Topics in Geometric Analysis

Analysis on Symmetric Spaces, Generalised Spectral Zetas, and the Hypergeometric Function

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Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

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Abstract

In this thesis we study topics pertaining to the analysis, geometry, and spectral theory of the Laplacian on Riemannian symmetric spaces of rank one. These spaces are quotients of the form $\mathscr{X} = \mathbf{G}/\mathbf{H}$ with $\mathbf{G}$ a Lie group and $\mathbf{H}$ a suitable subgroup of $\mathbf{G}$. In the compact case they entail the unit sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbb{R}\mathbf{P}^n = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$, the quaternionic projective space $\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$, and the Cayley Plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}^4/\mathbf{Spin}(9)$. In the non-compact case they entail the real hyperbolic space $\mathbb{R}\mathbf{H}^n = \mathbf{SO}_0(n,1)/\mathbf{SO}(n)$, the complex hyperbolic space $\mathbb{C}\mathbf{H}^n = \mathbf{SU}(n,1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$, the quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n = \mathbf{Sp}(n,1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$, and the hyperbolic Cayley plane $\mathbf{H}^2(\mathrm{Cay}) = \mathbf{F}^4_\star/\mathbf{Spin}(9)$.

At the heart of this study is the analysis of operators built out of the Laplacian, the spherical functions, the spectral projections and measure, and the functional calculus associated with the Laplacian. These build close connections with the theory of special functions (the hypergeometric function and orthogonal polynomials) on the one hand and representation theory and harmonic analysis on these spaces on the other.

The Maclaurin spectral coefficients are computed by using a class of differential-spectral identities developed here together with an explicit description of the spectrum, the multiplicity functions and Plancherel measures for each space. We closely examine the heat kernels and the spectral zeta functions, as two key tools, the latter being given in the compact spaces by the Dirichlet type series

$$\zeta(s;\mathscr{X}) = \sum_{k=1}^{\infty} \frac{M_k(\mathscr{X})}{[\lambda_k]^s}, \qquad \Re s > d/2,$$

and in the non-compact case, via the Mellin transform of the heat trace, by

$$\zeta(s;\mathscr{X}) = \mathsf{c}(\mathscr{X}) \int_0^\infty \frac{\mu(\lambda)}{(\lambda^2+\rho^2)^s}\, d\lambda.$$

We introduce and study some new generalisations of these spectral objects and among other things provide explicit representations for them in terms of the Hurwitz zeta function, Beta, Gamma and other special functions. These in particular lead to the discovery of the poles and residues of the zeta type functions and, more remarkably, deeper proportionality relations between individual pairs of compact and non-compact spaces in "*duality*".

We extend the classical Hecke-Funk identity (originally for spheres) to all compact symmetric spaces, where we show for suitable F with Schwartz kernel K_F that

$$F(-\Delta_{\mathscr{X}})\phi = [\widehat{K_F}]_k^{\alpha,\beta}\phi, \qquad \phi \in \mathcal{H}_k,$$

where $\mathcal{H}_k$ are the finite-dimensional eigenspaces of the Laplacian and

$$[\widehat{K_F}]_k^{\alpha,\beta} = \frac{(2\pi)^{d/2}}{2^\beta\Gamma(d/2)}\int_{-1}^{1} K_F(t)\mathscr{P}_k^{\alpha,\beta}(t)(1-t)^\alpha(1+t)^\beta dt.$$

Here $\mathscr{P}_k^{\alpha,\beta}$ are the Jacobi polynomials with $\alpha, \beta > -1$ fixed parameters associated to each space. A spectral-differential identity on the hypergeometric functions of the form

$$\mathscr{L}_{\mathsf{P}}\left[{}_2F_1(a,b;c;\mathscr{E}(\theta))\right]\Big|_{\theta=0} = \sum_{m=0}^{N} p_m \mathsf{H}_m(-ab),$$

is proved, where $\mathscr{L}_{\mathsf{P}}$ is a differential operator and $\mathsf{H}_m(X) = \mathsf{H}_m(a,b,c;\mathscr{E};X)$ is an explicitly computable polynomial, which with various specialisations and generalisations as stated above play a central role throughout the thesis.

Contents

Introduction

This thesis is formed by a collection of studies in the area of geometric analysis, primarily focused on symmetric spaces of rank one. Each chapter consists of a different study, and thus each is self-contained (bar a shared set of appendices and certain omissions to avoid excessive repetition) and has its own character and feel. Throughout we will be working with the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$, as well as the families of functions and polynomials that arise as special cases of it, in paricular the Jacobi and Gegenbauer families. Our interest in these families arises from how they act as the zonal spherical functions on rank-one symmetric spaces.

In Chapter 1 we present a differential identity on the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$, unifying and extending certain spectral results on the scale of Gegenbauer and Jacobi polynomials, and leading to a new class of hypergeometric related scalars $\mathsf{c}_j^m(a, b, c)$ and polynomials $\mathscr{R}_m = \mathscr{R}_m(X)$. We next consider the Laplacian on a compact rank one symmetric space and for operators of the Laplace transform type use an operator trace relation to describe the Maclaurin spectral coefficients of the Schwartz kernels of these operators. Other related representations as well as an extension of the differential identity to generalised hypergeometric functions are discussed.

In Chapter 2, we build on the differential identity in Chapter 1. Here we formulate and prove a generalised multi-variable differential-spectral identity on the hypergeometric function and discuss applications of this identity to the spectral representation of operator kernels and zonal spherical functions on symmetric spaces of rank-one. Some extensions of the main result of this chapter and a number of related examples are also presented.

In Chapter 3, a constant coefficient partial differential operator $\mathscr{L}_{\mathsf{P}} = \mathsf{P}(\partial)$ of order N is applied to the Gauss hypergeometric function ${}_2F_1 = {}_2F_1(a, b; c; z)$ with $2z = 1 - g(X_1) \dots g(X_q)$ and point evaluated at $(X_1, \dots, X_q) = 0$. Here g is an even smooth function satisfying $g(0) = 1$. A representation formula is obtained that completely classifies the action by a sequence of polynomials $\mathscr{R}_\gamma(a, b, c; g; X)$ and scalars $\mathsf{c}_j^\gamma(a, b, c; g)$ directly linking to the hypergeometric parameters a, b, c and the function g. Explicit and computable descriptions of these quantities in terms of the elementary symmetric polynomials and the exponential Bell polynomials are given and some applications to the analysis of Riemannian symmetric spaces are discussed. Extensions of the result to the generalised hypergeometric function ${}_pF_r = {}_pF_r(\mathbf{a}; \mathbf{b}; z)$ with vector parameters $\mathbf{a}, \mathbf{b}$ and the matrix hypergeometric function ${}_2F_1(\mathbf{A}, \mathbf{B}; \mathbf{C}; z)$ with matrix parameters $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are established and discussed.

In Chapter 4, we present applications of the differential identity outlined in the previous chapters to the analysis of invariant operators on compact rank-one symmetric spaces, including operator trace formulations pertaining to functions of the Laplace-Beltrami operator. A generalisation of the Funk-Hecke formula is stated. The Maclaurin spectral functions associated with various operator families are examined and exploited and a novel trace representation via ϕ-series encompassing Jacobi theta series and related function families are established.

In Chapter 5, we undertake a deeper study of the Maclaurin spectral coefficients $b^n_{2\ell}(t; \mathscr{X})$ introduced in earlier chapters, specifically in the case of the heat semi-group, where they are seen to be generalisations of the heat trace as $\Theta(t; \mathscr{X}) = b^n_0(t; \mathscr{X})$. We extend the well-known short-time asymptotics of the heat trace on symmetric spaces, providing explicit formulae for an analogous expansion of the so-called Maclaurin heat coefficients on each rank-one symmetric space $\mathscr{X}$. Comparing the expansions of spaces in duality (compact vs. non-compact), we arrive at a proportionality principle between the so-called *generalised* Minakshisundaram-Pleijel heat coefficients.

In Chapter 6, we study the spectral zeta function $\zeta(s; \mathscr{X})$ on rank-one symmetric spaces, motivated by their remarkable relation to the heat trace via an application of the Mellin transform. Although the form of these zeta functions differs between spaces of compact and non-compact type, they share the same set of poles, with the residues being represented using the Minakshisundaram-Pleijel heat coefficients $\mathsf{a}_j(\mathscr{X})$ of the asymptotic expansions of the heat trace as discussed in the previous chapter. With the aim of extending this, we take the Mellin transform of the Maclaurin heat coefficients $b^n_{2\ell}(t; \mathscr{X})$, deriving a generalised spectral zeta function $\zeta_\ell(s; \mathscr{X})$. We give explicit formulae for these functions on compact and non-compact spaces in terms of the Hurwitz zeta function and various special functions, enabling us to locate the poles. We finally derive and prove a proportionality principle on the level of the residues of the generalised zeta function on spaces in duality.

Tables

This section contains tables of geometric and spectral data referred to throughout the thesis.

Table 1: Spectral data for compact rank one symmetric spaces

$\mathscr{X}$	λ_k^n	$M_k(\mathscr{X})$
$\mathbb{S}^n$	$k(k+n-1)$	$\dfrac{(2k+n-1)(k+n-2)!}{k!(n-1)!}$
$\mathbb{R}\mathbf{P}^n$	$2k(2k+n-1)$	$\dfrac{(4k+n-1)(2k+n-2)!}{(2k)!(n-1)!}$
$\mathbb{C}\mathbf{P}^n$	$k(k+n)$	$\dfrac{2k+n}{n}\left(\dfrac{\Gamma(k+n)}{\Gamma(n)k!}\right)^2$
$\mathbb{H}\mathbf{P}^n$	$k(k+2n+1)$	$\dfrac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left(\dfrac{\Gamma(k+2n)}{k!\Gamma(2n)}\right)^2$
$\mathbf{P}^2(\mathrm{Cay})$	$k(k+11)$	$6(2k+11)\dfrac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}$

Table 2: Geometric quantities for compact rank-one symmetric spaces

$\mathscr{X}$	$\mathbb{S}^n$	$\mathbb{R}\mathbf{P}^n$	$\mathbb{C}\mathbf{P}^n$	$\mathbb{H}\mathbf{P}^n$	$\mathbf{P}^2(\mathrm{Cay})$
$\ell_p(\mathscr{X})$	2π	π	2π	2π	2π
$\mathrm{Vol}(\mathscr{X})$	$\dfrac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$	$\dfrac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$	$\dfrac{4^n\pi^n}{n!}$	$\dfrac{4^{2n}\pi^{2n}}{(2n+1)!}$	$\dfrac{3!(4\pi)^8}{11!}$

Table 3: Shared parameters for dual rank-one symmetric spaces

$\mathscr{X}_c$	d	α	β	ρ	$\mathscr{X}_{nc}$
$\mathbb{S}^n$	n	$(n-2)/2$	$(n-2)/2$	$(n-1)/2$	$\mathbb{R}\mathbf{H}^n$
$\mathbb{R}\mathbf{P}^n$	n	$(n-2)/2$	$(n-2)/2$	$(n-1)/2$	-
$\mathbb{C}\mathbf{P}^n$	$2n$	$n-1$	0	$n/2$	$\mathbb{C}\mathbf{H}^n$
$\mathbb{H}\mathbf{P}^n$	$4n$	$2n-1$	1	$(2n+1)/2$	$\mathbb{H}\mathbf{H}^n$
$\mathbf{P}^2(\mathrm{Cay})$	16	7	3	$11/2$	$\mathbf{H}^2(\mathrm{Cay})$

Table 4: Hypergeometric parameter values for compact rank one symmetric spaces

$\mathscr{X}$	a	b	c	$-ab$
$\mathbb{S}^n$	$-k$	$k+n-1$	$n/2$	$k(k+n-1)$
$\mathbb{R}\mathbf{P}^n$	$-2k$	$2k+n-1$	$n/2$	$2k(2k+n-1)$
$\mathbb{C}\mathbf{P}^n$	$-k$	$k+n$	n	$k(k+n)$
$\mathbb{H}\mathbf{P}^n$	$-k$	$k+2n+1$	$2n$	$k(k+2n+1)$
$\mathbf{P}^2(\mathrm{Cay})$	$-k$	$k+11$	8	$k(k+11)$

Table 5: The first few coefficients $\mathsf{c}_j^m(a,b,c)$

j	c_j^1	c_j^2	c_j^3
1	$-\dfrac{1}{2c}$	$-\dfrac{3(a+b)-2c+1}{4c\,(c+1)}$	$-\dfrac{15(a+b)^2-15(a+b)c}{4c\,(c+1)\,(c+2)}$ $-\dfrac{15(a+b)+2\,c^2-9\,c+4}{4c\,(c+1)\,(c+2)}$
2	0	$\dfrac{3}{4c(c+1)}$	$\dfrac{45(a+b)-30\,c+15}{8\,c\,(c+1)\,(c+2)}$
3	0	0	$-\dfrac{15}{8\,c\,(c+1)\,(c+2)}$

Chapter 1

1.1 Introduction

Let $(\mathscr{M}, g)$ be an d-dimensional $(d \geq 2)$ compact smooth Riemannian manifold without boundary and let $\Delta = \Delta_g$ denote the Laplace-Beltrami operator on $\mathscr{M}$, given in local coordinates, by $\Delta_g = 1/\sqrt{\det g} \sum \partial_j \left(\sqrt{\det g}\, g^{jk} \partial_k\right)$.

By basic spectral theory there exists a complete orthonormal basis $(f_j : j \geq 0)$ of eigenfunctions of $-\Delta_g$ in $L^2(\mathscr{M}, dv_g)$ with a spectrum $\Sigma = \Sigma(-\Delta_g)$ consisting purely of eigenvalues. Each eigenvalue has a finite multiplicity and the spectrum can be arranged as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with $\lambda_j \nearrow \infty$. Thus $-\Delta_g f_j = \lambda_j f_j$ and by suitably normalising $||f_j||_{L^2(\mathscr{M})} = 1$ for all $j \geq 0$ while $(f_j, f_k)_{L^2(\mathscr{M})} = 0$ for $0 \leq j \neq k$. Now for a given function $\Phi = \Phi(X)$ in the Borel functional calculus of $-\Delta_g$ the operator $\Phi(-\Delta_g)$ has a Schwartz kernel given by the spectral sum

$$K_\Phi(x, y) = \sum_{j=0}^{\infty} \Phi(\lambda_j)\, f_j(x) f_j(y), \qquad x, y \in \mathscr{M}. \tag{1.1.1}$$

If $\mathscr{M}$ is a compact rank one symmetric space of a Lie group then by using the addition formula for the matrix coefficients of irreducible unitary representations the above simplifies to

$$K_\Phi(\theta) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{M})}{\mathrm{Vol}(\mathscr{M})} \Phi(\lambda_k) \mathscr{F}_k(\theta; \mathscr{M}), \tag{1.1.2}$$

where $\mathscr{F}_k = \mathscr{F}_k(\theta; \mathscr{M})$ are the spherical functions on $\mathscr{M}$, $\lambda_k = \lambda_k(\mathscr{M})$ are the numerically *distinct* eigenvalues of $-\Delta$ on $\mathscr{M}$, $M_k = M_k(\mathscr{M})$ is the dimension of the eigenspace associated with λ_k, $\theta = \theta(x, y)$ is the distance between $x, y \in \mathscr{M}$ and $\mathrm{Vol}(\mathscr{M})$ denotes the volume of $\mathscr{M}$. (*See*, e.g., [6, 62, 83, 99, 100])

The families of compact rank one symmetric spaces of interest are the sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbb{R}\mathbf{P}^n = \mathbf{S}^n/\{\pm\} = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(1))$, the quaternionic projective space $\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ and the Cayley projective plane $\mathbf{P}^2(\mathrm{Cay}) =$

$\mathbf{F}_4/\mathbf{Spin}(9)$. These spaces with the exception of the real projective space $\mathbb{R}\mathbf{P}^n$ $(n \geq 1)$ and the circle $\mathbb{S}^1$ are simply-connected where $\pi_1(\mathbb{R}\mathbf{P}^n) \cong \mathbb{Z}_2$ $n \geq 2$ and $\pi_1(\mathbb{S}^1) \cong \pi_1(\mathbb{R}\mathbf{P}^1) \cong \mathbb{Z}$. (For further discussion and results see [6, 7, 99] and [28, 41, 98, 100]).

The scale of Jacobi polynomials $\mathscr{P}_k^{(\alpha,\beta)}$ (with $k \geq 0$, $\alpha, \beta > -1$) are intertwined with the spherical functions on these symmetric spaces (for suitable α, β) and here, in the simply-connected case, (1.1.2) can be rewritten as [1]

$$K_\Phi(\theta) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{M})}{\mathrm{Vol}(\mathscr{M})} \Phi(\lambda_k^n) \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta). \tag{1.1.3}$$

Tables 1 and 2 present some of the relevant spectral and geometric data for the rank one symmetric spaces described above and Table 3 gives among other things the parameters α, β for the spherical functions associated with these spaces. Now in view of the Schwartz kernel K_Φ in (1.1.3) being an even function of θ, subject to sufficient regularity, it admits a formal Maclaurin expansion about $\theta = 0$, given by

$$\sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j)!} \frac{\partial^{2j}}{\partial\theta^{2j}} K_\Phi \bigg|_{\theta=0} = \sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j)!} \frac{b_{2j}[\Phi]}{\mathrm{Vol}(\mathscr{M})}. \tag{1.1.4}$$

The Maclaurin spectral coefficients $b_{2j}[\Phi]$ defined above through the successive differentiation of the Schwartz kernel at the origin can be given an interesting trace formulation by applying the following statement, in essence, a differential-spectral identity on the Jacobi polynomials. [2]

Towards this end let $\mathsf{P}_N(X) = p_0 + \sum p_i X^i$ (summation for $1 \leq i \leq N$) be a polynomial of degree $N \geq 2$ and consider the differential operator

$$\mathscr{L}_\mathsf{P} = \mathsf{P}_N\,(d/d\theta) = p_0 + \sum_{i=1}^{N} p_i d^i/d\theta^i. \tag{1.1.5}$$

Theorem. *The action of $\mathscr{L}_\mathsf{P}$ as in (1.1.5) on the Jacobi polynomial $\mathscr{P}_k^{(\alpha,\beta)}$ satisfies the following identity,*

$$\begin{aligned} \mathscr{L}_\mathsf{P} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} &= p_0 + \sum_{m=1}^{\lfloor N/2 \rfloor} p_{2m} \sum_{j=1}^{m} \mathsf{c}_j^m(\alpha,\beta)[\lambda_k^{\alpha,\beta}]^j \\ &= p_0 + \sum_{m=1}^{\lfloor N/2 \rfloor} p_{2m} \mathscr{R}_m(\lambda_k^{\alpha,\beta}). \end{aligned} \tag{1.1.6}$$

[1] When $\mathscr{M} = \mathbb{R}\mathbf{P}^n$ it suffices to set $F_k(\theta) = \mathscr{P}_{2k}^{((n-2)/2,(n-2)/2)}(\cos\theta) = \mathscr{C}_{2k}^{(n-1)/2}(\cos\theta)$.

[2] This theorem is a special case of a more general result on hypergeometric functions appearing in Theorem 1.2.2.

Here $\lambda_k^{\alpha,\beta} = k(\alpha+\beta+k+1)$ *are the eigenvalues of the Jacobi operator* (*see* (A.1.6)), $\mathsf{c}_j^m(\alpha,\beta)$ *are suitable coefficients and* $\mathscr{R}_m = \mathscr{R}_m(X)$ *is the* m^{th} *degree polynomial*

$$\mathscr{R}_m(X) = \sum_{j=1}^{m} \mathsf{c}_j^m(\alpha,\beta) X^j. \tag{1.1.7}$$

Returning now to the discussion prior to the theorem it is seen that here the Maclaurin spectral coefficients associated with the Schwartz kernel K_Φ are given by $b_0[\Phi] = \mathrm{Tr}\{\Phi(-\Delta_{\mathscr{M}})\}$, and for $l \geq 1$ by

$$b_{2l}[\Phi] = \frac{\partial^{2l}}{\partial\theta^{2l}} K_\Phi(\theta)\bigg|_{\theta=0} \mathrm{Vol}(\mathscr{M}) = \mathrm{Tr}\{[\mathscr{R}_l\Phi](-\Delta_{\mathscr{M}})\}, \tag{1.1.8}$$

where Tr denotes the operator trace, in this case the operator being $[\mathscr{R}_l\Phi](-\Delta_{\mathscr{M}})$. As a particular example, $\Phi(X) = e^{-tX}$ in (1.1.3) gives the heat kernel on $\mathscr{M}$, with the coefficients in (1.1.8) corresponding to the Maclaurin heat coefficients.

In this chapter we specialise to functions $\Phi = \Phi(X)$ of Laplace transform type, namely, those that for a suitable L^1-summable f are given by integral

$$\Phi(X) = \int_0^\infty f(s) e^{-Xs}\, ds, \quad X \geq 0. \tag{1.1.9}$$

Applying (1.1.8), we can write the Maclaurin coefficients for K_Φ as

$$\begin{aligned} b_{2l}[\Phi(-\Delta)] &= \int_0^\infty f(s) \sum_{k=0}^{\infty} M_k \sum_{j=1}^{l} \mathsf{c}_j^l [\lambda_k^n]^j e^{-\lambda_k^n s}\, ds \\ &= \int_0^\infty f(s) \left[\mathscr{R}_l\left(-\frac{d}{ds}\right)\right] \mathrm{Tr}\, e^{s\Delta}\, ds = \int_0^\infty f(s) b_{2l}[e^{s\Delta}]\, ds. \end{aligned} \tag{1.1.10}$$

Here the polynomial $\mathscr{R}_l(X)$ is defined in (1.2.8) and $b_{2l}[e^{s\Delta}]$ are the Maclaurin *heat* coefficients given by $b_{2l}[e^{s\Delta}] = \mathscr{R}_l(-d/ds)\,\mathrm{Tr}\, e^{s\Delta}$. One particular example of (1.1.9) is when $f(s) = f_\sigma(s) = s^{a-1}e^{-s\sigma}/\Gamma(a)$ with $\mathrm{Re}(a) > 1$ in which case for Φ one recovers the resolvent operator $\mathbf{R}_\sigma$ raised to the power a. For related discussion and applications *see* [10, 46, 87, 90, 92, 94] and the references therein.

To describe the plan of the current chapter, Section 1.2 explores and extends the differential-spectral identity (1.1.6) to the more general context of hypergeometric functions (see Appendix A.1). The main theorem here (Theorem 1.2.2) unifies and extends these concepts to a setting where no immediate spherical function representation or spectral interpretation of the hypergeometric function $F(a,b;c;z)$ is applicable. This leads to a new characterisation of the scalars $\mathsf{c}_j^m(a,b,c)$ and polynomials $\mathscr{R}_m$ in (1.1.7), reducing to the Jacobi polynomial case when $a=-k$, $b=\alpha+\beta+k+1$, $c=\alpha+1$ (see Table 3 and

Table 4). As an application in the remaining sections we invoke these ideas along with the trace formulation of the Maclaurin spectral coefficients (1.1.8) to operators of Laplace transform type on a scale of compact rank one symmetric spaces to give a representation of these spectral coefficients via those of the heat semigroup and the Jacobi theta functions. Let us finish off this introduction by highlighting some important special cases of the hypergeometric function ${}_2F_1(a, b; c; z)$ for future reference (*cf.*, e.g., [1, 4, 47, 73]).

- The Legendre polynomial $P_k(t)$, $k \geq 0$,

$$\begin{aligned} P_k(t) &= {}_2F_1(-k, k+1; 1; (1-t)/2) \\ &= \frac{1}{2^k k!} \frac{d^k}{dt^k} \left[\left(t^2 - 1 \right)^k \right]. \end{aligned} \tag{1.1.11}$$

- The Gegenbauer polynomial $\mathscr{C}_k^\nu(t)$, $\nu > -1/2$, $k \geq 0$,

$$\begin{aligned} \mathscr{C}_k^\nu(t) &= {}_2F_1(-k, 2\nu + k; \nu + 1/2; (1-t)/2) \\ &= \frac{(-1)^k}{2^k (\nu + 1/2)_k} \left(1 - t^2 \right)^{-\nu + 1/2} \frac{d^k}{dt^k} \left[\left(1 - t^2 \right)^{k + \nu - 1/2} \right]. \end{aligned} \tag{1.1.12}$$

- The Jacobi polynomial $\mathscr{P}_k^{(\alpha,\beta)}(t)$, $k \geq 0$, $\alpha, \beta > -1$,

$$\begin{aligned} \mathscr{P}_k^{(\alpha,\beta)}(t) &= {}_2F_1(-k, \alpha + \beta + k + 1; \alpha + 1; (1-t)/2) \\ &= \frac{(-1)^k}{2^k (\alpha + 1)_k} (1-t)^{-\alpha} (1+t)^{-\beta} \frac{d^k}{dt^k} \left[(1-t)^\alpha (1+t)^\beta (1-t^2)^k \right]. \end{aligned} \tag{1.1.13}$$

- The incomplete Beta function $B(x; p, q)$,

$$\begin{aligned} B(x; p, q) &= \frac{x^p}{p} {}_2F_1(p, 1-q; p+q; x) \\ &= \int_0^x t^{p-1} (1-t)^{q-1} dt. \end{aligned} \tag{1.1.14}$$

Note in particular that in the Gegenbauer and Jacobi cases we have $\mathscr{P}_k^{(\alpha,\beta)}(1) = 1$ and $\mathscr{C}_k^\nu(1) = 1$ by the choice of normalisation.

1.2 Hypergeometric coefficients and a combinatorial identity

In this section we present the main result which uses a combinatorial identity together with a recursive formula to describe the action of the differential operator $\mathscr{L}_{\mathsf{P}}$ on the hypergeometric function ${}_2F_1(a, b; c; z)$. This naturally leads to the introduction of a class of the polynomials $\mathscr{R}_m = \mathscr{R}_m(X)$ (with $m \geq 1$) and a set of scalars, the hypergeometric coefficients, $\mathsf{c}_j^m(a,b,c)$ (with $1 \leq j \leq m$) that play a central role in the chapter.

Before presenting the main theorem, we introduce a set of scalars s_j^p, which we define as the coefficients of Y^j in the polynomial

$$\prod_{k=0}^{p-1} (Y + X_k) = \sum_{j=0}^{p} \mathsf{s}_j^p(X_0, \dots, X_{p-1}) Y^j, \tag{1.2.1}$$

for scalars $X_0, \dots, X_{p-1}$. In fact these scalars can be described by the elementary symmetric polynomials as $\mathsf{S}_{p-j}(X_0, \dots, X_{p-1}) = \mathsf{s}_j^p$, where $\mathsf{S}_j(X_0, \dots, X_{p-1})$ denotes the sum of the distinct products of length j of the variables $X_0, \dots, X_{p-1}$. In particular, we have

$$\mathsf{s}_j^j = 1, \quad \mathsf{s}_{j-1}^j = \sum_{\ell=0}^{j-1} X_\ell, \quad \mathsf{s}_0^j = \prod_{\ell=0}^{j-1} X_\ell. \tag{1.2.2}$$

Now we have the following lemma, which relates a product of Pochhammer symbols, which are essential to the hypergeometric function, to the scalars introduced above.

Lemma 1.2.1. *With the Pochhammer symbol $(a)_j$ defined in (A.1.2), the product of $(a)_j$ and $(b)_j$ can be written as a polynomial in ab as*

$$(a)_j(b)_j = \prod_{p=0}^{j-1} \left(ab + \underbrace{p(a+b+p)}_{\rho_p} \right) = \sum_{l=1}^{j} \mathsf{s}_l^j [ab]^l, \tag{1.2.3}$$

where the scalars $\mathsf{s}_l^j = \mathsf{s}_l^j(a+b)$ are defined in (1.2.1) by setting $\rho_p = p(a+b+p)$. Note that with $\rho_p = p(a+b+p)$ we have $\mathsf{s}_0^j = 0$.

Proof. Referring to (A.1.2) we can write

$$(a)_j = \prod_{p=0}^{j-1} (a+p), \tag{1.2.4}$$

and similarly for $(b)_j$. Applying this to the product $(a)_j(b)_j$, we have

$$(a)_j(b)_j = \prod_{k=0}^{j-1}(a+k) \prod_{l=0}^{j-1}(b+l) = \prod_{p=0}^{j-1}(a+p)(b+p) = \prod_{p=0}^{j-1} (ab + p(a+b+p)). \tag{1.2.5}$$

The conclusion follows by observing (1.2.1). $\square$

We can now present the main theorem, which shows the action of the differential operator $\mathscr{L}_\mathsf{P}$ on the hypergeometric function ${}_2F_1(a,b;c;z)$ and gives an explicit description of the associated coefficients $\mathsf{c}_j^m(a,b,c)$.

Theorem 1.2.2 (Hypergeometric coefficients)**.** *With $\mathscr{L}_{\mathsf{P}}$ the operator as in (1.1.5), for $|z| < 1$, and $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \dots,$ the hypergeometric function $F(a, b; c; z)$ satisfies the identity*

$$\mathscr{L}_{\mathsf{P}} \left[{}_2F_1 \left(a, b; c; \frac{1-\cos\theta}{2} \right) \right] \bigg|_{\theta=0} = p_0 + \sum_{m=1}^{\lfloor N/2 \rfloor} p_{2m} \sum_{j=1}^{m} \mathsf{c}_j^m(a, b, c)[-ab]^j. \tag{1.2.6}$$

The scalars $\{\mathsf{c}_j^m(a, b, c) : 1 \leq j \leq m\}$ are called the hypergeometric coefficients, given explicitly by

$$\mathsf{c}_j^m(a, b, c) = (-1)^j \sum_{i=j}^{m} (-2)^{-i} \mathsf{b}_i^m \mathsf{s}_j^i \prod_{p=0}^{i-1} (c+p)^{-1} \tag{1.2.7}$$

with b_i^m as in (4.2.1), and $\mathsf{s}_j^i = \mathsf{s}_j^i(a+b)$ are the scalars defined in (1.2.1) with $\rho_p = p(a+b+p)$.

Before stating the proof of this theorem, we introduce the m-degree polynomial $\mathscr{R}_m(X)$, defined as $\mathscr{R}_0(X) = 1$, and for $m \geq 1$

$$\mathscr{R}_m(X) = \sum_{j=1}^{m} \mathsf{c}_j^m(a, b, c) X^j. \tag{1.2.8}$$

This lets us write the statement of (1.2.6) as

$$\mathscr{L}_{\mathsf{P}} \left[{}_2F_1 \left(a, b; c; \frac{1-\cos\theta}{2} \right) \right] \bigg|_{\theta=0} = p_0 + \sum_{m=1}^{\lfloor N/2 \rfloor} p_{2m} \mathscr{R}_m(-ab). \tag{1.2.9}$$

Proof. We begin by noting that since ${}_2F_1\left(a, b; c; (1-\cos\theta)/2\right)$ is an even function of θ, evaluating its derivatives of odd order at zero will give zero. That is, when we apply $\mathscr{L}_{\mathsf{P}}$ to ${}_2F_1\left(a, b; c; (1-\cos\theta)/2\right)$ and evaluate at $\theta = 0$, we have

$$\begin{aligned}\mathscr{L}_{\mathsf{P}} \left[{}_2F_1 \left(a, b; c; \frac{1-\cos\theta}{2} \right) \right] \bigg|_{\theta=0} &= p_0 + \sum_{i=1}^{N} p_i \frac{d^i}{d\theta^i} {}_2F_1 \left(a, b; c; \frac{1-\cos\theta}{2} \right) \bigg|_{\theta=0} \\ &= p_0 + \sum_{m=1}^{\lfloor N/2 \rfloor} p_{2m} \frac{d^{2m}}{d\theta^{2m}} {}_2F_1 \left(a, b; c; \frac{1-\cos\theta}{2} \right) \bigg|_{\theta=0}.\end{aligned} \tag{1.2.10}$$

This allows us to use (4.2.1) with $f(\cos\theta) = {}_2F_1\left(a, b; c; (1-\cos\theta)/2\right)$, and then apply the differential identities defined in Appendix A.1 as follows.

$$\begin{aligned}\frac{d^{2m}}{d\theta^{2m}} {}_2F_1 \left(a, b; c; \frac{1-\cos\theta}{2} \right) \bigg|_{\theta=0} &= \left\{ \sum_{j=1}^{m} \frac{\mathsf{b}_j^m}{(-2)^j} \frac{d^j}{dz^j} F(a, b; c; z) \right\} \bigg|_{z=0} \\ \text{via (A.3.4)} \quad &= \sum_{j=1}^{m} \frac{\mathsf{b}_j^m}{(-2)^j} \frac{(a)_j (b)_j}{(c)_j} F(a+j, b+j; c+j, 0) \\ &= \sum_{j=1}^{m} \frac{\mathsf{b}_j^m}{(-2)^j} \frac{(a)_j (b)_j}{(c)_j}.\end{aligned} \tag{1.2.11}$$

Referring to (A.1.2), we can say

$$\frac{\mathsf{b}_j^m}{(-2)^j(c)_j} = \frac{\mathsf{b}_j^m}{(-2)^j\prod_{p=0}^{j-1}(c+p)} = \frac{\mathsf{b}_j^m}{\mathscr{C}^j(c)}, \tag{1.2.12}$$

where the product $(-2)^j\prod_{p=0}^{j-1}(c+p) = \mathscr{C}^j(c)$ is used for conciseness. Substituting (1.2.12) back into (1.2.11), we have

$$\begin{aligned}\frac{d^{2m}}{d\theta^{2m}}{}_2F_1\left(a,b;c;\frac{1-\cos\theta}{2}\right)\bigg|_{\theta=0} &= \sum_{j=1}^{m}\frac{\mathsf{b}_j^m}{\mathscr{C}^j(c)}(a)_j(b)_j \\ &= \sum_{j=1}^{m}\frac{\mathsf{b}_j^m}{\mathscr{C}^j(c)}\sum_{l=1}^{j}\mathsf{s}_l^j(ab)^l. \end{aligned}\tag{1.2.13}$$

Here we have used Lemma 1.2.1 to introduce the scalars s_l^j. Now expanding the sum and isolating powers of ab lets us rearrange to

$$\begin{aligned}\frac{d^{2m}}{d\theta^{2m}}{}_2F_1\left(a,b;c;\frac{1-\cos\theta}{2}\right)\bigg|_{\theta=0} &= \sum_{j=1}^{m}(ab)^j\left(\sum_{i=j}^{m}\frac{\mathsf{b}_i^m}{\mathscr{C}^i(c)}\mathsf{s}_j^i\right) \\ &= \sum_{j=1}^{m}(-ab)^j\left((-1)^j\sum_{i=j}^{m}\frac{\mathsf{b}_i^m}{\mathscr{C}^i(c)}\mathsf{s}_j^i\right) \\ &= \sum_{j=1}^{m}(-ab)^j\mathsf{c}_j^m(a,b,c), \end{aligned}\tag{1.2.14}$$

where we have written $(-1)^j\sum_{i=j}^{m}\mathsf{b}_i^m\mathsf{s}_j^i[\mathscr{C}^i(c)]^{-1} = \mathsf{c}_j^m(a,b,c)$. The conclusion follows when we substitute this back into (1.2.10). □

The first few hypergeometric coefficients can be found in Table 5. Using values of a, b, and c given in Table 4 yields the respective specialised coefficients for each rank one symmetric space.

1.3 Maclaurin spectral coefficients via Jacobi theta functions on rank one symmetric spaces

Returning to the Schwartz kernel $K_\Phi(\theta)$ from (1.1.3), where we described the Maclaurin coefficients $b_{2l}[\Phi]$ as in (1.1.8), we now specialise to functions $\Phi = \Phi(X)$ of the Laplace transform type. For a suitable function $f \in L^1$, we take Φ as

$$\Phi(X) = \int_0^\infty f(s)e^{-Xs}\,ds, \quad X \geq 0. \tag{1.3.1}$$

Applying the trace formulation (1.1.8) and taking advantage of (1.3.1) we can connect the Maclaurin spectral coefficients for K_Φ to those for the heat kernel by writing,

$$\begin{aligned} b_{2l}[\Phi(-\Delta)] &= \int_0^\infty f(s) \sum_{k=0}^{\infty} M_k \sum_{j=1}^{l} \mathsf{c}_j^l [\lambda_k^n]^j e^{-\lambda_k^n s}\, ds \\ &= \int_0^\infty f(s) \left[\mathscr{R}_l \left(-\frac{d}{ds} \right) \right] \operatorname{Tr} e^{s\Delta}\, ds = \int_0^\infty f(s) b_{2l}[e^{s\Delta}]\, ds. \end{aligned} \tag{1.3.2}$$

Here the polynomial $\mathscr{R}_l(X)$ is defined in (1.2.8) and $b_{2l}[e^{s\Delta}]$ are the Maclaurin heat coefficients given by $b_{2l}[e^{s\Delta}] = \mathscr{R}_l(-d/ds) \operatorname{Tr} e^{s\Delta}$. An interesting example is when $f(s) = f_\sigma(s) = s^{a-1}e^{-s\sigma}/\Gamma(a)$ with $\Re a > 1$ where one recovers the resolvent operator to the power a, that is, $\Phi(-\Delta) = \mathbf{R}_\sigma^a$. (*See* [46, 42, 92, 94] for more).

In this section we express the Maclaurin coefficients $b_{2l}[\Phi]$ in terms of the classical Jacobi theta functions of the first, second, and third kind, each defined in Appendix A.2, first on the unit sphere $\mathbb{S}^n$, then the real projective space $\mathbb{R}\mathbf{P}^n$, then the complex projective space $\mathbb{C}\mathbf{P}^n$, and finally the quaternionic projective space $\mathbb{H}\mathbf{P}^n$.

In the following subsections we consider the rank one symmetric spaces $\mathbb{S}^n$, $\mathbb{R}\mathbf{P}^n$, $\mathbb{C}\mathbf{P}^n$, and $\mathbb{H}\mathbf{P}^n$ respectively, and give an explicit formulation for the Maclaurin spectral coefficients for functions Φ as in (1.3.1). We refer to Table 1 for the spectral data on each of these spaces, and to Table 4 for the respective values of a, b, and c associated to each space, so that we can calculate explicitly the Maclaurin heat coefficients $b_{2l}[e^{s\Delta}]$.

1.3.1 On the unit sphere $\mathbb{S}^n$

In the following two theorems, for the odd and even dimensional case respectively, we will see how the Jacobi theta functions of the first and second kind naturally arise in the Maclaurin coefficients associated to functions $\Phi(-\Delta)$ of Laplace transform type.

Theorem 1.3.1 ($\mathbb{S}^n$, $n \geq 3$ odd)**.** *Take $\Phi(X)$ to be a function of Laplace transform type, as in* (1.3.1)*. Then the Maclaurin spectral coefficients $b_{2l}[\Phi]$ for odd $n \geq 3$ can be written as*

$$b_0[\Phi] = \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{a}_m^n (-1)^{m+1}}{(n-1)!} \int_0^\infty f(s) \vartheta_1^{(m+1)}(s)\, d\mu, \tag{1.3.3}$$

where $d\mu = e^{s(n-1)^2/4} ds$, and for $l > 0$ we have

$$b_{2l}[\Phi] = \sum_{m=0}^{\frac{n-3}{2}} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{a}_m^n (-1)^{j+m+1} \mathsf{c}_j^l}{(n-1)!} \binom{j}{i} \left(\frac{n-1}{2} \right)^{2i} \int_0^\infty f(s) \vartheta_1^{(m+j-i+1)}(s)\, d\mu, \tag{1.3.4}$$

where $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (1.2.7) *specialised to the unit sphere with a, b, c as in Table 4, and the scalars a_m^n are defined in* (A.4.1)*.*

Proof. We begin by writing the multiplicty function M_k in a form that lets us apply (A.4.1). Indeed, writing $X_k = k+(n-1)/2$, we can refer to Table 1 to write the multiplicity as

$$M_k = (2k+n-1)\frac{\Gamma(k+n-1)}{k!(n-1)!} = \frac{2X_k}{(n-1)!}\prod_{j=1}^{n-2}(k+j). \tag{1.3.5}$$

We now note that each term $(k+j)$, for $j = 1, \ldots, n-2$ of the product above can be written as $(X_k \pm j)$ for $j = 0, \ldots, (n-3)/2$. Taking the product of $(X_k - j)$ and $(X_k + j)$ lets us apply (A.4.1) to write

$$M_k = \frac{2}{(n-1)!}\prod_{j=0}^{\frac{n-3}{2}}(X_k^2 - j^2) = \sum_{m=0}^{\frac{n-3}{2}}\frac{2\mathsf{a}_m^n}{(n-1)!}X_k^{2m+2}. \tag{1.3.6}$$

Since the sum above vanishes when X_k is an integer between 1 and $(n-3)/2$, we can use the substitution $X_k \to p$ to write the heat trace $\operatorname{Tr} e^{s\Delta}$ as

$$\begin{aligned}
\operatorname{Tr} e^{s\Delta} = \sum_{k=0}^{\infty} M_k e^{-s\lambda_k^n} &= \sum_{m=0}^{\frac{n-3}{2}}\frac{2\mathsf{a}_m^n}{(n-1)!}\sum_{k=0}^{\infty}X_k^{2m+2}e^{-s\left(X_k^2-(n-1)^2/4\right)} \\
&= \sum_{m=0}^{\frac{n-3}{2}}\frac{2\mathsf{a}_m^n e^{s(n-1)^2/4}}{(n-1)!}\sum_{p=1}^{\infty}p^{2m+2}e^{-sp^2} \\
&= \sum_{m=0}^{\frac{n-3}{2}}\frac{\mathsf{a}_m^n(-1)^{m+1}e^{s(n-1)^2/4}}{(n-1)!}\vartheta_1^{(m+1)}(s).
\end{aligned} \tag{1.3.7}$$

Substituting the above into (1.3.2) and differentiating via Leibniz rule gives the results. □

In the even dimensional case, for $n \geq 2$, we have a similar formulation involving derivatives of the Jacobi theta function of the second kind.

Theorem 1.3.2 ($\mathbb{S}^n$, $n \geq 2$ even)**.** *Take $\Phi(X)$ as in (1.3.1). Then the Maclaurin spectral coefficients can be written as*

$$b_0[\Phi] = \sum_{m=0}^{\frac{n-2}{2}}\frac{\mathsf{b}_m^n(-1)^m}{(n-1)!}\int_0^{\infty} f(s)\vartheta_2^{(m)}(s)\,d\mu, \tag{1.3.8}$$

where $d\mu = e^{s(n-1)^2/4}ds$, and for $l > 0$ we have

$$b_{2l}[\Phi] = \sum_{m=0}^{\frac{n-2}{2}}\sum_{j=1}^{l}\sum_{i=0}^{j}\frac{\mathsf{b}_m^n(-1)^{j+m}\mathsf{c}_j^l}{(n-1)!}\binom{j}{i}\left(\frac{n-1}{2}\right)^{2i}\int_0^{\infty} f(s)\vartheta_2^{(m+j-i)}(s)\,d\mu. \tag{1.3.9}$$

where $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (1.2.7) specialised to the unit sphere with a, b, c as in Table 4, and the scalars b_m^n are as in (A.4.2).

Proof. This proof is similar to the proof of Theorem 1.3.1, so we skip some of the details. Taking $X_k = k + (n-1)/2$, we can express M_k in terms of the coefficients b^n_m from (A.4.2) (here we assume $n \geq 4$ as for $n = 2$ we can easily arrive at (1.3.10) below with $\mathsf{b}^2_0 = 1$ and without recourse to (A.4.2)) as

$$\begin{aligned} M_k = \frac{2X_k}{(n-1)!}\prod_{j=1}^{n-2}(k+j) &= \frac{2X_k}{(n-1)!}\prod_{j=\frac{1}{2}}^{\frac{n-2}{2}-\frac{1}{2}}(X_k^2 - j^2) \\ &= \sum_{m=0}^{\frac{n-2}{2}} \frac{2\mathsf{b}^n_m X_k}{(n-1)!} X_k^{2m}. \end{aligned} \tag{1.3.10}$$

Hence the heat trace $\operatorname{Tr} e^{s\Delta}$ can be written as

$$\begin{aligned} \operatorname{Tr} e^{s\Delta} = \sum_{k=0}^{\infty} M_k e^{-s\lambda_k^n} &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}^n_m e^{s(n-1)^2/4}}{(n-1)!} \sum_{p=\frac{1}{2}}^{\infty} 2p^{2m+1} e^{-sp^2} \\ &= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}^n_m (-1)^m e^{s(n-1)^2/4}}{(n-1)!} \vartheta_2^{(m)}(s). \end{aligned} \tag{1.3.11}$$

The result follows when we substitute this formulation of the heat trace into (1.3.2) and differentiate. □

The polynomial $\mathscr{R}_l$ on the sphere, for $l = 0, 1, 2$. We now present explicit values of $b_{2l}[e^{s\Delta}] = \mathscr{R}_l(-d/ds)\operatorname{Tr} e^{s\Delta}$ for $\mathbb{S}^n$, for $n = 2, 3, 4$. We have $\mathscr{R}_0(X) = 1$, $\mathscr{R}_1(X) = -X/n$, $\mathscr{R}_2(X) = (2-2n)/(n^2+2n)X + 3/(n^2+2n)X^2$. As a result we have:

- $\mathbb{S}^2$: $\operatorname{Tr} e^{s\Delta} = \vartheta_2 e^{s/4}$, $\mathscr{R}_1(-d/ds)\operatorname{Tr} e^{s\Delta} = [\vartheta_2'/2 + \vartheta_2/8]\, e^{s/4}$,
  $\mathscr{R}_2(-d/ds)\operatorname{Tr} e^{s\Delta} = [3\vartheta_2'' + 7/2\vartheta_2' + 11/16\vartheta_2]\, e^{s/4}/8$.

- $\mathbb{S}^3$: $\operatorname{Tr} e^{s\Delta} = -\vartheta_1' e^s/2$, $\mathscr{R}_1(-d/ds)\operatorname{Tr} e^{s\Delta} = -[\vartheta_1'' + \vartheta_1']\, e^s/6$,
  $\mathscr{R}_2(-d/ds)\operatorname{Tr} e^{s\Delta} = -[\vartheta_1'''/10 + \vartheta_1''/3 + 7/30\vartheta_1']\, e^s$.

- $\mathbb{S}^4$: $\operatorname{Tr} e^{s\Delta} = -[\vartheta_2'/6 + \vartheta_2/24]\, e^{9s/4}$,
  $\mathscr{R}_1(-d/ds)\operatorname{Tr} e^{s\Delta} = -[\vartheta_2'' + 5/2\vartheta_2' + 9/16\vartheta_2]\, e^{9s/4}/24$,
  $\mathscr{R}_2(-d/ds)\operatorname{Tr} e^{s\Delta} = -[\vartheta_2'''/3 + 9/4\vartheta_2'' + 179/48\vartheta_2' + 51/64\vartheta_2]\, e^{9s/4}/16$.

1.3.2 On the real projective space $\mathbb{R}\mathbf{P}^n$

Before stating the results below we introduce half series of the odd and even terms that make up the first and second theta functions. Taking $\vartheta_{1,o}(s)$ as the sum of the odd terms of ϑ_1, and $\vartheta_{1,e}(s)$ as the sum of the even terms, we can write $\vartheta_{1o}(s) + \vartheta_{1,e}(s) = \vartheta_1(s)$. We define these explicitly as

$$\vartheta_{1,o}(s) = \sum_{j\in\mathbb{Z}} e^{-s(2j+1)^2}, \quad \vartheta_{1,e}(s) = \sum_{j\in\mathbb{Z}} e^{-s(2j)^2}. \tag{1.3.12}$$

Likewise for ϑ_2, we define $\vartheta_{2,o}(s)$ and $\vartheta_{2,e}(s)$ so that $\vartheta_{2,o}(s) + \vartheta_{2,e}(s) = \vartheta_2(s)$ as

$$\vartheta_{2,o}(s) = \sum_{j=0}^{\infty}(4j+1)e^{-s(2j+1/2)^2}, \quad \vartheta_{2,e}(s) = \sum_{j=0}^{\infty}(4j+3)e^{-s(2j+3/2)^2}. \tag{1.3.13}$$

Theorem 1.3.3 ($\mathbb{R}\mathbf{P}^n$, $n \geq 3$ odd)**.** *Take $\Phi(X)$ as in (1.3.1). Then the Maclaurin spectral coefficients $b_{2l}[\Phi]$ can be written as*

$$b_0[\Phi] = \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{a}^n_m(-1)^{m+1}}{(n-1)!} \int_0^\infty f(s)\vartheta_{1,*}^{(m+1)}(s)\, d\mu, \tag{1.3.14}$$

where $d\mu = e^{s(n-1)^2/4}ds$, and for $l > 0$ we have

$$b_{2l}[\Phi] = \sum_{m=0}^{\frac{n-3}{2}} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{a}^n_m(-1)^{j+m+1}\mathsf{c}^l_j}{(n-1)!} \binom{j}{i} \left(\frac{n-1}{2}\right)^{2i} \int_0^\infty f(s)\vartheta_{1,*}^{(m+1+j-i)}(s)\, d\mu, \tag{1.3.15}$$

where we take $\vartheta_{1,o}$ when $(n-1)/2$ is odd, and $\vartheta_{1,e}$ when $(n-1)/2$ is even. Here $\mathsf{c}^l_j = \mathsf{c}^l_j(a,b,c)$ are the hypergeometric coefficients as in (1.2.7) specialised to the real projective space with a, b, c as in Table 4, and the scalars a^n_m the coefficients defined in (A.4.1).

Proof. Taking $X_k = 2k + (n-1)/2$, we can use (A.4.1) to express M_k in terms of the coefficients a^n_m,

$$\begin{aligned} M_k = \frac{2X_k}{(n-1)!}\prod_{j=1}^{n-2}(2k+j) &= \frac{2}{(n-1)!}\prod_{j=0}^{\frac{n-3}{2}}(X_k^2 - j^2) \\ &= \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{a}^n_m}{(n-1)!} X_k^{2m+2}. \end{aligned} \tag{1.3.16}$$

This lets us write the heat trace as

$$\begin{aligned} \operatorname{Tr} e^{s\Delta} = \sum_{k=0}^{\infty} M_k e^{-s\lambda^n_k} &= \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{a}^n_m e^{s(n-1)^2/4}}{(n-1)!} \sum_{k=0}^{\infty} X_k^{2m+2} e^{-sX_k^2} \\ &= \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{a}^n_m e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \frac{d^{m+1}}{ds^{m+1}} \sum_{k=0}^{\infty} e^{-sX_k^2}. \end{aligned} \tag{1.3.17}$$

Here we note that X_k takes odd integer values when $(n-1)/2$ is odd, and even integer values when $(n-1)/2$ is even. With this in mind we can consider both of these cases separately, and referring to (1.3.12) we have the following formulations.

(i) When $(n-1)/2$ is odd, we substitute $X_k \to 2j+1$ and extend the sums so they run over $\mathbb{Z}$ by noting that (1.3.16) gives that M_k vanishes when X_k is a integer between 0 and $(n-3)/2$.

$$\begin{aligned}
\operatorname{Tr} e^{s\Delta} &= \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{a}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \frac{d^{m+1}}{ds^{m+1}} \sum_{j=\frac{n-3}{4}}^{\infty} e^{-s(2j+1)^2} \\
&= \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{a}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \frac{d^{m+1}}{ds^{m+1}} \sum_{j\in\mathbb{Z}} e^{-s(2j+1)^2} \\
&= \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{a}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \vartheta_{1,o}^{(m+1)}(s).
\end{aligned} \tag{1.3.18}$$

(ii) When $(n-1)/2$ is even, we use the substitution $X_k = 2j$ and extend the sums as above to write

$$\begin{aligned}
\operatorname{Tr} e^{s\Delta} &= \sum_{m=0}^{\frac{n-3}{2}} \frac{2\mathsf{a}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \frac{d^{m+1}}{ds^{m+1}} \sum_{j=\frac{n-1}{4}}^{\infty} e^{-s(2j)^2} \\
&= \sum_{m=0}^{\frac{n-3}{2}} \frac{\mathsf{a}_m^n e^{s(n-1)^2/4}(-1)^{m+1}}{(n-1)!} \vartheta_{1,e}^{(m+1)}(s).
\end{aligned} \tag{1.3.19}$$

The result follows when we substitute these formulations of the heat trace into (1.3.2) and differentiate appropriately. □

Theorem 1.3.4 ($\mathbb{R}\mathbf{P}^n$, $n \geq 2$ even)**.** *Take $\Phi(X)$ as in (1.3.1). Then the Maclaurin spectral coefficients can be written as*

$$b_0[\Phi] = \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n}{(n-1)!} \int_0^\infty f(s)\vartheta_{2,*}^{(m)}(s)\, d\mu, \tag{1.3.20}$$

where $d\mu = e^{s(n-1)^2/4} ds$, and for $l > 0$ we have

$$b_{2l}[\Phi] = \sum_{m=0}^{\frac{n-2}{2}} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{b}_m^n (-1)^{j+m} \mathsf{c}_j^l}{(n-1)!} \binom{j}{i} \left(\frac{n-1}{2}\right)^{2i} \int_0^\infty f(s)\vartheta_{2,*}^{(m+j-i)}(s)\, d\mu. \tag{1.3.21}$$

where we take $\vartheta_{2,o}$ for $n/2$ odd, and $\vartheta_{2,e}$ for $n/2$ even. Here $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (1.2.7) specialised to the real projective space with a, b, c as in Table 4, and the scalars b_m^n are the coefficients defined in (A.4.2).

Proof. As before we begin by writing the multiplicity function in terms of a polynomial.

We set $X_k = 2k + (n-1)/2$ and for $n \geq 4$ write

$$M_k = \frac{2X_k}{(n-1)!}\prod_{j=1}^{n-2}(2k+j) = \frac{2X_k}{(n-1)!}\prod_{j=\frac{1}{2}}^{\frac{n-2}{2}-\frac{1}{2}}(X_k^2 - j^2)$$

$$= \frac{2X_k}{(n-1)!}\prod_{j=\frac{1}{2}}^{\frac{n-2}{2}-\frac{1}{2}}(X_k^2 - j^2)$$

$$= \sum_{m=0}^{\frac{n-2}{2}} \frac{2\mathsf{b}_m^n}{(n-1)!} X_k^{2m+1}, \tag{1.3.22}$$

(note that the last equation remains true for $n=2$). Next substituting for M_k

$$\operatorname{Tr} e^{s\Delta} = \sum_{k=0}^{\infty} M_k e^{-s\lambda_k^n} = \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n e^{s(n-1)^2/4}}{(n-1)!}\sum_{k=0}^{\infty} 2X_k (X_k)^{2m} e^{-s(X_k)^2}$$

$$= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n e^{s(n-1)^2/4}}{(n-1)!}\sum_{k=0}^{\infty} \frac{d^m}{ds^m} 2X_k e^{-s(X_k)^2}. \tag{1.3.23}$$

Here the sequence $2X_k$ for $k = 0, 1, 2, \ldots$ takes values $4j+1$ for $j = 0, 1, \ldots$ when $n/2$ is odd, and takes values $4j+3$ when $n/2$ is even.

(i) When $n/2$ is odd, we substitute $X_k \to 2j + 1/2$. We then extend the sums to $j = 0$ by noting the sum in (1.3.22) vanishes for $X_k = 1/2, 3/2, \ldots, (n-2)/2 - 1/2$.

$$\operatorname{Tr} e^{s\Delta} = \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n e^{s(n-1)^2/4}}{(n-1)!}\sum_{j=\frac{n-2}{4}}^{\infty} \frac{d^m}{ds^m}(4j+1)e^{-s(2j+1/2)^2}$$

$$= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n e^{s(n-1)^2/4}}{(n-1)!}\sum_{j=0}^{\infty} \frac{d^m}{ds^m}(4j+1)e^{-s(2j+1/2)^2}$$

$$= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n e^{s(n-1)^2/4}}{(n-1)!}\vartheta_{2,o}^{(m)}(s). \tag{1.3.24}$$

(ii) When $n/2$ is even, we use the substitution $X_k = 2j + 3/2$ and extend the sums as above to write

$$\operatorname{Tr} e^{s\Delta} = \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n e^{s(n-1)^2/4}}{(n-1)!}\sum_{j=\frac{n-4}{4}}^{\infty} \frac{d^m}{ds^m}(4j+3)e^{-s(2j+3/2)^2}$$

$$= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n e^{s(n-1)^2/4}}{(n-1)!}\sum_{j=0}^{\infty} \frac{d^m}{ds^m}(4j+3)e^{-s(2j+3/2)^2}$$

$$= \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n e^{s(n-1)^2/4}}{(n-1)!}\vartheta_{2,e}^{(m)}(s). \tag{1.3.25}$$

The heat trace formula can then be written as

$$\operatorname{Tr} e^{s\Delta} = \sum_{m=0}^{\frac{n-2}{2}} \frac{\mathsf{b}_m^n e^{s(n-1)^2/4}}{(n-1)!} \vartheta_{2,*}^{(m)}(s), \tag{1.3.26}$$

where we take $\vartheta_{2,o}$ when $n/2$ is odd, and $\vartheta_{2,e}$ when $n/2$ is even. Substituting (1.3.26) into (1.3.2) and differentiating gives the result. □

1.3.3 On the complex projective space $\mathbb{C}\mathbf{P}^n$

On $\mathbb{C}\mathbf{P}^n$ for n odd, we see that the theta function ϑ_2 naturally arises in the Maclaurin coefficients, and for n even we similarly see ϑ_3. This is in contrast to the earlier cases of $\mathbb{S}^n$ and $\mathbb{R}\mathbf{P}^n$, where we saw ϑ_1 and ϑ_2 instead.

Theorem 1.3.5 ($\mathbb{C}\mathbf{P}^n$, $n \geq 3$ odd)**.** *Let the function $\Phi(X)$ be of Laplace transform type as in (1.3.1). Then the Maclaurin coefficients $b_{2l}[\Phi]$ in odd dimensions $n \geq 3$ are given explicitly by*

$$b_0[\Phi] = \sum_{m=0}^{n-1} \frac{\mathsf{c}_m^n(-1)^m}{n!(n-1)!} \int_0^\infty \vartheta_2^{(m)}(s) f(s)\, d\mu, \tag{1.3.27}$$

where $d\mu = d\mu(s) = e^{sn^2/4} ds$, and for $l > 0$

$$b_{2l}[\Phi] = \sum_{m=0}^{n-1} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{c}_m^n(-1)^{j+m}\mathsf{c}_j^l}{n!(n-1)!} \binom{j}{i} \left(\frac{n}{2}\right)^{2i} \int_0^\infty f(s)\vartheta_2^{(m+j-i)}(s)\, d\mu, \tag{1.3.28}$$

where the coefficients $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (1.2.7) specialised to the complex projective space with a, b, c as in Table 4, and c_m^n are the scalars defined in (A.4.3).

Proof. Defining $X_k = k + n/2$, we can write the multiplicity function M_k of the eigenvalues of the Laplacian on $\mathbb{C}\mathbf{P}^n$ as

$$M_k = \frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2 = \frac{2X_k}{n!(n-1)!}\prod_{j=1}^{n-1}(k+j)^2. \tag{1.3.29}$$

We can write the product above in terms of X_k by noting pairs of terms of the form $(k+j)$ for $j = 1, 2, \ldots, n-1$ can be multiplied together to get terms of the form $(X_k^2 - j^2)$ for $j = 1/2, 3/2, \ldots \ldots (n-2)/2$. Applying (A.4.3), this leaves us with

$$M_k = \frac{2X_k}{n!(n-1)!} \prod_{j=\frac{1}{2}}^{\frac{n-3}{2}+\frac{1}{2}} (X_k^2 - j^2)^2 = \frac{2X_k}{n!(n-1)!} \sum_{m=0}^{n-1} \mathsf{c}_m^n X_k^{2m}, \tag{1.3.30}$$

Hence we can write the heat trace $\operatorname{Tr} e^{s\Delta}$ as

$$\begin{aligned}\operatorname{Tr} e^{s\Delta} = \sum_{k=0}^{\infty} M_k e^{-s\lambda_k^n} &= \sum_{m=0}^{n-1} \frac{2\mathsf{c}_m^n}{n!(n-1)!} \sum_{k=0}^{\infty} X_k^{2m+1} e^{-s(X_k^2 - n^2/4)} \\ &= \sum_{m=0}^{n-1} \frac{\mathsf{c}_m^n (-1)^m e^{sn^2/4}}{n!(n-1)!} \sum_{j=0}^{\infty} (2j+1) \frac{d^m}{ds^m} \left[e^{-s(j+1/2)^2} \right] \\ &= \sum_{m=0}^{n-1} \frac{\mathsf{c}_m^n (-1)^m e^{sn^2/4}}{n!(n-1)!} \vartheta_2^{(m)}(s). \end{aligned} \tag{1.3.31}$$

Here we have noted that (1.3.30) implies that the multiplicity vanishes when X_k is a half-integer in the range $1/2, 3/2, \ldots, (n-3)/2 + 1/2$. This lets us extend the sum in the third step above. Differentiating via Leibniz rule and substituting the result into (1.3.1) gives the solution. □

Theorem 1.3.6 ($\mathbb{C}\mathbf{P}^n$, $n \geq 2$ even)**.** *Take $\Phi(X)$ as in (1.3.1). Then for even $n \geq 2$, the Maclaurin coefficients $b_{2l}[\Phi]$ can be expressed as*

$$b_0[\Phi] = \sum_{m=0}^{n-2} \frac{(-1)^{m+1} \mathsf{d}_m^n}{n!(n-1)!} \int_0^{\infty} \vartheta_3^{(m+1)}(s) f(s)\, d\mu, \tag{1.3.32}$$

where $d\mu = d\mu(s) = e^{sn^2/4} ds$ and for $l > 0$,

$$b_{2l}[\Phi] = \sum_{m=0}^{n-2} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{\mathsf{d}_m^n (-1)^{j+m+1} \mathsf{c}_j^l}{n!(n-1)!} \binom{j}{i} \left(\frac{n}{2}\right)^{2i} \int_0^{\infty} f(s) \vartheta_3^{(m+j-i+1)}(s)\, d\mu, \tag{1.3.33}$$

where the coefficients $\mathsf{c}_j^l = \mathsf{c}_j^l(a,b,c)$ are the hypergeometric coefficients as in (1.2.7) specialised to the complex projective space with a, b, c as in Table 4, and d_m^n are the scalars defined in (A.4.4).

Proof. Taking $X_k = k + n/2$ as in the proof of the previous theorem, we can then write the multiplicity as

$$\begin{aligned} M_k = \frac{2k+n}{n!(n-1)!} \left[\frac{\Gamma(k+n)}{k!} \right]^2 &= \frac{2X_k}{n!(n-1)!} \prod_{j=1}^{n-1} (k+j)^2 \\ &= \frac{2X_k^3}{n!(n-1)!} \prod_{j=1}^{\frac{n-2}{2}} (X_k^2 - j^2)^2. \end{aligned} \tag{1.3.34}$$

In the last equality we have noted that each $(k+j)$ from the product over $j = 0, \ldots, n-1$ can be written as $(X_k \pm j)$ for $j = 1, \ldots, (n-2)/2$. Factoring out the $j = 0$ terms and taking the product of $(X_k - j)$ with $(X_k + j)$ gives the required result. Now applying (A.4.4) gives

$$M_k = \frac{2X_k^3}{n!(n-1)!} \sum_{m=0}^{n-2} \mathsf{d}_m^n X_k^{2m}, \tag{1.3.35}$$

Inserting this formulation into the heat trace, and using that the above sum vanishes for X_k an integer between 1 and $(n-2)/2$, we can apply the substitution $X_k \to p$ to write

$$\begin{aligned}\operatorname{Tr} e^{s\Delta} = \sum_{k=0}^{\infty} M_k e^{-s\lambda_k^n} &= \sum_{m=0}^{n-2} \frac{2\mathsf{d}_m^n}{n!(n-1)!} \sum_{p=1}^{\infty} p^{2m+3} e^{-s(p^2-n^2/4)} \\ &= \sum_{m=0}^{n-2} \frac{\mathsf{d}_m^n e^{sn^2/4}}{n!(n-1)!} (-1)^{m+1} \vartheta_3^{(m+1)}(s). \qquad (1.3.36)\end{aligned}$$

The result follows after differentiating this via Leibniz rule. □

The polynomial $\mathscr{R}_l$ on the $\mathbb{C}\mathbf{P}^n$, for $l = 0, 1, 2$. We now present explicit values of $b_{2l}[e^{s\Delta}] = \mathscr{R}_l(-d/ds)\operatorname{Tr} e^{s\Delta}$ on $\mathbb{C}\mathbf{P}^n$, for $n = 1, 2, 3, 4$. We have $\mathscr{R}_0(X) = 1$, $\mathscr{R}_1(X) = -X/2n$, $\mathscr{R}_2(X) = -X/4n + 3/(4n(n+1))X^2$. As a result we have:

- $\mathbb{C}\mathbf{P}^1$: $\Theta(s) = \operatorname{Tr} e^{s\Delta} = \vartheta_2 e^{s/4}$, $\mathscr{R}_1(-d/ds)\Theta(s) = [\vartheta_2'/2 + \vartheta_2/8]\, e^{s/4}$, $\mathscr{R}_2(-d/ds)\Theta(s) = [3\vartheta_2'' + 7/2\vartheta_2' + 11/16\vartheta_2]\, e^{s/4}/8$.
- $\mathbb{C}\mathbf{P}^2$: $\Theta(s) = \operatorname{Tr} e^{s\Delta} = -\vartheta_3' e^{s/4}/2$, $\mathscr{R}_1(-d/ds)\Theta(s) = -[\vartheta_3'' + \vartheta_3']\, e^{s}/8$, $\mathscr{R}_2(-d/ds)\Theta(s) = -[\vartheta_3''' + 3\vartheta_3'' + 2\vartheta_3']\, e^{s}/16$.
- $\mathbb{C}\mathbf{P}^3$: $\Theta(s) = \operatorname{Tr} e^{s\Delta} = [\vartheta_2''/4 + \vartheta_2'/8 + \vartheta_2/64]\, e^{9s/4}/3$, $\mathscr{R}_1(-d/ds)\Theta(s) = [\vartheta_2''' + 11/4\vartheta_2'' + 19/16\vartheta_2' + 9/64\vartheta_2]\, e^{9s/4}/72$, $\mathscr{R}_2(-d/ds)\Theta = [\vartheta_2'''' + 19/3\vartheta_2''' + 265/24\vartheta_2'' + 211/48\vartheta_2' + 129/256\vartheta_2] e^{9s/4}/192$.
- $\mathbb{C}\mathbf{P}^4$: $\Theta(s) = \operatorname{Tr} e^{s\Delta} = -[\vartheta_3''' + 2\vartheta_3'' + \vartheta_3']\, e^{4s}/144$, $\mathscr{R}_1(-d/ds)\Theta(s) = -[\vartheta_3''''/9 + 2/3\vartheta_3''' + \vartheta_3'' + 4/9\vartheta_3']\, e^{4s}/128$, $\mathscr{R}_2(-d/ds)\Theta(s) = -\left[3\vartheta_3^{(5)} + 35\vartheta_3'''' + 129\vartheta_3''' + 165\vartheta_3'' + 68\vartheta_3'\right] e^{4s}/11520$.

1.3.4 On the quaternionic projective space $\mathbb{H}\mathbf{P}^n$

Theorem 1.3.7 ($n \geq 1$, $\mathbb{H}\mathbf{P}^n$)**.** *Take $\Phi(X)$ to be a function of Laplace transform type as in (1.3.1). Then the Maclaurin spectral coefficients $b_{2l}[\Phi]$ are given by*

$$b_0[\Phi] = \sum_{m=0}^{2n-1} \frac{(-1)^m \mathsf{e}_m^n}{(2n-1)!(2n+1)!} \int_0^{\infty} f(s) \vartheta_2^{(m)}(s)\, d\mu, \qquad (1.3.37)$$

where $d\mu = d\mu(s) = e^{s(2n+1)^2/4} ds$, and for $l > 0$ we have

$$b_{2l}[\Phi] = \sum_{m=0}^{2n-1} \sum_{j=1}^{l} \sum_{i=0}^{j} \frac{(-1)^{m+j} \mathsf{e}_m^n \mathsf{c}_j^l}{(2n-1)!(2n+1)!} \binom{j}{i} \left(\frac{2n+1}{2}\right)^{2i} \int_0^{\infty} f(s) \vartheta_2^{(m+j-i)}(s)\, d\mu \qquad (1.3.38)$$

where the coefficients $\mathsf{c}_j^l = \mathsf{c}_j^l(a, b, c)$ are the hypergeometric coefficients as in (1.2.7) specialised to the quaternionic projective space with a, b, c as in Table 4, and e_m^n are the scalars defined in (A.4.5).

Proof. Setting $X_k = k+n+1/2$, we can write the multiplicity function M_k as

$$
\begin{aligned}
M_k &= \frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left(\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right)^2 \\
&= \frac{2X_k(k+2n)}{(2n-1)!(2n+1)!(k+1)}\prod_{j=1}^{2n-1}(k+j)^2 \\
&= \frac{2X_k}{(2n-1)!(2n+1)!}\left[X_k^2-(2n-1)^2/4\right]\prod_{j=1/2}^{n-3/2}(X_k^2-j^2)^2 \\
&= \frac{2X_k}{(2n-1)!(2n+1)!}\sum_{m=0}^{2n-1}\mathsf{e}_m^n X^{2m},
\end{aligned}
\tag{1.3.39}
$$

where we have used (A.4.5) to write this as a polynomial in X. Hence we can write the heat trace as

$$
\begin{aligned}
\operatorname{Tr}\, e^{s\Delta} = \sum_{k=0}^{\infty} M_k e^{-s\lambda_k^n} &= \sum_{m=0}^{2n-1}\frac{2\mathsf{e}_m^n e^{s(2n-1)^2/4}}{(2n-1)!(2n+1)!}\sum_{k=0}^{\infty}X_k^{2m+1}e^{-sX_k^2} \\
&= \sum_{m=0}^{2n-1}\frac{(-1)^m\mathsf{e}_m^n e^{s(2n-1)^2/4}}{(2n-1)!(2n+1)!}\frac{d^m}{ds^m}\sum_{k=0}^{\infty}2X_k e^{-sX_k^2}.
\end{aligned}
\tag{1.3.40}
$$

Here we can use the substitution $X_k \to p$, and the extend the sum to zero by noting that (1.3.39) shows that the multiplicity vanishes when $X_k - 1/2$ is an integer between 0 and $n-1$.

$$
\begin{aligned}
\operatorname{Tr}\, e^{s\Delta} &= \sum_{m=0}^{2n-1}\frac{(-1)^m\mathsf{e}_m^n e^{s(2n+1)^2/4}}{(2n-1)!(2n+1)!}\frac{d^m}{ds^m}\sum_{p=n+\frac{1}{2}}^{\infty}2pe^{-sp^2} \\
&= \sum_{m=0}^{2n-1}\frac{(-1)^m\mathsf{e}_m^n e^{s(2n+1)^2/4}}{(2n-1)!(2n+1)!}\vartheta_2^{(m)}(s).
\end{aligned}
\tag{1.3.41}
$$

Substituting this formulation of the trace into (1.3.2) and differentiating via Leibniz rule gives the result. □

The polynomial $\mathscr{R}_l$ on the $\mathbb{H}\mathbf{P}^n$, for $l = 0,1,2$. We now present explicit values of $\mathscr{R}_l(-d/ds)\operatorname{Tr} e^{s\Delta}$ on $\mathbb{H}\mathbf{P}^n$, for $n = 1,2$. We have $\mathscr{R}_0(X) = 1$, $\mathscr{R}_1(X) = -X/4n$, $\mathscr{R}_2(X) = -(n+2)/(8n^2-4n)X + 3/(16^2+8n)X^2$. As a result we can write:

- $\mathbb{H}\mathbf{P}^1$: $\Theta(s) = \operatorname{Tr} e^{s\Delta} = -\left[\vartheta_2' + \vartheta_2/4\right] e^{9s/4}/6$,
  $\mathscr{R}_1(-d/ds)\Theta(s) = -\left[\vartheta_2'' + 5/2\vartheta_2' + 9/16\vartheta_2\right] e^{9s/4}/24$,
  $\mathscr{R}_2(-d/ds)\Theta(s) = -\left[\vartheta_2'''/3 + 9/4\vartheta_2'' + 179/48\vartheta_2' + 51/64\vartheta_2\right] e^{9s/4}/16$.
- $\mathbb{H}\mathbf{P}^2$: $\Theta(s) = \operatorname{Tr} e^{s\Delta} = -\left[\vartheta_2''' + 11/4\vartheta_2'' + 19/16\vartheta_2' + 9/64\vartheta_2\right] e^{25s/4}/720$,
  $\mathscr{R}_1(-d/ds)\Theta(s) = -[16\vartheta_2'''' + 144\vartheta_2''' + 294\vartheta_2'' + 121\vartheta_2' + 225/16\vartheta_2]e^{25s/4}/92160$,

$$\mathscr{R}_2(-d/ds)\Theta(s) = -[3072\vartheta_2^{(5)} + 55040\vartheta_2'''' + 302976\vartheta_2''' + 526560\vartheta_2'' + 209852\vartheta_2' + 24075\vartheta_2]e^{25s/4}/58982400.$$

Remark 1.3.8. As a result of the well-known identifications $\mathbb{H}\mathbf{P}^1 \cong \mathbb{S}^4$ and $\mathbb{C}\mathbf{P}^1 \cong \mathbb{S}^2$ we have all the corresponding quantities calculated above agreeing in these special cases respectively.

1.4 Further extension to generalised hypergeometric functions

The hypergeometric function ${}_2F_1(z) = F(a, b; c; z)$ is in fact a special case of the generalised hypergeometric function ${}_pF_q(\mathbf{a}; \mathbf{b}; z)$ (see [47], pp. 182-198), where $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_q)$ with $a_i, b_j \in \mathbb{C}$ and none of the $b_1, \ldots, b_q$ are non-positive integers. The generalised hypergeometric function is defined as

$$ {}_pF_q(\mathbf{a}; \mathbf{b}; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{z^k}{k!}. \tag{1.4.1}$$

Defining the operator $\vartheta = z d/dz$, the generalised hypergeometric function $w = {}_pF_q(\mathbf{a}; \mathbf{b}; z)$ is a solution to the order $\max(p, q+1)$ differential equation

$$(\vartheta(\vartheta + b_1 - 1) \ldots (\vartheta + b_q - 1) - z(\vartheta + a_1) \ldots (\vartheta + a_p))\, w = 0, \tag{1.4.2}$$

although [73] gives examples where for certain values of p and q, a differential equation of lower order than expected can be satisfied by the generalised hypergeometric function.

It is straightforward to derive generalisations of the identities stated in Appendix A.1 for ${}_2F_1(a, b; c; z)$. Indeed, we have the differential identity

$$\begin{aligned} \frac{d^m}{dz^m} {}_pF_q(\mathbf{a}; \mathbf{b}; z) &= \frac{\prod_{i=1}^p (a_i)_m}{\prod_{j=1}^q (b_j)_m} \sum_{k=0}^{\infty} \frac{(a_1+m)_k (a_2+m)_k \ldots (a_p+m)_k}{(b_1+m)_k (b_2+m)_k \ldots (b_q+m)_k} \frac{z^k}{k!} \\ &= \frac{\prod_{i=1}^p (a_i)_m}{\prod_{j=1}^q (b_j)_m} {}_pF_q(\mathbf{a}+m; \mathbf{b}+m; z) \end{aligned} \tag{1.4.3}$$

where $(\mathbf{a} + m)$ denotes $(a_1 + m, \ldots, a_p + m)$.

We have more general convergence conditions for ${}_pF_q(\mathbf{a}; \mathbf{b}; z)$. Indeed, if any of $a_1, \ldots, a_p$ are non-positive integers then the series (1.4.1) is finite. This leads to a polynomial of degree $-a_l$, where a_l is the non-positive integer. Similarly if any of $b_1, \ldots, b_q$ are non-positive integers then the series diverges. If neither of these conditions are met, then we have the following three cases for convergence, depending on the relative values of p and q.

- If $p > q + 1$ then the series diverges everywhere except $z = 0$.

- If $p = q + 1$ (as with ${}_2F_1(z)$) then the series converges for $|z| < 1$ and diverges for $|z| > 1$. It may or may not diverge for $|z| = 1$.
- If $p < q + 1$ then the series converges for all z, and so ${}_pF_q$ is an entire function.

The statement of Lemma 1.2.1 becomes a product of p Pochhammer symbols,

$$(a_1)_j(a_2)_j \dots (a_p)_j = \prod_{k=0}^{j-1}\prod_{i=1}^{p}(a_i + k) = \sum_{l=0}^{j} \mathsf{d}_{l,j}(\mathbf{a}) \left[\prod_{i=1}^{p} a_i\right]^l . \tag{1.4.4}$$

In this case the scalars $\mathsf{d}_{l,j}(\mathbf{a})$ are the coefficients of the 'eigenvalue' $\prod_{i=1}^{p} a_i$ in the above polynomial. With the operator $\mathscr{L}_{\mathsf{P}} = \mathsf{P}_d(d/d\theta)$ as in (1.1.5), we can then state the following theorem.

Theorem 1.4.1. *Let $\mathscr{L}_{\mathsf{P}}$ be the differential operator as defined in (1.1.5). Then for $\mathbf{a} = (a_1, \dots, a_p)$ and $\mathbf{b} = (b_1, \dots, b_q)$, with each b_j not a non-positive integer, the generalised hypergeometric function $F(\mathbf{a}; \mathbf{b}; z)$ satisfies the differential identity*

$$\mathscr{L}_{\mathsf{P}}\left[{}_pF_q\right]\left(\mathbf{a}; \mathbf{b}; \frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0} = p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \sum_{j=1}^{m} \mathsf{c}_j^m(\mathbf{a}, \mathbf{b}) \left[-\prod_{i=1}^{p} a_i\right]^j , \tag{1.4.5}$$

where the scalars $\mathsf{c}_j^m(\mathbf{a}, \mathbf{b})$ are called the generalised hypergeometric coefficients, and are explicitly given by

$$\mathsf{c}_j^m(\mathbf{a}, \mathbf{b}) = (-1)^j \sum_{i=j}^{m} \frac{\mathsf{b}_i^m \mathsf{s}_j^i}{(-2)^i \prod_{k=0}^{i-1}\prod_{l=1}^{q}(b_l + k)}. \tag{1.4.6}$$

Here the scalars b_i^m are defined in (4.2.1), and the scalars $\mathsf{s}_j^i = \mathsf{s}_j^i(\mathbf{a})$ are defined in (A.1.9).

Chapter 2

A Differential-Spectral Identity on the Hypergeometric Function and Dual Polynomials

2.1 Statement of the Main Result

The Gauss hypergeometric function ${}_2F_1 = {}_2F_1(a, b; c; z)$ for $a, b, c \in \mathbb{C}$ unifies a large class of functions and orthogonal polynomials that play an important role in analysis and mathematical physics. One such class are the Jacobi functions and polynomials that have intimate links with zonal spherical functions and spectral projections on symmetric spaces of rank-one. In this chapter we establish a somewhat general differential-spectral identity on the hypergeometric function and discuss various refinements and applications of it to rank-one symmetric spaces of both compact and non-compact types.

To this end let $\mathsf{P}_N(X) = p_0 + p_1 X + \cdots + P_N X^N$ $(N \geq 2)$ and consider the differential operator $\mathscr{L}_{\mathsf{P}} = \mathsf{P}_N(d/d\theta) = p_0 + p_1 d/d\theta + \cdots + p_N d^N/d\theta^N$. In Theorem 2.1.1 we consider the action of $\mathscr{L}_{\mathsf{P}}$ on ${}_2F_1(a, b; c; \mathscr{E}(\theta))$ with $\mathscr{E} = \mathscr{E}(\theta)$ a sufficiently smooth function in a neighbourhood of the origin satisfying $\mathscr{E}(0) = 0$ and give a precise and explicit description of this action in terms of the hypergeometric parameter a, b, c and a closely related class of polynomials $\mathsf{H}_m = \mathsf{H}_m(X)$ $(m \geq 0)$ all whose coefficients are directly computed via certain values of the well known Bell polynomials $\mathsf{B}_{m,j} = \mathsf{B}_{m,j}(\mathbf{X})$ and the elementary symmetric polynomials $\mathsf{S} = \mathsf{S}_j(\mathbf{X})$. [1]

We then consider some special cases of the functions $\mathscr{E} = \mathscr{E}(\theta)$ and discuss the effect

[1]See the appendix at the end for the definition and a summary of the main properties of these families of polynomials. See also [4, 12, 23, 70, 71, 94] for related discussion and results.

of natural symmetries on the structure and certain cancellations of the polynomials $\mathsf{H}_m = \mathsf{H}_m(X)$ before specialising to the Jacobi and Gegenbauer functions and polynomials and discussing applications to analysis and spectral theory of invariant operators on rank-one symmetric spaces.

Theorem 2.1.1. *Let $\mathscr{E} = \mathscr{E}(\theta)$ be a smooth function in a neighbourhood of the origin with $\mathscr{E}(0) = 0$ and consider the function ${}_2F_1 = {}_2F_1(a, b; c; z)$ for $a, b, c \in \mathbb{C}$ and $c \notin \{0, -1, -2, \dots\}$. Then with $\mathscr{L}_{\mathsf{P}}$ as above we have the differential identity*

$$\mathscr{L}_{\mathsf{P}} \left[{}_2F_1(a, b; c; \mathscr{E}(\theta)) \right] \Big|_{\theta=0} = \sum_{m=0}^{N} p_m \mathsf{H}_m(-ab), \tag{2.1.1}$$

where $\mathsf{H}_m(X) = \mathsf{H}_m(a, b, c; \mathscr{E}; X)$ is an explicitly computable polynomial in X of degree (at most) m. In fact, $\mathsf{H}_0(X) = 1$ and for $m \geq 1$ we have

$$\mathsf{H}_m(X) = \sum_{\ell=1}^{m} \left\{ (-1)^\ell \sum_{j=\ell}^{m} \frac{\mathsf{b}_j^m[\mathscr{E}] \mathsf{s}_\ell^j}{(c)_j} \right\} X^\ell = \sum_{\ell=1}^{m} \mathsf{h}_\ell^m X^\ell. \tag{2.1.2}$$

Here $\mathsf{b}_j^m[\mathscr{E}] = \mathsf{B}_{m,j}(\mathscr{E}'(\theta), \mathscr{E}''(\theta), \dots, \mathscr{E}^{(m-j+1)}(\theta))|_{\theta=0}$ with $\mathsf{B}_{m,j}$ the incomplete Bell polynomials whilst $\mathsf{s}_\ell^j = \mathsf{S}_{j-\ell}(Y_0, \dots, Y_{j-1})$ where $Y_k = k(k + a + b)$ for $0 \leq k \leq j - 1$ and $\ell \leq j \leq m$ with S_n the nth elementary symmetric polynomials.

By inspection of the proof below, it is seen that the dependence of the scalars $\mathsf{b}_j^m[\mathscr{E}]$ $(1 \leq j \leq m)$ is on the function $\mathscr{E}$ and its successive derivatives at the origin, whilst the dependence of the scalars s_ℓ^j $(1 \leq \ell \leq j)$ is only on the sum $a+b$. This sees relevance later in Section 2.3 when considering dual polynomials $\mathscr{R}_l = \mathscr{R}_l(a, b, c; \mathscr{E}; X)$ and in Section 2.4 where we specialise ${}_2F_1(a, b; c; z)$ to the zonal spherical functions on rank-one symmetric spaces. There we will see that $a + b$ is independent of the index of these functions, and hence so are H_m.

Proof. Directly applying the operator $\mathscr{L}_{\mathsf{P}}$ to the hypergeometric function at $z = \mathscr{E}(\theta)$, we have

$$\mathscr{L}_{\mathsf{P}} \left[{}_2F_1(a, b; c; \mathscr{E}(\theta)) \right] \Big|_{\theta=0} = \sum_{m=0}^{N} p_m \frac{d^m}{d\theta^m} {}_2F_1(a, b; c; \mathscr{E}(\theta)) \Big|_{\theta=0}. \tag{2.1.3}$$

We can apply Faá di Bruno's theorem to the derivatives on the right-hand side above as

$$\begin{aligned}
&\frac{d^m}{d\theta^m} \left[{}_2F_1(a, b; c; \mathscr{E}(\theta)) \right] \Big|_{\theta=0} \\
&\quad = \sum_{j=1}^{m} \frac{d^j}{dz^j} {}_2F_1(a, b; c; z) \Big|_{z=0} \cdot \mathsf{B}_{m,j}(\mathscr{E}'(\theta), \mathscr{E}''(\theta), \dots, \mathscr{E}^{(m-j+1)}(\theta)) \Big|_{\theta=0}, \\
&\quad = \sum_{j=1}^{m} \mathsf{b}_j^m[\mathscr{E}] \frac{d^j}{dz^j} {}_2F_1(a, b; c; z) \Big|_{z=0},
\end{aligned} \tag{2.1.4}$$

where we have set the scalars $\mathsf{b}_j^m = \mathsf{B}_{m,j}(\mathscr{E}'(\theta), \mathscr{E}''(\theta), \ldots, \mathscr{E}^{(m-j+1)}(\theta))\big|_{\theta=0}$, with $\mathsf{B}_{m,j}$ the incomplete Bell polynomials (see Appendix A.5). Next we recall the well known recursive relation for derivatives of ${}_2F_1(a,b;c;z)$,

$$\frac{d^m}{dz^m}{}_2F_1(a,b;c;z) = \frac{(a)_m(b)_m}{(c)_m}{}_2F_1(a+m,b+m;c+m;z). \tag{2.1.5}$$

Applying this to the derivatives in (2.1.4), noting also that ${}_2F_1(a,b;c;0) = 1$ for any values of a, b, and c, we can write

$$\begin{aligned}\frac{d^m}{d\theta^m}\left[{}_2F_1(a,b;c;\mathscr{E}(\theta))\right]\Big|_{\theta=0} &= \sum_{j=1}^{m}\mathsf{b}_j^m[\mathscr{E}]\frac{(a)_j(b)_j}{(c)_j}\left[{}_2F_1(a+j,b+j;c+j;z)\right]\Big|_{z=0}\\ &= \sum_{j=1}^{m}\mathsf{b}_j^m[\mathscr{E}]\frac{(a)_j(b)_j}{(c)_j}. \end{aligned}\tag{2.1.6}$$

Here, $(x)_j = x(x+1)\ldots(x+j-1)$ is the Pochhammer symbol (in this case the rising factorial). Now expanding the product $(a)_j(b)_j$ in powers of $X = ab$ via the elementary symmetric polynomials $\mathsf{S}_\ell(Y_0,\ldots,Y_{j-1})$, with $Y_k = k(a+b+k)$, for $k = 0,\ldots,j-1$ (see Appendix A.5, and in particular (A.5.11)), we can write

$$\begin{aligned}(a)_j(b)_j &= \prod_{k=0}^{j-1}(a+k)(b+k) = \prod_{k=0}^{j-1}\left[ab + k(a+b+k)\right]\\ &= \prod_{k=0}^{j-1}(X+Y_k) = \sum_{\ell=0}^{j}\mathsf{S}_\ell(Y_0,\ldots,Y_{j-1})X^{j-\ell}\\ &= \sum_{\ell=0}^{j}\mathsf{S}_{j-\ell}(Y_0,\ldots,Y_{j-1})X^{\ell}\\ &= \sum_{\ell=0}^{j}\mathsf{s}_\ell^jX^\ell = \sum_{\ell=1}^{j}\mathsf{s}_\ell^jX^\ell. \end{aligned}\tag{2.1.7}$$

Note that here we have taken advantage of $\mathsf{s}_0^j = \mathsf{S}_j(Y_0,Y_1,\ldots,Y_{j-1}) = 0$ as a result of $Y_0 = 0$. Substituting this into (2.1.6) and isolating powers of $-ab$, we have

$$\begin{aligned}\frac{d^m}{d\theta^m}\left[{}_2F_1(a,b;c;\mathscr{E}(\theta))\right]\Big|_{\theta=0} &= \sum_{j=1}^{m}\frac{\mathsf{b}_j^m[\mathscr{E}]}{(c)_j}\sum_{\ell=1}^{j}\mathsf{s}_\ell^j[ab]^\ell\\ &= \sum_{\ell=1}^{m}[-ab]^\ell(-1)^\ell\sum_{j=\ell}^{m}\frac{\mathsf{b}_j^m[\mathscr{E}]\mathsf{s}_\ell^j}{(c)_j}\\ &= \mathsf{H}_m(-ab), \end{aligned}\tag{2.1.8}$$

where we have taken H_m as defined in (2.1.2). Returning to (2.1.3), we have shown

$$\mathscr{L}_{\mathsf{P}}\left[{}_2F_1(a,b;c;\mathscr{E}(\theta))\right]\Big|_{\theta=0} = \sum_{m=0}^{N}p_m\mathsf{H}_m(-ab). \tag{2.1.9}$$

This completes the proof. □

Using (2.1.2) in the above theorem, we can express the first few polynomials H_m in the sequence as $\mathsf{H}_0(X) = 1$ and

$$\mathsf{H}_1(X) = -\frac{\mathsf{b}_1^1[\mathscr{E}]}{c}X, \ \mathsf{H}_2(X) = -\left(\frac{\mathsf{b}_1^2[\mathscr{E}]}{c} + \frac{(a+b+1)\mathsf{b}_2^2[\mathscr{E}]}{c(c+1)}\right)X + \frac{\mathsf{b}_2^2[\mathscr{E}]}{c(c+1)}X^2,$$
$$\mathsf{H}_3(X) = -\left(\frac{\mathsf{b}_1^3[\mathscr{E}]}{c} + \frac{(a+b+1)\mathsf{b}_2^3[\mathscr{E}]}{c(c+1)} + 2\frac{((a+b)^2+3(a+b)+2)\mathsf{b}_3^3[\mathscr{E}]}{c(c+1)(c+2)}\right)X$$
$$+\left(\frac{\mathsf{b}_2^3[\mathscr{E}]}{c(c+1)} + \frac{(3(a+b)+5)\mathsf{b}_3^3[\mathscr{E}]}{c(c+1)(c+2)}\right)X^2 - \frac{\mathsf{b}_3^3[\mathscr{E}]}{c(c+1)(c+2)}X^3. \qquad (2.1.10)$$

2.2 Refinements of the Action Identity Resulting from Symmetries of $\mathscr{E} = \mathscr{E}(\theta)$

In this section we look more closely at the action identity (2.1.1) and the possible simplifications and cancellations resulting from the symmetries of $\mathscr{E} = \mathscr{E}(\theta)$. Of particular interest here is when $\mathscr{E} = \mathscr{E}(\theta)$ is an even or odd function of θ and the implication this may bear on the vanishing of certain coefficients $\mathsf{b}_j^m[\mathscr{E}]$ or the polynomials H_m.

The case $\mathscr{E} = \mathscr{E}(\theta)$ even. If $\mathscr{E}$ is an even function, then so is ${}_2F_1(a, b; c; \mathscr{E}(\theta))$ and hence any of its odd derivatives will vanish at the origin. Now referring to (2.1.1), this implies that in the action of $\mathscr{L}_\mathsf{P}$ on the left there is no contribution from the odd terms in P, whilst for the sum on the right, for odd m we have $\mathsf{H}_m \equiv 0$, and for m even, say $m = 2l$, we have no terms in H_m of order higher than $l = m/2$. To look more closely at this, let us proceed with the following two lemmas.

Lemma 2.2.1. *Let $m \geq 1$ be odd, $1 \leq j \leq m$ and let $\mathbf{X} = (X_1, \dots, X_{m-j+1})$, where $X_l = 0$ for all l odd. Then $\mathsf{B}_{m,j}(\mathbf{X}) = 0$.*

Proof. We show that every admissible $(k_1, \dots, k_{m-j+1})$ [*see* (A.5.2), (A.5.3)] has at least one $k_l \neq 0$ with l odd. It then follows from the assumption on $\mathbf{X}$ above that each of the summands in $\mathsf{B}_{m,j}(\mathbf{X})$ [*see* (A.5.2)] is zero. Towards this end assume the contrary and suppose there exists $(k_1, \dots, k_{m-j+1})$ admissible such that $k_l = 0$ for every l odd. Then it is easily seen that the second condition in (A.5.3) simplifies to $2k_2 + 4k_4 + \cdots = m$ which gives the desired contradiction as m is odd. □

Lemma 2.2.2. *Let $m \geq 2$ be even, $m \geq j > m/2$, and $\mathbf{X} = (0, X_2, \dots, X_{2m-j+1})$. Then $\mathsf{B}_{m,j}(\mathbf{X}) = 0$.*

Proof. Here it suffices to show that when $m \geq j > m/2$ then for every admissible $(k_1, \dots, k_{m-j+1})$ we have $k_1 \neq 0$. Again arguing indirectly assume there exists and admissible $(k_1, k_2, \dots, k_{m-j+1})$ with $k_1 = 0$. Then the first condition in (A.5.3) gives $\sum k_l = j > m/2$, whilst the second condition upon splitting the sum and taking advantage of $k_1 = 0$ gives

$$m = \sum_{l=2}^{m-j+1} lk_l = \sum_{l=2}^{m-j+1} (l-2)k_l + 2\sum_{l=2}^{m-j+1} k_l > m. \tag{2.2.1}$$

This however is a clear contradiction and so the assertion follows at once. □

From Lemma 2.2.1 it immediately follows that when $\mathscr{E}$ is an even function and m is odd then $\mathsf{b}_j^m[\mathscr{E}] = 0$ for all $1 \le j \le m$ as here

$$\begin{aligned}\mathsf{b}_j^m[\mathscr{E}] = \mathsf{B}_{\mathsf{m,j}}(\mathbf{X}) = 0, \qquad \mathbf{X} &= (\mathscr{E}'(\theta), \mathscr{E}''(\theta), \dots, \mathscr{E}^{(m-j+1)}(\theta))\big|_{\theta=0} \\ &= (0, \mathscr{E}''(0), 0, \mathscr{E}^{(4)}(0), \dots, \mathscr{E}^{(m-j+1)}(0)). \end{aligned} \tag{2.2.2}$$

Moreover, here, again by noting $\mathscr{E}'(0) = 0$ in particular, it follows from Lemma 2.2.2 that $\mathsf{b}_j^{2l}[\mathscr{E}] = 0$ when $j \ge l+1$. Thus, in this case, when m is odd, $\mathsf{H}_m \equiv 0$, and when m is even, say $m = 2l$, H_{2l} is a polynomial of degree $l = m/2$, more specifically,

$$\begin{aligned}\mathsf{H}_{2l}(X) &= \sum_{j=1}^{2l} (-1)^j \sum_{i=j}^{2l} \frac{\mathsf{b}_i^{2l}[\mathscr{E}]\mathsf{s}_j^i}{(c)_i} X^j \\ &= \sum_{j=1}^{l} (-1)^j \sum_{i=j}^{l} \frac{\mathsf{b}_i^{2l}[\mathscr{E}]\mathsf{s}_j^i}{(c)_i} X^j \\ &=: \mathscr{R}_l(X). \end{aligned} \tag{2.2.3}$$

As a result of this reduction in the even polynomials under the stated symmetry of $\mathscr{E}$, in the last identity and below we use the convenient and more suggestive notation $\mathscr{R}_l(X) = \mathsf{H}_{2l}(X)$. In this notation (2.1.1) can be rewritten as

$$\mathscr{L}_{\mathsf{P}}\left[{}_2F_1(a, b; c; \mathscr{E}(\theta))\right]\Big|_{\theta=0} = \sum_{l=0}^{\lfloor N \rfloor} p_{2l}\mathscr{R}_l(-ab), \tag{2.2.4}$$

For the sake of future applications we now consider two particular instances of even $\mathscr{E} = \mathscr{E}(\theta)$. These in turn correspond to the ordinary and hyperbolic cosine functions with immediate links to the spherical zonal functions on rank-one symmetric spaces of compact and non-compact types respectively.

- Let $\mathscr{E}_c(\theta) = (1 - \cos\theta)/2 = \sin^2(\theta/2)$. Then (2.2.4) holds with $\mathscr{R}_m$ as in (2.2.3), where the scalars $\mathsf{b}_j^m[\mathscr{E}_c] = 0$ for odd m and $j > m/2$, whilst for m even, say $m = 2l$, $\mathsf{b}_j^{2l}[\mathscr{E}_c] = (-2)^{-j}\mathsf{B}_{2l,j}(0, -1, 0, 1, \dots) = (-2)^{-j}\mathsf{b}_j^{2l}[\cos\theta]$ for $j = 1, \dots, l$, with $\mathsf{b}_j^{2l}[\cos\theta]$ as in (A.5.8). We also have the recursion

$$\mathsf{b}_j^{2l}\left[\mathscr{E}_c\right] = \begin{cases} (-1)^{l+1}/2 & \text{if } j=1 \\ 2^{-j}(-1)^{1-j}\left(j^2\mathsf{b}_j^{2(l-1)} + (2j-1)\mathsf{b}_{j-1}^{2(l-1)}\right) & \text{if } 1<j\le l \\ 0 & \text{if } j>l. \end{cases} \tag{2.2.5}$$

Here the first few polynomials $\mathscr{R}_l$ in the sequence can be seen to be $\mathscr{R}_0(X)=1$ and

$$\begin{aligned} \mathscr{R}_1(X) &= -\frac{1}{2c}X, \qquad \mathscr{R}_2(X) = \frac{3}{4(c)_2}X^2 - \frac{3(a+b)-2c+1}{4(c)_2}X, \\ \mathscr{R}_3(X) &= \frac{1}{8(c)_3}\Big[-15X + (45(a+b)-30c+15)X^2 \\ &\qquad - [30(a+b)(a+b+1-c)+4c^2-18c+8]X^3\Big]. \end{aligned} \tag{2.2.6}$$

- Let $\mathscr{E}_{nc}(\theta) = (1-\cosh\theta)/2 = \sinh^2(\theta/2)$. Then (2.2.4) holds with $\mathscr{R}_m$ as in (2.2.3), where $\mathsf{b}_j^m[\mathscr{E}_{nc}]=0$ for odd m and $j>m/2$, whilst for m even, say $m=2l$, $\mathsf{b}_j^{2l}[\mathscr{E}_{nc}] = 2^{-j}(-1)^{l-j}\mathsf{B}_{2l,j}(0,-1,0,1,\dots)$ for $j=1,\dots,l$, with $\mathsf{b}_j^{2l}[\cos\theta]$ as in (A.5.8). We also have the recursion

$$\mathsf{b}_j^{2l}\left[\mathscr{E}_{nc}\right] = \begin{cases} -1/2 & \text{if } j=1 \\ 2^{-j}(-1)^{l-j-1}\left(j^2\mathsf{b}_j^{2(l-1)} + (2j-1)\mathsf{b}_{j-1}^{2(l-1)}\right) & \text{if } 1<j\le l \\ 0 & \text{if } j>l. \end{cases} \tag{2.2.7}$$

Here the first few polynomials $\mathscr{R}_l$ in the sequence are given by $\mathscr{R}_0(X)=1$,

$$\begin{aligned} \mathscr{R}_1(X) &= \frac{1}{2c}X, \qquad \mathscr{R}_2(X) = \frac{3}{4(c)_2}X^2 - \frac{3(a+b)-2c+1}{4(c)_2}X, \\ \mathscr{R}_3(X) &= \frac{1}{8(c)_3}\Big[15X^3 - (45(a+b)-30c+15)X^2 \\ &\qquad + [30(a+b)(a+b+1-c)+4c^2-18c+8]X\Big]. \end{aligned} \tag{2.2.8}$$

The case $\mathscr{E}=\mathscr{E}(\theta)$ odd. Unlike the even case here we do not in general have the convenience of the scalars $\mathsf{b}_j^m[\mathscr{E}]$ vanishing for certain ranges of m and j as can be seen from the elementary examples $\mathscr{E}(\theta)=\sin\theta$ or $\mathscr{E}(\theta)=\sinh\theta$. Indeed here $\mathsf{b}_j^m[\mathscr{E}] = \mathsf{B}_{\mathsf{m},\mathsf{j}}(\mathbf{X})$ with $\mathbf{X} = (\mathscr{E}'(\theta), \mathscr{E}''(\theta), \dots, \mathscr{E}^{(m-j+1)}(\theta))\big|_{\theta=0}$, that is, $\mathbf{X} = (1, 0, \mp 1, 0, \mp 1, 0, \dots, \mathscr{E}^{(m-j+1)}(0))$. Thus in (2.1.1) all H_m are present.

2.3 The Jacobi and Gegenbauer Function Families $\mathscr{P}_\mu^{\alpha,\beta}$ and $\mathscr{C}_\mu^\nu$ and Dual Polynomials $\mathscr{R}_l$

In this section we specialise the differential action to the class of Jacobi and Gegenbauer functions $\mathscr{P}_\mu^{\alpha,\beta}$ and $\mathscr{C}_\mu^\nu$ respectively. In view of the close connection between these function families and dual symmetric spaces of compact and non-compact types a natural notion of dual polynomials $\mathscr{R}_l$ resulting from (2.1.1) is introduced and studied. For more on Jacobi functions and polynomials, Jacobi operator and transform see [62, 70, 99] and for related topics see [4, 12, 71, 94].

The Jacobi function. The Jacobi function $\mathscr{P}_\mu^{\alpha,\beta} = \mathscr{P}_\mu^{\alpha,\beta}(t)$, with $\alpha, \beta > -1$ and $\mu \in \mathbb{C}$, is defined via the hypergeometric function (see the appendix at the end) by setting

$$\mathscr{P}_\mu^{\alpha,\beta}(t) = {}_2F_1\left(-\mu, \mu + 2\rho; \alpha + 1; (1-t)/2\right), \quad \mu \in \mathbb{C}, \tag{2.3.1}$$

where $\rho = (\alpha + \beta + 1)/2$. Referring to (A.1.5) or upon directly differentiating, it is easily seen that the Jacobi function is a solution to the second order differential equation

$$\left[\mathscr{L}^{\alpha,\beta} + \mu(\mu + 2\rho)\right] y = 0, \tag{2.3.2}$$

where

$$\mathscr{L}^{\alpha,\beta} = (1-t^2)\frac{d^2}{dt^2} - [\alpha - \beta + (\alpha + \beta + 2)t]\frac{d}{dt}. \tag{2.3.3}$$

is the well known Jacobi operator. The particular interest in the Jacobi functions in this chapter stems from the fact that for suitable choices of z, and certain ranges of α, β and μ (see Table 3), they directly relate to the zonal spherical functions on rank-one symmetric spaces of both compact and non-compact types. We discuss this connection in more detail in Section 2.4 (see also the last part of the current section). Let us for now return to Theorem 2.1.1 and consider the action of the differential operator $\mathscr{L}_{\mathsf{P}}$ on Jacobi functions and prompted by its later application to symmetric spaces restrict this to the choices of even functions $\mathscr{E} = \mathscr{E}_c$ and $\mathscr{E} = \mathscr{E}_{nc}$ as introduced and discussed towards the end of Section 2.2.

- By taking $\mu = k$ a non-negative integer in (2.3.1) we obtain the normalised Jacobi polynomial $\mathscr{P}_k^{\alpha,\beta}(t) = {}_2F_1(-k, k + 2\rho; \alpha + 1; (1-t)/2)$ (with the normalisation $\mathscr{P}_k^{\alpha,\beta}(1) = 1$). Here (2.3.2) takes the form

$$(1-t^2)\frac{d^2y}{dt^2} - (\alpha - \beta + (\alpha + \beta + 2)t)\frac{dy}{dt} + k(k + \alpha + \beta + 1)y = 0. \tag{2.3.4}$$

For fixed α, β the Jacobi polynomials $y = \mathscr{P}_k^{\alpha,\beta}$ $(k \geq 0)$ form a complete orthogonal system of eigen-functions for the Jacobi operator $\mathscr{L}^{\alpha,\beta}$ in the weighted Hilbert space $L^2([-1,1];(1-t)^\alpha(1+t)^\beta dt)$. Theorem 2.1.1 with $\mathscr{E}_c(\theta) = (1-\cos\theta)/2$ now leads to the following result.

Proposition 2.3.1. *Let $\mathscr{L}_\mathsf{P}$ be as in Theorem 2.1.1 and $\mathscr{P}_k^{\alpha,\beta}$ with $k \geq 0$ and $\alpha, \beta > -1$ as above. Then*

$$\mathscr{L}_\mathsf{P}\left[\mathscr{P}_k^{\alpha,\beta}(\cos\theta)\right]\Big|_{\theta=0} = \sum_{l=0}^{\lfloor N/2 \rfloor} p_{2l}\mathscr{R}_l(\lambda_k^{\alpha,\beta}). \tag{2.3.5}$$

Here $\lambda_k^{\alpha,\beta} = k(k+\alpha+\beta+1) = -ab$ and $\mathscr{R}_l(X) = \mathscr{R}_l(a,b,c;(1-\cos\theta)/2;X)$ where $\mathscr{R}_l$ is defined as in (2.2.3) with the hypergeometric parameters $a = -k$, $b = k+2\rho$, and $c = \alpha + 1$.

- By taking $\mu = -(\rho + i\lambda)$, where $\rho = (\alpha+\beta+1)/2$ and $\lambda \in \mathbb{C}$, we obtain from (2.3.1) the Jacobi functions $\mathscr{P}_{-(\rho+i\lambda)}^{\alpha,\beta}(t) = {}_2F_1(\rho+i\lambda, \rho-i\lambda; \alpha+1; (1-t)/2)$.

$$(t^2-1)\frac{d^2y}{dt^2} + (\alpha - \beta + (\alpha+\beta+2)t)\frac{dy}{dt} + (\rho^2+\lambda^2)y = 0, \tag{2.3.6}$$

Theorem 2.1.1 with $\mathscr{E}_{nc}(\theta) = (1-\cosh\theta)/2$ now leads to the following result.

Proposition 2.3.2. *Let $\mathscr{L}_\mathsf{P}$ be as in Theorem 2.1.1 and $\mathscr{P}_{-(\rho+i\lambda)}^{\alpha,\beta}$ with $\alpha, \beta > -1$, $\rho = (\alpha+\beta+1)/2$ and $\lambda \in \mathbb{C}$ be as above. Then*

$$\mathscr{L}_\mathsf{P}\left[\mathscr{P}_{-(\rho+i\lambda)}^{\alpha,\beta}(\cosh\theta)\right]\Big|_{r=0} = \sum_{l=0}^{\lfloor N/2 \rfloor} p_{2l}\mathscr{R}_l^\star(\lambda^2+\rho^2). \tag{2.3.7}$$

Here $ab = \lambda^2 + \rho^2$ are the "*generalised*" eigenvalues of the Jacobi operator $\mathscr{L}^{\alpha,\beta}$ in (2.3.3). Note also that we have set $\mathscr{R}_l^\star(X) = \mathscr{R}_l(-X)$ with $\mathscr{R}_l(X) = \mathscr{R}_l(a,b,c;(1-\cosh\theta)/2;X)$ exactly as in (2.2.3). The hypergeometric parameters here are $a = \rho + i\lambda$, $b = \rho - i\lambda$, and $c = \alpha + 1$.

The Gegenbauer function. In the case $\alpha = \beta = \nu - 1/2$, the Jacobi function $\mathscr{P}_\mu^{\alpha,\beta}$ reduces to the Gegenbauer function $\mathscr{C}_\mu^\nu$. More specifically, here we have,

$$\mathscr{C}_\mu^\nu(t) = \mathscr{P}_\mu^{\nu-1/2,\nu-1/2}(t) = {}_2F_1\left(-\mu, \mu+2\nu; \nu+1/2; (1-t)/2\right). \tag{2.3.8}$$

It is easily seen that the Gegenbauer function $y = \mathscr{C}_\mu^\nu(t)$ arises as solution to the differential equation

$$(1-t^2)\frac{d^2y}{dt^2} - (2\nu+1)t\frac{dy}{dt} + \mu(\mu+\nu)y = 0. \tag{2.3.9}$$

These are linked to the zonal spherical functions on the sphere and the real projective space in the compact case as well as the real hyperbolic space in the non-compact case. Furthermore the action identity in Theorem 2.1.1 in this case can be formulated and described as follows.

- By taking $\mu = k$ a non-negative integer in (2.3.8) we obtain the normalised Gegenbauer polynomials $\mathscr{C}_k^\nu(t) = {}_2F_1(-k, k+2\nu; \nu+1/2; (1-t)/2)$. Hence with $\mathscr{E}(\theta) = (1-\cos\theta)/2$ we have

$$\mathscr{L}_{\mathsf{P}}\left[\mathscr{C}_k^\nu(\cos\theta)\right]\Big|_{\theta=0} = \sum_{l=0}^{\lfloor N/2\rfloor} p_{2l}\mathscr{R}_l(\lambda_k^\nu), \qquad \lambda_k^\nu = k(k+2\nu). \tag{2.3.10}$$

- By taking $\mu = -(\nu+i\lambda)$ and $\mathscr{E}(\theta) = (1-\cosh\theta)/2$ we have

$$\mathscr{L}_{\mathsf{P}}\left[\mathscr{C}_{-(\nu+i\lambda)}^\nu(\cosh\theta)\right]\Big|_{\theta=0} = \sum_{l=0}^{\lfloor N/2\rfloor} p_{2l}\mathscr{R}_l^\star(\lambda^2+\nu^2). \tag{2.3.11}$$

Relation between dual polynomials $\mathscr{R}_l$. As seen in Theorem 2.1.1, the form and structure of the polynomials H_m describing the action (2.1.1) depend on both the set of hypergeometric parameters a, b and c as well as the function $\mathscr{E} = \mathscr{E}(\theta)$. Moreover the dependence on the hypergeometric parameters is only through $a+b$ and c (*see* the comments after the statement of Theorem 2.1.1). Thus in particular the polynomials $\mathscr{R}_l^c(X) = \mathscr{R}_l(-k, k+2\rho, \alpha+1; (1-\cos\theta)/2; X)$ in (2.3.5) and $\mathscr{R}_l^{nc}(X) = \mathscr{R}(\rho+i\lambda, \rho-i\lambda, \alpha+1; (1-\cosh\theta)/2; X)$ in (2.3.7) are independent of the values k and λ respectively. For fixed α, β, and therefore fixed ρ, we refer to these polynomials as *dual* to one-another. The terminology is prompted by the fact that the Jacobi polynomials $\mathscr{P}_k^{\alpha,\beta}(\cos\theta)$ and functions $\mathscr{P}_{-\rho-i\lambda}^{\alpha,\beta}(\cosh\theta)$, for certain ranges of α, β, represent the zonal spherical functions on symmetric spaces that are *dual* to one-another.

More generally and motivated by the above discussion we say that a pair of polynomials $\mathscr{R}_l = \mathscr{R}_l(a, b, c; \mathscr{E}(\theta); X)$ are *dual* to one-another *iff* they have the same $a+b$ and c, but with the different $\mathscr{E} = \mathscr{E}_c(\theta)$ and $\mathscr{E} = \mathscr{E}_{nc}(\theta)$ respectively. Quite remarkably we now have the following result.

Theorem 2.3.3. *(Duality relation)* $\mathscr{R}_l^c(X) = (-1)^l\mathscr{R}_l^{nc}(X)$.

Proof. Since as stated the dependence of the coefficients of the polynomials $\mathscr{R}_l$ on the hypergeometric parameters is via $a+b$ and c the only structural difference between dual polynomials comes from the difference in the respective sequences $\mathsf{b}_j^m[\mathscr{E}_c]$ and $\mathsf{b}_j^m[\mathscr{E}_{nc}]$ (*see*

(2.1.2) and (2.2.3)). A close inspection of these coefficients (*see* (2.2.5) and (2.2.7)) and a reference to (2.2.3) gives the desired conclusion. $\square$

We return to Theorem 2.3.3 later when discussing applications to symmetric spaces. On passing we point out that throughout the subscripts in $\mathscr{E}_c$ and $\mathscr{E}_{nc}$ and the subsequent superscripts in $\mathscr{R}_l^c$ and $\mathscr{R}_l^{nc}$ are to highlight the relationships of these even functions and polynomials to *compact* and *non-compact* symmetric spaces respectively.

2.4 Applications to Rank-One Symmetric Spaces $\mathscr{X} = \mathbf{G}/\mathbf{H}$

Let $\mathscr{X} = \mathbf{G}/\mathbf{H}$ be a d-dimensional rank-one symmetric space and let $-\Delta_{\mathscr{X}}$ denote the Laplace-Beltrami operator in $L^2(\mathscr{X}; dv_g)$. A complete list of these spaces and their respective parameters α, β and ρ is given below (*see* Table 3). Now as a self-adjoint operator in the Hilbert space $L^2(\mathscr{X}; dv_g)$ the Laplacian $-\Delta_{\mathscr{X}}$ admits a resolution of the identity $(E_\lambda : \lambda \geq 0)$ such that for any function $F = F(X)$ in the Borel functional calculus of $-\Delta_{\mathscr{X}}$ we can write

$$\mathbf{F} = F(-\Delta_{\mathscr{X}}) = \int_0^\infty F(\lambda)\, dE_\lambda. \tag{2.4.1}$$

Now the Schwartz kernel of this operator, depending as to whether $\mathscr{X}$ is compact or non-compact, can be written by a spectral sum or integral respectively. More specifically in the former case we have

$$K_F(x,y) = \sum_{k=0}^\infty \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k)\mathscr{F}_k(\theta;\mathscr{X}), \qquad x,y \in \mathscr{X}, \tag{2.4.2}$$

where $\mathscr{F}_k = \mathscr{F}_k(\theta;\mathscr{X})$ are the zonal spherical functions on $\mathscr{X}$, $\lambda_k = \lambda_k(\mathscr{X})$ are the numerically distinct eigenvalues of $-\Delta_{\mathscr{X}}$, $M_k = M_k(\mathscr{X})$ is the (*finite*) dimension of the eigenspace of λ_k, $\theta = \theta(x,y)$ is the distance between $x, y \in \mathscr{X}$ and $\mathrm{Vol}(\mathscr{X})$ the volume of $\mathscr{M}$. We point out that in the simply-connected cases (see below) the associated multiplicity function $M_k(\mathscr{X})$ is given by

$$M_k(\mathscr{X}) = \frac{(\alpha+\beta+2k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\beta+1)\Gamma(k+d/2)}{\Gamma(k+1)\Gamma(\alpha+\beta+2)\Gamma(d/2)\Gamma(k+\beta+1)}. \tag{2.4.3}$$

Likewise the volume admits the formulation

$$\mathrm{Vol}(\mathscr{X}) = \frac{2^d\pi^{d/2}\Gamma(\beta+1)}{\Gamma(\beta+1+d/2)}. \tag{2.4.4}$$

On the other hand, for $\mathscr{X}$ non-compact, K_F is given by the spectral integral

$$K_F(x,y) = c_d \int_0^\infty F(\rho^2+\lambda^2)\mathscr{F}_\lambda(r;\mathscr{X})\mu(\lambda)\; d\lambda, \qquad x,y \in \mathscr{X}, \tag{2.4.5}$$

where $\mathscr{F}_\lambda = \mathscr{F}_\lambda(r;\mathscr{X})$ are the zonal spherical functions on $\mathscr{X}$, $\rho^2+\lambda^2$ are the generalised eigenvalues of $-\Delta_{\mathscr{X}}$, $\mu(\lambda)=\mu(\lambda;\mathscr{X})$ is the Plancherel measure on $\mathscr{X}$, $r=r(x,y)$ is the distance between $x,y\in\mathscr{X}$ and

$$c_d = \frac{2^{2\beta-1}\Gamma(\alpha+1)}{\pi^{\alpha+2}}. \tag{2.4.6}$$

Here the Plancherel measure is given by $\mu(\lambda) = [\mathsf{c}(\lambda)\mathsf{c}(-\lambda)]^{-1} = |\mathsf{c}(\lambda)|^{-2}$ where c is the Harish-Chandra function associated to $\mathscr{X}$ given by (at least for when $\mathrm{Im}(\lambda)<0$) by the formula

$$\mathsf{c}(\lambda) = \lim_{r\nearrow\infty} \mathscr{F}_\lambda(r;\mathscr{X})e^{(\rho-i\lambda)r} = \frac{4^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(2i\lambda)}{\Gamma(\rho+i\lambda)\Gamma(i\lambda+(\alpha+1-\beta)/2)}. \tag{2.4.7}$$

As for spherical functions we first note that the radial part of the Laplacian is given by the operator

$$[-\Delta_{\mathscr{X}}]_{rad} = -\frac{\partial^2}{\partial\theta^2} - \frac{A'(\theta)}{A(\theta)}\frac{\partial}{\partial\theta}, \tag{2.4.8}$$

where $A(\theta)$ is the area of the sphere of radius θ centered at the origin in $\mathscr{X}$. For spaces of compact type, $A(\theta)$ is given in (2.4.9), whilst for non-compact spaces see (2.4.17). For more on analysis on symmetric spaces relating to the discussion here see [28, 30, 62, 99] as well as [6, 27, 67, 70, 94]. See also Appendix A.7 for a description of the above spectral-geometric quantities for individual spaces.

Rank one symmetric spaces of compact type. These spaces are the sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbb{R}\mathbf{P}^n = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$, the quaternionic projective space $\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$ and finally the Cayley Plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}^4/\mathbf{Spin}(9)$. Note that all these spaces with the exception of $\mathbb{S}^1$ and $\mathbb{R}\mathbf{P}^n$, $n\geq 1$, are simply-connected. Now, referring to (2.4.8), in the compact case we have

$$A(\theta) = \omega_{d-1}\left[(\sin\theta/2)/2\right]^{2\alpha+1}(\cos\theta/2)^{2\beta+1}, \tag{2.4.9}$$

where $\omega_{d-1} = \mathrm{Vol}(\mathbb{S}^{d-1})$, and $\alpha,\beta>-1$ are real parameters associated to $\mathscr{X}$. This leads to the radial part of $-\Delta$ having the form

$$\begin{aligned}[\Delta_{\mathscr{X}}]_{rad} =& \frac{\partial^2}{\partial\theta^2} + \left[\frac{1}{2}(2\alpha+1)\cot\theta/2 - \frac{1}{2}(2\beta+1)\tan\theta/2\right]\frac{\partial}{\partial\theta}\\ =& \frac{\partial^2}{\partial\theta^2} + \left[(2\beta+1)\cot\theta + (\alpha-\beta)\cot\theta/2\right]\frac{\partial}{\partial\theta}.\end{aligned} \tag{2.4.10}$$

With a change of variables $t = \cos\theta$ one arrives at the Jacobi operator as encountered earlier

$$\mathscr{L}^{\alpha,\beta} = (1-t^2)\frac{d^2}{dt^2} - [\alpha-\beta+(\alpha+\beta+2)t]\frac{d}{dt}. \tag{2.4.11}$$

As a result it is seen that the zonal spherical functions $\mathscr{F}_k(\theta;\mathscr{X})$ $(k \geq 0)$ here can be expressed as

$$\mathscr{F}_k(\theta;\mathscr{X}) = \mathscr{P}_k^{\alpha,\beta}(\cos\theta) = {}_2F_1\left(-k, k+\alpha+\beta+1; \alpha+1; \sin^2(\theta/2)\right). \tag{2.4.12}$$

In light of this description of the zonal spherical functions, the spectral sum (2.4.2), with a slight abuse of notation, can be rewritten as

$$K_F(\theta) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k)\mathscr{P}_k^{\alpha,\beta}(\cos\theta). \tag{2.4.13}$$

Now proceeding formally it is seen that the Maclaurin expansion of the kernel K_F can be written as

$$\sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!}\frac{\partial^{2l}}{\partial\theta^{2l}} K_F(\theta)\Big|_{\theta=0} = \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} b_{2l}[K_F;\mathscr{X}]. \tag{2.4.14}$$

Proposition 2.4.1. *The Maclaurin spectral coefficients $b_{2l}[K_F;\mathscr{X}]$ for $l \geq 0$ are given by*

$$b_{2l}[K_F;\mathscr{X}] = \frac{1}{\mathrm{Vol}(\mathscr{X})}\mathrm{Tr}[F\mathscr{R}_l](-\Delta_{\mathscr{X}}), \quad l \geq 0, \tag{2.4.15}$$

where $\mathscr{R}_l = \mathscr{R}_l(-k, k+2\rho, \alpha+1; \mathscr{E}_c(\theta); X)$.

Proof. Upon referring to (2.4.14) it suffices to note that

$$\begin{aligned} b_{2l}[K_F;\mathscr{X}] = \frac{\partial^{2l}}{\partial\theta^{2l}} K_F(\theta)\Big|_{\theta=0} &= \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k^{\alpha,\beta}) \frac{\partial^{2l}}{\partial\theta^{2l}} \mathscr{P}_k^{\alpha,\beta}(\cos\theta)\Big|_{\theta=0} \\ &= \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k^{\alpha,\beta})\mathscr{R}_l(\lambda_k^{\alpha,\beta}) \end{aligned} \tag{2.4.16}$$

where in passing to the second line we have used Theorem 2.1.1 in the form (2.3.5) with $\mathsf{P}(X) = X^{2l}$. This completes the proof. $\square$

Rank one symmetric space of non-compact type. These spaces are the real hyperbolic space $\mathbb{R}\mathbf{H}^n = \mathbf{SO}_0(n,1)/\mathbf{SO}(n)$, the complex hyperbolic space $\mathbb{C}\mathbf{H}^n = \mathbf{SU}(n,1)/\mathbf{S}(\mathbf{U}(n)\times \mathbf{U}(1))$, the quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n = \mathbf{Sp}(n,1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$, and the Cayley plane $\mathbf{H}^2(\mathrm{Cay}) = \mathbf{F}_\star^4/\mathbf{Spin}(9)$. Now again, referring to (2.4.8), in the non-compact case we have

$$A(\theta) = \omega_{d-1}(\sinh\theta)^{2\alpha+1}(\cosh\theta)^{2\beta+1}. \tag{2.4.17}$$

where $\alpha, \beta > -1$ are real parameters associated with $\mathscr{X}$ as in Table 3. Therefore the radial part of the Laplacian is seen to have the form

$$[-\Delta_{\mathscr{X}}]_{rad} = -\frac{\partial^2}{\partial r^2} - [(2\alpha+1)\coth r + (2\beta+1)\tanh r]\,\frac{\partial}{\partial r}. \tag{2.4.18}$$

The change of variables $t = \cosh r$ leads again to the Jacobi operator $\mathscr{L}^{\alpha,\beta}$. Hence upon recalling the equation

$$(t^2-1)\frac{d^2y}{dt^2} + (\alpha - \beta + (\alpha+\beta+2)t)\frac{dy}{dt} + (\rho^2+\lambda^2)y = 0, \tag{2.4.19}$$

it follows that the zonal spherical functions $\mathscr{F}_\lambda(r;\mathscr{X})$ here can be expressed as

$$\begin{aligned} \mathscr{F}_\lambda(r;\mathscr{X}) &= \mathscr{P}^{\alpha,\beta}_{-(\rho+i\lambda)}(\cosh r) \\ &= {}_2F_1\left(\rho+i\lambda, \rho-i\lambda; \alpha+1; -\sinh^2(r/2)\right). \end{aligned} \tag{2.4.20}$$

With the aid of this description of the zonal spherical functions, the spectral integral (2.4.5), with a slight abuse of notation, can be rewritten as

$$K_F(r) = c_d \int_0^\infty F(\rho^2+\lambda^2)\mathscr{P}^{\alpha,\beta}_{-(\rho+i\lambda)}(\cosh r)\mu(\lambda)\,d\lambda. \tag{2.4.21}$$

Now proceeding formally it is seen that the Maclaurin expansion of the kernel K_F can be written as

$$\sum_{l=0}^\infty \frac{r^{2l}}{(2l)!}\frac{\partial^{2l}}{\partial r^{2l}}K_F(r)\Big|_{r=0} = \sum_{l=0}^\infty \frac{r^{2l}}{(2l)!} b_{2l}[K_F;\mathscr{X}]. \tag{2.4.22}$$

Proposition 2.4.2. *The Maclaurin spectral coefficients $b_{2l}[K_F;\mathscr{X}]$ for $l \geq 0$ are given by*

$$b_{2l}[K_F;\mathscr{X}] = \mathrm{Tr}[F\mathscr{R}^\star_l](-\Delta_{\mathscr{X}}), \quad l \geq 0, \tag{2.4.23}$$

where $\mathscr{R}_l = \mathscr{R}_l(\rho+i\lambda, \rho-i\lambda, \alpha+1; \mathscr{E}_{nc}(r); X)$, $\mathscr{R}^\star_l(X) = \mathscr{R}_l(-X)$ and the trace on the right is associated with the kernel

$$K_{F\mathscr{R}^\star_l}(r) = c_d \int_0^\infty F(\rho^2+\lambda^2)\mathscr{R}^\star_l(\rho^2+\lambda^2)\mathscr{P}^{\alpha,\beta}_{-(\rho+i\lambda)}(\cosh r)\mu(\lambda)\,d\lambda. \tag{2.4.24}$$

Proof. Upon referring to (2.4.22) it suffices to note that

$$\begin{aligned} b_{2l}[K_F;\mathscr{X}] &= \frac{\partial^{2l}}{\partial r^{2l}}K_F(r)\Big|_{r=0} \\ &= c_d\int_0^\infty F(\rho^2+\lambda^2)\frac{\partial^{2l}}{\partial r^{2l}}\mathscr{P}^{\alpha,\beta}_{-(\rho+i\lambda)}(\cosh r)\Big|_{r=0}\mu(\lambda)\,d\lambda \\ &= c_d\int_0^\infty F(\rho^2+\lambda^2)\mathscr{R}^\star_l(\rho^2+\lambda^2)\mu(\lambda)\,d\lambda, \end{aligned} \tag{2.4.25}$$

where in passing to the third line we have used Theorem 2.1.1 in the form (2.3.7) with $\mathsf{P}(X) = X^{2l}$. The proof is thus complete. $\square$

Chapter 3

A Representation Formula for a PDO Action on Even Compositions of the Hypergeometric Function and its Generalisations

3.1 Introduction and statement of the result

The Gauss hypergeometric function ${}_2F_1 = {}_2F_1(a,b;c;z)$ is defined for $|z| < 1$ and $a,b,c \in \mathbb{C}$ with c not a non-positive integer (i.e., $c \neq 0, -1, -2, \dots$) by the series

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}. \tag{3.1.1}$$

Here $(x)_k = x(x+1)\dots(x+k-1)$ for $k \geq 1$ and $(x)_0 = 1$ is the rising factorial. The series converges absolutely and locally uniformly inside the unit disk and can be seen to satisfy the second order differential equation, [1]

$$ z(1-z)\frac{d^2w}{dz^2}(z) + (c - (a+b+1)\,z)\,\frac{dw}{dz}(z) - abw(z) = 0. \tag{3.1.2}$$

Let $\mathsf{P} = \sum p_\gamma \mathbf{X}^\gamma$ be a polynomial in $\mathbf{X} = (X_1, \dots, X_q)$ with coefficients p_γ. Here $\mathbf{X}^\gamma = X_1^{\gamma_1} \dots X_q^{\gamma_q}$ and the sum extends over all multi-indices $\gamma = (\gamma_1, \dots, \gamma_q)$ of non-negative integers satisfying $|\gamma| = \gamma_1 + \cdots + \gamma_q \leq N$ for some fixed $N \in \mathbb{N}$. We associate with P the

[1] For further background see [4, 12, 93].

partial differential operator $\mathscr{L}_{\mathsf{P}}$ in q variables

$$\mathscr{L}_{\mathsf{P}} = \mathsf{P}(\partial) = p_0 + \sum_{|\gamma|=1}^{N} p_\gamma \frac{\partial^{|\gamma|}}{\partial X_1^{\gamma_1} \dots \partial X_q^{\gamma_q}}. \tag{3.1.3}$$

Setting $z = (1-t)/2$ with $t = \mathscr{G}(\mathbf{X}) = g(X_1)\dots g(X_q)$ where $g = g(X)$ is a smooth *even* function of the single variable X in a neighbourhood of the origin $X = 0$ satisfying $g(0) = 1$ and as before $\mathbf{X} = (X_1, \dots, X_q)$, we set ourselves the task of computing the action of $\mathscr{L}_{\mathsf{P}}$ on this multivariable function at $\mathbf{X} = 0$. We prove that for an explicitly computable set of scalars $\mathsf{c}_j^\gamma = \mathsf{c}_j^\gamma(a, b, c; \mathscr{G})$ – depending on the function g and the hypergeometric parameters a, b, c (this dependence will be presented below) – this action can be completely described by the formula (*see* Theorem 3.3.1)

$$\mathscr{L}_{\mathsf{P}} \left[{}_2F_1\left(a, b; c; [1 - \mathscr{G}(\mathbf{X})]/2\right)\right] \Big|_{\mathbf{X}=0} = p_0 + \sum_{|\gamma|=1}^{\lfloor N/2 \rfloor} p_{2\gamma} \sum_{j=1}^{|\gamma|} \mathsf{c}_j^\gamma \left(-ab\right)^j. \tag{3.1.4}$$

Introducing the polynomials $\mathscr{R}_\gamma(X) = \mathscr{R}_\gamma(a, b, c; \mathscr{G}; X) = \sum_{j=1}^{|\gamma|} \mathsf{c}_j^\gamma X^j$ of the single variable X and of degree $|\gamma|$ (with $\mathscr{R}_0(X) \equiv 1$) we can rewrite this as

$$\mathscr{L}_{\mathsf{P}} \left[{}_2F_1\left(a, b; c; [1 - \mathscr{G}(\mathbf{X})]/2\right)\right] \Big|_{\mathbf{X}=0} = \sum_{|\gamma|=0}^{\lfloor N/2 \rfloor} p_{2\gamma} \mathscr{R}_\gamma(-ab). \tag{3.1.5}$$

The choice of $z = [1 - g(X_1)\dots g(X_q)]/2$ is prompted by applications to spectral theory of Riemannian symmetric spaces of rank one. Here the zonal spherical functions can be described in terms of the Jacobi and Gegenbauer functions and polynomials (all being particular instances of the hypergeometric function) and with the choices of $g(X) = \cos X$ and $g(X) = \cosh X$ for the compact versus non-compact spaces respectively. Such applications serve as a main motivation for this work. For related results *see* [4, 10, 12, 23, 41, 53, 70, 71, 99] and for further reading and background *see* [36, 62, 64, 89, 94].

Let us finish off this introduction by giving a brief plan of the chapter. The above result is proved in Section 3.3 after going through some auxiliary results, generalities and essential notation in Section 3.2. Indeed as will be seen later the elementary symmetric polynomials and the exponential or incomplete Bell polynomials enter the scene by way of giving an explicit description of the polynomials $\mathscr{R}_\gamma = \mathscr{R}_\gamma(a, b, c; \mathscr{G}; X)$ and their coefficients $\mathsf{c}_j^\gamma = \mathsf{c}_j^\gamma(a, b, c; \mathscr{G})$ (*see* Theorem 3.3.1). For this natural reason we quickly go through these and prove an interesting extension of the Faá di Bruno formula to a multivariable context (*see* in particular Theorem 3.2.2 and the subsequent examples). In Section 3.4 we specialise the result to the two families of Jacobi and Gegenbauer functions and polynomials which are intimately tied with the zonal spherical functions on Riemannian symmetric spaces of rank one. In the last two sections we discuss further extensions of the main result, specifically, in Section 3.5 we establish the explicit form of the action identity for matrix hypergeometric functions, that is, the hypergeometric function ${}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z)$ with matrix parameters and in Section 3.6 for the generalised hypergeometric function ${}_pF_r(\mathbf{a};\mathbf{b};z)$ with vector parameters. These results can be seen as providing far reaching generalisations of certain analytic and spectral objects and identities to contexts way beyond their natural habitat of Riemannian symmetric spaces, the Laplacian spectrum and the zonal spherical functions and as such have potential for many further interesting applications.

3.2 The combinatorics of the scalars $\mathsf{b}_j^\gamma[\mathscr{G}(\mathbf{X})]$

To facilitate and formalise the application of the differential operator $\mathscr{L}_\mathsf{P}$ to the hypergeometric function we start by taking a closer look at the monomial action $\mathsf{P}(\partial)=\partial^\gamma$ at $\mathbf{X}=0$, specifically,

$$\partial^\gamma f(\mathscr{G}(\mathbf{X}))\Big|_{\mathbf{X}=0} := \frac{\partial^{|\gamma|}}{\partial^{\gamma_1}X_1\dots\partial^{\gamma_q}X_q} f(\mathscr{G}(\mathbf{X}))\Big|_{\mathbf{X}=0}. \tag{3.2.1}$$

Here $\mathscr{G}(\mathbf{X})=g(X_1)\dots g(X_q)$ where $g=g(X)$ is an even smooth function near $X=0$ satisfying $g(0)=1$ and $f=f(t)$ is a smooth function near $t=1$. Before proceeding further we recall the Faá di Bruno formula, a generalisation of the chain rule for derivatives (*cf.*, e.g., [36] pp. 137-9). This formula asserts that for sufficiently smooth functions f,g the mth order derivative of the composition $h(X)=f(g(X))$ can be written

$$\frac{d^m h}{dX^m}(X)=\sum_{j=1}^m f^{(j)}(g(X))\cdot\mathsf{B}_{m,j}(g'(X),g''(\theta),\dots,g^{(m-j+1)}(X)). \tag{3.2.2}$$

Here $\mathsf{B}_{m,j}=\mathsf{B}_{m,j}(\mathbf{Y})$ with $1\le j\le m$ and $\mathbf{Y}=(Y_1,Y_2,\dots,Y_{m-j+1})$ are the *incomplete* Bell polynomials, defined by

$$\mathsf{B}_{m,j}(\mathbf{Y})=\sum_{\mathscr{K}}\frac{m!}{k_1!k_2!\dots k_{m-j+1}!}\left(\frac{Y_1}{1!}\right)^{k_1}\left(\frac{Y_2}{2!}\right)^{k_2}\cdots\left(\frac{Y_{m-j+1}}{(m-j+1)!}\right)^{k_{m-j+1}} \tag{3.2.3}$$

where the sum is taken over the set $\mathscr{K}$ of all *admissible* $(k_1,k_2,\dots,k_{m-j+1})$, that is, finite sequences of non-negative integers $k_1,\dots,k_{m-j+1}$ such that

$$\sum_{l=1}^{m-j+1}k_l=j,\qquad \sum_{l=1}^{m-j+1}lk_l=m. \tag{3.2.4}$$

The incomplete Bell polynomials satisfy the generating function relation (for each fixed $j\ge 0$) (*cf.*, e.g., [36], pp. 133: [3a], [3a'])

$$\frac{1}{j!}\left[\sum_{l=1}^\infty Y_l\frac{t^l}{l!}\right]^j=\sum_{n=j}^\infty\mathsf{B}_{n,j}(Y_1,Y_2,\dots,Y_{n+j-1})\frac{t^n}{n!}. \tag{3.2.5}$$

It is also a straightforward consequence of the above that the incomplete Bell polynomials satisfy the scaling identity

$$\begin{aligned}\alpha^j \beta^m \mathsf{B}_{m,j}(\mathbf{Y}) &= \alpha^j \beta^m \mathsf{B}_{m,j}(Y_1, Y_2, \dots, Y_{m-j+1}) \\ &= \mathsf{B}_{m,j}(\alpha\beta Y_1, \alpha\beta^2 Y_2, \dots, \alpha\beta^{m-j+1} Y_{m-j+1}).\end{aligned} \tag{3.2.6}$$

Even composition in single and multivariables. We assume throughout this chapter that $g = g(X)$ is a smooth *even* function in a neighbourhood of $X = 0$ normalised by $g(0) = 1$. The task is now to look at the action $\partial^\gamma f(\mathscr{G}(\mathbf{X}))$ as in (3.2.1) by invoking (A.5.1) in the single and multivariable cases respectively.

The case $\mathscr{G}(\mathbf{X}) = g(X)$. In the single-variable case the differential action can be simplified greatly. We observe firstly that the derivative of an even function is odd and the derivative of an odd function is even and secondly that any odd function vanishes at the origin. Therefore all odd order derivatives of $g = g(X)$ and $h = f(g(X))$ must vanish at the origin and so in discussing (3.2.1) we can restrict to derivatives of even order only. As a matter of fact the differential identity (A.5.1) (with $2m$ replacing m) can here be shown to reduce to

$$\frac{d^{2m}}{dX^{2m}} f\left(g(X)\right)\Big|_{X=0} = \sum_{j=1}^{m} \mathsf{b}_j^m[g] \frac{d^j}{dt^j} f(t)\Big|_{t=1}, \tag{3.2.7}$$

where via (A.5.1) we have defined a set of scalars $\mathsf{b}_j^m[g]$ given for $j = 1, \dots, m$ by the values of the Bell polynomials $\mathsf{B}_{2m,j}$ at the vector of consecutive derivatives of g at $X = 0$, i.e., $\mathsf{b}_j^m[g] = \mathsf{B}_{2m,j}(0, g''(X), 0, g^{(4)}(X), \dots, g^{(2m-j+1)}(X))|_{X=0}$.

Remark 3.2.1. *For the sake of future applications to the hypergeometric function in Section 3.4, writing $f(t) = F(z)$ with $z = (1-t)/2$, we have the identity*

$$\begin{aligned}\frac{d^{2m}}{dX^{2m}} F\left(\frac{1-g(X)}{2}\right)\Bigg|_{X=0} &= \frac{d^{2m}}{dX^{2m}} f(g(X))\Bigg|_{X=0} = \sum_{j=1}^{m} \mathsf{b}_j^m \frac{d^j f}{dt^j}\Big|_{t=1} \\ &= \sum_{j=1}^{m} \mathsf{b}_j^m \frac{d^j}{dt^j} F\left(\frac{1-t}{2}\right)\Big|_{t=1} = \sum_{j=1}^{m} \frac{\mathsf{b}_j^m}{(-2)^j} \frac{d^j F}{dz^j}\Big|_{z=0},\end{aligned} \tag{3.2.8}$$

where $\mathsf{b}_j^m = \mathsf{b}_j^m[g]$ [compare also with (3.2.7)]. As the derivatives of $(1-g(X))/2$ are of the form $-g^{(j)}(X)/2$, the latter can be alternatively visualised, using (A.5.1) and (3.2.6), by noting

$$\begin{aligned}\mathsf{b}_j^m[(1-g)/2] =& \mathsf{B}_{2m,j}(0, -g''(X)/2, 0, -g^{(4)}(X)/2, \dots, -g^{(2m-j+1)}(X)/2)\Big|_{X=0} \\ =&(-2)^{-j} \mathsf{B}_{2m,j}(0, g''(X), 0, g^{(4)}(X), \dots, g^{(2m-j+1)}(X))\Big|_{X=0} \\ =&(-2)^{-j} \mathsf{b}_j^m[g].\end{aligned} \tag{3.2.9}$$

The case $\mathscr{G}(\mathbf{X}) = g(X_1)\dots g(X_q)$. We now aim to extend the identity (3.2.7) to our multivariable context and prove that for given multi-index $\gamma = (\gamma_1, \dots, \gamma_q)$ there is a computable set of scalars $\mathsf{b}_j^\gamma = \mathsf{b}_j^\gamma[\mathscr{G}(\mathbf{X})]$, for $j = 1, \dots, |\gamma|$, such that

$$\partial^{2\gamma} f(g(X_1)\dots g(X_q))\Big|_{\mathbf{X}=0} = \sum_{j=1}^{|\gamma|} \mathsf{b}_j^\gamma[\mathscr{G}(\mathbf{X})] \frac{d^j}{dt^j} f(t)\Big|_{t=1}. \tag{3.2.10}$$

Since $g = g(X)$ is even any of its odd order derivatives are odd and hence zero at $X = 0$. Applying this to the product $g(X_1)\dots g(X_q)$ it is plain that any odd order derivative of $f(g(X_1)\dots g(X_q))$ in any of the variable $X_1, \dots, X_q$ must vanish when we set each $X_i = 0$. Hence we can ignore any γ with odd elements, or equivalently, just consider those multi-indices of the form $2\gamma = (2\gamma_1, \dots, 2\gamma_q)$. To begin we apply a single variable derivative to $f(\mathscr{G}(\mathbf{X})) = f(g(X_1)\dots g(X_q))$. For a positive integer γ_1, we write this as

$$\begin{aligned}\frac{\partial^{2\gamma_1}}{\partial X_1^{2\gamma_1}} f(g(X_1)\dots g(X_q))\Big|_{X_i=0} &= \frac{\partial^{2\gamma_1}}{\partial X_1^{2\gamma_1}} f(g(X_1))\Big|_{X_1=0} \\ &= \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1}[g(X_1)] \frac{d^j}{dt^j} f(t)\Big|_{t=1}\end{aligned} \tag{3.2.11}$$

where we notice that for the variables that aren't differentiated (that is, all of the remaining $X_2, \dots, X_q$ in this case), we can freely set $X_i = 0$ in the first step so that $g(X_i) = 1$. Somewhat these extra variables have no effect on the outcome here. Next we try differentiating with respect to two variables, say X_1 and X_2. To this end we fix a multi-index $\gamma = (\gamma_1, \gamma_2)$, and look to simplify the evaluation of the derivative

$$\frac{\partial^{2\gamma_1}}{\partial X_1^{2\gamma_1}} \frac{\partial^{2\gamma_2}}{\partial X_2^{2\gamma_2}} f(g(X_1)\dots g(X_q))\Big|_{\mathbf{X}=0} = \partial^{2\gamma} f(g(X_1)\dots g(X_q))\Big|_{\mathbf{X}=0}. \tag{3.2.12}$$

As before, we can set the irrelevant X_j's to zero immediately, and then perform the derivative in X_1 first via the single-variable formula (3.2.11) as

$$\begin{aligned}\partial^{2\gamma} f(g(X_1)\dots g(X_q))\Big|_{\mathbf{X}=0} &= \frac{\partial^{2\gamma_1}}{\partial X_1^{2\gamma_1}} \frac{\partial^{2\gamma_2}}{\partial X_2^{2\gamma_2}} f(g(X_1)g(X_2))\Big|_{X_1=X_2=0} \\ &= \frac{\partial^{2\gamma_2}}{\partial X_2^{2\gamma_2}} \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} f^{(j)}(g(X_2))\,[g(X_2)]^j\,\Big|_{X_2=0}.\end{aligned} \tag{3.2.13}$$

Here $f^{(j)}$ denotes the j^{th} derivative of f. We now need to apply the derivative in X_2 to the product $f^{(j)}(g(X_2))\,[g(X_2)]^j$. To do this, we define $F_j(t) = f^{(j)}(t)t^j$. Substituting this into (3.2.13), we can again apply (3.2.11) and write

$$(3.2.13) = \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} \frac{\partial^{2\gamma_2}}{\partial X_2^{2\gamma_2}} F_j(g(X_2))\Big|_{X_2=0} = \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} \sum_{k=1}^{\gamma_2} \mathsf{b}_k^{\gamma_2} F_j^{(k)}(t)\Big|_{t=1}. \tag{3.2.14}$$

We can calculate the k^{th} derivative of $F_j(t) = f^{(j)}(t)t^j$ via the Leibniz rule as

$$F_j^{(k)}(t) = \sum_{\ell=0}^{k} \frac{\binom{k}{\ell}\Gamma(j+1)}{\Gamma(j-k+\ell+1)} f^{(j+\ell)}(t)t^{j+\ell-k}. \tag{3.2.15}$$

We note that the above sum is understood to be zero when $k-\ell > j$ due to the poles of the Gamma function at non-positive whole integers. Therefore we can freely set $t=1$ as

$$(3.2.14) = \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} \sum_{k=1}^{\gamma_2} \mathsf{b}_k^{\gamma_2} \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{\Gamma(j+1)}{\Gamma(j-k+\ell+1)} f^{(j+\ell)}(t)\Big|_{t=1}. \tag{3.2.16}$$

Rearranging the above as a sum over $j = 1, \ldots, |\gamma| = \gamma_1 + \gamma_2$ of $f^{(j)}(t)$, we have

$$\begin{aligned} (3.2.16) &= \sum_{j=1}^{|\gamma|} \sum_{\substack{p=j-\gamma_2 \\ p\geq 1}}^{\gamma_1} \mathsf{b}_p^{\gamma_1} \sum_{\substack{k=j-p \\ k\geq 1}}^{\gamma_2} \mathsf{b}_k^{\gamma_2} \binom{k}{j-p} \frac{\Gamma(p+1)}{\Gamma(j-k+1)} f^{(j)}(t)\Big|_{t=1} \\ &= \sum_{j=1}^{|\gamma|} \mathsf{b}_j^{(\gamma_1,\gamma_2)}[g(X_1)g(X_2)] f^{(j)}(t)\Big|_{t=1}, \end{aligned} \tag{3.2.17}$$

where we have written

$$\mathsf{b}_j^{(\gamma_1,\gamma_2)}[g(X_1)g(X_2)] = \sum_{\substack{p=j-\gamma_2 \\ p\geq 1}}^{\gamma_1} \mathsf{b}_p^{\gamma_1}[g(X_1)] \sum_{\substack{k=j-p \\ k\geq 1}}^{\gamma_2} \mathsf{b}_k^{\gamma_2}[g(X_2)] \binom{k}{j-p} \frac{\Gamma(p+1)}{\Gamma(j-k+1)}. \tag{3.2.18}$$

What we should observe from this description is that the coefficents $\mathsf{b}_j^{(\gamma_1,\gamma_2)}$ for a multi-index (γ_1, γ_2) are written as a nested sum of the coefficients $\mathsf{b}_p^{\gamma_1}$ and $\mathsf{b}_k^{\gamma_2}$ for the associated scalars γ_1, γ_2 with $p = 1, \ldots, \gamma_1$ and $k = 1, \ldots, \gamma_2$ respectively.

Theorem 3.2.2. *Let $g = g(X)$ be an even smooth function near the origin with $g(0) = 1$ and let $\mathscr{G}(\mathbf{X}) = g(X_1)\ldots g(X_q)$ with $\mathbf{X} = (X_1, \ldots, X_q)$. Then*

$$\partial^{2\gamma} f(\mathscr{G}(\mathbf{X}))\Big|_{\mathbf{X}=0} = \sum_{j=1}^{|\gamma|} \mathsf{b}_j^{\gamma}[\mathscr{G}(\mathbf{X})] f^{(j)}(t)\Big|_{t=1}. \tag{3.2.19}$$

The coefficients $\mathsf{b}_j^{\gamma}[\mathscr{G}(\mathbf{X})] = \mathsf{b}_j^{\gamma}[g(X_1)\ldots g(X_q)]$ are defined recursively as

$$\mathsf{b}_j^{\gamma}[\mathscr{G}(\mathbf{X})] = \sum_{\substack{p=j-\gamma_q \\ p\geq 1}}^{|\gamma|-\gamma_q} \mathsf{b}_p^{\tilde{\gamma}}[g(X_1)\ldots g(X_{q-1})] \sum_{\substack{k=j-p \\ k\geq 1}}^{\gamma_q} \frac{\binom{k}{j-p}\Gamma(p+1)}{\Gamma(j-k+1)} \mathsf{b}_k^{\gamma_q}[g(X_q)], \tag{3.2.20}$$

where $\mathsf{b}_j^{\tilde{\gamma}}[g(X_1)\ldots g(X_{q-1})]$ are the coefficients associated to $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_{q-1})$, and $\mathsf{b}_k^{\gamma_q}[g(X_q)]$ are the coefficients from (3.2.7) with $m = \gamma_q$.

Proof. We have already explored the two cases $\gamma = (\gamma_1)$, and $\gamma = (\gamma_1, \gamma_2)$ as motivation prior to the statement of the theorem, and arrived at the formula (3.2.18). We use this as the base case in an induction argument. We now assume that (3.2.19) holds for some arbitrary $\tilde{\gamma} = (\gamma_1, \dots, \gamma_{q-1})$, with the associated scalars $\mathsf{b}_j^{\tilde{\gamma}}[g(X_1)\dots g(X_{q-1})]$ defined by

$$\begin{aligned} &\mathsf{b}_j^{\tilde{\gamma}}[g(X_1)\dots g(X_{q-1})] \\ &= \sum_{\substack{p=j-\gamma_{q-1}\\ p\geq 1}}^{|\tilde{\gamma}|-\gamma_{q-1}} \mathsf{b}_p^{(\gamma_1,\dots,\gamma_{q-2})}[g(X_1)\dots g(X_{q-2})] \sum_{\substack{k=j-p\\ k\geq 1}}^{\gamma_{q-1}} \frac{\binom{k}{j-p}\Gamma(p+1)}{\Gamma(j-k+1)} \mathsf{b}_k^{\gamma_{q-1}}[g(X_{q-1})], \end{aligned} \tag{3.2.21}$$

which is equivalent to (3.2.20) rolled back by one iteration. We can then derive the coefficients $\mathsf{b}_j^{\gamma}[g(X_1)\dots g(X_q)] = \mathsf{b}_j^{\gamma}$ for $\gamma = (\gamma_1, \dots, \gamma_{q-1}, \gamma_q)$. Applying $\partial^{2\gamma}$ to $f(g(X_1)\dots g(X_q))$, we can evaluate the derivatives in the first $q-1$ variables by our assumption on $\tilde{\gamma}$, introducing the scalars $\mathsf{b}_j^{\tilde{\gamma}}[g(X_1)\dots g(X_{q-1})] = \mathsf{b}_j^{\tilde{\gamma}}$ given in (3.2.21) as

$$\begin{aligned} \partial^{2\gamma} f(\mathscr{G}(\mathbf{X}))\Big|_{\mathbf{X}=0} &= \frac{\partial^{2|\gamma|}}{\partial X_1^{2\gamma_1}\dots\partial X_q^{2\gamma_q}} f(g(X_1)\dots g(X_q))\Big|_{\mathbf{X}=0} \\ &= \frac{\partial^{2\gamma_q}}{\partial X_q^{2\gamma_q}} \left[\partial^{2\tilde{\gamma}} f(g(X_1)\dots g(X_q))\Big|_{\substack{X_i=0\\ i\neq q}} \right]\Bigg|_{X_q=0} \\ &= \frac{\partial^{2\gamma_q}}{\partial X_q^{2\gamma_q}} \left[\sum_{j=1}^{|\tilde{\gamma}|} \mathsf{b}_j^{\tilde{\gamma}} f^{(j)}(g(X_q))\,[g(X_q)]^j \right]\Bigg|_{X_q=0}. \end{aligned} \tag{3.2.22}$$

In the third step we have set the variables that aren't differentiated to zero, as we know they have no effect on the result. Again writing $F_j(t) = f^{(j)}(t)t^j$, with its derivatives given in (3.2.15), we follow on from (3.2.22) as

$$\begin{aligned} \partial^{2\gamma} f(\mathscr{G}(\mathbf{X}))\Big|_{\mathbf{X}=0} &= \sum_{j=1}^{|\tilde{\gamma}|} \mathsf{b}_j^{\tilde{\gamma}} \frac{\partial^{2\gamma_q}}{\partial X_q^{2\gamma_q}} F_j(g(X_q))\Big|_{X_q=0} \\ &= \sum_{j=1}^{|\gamma|-\gamma_q} \mathsf{b}_j^{\tilde{\gamma}} \sum_{k=1}^{\gamma_q} \mathsf{b}_k^{\gamma_q}[g(X_q)] \frac{d^k}{dt^k} F_j(t)\Big|_{t=1} \\ &= \sum_{j=1}^{|\gamma|-\gamma_q} \mathsf{b}_j^{\tilde{\gamma}} \sum_{k=1}^{\gamma_q} \mathsf{b}_k^{\gamma_q}[g(X_q)] \sum_{\ell=0}^{k} \frac{\binom{k}{\ell}\Gamma(j+1)}{\Gamma(j-k+\ell+1)} f^{(j+\ell)} t^{j+\ell-k}\Big|_{t=1}. \end{aligned} \tag{3.2.23}$$

Isolating the derivatives $f^{(j)}(t)$ and simplifying, we arrive at

$$\begin{aligned} \partial^{2\gamma} f(\mathscr{G}(\mathbf{X}))\Big|_{\mathbf{X}=0} =& f(g(X_1)\dots g(X_q))\Big|_{\mathbf{X}=0} \\ =& \sum_{j=1}^{|\gamma|} \sum_{\substack{p=j-\gamma_q\\ p\geq 1}}^{|\gamma|-\gamma_q} \mathsf{b}_p^{\tilde{\gamma}} \sum_{\substack{k=j-p\\ k\geq 1}}^{\gamma_q} \mathsf{b}_k^{\gamma_q} \frac{\Gamma(p+1)}{\Gamma(j-k+1)} \binom{k}{j-p} f^{(j)}(t)\Big|_{t=1}, \end{aligned} \tag{3.2.24}$$

where we have denoted $\mathsf{b}_j^{\gamma} = \mathsf{b}_j^{\gamma}[g(X_1)\dots g(X_q)]$, $\mathsf{b}_p^{\tilde{\gamma}} = \mathsf{b}_p^{\tilde{\gamma}}[g(X_1)\dots g(X_{q-1})]$, and $\mathsf{b}_k^{\gamma_q} = \mathsf{b}_k^{\gamma_q}[g(X_q)]$. This gives us the result by an induction argument. $\square$

Remark 3.2.3. *As a counterpart of what was stated earlier in Remark 3.2.1, and by writing $f(t) = F(z)$ with $z = (1-t)/2$ as before, it is easily seen that here, we have*

$$\partial^{2\gamma} F([1-\mathscr{G}(\mathbf{X})]/2)\Big|_{\mathbf{X}=0} = \partial^{2\gamma} f(\mathscr{G}(\mathbf{X}))\Big|_{\mathbf{X}=0} = \sum_{j=1}^{|\gamma|} \mathsf{b}_j^{\gamma}[\mathscr{G}(\mathbf{X})] \frac{d^j}{dt^j} f(t)\Big|_{t=1},$$
$$= \sum_{j=1}^{m} \mathsf{b}_j^{\gamma}[\mathscr{G}(\mathbf{X})] \frac{d^j}{dt^j} F([1-t]/2)\Big|_{t=1} = \sum_{j=1}^{m} (-2)^{-j} \mathsf{b}_j^{\gamma}[\mathscr{G}(\mathbf{X})] \frac{d^j F}{dz^j}\Big|_{z=0}.$$

Two Examples. For the sake of clarity and illustration of the above let us now discuss two relevant and useful examples.

- **When** $g(X) = \cos X$**.** A common example of an even function satisfying $g(0) = 1$ is $g(X) = \cos X$. The periodic pattern of the derivatives of g allows us to compute the coefficients b_j^m in (3.2.7) (*cf.*, e.g. [36, 89, 55]) as

$$\mathsf{b}_j^m[\cos X] = \mathsf{B}_{2m,j}(-\sin X, -\cos X, \dots)|_{X=0} = \mathsf{B}_{2m,j}(0, -1, 0, 1, \dots) \tag{3.2.25}$$

and so

$$\mathsf{b}_j^m[\cos X] = \frac{(-1)^{j+m}}{j!} \sum_{\ell=0}^{j} \frac{(-1)^\ell}{2^\ell} \binom{j}{\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q-\ell)^{2m}. \tag{3.2.26}$$

Alternatively we can express $\mathsf{b}_j^m[\cos X]$ via the recursive relation

$$\mathsf{b}_j^m[\cos X] = \begin{cases} (-1)^m & \text{for } j = 1 \\ -\left(j^2 \mathsf{b}_j^{m-1} + (2j-1)\mathsf{b}_{j-1}^{m-1}\right) & \text{for } 1 < j \le m \\ 0 & \text{for } j > m. \end{cases}$$

From (3.2.7) and Remark 3.2.1 and with $\mathsf{b}_j^m = \mathsf{b}_j^m[\cos X]$ we can write

$$\frac{d^{2m}}{dX^{2m}} F([1-\cos X]/2)\Big|_{X=0} = \sum_{j=1}^{m} (-2)^{-j} \mathsf{b}_j^m \frac{d^j F}{dz^j}\Big|_{z=0}. \tag{3.2.27}$$

We also see that the conclusion of Theorem 3.2.2 holds for $g(X) = \cos X$, and so together with Remark 3.2.3 we have

$$\partial^{2\gamma} F([1-\cos X_1 \dots \cos X_q]/2)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{m} (-2)^{-j} \mathsf{b}_j^{\gamma} \frac{d^j F}{dz^j}\Big|_{z=0}, \tag{3.2.28}$$

where in the last equation $\mathsf{b}_j^\gamma = \mathsf{b}_j^\gamma[\mathscr{G}(\mathbf{X})]$ with $\mathscr{G}(\mathbf{X}) = \cos X_1 \dots \cos X_q$.

- **When** $g(X) = \cosh X$**.** Related to the above example, if $g(X) = \cosh X$ then we can similarly write (*cf.*, again [36, 89] and the references therein) as $\mathsf{b}_j^m[\cosh X] =$

$\mathsf{B}_{2m,j}(\sinh X, \cosh X, \dots)|_{X=0} = \mathsf{B}_{2m,j}(0,1,0,1,\dots)$ and so

$$\mathsf{b}_j^m[\cosh X] = \frac{1}{2^j j!}\sum_{\ell=0}^{j}(-1)^\ell \binom{j}{\ell}(j-2\ell)^{2m}. \tag{3.2.29}$$

Similar to the case of the ordinary cosine here we can alternatively express $\mathsf{b}_j^m[\cosh X]$ by the recursive formula

$$\mathsf{b}_j^m[\cosh X] = \begin{cases} 1 & \text{for } j=1 \\ (-1)^{m+1}\left(j^2\mathsf{b}_j^{m-1} + (2j-1)\mathsf{b}_{j-1}^{m-1}\right) & \text{for } 1 < j \le m \\ 0 & \text{for } j > m. \end{cases}$$

We note in particular that here we have the remarkable identity $\mathsf{b}_j^m[\cosh X] = (-1)^m \mathsf{b}_j^m[\cos X]$. Again Remark 3.2.1 gives us

$$\frac{d^{2m}}{dX^{2m}} F([1-\cosh X]/2)\Big|_{X=0} = \sum_{j=1}^{m}(-2)^{-j}\mathsf{b}_j^m \frac{d^jF}{dz^j}\Big|_{z=0}, \tag{3.2.30}$$

with $\mathsf{b}_j^m = \mathsf{b}_j^m[\cosh X]$ whilst Theorem 3.2.2 and Remark 3.2.3 give us

$$\partial^{2\gamma} F([1-\cosh X_1 \dots \cosh X_q]/2)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{m}(-2)^{-j}\mathsf{b}_j^\gamma \frac{d^jF}{dz^j}\Big|_{z=0}, \tag{3.2.31}$$

where we have set $\mathsf{b}_j^\gamma = \mathsf{b}_j^\gamma[\mathscr{G}(\mathbf{X})]$ with $\mathscr{G}(\mathbf{X}) = \cosh X_1 \dots \cosh X_q$.

3.3 A differential identity on the hypergeometric function

Before presenting the main theorem, we introduce a set of scalars s_j^p, which we define as the coefficients of Y^j in the polynomial

$$\prod_{k=0}^{p-1}(Y + X_k) = \sum_{j=0}^{p}\mathsf{s}_j^p(X_0, \dots, X_{p-1})Y^j, \tag{3.3.1}$$

for scalars $X_0, \dots, X_{p-1}$. As a matter of fact these scalars can be described by the elementary symmetric polynomials as $\mathsf{S}_{p-j}(X_0, \dots, X_{p-1}) = \mathsf{s}_j^p$, where $\mathsf{S}_j(X_0, \dots, X_{p-1})$ denotes the sum of the distinct products of length j of the variables $X_0, \dots, X_{p-1}$. In particular, we have

$$\mathsf{s}_j^j = 1, \quad \mathsf{s}_{j-1}^j = \sum_{\ell=0}^{j-1} X_\ell, \quad \dots \quad \mathsf{s}_0^j = \prod_{\ell=0}^{j-1} X_\ell. \tag{3.3.2}$$

Theorem 3.3.1. *Let $g = g(X)$ be an even smooth function near the origin with $g(0) = 1$ and let $\mathscr{G}(\mathbf{X}) = g(X_1)\dots g(X_q)$. Consider the constant coefficient partial differential operator $\mathscr{L}_{\mathsf{P}}$ as in (3.1.3). Then for $a, b \in \mathbb{C}$ and $c \in \mathbb{C}\backslash\mathbb{N}_0^-$ we have*

$$\mathscr{L}_{\mathsf{P}}\left[{}_2F_1\left(a,b;c;[1-\mathscr{G}(\mathbf{X})]/2\right)\right]\Big|_{\mathbf{X}=0} = p_0 + \sum_{|\gamma|=1}^{\lfloor N/2 \rfloor} p_{2\gamma}\sum_{j=1}^{|\gamma|}\mathsf{c}_j^\gamma(a,b,c;\mathscr{G})\,(-ab)^j\,. \tag{3.3.3}$$

The scalars $\mathsf{c}_j^\gamma = \mathsf{c}_j^\gamma(a,b,c;\mathscr{G})$ *are given explicitly by the formula*

$$\mathsf{c}_j^\gamma(a,b,c;\mathscr{G}) = (-1)^{i+j}\sum_{i=j}^{|\gamma|} 2^{-i}\mathsf{b}_i^\gamma[\mathscr{G}(\mathbf{X})]\mathsf{s}_j^i \prod_{p=0}^{i-1}(c+p)^{-1}, \tag{3.3.4}$$

where $\mathsf{b}_i^\gamma[\mathscr{G}(\mathbf{X})] = \mathsf{b}_i^\gamma[g(X_1)\dots g(X_q)]$ *are defined in Theorem 3.2.2, and the scalars* $\mathsf{s}_j^i = \mathsf{s}_j^i(a+b)$ *are defined in* (3.3.1) *with* $X_k = k(k+a+b)$.

Proof. We begin by applying $\mathscr{L}_\mathsf{P}$ to ${}_2F_1(a,b;c;z)$ at $z = [1-\mathscr{G}(\mathbf{X})]/2$, where $\mathscr{G}(\mathbf{X}) = g(X_1)\dots g(X_q)$ noting that any odd derivative in any of the variables vanishes [see the paragraph following (3.2.10)]. Thus we have

$$\mathscr{L}_\mathsf{P}[{}_2F_1(a,b;c;[1-\mathscr{G}(\mathbf{X})]/2)]\Big|_{\mathbf{X}=0} = \sum_{|\gamma|=0}^{\lfloor N/2\rfloor} p_{2\gamma}\partial^{2\gamma}{}_2F_1(a,b;c;[1-\mathscr{G}(\mathbf{X})]/2)\Big|_{\mathbf{X}=0}.$$

Since ${}_2F_1\,(a,b;c;z)$ is a smooth function we can apply Theorem 3.2.2 and Remark 3.2.3 to the derivatives within the summation above. With this we have

$$\partial^{2\gamma}{}_2F_1(a,b;c;[1-\mathscr{G}(\mathbf{X})]/2)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{|\gamma|}(-2)^{-j}\mathsf{b}_j^\gamma[\mathscr{G}]\frac{d^j}{dz^j}{}_2F_1\,(a,b;c;z)\,\Big|_{z=0},$$

where we note $\mathsf{b}_j^\gamma[\mathscr{G}] = \mathsf{b}_j^\gamma[g(X_1)\dots g(X_q)]$. Next by differentiating (3.1.1) we can derive a recursive relation for the derivatives of the hypergeometric function

$$\frac{d^j}{dz^j}{}_2F_1(a,b;c;z) = \frac{(a)_j(b)_j}{(c)_j}F(a+j,b+j;c+j;z). \tag{3.3.5}$$

With this, and recalling that ${}_2F_1(a,b;c;0)=1$, we can continue by writing

$$\partial^{2\gamma}{}_2F_1(a,b;c;[1-\mathscr{G}(\mathbf{X})]/2)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{|\gamma|}(-2)^{-j}\mathsf{b}_j^\gamma[\mathscr{G}]\frac{(a)_j(b)_j}{(c)_j}. \tag{3.3.6}$$

The product $(a)_j(b)_j$ can be expanded into a polynomial in ab upon noting

$$(a)_j(b)_j = \prod_{p=0}^{j-1}(a+p)\prod_{q=0}^{j-1}(b+q) = \prod_{k=0}^{j-1}(ab+k\,[a+b+k]) = \sum_{l=1}^{j}\mathsf{s}_l^j[ab]^l, \tag{3.3.7}$$

where we have used (3.3.1) with $Y = ab$ and $X_k = k\,[a+b+k]$ to introduce the scalars s_l^j. Returning to (3.3.6), we can now assert that

$$\begin{aligned}\partial^{2\gamma}{}_2F_1(a,b;c;[1-\mathscr{G}(\mathbf{X})]/2)\Big|_{\mathbf{X}=0} &= \sum_{j=1}^{|\gamma|}(-2)^{-j}\frac{\mathsf{b}_j^\gamma[\mathscr{G}]}{(c)_j}\sum_{l=0}^{j}\mathsf{s}_l^j[ab]^l,\\ &= \sum_{j=1}^{|\gamma|}(-ab)^j\sum_{i=j}^{|\gamma|}(-1)^j\frac{\mathsf{b}_i^\gamma[\mathscr{G}]\mathsf{s}_j^i}{(-2)^i(c)_i}.\end{aligned} \tag{3.3.8}$$

Note that we have arranged the sum to be over powers of $-ab$. The reason for this will be clear in later applications. It now suffices to define the generalised hypergeometric coefficients as in (3.3.4) and the conclusion follows. □

3.4 The Jacobi and Gegenbauer function families $\mathscr{P}_\mu^{\alpha,\beta}(z)$ and $\mathscr{C}_\mu^{\nu}(z)$

The hypergeometric function ${}_2F_1 = {}_2F_1(a,b;c;z)$ unifies and generalises several families of orthogonal functions and polynomials. Of particular interest are the Jacobi family $\mathscr{P}_\mu^{\alpha,\beta}(z)$, which we define for $\alpha, \beta > -1$ by (*see* also [4, 55, 70])

$$\mathscr{P}_\mu^{\alpha,\beta}(z) = {}_2F_1\left(-\mu, \mu+\alpha+\beta+1; \alpha+1; (1-z)/2\right), \quad \mu \in \mathbb{C}. \tag{3.4.1}$$

For $\mu = k \in \mathbb{N}$ these become the normalised Jacobi polynomials, which act as the zonal spherical functions on rank-one symmetric spaces of compact type for certain ranges of α and β when $z = \cos\theta$. Similarly for $\mu = -(i\lambda + (\alpha+\beta+1)/2)$ we recover the zonal spherical functions on rank-one symmetric spaces of non-compact type when $z = \cosh\theta$ (*see*, e.g., [62, 70, 99]). Substituting for a, b, and c from (3.4.1) in Theorem 3.3.1, gives the following result.

Theorem 3.4.1. *Let $\mathscr{G}(\mathbf{X}) = g(X_1)\dots g(X_q)$ for an even and smooth function $g = g(X)$ near the origin with $g(0) = 1$. Let $\mathscr{L}_{\mathsf{P}} = \mathsf{P}(\partial)$ be as in (3.1.3) and let $\mathscr{P}_\mu^{\alpha,\beta}$ be as in (3.4.1) with $\alpha, \beta > -1$ and $\mu \in \mathbb{C}$. Then*

$$\mathscr{L}_{\mathsf{P}}\left[\mathscr{P}_\mu^{\alpha,\beta}(\mathscr{G}(\mathbf{X}))\right]\Big|_{\mathbf{X}=0} = p_0 + \sum_{|\gamma|=1}^{\lfloor N/2\rfloor} p_{2\gamma} \sum_{j=1}^{|\gamma|} \mathsf{c}_j^\gamma \left[\mu(\mu+\alpha+\beta+1)\right]^j, \tag{3.4.2}$$

where $\mu(\mu+\alpha+\beta+1)$ are the generalised eigenvalues of the Jacobi operator. The scalars $\mathsf{c}_j^\gamma = \mathsf{c}_j^\gamma(\alpha,\beta;\mathscr{G})$ are given explicitly by

$$\mathsf{c}_j^\gamma(\alpha,\beta;\mathscr{G}) = \sum_{i=j}^{|\gamma|} (-1)^{i+j} \frac{\mathsf{b}_i^\gamma[\mathscr{G}]\mathsf{s}_j^i}{2^i(\alpha+1)_i} \tag{3.4.3}$$

where $\mathsf{b}_i^\gamma[\mathscr{G}] = \mathsf{b}_i^\gamma[g(X_1)\dots g(X_q)]$ are defined in Theorem 3.2.2, and the scalars $\mathsf{s}_j^i = \mathsf{s}_j^i(\alpha+\beta)$ are defined in (3.3.1) with $X_k = k(k+\alpha+\beta+1)$.

The Jacobi function can be specialised further to the Gegenbauer function $\mathscr{C}_\mu^\nu$, which we arrive at by setting $\alpha = \beta = \nu - 1/2$ for $\nu > -1/2$ as

$$\mathscr{C}_\mu^\nu(z) = \mathscr{P}_\mu^{\nu-1/2,\nu-1/2}(z) = {}_2F_1\left(-\mu, \mu+2\nu; \nu+1/2; (1-z)/2\right). \tag{3.4.4}$$

As with the Jacobi function, setting $\mu = k \in \mathbb{N}$ results in the normalised Gegenbauer polynomials $\mathscr{C}_k^\nu$, which act as the zonal spherical functions on the sphere $\mathbb{S}^n$ and the real projective space $\mathbb{R}\mathbf{P}^n$ when $z = \cos\theta$. On the other hand, setting $\mu = -(i\lambda+\nu)$ gives us the zonal spherical functions on the hyperbolic upper-half space $\mathbf{H}^n$.

Theorem 3.4.2. *Let $\mathscr{G}(\mathbf{X}) = g(X_1)\dots g(X_q)$ for an even and smooth function $g = g(X)$ near the origin satisfying $g(0) = 1$. Let $\mathscr{L}_{\mathsf{P}}$ be as in (3.1.3) and let $\mathscr{C}^\nu_\mu$ be as in (3.4.4) with $\nu > -1/2$ and $\mu \in \mathbb{C}$. Then*

$$\mathscr{L}_{\mathsf{P}}\left[\mathscr{C}^\nu_\mu(\mathscr{G}(\mathbf{X}))\right]\Big|_{\mathbf{X}=0} = p_0 + \sum_{|\gamma|=1}^{\lfloor N/2\rfloor} p_{2\gamma}\sum_{j=1}^{|\gamma|}\mathsf{c}^\gamma_j\left[\mu(\mu+2\nu)\right]^j, \tag{3.4.5}$$

where $\mu(\mu+2\nu)$ are the generalised eigenvalues of the Gegenbauer operator. The scalars $\mathsf{c}^\gamma_j = \mathsf{c}^\gamma_j(\nu;\mathscr{G})$ are given explicitly given by

$$\mathsf{c}^\gamma_j(\nu;\mathscr{G}) = \sum_{i=j}^{|\gamma|}(-1)^{i+j}\frac{\mathsf{b}^\gamma_i[\mathscr{G}]\mathsf{s}^i_j}{2^i(\nu+1/2)_i} \tag{3.4.6}$$

where $\mathsf{b}^\gamma_i[\mathscr{G}] = \mathsf{b}^\gamma_i[g(X_1)\dots g(X_q)]$ are defined in Theorem 3.2.2, and the scalars $\mathsf{s}^i_j = \mathsf{s}^i_j(\nu)$ are defined in (3.3.1) with $X_k = k(k+2\nu)$.

3.5 The $\mathsf{P}(\partial)$ action on the matrix hypergeometric function ${}_2F_1 = {}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z)$

Here we consider the hypergeometric function with $n\times n$ matrix parameters $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, with $(\mathbf{C}+k\mathbf{I})$ an invertible matrix for every integer $k\geq 0$. (Here $\mathbf{I}=\mathbf{I}_n$ is the $n\times n$ identity matrix.) Recall that this is defined, for complex z, by the infinite series

$${}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z) := \sum_{k=0}^{\infty}(\mathbf{A})_k(\mathbf{B})_k(\mathbf{C})_k^{-1}\frac{z^k}{k!}, \tag{3.5.1}$$

where the matrix extension of the rising factorial is given by $(\mathbf{F})_0 = \mathbf{I}$ and for $k\geq 1$ by $(\mathbf{F})_k = \mathbf{F}(\mathbf{F}+\mathbf{I})\dots(\mathbf{F}+(k-1)\mathbf{I})$. The series can be shown to converge for all $|z|<1$ and similar to the case of the Gauss hypergeometric function conditions can be given on the parameters $\mathbf{A},\mathbf{B}$ and $\mathbf{C}$ to imply convergence for $|z|=1$ (*cf.*, e.g., [64]). Note that for $\mathbf{A}=a\mathbf{I}$, $\mathbf{B}=b\mathbf{I}$ and $\mathbf{C}=c\mathbf{I}$ we recover the Gauss hypergeometric function in that ${}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z) = {}_2F_1(a,b;c;z)\mathbf{I}$. Now, returning to (3.5.1), since the series is convergent for $|z|<1$, differentiation, and using the relation $(\mathbf{F})_{m+j} = (\mathbf{F})_m(\mathbf{F}+m\mathbf{I})_j$, leads to

$$\begin{aligned}\frac{d^j}{dz^j}{}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z) &= \sum_{k=j}^{\infty}(\mathbf{A})_k(\mathbf{B})_k(\mathbf{C})_k^{-1}[k(k-1)\dots(k-j+1)]\frac{z^{k-j}}{k!}\\ &= \sum_{k=0}^{\infty}(\mathbf{A})_{j+k}(\mathbf{B})_{j+k}(\mathbf{C})_{j+k}^{-1}\frac{z^k}{k!}\\ &= \sum_{k=0}^{\infty}(\mathbf{A})_j(\mathbf{A}+j\mathbf{I})_k(\mathbf{B})_j(\mathbf{B}+j\mathbf{I})_k(\mathbf{C}+j\mathbf{I})_k^{-1}(\mathbf{C})_j^{-1}\frac{z^k}{k!}.\end{aligned} \tag{3.5.2}$$

Evaluating at $z = 0$ then gives $d^j/dz^j {}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z)|_{z=0} = (\mathbf{A})_j(\mathbf{B})_j(\mathbf{C})_j^{-1}$. In passing we also note that if $\mathbf{A}$ and $\mathbf{B}$ or if $\mathbf{B}$ and $\mathbf{C}$ commute, then by a basic commutativity argument, we can rewrite the series on the right in (3.5.2) as a hypergeometric series (with shifted matrix parameters) and hence arrive at

$$\frac{d^j}{dz^j}{}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z) = (\mathbf{A})_j(\mathbf{B})_j\left[{}_2F_1(\mathbf{A}+j\mathbf{I},\mathbf{B}+j\mathbf{I};\mathbf{C}+j\mathbf{I};z)\right](\mathbf{C})_j^{-1},$$

for when $\mathbf{A}$ and $\mathbf{B}$ commute or alternatively for when $\mathbf{B}$ and $\mathbf{C}$ commute:

$$\frac{d^j}{dz^j}{}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z) = (\mathbf{A})_j\left[{}_2F_1(\mathbf{A}+j\mathbf{I},\mathbf{B}+j\mathbf{I};\mathbf{C}+j\mathbf{I};z)\right](\mathbf{B})_j(\mathbf{C})_j^{-1}.$$

Naturally we would like to see how the differential operator $\mathscr{L}_{\mathsf{P}}$ behaves when applied to this matrix parameter extension of the hypergeometric function. The following statement gives an answer to the question. (Compare with Theorem 3.3.1.)

Theorem 3.5.1. *Let $\mathscr{G}(\mathbf{X}) = g(X_1)\dots g(X_q)$ for an even smooth function $g = g(X)$ near the origin with $g(0) = 1$. Let $\mathscr{L}_{\mathsf{P}} = \mathsf{P}(\partial)$ be as in (3.1.3) and let $\mathbf{A},\mathbf{B}$ and $\mathbf{C} \in \mathbb{C}^{n\times n}$ with $(\mathbf{C}+k\mathbf{I})$ being invertible for every integer $k \geq 0$. Then*

$$\mathscr{L}_{\mathsf{P}}[{}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};[1-\mathscr{G}(\mathbf{X})]/2)]\Big|_{\mathbf{X}=0} = p_0\mathbf{I} + \sum_{|\gamma|=1}^{\lfloor N/2\rfloor} p_{2\gamma}\sum_{j=1}^{|\gamma|}\frac{\mathsf{b}_j^\gamma[\mathscr{G}]}{(-2)^j}(\mathbf{A})_j(\mathbf{B})_j(\mathbf{C})_j^{-1}. \tag{3.5.3}$$

Here $\mathsf{b}_j^\gamma[\mathscr{G}]$ are as defined in Theorem 3.2.2. If additionally $\mathbf{A},\mathbf{B}$ commute then,

$$(3.5.3) = p_0\mathbf{I} + \sum_{|\gamma|=1}^{\lfloor N/2\rfloor} p_{2\gamma}\sum_{j=1}^{|\gamma|}\sum_{i=j}^{|\gamma|}(-1)^{i+j}\frac{\mathsf{b}_i^\gamma[\mathscr{G}]}{2^i}\mathbf{S}_j^i[-\mathbf{AB}]^j(\mathbf{C})_i^{-1}, \tag{3.5.4}$$

where $\mathbf{S}_j^i = \mathsf{S}_{i-j}(X_0,\dots,X_{i-1})$ with $X_k = k(\mathbf{A}+\mathbf{B}+k\mathbf{I})$.

Proof. We closely follow the proof of Theorem 3.3.1, as the steps are very similar. We begin by noting that setting $z = 0$ in the definition (3.5.1) gives us the pointwise identity ${}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};0) = \mathbf{I}$. Now directly applying the operator $\mathscr{L}_{\mathsf{P}}$ to ${}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z)$ at $z = (1-\mathscr{G}(\mathbf{X}))/2$, where $\mathscr{G}(\mathbf{X}) = g(X_1)\dots g(X_q)$, and recalling that the odd derivatives vanish, we have

$$\begin{aligned}\text{LHS of (3.5.3)} &= \sum_{|\gamma|=0}^{\lfloor N/2\rfloor} p_{2\gamma}\partial^{2\gamma}{}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};[1-\mathscr{G}(\mathbf{X})]/2)\Big|_{\mathbf{X}=0}\\ &= p_0\mathbf{I} + \sum_{|\gamma|=1}^{\lfloor N/2\rfloor} p_{2\gamma}\sum_{j=1}^{|\gamma|}\frac{\mathsf{b}_j^\gamma[\mathscr{G}]}{(-2)^j}\frac{d^j}{dz^j}{}_2F_1(\mathbf{A},\mathbf{B};\mathbf{C};z)\Big|_{z=0}.\end{aligned} \tag{3.5.5}$$

An application of the recursive formula for the derivatives of the hypergeometric matrix function given prior to the theorem now leads to the result. Next if $\mathbf{A},\mathbf{B}$ commute then

$$(\mathbf{A})_j(\mathbf{B})_j = \prod_{p=0}^{j-1}[\mathbf{AB}+p(\mathbf{A}+\mathbf{B}+p\mathbf{I})] = \sum_{\ell=1}^{j}\mathbf{S}_\ell^j[\mathbf{AB}]^\ell \tag{3.5.6}$$

where the matrices $\mathbf{S}^j_\ell$ as stated are defined by the extension of the elementary symmetric polynomials $\mathsf{S}_{j-\ell}$ to $\mathbb{C}^{n\times n}$ (i.e., $n\times n$ matrix arguments) evaluated at X_k as described (*see* the start of Section 3.3). Substituting back in (3.5.3) and rearranging the sums gives to the conclusion. $\square$

3.6 Extension of Theorem 3.3.1 to the generalised hypergeometric function ${}_pF_r = {}_pF_r(\mathbf{a}; \mathbf{b}; z)$

The Gauss hypergeometric function ${}_2F_1 = {}_2F_1(a, b; c; z)$ can be seen as a special case of the generalised hypergeometric function ${}_pF_r = {}_pF_r(\mathbf{a}; \mathbf{b}; z)$, where, here $\mathbf{a} = (a_1, \dots, a_p)$ and $\mathbf{b} = (b_1, \dots, b_r)$ with no b_ℓ $(1 \le \ell \le r)$ a non-positive integer. Indeed this is defined, for complex z, by the series (see [4, 12, 93, 55])

$$ {}_pF_r(\mathbf{a}; \mathbf{b}; z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_k}{\prod_{j=1}^r (b_j)_k} \frac{z^k}{k!}. \tag{3.6.1} $$

The series converges for all finite values of z when $p \le r$ and all $|z| < 1$ when $p = r + 1$ but diverges for all $z \neq 0$ when $p > r + 1$. In the case $p = r + 1$ the series converges absolutely for all $|z| = 1$ if $\Re(\sum_i b_i - \sum_i a_i) > 0$ and converges conditionally for all $|z| = 1$ and $z \neq 1$ if $-1 < \Re(\sum_i b_i - \sum_i a_i) \le 0$ while the series diverges if $\Re(\sum_i b_i - \sum_i a_i) \le -1$. Clearly when any of the parameters a_ℓ (with $1 \le \ell \le p$) is a non-positive integer the series terminates and becomes a polynomial in z. The generalised hypergeometric series satisfies the differential recursive formula

$$ \frac{d^m}{dz^m} {}_pF_r(\mathbf{a}; \mathbf{b}; z) = \frac{(a_1)_m (a_2)_m \cdots (a_p)_m}{(b_1)_m (b_2)_m \cdots (b_r)_m} {}_pF_r(\mathbf{a} + \mathbf{m}; \mathbf{b} + \mathbf{m}; z) \tag{3.6.2} $$

where indiscriminately we have written $\mathbf{a}+\mathbf{m} = (a_1+m, \dots, a_p+m)$ and likewise $\mathbf{b}+\mathbf{m} = (b_1 + m, \dots, b_r + m)$. It also can be seen to satisfy the differential equation

$$ z \prod_{j=1}^{p} \left(z\frac{d}{dz} + a_j \right) {}_pF_r(\mathbf{a}; \mathbf{b}; z) = z\frac{d}{dz} \prod_{j=1}^{r} \left(z\frac{d}{dz} + b_j - 1 \right) {}_pF_r(\mathbf{a}; \mathbf{b}; z). \tag{3.6.3} $$

A general product of rising factorials can be written as

$$ \prod_{i=1}^{p} (a_i)_k = \prod_{j=0}^{k-1} \prod_{i=1}^{p} (a_i + j) = \sum_{j=0}^{k} \mathsf{s}^k_j(\mathbf{a}) \left(\prod_{i=1}^{p} a_i \right)^j, \tag{3.6.4} $$

giving us scalars $\mathsf{s}^k_j(\mathbf{a})$ defined as the coefficients for the sum on the right. Arguing now as in the proof of Theorem 3.3.1 with the suitable adjustments leads to the following result.

Theorem 3.6.1. *Let* $\mathscr{G}(\mathbf{X}) = g(X_1) \dots g(X_q)$ *for an even smooth* $g = g(X)$ *near the origin with* $g(0) = 1$. *Let* $\mathscr{L}_\mathsf{P} = \mathsf{P}(\partial)$ *be as in* (3.1.3) *and let* $\mathbf{a} = (a_1, \dots, a_p)$ *and*

$\mathbf{b} = (b_1, \ldots, b_r)$ *with no* b_ℓ $(1 \le \ell \le r)$ *a non-positive integer. Then*

$$\mathscr{L}_{\mathsf{P}} \left[{}_pF_r \left(\mathbf{a}; \mathbf{b}; [1 - \mathscr{G}(\mathbf{X})]/2 \right) \right] \Big|_{\mathbf{X}=0} = p_0 + \sum_{|\gamma|=1}^{\lfloor N/2 \rfloor} p_{2\gamma} \sum_{j=1}^{|\gamma|} \mathsf{c}_j^\gamma \left(- \prod_{i=1}^{p} a_i \right)^j \tag{3.6.5}$$

where the scalars $\mathsf{c}_j^\gamma = \mathsf{c}_j^\gamma(\mathsf{a}, \mathsf{b}; \mathscr{G})$ *are given explicitly by*

$$\mathsf{c}_j^\gamma = \mathsf{c}_j^\gamma(\mathbf{a}, \mathbf{b}; \mathscr{G}) = (-1)^{i+j} \sum_{i=j}^{|\gamma|} \frac{\mathsf{b}_i^\gamma[\mathscr{G}] \mathsf{s}_j^i(\mathbf{a})}{2^i} \prod_{k=0}^{i-1} \prod_{\ell=1}^{r} (b_\ell + k)^{-1}. \tag{3.6.6}$$

Chapter 4

Funk-Hecke Formula and Spectral Functions on Compact Symmetric Spaces of Rank one

4.1 Introduction

Let $\mathsf{P} = \sum p_\gamma \mathbf{X}^\gamma$ be a polynomial in $\mathbf{X} = (X_1, \dots, X_q)$ with coefficients p_γ. Here $\mathbf{X}^\gamma = X_1^{\gamma_1} \dots X_q^{\gamma_q}$ and the sum extends over all multi-indices $\gamma = (\gamma_1, \dots, \gamma_q)$ of non-negative integers satisfying $|\gamma| = \gamma_1 + \dots + \gamma_q \leq N$ for some fixed $N \in \mathbb{N}$. We associate with P the partial differential operator $\mathscr{L}_\mathsf{P}$ in q variables

$$\mathscr{L}_\mathsf{P} = \mathsf{P}(\partial) = p_0 + \sum_{|\gamma|=1}^{N} p_\gamma \frac{\partial^{|\gamma|}}{\partial X_1^{\gamma_1} \dots \partial X_q^{\gamma_q}}. \tag{4.1.1}$$

Let $\mathscr{P}_k^{\alpha,\beta} = \mathscr{P}_k^{\alpha,\beta}(t)$ with $\alpha, \beta > -1$ and integer $k \geq 0$ denote the normalised Jacobi polynomial (see Appendix A.3). By setting $t = \cos X_1 \dots \cos X_q$, with $X_1, \dots, X_q$ real variables we set ourselves the task of computing the action of $\mathscr{L}_\mathsf{P}$ on this multivariable function at the origin $(X_1, \dots, X_l) = 0$. We prove that for an explicitly computable set of scalars $\mathsf{c}_j^\gamma(\alpha, \beta)$ (that will be given) this action can be completely described by the formula (*cf.* Theorem 4.2.2)

$$\left[\mathscr{L}_\mathsf{P} \mathscr{P}_k^{\alpha,\beta}\right](\cos X_1 \dots \cos X_l)\Big|_{\mathbf{X}=0} = p_0 + \sum_{|\gamma|=1}^{\lfloor N/2 \rfloor} p_{2\gamma} \sum_{j=1}^{|\gamma|} \mathsf{c}_j^\gamma(\alpha, \beta) \left[\lambda_k^{\alpha,\beta}\right]^j, \tag{4.1.2}$$

where $\lambda_k^{\alpha,\beta} = k(k+\alpha+\beta+1)$ are the eigenvalues of the Jacobi operator $\mathscr{L}_{(\alpha,\beta)}$. Expressed differently, by introducing the polynomials $\mathscr{R}_\gamma = \mathscr{R}_\gamma(\alpha, \beta; X) = \sum_{j=1}^{|\gamma|} \mathsf{c}_j^\gamma(\alpha, \beta) X^j$ of the single variable X and of degree $|\gamma|$ we can rewrite this as

$$\left[\mathscr{L}_{\mathsf{P}}\mathscr{P}_k^{\alpha,\beta}\right](\cos X_1 \dots \cos X_l)\Big|_{\mathbf{X}=0} = p_0 + \sum_{|\gamma|=1}^{\lfloor N/2 \rfloor} p_{2\gamma}\mathscr{R}_\gamma(\lambda_k^{\alpha,\beta}). \tag{4.1.3}$$

Apart from being interesting in its own right the above representation formula has many nice applications to analysis as well as spectral geometry of compact rank-one symmetric spaces that are discussed in the rest of the chapter. Among these are the description and calculation of the Maclaurin spectral coefficients associated with Schwartz kernels of numerous functions of the Laplace-Beltrami operator (e.g., those of the heat kernel and Riesz spectral projections). Also as a result of this analysis we give an extension of the celebrated Funk-Hecke formula to all compact rank-one symmetric spaces and present some novel operator trace representation for functions of the Laplacian by linking to suitable theta series and other related function families. To the best of our knowledge this is the first formulation of the Funk-Hecke identity (originally for spheres [51, 60], [49]) in the context of compact symmetric spaces. For more on Jacobi polynomials and their significance in analysis on symmetric spaces *see* [4, 9, 12, 24, 55, 62, 68, 70, 71, 99] and for further related results see [6, 8, 21, 23, 27, 28, 41, 42, 61, 67, 79, 82, 94] as well as [30, 52, 54, 59, 87, 88] and the references therein.

4.2 A partial differential action $\mathscr{L}_{\mathsf{P}}$ on $\mathscr{P}_k^{\alpha,\beta}$

In this section we state two main theorems. Theorem 4.2.1 gives a combinatorial identity where we derive a set of scalars b_j^γ that classify a special case of a higher order chain rule for a multi-index partial derivative. Then in Theorem 4.2.2 we prove the differential identity relating to the action of the operator $\mathscr{L}_{\mathsf{P}}$ in (4.1.1) on the normalised Jacobi polynomials. Some applications of this action identity to analysis on compact symmetric spaces will be discussed later in Sections 4.3 and 4.5. Other application will be discussed in a forthcoming paper of the authors. To motivate the discussion let us take a smooth function $f = f(t)$ and consider the differentiation formula that holds for suitable scalars b_j^m $(1 \le j \le m)$:

$$\frac{d^{2m}}{dX^{2m}} f(\cos X)\bigg|_{X=0} = \sum_{j=1}^{m} \mathsf{b}_j^m \frac{d^j}{dt^j} f(t)\bigg|_{t=1}, \qquad m \ge 1. \tag{4.2.1}$$

This identity can be derived directly or else by using the classical Faà di Bruno's formula leading to the explicit description

$$\mathsf{b}_j^m = \mathsf{B}_{2m,j}\left(g'(X), \dots, g^{(2m-j+1)}(X)\right)\Big|_{X=0}, \quad g(X) = \cos X, \tag{4.2.2}$$

with $\mathsf{B}_{2m,j}$ denoting the incomplete Bell polynomial (see [3], pp. 204-207). Note that here we have restricted to the even derivatives only since any odd derivative of $f(\cos X)$ evaluated at $X = 0$ will vanish. The scalars b_j^m can also be seen by direct computation to satisfy the recursive formula

$$\mathsf{b}_j^m = \begin{cases} (-1)^m & \text{if } j = 1 \\ -\left(j^2\mathsf{b}_j^{m-1} + (2j-1)\mathsf{b}_{j-1}^{m-1}\right) & \text{if } 1 < j \leq m \\ 0 & \text{if } j > m. \end{cases} \tag{4.2.3}$$

In the following theorem, we generalise the formula (4.2.1) to several variables. Recall that for a multi-index $\gamma = (\gamma_1, \dots, \gamma_q)$ of non-negative integers we have

$$\partial^\gamma = \frac{\partial^{|\gamma|}}{\partial X_1^{\gamma_1} \dots \partial X_q^{\gamma_q}}, \qquad |\gamma| = \gamma_1 + \dots + \gamma_q. \tag{4.2.4}$$

Theorem 4.2.1. *For $f = f(t)$ a smooth function in a neighbourhood of $t = 1$ and with ∂^γ as above we have*

$$\partial^{2\gamma} f(\cos X_1 \dots \cos X_l)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{|\gamma|} \mathsf{b}_j^\gamma f^{(j)}(t)\Big|_{t=1}, \qquad l \geq q, \tag{4.2.5}$$

where the scalars b_j^γ on the right are defined recursively as

$$\mathsf{b}_j^\gamma = \sum_{\substack{p=j-\gamma_q \\ p \geq 1}}^{|\gamma|-\gamma_q} \mathsf{b}_p^{\tilde{\gamma}} \sum_{\substack{k=j-p \\ k \geq 1}}^{\gamma_q} \mathsf{b}_k^{\gamma_q} \frac{\binom{k}{j-p}\Gamma(p+1)}{\Gamma(j-k+1)}. \tag{4.2.6}$$

Here $\mathsf{b}_j^{\tilde{\gamma}}$ are the coefficients associated to $\tilde{\gamma} = (\gamma_1, \dots, \gamma_{q-1})$ and $\mathsf{b}_k^{\gamma_q}$ are the coefficients from (4.2.1) with $m = \gamma_q$.

Proof. Again we focus on the even derivatives only. Here we use an induction argument starting with the case $\gamma = (\gamma_1)$ where we only perform derivatives in a single variable X_1, and any other variable is just set to 0. Since $\cos 0 = 1$, the variables that aren't differentiated have no effect, and so this case is exactly as in (4.2.1) as

$$\frac{\partial^{2\gamma_1}}{\partial X_1^{2\gamma_1}} f\left(\cos X_1 \dots \cos X_l\right)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} f^{(j)}(t)\Big|_{t=1}, \tag{4.2.7}$$

where $f^{(j)}(t)$ denotes the j^{th} derivative of $f(t)$. Next we introduce a second derivative in the variable X_2 by writing $\gamma = (\gamma_1, \gamma_2)$. Evaluating the derivative in X_1 first, via (4.2.1), we have

$$\partial^{2\gamma} f(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} = \frac{\partial^{2\gamma_1+2\gamma_2}}{\partial X_1^{2\gamma_1} \partial X_2^{2\gamma_2}} f\left(\cos X_1 \ldots \cos X_l\right)\Bigg|_{X_i=0} \tag{4.2.8}$$
$$= \frac{\partial^{2\gamma_2}}{\partial X_2^{2\gamma_2}} \left[\frac{\partial^{2\gamma_1}}{\partial X_1^{2\gamma_1}} f(\cos X_1 \cos X_2)\Big|_{X_1=0} \right] \Bigg|_{X_2=0}$$
$$= \frac{\partial^{2\gamma_2}}{\partial X_2^{2\gamma_2}} \left[\sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} f^{(j)}(\cos X_2)[\cos X_2]^j \right] \Bigg|_{X_2=0}.$$

Note that we have freely set the irrelevant variables to zero above. Now defining a function $F_j(t) = f^{(j)}(t)t^j$, we can write (4.2.8) as

$$\partial^{2\gamma} f(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} \frac{\partial^{2\gamma_2}}{\partial X_2^{2\gamma_2}} F_j(\cos X_2)\Big|_{X_2=0} \tag{4.2.9}$$

where we can evaluate the X_2 derivative using (4.2.1) as

$$\partial^{2\gamma} f(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} \sum_{k=1}^{\gamma_2} \mathsf{b}_k^{\gamma_2} F_j^{(k)}(t)\Big|_{t=1}$$
$$= \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} \sum_{k=1}^{\gamma_2} \mathsf{b}_k^{\gamma_2} \frac{d^k}{dt^k} \left[f^{(j)}(t) t^j \right] \Big|_{t=1}. \tag{4.2.10}$$

We can calculate the k^{th} derivative of $F_j(t) = f^{(j)}(t)t^j$ as

$$\frac{d^k}{dt^k} \left[f^{(j)}(t) t^j \right] = \sum_{l=0}^{k} \frac{\binom{k}{l}\Gamma(j+1)}{\Gamma(j-k+l+1)} f^{(j+l)}(t) t^{j-k+l}, \tag{4.2.11}$$

and hence (4.2.10) becomes

$$\partial^{2\gamma} f(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{\gamma_1} \mathsf{b}_j^{\gamma_1} \sum_{k=1}^{\gamma_2} \mathsf{b}_k^{\gamma_2} \sum_{p=0}^{k} \frac{t^{j-k+p}\binom{k}{p}\Gamma(j+1)}{\Gamma(j-k+p+1)} f^{(j+p)}(t)\Big|_{t=1}. \tag{4.2.12}$$

Rearranging (4.2.12) to isolate the derivatives of $f(t)$, we have

$$\partial^{2\gamma} f(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{\gamma_1+\gamma_2} \mathsf{b}_j^{\gamma} f^{(j)}(t)\Big|_{t=1}, \tag{4.2.13}$$

where $\mathsf{b}_j^{\gamma} = \mathsf{b}_j^{(\gamma_1,\gamma_2)}$ are explicitly given by

$$\mathsf{b}_j^{\gamma} = \sum_{\substack{p=j-\gamma_2 \\ p\geq 1}}^{\gamma_1} \mathsf{b}_p^{\gamma_1} \sum_{\substack{k=j-p \\ k\geq 1}}^{\gamma_2} \mathsf{b}_k^{\gamma_2} \frac{\binom{k}{j-p}\Gamma(p+1)}{\Gamma(j-k+1)}, \tag{4.2.14}$$

which corresponds to the formulation (4.2.6) in the case where $\gamma = (\gamma_1, \gamma_2)$.

Now for a multi-index $\gamma = (\gamma_1, \ldots, \gamma_q)$ we set $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_{q-1})$. Assuming that (4.2.5) and (4.2.6) hold for $\tilde{\gamma}$, we can derive the result for γ by an identical argument as above.

Applying $\partial^{2\gamma}$ to $f(\cos X_1 \ldots \cos X_l)$ for $l \geq q$, we can use our assumption to write

$$\begin{aligned}\partial^{2\gamma} f(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} &= \frac{\partial^{2\gamma_q}}{\partial X_q^{2\gamma_q}} \frac{\partial^{2|\tilde{\gamma}|}}{\partial X_1^{2\gamma_1} \ldots \partial X_{q-1}^{2\gamma_{q-1}}} f(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} \\ &= \frac{\partial^{2\gamma_q}}{\partial X_q^{2\gamma_q}} \sum_{j=1}^{|\tilde{\gamma}|} \mathsf{b}_j^{\tilde{\gamma}} f^{(j)}(\cos X_q)[\cos X_q]^j \Big|_{\mathbf{X}=0}. \end{aligned} \tag{4.2.15}$$

This form is familiar from (4.2.8), and so again writing $F_j(t) = f^{(j)}(t)t^j$ we can follow the steps (4.2.9)-(4.2.14) to arrive at

$$\partial^{2\gamma} f(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} = \sum_{j=1}^{|\gamma|} \sum_{\substack{p=j-\gamma_q \\ p\geq 1}}^{|\gamma|-\gamma_q} \mathsf{b}_p^{\tilde{\gamma}} \sum_{\substack{k=j-p \\ k\geq 1}}^{\gamma_q} \mathsf{b}_k^{\gamma_q} \frac{\binom{k}{j-p}\Gamma(p+1)}{\Gamma(j-k+1)} f^{(j)}(t)\Big|_{t=1}, \tag{4.2.16}$$

where taking b_j^{γ} as in (4.2.6) gives us the result. □

Before stating the following theorem, we require some notation to be defined. For a set of scalars $\rho_1, \ldots, \rho_{i-1}$, let d_j^i be the coefficients of X^j in the polynomial

$$\prod_{p=0}^{i-1} (X - \rho_p) = \sum_{j=0}^{i} \mathsf{d}_j^i X^j, \qquad i \geq 1. \tag{4.2.17}$$

Theorem 4.2.2. *Let $\mathscr{L}_{\mathsf{P}}$ be the differential operator defined in (4.1.1). Then the normalised Jacobi polynomial $\mathscr{P}_k^{\alpha,\beta}$, $\alpha, \beta > -1$, satisfies the differential identity*

$$\left[\mathscr{L}_{\mathsf{P}} \mathscr{P}_k^{\alpha,\beta}\right] (\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} = p_0 + \sum_{|\gamma|=1}^{\lfloor N/2 \rfloor} p_{2\gamma} \sum_{j=1}^{|\gamma|} \mathsf{c}_j^{\gamma}(\alpha, \beta) \left[\lambda_k^{\alpha,\beta}\right]^j. \tag{4.2.18}$$

Here $\lambda_k^{\alpha,\beta} = k(k+\alpha+\beta+1)$ denotes the eigenvalues of the Jacobi operator (A.3.3). The scalar coefficients $\mathsf{c}_j^{\gamma}(\alpha,\beta)$ are explicitly given by

$$\mathsf{c}_j^{\gamma}(\alpha, \beta) = \sum_{i=j}^{|\gamma|} \frac{\mathsf{b}_i^{\gamma} \mathsf{d}_j^i}{2^i} \prod_{l=0}^{i-1} (\alpha + 1 + l)^{-1} \tag{4.2.19}$$

where b_i^{γ} are the scalars defined in Theorem 4.2.1, and d_j^i are the scalars defined in (4.2.17) by setting $X = \lambda_k^{\alpha,\beta}$ for fixed k and $\rho_p = \lambda_p^{\alpha,\beta}$.

Proof. We begin by directly applying $\mathscr{L}_{\mathsf{P}}$ to $\mathscr{P}_k^{\alpha,\beta}(\cos X_1 \ldots \cos X_l)$, then setting each X_i to zero. Recalling that $\mathscr{P}_k^{\alpha,\beta}(1) = 1$, we have

$$\begin{aligned} \text{LHS (4.2.18)} &= p_0 \mathscr{P}_k^{\alpha,\beta}(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} + \\ &\quad + \sum_{|\gamma|=1}^{N} p_\gamma \partial^\gamma \mathscr{P}_k^{\alpha,\beta}(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} \\ &= p_0 + \sum_{|\gamma|=1}^{N} p_\gamma \partial^\gamma \mathscr{P}_k^{\alpha,\beta}(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0}. \end{aligned} \tag{4.2.20}$$

The next step is to note that if any γ_i is odd that the associated derivative will vanish due to $\cos X_1 \ldots \cos X_q$ being an even function. Hence we can ignore any odd derivatives and consider only multi-indices 2γ. We know $\mathscr{P}_k^{\alpha,\beta}$ is smooth and so satisfies Theorem 4.2.1. This lets us rewrite (4.2.20) as

$$\begin{aligned} \text{LHS (4.2.18)} &= p_0 + \sum_{|\gamma|=1}^{\lfloor N/2 \rfloor} p_{2\gamma} \partial^{2\gamma} \mathscr{P}_k^{\alpha,\beta}(\cos X_1 \ldots \cos X_l)\Big|_{\mathbf{X}=0} \\ &= p_0 + \sum_{|\gamma|=1}^{\lfloor N/2 \rfloor} p_{2\gamma} \sum_{j=1}^{|\gamma|} \mathsf{b}_j^\gamma \frac{d^j}{dt^j} \mathscr{P}_k^{\alpha,\beta}(t)\Big|_{t=1}. \end{aligned} \tag{4.2.21}$$

We now refer to the recursive formula for derivatives of the Jacobi polynomial (A.3.4), which gives

$$\begin{aligned} \sum_{j=1}^{|\gamma|} \mathsf{b}_j^\gamma \frac{d^j}{dt^j} \mathscr{P}_k^{\alpha,\beta}\Big|_{t=1} &= \sum_{j=1}^{|\gamma|} \frac{\mathsf{b}_j^\gamma \Gamma(\alpha+1) k! \Gamma(k+j+\alpha+\beta+1)}{2^j \Gamma(\alpha+j+1)(k-j)! \Gamma(k+\alpha+\beta+1)} \mathscr{P}_{k-j}^{(\alpha+j,\beta+j)}\Bigg|_{t=1} \\ &= \sum_{j=1}^{|\gamma|} \frac{\mathsf{b}_j^\gamma \Gamma(\alpha+1) k! \Gamma(k+j+\alpha+\beta+1)}{2^j \Gamma(\alpha+j+1)(k-j)! \Gamma(k+\alpha+\beta+1)} \end{aligned} \tag{4.2.22}$$

The last equality follows by noting that $\mathscr{P}_{k-j}^{(\alpha+j,\beta+j)}(1) = 1$. Using the fact that $\Gamma(z+1) = z\Gamma(z)$, we can reduce (4.2.22) by taking advantage of

$$\frac{2^{-j}\Gamma(\alpha+1)}{\Gamma(\alpha+j+1)} = 2^{-j} \prod_{k=0}^{j-1} (\alpha+k+1)^{-1} \tag{4.2.23}$$

Similarly, we can reduce the remaining Gamma functions in (4.2.22) as

$$\begin{aligned} \frac{\Gamma(k+j+\alpha+\beta+1)\Gamma(k+1)}{\Gamma(k+\alpha+\beta+1)\Gamma(k+1-j)} &= \prod_{p=0}^{j-1} (k+\alpha+\beta+1+p) \prod_{q=0}^{j-1} (k-q) \\ &= \prod_{p=0}^{j-1} \left(k(k+\alpha+\beta+1) - p(p+\alpha+\beta+1) \right). \end{aligned}$$

This product is equal to (4.2.17) with $X = \lambda_k^{\alpha,\beta}$ and $\rho_p = \lambda_p^{\alpha,\beta}$, meaning we can write

$$\frac{\Gamma(k+j+\alpha+\beta+1)\Gamma(k+1)}{\Gamma(k+\alpha+\beta+1)\Gamma(k+1-j)} = \prod_{p=0}^{j-1} (\lambda_k^{\alpha,\beta} - \lambda_p^{\alpha,\beta}) = \sum_{l=0}^{j} \mathsf{d}_l^j [\lambda_k^{\alpha,\beta}]^l. \tag{4.2.24}$$

Substituting (4.2.23) and (4.2.24) into (4.2.22) gives the greatly reduced form

$$\text{LHS (4.2.22)} = \sum_{j=1}^{|\gamma|} 2^{-j} \mathsf{b}_j^\gamma \prod_{k=0}^{j-1} (\alpha+k+1)^{-1} \sum_{l=0}^{j} \mathsf{d}_l^j \left[\lambda_k^{\alpha,\beta}\right]^l = \sum_{j=1}^{|\gamma|} \left[\lambda_k^{\alpha,\beta}\right]^j \mathsf{c}_j^\gamma(\alpha,\beta),$$

where we have set $\mathsf{c}_j^\gamma(\alpha,\beta)$ as given by (4.2.19). This completes the proof. $\square$

4.3 Jacobi polynomials $\mathscr{P}_k^{\alpha,\beta}$ as spherical functions and Maclaurin spectral coefficients $b_{2l}^{\alpha,\beta}[K_F]$

One of the main interests in the normalised Jacobi polynomials $\{\mathscr{P}_k^{\alpha,\beta} : k \geq 0\}$ is that for suitable choices of parameters α and β they represent the zonal spherical functions on compact Riemannian symmetric spaces of rank-one $\mathscr{X} = \mathbf{G}/\mathbf{H}$ (*see* Table 3).

These spaces are completely classified and with $d = \dim(\mathscr{X})$ and $n \geq 1$ can be listed as: the sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$ with $d = n$, the real projective space $\mathbb{R}\mathbf{P}^n = \mathbf{SO}(n+1)/\mathbf{O}(n)$ with $d = n$, the complex projective space $\mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(1))$ with $d = 2n$, the quaternionic projective space $\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n) \times \mathbf{Sp}(1))$ with $d = 4n$ and $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}^4/\mathbf{Spin}(9)$ the Cayley plane with $d = 16$ (see, e.g., [21, 62, 99]).

Table 3 provides the parameters α and β associated to each of these spaces, hence identifying the associated normalised Jacobi polynomials as their zonal spherical functions. Likewise Table 1 and Table 2 provides a summary of relevant spectral-geometric data. Let us pause briefly to discuss some necessary background on the geometry and spectral theory of these spaces. Firstly, all these spaces with the exception of $\mathbb{S}^1$ and $\mathbb{R}\mathbf{P}^n$, $n \geq 1$, are simply-connected, whilst $\pi_1(\mathbb{R}\mathbf{P}^n) \cong \mathbb{Z}_2$ for $n \geq 2$ and $\pi_1(\mathbb{S}^1) \cong \pi_1(\mathbb{R}\mathbf{P}^1) \cong \mathbb{Z}$. Next, the geodesic flow on each of these spaces is periodic (all geodesics are closed when sufficiently continued) and more importantly all prime geodesics have equal length, hereafter, denoted by $\ell_p(\mathscr{X})$ (see, e.g., [66], Theorem 5.2.1).

Recall that on a compact closed Riemannian manifold $(\mathscr{M}, g)$, the Laplace-Beltrami operator $-\Delta_{\mathscr{M}}$, as a closed self-adjoint operator in $L^2(\mathscr{M}, dv_g)$ with $dv_g = \sqrt{\det g}\, dx_1 dx_2 \dots dx_d$, is given in local coordinates by

$$-\Delta_{\mathscr{M}} = -\frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{d} \partial_j \left(\sqrt{\det g} g^{jk} \partial_k \right). \tag{4.3.1}$$

It has a spectrum $\Sigma = \Sigma(\mathscr{M}; -\Delta_{\mathscr{M}}) \subset [0, \infty)$ consisting solely of (non-negative) eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ with $\lambda_j \nearrow \infty$ as $j \nearrow \infty$. Furthermore, the normalised eigenfunctions $\{\phi_j : j \geq 0\}$ can be chosen to form an orthonormal basis for the Hilbert space $L^2(\mathscr{M}, dv_g)$. Here $-\Delta_{\mathscr{M}} \phi_j = \lambda_j \phi_j$ while $||\phi_j||_{L^2} = 1$ for all j, and $\langle \phi_j, \phi_k \rangle_{L^2} = 0$ for $j \neq k$.

Now for a given $F = F(X)$ with $X \geq 0$ in the functional calculus of $-\Delta_{\mathscr{M}}$, by considering $F(-\Delta_{\mathscr{M}})$, we can write for $\varphi \in L^2(\mathscr{M})$ in the operator domain,

$$[F(-\Delta_{\mathscr{M}})\varphi](x) = \int_{\mathscr{M}} K_F(x,y)\varphi(y)\, dv_g(y), \quad x \in \mathscr{M}, \tag{4.3.2}$$

where $K_F = K_F(x, y)$, the Schwartz kernel of $F(-\Delta_{\mathscr{M}})$, is given by the spectral sum $K_F = \sum F(\lambda_j)\phi_j \otimes \phi_j$, specifically,

$$K_F(x, y) = \sum_{\lambda_j \in \Sigma} F(\lambda_j)\phi_j(x)\phi_j(y), \quad x, y \in \mathscr{M}. \tag{4.3.3}$$

For a compact rank-one symmetric space $\mathscr{X} = \mathbf{G}/\mathbf{H}$ of a compact Lie group $\mathbf{G}$, with $\mathbf{H}$ the isotropy group of a point in $\mathscr{X}$, one can use the well known addition formula for the matrix coefficients of irreducible unitary representations to write (4.3.3), with a slight abuse of notation $K_F(x, y) = K_F(\theta)$, in the form

$$\begin{aligned} K_F &= \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k^{\alpha,\beta})\mathscr{F}_k(\theta) \\ &= \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k^{\alpha,\beta})\mathscr{P}_k^{\alpha,\beta}(\cos\theta). \end{aligned} \tag{4.3.4}$$

Here $\{\mathscr{F}_k(\theta) : k \geq 0\}$ are the zonal spherical functions on $\mathscr{X}$, more specifically, $\mathscr{F}_k(\theta) = {}_2F_1(-k, \alpha + \beta + 1 + k; \alpha + 1; (1 - \cos\theta)/2) = \mathscr{P}_k^{\alpha,\beta}(\cos\theta)$, [1] where ${}_2F_1$ is the familliar Gaussian hypergeometric function. Furthermore, $\theta = \theta(x, y)$ is the geodesic distance between points x and y in $\mathscr{X}$, $\mathrm{Vol}(\mathscr{X})$ denotes the volume of $\mathscr{X}$ (see Table 2), and $M_k(\mathscr{X})$ denotes the multiplicity of the numerically distinct eigenvalues $\lambda_k^{\alpha,\beta} = k(k + \alpha + \beta + 1)$ (see Table 1). To comment on the zonal spherical functions further, recall that the radial part of the Laplacian on $\mathscr{X}$ can be written as

$$(-\Delta_{\mathscr{X}})_{rad} = -\frac{\partial^2}{\partial\theta^2} - \frac{A'(\theta)}{A(\theta)}\frac{\partial}{\partial\theta}, \tag{4.3.5}$$

where $A(\theta)$ stands for the area of the sphere of radius $\theta > 0$ in $\mathscr{X}$, specifically, given by

$$A(\theta) = \omega_{d-1} \left[(\sin k\theta)/k\right]^{2\alpha+1} (\cos k\theta)^{2\beta+1}. \tag{4.3.6}$$

Here $\omega_{d-1} = \mathrm{Vol}(\mathbb{S}^{d-1})$ and in the simply-connected case, $k = 1/2$, whilst in the non simply-connected case $k = 1$ (see, e.g., [99] pp. 28-9). Thus

$$\begin{aligned} (-\Delta_{\mathscr{X}})_{rad} &= -\frac{\partial^2}{\partial\theta^2} - \left[\frac{1}{2}(2\alpha + 1)\cot\theta/2 - \frac{1}{2}(2\beta + 1)\tan\theta/2\right]\frac{\partial}{\partial\theta} \\ &= -\frac{\partial^2}{\partial\theta^2} - \left[(2\beta + 1)\cot\theta + (\alpha - \beta)\cot\theta/2\right]\frac{\partial}{\partial\theta}, \end{aligned} \tag{4.3.7}$$

which after the change of variables $t = \cos\theta$ reduces to the Jacobi operator (A.3.3), justifying the appearance of $\mathscr{P}_k^{\alpha,\beta}$ in (4.3.4). Regarding the other geometric and spectral data associated to $\mathscr{X}$, in the simply-connected case, we have explicit formulae

$$\mathrm{Vol}(\mathscr{X}) = \frac{2^d \pi^{d/2}\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}, \tag{4.3.8}$$

[1] Note that in the case of $\mathscr{X} = \mathbb{R}\mathbf{P}^n$, we have $\mathscr{F}_k(\theta) = \mathscr{P}_{2k}^{\alpha,\beta}(\cos\theta)$.

noting $d = 2\alpha + 2$. Setting $\rho = (\alpha+\beta+1)/2$, we have $\lambda_k^{\alpha,\beta} = (\rho+k)^2 - \rho^2 = k(k+\alpha+\beta+1)$ for the numerically distinct eigenvalues whilst the multiplicity function takes the form $M_k(\mathscr{X}) = M_k^{\alpha,\beta}$ where

$$M_k^{\alpha,\beta} = \frac{(\alpha+\beta+2k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\beta+1)\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+\beta+2)\Gamma(\alpha+1)\Gamma(k+\beta+1)}. \tag{4.3.9}$$

We present the numerically distinct eigenvalues and associated multiplicities for each compact rank-one symmetric space in Table 1 at the end of this section.

Maclaurin spectral coefficients. For the purpose of future reference, we now introduce a sequence of scalars $\{b_{2l}^{\alpha,\beta} = b_{2l}^{\alpha,\beta}[K] : l \geq 0,\ \alpha,\beta > -1\}$, hereafter referred to as the Maclaurin spectral coefficients associated to a given kernel K. To this end, we first introduce an infinite scale of polynomials $\mathscr{R}_m^{\alpha,\beta}(X)$, which we build from the scalar coefficients $\mathsf{c}_j^{\gamma}(\alpha,\beta)$ from Theorem 4.2.2 at $l = q = 1$. Indeed here we set

$$\mathscr{R}_m^{\alpha,\beta}(X) = \sum_{j=1}^{m} \mathsf{c}_j^m(\alpha,\beta) X^j, \qquad m \geq 1, \tag{4.3.10}$$

and for the sake of convenience $\mathscr{R}_0^{\alpha,\beta}(X) \equiv 1$. Disassociating for the moment any connections between the parameters α,β and rank-one symmetric spaces we proceed to the following definition.

Definition 4.3.1. Assume $\alpha,\beta > -1$ and let $K \in L^2(]-1,1[;(1-t)^\alpha(1+t)^\beta dt)$. Then the Maclaurin spectral coefficients associated with K are defined by

$$b_{2l}^{\alpha,\beta}[K] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+\beta+2)}{(4\pi)^{\alpha+1}\Gamma(\beta+1)} M_k^{\alpha,\beta} F(\lambda_k^{\alpha,\beta}) \mathscr{R}_l^{\alpha,\beta}(\lambda_k^{\alpha,\beta}), \qquad l \geq 0. \tag{4.3.11}$$

Here $M_k^{\alpha,\beta}$ is the multiplicity function as in (4.3.9) and $\lambda_k^{\alpha,\beta} = k(k+\alpha+\beta+1)$.

Naturally when $K = K_F$ denotes the Schwartz kernel associated with the operator $F(-\Delta_{\mathscr{X}})$ then we can write the above spectral coefficients in operator trace form

$$b_{2l}^{\alpha,\beta}[K_F] = \frac{1}{\mathrm{Vol}(\mathscr{X})} \mathrm{Tr}[F\mathscr{R}_l^{\alpha,\beta}](-\Delta_{\mathscr{X}}), \quad l \geq 0. \tag{4.3.12}$$

In other words $b_{2l}^{\alpha,\beta}[K_F]$ is the trace of the operator $\mathrm{Vol}(\mathscr{X})^{-1}[F\mathscr{R}_l^{\alpha,\beta}](-\Delta_{\mathscr{X}})$. Let us give the motivation behind this definition and how the polynomials $\mathscr{R}_l^{\alpha,\beta}$ arise in the analysis. With a slight abuse of notation, we now formally write the Maclaurin expansion about $\theta = 0$ of the kernel K_F as in (4.3.4) as

$$\sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \frac{\partial^{2l}}{\partial\theta^{2l}} K_F(\theta)\Big|_{\theta=0} = \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k^{\alpha,\beta}) \frac{\partial^{2l}}{\partial\theta^{2l}} \mathscr{P}_k^{\alpha,\beta}(\cos\theta)\Big|_{\theta=0}. \tag{4.3.13}$$

Applying Theorem 4.2.2 with $l = q = 1$ to the derivatives on the right-hand side above, noting the appearance of the Maclaurin coefficients from (4.3.12),

$$
\begin{aligned}
\sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \frac{\partial^{2l}}{\partial \theta^{2l}} K_F(\theta)\Big|_{\theta=0} &= \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k^{\alpha,\beta}) \sum_{j=1}^{l} \mathsf{c}_j^l(\alpha,\beta) \left[\lambda_k^{\alpha,\beta}\right]^j \\
&= \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k^{\alpha,\beta}) \mathscr{R}_l^{\alpha,\beta}(\lambda_k^{\alpha,\beta}), \\
&= \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \frac{\mathrm{Tr}[F\mathscr{R}_l^{\alpha,\beta}](-\Delta_{\mathscr{X}})}{\mathrm{Vol}(\mathscr{X})} = \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} b_{2l}^{\alpha,\beta}[K_F]. \qquad (4.3.14)
\end{aligned}
$$

To clarify concepts we now present some examples of $F = F(X)$ along with their associated Maclaurin spectral coefficients $\{b_{2l}^{\alpha,\beta} = b_{2l}^{\alpha,\beta}[K_F] : l \geq 0\}$. Here we first consider the spectral projections P_k, that is, orthogonal projections of $L^2(\mathscr{X})$ onto the kth eigenspace $\mathcal{H}_k$ (with $k \geq 0$ fixed). Next we consider the heat semi-group $\{T_t = e^{t\Delta_{\mathscr{X}}} : t > 0\}$, and then with the aid of this we move on to considering an interesting and fairly large class of operators of the Laplace transform type.

- Choosing $F = F_k$ such that $F \equiv 0$ on $\Sigma(\mathscr{X}; -\Delta)\backslash\{\lambda_k^{\alpha,\beta}\}$ and $F(\lambda_k^{\alpha,\beta}) = 1$ we have $P_k = F(-\Delta_{\mathscr{X}})$ with $\mathrm{Vol}(\mathscr{X})K_F = M_k(\mathscr{X})\mathscr{P}_k^{\alpha,\beta}(\cos\theta)$. Here it is seen that

$$
b_{2\ell}^{\alpha,\beta}[K_F] = \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} \mathscr{R}_\ell(\lambda_k^{\alpha,\beta}), \qquad \ell \geq 0. \qquad (4.3.15)
$$

- For the heat semi-group $\{e^{t\Delta_{\mathscr{X}}} : t > 0\}$, we set $F(X) = F_t(X) = e^{-tX}$. Here the appearance of the exponential function results in a simplified description of the associated Maclaurin coefficients $b_{2l}^{\alpha,\beta}[\mathscr{H}_t]$, with $\mathscr{H}_t(\theta)$ denoting the heat kernel

$$
\mathscr{H}_t(\theta) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} e^{-\lambda_k^{\alpha,\beta} t} \mathscr{P}_k^{\alpha,\beta}(\cos\theta), \qquad t > 0. \qquad (4.3.16)
$$

As a matter of fact the Maclaurin spectral coefficients $b_{2l}^{\alpha,\beta}[\mathscr{H}_t]$, for $t > 0$, and with the aid of the differential operator $\mathscr{R}_l^{\alpha,\beta}(-d/dt)$ acting on the heat trace can be written in as

$$
\begin{aligned}
b_{2l}^{\alpha,\beta}[\mathscr{H}_t] &= \frac{1}{\mathrm{Vol}(\mathscr{X})} \mathrm{Tr}[F_t \mathscr{R}_l^{\alpha,\beta}](-\Delta_{\mathscr{X}}) \\
&= \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} \sum_{j=1}^{l} \mathsf{c}_j^l(\alpha,\beta) \left[\lambda_k^{\alpha,\beta}\right]^j e^{-\lambda_k^{\alpha,\beta} t} \\
&= \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} \sum_{j=1}^{l} \mathsf{c}_j^l(\alpha,\beta) \left[-d/dt\right]^j e^{-\lambda_k^{\alpha,\beta} t} \\
&= \frac{1}{\mathrm{Vol}(\mathscr{X})} \mathscr{R}_l^{\alpha,\beta}\left(-\frac{d}{dt}\right) \mathrm{Tr}\left[e^{t\Delta_{\mathscr{X}}}\right]. \qquad (4.3.17)
\end{aligned}
$$

- An interesting extension of the previous example arises when $F(X)$ is of Laplace transform type, specifically,

$$F(X) = \mathcal{L}[g](X) = \int_0^\infty g(t)e^{-tX}\,dt, \quad X \ge 0, \tag{4.3.18}$$

for some, say, L^1-integrable function g. In this case the Maclaurin spectral coefficients are linked via Fubini's theorem to those of the heat kernel with $b_{2l}^{\alpha,\beta} = b_{2l}^{\alpha,\beta}[K_F]$ as

$$\begin{aligned} b_{2l}^{\alpha,\beta} =& \frac{1}{\mathrm{Vol}(\mathscr{X})}\mathrm{Tr}[F\mathscr{R}_l^{\alpha,\beta}](-\Delta_{\mathscr{X}}) \\ =& \sum_{k=0}^\infty \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})}\mathscr{R}_l^{\alpha,\beta}(\lambda_k^{\alpha,\beta})\int_0^\infty g(t)e^{-t\lambda_k^{\alpha,\beta}}\,dt \\ =& \int_0^\infty g(t)\sum_{k=0}^\infty \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})}\sum_{j=1}^l \mathsf{c}_j^l(\alpha,\beta)[\lambda_k^{\alpha,\beta}]^j e^{-t\lambda_k^{\alpha,\beta}}\,dt \\ =& \int_0^\infty g(t)\sum_{k=0}^\infty \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})}\mathscr{R}_l^{\alpha,\beta}\left(-\frac{d}{dt}\right)e^{-t\lambda_k^{\alpha,\beta}}\,dt = \int_0^\infty g(t)b_{2l}^{\alpha,\beta}[\mathscr{H}_t]\,dt. \end{aligned} \tag{4.3.19}$$

4.4 Orthogonality in a weighted Hilbert space and a Funk-Hecke identity on $\mathscr{X}$

The normalised Jacobi polynomials $\{\mathscr{P}_k^{\alpha,\beta} : k \ge 0,\ \alpha,\beta > -1\}$ form a complete orthogonal system in the weighted Hilbert space $L^2_{\mathsf{w}} = L^2(]-1,1[; \mathsf{w}^{\alpha,\beta}(t)dt)$, with the weight function $\mathsf{w}^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ (*cf.* [71]). Note that due to the integrability of the weight function $\mathsf{w}^{\alpha,\beta}$ near the endpoints of the interval we have $\mathscr{C}([-1,1]) \subset L^2_{\mathsf{w}}$. Now for functions $f, g \in L^2_{\mathsf{w}}$, we have the inner-product on L^2_{w} defined as

$$\langle f, g\rangle_{L^2_{\mathsf{w}}} = \int_{-1}^1 f(t)\overline{g(t)}\mathsf{w}^{\alpha,\beta}(t)\ dt = \int_{-1}^1 f(t)\overline{g(t)}(1-t)^\alpha(1+t)^\beta\ dt. \tag{4.4.1}$$

We also define $\ell^2_\omega = \{\xi = (\xi_j : j \ge 0) : \sum |\xi_j|^2\omega_j^{\alpha,\beta} < \infty\}$ with $\omega^{\alpha,\beta} = (\omega_j^{\alpha,\beta})$ a sequence of positive weights. This is a weighted sequence space with associated inner product given by

$$\langle \xi, \eta\rangle_{\ell^2_\omega} = \sum_{j=0}^\infty \xi_j\overline{\eta_j}\omega_j^{\alpha,\beta}. \tag{4.4.2}$$

Now by basic Hilbert space theory and an orthogonality argument using the polynomials $\{\mathscr{P}_k^{\alpha,\beta} : k \ge 0,\ \alpha,\beta > -1\}$, for given $K \in L^2_{\mathsf{w}}$, we can write

$$K(t) = \sum_{k=0}^\infty a_k^{\alpha,\beta}\mathscr{P}_k^{\alpha,\beta}(t), \qquad a_k^{\alpha,\beta} = a_k^{\alpha,\beta}(K) = \frac{\langle K, \mathscr{P}_k^{\alpha,\beta}\rangle_{L^2_{\mathsf{w}}}}{||\mathscr{P}_k^{\alpha,\beta}||^2_{L^2_{\mathsf{w}}}}. \tag{4.4.3}$$

Here the sequence $(a_k^{\alpha,\beta} : k \geq 0)$ lies in the weighted sequence space ℓ^2_ω, where by direct calculation the weight ω is seen to be given via the induced L^2_{w}-norm

$$\begin{aligned} \omega_k^{\alpha,\beta} &= ||\mathscr{P}_k^{\alpha,\beta}||^2_{L^2_{\mathsf{w}}} = \int_{-1}^{1} |\mathscr{P}_k^{\alpha,\beta}(t)|^2 (1-t)^\alpha (1+t)^\beta \, dt \\ &= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)^2}{2k+\alpha+\beta+1} \frac{\Gamma(k+1)\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)\Gamma(\alpha+k+1)}. \end{aligned} \tag{4.4.4}$$

Note that the last equation uses (A.3.2) together with $\mathscr{P}_k^{\alpha,\beta}(t) = P_k^{\alpha,\beta}(t)/P_k^{\alpha,\beta}(1)$. Moreover a direct calculation using orthogonality gives the L^2_{w}-norm squared of K as

$$\begin{aligned} ||K||^2_{L^2_{\mathsf{w}}} &= \sum_{k=0}^{\infty} |a_k^{\alpha,\beta}|^2 \omega_k^{\alpha,\beta} = ||(a_k^{\alpha,\beta} : k \geq 0)||^2_{\ell^2_\omega} \\ &= \sum_{k=0}^{\infty} \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)^2}{2k+\alpha+\beta+1} \frac{\Gamma(k+1)\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)\Gamma(\alpha+k+1)} |a_k^{\alpha,\beta}|^2. \end{aligned} \tag{4.4.5}$$

Thus it is plain that $K \in L^2(]-1,1[: \mathsf{w}^{\alpha,\beta}(t)dt) \iff (a_k^{\alpha,\beta}(K) : k \geq 0) \in \ell^2_{\omega^{\alpha,\beta}}$. Now moving forward for given function F, we build an associated kernel K_F for the operator $F(-\Delta_{\mathscr{X}})$ by invoking the spectral sum (4.3.4). Upon comparing this with (4.4.3) it is then plain that we can write the associated $(a_k^{\alpha,\beta})$ as

$$a_k^{\alpha,\beta}(K_F) = \frac{\langle K_F, \mathscr{P}_k^{\alpha,\beta} \rangle_{L^2_{\mathsf{w}}}}{||\mathscr{P}_k^{\alpha,\beta}||^2_{L^2_{\mathsf{w}}}} = \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} F(\lambda_k^{\alpha,\beta}) = \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} [\widehat{K_F}]_k^{\alpha,\beta}. \tag{4.4.6}$$

This therefore leads us naturally to the following results, which presents an explicit formula for the eigenvalues of the operator $F(-\Delta_{\mathscr{X}})$ as well its trace.

Theorem 4.4.1. *The eigenspaces $(\mathcal{H}_k : k \geq 0)$ of $F(-\Delta_{\mathscr{X}})$ (with kernel K_F) are precisely the same as those of $-\Delta_{\mathscr{X}}$:*

$$F(-\Delta_{\mathscr{X}})\phi = [\widehat{K_F}]_k^{\alpha,\beta} \phi, \qquad \phi \in \mathcal{H}_k, \tag{4.4.7}$$

where the eigenvalues are given by

$$\begin{aligned} [\widehat{K_F}]_k^{\alpha,\beta} &= \frac{(2\pi)^{d/2}}{2^\beta \Gamma(d/2)} \int_{-1}^{1} K_F(t) {}_2F_1\left(-j, \alpha+\beta+j+1; \alpha+1; \frac{1-t}{2}\right) \mathsf{w}^{\alpha,\beta}(t) dt \\ &= \frac{(2\pi)^{d/2}}{2^\beta \Gamma(d/2)} \int_{-1}^{1} K_F(t) \mathscr{P}_k^{\alpha,\beta}(t)(1-t)^\alpha (1+t)^\beta dt. \end{aligned} \tag{4.4.8}$$

Proof. We arrive easily at (4.4.8) by rearranging (4.4.6) and substituting using (4.4.3). Indeed taking advantage of the explicit formulae (4.3.9) for the multiplicity and (4.4.4) for the L_{w}-norm, after immediately cancelling some terms we can write,

$$M_j(\mathscr{X}) ||\mathscr{P}_j^{\alpha,\beta}||^2_{L^2_{\mathsf{w}}} = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{4.4.9}$$

Combining this with (4.3.8), we then have

$$\frac{\mathrm{Vol}(\mathscr{X})}{M_j(\mathscr{X})||\mathscr{P}_j^{\alpha,\beta}||^2_{L^2_{\mathrm{w}}}} = \frac{2^{2\alpha+2}\pi^{\alpha+1}\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} = \frac{(2\pi)^{d/2}}{2^\beta\Gamma(d/2)}, \tag{4.4.10}$$

and so the conclusion follows at once. □

Theorem 4.4.2. *Under the above assumptions on the operator $F(-\Delta_{\mathscr{X}})$ and its kernel K_F, the trace of $F(-\Delta_{\mathscr{X}})$ is given by*

$$\frac{\mathrm{Tr}\,[F(-\Delta_{\mathscr{X}})]}{\mathrm{Vol}(\mathscr{X})} = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})}[\widehat{K_F}]_k^{\alpha,\beta} = \sum_{k=0}^{\infty} a_k^{\alpha,\beta}(K_F) = K_F(1) = b_0^{\alpha,\beta}[K_F]. \tag{4.4.11}$$

Furthermore, we have the following Plancherel type identity:

$$||K_F||^2_{L^2_{\mathrm{w}}} = \frac{\Gamma(d/2)\Gamma(\beta+1+d/2)}{2^{3d/2-\beta}\pi^d\Gamma(\beta+1)} \sum_{k=0}^{\infty} M_k(\mathscr{X}) \left|[\widehat{K_F}]_k^{\alpha,\beta}\right|^2. \tag{4.4.12}$$

Proof. By definition the trace of $F(-\Delta_{\mathscr{X}})$ is the sum of its eigenvalues (counting multiplicities). So referring to (4.4.6) the first two equalities in (4.4.11) follow. The next equality follows when we refer to (4.4.3) and recall that $\mathscr{P}_k^{\alpha,\beta}(1) = 1$, with the final equality following by setting $l = 0$ in the definition of the Maclaurin spectral coefficients (4.3.12). Finally (4.4.12) follows by acknowledging (4.4.5) together with (4.4.6) and (4.4.9). □

Remark 4.4.3. With $a_k^{\alpha,\beta}$ as in (4.4.6), we may formally write (4.3.13)-(4.3.14) as

$$\begin{aligned} \text{LHS (4.3.13)} &= \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \sum_{k=0}^{\infty} a_k^{\alpha,\beta} \frac{\partial^{2l}}{\partial\theta^{2l}} \mathscr{P}_k^{\alpha,\beta}(\cos\theta)\Big|_{\theta=0} \\ &= \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \sum_{k=0}^{\infty} a_k^{\alpha,\beta} \mathscr{R}_l^{\alpha,\beta}(\lambda_k^{\alpha,\beta}) = \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \frac{\mathrm{Tr}[F\mathscr{R}_l^{\alpha,\beta}](-\Delta_{\mathscr{X}})}{\mathrm{Vol}(\mathscr{X})}, \end{aligned} \tag{4.4.13}$$

with $\mathscr{R}_l^{\alpha,\beta}$ as defined in (4.3.12). Furthermore we write $b_{2l}^{\alpha,\beta}[K_F]$ as

$$\begin{aligned} b_{2l}^{\alpha,\beta}[K_F] &= \frac{\mathrm{Tr}[F\mathscr{R}_l^{\alpha,\beta}](-\Delta_{\mathscr{X}})}{\mathrm{Vol}(\mathscr{X})} \\ &= \sum_{k=0}^{\infty} \frac{\mathscr{R}_l^{\alpha,\beta}(\lambda_k^{\alpha,\beta})}{||\mathscr{P}_k^{\alpha,\beta}||^2_{L^2_{\mathrm{w}}}} \int_{-1}^{1} K_F(t)\mathscr{P}_k^{\alpha,\beta}(t)(1-t)^\alpha(1+t)^\beta\, dt. \end{aligned} \tag{4.4.14}$$

The ϕ-series. In what follows we specialise the above discussion to each of the compact rank-one symmetric spaces $\mathscr{X}$ encountered before. For future reference it is convenient to introduce a summability notation, namely,

$$\phi(\mathbf{X};\gamma;a,f) = \sum_{k=0}^{\infty} \gamma_k X_k^a f(X_k), \tag{4.4.15}$$

where $\mathbf{X} = (X_k : k \geq 0)$ is a sequence with $X_k \geq 0$, $\gamma = (\gamma_k : k \geq 0) \subset [0,1]$, a is a positive integer, and $f = f(X)$, $X \geq 0$ is a smooth function with sufficiently fast decay at infinity. We call (4.4.15) the ϕ-series associated with $(\mathbf{X};\gamma;a,f)$. When $\gamma_k \equiv 1$ we write $\phi(\mathbf{X};a,f) = \phi(\mathbf{X};\gamma;a,f)$. [2] Before moving forward, let us recall the classical Jacobi type ϑ-series $\vartheta_1, \vartheta_2, \vartheta_3$ as (*see* [28, 86])

$$\vartheta_1(t) = \sum_{j=-\infty}^{\infty} e^{-j^2 t} = 1 + 2\sum_{j=1}^{\infty} e^{-j^2 t}, \tag{4.4.16}$$

$$\vartheta_2(t) = \sum_{j=0}^{\infty}(2j+1)e^{-(j+1/2)^2 t}, \qquad \vartheta_3(t) = 2\sum_{j=0}^{\infty} je^{-j^2 t}. \tag{4.4.17}$$

These functions together with their derivatives can be used to describe various heat related quantities on compact rank-one symmetric spaces in a convenient way (*cf.*, e.g., [7, 23, 28]). Let us now present some basic examples of ϕ-series:

- Let $f(X) = f_t(X) = e^{-tX^2}$. Then with $\mathbf{X} = (k : k \geq 0)$, $\gamma_k \equiv 1$ we have $\phi(\mathbf{X};0,f_t) = [\vartheta_1(t) + 1]/2$, whilst for $a = 2m$ with $m > 0$ we have

$$\phi(\mathbf{X};2m,f_t) = \sum_{k=0}^{\infty} k^{2m} e^{-tk^2} = (-1)^m \vartheta_1^{(m)}(t)/2, \tag{4.4.18}$$

  where $\vartheta_1^{(m)}$ denotes the m^{th} derivative of ϑ_1 as in (4.4.16). In contrast, if $\mathbf{X} = (k + 1/2 : k \geq 0)$, then with ϑ_2 as in (4.4.17), we have $\phi(\mathbf{X};1,f_t) = \vartheta_2(t)/2$, whilst for $a = 2m+1$ with $m > 0$ we have

$$\begin{aligned}\phi(\mathbf{X};2m+1,f_t) &= \sum_{k=0}^{\infty}(k+1/2)^{2m+1}e^{-t(k+1/2)^2} \\ &= \sum_{j=1/2}^{\infty} j^{2m+1}e^{-tj^2} = (-1)^m \vartheta_2^{(m)}(t)/2. \end{aligned}\tag{4.4.19}$$

  Finally, if $\mathbf{X} = (k : k \geq 0)$, then with ϑ_3 as in (4.4.17) we have $\phi(\mathbf{X};1,f_t) = \vartheta_3(t)/2$, whilst for $a = 2m+1$ with $m > 0$ we have

$$\phi(\mathbf{X};2m+1,f_t) = \sum_{k=0}^{\infty} k^{2m+1}e^{-tk^2} = (-1)^m \vartheta_3^{(m)}(t)/2. \tag{4.4.20}$$

- Let $f(X)$ be of Laplace transform type, i.e., $f(X) = \mathcal{L}[g](X)$ as in (4.3.18). Then with $e_s(X) = e^{-sX}$, the ϕ-series can be written as

$$\begin{aligned}\phi(\mathbf{X};\gamma;a,f) &= \sum_{k=0}^{\infty}\gamma_k X_k^a \int_0^{\infty} g(s)e^{-sX_k}\,ds = \int_0^{\infty} g(s)\sum_{k=0}^{\infty}\gamma_k X_k^a e^{-sX_k}\,ds \\ &= \int_0^{\infty} g(s)(-d/ds)^a \phi(\mathbf{X};\gamma;0,e_s)\,ds. \end{aligned}\tag{4.4.21}$$

[2] In applications to symmetric spaces below we have $X_k = k + \rho$ so that $X_k^2 - \rho^2 = \lambda_k^n$.

4.5 Applications to compact rank-one symmetric spaces $\mathscr{X} = \mathbf{G/H}$: Trace computations and ϕ-series

As a further application of the ongoing discussion here we give explicit trace formulation of $F(-\Delta_{\mathscr{X}})$ for the spaces $\mathscr{X} = \mathbf{G/H}$ in Section 4.3. Notation and terminology used here are precisely as introduced earlier in Sections 4.3 and 4.4.

The case $\mathscr{X} = \mathbb{S}^n$. Here $\alpha = \beta = (n-2)/2$ and the zonal spherical functions are given through the normalised Gegenbauer polynomials (see Appendix A.3) as $\mathscr{F}_k(\theta) = {}_2F_1(-k, k+n-1; n/2; (1-\cos\theta)/2) = \mathscr{P}_k^{(n-2)/2,(n-2)/2}(\cos\theta) = \mathscr{C}_k^{(n-1)/2}(\cos\theta)$ with $k \geq 0$.

Furthermore in light of Theorem 4.4.1 the eigenvalues $F(\lambda_k^{\alpha,\beta}) = F(\lambda_k^n)$ can in turn be expressed as

$$F(\lambda_k^n) = [\widehat{K_F}]_k^{(n-2)/2,(n-2)/2} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{-1}^1 K_F(t)\mathscr{C}_k^{(n-1)/2}(t)(1-t^2)^{(n-2)/2} dt. \quad (4.5.1)$$

Proposition 4.5.1. *The operator trace of $\mathbf{F} = F(-\Delta)$ with $-\Delta$ the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^n$ is given by*

$$\mathrm{Tr}\,[F(-\Delta)] = \begin{cases} \displaystyle\sum_{m=0}^{(n-3)/2} \frac{2\mathsf{a}_m^n}{(n-1)!}\phi(\mathbf{X}; 2m+2, f) & \text{for } n \geq 3 \text{ odd}, \\ \displaystyle\sum_{m=0}^{(n-2)/2} \frac{2\mathsf{b}_m^n}{(n-1)!}\phi(\mathbf{X}; 2m+1, f) & \text{for } n \geq 2 \text{ even}, \end{cases} \quad (4.5.2)$$

where $\mathbf{X} = (X_k = k + (n-1)/2 : k \geq 0)$, $f(X) = F(X^2 - (n-1)^2/4)$, and the scalars a_m^n and b_m^n are defined in (A.4.1) and (A.4.2) respectively, with $\mathsf{b}_0^2 = 1$.

Proof. Extracting the multiplicity $M_k(\mathbb{S}^n)$ from Table 1, we write

$$M_k(\mathbb{S}^n) = (2k+n-1)\frac{\Gamma(k-1+n)}{(n-1)!k!} = \frac{2(k+(n-1)/2)}{(n-1)!}\prod_{j=1}^{n-2}(k+j). \quad (4.5.3)$$

Now fix a sequence $X_k = k + (n-1)/2$, reflecting the eigenvalues $\lambda_k^n = k(k+n-1)$ on the unit sphere so that $X_k^2 - (n-1)^2/4 = \lambda_k^n$. For $n \geq 3$ odd, we can then re-write this via (A.4.1) as

$$M_k(\mathbb{S}^n) = \frac{2}{(n-1)!}\prod_{j=0}^{(n-3)/2}(X_k^2 - j^2) = \frac{2}{(n-1)!}\sum_{m=0}^{(n-3)/2} \mathsf{a}_m^n X_k^{2m+2}. \quad (4.5.4)$$

This allows us to write the trace of $\mathbf{F} = F(-\Delta)$ for $n \geq 3$ odd as

$$\sum_{k=0}^{\infty} M_k(\mathbb{S}^n) F(\lambda_k^n) = \sum_{m=0}^{(n-3)/2} \frac{2\mathsf{a}_m^n}{(n-1)!} \sum_{k=0}^{\infty} X_k^{2m+2} F(X_k^2 - (n-1)^2/4)$$
$$= \sum_{m=0}^{(n-3)/2} \frac{2\mathsf{a}_m^n}{(n-1)!} \sum_{k=0}^{\infty} X_k^{2m+2} f(X_k), \tag{4.5.5}$$

where we have defined $f(X) = F(X^2 - (n-1)^2/4)$. Note that $f(X_k) = F(\lambda_k^n)$ as in (4.5.1). Next for $n \geq 2$ even, with X_k and $f(X)$ as before, we can proceed from (4.5.3) using (A.4.2), as

$$M_k(\mathbb{S}^n) = \frac{2X_k}{(n-1)!} \prod_{j=1}^{n-2} (k+j) = \frac{2X_k}{(n-1)!} \prod_{j=\frac{1}{2}}^{\frac{n-2}{2}-\frac{1}{2}} (X_k^2 - j^2)$$
$$= \sum_{m=0}^{(n-2)/2} \frac{2\mathsf{b}_m^n}{(n-1)!} X_k^{2m+1}, \tag{4.5.6}$$

where for $n = 2$ we can easily take $\mathsf{b}_0^2 = 1$. This leads to the trace of $\mathbf{F} = F(-\Delta)$ for $n \geq 2$ even being given by

$$\mathrm{Tr}\,[F(-\Delta)] = \sum_{k=0}^{\infty} M_k(\mathbb{S}^n) F(\lambda_k^n) = \sum_{m=0}^{(n-2)/2} \frac{2\mathsf{b}_m^n}{(n-1)!} \sum_{k=0}^{\infty} X_k^{2m+1} f(X_k). \tag{4.5.7}$$

Recalling the definition of $\phi(\mathbf{X}; a, f)$ as in (4.4.15), this completes the proof. □

The case $\mathscr{X} = \mathbb{R}\mathbf{P}^n$. As indicated earlier in Section 4.3 the real projective space is not simply-connected, however, for $n \geq 2$, it has the unit sphere $\mathbb{S}^n$ as its universal and double cover. This leads to the zonal spherical functions on $\mathbb{R}\mathbf{P}^n$ corresponding precisely to those of $\mathbb{S}^n$ for k even. In particular here we have we $\mathscr{F}_k(\theta) = {}_2F_1(-2k, 2k+n-1; n/2; (1-\cos\theta)/2) = \mathscr{P}_{2k}^{(n-2)/2,(n-2)/2}(\cos\theta) = \mathscr{C}_{2k}^{(n-1)/2}(\cos\theta)$.

Moreover by virtue of a similar relation between eigenvalues here we have the identity

$$F(\lambda_k^n) = [\widehat{K_F}]_{2k}^{(n-2)/2,(n-2)/2} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{-1}^{1} K_F(t) \mathscr{C}_{2k}^{(n-1)/2}(t)(1-t^2)^{(n-2)/2} dt. \tag{4.5.8}$$

Proposition 4.5.2. *The operator trace of $\mathbf{F} = F(-\Delta)$ with $-\Delta$ the Laplace-Beltrami operator on the real projective space $\mathbb{R}\mathbf{P}^n$ is given by*

$$\mathrm{Tr}\,[F(-\Delta)] = \begin{cases} \displaystyle\sum_{m=0}^{(n-3)/2} \frac{2\mathsf{a}_m^n}{(n-1)!} \phi(\mathbf{X}; 2m+2, f) & \text{for } n \geq 3 \text{ odd}, \\ \displaystyle\sum_{m=0}^{(n-2)/2} \frac{2\mathsf{b}_m^n}{(n-1)!} \phi(\mathbf{X}; 2m+1, f) & \text{for } n \geq 2 \text{ even}, \end{cases} \tag{4.5.9}$$

where $\mathbf{X} = (X_k = 2k + (n-1)/2; k \geq 0)$, $f(X) = F(X^2 - (n-1)^2/4)$, and the scalars a_m^n and b_m^n are defined in (A.4.1) and (A.4.2) respectively.

Proof. We omit the full proof as it follows from the proof of Proposition 4.5.1, with the notable differences being the multiplicity $M_k(\mathbb{R}\mathbf{P}^n)$ as in Table 1, the eigenvalues $\lambda_k^n = 2k(2k+n-1)$, and definition of $X_k = 2k + (n-1)/2$ so that we again have $X_k^2 - (n-1)^2/4 = \lambda_k^n$. This in turn means that $f(X_k) = F(\lambda_k^n)$ as in (4.5.8). $\square$

The case $\mathscr{X} = \mathbb{C}\mathbf{P}^n$. Here $\alpha = n-1$, $\beta = 0$ and the zonal spherical functions are given by $\mathscr{F}_k(\theta) = {}_2F_1(-k, n+k; n; (1-\cos\theta)/2) = \mathscr{P}_k^{n-1,0}(\cos\theta)$. Moreover in light of Theorem 4.4.1,

$$F(\lambda_k^n) = [\widehat{K_F}]_k^{n-1,0} = \frac{(2\pi)^n}{(n-1)!}\int_{-1}^{1} K_F(t)\mathscr{P}_k^{n-1,0}(t)(1-t)^{n-1}\,dt. \tag{4.5.10}$$

Proposition 4.5.3. *The operator trace of $\mathbf{F} = F(-\Delta)$ with $-\Delta$ the Laplace-Beltrami operator on the complex projective space $\mathbb{C}\mathbf{P}^n$ is given by*

$$\operatorname{Tr}[F(-\Delta)] = \begin{cases} \displaystyle\sum_{m=0}^{n-1} \frac{2\mathsf{c}_m^n}{n!(n-1)!}\phi(\mathbf{X}; 2m+1, f) & \text{for } n \geq 3 \text{ odd},\\ \displaystyle\sum_{m=0}^{n-2} \frac{2\mathsf{d}_m^n}{n!(n-1)!}\phi(\mathbf{X}; 2m+3, f) & \text{for } n \geq 2 \text{ even},\end{cases} \tag{4.5.11}$$

where $\mathbf{X} = (X_k = k + n/2 : k \geq 0)$, $f(X) = F(X^2 - n^2/4)$, and the scalars c_m^n and d_m^n are defined in (A.4.3) *and* (A.4.4) *respectively.*

Proof. With the multiplicity $M_k(\mathbb{C}\mathbf{P}^n)$ given as in Table 1, we fix $X_k = k + n/2$ so that $X_k^2 - n^2/4 = \lambda_k^n = k(k+n)$ and write

$$M_k(\mathbb{C}\mathbf{P}^n) = \frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2 = \frac{2X_k}{n!(n-1)!}\prod_{j=1}^{n-1}(k+j)^2. \tag{4.5.12}$$

Now for $n \geq 3$ odd we proceed from (4.5.12), noting that the terms $(k+j)$ in the range $j = 1, \ldots, n-1$ can be re-written in the form $(X_k^2 - j^2)$ in the range $j = 1/2, 3/2, \ldots, (n-2)/2$. Together with (A.4.3), this allows us to write

$$M_k(\mathbb{C}\mathbf{P}^n) = \frac{2X_k}{n!(n-1)!}\prod_{j=\frac{1}{2}}^{\frac{n-3}{2}+\frac{1}{2}}(X_k^2 - j^2)^2 = \frac{2}{n!(n-1)!}\sum_{m=0}^{n-1}\mathsf{c}_m^n X_k^{2m+1}. \tag{4.5.13}$$

Hence the trace of $\mathbf{F} = F(-\Delta)$ for $n \geq 3$ odd can be written as

$$\begin{aligned}\operatorname{Tr}[F(-\Delta)] = \sum_{k=0}^{\infty} M_k(\mathbb{C}\mathbf{P}^n)F(\lambda_k^n) &= \sum_{m=0}^{n-1}\frac{2\mathsf{c}_m^n}{n!(n-1)!}\sum_{k=0}^{\infty}X_k^{2m+1}F(X_k^2 - n^2/4)\\ &= \sum_{m=0}^{n-1}\frac{2\mathsf{c}_m^n}{n!(n-1)!}\sum_{k=0}^{\infty}X_k^{2m+1}f(X_k),\end{aligned} \tag{4.5.14}$$

where we define $f(X) = F(X^2 - n^2/4)$ resulting in $f(X_k) = F(\lambda_k^n)$, as in (4.5.10). Next for $n \geq 2$ even, with $f(X)$ and X_k as above, we have via (A.4.4)

$$M_k(\mathbb{C}\mathbf{P}^n) = \frac{2X_k}{n!(n-1)!}\prod_{j=1}^{n-1}(k+j)^2 = \frac{2X_k^3}{n!(n-1)!}\prod_{j=1}^{\frac{n-2}{2}}(X_k^2 - j^2)^2$$
$$= \frac{2}{n!(n-1)!}\sum_{m=0}^{n-2} \mathsf{d}_m^n X_k^{2m+3}.$$

This leads to the trace of $\mathbf{F} = F(-\Delta)$ for $n \geq 2$ even taking the form

$$\mathrm{Tr}\,[F(-\Delta)] = \sum_{k=0}^{\infty} M_k(\mathbb{C}\mathbf{P}^n)F(\lambda_k^n) = \sum_{m=0}^{n-2}\frac{2\mathsf{d}_m^n}{n!(n-1)!}\sum_{k=0}^{\infty} X_k^{2m+3} f(X_k),$$

where we have arrived at the result after substitution of (4.4.15). $\square$

The case $\mathscr{X} = \mathbb{H}\mathbf{P}^n$. Here $\alpha = 2n-1$, $\beta = 1$ and the zonal spherical functions are given by $\mathscr{F}_k(\theta) = {}_2F_1(-k, 2n+k+1; 2n; (1-\cos\theta)/2) = \mathscr{P}_k^{2n-1,1}(\cos\theta)$. Moreover

$$F(\lambda_k^n) = [\widehat{K_F}]_k^{2n-1,1} = \frac{(2\pi)^{2n}}{2\Gamma(2n)}\int_{-1}^{1} K_F(t)\mathscr{P}_k^{2n-1,1}(t)(1-t)^{2n-1}(1+t)\,dt. \qquad (4.5.15)$$

Proposition 4.5.4. *The operator trace of $\mathbf{F} = F(-\Delta)$ with $-\Delta$ the Laplace-Beltrami operator on the quaternionic projective space $\mathbb{H}\mathbf{P}^n$ is given by*

$$\mathrm{Tr}\,[F(-\Delta)] = \sum_{m=0}^{n-1}\frac{2\mathsf{e}_m^n}{\Gamma(2n)(2n+1)!}\phi(\mathbf{X}; 2m+1, f), \qquad (4.5.16)$$

where $\mathbf{X} = (X_k = k + (2n+1)/2 : k \geq 0)$, $f(X) = F(X^2 - (2n+1)^2/4)$, and the scalars e_m^n are defined in (A.4.5).

Proof. Let $X_k = k + (2n+1)/2$ with $X_k^2 - (2n+1)^2/4 = \lambda_k^n = k(k+2n+1)$. Then referring to Table 1 we can write the multiplicity as

$$M_k(\mathbb{H}\mathbf{P}^n) = \frac{2X_k(k+2n)}{(2n-1)!(2n+1)!(k+1)}\prod_{j=1}^{2n-1}(k+j)^2$$
$$= \frac{2X_k}{(2n-1)!(2n+1)!}\left[X^2 - (2n-1)^2/4\right]\prod_{j=1/2}^{n-3/2}(X^2 - j^2)^2$$
$$= \sum_{m=0}^{2n-1}\frac{2\mathsf{e}_m^n X_k^{2m+1}}{(2n-1)!(2n+1)!}, \qquad (4.5.17)$$

where we have applied (A.4.5). Substituting for this in the trace of $\mathbf{F} = F(-\Delta)$ results in the representation

$$\sum_{k=0}^{\infty} M_k(\mathbb{H}\mathbf{P}^n)F(\lambda_k^n) = \sum_{m=0}^{2n-1}\frac{2\mathsf{e}_m^n}{(2n-1)!(2n+1)!}\sum_{k=0}^{\infty} X_k^{2m+1}F(X_k^2 - (2n+1)^2/4)$$
$$= \sum_{m=0}^{2n-1}\frac{2\mathsf{e}_m^n}{(2n-1)!(2n+1)!}\sum_{k=0}^{\infty} X_k^{2m+1}f(X_k), \qquad (4.5.18)$$

where we have defined $f(X) = F(X^2 - (2n+1)^2/4)$ so that we have $f(X_k) = F(\lambda_k^n)$, as in (4.5.15). The result follows after acknowledging (4.4.15). $\square$

The case $\mathscr{X} = \mathbf{P}^2(\mathbf{Cay})$. Here $\mathscr{F}_k(\theta) = {}_2F_1(-k, k+11; 8; (1-\cos\theta)/2) = \mathscr{P}_k^{7,3}(\cos\theta)$ are the zonal spherical functions ($\alpha = 7$, $\beta = 3$) and the eigenvalues $F(\lambda_k^{\alpha,\beta}) = F(\lambda_k)$ are exactly

$$F(\lambda_k) = [\widehat{K_F}]_k^{7,3} = \frac{32\pi^8}{7!}\int_{-1}^{1} K_F(t)\mathscr{P}_k^{7,3}(t)(1-t^2)^5(1-t)^3\, dt. \tag{4.5.19}$$

Proposition 4.5.5. *The operator trace of $\mathbf{F} = F(-\Delta)$ with $-\Delta$ the Laplace-Beltrami operator on the Cayley plane $\mathscr{X} = \mathbf{P}^2(\mathrm{Cay})$ can be written*

$$\mathrm{Tr}\,[F(-\Delta)] = \sum_{m=0}^{7} \frac{12\mathsf{f}_m}{7!11!}\phi(\mathbf{X}; 2m+1, f), \tag{4.5.20}$$

where $\mathbf{X} = (X_k = k + 11/2 : k \geq 0)$, $f(X) = F(X^2 - 121/4)$, and the scalars f_m are defined in (A.4.6).

Proof. Let $X_k = k + 11/2$ with $X_k^2 - 121/4 = \lambda_k^n = k(k+11)$. The multiplicity on the Cayley plane $\mathscr{X} = \mathbf{P}^2(\mathrm{Cay})$ can be written as

$$M_k = \frac{12X_k}{7!11!}\frac{\Gamma(k+8)\Gamma(k+11)}{\Gamma(k+1)\Gamma(k+4)} = \frac{12X_k}{7!11!}\prod_{j=1}^{7}(k+j)\prod_{j=4}^{10}(k+j). \tag{4.5.21}$$

Transforming and writing each product on the right in terms of X_k, we have

$$\begin{aligned} M_k &= \frac{12X_k}{7!11!}\prod_{j=1/2}^{3/2}(X_k^2 - j^2)\prod_{j=1/2}^{9/2}(X_k^2 - j^2) \\ &= \frac{12X_k}{7!11!}(X_k^2 - 1/4)^2(X_k^2 - 9/4)^2(X_k^2 - 25/4)(X_k^2 - 49/4)(X_k^2 - 81/4) \\ &= \frac{12}{7!11!}\sum_{j=0}^{7}\mathsf{f}_j X_k^{2j+1}. \end{aligned} \tag{4.5.22}$$

where we have applied (A.4.6). Thus the trace of $\mathbf{F} = F(-\Delta)$ can be written as

$$\begin{aligned} \sum_{k=0}^{\infty} M_k F(\lambda_k) &= \sum_{m=0}^{7}\frac{12\mathsf{f}_m}{7!11!}\sum_{k=0}^{\infty}\gamma_k X_k^{2m+1}F(X_k^2 - 121/4) \\ &= \sum_{m=0}^{7}\frac{12\mathsf{f}_m}{7!11!}\sum_{k=0}^{\infty}\gamma_k X_k^{2m+1}f(X_k) = \sum_{m=0}^{7}\frac{12\mathsf{f}_m}{7!11!}\phi(\mathbf{X}; \gamma; 2m+1, f) \end{aligned} \tag{4.5.23}$$

where $f(X_k) = F(\lambda_k)$ as in (4.5.19). This completes the proof. $\square$

Remark 4.5.6. Replacing F with $F\mathscr{R}_l^{\alpha,\beta}$ in each proposition gives $b_{2l}^{\alpha,\beta}[K_F]$ by virtue of the trace formulation (4.3.12).

Chapter 5

The Proportionality Principle on Symmetric Spaces and Maclaurin Spectral Functions

5.1 Introduction

Let $(\mathscr{M}, g)$ be a d-dimensional Riemannian manifold without boundary, and let $-\Delta_{\mathscr{M}}$ denote the Laplace-Beltrami operator on $\mathscr{M}$. Then by basic spectral theory there exists a resolution of the identity $(E_\lambda : \lambda > 0)$ describing the spectral measure dE_λ, such that for a function $F = F(X)$ with $X \geq 0$ in the Borel functional calculus of $-\Delta_{\mathscr{M}}$ we may write the operator $F(-\Delta_{\mathscr{M}})$ via the integral

$$F(-\Delta_{\mathscr{M}}) = \int_0^\infty F(\lambda)\, dE_\lambda. \tag{5.1.1}$$

As stated earlier, if $\mathscr{M}$ is compact then there exists a complete orthonormal basis of eigenfunctions $(\phi_j : j \geq 0)$ in $L^2(\mathscr{M}; dv_g)$, specifically $\langle \phi_j, \phi_k \rangle_{L^2(\mathscr{M})} = 0$ for each $j \neq k \geq 0$, and $||\phi_j||_{L^2(\mathscr{M})} = 1$ for each $j \geq 0$. The associated spectrum of eigenvalues $\Sigma(-\Delta_{\mathscr{M}}) = (\lambda_j : j \geq 0)$ may be arranged in ascending order $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$, each having finite multiplicity M_k, whilst $-\Delta_{\mathscr{M}}\phi_j = \lambda_j \phi_j$. Here the spectral projection dE_λ becomes the orthogonal projection onto the eigenspace associated to $\lambda \in \Sigma(-\Delta_{\mathscr{M}})$, that is $dE_\lambda = \sum \phi_j \otimes \phi_j$. The integral (5.1.1) is thus seen to reduce to a spectral sum. In contrast, in the non-compact case we require the full integral (with no reduction) due to the presence of a continuous part of the spectrum.

In this chapter we study the heat semi-group $(T(t) : t > 0)$, given in the setting above by $F = F_t(X) = e^{-tX}$. Here (5.1.1) becomes

$$e^{t\Delta_{\mathscr{M}}} = \int_0^\infty e^{-t\lambda}\, dE_\lambda. \tag{5.1.2}$$

Furthermore, on the level of the heat kernel $H_{\mathscr{M}}(t; x, y)$, which is the Schwartz kernel of the operator (5.1.2), with x and y spatial variables on $\mathscr{M}$, we can write

$$H_{\mathscr{M}}(t; x, y) = \int_0^\infty e^{-t\lambda}\, dE_\lambda(x, y). \tag{5.1.3}$$

In the compact case $T(t)$ is of trace class, and it was shown in [82], using PDE techniques and the heat parametrix, that the heat trace $\mathrm{Tr}T(t) = \Theta(t; \mathscr{M})$ satisfies the short-time asymptotics

$$\Theta(t; \mathscr{M}) = \int_{\mathscr{M}} H_{\mathscr{M}}(t; x, x)\, dv_g \sim \frac{1}{(4\pi t)^{d/2}} \sum_{j=0}^{\infty} \mathsf{a}_j(\mathscr{M}) t^j, \qquad t \searrow 0. \tag{5.1.4}$$

Here the so-called Minakshisunaram-Pleijel heat coefficients ($\mathsf{a}_j = \mathsf{a}_j(\mathscr{M}) : j = 0, 1, \dots$) are a set of scalars that arise as integrals of polynomials of the Riemann curvature tensor and its associated derivatives.

We now specialise to the case of symmetric spaces $\mathscr{X} = \mathbf{G}/\mathbf{H}$ of rank one (of both compact and non-compact type). A complete list of these spaces was given earlier in Chapter 2, however for the sake of the reader's convenience this will appear again below (see Section 5.2). Naturally in the compact case , where we denote $\mathscr{X} = \mathscr{X}_c$, we still have (5.1.4) by default with $\mathscr{X}_c$ replacing $\mathscr{M}$. Interestingly, a similar expansion can be shown to hold in the non-compact case (see [29]). For the sake of future reference, this takes the form

$$\Theta(t; \mathscr{X}_{nc}) = \int_{\mathscr{X}_{nc}} H_{\mathscr{X}_{nc}}(t; x, x)\, dv_g = \frac{1}{(4\pi t)^{d/2}} \sum_{j=0}^{\infty} \mathsf{a}_j(\mathscr{X}_{nc}) t^j. \tag{5.1.5}$$

By a well-known proportionality principle (see [29]), for a pair of symmetric spaces $\mathscr{X}_c$ and $\mathscr{X}_{nc}$ in duality (compact vs. non-compact), we have the following relationship between the associated Minakshisundaram-Pleijel coefficients

$$\mathsf{a}_j(\mathscr{X}_c) = (-1)^j \mathsf{a}_j(\mathscr{X}_{nc}). \tag{5.1.6}$$

The Maclaurin heat coefficients $b^n_{2\ell}(t; \mathscr{X})$ for $\ell \geq 0$ were introduced and studied earlier in the thesis. Recall that in the context of symmetric spaces of rank one, the heat kernel is a function of the geodesic distance between x and y, specifically $H_{\mathscr{X}}(t; x, y) = H_{\mathscr{X}}(t; \psi)$. Then we have

$$b^n_{2\ell}(t; \mathscr{X}) = \frac{d^{2\ell}}{d\psi^{2\ell}} H_{\mathscr{X}}(t; \psi)\Big|_{\psi=0}. \tag{5.1.7}$$

One of the main aims of this chapter is to extend the proportionality principle (5.1.6) to the context of the $b^n_{2\ell}(t;\mathscr{X})$ functions. A prime motivation for this is the fact that the $b^n_{2\ell}(t;\mathscr{X})$ family generalises the heat trace as $b^n_0(t;\mathscr{X}) = \Theta(t;\mathscr{X})$.

In Section 5.3 we calculate the asymptotics ($t \searrow 0$) for the Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X}_c)$ on each rank-one symmetric space of compact type $\mathscr{X}_c$, and we show that they can be represented in the form

$$b^n_{2\ell}(t;\mathscr{X}_c) \sim \frac{e^{\rho^2 t}(4\pi)^\ell}{(4\pi t)^{d/2+\ell}} \sum_{j=0}^{\infty}\sum_{p=0}^{\ell-1}\sum_{m=0}^{\ell-p} \mathsf{p}^{m,\ell}_{j,p}(\mathscr{X}_c) t^{j+m+p}, \qquad t \searrow 0. \tag{5.1.8}$$

In each case, one can arrive at the asymptotics for the heat trace $b^n_0(t;\mathscr{X}_c)$ by setting $\ell = 0$, $p = 0$, and $m = 0$, together with removing the summations over p and m, resulting in $\mathsf{p}^{0,0}_{j,0}(\mathscr{X}_c) = \mathsf{p}_j(\mathscr{X}_c)$ where

$$\Theta(t;\mathscr{X}_c) = b^n_0(t;\mathscr{X}_c) \sim \frac{e^{\rho^2 t}}{(4\pi t)^{d/2}} \sum_{j=0}^{\infty} \mathsf{p}_j(\mathscr{X}_c) t^j, \qquad t \searrow 0. \tag{5.1.9}$$

Once the series expansion of the exponential term is incorporated into the summation in (5.1.9), using the Cauchy product for power series, one arrives at (5.1.4). Returning to the asymptotics for $b^n_{2\ell}(t;\mathscr{X}_c)$ in (5.1.8), we give explicit formulae for $\mathsf{p}^{m,\ell}_{j,p}(\mathscr{X}_c)$ for each of the compact symmetric spaces of rank-one. In Section 5.4 we mirror this by explicitly arranging the Maclaurin heat coefficients on each non-compact rank-one symmetric space $\mathscr{X}_{nc}$ as an infinite series of the form

$$b^n_{2\ell}(t;\mathscr{X}_{nc}) = \frac{e^{-\rho^2 t}(4\pi)^\ell}{(4\pi t)^{d/2+\ell}} \sum_{j=0}^{\infty}\sum_{p=0}^{\ell-1}\sum_{m=0}^{\ell-p} \mathsf{q}^{m,\ell}_{j,p}(\mathscr{X}_{nc}) t^{j+m+p}. \tag{5.1.10}$$

Again we note that setting $\ell = m = p = 0$ and removing the summation over p and m results in $\mathsf{q}^{0,0}_{j,0}(\mathscr{X}_{nc}) = \mathsf{q}_j(\mathscr{X}_{nc})$, where we have

$$\Theta(t;\mathscr{X}_{nc}) = b^n_0(t;\mathscr{X}_{nc}) = \frac{e^{-\rho^2 t}}{(4\pi t)^{d/2}} \sum_{j=0}^{\infty} \mathsf{q}_j(\mathscr{X}_{nc}) t^j, \tag{5.1.11}$$

where incorporating the exponential term results in (5.1.5).

In Section 5.5, which can be seen as the climax of this chapter, we estabish the proportionality principle for the Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X})$. As a matter of fact, defining the so-called *generalised* Minakshisundaram-Pleijel heat coefficients $\mathsf{a}_{j,\ell}(\mathscr{X})$, in the compact case through the short-time asymptotic expansion

$$b^n_{2\ell}(t;\mathscr{X}_c) \sim \frac{1}{(4\pi t)^{d/2+\ell}} \sum_{j=0}^{\infty} \mathsf{a}_{j,\ell}(\mathscr{X}_c) t^j, \qquad t \searrow 0, \tag{5.1.12}$$

and in the non-compact case through the infinite series

$$b^n_{2\ell}(t;\mathscr{X}_{nc}) = \frac{1}{(4\pi t)^{d/2+\ell}} \sum_{j=0}^{\infty} \mathsf{a}_{j,\ell}(\mathscr{X}_{nc}) t^j, \tag{5.1.13}$$

we show using explicit calculations that for a pair of spaces in duality (compact vs. non-compact) we have $\mathsf{p}^{m,\ell}_{j,p}(\mathscr{X}_c) = (-1)^{j+p+m}\mathsf{q}^{m,\ell}_{j,p}(\mathscr{X}_{nc})$, with the quantities $\mathsf{p}^{m,\ell}_{j,p}(\mathscr{X}_c)$ and $\mathsf{q}^{m,\ell}_{j,p}(\mathscr{X}_{nc})$ here referring to those introduced in (5.1.8) and (5.1.10) respectively. This will subsequently lead to the proportionality principle

$$\mathsf{a}_{j,\ell}(\mathscr{X}_c) = (-1)^j \mathsf{a}_{j,\ell}(\mathscr{X}_{nc}). \tag{5.1.14}$$

5.2 The Maclaurin heat coefficients on rank-one symmetric spaces

Let $\mathscr{X}$ be a d-dimensional rank-one symmetric space. The radial part of the Laplace-Beltrami operator has the form

$$[-\Delta_{\mathscr{X}}]_{rad} = -\frac{\partial^2}{\partial\psi^2} - \frac{A'(\psi)}{A(\psi)}\frac{\partial}{\partial\psi}, \tag{5.2.1}$$

where $A(\psi)$ is the area of the sphere of radius ψ centered at the origin in $\mathscr{X}$. A suitable change of variables reduces this to the well-known Jacobi operator, whose eigenfunctions are precisely given by the Jacobi function $\mathscr{P}^{\alpha,\beta}_\mu = \mathscr{P}^{\alpha,\beta}_\mu(z)$, with $\alpha, \beta > -1$ and $\mu \in \mathbb{C}$. For the sake of clarity, these are defined in terms of the hypergeometric function as

$$\mathscr{P}^{\alpha,\beta}_\mu(z) = {}_2F_1\left(-\mu, \mu+\alpha+\beta+1; \alpha+1; (1-z)/2\right), \quad \mu \in \mathbb{C}. \tag{5.2.2}$$

The particular interest in the Jacobi functions here stems from the fact that for suitable choices of z, and certain ranges of μ, α, and β, (see Table 3), (5.2.2) represents the zonal spherical functions on rank-one symmetric spaces of both compact and non-compact types.

Heat coefficients on a compact space $\mathscr{X}_c$. The rank-one symmetric spaces of compact type are the unit sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbb{R}\mathbf{P}^n = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$, the quaternionic projective space $\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$, and the Cayley Plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}^4/\mathbf{Spin}(9)$. In the compact case, referring to (5.2.1), one has

$$A = A(\theta) = \omega_{d-1}\left[(\sin\theta/2)/2\right]^{2\alpha+1}(\cos\theta/2)^{2\beta+1}, \tag{5.2.3}$$

where $\omega_{d-1} = \mathrm{Vol}(\mathbb{S}^{d-1})$, and $\alpha, \beta > -1$ are real parameters associated to $\mathscr{X}_c$ (see Table 3). This leads to the radial part of the Laplace-Beltrami operator on $\mathscr{X}_c$ having the form

$$\begin{aligned} [-\Delta_{\mathscr{X}_c}]_{rad} &= -\frac{\partial^2}{\partial\theta^2} - \left[\frac{1}{2}(2\alpha+1)\cot\theta/2 - \frac{1}{2}(2\beta+1)\tan\theta/2\right]\frac{\partial}{\partial\theta} \\ &= -\frac{\partial^2}{\partial\theta^2} - [(2\beta+1)\cot\theta + (\alpha-\beta)\cot\theta/2]\frac{\partial}{\partial\theta}, \end{aligned} \tag{5.2.4}$$

which after a change of variables $\theta \mapsto \arccos t$ becomes the well-known Jacobi operator

$$\mathcal{L}^{\alpha,\beta} = (1-t^2)\frac{d^2}{dt^2} - (\alpha - \beta + (\alpha+\beta+2)t)\frac{d}{dt}. \tag{5.2.5}$$

The heat kernel on a compact rank-one symmetric space is given by

$$H_{\mathscr{X}_c}(t;x,y) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X}_c)}{\mathrm{Vol}(\mathscr{X}_c)} \mathscr{P}_k^{\alpha,\beta}(\cos\theta) e^{-t\lambda_k^{\alpha,\beta}}, \tag{5.2.6}$$

where α, β are the parameters associated to $\mathscr{X}_c$, $M_k(\mathscr{X}_c)$ is the multiplicity function associated to the numerically distinct eigenvalues $\lambda_k^{\alpha,\beta} = k(k+\alpha+\beta+1)$ of the Jacobi operator (A.3.3), given by

$$M_k(\mathscr{X}) = \frac{(\alpha+\beta+2k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\beta+1)\Gamma(k+d/2)}{\Gamma(k+1)\Gamma(\alpha+\beta+2)\Gamma(d/2)\Gamma(k+\beta+1)}, \tag{5.2.7}$$

$\mathrm{Vol}(\mathscr{X}_c)$ denotes the volume of the space $\mathscr{X}_c$, and θ denotes the geodesic distance between the points $x, y \in \mathscr{X}_c$. Furthermore, $\mathscr{P}_k^{\alpha,\beta}$ are the zonal spherical functions associated to the space $\mathscr{X}_c$, which in the case of a rank-one symmetric space of compact type are given exactly by the normalised Jacobi polynomials $\mathscr{P}_k^{\alpha,\beta}(t) = P_k^{\alpha,\beta}(t)/P_k^{\alpha,\beta}(1)$. We define these via (5.2.2) as

$$\mathscr{P}_k^{\alpha,\beta}(\cos\theta) = {}_2F_1(-k, k+\alpha+\beta+1; \alpha+1; (1-\cos\theta)/2). \tag{5.2.8}$$

We now define the Maclaurin heat coefficients $b_{2\ell}^n(t;\mathscr{X}_c)$ for compact rank-one symmetric spaces. To motivate the name we abuse notation slightly to write $H_{\mathscr{X}_c}(t;\theta) = H_{\mathscr{X}_c}(t;x,y)$ as in (5.2.6), which we then take the Maclaurin expansion about $\theta = 0$ of so that the coefficients of $\theta^{2\ell}/(2\ell)!$ are given by

$$b_{2\ell}^n(t;\mathscr{X}_c) := \frac{d^{2\ell}}{d\theta^{2\ell}} H_{\mathscr{X}}(t;\theta)\Big|_{\theta=0}. \tag{5.2.9}$$

The name of these coefficients is motivated by how they are seen to arise as the coefficients of the Maclaurin expansion of the heat kernel about $\theta = 0$. We note that the derivatives will pass onto the Jacobi polynomial $\mathscr{P}_k^{\alpha,\beta}(\cos\theta)$ as in (5.2.6). Referring to Lemma A.6.3 and (5.2.10), the Maclaurin heat coefficients $b_{2\ell}^n(t;\mathscr{X}_c)$ can then be written as

$$b_{2\ell}^n(t;\mathscr{X}_c) = \mathscr{R}_\ell^{\alpha,\beta}(-d/dt)\Theta(t;\mathscr{X}_c). \tag{5.2.10}$$

In particular we have $b_0^n(t;\mathscr{X}_c)=\Theta(t;\mathscr{X}_c)$, and so the Maclaurin heat coefficients are seen to somewhat generalise the trace of the heat kernel.

On a non-compact space $\mathscr{X}_{nc}$. The rank-one symmetric spaces of non-compact type are the real hyperbolic space $\mathbb{R}\mathbf{H}^n=\mathbf{SO}_0(n,1)/\mathbf{SO}(n)$, the complex hyperbolic space $\mathbb{C}\mathbf{H}^n=\mathbf{SU}(n,1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$, the quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n=\mathbf{Sp}(n,1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$, and the hyperbolic Cayley plane $\mathbf{H}^2(\mathrm{Cay})=\mathbf{F}_\star^4/\mathbf{Spin}(9)$. Here we have

$$A(r)=\omega_{d-1}(\sinh(r))^{2\alpha+1}(\cosh(r))^{2\beta+1}, \tag{5.2.11}$$

and hence the radial part of the Laplace-Beltrami operator on $\mathscr{X}_{nc}$ is

$$[-\Delta_{\mathscr{X}_{nc}}]_{rad}=-\frac{\partial^2}{\partial r^2}-[(2\alpha+1)\coth r+(2\beta+1)\tanh r]\frac{\partial}{\partial r}. \tag{5.2.12}$$

We rescale this to define a new operator $\mathcal{L}^{\alpha,\beta}$ as

$$\mathcal{L}^{\alpha,\beta}=-\frac{\partial^2}{\partial r^2}-\frac{1}{2}\left[(2\alpha+1)\coth\frac{r}{2}+(2\beta+1)\tanh\frac{r}{2}\right]\frac{\partial}{\partial r} \tag{5.2.13}$$

Moreover, the heat kernel $H_{\mathscr{X}_{nc}}(t;x,y)$ is given in this case by the integral

$$H_{\mathscr{X}_{nc}}(t;x,y)=\frac{2^{2\beta-1}\Gamma(\alpha+1)}{\pi^{\alpha+2}}\int_0^\infty\Phi_\lambda^{\alpha,\beta}(r)e^{-t(\lambda^2+\rho^2)}\mu(\lambda)\,d\lambda. \tag{5.2.14}$$

where r is the geodesic distance between points $x,y\in\mathscr{X}_{nc}$, $\rho^2+\lambda^2$ are the eigenvalues of the operator $\mathcal{L}^{\alpha,\beta}$ as in (5.2.13), where we denote $\rho=(\alpha+\beta+1)/2$, and $\Phi_\lambda^{\alpha,\beta}(r)$ are the zonal spherical functions on $\mathscr{X}_{nc}$, given via the Jacobi function as

$$\Phi_\lambda^{\alpha,\beta}(r)=\mathscr{P}_{-i\lambda-\rho}^{\alpha,\beta}(\cosh r)={}_2F_1\left(\rho+i\lambda,\rho-i\lambda;\alpha+1;(1-\cosh r)/2\right). \tag{5.2.15}$$

Here $\mu(\lambda)=|\mathsf{C}(\lambda)|^{-2}$ denotes the Plancherel measure, where $\mathsf{C}(\lambda)$ is the Harish-Chandra function

$$\mathsf{C}(\lambda)=\lim_{r\nearrow\infty}\Phi_\lambda^{\alpha,\beta}(r)e^{(\rho-i\lambda)r}=\frac{4^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(2i\lambda)}{\Gamma(\rho+i\lambda)\Gamma(i\lambda+(\alpha+1-\beta)/2)}. \tag{5.2.16}$$

Next we define the Maclaurin heat coefficients $b_{2\ell}^n(t;\mathscr{X}_{nc})$ for a non-compact space. Writing $H_{\mathscr{X}_{nc}}(t;r)=H_{\mathscr{X}_{nc}}(t;x,y)$, with r the geodesic distance between x and y, we can define the Maclaurin heat coefficients by

$$b_{2\ell}^n(t;\mathscr{X}_{nc})=\frac{d^{2\ell}}{dr^{2\ell}}H_{\mathscr{X}_{nc}}(t;r)\Big|_{r=0}. \tag{5.2.17}$$

Similar to the compact case, the derivatives above fall directly onto the Jacobi function. Referring to Lemma A.6.4 and (A.6.25) in this case, we can write the Maclaurin heat coefficients $b_{2\ell}^n(t;\mathscr{X}_{nc})$ as

$$b_{2\ell}^n(t;\mathscr{X}_{nc})=\mathscr{Q}_\ell^{\alpha,\beta}(d/dt)\Theta(t;\mathscr{X}_{nc}). \tag{5.2.18}$$

Again we see that when $\ell = 0$ we have $b_0^n(t; \mathscr{X}_{nc}) = \Theta(t; \mathscr{X}_{nc})$.

Table 3 illustrates the duality of the spaces of compact and non-compact type, showing that the spaces that correspond to one-another share not only their dimension, but also the parameters α and β.

5.3 Expansion of $b_{2\ell}^n(t; \mathscr{X})$ as $t \searrow 0$ on compact $\mathscr{X}$

On a rank-one symmetric space $\mathscr{X} = \mathscr{X}_c$ of compact type, the trace of the heat kernel is given by

$$\Theta(t; \mathscr{X}_c) = \frac{1}{\mathrm{Vol}(\mathscr{X}_c)} \sum_{k=0}^{\infty} M_k(\mathscr{X}_c) e^{-\lambda_k^{\alpha,\beta} t}, \quad t > 0, \tag{5.3.1}$$

with the multiplicity $M_k(\mathscr{X}_c)$ given in (5.2.7). In what follows we specialse to each of the rank-one symmetric spaces $\mathscr{X}_c$ and provide an explicit asymptotic formula for the Maclaurin heat coefficients $b_{2\ell}^n(t; \mathscr{X}_c)$ by first writing the trace of the heat kernel $\Theta(t; \mathscr{X}_c)$ in terms of the classical Jacobi theta functions, before using the known asymptotics for these functions. We can then use the relation (5.2.10) to derive the asymptotics for $b_{2\ell}^n(t; \mathscr{X}_c)$.

We can arrange the asymptotics of the Maclaurin heat coefficients $b_{2\ell}^n(t; \mathscr{X}_c)$ in the form (5.1.8) where in the following propositions we present the explicit value of $\mathsf{p}_{j,p}^{m,\ell}(\mathscr{X}_c)$.

In each case we will first write the multiplicity function in terms of a polynomial of $X_k = k + \rho$, where $\rho = (\alpha + \beta = 1)/2$. This is because we then have $X_k^2 - \rho^2 = k(k + \alpha + \beta + 1) = \lambda_k^n$. We also define a general term

$$\mathscr{G}_{j,p}^{m,\ell} = \mathscr{G}_{j,p}^{m,\ell}(\mathscr{X}_c) = \binom{\ell - p}{m} \rho^{2m} \mathsf{h}_{\ell-p}^{\ell}(\alpha, \beta), \tag{5.3.2}$$

where $\mathsf{h}_p^\ell = \mathsf{h}_p^\ell(\alpha, \beta)$ are given in Lemma A.6.3, for suitable values of α and β depending on the space $\mathscr{X}_c$ (see Table 3).

The case $\mathscr{X}_c = \mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{O}(n)$. Here we have $\alpha = \beta = (n-2)/2$. Hence $\rho = (n-1)/2$ and $d = n$, $X_k = k + (n-1)/2$, whilst the volume and multiplicity are given in Table 1.

Proposition 5.3.1 ($\mathscr{X}_c = \mathbb{S}^n$)**.** *For $n \geq 3$ odd, $\mathsf{p}_{j,p}^{m,\ell}(\mathbb{S}^n) = 0$ for $j > (n-3)/2$, whilst for $0 \leq j \leq (n-3)/2$ we have*

$$\mathsf{p}_{j,p}^{m,\ell}(\mathbb{S}^n) = (-1)^m (n/2)_{\ell-p-m-j} \mathsf{a}_{(n-3)/2-j}^n \mathscr{G}_{j,p}^{m,\ell}, \tag{5.3.3}$$

with a_j^n given in (A.4.1). For $n \geq 2$ even, with b_j^n as in (A.4.2), we have

$$\mathsf{p}_{j,p}^{m,\ell}(\mathbb{S}^n) = \begin{cases} (-1)^m (n/2)_{\ell-p-m-j} \mathsf{b}^n_{n/2-1-j} \mathscr{G}_{j,p}^{m,\ell} & \text{for } 0 \leq j \leq n/2-1 \\ \displaystyle\sum_{k=0}^{(n-2)/2} \frac{(-1)^{j+\ell-p} \mathsf{b}_k^n \mathscr{B}_1(k+j-n/2)}{\Gamma(n/2)\Gamma(j+p-\ell+m-n/2)} \mathscr{G}_{j,p}^{m,\ell} & \text{for } j \geq n/2, \end{cases} \tag{5.3.4}$$

with $\mathscr{B}_1(X)$ is as in (A.6.3), and $\mathscr{G}_{j,p}^{m,\ell} = \mathscr{G}_{j,p}^{m,\ell}(\mathbb{S}^n)$ as in (5.3.2).

Proof. With $M_k(\mathbb{S}^n) = M_k$ as in Table 1, we can write for $n \geq 3$ odd

$$M_k = \frac{2X_k}{(n-1)!} \prod_{j=1}^{n-2} (k+j) = \prod_{j=0}^{(n-3)/2} \frac{2(X_k^2 - j^2)}{(n-1)!} = \sum_{j=0}^{(n-3)/2} \frac{2\mathsf{a}_j^n X_k^{2j+2}}{(n-1)!}, \tag{5.3.5}$$

where we have used (A.4.1) to write the multiplicity in a polynomial form. Now writing the eigenvalues $\lambda_k^n = k(k+n-1) = X_k^2 - \rho^2$, the trace of the heat kernel (5.3.1) is then given by

$$\begin{aligned} \Theta(t;\mathbb{S}^n) &= \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \sum_{k=0}^{\infty} \frac{1}{(n-1)!} \sum_{j=0}^{(n-3)/2} \mathsf{a}_j^n X_k^{2j+2} e^{-t(X_k^2-\rho^2)} \\ &= \sum_{j=0}^{(n-3)/2} e^{t\rho^2} \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{\mathsf{a}_j^n}{(n-1)!} \sum_{p=1}^{\infty} p^{2j+2} e^{-tp^2} \\ &= \frac{e^{t\rho^2}}{2\pi^{(n+1)/2}} \frac{\Gamma((n+1)/2)}{(n-1)!} \sum_{j=0}^{(n-3)/2} \mathsf{a}_j^n (-1)^{j+1} \vartheta_1^{(j+1)}(t). \end{aligned} \tag{5.3.6}$$

Here we have collected the inner sum into derivatives of the Jacobi theta function ϑ_1^{j+1} provided in Appendix A.6. We then substitute for its asymptotics to arrive at a formula for the asymptotics of the trace of the heat kernel. Next for $n \geq 2$ even, we write the multiplicity $M_k(\mathbb{S}^n)$ as

$$M_k(\mathbb{S}^n) = \frac{2X_k}{(n-1)!} \prod_{j=\frac{1}{2}}^{\frac{n-2}{2}-\frac{1}{2}} (X_k^2 - j^2) = \sum_{j=0}^{(n-2)/2} \frac{2\mathsf{b}_j^n}{(n-1)!} X_k^{2j+1}. \tag{5.3.7}$$

Therefore the heat trace $\Theta(t;\mathbb{S}^n)$ is given in this case as

$$\begin{aligned} \Theta(t;\mathbb{S}^n) &= \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \sum_{k=0}^{\infty} \sum_{j=0}^{\frac{n-2}{2}} \frac{2\mathsf{b}_j^n}{(n-1)!} X_k^{2j+1} e^{-t(X_k^2-\rho^2)} \\ &= \sum_{j=0}^{(n-2)/2} e^{t\rho^2} \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{\mathsf{b}_j^n}{(n-1)!} \sum_{k=0}^{\infty} X_k^{2j+1} e^{-tX_k^2} \\ &= \sum_{j=0}^{(n-2)/2} e^{t\rho^2} \frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}} \frac{\mathsf{b}_j^n}{(n-1)!} \sum_{p=1/2}^{\infty} 2p^{2j+1} e^{-tp^2}. \end{aligned} \tag{5.3.8}$$

Hence referring to the formula for $\vartheta_2(t)$ in Appendix A.6 we have

$$\Theta(t;\mathbb{S}^n) = \frac{\Gamma((n+1)/2)}{(n-1)!}\frac{e^{t\rho^2}}{2\pi^{(n+1)/2}}\sum_{j=0}^{(n-2)/2}\mathsf{b}_j^n(-1)^j\vartheta_2^{(j)}(t). \tag{5.3.9}$$

Substituting for the asymptotics for $\vartheta_2^{(j)}(t)$ given in the appendix, we arrive at a formula for the asymptotics of the trace of the heat kernel in this case also. Referring to (5.2.10), we then apply $\mathscr{R}_\ell(-d/dt)$ to these formulae for the trace of the heat kernel, resulting in a formula for $b_{2\ell}^n$ in each case. Reindexing these for positive powers of t, we may extract the desired values of $\mathsf{p}_{j,p}^{m,\ell}(\mathbb{S}^n)$. □

The case $\mathscr{X}_c = \mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$. Here we have $\alpha = n-1$ and $\beta = 0$. Hence $\rho = n/2$, $d = 2n$, $X_k = k+n/2$, and the volume and multiplicity are given in Table 1.

Proposition 5.3.2 ($\mathscr{X}_c = \mathbb{C}\mathbf{P}^n$)**.** *For $n \geq 3$ odd, we have*

$$\mathsf{p}_{j,p}^{m,\ell}(\mathbb{C}\mathbf{P}^n) = \begin{cases} (-1)^m(n)_{\ell-p-m-j}\mathsf{c}_{n-1-j}^n\mathscr{G}_{j,p}^{m,\ell} & \text{for } 0\leq j\leq n-1\\ \displaystyle\sum_{k=0}^{n-1}\frac{(-1)^{j+\ell-p}\mathsf{c}_k^n\mathscr{B}_1(k+j-n)}{\Gamma(n)\Gamma(j+p-\ell+m-n)}\mathscr{G}_{j,p}^{m,\ell} & \text{for } j\geq n,\end{cases} \tag{5.3.10}$$

with c_j^n given in (A.4.3)*, and $\mathscr{B}_1(X)$ given in* (A.6.3)*. For $n\geq 2$ even, we have*

$$\mathsf{p}_{j,p}^{m,\ell}(\mathbb{C}\mathbf{P}^n) = \begin{cases} (-1)^m\mathsf{d}_{n-2-j}^n(n)_{\ell-p-m-j}\mathscr{G}_{j,p}^{m,\ell} & \text{for } \ 0\leq j\leq n-2\\ \displaystyle\sum_{k=0}^{n-2}\frac{(-1)^{j+\ell-p}\mathsf{d}_k^n\mathscr{B}_2(k+j-n)}{\Gamma(n)\Gamma(j-\ell+m+p-1-n)}\mathscr{G}_{j,p}^{m,\ell} & \text{for } j\geq n,\end{cases}$$

with d_j^n as in (A.4.4)*, $\mathscr{B}_2(X)$ as in* (A.6.3)*, and $\mathscr{G}_{j,p}^{m,\ell} = \mathscr{G}_{j,p}^{m,\ell}(\mathbb{C}\mathbf{P}^n)$ as in* (5.3.2)*.*

Proof. For $n\geq 1$ odd, the multiplicity $M_k(\mathbb{C}\mathbf{P}^n) = M_k$ can be written using (A.4.3) as a polynomial in X_k as

$$M_k = \frac{2X_k}{n!(n-1)!}\prod_{j=1/2}^{\frac{n-3}{2}+\frac{1}{2}}(X_k^2-j^2)^2 = \frac{2}{n!(n-1)!}\sum_{j=0}^{n-1}\mathsf{c}_j^n X_k^{2j+1}. \tag{5.3.11}$$

Hence the trace of the heat kernel $\Theta(t;\mathbb{C}\mathbf{P}^n)$ is given by

$$\begin{aligned}\Theta(t;\mathbb{C}\mathbf{P}^n) &= \frac{2}{4^n\pi^n}\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}\frac{\mathsf{c}_j^n}{\Gamma(n)}X_k^{2j+1}e^{-t(X_k^2-\rho^2)}\\ &= \frac{2e^{t\rho^2}}{4^n\pi^n}\sum_{j=0}^{n-1}\frac{\mathsf{c}_j^n}{\Gamma(n)}\sum_{k=0}^{\infty}X_k^{2j+1}e^{-tX_k^2} = \frac{e^{t\rho^2}}{4^n\pi^n}\sum_{j=0}^{n-1}\frac{\mathsf{c}_j^n}{\Gamma(n)}(-1)^j\vartheta_2^{(j)}(t).\end{aligned} \tag{5.3.12}$$

On the other hand, for $n \geq 2$ even, we can use (A.4.4) to write

$$M_k = \frac{2X_k^3}{n!(n-1)!} \prod_{j=1}^{(n-2)/2} (X_k^2 - j^2) = \frac{2}{n!(n-1)!} \sum_{j=0}^{n-2} \mathsf{d}_j^n X_k^{2j+3}. \tag{5.3.13}$$

Therefore in this case the trace of the heat kernel $\Theta(t; \mathbb{C}\mathbf{P}^n) = \Theta(t)$ is given by

$$\begin{aligned}\Theta(t) &= \frac{2}{4^n \pi^n} \sum_{k=0}^{\infty} \sum_{j=0}^{n-2} \frac{\mathsf{d}_j^n}{\Gamma(n)} X_k^{2j+3} e^{-t(X_k^2-\rho^2)} = \frac{2e^{t\rho^2}}{4^n\pi^n} \sum_{j=0}^{n-2} \frac{\mathsf{d}_j^n}{\Gamma(n)} \sum_{k=0}^{\infty} X_k^{2j+3} e^{-tX_k^2} \\ &= \frac{e^{t\rho^2}}{4^n\pi^n} \sum_{j=0}^{n-2} \frac{\mathsf{d}_j^n}{\Gamma(n)} (-1)^{j+1} \vartheta_3^{(j+1)}(t),\end{aligned} \tag{5.3.14}$$

where this time we have arranged the inner sum as derivatives of the Jacobi theta function of the third kind, given in Appendix A.6. Again substituting for the asymptotics of $\vartheta_2^{(j)}(t)$ and $\vartheta_3^{(j+1)}(t)$ (also given in Appendix A.6) we arrive at the asypmptotic values of $b_0^n(t; \mathbb{C}\mathbf{P}^n)$. We then apply $\mathscr{R}_\ell(-d/dt)$ to these, as in the previous proof, resulting in $b_{2\ell}^n(t; \mathbb{C}\mathbf{P}^n)$. After a re-indexing for positive powers of t, we may then extract the values of $\mathsf{p}_{j,p}^{m,\ell}(\mathbb{C}\mathbf{P}^n)$. $\square$

The case $\mathscr{X}_c = \mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$. Here we have $\alpha = 2n-1$ and $\beta = 1$, and hence $\rho = (2n+1)/2$, $d = 4n$, and $X_k = k + (2n+1)/2$. Moreover, the volume and multiplicity are given in Table 1.

Proposition 5.3.3 ($\mathscr{X}_c = \mathbb{H}\mathbf{P}^n$). *For $n \geq 1$, we have*

$$\mathsf{p}_{j,p}^{m,\ell}(\mathbb{H}\mathbf{P}^n) = \begin{cases} (-1)^m (2n)_{\ell-p-m-j} \mathsf{e}_{2n-1-j}^n \mathscr{G}_{j,p}^{m,\ell} & \text{for } 0 \leq j \leq 2n-1 \\ \displaystyle\sum_{k=0}^{2n-1} \frac{(-1)^{j+\ell-p} \mathsf{e}_k^n \mathscr{B}_1(k+j-2n)}{\Gamma(2n)\Gamma(j+p-\ell+m-2n)} \mathscr{G}_{j,p}^{m,\ell} & \text{for } j \geq 2n, \end{cases} \tag{5.3.15}$$

with e_j^n as in (A.4.5), $\mathscr{B}_1(X)$ as in (A.6.3), and $\mathscr{G}_{j,p}^{m,\ell} = \mathscr{G}_{j,p}^{m,\ell}(\mathbb{H}\mathbf{P}^n)$ as in (5.3.2).

Proof. We begin by writing the multiplicity $M_k(\mathbb{H}\mathbf{P}^n) = M_k$ using (A.4.5) as

$$\begin{aligned} M_k &= \frac{2X_k(k+2n)}{(2n-1)!(2n+1)!(k+1)} \prod_{j=1}^{2n-1} (k+j)^2 \\ &= \frac{2X_k(X_k^2 - (2n-1)^2/4)}{(2n-1)!(2n+1)!} \prod_{j=1/2}^{n-3/2} (X_k^2 - j^2) = \sum_{j=0}^{2n-1} \frac{2\mathsf{e}_j^n X_k^{2j+1}}{\Gamma(2n)(2n+1)!}. \end{aligned} \tag{5.3.16}$$

Therefore the heat trace $\Theta(t; \mathbb{H}\mathbf{P}^n)$ is given in this case in terms of the Jacobi theta functions of the second kind as

$$\Theta(t;\mathbb{H}\mathbf{P}^n) = \frac{2}{4^{2n}\pi^{2n}} \sum_{k=0}^{\infty} \sum_{j=0}^{2n-1} \frac{\mathsf{e}_j^n X_k^{2j+1}}{\Gamma(2n)} e^{-t(X_k^2-\rho^2)}$$
$$= \frac{2e^{t\rho^2}}{(4\pi)^{2n}} \sum_{j=0}^{2n-1} \frac{\mathsf{e}_j^n}{\Gamma(2n)} \sum_{p=0}^{\infty} (p+1/2)^{2j+1} e^{-t(p+1/2)^2}$$
$$= \frac{e^{t\rho^2}}{(4\pi)^{2n}} \sum_{j=0}^{2n-1} \frac{\mathsf{e}_j^n}{\Gamma(2n)} (-1)^j \vartheta_2^{(j)}(t). \tag{5.3.17}$$

We then substitute for the asymptotics of $\vartheta_2^{(j)}(t)$ as in Appendix A.6 to arrive at the asymptotics for $b_0^n(t;\mathbb{H}\mathbf{P}^n)$. For the general $\ell \geq 0$ case we apply $\mathscr{R}_\ell(-d/dt)$ to this, before rearranging for a positive power of t to extract the value of $\mathsf{p}_{j,p}^{m,\ell}(\mathbb{H}\mathbf{P}^n)$. □

The case $\mathscr{X}_c = \mathbf{P}^2(Cay) = \mathbf{F}^4/\mathbf{Spin}(9)$. Here we have $\alpha = 7$ and $\beta = 3$, therefore $\rho = 11/2$, $d = 16$, and $X_k = k + 11/2$. We also have the volume and multiplicity in Table 1.

Proposition 5.3.4 ($\mathscr{X}_c = \mathbf{P}^2(Cay)$)**.** *We have*

$$\mathsf{p}_{j,p}^{m,\ell}(\mathbf{P}^2(Cay)) = \begin{cases} (-1)^m (8)_{\ell-p-m-j} \mathsf{f}_{7-j} \mathscr{G}_{j,p}^{m,\ell} & \text{for } 0 \leq j \leq 7 \\ \displaystyle\sum_{k=0}^{7} \frac{(-1)^{j+\ell-p} \mathsf{f}_k \mathscr{B}_1(k+j-8)}{\Gamma(8)\Gamma(j+p-\ell+m-8)} \mathscr{G}_{j,p}^{m,\ell} & \text{for } j \geq 8, \end{cases} \tag{5.3.18}$$

with f_j as in (A.4.6), $\mathscr{B}_1(X)$ as in (A.6.3), and $\mathscr{G}_{j,p}^{m,\ell} = \mathscr{G}_{j,p}^{m,\ell}(\mathbf{P}^2(Cay))$ as in (5.3.2).

Proof. In this case we can arrange the multiplicity using (A.4.6) as

$$M_k = \frac{12X_k}{7!11!}(X_k^2-1/4)^2(X_k^2-9/4)^2(X_k^2-25/4)(X_k^2-49/4)(X_k^2-81/4)$$
$$= \frac{12}{7!11!}\sum_{j=0}^{7} \mathsf{f}_j X_k^{2j+1}. \tag{5.3.19}$$

Hence the trace of the heat kernel $\Theta(t;\mathbf{P}^2(Cay)) = \Theta(t)$ is given by

$$\Theta(t) = \frac{12}{7!3!(4\pi)^8}\sum_{k=0}^{\infty}\sum_{j=0}^{7} \mathsf{f}_j X_k^{2j+1} e^{-t(X_k^2-\rho^2)} = \sum_{j=0}^{7} \frac{12\mathsf{f}_j e^{t\rho^2}}{7!3!(4\pi)^8} \sum_{k=0}^{\infty} X_k^{2j+1} e^{-tX_k^2}$$
$$= \sum_{j=0}^{7} \frac{12\mathsf{f}_j e^{t\rho^2}}{7!3!(4\pi)^8} \sum_{p=0}^{\infty} (p+1/2)^{2j+1} e^{-t(p+1/2)^2} = \frac{e^{t\rho^2}}{7!(4\pi)^8} \sum_{j=0}^{7} (-1)^j \mathsf{f}_j \vartheta_2^{(j)}(t). \tag{5.3.20}$$

We then substitute for the asymptotics of $\vartheta_2^{(j)}(t)$ as in Appendix A.6 for the asymptotics of $b_0^n(t;\mathbf{P}^2(Cay))$. For the general $\ell \geq 0$ case we apply $\mathscr{R}_\ell(-d/dt)$ to this to get the asymptotics of $b_{2\ell}^n(t;\mathbf{P}^2(Cay))$, at which point we arrange and re-index for positive powers of t, and extract the value of $\mathsf{p}_{j,p}^{m,\ell}(\mathbf{P}^2(Cay))$. □

5.4 Expansion of $b_{2\ell}^n(t;\mathscr{X})$ on non-compact $\mathscr{X}$

The trace of the heat kernel $\Theta(t;\mathscr{X}_{nc})$ on a rank-one symmetric space $\mathscr{X}_{nc}$ of non-compact type is given by

$$\Theta(t;\mathscr{X}_{nc}) = \frac{2^{2\beta-1}\Gamma(\alpha+1)}{\pi^{\alpha+2}} \int_0^\infty e^{-t(\lambda^2+\rho^2)}\mu(\lambda)\, d\lambda, \tag{5.4.1}$$

where $\rho = (\alpha+\beta+1)/2$, and $\mu(\lambda) = |\mathbf{C}(\lambda)|^{-\mathbf{2}}$ is the Plancherel measure, where $\mathbf{C}(\lambda)$ is the Harish-Chandra function given in (5.2.16), so that $\mu(\lambda)$ has the explicit form

$$\mu(\lambda) = \frac{|\Gamma(i\lambda+(\alpha+\beta+1)/2)|^2|\Gamma(i\lambda+(\alpha-\beta+1)/2)|^2}{|2^{\alpha+\beta+1-2i\lambda}|^2|\Gamma(\alpha+1)|^2|\Gamma(2i\lambda)|^2}. \tag{5.4.2}$$

As in the compact case, we can arrange the Maclaurin heat coefficients $b_{2\ell}^n(t;\mathscr{X}_{nc})$ as a series of the form (5.1.10) where in the following propositions we present the values of $\mathsf{q}_{j,p}^{m,\ell}(\mathscr{X}_{nc})$ for each space.

As in the compact case, we define a general term below that will aid in the following formulations of the values of $\mathsf{q}_{j,p}^{m,\ell}(\mathscr{X}_{nc})$.

$$\mathscr{G}_{j,p}^{m,\ell} = \mathscr{G}_{j,p}^{m,\ell}(\mathscr{X}_{nc}) = \binom{\ell-p}{m}\rho^{2m}\mathsf{H}_{\ell-p}^{\ell}(\alpha,\beta), \tag{5.4.3}$$

where $\mathsf{H}_p^\ell = \mathsf{H}_p^\ell(\alpha,\beta)$ are given in Lemma A.6.3, for suitable values of α and β depending on the space $\mathscr{X}_{nc}$. Moreover $\rho = (\alpha+\beta+1)/2$. Here we also make use of the identity

$$\frac{\Gamma(n/2-j)}{\Gamma(n/2)}\frac{\Gamma(j-n/2+1)}{\Gamma(j-m-n/2+1)} = (-1)^m\frac{\Gamma(n/2+m-j)}{\Gamma(n/2)}, \tag{5.4.4}$$

The case $\mathscr{X}_{nc} = \mathbb{R}\mathbf{H}^n = \mathbf{SO}_0(n,1)/\mathbf{SO}(n)$. Here we have $\alpha = \beta = (n-2)/2$. Hence $\rho = (n-1)/2$ and $d = n$. These values are shared with $\mathbb{S}^n$. Here the Plancherel measure is given for $n \geq 1$ odd by

$$\mu(\lambda) = \frac{\pi}{|\Gamma(n/2)2^{n-2}|^2}\prod_{j=0}^{(n-3)/2}(\lambda^2+j^2), \tag{5.4.5}$$

whilst for $n \geq 2$ even we have

$$\mu(\lambda) = \frac{\pi\lambda\tanh(\pi\lambda)}{|\Gamma(n/2)2^{n-2}|^2}\prod_{j=1/2}^{(n-3)/2}(\lambda^2+j^2). \tag{5.4.6}$$

Note that for $n = 2$, the product above is simply set to 1.

Proposition 5.4.1 ($\mathscr{X} = \mathbb{R}\mathbf{H}^n$)**.** *For $n \geq 3$ odd, $\mathsf{q}_{j,p}^{m,\ell}(\mathbb{R}\mathbf{H}^n) = 0$ for $j > (n-3)/2$, whilst for $0 \leq j \leq (n-3)/2$ we have*

$$\mathsf{q}_{j,p}^{m,\ell}(\mathbb{R}\mathbf{H}^n) = (-1)^{\ell-p}(n/2)_{\ell-p-m-j}\mathsf{A}_{(n-3)/2-j}^n\mathscr{G}_{j,p}^{m,\ell}, \tag{5.4.7}$$

where A_k^n *are given in* (A.4.7)*. For* $n \geq 2$ *even we have*

$$\mathsf{q}_{j,p}^{m,\ell}(\mathbb{R}\mathbf{H}^n) = \begin{cases} (-1)^{\ell-p}(n/2)_{\ell-p-m-j}\mathsf{B}_{n/2-1-j}^n\mathscr{G}_{j,p}^{m,\ell} & \text{for } 0 \leq j \leq n/2-1 \\ \displaystyle\sum_{k=0}^{(n-2)/2} \frac{\mathsf{B}_k^n\mathscr{B}_1^\star(k+j-n/2)(-1)^{j+n/2+m+1}}{\Gamma(n/2)\Gamma(j+p-\ell+m-n/2)}\mathscr{G}_{j,p}^{m,\ell} & \text{for } j \geq n/2, \end{cases} \tag{5.4.8}$$

with B_j^n *as in* (A.4.8)*,* $\mathscr{B}_1^\star(X)$ *as in* (A.6.3)*, and* $\mathscr{G}_{j,p}^{m,\ell} = \mathscr{G}_{j,p}^{m,\ell}(\mathbb{R}\mathbf{H}^n)$ *as in* (5.4.3)*.*

Proof. We first derive the trace of the heat kernel for $n \geq 3$ odd. Using (A.4.7) on the Plancherel measure (5.4.5), the trace $\Theta(t;\mathbb{R}\mathbf{H}^n)$ of the heat kernel as in (6.2.4) can be written as

$$\begin{aligned}
\Theta(t;\mathbb{R}\mathbf{H}^n) &= \frac{e^{-t(n-1)^2/4}}{\pi^{n/2}\Gamma(n/2)2^n}\int_0^\infty 2e^{-t\lambda^2}\prod_{j=0}^{(n-3)/2}(j^2+\lambda^2)\,d\lambda \\
&= \frac{e^{-t(n-1)^2/4}}{\pi^{n/2}\Gamma(n/2)2^n}\sum_{k=0}^{(n-3)/2}\mathsf{A}_k^n\int_0^\infty 2e^{-t\lambda^2}\lambda^{2k+2}\,d\lambda \\
&= \frac{e^{-t(n-1)^2/4}}{\pi^{n/2}\Gamma(n/2)2^n}\sum_{k=0}^{(n-3)/2}\mathsf{A}_k^n\frac{1}{t^{k+3/2}}\Gamma(k+3/2).
\end{aligned} \tag{5.4.9}$$

Therefore we have

$$\Theta(t;\mathbb{R}\mathbf{H}^n) = \frac{e^{-t(n-1)^2/4}}{(4\pi t)^{n/2}}\sum_{k=0}^{(n-3)/2}\frac{\Gamma(k+3/2)}{\Gamma(n/2)}\mathsf{A}_k^n t^{(n-3)/2-k}. \tag{5.4.10}$$

For $n \geq 2$ even, we use (A.4.8) on the Plancherel measure as in (5.4.6), which lets us write the trace of the heat kernel $\Theta(t;\mathbb{R}\mathbf{H}^n) = \Theta(t)$ as

$$\begin{aligned}
\Theta(t) &= \frac{e^{-t\rho^2}}{\pi^{n/2}\Gamma(n/2)2^n}\int_0^\infty 2\lambda e^{-t\lambda^2}\tanh(\pi\lambda)\prod_{j=1/2}^{(n-3)/2}(\lambda^2+j^2)\,d\lambda \\
&= \frac{e^{-t\rho^2}}{\pi^{n/2}\Gamma(n/2)2^n}\sum_{k=0}^{(n-2)/2}\mathsf{B}_k^n\int_0^\infty 2e^{-t\lambda^2}\tanh(\pi\lambda)\lambda^{2k+1}\,d\lambda \\
&= \frac{e^{-t\rho^2}}{\pi^{n/2}\Gamma(n/2)2^n}\sum_{k=0}^{(n-2)/2}\mathsf{B}_k^n\left[\frac{\Gamma(k+1)}{t^{k+1}}+\sum_{j=0}^\infty\frac{(-1)^{j+1}}{j!}\mathscr{B}_1^\star(j+k)t^j\right].
\end{aligned} \tag{5.4.11}$$

where we have used Lemma A.6.1 to evaluate the integral. This leads to the trace of the heat kernel $\Theta(t;\mathbb{R}\mathbf{H}^n) = \Theta(t)$ being given by

$$\Theta(t) = \frac{e^{-t(n-1)^2/4}}{(4\pi t)^{n/2}}\sum_{k=0}^{(n-2)/2}\frac{\mathsf{B}_k^n}{\Gamma(n/2)}\left[\frac{\Gamma(k+1)}{t^{k+1-n/2}}+\sum_{j=0}^\infty\frac{(-1)^{j+1}}{j!}\mathscr{B}_1^\star(j+k)t^{j+n/2}\right], \tag{5.4.12}$$

Next to calculate $b_{2\ell}^n(t;\mathbb{R}\mathbf{H}^n)$ from these, we apply $\mathscr{Q}_\ell(d/dt)$ as in (A.6.24) to the heat traces in odd and even dimensions. In evaluating this action, we use the Gamma function identity (5.4.4) to simplify the result, followed by a re-indexing for positive powers of t before extracting the values of $\mathsf{q}_{j,p}^{m,\ell}(\mathbb{R}\mathbf{H}^n)$. □

The case $\mathscr{X}_c = \mathbb{C}\mathbf{H}^n = \mathbf{SU}(n,1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$. Here we have $\alpha = n-1$ and $\beta = 0$, as with $\mathbb{C}\mathbf{P}^n$. Hence $\rho = n/2$, and $d = 2n$. The Plancherel measure $\mu(\lambda)$ as in (5.4.2) is given for $n \geq 1$ odd by by

$$\mu(\lambda) = \frac{\pi\lambda\tanh(\pi\lambda)}{(2^{n-1}\Gamma(n))^2}\prod_{j=1/2}^{(n-2)/2}(\lambda^2+j^2)^2, \tag{5.4.13}$$

and for $n \geq 2$ even by

$$\mu(\lambda) = \frac{\pi\lambda^3\coth(\pi\lambda)}{2^{2n-2}\Gamma(n)^2}\prod_{j=1}^{(n-2)/2}(\lambda^2+j^2)^2. \tag{5.4.14}$$

Proposition 5.4.2 ($\mathscr{X} = \mathbb{C}\mathbf{H}^n$)**.** *For $n \geq 1$ odd, we have*

$$\mathsf{q}_{j,p}^{m,\ell}(\mathbb{C}\mathbf{H}^n) = \begin{cases} (-1)^{\ell-p}(n)_{\ell-p-m-j}\mathsf{C}_{n-1-j}^n\mathscr{G}_{j,p}^{m,\ell} & \text{for } 0 \leq j \leq n-1 \\ \displaystyle\sum_{k=0}^{n-1}\frac{(-1)^{j+m}\mathsf{C}_k^n\mathscr{B}_1^\star(k+j-n)}{\Gamma(n)\Gamma(j+p-\ell+m-n)}\mathscr{G}_{j,p}^{m,\ell} & \text{for } j \geq n, \end{cases} \tag{5.4.15}$$

with C_j^n as in (A.4.9) and $\mathscr{B}_1^\star(X)$ as in (A.6.3). For $n \geq 2$ even we have

$$\mathsf{q}_{j,p}^{m,\ell}(\mathbb{C}\mathbf{H}^n) = \begin{cases} (-1)^{\ell-p}\mathsf{D}_{n-2-j}^n(n)_{\ell-p-m-j}\mathscr{G}_{j,p}^{m,\ell} & \text{for } \ 0 \leq j \leq n-2 \\ \displaystyle\sum_{k=0}^{n-2}\frac{(-1)^{k+m+1}\mathsf{D}_k^n\mathscr{B}_2(k+j-n)}{\Gamma(n)\Gamma(j-\ell+m+p-1-n)}\mathscr{G}_{j,p}^{m,\ell} & \text{for } j \geq n, \end{cases}$$

with D_j^n as in (A.4.10), $\mathscr{B}_2(X)$ as in (A.6.3), and $\mathscr{G}_{j,p}^{m,\ell} = \mathscr{G}_{j,p}^{m,\ell}(\mathbb{C}\mathbf{H}^n)$ as in (5.4.3).

Proof. With the Plancherel measure as in (5.4.13), we can write $\Theta_{\mathbb{C}\mathbf{H}^n}(t) = \Theta(t)$ via (A.4.9) as

$$\begin{aligned}\Theta(t) &= \frac{e^{-t\rho^2}}{2^{2n}\pi^n\Gamma(n)}\int_0^\infty 2\tanh(\pi\lambda)\lambda e^{-t\lambda^2}\prod_{j=1/2}^{(n-2)/2}(\lambda^2+j^2)^2\,d\lambda \\ &= \frac{e^{-t\rho^2}}{2^{2n}\pi^n\Gamma(n)}\sum_{k=0}^{n-1}\mathsf{C}_k^n\int_0^\infty 2\tanh(\pi\lambda)\lambda^{2k+1}e^{-t\lambda^2}\,d\lambda \\ &= \frac{e^{-t\rho^2}}{2^{2n}\pi^n\Gamma(n)}\sum_{k=0}^{n-1}\mathsf{C}_k^n\left[\frac{\Gamma(k+1)}{t^{k+1}}+\sum_{j=0}^\infty\frac{(-1)^{j+1}}{j!}\mathscr{B}_1^\star(j+k)t^j\right]. \end{aligned}\tag{5.4.16}$$

Here we have used Lemma A.6.1 to evaluate the integral. Rearranging this we may extract the statement of q_j for $n \geq 1$ odd. Next for $n \geq 2$ even, using (A.4.10), we can write the trace of the heat kernel as

$$\begin{aligned}\Theta(t;\mathbb{C}\mathbf{H}^n) &= \frac{e^{-t\rho^2}}{2^{2n}\Gamma(n)\pi^n}\int_0^\infty 2\lambda^3\coth(\pi\lambda)e^{-t\lambda^2}\prod_{j=1}^{(n-2)/2}(\lambda^2+j^2)^2\,d\lambda \\ &= \frac{e^{-t\rho^2}}{2^{2n}\Gamma(n)\pi^n}\sum_{k=0}^{n-2}\mathsf{D}_k^n\int_0^\infty 2e^{-t\lambda^2}\coth(\pi\lambda)\lambda^{2k+3}\,d\lambda \\ &= \frac{e^{-t\rho^2}}{2^{2n}\Gamma(n)\pi^n}\sum_{k=0}^{n-2}\mathsf{D}_k^n\left[\frac{\Gamma(k+2)}{t^{k+2}}-\sum_{j=k}^{\infty}\frac{(-1)^k}{(j-k)!}\mathscr{B}_2(j)t^{j-k}\right]. \end{aligned} \tag{5.4.17}$$

In evaluating the integral in the middle inequality, we have used Lemma A.6.1. A rearrangement then allows us to extract q_j in this case. We then apply $\mathscr{Q}_\ell(d/dt)$ to the above to obtain the statement for $b^n_{2\ell}(t;\mathbb{C}\mathbf{H}^n)$, and hence extract the statements of $\mathsf{q}^{m,\ell}_{j,p}(\mathbb{C}\mathbf{H}^n)$ in each case. $\square$

The case $\mathscr{X}_{nc} = \mathbb{H}\mathbf{H}^n = \mathbf{Sp}(n,1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$. Here we have $\alpha = 2n-1$ and $\beta = 1$, and so $\rho = (2n+1)/2$, and $d = 4n$. For $n \geq 1$, the Plancherel measure $\mu(\lambda)$ as in (5.4.2) is given by

$$\mu(\eta) = \frac{\pi\lambda\tanh(\pi\lambda)}{2^{4n}\Gamma(2n)^2}\left[\lambda^2+(2n-1)^2/4\right]\prod_{j=1/2}^{n-3/2}(\lambda^2+j^2)^2. \tag{5.4.18}$$

Proposition 5.4.3 ($\mathscr{X} = \mathbb{H}\mathbf{H}^n$). *For $n \geq 1$, we have*

$$\mathsf{q}^{m,\ell}_{j,p}(\mathbb{H}\mathbf{H}^n) = \begin{cases} (-1)^{\ell-p}(2n)_{\ell-p-m-j}\mathsf{E}^n_{2n-1-j}\mathscr{G}^{m,\ell}_{j,p} & \text{for } 0 \leq j \leq 2n-1 \\ \displaystyle\sum_{k=0}^{2n-1}\frac{(-1)^{j+m+1}\mathsf{E}^n_k\mathscr{B}^\star_1(k+j-2n)}{\Gamma(n)\Gamma(j+p-\ell+m-2n)}\mathscr{G}^{m,\ell}_{j,p} & \text{for } j \geq 2n, \end{cases} \tag{5.4.19}$$

with E^n_j as in (A.4.11), $\mathscr{B}^\star_1(X)$ as in (A.6.3), and $\mathscr{G}^{m,\ell}_{j,p} = \mathscr{G}^{m,\ell}_{j,p}(\mathbb{H}\mathbf{H}^n)$ as in (5.4.3).

Proof. With the Plancherel measure as in (5.4.18) in this case, we can use (A.4.11) to write the heat trace $\Theta(t;\mathbb{H}\mathbf{H}^n) = \Theta(t)$ as

$$\begin{aligned}\Theta(t) &= \frac{e^{-t\rho^2}}{2^{4n}\pi^{2n}\Gamma(2n)}\int_0^\infty 2e^{-t\lambda^2}\lambda\tanh(\pi\lambda)\left[\lambda^2+(2n-1)^2/4\right]\prod_{j=1/2}^{n-3/2}(\lambda^2+j^2)^2 d\lambda \\ &= \frac{e^{-t\rho^2}}{2^{4n}\pi^{2n}\Gamma(2n)}\sum_{k=0}^{2n-1}\mathsf{E}^n_k\int_0^\infty 2e^{-t\lambda^2}\tanh(\pi\lambda)\lambda^{2k+1}d\lambda \\ &= \frac{e^{-t\rho^2}}{2^{4n}\pi^{2n}\Gamma(2n)}\sum_{k=0}^{2n-1}\mathsf{E}^n_k\left[\frac{\Gamma(k+1)}{t^{k+1}}+\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}\mathscr{B}^\star_1(j+k)t^j\right]. \end{aligned} \tag{5.4.20}$$

We may then write

$$\Theta(t) = \frac{e^{-t\rho^2}}{(4\pi t)^{2n}}\sum_{k=0}^{2n-1}\frac{\mathsf{E}^n_k}{\Gamma(2n)}\left[\frac{\Gamma(k+1)}{t^{k+1-2n}}+\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}\mathscr{B}^\star_1(j+k)t^{j+2n}\right], \tag{5.4.21}$$

which after a rearrangement leads to the value of q_j. Applying $\mathscr{Q}_\ell(d/dt)$ to the above statement, we arrive at the value of $b_{2\ell}^n(t;\mathbb{H}\mathbf{H}^n)$, which after a re-indexing for a positive power of t provides the desired value of $\mathsf{q}_{j,p}^{m,\ell}(\mathbb{H}\mathbf{H}^n)$. $\square$

The case $\mathscr{X}_{nc} = \mathbf{H}^2(Cay) = \mathbf{F}_\star^4/\mathbf{Spin}(9)$. Here we have $\alpha = 7$ and $\beta = 3$, and so $\rho = 11/2$, and $d = 16$. The Plancherel measure $\mu(\lambda)$ as in (5.4.2) is given by

$$\mu(\lambda) = \frac{\pi\lambda\tanh(\pi\lambda)}{2^{20}\Gamma(8)^2}(\lambda^2+81/4)(\lambda^2+49/4)(\lambda^2+25/4)(\lambda^2+9/4)^2(\lambda^2+1/4)^2. \quad (5.4.22)$$

Proposition 5.4.4 ($\mathscr{X} = \mathbf{H}^2(Cay)$)**.** *Here we have*

$$\mathsf{q}_{j,p}^{m,\ell}(\mathbf{H}^2(Cay)) = \begin{cases} (-1)^{\ell-p}(8)_{\ell-p-m-j}\mathsf{F}_{7-j}\mathscr{G}_{j,p}^{m,\ell} & \text{for } 0 \le j \le 7 \\ \displaystyle\sum_{k=0}^{7} \frac{(-1)^{j+m+1}\mathsf{F}_k\mathscr{B}_1^\star(k+j-8)}{\Gamma(n)\Gamma(j+p-\ell+m-8)}\mathscr{G}_{j,p}^{m,\ell} & \text{for } j \ge 8, \end{cases} \quad (5.4.23)$$

with F_j as in (A.4.12), $\mathscr{B}_1^\star(X)$ as in (A.6.3), and $\mathscr{G}_{j,p}^{m,\ell} = \mathscr{G}_{j,p}^{m,\ell}(\mathbf{H}^2(Cay))$ as in (5.4.3).

Proof. Using (A.4.12) on the Plancherel measure (5.4.22) we can write the heat trace $\Theta(t;\mathbf{H}^2(Cay)) = \Theta(t)$ as

$$\begin{aligned} \Theta(t) &= \frac{2^5 e^{-121t/4}\Gamma(8)}{\pi^9}\int_0^\infty e^{-t\lambda^2}\mu(\lambda)\,d\lambda \\ &= \frac{e^{-121t/4}}{2^{16}\Gamma(8)\pi^8}\sum_{k=0}^{7}\mathsf{F}_k\int_0^\infty 2\lambda^{2k+1}\tanh(\pi\lambda)e^{-t\lambda^2}\,d\lambda \\ &= \frac{e^{-121t/4}}{2^{16}\Gamma(8)\pi^8}\sum_{k=0}^{7}\mathsf{F}_k\left[\frac{\Gamma(k+1)}{t^{k+1}} + \sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}\mathscr{B}_1^\star(j+k)t^j\right]. \end{aligned} \quad (5.4.24)$$

Rearranging for a positive power of t, we may extract the value of q_j. Next we apply $\mathscr{Q}_\ell(d/dt)$ to the above statement to arrive at the value of $b_{2\ell}^n(t;\mathbf{H}^2(Cay))$, which we then re-indexing for a positive power of t. This provides the desired value of $\mathsf{q}_{j,p}^{m,\ell}(\mathbf{H}^2(Cay))$. $\square$

5.5 Proportionality Principle for $b_{2\ell}^n(t;\mathscr{X})$

The trace of the heat kernel $\Theta(t;\mathscr{X}_{nc})$ on a compact d-dimensional rank-one symmetric space $\mathscr{X}_c$ has the asymptotics given by (5.1.4), whilst on a non-compact rank-one symmetric space $\mathscr{X}_{nc}$ the trace are given by (5.1.5). In the case that $\mathscr{X}_c$ is dual to $\mathscr{X}_{nc}$, the Minakshisundaram-Pleijel coefficients $\mathsf{a}_j(\mathscr{X})$ for each space satisfy the proportionality (5.1.6). In this section we extend this proportionality to the level of the Maclaurin heat coefficients $b_{2\ell}^n(t;\mathscr{X})$.

Generalised Minakshisundaram-Pleijel coefficients on $\mathscr{X}_c$. In (5.2.10) we defined the Maclaurin heat coefficients associated to a compact rank-one symmetric space. Considering the asymptotic expansion (5.1.4), we can apply $\mathscr{R}_\ell(-d/dt)$ with $\mathscr{R}_\ell(X)$ as in (A.6.20) to then write the asymptotics of the Maclaurin heat coefficients on a compact space as

$$\begin{aligned} b^n_{2\ell}(t;\mathscr{X}_c) &\sim \mathscr{R}^{\alpha,\beta}_\ell(-d/dt)\frac{1}{(4\pi t)^{d/2}}\sum_{j=0}^{\infty}\mathsf{a}_j(\mathscr{X}_c)t^j \\ &\sim \frac{1}{(4\pi)^{d/2}}\sum_{j=0}^{\infty}\mathsf{a}_j(\mathscr{X}_c)\sum_{p=1}^{\ell}\mathsf{h}^\ell_p(-1)^p\frac{d^p}{dt^p}t^{j-d/2} \\ &\sim \frac{(4\pi)^\ell}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\mathsf{a}_j(\mathscr{X}_c)\sum_{p=1}^{\ell}\mathsf{h}^\ell_p(-1)^p\frac{\Gamma(j-d/2+1)}{\Gamma(j-d/2+1-p)}t^{j+\ell-p}. \end{aligned} \tag{5.5.1}$$

After re-indexing this for a positive power of t, we have

$$\begin{aligned} b^n_{2\ell}(t;\mathscr{X}_c) &\sim \frac{(4\pi)^\ell}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\left[\mathsf{a}_j(\mathscr{X}_c)\sum_{p=0}^{\ell-1}\mathsf{h}^\ell_{\ell-p}\frac{(-1)^{\ell-p}\Gamma(j-d/2+1)}{\Gamma(j-d/2-\ell+p+1)}\right]t^{j+p} \\ &\sim \frac{(4\pi)^\ell}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\left[\sum_{p=0}^{\ell-1}\mathsf{h}^\ell_{\ell-p}\frac{(-1)^{\ell-p}\Gamma(j-d/2+1)}{\Gamma(j-d/2-\ell+p+1)}\mathsf{a}_j(\mathscr{X}_c)\right]t^{j+p} \\ &\sim \frac{1}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\sum_{p=0}^{\ell-1}\mathsf{b}^\ell_{j,p}(\mathscr{X}_c)t^{j+p}. \end{aligned} \tag{5.5.2}$$

where we have collected several terms into $\mathsf{b}^\ell_{j,p}(\mathscr{X}_c)$ as

$$\mathsf{b}^\ell_{j,p}(\mathscr{X}_c) = \mathsf{h}^\ell_{\ell-p}(4\pi)^\ell\frac{(-1)^{\ell-p}\Gamma(j-d/2+1)}{\Gamma(j-d/2-\ell+p+1)}\mathsf{a}_j(\mathscr{X}_c). \tag{5.5.3}$$

Now rearranging (5.5.2) so that we have increasing powers of t, we arrive at the following proposition.

Proposition 5.5.1. *On a compact rank-one symmetric space $\mathscr{X}_c$, the Maclaurin heat coefficients have the asymptotics*

$$b^n_{2\ell}(t;\mathscr{X}_c) \sim \frac{1}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\mathsf{a}_{j,\ell}(\mathscr{X}_c)t^j, \qquad t\searrow 0. \tag{5.5.4}$$

Here the generalised Minakshisundaram-Pleijel coefficients $\mathsf{a}_{j,\ell}(\mathscr{X}_c)$ for a compact space are given for $\mathsf{b}^\ell_{j,p}(\mathscr{X}_c)$ as in (5.5.3) by

$$\mathsf{a}_{j,\ell}(\mathscr{X}_c) = \sum_{q=0}^{j}\mathsf{b}^\ell_{q,j-q}(\mathscr{X}_c). \tag{5.5.5}$$

Proof. Here we have rearranged (5.5.2) to sum over increasing powers of t. We note that the sum over p there only runs over $p = 0, \ldots, \ell - 1$, and that the value $j - q$ in (5.5.5) may go outside this range. However in these cases the value of $\mathsf{b}^{\ell}_{q,j-q}(\mathscr{X})$ is zero because the value of $\mathsf{h}^{\ell}_{p}(\alpha, \beta)$ is zero by definition outside these ranges. $\square$

Generalised Minakshisundaram-Pleijel coefficients on $\mathscr{X}_{nc}$. In this case we define the Maclaurin heat coefficients associated to a non-compact rank-one symmetric space $\mathscr{X}_{nc}$ in (A.6.25). Considering the expansion (5.1.5), we can then apply $\mathscr{Q}_{\ell}(d/dt)$ to write a similar expansion of $b^{n}_{2\ell}(t; \mathscr{X}_{nc})$ as

$$\begin{aligned} b^{n}_{2\ell}(t; \mathscr{X}_{nc}) &= \mathscr{Q}^{\alpha,\beta}_{\ell}(d/dt) \frac{1}{(4\pi t)^{d/2}} \sum_{j=0}^{\infty} \mathsf{a}_j(\mathscr{X}_{nc}) t^j \\ &= \frac{1}{(4\pi)^{d/2}} \sum_{j=0}^{\infty} \mathsf{a}_j(\mathscr{X}_{nc}) \sum_{p=1}^{\ell} \mathsf{h}^{\ell}_{p} \frac{d^p}{dt^p} t^{j-d/2}. \end{aligned} \tag{5.5.6}$$

As above in the compact case before, this leads to the arrangement

$$b^{n}_{2\ell}(t; \mathscr{X}_{nc}) = \frac{1}{(4\pi t)^{d/2+\ell}} \sum_{j=0}^{\infty} \sum_{p=0}^{\ell-1} \mathsf{b}^{\ell}_{j,p}(\mathscr{X}_{nc}) t^{j+p} \tag{5.5.7}$$

where this time we have defined $\mathsf{b}^{\ell}_{j,p}(\mathscr{X}_{nc})$ as

$$\mathsf{b}^{\ell}_{j,p}(\mathscr{X}_{nc}) = \mathsf{H}^{\ell}_{p} (4\pi)^{\ell} \frac{\Gamma(j - d/2 + 1)}{\Gamma(j - d/2 - \ell + p + 1)} \mathsf{a}_j(\mathscr{X}_{nc}). \tag{5.5.8}$$

A similar re-arrangement into increasing powers of t leads to the following proposition.

Proposition 5.5.2. *On a non-compact rank-one symmetric space $\mathscr{X}_{nc}$, the Maclaurin heat coefficients exhibit the expansion*

$$b^{n}_{2\ell}(t; \mathscr{X}_{nc}) = \frac{1}{(4\pi t)^{d/2+\ell}} \sum_{j=0}^{\infty} \mathsf{a}_{j,\ell}(\mathscr{X}_{nc}) t^j, \tag{5.5.9}$$

where the generalised Minakshisundaram-Pleijel coefficients $\mathsf{a}_{j,\ell}(\mathscr{X}_{nc})$ for a non-compact space are given for $\mathsf{b}^{\ell}_{q,j-q}(\mathscr{X}_{nc})$ is given in (5.5.8) by

$$\mathsf{a}_{j,\ell}(\mathscr{X}_{nc}) = \sum_{q=0}^{j} \mathsf{b}^{\ell}_{q,j-q}(\mathscr{X}_{nc}). \tag{5.5.10}$$

Proof. Here we have arranged (5.5.7) into increasing powers of t, noting again that the range p there can be exceeded by $j - q$ in (5.5.10) since $\mathsf{H}^{\ell}_{p} = 0$ outside this range. $\square$

The generalised proportionality principle. We can compare (5.5.4) and (5.5.9) using

Remark A.6.5 together with the proportionality principle (5.1.6) to see that for a pair of dual spaces we have

$$\mathsf{b}^{\ell}_{j,p}(\mathscr{X}_c) = (-1)^{p+j}\mathsf{b}^{\ell}_{j,p}(\mathscr{X}_{nc}) \tag{5.5.11}$$

and hence via (5.5.5) and (5.5.10) we have the generalised proportionality

$$\mathsf{a}_{j,\ell}(\mathscr{X}_c) = \sum_{q=0}^{j}\mathsf{b}^{\ell}_{q,j-q}(\mathscr{X}_c) = \sum_{q=0}^{j}(-1)^j\mathsf{b}^{\ell}_{q,j-q}(\mathscr{X}_{nc}) = (-1)^j\mathsf{a}_{j,\ell}(\mathscr{X}_{nc}). \tag{5.5.12}$$

We summarise this in the following theorem.

Theorem 5.5.3. *Let $\mathscr{X}_c$ and $\mathscr{X}_{nc}$ be a dual pair of rank-one symmetric spaces. Then the generalised Minakshisundaram-Pleijel heat coefficients $\mathsf{a}_{j,\ell}(\mathscr{X})$ have the proportionality*

$$\mathsf{a}_{j,\ell}(\mathscr{X}_c) = (-1)^j\mathsf{a}_{j,\ell}(\mathscr{X}_{nc}), \tag{5.5.13}$$

where $\mathsf{a}_{j,\ell}(\mathscr{X}_c)$ is given in (5.5.4), and $\mathsf{a}_{j,\ell}(\mathscr{X}_{nc})$ in (5.5.9).

Proof. Although a proof is provided prior to the statement, we may also refer to the explicit calculations in Section 5.3 and Section 5.4. There we arrange the asymptotics for $b^n_{2\ell}(t;\mathscr{X}_c)$ as

$$b^n_{2\ell}(t;\mathscr{X}_c) \sim \frac{e^{\rho^2 t}(4\pi)^{\ell}}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\sum_{p=0}^{\ell-1}\sum_{m=0}^{\ell-p}\mathsf{p}^{m,\ell}_{j,p}(\mathscr{X}_c)t^{j+m+p} \tag{5.5.14}$$

and we arrange $b^n_{2\ell}(t;\mathscr{X}_{nc})$ as

$$b^n_{2\ell}(t;\mathscr{X}_{nc}) = \frac{e^{-\rho^2 t}(4\pi)^{\ell}}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\sum_{p=0}^{\ell-1}\sum_{m=0}^{\ell-p}\mathsf{q}^{m,\ell}_{j,p}(\mathscr{X}_{nc})t^{j+m+p}, \tag{5.5.15}$$

where we present the explicit values of $\mathsf{p}^{m,\ell}_{j,p}(\mathscr{X}_c)$ and $\mathsf{q}^{m,\ell}_{j,p}(\mathscr{X}_{nc})$ for each rank-one symmetric space. It is clear by inspection of these explicit values, and careful use of Remark A.6.5 and Remark A.4.1, that for a dual pair $\mathscr{X}_c$ and $\mathscr{X}_{nc}$ we have the deeper relationship

$$\mathsf{p}^{m,\ell}_{j,p}(\mathscr{X}_c) = (-1)^{m+p+j}\mathsf{q}^{m,\ell}_{j,p}(\mathscr{X}_{nc}). \tag{5.5.16}$$

Next we arrange (5.5.14) in increasing powers of t, resulting in

$$\begin{aligned} b^n_{2\ell}(t;\mathscr{X}_c) &\sim \frac{e^{\rho^2 t}(4\pi)^{\ell}}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\sum_{i=0}^{j-k}\mathsf{p}^{j-k-i,\ell}_{k,i}(\mathscr{X}_c)\right]t^j \\ &= \frac{e^{\rho^2 t}}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\mathsf{B}_{j,\ell}(\mathscr{X}_c)t^j = \frac{1}{(4\pi t)^{d/2+\ell}}\sum_{j=0}^{\infty}\mathsf{a}_{j,\ell}(\mathscr{X}_c)t^j, \end{aligned} \tag{5.5.17}$$

where $\mathsf{a}_{j,\ell}(\mathscr{X}_c)$ are the generalised Minakshisundaram-Pleijel heat coefficients on a compact space, and we have defined the coefficients $\mathsf{B}_{j,\ell}(\mathscr{X}_c)$ as

$$\mathsf{B}_{j,\ell}(\mathscr{X}_c) = (4\pi)^\ell \sum_{k=0}^{j} \sum_{i=0}^{j-k} \mathsf{p}_{k,i}^{j-k-i,\ell}(\mathscr{X}_c). \tag{5.5.18}$$

A similar rearrangement of (5.5.15) leads to the analogous coefficients for a non-compact space being defined by

$$\begin{aligned} b_{2\ell}^n(t;\mathscr{X}_{nc}) &= \frac{e^{-\rho^2 t}(4\pi)^\ell}{(4\pi t)^{d/2+\ell}} \sum_{j=0}^{\infty} \left[\sum_{k=0}^{j} \sum_{i=0}^{j-k} \mathsf{q}_{k,i}^{j-k-i,\ell}(\mathscr{X}_{nc}) \right] t^j \\ &= \frac{e^{-\rho^2 t}}{(4\pi t)^{d/2+\ell}} \sum_{j=0}^{\infty} \mathsf{B}_{j,\ell}(\mathscr{X}_{nc}) t^j = \frac{1}{(4\pi t)^{d/2+\ell}} \sum_{j=0}^{\infty} \mathsf{a}_{j,\ell}(\mathscr{X}_{nc}) t^j, \end{aligned} \tag{5.5.19}$$

where $\mathsf{a}_{j,\ell}(\mathscr{X}_{nc})$ are the generalised Minakshisundaram-Pleijel heat coefficients for a non-compact space, and we have written

$$\mathsf{B}_{j,\ell}(\mathscr{X}_{nc}) = (4\pi)^\ell \sum_{k=0}^{j} \sum_{i=0}^{j-k} \mathsf{q}_{k,i}^{j-k-i,\ell}(\mathscr{X}_{nc}). \tag{5.5.20}$$

With this, we invoke (5.5.16) for a dual pair $\mathscr{X}_c$ and $\mathscr{X}_{nc}$ and write

$$\begin{aligned} \mathsf{B}_{j,\ell}(\mathscr{X}_c) &= (4\pi)^\ell \sum_{k=0}^{j} \sum_{i=0}^{j-k} \mathsf{p}_{k,i}^{j-k-i,\ell}(\mathscr{X}_c) \\ &= (-1)^j (4\pi)^\ell \sum_{k=0}^{j} \sum_{i=0}^{j-k} \mathsf{q}_{k,i}^{j-k-i,\ell}(\mathscr{X}_{nc}) = (-1)^j \mathsf{B}_{j,\ell}(\mathscr{X}_{nc}). \end{aligned} \tag{5.5.21}$$

We note that it is enough to show that $\mathsf{B}_{j,\ell}(\mathscr{X}_c) = (-1)^j \mathsf{B}_{j,\ell}(\mathscr{X}_{nc})$, since the sign of the power of the exponential terms in (5.5.17) and (5.5.19) does not affect the resulting power of (-1). We see this by writing

$$e^{t\rho^2} \sum_{j=0}^{\infty} \mathsf{B}_{j,\ell}(\mathscr{X}_c) t^j = \left[\sum_{j=0}^{\infty} \frac{\rho^{2j}}{j!} t^j \right] \left[\sum_{j=0}^{\infty} \mathsf{B}_{j,\ell}(\mathscr{X}_c) t^j \right] = \sum_{j=0}^{\infty} \mathsf{a}_{j,\ell}(\mathscr{X}_c) t^j, \tag{5.5.22}$$

where using the Cauchy product $\mathsf{a}_{j,\ell}(\mathscr{X}_c)$ is given explicitly by

$$\mathsf{a}_{j,\ell}(\mathscr{X}_c) = \sum_{k=0}^{j} \frac{\rho^{2(j-k)}}{(j-k)!} \mathsf{B}_{k,\ell}(\mathscr{X}_c). \tag{5.5.23}$$

Now similarly for $\mathsf{B}_{j,\ell}(\mathscr{X}_{nc})$, we take a negative power of the exponential and write

$$e^{-t\rho^2} \sum_{j=0}^{\infty} \mathsf{B}_{j,\ell}(\mathscr{X}_{nc}) t^j = \left[\sum_{j=0}^{\infty} (-1)^j \frac{\rho^{2j}}{j!} t^j \right] \left[\sum_{j=0}^{\infty} \mathsf{B}_{j,\ell}(\mathscr{X}_{nc}) t^j \right] = \sum_{j=0}^{\infty} \mathsf{a}_{j,\ell}(\mathscr{X}_{nc}) t^j, \tag{5.5.24}$$

where $\mathsf{a}_{j,\ell}(\mathscr{X}_{nc})$ is given by

$$\mathsf{a}_{j,\ell}(\mathscr{X}_{nc}) = \sum_{k=0}^{j} (-1)^{j-k} \frac{\rho^{2(j-k)}}{(j-k)!} \mathsf{B}_{k,\ell}(\mathscr{X}_{nc}). \tag{5.5.25}$$

Hence given $\mathsf{B}_{j,\ell}(\mathscr{X}_{nc}) = (-1)^j \mathsf{B}_{j,\ell}(\mathscr{X}_{nc})$ as in (5.5.21), then from (5.5.23) and (5.5.25) we clearly see $\mathsf{a}_{j,\ell}(\mathscr{X}_c) = (-1)^j \mathsf{a}_{j,\ell}(\mathscr{X}_{nc})$ as desired. This completes the proof. $\square$

Chapter 6

Generalised Spectral Zeta Functions: The Residues, Poles, and the Proportionality Principle

6.1 Introduction

With the same notation for the Laplacian, and all its associated spectral quantities as in the previous chapter (in the interest of brevity, we do not repeat this here), we define the spectral zeta function $\zeta(s;\mathscr{M})$ for a compact Riemannian manifold $(\mathscr{M},g)$ by the Dirichlet-type series

$$\zeta(s;\mathscr{M}) = \sum_{k=1}^{\infty} \frac{M_k(\mathscr{M})}{[\lambda_k]^s}, \quad \mathrm{Re}s > d/2. \tag{6.1.1}$$

This can be extended by analytic continuation to a meromorphic function on the complex plane $\mathbb{C}$ with its poles $s = s_j$ lying on the real axis where, specifically depending on the dimension being even or odd, they occur only at points [1]

$$s_j = d/2 - j, \qquad j = \begin{cases} 0, 1, 2, \dots, d/2 - 1 & \text{for d even,} \\ 0, 1, 2, \dots & \text{for d odd.} \end{cases} \tag{6.1.2}$$

It is well-known that the spectral zeta function can be written as the Mellin transform of the trace of the heat kernel $H_{\mathscr{M}}(t;x,x) = \Theta(t;\mathscr{M})$, specifically

$$\begin{aligned} \zeta(s;\mathscr{M}) := \frac{\mathrm{Vol}(\mathscr{M})}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\Theta(t;\mathscr{M}) - 1\right) dt &= \sum_{k=1}^{\infty} \frac{M_k(\mathscr{M})}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda_k} dt \\ = \frac{1}{\Gamma(s)} \sum_{k=1}^{\infty} \frac{M_k(\mathscr{M})}{[\lambda_k]^s} \int_0^\infty u^{s-1} e^{-u} du &= \sum_{k=1}^{\infty} \frac{M_k(\mathscr{M})}{[\lambda_k]^s}. \end{aligned} \tag{6.1.3}$$

[1] We come to the residues at the poles of $\zeta(s;\mathscr{M})$ later on in the chapter.

This relationship links the residues of the spectral zeta functions to the short-time asymptotics of the heat trace as formulated via the Minakshisundaram-Pleijel heat coefficients $\mathsf{a}_j(\mathscr{M})$. In fact a direct analysis leads to the identity

$$\mathrm{Res}_{s=s_j}\zeta(s;\mathscr{M}) = \frac{\mathsf{a}_j(\mathscr{M})}{(4\pi)^{d/2}\Gamma(d/2-j)}. \tag{6.1.4}$$

Recall that the scalars $\mathsf{a}_j(\mathscr{M})$ are the quantities arising from the short-time asymptotic expansion

$$\Theta(t;\mathscr{M}) \sim \frac{1}{(4\pi t)^{d/2}}\sum_{j=0}^{\infty}\mathsf{a}_j(\mathscr{M})t^j, \qquad t\searrow 0. \tag{6.1.5}$$

Commenting further, in odd dimensions the spectral zeta function has zeros at negative integers whilst $\zeta(0,\mathscr{M}) = -1$. In contrast in even dimensions, we have

$$\zeta(0;\mathscr{M}) = \frac{\mathsf{a}_{d/2}(\mathscr{M})}{(4\pi)^{d/2}} - 1, \quad \zeta(-k;\mathscr{M}) = \frac{(-1)^k k!}{(4\pi)^{d/2}}\mathsf{a}_{d/2+k}(\mathscr{M}), \quad k = 1, 2, \dots. \tag{6.1.6}$$

From this point onwards we specialise the discussion to the setting of rank-one symmetric spaces $\mathscr{X} = \mathbf{G}/\mathbf{H}$. In the compact case, with $\mathscr{X} = \mathscr{X}_c$, we have all of the above statements subject to $\mathscr{M} = \mathscr{X}_c$. In the non-compact case $\mathscr{X} = \mathscr{X}_{nc}$, the spectrum of the Laplace-Beltrami operator has a continuous part, and so $\zeta(s;\mathscr{X}_{nc})$ cannot be defined by a Dirichlet-type series as in the compact case. However, motivated by the identity (6.1.3) and the functional calculus of the Laplace-Beltrami operator, one can formally define the spectral zeta function $\zeta(s;\mathscr{X}_{nc})$ by taking the Mellin transform of the heat trace on a non-compact space $H_{\mathscr{X}_{nc}}(t;x,x) = \Theta(t;\mathscr{X}_{nc})$ [2] as

$$\zeta(s;\mathscr{X}_{nc}) = \mathsf{c}(\mathscr{X}_{nc})\int_0^{\infty}\frac{\mu(\lambda)}{(\lambda^2+\rho^2)^s}\,d\lambda, \tag{6.1.7}$$

with $\mathsf{c}(\mathscr{X}_{nc})$ a scaling factor (see (6.2.8)), and $\rho = (\alpha+\beta+1)/2$, with the parameters α and β associated to each symmetric space of rank one given in Table 3.

Although the above integral (6.1.7) bares no immediate resemblance to the Dirichlet-type series (6.1.1), a remarkable fact is that they do share the same set of poles as in (6.1.2), and furthermore the residues at a given pole s_j of $\zeta(s;\mathscr{X}_{nc})$, for $\mathscr{X}_{nc}$ non-compact is exactly

$$\mathrm{Res}_{s=s_j}\zeta(s;\mathscr{X}_{nc}) = \frac{\mathsf{a}_j(\mathscr{X}_{nc})}{(4\pi)^{d/2}\Gamma(d/2-j)}, \tag{6.1.8}$$

where $\mathsf{a}_j(\mathscr{X}_{nc})$ are coefficients in a series expansion of $\Theta(t;\mathscr{X}_{nc})$ given by

$$\Theta(t;\mathscr{X}_{nc}) = \frac{1}{(4\pi t)^{d/2}}\sum_{j=0}^{\infty}\mathsf{a}_j(\mathscr{X}_{nc})t^j. \tag{6.1.9}$$

[2] Note that due to the space being symmetric the heat trace becomes independent of the variable x.

In [29] the expansions (6.1.5) and (6.1.9) were studied in depth, and the so-called Proportionality Principle was formulated, stating that for a compact symmetric space $\mathscr{X}_c$ and its non-compact dual $\mathscr{X}_{nc}$ we have the relation

$$\mathsf{a}_j(\mathscr{X}_c) = (-1)^j \mathsf{a}_j(\mathscr{X}_{nc}). \tag{6.1.10}$$

We can then deduce a similar proportionality between the residues of a shared pole s_j of $\zeta(s;\mathscr{X}_c)$ and $\zeta(s;\mathscr{X}_{nc})$ by invoking the identites (6.1.4) and (6.1.8), hence obtaining

$$\mathrm{Res}_{s=s_j}\zeta(s;\mathscr{X}_c) = (-1)^j \mathrm{Res}_{s=s_j}\zeta(s;\mathscr{X}_{nc}). \tag{6.1.11}$$

Following the approach, analysis, and results in Chapter 5 on the Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X})$, and the above motivating discussion, in this chapter, by defining a generalised spectral zeta function $\zeta_\ell(s;\mathscr{X})$ as the Mellin transform of $b^n_{2\ell}(t;\mathscr{X})$, we aim at formulating and proving a generalised proportionality principle analogous to (6.1.12) between the residues of $\zeta_\ell(s;\mathscr{X}_c)$ and $\zeta_\ell(s;\mathscr{X}_{nc})$ at their respective poles. More specifically the relation

$$\mathrm{Res}_{s=s_j}\zeta_\ell(s;\mathscr{X}_c) = (-1)^j \mathrm{Res}_{s=s_j}\zeta_\ell(s;\mathscr{X}_{nc}), \qquad \ell \geq 0. \tag{6.1.12}$$

6.2 Zeta functions on rank-one symmetric spaces

Compact rank-one symmetric spaces and their zeta functions. Here we specialise to the case where $\mathscr{X} = \mathscr{X}_c$ is a compact rank-one symmetric space. These are given by the unit sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbb{R}\mathbf{P}^n = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$, the quaternionic projective space $\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$, and the Cayley Plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}^4/\mathbf{Spin}(9)$.

We can define the heat kernel $H_{\mathscr{X}_c}(t;x,y)$ explicitly by the spectral sum

$$H_{\mathscr{X}_c}(t;x,y) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X}_c)}{\mathrm{Vol}(\mathscr{X}_c)} \mathscr{P}_k^{\alpha,\beta}(\cos\theta) e^{-t\lambda_k^{\alpha,\beta}}. \tag{6.2.1}$$

Here θ denotes the geodesic distance between x and y on $\mathscr{X}_c$, and $\mathscr{P}_k^{\alpha,\beta}$ are the normalised Jacobi polynomials satisfying $\mathscr{P}_k^{\alpha,\beta}(0) = 1$. In this case the spectral zeta function is given by

$$\zeta(s;\mathscr{X}_c) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X}_c)}{[k(k+\alpha+\beta+1)]^s}, \tag{6.2.2}$$

where $k(k+\alpha+\beta+1) = \lambda_k^{\alpha,\beta}$ are the numerically distinct eigenvalues of the Jacobi operator, and the associated multiplicity function $M_k(\mathscr{X}_c)$ is given explicitly by

$$M_k(\mathscr{X}_c) = \frac{(\alpha+\beta+2k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\beta+1)\Gamma(k+d/2)}{\Gamma(k+1)\Gamma(\alpha+\beta+2)\Gamma(d/2)\Gamma(k+\beta+1)}, \tag{6.2.3}$$

Here α and β are fixed parameters associated to each space (see Table 3), where $d = \dim(\mathscr{X}_c) = 2\alpha+2$.

Zeta functions on a non-compact rank-one symmetric space. Let $\mathscr{X}_{nc}$ be a rank-one symmetric space of non-compact type. The these are the real hyperbolic space $\mathbb{R}\mathbf{H}^n = \mathbf{SO}_0(n,1)/\mathbf{SO}(n)$, the complex hyperbolic space $\mathbb{C}\mathbf{H}^n = \mathbf{SU}(n,1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$, the quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n = \mathbf{Sp}(n,1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$, and the hyperbolic Cayley plane $\mathbf{H}^2(\mathrm{Cay}) = \mathbf{F}_\star^4/\mathbf{Spin}(9)$.

By taking the Mellin transform of the trace of the heat kernel $\Theta(t;\mathscr{X}_{nc}) = H_{\mathscr{X}_{nc}}(t;x,x)$ we aim to derive a spectral zeta function in this setting, analogous to that on the compact dual space $\mathscr{X}_c$. Here the heat kernel is given by the integral

$$H_{\mathscr{X}_{nc}}(t;x,y) = \frac{2^{2\beta-1}\Gamma(\alpha+1)}{\pi^{\alpha+2}}\int_0^\infty \Phi_\lambda^{\alpha,\beta}(r)e^{-t(\lambda^2+\rho^2)}\mu(\lambda)\,d\lambda, \tag{6.2.4}$$

where $\Phi_\lambda^{\alpha,\beta}(r) = \mathscr{P}_{-i\lambda-\rho}^{\alpha,\beta}(\cosh r)$ are the Jacobi functions, $(\lambda^2+\rho^2)$ are the eigenvalues of the Laplace-Beltrami operator on $\mathscr{X}_{nc}$, with $\lambda\in\mathbb{C}$, and $\mu(\lambda) = |C(\lambda)|^{-2}$ is the associated Plancherel measure given by

$$\mu(\lambda) = \frac{|\Gamma(i\lambda+(\alpha+\beta+1)/2)|^2|\Gamma(i\lambda+(\alpha-\beta+1)/2)|^2}{|2^{\alpha+\beta+1-2i\lambda}|^2|\Gamma(\alpha+1)|^2|\Gamma(2i\lambda)|^2}, \tag{6.2.5}$$

where $C(\lambda)$ denotes the Harish-Chandra function

$$C(\lambda) = \lim_{r\nearrow\infty}\Phi_\lambda^{\alpha,\beta}(r)e^{(\rho-i\lambda)r} = \frac{4^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(2i\lambda)}{\Gamma(\rho+i\lambda)\Gamma(i\lambda+(\alpha+1-\beta)/2)}. \tag{6.2.6}$$

Here α and β are fixed parameters associated to the space $\mathscr{X}_{nc}$, with $\rho = (\alpha+\beta+1)/2$, and we note that a pair of dual spaces share the same values for these parameters (see Table 3).

Setting $x=y$ in the heat kernel (6.2.4), we define its trace by the integral

$$\Theta(t;\mathscr{X}_{nc}) = H_{\mathscr{X}_{nc}}(t;x,x) = \frac{2^{2\beta-1}\Gamma(\alpha+1)}{\pi^{\alpha+2}}\int_0^\infty e^{-t(\lambda^2+\rho^2)}\mu(\lambda)\,d\lambda. \tag{6.2.7}$$

Taking the Mellin transform of this, we can then define a spectral zeta function on a non-compact rank-one symmetric space $\mathscr{X}_{nc}$ as

$$\begin{aligned}\zeta(s;\mathscr{X}_{nc}) &:= \frac{1}{\Gamma(s)}\left[\mathcal{M}\Theta(t;\mathscr{X}_{nc})\right](s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\Theta(t;\mathscr{X}_{nc})\,dt \\ &= \frac{2^{2\beta-1}\Gamma(\alpha+1)}{\pi^{\alpha+2}\Gamma(s)}\int_0^\infty t^{s-1}\int_0^\infty e^{-t(\lambda^2+\rho^2)}\mu(\lambda)\,d\lambda\,dt \\ &= \frac{2^{2\beta-1}\Gamma(\alpha+1)}{\pi^{\alpha+2}}\int_0^\infty \frac{\mu(\lambda)}{(\lambda^2+\rho^2)^s}\,d\lambda. \end{aligned} \tag{6.2.8}$$

As in the compact case, in odd dimensions $\zeta(s;\mathscr{X}_{nc})$ has zeros at negative integers, and also satisfies $\zeta(0;\mathscr{X}_{nc}) = -1$. Mirroring (6.1.6), in even dimensions we have the point values

$$\zeta(0;\mathscr{X}_{nc}) = \frac{\mathsf{a}_{d/2}(\mathscr{X}_{nc})}{(4\pi)^{d/2}} - 1, \quad \zeta(-k;\mathscr{X}_{nc}) = \frac{(-1)^k k!}{(4\pi)^{d/2}}\mathsf{a}_{d/2+k}(\mathscr{X}_{nc}), \quad k = 1,2,\ldots. \tag{6.2.9}$$

where $\mathsf{a}_j(\mathscr{X}_{nc})$ are the coefficients in the expansion (6.1.9). What is most interesting is that the poles of $\zeta(s;\mathscr{X}_{nc})$ occur at the same points as those of $\zeta(s;\mathscr{X}_c)$ as well. They are at $s = s_j = d/2 - j$, where for odd dimensional spaces - that is, the real hyperbolic space $\mathbb{R}\mathbf{H}^n$ with $n \geq 3$ odd - j runs over all non-negative integers, and for even dimensional spaces j runs over $0, 1, \ldots, d/2-1$. Furthermore, the residue at a given pole is given by (6.1.8), mirroring the residue on a compact space in (6.1.4).

The Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X})$. We focus on briefly defining the Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X})$, that are seen to generalise the heat trace on both compact and non-compact spaces. In both of these cases, $b^n_{2\ell}(t;\mathscr{X})$ are seen to arise in the Maclaurin expansion of the heat kernel $H_{\mathscr{X}}(t;x,y)$. If $\mathscr{X} = \mathscr{X}_c$ is a compact rank-one symmetric space, the heat kernel is given in (6.2.1). To first introduce the Maclaurin heat coefficients, we abuse notation slightly to write $H_{\mathscr{X}}(t;\theta) = H_{\mathscr{X}}(t;x,y)$, and define

$$b^n_{2\ell}(t;\mathscr{X}_c) := \frac{d^{2\ell}}{d\theta^{2\ell}}H_{\mathscr{X}}(t;\theta)\Big|_{\theta=0}. \tag{6.2.10}$$

The name of these coefficients is motivated by how they are seen to arise as the coefficients of the Maclaurin expansion of the heat kernel about $\theta = 0$. We note that the derivatives will pass onto the Jacobi polynomial $\mathscr{P}^{\alpha,\beta}_k(\cos\theta)$ as in (6.2.1). Referring to Lemma A.6.3 and (A.6.21), the Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X}_c)$ can then be written as

$$b^n_{2\ell}(t;\mathscr{X}_c) = \mathscr{R}^{\alpha,\beta}_\ell(-d/dt)\Theta(t;\mathscr{X}_c). \tag{6.2.11}$$

On a non-compact rank-one symmetric space, the story is similar. In this case the heat kernel $H_{\mathscr{X}_{nc}}(t;x,y)$ is given in (6.2.4), and so writing $H_{\mathscr{X}_{nc}}(t;r) = H_{\mathscr{X}_{nc}}(t;x,y)$, with r

the geodesic distance between x and y, we can define the Maclaurin heat coefficients by

$$b^n_{2\ell}(t;\mathscr{X}_{nc}) = \frac{d^{2\ell}}{dr^{2\ell}} H_{\mathscr{X}_{nc}}(t;r)\Big|_{r=0}. \tag{6.2.12}$$

Similar to the compact case, the derivatives above fall directly onto the Jacobi function. Referring to Lemma A.6.4 and (A.6.25) in this case, we can write the Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X}_{nc})$ as

$$b^n_{2\ell}(t;\mathscr{X}_{nc}) = \mathscr{Q}^{\alpha,\beta}_{\ell}(d/dt)\Theta(t;\mathscr{X}_{nc}). \tag{6.2.13}$$

We note that in both (6.2.11) and (6.2.13), when $\ell = 0$ we see the trace of the heat kernel arising as $b^n_0(t;\mathscr{X}) = \Theta(t;\mathscr{X})$.

The generalised spectral zeta function $\zeta_\ell(s;\mathscr{X})$. From the relation between the Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X})$ and the trace of the heat kernel given in (6.2.11) and (6.2.13) for compact and non-compact spaces respectively, we are motivated to take the Mellin transform of $b^n_{2\ell}(t;\mathscr{X})$ with the aim of defining a more general spectral zeta function. In the compact case first, we define $\zeta_\ell(s;\mathscr{X}_c) = \mathrm{Vol}(\mathscr{X}_c)[\mathcal{M}b^n_{2\ell}(t;\mathscr{X}_c)](s)/\Gamma(s)$ so that using the definition (6.2.13) in conjunction with Lemma A.6.3 we have

$$\begin{aligned}
\zeta_\ell(s;\mathscr{X}_c) &:= \frac{\mathrm{Vol}(\mathscr{X}_c)}{\Gamma(s)} \int_0^\infty t^{s-1}\left[\mathscr{R}_\ell(-d/dt)\Theta(t;\mathscr{X}_c)\right]\, dt \\
&= \frac{1}{\Gamma(s)} \sum_{p=1}^{\ell} (-1)^p \mathsf{h}^\ell_p \sum_{k=0}^{\infty} M_k(\mathscr{X}_c) \int_0^\infty t^{s-1} \frac{d^p}{dt^p} e^{-t\lambda_k}\, dt \\
&= \sum_{p=1}^{\ell} \mathsf{h}^\ell_p \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X}_c)}{[\lambda_k]^{s-p}} = \sum_{p=1}^{\ell} \mathsf{h}^\ell_p \zeta(s-p;\mathscr{X}_c).
\end{aligned} \tag{6.2.14}$$

Next on a non-compact space, we define $\zeta_\ell(s;\mathscr{X}_{nc}) = \left[\mathcal{M}b^n_{2\ell}(t;\mathscr{X}_{nc})\right](s)/\Gamma(s)$ so that using (6.2.13) and Lemma A.6.4 we can write

$$\frac{1}{\Gamma(s)}\left[\mathcal{M}b^n_{2\ell}(t;\mathscr{X}_{nc})\right](s) = \sum_{p=1}^{\ell} \frac{\mathsf{H}^\ell_p}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{d^p}{dt^p}\Theta(t;\mathscr{X}_{nc})\, dt. \tag{6.2.15}$$

Hence we may define a new zeta function $\zeta_\ell(s;\mathscr{E}_{nc})$, with $\ell \geq 0$, as

$$\begin{aligned}
\zeta_\ell(s;\mathscr{X}_{nc}) &= \sum_{p=1}^{\ell} (-1)^p \frac{\mathsf{H}^\ell_p}{\Gamma(s)} \mathbf{c}(\mathscr{X}_{nc}) \int_0^\infty t^{s-1} \int_0^\infty (\lambda^2+\rho^2)^p \mu(\lambda) e^{-t(\lambda^2+\rho^2)}\, d\lambda\, dt \\
&= \sum_{p=1}^{\ell} (-1)^p \mathsf{H}^\ell_p \mathbf{c}(\mathscr{X}_{nc}) \int_0^\infty \frac{\mu(\lambda)}{(\lambda^2+\rho^2)^{s-p}}\, d\lambda \\
&= \sum_{p=1}^{\ell} (-1)^p \mathsf{H}^\ell_p \zeta(s-p;\mathscr{X}_{nc}).
\end{aligned} \tag{6.2.16}$$

where for convenience we have denoted $\mathbf{c}(\mathscr{X}_{nc}) = 2^{2\beta-1}\Gamma(\alpha+1)/\pi^{\alpha+2}$ as the scaling factor in the definition of $\Theta(t;\mathscr{X}_{nc})$ in (6.2.4).

6.3 Explicit calculations of the zeta function $\zeta_\ell(s;\mathscr{X})$

In what follows we will provide explicit formulae for the spectral zeta function $\zeta_\ell(s;\mathscr{X})$ for each rank-one symmetric space of both compact and non-compact type. First we require some auxilliary results that will assist in evaluating each case. We define two functions $\mathscr{B}_1$ and $\mathscr{B}_2$ as

$$\mathscr{B}_1(X) = (1-2^{-1-2X})\frac{|B_{2(X+1)}|}{X+1}, \qquad \mathscr{B}_2(X) = \frac{B_{2(X+2)}}{X+2}, \tag{6.3.1}$$

where B_{2n} denotes the $2n^{th}$ Bernoulli number.

Lemma 6.3.1. *For any non-negative integer k, we have*

$$\begin{aligned}\int_0^\infty \frac{2\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+1}\ d\lambda &= \rho^{2k+2-2s}B(k+1,s-k-1)\\ &\quad -\sum_{j=0}^\infty \frac{1}{\rho^{2s+2j}}\frac{(-1)^j}{j!}\frac{\Gamma(s+j)}{\Gamma(s)}\mathscr{B}_1(k+j),\end{aligned} \tag{6.3.2}$$

with $B(x,y)$ denoting the well-known Beta function, and $\mathscr{B}_1(X)$ as in (A.6.3). Similarly we have

$$\begin{aligned}\int_0^\infty \frac{2\coth(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+3}\ d\lambda &= \rho^{2k+4-2s}B(k+2,s-k-2)\\ &\quad -\sum_{j=0}^\infty \frac{1}{\rho^{2s+2j}}\frac{(-1)^j}{j!}\frac{\Gamma(s+j)}{\Gamma(s)}\mathscr{B}_2(k+j+1),\end{aligned} \tag{6.3.3}$$

with $B(x,y)$ denoting the Beta function, and $\mathscr{B}_2(X)$ as in (A.6.3).

Proof. We make use of the identity

$$\tanh(\pi x) = 1 - \frac{2}{1+e^{2\pi x}} \tag{6.3.4}$$

which allows us to split the integral on the left-hand side of (A.6.7) as

$$\int_0^\infty \frac{2\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+1}d\lambda = \int_0^\infty \frac{2\lambda^{2k+1}}{(\lambda^2+\rho^2)^s}d\lambda - \int_0^\infty \frac{4(\lambda^2+\rho^2)^{-s}}{1+e^{2\pi\lambda}}\lambda^{2k+1}d\lambda. \tag{6.3.5}$$

The first integral above can be written as

$$\begin{aligned}\int_0^\infty \frac{2\lambda^{2k+1}}{(\lambda^2+\rho^2)^s}d\lambda &= \rho^{2k+1-2s}\int_0^\infty \frac{2[\lambda/\rho]^{2k+1}}{([\lambda/\rho]^2+1)^s}d\lambda\\ &= \rho^{2k+2-2s}\int_0^\infty \frac{2p^{2k+1}}{(p^2+1)^s}\ dp\end{aligned} \tag{6.3.6}$$

We now refer to the integral identity

$$\int_0^\infty p^{a-1}(1+p^2)^{b-1}\ dp = \frac{1}{2}B\left(\frac{a}{2},1-b-\frac{a}{2}\right), \tag{6.3.7}$$

which allows (A.6.11) to be evaluated as

$$\int_0^\infty \frac{2\lambda^{2k+1}}{(\lambda^2+\rho^2)^s}\,d\lambda = \rho^{2k+2-2s} B\left(k+1, s-k-1\right). \tag{6.3.8}$$

For the second integral, we apply the generalised binomial theorem to write

$$\begin{aligned}\frac{1}{(\lambda^2+\rho^2)^s} = \frac{\rho^{-2s}}{(\lambda^2/\rho^2+1)^s} &= \sum_{j=0}^{\infty}\binom{-s}{j}\rho^{-2s-2j}\lambda^{2j} \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{\Gamma(s+j)}{\Gamma(s)}\rho^{-2s-2j}\lambda^{2j}.\end{aligned} \tag{6.3.9}$$

This allows us to write the second integral explicitly as

$$\int_0^\infty \frac{4(\lambda^2+\rho^2)^{-s}}{1+e^{2\pi\lambda}}\lambda^{2k+1}d\lambda = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{\Gamma(s+j)}{\Gamma(s)}\rho^{-2s-2j}\int_0^\infty \frac{4\lambda^{2k+2j+1}}{1+e^{2\pi\lambda}}d\lambda. \tag{6.3.10}$$

To evaluate the integral here, we refer to the identity

$$\int_0^\infty \frac{x^{2n-1}}{1+e^{2\pi x}}\,dx = (1-2^{1-2n})\frac{|B_{2n}|}{4n} = \frac{\mathscr{B}_1(n-1)}{4}. \tag{6.3.11}$$

This completes the first result. The proof of the second result follows the above closely, using a different set of integral identites. Firstly instead of (A.6.9) we use

$$\coth(\pi x) = 1 - \frac{2}{1-e^{2\pi x}} \tag{6.3.12}$$

Next instead of (A.6.16) we use

$$\int_0^\infty \frac{x^{2n-1}}{1-e^{2\pi x}}\,dx = (-1)^n\frac{B_{2n}}{4n} := (-1)^n\frac{\mathscr{B}_2(n-2)}{4}. \tag{6.3.13}$$

The result follows. □

We will make use of the Hurwitz zeta function $\zeta_H(s,q)$, which for future reference is defined as

$$\zeta_H(s,q) = \sum_{m=0}^{\infty}\frac{1}{(q+m)^s}, \qquad \mathrm{Re(s)} > 1. \tag{6.3.14}$$

The Hurwitz zeta function can be extended via analytic continuation to a meromorphic function on $\mathbb{C}\backslash\{1\}$, with a simple pole at $s=1$ with residue 1. In each case below we formulate $\zeta_\ell(s;\mathscr{X})$ so that the $\ell=0$ case (that is, the spectral zeta function) follows by suitably setting the value $p=0$ throughout, with the sum over p disappearing. This also requires the knowledge that $\mathsf{h}_0^\ell = \mathsf{H}_0^\ell = 1$.

The case $\mathscr{X}_c = \mathbb{S}^n$ vs. $\mathscr{X}_{nc} = \mathbb{R}\mathbf{H}^n$. Here we have $\alpha = \beta = (n-2)/2$, and hence

$\rho = (n-1)/2$ and $d = n$. We begin with the compact case, where the multiplicity $M_k(\mathbb{S}^n)$ is given by

$$M_k(\mathbb{S}^n) = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} \tag{6.3.15}$$

We also fix a parameter $X_k = k + \rho$, so that the eigenvalues of $-\Delta_{\mathbb{S}^n}$ satisfy $\lambda_k^n = k(k+n-1) = X_k^2 - \rho^2$.

Proposition 6.3.2. *Let $\mathscr{X} = \mathbb{S}^n$. Then for $n \geq 3$ odd, the zeta function $\zeta_\ell(s;\mathbb{S}^n)$ can be written as*

$$\zeta_\ell(s;\mathbb{S}^n) = \sum_{m=0}^{\infty}\sum_{p=1}^{\ell}\sum_{j=0}^{\frac{n-3}{2}} \frac{2\mathsf{h}_p^\ell \mathsf{a}_j^n}{\Gamma(n)}\frac{\rho^{2m}}{m!}(s-p)_m \zeta_H(2(s-p+m-j)+2,\rho+1), \tag{6.3.16}$$

with a_j^n as in (A.4.1) and $\zeta_H(s,q)$ as in (6.3.14). For $n \geq 2$ even, we have

$$\zeta_\ell(s;\mathbb{S}^n) = \sum_{m=0}^{\infty}\sum_{p=1}^{\ell}\sum_{j=0}^{\frac{n-2}{2}} \frac{2\mathsf{h}_p^\ell \mathsf{b}_j^n \rho^{2m}}{\Gamma(n)m!}(s-p)_m \zeta_H(2(s-p+m-j)-1,\rho+1) \tag{6.3.17}$$

with b_k^n as in (A.4.2) and $\zeta_H(s,q)$ as in (6.3.14). In each case $\rho = (n-1)/2$, and $\mathsf{h}_p^\ell = \mathsf{h}_p^\ell(\alpha,\beta)$ are defined in Lemma A.6.3.

Proof. For $n \geq 3$ odd, with the multiplicity function $M_k(\mathbb{S}^n)$ given in (6.3.15) and $X_k = k + \rho = k + (n-1)/2$, we can write

$$M_k(\mathbb{S}^n) = \frac{2X_k}{(n-1)!}\prod_{j=1}^{n-2}(k+j) = \frac{2}{(n-1)!}\prod_{j=0}^{(n-3)/2}(X_k^2 - j^2) = \sum_{j=0}^{(n-3)/2}\frac{2\mathsf{a}_j^n X_k^{2j+2}}{(n-1)!}. \tag{6.3.18}$$

In the final step we have used (A.4.1) to represent $M_k(\mathbb{S}^n)$ as a polynomial. Substituting this into (6.2.2), $\zeta(s;\mathbb{S}^n)$ is given in this case by

$$\zeta(s;\mathbb{S}^n) = \sum_{j=0}^{(n-3)/2}\frac{2\mathsf{a}_j^n}{(n-1)!}\sum_{k=1}^{\infty}\frac{X_k^{2j+2}}{(X_k^2-\rho^2)^s} = \sum_{j=0}^{(n-3)/2}\frac{2\mathsf{a}_j^n}{(n-1)!}\sum_{k=1}^{\infty}\frac{X_k^{2j+2-2s}}{(1-[\rho/X_k]^2)^s}. \tag{6.3.19}$$

Given that $\rho/X_k < 1$ by definition, we can use the binomial expansion as

$$\begin{aligned}
\zeta(s;\mathbb{S}^n) &= \frac{2}{(n-1)!}\sum_{j=0}^{(n-3)/2}\mathsf{a}_j^n\sum_{k=1}^{\infty}X_k^{2j+2-2s}\sum_{m=0}^{\infty}\frac{\Gamma(s+m)}{m!\Gamma(s)}\left(\frac{\rho}{X_k}\right)^{2m}\\
&= \frac{2}{(n-1)!}\sum_{j=0}^{(n-3)/2}\mathsf{a}_j^n\sum_{m=0}^{\infty}\frac{\rho^{2m}}{m!}\frac{\Gamma(s+m)}{\Gamma(s)}\sum_{k=1}^{\infty}X_k^{-2s-2m+2j+2}\\
&= \frac{2}{(n-1)!}\sum_{j=0}^{(n-3)/2}\mathsf{a}_j^n\sum_{m=0}^{\infty}\frac{\rho^{2m}}{m!}\frac{\Gamma(s+m)}{\Gamma(s)}\zeta_H(2s+2m-2j+2,1+\rho),
\end{aligned}$$

where we have arranged the second infinite sum as a Hurwitz zeta function. This completes the $\ell = 0$ case for odd dimensions.

In the case that $n \geq 2$ is even, we can arrange the multiplicity $M_k(\mathbb{S}^n)$ differently as

$$M_k(\mathbb{S}^n) = \frac{2X_k}{(n-1)!} \prod_{j=\frac{1}{2}}^{\frac{n-2}{2}-\frac{1}{2}} (X_k^2 - j^2) = \sum_{j=0}^{(n-2)/2} \frac{2\mathsf{b}_j^n}{(n-1)!} X_k^{2j+1}, \tag{6.3.20}$$

where we have used (A.4.2). Substituting this into the spectral zeta function, we proceed as in the odd dimensional case above via the binomial expansion as

$$\begin{aligned}
\zeta(s;\mathbb{S}^n) &= \frac{2}{(n-1)!} \sum_{j=0}^{(n-2)/2} \mathsf{b}_j^n \sum_{k=1}^{\infty} \frac{X_k^{2j+1}}{(X_k^2-\rho^2)^s} \\
&= \frac{2}{(n-1)!} \sum_{j=0}^{(n-2)/2} \mathsf{b}_j^n \sum_{m=0}^{\infty} \frac{\rho^{2m}\Gamma(s+m)}{m!\Gamma(s)} \sum_{k=1}^{\infty} X_k^{2j+1-2s-2m},
\end{aligned}$$

where we can similarly arrange the inner sum as a Hurwitz zeta function. We can then substitute these into (6.2.14) to arrive at the results. □

Next for the non-compact case, depending on the dimension of $\mathbb{R}\mathbf{H}^n$ the Plancherel measure (6.2.5) is reduced to

$$\mu(\lambda) = \begin{cases} \dfrac{\pi}{|\Gamma(n/2)2^{n-2}|^2} \displaystyle\prod_{j=0}^{(n-3)/2} (\lambda^2+j^2), & n \geq 3 \text{ odd}, \\ \dfrac{\pi\lambda\tanh(\pi\lambda)}{|\Gamma(n/2)2^{n-2}|^2} \displaystyle\prod_{j=1/2}^{(n-3)/2} (\lambda^2+j^2), & n \geq 2 \text{ even}. \end{cases} \tag{6.3.21}$$

Note that for $n=3$ and $n=2$ respectively above, the products are set to 1.

Proposition 6.3.3. *Let $\mathscr{X} = \mathbb{R}\mathbf{H}^n$. Then for $n \geq 3$ odd, the zeta function $\zeta_\ell(s;\mathbb{R}\mathbf{H}^n)$ can be written as*

$$\zeta_\ell(s;\mathbb{R}\mathbf{H}^n) = \sum_{p=1}^{\ell} \sum_{k=0}^{(n-3)/2} \frac{(-1)^p \mathsf{H}_p^\ell \mathsf{A}_k^n}{(4\pi)^{n/2}\Gamma(n/2)} \frac{\Gamma(k+1/2)\Gamma(s-p-k-1/2)}{\rho^{2s-2k-1-2p}\Gamma(s-p)}, \tag{6.3.22}$$

with A_k^n as in (A.4.7). For $n \geq 2$ even, we have

$$\begin{aligned}
\zeta_\ell(s;\mathbb{R}\mathbf{H}^n) = \sum_{p=1}^{\ell} \sum_{k=0}^{(n-2)/2} \frac{(-1)^p \mathsf{H}_p^\ell}{(4\pi)^{n/2}\Gamma(n/2)} \mathsf{B}_k^n \Bigg[& \frac{\Gamma(k+1)\Gamma(s-p-k-1)}{\rho^{2s-2k-2-2p}\Gamma(s-p)} \\
& - \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{(s-p)_j}{\rho^{2s-2p+2j}} \mathscr{B}_1(k+j) \Bigg],
\end{aligned} \tag{6.3.23}$$

with B_k^n as in (A.4.8), and $\mathscr{B}_1(X)$ as in (A.6.3). In each case $\rho = (n-1)/2$, and $\mathsf{H}_p^\ell = \mathsf{H}_p^\ell(\alpha,\beta)$ are defined in Lemma A.6.4.

Proof. In the case $n \geq 3$ odd, we have the Plancherel measure $\mu(\lambda)$ as in (6.3.21), and hence using (A.4.7) we can write the spectral zeta function $\zeta_0(s; \mathbb{R}\mathbf{H}^n) = \zeta(s; \mathbb{R}\mathbf{H}^n)$ as

$$\begin{aligned}\zeta_0(s; \mathbb{R}\mathbf{H}^n) &= \frac{1}{2^{n-1}\pi^{n/2}\Gamma(n/2)} \int_0^\infty \frac{1}{(\lambda^2+\rho^2)^s} \prod_{j=0}^{(n-3)/2} (j^2+\lambda^2)\, d\lambda \\ &= \frac{2}{(4\pi)^{n/2}\Gamma(n/2)} \sum_{k=0}^{(n-3)/2} \mathsf{A}_k^n \int_0^\infty \frac{\lambda^{2k}}{(\lambda^2+\rho^2)^s}\, d\lambda. \end{aligned} \tag{6.3.24}$$

The integral above can be evaluated as in (A.6.13), and so we have

$$\zeta_0(s; \mathbb{R}\mathbf{H}^n) = \frac{1}{(4\pi)^{n/2}\Gamma(n/2)} \sum_{k=0}^{(n-3)/2} \mathsf{A}_k^n \rho^{2k+1-2s} B\left(k+\frac{1}{2}, s-k-\frac{1}{2}\right). \tag{6.3.25}$$

Now for $n \geq 2$ even, we have $\mu(\lambda)$ as in (6.3.21), and so via (A.4.8) we have

$$\begin{aligned}\zeta_0(s; \mathbb{R}\mathbf{H}^n) &= \frac{2}{(4\pi)^{n/2}\Gamma(n/2)} \int_0^\infty \frac{\lambda\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s} \prod_{j=1/2}^{(n-3)/2} (\lambda^2+j^2)\, d\lambda \\ &= \frac{2}{(4\pi)^{n/2}\Gamma(n/2)} \sum_{k=0}^{(n-2)/2} \mathsf{B}_k^n \int_0^\infty \frac{\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+1}\, d\lambda. \end{aligned} \tag{6.3.26}$$

Referring to Lemma A.6.2 to evaluate the integral above, we then substitute these into (6.2.16) to arrive at the results for odd and even dimensions. $\square$

The case $\mathscr{X}_c = \mathbb{C}\mathbf{P}^n$ vs. $\mathscr{X}_{nc} = \mathbb{C}\mathbf{H}^n$. Here we have $\alpha = n-1$ and $\beta = 0$, leading to $\rho = n/2$ and $d = 2n$. The multiplicity function $M_k(\mathbb{C}\mathbf{P}^n)$ is given by

$$M_k(\mathbb{C}\mathbf{P}^n) = \frac{2k+n}{n}\left(\frac{\Gamma(k+n)}{\Gamma(n)k!}\right)^2 \tag{6.3.27}$$

Moreover we again fix the parameter $X_k = k+\rho = k+n/2$ so that the eigenvalues of $-\Delta_{\mathbb{C}\mathbf{P}^n}$ satisfy $\lambda_k^n = k(k+n) = X_k^2 - \rho^2$.

Proposition 6.3.4. *Let $\mathscr{X} = \mathbb{C}\mathbf{P}^n$. Then for $n \geq 1$ odd, the zeta function $\zeta_\ell(s; \mathbb{C}\mathbf{P}^n)$ is given by*

$$\zeta_\ell(s; \mathbb{C}\mathbf{P}^n) = \sum_{m=0}^{\infty}\sum_{p=1}^{\ell}\sum_{j=0}^{n-1} \frac{2\mathsf{h}_p^\ell \mathsf{c}_j^n \rho^{2m}}{\Gamma(n)n!m!}(s-p)_m \zeta_H(2(s-p+m-j)-1, \rho+1), \tag{6.3.28}$$

with c_k^n as in (A.4.3) and $\zeta_H(s,q)$ as in (6.3.14). For $n \geq 2$ even, we have

$$\zeta_\ell(s; \mathbb{C}\mathbf{P}^n) = \sum_{m=0}^{\infty}\sum_{p=1}^{\ell}\sum_{j=0}^{n-2} \frac{2\mathsf{h}_p^\ell \mathsf{d}_j^n \rho^{2m}}{\Gamma(n)n!m!}(s-p)_m \zeta_H(2(s+m-p-j)-3, \rho+1) \tag{6.3.29}$$

with d_k^n as in (A.4.4) and $\zeta_H(s,q)$ as in (6.3.14). In each case $\rho = n/2$ and the scalars $\mathsf{h}_p^\ell = \mathsf{h}_p^\ell(\alpha,\beta)$ are defined in Lemma A.6.3.

Proof. For $n \geq 1$ odd, the multiplicity function $M_k(\mathbb{C}\mathbf{P}^n)$ as in (6.3.27) can be arranged as a polynomial in $X_k = k + n/2$ of the form

$$M_k(\mathbb{C}\mathbf{P}^n) = \frac{2X_k}{n!(n-1)!} \prod_{j=1/2}^{\frac{n-3}{2}+\frac{1}{2}} (X_k^2 - j^2)^2 = \frac{2}{n!(n-1)!} \sum_{j=0}^{n-1} \mathsf{c}_j^n X_k^{2j+1}, \tag{6.3.30}$$

where we have made use of (A.4.3). Substituting this into (6.2.2), the spectral zeta function $\zeta(s;\mathbb{C}\mathbf{P}^n) = \zeta_0(s;\mathbb{C}\mathbf{P}^n)$ is given by

$$\begin{aligned}\zeta_0(s;\mathbb{C}\mathbf{P}^n) &= \frac{2}{n!(n-1)!} \sum_{j=0}^{n-1} \mathsf{c}_j^n \sum_{k=1}^{\infty} \frac{X_k^{2j+1}}{(X_k^2 - \rho^2)^s} \\ &= \frac{2}{n!(n-1)!} \sum_{j=0}^{n-1} \mathsf{c}_j^n \sum_{m=0}^{\infty} \frac{\rho^{2m}\Gamma(s+m)}{\Gamma(s)m!} \sum_{k=1}^{\infty} X_k^{2j+1-2m-2s}\end{aligned}$$

We arrange the inner sum as a Hurwitz zeta function $\zeta_H(s,q)$, as in the previous proof. For $n \geq 2$ even, $M_k(\mathbb{C}\mathbf{P}^n)$ can be arranged as

$$M_k = \frac{2X_k^3}{n!(n-1)!} \prod_{j=1}^{(n-2)/2} (X_k^2 - j^2) = \frac{2}{n!(n-1)!} \sum_{j=0}^{n-2} \mathsf{d}_j^n X_k^{2j+3}, \tag{6.3.31}$$

where we have made use of (A.4.4). Substituing this into (6.2.2), we have

$$\begin{aligned}\zeta_0(s;\mathbb{C}\mathbf{P}^n) &= \frac{2}{n!(n-1)!} \sum_{j=0}^{n-2} \mathsf{d}_j^n \sum_{k=1}^{\infty} \frac{X_k^{2j+3}}{(X_k^2 - \rho^2)^s} \\ &= \frac{2}{n!(n-1)!} \sum_{j=0}^{n-2} \mathsf{d}_j^n \sum_{m=0}^{\infty} \frac{\rho^{2m}\Gamma(m+s)}{\Gamma(s)m!} \sum_{k=1}^{\infty} X_k^{2j+3-2s-2m},\end{aligned}$$

Arranging the inner sum as a Hurwitz zeta function, we can then substitute both cases into (6.2.14) to arrive at the statements of $\zeta_\ell(s;\mathbb{C}\mathbf{P}^n)$. □

Next for the complex hyperbolic space $\mathbb{C}\mathbf{H}^n$, the Plancherel measure (6.2.5) takes the form

$$\mu(\lambda) = \begin{cases} \dfrac{\pi\lambda\tanh(\pi\lambda)}{(2^{n-1}\Gamma(n))^2} \displaystyle\prod_{j=1/2}^{(n-2)/2} (\lambda^2 + j^2)^2, & n \geq 1 \text{ odd}, \\ \dfrac{\pi\lambda^3\coth(\pi\lambda)}{2^{2n-2}\Gamma(n)^2} \displaystyle\prod_{j=1}^{(n-2)/2} (\lambda^2 + j^2)^2, & n \geq 2 \text{ even}. \end{cases} \tag{6.3.32}$$

Proposition 6.3.5. *Let $\mathscr{X} = \mathbb{C}\mathbf{H}^n$. Then for $n \geq 1$ odd, the zeta function $\zeta_\ell(s;\mathbb{C}\mathbf{H}^n)$ can be written as*

$$\begin{aligned}\zeta_\ell(s;\mathbb{C}\mathbf{H}^n) = \sum_{p=1}^{\ell}\sum_{k=0}^{n-1} \frac{(-1)^p \mathsf{H}_p^\ell}{(4\pi)^n\Gamma(n)} \mathsf{C}_k^n \Bigg[&\frac{\Gamma(k+1)\Gamma(s-p-k-1)}{\rho^{2s-2k-2-2p}\Gamma(s-p)} \\ &- \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{(s-p)_j}{\rho^{2s-2p+2j}} \mathscr{B}_1(k+j) \Bigg],\end{aligned} \tag{6.3.33}$$

where C_k^n *is given in* (A.4.9), *and* $\mathscr{B}_1(X)$ *as in* (A.6.3), *whilst for* $n \geq 2$ *even*

$$\zeta_\ell(s;\mathbb{C}\mathbf{H}^n) = \sum_{p=1}^{\ell}\sum_{k=0}^{n-2}\frac{(-1)^p\mathsf{H}_p^\ell}{(4\pi)^n\Gamma(n)}\mathsf{D}_k^n\Bigg[\frac{\Gamma(k+2)\Gamma(s-p-k-2)}{\rho^{2s-2k-4-2p}\Gamma(s-p)} - \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{(s-p)_j}{\rho^{2s-2p+2j}}\mathscr{B}_2(k+j+1)\Bigg], \tag{6.3.34}$$

Here D_k^n *is given in* (A.4.10), *and* $\mathscr{B}_2(X)$ *is given in* (A.6.3). *In each case* $\rho = n/2$, *and* $\mathsf{H}_p^\ell = \mathsf{H}_p^\ell(\alpha,\beta)$ *are defined in Lemma A.6.4.*

Proof. For $n \geq 1$ odd, with $\mu(\lambda)$ given in (6.3.32) and using (A.4.9) we have

$$\begin{aligned}\zeta_0(s;\mathbb{C}\mathbf{H}^n) &= \frac{2}{(4\pi)^n\Gamma(n)}\int_0^\infty \frac{\lambda\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s}\prod_{j=1/2}^{(n-2)/2}(\lambda^2+j^2)^2\,d\lambda\\ &= \frac{2}{(4\pi)^n\Gamma(n)}\sum_{k=0}^{n-1}\mathsf{C}_k^n\int_0^\infty\frac{\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+1}\,d\lambda. \end{aligned}\tag{6.3.35}$$

When $n \geq 2$ even, the Plancherel measure $\mu(\lambda)$ is given in (6.3.32), and so via (A.4.10) we can write the zeta function as

$$\begin{aligned}\zeta_0(s;\mathbb{C}\mathbf{H}^n) &= \frac{2}{(4\pi)^n\Gamma(n)}\mathsf{D}_k^n\int_0^\infty \frac{\lambda^3\coth(\pi\lambda)}{(\lambda^2+\rho^2)^s}\prod_{j=1}^{(n-2)/2}(\lambda^2+j^2)^2\,d\lambda\\ &= \frac{2}{(4\pi)^n\Gamma(n)}\sum_{k=0}^{n-2}\mathsf{D}_k^n\int_0^\infty\frac{\coth(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+3}\,d\lambda. \end{aligned}\tag{6.3.36}$$

Referring to Lemma A.6.2 to evaluate each integral above, we then substitute the resulting formulae into (6.2.16) to arrive at the results in each case. □

The case $\mathscr{X}_c = \mathbb{H}\mathbf{P}^n$ **vs.** $\mathscr{X}_{nc} = \mathbb{H}\mathbf{H}^n$**.** Here we have $\alpha = 2n-1$ and $\beta = 1$. Hence $\rho = (2n+1)/2$ and $d = 4n$. The multiplicity function $M_k(\mathbb{H}\mathbf{P}^n)$ is given explicitly by

$$M_k(\mathbb{H}\mathbf{P}^n) = \frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left(\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right)^2 \tag{6.3.37}$$

In this case we fix $X_k = k+\rho = k+(2n+1)/2$, so that the eigenvalues of $-\Delta_{\mathbb{H}\mathbf{P}^n}$ satisfy $\lambda_k^n = k(k+2n+1) = X_k^2 - \rho^2$.

Proposition 6.3.6. *Let* $\mathscr{X} = \mathbb{H}\mathbf{P}^n$. *Then for* $n \geq 1$, *the zeta function* $\zeta_\ell(s;\mathbb{H}\mathbf{P}^n)$ *can be written as*

$$\zeta_\ell(s;\mathbb{H}\mathbf{P}^n) = \sum_{m=0}^{\infty}\sum_{p=1}^{\ell}\sum_{j=0}^{2n-1}\frac{2\mathsf{h}_p^\ell\mathsf{e}_j^n\rho^{2m}}{(2n+1)!}\frac{(s-p)_m}{\Gamma(2n)m!}\zeta_H(2(s-p+m-j)-1,\rho+1). \tag{6.3.38}$$

with e_k^n *as in* (A.4.5) *and* $\zeta_H(s,q)$ *as in* (6.3.14). *Here* $\rho = (2n+1)/2$, *and the scalars* $\mathsf{h}_p^\ell = \mathsf{h}_p^\ell(\alpha,\beta)$ *are defined in Lemma A.6.3.*

Proof. By suitably manipulating the Gamma functions, the multiplicity function $M_k(\mathbb{H}\mathbf{P}^n)$ given in (6.3.39) can be arranged as a polynomial of the form

$$M_k(\mathbb{H}\mathbf{P}^n) = \frac{2X_k(k+2n)}{(2n-1)!(2n+1)!(k+1)} \prod_{j=1}^{2n-1}(k+j)^2$$
$$= \frac{2X_k(X_k^2-(2n-1)^2/4)}{(2n-1)!(2n+1)!} \prod_{j=1/2}^{n-3/2}(X_k^2-j^2) = \sum_{j=0}^{2n-1} \frac{2\mathsf{e}_j^n X_k^{2j+1}}{(2n-1)!(2n+1)!}. \quad (6.3.39)$$

Here we have used (A.4.5). Substituting this into (6.2.2), the spectral zeta function $\zeta(s;\mathbb{H}\mathbf{P}^n) = \zeta_0(s;\mathbb{H}\mathbf{P}^n)$ can be written as

$$\zeta_0(s;\mathbb{H}\mathbf{P}^n) = \frac{2}{(2n-1)!(2n+1)!} \sum_{j=0}^{2n-1} \mathsf{e}_j^n \sum_{k=1}^{\infty} \frac{X_k^{2j+1}}{(X_k^2-\rho^2)^s}$$
$$= \frac{2}{(2n-1)!(2n+1)!} \sum_{j=0}^{2n-1} \mathsf{e}_j^n \sum_{m=0}^{\infty} \frac{\rho^{2m}\Gamma(s+m)}{\Gamma(s)m!} \sum_{k=1}^{\infty} X_k^{2j+1-2m-2s}.$$

We now arrange the inner sum as a Hurwitz zeta function, as in the previous proofs, before substituting into (6.2.14) to arrive at the formula for $\zeta_\ell(s;\mathbb{H}\mathbf{P}^n)$. □

In the non-compact case, the Plancherel measure (6.2.5) on $\mathbb{H}\mathbf{H}^n$ is given by

$$\mu(\eta) = \frac{\pi\lambda\tanh(\pi\lambda)}{2^{4n}\Gamma(2n)^2}\left[\lambda^2+(2n-1)^2/4\right] \prod_{j=1/2}^{n-3/2}(\lambda^2+j^2)^2. \quad (6.3.40)$$

Proposition 6.3.7. *Let $\mathscr{X} = \mathbb{H}\mathbf{H}^n$. Then for $n \geq 1$, the zeta function $\zeta_0(s;\mathbb{H}\mathbf{H}^n)$ can be written as*

$$\zeta_\ell(s;\mathbb{H}\mathbf{H}^n) = \sum_{p=1}^{\ell}\sum_{k=0}^{2n-1} \frac{(-1)^p\mathsf{H}_p^\ell}{(4\pi)^{2n}\Gamma(2n)} \mathsf{E}_k^n \left[\frac{\Gamma(k+1)\Gamma(s-p-k-1)}{\rho^{2s-2k-2-2p}\Gamma(s-p)} \right.$$
$$\left. - \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\frac{(s-p)_j}{\rho^{2s-2p+2j}}\mathscr{B}_1(k+j)\right], \quad (6.3.41)$$

where E_k^n is given in (A.4.11), and $\mathscr{B}_1(X)$ is defined in (A.6.3). Here $\rho = (2n+1)/2$, and $\mathsf{H}_p^\ell = \mathsf{H}_p^\ell(\alpha,\beta)$ are defined in Lemma A.6.4.

Proof. For $n \geq 1$, $\mu(\lambda)$ is given in (6.3.40), and so using (A.4.11) we can write

$$\zeta_0(s;\mathbb{H}\mathbf{H}^n) = \frac{2}{(4\pi)^{2n}\Gamma(2n)} \int_0^\infty \frac{\lambda\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s}\left[\lambda^2+(2n-1)^2/4\right] \prod_{j=1/2}^{n-3/2}(\lambda^2+j^2)^2\, d\lambda$$
$$= \frac{2}{(4\pi)^{2n}\Gamma(2n)} \sum_{k=0}^{2n-1} \mathsf{E}_k^n \int_0^\infty \frac{\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+1}\, d\lambda. \quad (6.3.42)$$

We refer to Lemma A.6.2 to evaluate the integral above, the result of which we then substitute into (6.2.16) to arrive at the statement of $\zeta_\ell(s;\mathbb{H}\mathbf{H}^n)$. □

The case $\mathscr{X}_c = \mathbf{P}^2(Cay)$ vs. $\mathscr{X}_{nc} = \mathbf{H}^2(Cay)$. Here we have $\alpha = 7$ and $\beta = 3$, and so $\rho = 11/2$. Here the multiplicity function $M_k(\mathbf{P}^2(Cay))$ is given by

$$M_k(\mathbf{P}^2(Cay)) = 6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}. \tag{6.3.43}$$

As in each of the previous cases, we fix the parameter $X_k = k+\rho = k+11/2$ so that the eigenvalues of $-\Delta_{\mathbf{P}^2(Cay)}$ satisfy $\lambda_k = k(k+11) = X_k^2 - \rho^2$.

Proposition 6.3.8. *Let $\mathscr{X} = \mathbf{P}^2(Cay)$. Then the zeta function $\zeta_\ell(s;\mathbf{P}^2(Cay))$ can be written as*

$$\zeta_\ell(s;\mathbf{P}^2(Cay)) = \sum_{p=0}^{\ell}\sum_{m=0}^{\infty}\sum_{j=0}^{7}\frac{12\mathsf{h}_p^\ell\mathsf{f}_j\rho^{2m}}{7!11!m!}(s-p)_m\zeta_H(2s-2p+2m-2j-1,\rho+1), \tag{6.3.44}$$

with f_k as in (A.4.6) and $\zeta_H(s,q)$ as in (6.3.14). Here $\rho = 11/2$, and the scalars $\mathsf{h}_p^\ell = \mathsf{h}_p^\ell(\alpha,\beta)$ are defined in Lemma A.6.3 for $\alpha = 11$ and $\beta = 3$.

Proof. With the multiplicity function $M_k(\mathbf{P}^2(Cay)) = M_k$ as in (6.3.45), we have

$$\begin{aligned} M_k &= \frac{12X_k}{7!11!}(X_k^2-1/4)^2(X_k^2-9/4)^2(X_k^2-25/4)(X_k^2-49/4)(X_k^2-81/4) \\ &= \frac{12}{7!11!}\sum_{j=0}^{7}\mathsf{f}_j X_k^{2j+1}. \end{aligned} \tag{6.3.45}$$

Substituting this into (6.2.2) and proceeding as in the previous case, we can write the spectral zeta function $\zeta(s;\mathbf{P}^2(Cay))$ as

$$\begin{aligned} \zeta_0(s;\mathbf{P}^2(Cay)) &= \frac{12}{7!11!}\sum_{j=0}^{7}\mathsf{f}_j\sum_{k=1}^{\infty}\frac{X_k^{2j+1}}{(X_k^2-\rho^2)^s} \\ &= \frac{12}{7!11!}\sum_{j=0}^{7}\mathsf{f}_j\sum_{m=0}^{\infty}\frac{\Gamma(s+m)\rho^{2m}}{\Gamma(s)m!}\sum_{k=1}^{\infty}X_k^{2j+1-2s-2m}, \end{aligned} \tag{6.3.46}$$

where we can arrange the inner sum as a Hurwitz zeta function. We then substitute this into (6.2.14) to complete the proof. $\square$

For the non-compact case, the Plancherel measure on $\mathbf{H}^2(Cay)$ is given by

$$\mu(\lambda) = \frac{\pi\lambda\tanh(\pi\lambda)}{2^{20}\Gamma(8)^2}(\lambda^2+81/4)(\lambda^2+49/4)(\lambda^2+25/4)(\lambda^2+9/4)^2(\lambda^2+1/4)^2. \tag{6.3.47}$$

Proposition 6.3.9. *Let $\mathscr{X} = \mathbf{H}^2(Cay)$. Then the spectral zeta function $\zeta(s;\mathbf{H}^2(Cay))$ can be written as*

$$\begin{aligned} \zeta_\ell(s;\mathbf{H}^2(Cay)) = \sum_{p=1}^{\ell}\sum_{k=0}^{7}\frac{(-1)^p\mathsf{H}_p^\ell}{(4\pi)^8\Gamma(8)}\mathsf{F}_k\Bigg[&\frac{\Gamma(k+1)\Gamma(s-p-k-1)}{\rho^{2s-2-2k-2p}\Gamma(s-p)} \\ &-\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{(s-p)_j}{\rho^{2s-2p+2j}}\mathscr{B}_1(k+j)\Bigg]. \end{aligned} \tag{6.3.48}$$

Here F_k *is given in* (A.4.12), $\mathscr{B}_1(X)$ *is given in* (A.6.3), $\rho = 11/2$, *and* $\mathsf{H}_p^\ell = \mathsf{H}_p^\ell(\alpha, \beta)$ *are defined in Lemma A.6.4.*

Proof. In this case we have the Plancherel measure $\mu(\lambda)$ as in (6.3.47), and hence the zeta function $\zeta(s; \mathbf{H}^2(Cay))$ can be written using (A.4.12) as

$$\zeta(s; \mathbf{H}^2(Cay)) = \frac{2}{(4\pi)^8 \Gamma(8)} \sum_{k=0}^{7} \mathsf{F}_k \int_0^\infty \frac{\tanh(\pi\lambda)}{(\lambda^2 + \rho^2)^s} \lambda^{2k+1}\, d\lambda. \tag{6.3.49}$$

Evaluating the integral above using Lemma A.6.2, we substitute into (6.2.16) to arrive at the result. □

6.4 The proportionality principle on $\zeta_\ell(s; \mathscr{X})$

The Minakshisundaram-Pleijel heat coefficients on a pair of dual spaces $\mathscr{X}_c$ and $\mathscr{X}_{nc}$ are related by the Proportionality principle

$$\mathsf{a}_j(\mathscr{X}_c) = (-1)^j \mathsf{a}_j(\mathscr{X}_{nc}), \tag{6.4.1}$$

with $\mathsf{a}_j(\mathscr{X}_c)$ as in (6.1.5) and $\mathsf{a}_j(\mathscr{X}_{nc})$ as in (6.1.9) This relation has strong and far-reaching implications in the analysis of Riemannian symmetric spaces. In particular, for a dual pair of compact and non-compact rank-one symmetric spaces $\mathscr{X}_c$ and $\mathscr{X}_{nc}$, we have at the following proportionality between the residues of the poles of the spectral zeta function.

Proposition 6.4.1. *For a dual pair* $\mathscr{X}_c$ *and* $\mathscr{X}_{nc}$ *of rank-one symmetric spaces of compact and non-compact type, the residues at the shared poles of the spectral zeta functions* $\zeta(s; \mathscr{X})$ *defined in* (6.2.2) *and* (6.2.8) *for the compact and non-compact spaces respectively satisfy the proportionality*

$$\mathrm{Res}_{s=s_j} \zeta(s; \mathscr{X}_c) = (-1)^j \mathrm{Res}_{s=s_j} \zeta(s; \mathscr{X}_{nc}). \tag{6.4.2}$$

Proof. We recall that the residues of the poles in the compact case are given in (6.1.4) in terms of the coefficients $\mathsf{a}_j(\mathscr{X}_c)$, whilst in the non-compact case they are given in terms of $\mathsf{a}_j(\mathscr{X}_{nc})$ in (6.1.8). Comparing these formulae and referring to the known proportionality (6.4.1), we arrive at the result. □

In Section 6.2 we introduced the zeta function $\zeta_\ell(s; \mathscr{X})$, defined on a rank-one symmetric space of compact or non-compact type in (6.2.14) and (6.2.16) respectively. In what follows we give the locations of the poles of $\zeta_\ell(s; \mathscr{X})$ and provide the residues in each case.

The poles and residues of $\zeta_\ell(s;\mathscr{X})$ in odd dimensions. In the odd-dimensional case - that is on $\mathbb{S}^n$ and $\mathbb{R}\mathbf{H}^n$ with $n \geq 3$ odd - we know that there are infinitely many points that can be poles of the spectral zeta function $\zeta(s;\mathscr{X})$. These are at $s = s_j = d/2 - j$, for all non-negative integers j. Given that $\zeta_\ell(s;\mathscr{X})$ is a weighted sum of translations of the spectral zeta function, its poles will be at points that are the same such translations of the poles of the spectral zeta function. In particular, the poles of $\zeta_\ell(s;\mathscr{X})$ in odd dimensions for both compact and non-compact cases are seen to be at points $s = s_j = d/2 + \ell - j$ for all non-negative integers j. We easily arrive at the following results detailing the residues at such poles.

Proposition 6.4.2. *Let $\zeta_\ell(s;\mathscr{X}_c)$ be defined as in (6.2.14) with $\mathscr{X}_c = \mathbb{S}^n$, $n \geq 3$ odd. Then the poles of $\zeta_\ell(s;\mathbb{S}^n)$ are at points $s = s_j = n/2 + \ell - j$ for $j = 0, 1, 2, \ldots$, where the residue at the pole $s_j = n/2 + \ell - j$ is given by*

$$\mathrm{Res}_{s=s_j}\zeta_\ell(s;\mathbb{S}^n) = \sum_{k=1}^{\ell} \frac{\mathsf{h}_k^\ell \mathsf{a}_{j+k-\ell}(\mathbb{S}^n)}{(4\pi)^{n/2}\Gamma(n/2+\ell-j-k)}, \qquad n \geq 3 \text{ odd}, \tag{6.4.3}$$

where $\mathsf{a}_j(\mathbb{S}^n)$ are the Minakshisundaram-Pleijel heat coefficients on $\mathbb{S}^n$.

Proof. The residues associated to a pole of $\zeta_\ell(s;\mathscr{X}_c)$ are given by the sum of the residues at the poles of the constituent elements of the sum (6.2.14), that is those of the spectral zeta function $\zeta(s;\mathscr{X}_c)$ weighted by h_k^ℓ. These are given in (6.1.4), and so the result follows. □

Proposition 6.4.3. *Let $\zeta_\ell(s;\mathscr{X}_{nc})$ be defined as in (6.2.16) with $\mathscr{X}_{nc} = \mathbb{R}\mathbf{H}^n$, $n \geq 3$ odd. Then the poles of $\zeta_\ell(s;\mathbb{R}\mathbf{H}^n)$ are at $s = s_j = d/2 + \ell - j$ for $j = 0, 1, 2, \ldots$, where the residue at the pole $s_j = n/2 + \ell - j$ is given by*

$$\mathrm{Res}_{s=s_j}\zeta_\ell(s;\mathbb{R}\mathbf{H}^n) = \sum_{k=1}^{\ell} \frac{(-1)^k \mathsf{H}_k^\ell \mathsf{a}_{j+k-\ell}(\mathbb{R}\mathbf{H}^n)}{(4\pi)^{n/2}\Gamma(n/2+\ell-j-k)}, \qquad n \geq 3 \text{ odd}, \tag{6.4.4}$$

where $\mathsf{a}_j(\mathbb{R}\mathbf{H}^n)$ are the Minakshisundaram-Pleijel heat coefficients on $\mathbb{R}\mathbf{H}^n$.

Proof. As in the previous proof, we take a sum as in the definition of $\zeta_\ell(s;\mathscr{X}_{nc})$ of residues of the spectral zeta function $\zeta(s;\mathscr{X}_{nc})$, which are given in (6.1.8). Here the sum is weighted by H_k^ℓ. □

The poles and residues of $\zeta_\ell(s;\mathscr{X})$ in even dimensions. Next we consider the more general even dimensional case, where as before we see that the potential poles of $\zeta_\ell(s;\mathscr{X})$ in both the compact and non-compact cases are at $s = s_j = d/2 + \ell - j$, however in this case j runs from $j = 1, 2, \ldots, d/2 - 1$. This is deduced immediately from knowing the poles of the associated spectral zeta function in each case.

Proposition 6.4.4. *Let $\zeta_\ell(s; \mathscr{X}_c)$ be defined as in (6.2.14) with $d = \dim(\mathscr{X}_c)$ even. Then the poles of $\zeta_\ell(s; \mathscr{X}_c)$ are at $s = s_j = d/2 + \ell - j$ for $j = 0, 1, \ldots d/2 - 1$, where the residue at the pole $s_j = d/2 + \ell - j$ is given by*

$$\mathrm{Res}_{s=s_j} \zeta_\ell(s; \mathscr{X}_c) = \sum_{k=1}^{\ell} \frac{\mathsf{h}_k^\ell \mathsf{a}_{j+k-\ell}(\mathscr{X}_c)}{(4\pi)^{d/2} \Gamma(d/2 + \ell - j - k)}, \qquad d \geq 2 \text{ even.} \tag{6.4.5}$$

Proof. As in the proof of Proposition 6.4.2, we sum the residues (6.1.4), weighted by h_k^ℓ, to arrive at the result. $\square$

Proposition 6.4.5. *Let $\zeta_\ell(s; \mathscr{X}_{nc})$ be defined as in (6.2.16) with $d = \dim(\mathscr{X}_c)$ even. Then the poles of $\zeta_\ell(s; \mathscr{X}_{nc})$ are at $s = s_j = d/2 + \ell - j$ for $j = 0, 1, \ldots d/2 - 1$, where the residue at the pole $s_j = d/2 + \ell - j$ is given by*

$$\mathrm{Res}_{s=s_j} \zeta_\ell(s; \mathscr{X}_{nc}) = \sum_{k=1}^{\ell} \frac{\mathsf{H}_k^\ell \mathsf{a}_{j+k-\ell}(\mathscr{X}_{nc})}{(4\pi)^{d/2} \Gamma(d/2 + \ell - j - k)}, \qquad d \geq 2 \text{ even.} \tag{6.4.6}$$

Proof. As in the proof of Proposition 6.4.2, we sum the residues (6.1.8), weighted by H_k^ℓ, to arrive at the result. $\square$

The following result ties together each of the previous sections, stating the Proportionality principle on the level of the residues of $\zeta_\ell(s; \mathscr{X}_c)$ vs. $\zeta_\ell(s; \mathscr{X}_{nc})$.

Proposition 6.4.6. *For a dual pair $\mathscr{X}_c$ and $\mathscr{X}_{nc}$ of rank-one symmetric spaces of compact and non-compact type, the residues at the shared poles of the zeta functions $\zeta_\ell(s; \mathscr{X})$ defined in (6.2.14) and (6.2.16) for the compact and non-compact spaces respectively satisfy the proportionality*

$$\mathrm{Res}_{s=s_j} \zeta_\ell(s; \mathscr{X}_c) = (-1)^j \mathrm{Res}_{s=s_j} \zeta_\ell(s; \mathscr{X}_{nc}). \tag{6.4.7}$$

Proof. With the residues on a compact space given in (6.4.3) and (6.4.5) (note that these are the same formula on different spaces), we compare with the residues on a non-compact space given in (6.4.4) and (6.4.6). Recalling that for a suitable dual pair $\mathscr{X}_c$ and $\mathscr{X}_{nc}$ of rank-one symmetric spaces we have $\mathsf{h}_k^\ell = (-1)^\ell \mathsf{H}_k^\ell$, with these scalars being defined in Lemma A.6.3 and Lemma A.6.4 respectively, we again refer to (6.4.1) to arrive at the result. $\square$

Point values of $\zeta_\ell(s; \mathscr{X})$ on compact and non-compact spaces. In both compact and non-compact cases, we see that the zeta function $\zeta_\ell(s;\mathscr{X})$ is given by a weighted sum of translates of the associated spectral zeta function $\zeta(s;\mathscr{X})$. Moreover, we see that since we have defined $\mathscr{R}_0(X)=1$ and $\mathscr{Q}_0(X)=1$, we may formally define $\zeta_0(s,\mathscr{X})=\zeta(s;\mathscr{X})$. That is, the $\ell=0$ case of $\zeta_\ell(s;\mathscr{X})$ on both the compact and non-compact spaces is given by the spectral zeta function.

By referring to the explicit formulae presented in Section 6.2, we arrive at the following results on point values of $\zeta_\ell(s;\mathscr{X})$. Firstly for the odd dimensional case: $\mathbb{S}^n$ and $\mathbb{R}\mathbf{H}^n$ with $n\geq 3$ odd.

Proposition 6.4.7. *Let $\zeta_\ell(s;\mathscr{X}_c)$ be defined as in (6.2.14) with $\mathscr{X}_c=\mathbb{S}^n$, $n\geq 3$ odd. Then we have:*

- $\zeta_\ell(-k;\mathbb{S}^n)=0$, *for* $k=0,1,2,\ldots$
- $\zeta_\ell(1;\mathbb{S}^n)=(-1)^{\ell+1}$.

Let $\zeta_\ell(s;\mathscr{X}_{nc})$ be as in (6.2.16) with $\mathscr{X}_{nc}=\mathbb{R}\mathbf{H}^n$, $n\geq 3$ odd. Then we have:

- $\zeta_\ell(-k;\mathbb{R}\mathbf{H}^n)=0$, *for* $k=0,1,2,\ldots$
- $\zeta_\ell(1;\mathbb{R}\mathbf{H}^n)=1$.

Proof. On both $\mathbb{S}^n$ and $\mathbb{R}\mathbf{H}^n$ with $n\geq 3$ odd, we know that $\zeta(s;\mathscr{X})$ has zeros at negative integers, and moreover satisfies $\zeta(0;\mathscr{X})=-1$. Given that $\zeta_\ell(s;\mathscr{X})$ is defined in both compact and non-compact cases as a sum of translates to the left (towards negative integers), we easily deduce that the zeros of $\zeta_\ell(s;\mathscr{X})$ are simply the zeros of $\zeta(s;\mathscr{X})$ translated one to the left. For the value of $\zeta_\ell(0;\mathscr{X})$, we substitute $s=0$ into the formulae (6.2.14) and (6.2.16), and notice that all but the first term of the sums vanish. In the compact case we are left with $\zeta(0;\mathbb{S}^n)\mathsf{h}_1^\ell=(-1)^{\ell+1}$, whilst in the non-compact case we are left with $\zeta(0;\mathbb{R}\mathbf{H}^n)\mathsf{H}_1^\ell=1$, where we have noted that $\mathsf{h}_1^\ell=(-1)^\ell$, and $\mathsf{H}_1^\ell=1$. □

Proposition 6.4.8. *Let $\zeta_\ell(s;\mathscr{X}_c)$ be defined as in (6.2.14), with $\dim(\mathscr{X}_c)$ even. Then we have:*

- $\zeta_\ell(1;\mathscr{X}_c)=(-1)^{\ell+1}+\dfrac{(-1)^\ell}{(4\pi)^{d/2}}\mathsf{a}_{d/2}(\mathscr{X}_c)+\displaystyle\sum_{p=1}^{\ell-1}\dfrac{(-1)^p p!}{(4\pi)^{d/2}}\mathsf{h}_{p+1}^\ell\mathsf{a}_{d/2+p}(\mathscr{X}_c)$
- $\zeta_\ell(-k;\mathscr{X}_c)=\displaystyle\sum_{p=1}^{\ell}\dfrac{(-1)^{k+p}(k+p)!}{(4\pi)^{d/2}}\mathsf{h}_p^\ell\mathsf{a}_{d/2+k+p}(\mathscr{X}_c)$, $k=0,1,2,\ldots$

where $\mathsf{a}_j(\mathscr{X}_c)$ are the Minakshisundaram-Pleijel heat coefficients given in the expansion (6.1.5). Similarly, let $\zeta_\ell(s;\mathscr{X}_{nc})$ be defined as in (6.2.16) with $\dim(\mathscr{X}_{nc})$ even. Then we have:

- $\zeta_\ell(1;\mathscr{X}_{nc}) = 1 - \frac{1}{(4\pi)^{d/2}}\mathsf{a}_{d/2}(\mathscr{X}_{nc}) + \sum_{p=1}^{\ell-1}\frac{(-1)^p p!}{(4\pi)^{d/2}}\mathsf{H}^\ell_{p+1}\mathsf{a}_{d/2+p}(\mathscr{X}_{nc})$

- $\zeta_\ell(-k;\mathscr{X}_{nc}) = \sum_{p=1}^{\ell}\frac{(-1)^k(k+p)!}{(4\pi)^{d/2}}\mathsf{H}^\ell_p\mathsf{a}_{d/2+k+p}(\mathscr{X}_{nc}),$

where $\mathsf{a}_j(\mathscr{X}_c)$ *are the coefficients given in the expansion* (6.1.9)

Proof. We arrive at the above result in each case by substituting for the appropriate value of s in the definitions of $\zeta_\ell(s;\mathscr{X})$ in (6.2.14) and (6.2.16), and then referring to the point values of the spectral zeta function in even dimensions given in (6.1.6) and (6.2.9) for compact and non-compact spaces respectively. We note again that $\mathsf{h}^\ell_1 = (-1)^\ell$, and $\mathsf{H}^\ell_1 = 1$, simplifying the terms outside the sums when $s = 1$ in each case. □

Appendix A

A.1 The hypergeometric function ${}_2F_1(z) = F(a, b; c; z)$

The hypergeometric function (see [4] pp. 61-123) is defined on the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ as

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{A.1.1}$$

with $a, b, c \in \mathbb{C}$ and c not a non-positive integer. The series converges uniformly inside the unit disk, therefore representing a holomorphic function (in the z variable) whilst beyond the unit disk it can be extended by usual analytic continuation. (For a discussion of the behaviour on the circle of convergence see below.) Here $(a)_m$ denotes the Pochhammer symbol

$$(a)_m = a(a+1)\dots(a+m-1) = \frac{\Gamma(a+m)}{\Gamma(a)}, \tag{A.1.2}$$

where we have the second equality if a is not a non-positive integer. The Pochhammer symbol is also known as the rising factorial (see [93], pp. 149-165). The hypergeometric function satisfies the differential identity

$$\frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a+1, b+1; c+1; z), \tag{A.1.3}$$

from which we easily derive that for m derivatives

$$\frac{d^m}{dz^m} F(a, b; c; z) = \frac{(a)_m (b)_m}{(c)_m} F(a+m, b+m; c+m; z). \tag{A.1.4}$$

From (A.1.1) we also have the point-wise identity $F(a, b; c; 0) = 1$.

The hypergeometric function arises as a solution to the hypergeometric differential equation

$$z(1-z)\frac{d^2 w}{dz^2} + (c - (a+b+1)\, z)\, \frac{dw}{dz} - abw = 0. \tag{A.1.5}$$

The hypergeometric differential equation can be reached from any second-order ordinary differential equation with at most three regular points by a suitable change of variables.

By the change of variables $z = (1-t)/2$, and setting $a = -k, b = \alpha+\beta+k+1$, and $c = \alpha+1$, one can transform (A.1.5) into the well known Jacobi differential equation,

$$(1-t^2)\frac{d^2w}{dt^2} + (\beta-\alpha-(\alpha+\beta+2)t)\frac{dw}{dt} + k(\alpha+\beta+k+1)w = 0, \tag{A.1.6}$$

which is solved by the Jacobi polynomial $w = \mathscr{P}_k^{(\alpha,\beta)}(t)$, a special case of the hypergeometric function. For more background reading and reference on this see [1, 4, 12].

The Generalised Hypergeometric Function ${}_pF_q(\mathbf{a}; \mathbf{b}; z)$. This generalises the Gauss hypergeometric function in an obvious way. Like there it is defined for $\mathbf{a} = (a_1, \dots, a_p)$ and $\mathbf{b} = (b_1, \dots, b_q)$ with no b_j $(1 \le j \le q)$ a non-positive integer, again initially by the series,

$${}_pF_q(\mathbf{a}; \mathbf{b}; z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_k}{\prod_{j=1}^q (b_j)_k} \frac{z^k}{k!}, \tag{A.1.7}$$

that converges for all finite values of z when $p \le q$ and all $|z| < 1$ when $p = q+1$. The series diverges for all non-zero z when $p > q+1$. In the case $p = q+1$ the series converges absolutely for all $|z| = 1$ if $\mathrm{Re}(\sum_i \mathrm{b_i} - \sum_i \mathrm{a_i}) > 0$ and converges conditionally for all $|z| = 1$ and $z \ne 1$ if $-1 < \mathrm{Re}(\sum_i \mathrm{b_i} - \sum_i \mathrm{a_i}) \le 0$ while the series diverges if $\mathrm{Re}(\sum_i \mathrm{b_i} - \sum_i \mathrm{a_i}) \le -1$. Clearly when any of the parameters a_i (with $1 \le i \le p$) is a non-positive integer the series terminates and becomes a polynomial in z. From the definition it is seen that ${}_pF_q(\mathbf{a}; \mathbf{b}; 0) = 1$. Moreover differentiating in z, we easily derive the recursive relation [1]

$$\frac{d^m}{dz^m} {}_pF_q(\mathbf{a}; \mathbf{b}; z) = \frac{\prod_{i=1}^p (a_i)_m}{\prod_{j=1}^q (b_j)_m} {}_pF_q(\mathbf{a}+m; \mathbf{b}+m; z). \tag{A.1.8}$$

Proceeding forward we can now consider applying the differential operator $\mathscr{L}_\mathsf{P}$ to the function ${}_pF_q(\mathbf{a}; \mathbf{b}; \mathscr{E}(\theta))$ as in Theorem 2.1.1. To this end first note that in this context equation (2.1.7) becomes a product of p Pochhammer symbols, and plainly can be written as,

$$(a_1)_j (a_2)_j \dots (a_p)_j = \sum_{\ell=1}^{j} \mathscr{S}_\ell^j(\mathbf{a}) \left[\prod_{i=1}^p a_i\right]^\ell. \tag{A.1.9}$$

Here the scalars $\mathscr{S}_\ell^j(\mathbf{a})$ are the coefficients of the factor $X = \prod_{i=1}^p a_i$ in the polynomial expansion of the product on the left in X. Indeed to justify (A.1.9) and give a description of the scalars $\mathscr{S}_\ell^j$ let

$$\mathsf{p}(\lambda) = \prod_{\ell=1}^p (\lambda + a_\ell) = \sum_{\ell=0}^p \mathsf{S}_{p-\ell}(a_1, a_2, \dots, a_p)\lambda^\ell. \tag{A.1.10}$$

[1] Here for brevity we have written $\mathbf{a}+m = (a_1+m, \dots, a_p+m)$, $\mathbf{b}+m = (b_1+m, \dots, b_q+m)$.

Now for $0 \le k \le j-1$ put $Y_k = \mathsf{p}(k) - \mathsf{p}(0)$. Then $\mathsf{p}(0) = \mathsf{S}_p(a_1, a_2, \dots, a_p) = X$ and it is plain that

$$Y_k = \mathsf{p}(k) - X = \sum_{\ell=1}^{p} \mathsf{S}_{p-\ell}(a_1, a_2, \dots, a_p) k^\ell. \tag{A.1.11}$$

As a result starting from the product on the left in (A.1.9) we can write

$$\begin{aligned}
(a_1)_j (a_2)_j \dots (a_p)_j &= \prod_{\ell=1}^{p} \prod_{k=0}^{j-1} (k + a_\ell) = \prod_{k=0}^{j-1} \mathsf{p}(k) = \prod_{k=0}^{j-1} [X + Y_k] \\
&= \sum_{\ell=0}^{j} \mathsf{S}_{j-\ell}(Y_0, Y_1, \dots, Y_{j-1}) X^\ell \\
&= \sum_{\ell=1}^{j} \mathsf{S}_{j-\ell}(Y_0, Y_1, \dots, Y_{j-1}) X^\ell \\
&= \sum_{\ell=1}^{j} \mathscr{S}_\ell^j(\mathbf{a}) X^\ell = \sum_{\ell=1}^{j} \mathscr{S}_\ell^j(\mathbf{a}) \left[\prod_{i=1}^{p} a_i \right]^\ell
\end{aligned} \tag{A.1.12}$$

which is (A.1.9) as required. Note that for $p = 2$ and $\mathbf{a} = (a, b)$ we have $\mathscr{S}_\ell^j(\mathbf{a}) = \mathsf{s}_\ell^j$ as in Theorem 2.1.1. Now with the operator $\mathscr{L}_{\mathsf{P}} = \mathsf{P}_N(d/d\theta)$ as before we can then state the following theorem.

Theorem A.1.1. *With the notation as in Theorem 2.1.1 and ${}_pF_q(\mathbf{a}; \mathbf{b}; z)$ as above we have the identity*

$$\begin{aligned}
\mathscr{L}_{\mathsf{P}} \left[{}_pF_q(\mathbf{a}; \mathbf{b}; \mathscr{E}(\theta)) \right] \Big|_{\theta=0} &= p_0 + \sum_{m=1}^{N} p_m \sum_{j=1}^{m} \mathsf{c}_j^m(\mathbf{a}, \mathbf{b}; \mathscr{E}) \left[- \prod_{i=1}^{p} a_i \right]^j \\
&= \sum_{m=0}^{N} p_m \mathsf{H}_m \left(- \prod_{i=1}^{p} a_i \right),
\end{aligned} \tag{A.1.13}$$

where $\mathsf{H}_m(X) = \mathsf{H}_m(\mathbf{a}, \mathbf{b}; \mathscr{E}; X)$ is defined as $\mathsf{H}_0(X) = 1$ and for $m \ge 1$, as

$$\mathsf{H}_m(X) = \sum_{j=1}^{m} \mathsf{c}_j^m(\mathbf{a}, \mathbf{b}; \mathscr{E}) X^j, \tag{A.1.14}$$

with the coefficients $\mathsf{c}_j^m(\mathbf{a}, \mathbf{b}; \mathscr{E})$ defined explicitly as

$$\mathsf{c}_j^m(\mathbf{a}, \mathbf{b}; \mathscr{E}) = (-1)^j \sum_{n=j}^{m} \frac{\mathsf{b}_n^m[\mathscr{E}] \mathscr{S}_j^n(\mathbf{a})}{\prod_{j=1}^{q} (b_j)_n}. \tag{A.1.15}$$

A.2 Asymptotics of Jacobi theta functions $\vartheta_1, \vartheta_2, \vartheta_3$ and their derivatives

Here we present asymptotic data for the classical Jacobi theta functions and their derivatives. For further details *see* [28, 86].

$$\vartheta_1(s) = 1 + 2\sum_{j=1}^{\infty} e^{-j^2 s}, \tag{A.2.1}$$

$$\vartheta_2(s) = \sum_{j=0}^{\infty}(2j+1)e^{-(j+1/2)^2 s}, \tag{A.2.2}$$

$$\vartheta_3(s) = 2\sum_{j=0}^{\infty} j e^{-j^2 s}. \tag{A.2.3}$$

For ϑ_1 we have

$$\vartheta_1(s) = \sqrt{\pi/2} + O(e^{-1/s}), \tag{A.2.4}$$

$$\vartheta_1^{(m+1)}(s) = (-1)^{m+1}\Gamma(m+3/2)s^{-m-3/2} + O(e^{-1/s}). \tag{A.2.5}$$

For $\vartheta_2(s)$ and $\vartheta_3(s)$, we use the well known Bernoulli numbers B_{2j} [6, 28]. Indeed, we have for ϑ_2

$$\vartheta_2(s) \sim \frac{1}{s} + \sum_{j=0}^{\infty} \frac{s^j}{j!}\frac{(-1)^j}{j+1}\left(1 - 2^{-2j-1}\right)B_{2j+2}, \tag{A.2.6}$$

$$\vartheta_2^{(m)}(s) \sim \frac{(-1)^m m!}{s^{m+1}} + \sum_{j=m}^{\infty} \frac{s^{j-m}}{(j-m)!}\frac{(-1)^j}{j+1}\left(1 - 2^{-2j-1}\right)B_{2j+2} \tag{A.2.7}$$

and for ϑ_3,

$$\vartheta_3(s) \sim \frac{1}{s} + \sum_{j=0}^{\infty} \frac{s^j}{j!}\frac{(-1)^j}{j+1}B_{2j+2}, \tag{A.2.8}$$

$$\vartheta_3^{(m+1)}(s) \sim \frac{(-1)^{m+1}(m+1)!}{s^{m+2}} + \sum_{j=m+1}^{\infty} \frac{s^{j-m-1}}{(j-m-1)!}\frac{(-1)^j}{j+1}B_{2j+2}. \tag{A.2.9}$$

A.3 Some identities on Jacobi polynomials

The Jacobi polynomials are defined via the well-known Gaussian hypergeometric function as (*see* [4, 12, 55, 70, 71] for backgroud and further readings)

$$\begin{aligned} P_k^{\alpha,\beta}(t) &= \frac{\Gamma(\alpha+k+1)}{k!\Gamma(\alpha+1)}\,{}_2F_1(-k, \alpha+\beta+k+1; \alpha+1; (1-t)/2) \\ &= \sum_{j=0}^{k} \frac{\Gamma(k+\alpha+\beta+1+j)}{\Gamma(k+\alpha+\beta+1)}\frac{\Gamma(\alpha+1+k)}{\Gamma(\alpha+j+1)}\frac{(t-1)^j}{2^j j!(k-j)!}, \end{aligned} \tag{A.3.1}$$

with $k \geq 0$ and $\alpha, \beta > -1$. The Jacobi polynomials form an orthogonal system in the weighted Hilbert space $L^2_{\mathsf{w}}(-1,1)$ with $\mathsf{w}(t) = (1-t)^\alpha(1+t)^\beta$. It can be seen that here we have the orthogonality relation

$$\int_{-1}^{1} P_k^{\alpha,\beta}(s) P_l^{\alpha,\beta}(s)\mathsf{w}(s)\, ds = \frac{2^\gamma}{2k+\gamma} \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+\gamma)\Gamma(k+1)} \delta_{kl}, \tag{A.3.2}$$

where we have denoted $\gamma = \alpha+\beta+1$. This orthogonality also implies the symmetry $P_k^{\alpha,\beta}(-t) = (-1)^k P_k^{\beta,\alpha}(t)$. The Jacobi polynomials $y = P_k^{\alpha,\beta}$ are solutions to the Jacobi differential equation

$$(1-t^2)\frac{d^2y}{dt^2} - (\alpha - \beta + (\alpha+\beta+2)t)\frac{dy}{dt} + k(k+\alpha+\beta+1)y = 0. \tag{A.3.3}$$

For $m > 0$, the derivatives of $P_k^{\alpha,\beta}(t)$ satisfy the recursive relation

$$\frac{d^m}{dt^m} P_k^{\alpha,\beta}(t) = \frac{\Gamma(k+m+\alpha+\beta+1)}{2^m \Gamma(k+\alpha+\beta+1)} P_{k-m}^{\alpha+m,\beta+m}(t). \tag{A.3.4}$$

Noting that $P_k^{\alpha,\beta}(1) = \Gamma(\alpha+k+1)/[\Gamma(\alpha+1)k!]$, we define the normalised Jacobi polynomials $\mathscr{P}_k^{\alpha,\beta}(t) = P_k^{\alpha,\beta}(t)/P_k^{\alpha,\beta}(1)$ so that $\mathscr{P}_k^{\alpha,\beta}(1) = 1$. These as seen are linked to the zonal spherical functions on compact rank-one symmetric spaces (see Sections 4.3 and 4.4). Specialising to $\alpha = \beta = \nu - 1/2$, for $\nu > -1/2$, one can recover the Gegenbauer polynomials $C_k^\nu(t)$ as

$$C_k^\nu(t) = \frac{\Gamma(\nu+1/2)}{\Gamma(2\nu)} \frac{\Gamma(k+2\nu)}{\Gamma(k+\nu+1/2)} P_k^{\nu-1/2,\nu-1/2}(t). \tag{A.3.5}$$

The normalised Gegenbauer polynomials $\mathscr{C}_k^\nu(t) = C_k^\nu(t)/C_k^\nu(1)$, where $C_k^\nu(1) = \Gamma(k+2\nu)/[k!\Gamma(2\nu)]$, are linked to the zonal spherical functions on the unit sphere $\mathbb{S}^n$ and real projective space $\mathbb{R}\mathbf{P}^n$ when $\nu = (n-1)/2$.

A.4 Polynomial expansions relating to $M_k(\mathscr{X})$

To facilitate the polynomial expansions of multiplicity functions $M_k(\mathscr{X})$ in Section 4.5, we define the sets of scalars $\mathsf{a}_m^n, \mathsf{b}_m^n, \mathsf{c}_m^n, \mathsf{d}_m^n$ respectively as the coefficients in the following polynomial expansions:

$$\prod_{j=0}^{\mathscr{S}}(X^2-j^2)=\sum_{m=0}^{\mathscr{S}}\mathsf{a}_m^n X^{2m+2},\quad \mathscr{S}=\frac{n-3}{2},\tag{A.4.1}$$

$$\prod_{j=\frac{1}{2}}^{\mathscr{S}-\frac{1}{2}}(X^2-j^2)=\sum_{m=0}^{\mathscr{S}}\mathsf{b}_m^n X^{2m},\quad \mathscr{S}=\frac{n-2}{2},\tag{A.4.2}$$

$$\prod_{j=\frac{1}{2}}^{\mathscr{S}+\frac{1}{2}}(X^2-j^2)^2=\sum_{m=0}^{2\mathscr{S}+2}\mathsf{c}_m^n X^{2m},\quad \mathscr{S}=\frac{n-3}{2},\tag{A.4.3}$$

$$\prod_{j=1}^{\mathscr{S}}(X^2-j^2)^2=\sum_{m=0}^{2\mathscr{S}}\mathsf{d}_m^n X^{2m},\quad \mathscr{S}=\frac{n-2}{2}.\tag{A.4.4}$$

Here n is a positive integer. In fact both (A.4.1) and (A.4.3) require $n\geq 3$ to be odd, and likewise (A.4.2) and (A.4.4) require $n\geq 4$ to be even. Note that the products in (A.4.2) and (A.4.3) run over (non-whole) half integers, that is, they iterate by $j\to j+1$ starting from $j=1/2$. Moreover, we define the scalars e_m^n and f_m as the coefficients of X^{2m} in the polynomials below.

$$\left[X^2-(2n-1)^2/4\right]\prod_{j=1/2}^{n-3/2}(X^2-j^2)^2=\sum_{m=0}^{2n-1}\mathsf{e}_m^n X^{2m},\quad n\geq 1,\tag{A.4.5}$$

$$(X^2-81/4)(X^2-49/4)(X^2-25/4)(X^2-9/4)^2(X^2-1/4)^2=\sum_{m=0}^{7}\mathsf{f}_m X^{2m}.\tag{A.4.6}$$

On a similar note, we define the scalars A_k^n, B_k^n, C_k^n, D_k^n, E_k^n, and F_k as the coefficients in the following polynomials:

$$\prod_{j=0}^{(n-3)/2}(X^2+j^2)=\sum_{k=0}^{(n-3)/2}\mathsf{A}_k^n X^{2k+2},\quad n\geq 3\text{ odd}\tag{A.4.7}$$

$$\prod_{j=1/2}^{(n-3)/2}(X^2+j^2)=\sum_{k=0}^{(n-2)/2}\mathsf{B}_k^n X^{2k},\quad n\geq 4\text{ even}\tag{A.4.8}$$

$$\prod_{j=1/2}^{(n-2)/2}(X^2+j^2)^2=\sum_{k=0}^{n-1}\mathsf{C}_k^n X^{2k},\quad n\geq 3\text{ odd}\tag{A.4.9}$$

$$\prod_{j=1}^{(n-2)/2}(X^2+j^2)^2=\sum_{k=0}^{n-2}\mathsf{D}_k^n X^{2k},\quad n\geq 4\text{ even},\tag{A.4.10}$$

$$\left[X^2+(2n-1)^2/4\right]\prod_{j=1/2}^{n-3/2}(X^2+j^2)^2=\sum_{k=0}^{2n-1}\mathsf{E}_k^n X^{2k},\quad n\geq 1,\tag{A.4.11}$$

$$(X^2+81/4)(X^2+49/4)(X^2+25/4)(X^2+9/4)^2(X^2+1/4)^2=\sum_{k=0}^{7}\mathsf{F}_k X^{2k}.\tag{A.4.12}$$

Note that the leading coefficient in each of the cases above, that is, the coefficient of the highest power of X, is always equal to 1.

Remark A.4.1. These polynomials bare a strong resemblance to those defined in (A.4.1)-(A.4.6), and in fact for admissible ranges of n we have

$$\mathsf{a}_k^n = (-1)^{(n+1)/2+k}\mathsf{A}_k^n, \qquad \mathsf{b}_k^n = (-1)^{(n-2)/2+k}\mathsf{B}_k^n, \qquad \mathsf{c}_k^n = (-1)^k\mathsf{C}_k^n,$$
$$\mathsf{d}_k^n = (-1)^k\mathsf{D}_k^n, \qquad \mathsf{e}_k^n = (-1)^{k+1}\mathsf{E}_k^n, \qquad \mathsf{f}_k = (-1)^{k+1}\mathsf{F}_k.$$

A.5 Faá di Bruno's Theorem and the Elementary Symmetric Polynomials

The Bell Polynomials B_j^m and Faá di Bruno's Theorem. The classical Faá di Bruno's theorem asserts that for sufficiently smooth functions f, g the mth order derivative of the composition $h(X) = f(g(X))$ is given by

$$\frac{d^m h}{dX^m}(X) = \sum_{j=1}^{m} f^{(j)}(g(X)) \cdot \mathsf{B}_{m,j}(g'(X), g''(X), \dots, g^{(m-j+1)}(X)). \tag{A.5.1}$$

Here $\mathsf{B}_{m,j} = \mathsf{B}_{m,j}(\mathbf{X})$ with $1 \le j \le m$ and $\mathbf{X} = (X_1, X_2, \dots, X_{m-j+1})$ are the *incomplete* Bell polynomials, defined

$$\mathsf{B}_{m,j}(x) = \sum_{\mathscr{K}} \frac{m!}{k_1!k_2!\dots k_{m-j+1}!}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\left(\frac{x_{m-j+1}}{(m-j+1)!}\right)^{k_{m-j+1}} \tag{A.5.2}$$

where the sum is taken over the set $\mathscr{K}$ of all *admissible* $(k_1, k_2, \dots, k_{m-j+1})$, that is, finite sequences of non-negative integers $k_1, \dots, k_{m-j+1}$ such that

$$\sum_{l=1}^{m-j+1} k_l = j, \qquad \sum_{l=1}^{m-j+1} lk_l = m. \tag{A.5.3}$$

The incomplete Bell polynomials satisfy the generating function relation (for each fixed $j \ge 0$)

$$\frac{1}{j!}\left[\sum_{l=1}^{\infty} X_l \frac{t^l}{l!}\right]^j = \sum_{n=j}^{\infty} \mathsf{B}_{n,j}(X_1, X_2, \dots, X_{n+j-1})\frac{t^n}{n!}, \tag{A.5.4}$$

and the computationally convenient recursive formula

$$\mathsf{B}_{m,j} = \sum_{l=1}^{m-j+1} \binom{m-1}{l-1} X_l \mathsf{B}_{m-l,j-1} \tag{A.5.5}$$

We point out that the value of $\mathsf{B}_{m,j}(\mathbf{X})$ on the finite sequence of factorials gives the (unsigned) Stirling numbers of the first kind

$$\mathsf{B}_{m,j}(0!, 1!, \dots, (m-j)!) = |\mathsf{s}(m,j)| \tag{A.5.6}$$

[recall that by expanding the *falling* factorial we have $x(x-1)\dots(x-m+1) = \sum_{j=0}^{m} \mathsf{s}(m,j)x^j$] whilst the value of $\mathsf{B}_{m,j}(\mathbf{X})$ on the finite sequence of ones gives the Stirling numbers of the second kind

$$\mathsf{B}_{m,j}(1,1,\dots,1) = \mathsf{S}(m,j) \tag{A.5.7}$$

[recall that $\mathsf{S}(m,j) = 1/j! \sum_{l=0}^{j} (-1)^l \binom{j}{l}(j-l)^m$].

To illustrate the above and also for the sake of its particular relevance in the earlier parts of the thesis we look at the specific example of $\mathscr{E}(\theta) = \cos\theta$. Then $\mathsf{b}_j^m[\cos\theta] = \mathsf{B}_{2m,j}(0,-1,0,1,\dots)$. Therefore $\mathsf{b}_j^m[\cos\theta] = 0$ for m odd, whilst for m even, say, $m = 2l$,

$$\mathsf{b}_j^{2l}[\cos\theta] = \begin{cases} (-1)^l & \text{if } j = 1 \\ -\left(j^2\mathsf{b}_j^{2(l-1)} + (2j-1)\mathsf{b}_{j-1}^{2(l-1)}\right) & \text{if } 1 < j \le l \\ 0 & \text{if } j > l. \end{cases} \tag{A.5.8}$$

The Elementary Symmetric Polynomials S_j. The elementary symmetric polynomial $\mathsf{S}_j = \mathsf{S}_j(\mathbf{X})$ in the vector variable $\mathbf{X} = (X_1, X_2, \dots, X_n)$ with $0 \le j \le n$ is defined as the sum of the distinct products of length j of the variables $X_1, X_2, \dots, X_n$. This specifically means that we have $\mathsf{S}_0(\mathbf{X}) = 1$, and for $1 \le j \le n$ in turn we have

$$\mathsf{S}_1(\mathbf{X}) = \sum_{k=1}^{n} X_k, \qquad \mathsf{S}_2(\mathbf{X}) = \sum_{1\le k<l\le n} X_k X_l, \tag{A.5.9}$$

$$\mathsf{S}_3(\mathbf{X}) = \sum_{1\le k<l<m\le n} X_k X_l X_m, \quad \dots \quad \mathsf{S}_n(\mathbf{X}) = \prod_{k=1}^{n} X_k. \tag{A.5.10}$$

We then have the well-known identity (with $n \ge 1$)

$$\prod_{k=1}^{n}(\lambda + X_k) = \sum_{j=0}^{n} \mathsf{S}_j(\mathbf{X})\lambda^{n-j} = \lambda^n + \mathsf{S}_1(\mathbf{X})\lambda^{n-1} + \dots + \mathsf{S}_{n-1}(\mathbf{X})\lambda + \mathsf{S}_n(\mathbf{X}). \tag{A.5.11}$$

A.6 Integral and differential identities

Lemma A.6.1. *For any non-negative integer k and $t > 0$, we have the integral identities*

$$\int_0^\infty 2\lambda^{2k+1} \tanh(\pi\lambda) e^{-t\lambda^2}\, d\lambda = \frac{\Gamma(k+1)}{t^{k+1}} + \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j!} \mathscr{B}_1^\star(j+k)t^j, \tag{A.6.1}$$

and similarly,

$$\int_0^\infty 2\lambda^{2k+3} \coth(\pi\lambda) e^{-t\lambda^2}\, d\lambda = \frac{\Gamma(k+2)}{t^{k+2}} - \sum_{j=k}^{\infty} \frac{(-1)^k}{(j-k)!} \mathscr{B}_2(j)t^{j-k}, \tag{A.6.2}$$

where for B_{2n} denoting the Bernoulli numbers we have defined

$$\mathscr{B}_1^\star(X) = (1 - 2^{-1-2X})\frac{|B_{2(X+1)}|}{X+1}, \qquad \mathscr{B}_2(X) = \frac{B_{2(X+2)}}{X+2}, \tag{A.6.3}$$

where we similarly denote $\mathscr{B}_1(X) = (-1)^{X+1}\mathscr{B}_1^\star(X)$.

Proof. The results follow by using the known identities

$$\tanh(\pi x) = 1 - \frac{2}{1+e^{2\pi x}}, \qquad \coth(\pi x) = 1 - \frac{2}{1-e^{2\pi x}}, \tag{A.6.4}$$

together with the integrals

$$\int_0^\infty \frac{x^{2n-1}}{1+e^{2\pi x}}\, dx = (1-2^{1-2n})\frac{|B_{2n}|}{4n} = \frac{\mathscr{B}_1^\star(n-1)}{4} \tag{A.6.5}$$

and similarly

$$\int_0^\infty \frac{x^{2n-1}}{1-e^{2\pi x}}\, dx = (-1)^n \frac{B_{2n}}{4n} := (-1)^n \frac{\mathscr{B}_2(n-2)}{4}. \tag{A.6.6}$$

The result follows. □

Lemma A.6.2. *For any non-negative integer k, we have*

$$\begin{aligned}\int_0^\infty \frac{2\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+1}\, d\lambda = \rho^{2k+2-2s} B(k+1, s-k-1)\\ - \sum_{j=0}^\infty \frac{1}{\rho^{2s+2j}} \frac{(-1)^j}{j!} \frac{\Gamma(s+j)}{\Gamma(s)} \mathscr{B}_1(k+j),\end{aligned} \tag{A.6.7}$$

with $B(x, y)$ denoting the well-known Beta function, and $\mathscr{B}_1(X)$ as in (A.6.3). Similarly we have

$$\begin{aligned}\int_0^\infty \frac{2\coth(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+3}\, d\lambda = \rho^{2k+4-2s} B(k+2, s-k-2)\\ - \sum_{j=0}^\infty \frac{1}{\rho^{2s+2j}} \frac{(-1)^j}{j!} \frac{\Gamma(s+j)}{\Gamma(s)} \mathscr{B}_2(k+j+1),\end{aligned} \tag{A.6.8}$$

with $B(x, y)$ denoting the Beta function, and $\mathscr{B}_2(X)$ as in (A.6.3).

Proof. We make use of the identity

$$\tanh(\pi x) = 1 - \frac{2}{1+e^{2\pi x}} \tag{A.6.9}$$

which allows us to split the integral on the left-hand side of (A.6.7) as

$$\int_0^\infty \frac{2\tanh(\pi\lambda)}{(\lambda^2+\rho^2)^s}\lambda^{2k+1} d\lambda = \int_0^\infty \frac{2\lambda^{2k+1}}{(\lambda^2+\rho^2)^s} d\lambda - \int_0^\infty \frac{4(\lambda^2+\rho^2)^{-s}}{1+e^{2\pi\lambda}}\lambda^{2k+1} d\lambda. \tag{A.6.10}$$

The first integral above can be written as

$$\begin{aligned}\int_0^\infty \frac{2\lambda^{2k+1}}{(\lambda^2+\rho^2)^s} d\lambda &= \rho^{2k+1-2s} \int_0^\infty \frac{2[\lambda/\rho]^{2k+1}}{([\lambda/\rho]^2+1)^s} d\lambda\\ &= \rho^{2k+2-2s} \int_0^\infty \frac{2p^{2k+1}}{(p^2+1)^s}\, dp\end{aligned} \tag{A.6.11}$$

We now refer to the integral identity

$$\int_0^\infty p^{a-1}(1+p^2)^{b-1}\, dp = \frac{1}{2} B\left(\frac{a}{2}, 1-b-\frac{a}{2}\right), \tag{A.6.12}$$

which allows (A.6.11) to be evaluated as

$$\int_0^\infty \frac{2\lambda^{2k+1}}{(\lambda^2+\rho^2)^s}\,d\lambda = \rho^{2k+2-2s} B\left(k+1, s-k-1\right). \tag{A.6.13}$$

For the second integral, we apply the generalised binomial theorem to write

$$\begin{aligned}\frac{1}{(\lambda^2+\rho^2)^s} = \frac{\rho^{-2s}}{(\lambda^2/\rho^2+1)^s} &= \sum_{j=0}^{\infty}\binom{-s}{j}\rho^{-2s-2j}\lambda^{2j} \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{\Gamma(s+j)}{\Gamma(s)}\rho^{-2s-2j}\lambda^{2j}.\end{aligned} \tag{A.6.14}$$

This allows us to write the second integral explicitly as

$$\int_0^\infty \frac{4(\lambda^2+\rho^2)^{-s}}{1+e^{2\pi\lambda}}\lambda^{2k+1}d\lambda = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{\Gamma(s+j)}{\Gamma(s)}\rho^{-2s-2j}\int_0^\infty \frac{4\lambda^{2k+2j+1}}{1+e^{2\pi\lambda}}d\lambda. \tag{A.6.15}$$

To evaluate the integral here, we refer to the identity

$$\int_0^\infty \frac{x^{2n-1}}{1+e^{2\pi x}}\,dx = (1-2^{1-2n})\frac{|B_{2n}|}{4n} = \frac{\mathscr{B}_1(n-1)}{4}. \tag{A.6.16}$$

This completes the first result. The proof of the second result follows the above closely, using a different set of integral identites. Firstly instead of (A.6.9) we use

$$\coth(\pi x) = 1 - \frac{2}{1-e^{2\pi x}} \tag{A.6.17}$$

Next instead of (A.6.16) we use

$$\int_0^\infty \frac{x^{2n-1}}{1-e^{2\pi x}}\,dx = (-1)^n\frac{B_{2n}}{4n} := (-1)^n\frac{\mathscr{B}_2(n-2)}{4}. \tag{A.6.18}$$

The result follows. □

Lemma A.6.3. *The normalised Jacobi polynomials $\mathscr{P}_k^{\alpha,\beta}(t)$, $k \geq 0$, $\alpha,\beta > -1$, satisfy the differential identity*

$$\frac{d^{2\ell}}{d\theta^{2\ell}}\mathscr{P}_k^{\alpha,\beta}(\cos\theta)\Big|_{\theta=0} = \sum_{j=1}^{\ell}\mathsf{h}_j^\ell(\alpha,\beta)[k(k+\alpha+\beta+1)]^j, \qquad \ell \geq 1. \tag{A.6.19}$$

Here the set of scalars $(\mathsf{h}_j^\ell(\alpha,\beta) : 1 \leq j \leq \ell)$ are explicitly computable, and $k(k+\alpha+\beta+1)$ are the eigenvalues of the Jacobi operator Note that with $\ell = 0$ no derivatives take place, so we may formally set $\mathsf{h}_0^0 = 1$.

With the previous lemma in mind, we now introduce a sequence of polynomials $\mathscr{R}_\ell^{\alpha,\beta}$ that will appear regularly throughout the rest of the thesis. We define $\mathscr{R}_0^{\alpha,\beta}(X)$, and for $\ell \geq 1$

$$\mathscr{R}_\ell^{\alpha,\beta}(X) = \sum_{j=1}^{\ell}\mathsf{h}_j^\ell(\alpha,\beta)X^j. \tag{A.6.20}$$

We then see that the Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X}_c)$ can be written as

$$\begin{aligned} b^n_{2\ell}(t;\mathscr{X}_c) &= \sum_{k=0}^{\infty}\frac{M_k(\mathscr{X}_c)}{\mathrm{Vol}(\mathscr{X}_c)}\mathscr{R}^{\alpha,\beta}_\ell(\lambda^{\alpha,\beta}_k)e^{-t\lambda^{\alpha,\beta}_k} = \sum_{k=0}^{\infty}\frac{M_k(\mathscr{X}_c)}{\mathrm{Vol}(\mathscr{X}_c)}\mathscr{R}^{\alpha,\beta}_\ell(-d/dt)e^{-t\lambda^{\alpha,\beta}_k} \\ &= \mathscr{R}^{\alpha,\beta}_\ell(-d/dt)H_{\mathscr{X}_c}(t;x,x) = \mathscr{R}^{\alpha,\beta}_\ell(-d/dt)\Theta(t;\mathscr{X}_c), \end{aligned} \tag{A.6.21}$$

where $H_{\mathscr{X}_c}(t;x,x)=\Theta(t;\mathscr{X}_c)$ is the trace of the heat kernel on a compact space.

Lemma A.6.4. *The Jacobi function $\mathscr{P}^{\alpha,\beta}_\mu(t)$, with $\mu=-(i\lambda+\rho)$, $\rho=(\alpha+\beta+1)/2$, and $\alpha,\beta>-1$, satisfies the differential identity*

$$\frac{d^{2\ell}}{dr^{2\ell}}\mathscr{P}^{\alpha,\beta}_\mu(\cosh r)\Big|_{r=0} = \sum_{j=1}^{\ell}\mathsf{H}^\ell_j(\alpha,\beta)[-(\lambda^2+\rho^2)]^j, \qquad \ell\geq 1. \tag{A.6.22}$$

Here the set of scalars $(\mathsf{H}^\ell_j(\alpha,\beta):1\leq j\leq\ell)$ are explicitly computable for $\alpha,\beta>-1$, and $\lambda^2+\rho^2$ are the eigenvalues of the Jacobi operator on a non-compact space. Note that with $\ell=0$ no derivatives take place, so we may formally set $\mathsf{H}^0_0=1$.

Remark A.6.5. With H^ℓ_j the scalars as described in Lemma A.6.4, and h^ℓ_j the analogous scalars described in Lemma A.6.3, we have the relation

$$\mathsf{H}^\ell_j(\alpha,\beta)=(-1)^\ell\mathsf{h}^\ell_j(\alpha,\beta). \tag{A.6.23}$$

We now introduce a set of polynomials $\mathscr{Q}^{\alpha,\beta}_\ell$ for future reference, defined as $\mathscr{Q}^{\alpha,\beta}_0(X)=1$ and for $\ell>0$

$$\mathscr{Q}^{\alpha,\beta}_\ell(X)=\sum_{j=1}^{\ell}\mathsf{H}^\ell_j(\alpha,\beta)X^j. \tag{A.6.24}$$

With these, we see that the Maclaurin heat coefficients $b^n_{2\ell}(t;\mathscr{X}_{nc})$ defined in (5.2.17) can be written as

$$\begin{aligned} b^n_{2\ell}(t;\mathscr{X}_{nc}) &= \frac{2^{2\beta-1}\Gamma(\alpha+1)}{\pi^{\alpha+2}}\int_0^\infty \mathscr{Q}_\ell(-\rho^2-\lambda^2)e^{-t(\rho^2+\lambda^2)}\mu(\lambda)\,d\lambda \\ &= \frac{2^{2\beta-1}\Gamma(\alpha+1)}{\pi^{\alpha+2}}\int_0^\infty \mathscr{Q}_\ell(d/dt)e^{-t(\rho^2+\lambda^2)}\mu(\lambda)\,d\lambda \\ &= \mathscr{Q}_\ell(d/dt)H_{\mathscr{X}_{nc}}(t;x,x)=\mathscr{Q}_\ell(d/dt)\Theta(t;\mathscr{X}_{nc}), \end{aligned} \tag{A.6.25}$$

where $H_{\mathscr{X}_{nc}}(t;x,x)=\Theta(t;\mathscr{X}_{nc})$ is the trace of the heat kernel on a non-compact space.

A.7 Plancherel Measure $\mu(\lambda;\mathscr{X})$ and Multiplicity Function $M(k;\mathscr{X})$ for $\mathscr{X}=\mathbf{G}/\mathbf{H}$

Earlier in the thesis we encountered the Plancherel measure $\mu(\lambda)=\mu(\lambda;\mathscr{X})$ in the case of the non-compact symmetric spaces, expressed in terms of the Harish-Chandra

c-function. In the first part of this appendix we give the explicit form of this measure for each individual family of such spaces. Then in the second part we turn to compact symmetric spaces and give similar explicit formulations of the multiplicity function $M(k;\mathscr{X}) = M_k(\mathscr{X})$ as well as some other relevant spectral-geometric objects including the spectrum $\Sigma(-\Delta_{\mathscr{X}}) = \{\lambda_k(\mathscr{X}) : k \ge 0\}$ and volume $\mathrm{Vol}(\mathscr{X})$.

A.7.1 The non-compact case $\mathscr{X} = \mathbf{G}/\mathbf{H}$

- $\mathscr{X} = \mathbb{R}\mathbf{H}^n = \mathbf{SO}_0(n,1)/\mathbf{SO}(n)$. For the *real* hyperbolic space we have $d = n$, $c_d = 2^{n-3}\Gamma(n/2)/\pi^{n/2+1}$ and the Plancherel measure depending as to whether n is odd or even is given respectively by

$$\mu(\lambda) = \pi \left[\frac{2^{2-n}}{\Gamma(n/2)}\right]^2 \prod_{j=0}^{\frac{n-3}{2}} (\lambda^2 + j^2), \qquad (n \ge 3 \text{ odd}), \tag{A.7.1}$$

$$\mu(\lambda) = \pi \left[\frac{2^{2-n}}{\Gamma(n/2)}\right]^2 \lambda \tanh(\pi\lambda) \prod_{j=\frac{1}{2}}^{\frac{n-3}{2}} (\lambda^2 + j^2), \qquad (n \ge 2 \text{ even}). \tag{A.7.2}$$

- $\mathscr{X} = \mathbb{C}\mathbf{H}^n = \mathbf{SU}(n,1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$. For the *complex* hyperbolic space $d = 2n$, $c_d = \Gamma(n)/(2\pi^{n+1})$ and the Plancherel measure depending as to whether n is odd or even is given by

$$\mu(\lambda) = \frac{\pi\lambda\tanh(\pi\lambda)}{[2^{n-1}\Gamma(n)]^2} \prod_{j=\frac{1}{2}}^{\frac{n-2}{2}} (\lambda^2 + j^2)^2, \qquad (n \ge 3 \text{ odd}), \tag{A.7.3}$$

$$\mu(\lambda) = \frac{\pi\lambda^3\coth(\pi\lambda)}{[2^{n-1}\Gamma(n)]^2} \prod_{j=1}^{\frac{n-2}{2}} (\lambda^2 + j^2)^2, \qquad (n \ge 2 \text{ even}). \tag{A.7.4}$$

- $\mathscr{X} = \mathbb{H}\mathbf{H}^n = \mathbf{Sp}(n,1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$. For the *quaternionic* hyperbolic space $d = 4n$, $c_d = 2\Gamma(2n)/\pi^{2n+1}$ and the Plancherel measure is given by

$$\mu(\lambda) = \frac{\pi\lambda\tanh(\pi\lambda)}{[2^{2n}\Gamma(2n)]^2} [\lambda^2 + (2n-1)^2/4] \prod_{j=\frac{1}{2}}^{\frac{2n-3}{2}} (\lambda^2 + j^2)^2 \tag{A.7.5}$$

  (Note that for $n = 1$ the product is omitted and we have $\mathbb{H}\mathbf{H}^1 = \mathbb{R}\mathbf{H}^4$.)

- $\mathscr{X} = \mathbf{H}^2(\mathbf{Cay}) = \mathbf{F}_\star^4/\mathbf{Spin}(9)$. For the *Cayley* hyperbolic space $d = 18$, $c_d = 2^5\Gamma(8)/\pi^9$ and the Plancherel measure is given by

$$\mu(\lambda) = \frac{\pi\lambda\tanh(\pi\lambda)}{2^{20}\Gamma(8)^2}\left(\lambda^2 + \frac{81}{4}\right)\left(\lambda^2 + \frac{49}{4}\right)\left(\lambda^2 + \frac{25}{4}\right) \times$$
$$\times \left(\lambda^2 + \frac{9}{4}\right)^2 \left(\lambda^2 + \frac{1}{4}\right)^2. \tag{A.7.6}$$

A.7.2 The compact case $\mathscr{X} = \mathbf{G}/\mathbf{H}$

- $\mathscr{X} = \mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$**,** $\mathbb{R}\mathbf{P}^n = \mathbf{SO}(n+1)/\mathbf{O}(n)$**.** For the sphere and the *real* projective space the spectrum, the volume and the multiplicity function are in turn as follows.

  1. $\mathscr{X} = \mathbb{S}^n$: The distinct eigenvalues are given by $\lambda_k(\mathscr{X}) = k(n+k-1)$ with $k \geq 0$, $\mathrm{Vol}(\mathscr{X}) = 2\pi^{(n+1)/2}/\Gamma\left((n+1)/2\right)$ and
  $$M_k(\mathscr{X}) = (2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}. \tag{A.7.7}$$
  2. $\mathscr{X} = \mathbb{R}\mathbf{P}^n$: As a result of $\mathbb{S}^n$ being a double cover of $\mathbb{R}\mathbf{P}^n$ $(n \geq 2)$ we have $\lambda_k(\mathscr{X}) = \lambda_{2k}(\mathbb{S}^n) = 2k(n+2k-1)$ with $k \geq 0$, $\mathrm{Vol}(\mathscr{X}) = \mathrm{Vol}(\mathbb{S}^n)/2 = \pi^{(n+1)/2}/\Gamma\left((n+1)/2\right)$ and
  $$M_k(\mathscr{X}) = M_{2k}(\mathbb{S}^n) = (4k+n-1)\frac{(2k+n-2)!}{(2k)!(n-1)!}. \tag{A.7.8}$$

- $\mathscr{X} = \mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$**.** For the *complex* projective space the distinct eigenvalues are given by $\lambda_k(\mathscr{X}) = k(k+n)$ with $k \geq 0$, $\mathrm{Vol}(\mathscr{X}) = 4^n\pi^n/n!$ and the multiplicity function is given by
$$M_k(\mathscr{X}) = \frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2 .\mathrm{Vol}(\mathscr{X}) = \frac{4^n\pi^n}{n!} \tag{A.7.9}$$

- $\mathscr{X} = \mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$**.** For the *quaternionic* projective space the distinct eigenvalues are given by $\lambda_k(\mathscr{X}) = k(k+2n+1)$ with $k \geq 0$, $\mathrm{Vol}(\mathscr{X}) = (4\pi)^{2n}/\Gamma\left(2n+2\right)$ and the multiplicity function is
$$M_k(\mathscr{X}) = \frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left[\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right]^2. \tag{A.7.10}$$

- $\mathscr{X} = \mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$**.** For the *Cayley* projective plane the distinct eigenvalues are given by $\lambda_k = k(k+11)$ with $k \geq 0$, $\mathrm{Vol}(\mathscr{X}) = 3!(4\pi)^8/11!$ and the multiplicity function is given by
$$M_k(\mathscr{X}) = 6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}. \tag{A.7.11}$$

A.8 Explicit formulae for the spectral zeta function

Here we collect the explicit formulae for the spectral zeta function, which occurs as a special case $\zeta(s;\mathscr{X}) = \zeta_0(s;\mathscr{X})$ of the zeta functions $\zeta_\ell(s;\mathscr{X})$ defined in (6.2.14) and (6.2.16). For convenience, we recall the Hurwitz zeta function as
$$\zeta_H(s,q) = \sum_{m=0}^{\infty}\frac{1}{(q+m)^s}, \qquad \mathrm{Re(s)} > 1. \tag{A.8.1}$$

The case $\mathscr{X}_c = \mathbb{S}^n$ vs. $\mathscr{X}_{nc} = \mathbb{R}\mathbf{H}^n$. Here we have for $n \geq 3$ odd

$$\zeta_0(s;\mathbb{S}^n) = \sum_{m=0}^{\infty} \sum_{j=0}^{(n-3)/2} \frac{2\mathsf{a}_j^n \rho^{2m}}{m!(n-1)!}(s)_m \zeta_H(2s+2m-2j+2,\rho+1), \tag{A.8.2}$$

$$\zeta_0(s;\mathbb{R}\mathbf{H}^n) = \sum_{k=0}^{(n-3)/2} \frac{\mathsf{A}_k^n \rho^{2k+1-2s}}{(4\pi)^{n/2}\Gamma(n/2)} \frac{\Gamma(k+1/2)\Gamma(s-k-1/2)}{\Gamma(s)}. \tag{A.8.3}$$

Next for $n \geq 2$ even, we have

$$\zeta_0(s;\mathbb{S}^n) = \sum_{m=0}^{\infty} \sum_{j=0}^{(n-2)/2} \frac{2\mathsf{b}_j^n \rho^{2m}}{m!(n-1)!}(s)_m \zeta_H(2s+2m-2j-1,\rho+1), \tag{A.8.4}$$

$$\zeta_0(s;\mathbb{R}\mathbf{H}^n) = \sum_{k=0}^{(n-2)/2} \frac{\mathsf{B}_k^n}{(4\pi)^{n/2}\Gamma(n/2)} \left[\frac{\Gamma(k+1)\Gamma(s-k-1)}{\rho^{2s-2k-2}\Gamma(s)} - \sum_{j=0}^{\infty} \frac{(-1)^j}{\rho^{2s+2j}} \frac{(s)_j}{j!} \mathscr{B}_1^{\star}(k+j) \right]. \tag{A.8.5}$$

The case $\mathscr{X}_c = \mathbb{C}\mathbf{P}^n$ vs. $\mathscr{X}_{nc} = \mathbb{C}\mathbf{H}^n$. Here we have for $n \geq 1$ odd

$$\zeta_0(s;\mathbb{C}\mathbf{P}^n) = \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{2\mathsf{c}_j^n \rho^{2m}}{\Gamma(n)} \frac{(s)_m}{n!m!} \zeta_H(2s+2m-2j-1,\rho+1), \tag{A.8.6}$$

$$\zeta_0(s;\mathbb{C}\mathbf{H}^n) = \sum_{k=0}^{n-1} \frac{\mathsf{C}_k^n}{(4\pi)^n\Gamma(n)} \left[\frac{\Gamma(k+1)\Gamma(s-k-1)}{\rho^{2s-2k-2}\Gamma(s)} - \sum_{j=0}^{\infty} \frac{(-1)^j}{\rho^{2s+2j}} \frac{(s)_j}{j!} \mathscr{B}_1^{\star}(k+j) \right]. \tag{A.8.7}$$

Next for $n \geq 2$ even, we have

$$\zeta_0(s;\mathbb{C}\mathbf{P}^n) = \sum_{m=0}^{\infty} \sum_{j=0}^{n-2} \frac{2\mathsf{d}_j^n \rho^{2m}}{(n-1)!} \frac{(s)_m}{n!m!} \zeta_H(2s+2m-2j-3,\rho+1), \tag{A.8.8}$$

$$\zeta_0(s;\mathbb{C}\mathbf{H}^n) = \sum_{k=0}^{n-2} \frac{\mathsf{D}_k^n}{(4\pi)^n\Gamma(n)} \left[\rho^{2k+4-2s} \frac{\Gamma(k+2)\Gamma(s-k-2)}{\Gamma(s)} - \sum_{j=0}^{\infty} \frac{(-1)^j}{\rho^{2s+2j}} \frac{(s)_j}{j!} \mathscr{B}_2(k+j+1) \right]. \tag{A.8.9}$$

The case $\mathscr{X}_c = \mathbb{H}\mathbf{P}^n$ vs. $\mathscr{X}_{nc} = \mathbb{H}\mathbf{H}^n$. Here we have for $n \geq 1$

$$\zeta_0(s;\mathbb{H}\mathbf{P}^n) = \sum_{m=0}^{\infty} \sum_{j=0}^{2n-1} \frac{2\mathsf{e}_j^n \rho^{2m}(s)_m}{\Gamma(2n)(2n+1)!m!} \zeta_H(2s+2m-2j-1,\rho+1), \tag{A.8.10}$$

$$\zeta_0(s;\mathbb{H}\mathbf{H}^n) = \sum_{k=0}^{2n-1} \frac{\mathsf{E}_k^n}{(4\pi)^{2n}\Gamma(2n)} \left[\frac{\Gamma(k+1)\Gamma(s-k-1)}{\rho^{2s-2k-2}\Gamma(s)} - \sum_{j=0}^{\infty} \frac{(-1)^j}{\rho^{2s+2j}} \frac{(s)_j}{j!} \mathscr{B}_1^{\star}(k+j) \right], \tag{A.8.11}$$

The case $\mathscr{X}_c = \mathbf{P}^2(Cay)$ vs. $\mathscr{X}_{nc} = \mathbf{H}^2(Cay)$. Here we have

$$\zeta_0(s;\mathbf{P}^2(Cay)) = \sum_{m=0}^{\infty}\sum_{j=0}^{7}\frac{12\mathsf{f}_j\rho^{2m}(s)_m}{7!11!m!}\zeta_H(2s+2m-2j-1,\rho+1), \tag{A.8.12}$$

$$\zeta_0(s;\mathbf{H}^2(Cay)) = \sum_{k=0}^{7}\frac{\mathsf{F}_k}{(4\pi)^8\Gamma(8)}\left[\frac{\Gamma(k+1)\Gamma(s-k-1)}{\rho^{2s-2k-2}\Gamma(s)} - \sum_{j=0}^{\infty}\frac{(-1)^j}{\rho^{2s+2j}}\frac{(s)_j}{j!}\mathscr{B}_1^\star(k+j)\right]. \tag{A.8.13}$$

A.9 Explicit formulae for the trace of the heat kernel

Here we collect the explicit formulae for $\mathsf{p}_j(\mathscr{X}_c)$ and $\mathsf{q}_j(\mathscr{X}_{nc})$ as they appear in the expansions (5.1.9) and (5.1.11) of the trace of the heat kernel.

The case $\mathscr{X}_c = \mathbb{S}^n$ vs. $\mathscr{X}_{nc} = \mathbb{R}\mathbf{H}^n$. Here we have for $n \geq 3$ odd $\mathsf{p}_j(\mathbb{S}^n) = 0$ and $\mathsf{q}_j(\mathbb{R}\mathbf{H}^n) = 0$ for $j > (n-3)/2$, whilst for $0 \leq j \leq (n-3)/2$ we have

$$\mathsf{p}_j(\mathbb{S}^n) = \frac{\Gamma(n/2-j)}{\Gamma(n/2)}\mathsf{a}^n_{(n-3)/2-j}, \tag{A.9.1}$$

$$\mathsf{q}_j(\mathbb{R}\mathbf{H}^n) = \frac{\Gamma(n/2-j)}{\Gamma(n/2)}\mathsf{A}^n_{(n-3)/2-j}. \tag{A.9.2}$$

Next for $n \geq 2$ even, we have

$$\mathsf{p}_j(\mathbb{S}^n) = \begin{cases} \dfrac{\Gamma(n/2-j)}{\Gamma(n/2)}\mathsf{b}^n_{n/2-1-j} & \text{for } \ 0 \leq j \leq \dfrac{n}{2}-1, \\ (-1)^j\displaystyle\sum_{k=0}^{n/2-1}\frac{\mathsf{b}^n_k}{\Gamma(n/2)}\frac{\mathscr{B}_1(k+j-n/2)}{(j-n/2)!} & \text{for } \ j \geq \dfrac{n}{2}. \end{cases} \tag{A.9.3}$$

$$\mathsf{q}_j(\mathbb{R}\mathbf{H}^n) = \begin{cases} \dfrac{\Gamma(n/2-j)}{\Gamma(n/2)}\mathsf{B}^n_{(n-2)/2-j} & \text{for } \ 0 \leq j \leq \dfrac{n}{2}-1, \\ \displaystyle\sum_{k=0}^{(n-2)/2}\frac{\mathsf{B}^n_k}{\Gamma(n/2)}\frac{\mathscr{B}^\star_1(k+j-n/2)}{(-1)^{j+n/2+1}(j-n/2)!} & \text{for } \ j \geq \dfrac{n}{2}. \end{cases} \tag{A.9.4}$$

The case $\mathscr{X}_c = \mathbb{C}\mathbf{P}^n$ vs. $\mathscr{X}_{nc} = \mathbb{C}\mathbf{H}^n$. Here we have for $n \geq 1$ odd

$$\mathsf{p}_j(\mathbb{C}\mathbf{P}^n) = \begin{cases} \dfrac{\Gamma(n-j)}{\Gamma(n)}\mathsf{c}^n_{n-1-j} & \text{for } \ 0 \leq j \leq n-1, \\ \displaystyle\sum_{k=0}^{n-1}(-1)^j\frac{\mathsf{c}^n_k}{\Gamma(n)}\frac{\mathscr{B}_1(k+j-n)}{(j-n)!} & \text{for } \ j \geq n. \end{cases} \tag{A.9.5}$$

$$\mathsf{q}_j(\mathbb{C}\mathbf{H}^n) = \begin{cases} \dfrac{\Gamma(n-j)}{\Gamma(n)}\mathsf{C}^n_{n-1-j} & \text{for } \ 0 \leq j \leq n-1, \\ \displaystyle\sum_{k=0}^{n-1}(-1)^j\frac{\mathsf{C}^n_k}{\Gamma(n)}\frac{\mathscr{B}^\star_1(k+j-n)}{(j-n)!} & \text{for } \ j \geq n. \end{cases} \tag{A.9.6}$$

Next for $n \geq 2$ even, we have

$$\mathsf{p}_j(\mathbb{C}\mathbf{P}^n) = \begin{cases} \dfrac{\Gamma(n-j)}{\Gamma(n)}\mathsf{d}^n_{n-2-j} & \text{for } \ 0 \leq j \leq n-2, \\ \displaystyle\sum_{k=0}^{n-2}(-1)^j \frac{\mathsf{d}^n_k}{\Gamma(n)} \frac{\mathscr{B}_2(k+j-n)}{(j-n)!} & \text{for } \ j \geq n. \end{cases} \tag{A.9.7}$$

$$\mathsf{q}_j(\mathbb{C}\mathbf{H}^n) = \begin{cases} \dfrac{\Gamma(j-n)}{\Gamma(n)}\mathsf{D}^n_{n-2-j} & \text{for } \ 0 \leq j \leq n-2, \\ \displaystyle\sum_{k=0}^{n-2}(-1)^{k+1} \frac{\mathsf{D}^n_k}{\Gamma(n)} \frac{\mathscr{B}_2(k+j-n)}{(j-n)!} & \text{for } \ j \geq n. \end{cases} \tag{A.9.8}$$

The case $\mathscr{X}_c = \mathbb{H}\mathbf{P}^n$ vs. $\mathscr{X}_{nc} = \mathbb{H}\mathbf{H}^n$. Here we have for $n \geq 1$

$$\mathsf{p}_j(\mathbb{H}\mathbf{P}^n) = \begin{cases} \dfrac{\Gamma(2n-j)}{\Gamma(2n)}\mathsf{e}^n_{2n-1-j} & \text{for } \ 0 \leq j \leq 2n-1, \\ \displaystyle\sum_{k=0}^{2n-1}(-1)^j \frac{\mathsf{e}^n_k}{\Gamma(2n)} \frac{\mathscr{B}_1(k+j-2n)}{(j-2n)!} & \text{for } \ j \geq 2n. \end{cases} \tag{A.9.9}$$

$$\mathsf{q}_j(\mathbb{H}\mathbf{H}^n) = \begin{cases} \dfrac{\Gamma(2n-j)}{\Gamma(2n)}\mathsf{E}^n_{2n-1-j} & \text{for } \ 0 \leq j \leq 2n-1, \\ \displaystyle\sum_{k=0}^{2n-1}(-1)^j \frac{\mathsf{E}^n_k}{\Gamma(2n)} \frac{\mathscr{B}^\star_1(k+j-2n)}{(j-2n)!} & \text{for } \ j \geq 2n. \end{cases} \tag{A.9.10}$$

The case $\mathscr{X}_c = \mathbf{P}^2(Cay)$ vs. $\mathscr{X}_{nc} = \mathbf{H}^2(Cay)$. Here we have

$$\mathsf{p}_j(\mathbf{P}^2(Cay)) = \begin{cases} \dfrac{\Gamma(8-j)}{\Gamma(8)}\mathsf{f}_{7-j} & \text{for } \ 0 \leq j \leq 7, \\ (-1)^j \displaystyle\sum_{k=0}^{7} \frac{\mathsf{f}_k}{\Gamma(8)} \frac{\mathscr{B}_1(k+j-8)}{(j-8)!} & \text{for } \ j \geq 8. \end{cases} \tag{A.9.11}$$

$$\mathsf{q}_j(\mathbf{H}^2(Cay)) = \begin{cases} \dfrac{\Gamma(8-j)}{\Gamma(8)}\mathsf{F}_{7-j} & \text{for } \ 0 \leq j \leq 7, \\ \displaystyle\sum_{k=0}^{7}(-1)^j \frac{\mathsf{F}_k}{\Gamma(8)} \frac{\mathscr{B}^\star_1(k+j-8)}{(j-8)!} & \text{for } \ j \geq 8. \end{cases} \tag{A.9.12}$$

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