_{u_2}^c) \times \mathbb{R}]\right) &= \int_{\partial^*\Sigma_{u_1} \cap \left[(B \cap J_{u_1}^c \cap J_{u_2}^c \cap \{\widetilde{u_1} < \widetilde{u_2}\}) \times \mathbb{R}\right]} \phi_K(-\nu^{\Sigma^{u_1}}(x)) d\mathcal{H}^{n-1}(x) \\ &= \int_{B \cap \{\widetilde{u_1} < \widetilde{u_2}\}} \phi_K(\nabla u_1(x), 1) dx + |(D^c u_1, 0)|_K(B \cap \{\widetilde{u_1} < \widetilde{u_2}\}). \end{aligned}$$

Using Fubini theorem and (4.4.21) we get

$$\begin{aligned}
&P_K\left(\Sigma_{u_1};(\Sigma^{u_2})^{(1)}\cap[(B\cap J_{u_1}\cap J^c_{u_2})\times\mathbb{R}]\right)\\
&=\int_{\partial^*\Sigma_{u_1}\cap[((\Sigma^{u_2})^{(1)}\cap B\cap J_{u_1}\cap J^c_{\tilde{u}_2})\times\mathbb{R}]}\phi_K(-\nu^{\Sigma^{u_1}}(y))d\mathcal{H}^{n-1}(y)\\
&=\int_{\{x\in\mathbb{R}^n:\,\mathbf{p}x\in B\cap J_{u_1}\cap J^c_{\tilde{u}_2},\,u_1^\wedge(\mathbf{p}x)<\mathbf{q}x<\min(u_1^\vee(\mathbf{p}x),\widetilde{u}_2(\mathbf{p}x))\}}\phi_K(-\nu^{\Sigma^{u_1}}(y))d\mathcal{H}^{n-1}(y)\\
&=\int_{(B\cap J_{u_1}\cap J^c_{\tilde{u}_2})\times\mathbb{R}}\phi_K(-\nu^{\Sigma^{u_1}}(y))1_{\{\mathbf{q}x>u_1^\wedge(\mathbf{p}x)\}}(y)1_{\{\mathbf{q}x<\min(u_1^\vee(\mathbf{p}x),\widetilde{u}_2(\mathbf{p}x))\}}(y)d\mathcal{H}^{n-1}(y)\\
&=\int_{B\cap J_{u_1}\cap J^c_{\tilde{u}_2}}d\mathcal{H}^{n-2}(z)\int_{\mathbb{R}}\phi_K(-\nu^{\Sigma^{u_1}}(z,t))1_{\{s>u_1^\wedge(z)\}}(z,t)1_{\{s<\min(u_1^\vee(z),\widetilde{u}_2(z))\}}(z,t)d\mathcal{H}^1(t)\\
&=\int_{B\cap J_{u_1}\cap J^c_{\tilde{u}_2}}d\mathcal{H}^{n-2}(z)\int_{\mathbb{R}}\phi_K(\nu_{u_1}(z),0)1_{\{t>u_1^\wedge(z)\}}(z,t)1_{\{t<\min(u_1^\vee(z),\widetilde{u}_2(z))\}}(z,t)d\mathcal{H}^1(t)\\
&=\int_{B\cap J_{u_1}\cap J^c_{\tilde{u}_2}}\phi_K\left(\nu_{u_1}(z),0\right)\left(\min(u_1^\vee(z),\widetilde{u}_2(z))-u_1^\wedge(z)\right)d\mathcal{H}^{n-2}(z).
\end{aligned}$$

Observe that we could have used $u_2^\wedge$ or $u_2^\vee$ instead of $\widetilde{u_2}$ since we are working in $B\cap J_{u_1}\cap J^c_{\tilde{u}_2}$. For similar arguments, using (4.4.22) we get that

$$\begin{aligned}
&P_K\left(\Sigma_{u_1};(\Sigma^{u_2})^{(1)}\cap[(B\cap J_{u_1}\cap J_{u_2})\times\mathbb{R}]\right)\\
&=\int_{B\cap J_{u_1}\cap J_{u_2}}\phi_K\left(\nu_{u_1}(z),0\right)\left(\min(u_1^\vee(z),u_2^\wedge(z))-u_1^\wedge(z)\right)d\mathcal{H}^{n-2}(z).
\end{aligned}$$

Furthermore, thanks to (4.4.23) we deduce that $\mathcal{H}^{n-1}\left(\partial^*\Sigma_{u_1}\cap(\Sigma^{u_2})^{(1)}\cap(J^c_{u_1}\cap J_{u_2})\times\mathbb{R}]\right)=0$. Thus, we have that

$$P_K\left(\Sigma_{u_1};(\Sigma^{u_2})^{(1)}\cap[(B\cap J^c_{u_1}\cap J_{u_2})\times\mathbb{R}]\right)=0.$$

Therefore,

$$\begin{aligned}
P_K\left(\Sigma_{u_1};(\Sigma^{u_2})^{(1)}\cap(B\times\mathbb{R})\right)=&\int_{B\cap\{\widetilde{u_1}<\widetilde{u_2}\}}\phi_K(\nabla u_1(x),-1)dx &(4.4.24)\\
&+\int_{B\cap J_{u_1}}\phi_K\left(\nu_{u_1}(z),0\right)\left(\min(u_1^\vee(z),u_2^\wedge(z))-u_1^\wedge(z)\right)d\mathcal{H}^{n-2}(z)\\
&+|(D^cu_1,0)|_K(B\cap\{\widetilde{u_1}<\widetilde{u_2}\}). &(4.4.25)
\end{aligned}$$

By symmetry, we got that

$$\begin{aligned}
P_K\left(\Sigma^{u_2};(\Sigma_{u_1})^{(1)}\cap(B\times\mathbb{R})\right)=&\int_{B\cap\{\widetilde{u_1}<\widetilde{u_2}\}}\phi_K(-\nabla u_2(x),1)dx &(4.4.26)\\
&+\int_{B\cap J_{u_2}}\phi_K\left(\nu_{u_2}(z),0\right)\left(u_2^\vee(z)-\max(u_2^\wedge(z),u_1^\vee(z))\right)d\mathcal{H}^{n-2}(z)\\
&+|(-D^cu_2,0)|_K(B\cap\{\widetilde{u_1}<\widetilde{u_2}\}).
\end{aligned}$$

Putting together (4.4.24) and (4.4.26) we obtain the formula for $P_K(E;B\times\mathbb{R})$. □

We now extend Lemma 4.4.6 to the case of $GBV$ functions.

**Theorem 4.4.7.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3). If $u_1, u_2 \in GBV(\mathbb{R}^{n-1})$ with $u_1 \le u_2$ and $E = \Sigma_{u_1} \cap \Sigma^{u_2}$ has finite volume, then $E$ is a set of locally finite perimeter and for every Borel set $B \subset \mathbb{R}^{n-1}$*

$$\begin{aligned} P_K(E; B\times\mathbb{R}) = &\int_{B\cap\{u_1<u_2\}} \phi_K(\nabla u_1(x), -1)dx + \int_{B\cap\{u_1<u_2\}} \phi_K(-\nabla u_2(x), 1)dx \\ &+ \int_{B\cap J_{u_1}} \phi_K\left(\nu_{u_1}(z), 0\right)\left(\min(u_1^\vee(z), u_2^\wedge(z)) - u_1^\wedge(z)\right) d\mathcal{H}^{n-2}(z) \\ &+ \int_{B\cap J_{u_2}} \phi_K\left(-\nu_{u_2}(z), 0\right)\left(u_2^\vee(z) - \max(u_2^\wedge(z), u_1^\vee(z))\right) d\mathcal{H}^{n-2}(z) \\ &+ |(D^c u_1, 0)|_K(B\cap\{\tilde{u_1} < \tilde{u_2}\}) + |(-D^c u_2, 0)|_K(B\cap\{\tilde{u_1} < \tilde{u_2}\}). \end{aligned} \tag{4.4.27}$$

*Proof.* To prove (4.4.27) it suffices to consider the case where $B$ is bounded since (4.4.27) is an identity between Borel measures on $\mathbb{R}^{n-1}$. Given $M > 0$, let $E_M = \Sigma_{\tau_M(u_1)} \cap \Sigma^{\tau_M(u_2)}$. Since $\tau_M(u_i) \in BV_{loc}(\mathbb{R}^{n-1})$ for every $M > 0$, $i = 1, 2$, by Lemma 4.4.6 we find that $E_M$ is a set of locally finite perimeter and that (4.4.11) holds true on $E_M$ with $\tau_M(u_1)$ and $\tau_M(u_2)$ in place of $u_1$ and $u_2$. To complete the proof of the theorem we are going to show the following identities

$$P_K(E; B\times\mathbb{R}) = \lim_{M\to+\infty} P_K(E_M; B\times\mathbb{R}) \tag{4.4.28}$$

$$\int_{B\cap\{u_1<u_2\}} \phi_K(\nabla u_1(x), -1)dx = \lim_{M\to+\infty} \int_{B\cap\{\tau_M(u_1)<\tau_M(u_2)\}} \phi_K(\nabla\tau_M(u_1)(x), -1)dx \tag{4.4.29}$$

$$\int_{B\cap\{u_1<u_2\}} \phi_K(-\nabla u_2(x), 1)dx = \lim_{M\to+\infty} \int_{B\cap\{\tau_M(u_1)<\tau_M(u_2)\}} \phi_K(-\nabla\tau_M(u_2)(x), 1)dx \tag{4.4.30}$$

$$\begin{aligned} &|(D^c u_1, 0)|_K(B\cap\{\tilde{u_1} < \tilde{u_2}\}) = \\ &\lim_{M\to+\infty} \int_{B\cap\{\widetilde{\tau_M(u_1)}<\widetilde{\tau_M(u_2)}\}} \phi_K\left(\frac{dD^c\tau_M(u_1)}{d|D^c\tau_M(u_1)|}(x), 0\right) d|D^c\tau_M(u_1)|(x) \end{aligned} \tag{4.4.31}$$

$$\begin{aligned} &|(-D^c u_2, 0)|_K(B\cap\{\tilde{u_1} < \tilde{u_2}\}) = \\ &\lim_{M\to+\infty} \int_{B\cap\{\widetilde{\tau_M(u_1)}<\widetilde{\tau_M(u_2)}\}} \phi_K\left(-\frac{dD^c\tau_M(u_2)}{d|D^c\tau_M(u_2)|}(x), 0\right) d|D^c\tau_M(u_2)|(x) \end{aligned} \tag{4.4.32}$$

$$\begin{aligned} &\int_{B\cap J_{u_1}} \phi_K\left(\nu_{u_1}(z), 0\right)\left(\min(u_1^\vee(z), u_2^\wedge(z)) - u_1^\wedge(z)\right) d\mathcal{H}^{n-2}(z) = \\ &\lim_{M\to+\infty} \int_{B\cap J_{\tau_M(u_1)}} \phi_K\left(\frac{dD^j\tau_M(u_1)}{d|D^j\tau_M(u_1)|}(z), 0\right)\left(\min(\tau_M(u_1)^\vee(z), \tau_M(u_2)^\wedge(z)) - \tau_M(u_1)^\wedge(z)\right) d\mathcal{H}^{n-2}(z) \end{aligned} \tag{4.4.33}$$

$$\int_{B\cap J_{u_2}} \phi_K\left(-\nu_{u_2}(z),0\right)\left(u_2^\vee(z)-\max(u_2^\wedge(z),u_1^\vee(z))\right)d\mathcal{H}^{n-2}(z)= \tag{4.4.34}$$
$$\lim_{M\to+\infty}\int_{B\cap J_{\tau_M(u_2)}} \phi_K\left(-\frac{dD^j\tau_M(u_2)}{d|D^j\tau_M(u_2)|}(z),0\right)\left(\tau_M(u_2)^\vee(z)-\max(\tau_M(u_2)^\wedge(z),\tau_M(u_1)^\vee(z))\right)d\mathcal{H}^{n-2}(z).$$

Observe that by [2, Theorem 3.99] with $f=\tau_M$ we have for $i=1,2$

$$D\left(\tau_M(u_i)\right)=1_{\{|u_i|<M\}}\nabla u_i\,\mathcal{L}^{n-1}+\left(\tau_M(u_i^\vee)-\tau_M(u_i^\wedge)\right)\nu_{u_i}\,\mathcal{H}^{n-2}\llcorner S_{u_i}+1_{\{|\tilde{u_i}|<M\}}\,D^c u_i \tag{4.4.35}$$

We divide the proof in few steps.

**Step 1 (Jump part)** By relations (2.0.7)-(2.0.10) and relation (4.4.35) we get that $\{J_{\tau_M(u_i)}\}_{M>0}$ is a monotone increasing family of sets whose union is $J_{u_i}$, $i=1,2$. Moreover, observing that

$$\min\left(\tau_M(s);\tau_M(t)\right)=\tau_M\left(\min(s;t)\right)\qquad\forall\, s,t\in\mathbb{R}$$
$$\max\left(\tau_M(s);\tau_M(t)\right)=\tau_M\left(\max(s;t)\right)\qquad\forall\, s,t\in\mathbb{R}$$

and taking into account relation (2.0.10) we deduce that both

$$\left(\min(\tau_M(u_1)^\vee(z),\tau_M(u_2)^\wedge(z))-\tau_M(u_1)^\wedge(z)\right)_{M>0},$$
$$\left(\tau_M(u_2)^\vee(z)-\max(\tau_M(u_2)^\wedge(z),\tau_M(u_1)^\vee(z))\right)_{M>0}$$

are increasing family of functions. Thus, the proof of (4.4.33) and (4.4.34) is completed.

**Step 2 (Cantor part)** Firstly, let us notice that by definition of *approximate average* (see Section 2) and relation (2.0.7)

$$\left\{\widetilde{\tau_M(u_1)}<\widetilde{\tau_M(u_2)}\right\}=\left\{\tau_M(u_2^\vee)-\tau_M(u_1^\vee)>0\right\}\cup\left\{\tau_M(u_2^\wedge)-\tau_M(u_1^\wedge)>0\right\}.$$

Thus, by relation (2.0.11) we deduce that $\{\widetilde{\tau_M(u_1)}<\widetilde{\tau_M(u_2)}\}_{M>0}$ is a monotone increasing family of sets whose union is $\{\widetilde{u_1}<\widetilde{u_2}\}$. Let us call $A_M=\{\widetilde{\tau_M(u_1)}<\widetilde{\tau_M(u_2)}\}$ and $A=\{\widetilde{u_1}<\widetilde{u_2}\}$. By relation (2.0.25) and by the monotonicity of the sets $\{A_M\}_{M>0}$ we have that

$$\lim_{M\to+\infty}|D^c u_i|\left(B\cap\{A_M\}\right)=|D^c u_i|(B\cap A)=\lim_{M\to+\infty}|D^c\tau_M u_i|(B\cap A). \tag{4.4.36}$$

Again by the monotonicity of the family of sets $\{A_M\}_{M>0}$ and by (4.4.35) we have

$$|D^c u_i|(A_M)\le|D^c\tau_M u_i|(A_M)\le|D^c\tau_M u_i|(A).$$

Thus, taking the limit for $M\to+\infty$ in the above relation we obtain

$$|D^c u_i|(A)\le\liminf_{M\to\infty}|D^c\tau_M u_i|(A_M)\le\limsup_{M\to\infty}|D^c\tau_M u_i|(A_M)\le|D^c u_i|(A),$$

proving that

$$\lim_{M\to+\infty} |D^c\tau_M u_i|(A_M) = |D^c u_i|(A).$$

Analogously, having in mind Remark 4.1.13 we get that

$$|(D^c u_1, 0)|_K(B\cap A) = \lim_{M\to+\infty} |(D^c\tau_M u_1, 0)|_K(B\cap\{A_M\}),$$
$$|(-D^c u_2, 0)|_K(B\cap A) = \lim_{M\to+\infty} |(-D^c\tau_M u_2, 0)|_K(B\cap\{A_M\}).$$

This concludes the proof for both (4.4.31) and (4.4.32).

**Step 3 (Absolutely Continuos part)** By (4.4.35) we get

$$\begin{aligned}\int_{B\cap\{\tau_M(u_1)<\tau_M(u_2)\}} \phi_K(\nabla\tau_M(u_1)(x), -1)dx &= \int_{B\cap\{\tau_M(u_1)<\tau_M(u_2)\}\cap\{|u_1|<M\}} \phi_K(\nabla u_1(x), -1)\,dx \\ &+ \int_{B\cap\{\tau_M(u_1)<\tau_M(u_2)\}\cap\{|u_1|\geq M\}} \phi_K(0, -1)\,dx \\ &= I_1^M + I_2^M.\end{aligned}$$

Notice that

$$\begin{aligned}|I_2^M| &= \phi_K(0,-1)\mathcal{L}^{n-1}\left(B\cap\{\tau_M(u_1)<\tau_M(u_2)\}\cap\{|u_1|\geq M\}\right) \\ &\leq \phi_K(0,-1)\mathcal{L}^{n-1}\left(B\cap\{|u_1|\geq M\}\right).\end{aligned}$$

By the fact that $\{|u_1| \geq M\}_{M>0}$ is a decreasing family of sets whose intersection is $\{|u_1| = +\infty\}$ we deduce that

$$\lim_{M\to\infty} |I_2^M| = 0.$$

Since both $\{|u| < M\}_{M>0}$ and $\{\tau_M(u_1) < \tau_M(u_2)\}_{M>0}$ are increasing family of sets, we apply the monotone convergence theorem to get that

$$\lim_{M\to\infty} I_1^M = \int_{B\cap\{u_1<u_2\}} \phi_K(\nabla u_1(x), -1)\,dx.$$

An analogous argument can be used for relation (4.4.30) and so this concludes the proof for both (4.4.29) and (4.4.30).

**Step 4 (Perimeter functional part)** Lastly, let us consider the family of sets $E_{M_h} = E\cap\{|x_n| < M_h\}$ where the sequence of real numbers $\{M_h\}_{h\in\mathbb{N}}$ has been chosen s.t.

$$\lim_{h\to+\infty} \mathcal{H}^{n-1}\left(E^{(1)}\cap\{|\mathbf{q}x| = M_h\}\right) = 0, \quad \mathcal{H}^{n-1}\left(\partial^e E\cap\{|\mathbf{q}x| = M_h\}\right) = 0 \quad \forall h\in\mathbb{N}. \tag{4.4.37}$$

Observe that the the existence of such a sequence $\{M_h\}_{h\in\mathbb{N}}$ is guaranteed by the fact that $|E| < \infty$ and by the fact that $\mathcal{H}^{n-1} \llcorner \partial^e E$ is a Radon measure. Thanks to the above two relations and [32, Theorem 16.3] we have that

$$\begin{aligned} P_K\left(E_{M_h}; B\times\mathbb{R}\right) &= \int_{\partial^e E_{M_h}\cap(B\times\mathbb{R})} \phi_K(\nu^{E_{M_h}}(x))d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial^e E_{M_h}\cap(B\times\mathbb{R})\cap\{|\mathbf{q}x|<M_h\}} \phi_K(\nu^{E_{M_h}}(x))d\mathcal{H}^{n-1}(x) \\ &+ \int_{E^{(1)}\cap\{|\mathbf{q}x|=M_h\}\cap(B\times\mathbb{R})} \phi_K(\nu^{E_{M_h}}(x))d\mathcal{H}^{n-1}(x). \end{aligned}$$

Observing that,

$$\int_{E^{(1)}\cap\{|\mathbf{q}x|=M_h\}\cap(B\times\mathbb{R})} \phi_K(\nu^{E_{M_h}}(x))d\mathcal{H}^{n-1}(x) \le C\mathcal{H}^{n-1}(E^{(1)}\cap\{|\mathbf{q}x| = M_h\}),$$

and considering the first relation in (4.4.37) we finally get

$$\lim_{h\to+\infty} \int_{\partial^e E_{M_h}\cap(B\times\mathbb{R})\cap\{|\mathbf{q}x|<M_h\}} \phi_K(\nu^{E_{M_h}}(x))d\mathcal{H}^{n-1}(x) = P_K(E; B\times\mathbb{R}).$$

This concludes the proof. □

**Lemma 4.4.8.** *If $v \in (BV\cap L^\infty)(\mathbb{R}^{n-1}; [0,\infty))$, $b \in GBV(\mathbb{R}^{n-1})$ and we set $u_1 = b-(v/2) \in GBV(\mathbb{R}^{n-1})$, $u_2 = b+(v/2) \in GBV(\mathbb{R}^{n-1})$ then for $\mathcal{H}^{n-2}$-a.e. $x \in J_v \cap J_b$ we have*

$$\text{if } x \in \left\{[b] < \left[\frac{v}{2}\right] : \nu_b = \nu_v\right\} \cup \{\nu_b = -\nu_v\} \quad \text{then} \quad \frac{dD^j u_1}{d|D^j u_1|}(x) = -\nu_v(x) \tag{4.4.38}$$

$$\text{if } x \in \left\{[b] > \left[\frac{v}{2}\right] : \nu_b = \nu_v\right\} \quad \text{then} \quad \frac{dD^j u_1}{d|D^j u_1|}(x) = +\nu_v(x) \tag{4.4.39}$$

$$\text{if } x \in \left\{[b] < \left[\frac{v}{2}\right] : \nu_b = -\nu_v\right\} \cup \{\nu_b = \nu_v\} \quad \text{then} \quad \frac{dD^j u_2}{d|D^j u_2|}(x) = +\nu_v(x) \tag{4.4.40}$$

$$\text{if } x \in \left\{[b] > \left[\frac{v}{2}\right] : \nu_b = -\nu_v\right\} \quad \text{then} \quad \frac{dD^j u_2}{d|D^j u_2|}(x) = -\nu_v(x). \tag{4.4.41}$$

*Moreover,*

$$\text{if } x \in \left\{[b] = \frac{1}{2}[v] : \nu_b = \nu_v\right\} \quad \text{then} \quad x \notin J_{u_1} \tag{4.4.42}$$

$$\text{if } x \in \left\{[b] = \frac{1}{2}[v] : \nu_b = -\nu_v\right\} \quad \text{then} \quad x \notin J_{u_2}. \tag{4.4.43}$$

*Proof.* Firstly, let us notice that thanks to [32, Proposition 10.5] we already know that for $\mathcal{H}^{n-2}$-a.e. $x \in J_v \cap J_b$ either we have

$$\nu_v(x) = \nu_b(x) \qquad \text{or} \qquad \nu_v(x) = -\nu_b(x).$$

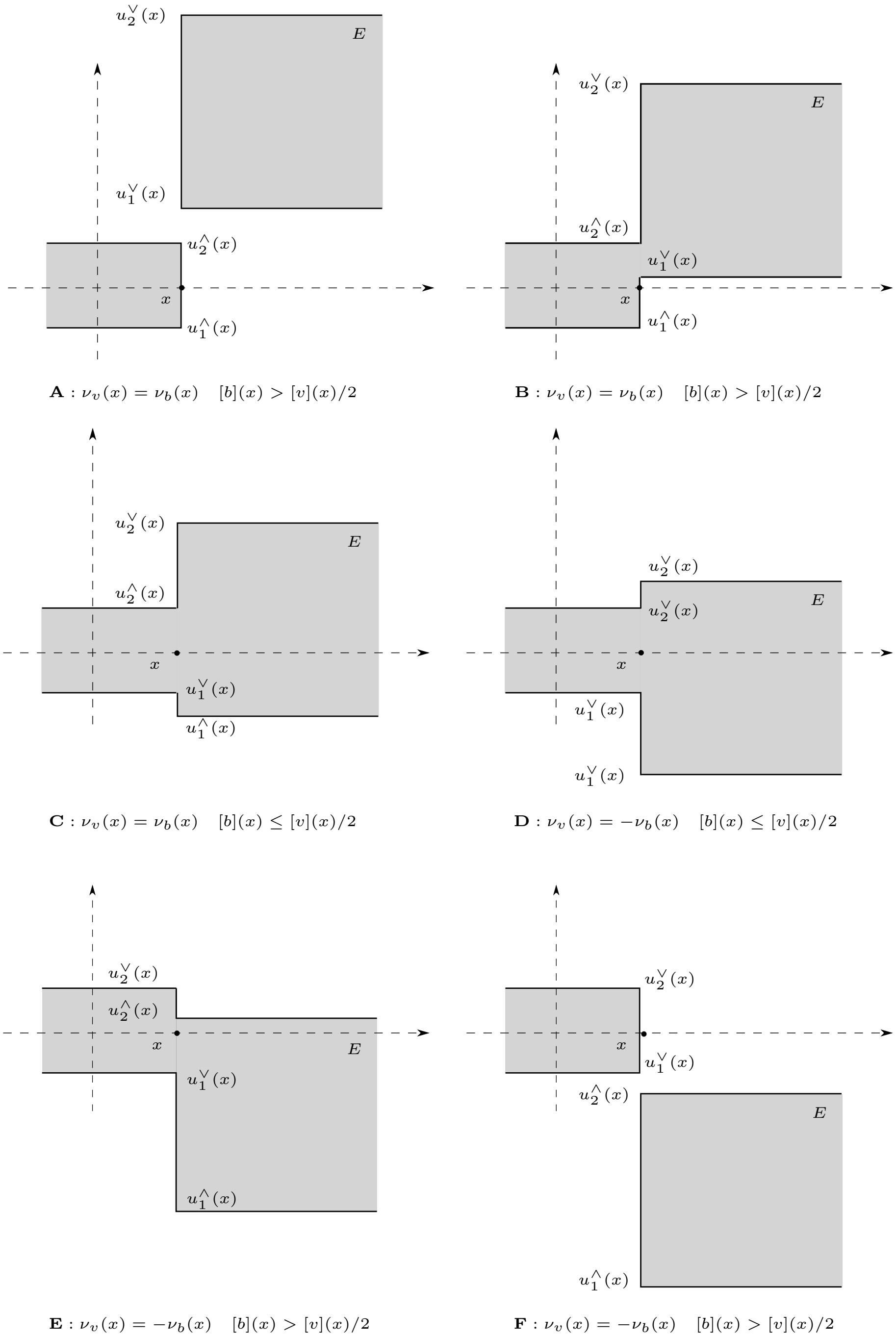


Figure 4.4.4

Let us start by proving relation (4.4.38). In particular, using the definition of upper and lower limits, we want to prove that when $x \in \left\{[b] < \left[\frac{v}{2}\right] : \nu_b = \nu_v\right\}$ (see figure 4.4.4 **C**) then

$$u_1^\vee(x) = -\left(\frac{v}{2}\right)^\wedge(x) + b^\wedge(x), \quad u_1^\wedge(x) = -\left(\frac{v}{2}\right)^\vee(x) + b^\vee(x), \quad \nu_{u_1}(x) = -\nu_v(x). \tag{4.4.44}$$

As we said, we just need to verify if the definition of jump direction for the upper and lower limit is satisfied, namely if for every $\epsilon > 0$ we have that

$$\lim_{\rho\to+\infty} \frac{\mathcal{H}^{n-1}\left(\left\{y \in \mathbb{R}^{n-1} :\, \left|u_1(y) - \left(-\left(\frac{v}{2}\right)^\wedge(x) + b^\wedge(x)\right)\right| > \epsilon\right\} \cap H^+_{x,-\nu_v} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}} = 0. \tag{4.4.45}$$

Let us substitute in the numerator of (4.4.45) $u_1 = b - \frac{v}{2}$ and observe that by the triangular inequality we have that

$$\begin{aligned}&\left\{y \in \mathbb{R}^{n-1} :\, \left|b(y) - \frac{v}{2} + \left(\frac{v}{2}\right)(y)^\wedge(x) - b^\wedge(x)\right| > \epsilon\right\}\\ &\subseteq \left\{y \in \mathbb{R}^{n-1} :\, |b(y) - b^\wedge(x)| + \left|\frac{v}{2}(y) - \left(\frac{v}{2}\right)^\wedge(x)\right| > \epsilon\right\} := A.\end{aligned}$$

Consider now the following partition of $A$,

$$\left\{y \in \mathbb{R}^{n-1} :\, |b(y) - b^\wedge(x)| > \frac{\epsilon}{2}\right\} \cap A := A_{>\epsilon}, \tag{4.4.46}$$

$$\left\{y \in \mathbb{R}^{n-1} :\, |b(y) - b^\wedge(x)| \le \frac{\epsilon}{2}\right\} \cap A := A_{<\epsilon}, \tag{4.4.47}$$

$$\left\{y \in \mathbb{R}^{n-1} :\, |b(y) - b^\wedge(x)| = \frac{\epsilon}{2}\right\} \cap A := A_{=\epsilon}. \tag{4.4.48}$$

So, using the above partition we can estimate the quantity in the limit relation (4.4.45) as follows

$$\begin{aligned}&\frac{\mathcal{H}^{n-1}\left(\left\{y \in \mathbb{R}^{n-1} :\, \left|u_1(y) - \left(-\left(\frac{v}{2}\right)^\wedge(x) + b^\wedge(x)\right)\right| > \epsilon\right\} \cap H^+_{x,-\nu_v} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}\\ &\le \frac{\mathcal{H}^{n-1}\left(A \cap H^+_{x,-\nu_v} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}} \le \frac{\mathcal{H}^{n-1}\left(A_{>\epsilon} \cap H^+_{x,-\nu_v} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}\\ &+ \frac{\mathcal{H}^{n-1}\left(A_{<\epsilon} \cap H^+_{x,-\nu_v} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}} + \frac{\mathcal{H}^{n-1}\left(A_{=\epsilon} \cap H^+_{x,-\nu_v} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}.\end{aligned} \tag{4.4.49}$$

By relation (4.4.46) we have that

$$A_{>\epsilon} \subseteq \left\{y \in \mathbb{R}^{n-1} :\, |b(y) - b^\wedge(x)| > \frac{\epsilon}{2}\right\}.$$

Thus,

$$\begin{aligned}&\lim_{\rho\to+\infty} \frac{\mathcal{H}^{n-1}\left(A_{>\epsilon} \cap H^+_{x,-\nu_v} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}\\ &\le \lim_{\rho\to+\infty} \frac{\mathcal{H}^{n-1}\left(\{y \in \mathbb{R}^{n-1} :\, |b(y) - b^\wedge(x)| > \frac{\epsilon}{2}\} \cap H^+_{x,-\nu_v} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}} = 0,\end{aligned} \tag{4.4.50}$$

where the latter equality holds true by definition of $b^\wedge(x)$ having in mind that $\nu_b = \nu_v$ by assumption. Concerning $A_{<\epsilon}$ we have that

$$\begin{aligned}A_{<\epsilon} &= \left\{y \in \mathbb{R}^{n-1} :\, \left|\frac{v}{2}(y) - \left(\frac{v}{2}\right)^\wedge(x)\right| > \epsilon - |b(y) - b^\wedge(x)| \ge \frac{\epsilon}{2}\right\}\\ &\subseteq \left\{y \in \mathbb{R}^{n-1} :\, \left|\frac{v}{2}(y) - \left(\frac{v}{2}\right)^\wedge(x)\right| > \frac{\epsilon}{2}\right\}.\end{aligned}$$

Thus,

$$
\begin{aligned}
&\lim_{\rho\to+\infty}\frac{\mathcal{H}^{n-1}\left(A_{<\epsilon}\cap H^+_{x,-\nu_v}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}\\
&\le \lim_{\rho\to+\infty}\frac{\mathcal{H}^{n-1}\left(\left\{y\in\mathbb{R}^{n-1}:\ \left|\frac{v}{2}(y)-\left(\frac{v}{2}\right)^{\wedge}(x)\right|>\frac{\epsilon}{2}\right\}\cap H^+_{x,-\nu_v}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0.
\end{aligned}
\tag{4.4.51}
$$

Thanks to the estimate (4.4.49), putting together (4.4.50) and (4.4.51) we get that (4.4.45) holds true for every $\epsilon>0$. To conclude we have to prove estimate (4.4.45) for $u_1^{\wedge}(x)$ namely we have to prove that

$$
\lim_{\rho\to+\infty}\frac{\mathcal{H}^{n-1}\left(\left\{y\in\mathbb{R}^{n-1}:\ \left|u_1(y)-\left(-\left(\frac{v}{2}\right)^{\vee}(x)+b^{\vee}(x)\right)\right|>\epsilon\right\}\cap H^-_{x,-\nu_v}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0\quad\forall\,\epsilon>0.
$$

In order to prove that, just use the same argument used for (4.4.45), noticing that $H^-_{x,-\nu_v}=H^+_{x,\nu_v}=H^+_{x,\nu_b}$. To prove the remaining statements (4.4.39)-(4.4.41), it is sufficient to consider the same argument adopted for (4.4.45), considering in each case the right function either $\frac{v}{2}$ or $b$ with which construct the partition $A_{>\epsilon}$ and $A_{<\epsilon}$.

Let us now prove relation (4.4.42). Let $x\in\{[b]=\frac{1}{2}[v]:\ \nu_b=\nu_v\}$ and let us consider the functions $b_k,u_{1,k}\in GBV(\mathbb{R}^{n-1})$, $k\in\mathbb{N}$ defined as

$$
b_k(z)=\begin{cases} b(z), & \text{if } z\in H^-_{x,\nu_b(x)}\\[2em] b(z)-\frac{1}{k}[b](x), & \text{if } z\in H^+_{x,\nu_b(x)}.\end{cases}
\qquad
u_{1,k}(z)=\begin{cases} u_1(z), & \text{if } z\in H^-_{x,\nu_b(x)}\\[2em] u_1(z)-\frac{1}{k}[b](x), & \text{if } z\in H^+_{x,\nu_b(x)}.\end{cases}
$$

Let us note that $u_{1,k}=b_k-\frac{1}{2}v$. Moreover, note that, $b_k^{\wedge}(x)=b^{\wedge}(x)$, $b_k^{\vee}(x)=b^{\vee}(x)-\frac{1}{k}[b](x)$ and so $[b_k](x)=[b](x)-\frac{1}{k}[b](x)$. In particular, we have that $x\in\{[b_k]<1/2[v]:\ \nu_b=\nu_v\}$. Thus, by relations (4.4.38) and (4.4.44) applied to $u_{1,k}$ we get that

$$
u_{1,k}^{\vee}(x)=-\frac{1}{2}v^{\wedge}(x)+b_k^{\wedge}(x)=-\frac{1}{2}v^{\wedge}(x)+b^{\wedge}(x),\tag{4.4.52}
$$

$$
\begin{aligned}
u_{1,k}^{\wedge}(x)&=-\frac{1}{2}v^{\vee}(x)+b_k^{\vee}(x)=-\frac{1}{2}v^{\vee}(x)+b^{\vee}(x)-\frac{1}{k}[b](x)\\
&=-\frac{1}{2}v^{\vee}(x)+b^{\wedge}(x)+\left(1-\frac{1}{k}\right)[b](x)
\end{aligned}
\tag{4.4.53}
$$

Moreover, by (2.0.3) and (2.0.4) we have that

$$
u_{1,k}^{\vee}(x)=\inf\left\{t\in\mathbb{R}:\ \lim_{\rho\to0^+}\frac{\mathcal{H}^{n-1}\left(\{u_{1,k}>t\}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0\right\}\tag{4.4.54}
$$

$$
u_{1}^{\vee}(x)=\inf\left\{t\in\mathbb{R}:\ \lim_{\rho\to0^+}\frac{\mathcal{H}^{n-1}\left(\{u_{1}>t\}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0\right\}\tag{4.4.55}
$$

$$
u_{1,k}^{\wedge}(x)=\sup\left\{t\in\mathbb{R}:\ \lim_{\rho\to0^+}\frac{\mathcal{H}^{n-1}\left(\{u_{1,k}<t\}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0\right\}\tag{4.4.56}
$$

$$
u_{1}^{\wedge}(x)=\sup\left\{t\in\mathbb{R}:\ \lim_{\rho\to0^+}\frac{\mathcal{H}^{n-1}\left(\{u_{1}<t\}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0\right\}.\tag{4.4.57}
$$

Observe that the sequence $(u_{1,k})_{k\in\mathbb{N}}$ is non decreasing in $k$. Thus, we can deduce the following inclusions $\forall\, k > 1$

$$\left\{t\in\mathbb{R}:\ \lim_{\rho\to 0^+}\frac{\mathcal{H}^{n-1}\left(\{u_{1,k}>t\}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0\right\}\subset\left\{t\in\mathbb{R}:\ \lim_{\rho\to 0^+}\frac{\mathcal{H}^{n-1}\left(\{u_{1}>t\}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0\right\}$$
$$\left\{t\in\mathbb{R}:\ \lim_{\rho\to 0^+}\frac{\mathcal{H}^{n-1}\left(\{u_{1,k}<t\}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0\right\}\subset\left\{t\in\mathbb{R}:\ \lim_{\rho\to 0^+}\frac{\mathcal{H}^{n-1}\left(\{u_{1}<t\}\cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}}=0\right\}.$$

Thanks to the above inclusions, having in mind definitions (4.4.54)-(4.4.57) together with relations (4.4.52), (4.4.53) we get

$$-\frac{1}{2}v^{\vee}(x)+b^{\wedge}(x)+\left(1-\frac{1}{k}\right)[b](x)=u_{1,k}^{\wedge}(x)\le u_1^{\wedge}(x)\le u_1^{\vee}(x)\le u_{1,k}^{\vee}(x)=-\frac{1}{2}v^{\wedge}(x)+b^{\wedge}(x).$$

Since $-\frac{1}{2}v^{\vee}(x)=-\frac{1}{2}v^{\wedge}(x)-\frac{1}{2}[v](x)$, passing through the limit as $k\to+\infty$ in the above relation, we conclude that $u_1^{\wedge}(x)=u_1^{\vee}(x)$ and so $x\notin J_{u_1}$. This concludes the proof of (4.4.42). Using a similar argument as the one used for (4.4.42), we can prove (4.4.43). □

**Remark 4.4.9.** *The cases where $[b](x)=0$ i.e. $x\in J_v\setminus J_b$ can be seen as degenerate situations in Lemma 4.4.8 considering in those characterizations $[b]=0$. A similar argument can be applied to show that for $\mathcal{H}^{n-2}$-a.e. $x\in J_b\setminus J_v$ we have $\nu_{u_i}=\nu_b$, $i=1,2$.*

**Remark 4.4.10.** *Let us introduce the following compact notation.*

$$\begin{aligned}
&\boldsymbol{A}=J_v\setminus J_b,\\
&\boldsymbol{B}_1=\left\{J_v\cap J_b:\ \nu_v=\nu_b,\ [b]<\frac{1}{2}[v]\right\},\quad \boldsymbol{B}_2=\left\{J_v\cap J_b:\ \nu_v=\nu_b,\ [b]=\frac{1}{2}[v]\right\},\\
&\boldsymbol{B}_3=\left\{J_v\cap J_b:\ \nu_v=\nu_b,\ [b]>\frac{1}{2}[v]\right\},\\
&\boldsymbol{B}_4=\left\{J_v\cap J_b:\ \nu_v=-\nu_b,\ [b]<\frac{1}{2}[v]\right\},\quad \boldsymbol{B}_5=\left\{J_v\cap J_b:\ \nu_v=-\nu_b,\ [b]=\frac{1}{2}[v]\right\},\\
&\boldsymbol{B}_6=\left\{J_v\cap J_b:\ \nu_v=-\nu_b,\ [b]>\frac{1}{2}[v]\right\},\\
&\boldsymbol{C}=J_b\setminus J_v.
\end{aligned}$$

*Note that we have*

$$J_v\cup J_b=\boldsymbol{A}\cup\left(\bigcup_{i=1}^{6}\boldsymbol{B}_i\right)\cup\boldsymbol{C}. \tag{4.4.58}$$

*Moreover, following the argument explained in the proof of Lemma 4.4.8 we can prove the following relations*

$$\textit{if } x\in\boldsymbol{A}\quad \textit{then}\quad u_1^{\vee}(x)=-\frac{1}{2}v^{\wedge}(x)+\tilde{b}(x);\ u_1^{\wedge}(x)=-\frac{1}{2}v^{\vee}(x)+\tilde{b}(x) \tag{4.4.59}$$
$$u_2^{\vee}(x)=\frac{1}{2}v^{\vee}(x)+\tilde{b}(x);\ u_2^{\wedge}(x)=\frac{1}{2}v^{\wedge}(x)+\tilde{b}(x). \tag{4.4.60}$$

$$\textit{if } x \in \boldsymbol{B}_1 \cup \boldsymbol{B}_2 \quad \textit{then} \quad u_1^\vee(x) = -\frac{1}{2}v^\wedge(x) + b^\wedge(x);\; u_1^\wedge(x) = -\frac{1}{2}v^\vee(x) + b^\vee(x) \tag{4.4.61}$$

$$u_2^\vee(x) = \frac{1}{2}v^\vee(x) + b^\vee(x);\; u_2^\wedge(x) = \frac{1}{2}v^\wedge(x) + b^\wedge(x). \tag{4.4.62}$$

$$\textit{if } x \in \boldsymbol{B}_3 \quad \textit{then} \quad u_1^\vee(x) = -\frac{1}{2}v^\vee(x) + b^\vee(x);\; u_1^\wedge(x) = -\frac{1}{2}v^\wedge(x) + b^\wedge(x) \tag{4.4.63}$$

$$u_2^\vee(x) = \frac{1}{2}v^\vee(x) + b^\vee(x);\; u_2^\wedge(x) = \frac{1}{2}v^\wedge(x) + b^\wedge(x). \tag{4.4.64}$$

$$\textit{if } x \in \boldsymbol{B}_4 \cup \boldsymbol{B}_5 \quad \textit{then} \quad u_1^\vee(x) = -\frac{1}{2}v^\wedge(x) + b^\vee(x);\; u_1^\wedge(x) = -\frac{1}{2}v^\vee(x) + b^\wedge(x) \tag{4.4.65}$$

$$u_2^\vee(x) = \frac{1}{2}v^\vee(x) + b^\wedge(x);\; u_2^\wedge(x) = \frac{1}{2}v^\wedge(x) + b^\vee(x). \tag{4.4.66}$$

$$\textit{if } x \in \boldsymbol{B}_6 \quad \textit{then} \quad u_1^\vee(x) = -\frac{1}{2}v^\wedge(x) + b^\vee(x);\; u_1^\wedge(x) = -\frac{1}{2}v^\vee(x) + b^\wedge(x) \tag{4.4.67}$$

$$u_2^\vee(x) = \frac{1}{2}v^\wedge(x) + b^\vee(x);\; u_2^\wedge(x) = \frac{1}{2}v^\vee(x) + b^\wedge(x). \tag{4.4.68}$$

$$\textit{if } x \in \boldsymbol{C} \quad \textit{then} \quad u_1^\vee(x) = -\frac{1}{2}\tilde{v}(x) + b^\vee(x);\; u_1^\wedge(x) = -\frac{1}{2}\tilde{v}(x) + b^\wedge(x) \tag{4.4.69}$$

$$u_2^\vee(x) = \frac{1}{2}\tilde{v}(x) + b^\vee(x);\; u_2^\wedge(x) = \frac{1}{2}\tilde{v}(x) + b^\wedge(x). \tag{4.4.70}$$

**Corollary 4.4.11.** *If $v \in (BV \cap L^\infty)(\mathbb{R}^{n-1};[0,\infty))$, $b \in GBV(\mathbb{R}^{n-1})$ and*

$$W = W[v,b] = \left\{ x \in \mathbb{R}^n : |\boldsymbol{q}x - b(\boldsymbol{p}x)| < \frac{v(\boldsymbol{p}x)}{2} \right\}, \tag{4.4.71}$$

*then $u_1 = b - (v/2) \in GBV(\mathbb{R}^{n-1})$, $u_2 = b + (v/2) \in GBV(\mathbb{R}^{n-1})$, $W$ is a set of locally finite perimeter with finite volume and for every Borel set $B \subset \mathbb{R}^{n-1}$ we have*

$$P_K(W; B \times \mathbb{R}) = \int_{B \cap \{v>0\}} \phi_K\left(\nabla\left(b - \frac{v}{2}\right), -1\right) + \phi_K\left(-\nabla\left(b + \frac{v}{2}\right), 1\right) d\mathcal{H}^{n-1} \tag{4.4.72}$$

$$+ \int_{B \cap J_v} \min\left(v^\vee, \left(\left[\frac{v}{2}\right] + [b] + \max\left(\left[\frac{v}{2}\right] - [b], 0\right)\right)\right) \phi_K(-\nu_v^j, 0)\, d\mathcal{H}^{n-2} \tag{4.4.73}$$

$$+ \int_{B \cap J_v} \min\left(v^\wedge, \max\left(0, [b] - \left[\frac{v}{2}\right]\right)\right) \phi_K(\nu_v^j, 0)\, d\mathcal{H}^{n-2} \tag{4.4.74}$$

$$+ \int_{B \cap (J_b \setminus J_v)} \min\left([b], \tilde{v}\right) \left(\phi_K(-\nu_b^j, 0) + \phi_{K^s}(\nu_b^j, 0)\right) d\mathcal{H}^{n-2} \tag{4.4.75}$$

$$+ \left| \left(D^c\left(b - \frac{v}{2}\right), 0\right) \right|_K (B \cap \{\tilde{v} > 0\}) \tag{4.4.76}$$

$$+ \left| \left(-D^c\left(b + \frac{v}{2}\right), 0\right) \right|_K (B \cap \{\tilde{v} > 0\}). \tag{4.4.77}$$

*Proof.* The absolutely continuous part and the Cantor parts of the formula, namely relations (4.4.72), (4.4.76) and (4.4.77) are obtained directly by substitution of $u_1 = b - \frac{1}{2}v$

and $u_2 = b + \frac{1}{2}v$ in the formula (4.4.27). To prove the jump parts of the formula i.e. (4.4.73), (4.4.74) and (4.4.75) we have first to notice that (see (4.4.58))

$$J_{u_1} \cup J_{u_2} = J_v \cup J_b = J_v \setminus J_b \cup (J_v \cap J_b) \cup J_b \setminus J_v = \mathbf{A} \cup \left( \bigcup_{i=1}^{6} \mathbf{B}_i \right) \cup \mathbf{C}.$$

Thanks to this relation, we can rewrite the second and third line of the formula (4.4.27) as

$$\begin{aligned}
&\int_{B \cap (J_{u_1} \cup J_{u_2})} \phi_K \left( \nu_{u_1}(z), 0 \right) \left( \min(u_1^{\vee}(z), u_2^{\wedge}(z)) - u_1^{\wedge}(z) \right) \\
&\quad + \phi_K \left( -\nu_{u_2}(z), 0 \right) \left( u_2^{\vee}(z) - \max(u_2^{\wedge}(z), u_1^{\vee}(z)) \right) d\mathcal{H}^{n-2}(z) \\
&= \int_{B \cap (J_{u_1} \cup J_{u_2})} I_1(z) + I_2(z) d\mathcal{H}^{n-2}(z) = \int_{\mathbf{A}} I_1(z) + I_2(z) d\mathcal{H}^{n-2}(z) \\
&\quad + \sum_{i=1}^{6} \int_{\mathbf{B}_i} I_1(z) + I_2(z) d\mathcal{H}^{n-2}(z) + \int_{\mathbf{C}} I_1(z) + I_2(z) d\mathcal{H}^{n-2}(z).
\end{aligned}$$

Using then Lemma 4.4.8, Remark 4.4.9 and Remark 4.4.10 we deduce relations (4.4.73), (4.4.74) and (4.4.75). This concludes the proof. □

**Corollary 4.4.12.** *If $v$ as in (1.1.3), then*

$$\begin{aligned}
P_K(F[v]; G \times \mathbb{R}) = &\int_{G \cap \{v > 0\}} \phi_K \left( -\frac{1}{2} \nabla(v), -1 \right) d\mathcal{H}^{n-1} + \int_{G \cap \{v > 0\}} \phi_K \left( -\frac{1}{2} \nabla(v), 1 \right) d\mathcal{H}^{n-1} \\
&+ \int_{G \cap J_v} [v] \phi_{K^s}(-\nu_v^j, 0) d\mathcal{H}^{n-2} + 2 \left| \left( -\frac{1}{2} D^c v, 0 \right) \right|_K (G).
\end{aligned}$$

*Proof.* The proof follows by applying Corollary 4.4.11 with $u_1 = -\frac{1}{2}v$ and $u_2 = \frac{1}{2}v$. □

## 4.5 Characterization of equality cases for the anisotropic perimeter inequality

This section is dedicated to the proof of Theorem 1.3.2. This proof is on the spirit of the proof of Theorem 1.1.4 (see [11, Theorem 1.9]). We split the proof of Theorem 1.3.2 in the *necessary part* and in the *sufficient part*.

*Proof of Theorem 1.3.2: Necessary conditions.* Let $E \in \mathcal{M}_{K^s}(v)$. This implies that all inequalities in relation (4.3.5) must hold as equalities. In particular, by the latter of these equalities we get that $N(z) = 2$ for $\mathcal{H}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$ implying that $E_z$ is $\mathcal{H}^1$-equivalent to a segment for $\mathcal{H}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$ that is condition (1.1.8). As a consequence , by Theorem 1.1.3, we have that $b_\delta = 1_{\{v > \delta\}} b_E \in GBV(\mathbb{R}^{n-1})$ for every $\delta > 0$ such that

$\{v > \delta\}$ is a set of finite perimeter in $\mathbb{R}^{n-1}$. Let us consider the same sets defined in [11, page 1568] namely

$$I = \{\delta > 0 : \{v < \delta\} \text{ and } \{v > \delta\} \text{ are sets of finite perimeter}\}, \tag{4.5.1}$$

$$J_\delta = \{M > 0 : \{b_\delta < M\} \text{ and } \{b_\delta > -M\} \text{ are sets of finite perimeter}\}. \tag{4.5.2}$$

Let us observe that $\mathcal{H}^1((0,\infty)\setminus I) = 0$ since $v \in BV(\mathbb{R}^{n-1})$ and that $\mathcal{H}^1((0,\infty)\setminus J_\delta) = 0$ for every $\delta \in I$, as for every $\delta \in I$ we have $b_\delta \in GBV(\mathbb{R}^{n-1})$. Let us fix $\delta, L \in I$ and $M \in J_\delta$ and set

$$\Sigma_{\delta,L,M} = \{\delta < v < L\} \cap \{|b_E| < M\} = \{|b_\delta| < M\} \cap \{\delta < v < L\},$$

so that $\Sigma_{\delta,L,M}$ is a set of finite perimeter. Since $\tau_M b_\delta \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$, $1_{\Sigma_{\delta,L,M}} \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$ and $\tau_M b_\delta = b_\delta = b_E$ on $\Sigma_{\delta,L,M}$, we set

$$b_{\delta,L,M} = 1_{\Sigma_{\delta,L,M}} b_E.$$

Note that $b_{\delta,L,M} \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$.

**Step 1** In this step we are going to prove that for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^{n-1}$ there exists $z(x) \in \partial K$ such that

$$\left\{\left(-\frac{1}{2}\nabla v(x) + t\nabla b_{\delta,L,M}(x), 1\right) : t \in [-1,1]\right\} \subset C^*_{K^s}(z(x)). \tag{4.5.3}$$

Indeed, let us set $v_{\delta,L,M} = 1_{\Sigma_{\delta,L,M}} v$. Since $v_{\delta,L,M}, b_{\delta,L,M} \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$, we can apply Corollary 4.4.11 and Remark 4.3.1 to $W = W[v_{\delta,L,M}, b_{\delta,L,M}]$. Moreover observe that $W[v_{\delta,L,M}, b_{\delta,L,M}] = E \cap (\Sigma_{\delta,L,M} \times \mathbb{R})$ and thus

$$\partial^e E \cap (\Sigma^{(1)}_{\delta,L,M} \times \mathbb{R}) = \partial^e W[v_{\delta,L,M}, b_{\delta,L,M}] \cap (\Sigma^{(1)}_{\delta,L,M} \times \mathbb{R}),$$

and so, for every Borel set $G \subset \Sigma^{(1)}_{\delta,L,M} \setminus (S_{v_{\delta,L,M}} \cup S_{b_{\delta,L,M}})$ we find that

$$\begin{aligned}
&P_{K^s}(E; G \times \mathbb{R}) = P_{K^s}(W[v_{\delta,L,M}, b_{\delta,L,M}]; G \times \mathbb{R}) \\
&= \int_G \phi_{K^s}\left(\nabla\left(b_{\delta,L,M} - \frac{v_{\delta,L,M}}{2}\right), 1\right) + \phi_{K^s}\left(-\nabla\left(b_{\delta,L,M} + \frac{v_{\delta,L,M}}{2}\right), 1\right) d\mathcal{H}^{n-1} \\
&+ \left|\left(D^c\left(b_{\delta,L,M} - \frac{v_{\delta,L,M}}{2}\right), 0\right)\right|_{K^s}(G) + \left|\left(-D^c\left(b_{\delta,L,M} + \frac{v_{\delta,L,M}}{2}\right), 0\right)\right|_{K^s}(G).
\end{aligned}$$

We can use Lemma 2.0.3 applied with $v_{\delta,L,M} = 1_{\Sigma_{\delta,L,M}} v$, to find that

$$\begin{aligned}
\nabla v_{\delta,L,M} &= 1_{\Sigma_{\delta,L,M}} \nabla v, \quad \mathcal{H}^{n-1}\text{-a.e. on } \mathbb{R}^{n-1}, \\
D^c v_{\delta,L,M} &= D^c v \llcorner \Sigma^{(1)}_{\delta,L,M}, \\
S_{v_{\delta,L,M}} \cap \Sigma^{(1)}_{\delta,L,M} &= S_v \cap \Sigma^{(1)}_{\delta,L,M}.
\end{aligned}$$

Thus,

$$P_{K^s}(E;G\times\mathbb{R}) = \int_G \phi_{K^s}\left(\nabla\left(b_{\delta,L,M}-\frac{v}{2}\right),1\right)+\phi_{K^s}\left(-\nabla\left(b_{\delta,L,M}+\frac{v}{2}\right),1\right)d\mathcal{H}^{n-1}$$
$$+\left|\left(D^c\left(b_{\delta,L,M}-\frac{v}{2}\right),0\right)\right|_{K^s}(G)+\left|\left(-D^c\left(b_{\delta,L,M}+\frac{v}{2}\right),0\right)\right|_{K^s}(G),$$

for every Borel set $G\subset\Sigma^{(1)}_{\delta,L,M}\setminus(S_{v_{\delta,L,M}}\cup S_{b_{\delta,L,M}})$. By assumptions we are assuming that $E\in\mathcal{M}_{K^s}(v)$ and so for every Borel set $G\subset\mathbb{R}^{n-1}$ we have that $P_{K^s}(E;G\times\mathbb{R}) = P_{K^s}(F[v];G\times\mathbb{R})$. In particular, having in mind the formula for $P_{K^s}(F[v];G\times\mathbb{R})$ given by Corollary 4.4.12, for every Borel set $G\subset\Sigma^{(1)}_{\delta,L,M}\setminus(S_{v_{\delta,L,M}}\cup S_{b_{\delta,L,M}})$ we get

$$0=\int_G \phi_{K^s}\left(\nabla\left(b_{\delta,L,M}-\frac{v}{2}\right),1\right)+\phi_{K^s}\left(-\nabla\left(b_{\delta,L,M}+\frac{v}{2}\right),1\right)-2\phi_{K^s}\left(-\nabla\left(\frac{v}{2}\right),1\right)d\mathcal{H}^{n-1} \tag{4.5.4}$$

$$+\left|\left(D^c\left(b_{\delta,L,M}-\frac{v}{2}\right),0\right)\right|_{K^s}(G)+\left|\left(-D^c\left(b_{\delta,L,M}+\frac{v}{2}\right),0\right)\right|_{K^s}(G)-2\left|\left(-D^c\left(\frac{v}{2}\right),0\right)\right|_{K^s}(G) \tag{4.5.5}$$

Let us notice that the first line in the above relation, namely (4.5.4) is greater or equal to zero by the sub additivity of $\phi_K$. Also the second line in the above relation, namely (4.5.5), is greater or equal to zero thanks to Lemma 4.2.5 with $\mu=\left(-\frac{1}{2}D^cv,0\right)$ and $\nu=(D^cb_{\delta,L,M},0)$. Thus, we have that

$$0=\int_G \phi_{K^s}\left(\nabla\left(b_{\delta,L,M}-\frac{v}{2}\right),1\right)+\phi_{K^s}\left(-\nabla\left(b_{\delta,L,M}+\frac{v}{2}\right),1\right)-2\phi_{K^s}\left(-\nabla\left(\frac{v}{2}\right),1\right)d\mathcal{H}^{n-1} \tag{4.5.6}$$

$$0=\left|\left(D^c\left(b_{\delta,L,M}-\frac{v}{2}\right),0\right)\right|_{K^s}(G)+\left|\left(-D^c\left(b_{\delta,L,M}+\frac{v}{2}\right),0\right)\right|_{K^s}(G)-2\left|\left(-D^c\left(\frac{v}{2}\right),0\right)\right|_{K^s}(G). \tag{4.5.7}$$

Let us observe that the relation (4.5.6) is satisfied if and only if $\mathcal{H}^{n-1}$-a.e. in $G$ we have

$$\phi_{K^s}\left(\nabla\left(b_{\delta,L,M}-\frac{v}{2}\right)(x),1\right)+\phi_{K^s}\left(-\nabla\left(b_{\delta,L,M}+\frac{v}{2}\right)(x),1\right)=2\phi_{K^s}\left(-\frac{\nabla v(x)}{2},1\right).$$

Thanks to Proposition 4.1.22 the condition above is satisfied if and only if for $\mathcal{H}^{n-1}$-a.e. $x\in G$, $\exists z(\bar{x})\in\partial K^s$ s.t.

$$\frac{\left(\nabla\left(b_{\delta,L,M}-\frac{v}{2}\right)(x),1\right)}{\phi_{K^s}\left(\nabla\left(b_{\delta,L,M}-\frac{v}{2}\right)(x),1\right)},\frac{\left(-\nabla\left(b_{\delta,L,M}+\frac{v}{2}\right)(x),1\right)}{\phi_{K^s}\left(-\nabla\left(b_{\delta,L,M}+\frac{v}{2}\right)(x),1\right)}\in\partial\phi^*_{K^s}(\bar{z}).$$

As we observed in Remark 4.1.23, and in particular using relation (4.1.22) with $y_1=\left(-\frac{1}{2}\nabla(x)+\nabla b_{\delta,L,M},1\right)$ and $y_2=\left(-\frac{1}{2}\nabla(x)-\nabla b_{\delta,L,M},1\right)$ the condition above is equivalent to say that for $\mathcal{H}^{n-1}$-a.e. $x\in G$, there exists $z(\bar{x})\in\partial K^s$ s.t.

$$\left\{\left(-\frac{1}{2}\nabla(x)+t\nabla b_{\delta,L,M},1\right)\,:\,t\in[-1,1]\right\}\subset C^*_K(z(x)). \tag{4.5.8}$$

This concludes the first step.

**Step 2** In this step we prove that there exist a Borel measurable function $g_{\delta,L,M}:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ such that

$$D^cb_{\delta,L,M}\llcorner\Sigma^{(1)}_{\delta,L,M}=g_{\delta,L,M}\left|\frac{1}{2}D^cv\right|_{K^s}\llcorner\Sigma^{(1)}_{\delta,L,M}.$$

We prove also an intermediate relation for (1.3.14). Indeed, let us rewrite relation (4.5.7) as

$$|(-D^c v,0)|_{K^s}(G)=\left|\left(D^c\left(b_{\delta,L,M}-\frac{v}{2}\right),0\right)\right|_{K^s}(G)+\left|\left(-D^c\left(b_{\delta,L,M}+\frac{v}{2}\right),0\right)\right|_{K^s}(G).$$

As already observed, by calling

$$\mu=\left(-\frac{D^c v}{2},0\right),$$
$$\nu=(D^c b_{\delta,L,M},0)$$

the above equality can be written as

$$2|\mu|_{K^s}(G)=|\mu+\nu|_{K^s}(G)+|\mu-\nu|_{K^s}(G).$$

Observe that we are in a case of equality in Lemma 4.2.5. Thus, by Remark 4.2.7, for $|D^c v|$-a.e. $x\in G$ we define

$$g_{\delta,L,M}(x)=\frac{dD^c b_{\delta,L,M}}{d\,|(D^c v/2,0)|_{K^s}},\quad h(x)=\frac{-dD^c v/2}{d\,|(D^c v/2,0)|_{K^s}},$$

and we conclude that for $|D^c v|$-a.e. $x\in G$ there exists $z(x)\in\partial K$ s.t.

$$\{h(x)+tg_{\delta,L,M}(x):\,t\in[-1,1]\}\subset C^*_K(z(x)). \tag{4.5.9}$$

This concludes the second step.

**Step 3** In this step we prove (1.3.13). We fix $\delta,L\in I$ and we define $\Sigma_{\delta,L}=\{\delta<v<L\}$, $b_{\delta,L}=1_{\Sigma_{\delta,L}}b_E$ and $v_{\delta,L}=1_{\Sigma_{\delta,L}}v$. Since $\Sigma_{\delta,L}$ is a set of finite perimeter, it turns out that $b_{\delta,L}\in GBV(\mathbb{R}^{n-1})$, while, by construction, $v_{\delta,L}\in(BV\cap L^\infty)(\mathbb{R}^{n-1})$. So, we can apply the formula of Corollary 4.4.11 to the set $W[v_{\delta,L},b_{\delta,L}]$. In particular, if $G\subset\Sigma^{(1)}_{\delta,L}\cap(S_{v_{\delta,L}}\cup S_{b_{\delta,L}})$, then

$$\begin{aligned}
&P_{K^s}(E;G\times\mathbb{R})=P_{K^s}(W[v_{\delta,L},b_{\delta,L}];G\times\mathbb{R})\\
&=\int_{G\cap J_v}\min\left(v^\vee,\left(\left[\frac{v}{2}\right]+[b_{\delta,L}]+\max\left(\left[\frac{v}{2}\right]-[b_{\delta,L}],0\right)\right)\right)\phi_{K^s}(-\nu_v,0)d\mathcal{H}^{n-2}\\
&+\int_{G\cap J_v}\min\left(v^\wedge,\max\left(0,[b_{\delta,L}]-\left[\frac{v}{2}\right]\right)\right)\phi_{K^s}(\nu_v,0)d\mathcal{H}^{n-2}\\
&+\int_{G\cap(J_{b_{\delta,L}}\setminus J_v)}\min\left([b_{\delta,L}],\tilde{v}\right)\left(\phi_{K^s}(-\nu_{b_{\delta,L}},0)+\phi_{K^s}(\nu_{b_{\delta,L}},0)\right)d\mathcal{H}^{n-2},
\end{aligned}\tag{4.5.10}$$

where we used the fact that, thanks to (2.0.15)

$$\Sigma^{(1)}_{\delta,L}\cap S_{v_{\delta,L}}=\Sigma^{(1)}_{\delta,L}\cap S_v,\quad v^\vee_{\delta,L}=v^\vee\quad v^\wedge_{\delta,L}=v^\wedge,\quad [v_{\delta,L}]=[v]\quad\forall x\in\Sigma^{(1)}_{\delta,L}.$$

Let us observe that, calling $I$ the argument of the integral in relation (4.5.10) i.e.

$$I=\min\left(v^\vee,\left(\left[\frac{v}{2}\right]+[b_{\delta,L}]+\max\left(\left[\frac{v}{2}\right]-[b_{\delta,L}],0\right)\right)\right)$$

we have that

$$\textit{if} \quad [b_{\delta,L}] = 0 \quad \textit{then} \quad I = [v], \tag{4.5.11}$$

$$\textit{if} \quad [b_{\delta,L}] \leq \frac{1}{2}[v] \quad \textit{then} \quad I = [v], \tag{4.5.12}$$

$$\textit{if} \quad [b_{\delta,L}] > \frac{1}{2}[v] \quad \textit{then} \quad I > [v]. \tag{4.5.13}$$

Recall that

$$P_{K^s}(F[v]; G \times \mathbb{R}) = \int_{G \cap J_v} [v]\phi_{K^s}(-\nu_v, 0) d\mathcal{H}^{n-2}.$$

Thus, since $\phi_{K^s} \geq 0$, imposing that $P_{K^s}(F[v]; G \times \mathbb{R}) = P_{K^s}(E; G \times \mathbb{R})$ and having in mind relations (4.5.11)-(4.5.13) we obtain that

$$\min\left([b_{\delta,L}], \tilde{v}\right) = 0, \quad \mathcal{H}^{n-2}\text{-a.e. in } G \cap (S_{b_{\delta,L}} \backslash S_v) \tag{4.5.14}$$

$$\min\left(v^\wedge, \max\left(0, [b_{\delta,L}] - \left[\frac{v}{2}\right]\right)\right) = 0, \quad \mathcal{H}^{n-2}\text{-a.e. in } G \cap S_v \tag{4.5.15}$$

$$I = \min\left(v^\vee, \left(\left[\frac{v}{2}\right] + [b_{\delta,L}] + \max\left(\left[\frac{v}{2}\right] - [b_{\delta,L}], 0\right)\right)\right) = [v] \quad \mathcal{H}^{n-2}\text{-a.e. in } G \cap S_v. \tag{4.5.16}$$

Since $\tilde{v} \geq \delta > 0$ in $\Sigma^{(1)}_{\delta,L}$, from (4.5.14) it follows that $S_{b_{\delta,L}} \cap \Sigma^{(1)}_{\delta,L} \subset_{\mathcal{H}^{n-2}} S_v$. Moreover, from (4.5.11), (4.5.12) together with (4.5.14) and (4.5.15) it follows that

$$[b_{\delta,L}] \leq \frac{[v]}{2} \quad \mathcal{H}^{n-2}\text{-a.e. } x \in G \cap S_v. \tag{4.5.17}$$

By (2.0.15), $[b_{\delta,L}] = [b_E]$ on $\Sigma^{(1)}_{\delta,L}$. By taking the union of $\Sigma^{(1)}_{\delta,L}$ on $\delta, L \in I$ and by taking (2.0.13), (2.0.14) into account we thus find that

$$[b_E] \leq \frac{[v]}{2} \quad \mathcal{H}^{n-2}\text{-a.e. on } \{v^\wedge > 0\} \cup \{v^\vee < \infty\}.$$

Since, by [22, 4.5.9(3)] $\{v^\vee = \infty\}$ is $\mathcal{H}^{n-2}$-negligible, we have proved (1.3.13).

**Step 4** In this step we prove (1.3.12). Let $\delta, L \in I$ and $M \in J_\delta$. Since $b_{\delta,L,M} = b_E$ $\mathcal{H}^{n-1}$-a.e. on $\Sigma_{\delta,L,M}$ by (4.5.3) and by (2.0.19) we find that for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_{\delta,L,M}$, there exists $z(x) \in \partial K^s$ s.t.

$$\left\{\left(-\frac{1}{2}\nabla v(x) + t\nabla b_E(x), 1\right) : t \in [-1, 1]\right\} \subset C^*_{K^s}(z(x)).$$

By taking a union first on $M \in J_\delta$ and then on $\delta, L \in I$, we find that for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$, there exists $z(x) \in \partial K^s$ s.t.

$$\left\{\left(-\frac{1}{2}\nabla v(x) + t\nabla b_E(x), 1\right) : t \in [-1, 1]\right\} \subset C^*_{K^s}(z(x)).$$

At the same time, by definition, $b_E = 0$ on $\{v = 0\}$. Thus, by (2.0.19), we have that $\nabla b_E = 0$ $\mathcal{H}^{n-1}$-a.e. on $\{v = 0\}$ and so, we deduce that for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^{n-1}$, there exists $z(x) \in \partial K^s$ s.t.

$$\left\{ \left( -\frac{1}{2}\nabla v(x) + t\nabla b_E(x), 1 \right) : t \in [-1,1] \right\} \subset C^*_{K^s}(z(x)).$$

This concludes the proof of (1.3.12).

**Step 5** In this step we prove (1.3.14). Let $\delta, L \in I$ and $M \in J_\delta$. Since $b_{\delta,L,M} = 1_{\Sigma_{\delta,L,M}} \tau_M b_\delta$, by Lemma 2.0.3 we have

$$D^c b_{\delta,L,M} = D^c(\tau_M b_\delta) \llcorner \Sigma^{(1)}_{\delta,L,M}.$$

Combining this fact with (4.5.9) we find that for every $G \subset \Sigma^{(1)}_{\delta,L,M}$, for $|D^c v|$-a.e. $x \in G$ there exists $z(x) \in \partial K$ s.t.

$$\{h(x) + t g_{\delta,M}(x) : t \in [-1,1]\} \subset C^*_K(z(x)),$$

where for $|D^c v|$-a.e. $x \in G$ the functions $g_{\delta,M}$ and $h$ are given by

$$g_{\delta,M}(x) = \frac{dD^c(\tau_M b_\delta)}{d\,|(D^c v/2, 0)|_{K^s}}, \quad h(x) = \frac{-dD^c v/2}{d\,|(D^c v/2, 0)|_{K^s}}.$$

Observe now that

$$\begin{aligned}
\bigcup_{L\in I} \Sigma^{(1)}_{\delta,L,M} &= \bigcup_{L\in I} \{|b_\delta| < M\}^{(1)} \cap \{v > \delta\}^{(1)} \cap \{v < L\}^{(1)} \\
&= \left( \{|b_\delta| < M\}^{(1)} \cap \{v > \delta\}^{(1)} \right) \cap \bigcup_{L\in I} \{v < L\}^{(1)} \\
&= \{|b_\delta| < M\}^{(1)} \cap \{v > \delta\}^{(1)} \cap \{v^\vee < \infty\},
\end{aligned}$$

where in the last identity we used (2.0.13). Note that, as we pointed out at the end of step 3, $\mathcal{H}^{n-2}(\{v^\vee = \infty\}) = 0$, so the set $\{v^\vee = \infty\}$ is negligible with respect to both $|D^c \tau_M b_\delta|$ and $|D^c v|$. Thus, we proved that for every bounded Borel set $G \subset \{|b_\delta| < M\}^{(1)} \cap \{v > \delta\}^{(1)}$, for $|D^c v|$-a.e. $x \in G$ there exists $z(x) \in \partial K$ s.t.

$$\{h(x) + t g_{\delta,M}(x) : t \in [-1,1]\} \subset C^*_K(z(x)). \tag{4.5.18}$$

Observe that for every $M' > M$ and $\delta' < \delta$ we have that $\tau_M b_\delta = \tau_{M'} b_{\delta'}$ on $\{|b_\delta| < M\} \cap \{v > \delta\}$. So, by Lemma 2.0.3 we get that

$$D^c(\tau_M b_\delta) \llcorner \{|b_\delta| < M\}^{(1)} \cap \{v > \delta\}^{(1)} = D^c(\tau_{M'} b_{\delta'}) \llcorner \{|b_\delta| < M\}^{(1)} \cap \{v > \delta\}^{(1)},$$

and therefore the function $g_{\delta,M}$ actually does not depend on $\delta, M$. So taking into account (4.5.18) we have that for $|D^c v|$-a.e. $x \in G$ there exists $z(x) \in \partial K$ s.t.

$$\{h(x) + t g(x) : t \in [-1,1]\} \subset C^*_K(z(x)). \tag{4.5.19}$$

Lastly, let us notice that

$$\tau_M b_\delta = M 1_{\{b_\delta \geq M\}} - M 1_{\{b_\delta \leq -M\}} + 1_{\{|b_\delta|<M\}\cap\{v>\delta\}} \tau_M b_\delta, \quad on\ \mathbb{R}^{n-1}$$

is an identity between BV functions. Thus, thanks to [2, Example 3.97] we find that

$$D^c \tau_M b_\delta = D^c(\tau_M b_\delta) \llcorner \left(G \cap \{|b_\delta| < M\}^{(1)} \cap \{v > \delta\}^{(1)}\right)$$

i.e. the measure $D^c \tau_M b_\delta$ is concentrated on $\{|b_\delta| < M\}^{(1)} \cap \{v > \delta\}^{(1)}$. Therefore, we deduce that for every bounded Borel set $G \subset \mathbb{R}^{n-1}$, for $|D^c v|$-a.e. $x \in G \cap \{|b_\delta| < M\}^{(1)} \cap \{v > \delta\}^{(1)}$ there exists $z(x) \in \partial K$ s.t.

$$\{h(x) + tg(x) : t \in [-1,1]\} \subset C_K^*(z(x)). \tag{4.5.20}$$

□

Before entering into the details of the proof for the *sufficient conditions* part, we need a couple of technical results.

**Proposition 4.5.1.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3)and let $v$ be as in (1.1.3). Then, if $E$ is a $v$-distributed set of finite perimeter with sections $E_z$ as segments $\mathcal{H}^{n-1}$-a.e on $\{v > 0\}$ we have that*

$$P_K(E; \{v^\wedge = 0\} \times \mathbb{R}) = P_K(F[v]; \{v^\wedge = 0\} \times \mathbb{R}) = \int_{\{v^\wedge=0\}} v^\vee \phi_K(-\nu_v, 0) d\mathcal{H}^{n-2}. \tag{4.5.21}$$

*Proof.* The proof of this result follows from a careful inspection of the proof of [11, Proposition 3.8], and for this reason is omitted. □

**Lemma 4.5.2.** *If $v \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$, $b : \mathbb{R}^{n-1} \to \mathbb{R}$ is such that $\tau_M b \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$ for a.e. $M > 0$ and $\mu$ is a $\mathbb{R}^{n-1}$-valued Radon measure such that*

$$\lim_{M\to\infty} |\mu - D^c \tau_M b|(G) = 0 \quad \textit{for every bounded Borel set } G \subset \mathbb{R}^{n-1}, \tag{4.5.22}$$

*then,*

$$|(D^c(b+v), 0)|_{K^s}(G) \leq |(\mu + D^c v), 0)|_{K^s}(G) \quad \textit{for every bounded Borel set } G \subset \mathbb{R}^{n-1}. \tag{4.5.23}$$

*Proof.* Let $L > 0$ such that $|v| \leq L$ $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$. If $f \in BV(\mathbb{R}^{n-1})$, then

$$\tau_M f = M 1_{\{f>M\}} - M 1_{\{f<-M\}} + 1_{\{|f|<M\}} f \in (BV \cap L^\infty)(\mathbb{R}^{n-1}),$$

for every $M$ such that $\{f > M\}$ and $\{f < -M\}$ are of finite perimeter and thus, by [2, Theorem 3.96]

$$D^c \tau_M f = D^c \left(1_{\{|f|<M\}} f\right) = 1_{\{|f|<M\}^{(1)}} D^c f = D^c f \llcorner \{|f| < M\}^{(1)};$$

in particular,

$$|(D^c \tau_M f, 0)|_{K^s} = |(D^c f, 0)|_{K^s} \llcorner \{|f| < M\}^{(1)} \leq |(D^c f, 0)|_{K^s}. \tag{4.5.24}$$

From the equality $\tau_M(\tau_{M+L}(b) + v) = \tau_M(b + v)$ and from (4.5.24) applied with $f = \tau_{M+L}(b) + v$ it follows that, for every Borel set $G \subset \mathbb{R}^{n-1}$,

$$\begin{aligned} |(D^c(\tau_M(b+v)), 0)|_{K^s}(G) &= |(D^c(\tau_M(\tau_{M+L}(b) + v)), 0)|_{K^s}(G) \\ &\leq |(D^c(\tau_{M+L}(b) + v), 0)|_{K^s}(G). \end{aligned} \tag{4.5.25}$$

Now observe that (4.5.22) implies that

$$\lim_{M \to \infty} |-(\mu - D^c \tau_M b)|(G) = 0 \quad \textit{for every bounded Borel set } G \subset \mathbb{R}^{n-1}. \tag{4.5.26}$$

Thanks to Remark 4.1.12 together with (4.5.22) and (4.5.26), for every bounded Borel set $G \subset \mathbb{R}^{n-1}$ we get

$$\lim_{M \to \infty} |-(\mu - D^c \tau_M b, 0)|_{K^s}(G) = \lim_{M \to \infty} |(\mu - D^c \tau_M b, 0)|_{K^s}(G) = 0. \tag{4.5.27}$$

Since we can always write $D^c(\tau_M b) + D^c v = (D^c(\tau_M b) - \mu) + (\mu + D^c v)$ by applying relations (4.2.2) and (4.2.3) we obtain

$$|(\mu + D^c v, 0)|_K(G) - |-(D^c(\tau_{M+L} b) - \mu, 0)|_K(G) \leq |(D^c(\tau_{M+L} b) + D^c v, 0)|_K(G) \tag{4.5.28}$$

$$\leq |(D^c(\tau_{M+L} b) - \mu, 0)|_K(G) + |(\mu + D^c v, 0)|_K(G). \tag{4.5.29}$$

So, by (4.5.27) we get

$$\lim_{M \to \infty} |(D^c(\tau_{M+L}(b) + v), 0)|_{K^s}(G) = |(\mu + D^c v, 0)|_{K^s}(G).$$

By (4.5.25) we get that

$$|(D^c(\tau_M(b+v)), 0)|_{K^s}(G) \leq |(\mu + D^c v, 0)|_{K^s}(G).$$

Lastly, by relation (4.1.11), we let $M \to \infty$ and we conclude the proof. □

*Proof of Theorem 1.3.2: sufficient conditions.* Let $E$ be a $v$-distributed set of finite perimeter satisfying (1.1.5), (1.3.12), (1.3.13) and (1.3.14). Let $I$ and $J_\delta$ be defined as in (4.5.1) and (4.5.2). Let $\delta, S \in I$ and let us set $b_{\delta,S} = 1_{\{\delta<v<S\}} b_E = 1_{\{\delta<v<S\}} b_\delta$. Then, for every $M \in J_\delta$, we have $\tau_M b_\delta \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$ and so we obtain that $\tau_M b_{\delta,S} \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$. Let us consider the $\mathbb{R}^{n-1}$-valued Radon measure $\mu_{\delta,S}$ on $\mathbb{R}^{n-1}$ defined as

$$\mu_{\delta,S}(G) = \int_{G\cap\{\delta<v<S\}^{(1)}\cap\{|b_E|^\vee<\infty\}} g(x) d\left|\left(\frac{1}{2}D^c v, 0\right)\right|_K,$$

for every bounded Borel set $G \subset \mathbb{R}^{n-1}$, where $g(x)$ is the function that appears in condition (1.3.14), namely

$$D^c(\tau_M(b_\delta))(G) = \int_{G\cap\{|b_\delta|<M\}^{(1)}\cap\{v>\delta\}^{(1)}} g(x) d\left|\left(\frac{1}{2}D^c v, 0\right)\right|_K.$$

Since $\tau_M b_{\delta,S} = 1_{\{v<S\}} \tau_M b_\delta$, by Lemma 2.0.3 we have $D^c(\tau_M b_{\delta,S}) = 1_{\{v<S\}^{(1)}} D^c(\tau_M b_\delta)$ and thus, for every Borel set $G \subset \mathbb{R}^{n-1}$,

$$\begin{aligned}
\lim_{M\to\infty} |\mu_{\delta,S} - D^c(\tau_M b_{\delta,S})|(G) &= \lim_{M\to\infty} |\mu_{\delta,S} - D^c(\tau_M b_\delta)|(G \cap \{v < S\}^{(1)}) \\
&\le \lim_{M\to\infty} \int_{G\cap\{\delta<v<S\}^{(1)}\cap\left(\{|b_E|^\vee<\infty\}\setminus\{|b_E|<M\}^{(1)}\right)} |g(x)| d|(D^c v/2, 0)|_{K^s}(x) \\
&= 0,
\end{aligned}$$

where the last equality follows from the fact that $\{|b_E| < M\}^{(1)}_{M\in I}$ is an increasing family of sets whose union is $\{|b_E|^\vee < \infty\}$. Thus, for every bounded Borel set $G \subset \mathbb{R}^{n-1}$, we get

$$\begin{aligned}
&\left|\left(-D^c(b_{\delta,S} + \frac{1}{2}v_{\delta,S}), 0\right)\right|_{K^s}(G) + \left|\left(D^c(b_{\delta,S} - \frac{1}{2}v_{\delta,S}), 0\right)\right|_{K^s}(G) \le \left|\left(-\mu_{\delta,S} - \frac{1}{2}D^c v_{\delta,S}, 0\right)\right|_{K^s}(G) \\
&+ \left|\left(\mu_{\delta,S} - \frac{1}{2}D^c v_{\delta,S}, 0\right)\right|_{K^s}(G) = |(-D^c v_{\delta,S}), 0)|_{K^s}(G),
\end{aligned} \tag{4.5.30}$$

where the inequality in the first line comes from Lemma 4.5.2 applied to $b_{\delta,S} - \frac{1}{2}v_{\delta,S}$ and $-b_{\delta,S} - \frac{1}{2}v_{\delta,S}$ with $v_{\delta,S} = 1_{\{\delta<v<S\}}v$), (see in particular (4.5.23)), whereas the equality is a consequence of Lemma 4.2.5 applied to the two Radon measures $\mu_{\delta,S} - \frac{1}{2}D^c v_{\delta,S}$ and $-\mu_{\delta,S} - \frac{1}{2}D^c v_{\delta,S}$ together with Remark 4.2.7 having in mind (1.3.14). Since $b_{\delta,S} \in GBV(\mathbb{R}^{n-1})$ and $v_{\delta,S} \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$, if $W = W[v_{\delta,S}, b_{\delta,S}]$, then we can compute $P_{K^s}(W; G \times \mathbb{R})$ for every Borel set $G \subset \mathbb{R}^{n-1}$ by Corollary 4.4.11. In particular, if $G \subset \{\delta < v < S\}^{(1)}$, then by $E \cap (\{\delta < v < S\} \times \mathbb{R}) = W \cap (\{\delta < v < S\} \times \mathbb{R})$, we find that

$$P_{K^s}(E; G \times \mathbb{R}) = P_{K^s}(W; G \times \mathbb{R}) \tag{4.5.31}$$

$$= \int_G \phi_{K^s}\left(\nabla\left(b_{\delta,S} - \frac{v_{\delta,S}}{2}\right), 1\right) + \phi_{K^s}\left(-\nabla\left(b_{\delta,S} + \frac{v_{\delta,S}}{2}\right), 1\right) d\mathcal{H}^{n-1} \tag{4.5.32}$$

$$+ \int_{G \cap J_v} \min\left(v^{\vee}_{\delta,S}, \left(\left[\frac{v_{\delta,S}}{2}\right] + [b_{\delta,S}] + \max\left(\left[\frac{v_{\delta,S}}{2}\right] - [b_{\delta,S}], 0\right)\right)\right) \phi_{K^s}(-\nu_v, 0) d\mathcal{H}^{n-2} \tag{4.5.33}$$

$$+ \int_{G \cap J_v} \min\left(v^{\wedge}_{\delta,S}, \max\left(0, [b_{\delta,S}] - \left[\frac{v_{\delta,S}}{2}\right]\right)\right) \phi_{K^s}(\nu_v, 0) d\mathcal{H}^{n-2} \tag{4.5.34}$$

$$+ \int_{G \cap (J_b \setminus J_v)} \min\left([b_{\delta,S}], \tilde{v}\right) \left(\phi_{K^s}(-\nu_b, 0) + \phi_{K^s}(\nu_b^j, 0)\right) d\mathcal{H}^{n-2} \tag{4.5.35}$$

$$+ \left|\left(D^c\left(b_{\delta,S} - \frac{v_{\delta,S}}{2}\right), 0\right)\right|_{K^s}(G) \tag{4.5.36}$$

$$+ \left|\left(-D^c\left(b_{\delta,S} + \frac{v_{\delta,S}}{2}\right), 0\right)\right|_{K^s}(G) \tag{4.5.37}$$

We can also compute $P_{\phi_{K_s}}(F[v_{\delta,S}]; G \times \mathbb{R})$. Taking also into account that $F[v] \cap (\{\delta < v < S\} \times \mathbb{R}) = F[v_{\delta,S}] \cap (\{\delta < v < S\} \times \mathbb{R})$ we obtain that

$$P_{K^s}(F[v]; G \times \mathbb{R}) = P_{K^s}(F[v_{\delta,S}]; G \times \mathbb{R}) = 2\int_G \phi_{K^s}\left(-\nabla\left(\frac{v_{\delta,S}}{2}\right), 1\right) d\mathcal{H}^{n-1}$$
$$+ \int_{G \cap J_{v_{\delta,S}}} [v]\phi_{K^s}(-\nu_v, 0) d\mathcal{H}^{n-2} + 2\int_G \phi_{K^s}\left(-\frac{dD^c\left(\frac{v_{\delta,S}}{2}\right)}{d\left|D^c\left(\frac{v_{\delta,S}}{2}\right)\right|}, 0\right) d\left|D^c\left(\frac{v_{\delta,S}}{2}\right)\right|.$$

Firstly, applying (2.0.19) to $b_E$ and (2.0.15) and $v$ we get

$$\nabla b_{\delta,S}(x) = \nabla b_E(x), \quad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x \in \{\delta < v < S\},$$
$$[v] = [v_{\delta,S}], \quad \textit{for } \mathcal{H}^{n-2}\textit{-a.e. on } \{\delta < v < S\}^{(1)}.$$

Putting together the above relations with the assumptions (1.3.12) and (1.3.13) we deduce that, for $\mathcal{H}^{n-1}$-a.e. $x \in \{\delta < v < S\}$ there exists $z(x) \in \partial K^s$ s.t.

$$\left\{\left(-\frac{1}{2}\nabla v(x) + t\nabla b_E(x), 1\right) : t \in [-1, 1]\right\} \subset C^*_{K^s}(z(x)), \tag{4.5.38}$$

$$2[b_{\delta,S}] = 2[b_E] \leq [v] = [v_{\delta,S}], \quad \textit{for } \mathcal{H}^{n-2}\textit{-a.e. on } \{\delta < v < S\}^{(1)}. \tag{4.5.39}$$

Thanks to Proposition 4.1.22 and Remark 4.1.23, condition (4.5.38) is equivalent to say that we can rewrite (4.5.32) in the following way

$$\int_G \phi_{K^s}\left(\nabla\left(b_{\delta,S} - \frac{v_{\delta,S}}{2}\right), 1\right) + \phi_{K^s}\left(-\nabla\left(b_{\delta,S} + \frac{v_{\delta,S}}{2}\right), 1\right) d\mathcal{H}^{n-1}$$
$$\int_G \phi_{K^s}\left(\nabla\left(b_E - \frac{v}{2}\right), 1\right) + \phi_{K^s}\left(-\nabla\left(b_E + \frac{v}{2}\right), 1\right) d\mathcal{H}^{n-1}$$
$$= 2\int_G \phi_{K^s}\left(-\nabla\left(\frac{v}{2}\right), 1\right) d\mathcal{H}^{n-1}. \tag{4.5.40}$$

Furthermore, substituting (4.5.39) into (4.5.33),(4.5.34) and (4.5.35), and using (4.5.30) applied to (4.5.36) and (4.5.37), we find that

$$P_{K^s}(E; \{\delta < v < S\}^{(1)} \times \mathbb{R}) \leq P_{K^s}(F[v]; \{\delta < v < S\}^{(1)} \times \mathbb{R}), \tag{4.5.41}$$

where, actually, equality holds thanks to (AS). Recalling that by [22, 69, 4.5.9(3)] we have that $\mathcal{H}^{n-2}(\{v^\vee = \infty\}) = 0$, thanks to (2.0.14) it follows that

$$\bigcup_{M \in I} \{v < M\}^{(1)} = \{v^\vee < \infty\} =_{\mathcal{H}^{n-2}} \mathbb{R}^{n-1}. \tag{4.5.42}$$

By (2.0.14) if we consider the sequences $\delta_h \in I$ and $S_h \in I$ such that $\delta_h \to 0$ and $S_h \to 0$ as $h \to \infty$ we get

$$\{v^\vee > 0\} = \bigcup_{h \in \mathbb{N}} \{\delta_h < v^\vee < S_h\}^{(1)}.$$

So, by the above relation together with (4.5.41), and (4.5.42) we get that

$$P_{K^s}(E; \{v^\wedge > 0\} \times \mathbb{R}) \leq P_{K^s}(F[v]; \{v^\wedge > 0\} \times \mathbb{R}).$$

By Proposition 4.5.1 $P_{K^s}(E; \{v^\wedge = 0\} \times \mathbb{R}) = P_{K^s}(F[v]; \{v^\wedge = 0\} \times \mathbb{R})$ and thus $P_{K^s}(E) = P_{K^s}(F[v])$. This concludes the proof. $\square$

## 4.6 Rigidity for the Steiner's inequality for the anisotropic perimeter

Let us start the section with the proof of Theorem 1.3.5.

*(Proof of Theorem 1.3.5).* By Theorem 1.1.4 we have to prove that conditions (1.1.9)-(1.1.11) holds true. We divide the proof in few steps.

**Step 1** In this step we prove that (1.1.9) holds true. Since $E \in \mathcal{M}_{K^s}(v)$, by Theorem 1.3.2 we have that condition (1.3.12) holds true, namely for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$ there exists $z(x) \in \partial K^s$ s.t.

$$\left(-\frac{1}{2}\nabla v(x) + t\nabla b_E(x), 1\right) \in C^*_{K^s}(z(x)) \quad \forall\, t \in [-1, 1].$$

By condition **R1** we have that for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$ there exists $z(x) \in \partial K^s$ s.t. $\forall\, t \in [-1, 1]$ there exists $\lambda = \lambda(t, x) \in [0, 1]$ such that

$$(t\nabla b_E(x), 0) = \lambda\left(-\frac{1}{2}\nabla v(x), 1\right).$$

that implies $\nabla b_E = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$, that implies $\nabla b_E = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^{n-1}$.

**Step 2** In this step we prove that (1.1.11) holds true. Again, since $E \in \mathcal{M}_{K^s}(v)$ we know that condition (1.3.14) holds true, namely we know that for $|D^c v|$-a.e. $x \in \{v^\wedge > 0\}$ there exists $z(x) \in \partial K$ s.t.

$$h(x) + tg(x) \in C^*_K(z(x)), \quad \forall t \in [-1,1]. \tag{4.6.1}$$

So, by condition **R2** we know that for $|D^c v|$-a.e. $x \in \{v^\wedge > 0\}$ there exists $\lambda = \lambda(x) \in [-1,1]$ such that $g(x) = \lambda h(x)$. By definition of $g(x)$ and $h(x)$, for every Borel set $G \subset \mathbb{R}^{n-1}$, every $M > 0$, and $\mathcal{H}^1$-a.e. $\delta > 0$ we have

$$\begin{aligned} D^c(\tau_M(b_\delta))(G) &= \int_{G\cap\{|b_\delta|<M\}^{(1)}\cap\{v>\delta\}^{(1)}} g(x) d\left|\left(\frac{1}{2}D^c v, 0\right)\right|_K(x) \\ &= \int_{G\cap\{|b_\delta|<M\}^{(1)}\cap\{v>\delta\}^{(1)}} \lambda(x)h(x) d\left|\left(\frac{1}{2}D^c v, 0\right)\right|_K(x) \\ &= \int_{G\cap\{|b_\delta|<M\}^{(1)}\cap\{v>\delta\}^{(1)}} -\frac{1}{2}\lambda(x) dD^c v(x). \end{aligned}$$

Since $-\frac{1}{2}\lambda(x) \in [-1/2, 1/2]$ for $|D^c v|$-a.e. $x \in \{v^\wedge > 0\}$, we conclude the proof of step 2.
**Step 3** In this step we prove that (1.1.12) and (1.1.13) holds true. By step 2 we have that (1.1.11) holds true. By taking the total variation in (1.1.11) we find that $2|D^c(\tau_M(b_\delta))|(G) \leq |D^c v|(G)$ for every bounded Borel set $G \subset \mathbb{R}^{n-1}$. By passing to the limit for $M \to +\infty$ (in $J_\delta$) and then $\delta \to 0$ (in $I$) we prove (1.1.12). As observed in [11, Remark 1.10], note that (1.1.13) is a consequence of (1.1.7), taking into account (1.1.9), (1.1.11) and (1.1.12). This concludes the proof. □

Studying whether conditions **R1** and **R2** hold true leads us to the following result, that, roughly speaking, provides a geometric characterization for those conditions to hold true. In the following, given any set $G \subset \mathbb{R}^n$ we denote by $\overline{G}$ its topological closure. Having in mind definitions of exposed and extreme points (see Definitions 4.1.31 and 4.1.30 respectively), we can now prove the following Proposition.

**Proposition 4.6.1.** *Let $v$ be as in (1.1.3) and let $K \subset \mathbb{R}^n$ be as in (1.3.3). For $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$ let us call $\nu(x) = \left(-\frac{1}{2}\nabla v(x), 1\right)$. Then,*

$$\textbf{\textit{R1}} \text{ holds true} \iff \frac{\nu(x)}{\phi_{K^s}(\nu(x))} \text{ is an extreme point of } \overline{(K^s)^*} \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \{v > 0\}. \tag{4.6.2}$$

$$\textbf{\textit{R2}} \text{ holds true} \iff \frac{h(x)}{\phi_{K^s}(h(x))} \text{ is an extreme point of } \overline{(K^s)^*} \text{ for } |D^c v|\text{-a.e. } x \in \{v^\wedge > 0\}, \tag{4.6.3}$$

*where $h$ has been defined in (1.3.15).*

*Proof.* Let us prove that (4.6.2) holds true, then statement (4.6.3) follows using an identical argument.

**Step 1** Let us assume that **R1** holds true and suppose by contradiction that there exist $G \subset \{v > 0\}$ such that $\mathcal{H}^{n-1}(G) > 0$ and $\nu(x)/\phi_{K^s}(\nu(x))$ is not an extreme point for $\mathcal{H}^{n-1}$-a.e. $x \in G$. In particular there exist $y(x) \neq z(x) \in \overline{(K^s)^*}$ and $\lambda(x) \in (0,1)$ such that

$$\frac{\nu(x)}{\phi_{K^s}(\nu(x))} = (1-\lambda(x))z(x) + \lambda(x)(y(x)).$$

By Lemma 4.1.27 this implies that

$$(1-\lambda)z(x) + \lambda y(x) \in \partial\phi^*_{K^s}(z) \quad \forall\, \lambda \in [0,1],\, \forall\, z \in \mathcal{Z}_{K^s}\left(\frac{\nu(x)}{\phi_{K^s}(\nu(x))}\right).$$

In particular this implies that

$$(1-\lambda)\phi_{K^s}(\nu(x))z(x) + \lambda\phi_{K^s}(\nu(x))y(x) \in C^*_{K^s}(z) \quad \forall\, \lambda \in [0,1],\, \forall\, z \in \mathcal{Z}_{K^s}\left(\frac{\nu(x)}{\phi_{K^s}(\nu(x))}\right), \tag{4.6.4}$$

where recall that $\mathcal{Z}_{K^s}\left(\nu(x)/\phi_{K^s}(\nu(x))\right) = \mathcal{Z}_{K^s}\left(\nu(x)\right)$. Since (4.6.4) holds true for $\mathcal{H}^{n-1}$-a.e. $x \in G$ and $\mathcal{H}^{n-1}(G) > 0$, we contradicted our assumptions.

**Step 2** Let us now assume that $\nu(x)/\phi_{K^s}\left(\nu(x)\right)$ is an extreme point of $\overline{(K^s)^*}$ for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$, and suppose by contradiction that **R1** is not verified, namely that there exists $y \in \mathbb{R}^n$, and $G \subset \{v > 0\}$ with $\mathcal{H}^{n-1}(G) > 0$ such that, for $\mathcal{H}^{n-1}$-a.e. $x \in G$ there exists $z \in \mathcal{Z}_{K^s}(\nu(x))$ such that,

$$\textit{if } \nu(x) \pm y \in C^*_{K^s}(z) \quad \Rightarrow \quad y \neq \lambda\nu(x), \quad \textit{for every } \lambda \in [-1,1].$$

In particular, by convexity,

$$(1-\lambda)\left(\nu(x)+y\right) + \lambda\left(\nu(x)-y\right) \in C^*_{K^s}(z), \quad \forall\, \lambda \in [0,1].$$

But this implies that the projection of this segment over $\partial\phi^*_{K^s}(z)$ contains in its relative interior the point $\nu(x)/\phi_{K^s}(\nu(x))$, namely there exists $\lambda(x) \in (0,1)$ such that

$$\frac{\nu(x)}{\phi_{K^s}(\nu(x))} = (1-\lambda(x))\frac{(\nu(x)+y)}{\phi_{K^s}\left(\nu(x)+y\right)} + \lambda(x)\frac{(\nu(x)-y)}{\phi_{K^s}\left(\nu(x)-y\right)}. \tag{4.6.5}$$

Since (4.6.5) holds true for $\mathcal{H}^{n-1}$-a.e. $x \in G$ and $\mathcal{H}^{n-1}(G) > 0$ we contradicted our assumptions. This concludes the proof. $\square$

**Lemma 4.6.2.** *Let $K \subset \mathbb{R}^n$ be as in (1.3.3) and let us consider the following set:*

$$\mathbb{V}_{K^s} := \left\{\nu^{K^s}(x) : \, x \in \partial^* K^s\right\}. \tag{4.6.6}$$

*Then, $y$ is an exposed point of $\overline{(K^s)^*}$ if and only if $y = \eta/\phi_{K^s}(\eta)$ for some $\eta \in \mathbb{V}_{K^s}$.*

*Proof.* This result is the direct consequence of Lemma 4.1.33 using $g = \phi^*_{(K^s)}$ and observing that $\partial\phi^*_{K^s}(x) = \nu^{K^s}(x)/\phi_{K^s}(\nu^{K^s}(x))$ for every $x \in \partial^* K^s$. □

**Lemma 4.6.3.** *Let $v$ be as in (1.1.3) and let $K \subset \mathbb{R}^n$ be as in (1.3.3). Moreover, assume that for $\mathcal{H}^{n-1}$-a.e. $z \in \{v > 0\}$, and for $|D^c v|$-a.e. $z \in \{v^\wedge > 0\}$ there exists a sequence $(\nu_h)_{h\in\mathbb{N}} \subset \mathbb{V}_{K^s}$ such that*

$$\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right) = \lim_{h\to+\infty} \nu_h. \tag{4.6.7}$$

*Then, conditions **R1**, **R2** hold true.*

*Proof.* By the positivity and continuity of the function $\phi_{K^s}$, together with the fact that $|\nu_h| = 1$ for every $h \in \mathbb{N}$, we know that condition (4.6.7) is equivalent to

$$\frac{\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right)}{\phi_{K^s}\left(\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right)\right)} = \lim_{h\to+\infty} \frac{\nu_h}{\phi_{K^s}(\nu_h)}.$$

Thus, by Remark 4.1.32,

$$\frac{\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right)}{\phi_{K^s}\left(\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right)\right)} \quad \textit{is an extreme point of } \overline{(K^s)^*}. \tag{4.6.8}$$

By Theorem 4.4.2, together with the 1-homogeneity of $\phi_{K^s}$ we know that

$$\frac{\left(-\frac{1}{2}\nabla v(z), 1\right)}{\phi_{K^s}\left(\left(-\frac{1}{2}\nabla v(z), 1\right)\right)} = \frac{\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right)}{\phi_{K^s}\left(\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right)\right)} \quad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } z \in \mathbb{R}^{n-1},$$

and,

$$\frac{(h(z), 0)}{\phi_{K^s}((h(z), 0))} = \frac{\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right)}{\phi_{K^s}\left(\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right)\right)} \quad \textit{for } |D^c v|\textit{-a.e. } z \in \{v^\wedge > 0\},$$

where we recall that

$$(h(z), 0) = \left(\frac{-dD^c v}{d|(D^c v, 0)|_{K^s}}(z), 0\right) \qquad \textit{and} \qquad \nu^{F[v]}\left(z, \frac{1}{2}v(z)\right) = \left(-\frac{dD^c v}{d|D^c v|}(z), 0\right).$$

Therefore, thanks to the above relations together with condition (4.6.8) and Proposition 4.6.1 we conclude. □

*Proof of Corollary 1.3.7.* Thanks to the above result, the proof of Corollary 1.3.7 follows as a direct consequence. □

*Proof of Corollary 1.3.8.* To prove Corollary 1.3.8 we have to notice that thanks to [28, Corollary 3, Theorem 1]), every point in $\partial(K^s)^*$ is an exposed point, so by Lemma 4.6.2 we have that $\mathbb{V}_{K^s}$ coincides with $\mathbb{S}^{n-1}$. Therefore, the assumption of Corollary 1.3.7, namely for $\mathcal{H}^{n-1}$-a.e. $z \in \{v > 0\}$, and for $|D^c v|$-a.e. $z \in \{v^\wedge > 0\}$ there exists $x \in \partial^* K^s$ such that $\nu^{F[v]}\left(z, \frac{1}{2}v(z)\right) = \nu^{K^s}(x)$, is always verified. This concludes the proof. □

**Remark 4.6.4.** *Given any $K \subset \mathbb{R}^n$ as in (1.3.3), thanks to Corollary 1.3.7, it is possible to construct simple examples of functions $v$ defined as in (1.1.3) such that $\mathcal{M}_{K^s}(v) \subset \mathcal{M}(v)$ (see for instance Figure 1.3.6). Indeed, let $K \subset \mathbb{R}^n$ be as (1.3.3) and let $x \in \partial^* K^s$, with $\boldsymbol{q}(x) > 0$ such that $\boldsymbol{q}(\nu^{K^s}(x)) > 0$. Recall that such a point always exists. In fact, by Theorem 4.1.1 applied to $K^s$, we know that for $\mathcal{H}^{n-1}$-a.e. $z \in \boldsymbol{p}(K^s)$, $\boldsymbol{q}(\nu^{K^s}(z,t)) \neq 0$ provided $(z,t) \in \partial^* K^s$. Moreover, the fact that we chose $\boldsymbol{q}(x) > 0$, by the convexity of $K^s$, implies that $\boldsymbol{q}(\nu^{K^s}(x)) > 0$. Let us call $\omega = \boldsymbol{p}(\nu^{K^s}(x))/|\boldsymbol{p}(\nu^{K^s}(x))| \in \mathbb{R}^{n-1}$ and let $\Omega \subset H^-_\omega \cap \mathbb{R}^{n-1}$ be an open bounded set. Let us now consider the function $v : \Omega \to (0,+\infty)$ defined as*

$$v(z) := 2\frac{\boldsymbol{p}(\nu^{K^s}(x)) \cdot z}{\boldsymbol{q}(\nu^{K^s}(x))}.$$

*By construction, such a function satisfies both (1.1.3) and the assumptions of Corollary 1.3.7. Therefore, $\mathcal{M}_{K^s}(v) \subset \mathcal{M}(v)$.*

# Bibliography

[1] L. Ambrosio, V. Caselles, S. Masnou, and J. Morel, *Connected components of sets of finite perimeter and applications to image processing*, J. Eur. Math. Soc. (JEMS), 3, pp. 39-92, 2001.

[2] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Science Publications, 2000. 23, 26, 29, 30, 32, 34, 39, 65, 66, 70, 79, 93, 110, 124, 125

[3] M. Barchiesi, F. Cagnetti, N. Fusco, *Stability of the Steiner symmetrization of convex sets*, J. Eur. Math. Soc. 15, 2013. 1, 9, 12, 41, 45, 49

[4] T. Bartsch, T. Weth, M. Willem, *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math., 96, pp. 1-18, 2005. 6

[5] A. Baernstein, *A unified approach to symmetrization*, Symposia Matematica, 35, pp. 47-91, 1995. 6

[6] I , Birindelli, F Leoni, F Pacella, *Symmetry and spectral properties for viscosity solutions of fully nonlinear equations*, J. Math. Pures Appl. 107(9), pp. 409-428, 2017. 6

[7] V. Bögelein, F. Duzar and N. Fusco, *A quantitative isoperimetric inequality on the sphere*, Adv. Calc. Var. (3), pp 223-265, 2017. 35

[8] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011.

[9] J. E. Brothers and W. P. Ziemer, *Minimal rearrangements of Sobolev functions*, J. Reine Angew. Math., 384, pp. 153-179, 1988.

[10] F. Cagnetti, M. Colombo, G. De Philippis, and F. Maggi, *Essential connectedness and the rigidity problem for Gaussian symmetrization*, Journal of the European Mathematical Society, 19 (2). pp. 395-439, 2017. 2, 12

[11] F. Cagnetti, M. Colombo, G. De Philippis, and F. Maggi, *Rigidity of equality cases in Steiner's perimeter inequality*, Anal. PDE 7 1535-1593, 2014. 1, 3, 4, 5, 6, 12, 29, 94, 106, 118, 119, 124, 129

[12] F. Cagnetti, M. Perugini, and D. Stöger, *Rigidity for perimeter inequality under spherical symmetrisation*, (in preparation), 2018. 1

[13] G. M. Capriani, *The Steiner rearrangement in any codimension*, Calc. Var. Partial Differential Equations, 49, pp. 517-548, 2014.

[14] M. Chlebík, A. Cianchi and N. Fusco, *The perimeter inequality under Steiner symmetrization: Cases of equality*, Annals of Mathematics, 162, 525-555., 2005. 1, 3, 4, 9, 12, 41, 73, 74, 75, 96, 98, 99

[15] A. Cianchi and N. Fusco, *Functions of bounded variation and rearrangements*, Arch. Ration. Mech. Anal., 165, pp. 1-40, 2002.

[16] ——, *Steiner symmetric extremals in Pólya-Szegö type inequalities*, Adv. Math., 203, pp. 673-728, 2006. 17, 20, 22, 98

[17] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli, *On the isoperimetric deficit in Gauss space*, Amer. J. Math., 133, pp. 131-186, 2011. 2

[18] J. Dalphin, *Some Characterizations of a uniform ball property*, ESAIM Proc. Surveys, 45, EDP Sci., Les Ulis, 2014. 91

[19] E. De Giorgi, *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita*, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. Sez. I, 1958. 4

[20] ——, *Selected papers*, Springer Collected Works in Mathematics, Springer, Heidelberg, 2013. [Author name on title page: Ennio Giorgi], Edited by Luigi Ambrosio, Gianni Dal Maso, Marco Forti, Mario Miranda and Sergio Spagnolo, Reprint of the 2006 edition.

[21] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, revised edition, 2015. 3, 7

[22] H. Federer, *Geoemtric measure theory*, volume 153 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag New york Inc., New York, xiv+676 pp pp, 1969. 122, 128

[23] I. Fonseca, S. Müller, *A uniqueness proof for the Wulff theorem.* Proc. R. Soc. Edinb. Sect. A, 119/1-2, pp. 125-136, 1991. 16

[24] R. F. Gariepy, W. P. Ziemer, *Modern real analysis*, PWS publishing company, 1995.

[25] M. Giaquinta, G. Modica, J. Soucek, *Cartesian Currents in the Calculus of Variations. I. Cartesian currents*, vol. 37 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 1998. 23, 70, 101, 102, 107

[26] B. Gidas, W.M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., 68, pp. 209-243, 1979. 6

[27] M.E. Gurtin, *On a theory of phase transitions with interfacial energy*, Arch. Ration. Mech. Anal., 87/3, pp. 187-212, 1985. 16

[28] H. Han and T. Nishimura, *Strictly convex Wulff shapes and $C^1$ convex integrands*, Proc. Amer. Math. Soc. 145, pp. 3997-4008, 2017. 131

[29] C. Herring, *Some theorems on the free energies of crystal surfaces*, Phys. Rev., 82, pp. 87-93, 1951. 16

[30] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, vol. 1150 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1985. 1

[31] ——, *On the isoperimetric nature of a rearrangement inequality and its consequences for some variational problems*, Arch. Rational Mech. Anal., 94, pp. 227-243, 1986. 1

[32] F. Maggi, *Sets of Finite Perimeter and Geometrical Variational Problems. An Introduction to Geometric Measure Theory*, Cambridge University Press, 2012. 4, 15, 23, 25, 26, 37, 68, 75, 78, 79, 99, 106, 112

[33] I. McGillivray, *An isoperimetric inequality in the plane with a log-convex density*, Ric. Mat., 67, No. 2, pp. 817-874, 2018. 9

[34] F. Morgan, A. Pratelli, *Existence of isoperimetric regions in $\mathbb{R}^n$ with density* Ann. Global Anal. Geom. 43, no. 4, pp. 331-365, 2013. 9, 12

[35] M. Perugini, *Rigidity of equality cases in Steiner's anisotropic perimeter inequality* (in preparation), (2018). 1

[36] G. Pólya, *Sur la symétrisation circulaire.* (French) C. R. Acad. Sci. Paris 230, pp 25-27, 1950. 6, 9, 12

[37] R. Tyrrell Rockafellar, *Convex analysis*, Princeton University Press, 1970. 77, 88, 89

[38] J. Serrin, *A symmetry property in potential theory*, Arch. Rational Mech. Anal., 4, pp 304-318, 1971. 6

[39] L. Simon, *Lectures on Geometric Measure Theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, Volume 3, 1983. 23, 35, 36

[40] D. Smets, M. Willem, *Partial symmetry and asymptotic behavior for some elliptic variational problems*, Calc. Var. Partial Differ. Equ., 18, pp 57-75, 2003. 6

[41] E. Schmidt, *Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und sphärischen Raum jeder Dimensionszahl*, Math. Z. 49, pp 1-109, 1943/44. 36

[42] D. Spector, *Simple proofs of some results of some results of Reshetnyak*, Proceedings of the American Mathematical Society, Volume 139, Number 5, pages 1681-1690, May 2011.

[43] J.E. Taylor, *Crystalline variational problems*, Bull. Am. Math. Soc., 84/4, 568–588, 1978. 16

[44] A. I. Vol′ pert, *Spaces* BV *and quasilinear equations*, Mat. Sb. (N.S.), 73 (115), pp. 255-302, 1967. 36

[45] G. Wulff, *Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Kristallflächen*, Z. Kristallogr., 34, 449–530, 1901. 16