Trace formulas for Dirac operators with applications to resonances

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Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

Most of the work presented in this thesis was done in collaboration with my supervisor Professor Michael Melgaard. The material contained in this thesis is original and my own work except where otherwise stated.

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Bobby Ho Yene Cheng, Doctor of Philosophy

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Summary

Motivated by their appearance in the physical sciences, scattering resonances of the three-dimensional Dirac operator perturbed by a real-valued, smooth, compactly supported, electric potential are studied. The potentials are 4×4 matrix-valued, multiplication operators. Under a prescribed mapping, the cut-off full resolvent is extended meromorphically from the physical half-plane to the whole complex plane. The poles that lie in the unphysical plane are defined as resonances for the perturbed Dirac operator.

This thesis presents basic properties of the free and full Dirac resolvents and introduces the resonances that the latter creates. Particular attention is paid to the resonances appearing at the threshold points when the full resolvent is studied near these limits.

The scattering matrix is analysed as a mapping between solutions of the Dirac eigenvalue problem and then used to establish the Birman-Kreĭn formula, which relates the trace difference between functions of the full and free Dirac operators. In turn, a Poisson wave trace formula in the distributional sense is established via an upper bound counting function and factorization of the scattering matrix determinant.

Both trace formulas are generalized such that resonances appearing at the threshold points are considered. Finally, under further restrictions on the potential, the existence of infinitely many Dirac resonances is proved as an application of our trace formulas.

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Chapter 1

Introduction

1.1 Resonances in the physical sciences

In physical scattering processes between two subatomic particles, we can for simplicity, consider three outcomes: the two particles could first bind to form a new and stable composite particle, secondly collide and scatter apart, or finally bind together for a short time before decaying into smaller particles.

In the final case, the short-lived composite particle is considered a scattering resonance if the lifetime τ is typically of order 10^{-23} seconds or less. Experimentally, resonances are identified with a large cross-sectional peak at a characteristic energy ξ and associated half-height width Γ (see Figure 1.1). Here the cross section, σ, is a measure of the effective area for scattering to occur and, in the case for resonances, is typically modelled by a Breit-Wigner distribution. This can be written as

$$\sigma(E) = \frac{1}{2\pi} \frac{\Gamma}{(E-\xi)^2 + (\Gamma/2)^2},$$

where E is the laboratory energy.

Since resonances are characterized by a peak energy ξ and lifetime $\tau \propto \Gamma^{-1}$, it is useful to denote them by the complex number $\rho = \xi - i\Gamma/2$. This is motivated by the following observation in quantum mechanics. The motion of a quantum particle of mass $m > 0$ under the influence of a potential V is described by the wavefunction $\phi(\boldsymbol{x}, t)$ that solves the time-dependent Schrödinger equation

$$i \frac{\partial}{\partial t} \phi(\boldsymbol{x}, t) = H_V \phi(\boldsymbol{x}, t),$$

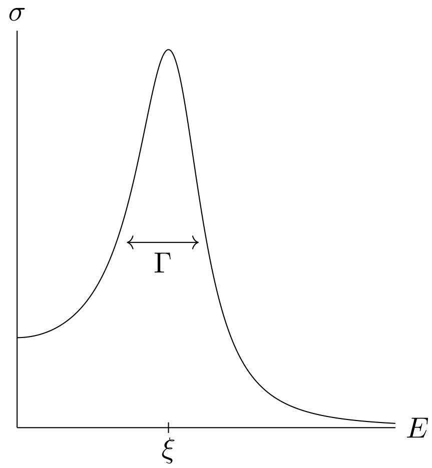

Figure 1.1: Idealized cross section profile for a scattering process, plotted against laboratory energy E. The resonance corresponds to the energy peak ξ with width Γ.

where $H_V = H_0 + V = -\Delta + V = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + V$ is the perturbed Schrödinger operator on $L^2(\mathbb{R}^3)$. This is analogous to solving Newton's second law of motion in classical mechanics. Furthermore if $\phi_0(\boldsymbol{x}) = \phi(\boldsymbol{x}, 0) \in L^2(\mathbb{R}^3)$ solves the time-independent Schrödinger equation, $H_V\phi_0 = \xi\phi$, then we have

$$\phi(\boldsymbol{x}, t) = e^{-it\xi}\phi_0(\boldsymbol{x}).$$

The quantum mechanical interpretation of $|\phi(\boldsymbol{x}, t)|^2$ is the probability density of finding the particle at position $\boldsymbol{x}$ and at time t. Moreover $\int_\Omega |\phi(\boldsymbol{x}, t)|^2 \,\mathrm{d}\boldsymbol{x}$ is the probability of locating the particle inside Ω at a given t. If instead we consider a resonance state where $\phi_0 \notin L^2(\mathbb{R}^3)$ then the evolved state is written as

$$\phi(\boldsymbol{x}, t) = e^{-it(\xi - i\Gamma/2)}\phi_0(\boldsymbol{x})$$

such that

$$\frac{|\phi(\boldsymbol{x}, t)|^2}{|\phi_0(\boldsymbol{x})|^2} = e^{-\Gamma t}.$$

Hence for increasing values of time, the Breit-Wigner model suggests that the probability density of the resonant state decreases exponentially. In practice it has decayed in 10^{-23} seconds or less.

Despite their short lives, resonances play a central role in particle physics. In fact many of the subatomic particles discovered as a result of scattering experiments are indeed resonances. Like their stable counterparts, they may possess well defined properties such as mass, electric charge and quantum spin. Amongst the high energy hadrons, these include the delta baryons and rho mesons (see for instance [50, section 3.5]). The Feynman diagram in Figure 1.2 demonstrates the creation of a delta baryon resonance as a result of a pion-proton scattering process before separating back into its two constituent particles.

More generally, resonances have been studied in various applications from atomic physics and quantum chemistry. These range from Stark and Zeeman effects (see [7, 13]) to numerical models of chaotic scattering (see [26, 49])

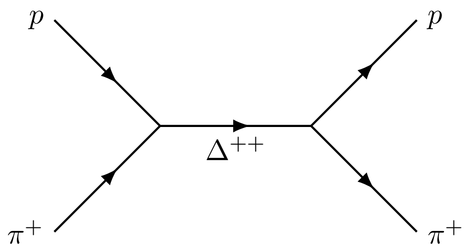

Figure 1.2: Feynman diagram of a pion (π^+) - proton (p) scattering process which temporarily creates a delta baryon resonance (Δ^{++}).

1.2 Resonances in mathematical physics

As described in the previous section, resonances are metastable states with a well defined energy and rate of decay. These physical attributes can be encapsulated mathematically as generalizations of eigenvalues with energies that can scatter to infinity (see for instance [88]). Such studies of resonances using the spectral properties of the system can be conducted with numerous methods. These include perturbations of eigenvalues, or studies of various mathematical objects such as the cut-off resolvent, the extended scattering matrix, or even the zeta function. See [54], [12], [67], and [22] respectively for examples of these studies. Despite the numerous definitions and approaches to studying resonances, the following two key questions are often explored:

1. Existence: Do resonances exist for a given Hamiltonian and class of potentials?
2. Counting: If resonances do exist, then how many are there? In particular, is it possible to establish upper and lower bounds?

The answers to these problems are richly studied in the case of Schrödinger operators. One important area of research stems from the method of complex scaling. This sees the previously selfadjoint Hamiltonian, H_V, modified by a complex parameter θ to a family of non-selfadjoint operators, $H_V(\theta)$, such that $H_V = H_V(0)$. Then $\operatorname{Im}\theta > 0$ has the effect of rotating the essential spectrum of H_V about the origin. This framework is based upon the work of Aguilar and Combes [2] and Balslev and Combes [3] for 2- and N-body operators respectively. This led Simon [69] to identify resonances of H_V as the eigenvalues of $H_V(\theta)$ in what is known as the Aguilar, Balslev, Combes and Simon (ABCS) theory. A key implementation of semiclassical analysis towards resonances was provided by Helffer and Sjöstrand [32]. The relationship between this and ABCS theory was established by Helffer and Martinez [31].

An interesting application of these techniques is to the shape resonance model originally proposed by Gamow [25] and Gurney and Condon [30]. Consider a quantum particle of energy, E, supposedly trapped inside a potential well, V, of maximum energy $V_{\max}$ where $V_{\max} > E$. Assuming $\lim_{|\boldsymbol{x}|\to\infty} V(\boldsymbol{x}) = 0$, then by quantum tunnelling effects, the particle will travel through the so-called *classically forbidden region* and eventually escape the well. This model has been used to study alpha particle emission from unstable nuclei. Notable works in this area include the contributions by Combes et al. [18] who proved the existence of shape resonances near the real axis, and Sigal [68] who provided bounds on shape resonance lifetimes.

The task of approximating resonance energies E and widths Γ can be achieved by the so-called *complex absorbing potential* (CAP) method. Here the full semiclassical Hamiltonian is perturbed further by a bounded, complex-valued potential that encompasses the bounded support of V and absorbs the diverging part of the solution. The benefits of this are twofold; resonances can then be analysed as bound states and numerical calculations can be considered in finite domains. Despite the CAP method being used extensively in the physical sciences (see for instance [56]), it was not until Stefanov [78] that a rigorous justification was given for its usage.

An alternative approach to studying resonances that this thesis considers exclusively is via the so-called cut-off resolvent $\rho(H_V - \lambda^2)^{-1}\rho$. Given ρ is a compactly supported bump function, we consider the meromorphic extension of the cut-off resolvent from the *physical* $\lambda = \sqrt{z}$ upper half-plane, across the real line, to $\mathbb{C}$. In the lower *unphysical* plane, the first bump function enables us to consider the resolvent as a mapping from $L^2_{\text{comp}}(\mathbb{R}^3)$ to $L^2_{\text{loc}}(\mathbb{R}^3)$. The second bump function therefore ensures the resultant mapping is back to $L^2_{\text{comp}}(\mathbb{R}^3)$. The isolated poles of $\rho(H_V - \lambda^2)^{-1}\rho$ in the unphysical plane are in turn defined as resonances.

In this framework Melrose [52] proved the existence of infinitely many resonances for odd dimension $d \geq 3$ and $V \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$. Here $C_0^\infty(\Omega;\mathbb{R})$ denotes the set of real-valued functions with compact support inside Ω and differentiable to any order. This remarkable result has since been extended to more general classes of potentials (see Sá Barreto and Zworski [63, 64]). Later Smith and Zworski [77] proposed an alternative proof by considering the trace of the difference of Schrödinger heat semigroups (see Hitrik and Polterovich [35, 36] for the heat trace asymptotics). In this thesis (see chapter 7) we prove a similar result for Dirac operators perturbed by potentials with diagonal elements in $C_0^\infty(\mathbb{R}^3,\mathbb{R})$.

Other notable results include the class of complex-valued potentials introduced by Christiansen [16] that have no associated resonances. We also mention here the work of Sjöstrand [72, 73] who presented an alternative trace formula valid for any dimension and lower resonance number bounds using the complex scaling framework. Later Nédélec [57, 58] used similar methods to obtain lower bounds for matrix-valued Schrödinger operators.

Answering questions on resonance existence and number bounds employs some key results that are interesting in their own right. For instance, the expansion of the full resolvent near the threshold is used to study resonances (or half-bound states) at that point. However these expansions were originally used to study the asymptotic behaviour of e^{-itH_V} and can be applied to other aspects of scattering theory such as reflection and transmission coefficients (see for instance [10] and references therein). In chapter 3 we establish the exact form of the full Dirac resolvent near the threshold points. Along with our development of the scattering matrix (see chapter 4), these will be used to prove our two trace formulas.

Motivated by the work of Lifshits [48] in crystal lattice fields, the trace formula attributed to Birman and Kreĭn [9] relates the trace difference between suitable functions of two selfadjoint operators H_V, H_0 and the spectral shift function (see also [39]). Furthermore if $H_V - H_0$ is trace class then it can be shown that the spectral shift function is related to the scattering matrix in a simple manner (see for instance [83] or [62]). Our first trace formula in this thesis is presented in chapter 5 with our interpretation of the Birman-Kreĭn trace formula for the full and free Dirac operators, $\mathbb{D}_V$ and $\mathbb{D}_0$, respectively.

This naturally leads to the Poisson wave trace formula whose first incarnation is attributed to Lax and Phillips [45]. Initially developed for compactly supported obstacle scattering cases in $\mathbb{R}^3$, the original trace formula was also limited to large values of the time-like parameter t. However it was later generalised to lower values of t by Bardos et al. [6], to higher odd dimensions by Menzala and Schonbek [53], and for all $t \neq 0$ by Melrose [51] when applied to resonances. A new approach developed by Zworski [84] based on the scattering matrix determinant provided a new proof that was valid for all $t \in \mathbb{R}$. The Poisson wave trace formula is instrumental to Melrose's proof in [52] as well as the lower bounds on resonance number by Sjöstrand and Zworski [75, 76]. Our Poisson wave trace formula (and second trace formula) is proved in chapter 6 and, similar to the Schrödinger case, is instrumental to our proof for the existence of infinitely many Dirac resonances in chapter 7.

We finish this section by citing a few review publications. A qualitative introduction to the ABCS theory and its application to the shape resonance model can be found in [34]. Simon [70] provides a rigorous overview on complex scaling whilst a review of its implementation in the physical sciences can be found in [55]. The reader is directed to Dyatlov and Zworski [23] (see also [88]) for an in-depth analysis of resonances in the cut-off resolvent framework, as well as [17, 81] and the notes by Sjöstrand [74]. In the same framework, Hislop [33] highlights and proves some of the fundamental theorems whereas the short surveys by Zworski [85, 86] provide motivation for these studies.

1.3 Resonances in relativistic quantum mechanics

The afore-mentioned work on resonances in section 1.2 are based upon perturbations of the Schrödinger operator. However the theory is incomplete since Schrödinger's equation ignores the intrinsic quantum spin of a particle and fails completely in the relativistic regime where particle energies are far greater than its rest mass.

One attempt to solve the latter issue was with the Klein-Gordon equation, a second order differential equation in both temporal and spatial coordinates. However as spin was again not considered, this was applicable only to spinless particles such as pions. Further problems arise from the Klein-Gordon equation in the form of negative energy solutions and the possibility of negative probability densities.

The conundrum was finally solved by Dirac [21] with his celebrated equation now named in his honour. By considering a wave equation with first order derivatives, the relativistic motion of massive spin$-\frac{1}{2}$ particles such as electrons and protons could be analysed. The Dirac equation had the extra benefit of overcoming the negative probability densities observed in the Klein Gordon equation and, more significantly, provided further theoretical justification of negative energy solutions. Indeed we now interpret these solutions as anti-particles, the mirror image of the subatomic particles we see except for an opposite electric charge. For instance the electron and positron are antiparticles of each other.

Therefore to study resonances whilst incorporating the special theory of relativity on spin$-\frac{1}{2}$ particles, perturbations of the Dirac operator are considered. However despite the rich variety of research on Schrödinger resonances, there exists limited studies in extending these results to the relativistic setting.

Amongst the first studies of three-dimensional Dirac operators were Weder [82] and Šeba [66] who adapted the complex scaling method for Schrödinger operators to associate Dirac resonances with eigenvalues of the complex scaled perturbed Dirac Hamiltonian. The microlocal approach initially developed by Helffer and Sjöstrand [32] was extended to the Dirac operator by Parisse [59, 60] who proved the existence of shape resonances in the semiclassical limit near the real axis. We also mention the work of Khochman [38] who established a local trace formula via complex scaling and consequently obtained upper bounds on resonance numbers. Note that this approach does not guarantee the existence of resonances. In the spirit of Sjöstrand [73] and Nédélec [57], Kungsman and Melgaard [43] considered Dirac Hamiltonians perturbed by a smoothly decaying scalar potential. This led to the existence of resonances near the potential extrema, and furthermore to a lower bound on their number.

Balslev and Helffer [4] presented an extended limiting absorption principle for Dirac operators that provides continuation properties of the resolvent and scattering matrix. These ideas, which extend from similar properties for Schrödinger operators with short range potentials by Balslev and Skibsted [5], are used to study local analyticity properties of Dirac resonances.

More recently, inspired by the work of Stefanov [78] for Schrödinger operators, Kungsman and Melgaard [41] likewise rigorously established the CAP method for the Dirac operator. As in the non-relativistic case, it was shown in the semi-classical limit that resonances near the real axis coincide with eigenvalues of the CAP adjusted Hamiltonian, and vice versa. The study was extended to clusters of resonances by the same authors in [42].

We also mention Kungsman and Melgaard [44] who used the Dirac cut-off resolvent to define resonances and subsequently obtained a Poisson wave trace formula. However it is assumed there that the threshold points $\pm m$ are not resonances.

1.4 Thesis overview

We study the Dirac operator $\mathbb{D}_0$ perturbed by an electric potential V. The goal of this thesis is to develop the necessary theory and establish for suitable V, two new trace formulas for the Dirac operator and their relationship to Dirac resonances. To achieve this we introduce a change of variable that allows us to define resonances as poles of the cut-off Dirac resolvent. The work is generalised such that the threshold points may also be resonances. Under our transformation, aside from these threshold points, we assume that resonances of the Dirac operator reside in the lower complex plane. As far as the author is aware, this method has not been previously used to study Dirac resonances. The work in chapters 3, 5, 6 and most of chapter 4 is new in the context of Dirac operators, and culminates with our two trace formulas. A significant application of these trace formulas is presented in chapter 7 whereby the existence of infinitely many resonances is proved. Again this is a new result and, along with the work from the preceding chapters, can be found in [15]. This thesis is divided into the following chapters:

In chapter 2, the Dirac operator is introduced before we develop fundamental properties of the free and full Dirac resolvents including their holomorphic and meromorphic continuations to the whole k-plane respectively. Assumption 2.4.1 summarises the properties of V that we use throughout the thesis, and from which we define resonances in Definition 2.4.3.

We derive properties of the full Dirac resolvent near the threshold points, $\pm m$, in chapter 3. The purpose of this analysis is so that we may account for the possibility of resonances at these points in our trace formulas.

In chapter 4 we define the scattering matrix, $S^{\pm}(k)$, as a mapping between the incoming and outgoing solutions to the Dirac eigenvalue problem. We prove that it can be written as the sum of the identity operator and a trace class operator, and furthermore establish a series of properties for $\det S^{\pm}(k)$ including its logarithmic derivative.

The first of our two trace formulas is derived in chapter 5 where we use the properties of the scattering matrix to form our Birman-Kreĭn trace formula. This follows from a series of resolvent and trace estimates. The threshold resonances are treated explicitly by employing the main result of chapter 3.

In chapter 6 we prove our second trace formula. The Poisson wave trace formula relates resonances to the trace difference of the wave groups. Its construction depends upon the determinant of the scattering matrix, an upper bound on resonance number inside a disc of radius $R > 0$, and the main result of chapter 5.

We present an immediate application of our trace formulas in chapter 7. Under further restrictions on V in Assumption 7.1.1 we prove that there exists infinitely many resonances of the perturbed Dirac operator.

In chapter 8 we summarize the key findings of this thesis and suggest a few open questions that could be studied as an extension to our work. Finally in the Appendix, numerous results used in this thesis from complex analysis, spectral theory, and Fredholm theory are listed.

Chapter 2

Resonances of the Dirac operator

We begin by recalling some basic properties of the Dirac operator in sections 2.1 and 2.2. Their proofs can be found in [79]. For convenience, we use the *natural units* $c = \hbar = 1$ throughout this thesis. The Dirac operator forms part of the celebrated Dirac theory, describing the relativistic motion of spin$-\frac{1}{2}$ particles of mass $m > 0$ free from the influence of any external forces. It was derived formally by Dirac [21] as a result of his attempts to linearise the energy-momentum relation $E = \sqrt{p^2 + m^2}$ before substituting the quantum energy and momentum operators,

$$E \to i\partial_t, \quad p_j \to -i\partial_j,$$

where $\partial_t = \partial/\partial_t$, $\partial_j = \partial/\partial_{x_j}$ for $j = 1, 2, 3$. In quantum field theory, all particles obeying the Dirac equation (or fermions) are interpreted as quantised excitations of a fermionic field. This framework forms the basis of the Standard Model of particle physics.

In section 2.3 we define the resolvent of the free Dirac operator and its integral kernel under the transformation $z \mapsto k(z) = \sqrt{z^2 - m^2}$. This change of variable provides a new interpretation of the resolvent, thus enabling us to study the subsequent properties of the Dirac resolvent and its associated resonances. This includes establishing its far field behaviour and proving its holomorphic extension from the upper *physical* half-plane to $\mathbb{C}$.

Finally in section 2.4 we define the resolvent of the Dirac operator perturbed by an electric potential V. For the remainder of the thesis we assume that V satisfies the conditions set forth in Assumption 2.4.1. We subsequently prove how the cut-off resolvent extends meromorphically to $\mathbb{C}$, with the poles leading directly to our definition of resonances of the perturbed Dirac operator.

2.1 The Dirac operator

The Dirac equation may be written as

$$i\partial_t\psi(\boldsymbol{x},t) = \mathbb{D}_0\psi(\boldsymbol{x},t).$$

The wavefunction $\psi(\boldsymbol{x},t)$ is a complex-valued, n-dimensional vector (or spinor) and, written in its original form,

$$\mathbb{D}_0 = -i\sum_{j=1}^{3}\alpha_j\partial_j + m\beta = -i\boldsymbol{\alpha}\cdot\nabla + m\beta, \qquad \boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3),$$

is the Dirac operator. The n-by-n matrices α_j and β satisfy the relations

$$\begin{aligned} \alpha_j\alpha_k + \alpha_k\alpha_j &= 2\delta_{jk}I_n, \quad j,k = 1,2,3, \\ \alpha_j\beta + \beta\alpha_j &= 0_n, \quad j = 1,2,3, \\ \beta^2 &= I_n, \end{aligned} \tag{2.1}$$

where δ_{jk}, I_n, and 0_n are the Kronecker delta, the n-by-n identity matrix, and the n-by-n null matrix respectively. It can be shown that the dimension n must be even and at least equal to 4. In this thesis we assume that $n = 4$ and for brevity use I to denote the identity matrix. Consequently $\mathbb{D}_0$ and any given scalar perturbation V such that $\mathbb{D}_V = \mathbb{D}_0 + V$ are 4-by-4 matrix operators. In addition, the wavefunction $\psi(\boldsymbol{x},t)$ is a 4-component column vector

$$\psi(\boldsymbol{x},t) = (\psi_j(\boldsymbol{x},t))_{1\le j\le 4} \in \mathbb{C}^4.$$

For completeness we write the 'standard representation' of matrices α_j and β:

$$\alpha_j = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}, \quad j = 1,2,3,$$

where the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ are defined

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Although several choices of α_j and β exist for $n = 4$ (see for instance Thaller [79]), this study is not dependent upon any particular representation, only that the relations in (2.1) hold.

We consider the selfadjoint Dirac operator $\mathbb{D}_0$ acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)^4$ endowed with the inner product

$$\langle \phi\,,\psi\rangle = \int_{\mathbb{R}^3}\sum_{j=1}^{4}\phi(\boldsymbol{x})\overline{\psi(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}, \qquad \phi = (\phi_j)_{1\le j\le 4},\ \psi = (\psi_j)_{1\le j\le 4}.$$

The domain of $\mathbb{D}_0$ is the first order Sobolev space $H^1(\mathbb{R}^3)^4$ with norm

$$\|\psi\|_{H^1(\mathbb{R}^3)^4} = \sqrt{\|\psi\|^2_{L^2(\mathbb{R}^3)^4} + \|\nabla\psi\|^2_{L^2(\mathbb{R}^3)^4}},$$

$$\|\psi\|^2_{L^2(\mathbb{R}^3)^4} = \sum_{i=1}^{4} \|\psi_i\|^2_{L^2(\mathbb{R}^3)}, \quad \|\nabla\psi\|^2_{L^2(\mathbb{R}^3)^4} = \sum_{j=1}^{3}\sum_{i=1}^{4} \|\partial_j\psi_i\|^2_{L^2(\mathbb{R}^3)}.$$

The operator $\mathbb{D}_0$ is itself essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)^4$. In the above we have used the notation for 4-component complex-valued spaces

$$L^2(\mathbb{R}^3)^4 = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 = L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3).$$

The spaces $H^1(\mathbb{R}^3)^4$ and $C_0^\infty(\mathbb{R}^3)^4$ are defined similarly. For brevity we may write $L^2 = L^2(\mathbb{R}^3)^4$ and $H^1 = H^1(\mathbb{R}^3)^4$ when the context is clear. The spectrum of $\mathbb{D}_0$, denoted $\mathrm{spec}(\mathbb{D}_0)$, is

$$\mathrm{spec}(\mathbb{D}_0) = (-\infty, -m] \cup [m, \infty),$$

and is absolutely continuous. The resolvent set $\mathbb{C} \setminus \mathrm{spec}(\mathbb{D}_0)$ is denoted $\rho(\mathbb{D}_0)$.

2.2 Eigenvalues, eigenspaces and diagonalization of the free Dirac operator

Manipulating $\mathbb{D}_0$ in momentum space enables us to treat it as a matrix multiplication operator. This permits easier computations of the spectrum of $\mathbb{D}_0$, particularly when it is diagonalizable. To do this we introduce the Fourier transform

$$(\mathscr{F}\psi)(\boldsymbol{p}) = \widehat{f}(\boldsymbol{p}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

acting upon a suitable integrable function ψ uniquely extended to $L^2(\mathbb{R}^3)^4 = L^2(\mathbb{R}^3, \mathrm{d}\boldsymbol{x})^4$. We also introduce the concept of unitary equivalence,

Definition 2.2.1 *Let A, B be linear operators in a Hilbert space $\mathcal{H}$ with domains* $\mathrm{Dom}(A)$ *and* $\mathrm{Dom}(B)$ *respectively. If U is a unitary operator, then A and B are unitarily equivalent provided* $U(\mathrm{Dom}(A)) = \mathrm{Dom}(B)$ *and* $UAU^{-1} = B$.

In the momentum space $(\mathscr{F}L^2(\mathbb{R}^3, \mathrm{d}\boldsymbol{x})^4) = L^2(\mathbb{R}^3, \mathrm{d}\boldsymbol{p})^4$, the free Dirac operator $\mathbb{D}_0$ acts as a multiplication matrix in the form

$$(\mathscr{F}\mathbb{D}_0\mathscr{F}^{-1})(\boldsymbol{p}) = \boldsymbol{\alpha}\cdot\boldsymbol{p} + m\beta. \tag{2.2}$$

For each $\boldsymbol{p}$, this is a 4×4 Hermitian matrix with eigenvalues given by

$$\lambda_{1,2} = -\lambda_{3,4} = \sqrt{|\boldsymbol{p}|^2 + m^2} =: \lambda(\boldsymbol{p}),$$

and the projections onto the corresponding eigenspaces given by

$$\Pi_{\pm}(\boldsymbol{p}) = \frac{1}{2}\left(I \pm \frac{\boldsymbol{\alpha} \cdot \boldsymbol{p} + m\beta}{\sqrt{|\boldsymbol{p}|^2 + m^2}}\right). \tag{2.3}$$

Introducing

$$U = \frac{(m + \lambda(\boldsymbol{p}))I_4 + m\beta\boldsymbol{\alpha} \cdot \boldsymbol{p}}{\sqrt{2\lambda(\boldsymbol{p})(m + \lambda(\boldsymbol{p}))}},$$

then it can be shown that $\mathbb{D}_0$ and $\beta\lambda(\boldsymbol{p})$ are unitarily equivalent under the unitary transformation $U\mathscr{F}$. In the standard representation, this diagonalizes the free Dirac operator,

$$U\mathscr{F}\mathbb{D}_0(U\mathscr{F})^{-1}(\boldsymbol{p}) = \beta\lambda(\boldsymbol{p}).$$

2.3 Resolvent of the free Dirac operator

To study resonances associated with the Dirac operator, the work in this thesis centres on switching from the spectral parameter z to the variable

$$k(z) = \sqrt{z^2 - m^2}. \tag{2.4}$$

This is motivated by writing the free resolvent, defined on $\rho(\mathbb{D}_0)$, as

$$(\mathbb{D}_0 - z)^{-1} = (\mathbb{D}_0 + z)(-\Delta + m^2 - z^2)^{-1}, \tag{2.5}$$

since $\mathbb{D}_0^2 = -\Delta + m^2$ by the anticommutation relations (2.1). Writing $k^2 = z^2 - m^2$, we then recognise

$$R_{00}(k) := (-\Delta - k^2)^{-1}$$

as the free resolvent of the three-dimensional Laplacian operator (see for instance [23]). By using the parameter k we can then take advantage of the Laplace resolvent and its properties.

This change of variable has the effect of mapping $\rho(\mathbb{D}_0)$ to a pair of half-planes in the k variable. Since we choose the branch of the square root such that $\operatorname{Im} k > 0$, then $\rho(\mathbb{D}_0)$ maps to the upper (or *physical*) half-plane. The lower half-plane will be named *unphysical*.

We note here that the mapping k cancels any negative sign of the spectral parameter z. Therefore for the inverse map $k \mapsto z(k)$, we use the negative prefactor to restore this negativity and write $z = \pm\sqrt{k^2+m^2}$.

We are now in a position to define the free resolvent via the k parameter

Definition 2.3.1 *Let* $\operatorname{Im} k > 0$. *Then the resolvent of the free Dirac operator is defined by*

$$R_0^{\pm}(k) := (\mathbb{D}_0 \mp \sqrt{k^2+m^2})^{-1} : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4.$$

We clarify our notation here when $k \in \mathbb{R} \setminus \{0\}$. In this case Balslev and Helffer [4] proved that the following limits, rewritten in the k-plane, exist

$$\begin{aligned} R_0^{\pm}(k) &= \lim_{\epsilon\to 0^+} (\mathbb{D}_0 \mp \sqrt{k^2+m^2} - i\epsilon)^{-1}, \\ R_0^{\pm}(-k) &= \lim_{\epsilon\to 0^+} (\mathbb{D}_0 \mp \sqrt{k^2+m^2} + i\epsilon)^{-1}, \end{aligned} \tag{2.6}$$

where $\epsilon > 0$ (Indeed Balslev and Helffer [4] prove in detail when these limits exist in terms of the uniform operator topology on weighted Sobolev spaces). This is analogous to the limiting absorption principle for free Laplacians (see [1] for exact details on the weighted Sobolev spaces that these exist):

$$R_{00}(\lambda) = \lim_{\epsilon\to 0^+} (-\Delta - \lambda^2 - i\epsilon)^{-1}, \quad R_{00}(-\lambda) = \lim_{\epsilon\to 0^+} (-\Delta - \lambda^2 + i\epsilon)^{-1},$$

where again $\epsilon > 0$. It can then be shown for $\operatorname{Im} k > 0$ that the resolvent acting on $u \in \mathscr{S}(\mathbb{R}^3)$, the Schwartz space of rapidly decreasing functions, can be written

$$R_0^{\pm}(k)u(\boldsymbol{x}) = \int_{\mathbb{R}^3} G_0^{\pm}(k; \boldsymbol{x}-\boldsymbol{y})u(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

with integral kernel

$$G_0^{\pm}(k;\boldsymbol{x}) = \left(i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{x}}{|\boldsymbol{x}|^2} + k\frac{\boldsymbol{\alpha}\cdot\boldsymbol{x}}{|\boldsymbol{x}|} + \beta m \pm \sqrt{k^2+m^2} \right) \frac{e^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}. \tag{2.7}$$

Since L^2_{comp} and L^2_{loc} are not Hilbert nor Banach spaces, it is necessary to clarify what is meant by the notion of holomorphic and meromorphic operator-valued functions between such spaces. Suppose $\Omega \subset \mathbb{C}$ is an open set. A holomorphic function with bounded values in $\mathcal{B}(L^2_{\text{comp}}, L^2_{\text{loc}})$ is a function $A(z)$ with values in the space of linear bounded operators $L^2_{\text{comp}} \to L^2_{\text{loc}}$ such that $\rho_1 A(z)\rho_2$ is holomorphic for $\rho_1, \rho_2 \in C_0^\infty(\mathbb{R}^3)$. It is assumed that such bump functions are contained in L^2_{comp} and L^2_{loc}.

Similarly an operator-valued function $A(z)$ is a meromorphic function on Ω if it is holomorphic on $\Omega \setminus S$, where $S \subset \Omega$ is discrete, and such that if $z_0 \in S$, then near z_0 we have

$$A(z) = \sum_{j=1}^{J} \frac{A_j}{(z - z_0)^j} + B(z),$$

with $A_j : L^2_{\text{comp}} \to L^2_{\text{loc}}$ (bounded in the sense $\rho_1 A_j \rho_2$ is bounded for all ρ_j as above) of finite rank, and $B(z)$ holomorphic with values in $\mathcal{B}(L^2_{\text{comp}}, L^2_{\text{loc}})$ for z in a neighbourhood of z_0. The integer J is assumed finite to signify poles of finite order.

We note in the following and indeed the rest of the thesis that C will denote a positive constant whose numerical value is not important. Furthermore C may correspond to different values from line to line.

Theorem 2.3.2 *The free Dirac resolvent defined in Definition 2.3.1 is a holomorphic family of operators in the upper half-plane with operator norm*

$$\|R_0^{\pm}(k)\|_{L^2 \to L^2} \leq \frac{|\sqrt{k^2 + m^2} + m|}{|k|^2}, \qquad \operatorname{Im} k > 0. \tag{2.8}$$

It continues analytically to the entire family of operators by the mapping

$$R_0^{\pm}(k) : L^2_{\text{comp}}(\mathbb{R}^3)^4 \to L^2_{\text{loc}}(\mathbb{R}^3)^4.$$

Moreover for any $\rho \in C_0^{\infty}(\mathbb{R}^3)$ where $\operatorname{diam}(\operatorname{supp} \rho) < L$ is finite, we have the estimates

$$\|\rho R_0^{\pm}(k) \rho\|_{L^2 \to H^j} \leq C \langle k \rangle^j e^{L(\operatorname{Im} k)_-}, \qquad j = 0, 1, \tag{2.9}$$

where $(x)_- = \max\{-x, 0\}$ and $\langle x \rangle = \sqrt{1 + |x|^2}$.

Proof. We begin by stating the following estimate for the free cut-off Laplacian resolvent (see for instance [23, section 3.1])

$$\|\rho R_{00}(\lambda) \rho\|_{L^2(\mathbb{R}^3) \to H^j(\mathbb{R}^3)} \leq C(1 + |\lambda|)^{j-1} e^{L(\operatorname{Im} \lambda)_-}, \qquad j = 0, 1, 2. \tag{2.10}$$

Then by the standard resolvent norm for selfadjoint operators (see Theorem A.2.1) and assuming $\operatorname{Im} k > 0$ (or equivalently $z \in \rho(\mathbb{D}_0)$) we have

$$\|R_0^{\pm}(k)\| = \sup_{\mu \in \operatorname{spec}(\mathbb{D}_0)} \frac{1}{|\mu - z(k)|} = \frac{1}{|\pm m \mp \sqrt{k^2 + m^2}|} \leq \frac{|m + \sqrt{k^2 + m^2}|}{|k|^2}.$$

From the integral kernel in (2.7), it is clear that if $\operatorname{Im} k < 0$, then the exponential term will deem $G_0^{\pm}(k; \boldsymbol{x} - \boldsymbol{y})$ to be only locally in $L^2(\mathbb{R}^3)^4$.

To estimate the cut-off resolvent norm we use (2.5) to write

$$
\begin{aligned}
\|\rho R_0^{\pm}(k)\rho\|_{L^2\to H^j} &= \|\rho(\mathbb{D}_0+z)R_{00}(k)\rho\|_{L^2\to H^j}\\
&= \|\rho(-i\boldsymbol{\alpha}\cdot\nabla+m\beta)R_{00}(k)\rho+z\rho R_{00}(k)\rho\|_{L^2\to H^j}\\
&= \|-i\boldsymbol{\alpha}\cdot\nabla\rho R_{00}(k)\rho+i\boldsymbol{\alpha}\cdot(\nabla\rho)R_{00}(k)\rho+(m\beta+z)\rho R_{00}(k)\rho\|_{L^2\to H^j}\\
&\le \|\nabla\rho R_{00}(k)\rho\|_{L^2\to H^j}+C(1+m\beta+|z|)\|\rho R_{00}(k)\rho\|_{L^2\to H^j}\\
&\le \|\rho R_{00}(k)\rho\|_{L^2\to H^{j+1}}+C\langle k\rangle\|\rho R_{00}(k)\rho\|_{L^2\to H^j}.
\end{aligned}
$$

For $j=0$ we then have from (2.10)

$$
\|\rho R_0^{\pm}(k)\rho\|_{L^2\to L^2}\le Ce^{L(\operatorname{Im}k)_-}+\frac{C\langle k\rangle}{(1+|k|)}e^{L(\operatorname{Im}k)_-}\le Ce^{L(\operatorname{Im}k)_-},
$$

whereas $j=1$ gives us

$$
\|\rho R_0^{\pm}(k)\rho\|_{L^2\to H^1}\le C(1+|k|)e^{L(\operatorname{Im}k)_-}+C\langle k\rangle e^{L(\operatorname{Im}k)_-}\le C\langle k\rangle e^{L(\operatorname{Im}k)_-}. \qquad \square
$$

The asymptotic behaviour of the free resolvent is captured in the following theorem.

Theorem 2.3.3 *Let $f\in\mathscr{S}(\mathbb{R}^3)$ and suppose $k\in\mathbb{R}\setminus\{0\}$. If $\boldsymbol{x}=r\boldsymbol{\theta}$ then*

$$
R_0^{\pm}(k)f(r\boldsymbol{\theta})=\frac{e^{ikr}}{4\pi r}\left[\beta m\pm\sqrt{k^2+m^2}+k\alpha\cdot\boldsymbol{\theta}\right]\int_{\mathbb{R}^3}e^{-ik\langle\boldsymbol{\theta},\boldsymbol{y}\rangle}f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}+\mathcal{O}\left(r^{-2}\right)
$$

as $r\to\infty$.

Proof. Writing $\sqrt{1+s}=1+s/2-s^2/8+\mathcal{O}\left(s^3\right)$, we have the expansion

$$
\begin{aligned}
|\boldsymbol{x}-\boldsymbol{y}| &= |\boldsymbol{x}|\sqrt{1-\frac{2\langle\boldsymbol{x},\boldsymbol{y}\rangle}{|\boldsymbol{x}|^2}+\frac{|\boldsymbol{y}|^2}{|\boldsymbol{x}|^2}}=r\left(1+\frac{1}{2}\left(\frac{|\boldsymbol{y}|^2}{r^2}-\frac{2\langle\boldsymbol{\theta},\boldsymbol{y}\rangle}{r}\right)+\mathcal{O}\left(r^{-2}\right)\right)\\
&= r-\langle\boldsymbol{\theta},\boldsymbol{y}\rangle+\mathcal{O}\left(r^{-1}\right),
\end{aligned}
$$

where we have used $\boldsymbol{x}/|\boldsymbol{x}|=\boldsymbol{\theta}$ and $|\boldsymbol{x}-\boldsymbol{y}|=\sqrt{|\boldsymbol{x}-\boldsymbol{y}|^2}=\sqrt{\langle\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y}\rangle}$. This expansion is used to write

$$
e^{ik|\boldsymbol{x}-\boldsymbol{y}|}=e^{ik(r-\langle\boldsymbol{\theta},\boldsymbol{y}\rangle)}e^{\mathcal{O}(1/r)}=e^{ik(r-\langle\boldsymbol{\theta},\boldsymbol{y}\rangle)}\left(1+\mathcal{O}\left(r^{-1}\right)\right).
$$

We also have

$$
\begin{aligned}
|\boldsymbol{x}-\boldsymbol{y}|^{-1} &= \frac{1}{|\boldsymbol{x}|\sqrt{1-2\langle\boldsymbol{x},\boldsymbol{y}\rangle/|\boldsymbol{x}|^2+|\boldsymbol{y}|^2/|\boldsymbol{x}|^2}}\\
&= \frac{1}{r}\left(1-\frac{1}{2}\left(\frac{|\boldsymbol{y}|^2}{r^2}-\frac{2\langle\boldsymbol{\theta},\boldsymbol{y}\rangle}{r}\right)+\mathcal{O}\left(r^{-2}\right)\right)\\
&= \frac{1}{r}\left(1+\frac{\langle\boldsymbol{\theta},\boldsymbol{y}\rangle}{r}\right)+\mathcal{O}\left(r^{-3}\right),
\end{aligned}
$$

where $1/\sqrt{1+s} = 1 - s/2 + 3s^2/8 + \mathcal{O}\left(s^3\right)$. Then for $p \in \mathbb{N}$ we write

$$|\boldsymbol{x}-\boldsymbol{y}|^{-p} = \frac{1}{r^p}\left[1 + \frac{\langle\boldsymbol{\theta},\boldsymbol{y}\rangle}{r} + \mathcal{O}\left(r^{-2}\right)\right]^p = \frac{1}{r^p}\left[1 + p\frac{\langle\boldsymbol{\theta},\boldsymbol{y}\rangle}{r}\right] + \mathcal{O}\left(r^{-(p+2)}\right).$$

We will use these expansions to rewrite the free resolvent kernel (2.7) in powers of r. Component-wise we have

$$\begin{aligned}
\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} &= \frac{e^{ik(r-\langle\boldsymbol{\theta},\boldsymbol{y}\rangle)}}{4\pi r} + \mathcal{O}\left(r^{-2}\right)\\
\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi} &= \frac{\alpha\cdot\boldsymbol{\theta}}{4\pi r}e^{ik(r-\langle\boldsymbol{\theta},\boldsymbol{y}\rangle)} + \mathcal{O}\left(r^{-2}\right)\\
\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi} &= \mathcal{O}\left(r^{-2}\right).
\end{aligned}$$

Hence we obtain for $f \in \mathscr{S}(\mathbb{R}^3)$ and $r \to \infty$

$$R_0^{\pm}(k)f(r\boldsymbol{\theta}) = \frac{e^{ikr}}{4\pi r}\left[\beta m \pm \sqrt{k^2+m^2} + k\alpha\cdot\boldsymbol{\theta}\right]\int_{\mathbb{R}^3} e^{-ik\langle\boldsymbol{\theta},\boldsymbol{y}\rangle}f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \mathcal{O}\left(r^{-2}\right). \qquad \square$$

2.4 The perturbed Dirac operator and resonances

Define $M_4(\mathbb{C})$ as the set of complex-valued 4×4 matrices. In this thesis we consider electric potentials V in $M_4(\mathbb{R})$ that act by multiplication. In particular we assume the following

Assumption 2.4.1 *Let $V : \mathbb{R}^3 \to M_4(\mathbb{R})$ be a smooth, compactly supported Hermitian matrix potential acting by multiplication.*

We do not consider perturbations by a magnetic vector potential. We write the Dirac operator perturbed by a scalar potential V satisfying Assumption 2.4.1 as

$$\mathbb{D}_V = \mathbb{D}_0 + V.$$

By the Kato-Rellich theorem (see for instance [34, chapter 13]), $\mathbb{D}_V$ is a selfadjoint operator in $L^2(\mathbb{R}^3)^4$ with domain $H^1(\mathbb{R}^3)^4$. The spectrum of $\mathbb{D}_V$ is composed of an essential spectrum $\mathrm{spec}_{\mathrm{ess}}(\mathbb{D}_V)$ which coincides with $\mathrm{spec}(\mathbb{D}_0)$, and its eigenvalues (or discrete spectrum, $\mathrm{spec}_d(\mathbb{D}_V)$) are located within $(-m,m)$ (see [8]). Under the transformation $z \mapsto k(z)$ defined in (2.4), any eigenvalues residing in $(-m,m)$ map to wholly imaginary points $\{iE'_j\}$ where $0 < E'_j < m$.

Under the assumptions of the previous section regarding the meromorphy of functions from $L^2_{\text{comp}}(\mathbb{R}^3)^4$ to $L^2_{\text{loc}}(\mathbb{R}^3)^4$, we define the full resolvent of the perturbed Dirac operator and its meromorphic extension.

Theorem 2.4.2 *Let V satisfy Assumption 2.4.1. Then the perturbed resolvent*

$$R_V^{\pm}(k) := (\mathbb{D}_V \mp \sqrt{k^2+m^2})^{-1} : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4,$$

defined for $\operatorname{Im} k > 0$ is a meromorphic family of operators with a finite number of poles, corresponding to the eigenvalues of $\mathbb{D}_V$. It extends to a meromorphic family of operators for $k \in \mathbb{C}$

$$R_V^{\pm}(k) : L^2_{\text{comp}}(\mathbb{R}^3)^4 \to L^2_{\text{loc}}(\mathbb{R}^3)^4.$$

Similar to the limits in (2.6), it was proved in [4] that if $k \in \mathbb{R} \setminus \{0\}$ then on suitable weighted Sobolev spaces the following exist where $\epsilon > 0$,

$$\begin{aligned} R_V^{\pm}(k) &= \lim_{\epsilon \to 0^+} (\mathbb{D}_0 + V \mp \sqrt{k^2+m^2} - i\epsilon)^{-1}, \\ R_V^{\pm}(-k) &= \lim_{\epsilon \to 0^+} (\mathbb{D}_0 + V \mp \sqrt{k^2+m^2} + i\epsilon)^{-1}. \end{aligned}$$

Proof of Theorem 2.4.2. We divide the proof into 3 steps.

1. First we show that $R_V^{\pm}(k)$ is a family of meromorphic operators when $\operatorname{Im} k > 0$. To this end write

$$(\mathbb{D}_V \mp \sqrt{k^2+m^2}) R_0^{\pm}(k) = I + V R_0^{\pm}(k), \tag{2.11}$$

and choose k where $\operatorname{Im} k$ is sufficiently large such that by (2.8),

$$\|V R_0^{\pm}(k)\| \le \|V\| \|R_0^{\pm}(k)\| < 1.$$

Then by the Neumann series theorem, $(-1)V R_0^{\pm}(k)$ is invertible

$$(I + V R_0^{\pm}(k))^{-1} = \sum_{n=0}^{\infty} (-V R_0^{\pm}(k))^n.$$

We can therefore invert (2.11) so that

$$R_V^{\pm}(k) := (\mathbb{D} \mp \sqrt{k^2+m^2})^{-1} = R_0^{\pm}(k)(I + V R_0^{\pm}(k))^{-1}. \tag{2.12}$$

Suppose $\rho \in C_0^{\infty}(\mathbb{R}^3)$ such that $\rho V = V$ on $\operatorname{supp} V$. For $\operatorname{Im} k > 0$ then $\rho R_0^{\pm}(k)$: $L^2(\mathbb{R}^3)^4 \to H^1(\operatorname{supp} V; \mathbb{C}^4)$ (and likewise $V R_0^{\pm}(k)$) is a compact operator on L^2 by the

Rellich-Kondrachov theorem (see for instance [24, section 5.7]). Then by analytic Fredhom theory (see Theorem A.3.1), $R_V^{\pm}(k)$ is a meromorphic family of operators in the upper k-plane.

2. To show the meromorphy of $R_V^{\pm}(k)$ in $\mathbb{C}$, again assume $\rho \in C_0^{\infty}(\mathbb{R}^3)$ such that $\rho V = V$ on $\operatorname{supp} V$ and consider

$$(I - VR_0^{\pm}(k)(I-\rho))(I + VR_0^{\pm}(k)) = I + VR_0^{\pm}(k)\rho - VR_0^{\pm}(k)(I-\rho)VR_0^{\pm}(k).$$

Since $(I-\rho)VR_0^{\pm}(k) = 0$ by the assumptions on ρ then

$$(I - VR_0^{\pm}(k)(I-\rho))(I + VR_0^{\pm}(k)) = I + VR_0^{\pm}(k)\rho. \tag{2.13}$$

Again by the arguments above choose $\operatorname{Im} k \gg 1$ such that $\|VR_0^{\pm}(k)\rho\| < 1$. Application of the Neumann series theorem means that $(-1)VR_0^{\pm}(k)$ is invertible and hence we may invert (2.13)

$$(I + VR_0^{\pm}(k))^{-1} = (I + VR_0^{\pm}(k)\rho)^{-1}(I - VR_0^{\pm}(k)(I-\rho)),$$

which we use to rewrite (2.12)

$$R_V^{\pm}(k) = R_0^{\pm}(k)(I + VR_0^{\pm}(k)\rho)^{-1}(I - VR_0^{\pm}(k)(I-\rho)). \tag{2.14}$$

By (2.9), $\rho R_0^{\pm}(k)\rho : L^2(\mathbb{R}^3)^4 \to H^1(\operatorname{supp} V; \mathbb{C}^4)$ and therefore $\rho R_0^{\pm}(k)\rho$ is compact on $L^2(\mathbb{R}^3)^4$ by the Rellich-Kondrachov theorem. Since $V = V\rho$ is bounded then $VR_0^{\pm}(k)\rho = V\rho R_0^{\pm}(k)\rho$ is also compact on $L^2(\mathbb{R}^3)^4$. In turn by the analytic Fredholm theorem, we have a meromorphic continuation of $(I + VR_0^{\pm}(k)\rho)^{-1}$ to $\mathbb{C}$.

3. We now show that

$$(I + VR_0^{\pm}(k)\rho)^{-1} : L^2_{\mathrm{comp}}(\mathbb{R}^3)^4 \to L^2_{\mathrm{comp}}(\mathbb{R}^3)^4.$$

Let $\chi, \tilde{\chi} : L^2_{\mathrm{comp}}(\mathbb{R}^3) \to L^2_{\mathrm{comp}}(\mathbb{R}^3)$ where $\chi\rho = \rho$ and $\tilde{\chi}\chi = \chi$. Then for $\operatorname{Im} k \gg 1$, $(I + VR_0^{\pm}(k)\rho)^{-1}$ exists as a Neumann series. Hence

$$\begin{aligned}(I + VR_0^{\pm}(k)\rho)^{-1}\chi &= (I + VR_0^{\pm}(k)\rho)^{-1}\tilde{\chi}(I + VR_0^{\pm}(k)\rho)(I + VR_0^{\pm}(k)\rho)^{-1}\chi \\ &= (I + VR_0^{\pm}(k)\rho)^{-1}(I + VR_0^{\pm}(k)\rho)\tilde{\chi}(I + VR_0^{\pm}(k)\rho)^{-1}\chi \\ &= \tilde{\chi}(I + VR_0^{\pm}(k)\rho)^{-1}\chi,\end{aligned} \tag{2.15}$$

as required. Given also that

$$I - VR_0^{\pm}(k)(I-\rho) : L^2_{\mathrm{comp}}(\mathbb{R}^3)^4 \to L^2_{\mathrm{comp}}(\mathbb{R}^3)^4,$$

we combine these with the form of $R_V^{\pm}(k)$ in (2.14) to obtain the meromorphy of $R_V^{\pm}(k)$ for $k \in \mathbb{C}$ as a family of operators $L^2_{\mathrm{comp}}(\mathbb{R}^3)^4 \to L^2_{\mathrm{loc}}(\mathbb{R}^3)^4$. □

We now define resonances of the perturbed Dirac operator and their multiplicities.

Definition 2.4.3 *Let V satisfy Assumption 2.4.1. Then,*

1. *The poles of the meromorphic extension of $R_V^{\pm}(k)$ coincide with the poles of $(I + VR_0^{\pm}(k)\rho)^{-1}$ and are referred to as (scattering) resonances of $\mathbb{D}_V$. The two sets of resonances are denoted by $\mathcal{R}_{\pm}$, with their union denoted $\mathcal{R} := \mathcal{R}_- \cup \mathcal{R}_+$.*

2. *If k is a resonance of $\mathbb{D}_V$ then the multiplicity $m_R(k)$ is defined by*

$$m_R(k) := \dim \operatorname{span}\{A_j^{\pm}(L^2_{\text{comp}})\}_{1\le j\le J}, \tag{2.16}$$

where

$$R_V^{\pm}(\zeta) = \sum_{j=1}^{J} \frac{A_j^{\pm}}{(\zeta - k)^j} + A^{\pm}(\zeta, k), \tag{2.17}$$

and $A^{\pm}(\zeta, k)$ is holomorphic for ζ near k.

Summary

In this chapter, we have introduced the Dirac operator and presented some of its basic spectral properties. Under our change of variable we have rewritten the free and full Dirac resolvents and shown that the free cut-off Dirac resolvent continues analytically from the upper k half-plane to $\mathbb{C}$. Moreover the same extension for the full cut-off resolvent is meromorphic and the poles have in turn been defined as resonances of the perturbed Dirac operator.

Chapter 3

Resolvent near threshold energies

In this chapter we study the full resolvent $R_V^{\pm}(k)$ near $k = 0$. In section 3.1 we obtain the expansion resembling (2.17) and determine how the operators in each term act. We will use this expansion to analyse the threshold resonances in chapter 5 when we construct our Birman-Kreĭn trace formula. Moreover we study the exact form of the operator $A_{\pm}$ in section 3.2 with a summary of all results presented in Theorem 3.2.5.

3.1 Expansion of the full resolvent near 0

We define the following spaces that the operators in the first two terms of the resolvent expansion in (3.1) map into.

Definition 3.1.1 *Define the following spaces*

$$V_{\pm} := \{v^{\pm} \in H^1(\mathbb{R}^3)^4 \mid (\mathbb{D}_V \mp m)v^{\pm} = 0\},$$
$$U_{\pm} := \{u^{\pm} \in H^1_{\text{loc}}(\mathbb{R}^3)^4 \mid (\mathbb{D}_V \mp m)u^{\pm} = 0\},$$

and the orthogonal projection $\Pi_{\pm} : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4$ *which maps* $L^2(\mathbb{R}^3)^4$ *functions into* $V_{\pm}$.

Theorem 3.1.2 *The full resolvent* $R_V^{\pm}(k) : L^2_{\text{comp}}(\mathbb{R}^3)^4 \to L^2_{\text{loc}}(\mathbb{R}^3)^4$ *near* $k = 0$ *can be expressed*

$$R_V^{\pm}(k) = \mp\frac{\Pi_{\pm}}{k^2}(\sqrt{k^2+m^2}+m) + \frac{iA_{\pm}}{k}\sqrt{\sqrt{k^2+m^2}+m} + B_{\pm}(k), \tag{3.1}$$

where $k \mapsto B_\pm(k)$ *is holomorphic near* 0*, and the operators* $\Pi_\pm : L^2(\mathbb{R}^3)^4 \to V_\pm,\ A_\pm : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ *satisfying* $(\mathbb{D}_V \mp m)\Pi_\pm = (\mathbb{D}_V \mp m)A_\pm = 0$ *are symmetric.*

Proof. We divide the proof into 4 steps.

1. For $\operatorname{Im} k > 0$ and assuming $|k| \ll \{E'_j\}$, where $\{E'_j\}$ is the set of eigenvalues of $\mathbb{D}_V$ mapped onto the positive imaginary axis, then the first result of Theorem 2.3.2 and analyticity of k near 0 suggests the decomposition

$$R_V^\pm(k) = \mp\frac{\tilde{A}_\pm}{k^2}(\sqrt{k^2+m^2}+m) + i\frac{A_\pm}{k}\sqrt{\sqrt{k^2+m^2}+m} + B_\pm(k), \tag{3.2}$$

where $A_\pm, \tilde{A}_\pm : L^2_{\mathrm{comp}}(\mathbb{R}^3)^4 \to L^2_{\mathrm{loc}}(\mathbb{R}^3)^4$ are finite rank operators and $B_\pm(k)$ is holomorphic near $k = 0$. The $\mp$ coefficients will be apparent in the final step of this proof when we show that $\tilde{A}_\pm = \Pi_\pm$.

2. The property $(\mathbb{D}_V \mp m)\tilde{A}_\pm = (\mathbb{D}_V \mp m)A_\pm = 0$ follows from the identity $I = (\mathbb{D}_V \mp \sqrt{k^2+m^2})R_V^\pm(k)$, using the decomposition (3.2), and equating the coefficients of $k^{-\alpha}$ where $\alpha = \{1,2\}$:

$$\begin{aligned}
I &= (\mathbb{D}_V \mp \sqrt{k^2+m^2})R_V^\pm(k)\\
&= (\mathbb{D}_V \mp m)\left[\mp\frac{\tilde{A}_\pm}{k^2}(\sqrt{k^2+m^2}+m) + i\frac{A_\pm}{k}\sqrt{\sqrt{k^2+m^2}+m}\right]\\
&\quad \mp(\sqrt{k^2+m^2}-m)\left[\mp\frac{\tilde{A}_\pm}{k^2}(\sqrt{k^2+m^2}+m) + i\frac{A_\pm}{k}\sqrt{\sqrt{k^2+m^2}+m}\right]\\
&\quad + (\mathbb{D}_V \mp \sqrt{k^2+m^2})B_\pm(k).
\end{aligned}$$

The denominators for the terms containing the prefactor $(\sqrt{k^2+m^2}-m)$ cancel:

$$\begin{aligned}
\frac{1}{k^2}(\sqrt{k^2+m^2}-m)(\sqrt{k^2+m^2}+m) &= 1,\\
\frac{1}{k}(\sqrt{k^2+m^2}-m)\sqrt{\pm(\sqrt{k^2+m^2}+m)} &= \sqrt{\pm1}\sqrt{(\sqrt{k^2+m^2}-m)},
\end{aligned}$$

and we therefore require the stated property.

3. To show symmetry of $\tilde{A}_\pm$ and $A_\pm$, set $\psi, \phi \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ and $0 < t \ll E'_j$ so that we have by selfadjointness of $R_V^{\pm}(it)$

$$
\begin{aligned}
\mp 2m\langle \tilde{A}_{\pm}\psi\,,\phi\rangle &= \mp\lim_{t\to 0}(\sqrt{m^2-t^2}+m)\langle \tilde{A}_{\pm}\psi\,,\phi\rangle \\
&= \lim_{t\to 0}\left\langle -t^2\left(R_V^{\pm}(it)-\frac{A_{\pm}}{t}\sqrt{\sqrt{m^2-t^2}+m}-B(it)\right)\psi\,,\phi\right\rangle \\
&= \lim_{t\to 0}\langle -t^2R_V^{\pm}(it)\psi\,,\phi\rangle = \lim_{t\to 0}\langle \psi\,,-t^2R_V^{\pm}(it)\phi\rangle \qquad (3.3)\\
&= \lim_{t\to 0}\left\langle \psi\,,-t^2\left(R_V^{\pm}(it)-\frac{A_{\pm}}{t}\sqrt{\sqrt{m^2-t^2}+m}-B(it)\right)\right\rangle \\
&= \mp\lim_{t\to 0}\overline{(\sqrt{m^2-t^2}+m)}\langle\psi,\tilde{A}_{\pm}\phi\rangle = \mp 2m\langle\psi,\tilde{A}_{\pm}\phi\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
i\sqrt{2m}\langle A_{\pm}\psi\,,\phi\rangle &= \lim_{t\to 0} i\sqrt{(\sqrt{m^2-t^2}+m}\langle A_{\pm}\psi\,,\phi\rangle \\
&= \lim_{t\to 0}\langle it\left(R_V^{\pm}(it)\pm\frac{\tilde{A}_{\pm}}{(it)^2}(\sqrt{m^2-t^2}+m)-B_{\pm}(it)\right)\psi\,,\phi\rangle \\
&= \lim_{t\to 0} it\langle\psi\,,\left(R_V^{\pm}(it)\pm\frac{\tilde{A}_{\pm}}{(it)^2}(\sqrt{m^2-t^2}+m)-B_{\pm}(it)\right)\phi\rangle \\
&= \lim_{t\to 0} it\langle\psi\,,\frac{iA_{\pm}}{it}\sqrt{\sqrt{-t^2+m^2}+m\phi}\rangle = i\sqrt{2m}\langle\psi\,,A_{\pm}\phi\rangle.
\end{aligned}
$$

4. Finally we explore the properties of $\tilde{A}_{\pm}$. First $\tilde{A}_{\pm} : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4$ is a bounded operator by Theorem 2.3.2 and so (3.3) holds for all ψ, $\phi\in L^2(\mathbb{R}^3)^4$. As we have also seen, $(\mathbb{D}_V\mp m)\tilde{A}_{\pm}=0$ and so the range of $\tilde{A}_{\pm}$ is contained in $V_{\pm}$ as per Definition (3.1.1). To show that indeed $\tilde{A}_{\pm}=\Pi_{\pm}$ then for any $v_{\pm}\in V_{\pm}$, $\phi\in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ and $|t|\ll\{E_j'\}$ we have by using $(\mathbb{D}_V\mp m)v_{\pm}=0$,

$$
\begin{aligned}
\langle v_{\pm}\,,\phi\rangle &= \langle R_V^{\pm}(it)\left[(\mathbb{D}_V\mp m)\mp(\sqrt{m^2-t^2}-m)\right]v_{\pm}\,,\phi\rangle \\
&= \mp(\sqrt{m^2-t^2}-m) \\
&\quad \left\langle\left[\pm\frac{\tilde{A}_{\pm}}{t^2}(\sqrt{m^2-t^2}+m)+\frac{A_{\pm}}{t}\sqrt{\sqrt{m^2-t^2}+m}+B_{\pm}(it)\right]v_{\pm}\,,\phi\right\rangle \\
&\xrightarrow[t\to 0]{} \langle\tilde{A}_{\pm}v_{\pm}\,,\phi\rangle.
\end{aligned}
$$

□

Theorem 3.1.3 *For the free Dirac resolvent kernel in (2.7) we list the following useful relations*

$$
\begin{aligned}
G_0^{\pm}(k;\boldsymbol{x}-\boldsymbol{y}) &= \left[\frac{i\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}+k\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}+\beta m\pm\sqrt{k^2+m^2}\right]\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}, \\
G_0^{\pm}(0;\boldsymbol{x}-\boldsymbol{y}) &= \left[\frac{i\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}+m(\beta\pm I)\right]\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}, \\
\partial_k G_0^{\pm}(k;\boldsymbol{x}-\boldsymbol{y}) &= \left[im\beta\pm i\sqrt{k^2+m^2}+\frac{k}{|\boldsymbol{x}-\boldsymbol{y}|}\left(i\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})\pm\frac{1}{\sqrt{k^2+m^2}}\right)\right]\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi},
\end{aligned}
$$

$$\partial_k G_0^{\pm}(0;\boldsymbol{x}-\boldsymbol{y}) = (\beta \pm I)\frac{im}{4\pi}.$$

Proof. The first relation was introduced in (2.7) and setting $k=0$ immediately gives the second. For $\partial_k G_0^{\pm}(k;\boldsymbol{x}-\boldsymbol{y})$ we have

$$\begin{aligned}
\partial_k G_0^{\pm}(k;\boldsymbol{x}-\boldsymbol{y}) &= \left[\frac{i\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + k\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} + \beta m \pm \sqrt{k^2+m^2}\right]\partial_k \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \\
&\quad + \left[\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} \pm \partial_k\sqrt{k^2+m^2}\right]\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \left[\frac{i\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + k\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} + \beta m \pm \sqrt{k^2+m^2}\right]\frac{i|\boldsymbol{x}-\boldsymbol{y}|e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \\
&\quad + \left[\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} \pm \frac{k}{\sqrt{k^2+m^2}}\right]\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi}\Bigg[-\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + i\beta m \pm i\sqrt{k^2+m^2} + \frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} \\
&\quad + \frac{k}{|\boldsymbol{x}-\boldsymbol{y}|}\left(i\alpha\cdot(\boldsymbol{x}-\boldsymbol{y}) \pm \frac{1}{\sqrt{k^2+m^2}}\right)\Bigg],
\end{aligned}$$

which provides us with the third relation after cancelling the $|\boldsymbol{x}-\boldsymbol{y}|^{-2}$ terms. Again setting $k=0$ we obtain $\partial_k G_0^{\pm}(0;\boldsymbol{x}-\boldsymbol{y})$ for the final result. □

Theorem 3.1.4 *Let $A_\pm$ be defined as Theorem 3.1.2. Then the image of $A_\pm$ is not contained in $L^2(\mathbb{R}^3)^4$ but in $U_\pm$. Moreover if $u_\pm \in U_\pm$ and V satisfies Assumption 2.4.1 then $u_\pm = R_0^{\pm}(0)g_\pm$ where $g_\pm = (\mathbb{D}_0 \mp m)u_\pm = -Vu_\pm \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$.*

Proof. We divide the proof into 3 steps.

1. Injectivity of $R_0^{\pm}(k)$ on $L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ (the left inverse being $\mathbb{D}_0 \mp \sqrt{k^2+m^2}$) implies for k near 0

$$R_V^{\pm}(k) = R_0^{\pm}(k)\left(\frac{C_\pm}{k^2}(\sqrt{k^2+m^2}+m) + \frac{D_\pm}{k}\sqrt{\sqrt{k^2+m^2}+m} + E_\pm(k)\right), \tag{3.4}$$

where $C_\pm$, $D_\pm$, $E_\pm(k) : L^2_{\mathrm{comp}}(\mathbb{R}^3)^4 \to L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ and $E_\pm(k)$ is holomorphic near $k=0$. This form resembles the identity in (2.14). Let $\psi \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$. Then we have the expansion

$$\begin{aligned}
i\sqrt{2m}A_\pm\psi &= i\left[\sqrt{\sqrt{k^2+m^2}+m}A_\pm\psi\right]_{k=0} \\
&= \left[\left(kR_V^{\pm}(k) \pm \frac{\Pi_\pm}{k}(\sqrt{k^2+m^2}+m) - kB_\pm(k)\right)\psi\right]_{k=0} \\
&\quad + \sum_{j=1}\frac{k^j}{j!}\partial_k^j\left[\left(kR_V^{\pm}(k) \pm \frac{\Pi_\pm}{k}(\sqrt{k^2+m^2}+m) - kB(k)\right)\psi\right]_{k=0}.
\end{aligned} \tag{3.5}$$

If we consider only the coefficients of k^0 and use (3.4) then the first term on the right-hand side of (3.5) becomes

$$\begin{aligned}
&\left[kR_V^{\pm}(k)\psi\right]_{k=0}\\
&\quad= \left[R_0^{\pm}(k)\left(\frac{C_\pm}{k}(\sqrt{k^2+m^2}+m) + D_\pm\sqrt{\sqrt{k^2+m^2}+m}\right)\psi\right]_{k=0}\\
&\quad= \left[\int_{\mathbb{R}^3}(\sqrt{k^2+m^2}+m)\left(\frac{i\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}+m(\beta\pm I)\right)\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}(D_\pm\psi)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right]_{k=0}\\
&\qquad+\left[\int_{\mathbb{R}^3}\sqrt{\sqrt{k^2+m^2}+m}\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi}(C_\pm\psi)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right]_{k=0}.
\end{aligned}$$

Taking the limit we find that (3.5) becomes

$$\begin{aligned}
i\sqrt{2m}A_\pm\psi &= 2m\int_{\mathbb{R}^3}\left(\frac{i\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}+m(\beta\pm I)\right)\frac{(D_\pm\psi)(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\\
&\quad+\sqrt{2m}\int_{\mathbb{R}^3}\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\frac{(C_\pm\psi)(\boldsymbol{y})}{4\pi}\,\mathrm{d}\boldsymbol{y}+\sum_{j=1}\frac{k^j}{j!}\partial_k^j\left[kR_V^{\pm}(k)\psi\right]_{k=0},
\end{aligned}$$

where $\int_{\mathbb{R}^3}(D_\pm\psi)(\boldsymbol{y})/|\boldsymbol{x}-\boldsymbol{y}|\,\mathrm{d}\boldsymbol{y}$ is not in $L^2(\mathbb{R}^3)^4$. We now show that this term does not cancel from the remaining derivative terms in the Taylor expansion. Hence for any $m\in\mathbb{N}$

$$\begin{aligned}
&\frac{k^m}{m!}\partial_k^m\left[kR_V^{\pm}(k)\psi\right]_{k=0}\\
&\quad=\frac{k^m}{m!}\partial_k^m\left[R_0^{\pm}(k)\left(\frac{C_\pm}{k}(\sqrt{k^2+m^2}+m)+D_\pm\sqrt{\sqrt{k^2+m^2}+m}\right)\psi\right]_{k=0}+\mathcal{O}(k).
\end{aligned}$$

Equating terms of order k^0, we see no term containing $D_\pm$ as $k\to 0$ since we see from the first relation in Theorem 3.1.3 that $\partial_k^j R_0^{\pm}(k)=\mathcal{O}(k)$ for $j=0,\dots,m$.

2. We now show that the range of $A_\pm$ lies in $U_\pm$. For all $k\in\mathbb{C}$, where $(\mathbb{D}_0\mp\sqrt{k^2+m^2})R_V^{\pm}(k):L^2_{\text{comp}}\to L^2_{\text{comp}}$ we have

$$R_V^{\pm}(k)=R_0^{\pm}(k)(\mathbb{D}_0\mp\sqrt{k^2+m^2})R_V^{\pm}(k), \tag{3.6}$$

and again expand as in (3.5) for $\psi\in L^2_{\text{comp}}(\mathbb{R}^3)^4$ such that by using (3.6) and collecting k^0 terms we have

$$i\sqrt{2m}A_\pm\psi=\left[kR_0^{\pm}(k)(\mathbb{D}_0\mp\sqrt{k^2+m^2})R_V^{\pm}(k)\psi\right]_{k=0}+\sum_{j=1}\frac{k^j}{j!}\partial_k^j\left[kR_V^{\pm}(k)\psi\right]_{k=0}. \tag{3.7}$$

Taking the non derivative term in (3.7), collecting k^0 terms and taking the limit we simply have

$$\left[kR_0^{\pm}(k)(\mathbb{D}_0\mp\sqrt{k^2+m^2})R_V^{\pm}(k)\psi\right]_{k=0}=i\sqrt{2m}R_0^{\pm}(0)(\mathbb{D}_0\mp m)A_\pm\psi,$$

whilst for each $j \geq 1$ derivative we have by collecting k^0 terms:

$$\begin{aligned}
\frac{k^j}{j!}\partial_k^j \left[kR_V^\pm(k)\psi\right]_{k=0} &= \frac{k^j}{j!}\partial_k^j \left[\mp\frac{\Pi_\pm}{k}(\sqrt{k^2+m^2}+m)\psi\right]_{k=0} + \mathcal{O}(k) \\
&= \mp\frac{k^j}{j!}\left[(\partial_k^{j-1}k^{-1})\partial_k(\sqrt{k^2+m^2}+m)\binom{j}{j-1}\Pi_\pm\psi\right]_{k=0} \\
&= \mp(-1)^j\left[\frac{k}{(\sqrt{k^2+m^2}+m)}\Pi_\pm\psi\right]_{k=0} = 0.
\end{aligned}$$

Hence (3.7) simplifies to

$$A_\pm\psi = R_0^\pm(0)(\mathbb{D}_0 \mp m)A_\pm\psi. \tag{3.8}$$

This together with the property $(\mathbb{D}_V \mp m)A_\pm = 0$ from Theorem 3.1.2 is enough to satisfy Definition 3.1.1 that the range of $A_\pm$ lies in $U_\pm$.

3. Let $u_\pm \in U_\pm$. By (3.8), we write $u_\pm = R_0^\pm(0)g_\pm \in U_\pm$ where $g_\pm = (\mathbb{D}_0 \mp m)u_\pm \in L^2_{\text{comp}}(\mathbb{R}^3)^4$. Proving $g_\pm = -Vu_\pm$ follows from Definition 3.1.1 and the result from the previous step: $0 = (\mathbb{D}_V \mp m)A_\pm = (\mathbb{D}_0 \mp m)A_\pm + VA_\pm$. □

3.2 Exact form of the full resolvent

In this section we find, for completeness, an explicit form of the operator $A_\pm$. To achieve this we analyse the far field behaviour of functions in $V_\pm$ before determining how they differ from those in $U_\pm$. This enables us to find a relationship between elements in $V_\pm$ and $U_\pm$, and moreover, ascertain that there is only one unique element of $U_\pm$. Since the image of $A_\pm$ is contained in $U_\pm$, this will form our basis for the image of $A_\pm$.

Theorem 3.2.1 *Let $v_\pm \in V_\pm$ and assume V satisfies Assumption 2.4.1. Then,*

1. $v_\pm = R_0^\pm(0)f_\pm$ *where* $f_\pm = (\mathbb{D}_0 \mp m)v_\pm = -Vv_\pm \in L^2_{\text{comp}}(\mathbb{R}^3)^4$ *and* $\int_{\mathbb{R}^3} f_\pm = 0$.

2. *Uniformly in $\boldsymbol{\omega} \in \mathbb{S}^2$ and locally uniformly in $\boldsymbol{y} \in \mathbb{R}^3$,*

$$\begin{aligned}
&v_\pm(\boldsymbol{y}+r\boldsymbol{\omega}) \\
&\quad= \frac{i}{4\pi r^3}\left[3\sum_{i,j=1}^{3}\alpha_i\omega_i\omega_j b_j^\pm(v_\pm) - \sum_{j=1}^{3}\alpha_j b_j^\pm(v_\pm)\right] \\
&\qquad+ \frac{m(\beta\pm I)}{4\pi r^2}\sum_j b_j^\pm(v_\pm)\omega_j + \frac{3m(\beta\pm I)}{8\pi r^3}\sum_{j,k}(B_{jk}^\pm(v_\pm) - 2b_j^\pm(v_\pm)y_k)\omega_j\omega_k \\
&\qquad+ \frac{m(\beta\pm I)}{8\pi r^3}\left(-\sum_j B_{jj}^\pm(v_\pm) + 2\sum_\ell b_\ell^\pm(v_\pm)y_\ell\right) + \mathcal{O}\left(r^{-4}\right), \quad r\to+\infty,
\end{aligned} \tag{3.9}$$

where

$$b_j^\pm(v_\pm) = \int_{\mathbb{R}^3} x_j(\mathbb{D}_0 \mp m)v_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad B_{jk}^\pm(v_\pm) = \int_{\mathbb{R}^3} x_j x_k(\mathbb{D}_0 \mp m)v_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{3.10}$$

3. *For $r>0$ and locally uniformly in $\boldsymbol{y} \in \mathbb{R}^3$ then*

$$\begin{aligned}
I_v^\pm(r,\boldsymbol{y}) &:= \int_{\mathbb{S}^2} v_\pm(\boldsymbol{y}+r\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega} = \mathcal{O}\left(r^{-4}\right), \\
\tilde{I}_v^\pm(r,\boldsymbol{y}) &:= \int_{\mathbb{S}^2} (\overline{\alpha}\cdot\boldsymbol{\omega})v_\pm(\boldsymbol{y}+r\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega} = \mathcal{O}\left(r^{-2}\right),
\end{aligned} \tag{3.11}$$

as $r\to\infty$.

Proof. We divide the proof into 4 steps.

1. Using (3.6) we have by equating k^{-2} factors

$$\Pi_\pm = R_0^\pm(0)(\mathbb{D}_0 \mp m)\Pi_\pm, \quad (\mathbb{D}_0 \mp m)\Pi_\pm : L^2_{\mathrm{comp}} \to L^2_{\mathrm{comp}}.$$

The range of this operator lies in $V_\pm$ by Theorem 3.1.2. If we set $f_\pm = (\mathbb{D}_0 \mp m)v_\pm \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ then $v_\pm = R_0^\pm(0)f_\pm$ is in the range of $\Pi_\pm$. Again by Theorem 3.1.2 we have $(\mathbb{D}_V \mp m)\Pi_\pm = 0$ and so we see that $f_\pm = (\mathbb{D}_0 \mp m)v_\pm = -Vv_\pm$.

2. Let $r>0$ and $\boldsymbol{\omega}\in\mathbb{S}^2$ so $r\boldsymbol{\omega}\in\mathbb{R}^3$. Using the second relation in Theorem 3.1.3 we then write for $v_\pm = R_0^\pm(0)f_\pm \in V_\pm$

$$\begin{aligned}
v_\pm(\boldsymbol{y}+r\boldsymbol{\omega}) &= R_0^\pm(0)f_\pm(\boldsymbol{y}+r\boldsymbol{\omega}) \\
&= \frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{-i\alpha\cdot(\boldsymbol{x}-(\boldsymbol{y}+r\boldsymbol{\omega}))}{|\boldsymbol{x}-(\boldsymbol{y}+r\boldsymbol{\omega})|^2} + m(\beta\pm I)\right)\frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{x}-(\boldsymbol{y}+r\boldsymbol{\omega})|}\,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

where the -1 factor occurs due to the fact that the second relation in Theorem (3.1.3) is with respect to $\boldsymbol{y}$ and not $\boldsymbol{x}$ as is the case here. Writing $\boldsymbol{x}-(\boldsymbol{y}+r\boldsymbol{\omega}) = -r[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]$

we have

$$v_\pm(\boldsymbol{y}+r\boldsymbol{\omega}) = \frac{i}{4\pi r^2}\int_{\mathbb{R}^3}\frac{\alpha\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^3}f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \frac{m(\beta\pm I)}{4\pi r}\int_{\mathbb{R}^3}\frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|}\,\mathrm{d}\boldsymbol{x}. \tag{3.12}$$

Next we consider Taylor expansions of the denominators on the right-hand side of (3.12). Since $(1+s)^{-1/2} = 1 - s/2 + 3s^2/8 + \mathcal{O}\left(s^3\right)$ and $\langle\boldsymbol{\omega}\,,\boldsymbol{\omega}\rangle = 1$ then

$$\begin{aligned}\frac{1}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|} &= \frac{1}{\sqrt{1-2\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle/r + |\boldsymbol{x}-\boldsymbol{y}|^2/r^2}} \\ &= 1 + \frac{\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r} - \frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2r^2} + \frac{3\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle^2}{2r^2} + \mathcal{O}\left(r^{-3}\right).\end{aligned}$$

Hence the second term on the right-hand side of (3.12) is

$$\begin{aligned}&\frac{1}{r}\int_{\mathbb{R}^3}\frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|}\,\mathrm{d}\boldsymbol{x} \\ &\quad= \frac{1}{r}\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\left[1 + \frac{\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r} - \frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2r^2} + \frac{3\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle^2}{2r^2}\right]\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-4}\right).\end{aligned} \tag{3.13}$$

We use the previous expansion to write

$$\frac{1}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^3} = \left[1 + \frac{\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r} + \mathcal{O}\left(r^{-2}\right)\right]^3 = 1 + 3\frac{\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r} + \mathcal{O}\left(r^{-2}\right).$$

For the first term on the right-hand side of (3.12) we therefore have

$$\begin{aligned}&\frac{1}{r^2}\int_{\mathbb{R}^3}\frac{\alpha\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^3}f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad= \frac{1}{r^2}\int_{\mathbb{R}^3}\alpha\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]\left[1+\frac{3\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r}\right]f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-4}\right).\end{aligned} \tag{3.14}$$

Bringing together (3.13) and (3.14) with (3.12) we see that

$$v_\pm(\boldsymbol{y}+r\boldsymbol{\omega}) = \frac{m(\beta\pm I)}{4\pi r}\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-2}\right). \tag{3.15}$$

By definition $v_\pm \in L^2(\mathbb{R}^3)^4$ but due to the prefactor r^{-1} we have that $\frac{1}{r}\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \notin L^2(\mathbb{R}^3)^4$. We therefore require

$$\int_{\mathbb{R}^3} f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0. \tag{3.16}$$

3. Equation (3.16) greatly simplifies our expansion of $v_\pm(\boldsymbol{y}+r\boldsymbol{\omega})$ in (3.12). To expand the first term on the right-hand side of (3.12) we use (3.14) to write

$$\begin{aligned}&\frac{1}{r^2}\int_{\mathbb{R}^3}\frac{\alpha\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^3}f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad= \frac{1}{r^2}\int_{\mathbb{R}^3}\alpha\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]\left[1+\frac{3\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r}\right]f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-4}\right) \\ &\quad= \frac{3}{r^3}\int_{\mathbb{R}^3}\sum_{i,j}\alpha_i\omega_i\omega_j x_j f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{r^3}\int_{\mathbb{R}^3}\sum_i \alpha_i x_i f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-4}\right).\end{aligned}$$

Hence by (3.10) we obtain the first and second terms on the right-hand side of (3.9)

$$\frac{i}{4\pi r^2}\int_{\mathbb{R}^3}\frac{\alpha\cdot[\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r]}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|^3}f_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\frac{i}{4\pi r^3}\left[3\sum_{i,j}\alpha_i\omega_i\omega_j b_i^\pm(v_\pm)-\sum_i\alpha_i b_i^\pm(v_\pm)\right] + \mathcal{O}\left(r^{-4}\right).$$

To expand the second term on the right-hand side of (3.12) consider first (3.13)

$$\begin{aligned}
&\frac{1}{r}\int_{\mathbb{R}^3}\frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|}\,\mathrm{d}\boldsymbol{x}\\
&\quad=\frac{1}{r}\int_{\mathbb{R}^3}f_\pm(\boldsymbol{x})\left[\frac{\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle}{r}-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2r^2}+\frac{3\langle\boldsymbol{\omega}\,,\boldsymbol{x}-\boldsymbol{y}\rangle^2}{2r^2}\right]\mathrm{d}\boldsymbol{x}+\mathcal{O}\left(r^{-4}\right)\\
&\quad=\frac{1}{r^2}\int_{\mathbb{R}^3}f_\pm(\boldsymbol{x})\sum_i\omega_i x_i\,\mathrm{d}\boldsymbol{x}-\frac{1}{2r^3}\int_{\mathbb{R}^3}f_\pm(\boldsymbol{x})\sum_i\left[x_i^2-2x_iy_i\right]\,\mathrm{d}\boldsymbol{x}\\
&\qquad+\frac{3}{2r^3}\int_{\mathbb{R}^3}f_\pm(\boldsymbol{x})\sum_{i,j}\left[\omega_i x_i\omega_j x_j-2\omega_i x_i\omega_j y_j\right]\,\mathrm{d}\boldsymbol{x}+\mathcal{O}\left(r^{-4}\right),
\end{aligned}$$

and by (3.10)

$$\begin{aligned}
&\frac{m(\beta\pm I)}{4\pi r}\int_{\mathbb{R}^3}\frac{f_\pm(\boldsymbol{x})}{|\boldsymbol{\omega}-(\boldsymbol{x}-\boldsymbol{y})/r|}\,\mathrm{d}\boldsymbol{x}\\
&\quad=\frac{m(\beta\pm I)}{4\pi r^2}\sum_i b_i^\pm(v_\pm)\omega_i-\frac{m(\beta\pm I)}{8\pi r^3}\left[\sum_i B_{ii}^\pm(v_\pm)-2\sum_i y_i b_i^\pm(v_\pm)\right]\\
&\qquad+\frac{3m(\beta\pm I)}{8\pi r^3}\sum_{i,j}\left[B_{ij}^\pm(v_\pm)-2y_i b_j^\pm(v_\pm)\right]\omega_i\omega_j+\mathcal{O}\left(r^{-4}\right),
\end{aligned}$$

which provides us with the remaining explicit terms on the right-hand side of (3.9).

4. To show (3.11) we note that

$$\int_{\mathbb{S}^2}\omega_j\,\mathrm{d}\boldsymbol{\omega}=0,\qquad\int_{\mathbb{S}^2}\omega_j\omega_k\,\mathrm{d}\boldsymbol{\omega}=\frac{4\pi}{3}\delta_{jk},\qquad\int_{\mathbb{S}^2}\omega_j\omega_k\omega_\ell\,\mathrm{d}\boldsymbol{\omega}=0,$$

where δ_{jk} is the Kronecker delta. For $I_\pm(r,\boldsymbol{y})$ the spherical integrals of all terms on the right-hand side of (3.9) either cancel or are equal to zero up to the power r^{-4}. Explicitly we have

$$\sum_j b_j^\pm(v_\pm)\int_{\mathbb{S}^2}\omega_j\,\mathrm{d}\boldsymbol{\omega}=0,$$

whilst we have cancellation amongst the r^{-3} terms. This occurs since the first 2 terms on the right-hand side of (3.9) cancel. Moreover writing

$$\begin{aligned}
&\frac{3m(\beta\pm I)}{8\pi r^3}\sum_{i,j}\left[B_{ij}^\pm(v_\pm)-2b_i^\pm(v_\pm)y_j\right]\int_{\mathbb{S}^2}\omega_i\omega_j\,\mathrm{d}\boldsymbol{\omega}\\
&\quad=\frac{3m(\beta\pm I)}{8\pi r^3}\sum_{i,j}\left[B_{ij}^\pm(v_\pm)-2b_i^\pm(v_\pm)y_j\right]\left(\frac{4\pi}{3}\delta_{ij}\right)\\
&\quad=\frac{m(\beta\pm I)}{2\pi r^3}\left[\sum_i B_{ii}^\pm(v_\pm)-2\sum_i y_i b_i^\pm(v_\pm)\right],
\end{aligned}$$

cancels the final two r^{-3} terms on the right-hand side of (3.9). For $\tilde{I}_{\pm}(r,\boldsymbol{y})$ we retain only one term,

$$\begin{aligned}
\tilde{I}_v^{\pm}(r,\boldsymbol{y}) &= \int_{\mathbb{S}^2} \sum_i \overline{\alpha_i}\omega_i v(\boldsymbol{y}+r\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega} \\
&= \sum_{i,j} \overline{\alpha_i} \int_{\mathbb{S}^2} \omega_i \left[\frac{m(\beta \pm I)}{4\pi r^2} b_j^{\pm}(v_{\pm})\omega_j\right] \mathrm{d}\boldsymbol{\omega} + \mathcal{O}\left(r^{-4}\right) \\
&= \frac{m(\beta \pm I)}{3r^2} \sum_i \overline{\alpha_i} b_i^{\pm}(v_{\pm}) + \mathcal{O}\left(r^{-4}\right).
\end{aligned}$$

Therefore $\tilde{I}_{\pm}(r,\boldsymbol{y}) = \mathcal{O}\left(r^{-2}\right)$ as $r \to \infty$. $\square$

Theorem 3.2.2 *Let $v \in V_{\pm}$, $\phi \in L^2_{\mathrm{comp}}(\mathbb{R}^3;\mathbb{R}^4)$ and set $u_{\pm} := R_0^{\pm}(0)\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)^4$. Then the following limit, independent of $\boldsymbol{y} \in \mathbb{R}^3$ exists:*

$$\langle v_{\pm}, u_{\pm}\rangle_0 := \lim_{R\to\infty} \int_{B(\boldsymbol{y};R)} v_{\pm}(\boldsymbol{x})\overline{u_{\pm}(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} = -i\langle H_v^{\pm},\phi\rangle + m(\overline{\beta} \pm I)\langle K_v^{\pm},\phi\rangle, \tag{3.17}$$

where we define

$$H_v^{\pm}(\boldsymbol{y}) := \frac{1}{4\pi}\int_0^{\infty} \tilde{I}_v^{\pm}(r,\boldsymbol{y})\,\mathrm{d}r, \qquad K_v^{\pm}(\boldsymbol{y}) := \frac{1}{4\pi}\int_0^{\infty} r I_v^{\pm}(r,\boldsymbol{y})\,\mathrm{d}r. \tag{3.18}$$

Proof. We divide the proof into 2 steps.

1. We first show locally uniformly in $\boldsymbol{y}, \boldsymbol{y}' \in \mathbb{R}^3$ and for large R that

$$\partial_{y_j} \int_{B(\boldsymbol{y}';R)} \left[\frac{-i\overline{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta} \pm I)\right] \frac{v_{\pm}(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x} = \mathcal{O}\left(r^{-1}\right). \tag{3.19}$$

Indeed using the fundamental theorem of calculus, $\partial_{y_j} r = \omega_j$, and $\boldsymbol{x} = \boldsymbol{y}' + r\boldsymbol{\omega}$ we have

$$\begin{aligned}
&\partial_{y_j} \int_{B(\boldsymbol{y}';R)} \frac{v_{\pm}(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|} \left[\frac{-i\overline{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta} \pm I)\right] \mathrm{d}\boldsymbol{x} \\
&= \partial_{y_j} \int_{\mathbb{S}^2}\int_0^R \frac{v_{\pm}(\boldsymbol{y}'+r\boldsymbol{\omega})}{|\boldsymbol{y}'+r\boldsymbol{\omega}-\boldsymbol{y}|} \left[\frac{-i\overline{\alpha}\cdot(\boldsymbol{y}'+r\boldsymbol{\omega}-\boldsymbol{y})}{|\boldsymbol{y}'+r\boldsymbol{\omega}-\boldsymbol{y}|^2} + m(\overline{\beta} \pm I)\right] r^2\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\omega} \\
&= R^2 \int_{\mathbb{S}^2} \omega_j \frac{v_{\pm}(\boldsymbol{y}'+R\boldsymbol{\omega})}{|\boldsymbol{y}'+R\boldsymbol{\omega}-\boldsymbol{y}|} \left[\frac{-i\overline{\alpha}\cdot(\boldsymbol{y}'+R\boldsymbol{\omega}-\boldsymbol{y})}{|\boldsymbol{y}'+R\boldsymbol{\omega}-\boldsymbol{y}|^2} + m(\overline{\beta} \pm I)\right] \mathrm{d}\boldsymbol{\omega} \\
&= R^2 \int_{\mathbb{S}^2} \omega_j v_{\pm}(\boldsymbol{y}'+R\boldsymbol{\omega}) \frac{-i\overline{\alpha}\cdot(\boldsymbol{y}'-\boldsymbol{y}+R\boldsymbol{\omega})}{|\boldsymbol{y}'-\boldsymbol{y}+R\boldsymbol{\omega}|^3}\,\mathrm{d}\boldsymbol{\omega} \\
&\quad + m(\overline{\beta} \pm I)R^2 \int_{\mathbb{S}^2} \omega_j \frac{v_{\pm}(\boldsymbol{y}'+R\boldsymbol{\omega})}{|\boldsymbol{y}'-\boldsymbol{y}+R\boldsymbol{\omega}|}\,\mathrm{d}\boldsymbol{\omega}.
\end{aligned}$$

Since $v_{\pm}(\boldsymbol{x}) = \mathcal{O}\left(|\boldsymbol{x}|^{-2}\right)$ then the two terms above on the right-hand side are $\mathcal{O}\left(R^{-2}\right)$ and $\mathcal{O}\left(R^{-1}\right)$ respectively. Then (3.19) holds true locally uniformly in $\boldsymbol{y}, \boldsymbol{y}'$ and implies that

$$\int_{B(\boldsymbol{y};R)} \left[\frac{-i\overline{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta}\pm I)\right] \frac{v_\pm(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x}$$
$$= \int_{B(\boldsymbol{y}';R)} \left[\frac{-i\overline{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta}\pm I)\right] \frac{v_\pm(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(R^{-1}\right).$$

2. For each fixed $\boldsymbol{y}' \in \mathbb{R}^3$ where $\boldsymbol{x}-\boldsymbol{y} = r\boldsymbol{\omega}$ and $\phi \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ we have

$$\begin{aligned}
&-i\langle H_v^\pm, \phi\rangle + m(\overline{\beta}\pm I)\langle K_v^\pm, \phi\rangle \\
&= \lim_{R\to\infty} \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_0^R\int_{\mathbb{S}^2} \left[\frac{-i\overline{\alpha}\cdot r\boldsymbol{\omega}}{r^2} + m(\overline{\beta}\pm I)\right] \frac{v_\pm(\boldsymbol{y}+r\boldsymbol{\omega})}{r} r^2\,\mathrm{d}\boldsymbol{\omega}\,\mathrm{d}r\overline{\phi(\boldsymbol{y})}\,\mathrm{d}\boldsymbol{y} \\
&= \lim_{R\to\infty} \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{B(\boldsymbol{y};R)} \left[\frac{-i\overline{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta}\pm I)\right] \frac{v_\pm(\boldsymbol{y}+r\boldsymbol{\omega})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x}\overline{\phi(\boldsymbol{y})}\,\mathrm{d}\boldsymbol{y} \\
&= \lim_{R\to\infty} \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{B(\boldsymbol{y}';R)} \left[\frac{-i\overline{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta}\pm I)\right] \frac{v_\pm(\boldsymbol{y}+r\boldsymbol{\omega})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x}\overline{\phi(\boldsymbol{y})}\,\mathrm{d}\boldsymbol{y} + \mathcal{O}\left(R^{-1}\right) \\
&= \lim_{R\to\infty} \frac{1}{4\pi}\int_{B(\boldsymbol{y}';R)} v_\pm(\boldsymbol{x})\left[\int_{\mathbb{R}^3}\left(\frac{\overline{i\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + m(\overline{\beta}\pm I)\right)\frac{\overline{\phi(\boldsymbol{y})}}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\right]\mathrm{d}\boldsymbol{x} \\
&= \lim_{R\to\infty}\int_{B(\boldsymbol{y}';R)} v_\pm(\boldsymbol{x})\overline{u_\pm(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} = \langle v_\pm, u_\pm\rangle_0.
\end{aligned}$$
□

Theorem 3.2.3 *The spaces $V_\pm$ and $U_\pm$ are related by*

$$V_\pm = \left\{v_\pm \in U_\pm : \int_{\mathbb{R}^3} (\mathbb{D}_0 \mp m)v_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0\right\}, \tag{3.20}$$

and we define the multiplicity

$$\widetilde{m}_R(\pm m) := \dim(U_\pm/V_\pm) = \{0,1\}. \tag{3.21}$$

Moreover if $u_\pm \in U_\pm$ then

$$u_\pm + \frac{m\sqrt{2m}}{4\pi}(\beta\pm I)A_\pm V\int_{\mathbb{R}^3}(\mathbb{D}_0 \mp m)u_\pm \in V_\pm. \tag{3.22}$$

Proof. We divide the proof into 2 steps.

1. Clearly $V_\pm \subset U_\pm$. Theorem 3.2.1 indicates that $0 = \int_{\mathbb{R}^3} f_\pm = \int_{\mathbb{R}^3}(\mathbb{D}_0 \mp m)v_\pm$ which gives rise to (3.20). However the method leading to the expansion (3.15) can also be applied to $u_\pm \in U_\pm$ in which case the $1/r$ term is non-zero. This leads to (3.21).

2. Extending the notation in Theorem 3.2.2 and analogous to that in Theorem 3.2.1, set $u_\pm = R_0^\pm(0)g_\pm$ such that $g_\pm = (\mathbb{D}_0 \mp m)u_\pm = -Vu_\pm \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$. If $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho = 1$ on $\operatorname{supp} V$ then we consider the Taylor expansion of $u_\pm$:

$$u_\pm = R_V^\pm(k)\rho(\mathbb{D}_V \mp \sqrt{k^2+m^2})\sum_{j=0}\frac{k^j}{j!}[\partial_k^j R_0^\pm(k)g_\pm]_{k=0}. \tag{3.23}$$

Justification for inserting the bump function ρ follows from (2.11). Since (3.23) is holomorphic on the left-hand side, then the poles on the right-hand side must cancel. Then collecting k^0 terms in the non-derivative term (3.23) becomes

$$R_V^{\pm}(k)\rho(\mathbb{D}_V \mp \sqrt{k^2+m^2})\left[R_0^{\pm}(k)g_{\pm}\right]_{k=0} \xrightarrow[k\to 0]{} 0,$$

since $(\mathbb{D}_V \mp m)u_{\pm} = 0$. Next consider the first derivative term on the right-hand side of (3.23). Collecting k^0 terms again and using the final relation in Theorem 3.1.3 we have

$$kR_V^{\pm}(k)(\mathbb{D}_V \mp \sqrt{k^2+m^2})\left[\partial_k R_0^{\pm}(k)g_{\pm}\right]_{k=0} \xrightarrow[k\to 0]{} iA_{\pm}(\mathbb{D}_V \mp m)\sqrt{2m}\int \frac{im}{4\pi}(\beta \pm I)g_{\pm}.$$

This remaining term lies in the range of $U_{\pm}$. From (3.23) we note that any further k^0 terms will be mapped into $\Pi_{\pm}$. Hence

$$u_{\pm} + \frac{m\sqrt{2m}}{4\pi}(\beta \pm I)A_{\pm}(\mathbb{D}_V \mp m)\int g_{\pm} \in V_{\pm}.$$

To obtain (3.22) we show that

$$(\beta \pm I)(\mathbb{D}_0 \mp m)\int_{\mathbb{R}^3} g_{\pm}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = (\beta \pm I)(-i\alpha\cdot\nabla + \beta m \mp m)\int_{\mathbb{R}^3} g_{\pm}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0,$$

which follows from $\alpha\cdot\nabla_{\boldsymbol{x}}\int_{\mathbb{R}^3} g_{\pm}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0$ and $(\beta \pm I)(\beta \mp I) = \beta^2 - I = 0$ by the conditions in (2.1). $\square$

Theorem 3.2.4 *The operator $A_{\pm}$ has the explicit form*

$$A_{\pm} = \widetilde{m}_R(\pm m)(w_{\pm}\otimes \overline{w}_{\pm}),$$

where $w_{\pm}$ is the unique element of $U_{\pm}$ satisfying

$$w_{\pm}(\boldsymbol{x}) = -\frac{1}{\sqrt{2m}}\frac{h}{|\boldsymbol{x}|} + \mathcal{O}\left(|\boldsymbol{x}|^{-2}\right), \qquad h = (1,1,1,1)^{\top}. \tag{3.24}$$

Proof. We divide the proof into 2 steps.

1. Let $v_{\pm} \in V_{\pm}$, $\psi \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ and $\rho \in C_0^{\infty}(\mathbb{R}^3)$ such that $\rho V = V$ and $\rho\psi = \psi$. If $t \ll E_j'$ then we write

$$\begin{aligned}\langle v_{\pm}\,,\psi\rangle &= \langle R_V^{\pm}(it)(\mathbb{D}_V \mp \sqrt{m^2-t^2})v_{\pm}\,,\psi\rangle \\ &= \langle[(\mathbb{D}_V \mp m) \mp (\sqrt{m^2-t^2}-m)]v_{\pm}\,, R_V^{\pm}(it)\psi\rangle \\ &= \mp(\sqrt{m^2-t^2}-m)\langle v_{\pm}\,, R_0^{\pm}(it)(\mathbb{D}_0 \mp \sqrt{m^2-t^2})R_V^{\pm}(it)\psi\rangle,\end{aligned}$$

where we have used $(\mathbb{D}_V \mp m)A_\pm = 0$ from Theorem 3.1.2. Explicitly we write

$$\begin{aligned}
&\langle v_\pm\,, \psi\rangle \\
&= \mp(\sqrt{m^2-t^2}-m)\int_{\mathbb{R}^3} v_\pm(\boldsymbol{x})\overline{\int_{\mathbb{R}^3}\left[\frac{i\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}+it\frac{\alpha\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}+\beta m\pm\sqrt{m^2-t^2}\right]} \\
&\quad \overline{\times\frac{e^{-t|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}(\mathbb{D}_0\mp\sqrt{m^2-t^2})R_V^\pm(it)\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}}\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Let $r\boldsymbol{\omega} = \boldsymbol{x}-\boldsymbol{y}$ where $r>0$, $\boldsymbol{\omega}\in\mathbb{S}^2$. Hence

$$\begin{aligned}
&\langle v_\pm\,, \psi\rangle \\
&= \mp(\sqrt{m^2-t^2}-m)\int_{\mathbb{R}^3}\lim_{R\to\infty}\int_0^R\int_{\mathbb{S}^2} v_\pm(\boldsymbol{y}+r\boldsymbol{\omega}) \\
&\quad\times\left[-i\overline{\alpha}\cdot\boldsymbol{\omega}(1+rt)+r(\overline{\beta}m\pm\sqrt{m^2-t^2})\right]\frac{e^{-tr}}{4\pi}\overline{(\mathbb{D}_0\mp\sqrt{m^2-t^2})R_V^\pm(it)\psi(\boldsymbol{y})}\,\mathrm{d}\boldsymbol{\omega}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{y} \\
&= \mp(\sqrt{m^2-t^2}-m)\int_{\mathbb{R}^3}\lim_{R\to\infty}\int_0^R\left[-i\tilde{I}_v^\pm(r,\boldsymbol{y})(1+rt)+r(\overline{\beta}m\pm\sqrt{m^2-t^2})I_v^\pm(r,\boldsymbol{y})\right] \\
&\quad\times\frac{e^{-tr}}{4\pi}\overline{(\mathbb{D}_0\mp\sqrt{m^2-t^2})R_V^\pm(it)\psi(\boldsymbol{y})}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{y},
\end{aligned}$$

where we have used (3.11). We proceed with the expansion $e^x = \sum_{j=0} x^j/j!$. For convergence we require for

$$\int_0^\infty r^\alpha I_v^\pm(r,\boldsymbol{y})\,\mathrm{d}r, \qquad \int_0^\infty r^\beta \tilde{I}_v^\pm(r,\boldsymbol{y})\,\mathrm{d}r,$$

that $\alpha<3$ and $\beta<1$ by (3.11). If we introduce

$$J_v^\pm(\boldsymbol{y}) := -\frac{1}{4\pi}\int_0^\infty r^2 I_v^\pm(r,\boldsymbol{y})\,\mathrm{d}r,$$

and use (3.18) we write

$$\begin{aligned}
&\langle v_\pm\,, \psi\rangle \\
&= \mp\frac{(\sqrt{m^2-t^2}-m)}{4\pi}\int_{\mathbb{R}^3}\lim_{R\to\infty}\int_0^R\left[-i\tilde{I}_v^\pm(r,\boldsymbol{y})+r(\overline{\beta}m\pm\sqrt{m^2-t^2})I_v^\pm(r,\boldsymbol{y})\,(1-rt)\right] \\
&\quad\times\overline{(\mathbb{D}_0\mp\sqrt{m^2-t^2})R_V^\pm(it)\psi(\boldsymbol{y})}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{y}+\mathcal{O}\left(t^{3/2}\right) \\
&= \mp(\sqrt{m^2-t^2}-m)\int_{\mathbb{R}^3}\left[-iH_v^\pm(\boldsymbol{y})+(\overline{\beta}m\pm\sqrt{m^2-t^2})\left(K_v^\pm(\boldsymbol{y})+tJ_v^\pm(\boldsymbol{y})\right)\right] \\
&\quad\times\overline{(\mathbb{D}_0\mp\sqrt{m^2-t^2})R_V^\pm(it)\psi(\boldsymbol{y})}\,\mathrm{d}\boldsymbol{y}+\mathcal{O}\left(t^{3/2}\right).
\end{aligned} \tag{3.25}$$

From (3.1) we note

$$\mp(\sqrt{m^2-t^2}-m)R_V^\pm(it) = \Pi_\pm \mp iA_\pm\sqrt{\sqrt{m^2-t^2}-m}+\mathcal{O}\left(\sqrt{m^2-t^2}-m\right),$$

such that we rewrite (3.25) as

$$\begin{aligned}\langle v_\pm\,,\psi\rangle &= -i\langle H_v^\pm\,,(\mathbb{D}_0 \mp \sqrt{m^2-t^2})\Pi_\pm\psi\rangle\\
&\quad - i\langle H_v^\pm\,,(\mp i)\sqrt{\sqrt{m^2-t^2}-m}(\mathbb{D}_0 \mp \sqrt{m^2-t^2})A_\pm\psi\rangle\\
&\quad + (\overline{\beta}m \pm \sqrt{m^2-t^2})\langle K_v^\pm\,,(\mathbb{D}_0 \mp \sqrt{m^2-t^2})\Pi_\pm\psi\rangle\\
&\quad + (\overline{\beta}m \pm \sqrt{m^2-t^2})\langle K_v^\pm\,,(\mp i)\sqrt{\sqrt{m^2-t^2}-m}(\mathbb{D}_0 \mp \sqrt{m^2-t^2})A_\pm\psi\rangle\\
&\quad + t(\overline{\beta}m \pm \sqrt{m^2-t^2})\langle J_v^\pm\,,(\mathbb{D}_0 \mp \sqrt{m^2-t^2})\Pi_\pm\psi\rangle\\
&\quad + t(\overline{\beta}m \pm \sqrt{m^2-t^2})\langle J_v^\pm\,,(\mp i)\sqrt{\sqrt{m^2-t^2}-m}(\mathbb{D}_0 \mp \sqrt{m^2-t^2})A_\pm\psi\rangle\\
&\quad + \mathcal{O}\left(\overline{\sqrt{m^2-t^2}-m}\right).\end{aligned}$$

Collect $\sqrt{\sqrt{m^2-t^2}-m}$ terms, let $t \to 0$ and use (3.17) so that we have

$$\begin{aligned}0 &= -i\langle H_v^\pm\,,(\mathbb{D}_0 \mp m)A_\pm\psi\rangle + m(\overline{\beta} \pm I)\langle K_v^\pm\,,(\mathbb{D}_0 \mp m)A_\pm\psi\rangle\\
&= \langle v_\pm\,,R_0^\pm(0)(\mathbb{D}_0 \mp m)A_\pm\psi\rangle_0 = \langle v_\pm\,,A_\pm\psi\rangle_0.\end{aligned} \tag{3.26}$$

2. We now concentrate on (3.26) which suggests that we have some form of orthogonality between $v_\pm \in V_\pm$ and $A_\pm\psi \in U_\pm$ on the inner product defined by (3.17). Trivially if $\widetilde{m}(\pm m) = 0$ then the sets $U_\pm$ and $V_\pm$ coincide exactly and we imply $A_\pm = 0$. Now consider when $\widetilde{m}(\pm m) = 1$. Existence and uniqueness of the element $w_\pm$ defined in (3.24) follow the arguments as in the proof of Theorem 3.2.3. The range of $A_\pm$ is contained in the span of $w_\pm$. As $A_\pm$ is symmetric (see Theorem 3.1.2) we have for some $c \in \mathbb{C}$

$$A_\pm = c(w_\pm \otimes \overline{w}_\pm).$$

The constant c can be determined by inserting $w_\pm$ into (3.22):

$$V_\pm \ni w_\pm + \sqrt{2m}\frac{m}{4\pi}(\beta \pm I)A_\pm V\int_{\mathbb{R}^3}\tilde{g}_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{3.27}$$

where we have set $\tilde{g}_\pm = (\mathbb{D}_0 \mp m)w_\pm \in L^2_{\mathrm{comp}}(\mathbb{R}^3)^4$ and $w_\pm = R_0^\pm(0)\tilde{g}_\pm \in L^2_{\mathrm{loc}}(\mathbb{R}^3)^4$. To determine $\int_{\mathbb{R}^3}\tilde{g}_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ we expand $w_\pm$ as in the proof of Theorem 3.2.1. Analogous to (3.15) we obtain

$$w_\pm(\boldsymbol{x}) = \frac{m(\beta \pm I)}{4\pi r}\int_{\mathbb{R}^3}\tilde{g}_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{O}\left(r^{-2}\right). \tag{3.28}$$

Unlike the case for $v_\pm \in V_\pm$, the first term on the right-hand side is non-zero since we have shown that we require a r^{-1} term for $w_\pm$. Comparing the r^{-1} terms in (3.24) and (3.28) we have

$$-\frac{h}{\sqrt{2m}} = \frac{m(\beta \pm I)}{4\pi}\int_{\mathbb{R}^3}\tilde{g}_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

or

$$\int_{\mathbb{R}^3} \tilde{g}_\pm(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\frac{4\pi}{m\sqrt{2m}}(\beta \pm I)^{-1}h. \tag{3.29}$$

Insert (3.29) into (3.27) and using $A_\pm V h = c w_\pm$ we find

$$V_\pm \ni w_\pm - c w_\pm,$$

or $c = 1$. □

Summary

In this chapter, we constructed the full resolvent of the Dirac operator near the threshold points. In section 3.1 we proved the basic properties of Theorem 3.1.2 and why that form exists. In section 3.2 we studied the exact form of the operator $A_\pm$. The final result is summarized below and will be used in the proof for our Birman-Kreĭn trace formula in chapter 5 when dealing with threshold resonances.

Theorem 3.2.5 *Let V satisfy Assumption 2.4.1 and assume $m_R(\pm m) > 0$. Then near $k = 0$ we have the decomposition for the full resolvent*

$$R_V^\pm(k) = \mp\frac{\Pi_\pm}{k^2}(\sqrt{k^2+m^2}+m) + \frac{\widetilde{m}_R(\pm m)}{k}(w_\pm \otimes \overline{w}_\pm)\sqrt{\sqrt{k^2+m^2}+m} + B_\pm(k),$$

where $k \to B_\pm(k)$ is holomorphic near $k = 0$, $\Pi_\pm$ is the orthogonal and symmetric projection defined in Definition 3.1.1, and the multiplicity $\widetilde{m}_R(\pm m)$ and $w_\pm$ are defined respectively by (3.21) and (3.24).

Chapter 4

The Dirac scattering matrix

In this chapter we introduce some concepts from scattering theory that will be used throughout the remainder of this thesis. The key ingredient is the scattering operator that maps between the initial and final states of a system perturbed by a potential V. In general there are two main approaches to scattering theory; the *inverse problem* assumes that the scattering operator is known and hence used to determine V. If, on the contrary, V is known, then studying the *direct problem* determines the scattering operator. Since we have outlined the properties of V in Assumption 2.4.1 we use the latter approach here.

In section 4.1 we introduce the concept of scattering states and the scattering operator for the Dirac system. These standard definitions and results can be found in [79] and references therein. With this in mind, in section 4.2 we define the scattering matrix, $S^{\pm}(k)$, as a mapping between incoming and outgoing terms of the solutions to the Dirac eigenvalue problem

$$(\mathbb{D}_V \mp \sqrt{k^2+m^2})w^{\pm} = 0. \tag{4.1}$$

We finish the chapter by proving several properties of $S^{\pm}(k)$ and $\det S^{\pm}(k)$ which will be used later in the thesis.

4.1 Scattering states

Standard arguments in spectral theory state that the Hilbert space $\mathcal{H}$ can be decomposed into the orthogonal spectral subspaces $\mathcal{H} = \mathcal{H}_{\mathrm{pp}} \oplus \mathcal{H}_{\mathrm{cont}}$ where $\mathcal{H}_{\mathrm{pp}}$ is the closure of the span of eigenvectors of $\mathbb{D}_V$ and $\mathcal{H}_{\text{cont}}$ corresponds to scattering states. If $\psi \in \mathcal{H}_{\text{cont}}$ then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|\chi(|\boldsymbol{x}| < R) e^{-i\mathbb{D}_V t}\psi\|^2 \,\mathrm{d}t = 0,$$

where χ is the indicator function. If we consider a particle state $\psi(t)$ passing through the interaction region of the potential V, then we expect the external forces to affect the direction that $\psi(t)$ leaves the region of influence (see Figure 4.1). Our assumption that V has compact support suggests that the state $\psi(t) = e^{-i\mathbb{D}_V t}\psi$ may be approximated by solutions of the free Dirac equation $\phi_\pm(t) = e^{-i\mathbb{D}_0 t}\phi_\pm$, $\phi_\pm \in \mathcal{H}$ as $t \to \pm\infty$:

$$\lim_{t\to\pm\infty} \|e^{-i\mathbb{D}_V t}\psi - e^{-i\mathbb{D}_0 t}\phi_\pm\| = 0. \tag{4.2}$$

We introduce the Møller wave operators

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{i\mathbb{D}_V t} e^{-i\mathbb{D}_0 t} : \mathcal{H} \to \mathcal{H}_{\text{cont}},$$

that map the initial (W_-) and final (W_+) states to $\psi(t)$. A key question in scattering theory is whether these operators exist for a pair of Hamiltonians (H, H_0). Indeed, under our assumptions on V, $W_\pm$ exist for the pair $(\mathbb{D}_V, \mathbb{D}_0)$ and, moreover, the completeness property in (4.2) holds (see [79]). Such a system is termed *asymptotically complete* and as a consequence we can use the Møller operators to define the scattering operator

$$S := W_+^* W_- : \mathcal{H} \to \mathcal{H},$$

that maps scattering states from $t = -\infty$ to $t = +\infty$. It can be shown that S commutes with $\mathbb{D}_0$ and as a result, can be represented by the scattering matrix $S^\pm(k)$. In the next section we explore further the properties of $S^\pm(k)$.

4.2 The scattering matrix and its properties

We first define the concepts of incoming and outgoing solutions. These will be used in the definition of the scattering matrix. The reader should note these definitions and not to confuse them with the $\pm$ notation introduced in section 2.3 for the sign of z.

Definition 4.2.1 *Any solution $u^\pm$ to $(\mathbb{D}_V \mp \sqrt{k^2+m^2})u^\pm = f^\pm$ where V satisfies Assumption 2.4.1, for $k \in \mathbb{R} \setminus \{0\}$ and $f^\pm \in L^2_{\text{comp}}(\mathbb{R}^3)^4$ is* outgoing *if*

$$u^\pm = R_0^\pm(k) g^\pm, \tag{4.3}$$

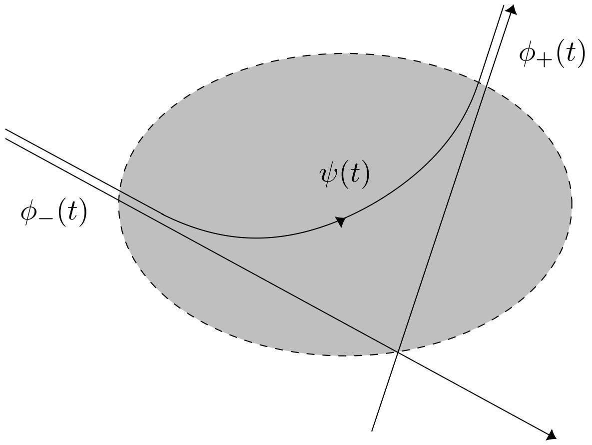

Figure 4.1: Schematic of a scattering process where the grey area indicates the support of V. Outside this interaction region, as $t \to -\infty$ the state $\psi(t)$ is approximated by $\phi_-(t)$ and likewise by $\phi_+(t)$ as $t \to \infty$.

holds for some $g^{\pm} \in L^2_{\text{comp}}(\mathbb{R}^3)^4$. *Similarly* $u^{\pm}$ *is* incoming *if* $u^{\pm} = R_0^{\pm}(-k)g^{\pm}$ *provided* $k \in \mathbb{R} \setminus \{0\}$ *and some* $g^{\pm} \in L^2_{\text{comp}}(\mathbb{R}^3)^4$ *exists.*

Using (2.12) we can rewrite (4.3) such that

$$\begin{aligned} u^{\pm} &= -R_V^{\pm}(k)Ve^{-ik\langle \bullet\,,\boldsymbol{\omega}\rangle} = R_0^{\pm}(k)g^{\pm}, \\ g^{\pm} &= -(I + VR_0^{\pm}(k))^{-1}Ve^{-ik\langle \bullet\,,\boldsymbol{\omega}\rangle} \in L^2_{\text{comp}}(\mathbb{R}^3)^4. \end{aligned} \tag{4.4}$$

To introduce the scattering matrix and provide motivation for its usage, we make use of the following theorem:

Theorem 4.2.2 *Let* $\boldsymbol{x} = r\boldsymbol{\theta} \in \mathbb{R}^3$. *Then for* $k \in \mathbb{R} \setminus \{0\}$ *in distributional sense we have*

$$e^{-ik\langle \boldsymbol{x}\,,\boldsymbol{\omega}\rangle} \sim \frac{2\pi i}{kr}\Big[e^{-ikr}\delta(\boldsymbol{\theta} - \boldsymbol{\omega}) - e^{ikr}\delta(\boldsymbol{\theta} + \boldsymbol{\omega})\Big], \qquad r \to \infty. \tag{4.5}$$

Proof. Assume $\phi \in C^{\infty}(\mathbb{S}^2)$ and $\boldsymbol{\omega} = (1,0,0)$. Since $\boldsymbol{\theta} \in \mathbb{S}^2$ then the scalar product $\langle \boldsymbol{\theta}\,,\boldsymbol{\omega}\rangle = \theta_1$ has stationary points at $\theta_1 = \pm 1$, corresponding to opposite poles. We assume further that $\phi(\boldsymbol{\theta})$ will have compact support at these poles. Then writing $\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2$ such that $\boldsymbol{\theta} = (\pm\sqrt{1 - |\boldsymbol{t}|^2}, \boldsymbol{t})$ we have

$$I(r) = \int_{\mathbb{S}^2} e^{-ikr\langle \boldsymbol{\theta}\,,\boldsymbol{\omega}\rangle}\phi(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} = \sum_{\pm}\int_{B_{\mathbb{R}^2}(0;1)} e^{\mp ik\sqrt{1-|\boldsymbol{t}|^2}r}\phi(\pm\sqrt{1 - |\boldsymbol{t}|^2}, \boldsymbol{t})J(t)\,\mathrm{d}\boldsymbol{t},$$

where $J(t) = 1 + \mathcal{O}\left(t^2\right)$ and the summation is over each pole. Using the method of stationary phase (see for instance [29, chapter I]) we therefore have as $kr \to \infty$

$$\begin{aligned} I(r) &\sim \frac{2\pi i}{kr}\left[e^{-ikr}\phi(1,0) - e^{ikr}\phi(-1,0)\right] + \mathcal{O}\left((kr)^{-2}\right) \\ &= \frac{2\pi i}{kr}\left[e^{-ikr}\int_{\mathbb{S}^2}\phi(\boldsymbol{\theta})\delta(\boldsymbol{\theta}-\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\theta} - e^{ikr}\int_{\mathbb{S}^2}\phi(\boldsymbol{\theta})\delta(\boldsymbol{\theta}+\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\theta}\right] + \mathcal{O}\left((kr)^{-2}\right). \end{aligned}$$

Hence we obtain (4.5) in distributional sense for the special case $\boldsymbol{\omega} = (1,0,0)$, ignoring the higher powers of $1/kr$ as $r \to \infty$. In general any ω can then be constructed as a sum of such functions. □

We now introduce the scattering matrix which maps the incoming components of $w^{\pm}$ solving (4.1) to the outgoing components.

Definition 4.2.3 *Solutions to the eigenvalue equation*

$$(\mathbb{D}_V \mp \sqrt{k^2+m^2})w^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}) = 0,$$

where V satisfies Assumption 2.4.1, will be considered of the form

$$w^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}) = e^{-ik\langle\boldsymbol{x},\boldsymbol{\omega}\rangle} + u^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}),$$

and the outgoing $u^{\pm}$ satisfies (4.3). If $b^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega})$ is the leading asymptotic term of $u^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega})$ such that

$$u^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}) = -\frac{2\pi i}{kr}e^{ikr}b^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}) + \mathcal{O}\left(r^{-2}\right), \tag{4.6}$$

then we define the absolute scattering matrix as

$$S^{\pm}_{\mathrm{abs}}(k) : \delta(\boldsymbol{\theta}-\boldsymbol{\omega}) \mapsto -(\delta(\boldsymbol{\theta}+\boldsymbol{\omega}) + b^{\pm}(k,\boldsymbol{\theta},\boldsymbol{\omega})), \tag{4.7}$$

which we normalize to define the scattering matrix

$$S^{\pm}(k) : \delta(\boldsymbol{\theta}-\boldsymbol{\omega}) \mapsto \delta(\boldsymbol{\theta}-\boldsymbol{\omega}) + b^{\pm}(k,\boldsymbol{\theta},-\boldsymbol{\omega}). \tag{4.8}$$

To motivate the concepts of $S^{\pm}_{\mathrm{abs}}(k)$ and $S^{\pm}(k)$, we use the distributional form presented in Theorem 4.2.2 and (4.6) to write the leading term of $w^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega})$ for $r \to \infty$ as

$$\begin{aligned} w^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}) &= e^{-ik\langle\boldsymbol{x},\boldsymbol{\omega}\rangle} + u^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}) \\ &\sim \frac{2\pi i}{kr}\left[e^{-ikr}\delta(\boldsymbol{\theta}-\boldsymbol{\omega}) - e^{ikr}\delta(\boldsymbol{\theta}+\boldsymbol{\omega})\right] + u^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}) \\ &\sim \frac{2\pi i}{kr}\left[e^{-ikr}\delta(\boldsymbol{\theta}-\boldsymbol{\omega}) - e^{ikr}(\delta(\boldsymbol{\theta}+\boldsymbol{\omega}) + b^{\pm}(\boldsymbol{x},k,\boldsymbol{\omega}))\right]. \end{aligned}$$

The e^{ikr} prefactor is due to (4.4) and Theorem 2.3.3. Then the absolute scattering matrix maps the incoming terms (those containing the prefactor e^{-ikr}) to the outgoing terms (those containing e^{ikr}) above, thus leading to (4.7). Note that if $V = 0$ then $S^{\pm}_{\text{abs}} f(\theta) = -f(-\theta)$ for suitable f. It is therefore more natural to consider an alternative scattering matrix, $S^{\pm}(k)$, such that for $V = 0$, $S^{\pm} f(\theta) = f(\theta)$. This is our chosen normalization of the absolute scattering matrix as defined in (4.8). These two notions of the scattering matrix are therefore related by

$$S^{\pm}(k) = -S^{\pm}_{\text{abs}}(k) J, \qquad J f(\boldsymbol{\theta}) = f(-\boldsymbol{\theta}). \tag{4.9}$$

Theorem 4.2.4 *Let V satisfy Assumption 2.4.1. Then the scattering matrix can be written as the operator*

$$S^{\pm}(k) = I - A^{\pm}(k) : L^2(\mathbb{S}^2)^4 \to L^2(\mathbb{S}^2)^4, \tag{4.10}$$

where

$$\begin{aligned} A^{\pm}(k) &= \pm \frac{ik\sqrt{k^2+m^2}}{4\pi^2} E_{\pm}(k)(I + V R_0^{\pm}(k)\rho)^{-1} V E_{\pm}(k)^*, \\ E_{\pm}(k, \boldsymbol{x}, \boldsymbol{\omega}) &= \Pi_{\pm}(k\boldsymbol{\theta})\rho(\boldsymbol{x}) e^{-ik\langle \boldsymbol{x}, \boldsymbol{\omega}\rangle} : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{S}^2)^4, \end{aligned} \tag{4.11}$$

the projection $\Pi_{\pm}(k\boldsymbol{\omega})$ is defined in (2.3) and $E_{\pm}(k, \boldsymbol{x}, \boldsymbol{\omega})$ is the kernel of $E_{\pm}(k)$. Moreover $A^{\pm}(k)$ is a trace class operator.

Proof. We divide the proof into 2 steps.

1. From (4.4) and Theorem 2.3.3 we write

$$\begin{aligned} u^{\pm}(r\boldsymbol{\theta}, k, \boldsymbol{\omega}) &= -R_0^{\pm}(k)(I + V R_0^{\pm}(k)\rho)^{-1} V e^{-ik\langle \bullet, \boldsymbol{\omega}\rangle} \\ &= -\frac{e^{ikr}}{4\pi r}(k\alpha \cdot \boldsymbol{\theta} + m\beta \pm \sqrt{k^2+m^2}) \\ &\quad \times \int_{\mathbb{R}^3} e^{-ik\langle \boldsymbol{y}, \boldsymbol{\theta}\rangle}(I + V R_0^{\pm}(k)\rho)^{-1} V e^{-ik\langle \bullet, \boldsymbol{\omega}\rangle}\, \mathrm{d}\boldsymbol{y} + \mathcal{O}\left(r^{-2}\right). \end{aligned}$$

Setting $\boldsymbol{\omega} \to -\boldsymbol{\omega}$ in accordance with (4.8), we write the Schwartz kernel of $A^{\pm}(k)$ by comparing $1/r$ terms in (4.6):

$$\begin{aligned} &b^{\pm}(r\boldsymbol{\theta}, k, -\boldsymbol{\omega}) \\ &= \left(-\frac{kr}{2\pi i} e^{-ikr}\right)\left(-\frac{e^{ikr}}{4\pi}(k\alpha \cdot \boldsymbol{\theta} + m\beta \pm \sqrt{k^2+m^2})\right) \\ &\quad \times \int_{\mathbb{R}^3} e^{-ik\langle \boldsymbol{y}, \boldsymbol{\theta}\rangle}(I + V R_0^{\pm}(k)\rho)^{-1} V e^{ik\langle \bullet, \boldsymbol{\omega}\rangle}\, \mathrm{d}\boldsymbol{y} \\ &= -\frac{ik\sqrt{k^2+m^2}}{4\pi^2}\frac{1}{2}\left(\frac{k\alpha \cdot \boldsymbol{\theta} + m\beta}{\sqrt{k^2+m^2}} + I\right)\int_{\mathbb{R}^3} e^{-ik\langle \boldsymbol{y}, \boldsymbol{\theta}\rangle}(I + V R_0^{\pm}(k)\rho)^{-1} V e^{ik\langle \bullet, \boldsymbol{\omega}\rangle}\, \mathrm{d}\boldsymbol{y} \end{aligned}$$

$$= -(\pm 1)\frac{ik\sqrt{k^2+m^2}}{4\pi^2}\Pi_\pm(k\boldsymbol{\theta})\int_{\mathbb{R}^3} e^{-ik\langle \boldsymbol{y},\boldsymbol{\theta}\rangle}\rho(\boldsymbol{y})(I+VR_0^\pm(k)\rho)^{-1}V\rho(\boldsymbol{y})e^{ik\langle \bullet,\boldsymbol{\omega}\rangle}\,\mathrm{d}\boldsymbol{y},$$

where $(I+VR_0^\pm(k)\rho)^{-1}V = \rho(I+VR_0^\pm(k)\rho)^{-1}V$ by (2.15). This proves (4.10).

2. To prove for a given k, where $k \notin \mathcal{R}_\pm$ (see Definition 2.4.3), that $A^\pm(k)$ is a trace class operator, we use (4.10) and (A.4) to write

$$\|A^\pm(k)\|_{\mathcal{B}_1} \le Ce^{C|k|}\|E_\pm(k)\|\|(I+VR_0^\pm(k)\rho)^{-1}\|\sum_{j=1} s_j[E_\pm(k)]. \tag{4.12}$$

Since $k \notin \mathcal{R}_\pm$, then $\|(I+VR_0^\pm(k)\rho)^{-1}\|$ is bounded. In addition we have for $u \in L^2(\mathbb{R}^3)^4$

$$\|E_\pm(k)u\|_{L^2(\mathbb{S}^2)^4} \le Ce^{C|k|}\|u\|_{L^2(\mathbb{R}^3)^4}. \tag{4.13}$$

To estimate $s_j(E_\pm(k))$ in (4.12) we denote by Δ_ω the Laplace-Beltrami operator on $\mathbb{S}^2$ so that by (A.4) we see

$$s_j[E_\pm(k)] \le s_j[(I-\Delta_\omega)^{-\ell}]\|(I-\Delta_\omega)^\ell E_\pm(k)\|_{L^2(\mathbb{R}^3)^4\to L^2(\mathbb{S}^2)^4}. \tag{4.14}$$

Evaluating $s_j[(I-\Delta_\omega)^{-\ell}]$ follows from (A.10) on the two-dimensional surface of $\mathbb{S}^2$. Hence

$$s_j[(I-\Delta_\omega)^{-\ell}] \le Cj^{-\ell}. \tag{4.15}$$

Similar to (4.13) we also have

$$\|(I-\Delta_\omega)^\ell E_\pm(k)u\|_{L^2(\mathbb{S}^2)^4} \le (2\ell)!e^{C|k|}\|u\|_{L^2(\mathbb{R}^3)^4}, \tag{4.16}$$

where the $(2\ell)!$ factorial is due to differentiating the exponential. Using (4.15), (4.16) and Stirling's approximation (see for instance [80, section 1.87]), $n! \le Cn^n$ alongside setting the free parameter $\ell = (\sqrt{j/e})/2$, we estimate (4.14) as

$$s_j[E_\pm(k)] \le Ce^{C|k|}j^{-\ell}(2\ell)! \le Ce^{C|k|}e^{-\sqrt{j}/C}. \tag{4.17}$$

We therefore conclude that the summation in (4.12) is finite and so $A^\pm(k)$ is a trace class operator. $\square$

Theorem 4.2.5 *Let V satisfy Assumption 2.4.1 and suppose $k \in \mathbb{R}\setminus\{0\}$. Then for a given $g^\pm \in C^\infty(\mathbb{S}^2)^4$ there exists a $f^\pm \in C^\infty(\mathbb{S}^2)^4$ and $v^\pm \in H^1_{\mathrm{loc}}(\mathbb{R}^3)^4$ such that*

$$(\mathbb{D}_V \mp \sqrt{k^2+m^2})v^\pm = 0,$$
$$v^\pm(r\boldsymbol{\theta}) = \frac{C}{r}\left(e^{-ikr}g^\pm(\boldsymbol{\theta}) + e^{ikr}f^\pm(\boldsymbol{\theta})\right) + \mathcal{O}\left(r^{-2}\right).$$

Moreover we have

$$\begin{aligned} S^\pm_{\mathrm{abs}}(k) &: g^\pm(\boldsymbol{\theta}) \mapsto f^\pm(\boldsymbol{\theta}),\\ S^\pm(k) &: -g^\pm(-\boldsymbol{\theta}) \mapsto f^\pm(\boldsymbol{\theta}). \end{aligned} \tag{4.18}$$

Proof. For $g^{\pm} \in C^{\infty}(\mathbb{S}^2)^4$, we define

$$u_0^{\pm}(\boldsymbol{x}) := \int_{\mathbb{S}^2} g^{\pm}(\boldsymbol{\omega}) e^{-ik\langle \boldsymbol{x}, \boldsymbol{\omega}\rangle} \, \mathrm{d}\boldsymbol{\omega},$$

$$\widetilde{u}_0^{\pm}(\boldsymbol{x}) := \int_{\mathbb{S}^2} g^{\pm}(\boldsymbol{\omega})(\alpha \cdot \boldsymbol{\omega}) e^{-ik\langle \boldsymbol{x}, \boldsymbol{\omega}\rangle} \, \mathrm{d}\boldsymbol{\omega},$$

$$v^{\pm}(\boldsymbol{x}) := u_0^{\pm}(\boldsymbol{x}) - R_V^{\pm}(k)[V + m\beta \mp \sqrt{k^2+m^2}] u_0^{\pm}(\boldsymbol{x}) + R_V^{\pm}(k) k \widetilde{u}_0^{\pm}(\boldsymbol{x}).$$

Then $\mathbb{D}_0 u_0^{\pm} = -k\widetilde{u}_0^{\pm} + m\beta u_0^{\pm}$ and so

$$(\mathbb{D}_V \mp \sqrt{k^2+m^2}) u_0^{\pm} = -k\widetilde{u}_0^{\pm} + m\beta u_0^{\pm} + V u_0^{\pm} \mp \sqrt{k^2+m^2} u_0^{\pm},$$

which satisfies

$$(\mathbb{D}_V \mp \sqrt{k^2+m^2}) v^{\pm} = [\mathbb{D}_V \mp \sqrt{k^2+m^2}] u_0^{\pm} - [V + m\beta \mp \sqrt{k^2+m^2}] u_0^{\pm} + k\widetilde{u}_0^{\pm} = 0.$$

Using (4.5) and Theorem 2.3.3 we finally have

$$\begin{aligned}
&v^{\pm}(r\boldsymbol{\theta}) \\
&\quad = u_0^{\pm}(\boldsymbol{x}) - R_V^{\pm}(k)[V + m\beta \mp \sqrt{k^2+m^2}] u_0^{\pm}(\boldsymbol{x}) + R_V^{\pm}(k) k \widetilde{u}_0^{\pm}(\boldsymbol{x}) \\
&\quad = \int_{\mathbb{S}^2} g^{\pm}(\boldsymbol{\omega}) e^{-ik\langle \boldsymbol{x}, \boldsymbol{\omega}\rangle} \, \mathrm{d}\boldsymbol{\omega} \\
&\qquad - \int_{\mathbb{S}^2} g^{\pm}(\boldsymbol{\omega}) R_V^{\pm}(k)[V - k\alpha \cdot \boldsymbol{\omega} + m\beta \mp \sqrt{k^2+m^2}] e^{-ik\langle \bullet, \boldsymbol{\omega}\rangle} \, \mathrm{d}\boldsymbol{\omega} \\
&\quad \sim \int_{\mathbb{S}^2} g^{\pm}(\boldsymbol{\omega}) \left(\frac{2\pi i}{kr}\right) \left(e^{-ikr}\delta(\boldsymbol{\theta} - \boldsymbol{\omega}) - e^{ikr}\delta(\boldsymbol{\theta} + \boldsymbol{\omega})\right) \mathrm{d}\boldsymbol{\omega} \\
&\qquad - \int_{\mathbb{S}^2} g^{\pm}(\boldsymbol{\omega}) \frac{e^{ikr}}{4\pi r} (k\alpha \cdot \boldsymbol{\theta} + m\beta \pm \sqrt{k^2+m^2})(2\pi)^{3/2} \\
&\qquad\quad \mathscr{F}\left[(I + VR_0^{\pm}(k)\rho)^{-1}[V - k\alpha \cdot \boldsymbol{\omega} + m\beta \mp \sqrt{k^2+m^2}] e^{-ik\langle \bullet, \boldsymbol{\omega}\rangle}\right](k\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\omega} + \mathcal{O}\left(r^{-2}\right) \\
&\quad = \frac{C}{r}\left(e^{-ikr} g^{\pm}(\boldsymbol{\theta}) + e^{ikr} f^{\pm}(\boldsymbol{\theta})\right) + \mathcal{O}\left(r^{-2}\right), \qquad r \to \infty,
\end{aligned}$$

and so (4.18) holds. ∎

Akin to Stone's formula for the free Laplacian (4.21) (see also [11, section A.3]), the next theorem describes the difference above and below $\operatorname{spec}(\mathbb{D}_0)$ of the cut-off free resolvent. This will prove instrumental in proving the trace properties of $\det S^{\pm}(k)$, the determinant defined in (A.7).

Theorem 4.2.6 *Let V satisfy Assumption 2.4.1 and assume $\rho \in C_0^{\infty}(\mathbb{R}^3)$ such that $V = \rho V$ on the support of V. Then the analytic extension of the free cut-off free resolvent satisfies*

$$[(\rho R_0^{\pm}(k)\rho - \rho R_0^{\pm}(-k)\rho)] = \pm \frac{ik\sqrt{k^2+m^2}}{4\pi^2} E_{\pm}(k)^* E_{\pm}(k), \tag{4.19}$$

where $E_{\pm}(k, \boldsymbol{x}, \boldsymbol{\omega})$ is defined in (4.11) and the projections $\Pi_{\pm}(k\boldsymbol{\theta})$ are defined in (2.3).

Proof. We divide the proof into 2 steps.

1. We first prove the following useful relationships for the Schwartz kernel of the free Laplacian resolvent satisfying $(R_{00}(\lambda)u)(\boldsymbol{x}) = \int_{\mathbb{R}^3} G_{00}(\boldsymbol{x}-\boldsymbol{y};\lambda)u(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$. Assuming $\operatorname{Im}\lambda > 0$, $z = \sqrt{\lambda}$ then

$$G_{00}(\boldsymbol{x};\lambda) = \frac{e^{i\lambda|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}, \tag{4.20}$$

$$G_{00}(\boldsymbol{x};\lambda) - G_{00}(\boldsymbol{x};-\lambda) = \frac{i\lambda}{8\pi^2}\int_{\mathbb{S}^2} e^{i\lambda\boldsymbol{\omega}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{\omega}, \tag{4.21}$$

where $\boldsymbol{\omega} \in \mathbb{S}^2$ in (4.21). To prove (4.20), first let $r = |\boldsymbol{k}|$ and $a = \cos\theta$. Then consider

$$\int_{\mathbb{R}^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{|\boldsymbol{k}|^2-\lambda^2}\,\mathrm{d}\boldsymbol{k} = 2\pi\int_0^\infty\int_{-1}^1 \frac{e^{i|\boldsymbol{x}|ra}}{r^2-\lambda^2}r^2\,\mathrm{d}a\,\mathrm{d}r = \frac{2\pi}{i|\boldsymbol{x}|}\int_0^\infty \frac{r}{r^2-\lambda^2}(e^{i|\boldsymbol{x}|r} - e^{-i|\boldsymbol{x}|r})\,\mathrm{d}r.$$

This can be rewritten so that the integral is considered over the whole real line and solved by standard contour methods

$$\int_{\mathbb{R}^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{|\boldsymbol{k}|^2-\lambda^2}\,\mathrm{d}\boldsymbol{k} = \frac{2\pi}{i|\boldsymbol{x}|}\int_{-\infty}^\infty \frac{re^{i|\boldsymbol{x}|r}}{r^2-\lambda^2}\,\mathrm{d}r = \frac{2\pi^2}{|\boldsymbol{x}|}e^{i|\boldsymbol{x}|\lambda}.$$

Finally we recognise that for suitable f, we have $(R_{00}(\lambda)f)(\boldsymbol{x}) = (G_{00} * f)(\boldsymbol{x})$, the convolution between the kernel and f. This follows from standard arguments that utilise the fact that the Fourier transform acting upon a differential operator is akin to multiplication by the momenta variable. Hence

$$G_{00}(\boldsymbol{x};\lambda) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{|\boldsymbol{k}|^2-\lambda^2}\,\mathrm{d}\boldsymbol{k} = \frac{e^{i|\boldsymbol{x}|\lambda}}{4\pi|\boldsymbol{x}|}.$$

To prove (4.21), let $\boldsymbol{\omega} = (r,\theta,\phi) \in \mathbb{S}^2$ where $r = 1$ such that setting $a = \cos\theta$ gives

$$\int_{\mathbb{S}^2} e^{-i\lambda\boldsymbol{\omega}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{\omega} = 2\pi\int_0^\pi e^{i\lambda|\boldsymbol{x}|\cos\theta}\sin\theta\,\mathrm{d}\theta = 2\pi\int_{-1}^1 e^{i\lambda|\boldsymbol{x}|a}\,\mathrm{d}a = \frac{4\pi}{\lambda|\boldsymbol{x}|}\sin(\lambda|\boldsymbol{x}|).$$

Hence by (4.20)

$$G_{00}(\boldsymbol{x};\lambda) - G_{00}(\boldsymbol{x};-\lambda) = \frac{1}{4\pi|\boldsymbol{x}|}(e^{i\lambda|\boldsymbol{x}|} - e^{-i\lambda|\boldsymbol{x}|}) = \frac{i}{2\pi|\boldsymbol{x}|}(\sin(\lambda|\boldsymbol{x}|)) = \frac{i\lambda}{8\pi^2}\int_{\mathbb{S}^2} e^{i\lambda\boldsymbol{\omega}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{\omega}.$$

2. Using (4.21) alongside (2.5), (2.2) and (2.3) we write for $f \in L^2(\mathbb{R}^3)^4$

$$
\begin{aligned}
&[\rho(R_0^{\pm}(k) - R_0^{\pm}(-k))\rho f](\boldsymbol{x}) \\
&\quad = \rho(\boldsymbol{x}) \left[(\mathbb{D}_0 \pm \sqrt{k^2+m^2}) \left(R_{00}(k) - R_{00}(-k)\right) \rho f \right](\boldsymbol{x}) \\
&\quad = \frac{ik}{8\pi^2} \rho(\boldsymbol{x})(\mathbb{D}_0 \pm \sqrt{k^2+m^2}) \int_{\mathbb{S}^2} e^{ik\boldsymbol{\omega}\cdot\boldsymbol{x}} \int_{\mathbb{R}^3} e^{-ik\boldsymbol{\omega}\cdot\boldsymbol{y}} \rho(\boldsymbol{y}) f(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{\omega} \\
&\quad = \frac{ik\sqrt{k^2+m^2}}{8\pi^2} \rho(\boldsymbol{x}) \int_{\mathbb{S}^2} e^{ik\boldsymbol{\omega}\cdot\boldsymbol{x}} \left(\frac{k\alpha\cdot\boldsymbol{\omega} + m\beta}{\sqrt{k^2+m^2}} \pm I \right) \int_{\mathbb{R}^3} e^{-ik\boldsymbol{\omega}\cdot\boldsymbol{y}} \rho(\boldsymbol{y}) f(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{\omega} \\
&\quad = \pm \frac{ik\sqrt{k^2+m^2}}{4\pi^2} \int_{\mathbb{S}^2} e^{ik\boldsymbol{\omega}\cdot\boldsymbol{x}} \rho(\boldsymbol{x}) \Pi_{\pm}(k\boldsymbol{\omega}) \int_{\mathbb{R}^3} e^{-ik\boldsymbol{\omega}\cdot\boldsymbol{y}} \rho(\boldsymbol{y}) f(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{\omega} \\
&\quad = \pm \frac{ik\sqrt{k^2+m^2}}{4\pi^2} \left(E_{\pm}(k)^* E_{\pm}(k) f\right)(\boldsymbol{x}). \qquad \square
\end{aligned}
$$

Theorem 4.2.7 *Let V satisfy Assumption 2.4.1. Then for $k \in \mathbb{C}$ the scattering matrix $S^{\pm}(k)$ is meromorphic and whose poles coincide with the poles of $R_V^{\pm}(k)$. Moreover*

$$S^{\pm}(k)^{-1} = S^{\pm}(k)^* = JS^{\pm}(-k)J, \qquad Jf(\boldsymbol{\theta}) = f(-\boldsymbol{\theta}), \tag{4.22}$$

and

$$(\det S^{\pm}(k))^{-1} = \det S^{\pm}(-k). \tag{4.23}$$

Proof. We divide the proof into 3 steps.

1. We first prove $S^{\pm}(k)^{-1} = JS^{\pm}(-k)J$. For $k \in \mathbb{R} \setminus \{0\}$, we have from (4.18) that $S_{\text{abs}}^{\pm}(-k) = S_{\text{abs}}^{\pm}(k)^{-1}$. Since $R_V^{\pm}(k)$ extends in a meromorphic manner to $\mathbb{C}$ (see Theorem 2.4.2) then by (2.14) and (4.11), $S^{\pm}(k)$ also extends meromorphically to $\mathbb{C}$. The poles of $R_V^{\pm}(k)$ and $S^{\pm}(k)$ hence coincide. Since $S^{\pm}(k) = -S_{\text{abs}}^{\pm}(k)J$ from (4.9) then $-S_{\text{abs}}^{\pm}(k)^{-1} = JS^{\pm}(k)^{-1}$ and so

$$S^{\pm}(-k) = -S_{\text{abs}}^{\pm}(-k)J = -S_{\text{abs}}^{\pm}(k)^{-1}J = JS^{\pm}(k)^{-1}J.$$

2. To prove $S^{\pm}(k)^{-1} = S^{\pm}(k)^*$, we note for $k \in \mathbb{R}$ that $R_0^{\pm}(k)^* = R_0^{\pm}(-k)$. Then taking the adjoint of (4.10) we have

$$
\begin{aligned}
&\mp \frac{4\pi^2}{ik\sqrt{k^2+m^2}} (S^{\pm}(k)S^{\pm}(k)^* - I) \\
&= E_{\pm}(k)(I + VR_0^{\pm}(k)\rho)^{-1} V E_{\pm}(k)^* - E_{\pm}(k) V (I + \rho R_0^{\pm}(-k)V)^{-1} E_{\pm}(k)^* \\
&\quad \pm \frac{ik\sqrt{k^2+m^2}}{4\pi^2} E_{\pm}(k)(I + VR_0^{\pm}(k)\rho)^{-1} V E_{\pm}(k)^* E_{\pm}(k) V (I + \rho R_0^{\pm}(-k)V)^{-1} E_{\pm}(k)^* \\
&= E_{\pm}(k)(I + VR_0^{\pm}(k)\rho)^{-1} V \Big[\rho R_0^{\pm}(-k)\rho - \rho R_0^{\pm}(k)\rho
\end{aligned}
$$

$$\pm \frac{ik\sqrt{k^2+m^2}}{4\pi^2} E_\pm(k)^* E_\pm(k)\Bigg] V(I+\rho R_0^\pm(-k)V)^{-1}E_\pm(k)^*,$$

which is equal to zero by (4.19). Hence $S^\pm(k)^{-1} = S^\pm(k)^*$ as stated.

3. Finally to prove (4.23) we use (A.7) and (A.8) (with $A_1 = A^\pm(k)$ and $A_2 = A^\pm(k)^*$) so that $(\det S^\pm(k))^{-1} = \det(S^\pm(k)^*)$. Using this and (4.22) we write

$$(\det S^\pm(k))^{-1} = \det[JS^\pm(-k)J] = \det[I - JA^\pm(-k)J] = \det[I - A^\pm(-k)],$$

where we used the second property in (A.8) for the last step. □

Theorem 4.2.8 *Let V satisfy Assumption 2.4.1. If $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho V = V$ on* $\operatorname{supp} V$*, then the scattering matrix satisfies*

$$\begin{aligned}
\operatorname{Tr}[S^\pm(k)^{-1}\partial_k S^\pm(k)] &= \operatorname{Tr} F^\pm(k) + \operatorname{Tr} F^\pm(-k),\\
F^\pm(k) &= \mp\frac{k}{\sqrt{k^2+m^2}} R_0^\pm(k)(I+VR_0^\pm(k)\rho)^{-1}VR_0^\pm(k),\\
F^\pm(-k) &= \pm\frac{k}{\sqrt{k^2+m^2}} R_0^\pm(-k)(I+VR_0^\pm(-k)\rho)^{-1}VR_0^\pm(-k).
\end{aligned} \tag{4.24}$$

Proof. From Theorem 4.2.4 we use (A.8) and (A.9) continued analytically from small μ until $\mu = 1$ so that

$$\begin{gathered}
\det S^\pm(k) = \det\big(I - T^\pm(k)\big),\\
T^\pm(k) = (I+VR_0^\pm(k)\rho)^{-1}V\left[R_0^\pm(k) - R_0^\pm(-k)\right]\rho : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4,
\end{gathered}$$

where we substituted in (4.19). Then

$$\begin{aligned}
\det S^\pm(k) &= \det\left[(I+VR_0^\pm(k)\rho)^{-1}\big((I+VR_0^\pm(k)\rho) - VR_0^\pm(k)\rho + VR_0^\pm(-k)\rho\big)\right]\\
&= \det\left[(I+VR_0^\pm(k)\rho)^{-1}(I+VR_0^\pm(-k)\rho)\right].
\end{aligned}$$

Taking the logarithmic derivative and using the Jacobi determinant formula (see for instance [27, section IV.1]) gives

$$\begin{aligned}
\partial_k \log\det S^\pm(k) &= \partial_k \log\det\left[(I+VR_0^\pm(k)\rho)^{-1}(I+VR_0^\pm(-k)\rho)\right]\\
&= \operatorname{Tr}\Bigg[\left[(I+VR_0^\pm(k)\rho)^{-1}(I+VR_0^\pm(-k)\rho)\right]^{-1}\\
&\qquad \partial_k\left((I+VR_0^\pm(k)\rho)^{-1}(I+VR_0^\pm(-k)\rho)\right)\Bigg]\\
&= -\operatorname{Tr}\left[(I+VR_0^\pm(-k)\rho)^{-1}\partial_{-k}\big(VR_0^\pm(-k)\rho\big)\right]\\
&\quad + \operatorname{Tr}\left[(I+VR_0^\pm(k)\rho)\big(\partial_k(I+VR_0^\pm(k)\rho)^{-1}\big)\right],
\end{aligned}$$

where we have used the trace cyclicity. We note that by the spectral theorem (see Theorem A.2.3)

$$\partial_k \rho R_0^{\pm}(k)\rho = \pm \frac{k}{\sqrt{k^2+m^2}} \int \rho(\lambda \mp \sqrt{k^2+m^2})^{-2} \rho \,\mathrm{d}E(\lambda) = \pm \frac{k}{\sqrt{k^2+m^2}} \rho R_0^{\pm}(k)^2 \rho, \tag{4.25}$$

and, similarly,

$$\partial_{-k} \rho R_0^{\pm}(-k)\rho = \mp \frac{k}{\sqrt{k^2+m^2}} \rho R_0^{\pm}(-k)^2 \rho,$$
$$\partial_k (I + V R_0^{\pm}(k)\rho)^{-1} = \mp \left(\frac{k}{\sqrt{k^2+m^2}} \right) (I + V R_0^{\pm}(k)\rho)^{-2} V R_0^{\pm}(k)^2 \rho.$$

Hence by trace cyclicity we have

$$\partial_k \log\det S^{\pm}(k) = \operatorname{Tr} F^{\pm}(k) + \operatorname{Tr} F^{\pm}(-k),$$
$$F^{\pm}(k) = \mp \frac{k}{\sqrt{k^2+m^2}} R_0^{\pm}(k)(I + V R_0^{\pm}(k)\rho)^{-1} V R_0^{\pm}(k),$$
$$F^{\pm}(-k) = \pm \frac{k}{\sqrt{k^2+m^2}} R_0^{\pm}(-k)(I + V R_0^{\pm}(-k)\rho)^{-1} V R_0^{\pm}(-k),$$

where

$$R_0^{\pm}(k)(I + V R_0^{\pm}(k)\rho)^{-1} V R_0^{\pm}(k) = R_0^{\pm}(k)\rho(I + V R_0^{\pm}(k)\rho)^{-1} V \rho R_0^{\pm}(k),$$

which follows from (2.15). We obtain the desired result by using the Jacobi determinant formula once again

$$\partial_k \log\det S^{\pm}(k) = \operatorname{Tr}[S^{\pm}(k)^{-1} \partial_k S^{\pm}(k)]. \qquad \square$$

Summary

In this chapter, we have, from basic principles, constructed the Dirac scattering matrix as a mapping between the incoming and outgoing components of the solution to (4.1). We have proved that it can be written as the sum of the identity operator and a trace class operator. Moreover, we have shown that the logarithmic derivative of the scattering matrix determinant is even in k. This will be directly used in obtaining our Birman-Kreĭn trace formula, and also in chapter 7 where we prove the existence of infinitely many resonances.

Chapter 5

The Birman-Kreĭn trace formula

In this chapter we present the first of our two trace formulas for the Dirac operator. In section 5.1 we prove that the difference between Schwartz functions of $\mathbb{D}_V$ and $\mathbb{D}_0$ is trace class. This leads to section 5.2 where our trace formula shows the relationship between the trace difference and the scattering matrix. To calculate the contribution from the threshold resonances, our main result from chapter 3 (Theorem 3.2.5) is employed.

5.1 Trace estimates of the resolvent

Theorem 5.1.1 *Let V satisfy Assumption 2.4.1. Then*

1. *If $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho V = V$ on $\operatorname{supp} V$ we have the following singular value estimates*

$$\begin{aligned} s_{j/\alpha}\left[\rho(\mathbb{D}_0 - i)^{-\beta}\right],\, s_{j/\alpha}\left[(\mathbb{D}_0 - i)^{-\beta}\rho\right] &\le Cj^{-\beta/3}, \\ s_{j/\alpha}\left[\rho(\mathbb{D}_V - i)^{-\beta}\right],\, s_{j/\alpha}\left[(\mathbb{D}_V - i)^{-\beta}\rho\right] &\le Cj^{-\beta/3}. \end{aligned} \tag{5.1}$$

2. *If $z \in \rho(\mathbb{D}_V) \cap \{|\operatorname{Im} z| < 1\}$ we have the trace estimates*

$$\left\|(\mathbb{D}_V - z)^{-1}(\mathbb{D}_V - i)^{-4} - (\mathbb{D}_0 - z)^{-1}(\mathbb{D}_0 - i)^{-4}\right\|_{\mathcal{B}_1} \le \begin{cases} A(z), & |\operatorname{Re} z| < m, \\ B(z), & |\operatorname{Re} z| \ge m, \end{cases}$$
$$A(z) = \max_{E \in \operatorname{spec}(\mathbb{D}_V)} \frac{C}{|\operatorname{Re} z - E|^2 + |\operatorname{Im} z|^2}, \qquad B(z) = \frac{C}{|\operatorname{Im} z|^2}. \tag{5.2}$$

3. *If $f \in \mathscr{S}(\mathbb{R})$, then $f(\mathbb{D}_V) - f(\mathbb{D}_0)$ is a trace class operator.*

Proof. We divide the proof into 4 steps.

1. For $z \in \rho(\mathbb{D}_0)$ we first show for the free resolvent norm

$$\begin{aligned}
\|(\mathbb{D}_0 - z)^{-1}\|_{L^2\to L^2} &= \frac{1}{|\operatorname{Im} z|}, && |\operatorname{Re} z| \geq m,\\
\|(\mathbb{D}_0 - z)^{-1}\|_{L^2\to L^2} &= \frac{1}{\sqrt{|\operatorname{Re} z - m|^2 + |\operatorname{Im} z|^2}}, && |\operatorname{Re} z| < m,
\end{aligned} \tag{5.3}$$

and, likewise, for the full resolvent norm, where $z \in \rho(\mathbb{D}_V)$,

$$\begin{aligned}
\|(\mathbb{D}_V - z)^{-1}\|_{L^2\to L^2} &= \frac{1}{|\operatorname{Im} z|}, && |\operatorname{Re} z| \geq m,\\
\|(\mathbb{D}_V - z)^{-1}\|_{L^2\to L^2} &= \max_{E\in\operatorname{spec}(\mathbb{D}_V)} \frac{1}{\sqrt{|\operatorname{Re} z - E|^2 + |\operatorname{Im} z|^2}}, && |\operatorname{Re} z| < m.
\end{aligned} \tag{5.4}$$

We use these to show that

$$\|(\mathbb{D}_0 - i)^{-1}\|_{L^2\to H^1}, \quad \|(\mathbb{D}_V - i)^{-1}\|_{L^2\to H^1} \leq C. \tag{5.5}$$

For the $L^2 \to L^2$ norms in (5.3) and (5.4) we use Theorem A.2.1. By the geometry of the $\mathbb{C}$ plane, if $|\operatorname{Re} z| \geq m$ then $\operatorname{dist}(\operatorname{spec}(\mathbb{D}_V), z) = \operatorname{dist}(\operatorname{spec}(\mathbb{D}_0), z) = |\operatorname{Im} z|$. Alternatively for $|\operatorname{Re} z| < m$, if $E \in \operatorname{spec}(\mathbb{D}_V)$ is the nearest point to $z \in \rho(\mathbb{D}_V)$ then

$$\frac{1}{\operatorname{dist}(\operatorname{spec}(\mathbb{D}_V), z)} = \frac{1}{\sqrt{|\operatorname{Im} z|^2 + |\operatorname{Re} z - E|^2}}.$$

For $(\mathbb{D}_0 - z)^{-1}$ then there are no discrete spectra in $(-m, m)$ and we require $E = m$. In this case we obtain the second estimate in (5.3). For the first estimate in (5.5) we use (2.5) and the Laplacian resolvent norm estimates (see for instance [23])

$$\|R_{00}(\lambda)\| \leq \frac{\langle\lambda\rangle^k}{|\lambda||\operatorname{Im}\lambda|}, \qquad \operatorname{Im}\lambda > 0,\ 0 \leq k \leq 2,$$

to show that

$$\begin{aligned}
\|(\mathbb{D}_0 - z)^{-1}\|_{L^2\to H^1} &= \|(\mathbb{D}_0 + z)(-\Delta - k^2)^{-1}\|_{L^2\to H^1}\\
&\leq C\left(\sum_{j=1} |\alpha_j| + |\beta| + I\right) \|(-\Delta - k^2)^{-1}\|_{L^2\to H^2}\\
&\leq C\left(\sum_{j=1} |\alpha_j| + |\beta| + I\right) \frac{\langle z\rangle^2}{|\operatorname{Im} z|^2},
\end{aligned}$$

where $|T| = \sqrt{T^*T}$ is the absolute value of each element in T, $\langle k(z)\rangle^2 = 1 + |z^2 - m^2| \leq C\langle z\rangle^2$ and $C|\operatorname{Im} k| \geq |\operatorname{Im} z|$. Set $z = i$ to obtain the first bound in (5.5). To show $C|\operatorname{Im} k| \geq |\operatorname{Im} z|$ we assume $z = x + iy$ and use the complex square root relation

$$\sqrt{z} = \frac{1}{\sqrt{2}}\left[\sqrt{|z| + \operatorname{Re} z} \pm i\sqrt{|z| - \operatorname{Re} z}\right],$$

so that

$$
\begin{aligned}
|\operatorname{Im} k| &= |\operatorname{Im}\sqrt{(x+iy)^2-m^2}| = |\operatorname{Im}\sqrt{x^2-y^2-m^2+2ixy}| \\
&= \frac{1}{\sqrt{2}}\left|\sqrt{\sqrt{(x^2-y^2-m^2)^2+(2xy)^2}-(x^2-y^2-m^2)}\right| \\
&= \frac{1}{\sqrt{2}}\left|\sqrt{\sqrt{x^4+y^4+m^4+2x^2y^2-2x^2m^2+2y^2m^2}-(x^2-y^2-m^2)}\right| \\
&\geq \frac{1}{\sqrt{2}}\left|\sqrt{\sqrt{x^4+m^4-2x^2m^2}-(x^2-y^2-m^2)}\right| \\
&= \frac{1}{\sqrt{2}}\left|\sqrt{x^2-m^2-(x^2-y^2-m^2)}\right| = \frac{|y|}{\sqrt{2}}.
\end{aligned}
$$

For the second bound in (5.5) we use (2.12) to write

$$
\|(\mathbb{D}_V-z)^{-1}\|_{L^2\to H^1} = \|(\mathbb{D}_0-z)^{-1}(I+V(\mathbb{D}_0-z)^{-1})^{-1}\|_{L^2\to H^1}. \tag{5.6}
$$

As in the proof of Theorem 2.4.2, $(I+V(\mathbb{D}_0-z)^{-1})^{-1}$ exists as a Neumann series provided (2.8) is true for large $|k|$. Given (5.3) then there is a large enough z for this to occur. By the Rellich-Kondrachov theorem, $V(\mathbb{D}_0-z)^{-1}$ is compact on $L^2(\mathbb{R}^3)^4$ and in turn, application of Theorem A.3.1 proves the meromorphic continuation of $(I+V(\mathbb{D}_0-z)^{-1})^{-1}$ to the rest of $\rho(\mathbb{D}_V)$. Set $z=i$ and using the first bound in (5.5) on (5.6) proves the second bound (5.5).

2. We next consider the singular value estimates on $(\mathbb{D}_0-i)^{-1}$ in (5.1). The proof follows from (A.10) with $m=1$, (A.6) and (5.5). If $\lceil\cdot\rceil$ denotes the ceiling function then

$$
\begin{aligned}
s_{\lceil j/\alpha\rceil}\left[\rho(\mathbb{D}_0-i)^{-\beta}\right] &\leq \|(-\Delta-1)^{1/2}\rho'(\mathbb{D}_0-i)^{-1}\|^{\beta}_{L^2\to L^2}\left[s_{\lceil j/\alpha\beta\rceil}\left[(-\Delta-1)^{-1/2}\right]\right]^{\beta} \\
&\leq \|\rho'(\mathbb{D}_0-i)^{-1}\|^{\beta}_{L^2\to H^1}\left[C\left\lceil\frac{j}{\alpha\beta}\right\rceil^{-1/3}\right]^{\beta} \leq C' j^{-\beta/3},
\end{aligned} \tag{5.7}
$$

where $\rho=(\rho')^{\beta}$ also satisfies $V=\rho' V$. A similar proof holds for $s_{\lceil j/\alpha\rceil}[(\mathbb{D}_0-i)^{-\beta}\rho]$. For the estimates involving $(\mathbb{D}_V-i)^{-1}$ in (5.1) we prove by induction. The $\beta=0$ case follows immediately: $s_{\lceil j/\alpha\rceil}[\rho((\mathbb{D}_V-i)^{-1})^0] = s_{\lceil j/\alpha\rceil}[((\mathbb{D}_V-i)^{-1})^0\rho] \leq C$. Assuming the estimate holds then we note by (A.5) and (A.6) that

$$
\begin{aligned}
& s_{\lceil j/2\alpha\rceil}\left[\sum_{k=1}^{\beta+1}\rho(\mathbb{D}_V-i)^{-\beta+k-2}(-V)(\mathbb{D}_0-i)^{-k}\right] \\
&\quad \le \sum_{k=1}^{\beta+1} s_{\lceil j/2(\beta+1)\alpha\rceil}\left[\rho(\mathbb{D}_V-i)^{-\beta+k-2}(-V)(\mathbb{D}_0-i)^{-k}\right] \\
&\quad \le \sum_{k=i}^{\beta+1} s_{\lceil j/2(\beta+1)\alpha\rceil}\left[\rho(\mathbb{D}_V-i)^{-\beta+k-2}\rho(-V)\rho(\mathbb{D}_0-i)^{-k}\right] \\
&\quad \le \sum_{k=1}^{\beta+1} s_{\lceil j/4(\beta+1)\alpha\rceil}\left[\rho(\mathbb{D}_V-i)^{-\beta+k-1}\right]\|(\mathbb{D}_V-i)^{-1}\|\|V\| s_{\lceil j/4(\beta+1)\alpha\rceil}\left[\rho(\mathbb{D}_0-i)^{-k}\right] \\
&\quad \le C\sum_{k=1}^{\beta+1}\left\lceil\frac{j}{4\alpha(\beta+1)}\right\rceil^{(-\beta+k-1)/3}\left\lceil\frac{j}{4\alpha(\beta+1)}\right\rceil^{-k/3} \le Cj^{-(\beta+1)/3}.
\end{aligned} \tag{5.8}
$$

Rewriting using the second resolvent identity (see Theorem A.2.2), we have for general z

$$
\begin{aligned}
&(\mathbb{D}_V-z)^{-\beta}-(\mathbb{D}_0-z)^{-\beta} \\
&\quad =\sum_{k=1}^{\beta}\left[(\mathbb{D}_V-z)^{-\beta+k-1}(\mathbb{D}_0-z)^{-k+1}-(\mathbb{D}_V-z)^{-\beta+k}(\mathbb{D}_0-z)^{-k}\right] \\
&\quad =\sum_{k=1}^{\beta}\left[(\mathbb{D}_V-z)^{-\beta+k}\left[(\mathbb{D}_V-z)^{-1}-(\mathbb{D}_0-z)^{-1}\right](\mathbb{D}_0-z)^{-k+1}\right] \\
&\quad =\sum_{k=1}^{\beta}\left[(\mathbb{D}_V-z)^{-\beta+k}\left[(\mathbb{D}_V-z)^{-1}(-V)(\mathbb{D}_0-z)^{-1}\right](\mathbb{D}_0-z)^{-k+1}\right],
\end{aligned} \tag{5.9}
$$

and hence

$$
\begin{aligned}
& s_{\lceil j/\alpha\rceil}\left[\rho(\mathbb{D}_V-i)^{-\beta-1}\right] \\
&\quad \le s_{\lceil j/2\alpha\rceil}\left[\rho(\mathbb{D}_0-i)^{-\beta-1}\right]+s_{\lceil j/2\alpha\rceil}\left[\sum_{k=1}^{\beta+1}\rho(\mathbb{D}_V-i)^{-\beta+k-2}(-V)(\mathbb{D}_0-i)^{-k}\right] \\
&\quad \le Cj^{-(\beta+1)/3}.
\end{aligned}
$$

This completes the inductive proof. A similar argument holds for $s_{j/\alpha}[(\mathbb{D}_V-i)^{-\beta-1}\rho]$.

3. For (5.2) we prove the first inequality. The second inequality follows similarly. First we write using (5.9)

$$
\begin{aligned}
&(\mathbb{D}_V-z)^{-1}(\mathbb{D}_V-i)^{-4}-(\mathbb{D}_0-z)^{-1}(\mathbb{D}_0-i)^{-4} \\
&\quad =\left[(\mathbb{D}_V-z)^{-1}-(\mathbb{D}_0-z)^{-1}\right](\mathbb{D}_0-i)^{-4}+(\mathbb{D}_V-z)^{-1}\left[(\mathbb{D}_V-i)^{-4}-(\mathbb{D}_0-i)^{-4}\right] \\
&\quad =-(\mathbb{D}_V-z)^{-1}V(\mathbb{D}_0-z)^{-1}(\mathbb{D}_0-i)^{-4}-(\mathbb{D}_V-z)^{-1}\sum_{k=1}^{4}(\mathbb{D}_V-i)^{k-5}V(\mathbb{D}_0-i)^{-k},
\end{aligned}
$$

where we have used the second resolvent identity once again. Using the method in (5.8) we estimate the singular value as

$$
\begin{aligned}
s_j\big[(\mathbb{D}_V - z)^{-1}(\mathbb{D}_V - i)^{-4} &- (\mathbb{D}_0 - z)^{-1}(\mathbb{D}_0 - i)^{-4}\big] \\
&\le s_{\lceil j/2\rceil}\left[-(\mathbb{D}_V - z)^{-1} V (\mathbb{D}_0 - z)^{-1}(\mathbb{D}_0 - i)^{-4}\right] \\
&\quad + s_{\lceil j/2\rceil}\left[-(\mathbb{D}_V - z)^{-1}\sum_{k=1}^{4}(\mathbb{D}_V - i)^{k-5} V (\mathbb{D}_0 - i)^{-k}\right] \\
&\le \|(\mathbb{D}_V - z)^{-1}\|\|(\mathbb{D}_0 - z)^{-1}\|\|V\| s_{\lceil j/2\rceil}[\rho(\mathbb{D}_0 - i)^{-4}] \\
&\quad + \|(\mathbb{D}_V - z)^{-1}\|\sum_{k=1}^{4} s_{\lceil j/8\rceil}\left[(\mathbb{D}_V - i)^{k-5} V (\mathbb{D}_0 - i)^{-k}\right] \\
&\le C\left(\|(\mathbb{D}_0 - z)^{-1}\| + 1\right)\|(\mathbb{D}_V - z)^{-1}\| j^{-4/3}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\big\|(\mathbb{D}_V - z)^{-1}(\mathbb{D}_V - i)^{-4} &- (\mathbb{D}_0 - z)^{-1}(\mathbb{D}_0 - i)^{-4}\big\|_{\mathcal{B}_1} \\
&= \sum_{j=1}^{\infty} s_j\left[(\mathbb{D}_V - z)^{-1}(\mathbb{D}_V - i)^{-4} - (\mathbb{D}_0 - z)^{-1}(\mathbb{D}_0 - i)^{-4}\right] \\
&\le C\left[\|(\mathbb{D}_0 - z)^{-1}\| + 1\right]\|(\mathbb{D}_V - z)^{-1}\|\sum_{j=1}^{\infty} j^{-4/3},
\end{aligned}
$$

where the summation is finite. It remains to use the estimates in (5.3) and (5.4) alongside the assumption $|\operatorname{Im} z| < 1$ to obtain (5.2).

4. To show the final part of the theorem, we write $f(z) = (z - i)^{-4} g(z)$ where $g \in \mathscr{S}(\mathbb{R})$ since $f \in \mathscr{S}(\mathbb{R})$. Introduce also $\tilde{g}$ as an analytic extension of g satisfying (A.12). Using the generalized Helffer-Sjöstrand formula in Theorem A.2.7 with $N = 4$ and $z_0 = i$ we write

$$
\begin{aligned}
&\|f(\mathbb{D}_V) - f(\mathbb{D}_0)\|_{\mathcal{B}_1} \\
&\quad = \left\|\frac{1}{\pi i}\int_{\mathbb{C}}\left[(\mathbb{D}_V - z)^{-1}(\mathbb{D}_V - i)^{-4} - (\mathbb{D}_0 - z)^{-1}(\mathbb{D}_0 - i)^{-4}\right]\overline{\partial}_z\tilde{g}(z)\,\mathrm{d}m(z)\right\|_{\mathcal{B}_1} \\
&\quad = C\int_{\{|\operatorname{Re} z|\ge m\}}\|(\mathbb{D}_V - z)^{-1}(\mathbb{D}_V - i)^{-4} - (\mathbb{D}_0 - z)^{-1}(\mathbb{D}_0 - i)^{-4}\|_{\mathcal{B}_1}|\overline{\partial}_z\tilde{g}(z)|\,\mathrm{d}m(z) \\
&\qquad + C\int_{\{|\operatorname{Re} z|< m\}}\|(\mathbb{D}_V - z)^{-1}(\mathbb{D}_V - i)^{-4} - (\mathbb{D}_0 - z)^{-1}(\mathbb{D}_0 - i)^{-4}\|_{\mathcal{B}_1}|\overline{\partial}_z\tilde{g}(z)|\,\mathrm{d}m(z).
\end{aligned}
$$

It remains to use (5.2) and note that for $N \ge 4$ in (A.12) of Theorem A.2.6 that this is finite. $\square$

5.2 The Birman-Kreĭn trace formula for the Dirac operator

We now state and prove our first trace formula.

Theorem 5.2.1 *Let V satisfy Assumption 2.4.1 and $f \in \mathscr{S}(\mathbb{R})$. If $S^{\pm}(k)$ is the scattering matrix as outlined in Definition 4.2.3 then*

$$\begin{aligned}\operatorname{Tr}\left(f(\mathbb{D}_V) - f(\mathbb{D}_0)\right) = & \frac{1}{2\pi i}\int_0^\infty f(\sqrt{k^2+m^2})\operatorname{Tr}\left[S^+(k)^{-1}\partial_k S^+(k)\right]\,\mathrm{d}k \\ & - \frac{1}{2\pi i}\int_0^\infty f(-\sqrt{k^2+m^2})\operatorname{Tr}\left[S^-(k)^{-1}\partial_k S^-(k)\right]\,\mathrm{d}k \\ & + \sum_{E_j} m_j f(E_j) + \frac{1}{2}\sum_{\pm} \pm\widetilde{m}_R(\pm m) f(\pm m),\end{aligned} \tag{5.10}$$

where E_j are the eigenvalues of $\mathbb{D}_V$ with associated multiplicity m_j, and the resonance multiplicity $\widetilde{m}_R(\pm m)$ is defined by (3.21).

Proof. We divide the proof into 6 steps.

1. By the spectral theorem of selfadjoint operators (see Theorem A.2.3) and Stone's formula (see [11, section A.3]) we have

$$\begin{aligned} f(\mathbb{D}_V) &= \int_{-\infty}^{-m} f(z)\,\mathrm{d}E(z) + \int_m^\infty f(z)\,\mathrm{d}E(z) + \sum_{E_j\in\operatorname{spec}_d(\mathbb{D}_V)} m_j f(E_j) u_j\otimes\bar{u}_j \\ &= \frac{1}{2\pi i}\int_{-\infty}^{-m} f(z)\left[(\mathbb{D}_V - (z+i0))^{-1} - (\mathbb{D}_V-(z-i0))^{-1}\right]\mathrm{d}z \\ &\quad + \frac{1}{2\pi i}\int_m^\infty f(z)\left[(\mathbb{D}_0 - (z+i0))^{-1} - (\mathbb{D}_0-(z-i0))^{-1}\right]\mathrm{d}z \\ &\quad + \sum_{E_j\in\operatorname{spec}_d(\mathbb{D}_V)} m_j f(E_j) u_j\otimes\bar{u}_j, \end{aligned} \tag{5.11}$$

where E_j are the eigenvalues of $\mathbb{D}_V$ with corresponding eigenvectors u_j. We concentrate first on the continuous part of the spectrum. By the final result of Theorem 5.1.1 we take the trace difference between functions of the full and free Dirac operator and rewrite in the k variable

$$\begin{aligned} & 4\pi i\operatorname{Tr}\left[f(\mathbb{D}_V) - f(\mathbb{D}_0)\right] \\ &= -\operatorname{Tr}\int_\infty^{-\infty} f(-\sqrt{k^2+m^2})\left[R_V^-(k) - R_V^-(-k) - R_0^-(k) + R_0^-(-k)\right]\frac{k}{\sqrt{k^2+m^2}}\,\mathrm{d}k \\ &\quad + \operatorname{Tr}\int_{-\infty}^\infty f(\sqrt{k^2+m^2})\left[R_V^+(k) - R_V^+(-k) - R_0^+(k) + R_0^+(-k)\right]\frac{k}{\sqrt{k^2+m^2}}\,\mathrm{d}k. \end{aligned} \tag{5.12}$$

Using (2.14) introduce

$$
\begin{aligned}
B^{\pm}(k) &= \pm \frac{k}{\sqrt{k^2+m^2}} \left(R_V^{\pm}(k) - R_0^{\pm}(k) \right) \\
&= \pm \frac{k}{\sqrt{k^2+m^2}} \left(-R_V^{\pm}(k) V R_0^{\pm}(k) \right) \\
&= \mp \frac{k}{\sqrt{k^2+m^2}} \left(R_0^{\pm}(k)(I + VR_0^{\pm}(k)\rho)^{-1} V R_0^{\pm}(k) \right) + \frac{\Pi_{\pm}}{k} \frac{(\sqrt{k^2+m^2}+m)}{\sqrt{k^2+m^2}},
\end{aligned}
$$

where we have used the second resolvent identity (see Theorem A.2.2) and made explicit the threshold singularities of the full resolvent from Theorem 3.2.5. Note that the simple pole there is cancelled by the k prefactor here, and that we do not include any contribution from the opposite resonance. We similarly introduce $B^{\pm}(-k)$ such that

$$
\begin{aligned}
B^{\pm}(k) &= F^{\pm}(k) + \frac{\Pi_{\pm}}{k} \frac{(\sqrt{k^2+m^2}+m)}{\sqrt{k^2+m^2}}, \\
B^{\pm}(-k) &= F^{\pm}(-k) - \frac{\Pi_{\pm}}{k} \frac{(\sqrt{k^2+m^2}+m)}{\sqrt{k^2+m^2}}, \\
F^{\pm}(k) &= \mp \frac{k}{\sqrt{k^2+m^2}} \left(R_0^{\pm}(k)(I + VR_0^{\pm}(k)\rho)^{-1} V R_0^{\pm}(k) \right), \\
F^{\pm}(-k) &= \pm \frac{k}{\sqrt{k^2+m^2}} \left(R_0^{\pm}(-k)(I + VR_0^{\pm}(-k)\rho)^{-1} V R_0^{\pm}(-k) \right).
\end{aligned}
\tag{5.13}
$$

Note that $F^{\pm}(k)$ and $F^{\pm}(-k)$ match the definitions in (4.24). We introduce $\tilde{f} \in \mathscr{S}(\mathbb{C})$ as an almost analytic extension of f satisfying (A.12). Hence if $0 < \epsilon \ll m$ then we write

$$
\begin{aligned}
&4\pi i \operatorname{Tr}[f(\mathbb{D}_V) - f(\mathbb{D}_0)] \\
&\quad = \sum_{\pm} \pm \lim_{\epsilon \to \infty} \int_{\mathbb{R} \setminus [-\epsilon, \epsilon]} f(\pm\sqrt{k^2+m^2}) \left(\operatorname{Tr} B^{\pm}(k) + \operatorname{Tr} B^{\pm}(-k) \right) \mathrm{d}k \\
&\qquad + \sum_{\pm} \pm \lim_{\epsilon \to \infty} \int_{\gamma_{\pm}(\epsilon)} \tilde{f}(-\sqrt{k^2+m^2}) \operatorname{Tr} B^{-}(\pm k) \, \mathrm{d}k \\
&\qquad + \sum_{\pm} \mp \lim_{\epsilon \to \infty} \int_{\gamma_{\pm}(\epsilon)} \tilde{f}(\sqrt{k^2+m^2}) \operatorname{Tr} B^{+}(\pm k) \, \mathrm{d}k \\
&\qquad + \sum_{\pm} \pm \lim_{\epsilon \to \infty} \int_{\partial\Gamma_{\mp}(\epsilon)} \tilde{f}(-\sqrt{k^2+m^2}) \operatorname{Tr} B^{-}(\mp k) \, \mathrm{d}k \\
&\qquad + \sum_{\pm} \pm \lim_{\epsilon \to \infty} \int_{\partial\Gamma_{\pm}(\epsilon)} \tilde{f}(\sqrt{k^2+m^2}) \operatorname{Tr} B^{\pm}(\pm k) \, \mathrm{d}k,
\end{aligned}
\tag{5.14}
$$

using the notation (see also Figure 5.1)

$$
\Gamma_{\pm}(\epsilon) = D(0;\epsilon) \cap \mathbb{C}_{\pm}, \qquad \mathbb{C}_{\pm} = \{k \in \mathbb{C} : \pm \operatorname{Im} k > 0\},
$$
$$
\gamma_{\pm}(\epsilon) = \{\partial\Gamma_{\pm}(\epsilon) : \pm \operatorname{Im} k > 0\},
$$

where the positively orientated contours $\partial\Gamma_{\pm}(\epsilon)$ enclose the open regions $\Gamma_{\pm}(\epsilon)$, and $D(0;\epsilon)$ is the open disc of radius ϵ centred at the origin. We number the terms on the right-hand side of (5.14) from 1 to 5 and evaluate each in the remaining steps.

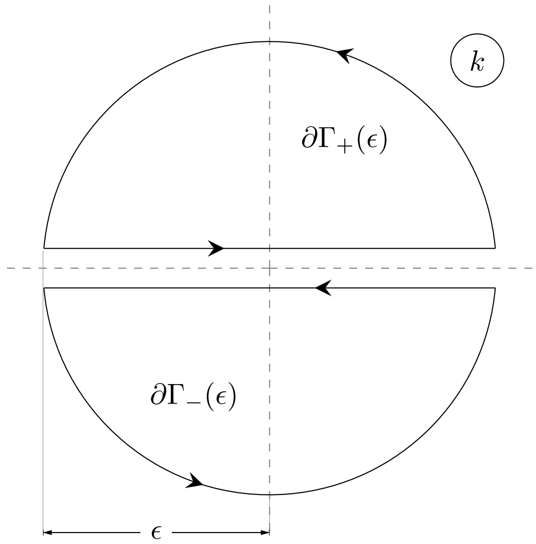

Figure 5.1: The contours $\partial\Gamma_+(\epsilon)$ and $\partial\Gamma_-(\epsilon)$ in the k plane. We have assumed that $\epsilon \ll m$.

2. In this step we examine term 1 on the right-hand side of (5.14). The explicit simple poles at the origin in (5.13) cancel and so we can take the limit $\epsilon \to 0$. By employing (4.24) we show for term 1

$$\begin{aligned}\lim_{\epsilon\to\infty} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} & f(\pm\sqrt{k^2+m^2}) \left(\operatorname{Tr} B^{\pm}(k) + \operatorname{Tr} B^{\pm}(-k)\right) \, \mathrm{d}k \\ &= \lim_{\epsilon\to\infty} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} f(\pm\sqrt{k^2+m^2}) \left(\operatorname{Tr} F^{\pm}(k) + \operatorname{Tr} F^{\pm}(-k)\right) \, \mathrm{d}k \\ &= \int_{\mathbb{R}} f(\pm\sqrt{k^2+m^2}) \operatorname{Tr}\left[S^{\pm}(k)^{-1}\partial_k S^{\pm}(k)\right] \, \mathrm{d}k.\end{aligned}$$

By (4.24), the integrand is even in k and we evaluate the integral on $[0,\infty)$.

3. To examine the remaining integrals in (5.14) located near the origin, we use Gohberg-Sigal theory (see Theorem A.3.2) to study the structure of $F^{\pm}(k)$ near this point. To this end we have

$$I + VR_0^{\pm}(k)\rho = U_1^{\pm}(k)\left(Q_2^{\pm}k^2 + Q_1^{\pm}k + Q_0^{\pm}\right)U_2^{\pm}(k),$$

where $U_j^{\pm}$ are holomorphic and invertible. Furthermore the projection operators $Q_j^{\pm}$ satisfy (A.13) and

$$\operatorname{rank} Q_2^{\pm} = \operatorname{Tr}\Pi_{\pm} = m_R(\pm m) - \widetilde{m}_R(\pm m), \qquad \operatorname{rank} Q_1^{\pm} = \widetilde{m}_R(\pm m).$$

Since the free and perturbed resolvents meromorphically extend from the upper k-plane to all $\mathbb{C}$ (see Theorems 2.3.2 and 2.4.2), then we can apply the generalized argument principle in (A.14) with

$$N_0(I + VR_0^{\pm}(k)\rho) = 2\operatorname{Tr}\Pi_{\pm} + \widetilde{m}_R(\pm m), \quad N_0((I + VR_0^{\pm}(k)\rho)^{-1}) = 0,$$

which count, with multiplicity, the number of zeros and poles of $I+VR_0^{\pm}(k)\rho$ respectively (see Theorem A.3.2). Using the definitions in (5.13) we therefore have

$$\begin{aligned}\oint \operatorname{Tr} F^{\pm}(k)\,\mathrm{d}k &= \mp \operatorname{Tr} \oint \left(\frac{k}{\sqrt{k^2+m^2}}\right) R_0^{\pm}(k)(I+VR_0^{\pm}(k)\rho)^{-1}VR_0^{\pm}(k)\,\mathrm{d}k \\ &= -\operatorname{Tr} \oint \partial_k(I+VR_0^{\pm}(k)\rho)(I+VR_0^{\pm}(k)\rho)^{-1}\,\mathrm{d}k \\ &= -2\pi i\,(2\operatorname{Tr}\Pi_{\pm}+\widetilde{m}_R(\pm m)) \\ &= -\oint \frac{(2\operatorname{Tr}\Pi_{\pm}+\widetilde{m}_R(\pm m))}{k}\,\mathrm{d}k,\end{aligned}$$

where we have used (4.25). A similar argument holds for $\operatorname{Tr} F^{\pm}(-k)$. Hence near $k=0$ we have

$$\operatorname{Tr} F^{\pm}(k) = -\frac{1}{k}\left[2\operatorname{Tr}\Pi_{\pm}+\widetilde{m}_R(\pm m)\right]+\varphi_{\pm}(k), \tag{5.15}$$

where $\varphi_{\pm}(k)$ is holomorphic for $\operatorname{Im} k \geq 0$.

4. In this step we consider how terms 4 and 5 on the right-hand side of (5.14) behave as we take the limit $\epsilon \to 0$. In fact we take one particular case below with the method applicable to the remaining integrals. First write using Green's formula (see for instance [46, chapter 16])

$$\begin{aligned}\left|\int_{\partial\Gamma_+(\epsilon)} \tilde{f}(-\sqrt{k^2+m^2})\operatorname{Tr} B^-(k)\,\mathrm{d}k\right| &= \left|2\int_{\Gamma_+(\epsilon)} \overline{\partial}_k\left[\tilde{f}(-\sqrt{k^2+m^2})\operatorname{Tr} B^-(k)\right]\mathrm{d}m\right| \\ &= \left|2\int_{\Gamma_+(\epsilon)} \left(\overline{\partial}_k\tilde{f}(-\sqrt{k^2+m^2})\right)\operatorname{Tr} B^-(k)\,\mathrm{d}m\right|,\end{aligned}$$

where m denotes the Lebesgue measure on $\mathbb{C}$ and $\operatorname{Tr} B^-(k)$ defined by (5.13) and (5.15) is analytic. Given

$$\overline{\partial}_k\tilde{f}(-\sqrt{k^2+m^2}) \leq C_N|\operatorname{Im} k|^N, \quad \forall N\in\mathbb{N},$$

and $\operatorname{Tr} B^-(k) = \mathcal{O}\left(k^{-1}\right)$, then

$$\left|\int_{\partial\Gamma_+(\epsilon)} \tilde{f}(-\sqrt{k^2+m^2})\operatorname{Tr} B^-(k)\,\mathrm{d}k\right| \leq C_N\epsilon^{N-1}\left|\int_{\Gamma_+(\epsilon)}\mathrm{d}m\right| = C_N\epsilon^{N+1}, \qquad \forall N\in\mathbb{N}.$$

We conclude that this and indeed all contributions from terms 4 and 5 in the right-hand side of (5.14) tend to 0 as $\epsilon \to 0$.

5. For terms 2 and 3 in the right-hand side of (5.14), we use the indentation lemma (see for instance [61]) to compute the integrals along circular arcs. Using (5.13) and (5.15) we have for term 2

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\gamma_+(\epsilon)} \tilde{f}(-\sqrt{k^2+m^2})\operatorname{Tr} B^-(k)\,\mathrm{d}k - \lim_{\epsilon\to 0}\int_{\gamma_-(\epsilon)} \tilde{f}(-\sqrt{k^2+m^2})\operatorname{Tr} B^-(-k)\,\mathrm{d}k \\
&\quad= \lim_{\epsilon\to 0}\int_{\gamma_+(\epsilon)} \frac{\tilde{f}(-\sqrt{k^2+m^2})}{k}\left[-(2\operatorname{Tr}\Pi_- + \widetilde{m}_R(-m)) + \operatorname{Tr}\Pi_-\frac{\sqrt{k^2+m^2}+m}{\sqrt{k^2+m^2}}\right]\mathrm{d}k \\
&\qquad - \lim_{\epsilon\to 0}\int_{\gamma_-(\epsilon)} \frac{\tilde{f}(-\sqrt{k^2+m^2})}{k}\left[(2\operatorname{Tr}\Pi_- + \widetilde{m}_R(-m)) - \operatorname{Tr}\Pi_-\frac{\sqrt{k^2+m^2}+m}{\sqrt{k^2+m^2}}\right]\mathrm{d}k \\
&\quad= -2\pi i\widetilde{m}_R(-m)f(-m).
\end{aligned}
$$

This completes the analysis on term 2 on the right-hand side of (5.14). Term 3 similarly follows

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\gamma_-(\epsilon)} \tilde{f}(\sqrt{k^2+m^2})\operatorname{Tr} B^+(k)\,\mathrm{d}k - \lim_{\epsilon\to 0}\int_{\gamma_+(\epsilon)} \tilde{f}(\sqrt{k^2+m^2})\operatorname{Tr} B^+(-k)\,\mathrm{d}k \\
&\qquad= 2\pi i\widetilde{m}_R(-m)f(m).
\end{aligned}
$$

6. Bringing together all the previous steps with (5.12) produces the first, second and fourth terms on the right-hand side of (5.10). For the third term, it remains to bring back the contribution from the discrete eigenvalues. This follows immediately from the form in (5.11) since $\operatorname{Tr} u_j \otimes \overline{u}_j = 1$. □

Summary

In this chapter, we have brought together our resolvent expansion and scattering matrix determinant from chapters 3 and 4 respectively to derive our first trace formula; a reinterpretation of the Birman-Kreĭn formula. Using various trace estimates from section 5.1, we proved in section 5.2 how the trace difference between Schwartz functions of $\mathbb{D}_0$ and $\mathbb{D}_V$ relate to the scattering matrix, the eigenvalues of $\mathbb{D}_V$ and the threshold resonances. Our Birman-Kreĭn trace formula will be used in the proof of our Poisson wave trace formula in chapter 6.

Chapter 6

The Poisson wave trace formula

In this chapter we present our second trace formula. In preparation for this, we prove two pre-requisite theorems in section 6.1. The first estimates an upper bound on the number of resonances contained within a disc located at $k = 0$ of radius $r > 0$. In the second, a factorization of $\det S^{\pm}(k)$ in terms of Weierstrauss products is presented. In turn, these results alongside the Birman-Kreĭn formula (Theorem 5.2.1) are used to construct our Poisson wave trace formula in section 6.2. This is valid in distributional sense for all $t \in \mathbb{R}$ and, like the previous chapter, the threshold resonances are treated explicitly.

6.1 A resonance counting function and factorization of the scattering matrix

As a reminder, the Dirac resonances occur as poles of the full resolvent, $R_V^{\pm}(k)$, when extended to $\mathbb{C}$. As in Definition 2.4.3, the two sets of resonances are denoted $\mathcal{R}_{\pm}$, with the total set of resonances formed by their union $\mathcal{R} := \mathcal{R}_{-} \cup \mathcal{R}_{+}$. We will continually use this notation for the remainder of the thesis.

Theorem 6.1.1 *Let V satisfy Assumption 2.4.1 and assume $\rho \in C_0^{\infty}(\mathbb{R}^3)$ such that $\rho V = V$ holds. If we also define*

$$H^{\pm}(k) := \det(I - (V R_0^{\pm}(k)\rho)^4),$$

then

$$|H^{\pm}(k)| \leq C \exp(C|k|^4). \tag{6.1}$$

Moreover the number of resonances inside the disc $D(0;r)$ denoted

$$N(r) := \#\{k \in \mathcal{R} : 0 < |k| \leq r\},$$

satisfies

$$N(r) \leq Cr^4. \tag{6.2}$$

Proof. We divide the proof into 4 steps.

1. Recall from Definition 2.4.3 that the resonances of $\mathbb{D}_V$ coincide with the poles of $(I + VR_0^{\pm}(k)\rho)^{-1}$. Since we have

$$I - (VR_0^{\pm}(k)\rho)^4 = \big(I + VR_0^{\pm}(k)\rho\big)\,\big(I - VR_0^{\pm}(k)\rho + (VR_0^{\pm}(k)\rho)^2 - \big(VR_0^{\pm}(k)\rho\big)^3\big), \tag{6.3}$$

and provided $(VR_0^{\pm}(k)\rho)^4 \in \mathcal{B}_1$, then the resonances where $k \neq 0$ correspond to the zeros of $H^{\pm}(k) = \det(I - (VR_0^{\pm}(k)\rho)^4)$. Using (A.3), (A.6) and Theorem A.2.4 we have

$$|H(k)^{\pm}| \leq \prod_{j=1}^{\infty}\Big[1 + \|V\|_{L^\infty}^4 \big(s_{\lceil j/4\rceil}\left[\rho R_0^{\pm}(k)\rho\right]\big)^4\Big]. \tag{6.4}$$

In the next two steps we will show that $(VR_0^{\pm}(k)\rho)^4 \in \mathcal{B}_1$ for $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$ respectively.

2. To estimate the singular values of $\rho R_0^{\pm}(k)\rho$ for $\operatorname{Im} k > 0$, we argue as in (5.7) and use (2.9) to write

$$\begin{aligned} s_j\left[\rho R_0^{\pm}(k)\rho\right] &\leq s_j\left[(-\Delta - 1)^{-1/2}\right]\|(-\Delta-1)^{1/2}\rho R_0^{\pm}(k)\rho\|_{L^2\to L^2} \\ &\leq Cj^{-1/3}\langle k\rangle e^{C(\operatorname{Im} k)_-}. \end{aligned} \tag{6.5}$$

Then $\sum_j s_j[(\rho_1 R_0^{\pm}(k)\rho_1)^4]$ converges. Back to (6.4) we find $H^{\pm}(k)$ for $\operatorname{Im} k > 0$ is indeed trace class

$$\begin{aligned} |H^{\pm}(k)| &\leq \prod_{j=1}^{\infty}\left[1 + Cj^{-4/3}\langle k\rangle^4\right] \leq \prod_{j=1}^{\infty}\exp[C\langle k\rangle^4 j^{-4/3}] = \exp\left(C\langle k\rangle^4\sum_j j^{-4/3}\right) \\ &\leq \exp(C\langle k\rangle^4) \leq C\exp(C|k|^4). \end{aligned}$$

3. In estimating the singular value $s_j((VR_0^{\pm}(k)\rho)^4)$ for $\operatorname{Im} k < 0$ we use (4.19). By (A.3) and (A.5) plus the fact that $\|A\| = \|A^*\|$ for bounded A (see for instance [40, Theorem 3.9-2]), we obtain

$$\begin{aligned} s_j[\rho R_0^{\pm}(k)\rho] &\leq s_{\lceil j/2\rceil}[Ck\sqrt{k^2+m^2}E_{\pm}(k)^*E_{\pm}(k)] + s_{\lceil j/2\rceil}[\rho R_0^{\pm}(-k)\rho] \\ &\leq Ce^{C|k|}s_{\lceil j/2\rceil}[E_{\pm}(k)]\|E_{\pm}(k)\|_{L^2(\mathbb{R}^3)^4\to L^2(\mathbb{S}^2)^4} + Cs_{\lceil j/2\rceil}[\rho R_0^{\pm}(-k)\rho]. \end{aligned} \tag{6.6}$$

For the first term on the right-hand side of (6.6) recall (4.13) and (4.17) whereas the second term can be estimated using (6.5). Hence

$$s_j[(\rho R_0^{\pm}(k)\rho)] \leq Ce^{C|k|}e^{-\sqrt{j}/C} + Cj^{-1/3}\langle k\rangle. \tag{6.7}$$

4. We now aim to estimate (6.7) depending upon the value of j. First let $j \leq 2C^4|k|^2$, then

$$s_j[(\rho R_0^{\pm}(k)\rho)] \leq Ce^{C|k|},$$

since $e^{-\sqrt{j}/C}$ and $j^{-1/2}$ are monotonically decreasing functions. Moreover, up to a constant, the latter is greater which enables us to also estimate (6.7) for $j > 2C^4|k|^2$

$$s_j[(\rho R_0^{\pm}(k)\rho)] \leq Ce^{-C\sqrt{j}} + Cj^{-1/3}\langle k\rangle \leq Cj^{-1/3}\langle k\rangle.$$

In summary, taking the quartic power we have

$$s_j[(\rho R_0^{\pm}(k)\rho)^4] \leq \left(s_{\lceil j/4\rceil}[\rho R_0^{\pm}(k)\rho]\right)^4 \leq \begin{cases} Ce^{C|k|}, & j \leq 2C^4|k|^2, \\ Cj^{-4/3}\langle k\rangle^4, & j > 2C^4|k|^2. \end{cases}$$

Substitute into (6.4) and we have

$$|H^{\pm}(k)| \leq \prod_{j\leq 2C^4|k|^2}(1+Ce^{C|k|})\prod_{j>2C^4|k|^2}(1+Cj^{-4/3}\langle k\rangle^4).$$

Evaluating

$$\prod_{j\leq 2C^4|k|^2}(1+Ce^{C|k|}) \leq (2Ce^{C|k|})^{2C^4|k|^2} \leq Ce^{C|k|^3},$$

and

$$\prod_{j>2C^4|k|^2}(1+Cj^{-4/3}\langle k\rangle^4) \leq \prod_{j>2C^4|k|^2}\exp(Cj^{-4/3}\langle k\rangle^4) \leq Ce^{C|k|^4},$$

proves (6.1). It remains to insert (6.1) into Jensen's formula (see Theorem A.1.3) to obtain (6.2)

$$\log(2)N(r) \leq \frac{1}{2\pi}\int_0^{2\pi}\log|\exp(|2Cre^{i\theta}|^4)|\,\mathrm{d}\theta \leq C|r|^4,$$

where $\log(2)N(r) \leq N(r)\int_r^{2r}\frac{1}{t}\,\mathrm{d}t \leq \int_0^{2r}\frac{n(t)}{r}\,\mathrm{d}t.$ □

Theorem 6.1.2 (Scattering matrix factorization) *Let V satisfy Assumption 2.4.1. Then the scattering matrix determinant may be written*

$$\det S^{\pm}(k) = (-1)^{\widetilde{m}_R(\pm m)}e^{g(k)}\frac{P_{\pm}(-k)}{P_{\pm}(k)}, \tag{6.8}$$

where

$$P_{\pm}(k) := \prod_{k_j\in\mathcal{R}_{\pm}\setminus\{0\}}E_4(k/k_j)^{m_R(k_j)}, \quad E_p(k) := (1-k)\exp\left(\sum_{\ell=1}^{p}k^{\ell}/\ell\right),$$

$$g(k) = a_3k^3 + a_1k.$$

Proof. We divide the proof into 4 steps.

1. First we shall use the following bound on the determinant of the scattering matrix outside the union of discs surrounding each resonance

$$\begin{gathered}|\det S^{\pm}(k)| \leq Ce^{C|k|^{12+\epsilon}},\\ k \notin \bigcup_{k_j\in\mathcal{R}_{\pm}} D(k_j;\langle k_j\rangle^{-4-\epsilon}), \quad \epsilon>0,\ |z|>r_0.\end{gathered} \tag{6.9}$$

Indeed by Theorems A.2.4 and 4.2.4 we write

$$\begin{aligned}|\det S^{\pm}(k)| = |\det(I-A^{\pm}(k))| &\leq \prod_{j=1}^{\infty}(1+s_j[A^{\pm}(k)])\\ &\leq \prod_{j=1}^{\infty}\Big(1+e^{C|k|}\|E_{\pm}(k)\|\|(I+VR_0^{\pm}(k)\rho)^{-1}V\|s_j\left[E_{\pm}(k)\right]\Big).\end{aligned} \tag{6.10}$$

Using (2.9), (6.3) and the second part of Theorem A.2.4 we have

$$\begin{aligned}\|(I+VR_0^{\pm}(k)\rho)^{-1}\| &\leq \sum_{j=0}^{3}\|-VR_0^{\pm}(k)\rho\|^{j}_{L^2\to L^2}\|(I-(VR_0^{\pm}(k)\rho)^4)^{-1}\|\\ &\leq Ce^{C|k|}\frac{\det(I+|\rho R_0^{\pm}(k)\rho|^4)}{|\det(I-(\rho R_0^{\pm}(k)\rho)^4)|}.\end{aligned} \tag{6.11}$$

The numerator of the right-hand side of (6.11) can be estimated in the same vein as (6.1), namely

$$\det(I+|\rho R_0^{\pm}(k)\rho|^4) \leq C\exp(C|k|^4).$$

To estimate the denominator of (6.11), we can apply Cartan's minimum modulus theorem (see Theorem A.1.7):

$$\begin{gathered}|\det(I-(\rho R_0^{\pm}(k)\rho)^4)| \geq Ce^{-C|k|^{4+\epsilon}},\\ k \notin \bigcup_{k_j\in\mathcal{R}_{\pm}} D(k_j;\langle k_j\rangle^{-4-\epsilon}), \quad \epsilon>0,\ |z|>r_0.\end{gathered} \tag{6.12}$$

On the same set, we insert Equations (4.13), (4.17), (6.11), and (6.12) into (6.10) to find

$$\begin{aligned}|\det S^{\pm}(k)| &\leq \prod_{j=1}^{\infty}\left[1+Ce^{C|k|^{4+\epsilon}}e^{-\sqrt{j}/C}\right]\\ &= \prod_{j\leq 4C^4|k|^{2(4+\epsilon)}}\left[1+Ce^{C|k|^{4+\epsilon}}e^{-\sqrt{j}/C}\right]\prod_{j>4C^4|k|^{2(4+\epsilon)}}\left[1+Ce^{C|k|^{4+\epsilon}}e^{-\sqrt{j}/C}\right].\end{aligned}$$

By the same arguments as in the last step of the proof for Theorem 6.1.1 we estimate

$$\prod_{j\leq 4C^4|k|^{2(4+\epsilon)}}\left[1+Ce^{C|k|^{4+\epsilon}}e^{-\sqrt{j}/C}\right] \leq C\prod_{j\leq 4C^4|k|^{2(4+\epsilon)}}\left[e^{C|k|^{4+\epsilon}}\right] \leq Ce^{C|k|^{12+\epsilon}},$$

whilst for $j > 4C^4|k|^{2(4+\epsilon)} \leq \sqrt{j}/2$ we have

$$\prod_{j>4C^4|k|^{2(4+\epsilon)}} \left[1 + Ce^{C|k|^{4+\epsilon}} e^{-\sqrt{j}/C}\right] \leq \prod_{j>4C^4|k|^{2(4+\epsilon)}} \exp\left(Ce^{-C\sqrt{j}}\right) \leq C.$$

This concludes the proof of (6.9).

2. The zeros of $H^{\pm}(k)$ defined in Theorem 6.1.1 coincide with the non-zero resonances of $\mathbb{D}_V$. However (4.23) implies that zeros of $\det S^{\pm}(k)$ when $\operatorname{Im} k > 0$ correspond to the resonances of $\mathbb{D}_V$ when $\operatorname{Im} k < 0$. By the Weierstrass factorization theorem (see Theorem A.1.5) we then obtain (6.8). The genus of E_p is equal to 4 due to (6.2) plus Theorems A.1.4 and A.1.5. Note that $(-1)^{\widetilde{m}_R(\pm m)}$ appears due to the cancellation between $(-k)^{\widetilde{m}_R(\pm m)}$ and $(k)^{\widetilde{m}_R(\pm m)}$ when accounting for zeros at the origin (see (A.2)).

3. Next we show that $g(k)$ is a polynomial. By virtue of the estimate (6.2), then (A.1.6) and (A.1.7) imply that

$$e^{-C|k|^{4+\epsilon}} \leq |P(\pm k)| \leq e^{C|k|^{4+\epsilon}}. \tag{6.13}$$

We use this alongside (6.8) and (6.9) so that

$$|e^{g(k)}| = |\det S^{\pm}(k)| \frac{|P_{\pm}(k)|}{|P_{\pm}(-k)|} \leq Ce^{C|k|^{12+\epsilon}}, \tag{6.14}$$

on the set defined in (6.9). The upper bound of resonance number in (6.2) implies for $\epsilon > 0$ that we have a sequence $r_j \to \infty$ such that the circles $\partial D(0; r_j)$ do not intersect circles around a resonance at k. That is for all j

$$\partial D(0; r_j) \cap \bigcup_{k_j \in \mathcal{R}_{\pm}} D\left(k_j; \langle k_j \rangle^{-4-\epsilon}\right) = \emptyset.$$

The maximum modulus principle (see Theorem A.1.1) implies that the estimate in (6.14) holds on the circles r_k and thus everywhere as $r_k \to \infty$. Since $|e^{g(k)}| \leq Ce^{\operatorname{Re} g(k)}$ by (6.14) we have $\operatorname{Re} g(k) \leq C|k|^{12+\epsilon}$. For the entire function g, we apply the Borel–Carathéodory theorem (see Theorem A.1.2) such that

$$|g(k)| \leq C|k|^{12+\epsilon},$$

which implies g is a polynomial of maximum order 12.

4. To show that g is a polynomial of degree no greater than 3, we first show that

$$|\det S^{\pm}(k)| \leq C \exp(C|k|^3), \qquad \operatorname{Im} k \geq 0,\ |k| > C, \tag{6.15}$$

where $|k|$ is sufficiently large to avoid the eigenvalues of $\mathbb{D}_V$. By retracing the steps from (6.10) onward and estimating $\|(I + VR_0^{\pm}(k)\rho)^{-1}\| \leq C$ for $\operatorname{Im} k > 0$, $|k| \geq C$, we have

$$\begin{aligned}
|\det S^{\pm}(k)| &\leq \prod_{j=1}^{\infty} \left(1 + C\|E_{\pm}(k)\| \|(I + VR_0^{\pm}(k)\rho)^{-1}\| s_j[E_{\pm}(k)]\right) \\
&\leq \prod_{j=1}^{\infty} \left(1 + Ce^{C|k|} e^{-\sqrt{j}/C}\right) \\
&= \prod_{j \leq 2C^4|k|^2} (1 + Ce^{C|k|} e^{-\sqrt{j}/C}) \prod_{j > 2C^4|k|^2} (1 + Ce^{C|k|} e^{-\sqrt{j}/C}) \\
&\leq Ce^{C|k|^3},
\end{aligned}$$

which proves (6.15). Insert into (6.14) for $\operatorname{Im} k > 0$, $|k| \geq C$ whilst using (6.13) we have

$$|e^{g(k)}| \leq Ce^{C|k|^{4+\epsilon}}.$$

As in the previous step this suggests that g is a polynomial of degree no greater than 4. However (4.23) implies $\exp(-g(k)) = \exp(g(-k))$, or that g is an odd polynomial: $g(k) = a_3 k^3 + a_1 k$. $\square$

6.2 The Poisson wave trace formula for the Dirac operator

In this section we state and prove our second trace formula.

Theorem 6.2.1 *Let V satisfy Assumption 2.4.1. Then in distributional sense on $\mathbb{R}_t$,*

$$\begin{aligned}
&2t^4 \operatorname{Tr}(\cos(t\sqrt{\mathbb{D}_V^2 - m^2}) - \cos(t\sqrt{\mathbb{D}_0^2 - m^2})) \\
&\quad = t^4 \sum_{\pm} \sum_{k_j \in \mathcal{R}_{\pm}} \pm m(k_j)\, e^{-i|t|k_j} + 2t^4 \sum_{E_j} m_j \cos(tk_j),
\end{aligned}$$

where in accordance with (2.16) and (3.21) we have the multiplicities

$$m(k) = \begin{cases} m_R(k), & k \neq 0, \\ \widetilde{m}_R(k), & k = 0, \end{cases}$$

and E_j are the eigenvalues of $\mathbb{D}_V$ with multiplicities m_j.

Proof. We divide the proof into 7 steps.

1. By the relations in Theorem 6.1.2 we first show that

$$\partial_k^5 \log \det S^{\pm}(k) = \sum_{k_j \in \mathcal{R}_{\pm} \setminus \{0\}} m_R(k_j) \partial_k^4 \left(\frac{1}{k + k_j} - \frac{1}{k - k_j} \right). \tag{6.16}$$

Indeed by recalling the properties of $P_\pm(k)$ from Theorem 6.1.2 then

$$\begin{aligned}
&\partial_k^5 \log\det S^\pm(k)\\
&= \partial_k^5\left[\log\left(e^{g(k)}\frac{P_\pm(-k)}{P_\pm(k)}\right)\right] = \partial_k^5\left[g(k)+\log P_\pm(-k)-\log P_\pm(k)\right]\\
&= \sum_{k_j\in\mathcal{R}_\pm\setminus\{0\}} m_R(k_j)\partial_k^5\left[\log\left(1+\frac{k}{k_j}\right)+\sum_{\ell=1}^4\frac{1}{\ell}\left(-\frac{k}{k_j}\right)^\ell-\log\left(1-\frac{k}{k_j}\right)-\sum_{\ell=1}^4\frac{1}{\ell}\left(\frac{k}{k_j}\right)^\ell\right]\\
&= \sum_{k_j\in\mathcal{R}_\pm\setminus\{0\}} m_R(k_j)\partial_k^4\left[\frac{1}{k+k_j}-\frac{1}{k-k_j}\right].
\end{aligned}$$

2. Let $u(t) = 2t^4\,\mathrm{Tr}\left[\cos\left(t\sqrt{\mathbb{D}_V^2-m^2}\right)-\cos\left(t\sqrt{\mathbb{D}_0^2-m^2}\right)\right] \in \mathscr{D}'(\mathbb{R})$ and $\phi \in C_0^\infty(\mathbb{R})$. Then

$$\begin{aligned}
\langle u,\phi\rangle &= \int 2t^4\,\mathrm{Tr}\left[\cos\left(t\sqrt{\mathbb{D}_V^2-m^2}\right)-\cos\left(t\sqrt{\mathbb{D}_0^2-m^2}\right)\right]\phi(t)\,\mathrm{d}t\\
&= \int t^4\,\mathrm{Tr}\left[e^{it\sqrt{\mathbb{D}_V^2-m^2}}+e^{-it\sqrt{\mathbb{D}_V^2-m^2}}-e^{it\sqrt{\mathbb{D}_0^2-m^2}}-e^{-it\sqrt{\mathbb{D}_0^2-m^2}}\right]\phi(t)\,\mathrm{d}t\\
&= \sqrt{2\pi}\,\mathrm{Tr}\left[\sum_\pm \widehat{t^4\phi}\left(\pm\sqrt{\mathbb{D}_V^2-m^2}\right)-\sum_\pm \widehat{t^4\phi}\left(\pm\sqrt{\mathbb{D}_0^2-m^2}\right)\right]\\
&= \sqrt{2\pi}\,\mathrm{Tr}\left[f(\mathbb{D}_V)-f(\mathbb{D}_0)\right],
\end{aligned}$$

where

$$f(z) = \widehat{t^4\phi}(\sqrt{z^2-m^2})+\widehat{t^4\phi}(-\sqrt{z^2-m^2}). \tag{6.17}$$

Using Theorem 5.2.1 we obtain

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\langle u\,,\phi\rangle = {}& \frac{1}{2\pi i}\int_0^\infty f(\sqrt{k^2+m^2})\,\mathrm{Tr}\left(S^+(k)^{-1}\partial_k S^+(k)\right)\,\mathrm{d}k\\
&-\frac{1}{2\pi i}\int_0^\infty f(-\sqrt{k^2+m^2})\,\mathrm{Tr}\left(S^-(k)^{-1}\partial_k S^-(k)\right)\,\mathrm{d}k\\
&+\sum_{E_j} m_j f(E_j)+\frac{1}{2}\sum_\pm \pm\widetilde{m}_R(\pm m)f(\pm m).
\end{aligned}$$

To aid later calculations we label these terms as

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\langle u\,,\phi\rangle &= A+B+C+D,\\
A &= \frac{1}{2\pi i}\int_0^\infty f(\sqrt{k^2+m^2})\,\mathrm{Tr}\left(S^+(k)^{-1}\partial_k S^+(k)\right)\,\mathrm{d}k,\\
B &= -\frac{1}{2\pi i}\int_0^\infty f(-\sqrt{k^2+m^2})\,\mathrm{Tr}\left(S^-(k)^{-1}\partial_k S^-(k)\right)\,\mathrm{d}k,\\
C &= \sum_{E_j} m_j f(E_j), \qquad D = \sum_\pm \pm\frac{1}{2}\widetilde{m}_R(\pm m)f(\pm m).
\end{aligned} \tag{6.18}$$

3. Consider first the continuous part of the spectrum. Define $h(k) := \widehat{\phi}(k)$ so that

$\partial_k^4 h(k) = \widehat{t^4\phi}(k)$. Then

$$
\begin{aligned}
2\pi i A &= \int_0^\infty f(\sqrt{k^2+m^2}) \operatorname{Tr}\left[S^+(k)^{-1}\partial_k S^+(k)\right] \mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}} f(\sqrt{k^2+m^2})\partial_k \log\det S^+(k)\,\mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[\widehat{t^4\phi}(k) + \widehat{t^4\phi}(-k)\right] \partial_k \log\det S^+(k)\,\mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[\partial_k^4 h(k) + \partial_k^4 h(-k)\right] \partial_k \log\det S^+(k)\,\mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[h(k) + h(-k)\right] \partial_k^5 \log\det S^+(k)\,\mathrm{d}k,
\end{aligned}
$$

where the surface terms disappear since $h \in \mathscr{S}(\mathbb{R})$. Using (6.16) then

$$
\begin{aligned}
2\pi i A &= \frac{1}{2}\int_{\mathbb{R}} [h(k)+h(-k)] \sum_{k_j\in\mathcal{R}_+\setminus\{0\}} m_R(k_j)\partial_k^4\left[\frac{1}{k+k_j} - \frac{1}{k-k_j}\right] \mathrm{d}k \\
&= \sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2}\int_{\mathbb{R}} \partial_k^4[h(k)+h(-k)]\left[\frac{1}{k+k_j} - \frac{1}{k-k_j}\right] \mathrm{d}k \\
&= \sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2}\int_{\mathbb{R}} \left[\widehat{t^4\phi}(k) + \widehat{t^4\phi}(-k)\right]\left[\frac{1}{k+k_j} - \frac{1}{k-k_j}\right] \mathrm{d}k.
\end{aligned}
$$

Explicitly we have

$$
\begin{aligned}
2\pi i A = \sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2\sqrt{2\pi}}\Bigg[&\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-itk}\phi(t)t^4\left(\frac{1}{k+k_j} - \frac{1}{k-k_j}\right)\mathrm{d}t\,\mathrm{d}k \\
&+ \int_{\mathbb{R}}\int_{\mathbb{R}} e^{itk}\phi(t)t^4\left(\frac{1}{k+k_j} - \frac{1}{k-k_j}\right)\mathrm{d}t\,\mathrm{d}k\Bigg] \\
= \sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2\sqrt{2\pi}}\Bigg[&\int_{\mathbb{R}}\Bigg(\int_0^\infty e^{-i|t|k}t^4\phi(t)\,\mathrm{d}t + \int_{-\infty}^0 e^{i|t|k}t^4\phi(t)\,\mathrm{d}t \\
&+ \int_0^\infty e^{i|t|k}t^4\phi(t)\,\mathrm{d}t + \int_{-\infty}^0 e^{-i|t|k}t^4\phi(t)\,\mathrm{d}t\Bigg)\left(\frac{1}{k+k_j} - \frac{1}{k-k_j}\right)\mathrm{d}k\Bigg].
\end{aligned}
$$

Hence

$$
\begin{aligned}
2\pi i A = \sum_{k_j\in\mathcal{R}_+\setminus\{0\}} \frac{m_R(k_j)}{2\sqrt{2\pi}}\Bigg[&\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i|t|k}t^4\phi(t)\left(\frac{1}{k+k_j} - \frac{1}{k-k_j}\right)\mathrm{d}t\,\mathrm{d}k \\
&+ \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i|t|k}t^4\phi(t)\left(\frac{1}{k+k_j} - \frac{1}{k-k_j}\right)\mathrm{d}t\,\mathrm{d}k\Bigg].
\end{aligned}
\tag{6.19}
$$

To utilise Jordan's lemma (see for instance [61, chapter 19]) and obtain semi-circular arcs that produce closed loops, we note that we require $\operatorname{Im} k < 0$ and $\operatorname{Im} k > 0$ for the first and second terms on the right-hand side of (6.19). Since we defined the resonances to lie in the lower k-plane, then we remove half of the terms in (6.19). Define

$$
C_1(R) = [-R,R] \cup \{Re^{i\theta} \mid \theta \in (\pi, 2\pi)\}, \qquad C_2(R) = [-R,R] \cup \{Re^{i\theta} \mid \theta \in (0, \pi)\},
$$

where $C_1(R)$ and $C_2(R)$ are both orientated positively. Hence

$$\begin{aligned} 2\pi i A &= \sum_{k_j \in \mathcal{R}_+ \setminus \{0\}} \frac{m_R(k_j)}{2\sqrt{2\pi}} \lim_{R\to\infty} \left[\int_{\mathbb{R}} \int_{C_1(R)} \frac{e^{-i|t|k}}{k-k_j} t^4 \phi(t)\,\mathrm{d}k\,\mathrm{d}t \right. \\ &\quad \left. + \int_{\mathbb{R}} \int_{C_2(R)} \frac{e^{i|t|k}}{k+k_j} t^4 \phi(t)\,\mathrm{d}k\,\mathrm{d}t \right] \\ &= 2\pi i \sum_{k_j \in \mathcal{R}_+ \setminus \{0\}} \frac{m_R(k_j)}{2\sqrt{2\pi}} \int_{\mathbb{R}} \left[e^{-i|t|k_j} + e^{i|t|(-k_j)} \right] t^4 \phi(t)\,\mathrm{d}t \\ &= \frac{2\pi i}{\sqrt{2\pi}} \left\langle \sum_{k_j \in \mathcal{R}_+ \setminus \{0\}} m_R(k_j) t^4 e^{-i|t|k_j}\,,\phi \right\rangle. \end{aligned}$$

4. We follow the same method as in the previous step to find

$$B = -\frac{2\pi i}{\sqrt{2\pi}} \left\langle \sum_{k_j \in \mathcal{R}_- \setminus \{0\}} m_R(k_j) t^4 e^{-i|t|k_j}\,,\phi \right\rangle.$$

5. For the discrete spectra we write using (6.17)

$$\begin{aligned} C &= \sum_{E_j} m_j f(E_j) = \sum_{E_j} m_j \left[\widehat{t^4\phi}\left(\sqrt{E_j^2 - m^2}\right) + \widehat{t^4\phi}\left(-\sqrt{E_j^2 - m^2}\right) \right] \\ &= \sum_{E_j} \frac{m_j}{\sqrt{2\pi}} \int_{\mathbb{R}} t^4 \left(e^{-itk_j} + e^{itk_j} \right) \phi(t)\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \left\langle \sum_{E_j} 2m_j t^4 \cos(tk_j)\,,\phi \right\rangle. \end{aligned}$$

6. For the threshold resonances we similarly use (6.17) to write

$$D = \sum_{\pm} \pm\frac{1}{2} \widetilde{m}_R(\pm m) f(\pm m) = \frac{1}{\sqrt{2\pi}} \left\langle \sum_{\pm} \pm \widetilde{m}_R(\pm m) t^4\,,\phi \right\rangle.$$

7. Insert the results from the previous steps into (6.18) to obtain the desired result.

□

Summary

In this chapter, we have obtained the second of our trace formulas; a Poisson wave trace formula for the Dirac operator. The proof was dependent on our Birman-Kreĭn trace formula as well as two other theorems from section 6.1. The first concerned a resonance counting function that estimated the upper bound of resonances inside a circle centred at the origin. The second pre-requisite is a factorization of the scattering matrix, made possible by the fact that zeros of $\det S^{\pm}(k)$ coincide with the resonances of $\mathbb{D}_V$. Our

Poisson wave trace formula in section 6.2 holds in distributional sense for all t and relates the trace difference between cosine operator-valued functions of $\mathbb{D}_V$ and $\mathbb{D}_0$ to the sum of all resonances and eigenvalues of $\mathbb{D}_V$. In chapter 7 we will use a less general version of our Poisson wave trace formula to prove that infinitely many resonances exist in certain circumstances.

Chapter 7

Existence of infinitely many resonances

This chapter presents a significant application of our trace formulas from chapters 5 and 6. Under certain conditions we prove that there exists infinitely many resonances of the perturbed Dirac operator. As a prerequisite we use an asymptotic expansion of the Dirac scattering phase in section 7.1 and show its relationship to an amended version of our Poisson wave trace formula. We then prove by contradiction in section 7.2 that under further assumptions on V we have infinitely many resonances.

7.1 The Dirac scattering phase

We use asymptotics of the Dirac scattering phase accredited to Bruneau and Robert [14]. We therefore further restrict our class of real-valued potentials to satisfy

Assumption 7.1.1 *Let* $V : \mathbb{R}^3 \to M_4(\mathbb{R})$ *take the form*

$$V = \begin{pmatrix} V_+ I_2 & 0 \\ 0 & V_- I_2 \end{pmatrix},$$

where I_2 *is the* 2×2 *identity matrix.*

Theorem 7.1.2 *Let* V *satisfy Assumptions 2.4.1 and 7.1.1. Also, define the scattering* *phase, $\sigma_\pm(k)$, by*

$$\sigma'_\pm(k) := \frac{1}{2\pi i}\partial_k \log\det S^\pm(k).$$

Then in the far field limit there exists a sequence a_j such that

$$\sigma'_+(k) - \sigma'_-(k) \sim \sum_{j=1}^{\infty} \frac{a_j(V)}{k^{2(j-1)}}, \qquad k \to \infty.$$

Moreover around $t = 0$ we have the expansion

$$\widehat{\sigma'_+}(t) - \widehat{\sigma'_-}(t) = C_1\delta(t) + \sum_{j=2} C_j|t|^{2j-3}, \tag{7.1}$$

where

$$C_1 = -2\sqrt{2\pi}\gamma_2(V), \qquad C_j = (-1)^j\sqrt{2\pi}\frac{\gamma_{2j}(V)}{(2j-3)!}, \quad j \geq 2,$$

$$\gamma_2(V) = \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\left[\left(\frac{V_+ - V_-}{2}\right)^2 + (V_+ - V_-) - 2\left(\frac{V_+ + V_-}{2}\right)^2\right](\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Proof. We divide the proof into 2 steps.

1. From [14, Theorem 2.1] we have the asymptote of the Dirac scattering phase in the spectral parameter. Since $z \to \pm\infty$ corresponds to $k \to \infty$, then changing variable gives

$$\begin{aligned}\sigma'_\pm(k) &\sim \mp\frac{k}{\sqrt{k^2+m^2}}\sum_{j=1}\gamma_j(V)\left[\pm\sqrt{k^2+m^2}\right]^{2-j} \\ &\sim -\gamma_1(V)k \mp \gamma_2(V) - \frac{\gamma_3(V)}{k} \mp \frac{\gamma_4(V)}{k^2} + \dots.\end{aligned}$$

If $a_{2j}(V) = -2\gamma_{2j}$ then we obtain as required

$$\sigma'_+(k) - \sigma'_-(k) \sim \sum_{j=1} \frac{a_{2j}(V)}{k^{2(j-1)}}, \qquad k \to \infty. \tag{7.2}$$

2. Write $k = \alpha\kappa$, $t = \beta\tau$ and $\kappa\tau = 1$ so that as $\kappa \to \infty$, then $\tau \to 0$. Then we take the Fourier transform of (7.2) (see [37, section 7.3]) so that we have as $\tau \to 0$,

$$\begin{aligned}\widehat{\sigma'_+}(\beta\tau) - \widehat{\sigma'_-}(\beta\tau) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\sigma'_+(\alpha\kappa) - \sigma'_-(\alpha\kappa)\right)e^{-i\alpha\kappa\beta\tau}\,\mathrm{d}(\alpha\kappa) \\ &\sim \sum_{j=1}\frac{a_{2j}(V)}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{-i\alpha\kappa\beta\tau}}{(\alpha\kappa)^{2(j-1)}}\,\mathrm{d}(\alpha\kappa) \\ &= \sqrt{2\pi}a_2(V)\delta(\beta\tau) + \sum_{j=2}a_{2j}(V)\sqrt{\frac{\pi}{2}}\frac{(-i)^{2(j-1)}(\beta\tau)^{2(j-1)-1}}{(2(j-1)-1)!}\operatorname{sgn}(\beta\tau).\end{aligned}$$

Let $\beta = 1$ and use $a_j(V) = -2\gamma_{2j}(V)$. Hence

$$\widehat{\sigma'_+}(t) - \widehat{\sigma'_-}(t) \sim -2\sqrt{2\pi}\gamma_2(V)\delta(t) + \sqrt{2\pi}\sum_{j=2}(-1)^j\frac{\gamma_{2j}(V)}{(2j-3)!}|t|^{2j-3}, \qquad t \to 0. \qquad \square$$

In the following theorem we use a less general result of Theorem 6.2.1 whereby we do not use the prefactor t^4. Hence the trace formula in this instance would hold in distributional sense on $\mathbb{R}\setminus\{0\}$. Only minor changes to the proof of Theorem 6.2.1 are required to show that the trace formula still holds.

Theorem 7.1.3 *Let V satisfy Assumptions 2.4.1 and 7.1.1. Near $t=0$, the less general distributional trace formula in Theorem 6.2.1 on $\mathbb{R}\setminus\{0\}$ satisfies*

$$\begin{aligned} 2\,\mathrm{Tr}\left[\cos(t\sqrt{\mathbb{D}_V^2-m^2})-\cos(t\sqrt{\mathbb{D}_0^2-m^2})\right]-T(t) \\ = -4\pi\gamma_2(V)\delta(t)+2\pi\sum_{j=2}(-1)^j\frac{\gamma_{2j}}{(2j-3)!}|t|^{2j-3}, \end{aligned} \tag{7.3}$$

where $T(t)=2\sum_{E_j}m_j\cos(tk_j)+\sum_{\pm}\pm\widetilde{m}_R(\pm m)$.

Proof. Similar to the proof of Theorem 6.2.1, we use

$$u(t)=2\,\mathrm{Tr}\left[\cos(t\sqrt{\mathbb{D}_V^2-m^2})-\cos(t\sqrt{\mathbb{D}_0^2-m^2})\right]-T(t),$$

plus $\phi\in C_0^\infty(\mathbb{R})$ and $f(z)=\widehat{\phi}(\sqrt{z^2-m^2})+\widehat{\phi}(-\sqrt{z^2-m^2})$ to write

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\langle u,\phi\rangle &= \int_0^\infty f(\sqrt{k^2+m^2})\sigma_+'(k)\,\mathrm{d}k-\int_0^\infty f(\sqrt{k^2+m^2})\sigma_-'(k)\,\mathrm{d}k \\ &= \int_0^\infty[\widehat{\phi}(k)+\widehat{\phi}(-k)]\sigma_+'(k)\,\mathrm{d}k-\int_0^\infty[\widehat{\phi}(k)+\widehat{\phi}(-k)]\sigma_-'(k)\,\mathrm{d}k \\ &= \left\langle\int_0^\infty e^{-ikt}\sigma_+'(k)\,\mathrm{d}k,\phi\right\rangle+\left\langle\int_0^\infty e^{ikt}\sigma_+'(k)\,\mathrm{d}k,\phi\right\rangle \\ &\quad -\left\langle\int_0^\infty e^{-ikt}\sigma_-'(k)\,\mathrm{d}k,\phi\right\rangle+\left\langle\int_0^\infty e^{ikt}\sigma_-'(k)\,\mathrm{d}k,\phi\right\rangle \\ &= \left\langle\int_{\mathbb{R}} e^{-ikt}\sigma_+'(k)\,\mathrm{d}k,\phi\right\rangle-\left\langle\int_{\mathbb{R}} e^{-ikt}\sigma_-'(k)\,\mathrm{d}k,\phi\right\rangle \\ &= \langle(\widehat{\sigma_+'}-\widehat{\sigma_-'}),\phi\rangle, \end{aligned}$$

where (4.24) dictates that $\sigma_\pm'(k)$ is even. Then using (7.1) we obtain (7.3) as required. □

7.2 Existence of infinitely many resonances

We now reach our final theorem which establishes the required conditions for the existence of infinitely many resonances associated with $\mathbb{D}_V$. This result is an application of our two trace formulas and is inspired by Melrose [52] and his result for Schrödinger resonances.

Theorem 7.2.1 *Let V satisfy Assumptions 2.4.1 and 7.1.1 such that $\gamma_2(V) \neq 0$ and $\gamma_{2j}(V) \neq 0$ for at least one $j \geq 2$. Then there exists infinitely many scattering resonances of the perturbed Dirac operator.*

Proof. We divide the proof into 2 steps.

1. We first prove that there are exists at least one resonance. By contradiction we assume that there are a finite number of eigenvalues and that the only resonances are at $z = \pm m$. Then

$$\sum_{k_j \in \mathcal{R}_+} m_R(k_j) e^{-i|t|k_j} - \sum_{k_j \in \mathcal{R}_-} m_R(k_j) e^{-i|t|k_j} = 0. \tag{7.4}$$

Incidentally (7.4) is also zero if resonances in $\mathcal{R}_+$ and $\mathcal{R}_-$ at k_j with equal multiplicities cancel out but this automatically implies that there exists at least two resonances. Inserting (7.4) into the Poisson wave equation in Theorem 6.2.1 we therefore have

$$2\,\mathrm{Tr}[\cos(t\sqrt{\mathbb{D}_V^2 - m^2}) - \cos(t\sqrt{\mathbb{D}_0^2 - m^2})] - T(t) = 0,$$

where $T(t)$ is defined in Theorem 7.1.3. It also implies that

$$0 = -4\pi\gamma_2(V)\delta(t) + 2\pi \sum_{j=2} (-1)^j \frac{\gamma_{2j}(V)}{(2j-3)!} |t|^{2j-3}.$$

For any small $t > 0$ the delta distribution is equal to zero but for any $\gamma_{2j}(V) \neq 0$, $j \geq 2$ the right-hand side is not zero and hence gives a contradiction. Therefore there exists at least one resonance not at $z = \pm m$.

2. Next assume that there are only a finite number of resonances. Then rearranging the Poisson wave equation in Theorem 6.2.1, the right-hand side of

$$\begin{aligned} &2\,\mathrm{Tr}\left[\cos(t\sqrt{\mathbb{D}_V^2 - m^2}) - \cos(t\sqrt{\mathbb{D}_0^2 - m^2})\right] - T(t) \\ &= \sum_{k_j \in \mathcal{R}_+} m_R(k_j) e^{-i|t|k_j} - \sum_{k_j \in \mathcal{R}_-} m_R(k_j) e^{-i|t|k_j}, \end{aligned} \tag{7.5}$$

is finite (possibly zero). Then we may continuously extend (7.5) to $t = 0$ such that the right-hand side is equal to or between $-\sum_{k_j \in \mathcal{R}_-} m_R(k_j)$ and $\sum_{k_j \in \mathcal{R}_+} m_R(k_j)$. However this contradicts (7.3) which is not continuous at $t = 0$ due to $\gamma_2(V) \neq 0$ and the delta distribution. We therefore conclude that there are infinitely many resonances. $\square$

Summary

In this chapter, we have applied our trace formulas to prove that it is possible to have infinitely many resonances. With further restrictions on the potential, we found a far field asymptotic expansion of the scattering phase. Using its Fourier transform, we then studied how the Poisson wave trace formula, when restricted to $\mathbb{R} \setminus \{0\}$, behaves as $t \to 0$. Further considerations in this limit were then used to prove the existence of infinitely many resonances associated with the perturbed Dirac operator.

Chapter 8

Concluding Remarks

Motivated by their appearance in the physical sciences, mathematical resonances have been studied extensively using Schrödinger operators. Numerous methods have been utilised to study their existence and number bounds. One drawback of Schrödinger operators is that they do not account for relativistic effects. This can be corrected by instead considering Dirac operators. However the literature concerning Dirac resonances is limited in comparison with the non-relativistic case.

To this end, in this thesis we have studied resonances of the Dirac operator perturbed by a smooth, compactly supported electric potential. They are defined as poles of the cut-off full resolvent when extended from the physical k half-plane to $\mathbb{C}$. The author is not aware of Dirac resonances being studied in this manner previously.

By considering our change of variable, we have had to construct from first principles various concepts such as operator resolvents, the scattering matrix, and how resonances are linked to the bigger picture. This enabled us to prove our Birman-Kreĭn and Poisson wave trace formulas that explicitly deal with any threshold resonances on a separate basis. Finally we applied our trace formulas to prove that it is indeed possible to have infinitely many resonances associated with the perturbed Dirac operator.

8.1 Future work

This thesis leads to various open questions which could be addressed as extensions to this work

- An immediate question is whether our proof for the existence of infinitely many Dirac resonances (see chapter 7) can be amended to avoid using our less general Poisson wave trace formula. This could for instance follow the method described by Smith and Zworski [77] who expand further on scattering phase asymptotics and factorization of the scattering matrix determinant.

- The dimensionality was fixed to $\mathbb{R}^3$. For Schrödinger operators it is an interesting observation that many results concerning resonances hold for odd dimension $d \geq 3$, with generalized proofs that hold for all of those dimensions. Different considerations are necessary for the $d = 1$ and even d cases. One would expect that a similar generalization would hold for Dirac operators. It would also be interesting if our result on the existence of infinitely many resonances holds if we increase from 4, the $n \times n$ dimension of the Dirac operator.

- Throughout the thesis, we have consistently assumed favourably smooth properties of the potential. For the Schrödinger case, this too was initially considered for proving the existence of infinitely many resonances. However the proof was later amended to hold true for more general classes of potentials. With respect to this thesis, this would require reworking large sections of our work to allow this. Likewise, studying the effect of a magnetic potential on Dirac resonances would also require a major recalculation of our work.

- It would stand to reason that any open questions on resonances of the Schrödinger operator would also be open for the Dirac operator. One area still under investigation in the former case is whether there exists an optimal, lower bound on resonance number for smooth, compactly supported potentials. In the Schrödinger case only partial results have been obtained with restrictive conditions imposed on V (see for instance [33, section 5.4]).

Appendix

A.1 Entire functions

All theorems in this section are used to define properties of $\det S^{\pm}(k)$ in chapter 6. For any given analytic function, Theorem A.1.1 describes how its maximum can be found on the boundary of a compact region whereas Theorem A.1.2 shows how such a maximum can be bounded by its real part. Both theorems can be found in [80, chapter V].

Theorems A.1.3 to A.1.5 culminate in a product description for an entire function with growth restrictions on its zeros. These have been adapted from [19, chapter XI] and [47, chapters 2 and 4]). See also section 4.3 of the latter reference for a proof of Theorem A.1.6.

Theorem A.1.7 provides a lower bound for holomorphic functions on a disc $D(0; R)$ that excludes the family of internal discs centred at the zeros of f (see for instance [80, section 8.71]).

Theorem A.1.1 (Maximum modulus theorem) *Let f be a non-constant analytic function on a bounded region $\Omega \subset \mathbb{C}$. If $|f(z)| \le M$ on $\partial\Omega$ then $|f(z)| < M$ on all interior points of Ω.*

Theorem A.1.2 (Borel–Carathéodory theorem) *Let f be analytic on a closed disc $\overline{D(0; R)} \subset \mathbb{C}$. Then for $0 < r < R$*

$$\sup_{|z|=r} |f(z)| \le \frac{2r}{R-r} \sup_{|z|\le R} \operatorname{Re} f(z) + \frac{R+r}{R-r}|f(0)|.$$

Theorem A.1.3 (Jensen's formula) *Let f be holomorphic in $\Omega \subset \mathbb{C}$ with zeros $\{a_j\}_j$*

inside the disc $D(0;R)$ of radius R centred at the origin. If $f(0) \neq 0$ then

$$\int_0^R \frac{n(t)}{t}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,\mathrm{d}\theta - \log|f(0)|,$$

where $n(t)$ counts the zeros of f inside $D(0;t)$.

Theorem A.1.4 (Hadamard factorization theorem) *Let f be an entire function with zeros $\{z_1, z_2, \ldots\}$. If the zero counting function satisfies $n(r) \leq r^p$ for $r > r_0$ then the summation*

$$\sum_{j=1}^{\infty} \frac{1}{|z_j|^{p+1}}, \tag{A.1}$$

is finite.

Theorem A.1.5 (Weierstrauss factorization theorem) *Let $\{z_k : z_k \neq 0\}_{k=1}^{\infty}$ satisfy $|z_k| \to \infty$, $k \to \infty$. Suppose also $p_k \in \mathbb{N}$ and $r > 0$ such that (A.1) is finite. Then the function*

$$P(z) = \prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{z_k}\right),$$

is entire and the Weierstrauss factors, E_n, for $n \in \mathbb{N}$ are defined

$$E_n(z) = \begin{cases} (1-z), & n = 0, \\ (1-z)\exp\left(\displaystyle\sum_{j=1}^{n} \frac{z^j}{j}\right), & n \neq 0. \end{cases}$$

Moreover let f be an entire function with zeros $\{z_j : z_j \neq 0\}$, repeated according to its multiplicity, and such that $f(0)$ is a zero of multiplicity $m \geq 0$. Then there exists an entire function g and sequence of integers $\{p_k\}$ such that

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{z_k}\right). \tag{A.2}$$

Theorem A.1.6 *Let p be the smallest integer such that (A.1) is finite. Then the growth order for a Weierstrauss product is also equal to p.*

Theorem A.1.7 (Minimum modulus theorem) *Let $P(z)$ be a canonical product of order p with zeros $\{z_j\}$. Then for $\epsilon > 0$*

$$\log|P(z)| \geq -|z|^{p+\epsilon}, \qquad z \notin \bigcup_{\{z_j\}} D(z_j; \langle z_j\rangle^{-p-\epsilon}),\ |z| \geq r_0.$$

A.2 Spectral theory

Theorems A.2.1 to A.2.3 concern standard results in spectral theory (see for instance [34] and [65]). We then introduce the concept of singular values and trace class operators. The details for the properties listed from (A.3) to (A.9) can be found in [71, chapters 1 and 3] and also [28, chapter VII]).

We direct the reader to [27, chapter V, section 5.1] for proofs of the estimates in Theorem A.2.4. Weyl's asymptotic counting law for eigenvalues of the Laplace-Beltrami operator on compact Riemannian manifolds in Theorem A.2.5 can be found in [11, section A.3].

Finally Theorems A.2.6 and A.2.7 are generalized versions of almost analytic extensions and the Helffer-Sjöstrand formula (see for instance [20, chapter 8] and [87, section 14.3] respectively).

Theorem A.2.1 *Let A be a selfadjoint operator. Then if $\lambda \in \rho(A)$, we have the resolvent norm*

$$\|(A-z)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \operatorname{spec}(A))}.$$

Theorem A.2.2 *The following are known as the first and second resolvent identities respectively.*

1. *Let A be a linear operator on a Hilbert space $\mathcal{H}$. Then for $z, \zeta \in \rho(A)$,*

$$(A-z)^{-1} - (A-\zeta)^{-1} = (z-\zeta)(A-z)^{-1}(A-\zeta)^{-1}.$$

2. *Let A and B be closed operators whereby $z \in \rho(A) \cap \rho(B)$. Then*

$$(A-z)^{-1} - (B-z)^{-1} = (A-z)^{-1}(B-A)(B-z)^{-1}.$$

Theorem A.2.3 (Spectral theorem of selfadjoint operators) *Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$. If $f(\lambda)$ is a complex-valued function and $u \in \mathcal{H}$ such that*

$$\int |f(\lambda)|^2 \, \mathrm{d}\langle E(\lambda)u \, , E(\lambda)u\rangle < \infty,$$

then we have the operator

$$f(H) = \int f(\lambda) \, \mathrm{d}E(\lambda).$$

Next we consider the singular values of compact operators. For a compact operator C, then its singular values, $s_j[A]$ where $\|A\| = s_1[A] \geq s_2[A] \geq, \ldots$ are defined as the non-zero eigenvalues of $|C| := (C^*C)^{1/2}$. Moreover if C_1, C_2 and B are compact and bounded operators respectively then

$$s_j[BC_1] \leq \|B\| s_j[C_1], \tag{A.3}$$

$$s_j[C_1 B] \leq \|B\| s_j[C_1], \tag{A.4}$$

$$s_{j+k+1}[C_1 + C_2] \leq s_{j+1}[C_1] + s_{k+1}[C_2], \tag{A.5}$$

$$s_{j+k+1}[C_1 C_2] \leq s_{j+1}[C_1] s_{k+1}[C_2]. \tag{A.6}$$

An operator is trace class (denoted $\mathcal{B}_1$) if its trace norm

$$\|A\|_{\mathcal{B}_1} = \sum_{j=1} s_j(A),$$

is finite. In that case, if $\{v_k\}$ is an orthonormal basis in the Hilbert space, $\mathcal{H}$, the trace of A is defined

$$\operatorname{Tr} A := \sum_{k=1} \langle v_k, A v_k \rangle.$$

Moreover if A is trace class and B is a bounded operator, then the trace satisfies

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA).$$

If A is a trace class operator, then we define the determinant of $I - A$ as

$$\det(I - A) = \prod_{j=1}^{\infty} (I - \lambda_j(A)), \tag{A.7}$$

where λ_j are the eigenvalues of A counted with multiplicity. Denoting A_1, A_2 and B as trace class and bounded operators respectively then we have the following properties

$$\begin{aligned} \det[(I - A_1)(I - A_2)] &= \det(I - A_1)\det(I - A_2), \\ \det(I - A_1 B) &= \det(I - B A_1). \end{aligned} \tag{A.8}$$

Furthermore for small values of μ then

$$\det(I - \mu A) = \exp\left[-\sum_{j=1}^{\infty} \mu^j \frac{\operatorname{Tr}[A^j]}{j} \right]. \tag{A.9}$$

Theorem A.2.4 *Let $A \in \mathcal{B}_1$. If $\lambda \in \mathbb{C}$ is such that the inverse $(I - \lambda A)^{-1}$ exists, then*

$$|\det(I - \lambda A)| \leq \prod_{j=1}^{\infty} (1 + |\lambda| s_j(A)),$$

$$\|(I - \lambda A)^{-1}\| \leq \frac{\det(I + \sqrt{A^* A})}{|\det(I - \lambda A)|}.$$

Theorem A.2.5 (Weyl law) *Let (M,g) be a compact n-dimensional Riemannian manifold and suppose that $-\Delta_M$ is the Laplace-operator on M. Furthermore denote the (discrete) spectrum as*

$$\operatorname{spec}(-\Delta_M) = \{0 = z_0 < z_1 \leq z_2 \leq \dots\},$$

with corresponding eigenfunctions ψ_j that form an orthonormal basis in $L^2(M)$. Then the counting function for the eigenvalues of $-\Delta_M$ satisfies

$$\#\{z_j \in \operatorname{spec}(-\Delta_M) : z_j \leq r\} \sim \frac{\operatorname{vol}_g(M)}{\Gamma(n/2+1)(4\pi)^{n/2}}\, r^{n/2},$$

or equivalently

$$z_j \sim \frac{\operatorname{vol}_g(M)}{\Gamma(n/2+1)(4\pi)^{n/2}}\, j^{2/n}.$$

Using Theorem A.2.5 where n is the number of dimensions of (M,g), it can then be shown that

$$s_j[(-\Delta_M + I)^{-m/2}] \leq C j^{-m/n}. \tag{A.10}$$

Theorem A.2.6 (Almost analytic extension) *Let $f \in \mathscr{S}(\mathbb{R})$ and $\chi \in C_0^\infty((-1,1))$ such that $\chi \equiv 1$ on $[-1/2,1/2]$. Assume also $z = x+iy$ such that $\overline{\partial}_z = (\partial_x + i\partial_y)/2$. Then*

$$\tilde{f}(x+iy) := \frac{1}{\sqrt{2\pi}}\chi(y)\int_{\mathbb{R}} \chi\left(\frac{y\lambda}{1+x^2}\right)\widehat{f}(\lambda)e^{i\lambda(x+iy)}\,\mathrm{d}\lambda \in C^\infty(\mathbb{C}) \tag{A.11}$$

is an almost analytic extension of f to $\mathbb{C}$ satisfying

$$\begin{aligned} \tilde{f}|_{\mathbb{R}} &= f, \qquad \operatorname{supp}\tilde{f} \subset \{z : |\operatorname{Im} z| \leq 1\}, \\ \overline{\partial}_z \tilde{f} &\leq C_N \frac{|\operatorname{Im} z|^N}{\langle x\rangle^{2(N+1)}}, \qquad \forall N \in \mathbb{N}. \end{aligned} \tag{A.12}$$

Proof. For $y = 0$ on the real line, then $\tilde{f} = f$ as required. The cut-off function χ outside of the integral in (A.11) ensures the support of $\tilde{f}$ is restricted to $\{|\operatorname{Im} z| \leq 1\}$. For the final property in (A.12) we calculate

$$\begin{aligned}
&\overline{\partial}_z \int_{\mathbb{R}} \chi\left(\frac{y\lambda}{1+x^2}\right)\widehat{f}(\lambda)e^{i\lambda(x+iy)}\,\mathrm{d}\lambda \\
&= \left(\frac{i}{2} - \frac{xy}{(1+x^2)}\right)\frac{1}{(1+x^2)}\int_{\mathbb{R}} \chi'\left(\frac{y\lambda}{1+x^2}\right)\lambda\widehat{f}(\lambda)e^{i\lambda(x+iy)}\,\mathrm{d}\lambda \\
&= \left(\frac{i}{2} - \frac{xy}{(1+x^2)}\right)\frac{y^N}{(1+x^2)^{N+1}}\int_{\mathbb{R}} \chi_N'\left(\frac{y\lambda}{1+x^2}\right)\lambda^{N+1}\widehat{f}(\lambda)e^{i\lambda(x+iy)}\,\mathrm{d}\lambda \\
&\leq C|1+y|\frac{|y|^N}{\langle x\rangle^{2(N+1)}}\int_{\operatorname{supp}\chi_N'} \chi_N'\left(\frac{y\lambda}{1+x^2}\right)\lambda^{N+1}\widehat{f}(\lambda)e^{-\lambda y}\,\mathrm{d}\lambda \\
&\leq C_N \frac{|y|^N}{\langle x\rangle^{2(N+1)}},
\end{aligned}$$

where we have assumed $|\operatorname{Im} z| \leq 1$ and used

$$\chi_N'\left(\frac{y\lambda}{1+x^2}\right) = \left(\frac{y\lambda}{1+x^2}\right)^{-N} \chi'\left(\frac{y\lambda}{1+x^2}\right) \in C_0^\infty((-1,1)).$$

For $\chi'(y)\int_{\mathbb{R}} \chi\left(\frac{y\lambda}{1+x^2}\right) \widehat{f}(\lambda) e^{i\lambda(x+iy)}\, \mathrm{d}\lambda$ we use the substitution

$$\chi\left(\frac{y\lambda}{1+x^2}\right) = (1+x^2)^{-N}\chi_N\left(\frac{y\lambda}{1+x^2}\right),$$

and note that $\chi'(y) \leq C_N|y|^N$ in the assumed range of $\operatorname{Im} z$. This completes the proof. $\square$

Theorem A.2.7 (Generalized Helffer-Sjöstrand formula) *Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$. Suppose $f \in \mathscr{S}(\mathbb{R})$ such that $f(z) = (z-z_0)^{-N} g(z)$ where $\operatorname{Im} z_0 > 0$ and $g \in \mathscr{S}(\mathbb{R})$ has analytic extension $\tilde{g}$. Then*

$$f(H) = \frac{1}{\pi}\int_{\mathbb{C}} (H-z)^{-1}(H-z_0)^{-N}\overline{\partial}_z \tilde{g}(z)\, \mathrm{d}m(z),$$

where m denotes the Lebesgue measure on $\mathbb{C}$.

Proof. Let $u, v \in \mathcal{H}$ and $Q = \frac{1}{\pi}\int_{\mathbb{C}}(H-z)^{-1}(H-z_0)^{-N}\overline{\partial}_z\tilde{g}(z)\,\mathrm{d}m(z)$. Then using Theorem A.2.3 we have

$$\begin{aligned}
\langle Qu\,, v\rangle &= \frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}_z \tilde{g}(z) \int_{\operatorname{spec}(H)} \frac{1}{(t-z)}\frac{1}{(t-z_0)^N}\, \mathrm{d}\langle E(t)u\,, v\rangle\, \mathrm{d}m(z) \\
&= \int_{\operatorname{spec}(H)} \frac{g(t)}{(t-z_0)^N}\, \mathrm{d}\langle E(t)u\,, v\rangle \\
&= \int_{\operatorname{spec}(H)} f(t)\, \mathrm{d}\langle E(t)u\,, v\rangle = \langle f(H)u\,, v\rangle,
\end{aligned}$$

as required. $\square$

A.3 Fredholm theory

Theorem A.3.1 (see for instance [87, section D.4]) is used to prove the meromorphic continuation of $R_0^\pm(k)$ and $R_V^\pm(k)$ from the upper k plane to $\mathbb{C}$. To study the structure of $F^\pm(k)$ near $k=0$ in Theorem 5.2.1 we use the method described in Theorem A.3.2 (see [28, chapter XI]).

Theorem A.3.1 (Analytic Fredholm theory) *Let $U \subset \mathbb{C}$ be a connected open set and suppose $\{A(z)\}_{z\in U}$ is a holomorphic family of Fredholm operators. If $A(z_0)^{-1}$ exists at* *some point* $z_0 \in U$*, then* $\{A(z)^{-1}\}_{z \in U}$ *is a meromorphic family of operators with poles of finite rank.*

Theorem A.3.2 (Gohberg-Sigal theory) *Let* $A(\lambda)$ *be a family of meromorphic Fredholm operators. If* $\mu \in \Omega \subset \mathbb{C}$ *and*

$$A(\lambda) = \sum_{j=1}^{J} \frac{A_j}{(\lambda - \mu)^j} + A_0(\lambda),$$

with $A_0(\lambda)$ *holomorphic near* μ *and zero Fredholm index, then there exists operators* $U_{1,2}(\lambda)$ *and* Q_k*,* $1 \leq k \leq K$ *such that near* μ*,*

$$A(\lambda) = U_1(\lambda) \left(\sum_{k=-K_1}^{K_2} (\lambda - \mu)^k Q_k \right) U_2(\lambda), \qquad k \in \mathbb{Z}.$$

Here $U_{1,2}(\lambda)$ *are holomorphic and invertible near* μ *whilst* Q_k *are disjoint projection operators satisfying*

$$\operatorname{rank}(I - Q_0) < \infty, \qquad \operatorname{rank} Q_k = 1, \; k > 0. \tag{A.13}$$

Moreover if $\sum_{j=0}^{K} Q_k = I$*, then the inverse satisfies*

$$A(\lambda)^{-1} = U_2(\lambda)^{-1} \left(\sum_{k=-K_1}^{K_2} (\lambda - \mu)^{-k} Q_k \right) U_1(\lambda)^{-1},$$

and we have

$$\frac{1}{2\pi i} \operatorname{Tr} \oint \left[A(\lambda)^{-1} \partial_\lambda A(\lambda) \right] \mathrm{d}\lambda = N_\mu(A) - N_\mu(A^{-1}), \tag{A.14}$$

where the positively orientated contour is around the single pole μ *of* $A(\lambda)^{-1}\partial_\lambda A(\lambda)$*, and* $N_\mu(A)$ *and* $N_\mu(A^{-1})$ *count, with multiplicity, the number of zeros and poles of* A *respectively.*

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