Structure preserving schemes and kinetic models for approximating measure valued solutions of hyperbolic equations

By

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Declaration

I hereby declare that this thesis has not been and will not be, submitted in whole or in part another University for the award of any other degree.

Signature . . . .

Ioannis Gkanis

Dedication

To my parents, brother and grandmother;

Euaggelia, Dimitris, Thodoris and Panagiota

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SUMMARY

In this thesis we consider approximate schemes and models for *hyperbolic conservation laws*. Systems of conservation laws are fundamental mathematical models and have received a lot of attention from the point of view of analysis, modelling and computations. They include the wave equations in elastic media and fundamental equations in fluid mechanics. We consider structure preserving schemes and kinetic models for approximating measure valued solutions of hyperbolic equations. Such solutions are of interest given their application to problems in *uncertainty quantification* and in *statistical inference*. This thesis contains new results on (i) the design of new schemes for the computation of entropy consistent approximations, with particular emphasis on the consistency of the computational algorithms to entropic measure valued solutions for HCL, (ii) the introduction of discrete and generalised kinetic models designed to directly approximate measure valued solutions by using a combination of approximate Young measures and the kinetic formulation of the conservation law and (iii) stability analysis of generalised viscus kinetic models. We obtain uniqueness within a particular class of vanishing viscosity limits of these models and of their corresponding measure valued solutions.

Contents

List of Abbrevations

HCL	Hyperbolic Conservation Laws
MHD	Magnetohydrodynamics
DCT	Dominated Convergence Theorem
BGK	Bhatnagar-Gross-Krook kinetic model

Notation

Here U is a bounded domain or $\mathbb{R}^d$, $S \subset \mathbb{R}^d$ where $d \in \mathbb{N}$ and $p \in \mathbb{N}$.

(i) $L^p(U) = \{u : U \mapsto \mathbb{R}^d | u$, is Lebesque measurable, $\|u\|_{L^p(U)} < \infty\}$

(ii) $L^\infty(U) = \{u : U \mapsto \mathbb{R}^d | u$, is Lebesque measurable, $\|u\|_{L^\infty(U)} < \infty\}$

(iii) Du is the weak derivative of u

(iv) $H^1(U) = \{u : U \mapsto \mathbb{R}^d | u \in L^2(U)$ and $Du \in L^2(U)\}$

(v) $W^1_\infty(U) = \{u : U \mapsto \mathbb{R}^d | u \in L^\infty(U)$ and $Du \in L^\infty(U)\}$

(vi) $C^p(U) = \{u : U \mapsto \mathbb{R}^d | u$ is p many times continuously differentiable$\}$

(vii) $C^{0,\gamma}$ is the space of Holder continuous functions with expenent γ

(viii) $C^p_c(U) = \{u : U \mapsto \mathbb{R}^d | u$ is p times continuously differentiable with compact support$\}$

(ix) $C_0(\mathbb{R}^d) = \{u : \mathbb{R}^d \mapsto \mathbb{R}^d | u$ is a continuous function which vanishes at infinity$\}$

(x) $C^\infty(U) = \{u : U \mapsto \mathbb{R}^d | u$ is infinitely many times continuously differentiable$\}$

(xi) $C^\infty_c(U) = \{u : U \mapsto \mathbb{R}^d | u$ is infinitely many times continuously differentiable with compact support$\}$

(xii) $L^p((0,+\infty);L^p(U)) = \{u : (0,+\infty) \mapsto L^p(U)|u(.,x)$ is a measurable function and

$\int_0^\infty \|u(t)\|_{L^p(U)}dt < \infty\}$

(xiii) $L^\infty((0,+\infty);L^p(U)) = \{u : (0,+\infty) \mapsto L^p(U)|u(.,x)$ is a measurable function and

$\operatorname{ess\,sup}_{t\in(0,+\infty)} \|u(t)\|_{L^p(U)} < \infty\}$

(xiv) $L^1(U;C_c(S)) = \{u : U \mapsto C_c(S)|u(.,x)$ is a measurable function and

$\int\limits_U \max\limits_{y\in S} \|u(y)\|_{L^p(U)}dy < \infty\}$

(xv) $C_c((0,+\infty);L^p(U)) = \{u : (0,+\infty) \mapsto L^p(U)|u(.,x)$ is continuous function with compact support and $\max\limits_{t\in(0,+\infty)} \|u(t)\|_{L^p(U)} < \infty\}$

(xvi) $L^\infty_w(U;\mathbf{M}^{\mathbb{P}}(S)) = \{\mu : U \mapsto M^{\mathbb{P}}(S)$ such that the function $x \mapsto< \mu(x), \phi >$ is measurable for all $\phi \in C_0(S)\}$

(xvii) $L^\infty((0,+\infty);H^1(U)) = \{u : (0,+\infty) \mapsto H^1(U)|u(.,x)$ is a measurable function and

$\operatorname{ess\,sup}_{t\in(0,+\infty)} \|u(t)\|_{L^p(U)} < \infty\}$

(xviii) $\mathbf{M}^+(\mathbb{R}^d)$ is the set of all positive Radon measures on $\mathbb{R}^d$

(xix) $\mathbf{M}^{\mathbb{P}}(\mathbb{R}^d) = \{\mu \in \mathbf{M}^+(\mathbb{R}^d), \mu(\mathbb{R}^d) = 1\}$

(xx) $\mathbf{Y}(U,\mathbb{R}^d)$ is the set of all Young measures

(xxi) $\mathbf{Y}_h(U,\mathbb{R}^d)$ is the set of all computational Young measures

(xxii) δ_u is the Dirac measure

(xxiii) $\mathcal{D}'$ is the dual space of distributions

(xxiv) S_h finite dimensional subspace of $C(S)$

(xxv) X_h is a standard conforming finite element space consisting of continuous piecewise polynomial functions

(xxvi) $\mathbb{P}_\ell$ is the set of all polynomials of degree ℓ

(xxvii) $P_{X_h} : L^2 \mapsto X_h$ is the L^2 projection operator onto X_h

(xxviii) $f_{\#}\mu$ is the pushforward measure defined by a given measure μ and a measurable function f

(xxix) $\mathcal{X}_A$ is the characteristic function of a given set A

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Chapter 1

Introduction

In this thesis we consider approximate schemes and models for *hyperbolic conservation laws* (HCL) with emphasis to approximations of measure valued solutions. HCL are partial differential equations of the form

$$u_t + \nabla_x \cdot A(u) = 0 \tag{1.1}$$

where $u = u(x,t) : \overline{\Omega} \mapsto \mathbb{R}^m$ with $\overline{\Omega} = \mathbb{R}^d \times [0, +\infty)$. Systems of conservation laws are fundamental mathematical models and have received a lot of attention from the point of view of analysis, modelling and computations. HCL include, for instance, the wave equations in elastic media, fundamental equations in fluid mechanics (Euler system, shallow water equations), and the magnetohydrodynamics (MHD) equations of plasma physics.

It is well known that the solutions of system (1.1) may develop discontinuities in finite time even in the case where the initial condition is smooth-see[15, 39, 34]. Thus *weak solutions*, i.e., solutions with lower regularity requirements, are introduced. However, weak solutions *are not necessarily unique*. In fact, small perturbations inducing smoothness into (1.1), yield approximating models which capture possibly, completely different "solutions" at the limit as pertrubations tend to zero. This is one of the most important and interesting challenges related to such problems. Generally, it is hoped that we can possibly distinguish a physically relevant solution out of, in some cases, infinitely many weak solutions, by enforcing additional eligibility criteria. To this end, it is typical to augment the system (2.1) with additional conditions. In the scalar case (i.e. when $m = 1$), additional *entropy conditions* provide a complete theory of existence and uniqueness, stability estimates and characterisation of viscosity limits, see [15]. On the contrary, for systems ($m \geq 2$), well-posedness results after the formation of the singularities, are not yet available with the exception of the one dimensional systems ($d = 1$). However, still the notion of entropy solutions remains important, especially in applications where mathematical justifications are necessary in computational modelling.

In addition to weak solutions, one may consider a more general framework of solutions for (2.2) based on parametrised measures, with respect to x and t. *Measure-valued solutions* were introduced by DiPerna [19] based on the theory of Young measures. Already important in other areas of applied analysis and notably in the calculus of variations [5], Young measures and compensated compactness become quite useful tools for the study of HCL, yielding exciting analytical results, mainly by studying the possibility of the measure μ to collapse to a parametrised Dirac mass of the form $\mu_{x,t} = \delta_{u(x,t)}$, (atomic measure) where in this case one would like to link the function u to an entropy solution of the conservation law-see [15] for a review.

In addition to their novelty as a mathematical theory, measure valued solutions, play a crucial role in *uncertainty quantification* and *statistical inference* for hyperbolic systems. One of the important aspects of the computational modelling associated to problems of the form (1.1) originates in the behaviour of approximations of solutions not always being certain. Uncertainties in the solution can be caused, for instance, by the initial data, or the parameters appearing in the model. A similar problem from a mathematical perspective relates to statistical inference on the solutions when we study the behaviour of an assembly of variable data of the model (e.g. we would like to have a statistical information on the behavior of the solution with many different initial conditions). Several works have been devoted to algorithms computing measure valued and statistical solutions, e.g., [38, 24, 18, 21, 2, 3, 1], giving rise to new mathematical problems. One of them is related to the fact that the notion of entropic measure-valued solutions is rather weak when non-atomic measures are considered. In fact, uniqueness is lost even in the scalar case, when non-atomic measures were allowed in the initial data, [15, 21]. Recent results suggest the use of a stronger notion of measure valued solutions, namely of *statistical solutions,* where one considers solutions which are probability measures on *function spaces,* such as $L^2(\Omega)$, [22], see also [25] for statistical solutions in the Navier-Stokes setting. In [22] was proved that statistical solutions can be viewed as well as an infinite sequence of correlated parametrised Young measures. In addition, a uniqueness result for appropriate statistical solutions was established. The numerical algorithms for the computation of measure valued and statistical solutions for HCL are, to date, mainly based on Monte Carlo sampling, i.e., on solving several deterministic problems and sampling the results. For such algorithms, it is croucial to guarantee that the deterministic solvers retain certain stability and entropy consistency properties.

This thesis contains new results mainly on (i) the design of new schemes for the computation of entropy consistent approximations, with particular emphasis on the consistency of the computational algorithms to entropic measure valued solutions for HCL, (ii) the introduction of discrete and generalised kinetic models designed to directly approximate measure valued solutions by using a combination of approximate Young measures and the kinetic formulation of the conservation law and (iii) stability analysis of generalised viscus kinetic models. We obtain uniqueness within a particular class of vanishing viscosity limits of these models and of their corresponding measure valued solutions.

Outline and summary of the results

In Chapter 2 we present key notions and corresponding notation for the study of HCL, including week and entropy solutions, measure valued solutions, kinetic formulations of the scalar conservation law, generalised kinetic formulations and connections to measure valued solutions.

Chapter 3 is devoted to the introduction and and analysis of new *entropy consistent* finite element schemes for time dependent systems of hyperbolic conservation laws. Entropy stability is a key property of the numerical scheme which is the discrete analog of the entropy inequality assumed for the system. Such schemes developed so far in the classic works of Tadmor [49, 50] (see also the surveys [51, 52]), Johnson, Hansbo and Szepessy [30, 31, 28], and their collaborators and start from a class of appropriate entropy conservative schemes. Then entropy diminishing schemes are obtained by adding appropriate artificial diffusion terms. This program was based on the reformulation of the HCL using the *entropy variables*, and this is the approach taken, as well, in modern works on the subject [23, 29, 21]. Our approach has a starting point a new mixed-type formulation of the hyperbolic system which does not replace the original variables. This formulation allows direct discretisation of the original variables and at the same time leads naturally to entropy conservative schemes. Significant flexibility is allowed in the design of the corresponding entropy stable computational algorithms. New finite element schemes are introduced and analysed. It is shown that the resulting approximations converge to an entropy week and when appropriate to an entropy measure valued solution. We consider approximating methods, which are based on a space-time finite element discretization with piecewise polynomials of degree one and zero in time. Our schemes and results can be extended to high-order elements as well. They can be also extended when discretisation in space is done using discontinuous Galerkin methods, e.g., [13, 14, 12]. In Section 3.6 below we show stability estimates yielding the entropy stability of the scheme. In Section 3.6 we derive first the entropy stability estimate for our scheme, Lemma 3.6.1. In Section 3.7, Theorem 3.7.1, we prove that assuming that $u_h \to u$ then u is an entropy solution of the conservation law. Section 3.8 is devoted to measure valued solutions of HCL. Our focus is the notion of measure valued solution of Di Perna and we prove that the numerical method is indeed compatible with this notion at the limit. In fact, Theorem 3.8.2 shows that approximating sequences obtained by our scheme generate an entropic measure valued solution of HCL. The results in Section 3.8 were obtained under the assumption that the approximations were uniformly bounded in L^∞. A more refined approach is possible for certain entropies. In fact, for entropies with u^p growth in Section 3.9 we show that our approximations are uniformly bounded in L^p and thus one can apply the L^p- based theory of measure valued solutions, [4], [17], to show that still the approximating sequences obtained by our scheme generate an entropic measure valued solution of (3.1). As far as we know, this is the first fully discrete numerical method for which such properties can be proved.

In Chapter 4, our aim is to develop a new approach to the computation of measure valued solutions and to quantify uncertainties for nonlinear hyperbolic problems, based on two key ingredients : *approximate Young measures* and *kinetic models*. We first present a framework for constructing *approximate Young measures,* based on earlier results by [48, 43]. Approximate Young measures were developed having in mind applications to calculus of variations and to energy minimisation, see [6]. We show in Section 4.4 that in the framework of conservation laws the approximation of the equation for measure valued solutions by such approximate measures, gives rise in a natural way to discrete kinetic models. These models are, however, severely under-determined. We overcome this issue by using tools from the kinetic formulation of conservation laws, see Chapter 2, [40, 45]. In the scalar case the kinetic formulation of conservation laws, [40, 45], provides an interesting connection to parametrised Young measures and to compensated compactness. This connection was further developed in [42, 46, 45] for scalar laws and in [16] where kinetic formulations were the analytical basis to study conservation laws with stochastic forcing. By using viscosity approximations and appropriate discrete defect measures we construct new discrete kinetic models; their solutions will provide approximations to entropic measure valued solutions. We further note that this approach can be extended to design a hierarchy of discrete kinetic models approximating statistical solutions for scalar conservation laws based on correlation measures, see [27]. Up to our knowledge, this approach provides the first systematic alternative to Monte-Carlo sampling for approximating measure-valued solutions to conservation laws. The approximate models, at all cases, rely on solving discretised kinetic models with prescribed defect measures on the right hand side. There are several emerging questions for future research related to the mathematical analysis of such problems, such as uniqueness and stability issues mainly for kinetic (and systems thereof) approximations to continuum macroscopic models. These models are partial differential equations with discrete kinetic velocities and their design is based on sharp consistency error bounds. In Section 2.2 we show that they satisfy a discrete form of entropy inequality.

In Chapter 5 we derive a stability result for viscous generalised kinetic formulations. Although, as mentioned, entropic measure valued solutions for scalar HCL are not unique, generalised kinetic formulations cary more information due to the explicit presence of the defect measure in the model, which in the case of viscosity approximation is known explicitly. It is therefore interesting to quest whether by introducing a form of artificial diffusion at the kinetic level, and considering the defect measure on the right hand side of the equation as an appropriate nonlinear function of f, it is possible to have some guarantees that we compute in the limit a unique measure. This chapter is devoted to stability analysis for generalised kinetic models including small diffusion terms and general initial data not necessarily restricted to $\chi_{u^0(x)}$, for some function u^0 . We consider models associated to the scalar multidimensional conservation law. Generalised kinetic formulations were introduced by Perthame [44, 45] and are generalisations of kinetic formulations of conservation law [40]. In Chapter 4, we observed that such models are relevant when one would like to approximate measure valued solutions through approximate Young measures. However, the study of generalised viscous kinetic formulations is of interest, even one considers alternative approaches, such as Monte Carlo sampling, based on standard schemes for approximating the conservation law, when such schemes include a form of artificial diffusion. Our main result, Theorem 5.2.2, implies *uniqueness within a class* under structural assumption hypotheses on the defect measure m'. This result essentially states that all viscous generalised kinetic functions have the same limit as soon as $\|B_\epsilon\|_{W_1^\infty(\mathbb{R}^d)} \to 0$, $\epsilon \to 0$, where B_ϵ is the diffusion tensor and the defect measures satisfy a dissipative structural assumption. These assumptions are to some extend generalisations of properties appearing in the analysis of [44, 45] for initial data $\chi_{u^0(x)}$.

Chapter 2

Hyperbolic conservation laws

Partial differential equations of the form

$$u_t + \nabla_x \cdot A(u) = 0 \tag{2.1a}$$

where $u = u(x,t) : \overline{\Omega} \mapsto \mathbb{R}^m$ with $\overline{\Omega} = \mathbb{R}^d \times [0, +\infty)$, are called a *systems of conservation laws.* Such systems are one of the most important class of mathematical models and has received a lot of attention both from the point of view of analysis and from the point of view of computational modelling. Formally, assuming for the moment that u is a smooth function, if we integrate (2.1a) over a bounded domain $K \subset \mathbb{R}^d$ we get

$$\frac{d}{dt}\int\limits_K u(x,t)dx + \int\limits_K \nabla_x \cdot A(u(x,t))dx = 0,$$

and therefore using the divergence theorem of vector calculus we obtain

$$\frac{d}{dt}\int\limits_K u(x,t)dx + \int\limits_{\partial K} (A(u(x,t)) \cdot \mathbf{n})ds = 0$$

with n to be the outward pointing unit normal at each point on the boundary ∂K. If u represents a physical quantity per unit , then the last relation indicates us that the rate of change with respect to time of the total amount of the quantity over K is equal and dependent only to the flux on the boundary ∂K. We call $A : \mathbb{R}^m \mapsto \mathbb{R}^{m\times d}$ the flux of the conservation law. The system (2.1a) is called hyperbolic if also the flux Jacobian $\nabla_u(A \cdot n)$ has real eigenvalues for each direction n. In addition we complement the conservation law with an initial condition of the form

$$u(x, 0) = u^0(x). \tag{2.1b}$$

In nature there exists a huge variety of physical phenomena that can be modelled and understood completely or to some extend through systems conservation laws. We mention for instance, the wave equations in elastic media, the shallow water equations of oceanography, the Euler equations for gas dynamics and the magnetohydrodynamics (MHD) equations of plasma physics. These equations are hyperbolic systems of the form (2.1), see [15] for a comprehensive exposition of the mathematical theory for such systems.

2.1 Weak and entropy solutions

It is well known-see[15, 39, 34] that the solutions of system (2.1) can develop discontinuities in finite time even in the case where the initial condition is smooth. For this reason we need to introduce *weak solutions*. This framework equips us with the suitable notions that enable us, at least in some important cases, to successfully interpret and understand discontinuous solutions of (2.1).

Definition 2.1.1 *We call a function $u \in L^\infty(\Omega)$ weak solution of the problem (2.1) if it fulfils*

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \partial_t\phi(x,t)\, u(x,t) + \nabla_x\phi(x,t) \cdot A(u(x,t))dxdt + \int_{\mathbb{R}^d} \phi(x,0)u_0(x)dx = 0 \tag{2.2}$$

for all test functions $\phi(x,t) \in C_c^\infty(\overline{\Omega})$. Weak solutions *are not necessarily unique* though. This is one of the most important and interesting challenges related to such problems. It is hoped that we can possibly distinguish a physically relevant solution out of, in some cases, infinitely many weak solutions, by enforcing additional eligibility criteria. To this end, it is typical to augment the system (2.1) with an additional condition.

Definition 2.1.2 *A pair of functions $\eta - Q$ with $\eta : \mathbb{R}^m \mapsto \mathbb{R}$ convex and $Q : \mathbb{R}^m \mapsto \mathbb{R}^d$ is called entropy pair if the following relation holds*

$$\eta_u(u)\nabla_u A(u) = \nabla_u Q(u). \tag{2.3}$$

Definition 2.1.3 *A weak solution of (2.2) is called an entropy solution if in addition satisfies*

$$-\int\limits_{\Omega} \Big(\eta(u)\phi_t + Q(u)\nabla_x \phi\Big) dxdt - \int\limits_{\mathbb{R}^d} \eta(u_0) \cdot \phi(0,x) dx \leq 0, \tag{2.4}$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$.

In the scalar case (i.e. when $m = 1$) every convex function paired with the entropy flux $Q(u) = \int^u \eta'(\xi)A'(\xi)d\xi$ can be considered as an admissible entropy pair for the problem (2.2)-(2.4). The fact that any convex function is an admissible entropy in the scalar case led to a complete theory of existence and uniqueness, Kružkov L^1-stability estimates and characterisation of viscosity limits, see [15]. On the contrary, in the case of systems this plethora of entropy functions does not exist and thus results for existence and uniqueness of weak entropy solutions in one dimension have been achieved only for initial conditions whose total variation is sufficiently small, while in the multi-dimensional case well-posedness results do not yet exist. However, still the notion of entropy solutions remains important, especially in applications where mathematical justifications are necessary in computational modelling. See Chapter 2, for detailed discussion on approximating schemes consistent with entropy inequalities.

2.2 Measure valued solutions

One may consider a more general framework of solutions for (2.2) based on parametrised, with respect to x, t, measures. To this end, we consider the notion of *measure-valued solutions* introduced by DiPerna [19] based on Young measures. Therefore, [4], let $\mathbf{M}^+(\mathbb{R}^m)$ be the set of all positive Radon measures on $\mathbb{R}^m$, and $\mathbf{M}^{\mathbb{P}}(\mathbb{R}^m) = \{\mu \in \mathbf{M}^+(\mathbb{R}^m), \mu(\mathbb{R}^m) = 1\}$ the corresponding set of probability measures.

Definition 2.2.1 *We call Young measure a weakly* measurable mapping from Ω into $\mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$.*

The set of all Young measures is denoted by $\mathbf{Y}(\Omega, \mathbb{R}^m)$.

Definition 2.2.2 *A parametrised measure $\mu \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ is said to be a measure-valued solution of the conservation law* (4.1) *if, [19],*

$$\int\limits_{\Omega} \big(\langle id, \mu_{x,t}\rangle \cdot \phi_t + \langle A, \mu_{x,t}\rangle \cdot \nabla_x \phi\big) dxdt + \int\limits_{\mathbb{R}} u_0 \cdot \phi(0,x) dx = 0, \tag{2.5}$$

for all $\phi \in C_0^\infty(\overline{\Omega})$ where by $\langle A, \mu_{x,t}\rangle$ we denote

$$\langle A, \mu_{x,t}\rangle = \int_{\mathbb{R}^m} A(\lambda) d\mu_{x,t}(\lambda).$$

In a similar fashion as for weak solutions, an *entropy measure-valued solution* satisfies the additional relation

$$\int\limits_{\Omega} \big(\langle \eta, \mu_{x,t}\rangle \cdot \phi_t + \langle Q, \mu_{x,t}\rangle \cdot \nabla_x \phi\big) dxdt \geq 0 \tag{2.6}$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$, where η is convex and (η, Q) an entropy entropy-flux pair, [15].

Already important in other areas of applied analysis and notably in the calculus of variations, see e.g., [5], Young measures and compensated compactness become quite popular tools, yielding exciting analytical results, mainly by studying the possibility of the measure μ to collapse to a parametrised Dirac mass of the form $\mu_{x,t} = \delta_{u(x,t)}$, (atomic measure) where in this case one would like to link the function

u to an entropy solution of the conservation law, see [15] for a review.

Beyond the exciting mathematical theory related to the study of measure valued solutions, such solutions become relevant in *uncertainty quantification* and *statistical inference* for hyperbolic systems. One of the important aspects of the computational modelling associated to problems of the form (2.1a) is originated by the fact that the behaviour of approximations of solutions is not always certain. Uncertainties in the solution can be caused, for instance, by the initial data, or the parameters appearing in the model. A similar problem from a mathematical perspective relates to statistical inference on the solutions when we study the behaviour of an assembly of variable data of the model. Several works have been devoted to algorithms computing measure valued and statistical solutions, e.g., [38, 24, 18, 21, 2, 3, 1], giving rise to new mathematical problems. One of them is related to the fact that the notion of entropic measure-valued solutions is rather weak when non-atomic measures are considered. In fact, uniqueness is lost even in the scalar case, when non-atomic measures were allowed in the initial data, [15, 21]. See more related details in Chapters 4 and 5.

2.3 Kinetic Formulation of HCL

In this section we summarize the kinetic theory for scalar conservation laws ($m = 1$), following, [45], [15]. In the kinetic theory the state at a point x and time t is described by the density function $f(\xi, x, t)$ of the velocity ξ. *Boltzmann equation* describes the evolution of $f(\xi, x, t)$ and provides an approximate model for various macroscopic (continuum) equations, see [9]. Motivated by kinetic models, Perthame and Tadmor [47], considered an ("artificial") kinetic equation to be associated to scalar conservation laws. In the spirit of the kinetic theory one would like to consider a density function f, which, however, is allowed to take negative values by introducing a scalar-valued artificial "velocity" ξ and express the "macroscopic" variable u as

$$u(x,t) = \int_{-\infty}^{+\infty} f(\xi, x, t) d\xi. \tag{2.7}$$

The function f is determined as the $\mu \to 0$ limit of solutions of the transport equation, [47],

$$\frac{\partial}{\partial t} f(\xi, x, t) + \nabla_u A(\xi) \cdot \nabla_x f(\xi, x, t) = \frac{1}{\mu}[\chi_{u(x,t)}(\xi) - f(\xi, x, t)], \tag{2.8}$$

where μ is a small positive parameter and we are employing the notation

$$\chi_\omega(\xi) = \begin{cases} 1 & \text{if } \; 0 < \xi \leq \omega \\ -1 & \text{if } \; \omega \leq \xi < 0 \\ 0 & \text{otherwise.} \end{cases}$$

The model (2.8) is similar to BGK model approximating the classical Boltzmann equation, [45]. Formally at least, the $\mu \to 0$ limits of solutions of (2.8) will satisfy

$$f(\xi, x, t) = \chi_{u(x,t)}(\xi), \quad \xi \in \mathbb{R} \ , \quad t \in [0, +\infty) \ , \tag{2.9}$$

so that f will be uniformly distributed on the interval with end-points 0 and u, with value -1 or $+1$. The statement of the following result is from [15]; it was first derived in [47].

Theorem 2.3.1 *Assume* $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. *For any* $\mu > 0$, *there exist bounded measurable functions* (f, u), *with*

$$f(\cdot, \cdot, t) \in C^0([0, +\infty); L^1(\mathbb{R} \times \mathbb{R}^d)), \quad u(\cdot, t) \in C^0([0, +\infty); L^1(\mathbb{R}^d)), \tag{2.10}$$

which provide the unique solution of (2.7), (2.8) *under the initial condition*

$$f(\xi, x, 0) = \chi_{u_0(x)}(\xi)\,, \quad \xi \in \mathbb{R}\,, \quad x \in \mathbb{R}^d. \tag{2.11}$$

Moreover,

$$0 \leq f(\xi, x, t) \leq 1 \ \ for \ \ \xi \geq 0, \qquad -1 \leq f(\xi, x, t) \leq 0 \ \ for \ \ \xi \leq 0. \tag{2.12}$$

If $\bar{u_0} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *are other initial data inducing the solution* $(\bar{f}, \bar{u})$*, then, for any* $t > 0$,

$$\|f(\cdot,\cdot,t) - \bar{f}(\cdot,\cdot,t)\|_{L^1(\mathbb{R}\times\mathbb{R}^d)} \leq \|f(\cdot,\cdot,0) - \bar{f}(\cdot,\cdot,0)\|_{L^1(\mathbb{R}\times\mathbb{R}^d)}, \tag{2.13}$$

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u(\cdot,0) - \bar{u}(\cdot,0)\|_{L^1(\mathbb{R}^d)}. \tag{2.14}$$

As $\mu \to 0$ the family (f_μ, u_μ) converges in L^1_{loc} to bounded measurable functions (f, u) such that f satisfies the transport equation

$$\frac{\partial}{\partial t} f(\xi, x, t) + \nabla_u A(\xi) \cdot \nabla_x f(\xi, x, t) = \frac{\partial m}{\partial \xi} \tag{2.15}$$

in $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R} \times (0, +\infty))$ for some nonnegative measure m with (2.11), (2.9) to hold and u is the admissible weak solution of (2.5) and (2.6). In setting up the transport equation (2.8), the role of the stiff term $\mu^{-1}(\chi_u - f)$ is to enforce, in the limit, (2.9). Other "collision" terms can lead to solutions of (2.5) , (2.6) as well. The equivalence of (2.15) to (2.5), (2.6) was studied in detail by Lions, Perthame and Tadmor, [40], see also [45] for a very detailed account on the theory.

Theorem 2.3.2 (Kinetic formulation of the scalar CL) *A bounded measurable function* u *on* $\mathbb{R}^d \times [0, +\infty)$ *with* $u(\cdot, t) \in C^0([0, +\infty); L^1(\mathbb{R}^d)$*, is the admissible weak solution of* (2.5) , (2.6) *if and only if the function* f *defined through* (2.9) *satisfies the transport equation* (2.15)*, for some nonnegative measure* m*, together with the initial condition* (2.11).

2.4 Generalised kinetic solutions and Young measures

The kinetic formulation can be generalised to handle distributional solutions of (2.15). The following definition from [45] will be very useful.

Definition 2.4.1 *A function $f(x,t,\xi) \in L^\infty(0,+\infty; L^1(\mathbb{R}^{d+1}))$ is called a* generalized kinetic solution *of the scalar conservation law with initial data f_0, if the following holds. For all $\phi \in D([0,+\infty) \times \mathbb{R}^d \times \mathbb{R})$ we have*

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^{d+1}} f(t,x,\xi)\left[\frac{\partial}{\partial t}\phi(x,t,\xi) + \nabla_u A(\xi)\cdot\nabla_x\phi(x,t,\xi)\right] dx d\xi dt \\ = \int_0^\infty \int_{\mathbb{R}^{d+1}} m(t,x,\xi)\frac{\partial\phi(x,t,\xi)}{\partial\xi} dx d\xi dt - \int_{\mathbb{R}^{d+1}} f_0(x,\xi)\phi(0,x,\xi) dx d\xi\end{aligned} \tag{2.16}$$

for some nonnegative measure m, which for a bounded function which vanishes at infinity $\mu(\xi)$ satisfies $\int_{\mathbb{R}^d\times\mathbb{R}} m(t,x,\xi)dtdx \le \mu(\xi)$. In addition there exists a nonnegative measure ν such that

$$|f(x,t,\xi)| = sgn(\xi) f(x,t,\xi) \le 1 \tag{2.17a}$$

$$\partial_\xi f(t,x,\xi) = \delta_0(\xi) - \nu(t,x,\xi) \tag{2.17b}$$

where

$$sgn(\xi) = \begin{cases} 1 & \text{if} \quad 0 < \xi \\ -1 & \text{if} \quad \xi < 0. \end{cases}$$

The relationship between f and ν, [40][Remark, p. 178], establishes a connection between the kinetic formulation and the entropic measure valued solutions in the scalar case. This relationship was studied in detail by Perthame and Tzavaras [46]. Notice that (2.17b) can be written for any test function φ as

$$\int f\varphi'(\xi)d\xi = \int_{\mathbb{R}} \varphi(\xi) d\nu_{x,t}(\lambda) - \varphi'(0)\,, \tag{2.18}$$

and

$$f = \int_{\mathbb{R}} \chi_\lambda(\xi) d\nu_{x,t}(\lambda). \tag{2.19}$$

It is quite interesting to note that $\nu_{x,t}$ being atomic, and in particular $\nu_{x,t} = \delta_{u(t,x)}$ corresponds to the case where $f(t,x,\xi) = \chi_{u(t,x)}(\xi)$. Generalised kinetic formulations and relationship (5.6c) will be instrumental in Chapters 4 and 5.

Chapter 3

A New Class of Entropy Stable Schemes for Hyperbolic Systems

3.1 Chapter overview

In this chapter we introduce and analyse new finite element schemes for time dependent systems of hyperbolic conservation laws (HCL)

$$\begin{aligned} \partial_t u(x,t)+\partial_x A(u(x,t)) &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where the solution $u(x,t) = (u_1(x,t), \ldots, u_m(x,t)), m \geq 1$ is vector valued, $u : \mathbb{R} \to \mathbb{R}^m$. The flux $A : \mathbb{R}^m \to \mathbb{R}^m$ is assumed to belong to $C^1(\mathbb{R}^m)$ and the initial value $u_0 \in [L_2(\mathbb{R})]^m$ is a given function with compact support. At this point we are not precise regarding the notion of the solution considered for the system. Our results apply to weak and measure valued solutions. Further, the schemes and most results can be extended to the multidimensional case as well, however we have restricted our attention to the one dimensional case for simplicity in the exposition.

Our aim is to propose a new class of entropy stable schemes for the system (3.1). Entropy stability is a key property of the numerical scheme which is the discrete analog of the entropy inequality assumed for the system. Such schemes developed so far in the classic works of Tadmor [49, 50] (see also the surveys [51, 52]), Johnson, Hansbo and Szepessy [30, 31, 28], and their collaborators and start from a class of appropriate entropy conservative schemes. Then entropy diminishing schemes are obtained by adding appropriate artificial diffusion terms. This program is based on the reformulation of the HCL using the entropy variables, and this is the approach taken, as well, in modern works on the subject [23, 29, 21]. Entropy conservative schemes are quite interesting on their own, since among others, provide physically relevant approximations to dispersive shocks, nonclassical shocks and other important systems, [37, 10, 35, 36, 20].

Our approach has a starting point a new mixed-type formulation of the hyperbolic system which does not replace the original variables. This formulation allows direct discretisation of the original variables and at the same time leads naturally to entropy conservative schemes. Significant flexibility is allowed in the design of the corresponding entropy stable computational algorithms. New finite element schemes are introduced and analysed. It is shown that the resulting approximations converge to an entropy weak and when appropriate to an entropy measure valued solution.

3.2 Motivation.

Consider the scalar conservation law

$$u_t + A(u)_x = 0.$$

Assume for the moment that u is smooth and η, Q is an entropy-entropy flux pair, i.e., $Q' = A'\eta'$. Then

$$u_t(t)\eta'(u) + A'(u)\, u_x\, \eta'(u) = u_t(t)\eta'(u) + \, u_x\, Q'(u) \, = \eta(u)_t + Q(u)_x$$

and thus

$$\eta(u)_t + Q(u)_x = 0.$$

Integrating now with respect to x and t while assuming that Q tents to zero at infinity we have

$$\int_0^t \int_{\mathbb{R}} \eta(u)_{t'} dx dt' + \int_0^t \int_{\mathbb{R}} Q(u)_x dx dt' = 0$$

which implies

$$\int_0^t \int_{\mathbb{R}} \eta(u)_{t'} dx dt' = 0.$$

Hence

$$\int_{\mathbb{R}} \eta(u(x,t))\, dx = \int_{\mathbb{R}} \eta(u(x,0))\, dx\,. \tag{3.2}$$

Consider a Galerkin scheme (discretisation only in x): Seek $u_h : [0,T] \to X_h$ such that

$$\int_{\mathbb{R}} (u_{h,t}(x,t),\, \varphi(x))\, dx + \int_{\mathbb{R}} (A(u_h(x,t))_x,\, \varphi(x))\, dx = 0 \qquad \text{for all } \varphi \in X_h. \tag{3.3}$$

Here X_h is a standard conforming finite element space consisting of continuous piecewise polynomial functions (for precise definitions see below). A key observation is that this scheme *is not entropy conservative,* i.e., it does not satisfy the discrete analog of (3.2). The main reason is that $\eta'(u_h)$ is not eligible test function anymore. Using a projection will inevitably introduce errors, which at the end will destroy the equality in (3.2).

The reformulation of the conservation law using entropy variables fixes this problem and yields entropy conservation at the discrete level. In fact, since η is convex, η' is invertible. Define a new variable v such that $v = \eta'(u)$. Then for $\kappa = \left[\eta'\right]^{-1}$ we have

$$u = \kappa(v), \qquad v = \eta'(u)\,.$$

The formulation of the HCL in entropy variables reads,

$$\kappa(v)_t + G\,(v)_x = 0\,, \qquad G(v) = A(\kappa(v)).$$

Now it is a simple matter to check that the corresponding Galerkin discretisation : Seek $v_h : [0,T] \to X_h$ such that

$$\int_{\mathbb{R}} (\kappa(v_h)_t,\, \varphi)\, dx + \int_{\mathbb{R}} (G\,(v_h)_x,\, \varphi)\, dx = 0 \qquad \text{for all } \varphi \in X_h,$$

is entropy conservative. Although simplified, this discussion highlights what one gains using the entropy variables at the discrete level. As mentioned earlier these variables are used in the classical works both for finite difference/volume as well as for finite element methods, [49, 50, 30, 31], and it seems that it is the only viable way to obtain entropy conservative schemes to day. There are some limitations to this approach. One can mention that although it can be recovered, the original variable does not belong to the discrete space and it is available only through v_h, the convexity of η is absolutely essential and considering natural viscosity in the schemes is rather involved.

Entropy conservative schemes revised: A mixed formulation. Our approach is to based to a formulation that keep both variables:

$$\begin{aligned} u_t + G\,(\tau)_x &= 0, \qquad G(\tau) = A(\kappa(\tau)), \\ \tau &= \eta'(u)\,. \end{aligned}$$

Of course, this writing is equivalent to HCL, but the corresponding discretisation scheme is new: Seek $u_h, \tau_h : [0,T] \to X_h$ such that

$$\begin{aligned} \int_{\mathbb{R}} (u_{h,t},\, \varphi)\, dx + \int_{\mathbb{R}} (G\,(\tau_h)_x,\, \varphi)\, dx &= 0 \qquad \text{for all } \varphi \in X_h, \\ \int_{\mathbb{R}} (\tau_h,\, \psi)\, dx - \int_{\mathbb{R}} (\eta'(u_h),\, \psi)\, dx &= 0 \qquad \text{for all } \psi \in X_h. \end{aligned} \tag{3.4}$$

One can check that this scheme is entropy conservative (the proof follows by modifying appropriately the proof of Lemma 3.6.1 below and it is omitted). This scheme an be used as basis for the design of several “advanced” high-order discretisations.

By keeping both variables we allow for flexible flux-function choices. In certain systems is possible to rewrite the HCL in a form where the flux depends on both variables,

$$\begin{aligned}\tilde{u}_t + H\left(\tilde{\tau}, \tilde{u}\right)_x =& 0,\\ \tilde{\tau} =& \eta'(\tilde{u}).\end{aligned}$$

The corresponding Galerkin discretisation is then: Seek $u_h, \tau_h : [0,T] \to V_h$ such that

$$\begin{aligned}\int_{\mathbb{R}} (u_{h,t},\, \varphi)\, dx + \int_{\mathbb{R}} (H\left(\tau_h, u_h\right)_x,\, \varphi)\, dx =& 0 \qquad \text{for all } \varphi \in V_h,\\ \int_{\mathbb{R}} (\tau_h,\, \psi)\, dx - \int_{\mathbb{R}} (\eta'(u_h),\, \psi)\, dx =& 0 \qquad \text{for all } \psi \in V_h.\end{aligned} \tag{3.5}$$

A key advantage of the later formulation is that the inversion of η' is not required. Mixed formulations of the form similar to (3.5) go back to [26] where the first energy consistent numerical methods for the Navier- Stokes-Korteweg system were introduced. Notably, for this system the energy function is not convex. In this work we focus on schemes whose design is based on (3.4). It is evident however that our results can be extended to systems of the form (3.5) under appropriate structural hypotheses.

3.3 Notation.

Going back to the system case we assume that (3.1) is equipped with an entropy-entropy flux pair (η, Q) where $Q : \mathbb{R}^m \to \mathbb{R}, \eta : \mathbb{R}^m \to \mathbb{R}$ convex and the following relation holds

$$\eta_u(u)\nabla_u A(u) = Q_u(u), \tag{3.6}$$

where by $\nabla_u A$ we denote the Jacobian of A. In addition to (3.6) we assume further that $\eta_u(0) = 0$ and $Q(0) = 0$. A function $u \in [L^\infty(\Omega)]^m$, $\Omega = (0, +\infty) \times \mathbb{R}$ that fulfils the relation

$$\int\limits_{\Omega} \big(u \cdot \phi_t + A(u) \cdot \phi_x\big) dx dt + \int\limits_{\mathbb{R}} u_0 \cdot \phi(0, x) dx = 0, \tag{3.7}$$

for all $\phi \in C_0^\infty(\overline{\Omega})$ is called weak solution of (3.1).

It is well known that weak solutions of (2.1.1) are not generally unique. In the scalar case uniqueness is established by imposing extra entropy inequalities. Nevertheless, in all cases it is relevant to consider additional criteria which identify physically relevant solutions, [15]. To this end, we call *entropy solutions*, weak solutions satisfying in addition

$$-\int\limits_{\Omega} \Big(\eta(u)\phi_t + Q(u)\phi_x\Big) dx dt - \int\limits_{\mathbb{R}} \eta(u_0) \cdot \phi(0, x) dx \leq 0, \tag{3.8}$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$.

A main concern in the numerical analysis of these problems is to construct approximations consistent with (3.8) in the sense that in the limit converge in a suitable sense towards an entropy solution.

3.4 A space-time finite element method

We consider approximating methods, which are based on a space-time finite element discretization with piecewise polynomials of degree one and zero in time. Our schemes and results can be extended to high-order elements without notable technical obstructions.

Let $0 = t_0 < t_1 < t_2 < \ldots$ be a sequence of time levels, define $I_n = (t_n, t_{n+1})$ and introduce the 'slabs', $S_n = \mathbb{R} \times I_n$. To simplify matters we shall assume that all solutions and approximations have compact support. For the space discretisation we shall consider a fixed finite element decomposition of a given interval $[-B, B]$. Let X_h be the spatial finite element space consisting of continuous piecewise polynomials of degree one extended to zero outside $[-B, B]$. The solutions and approximations are implicitly assumed to vanish outside $[-B, B]$. For $n = 0, 1, 2, \ldots$, let T_h^n be a mesh of S_n into space-time rectangles K and define

$$V_h^n = \{v \in [H^1(S_n)]^m : v|_K = v_1(x) \cdot v_2(t), v_1 \in X_h,\ v_2 \in \mathbb{P}_1\}.$$

We denote by $\bar{h} = \max_{K \in S_n,\ n \geq 0} \operatorname{diam} K$, and by $h,\ h > 0$ a mesh discretisation parameter proportional to $\bar{h}$. We shall restrict to quasi-uniform meshes and we shall use the fact $h \leq C \min_{K \in S_n,\ n \geq 0} \operatorname{diam} K$. We will seek an approximate solution u_h in the space $V_h = \{\varphi : \mathbb{R} \times (0, +\infty) \to \mathbb{R} : \varphi|_{S_n} \in V_h^n, n \geq 0\}$ we will have

$$u_h|_{S_n} \in V_h^n.$$

Further,we consider the finite element space:

$$W_h^n = \{\psi \in [H^1(S_n)]^m : \psi|_K = \psi_1(x)\psi_2(t),\ \psi_1 \in X_h,\ \psi_2 \in \mathbb{P}_0\},$$

and

$$W_h = \{w : \mathbb{R} \times (0, +\infty) \to \mathbb{R} : w|_{S_n} \in W_h^n, n \geq 0\}.$$

The space W_h is needed for our mixed formulation given that by construction $\partial_t v \in W_h^n$ for all $v \in V_h^n$. In fact, although both variables in our mixed formulation belong to the same space V_h we shall use two different test spaces. The method is a discontinuous Galerkin in time-type of method much the spirit of [30, 31, 28] adapted to our mixed formulation (3.4). It is important to note that we shall require an extra compatibility condition at the discrete time nodes, see (3.9c) below. We can now define our numerical scheme: Seek $u_h,\ \tau_h \in V_h$ such that for $n = 0, 1, \ldots,$

$$\int\limits_{S_n} \Big((u_h)_t + A\big(\kappa(\tau_h)\big)_x \Big) \cdot \phi_h \, dxdt + \gamma\,(h) \int\limits_{S_n} (\tau_h)_x \cdot (\phi_h)_x \, dxdt$$
$$+ \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \phi_h^{n+} \, dx = 0, \tag{3.9a}$$

$$\int\limits_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot \psi_h \, dxdt = 0, \tag{3.9b}$$

$$\int\limits_{\mathbb{R}} \big(\tau_h^{n+} - \eta_u(u_h^{n+})\big) \cdot \psi_h^{n+} \, dx = 0, \tag{3.9c}$$

for all $\phi_h \in V_h^n(S^n)$ and $\psi_h \in W_h^n(S^n)$, where $\kappa = \eta_u^{-1}$, $u_h^{n\pm} = \lim\limits_{s \to 0^{\pm}} u_h(t^n + s)$, $\phi_h^{n+} = \lim\limits_{s \to 0^{+}} \phi_h(t^n + s)$, $\psi_h^{n+} = \lim\limits_{s \to 0^{+}} \psi_h(t^n + s)$ and $u_h^{0-} = u_0$. Notice that although the dimension of the space W_h is lower than V_h, the final system (3.9a)-(3.9c) is balanced in terms of degrees of freedom due to the presence of (3.9c). Here $\gamma = \gamma(h) > 0$, is a positive parameter, used to tune the amount of the artificial diffusion induced in the scheme with $\gamma(h) \to 0$ as $h \to 0$. A few comments are in order. First, the above scheme is quite simplified and it serves as a model to verify that our approach can indeed be theoretically justified at the finite element level. One can add various stabilisation terms (shock capturing, diffusion at combined space-time direction, etc) to it, but these technical additions do not offer much on our qualitative understanding as far as behaviour of the scheme is concerned. Further, the choice of the artificial diffusion parameter $\gamma = \gamma(h)$ is also subtle. One may include a spatial dependent term within the integral which can be tuned in an adaptive manner, or/and to include nonlinear dependence on the approximate solution, in order to ensure consistency and flexibility, but again these technical alterations which might improve the quality of the approximations, do not add much in our understanding at the theoretical level. Thus we have decided to retain the scheme in a simple form.

3.5 Main results–Remarks.

As mentioned, the strategy to design entropy stable schemes for hyperbolic conservation laws, is to build on entropy conservative discretisation. Our approach is based on a novel entropy conservative discretisation, (3.4), which was the basis of the fully discrete space-time finite element scheme presented above. The choice of different test functions provides the right discretisation at the mixed level which leads to the desired properties. This scheme introduces artificial dissipation both in space and time (explicitly in the space variable and implicitly in the time variable through the discontinuous Galerkin time discretisation). In Section 3.6 below we show stability estimates yielding the entropy stability of the scheme. We have opted for conforming space discretisation, just to make the comparison with the results of [30, 31, 28] more tractable. We have chosen to include simple artificial diffusion terms, in order to keep the analysis as simple as possible and thus to highlight the ideas needed to verify the energy consistency for the scheme. Although the general plan of the proofs follows [30, 31, 28], we had to overcome several technical obstacles due to the new formulation introduced herein. The results of this work can be extended when discretisation in space is done using discontinuous Galerkin methods, e.g., [13, 14, 12], but the formulation becomes quite technical and special care should be given on the design of the discrete fluxes, compare to [26, 29]. In Section 3.6 we derive first the entropy stability estimate for our scheme, Lemma 3.6.1. In Section 3.7, Theorem 3.7.1, we prove that assuming that $u_h \to u$ then u is an entropy solution of the conservation law. A crucial result towards this goal is the compatibility of the mixed variables at the limit proved in Lemma 3.6.4. Section 3.8 is devoted to measure valued solutions of (3.1). Our focus is the notion of measure valued solution of Di Perna and we prove that the numerical method is indeed compatible with this notion at the limit. In fact, Theorem 3.8.2 shows that approximating sequences obtained by our scheme generate an entropic measure valued solution of (3.1). Towards this goal, an important intermediate step is Lemma 3.8.1 which connects the limiting behaviour of τ_h with the limiting measure extracted from the sequence $\{u_h\}$. The results in Section 3.8

were obtained under the assumption that the approximations were uniformly bounded in L^∞. A more refined approach is possible for certain entropies. In fact for certain entropies with u^p growth in Section 3.9 we show that our approximations are uniformly bounded in L^p and thus one can apply the L^p- based theory of measure valued solutions, [4], to show that still the approximating sequences obtained by our scheme generate an entropic measure valued solution of (3.1). As far as we know, this is the first numerical method for which such property can be proved.

3.6 Stability-Preliminary results

3.6.1 Stability Estimate

We shall prove now, a stability estimate on which is essential for the forthcoming results. We shall use that the convexity of η, implies that there is a compact set $D \subset \mathbb{R}^m$ and a constant $\sigma > 0$ such that the relation

$$\eta(v) - \eta(w) - \eta_u(w) \cdot (v - w) \geq \sigma |v - w|^2 \tag{3.10}$$

holds for all $v, w \in D$ where $|.|$ is the Euclidean norm.

Lemma 3.6.1 *Suppose that the ranges of a sequence of finite element solutions of* (3.9) *lies in the compact set D i.e.*

$$\{u_h\}_{h>0} \subset D. \tag{3.11}$$

Then for $N = 1, 2, \ldots,$ we have the estimate

$$\gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L_2(S_n)} + \sigma \sum_{n=0}^{N-1} \|u_h^{n-} - u_h^{n+}\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} \eta(u_h^{N-}) \leq \int_{\mathbb{R}} \eta(u_0) \tag{3.12}$$

Remark 3.6.2 *Observe now that having in mind* (3.23)*, assumption* (3.11) *guarantees that τ_h is bounded* *on D, provided that η_u is also a bounded function on D and also $\tau_h = 0$ when $u_h = 0$.*

Proof: We achieve this by setting $\phi_h = \tau_h$ in (3.9a). We have

$$\int\limits_{S_n} \left((u_h)_t + A\big(\kappa(\tau_h)\big)_x \right) \cdot \tau_h \, dxdt + \gamma \int\limits_{S_n} \big((\tau_h)_x\big)^2 dxdt + \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} \, dx = 0$$

Thus

$$\int\limits_{S_n} (u_h)_t \cdot \tau_h dxdt + \int\limits_{S_n} \nabla_u A(\kappa(\tau_h))\, \kappa(\tau_h)_x \cdot \eta_u(\kappa(\tau_h)) dxdt + \gamma \int\limits_{S_n} \big((\tau_h)_x\big)^2 dxdt + \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} dx = 0$$

and

$$\int\limits_{S_n} (u_h)_t \cdot \tau_h dxdt + \int\limits_{S_n} \eta_u(\kappa(\tau_h)) \nabla_u A(\kappa(\tau_h)) \cdot \kappa(\tau_h)_x dxdt + \gamma \int\limits_{S_n} \big((\tau_h)_x\big)^2 dxdt + \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} dx = 0.$$

Hence, by (3.6),

$$\int\limits_{S_n} (u_h)_t \cdot \tau_h dxdt + \int\limits_{S_n} Q\big(\kappa(\tau_h)\big)_x dxdt + \gamma \int\limits_{S_n} \big((\tau_h)_x\big)^2 dxdt + \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} dx = 0.$$

Since $\kappa(0) = 0$ we conclude

$$\int\limits_{S_n} (u_h)_t \cdot \tau_h dxdt + \gamma \int\limits_{S_n} \big((\tau_h)_x\big)^2 dxdt + \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} dx = 0. \tag{3.13}$$

Setting $\psi_h = (u_h)_t$ in (3.9b) we get,

$$\int\limits_{S_n} \tau_h \cdot \partial_t u_h dxdt = \int\limits_{S_n} \eta_u(u_h) \cdot \partial_t u_h dxdt = \int\limits_{S_n} \eta(u_h)_t dxdt$$

Thus, if we substitute the first term of (3.13) with the above we get,

$$\int\limits_{S_n} \eta(u_h)_t dxdt + h \int\limits_{S_n} \left((\tau_h)_x\right)^2 dxdt + \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} dx = 0$$

and (3.9c) finally implies

$$\int\limits_{S_n} \eta(u_h)_t dxdt + \gamma \int\limits_{S_n} \left((\tau_h)_x\right)^2 dxdt + \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \eta_u(u_h^{n+}) dx = 0.$$

Summation over n implies,

$$\sum_{n=0}^{N-1} \int\limits_{\mathbb{R}} \left(\eta(u_h^{n+1-}) - \eta(u_h^{n+}) + (u_h^{n+} - u_h^{n-}) \cdot \eta_u(u_h^{n+})\right) dx + \gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L^2(S_n)} = 0.$$

Hence,

$$\sum_{n=0}^{N-1} \int\limits_{\mathbb{R}} \left(\eta(u_h^{n-}) - \eta(u_h^{n+}) - (u_h^{n-} - u_h^{n+}) \cdot \eta_u(u_h^{n+})\right) dx + \gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L^2(S_n)} + \int\limits_{\mathbb{R}} \eta(u_h^{N-}) dx = \int\limits_{\mathbb{R}} \eta(u_0) dx. \tag{3.14}$$

Using (3.10) in (3.14) we obtain (3.12).

□

A crucial step forward in order to prove the convergence of (3.9) towards an entropy solution, is to show that as

$$u_h \to u \ , \ \tau_h \to \tau \ a.e. \text{ when } h \to 0 \tag{3.15}$$

then the following statement holds,

$$\tau = \eta_u(u) \; a.e. \tag{3.16}$$

in Ω. For this purpose we need also the following estimates for the L^2 projections.

Lemma 3.6.3 *For S being a discrete space, $S = V_h, W_h$ or X_h let P_S the L^2-projection onto S. There exist constants C such that for $\omega \in [H^1(S_n) \cap C(S_n)]^m$, $\phi \in H^1(S_n) \cap C(S_n)$, $u \in V_h^n$(or W_h^n), $n = 0, 1, 2, \dots,$ and $k = 0, 1$*

$$h^k \|\omega - P_{V_h}\omega\|_{H^k(S_n)} + \sqrt{h}\|\omega^{n+} - (P_{V_h}\omega)^{n+}\|_{L^2(\mathbb{R})} \leq Ch^2|\omega|_{H^2(S_n)},$$

$$h^k \|u\phi - P_{V_h}(u\phi)\|_{H^k(S_n)} + \sqrt{h}\|(u\phi)^{n+} - (P_{V_h}(u\phi))^{n+}\|_{L^2(\mathbb{R})}$$
$$\leq Ch\|u\|_{L_\infty(S_n)}(|\phi|_{H^1(S_n)} + h|\phi|_{H^2(S_n)}) \,.$$

Further, for P_{W_h} there holds

$$\|\omega - P_{W_h}\omega\|_{L^2(S_n)} \leq Ch|\omega|_{H^2(S_n)}.$$

In addition, for a $\phi \in C_c^\infty(\Omega)$ and $u \in V_h^n$, there is a constant $C = C(\phi)$ such that, with $n = 0, 1, 2, \dots,$ there holds that

$$\|(u\phi)^{n+} - P_{X_h}(u\phi)^{n+}\|_{L^2(\mathbb{R})} \leq Ch\|u^{n+}\|_{L^2(\mathbb{R})} \,.$$

Proof: For the first two estimates see [31]. By the best approximation property of the L^2 projection and the discontinuity of the functions of W_h in time one observes

$$\begin{aligned} \|\omega - P_{W_h}\omega\|_{L^2(S_n)} \leq & \|\omega - P_{0,t}\omega\|_{L^2(S_n)} + \|P_{0,t}(\omega - P_{X_h}\omega)\|_{L^2(S_n)} \\ \leq & \|\omega - P_{0,t}\omega\|_{L^2(S_n)} + \|\omega - P_{X_h}\omega\|_{L^2(S_n)} \\ \leq & \|\omega - P_{0,t}\omega\|_{L^2(S_n)} + \|\omega - I_{X_h}\omega\|_{L^2(S_n)} \end{aligned}$$

where I_{X_h} the standard interpolation operator onto X_h, and $P_{0,t}$ the L^2 projection in time operator onto the piecewise constant functions. The second relation then follows by standard interpolation estimates and the Poincaré - Friedrichs inequality. For the last estimate, observe first that for I' being a typical element of the decomposition of the finite element space X_h, there holds,

$$\|\omega^{n+} - I_{V_h}\omega^{n+}\|_{L^2(I')} \leq Ch^2|\omega^{n+}|_{H^2(I')}$$

see [8]. Clearly $(u\phi)^{n+} \in H^2(I')$. We will denote for the rest of the proof $(u\phi)^{n+}$ as $u\phi$. The above inequality yields

$$\|(u\phi) - P_{X_h}(u\phi)\|^2_{L^2(\mathbb{R})} \leq \sum_{I'} \|(u\phi) - I_{X_h}(u\phi)\|^2_{L^2(I')} \leq Ch^4 \sum_{I'} |u\phi|^2_{H^2(I')}. \tag{3.17}$$

Next, since,

$$\partial_x^2(u\phi) = 2\partial_x\phi\partial_x u + u\partial_x^2\phi$$

we have

$$|u\phi|^2_{H^2(I')} = \int_{I'} (2\partial_x\phi\partial_x u + u\partial_x^2\phi)^2 dx \leq \int_{I'} 6(\partial_x\phi)^2(\partial_x u)^2 dx + \int_{I'} 3u^2(\partial_x^2\phi)^2 dx.$$

Therefore

$$\begin{aligned} |u\phi|^2_{H^2(I')} &\leq 6\|\partial_x\phi\|^2_{L^\infty(\mathbb{R})}\|\partial_x u\|^2_{L^2(I')} + 3\|\partial_x^2\phi\|^2_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(I')} \\ &\leq C(\|\partial_x u\|^2_{L^2(I')} + \|u\|^2_{L^2(I')}) \end{aligned} \tag{3.18}$$

where $C = \max\{6\|\partial_x\phi\|^2_{L^\infty(\mathbb{R})}, 3\|\partial_x^2\phi\|^2_{L^\infty(\mathbb{R})}\}$. Furthermore, standard inverse inequalities imply, [8], $\|\partial_x u\|_{L^2(I')} \leq C_2 h^{-1}\|u\|_{L^2(I')}$. Hence, inequality (3.18) implies

$$|u\phi|^2_{H^2(I')} \leq C(C_2 h^{-2} + 1)\|u\|^2_{L^2(I')}\,.$$

Substituting this result in (3.17) we get

$$\|(u\phi) - P_{X_h}(u\phi)\|^2_{L^2(\mathbb{R})} \leq C(C_2h^2 + h^4)\sum_{I'} \|u\|^2_{L^2(I')},$$

and the proof is complete.

□

Lemma 3.6.4 *Under the assumptions* (3.11) *and* (3.15), $\tau = \eta_u(u)$ *almost everywhere in* Ω.

Proof: We prove (3.16) by employing (3.9b). For a function $\phi \in [C_c^\infty(\Omega)]^m$ we have

$$\int_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot \phi dxdt = \int_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot \big(\phi - P_{W_h}\phi\big) dxdt \tag{3.19}$$

where by P_{W_h} we denote the L_2-projection onto W_h space. Summing over n in (3.19) and considering that Ω_1 is a finite domain such that $supp(\phi) \subset \Omega_1$,we can see that

$$\begin{aligned}
&\Big|\sum_n \int_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot \big(\phi - P_{W_h}\phi\big) dxdt\Big| = \Big|\int_{\Omega} \big(\tau_h - \eta_u(u_h)\big) \cdot \big(\phi - P_{W_h}\phi\big) dxdt\Big| \\
&\leq \int_{\Omega_1} \Big|\big(\tau_h - \eta_u(u_h)\big) \cdot \big(\phi - P_{W_h}\phi\big)\Big| dxdt \leq \|\tau_h - \eta_u(u_h)\|_{L^\infty(\Omega_1)} \int_{\Omega_1} \Big|\big(\phi - P_{W_h}\phi\big)\Big| dxdt \\
&\leq \|\tau_h - \eta_u(u_h)\|_{L^\infty(\Omega_1)} \|\phi - P_{W_h}\phi\|_{L^2(\Omega_1)} |\Omega_1|^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 3.6.3 we infer that $\|\phi - P_{W_h}\phi\|_{L_2(\Omega_1)} \leq Ch\|\phi\|_2$ for some constant C. Therefore by (3.19) we obtain,

$$\Big|\int_{\Omega} (\tau_h - \eta_u(u_h)) \cdot \phi dxdt\Big| \leq C_1 h \|\phi\|_2. \tag{3.20}$$

Here we used $\|\tau_h\|_{L^\infty(\Omega_1)}, \|\eta_u(u_h)\|_{L^\infty(\Omega_1)} < C$, see Remark 3.6.5 below. Moreover, since $(\tau_h - \eta_u(u_h)) \cdot \phi$ are uniformly bounded and having in mind

$$(\tau_h - \eta_u(u_h)) \cdot \phi \to (\tau - \eta_u(u)) \cdot \phi \; a.e.$$

we obtain, by the dominated convergence theorem that

$$\int_\Omega (\tau_h - \eta_u(u_h)) \cdot \phi dxdt \to \int_\Omega (\tau - \eta_u(u)) \cdot \phi dxdtx. \tag{3.21}$$

Letting $h \to 0$ in (3.20) we see then using (3.21), we conclude

$$\int_\Omega (\tau - \eta_u(u)) \cdot \phi dxdt = 0 \tag{3.22}$$

for all $\phi \in [C_c^\infty(\Omega)]^m$ and our assertion follows. $\square$

Remark 3.6.5 (Alternative scheme definition) *It is interesting to note that the second and third equation in the definition of the scheme* (3.9) *can lead to a relationship between τ_h and u_h, and thus to the possibility of eliminating the extra variable τ_h . In fact, a simple calculation gives that τ_h can be expressed in terms of u_h through the relation*

$$\begin{aligned}\tau_h(x,t)|_{S_n} = {} & P_{X_h}(\eta_u(u_h^{n+}))\frac{t^{n+1}-t}{|I_n|} \\ & + \frac{t-t^n}{|I_n|}\left(P_{X_h}\left(\int_{I_n}\eta_u(u_h(x,t))dt\right)\frac{2}{|I_n|} - P_{X_h}(\eta_u(u_h^{n+}))\right).\end{aligned} \tag{3.23}$$

The original system form of the scheme is however more convenient in the analysis, and it is used throughout the paper. One of the consequences of this expression is when u_h is uniformly bounded, the same is true for τ_h .

Alternatively, one can observe that $\tau_h = I_1 P_{X_h} \eta_u(u_h(x,t))$, where the time interpolant I_q :

$C(I_n) \to \mathbb{P}_q(I_n)$ is defined as

$$\int_{I_n} I_q v\, w\, dt = \int_{I_n} v\, w\, dt, \qquad \forall w \in \mathbb{P}_{q-1}(I_n)\,, \tag{3.24a}$$

$$I_q v^{n+} = v^{n+}\,. \tag{3.24b}$$

Notice that such an interpolant preserves the optimal order of convergence of the time-discontinuous Galerkin method since one can show, $\|I_q v - v\|_{L^p(I_n)} \leq k_n^{q+1} \|v^{(q+1)}\|_{L^p(I_n)}$. Similar bounds, have been derived in [53, (12.10)].

3.7 Convergence

We can now proceed to the proof of convergence of u_h towards an entropy solution.

Theorem 3.7.1 *Assume that $\{u_h\}_{h>0} \subset D$ and $u_h \to u$, $\tau_h \to \tau$ a.e. as $h \to 0$. Let further that $\gamma(h)^{-1/2}\, h \to 0$. Then the limit of the solution u_h of the numerical scheme* (3.9)*, u, is an entropy solution of* (3.1).

Proof: First, we show that u is a weak solution of (3.1). Taking $\phi_h = P_{V_h}(\phi)$ in (3.9a) with $\phi \in [C_c^\infty(\overline{\Omega})]^m$, we will have

$$\begin{aligned}
&\int\limits_{S_n} \Big((u_h)_t + A\big(\kappa(\tau_h)\big)_x \Big) \cdot \phi \; dxdt + \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \phi^{n+} \; dx \\
&= \int\limits_{S_n} \Big((u_h)_t + A\big(\kappa(\tau_h)\big)_x \Big) \cdot \Big(\phi - P_{V_h}(\phi) \Big) dxdt \\
&+ \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot (\phi^{n+} - (P_{V_h}\phi)^{n+}) dx - \gamma \int\limits_{S_n} (\tau_h)_x \cdot (P_{V_h}(\phi))_x \; dxdt.
\end{aligned}$$

Notice that since $\phi \in [C_c^\infty(\overline{\Omega})]^m$ there exist n_0 such that $\phi(t,\cdot) = 0,\ t \geq t_{n_0}$. Therefore summing with respect to n we obtain

$$
\begin{aligned}
&-\sum_{n\geq 0}\int_{S_n}\Big((u_h)\cdot\phi_t + A\big(\kappa(\tau_h)\big)\cdot\phi_x\Big)dxdt + \sum_{n\geq 0}\int_{\mathbb{R}}(u_h^{n+} - u_h^{n-})\cdot\phi^{n+}\,dx \\
&+\sum_{n\geq 0}\int_{\mathbb{R}}(u_h^{n+1-} - u_h^{n+})\cdot\phi^{n+}\,dx \\
&=\sum_{n\geq 0}\int_{S_n}\Big((u_h)_t + A\big(\kappa(\tau_h)\big)_x\Big)\cdot\Big(\phi - P_{V_h}(\phi)\Big)dxdt \\
&+\sum_{n\geq 0}\int_{\mathbb{R}}(u_h^{n+} - u_h^{n-})\cdot(\phi^{n+} - (P_{V_h}\phi)^{n+})dx - \gamma\sum_{n\geq 0}\int_{S_n}(\tau_h)_x\cdot(P_{V_h}(\phi))_x\,dxdt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left|\int_{\Omega}\big(u_h\cdot\phi_t + A(\kappa(\tau_h))\cdot\phi_x\big)dxdt + \int_{\mathbb{R}}u_0\cdot\phi(0,x)dx\right| \\
&\leq \max_{\tau_h}\left|\nabla_u A\big(\kappa(\tau_h)\big)\right|\int_{\Omega}\left|(\tau_h)_x\right|\left|\phi - P_{V_h}(\phi)\right|dxdt \\
&+\sum_{n\geq 0}\int_{\mathbb{R}}\left|u_h^{n+} - u_h^{n-}\right|\left|\phi^{n+} - (P_{V_h}\phi)^{n+}\right|dx + \gamma\int_{\Omega}\left|(\tau_h)_x\right|\left|P_{V_h}(\phi))_x\right|dxdt \\
&\leq \max_{\tau_h}\left|\nabla_u A\big(\kappa(\tau_h)\big)\right|\,\|(\tau_h)_x\|_{L_2(\Omega)}\|\phi - P_{V_h}\phi\|_{L_2(\Omega)} \\
&+\left(\sum_{n=0}^{n_0}\|u_h^{n-} - u_h^{n+}\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}\left(\sum_{n=0}^{n_0}\|\phi^{n+} - (P_{V_h}\phi)^{n+}\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} \\
&+\gamma\|(\tau_h)_x\|_{L_2(\Omega)}\|(P_{V_h}\phi)_x\|_{L_2(\Omega)}.
\end{aligned}
$$

Again, from Lemma 3.6.3 it holds that $\|\phi - P_{V_h}\phi\|_{L_2(\Omega)} \leq Ch^2\|\phi\|_{H^2(\Omega)}$ and further $\sum_{n\geq 0} h\|\phi^{n+} - (P_{V_h}\phi)^{n+}\|^2_{L_2(\mathbb{R})} \leq Ch^4\|\phi\|^2_{H^2(\Omega)}$ for some constant C. Thus

$$
\begin{aligned}
&\left|\int_{\Omega}\big(u_h\cdot\phi_t + A(\kappa(\tau_h))\cdot\phi_x\big)dxdt + \int_{\mathbb{R}}u_0\cdot\phi(0,x)dx\right| \\
&\leq \max_{\tau_h}|\nabla_u A(\kappa(\tau_h))|\,\gamma^{\frac{1}{2}}\|(\tau_h)_x\|_{L^2(\Omega)}C\gamma^{-\frac{1}{2}}h^2\|\phi\|_{H^2(\Omega)} + C_3h^{\frac{3}{2}}\|\phi\|_{H^2(\Omega)} \\
&+\gamma\|(\tau_h)_x\|_{L_2(\Omega)}\|(P_{V_h}\phi)_x\|_{L^2(\Omega)}
\end{aligned}
\tag{3.25}
$$

for some constant C_3. Recalling now Lemma 3.6.4, we know that $A(\kappa(\tau_h)) \to A(u)$ a.e. Further, stability properties of the L_2-projection stability imply $\|(P_{V_h}\phi)_x\|_{L_2(\Omega)} \leq \|(\phi)_x\|_{L_2(\Omega)}$. Thus, letting

$h \to 0$ in (3.25) while choosing a proper γ such that $\gamma^{-\frac{1}{2}}h^2 \to 0$ and having in mind that (3.12) implies $(\gamma)^{\frac{1}{2}}\|(\tau_h)_x\|_{L^2(\Omega)} \le C$ we conclude that,

$$\int_{\Omega} \big(u \cdot \phi_t + A(u) \cdot \phi_x\big) dxdt + \int_{\mathbb{R}} u_0 \cdot \phi(0,x) dx = 0$$

and thus u is a weak solution for (3.1).

Next, we need to show that u is also an entropy solution. Setting $\phi_h = P_{V_h}(\tau_h \phi)$ in (3.9a) where $\phi \in C_c^{\infty}(\Omega)$, $\phi \ge 0$ we have

$$\begin{aligned}
&\int_{S_n} \Big((u_h)_t + A\big(\kappa(\tau_h)\big)_x\Big) \cdot \tau_h \phi \; dxdt + \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} \phi^{n+} \; dx \\
&= \int_{S_n} \Big((u_h)_t + A\big(\kappa(\tau_h)\big)_x\Big) \cdot \Big(\tau_h \phi - P_{V_h}(\tau_h \phi)\Big) dx \\
&+ \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot (\tau_h^{n+} \phi^{n+} - (P_{V_h}(\tau_h \phi))^{n+}) dx - \gamma \int_{S_n} (\tau_h)_x \cdot (P_{V_h}(\tau_h \phi))_x \; dxdt.
\end{aligned}$$

Since

$$\int_{S_n} \Big((u_h)_t + A\big(\kappa(\tau_h)\big)_x\Big) \cdot \tau_h \phi \; dxdt = \int_{S_n} (u_h)_t \cdot \tau_h \phi dxdt + \int_{S_n} \nabla_u A\big(\kappa(\tau_h)\big)\, \kappa(\tau_h)_x \cdot \eta_u(\kappa(\tau_h)) \phi dxdt$$

we conclude

$$\begin{aligned}
\int_{S_n} (u_h)_t \cdot \tau_h \phi dxdt + \int_{S_n} Q\big(\kappa(\tau_h)\big)_x \phi dxdt + \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} \phi^{n+} \; dx& \\
= \int_{S_n} A\big(\kappa(\tau_h)\big)_x \cdot \Big(\tau_h \phi - P_{V_h}(\tau_h \phi)\Big) dxdt - \gamma \int_{S_n} (\tau_h)_x \cdot (P_{V_h}(\tau_h \phi))_x \; dxdt& \\
+ \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot (\tau_h^{n+} \phi^{n+} - (P_{V_h}(\tau_h \phi))^{n+}) dx.&
\end{aligned} \tag{3.26}$$

Setting now $\psi_h = P_{W_h}((u_h)_t \phi)$ in (3.9b) we have

$$\int_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot (u_h)_t \phi dxdt = \int_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot \big((u_h)_t\phi - P_{W_h}((u_h)_t\phi)\big) dxdt,$$

or

$$\int_{S_n} \tau_h \cdot (u_h)_t \phi dxdt = \int_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot \big((u_h)_t\phi - P_{W_h}((u_h)_t\phi)\big) dxdt + \int_{S_n} [\eta(u_h)]_t \phi dxdt$$

Substituting this in (3.26) we get

$$\begin{aligned}
&\int_{S_n} [\eta(u_h)]_t \phi dxdt + \int_{S_n} Q\big(\kappa(\tau_h)\big)_x \phi dxdt + \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} \phi^{n+}\, dx \\
&= \int_{S_n} A\big(\kappa(\tau_h)\big)_x \cdot \Big(\tau_h\phi - P_{V_h}(\tau_h\phi)\Big) dxdt \\
&+ \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot (\tau_h^{n+}\phi^{n+} - (P_{V_h}(\tau_h\phi))^{n+}) dx - \gamma \int_{S_n} (\tau_h)_x \cdot (P_{V_h}(\tau_h\phi))_x\, dxdt \\
&- \int_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot \big((u_h)_t\phi - P_{W_h}((u_h)_t\phi)\big) dxdt.
\end{aligned}$$

Summing over n and integrating by parts on the left hand side of the equation we see that

$$\begin{aligned}
&- \int_{\Omega} \big(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x\big)\, dxdt \\
&+ \sum_n \int_{\mathbb{R}} \Big(\eta((u_h^{n-}) - \eta((u_h^{n+})) - (u_h^{n-} - u_h^{n+}) \cdot \tau_h^{n+}\Big) \phi^{n+}\, dx \\
&\quad = \int_{\Omega} A\big(\kappa(\tau_h)\big)_x \cdot \Big(\tau_h\phi - P_{V_h}(\tau_h\phi)\Big) dxdt \\
&\quad + \sum_n \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot (\tau_h^{n+}\phi^{n+} - (P_{V_h}(\tau_h\phi))^{n+}) dx - \gamma \int_{\Omega} (\tau_h)_x \cdot (P_{V_h}(\tau_h\phi))_x\, dxdt \\
&\quad - \int_{\Omega} \big(\tau_h - \eta_u(u_h)\big) \cdot \big((u_h)_t\phi - P_{W_h}((u_h)_t\phi)\big) dxdt.
\end{aligned} \tag{3.27}$$

Since $\phi \geq 0$ we have

$$
\begin{aligned}
&-\gamma \int_{\Omega} (\tau_h)_x (\tau_h \phi)_x dxdt \\
&\quad = -\gamma \int_{\Omega} (\tau_h)_x (\tau_h)_x \phi dxdt - \gamma \int_{\Omega} (\tau_h)_x \tau_h \phi_x dxdt \leq -\gamma \int_{\Omega} (\tau_h)_x \tau_h \phi_x dxdt.
\end{aligned}
$$

and thus equality 3.27 becomes

$$
\begin{aligned}
&- \int_{\Omega} \left(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x\right) dxdt \\
&+ \sum_n \int_{\mathbb{R}} \left(\eta((u_h^{n-}) - \eta((u_h^{n+})) - (u_h^{n-} - u_h^{n+}) \cdot \tau_h^{n+}\right)\phi^{n+}\, dx \\
&\quad \leq \int_{\Omega} A\big(\kappa(\tau_h)\big)_x \cdot \left(\tau_h \phi - P_{V_h}(\tau_h \phi)\right) dxdt \\
&\quad + \sum_n \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot (\tau_h^{n+}\phi^{n+} - (P_{V_h}(\tau_h\phi))^{n+}) dx \qquad (3.28) \\
&\quad - \left\{ \gamma \int_{\Omega} (\tau_h)_x \cdot (P_{V_h}(\tau_h \phi))_x \, dxdt - \gamma \int_{\Omega} (\tau_h)_x (\tau_h \phi)_x dxdt + \gamma \int_{\Omega} (\tau_h)_x \tau_h \phi_x dxdt \right\} \\
&\quad - \int_{\Omega} \big(\tau_h - \eta_u(u_h)\big) \cdot \big((u_h)_t \phi - P_{W_h}((u_h)_t \phi)\big) dxdt \\
&\quad =: A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

We will estimate the right hand side of (3.28). Our aim is to show

$$
\begin{aligned}
&- \int_{\Omega} \left(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x\right) dxdt \\
&\qquad + \sum_n \int_{\mathbb{R}} \left(\eta(u_h^{n-}) - \eta(u_h^{n+}) - (u_h^{n-} - u_h^{n+}) \cdot \tau_h^{n+}\right)\phi^{n+}\, dx \qquad (3.29) \\
&\qquad\quad \leq C(h^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + h\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}} h).
\end{aligned}
$$

To this end we first notice that Lemma (3.6.3) implies

$$\begin{aligned}\|\tau_h\phi - P_{V_h}(\tau_h\phi)\|_{L_2(\Omega)} &\leq C(h+h^2)\|\phi\|_{H^2(\Omega)},\\ \sum_n h\|\tau_h^{n_+}\phi^{n_+} - (P_{V_h}(\tau_h\phi))^{n_+}\|^2_{L_2(\mathbb{R})} &\leq C(h+h^2)^2\|\phi\|^2_{H^2(\Omega)}\end{aligned} \tag{3.30}$$

hence $|A_1| \leq C\gamma^{-\frac{1}{2}}h$ and $|A_2| \leq Ch^{\frac{1}{2}}$. It remains to estimate the third and the forth terms of the right-hand side of (3.28).

For the third term we notice that, the stability estimate in Lemma 3.6.1 implies

$$\begin{aligned}&\left|-\gamma\int_\Omega (\tau_h)_x \cdot ((P_{V_h}(\tau_x))_x - (\tau_h\phi)_x)\,dxdt\right|\\ &\qquad\leq \gamma\|(\tau_h)_x\|_{L^2(\Omega)}\|(P_{V_h}(\tau\phi))_x - (\tau_h\phi)_x\|_{L^2(\Omega)}\\ &\qquad\leq \gamma\|(\tau_h)_x\|_{L^2(\Omega)}\|\tau_h\|_{L^\infty(\Omega)}(\|\phi\|_{H^1(\Omega)} + h\|\phi\|_{H^2(\Omega)})\\ &\qquad\leq C(1+h)\gamma^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\left|\gamma\int_\Omega (\tau_h)_x \cdot \tau_h\phi_x dxdt\right| &\leq \gamma\|(\tau_h)_x\|_{L^2(\Omega)}\|\tau_h\|_{L^\infty(\Omega)}\,\|\phi\|_{H^1(\Omega)}\\ &\leq C\gamma^{\frac{1}{2}}\,.\end{aligned}$$

Hence $|A_3| \leq C(1+h)\gamma^{\frac{1}{2}}$. In order to bound A_4 we first need a bound for the term $(u_h)_t$ in $L^2(\Omega)$. We assume $\|(u_h)_t\|_{L^2(S_n)} > 0$ since if $\|(u_h)_t\|_{L^2(S_{n_1})} = 0$ for some index n_1 then it does not contribute to

$\sum\limits_{n\geq 0} \|(u_h)_t\|^2_{L^2(S_n)}$. Using $(3.9a)$ we have

$$
\begin{aligned}
\|(u_h)_t\|^2_{L^2(S_n)} &= \int\limits_{S_n} (u_h)_t \cdot (u_h)_t dxdt \\
&= -\int\limits_{S_n} A(\kappa(\tau_h))_x \cdot (u_h)_t dxdt - \gamma \int\limits_{S_n} (\tau_h)_x \cdot (u_h)_{tx} dxdt \\
&\qquad - \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot (u_h)_t^{n+} dx \\
&= -\int\limits_{S_n} \nabla_u A(\kappa\,(\tau_h))\,(\kappa\,(\tau_h))_x \cdot (u_h)_t dxdt - \gamma \int\limits_{S_n} (\tau_h)_x \cdot (u_h)_{tx} dxdt \\
&\qquad - \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot (u_h)_t^{n+} dx
\end{aligned}
$$

Hence, since τ_h is uniformly bounded, standard inverse inequalities imply

$$
\begin{aligned}
\|(u_h)_t\|^2_{L^2(S_n)} &\leq C \int\limits_{S_n} |(\tau_h)_x \cdot (u_h)_t| dxdt + \gamma \int\limits_{S_n} |(\tau_h)_x \cdot (u_h)_{tx}| dxdt \\
&\qquad + \int\limits_{\mathbb{R}} |(u_h^{n+} - u_h^{n-}) \cdot (u_h)_t^{n+}| dx \\
&\leq C \|(\tau_h)_x\|_{L^2(S_n)} \|(u_h)_t\|_{L^2(S_n)} + \gamma \|(\tau_h)_x\|_{L^2(S_n)} \|(u_h)_{tx}\|_{L^2(S_n)} \\
&\qquad + \|u_h^{n+} - u_h^{n-}\|_{L^2(\mathbb{R})} \|(u_h)_t^{n+}\|_{L^2(\mathbb{R})} \\
&\leq C \|(\tau_h)_x\|_{L^2(S_n)} \|(u_h)_t\|_{L^2(S_n)} + \gamma h^{-1} \|(\tau_h)_x\|_{L^2(S_n)} \|(u_h)_t\|_{L^2(S_n)} \\
&\qquad + C h^{-\frac{1}{2}} \|u_h^{n+} - u_h^{n-}\|_{L^2(\mathbb{R})} \|(u_h)_t\|_{L^2(S_n)} \\
&= C \gamma^{-\frac{1}{2}} \gamma^{\frac{1}{2}} \|(\tau_h)_x\|_{L^2(S_n)} \|(u_h)_t\|_{L^2(S_n)} + \gamma^{\frac{1}{2}} h^{-1} \gamma^{\frac{1}{2}} \|(\tau_h)_x\|_{L^2(S_n)} \|(u_h)_t\|_{L^2(S_n)} \\
&\qquad + C h^{-\frac{1}{2}} \|u_h^{n+} - u_h^{n-}\|_{L^2(\mathbb{R})} \|(u_h)_t\|_{L^2(S_n)}
\end{aligned}
$$

Finally, Lemma 3.6.1 implies

$$\begin{aligned}\sum_{n=0}^{N-1} \|(u_h)_t\|^2_{L^2(S_n)} &\leq C\gamma^{-1}\gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L^2(S_n)} + \gamma h^{-2}\gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L^2(S_n)} + h^{-1} \sum_{n=0}^{N-1} \|u_h^{n_+} - u_h^{n_-}\|^2_{L^2(\mathbb{R})} \\ &\leq C(\gamma^{-1} + \gamma h^{-2} + h^{-1}).\end{aligned}$$

Therefore

$$\Big(\sum_{n=0}^{N-1} \|(u_h)_t\|^2_{L^2(S_n)}\Big)^{\frac{1}{2}} \leq C(\gamma^{-\frac{1}{2}} + \gamma^{\frac{1}{2}} h^{-1} + h^{-\frac{1}{2}}). \tag{3.31}$$

Subsequently we will use the fact that

$$P_{W_h}\upsilon = P_{0,t}P_{X_h}\upsilon = P_{X_h}P_{0,t}\upsilon \tag{3.32}$$

for a $\upsilon \in H^1(S_n)$. This is a simple consequence of the tensor product nature of the elements of W_h at each S_n . Then (3.32) implies

$$\begin{aligned}|A_4| \leq &\left|\sum_{n\geq 0}\int_{S_n} \big(\tau_h - \eta_u(u_h)\big)\cdot\big((u_h)_t\phi - (u_h)_t P_{0,t}\phi\big)dxdt\right| \\ &\qquad + \left|\sum_{n\geq 0}\int_{S_n} \big(\tau_h - \eta_u(u_h)\big)\cdot\Big((u_h)_t P_{0,t}\phi - P_{X_h}\big((u_h)_t P_{0,t}\phi\big)\Big)dxdt\right| \\ &=: |\widetilde{A_1}| + |\widetilde{A_2}|\end{aligned}$$

where we used the fact $(u_h(.,x))_t \in \mathbb{P}_0(I_n)$ and thus $P_{0,t}((u_h)_t\phi) = (u_h)_t P_{0,t}\phi$.

The first term $\widetilde{A_1}$ is estimated as

$$
\begin{aligned}
|\widetilde{A_1}| &\le C \sum_{n\ge 0} \left(\|(u_h)_t\|_{L^2(S_n)}\right)\left(\|\phi - P_{0,t}\phi\|_{L^2(S_n)}\right) \\
&\le C \left(\sum_{n\ge 0} \|(u_h)_t\|^2_{L^2(S_n)}\right)^{\frac{1}{2}} \left(\sum_{n\ge 0} \|\phi - P_{0,t}\phi\|^2_{L^2(S_n)}\right)^{\frac{1}{2}} \\
&\le C \left(\gamma^{-\frac{1}{2}} + \gamma^{\frac{1}{2}} h^{-1} + h^{-\frac{1}{2}}\right) h \|\phi\|_{H^1(\Omega)} \\
&= C \left(\gamma^{-\frac{1}{2}} h + \gamma^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \|\phi\|_{H^1(\Omega)}.
\end{aligned}
$$

Notice that since $P_{0,t}$ commutes with differentiation with respect to x and it is stable in $L^2(I_n)$ and in $L^\infty(I_n)$, one can verify as in Lemma (3.6.3), that

$$
\|(u_h)_t P_{0,t}\phi - P_{X_h}\big((u_h)_t P_{0,t}\phi\big)\|_{L^2(S_n)} \le Ch\|(u_h)_t\|_{L^2(S_n)}.
$$

Hence,

$$
|\widetilde{A_2}| \le Ch\|(u_h)_t\|_{L^2(\Omega)} \le C \left(\gamma^{-\frac{1}{2}} h + \gamma^{\frac{1}{2}} + h^{\frac{1}{2}}\right).
$$

and combined with the previous bounds yields (3.29) . We conclude therefore,

$$
\begin{aligned}
&-\int_\Omega \big(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x\big)\, dxdt \\
&\quad + \sum_n \int_{\mathbb{R}} \Big(\eta(u_h^{n-}) - \eta(u_h^{n+}) - \eta_u(u_h^{n+})\cdot(u_h^{n-} - u_h^{n+})\Big)\phi^{n+} dx \\
&\quad \le C(h^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + h\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}} h) - \sum_n \int_{\mathbb{R}} \Big((\eta_u(u_h^{n+}) - \tau_h^{n+})\cdot(u_h^{n-} - u_h^{n+})\Big)\phi^{n+}\, dx.
\end{aligned}
\tag{3.33}
$$

Notice that (3.9c) implies (see (3.23))

$$
\tau_h^{n+} = P_{X_h}(\eta_u(u_h^{n+})).
$$

Therefore

$$
\begin{aligned}
&\sum_n \int_{\mathbb{R}} \Big((\eta_u(u_h^{n+}) - \tau_h^{n+}) \cdot (u_h^{n-} - u_h^{n+}) \Big) \phi^{n+} dx \\
=&\sum_n \int_{\mathbb{R}} \Big((\eta_u(u_h^{n+}) - P_{X_h}(\eta_u(u_h^{n+})) \cdot (u_h^{n-} - u_h^{n+}) \Big) \phi^{n+} dx \\
=&\sum_n \int_{\mathbb{R}} \eta_u(u_h^{n+}) \cdot \Big((u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big) \Big) dx.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&-\int_{\Omega} \big(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x \big)\, dxdt \\
&+\sum_n \int_{\mathbb{R}} \Big(\eta(u_h^{n-}) - \eta(u_h^{n+}) - \eta_u(u_h^{n+}) \cdot (u_h^{n-} - u_h^{n+}) \Big) \phi^{n+} dx \\
&\le (h^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + h\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}} h) + \sum_n \int_{\mathbb{R}} \Big| \eta_u(u_h^{n+}) \cdot \Big((u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big) \Big) \Big| dx.
\end{aligned}
$$

To conclude the proof, it remains to estimate the last term of the right-hand side. To this end, let $D_\phi \subset \mathbb{R}$ is a domain such that $supp(\phi(t^n)) \subset D$ for all n, and notice that

$$
\begin{aligned}
&\sum_n \int_{\mathbb{R}} \Big| \eta_u(u_h^{n+}) \cdot \Big((u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big) \Big) \Big| dx \\
&\le \max_n \|\eta_u(u_h^{n+})\|_{L^\infty(\mathbb{R})} \sum_n \int_{\mathbb{R}} \Big| (u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big) \Big| dx \\
&\le \max_n \|\eta_u(u_h^{n+})\|_{L^\infty(\mathbb{R})} \sum_n \Big(\int_{D_\phi} \Big| (u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big) \Big|_{\mathbb{R}}^2 dx \Big)^{\frac{1}{2}} |D_\phi|^{\frac{1}{2}} \\
&\le \max_n \|\eta_u(u_h^{n+})\|_{L^\infty(\mathbb{R})} \Big(\sum_{n \le n_0} \int_{D_\phi} \Big| (u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big) \Big|_{\mathbb{R}}^2 dx \Big)^{\frac{1}{2}} \Big(\sum_{n \le n_0} |D_\phi| \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Here, Lemma 3.6.3 implies

$$
\sum_{n \le n_0} \|(u_h^{n-} - u_h^{n+})\phi^{n^+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big)\|_{L^2(\mathbb{R})}^2 \le Ch^2 \sum_{n \le n_0} \|u_h^{n-} - u_h^{n+}\|_{L^2(\mathbb{R})}^2.
$$

The last inequality and (3.12) imply

$$\Big(\sum_{n\le n_0}\int_{D_\phi}\Big|(u_h^{n-}-u_h^{n+})\phi^{n+}-P_{X_h}\Big((u_h^{n-}-u_h^{n+})\phi^{n+}\Big)\Big|_{\mathbb{R}}^2 dx\Big)^{\frac{1}{2}}\Big(\sum_{n\le n_0}|D_\phi|\Big)^{\frac{1}{2}}$$
$$\le |D_\phi|^{\frac{1}{2}}Ch\,n_0^{\frac{1}{2}}\Big(\sum_{n\le n_0}\|u_h^{n-}-u_h^{n+}\|^2_{L^2(\mathbb{R})}\Big)^{\frac{1}{2}}\le |D_\phi|^{\frac{1}{2}}Ch\,h^{-\frac{1}{2}}\Big(\sum_{n\le n_0}\|u_h^{n-}-u_h^{n+}\|^2_{L^2(\mathbb{R})}\Big)^{\frac{1}{2}}$$
$$\le C_6h^{\frac{1}{2}}$$

where C_6 is an appropriate constant. We conclude therefore that

$$-\int_{\Omega}\big(\eta(u_h)\phi_t+Q\big(\kappa(\tau_h)\big)\phi_x\big)\,dxdt$$
$$+\sum_n\int_{\mathbb{R}}\Big(\eta(u_h^{n-})-\eta(u_h^{n+})-\eta_u(u_h^{n+})\cdot(u_h^{n-}-u_h^{n+})\Big)\phi^{n+}dx$$
$$\le C_4(h^{\frac{1}{2}}+\gamma^{\frac{1}{2}}+h\gamma^{\frac{1}{2}}+\gamma^{-\frac{1}{2}}h)+C_6h^{\frac{1}{2}}.$$

Finally, letting $h\to 0$ while using the convexity of η and lemma (3.6.1) we obtain the desired result. □

3.8 Measure-valued solutions

In this section we show that the scheme introduced herein is compatible with the notion of entropic measure valued solutions. This notion of very weak solutions hinges on the theory of Young measures, i.e., of parametrised with respect to x and t, appropriate probability measures in the phase space, e.g., [4], and was introduced by DiPerna, [19] . Modern uses of this notion relate to uncertainty quantification and to statistical inference when an assembly of solutions of the conservation law are considered, e.g., [2, 21]. Next we shall show that when approximating sequences obtained by our scheme generate a Young measure, this measure is indeed an entropic measure valued solution of (3.1).

Young measures

Let $\mathbf{M}(\mathbb{R}^m)$ be the set of all signed Radon measures on $\mathbb{R}^m$. We denote by $\mathbf{M}^+(\mathbb{R}^m)$ the set of all positive Radon measures and by $\mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$ the set of all probability measures over $\mathcal{B}(\mathbb{R}^m)$ that is,

$$\mathbf{M}^{\mathbb{P}}(\mathbb{R}^m) = \{\mu \in \mathbf{M}^+(\mathbb{R}^m), \mu(\mathbb{R}^m) = 1\}.$$

We call *Young measure* a weakly* measurable mapping from Ω into $\mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$, [4]. The set of all Young measures is denoted by $\mathbf{Y}(\Omega, \mathbb{R}^m)$.

Measure-valued solutions

A young measure $\mu \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ is said to be a measure-valued solution of the conservation law (3.1), DiPerna [19] , if it satisfies the expression

$$\int\limits_{\Omega} \big(\langle id, \mu_{x,t}\rangle \cdot \phi_t + \langle A, \mu_{x,t}\rangle \cdot \phi_x\big) dxdt + \int\limits_{\mathbb{R}} u_0 \cdot \phi(0,x) dx = 0, \tag{3.34}$$

for all $\phi \in C_0^\infty(\overline{\Omega})$ where by $\langle A, \mu_{x,t}\rangle$ we mean

$$\langle A, \mu_{x,t}\rangle = \int_{\mathbb{R}^m} A(\lambda) d\mu_{x,t}(\lambda).$$

Similarly as in the concept framework of weak solutions, a Young measure $\mu \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ which fulfils the additional relation

$$\int\limits_{\Omega} \big(\langle \eta, \mu_{x,t}\rangle \cdot \phi_t + \langle Q, \mu_{x,t}\rangle \cdot \phi_x\big) dxdt + \int\limits_{\mathbb{R}} \eta(u_0) \cdot \phi(0,x) dx \geq 0, \tag{3.35}$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$ is called an *entropy measure-valued solution* of the conservation law (3.1).

3.8.1 Convergence towards entropy measure-valued solutions

We turn now our attention to our main task in this section, that is to show convergence of the numerical scheme (3.9) to an entropy measure-valued solution as $h \to 0$, in the sense that, when approximating sequences obtained by our scheme generate a Young measure, this measure is indeed an entropic measure valued solution of the conservation law. In order to do that so, we need first the following lemma which connects the limiting behaviour of τ_h with μ.

Lemma 3.8.1 *Let $g : \mathbb{R}^m \mapsto \mathbb{R}$ be any continuous function that converges to zero at infinity. Assume that*

$$\|\tau_h - \eta_u(u_h)\|_{L^2} \to 0, \qquad h \to 0\,. \tag{3.36}$$

Then if (3.11) *holds, then there exist a subsequence of τ_h (still denoted as τ_h) for which it holds that*

$$g(\tau_h) \overset{*}{\rightharpoonup} \langle g, (\eta_u)_{\#}\mu\rangle \ \ in\ L^\infty(\Omega) \tag{3.37}$$

where μ is a young measure associated with u_h.

Proof: Since we assume that (3.11) holds, the fundamental theorem of Young measures(see [4]), implies that there a exist a subsequence u_{h_l} (which we relabel here u_h) and a young measure $\mu \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ such that

$$g(\eta_u(u_h)) \overset{*}{\rightharpoonup} \langle g, (\eta_u)_{\#}\mu\rangle \text{ in } L^\infty(\Omega) \tag{3.38}$$

as $h \to 0$. And further, on account to the fact that τ_h is also uniformly bounded (see remark (3.2)) we deduce that there exist a subsequence $\tau_{h_{l_\epsilon}}$ of the sequence τ_{h_l} (denoted as τ_h) and a young measure $\mu' \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ such that

$$g(\tau_h) \overset{*}{\rightharpoonup} \langle g, \mu'\rangle \text{ in } L^\infty(\Omega). \tag{3.39}$$

Subsequently, assume for the moment that $\bar{g} \in C^{\infty}(\mathbb{R}^m)$. Given a function $\phi \in C_c^{\infty}(\Omega)$ and using the mean value theorem for the function $\bar{g}$ we see that

$$\int\limits_{\Omega} \big(\bar{g}(\tau_h) - \bar{g}(\eta_u(u_h))\big)\phi dxdt = \int\limits_{\Omega} \int_0^1 \nabla \bar{g}(\xi(x,t,s))\, ds \cdot \big(\tau_h - \eta_u(u_h)\big)\phi dxdt$$

where for each x and t, the function $\xi(x,t,s)$ is defined as $\xi(x,t,s) = \tau_h - s((\tau_h - \eta_u(u_h))$. Let now Ω_1 be a finite domain such that $supp(\phi) \subset \Omega_1$. In addition, observe that the product of the functions $\phi(x,t)$ and $\nabla \bar{g}(\xi(x,t,s))$ is a smooth enough function in Ω_1. Now set $\tilde{G}(x,t) = \int_0^1 \nabla \bar{g}(\xi(x,t,s))\, ds$. Then,

$$\begin{aligned}
\Big|\int\limits_{\Omega} \big(\bar{g}(\tau_h) - \bar{g}(\eta_u(u_h))\big)\phi dxdt\Big| &= \Big|\int\limits_{\Omega} \big(\tau_h - \eta_u(u_h)\big) \cdot \big(\phi\tilde{G}(x,t) - P_{W_h}(\phi\tilde{G}(x,t))\big) dxdt\Big| \\
&\leq \int\limits_{\Omega_1} \Big|\big(\tau_h - \eta_u(u_h)\big) \cdot \big(\phi\tilde{G}(x,t) - P_{W_h}(\phi\tilde{G}(x,t))\big)\Big| dxdt \\
&\leq \|\tau_h - \eta_u(u_h)\|_{L^2(\Omega_1)} \|\phi\tilde{G}(x,t) - P_{W_h}(\phi\tilde{G}(x,t))\|_{L^2(\Omega_1)}\,.
\end{aligned}$$

Next we show that

$$\|\phi\tilde{G} - P_{W_h}(\phi\tilde{G})\|_{L^2(\Omega_1)} \leq C, \qquad h \to 0. \tag{3.40}$$

Assuming for a moment the validity of (3.40) we conclude,

$$\int\limits_{\Omega_1} \big(\bar{g}(\tau_h) - \bar{g}(\eta_u(u_h))\big)\phi dxdt \to 0$$

as $h \to 0$. Define now a sequence of functions $g^k \in C^{\infty}(\mathbb{R}^m)$ that converges uniformly at g on a compact domain that contains the ranges of τ_h and $\eta_u(u_h)$. From the above limit we have that for all k

given an $\epsilon > 0$, there exist a $\delta > 0$ such that

$$\left| \int_{\Omega_1} \big(g^k(\tau_h) - g^k(\eta_u(u_h))\big)\phi dxdt \right| < \epsilon \tag{3.41}$$

when $h < \delta$. Since,

$$\big(g^k(\tau_h) - g^k(\eta_u(u_h))\big)\phi \to \big(g(\tau_h) - g(\eta_u(u_h))\big)\phi$$

a.e. in Ω_1 as $k \to \infty$, we know by the DCT that

$$\int_{\Omega_1} \big(g^k(\tau_h) - g^k(\eta_u(u_h))\big)\phi dxdt \to \int_{\Omega_1} \big(g(\tau_h) - g(\eta_u(u_h))\big)\phi dxdt.$$

Thus, passing to the limit in (3.41) as $k \to \infty$ we obtain

$$\int_{\Omega} \big(g(\tau_h) - g(\eta_u(u_h))\big)\phi dxdt = \int_{\Omega_1} \big(g(\tau_h) - g(\eta_u(u_h))\big)\phi dxdt \to 0. \tag{3.42}$$

On the other hand from (3.38) and (3.39) we also have

$$\int_{\Omega} (g(\tau_h) - g(\eta_u(u_h)))\phi dxdt \to \int_{\Omega} \big(\langle g, \mu'_{x,t}\rangle - \langle g, (\eta_u)_{\#}\mu_{x,t}\rangle\big)\phi dxdt. \tag{3.43}$$

Finally, combining (3.42) and (3.43) we conclude that

$$\int_{\Omega} \big(\langle g, \mu'_{x,t}\rangle - \langle g, (\eta_u)_{\#}\mu_{x,t}\rangle\big)\phi dxdt = 0, \tag{3.44}$$

for all $\phi \in C_c^\infty(\Omega)$. Therefore, for almost all x and t in Ω we have

$$\langle g, \mu'_{x,t}\rangle = \langle g, (\eta_u)_{\#}\mu_{x,t}\rangle.$$

Thus,

$$\mu'_{x,t} = (\eta_u)_{\#}\mu_{x,t},$$

in the sense of measures a.e. in Ω and hence (3.37) is proved.

It thus remains to prove (3.40). To this end, we first observe, as in the proof of Lemma 3.6.3 that

$$\|\phi\tilde{G} - P_{W_h}(\phi\tilde{G})\|_{L^2(S_n)} \leq \|\phi\tilde{G} - P_{0,t}(\phi\tilde{G})\|_{L^2(S_n)} + \|\phi\tilde{G} - I_{X_h}(\phi\tilde{G})\|_{L^2(S_n)} .$$

Then, by the uniform boundedness of $u_h,\ \tau_h$, and the fact that τ_h is piecewise constant in time, we have

$$\begin{aligned}
\|\phi\tilde{G} - P_{0,t}(\phi\tilde{G})\|_{L^2(S_n)} \leq& ch\|\partial_t(\phi\tilde{G})\|_{L^2(S_n)} \\
\leq& ch\|(\partial_t\phi)\tilde{G}\|_{L^2(S_n)} + ch\|\phi(\partial_t\tilde{G})\|_{L^2(S_n)} \\
\leq& Ch\|\partial_t\phi\|_{L^2(S_n)} + Ch\|\phi\|_{L^\infty(S_n)}\|\partial_t u_h\|_{L^2(S_n)}, .
\end{aligned}$$

Thus in view of the bound (3.31) we will get that the contribution of this term to (3.40) yields a term converging to zero. It remains to estimate $\|\phi\tilde{G} - I_{X_h}(\phi\tilde{G})\|_{L^2(S_n)}$. By modifying the arguments in the proof of Lemma 3.6.3 and restricting our attention to a typical spacial element I' of the decomposition of the finite element space X_h, there holds,

$$\|\phi\tilde{G} - I_{X_h}(\phi\tilde{G})\|_{L^2(I')} \leq Ch^2|\phi\tilde{G}|_{H^2(I')}$$

see [8]. Locally, $\phi\tilde{G} \in H^2(I')$. For each fixed t in S_n the spatial estimate holds

$$\|(\phi\tilde{G}) - I_{X_h}(\phi\tilde{G})\|^2_{L^2(\mathbb{R})} \leq \sum_{I'} \|(\phi\tilde{G}) - I_{X_h}(\phi\tilde{G})\|^2_{L^2(I')} \leq Ch^4 \sum_{I'} |\phi\tilde{G}|^2_{H^2(I')}. \tag{3.45}$$

Next, since,

$$\partial_x^2(\phi\tilde{G}) = 2\partial_x\phi\partial_x\tilde{G} + \tilde{G}\partial_x^2\phi + \phi\partial_x^2\tilde{G}\,.$$

Using the definition of $\tilde{G}$ and the fact that piecewise in each element $\partial_x^2\tau_h = \partial_x^2 u_h = 0$ we observe

$$\begin{aligned}|\phi\tilde{G}|_{H^2(I')} \le C\|\partial_x\phi\|_{L^\infty(\mathbb{R})}(\|\partial_x u_h\|_{L^2(I')} + \|\partial_x\tau_h\|_{L^2(I')}) + C\|\partial_x^2\phi\|_{L^\infty(\mathbb{R})}\\ + C\|\phi\|_{L^\infty(\mathbb{R})}(\|\partial_x u_h\|^2_{L^4(I')} + \|\partial_x\tau_h\|^2_{L^4(I')})\,.\end{aligned}$$

Therefore, using the inverse inequalities, $\|\partial_x\chi\|_{L^2(I')} \le Ch^{-1}\|\chi\|_{L^2(I')}$, and $\|\partial_x\chi\|_{L^4(I')} \le Ch^{-3/4}\|\chi\|_{L^\infty(I')}$, [8, Lemma 4.5.3],

$$\begin{aligned}|\phi\tilde{G}|_{H^2(I')} &\le C\Big(\|\partial_x u_h\|_{L^2(I')} + \|\partial_x\tau_h\|_{L^2(I')} + \|\partial_x u_h\|^2_{L^4(I')} + \|\partial_x\tau_h\|^2_{L^4(I')} + 1\Big)\\ &\le C\Big(h^{-1} + h^{-3/2} + 1\Big)\,.\end{aligned} \tag{3.46}$$

In view of (3.76), we conclude therefore that $\|\phi\tilde{G} - I_{X_h}(\phi\tilde{G})\|_{L^2(S_n)} \le C$ as $h \to 0$ and the proof is complete. $\square$

We can now state the two main theorems of this section.

Theorem 3.8.2 *If assumption* (3.11) *holds then the numerical scheme* (3.9) *converges towards a measure-valued solution of* (3.1).

Proof: Using relation (3.25) from the proof of Theorem 3.7.1 we see that

$$\int_{\Omega} \big(u_h \cdot \phi_t + A(\kappa(\tau_h)) \cdot \phi_x\big) dx dt + \int_{\mathbb{R}} u_0 \cdot \phi(0,x) dx \to 0,$$

as $h \to 0$. But by combination of the fundamental theorem of Young measures and of Lemma 3.8.1 we obtain by letting $h \to 0$

$$\int_{\Omega} \big(u_h \cdot \phi_t + A(\kappa(\tau_h)) \cdot \phi_x\big) dx dt + \int_{\mathbb{R}} u_0 \cdot \phi(0,x) dx \to$$
$$\int_{\Omega} \big(\langle id, \mu_{x,t}\rangle \cdot \phi_t + \langle A, \mu_{x,t}\rangle \cdot \phi_x\big) dx dt + \int_{\mathbb{R}} u_0 \cdot \phi(0,x) dx = 0$$

which proves the theorem. □

Theorem 3.8.3 *If assumption* (3.11) *holds then the approached measure-valued solution of Theorem* 3.8.2 *is entropy consistent i.e. it fulfils the relation* (3.35).

Proof: Picking $\phi_h = P_{V_h}(\tau_h \phi)$ in (3.9a) where $\phi \in C_c^\infty(\overline{\Omega})$, $\phi \geq 0$ and following similar arguments as in the second part of the proof of Theorem 3.7.1 we conclude that

$$\begin{aligned}
&-\int_{\Omega} \big(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x - \int_{\mathbb{R}} \eta(u_0) \cdot \phi(x,0) \\
&+\sum_n \int_{\mathbb{R}} \Big(\eta(u_h^{n-}) - \eta(u_h^{n+}) - \eta_u(u_h^{n+}) \cdot (u_h^{n-} - u_h^{n+})\Big) \phi^{n+} dx \\
&\leq C_4(h^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + h\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}} h) + C_6 h^{\frac{1}{2}}.
\end{aligned}$$

Using the convexity of η we deduce

$$-\int\limits_{\Omega} \left(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x - \int\limits_{\mathbb{R}} \eta(u_0)\cdot\phi(x,0)\right.$$
$$\leq C_4(h^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + h\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}}h) + C_6 h^{\frac{1}{2}}.$$

Therefore, letting $h \to 0$ in the above expression and using Lemma 3.8.1 we conclude that

$$-\int\limits_{\Omega} \big(\langle\eta,\mu_{x,t}\rangle\phi_t + \langle Q,\mu_{x,t}\rangle\phi_x - \int\limits_{\mathbb{R}} \eta(u_0)\cdot\phi(x,0) \leq 0$$

and the proof is complete. $\square$

3.9 L^p controlled measure-valued solutions

In this section we show that in certain cases the assumption that the approximating sequences are uniformly bounded can be relaxed by employing the L^p theory of Young measures, [4]. In fact, uniform bounded sequences in L^p still generate Young measures. Roughly speaking, let u_j a bounded sequence of approximations in $L^p(\Omega,\mathbb{R}^m)$. Then there exists a subsequence and a measure $\mu \in \mathbf{Y}(\Omega,\mathbb{R}^m)$, $\mu = \mu_{x,t}$, $(x,t)\in\Omega$, such that for $G \in C_p(\mathbb{R}^m)$,

$$G(u_j) \rightharpoonup \overline{G}, \quad \text{where} \quad \overline{G}(x,t) = \langle G,\mu_{x,t}\rangle = \int_{\mathbb{R}^m} G(\lambda)d\mu_{x,t}(\lambda), \tag{3.47}$$

where $C_p(\mathbb{R}^m) = \{g \in C(\mathbb{R}^m) : \lim_{|\zeta|\to\infty} \frac{g(\zeta)}{|\zeta|^p} = 0\}$. The requirement that $G \in C_p(\mathbb{R}^m)$, in order to pass to the limit in (3.47) restricts its applicability of the limiting process (3.47) to nonlinear functions with limited growth at infinity. Typically, in applications to hyperbolic systems, assuming that the entropy η behaves as $|u|^p$, we can control the L^p norm of the entropy solutions. But then, when we would like to show that the limiting measure is consistent with the entropy, see e.g. Theorem 3.8.3, we need to pass to the limit for $G=\eta$. Such G just fails to belong to $C_p(\mathbb{R}^m)$. This is a quite subtle issue which can be bypassed by delicate analysis arguments and an appropriate alteration of the definition of the measure valued solutions introduced by Demoulini, Stuart and Tzavaras in [17], see also [7, 21, 33]. This approach leads to modifying (3.47) by amending the effect of $\mu_{x,t}$ at the limit adding an additional positive measure γ accounting for concentration effects.

Next, we first exploit properties of the relative entropy to show that approximations obtained by our scheme yield uniformly bounded sequences in L^p. Then under certain assumptions we show the entropy compatibility at the limit of the measure valued solution as in Theorem 3.8.3. To simplify the exposition we assume that our sequence does not create concentration effects at the limit, and thus the measure γ above is zero.

3.9.1 Stability for p-entropies via relative entropy

Our first task is to prove the entropy stability of the scheme. We assume that the entropy η possesses the following properties: There exist positive constants $\alpha_0, \alpha_1, c_0, \tilde{\alpha}_0, \tilde{\alpha}_1$ such that

(η –1) $\quad \alpha_0|v|^p \le |\eta(v)| \le \alpha_1|v|^p, \qquad v \in \mathbb{R}^m\,,$

(η –2) $\quad D^2\eta(z)\xi \cdot \xi \ge c_0\,|z|^{p-2}\,|\xi|^2, \qquad \xi, z \in \mathbb{R}^m\,,$

(η –3) $\quad \tilde{\alpha}_0|v|^{p-1} \le |\eta_u(v)| \le \tilde{\alpha}_1|v|^{p-1}, \qquad v \in \mathbb{R}^m\,,$

A consequence of (η –2) is that the *relative entropy* is non-negative:

$$\eta(v\,|\,w) := \eta(v) - \eta(w) - \eta_u(w)\cdot(v-w) \ge 0, \qquad v, w \in \mathbb{R}^m\,. \tag{3.48}$$

The following lower bound of the relative entropy follows from the above properties, [32],

Lemma 3.9.1 *Assume that the convex entropy η satisfies (η –2) with p such that $p-2 \ge 1$. Then there is a positive constant β such that the* relative entropy *satisfies:*

$$\eta(v\,|\,w) := \eta(v) - \eta(w) - \eta_u(w)\cdot(v-w) \ge \beta\Big(|v-w|^p - |w|^p\Big), \qquad v, w \in \mathbb{R}^m\,. \tag{3.49}$$

Proof: Notice that (η –2) implies

$$\begin{aligned}\eta(v\,|\,w) =&\eta(v) - \eta(w) - \eta_u(w)\cdot(v-w)\\ =&\int_0^1 s\, D^2\eta(v+s(w-v)\,)\, ds(v-w)\cdot(v-w)\\ \geq& c_0\int_0^1 s\,|\,v+s(w-v)\,|^{p-2}\, ds|v-w|^2\end{aligned}$$

Since, obviously,

$$|s(w-v)\,| = |v+s(w-v)-v| \leq |v+s(w-v)| + |v|\,,$$

and $|\cdot|^{p-2}$ is convex,

$$|s(w-v)\,|^{p-2} \leq \Big(|v+s(w-v)|+|v|\Big)^{p-2} \leq \tilde{c}\,\Big(|v+s(w-v)|^{p-2}+|v|^{p-2}\Big)\,,$$

i.e.,

$$\frac{1}{\tilde{c}}|s(w-v)\,|^{p-2} - |v|^{p-2} \leq |v+s(w-v)|^{p-2}\,.$$

Thus, using Young's inequality $ab \leq \frac{a^{\tilde{p}}}{\tilde{p}} + \frac{b^{\tilde{q}}}{\tilde{q}}$, with $\tilde{p} = p/2$ and $\tilde{q} = p/(p-2)$ we finally obtain for any δ small enough,

$$\begin{aligned}\eta(v\,|\,w) =&\eta(v) - \eta(w) - \eta_u(w)\cdot(v-w)\\ \geq& c\int_0^1 s\,|\,v+s(w-v)\,|^{p-2}\, ds|v-w|^2\\ \geq& c\int_0^1 s\,|s(w-v)\,|^{p-2} - |v|^{p-2}\, ds|v-w|^2\\ =& c\int_0^1 s^{p-1}\,|v-w\,|^p - s|v|^{p-2}|v-w|^2\, ds\\ \geq& c|v-w\,|^p - \delta\,|v-w\,|^p - C(\delta)|v|^p\,.\end{aligned}$$

The proof is complete upon selecting δ appropriately small. □

We are ready now to prove the main stability result for our scheme.

Lemma 3.9.2 *Assume that the entropy η satisfies (η –1, 2, 3) with p such that $p-2 \geq 1$. Then for $N = 1, 2, \ldots$, we have the estimate*

$$\gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L_2(S_n)} + \sum_{n=0}^{N-1} \int_{\mathbb{R}} \eta(u_h^{n-} \mid u_h^{n+}) + \int_{\mathbb{R}} \eta(u_h^{N-}) \leq \int_{\mathbb{R}} \eta(u_0)\,. \tag{3.50}$$

Furthermore, there exists a constant C independent of h such that

$$\sup_{t\geq 0} \|u_h(t,\cdot)\|_{L^p(\mathbb{R})} \leq C\,. \tag{3.51}$$

If instead of (η –2), the stronger condition

(η –2′) $\quad D^2\eta(z)\xi\cdot\xi \geq c_0\,(|z|^{p-2}+1)\,|\xi|^2, \qquad \xi, z \in \mathbb{R}^m\,,$

holds, then

$$\gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L_2(S_n)} + c_0 \sum_{n=0}^{N-1} \|u_h^{n-} - u_h^{n+}\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} \eta(u_h^{N-}) \leq \int_{\mathbb{R}} \eta(u_0)\,. \tag{3.52}$$

Remark 3.9.3 *Notice that (η –2′) is satisfied for entropies behaving as $|u|^p + |u|^2$.*

Proof: As before we select $\phi_h = \tau_h$ in (3.9a). We have

$$\int_{S_n} \Big((u_h)_t + A\big(\kappa(\tau_h)\big)_x \Big) \cdot \tau_h \, dxdt + \gamma \int_{S_n} \big((\tau_h)_x\big)^2 dxdt$$
$$+ \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} \, dx = 0$$

Thus,

$$\int_{S_n} (u_h)_t \cdot \tau_h dxdt + \int_{S_n} \eta_u(\kappa(\tau_h)) \nabla_u A(\kappa(\tau_h)\big) \cdot \kappa(\tau_h)_x dxdt + \gamma \int_{S_n} \big((\tau_h)_x\big)^2 dxdt$$
$$+ \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \tau_h^{n+} dx = 0.$$

As in Lemma 3.6.1 we finally obtain

$$\int\limits_{S_n} \eta(u_h)_t dxdt + \gamma \int\limits_{S_n} \big((\tau_h)_x\big)^2 dxdt + \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \eta_u(u_h^{n+}) dx = 0.$$

Summation over n implies,

$$\sum_{n=0}^{N-1} \int\limits_{\mathbb{R}} \Big(\eta(u_h^{n+1-}) - \eta(u_h^{n+}) + (u_h^{n+} - u_h^{n-}) \ \cdot \eta_u(u_h^{n+})\Big)\, dx \\ + \gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L^2(S_n)} = 0.$$

Hence,

$$\begin{aligned} &\sum_{n=0}^{N-1} \int\limits_{\mathbb{R}} \big(\eta(u_h^{n-}) \ - \eta(u_h^{n+}) - (u_h^{n-} - u_h^{n+}) \cdot \eta_u(u_h^{n+})\big)\, dx \\ &\qquad + \gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L^2(S_n)} + \int\limits_{\mathbb{R}} \eta(u_h^{N-}) dx \\ &= \sum_{n=0}^{N-1} \int\limits_{\mathbb{R}} \eta(u_h^{n-} \,|\, u_h^{n+})\, dx \\ &\qquad + \gamma \sum_{n=0}^{N-1} \|(\tau_h)_x\|^2_{L^2(S_n)} + \int\limits_{\mathbb{R}} \eta(u_h^{N-}) dx = \int\limits_{\mathbb{R}} \eta(u_0) dx. \end{aligned} \tag{3.53}$$

and the first assertion follows. Since $\int\limits_{\mathbb{R}} \eta(u_0)dx$ are uniformly bounded and the relative entropy $\int\limits_{\mathbb{R}} \eta(u_h^{n-} \,|\, u_h^{n+})\, dx$ is always nonnegative, the above estimate implies that for all N, $\int\limits_{\mathbb{R}} \eta(u_h^{N-})dx$ are uniformly bounded. Thus $\|u_h^{N-}\|_{L^p(\mathbb{R})} \le C$ for all N. We conclude therefore that for all N,

$$\sum_{n=0}^{N-1} \int\limits_{\mathbb{R}} \eta(u_h^{n-} \,|\, u_h^{n+})\, dx \le C\,, \tag{3.54}$$

and in particular,

$$\int\limits_{\mathbb{R}} \eta(u_h^{n-} \,|\, u_h^{n+})\, dx \le C\,, \tag{3.55}$$

Now Lemma 3.9.1 implies

$$\|u_h^{n-} - u_h^{n+}\|_{L^p(\mathbb{R})}^p \le C + c\,\|u_h^{n-}\|_{L^p(\mathbb{R})}^p . \tag{3.56}$$

But we have proved that $\|u_h^{n-}\|_{L^p(\mathbb{R})}$ is uniformly bounded and thus $\|u_h^{n-} - u_h^{n+}\|_{L^p(\mathbb{R})}$ is uniformly bounded as well. By the triangle inequality we obtain that $\|u_h^{n+}\|_{L^p(\mathbb{R})}$ is uniformly bounded. Since u_h is piecewise linear in time and its end points are uniformly bounded in $L^p(\mathbb{R})$, the second assertion follows. Finally (η –$2'$) and (3.9.2) implies (3.52) and the proof is complete. $\square$

Recall now that

$$\begin{aligned}\tau_h(x,t)|_{S_n} &= P_{X_h}(\eta_u(u_h^{n+}))\frac{t^{n+1}-t}{|I_n|}\\ &\quad + \frac{t-t^n}{|I_n|}\left(P_{X_h}\left(\int_{I_n}\eta_u(u_h(x,t))dt\right)\frac{2}{|I_n|} - P_{X_h}(\eta_u(u_h^{n+}))\right) .\end{aligned}$$

Let $q > 1$ be the conjugate of p. Then $(p-1)q = p$. Using the growth of η_u, see (η –3), and the stability of P_{X_h} in L^q we obtain, for each S_n,

$$\begin{aligned}\|\tau_h(\cdot,t)\|_{L^q(\mathbb{R})}^q|_{I_n} &\le C(\|\eta_u(u_h^{n+})\|_{L^q(\mathbb{R})}^q + \|\eta_u(u_h^{n-})\|_{L^q(\mathbb{R})}^q)\\ &\le C(\|u_h^{n+}\|_{L^p(\mathbb{R})}^p + \|u_h^{n-}\|_{L^p(\mathbb{R})}^p) \le C .\end{aligned}$$

Thus,

$$\sup_{t\ge 0}\|\tau_h(t,\cdot)\|_{L^q(\mathbb{R})} \le C . \tag{3.57}$$

3.9.2 Convergence

We can now state the two main theorems of this section.

Theorem 3.9.4 *Let η satisfies (η –1), (η –2′), and (η –3). If the flux A satisfies the growth condition $A \in C_p$ and furthermore the following compatibility condition is satisfied*

$$A(\kappa(\tau_h)) \stackrel{*}{\rightharpoonup} \langle A, \mu \rangle, \qquad \textit{in } L^1(\Omega) \tag{3.58}$$

where μ is a Young measure associated to u_h, then the numerical scheme (3.9) *converges towards a measure-valued solution of* (3.1).

Proof: The uniform bound of u_h in L^p, [4], implies that there a exist a subsequence u_{h_l} (which we relabel here u_h) and a Young measure $\mu \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ such that

$$G((u_h)) \stackrel{*}{\rightharpoonup} \langle G, \mu \rangle, \qquad \text{in } L^1(\Omega) \tag{3.59}$$

as $h \to 0$. And further, on account to the fact that τ_h is also uniformly bounded in L^q we deduce that there exist a subsequence $\tau_{h_{l_\epsilon}}$ of the sequence τ_{h_l} (denoted as τ_h) and a young measure $\mu' \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ such that

$$G(\kappa(\tau_h)) \stackrel{*}{\rightharpoonup} \langle G(\kappa), \mu' \rangle \text{ in } L^1(\Omega). \tag{3.60}$$

The compatibility assumption essentially says that $\langle A(\kappa), \mu' \rangle = \langle A, \mu \rangle$. On the other hand by modifying arguments of the previous section, see the proof of Theorem 3.7.1, we notice,

$$
\begin{aligned}
&\left| \int\limits_{\Omega} \big(u_h \cdot \phi_t + A(\kappa(\tau_h)) \cdot \phi_x\big) dx dt + \int\limits_{\mathbb{R}} u_0 \cdot \phi(0,x) dx \right| \\
&\leq \int\limits_{\Omega} \left|\nabla_u A\big(\kappa(\tau_h)\big)\right| \left|(\tau_h)_x\right| \left|\phi - P_{V_h}(\phi)\right| dx dt \\
&+ \sum_{n \geq 0} \int\limits_{\mathbb{R}} \left|u_h^{n+} - u_h^{n-}\right| \left|\phi^{n+} - (P_{V_h}\phi)^{n+}\right| dx + \gamma \int\limits_{\Omega} \left|(\tau_h)_x\right| \left|P_{V_h}(\phi))_x\right| dx dt \\
&\leq \ \|\nabla_u A\big(\kappa(\tau_h)\big)\|_{L_2(\Omega)} \|(\tau_h)_x\|_{L_2(\Omega)} \|\phi - P_{V_h}\phi\|_{L_\infty(\Omega)} \\
&+ \left(\sum_{n=0}^{n_0} \|u_h^{n-} - u_h^{n+}\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{n_0} \|\phi^{n+} - (P_{V_h}\phi)^{n+}\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} \\
&+ \gamma \|(\tau_h)_x\|_{L_2(\Omega)} \|(P_{V_h}\phi)_x\|_{L_2(\Omega)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left| \int\limits_{\Omega} \big(u_h \cdot \phi_t + A(\kappa(\tau_h)) \cdot \phi_x\big) dx dt + \int\limits_{\mathbb{R}} u_0 \cdot \phi(0,x) dx \right| \\
&\leq \|\tau_h\|_{L^2(\Omega)} \gamma^{\frac{1}{2}} \|(\tau_h)_x\|_{L^2(\Omega)} C \gamma^{-\frac{1}{2}} h^2 \|\phi\|_{W^{2,\infty}(\Omega)} + C h^{\frac{3}{2}} \|\phi\|_{H^2(\Omega)} \\
&+ \gamma \|(\tau_h)_x\|_{L_2(\Omega)} \|(P_{V_h}\phi)_x\|_{L^2(\Omega)}
\end{aligned}
\tag{3.61}
$$

Therefore, using the inverse inequality, $\|\chi\|_{L^2} \leq C h^{1/2-1/q} \|\chi\|_{L^q}$, [8, Lemma 4.5.3], while choosing a proper γ such that $\gamma^{-\frac{1}{2}} h^2 h^{1/2-1/q} \to 0$ and having in mind that (3.12) implies $(\gamma)^{\frac{1}{2}} \|(\tau_h)_x\|_{L^2(\Omega)} \leq C$ we conclude that,

$$
\int\limits_{\Omega} \big(u_h \cdot \phi_t + A(\kappa(\tau_h)) \cdot \phi_x\big) dx dt + \int\limits_{\mathbb{R}} u_0 \cdot \phi(0,x) dx \to 0,
$$

as $h \to 0$. But by our compatibility assumption on Young measures we obtain by letting $h \to 0$

$$\int\limits_{\Omega} \big(u_h \cdot \phi_t + A(\kappa(\tau_h)) \cdot \phi_x\big) dxdt + \int\limits_{\mathbb{R}} u_0 \cdot \phi(0,x)dx \to$$

$$\int\limits_{\Omega} \big(\langle id, \mu_{x,t}\rangle \cdot \phi_t + \langle A, \mu_{x,t}\rangle \cdot \phi_x\big) dxdt + \int\limits_{\mathbb{R}} u_0 \cdot \phi(0,x)dx = 0$$

which proves the theorem. □

To show that the limiting measure-valued solution is entropic, we shall neglect the possible concentration effects corresponding to the Young measures generated by u_h and τ_h. We thus have the following

Theorem 3.9.5 *Let η satisfies (η –1), (η –2′), and (η –3). Assume that the flux A is such that the following compatibility condition is satisfied*

$$Q(\kappa(\tau_h)) \stackrel{*}{\rightharpoonup} \langle Q, \mu\rangle, \qquad \textit{in } L^1(\Omega) \tag{3.62}$$

where μ is a Young measure associated to u_h, and furthermore

$$\eta(u_h) \stackrel{*}{\rightharpoonup} \langle \eta, \mu\rangle, \qquad \textit{in } L^1(\Omega) \tag{3.63}$$

holds. If τ_h are uniformly bounded in L^2 then the measure-valued solution established in Theorem 3.9.4 *is entropy consistent i.e. it fulfils the relation* (3.35).

Proof: Selecting $\phi_h = P_{V_h}(\tau_h \phi)$ in (3.9a) where $\phi \in C_c^\infty(\overline{\Omega})$, $\phi \geq 0$ and following similar arguments as in the second part of the proof of Theorem 3.7.1 we conclude

$$
\begin{aligned}
&-\int\limits_{\Omega} \big(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x\big)\,dxdt - \int\limits_{\mathbb{R}} \eta(u_0)\cdot\phi(x,0) \\
&+\sum_n \int\limits_{\mathbb{R}} \bigg(\eta((u_h^{n-}) - \eta((u_h^{n+})) - (u_h^{n-} - u_h^{n+})\cdot\tau_h^{n+}\bigg)\phi^{n+}\,dx \\
&\quad \leq \int\limits_{\Omega} A\big(\kappa(\tau_h)\big)_x \cdot \bigg(\tau_h\phi - P_{V_h}(\tau_h\phi)\bigg)dxdt \\
&\quad + \sum_n \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-})\cdot(\tau_h^{n+}\phi^{n+} - (P_{V_h}(\tau_h\phi))^{n+})dx \\
&\quad - \Big\{\gamma\int\limits_{\Omega}(\tau_h)_x\cdot(P_{V_h}(\tau_h\phi))_x\,dxdt - \gamma\int\limits_{\Omega}(\tau_h)_x(\tau_h\phi)_x dxdt + \gamma\int\limits_{\Omega}(\tau_h)_x\tau_h\phi_x dxdt\Big\} \\
&\quad - \int\limits_{\Omega}\big(\tau_h - \eta_u(u_h)\big)\cdot\big((u_h)_t\phi - P_{W_h}((u_h)_t\phi)\big)dxdt \\
&\quad =: A_1 + A_2 + A_3 + A_4.
\end{aligned}
\tag{3.64}
$$

We will proceed in a similar fashion as in the proof of Theorem 3.7.1, but using only the available bounds for u_h and τ_h. We will thus highlight only the points where different treatment is required. We estimate the terms A_i in the right hand side of (3.64). Notice first

$$
\begin{aligned}
|A_1| &\leq \int\limits_{\Omega} \big|\nabla_u A\big(\kappa(\tau_h)\big)\big|\,\big|(\tau_h)_x\big|\big|\tau_h\phi - P_{V_h}(\tau_h\phi)\big|dxdt \\
&\leq \|\nabla_u A\big(\kappa(\tau_h)\big)\|_{L_\infty(\Omega)}\|(\tau_h)_x\|_{L_2(\Omega)}\|\tau_h\phi - P_{V_h}(\tau_h\phi)\|_{L_2(\Omega)} \\
&\leq C\,\|\tau_h)\|_{L_\infty(\Omega)}\|(\tau_h)_x\|_{L_2(\Omega)}(h^2\|\tau_h\|_{L_2(\Omega)} + h^2\|(\tau_h)_x\|_{L_2(\Omega)})\,,
\end{aligned}
$$

where we used the fact $\|\tau_h\phi - P_{V_h}(\tau_h\phi)\|_{L_2(\Omega)} \leq C(h^2\|\tau_h\|_{L_2(\Omega)} + h^2\|(\tau_h)_x\|_{L_2(\Omega)})$ which can be derived by similar arguments to the proof of Lemma 3.6.3. Further, using the inverse inequality,

$\|\chi\|_{L^\infty} \leq Ch^{-1/2}\|\chi\|_{L^2}$, [8, Lemma 4.5.3], and the stability bound for $\|(\tau_h)_x\|_{L_2(\Omega)}$ we conclude

$$|A_1| \leq \, C\,(h^{3/2}\,\gamma^{-1} + h^{3/2}\,\gamma^{-1/2})\,. \tag{3.65}$$

Further, estimate $\|\tau_h^{n+}\phi^{n+} - (P_{V_h}(\tau_h\phi))^{n+}\|_{L_2(\Omega)} \leq Ch\|\tau_h^{n+}\|_{L_2(\Omega)}$ and the stability bound for the jumps in time of u_h imply,

$$|A_2| \leq \, C\,h^{1/2}\,. \tag{3.66}$$

Similarly, we observe,

$$\begin{aligned}
&\left| -\gamma \int_\Omega (\tau_h)_x \cdot ((P_{V_h}(\tau_x))_x - (\tau_h\phi)_x)\,dxdt \right| \\
&\qquad \leq \gamma\|(\tau_h)_x\|_{L^2(\Omega)}\|(P_{V_h}(\tau\phi))_x - (\tau_h\phi)_x\|_{L^2(\Omega)} \\
&\qquad \leq \gamma\|(\tau_h)_x\|_{L^2(\Omega)}\|\tau_h\|_{L^2(\Omega)}(\|\phi\|_{W^{1,\infty}(\Omega)} + h\|\phi\|_{W^{2,\infty}(\Omega)}) \\
&\qquad \leq C(1+h)\gamma^{\frac{1}{2}}\,.
\end{aligned}$$

Thus,

$$|A_3| \leq C(1+h)\gamma^{\frac{1}{2}}. \tag{3.67}$$

As in the proof of Theorem 3.7.1 one can show, using the stability properties of the scheme,

$$\Big(\sum_{n=0}^{N-1} \|(u_h)_t\|^2_{L^2(S_n)}\Big)^{\frac{1}{2}} \leq C(\gamma^{-\frac{1}{2}} + \gamma^{\frac{1}{2}}h^{-1} + h^{-\frac{1}{2}}). \tag{3.68}$$

and

$$
\begin{aligned}
|A_4| \leq & \left| \sum_{n\geq 0} \int_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot \big((u_h)_t \phi - (u_h)_t P_{0,t}\phi\big) dxdt \right| \\
& \qquad + \left| \sum_{n\geq 0} \int_{S_n} \big(\tau_h - \eta_u(u_h)\big) \cdot \Big((u_h)_t P_{0,t}\phi - P_{X_h}\big((u_h)_t P_{0,t}\phi\big)\Big) dxdt \right| \\
& =: |\widetilde{A_1}| + |\widetilde{A_2}|
\end{aligned}
$$

where we used the fact $(u_h(.,x))_t \in \mathbb{P}_0(I_n)$ and thus $P_{0,t}((u_h)_t\phi) = (u_h)_t P_{0,t}\phi$.

The term $\widetilde{A_1}$ is estimated, using the inverse inequality, $\|\chi\|_{L^p} \leq Ch^{1/p-1/2}\|\chi\|_{L^2}$, as

$$
\begin{aligned}
|\widetilde{A_1}| &\leq C \sum_{n\geq 0} \big((\|\tau_h - \eta_u(u_h)\|_{L^q(S_n)} \|(u_h)_t\|_{L^p(S_n)}\big) \big(\|\phi - P_{0,t}\phi\|_{L^\infty(S_n)}\big) \\
&\leq Ch^{1/p-1/2} \left(\sum_{n\geq 0} \|(u_h)_t\|^2_{L^2(S_n)} \right)^{\frac{1}{2}} |\phi - P_{0,t}\phi\|^2_{L^\infty(\Omega)} \\
&\leq C \left(\gamma^{-\frac{1}{2}} + \gamma^{\frac{1}{2}} h^{-1} + h^{-\frac{1}{2}}\right) h\, h^{1/p-1/2} \, \|\phi\|_{W^{1,\infty}(\Omega)} \, .
\end{aligned}
$$

Notice that since $P_{0,t}$ commutes with differentiation with respect to x and it is stable in $L^p(I_n)$ and in $L^\infty(I_n)$, one can verify as in Lemma 2.3, that

$$
\|(u_h)_t P_{0,t}\phi - P_{X_h}\big((u_h)_t P_{0,t}\phi\big)\|_{L^p(S_n)} \leq Ch\|(u_h)_t\|_{L^p(S_n)}.
$$

Hence by the same arguments,

$$
|\widetilde{A_2}| \leq Ch\|(u_h)_t\|_{L^2(\Omega)} \leq C \left(\gamma^{-\frac{1}{2}} h + \gamma^{\frac{1}{2}} + h^{\frac{1}{2}}\right) h^{1/p-1/2},
$$

and thus

$$
|A_4| \leq C \left(\gamma^{-\frac{1}{2}} h + \gamma^{\frac{1}{2}} + h^{\frac{1}{2}}\right) h^{1/p-1/2}. \tag{3.69}
$$

Summarising,

$$
\begin{aligned}
|A_1| + |A_2| + |A_3| + |A_4| &\le C\,\left(\gamma^{-\frac{1}{2}} h + \gamma^{\frac{1}{2}} + h^{\frac{1}{2}}\right)\, h^{1/p-1/2} + C(h^{3/2}\,\gamma^{-1} + h^{3/2}\,\gamma^{-1/2}). \\
&\le C\,(\gamma^{-1/2} h^{1/p+1/2} + \gamma^{1/2} h^{1/p-1/2} + h^{1/p} + h^{3/2}\,\gamma^{-1} + h^{3/2}\,\gamma^{-1/2}) \\
&=: \Theta(h,\gamma).
\end{aligned}
\tag{3.70}
$$

Following again Theorem 3.7.1 we conclude,

$$
\begin{aligned}
&-\int_{\Omega} \big(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x\big)\, dxdt - \int_{\mathbb{R}} \eta(u_0)\cdot\phi(x,0) \\
&+ \sum_n \int_{\mathbb{R}} \Big(\eta(u_h^{n-}) - \eta(u_h^{n+}) - \eta_u(u_h^{n+})\cdot(u_h^{n-} - u_h^{n+})\Big)\phi^{n+} dx \\
&\le \Theta(h,\gamma) + \sum_n \int_{\mathbb{R}} \Big|\eta_u(u_h^{n+})\cdot\Big((u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big)\Big)\Big| dx.
\end{aligned}
$$

Since the relative entropy is always positive, it remains to estimate the last term of the right-hand side.

To this end, let $D_\phi \subset \mathbb{R}$ is a domain such that $supp(\phi(t^n)) \subset D$ for all n, and notice that

$$
\begin{aligned}
&\sum_n \int_{\mathbb{R}} \Big|\eta_u(u_h^{n+})\cdot\Big((u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big)\Big)\Big| dx \\
&\le \max_n \|\eta_u(u_h^{n+})\|_{L^q(\mathbb{R})} \sum_n \|u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big)\|_{L^p(\mathbb{R})} \\
&\le \max_n \|u_h^{n+}\|_{L^p(\mathbb{R})} \sum_n \|u_h^{n-} - u_h^{n+})\phi^{n+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big)\|_{L^p(\mathbb{R})}
\end{aligned}
$$

Using similar arguments as in Lemma 3.6.3 we get

$$
\sum_{n\le n_0} \|(u_h^{n-} - u_h^{n+})\phi^{n^+} - P_{X_h}\Big((u_h^{n-} - u_h^{n+})\phi^{n+}\Big)\|^2_{L^p(\mathbb{R})} \le Ch^2 \sum_{n\le n_0} \|u_h^{n-} - u_h^{n+}\|^2_{L^p(\mathbb{R})}.
$$

Hence, by using inverse inequalities and the stability bound,

$$\sum_{n\le n_0} \|(u_h^{n-}-u_h^{n+})\phi^{n^+} - P_{X_h}\Big((u_h^{n-}-u_h^{n+})\phi^{n+}\Big)\|^2_{L^p(\mathbb{R})}$$
$$\le C h\, n_0^{\frac{1}{2}}\Big(\sum_{n\le n_0}\|u_h^{n-}-u_h^{n+}\|^2_{L^p(\mathbb{R})}\Big)^{\frac{1}{2}} \le C h\, h^{-\frac{1}{2}}\Big(\sum_{n\le n_0}\|u_h^{n-}-u_h^{n+}\|^2_{L^p(\mathbb{R})}\Big)^{\frac{1}{2}}$$
$$\le Ch^{1/2}\, h^{1/p-1/2}\Big(\sum_{n\le n_0}\|u_h^{n-}-u_h^{n+}\|^2_{L^2(\mathbb{R})}\Big)^{\frac{1}{2}} \le C\, h^{1/p}\,.$$

We conclude therefore that

$$-\int_\Omega \big(\eta(u_h)\phi_t + Q\big(\kappa(\tau_h)\big)\phi_x - \int_\mathbb{R} \eta(u_0)\cdot\phi(x,0)$$
$$\le \Theta(h,\gamma) + Ch^{1/p}.$$

Therefore, letting $h\to 0$, by selecting, e.g., $\gamma = ch$, in the above expression and using our hypotheses for the limiting measure we conclude that

$$-\int_\Omega \big(\langle\eta,\mu_{x,t}\rangle\phi_t + \langle Q,\mu_{x,t}\rangle\phi_x - \int_\mathbb{R} \eta(u_0)\cdot\phi(x,0) \le 0$$

and the proof is complete. $\square$

Remark 3.9.6 *Let η satisfies (η –1), (η –2′), and (η –3). In this remark we highlight that if the growth at infinity of the functions considered is neglected then* (3.58)*,* (3.62)*,* (3.63) *can be verified with some additional hypothesis on the approximating sequences. To fix ideas, assume*

$$\|\tau_h - \eta_u(u_h)\|_{L^q} \to 0, \qquad h\to 0\,, \tag{3.71}$$

and let $g:\mathbb{R}^m \mapsto \mathbb{R}$ be a continuous function that converges to zero at infinity. Then since (3.51)*,* (3.57) *hold, then there exist a subsequence of τ_h (still denoted as τ_h) for which it holds that*

$$g(\tau_h) \overset{*}{\rightharpoonup} \langle g, (\eta_u)_{\#}\mu \rangle = \langle g(\eta_u), \mu \rangle \ \text{ in } L^1(\Omega) \tag{3.72}$$

where μ is a young measure associated with u_h. As mentioned in the beginning of this section the uniform bound of u_h in L^p, [4], implies that there a exist a subsequence u_{h_l} (which we relabel here u_h) and a young measure $\mu \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ such that

$$g(\eta_u(u_h)) \overset{*}{\rightharpoonup} \langle g, (\eta_u)_{\#}\mu \rangle = \langle g(\eta_u), \mu \rangle, \qquad \text{in } L^1(\Omega) \tag{3.73}$$

as $h \to 0$. And further, on account to the fact that τ_h is also uniformly bounded in L^q we deduce that there exist a subsequence $\tau_{h_{l_\epsilon}}$ of the sequence τ_{h_l} (denoted as τ_h) and a Young measure $\mu' \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ such that

$$g(\tau_h) \overset{*}{\rightharpoonup} \langle g, \mu' \rangle \text{ in } L^1(\Omega). \tag{3.74}$$

As in Lemma 3.8.1, we consider first a smooth $\bar{g}$ and given a function $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \big(\bar{g}(\tau_h) - \bar{g}(\eta_u(u_h))\big)\phi dxdt = \int_\Omega \int_0^1 \nabla\bar{g}(\xi(x,t,s))\, ds \cdot \big(\tau_h - \eta_u(u_h)\big)\phi dxdt$$

where for each x and t, the function $\xi(x,t,s)$ is defined as $\xi(x,t,s) = \tau_h - s((\tau_h - \eta_u(u_h))$. Let now Ω_1 be a finite domain such that $supp(\phi) \subset \Omega_1$. In addition, observe that the product of the functions $\phi(x,t)$ and $\nabla\bar{g}(\xi(x,t,s))$ is a smooth enough function in Ω_1. Now set $\tilde{G}(x,t) = \int_0^1 \nabla\bar{g}(\xi(x,t,s))\, ds$. Then, as in Lemma 3.8.1, we conclude

$$\begin{aligned}\left|\int_\Omega \big(\bar{g}(\tau_h) - \bar{g}(\eta_u(u_h))\big)\phi dxdt\right| &= \left|\int_\Omega \big(\tau_h - \eta_u(u_h)\big) \cdot \big(\phi\tilde{G}(x,t) - P_{W_h}(\phi\tilde{G}(x,t))\big)dxdt\right| \\ &\leq \|\tau_h - \eta_u(u_h)\|_{L^q(\Omega_1)} \|\phi\tilde{G}(x,t) - P_{W_h}(\phi\tilde{G}(x,t))\|_{L^p(\Omega_1)}\,.\end{aligned}$$

We would like to show that

$$\|\phi\tilde{G} - P_{W_h}(\phi\tilde{G})\|_{L^p(\Omega_1)} \leq C, \qquad h \to 0. \tag{3.75}$$

Assuming for a moment the validity of (3.75) *we conclude,*

$$\int_{\Omega_1} \big(\bar{g}(\tau_h) - \bar{g}(\eta_u(u_h))\big)\phi dxdt \to 0$$

as $h \to 0$*. and the proof is concluded by repeating the arguments of Lemma 3.8.1. It thus remains to prove* (3.75)*. To this end, we first observe, as in the proof of Lemma 3.6.3 that*

$$\|\phi\tilde{G} - P_{W_h}(\phi\tilde{G})\|_{L^p(S_n)} \leq c\,\|\phi\tilde{G} - P_{0,t}(\phi\tilde{G})\|_{L^p(S_n)} + c\,\|\phi\tilde{G} - I_{X_h}(\phi\tilde{G})\|_{L^p(S_n)}\,.$$

Then, by the uniform boundedness of $u_h,\ \tau_h$*, and the fact that* τ_h *is piecewise constant in time, we have*

$$\begin{aligned}\|\phi\tilde{G} - P_{0,t}(\phi\tilde{G})\|_{L^p(S_n)} \leq& ch\|\partial_t(\phi\tilde{G})\|_{L^p(S_n)} \\ \leq& ch\|(\partial_t\phi)\tilde{G}\|_{L^p(S_n)} + ch\|\phi(\partial_t\tilde{G})\|_{L^p(S_n)}\,.\end{aligned}$$

By modifying the arguments in the proof of Lemma 3.6.3 and for a typical spacial element I' *of the decomposition of the finite element space* X_h*, we have,*

$$\|\phi\tilde{G} - I_{X_h}(\phi\tilde{G})\|_{L^p(I')} \leq Ch^2|\phi\tilde{G}|_{W^{2,p}(I')}$$

see [8]. For each fixed t *in* S_n *the spatial estimate holds*

$$\|(\phi\tilde{G}) - I_{X_h}(\phi\tilde{G})\|^2_{L^2(\mathbb{R})} \leq \sum_{I'} \|(\phi\tilde{G}) - I_{X_h}(\phi\tilde{G})\|^2_{L^2(I')} \leq Ch^4\sum_{I'}|\phi\tilde{G}|^2_{H^2(I')}. \tag{3.76}$$

Next, one may use, $\partial_x^2(\phi\tilde{G}) = 2\partial_x\phi\partial_x\tilde{G} + \tilde{G}\partial_x^2\phi + \phi\partial_x^2\tilde{G}$ *, and the fact that piecewise in each element*

$\partial_x^2 \tau_h = \partial_x^2 u_h = 0$ *to control* $|\phi \tilde{G}|_{W^{2,p}(I')}$ *. To complete the boundedness of* (3.75) *a combination of stability bounds, inverse estimates and growth assumptions on G will be required.*

3.10 Numerical Results

We will present now some numerical results of an implementation of the numerical scheme (3.9a)-(3.9c). For the purposes of the experiment we use the finite element spaces

$$V_h^n = \{v \in H^1(S_n) : v|_K = v_1(x)v_2(t), v_1(x)|_K \in P_1, v_2(t)|_K \in P_1, K \in T_h^n\}.$$

and

$$W_h^n = \{\psi \in H^1(S_n) : \psi|_K = \psi_1(x)\psi_2(t), \psi_1(x)|_K \in P_1, \psi_2(t)|_K \in P_0, K \in T_h^n\},$$

on each T_h^n for the following problem

$$\partial_t u + \partial_x (\frac{u^2}{2}) = 0, \quad x \in [0,1], \quad t > 0,$$
$$u(0) = u(1) = 0.$$

with initial conditions

$$u(0,x) = \begin{cases} 0 & \text{for } x < 2.05, \\ 1 & \text{for } 2.05 < x \le 5, \\ (7.9 - x)/2.9 & \text{for } 5 \le x \le 7.9, \\ 0 & \text{for } x > 7.9. \end{cases}$$

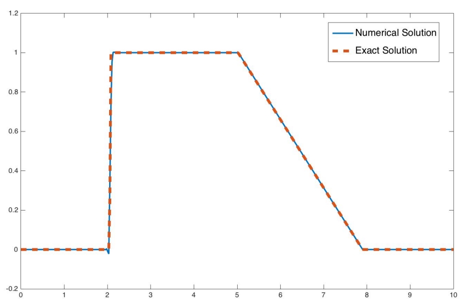
In the next result, we pick $\eta(u) = e^u$ to be the entropy function of our problem. Thus the numerical scheme (3.9a)-(3.9c) takes the form

$$\int_{S_n} \left((u_h)_t + \frac{1}{2}\left[\left(ln(\tau_h) \right)^2 \right]_x \right) \cdot \phi_h \, dxdt + \delta \int_{S_n} (\tau_h)_x \cdot (\phi_h)_x \, dxdt$$
$$+ \int_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \phi_h^{n+} \, dxdt = 0,$$

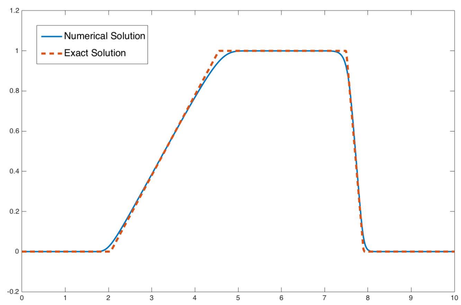
$$\int_{S_n} \left(\tau_h - (e^{u_h}) \right) \cdot \psi_h \, dxdt = 0,$$

$$\int_{\mathbb{R}} \left(\tau_h^{n+} - (e^{u_h^{n+}}) \right) \cdot \psi_h^{n+} \, dx = 0.$$

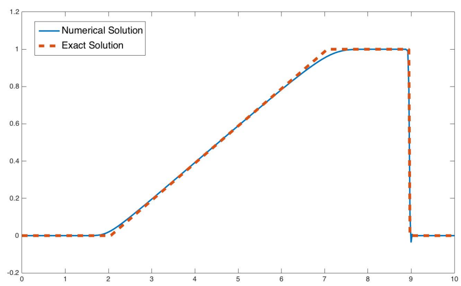
The partitions of both x and t axis in our experiment are consisted of 400 nodes. The following graphs show the results which have been obtained when taking different values of δ after 1, 100 and 200 time steps.



(a) $t = 0.02$

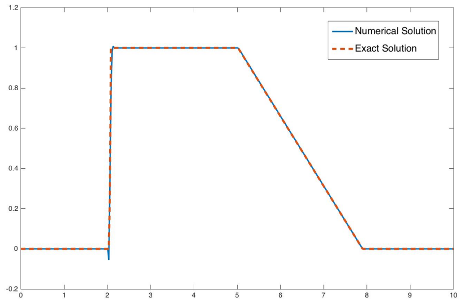


(b) $t = 2.50$

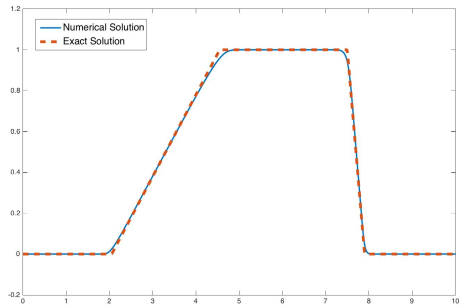


(c) $t = 5.01$

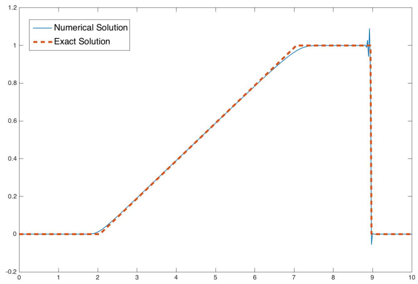
Figure 3.1: Numerical experiment showing the solution when $\delta = h/10$



(a) $t = 0.02$

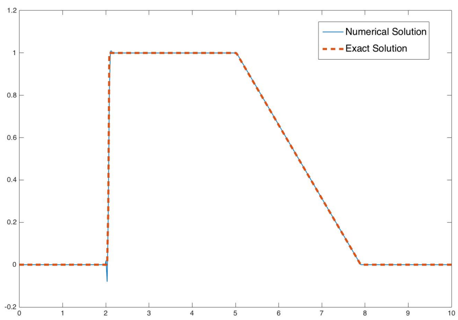


(b) $t = 2.50$

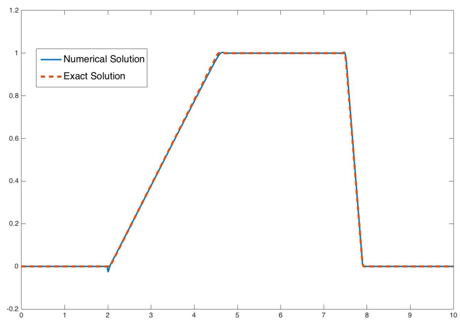


(c) $t = 5.01$

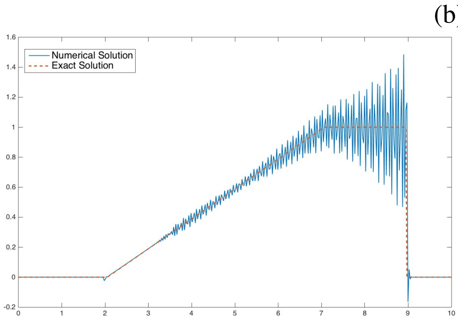
Figure 3.2: Numerical experiment showing the solution when $\delta = h/20$



(a) $t = 0.02$



(b) $t = 2.50$



(c) $t = 5.01$

Figure 3.3: Numerical experiment showing the solution when $\delta = 0$

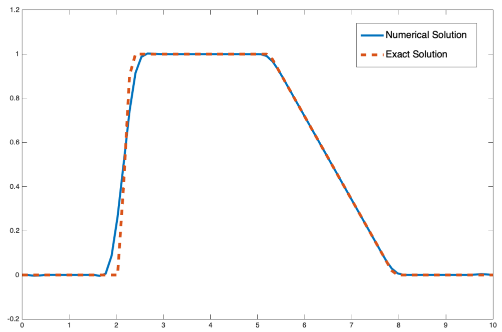
Next we present one more experiment, where we pick $\eta(u) = \frac{u^2}{2}$ to be the entropy function of our problem. Thus the numerical scheme (3.9a)-(3.9c) takes the form

$$\int\limits_{S_n} \left((u_h)_t + \frac{1}{2}\left(\tau_h^2\right)_x \right) \cdot \phi_h \, dxdt + \delta \int\limits_{S_n} (\tau_h)_x \cdot (\phi_h)_x \, dxdt$$
$$+ \int\limits_{\mathbb{R}} (u_h^{n+} - u_h^{n-}) \cdot \phi_h^{n+} \, dxdt = 0,$$

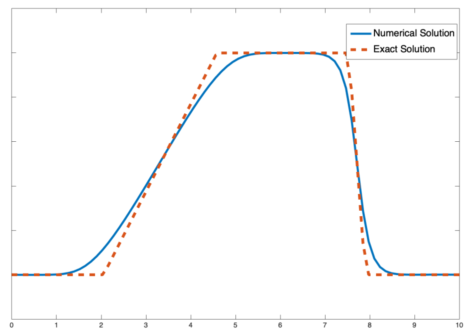
$$\int\limits_{S_n} \left(\tau_h - u_h\right) \cdot \psi_h \, dxdt = 0,$$

$$\int\limits_{\mathbb{R}} \left(\tau_h^{n+} - u_h^{n+}\right) \cdot \psi_h^{n+} \, dx = 0.$$

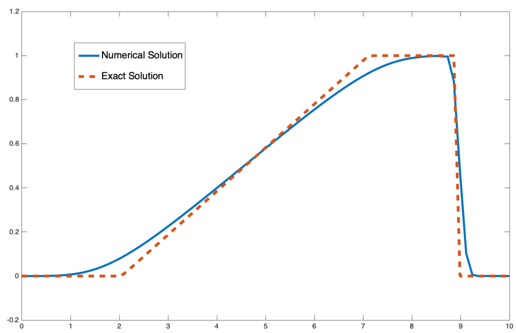
The experiment has been implemented three times where at each time we have doubled the number of the nodes of the previous partition. Thus, the first result includes for both x and t axis 80 nodes , the partition of the second result consist of 160 nodes and third result of 320 nodes. The graphs following show the results which have been obtained when taking a constant $\delta = \frac{h}{4}$.



(a) $t = 0.25$

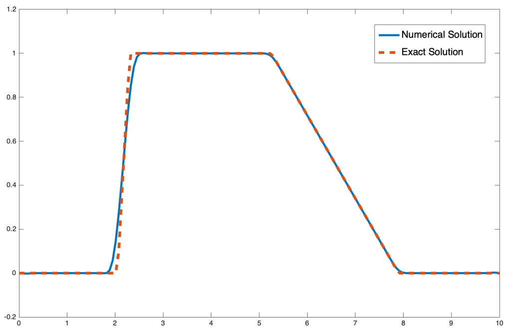


(b) $t = 2.50$

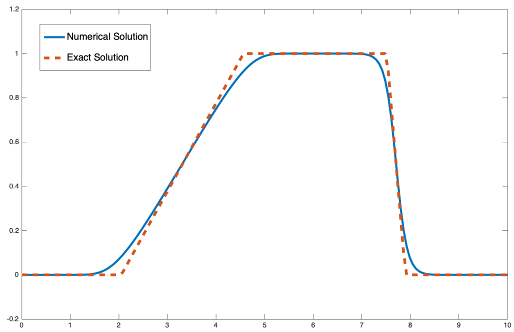


(c) $t = 5.06$

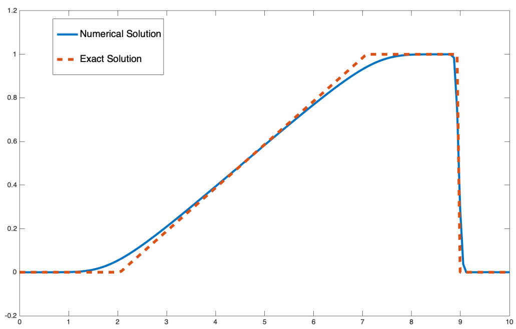
Figure 3.4: Results which have been obtained using a partition of 80 nodes



(a) $t = 0.25$

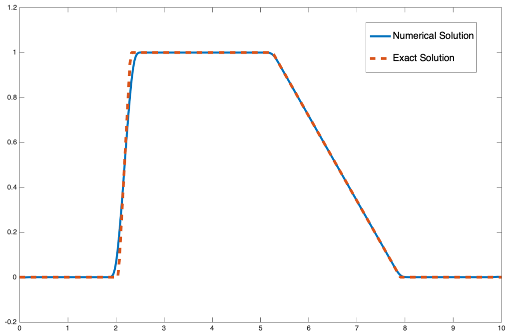


(b) $t = 2.50$

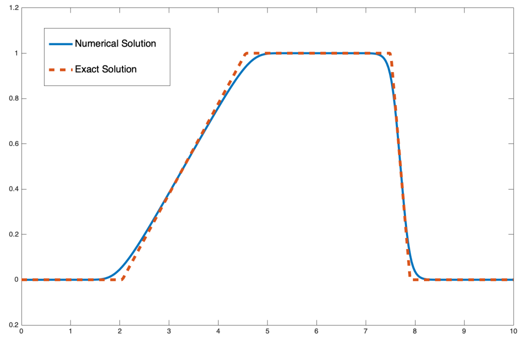


(c) $t = 5.03$

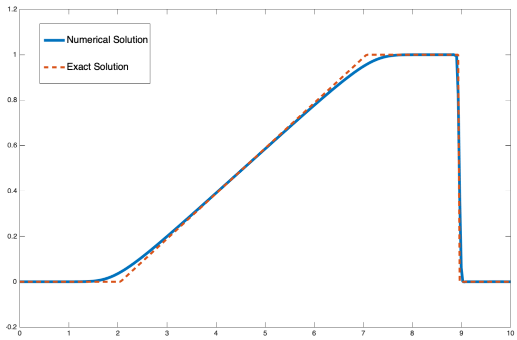
Figure 3.5: Results which have been obtained using a partition of 160 nodes



(a) $t = 0.25$



(b) $t = 2.50$



(c) $t = 5.01$

Figure 3.6: Results which have been obtained using a partition of 320 nodes

Chapter 4

Approximation of Measure-Valued solutions of HCL

4.1 Chapter overview

Our aim is to develop a new approach to the computation of measure valued solutions and to quantify uncertainties for nonlinear hyperbolic problems, based on two key ingredients : *approximate Young measures* and *kinetic models*. We first present a framework for constructing *approximate Young measures,* based on earlier results by [48, 43]. Approximate Young measures were developed having in mind applications to calculus of variations and to energy minimisation, see [6]. We show below that in the framework of conservation laws the approximation of the equation for measure valued solutions by such approximate measures, gives rise in a natural way to discrete kinetic models. These models are, however, severely under-determined. We overcome this issue by using tools from the kinetic formulation of conservation laws, see Chapter 2, [40, 45]. By using viscosity approximations and appropriate discrete defect measures we construct new discrete kinetic models; their solutions will provide approximations to entropic measure valued solutions. We further note, see [27], that this approach can be extended to design a hierarchy of discrete kinetic models approximating statistical solutions for scalar conservation laws based on correlation measures, [22]. Up to our knowledge, this approach provides the first systematic alternative to Monte Carlo sampling for approximating measure-valued solutions to conservation laws. The approximate models, in all cases, rely on solving discretised kinetic equations with prescribed approximate defect measures on the right hand side.

4.2 Computation of measure-valued solutions for hyperbolic problems

We focus on the scalar conservation law

$$u_t(x,t) + \mathrm{div} A(u(x,t)) = 0, \quad x \in \mathbb{R}^d, t > 0. \tag{4.1}$$

As mentioned we shall focus on approximating models and schemes for the computation of measure valued and statistical solutions of this equation. In this problem the behaviour of approximations of solutions is not always certain. Uncertainties in the solution can be caused, for instance, by the initial data, or the parameters appearing in the model. One of the reasons is that in practice it is impossible to obtain exact measurements. Hence, we are interested in studying and computing solutions that deal with the problem of uncertainty in PDEs. Furthermore, a similar problem from a mathematical perspective relates to statistical inference on the solutions when we study an assembly of variable data of the model. Statistics is a discrete endeavour and when it comes to complicated models such as nonlinear PDEs there are more than one (continuous) mathematical settings to formulate problems. A possible way to access uncertainty in nonlinear hyperbolic systems is to use the concept of measure-valued or statistical solutions, [38, 24, 18, 21, 2, 3, 1].

For simplicity of the exposition, we will present our approximate models in the one-dimensional ($d = 1$) case. The extension to the multidimensional scalar case is straightforward. For convenience and to fix the notation for $d = 1$ we repeat the definitions of measure valued solutions next.

*W*eakly* measurable functions. Let V be a normed space. A function $\mu : \Omega \to V^\star$ is called weakly* measurable if for $x \in \Omega$ the function $x \mapsto \langle \mu(x), \psi \rangle$ is measurable for all $\psi \in V$ where $V^\star$ is the dual space of V.

Measure valued solutions. Let now $\mathbf{M}^+(\mathbb{R}^m)$ be the set of all positive Radon measures on $\mathbb{R}^m$, and $\mathbf{M}^{\mathbb{P}}(\mathbb{R}^m) = \{\mu \in \mathbf{M}^+(\mathbb{R}^m), \mu(\mathbb{R}^m) = 1\}$ the corresponding set of probability measures. We call *Young measure* a weakly* measurable mapping from Ω into $\mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$. The set of all Young measures is denoted by $\mathbf{Y}(\Omega, \mathbb{R}^m)$. A parametrised measure $\mu \in \mathbf{Y}(\Omega, \mathbb{R}^m)$ is said to be a measure-valued solution of the conservation law (4.1) if, [19],

$$\int_\Omega \big(\langle id, \mu_{x,t} \rangle \cdot \phi_t + \langle A, \mu_{x,t} \rangle \cdot \phi_x \big) dxdt + \int_{\mathbb{R}} u_0 \cdot \phi(0,x) dx = 0, \tag{4.2}$$

for all $\phi \in C_0^\infty(\overline{\Omega})$ where by $\langle A, \mu_{x,t} \rangle$ and $\langle \lambda, \mu_{x,t} \rangle$ we denote

$$\langle A, \mu_{x,t} \rangle = \int_{\mathbb{R}^m} A(\lambda) d\mu_{x,t}(\lambda),$$

$$\langle id, \mu_{x,t} \rangle = \int_{\mathbb{R}^m} \lambda d\mu_{x,t}(\lambda).$$

In a similar fashion as for weak solutions, an *entropy measure-valued solution* satisfies the additional relation

$$\int_\Omega \big(\langle \eta, \mu_{x,t} \rangle \cdot \phi_t + \langle Q, \mu_{x,t} \rangle \cdot \phi_x \big) dxdt + \int_{\mathbb{R}} \eta(u_0) \cdot \phi(x,0) \geq 0, \tag{4.3}$$

for all $\phi \in C_0^\infty(\overline{\Omega})$ where η is convex and (η, Q) an entropy entropy-flux pair, [15]. The notion of entropic measure-valued solutions which was originally proposed by DiPerna [19] is rather weak when non-atomic measures are considered. A manifestation of this fact is the loss of uniqueness, even in the scalar case, when non-atomic measures were allowed in the initial data, [15]; an alternative definition was proposed recently in [22] which leads to a uniqueness result within a certain class of *statistical solutions*.

4.3 Approximation theory of Young measures

As mentioned, we first present a framework for constructing *approximate Young measures,* based on earlier results by [48, 43]. To this end, suppose if our solutions take values on a set S (generally $S \subset \mathbb{R}^m$ where in the scalar case $m = 1$). For every $h > 0$, we assume that we are given a finite dimensional space S_h, subspace of $C(S)$. In addition we assume that there exist a continuous linear projector $P_h : L^1(\Omega; C_0(S)) \to L^1(\Omega; S_h) = P_h(L^1(\Omega; C_0(S)))$. Let further $\mathbf{Y}_h(\Omega, S)$ be the set of all Young measures which map Ω into $(S_h)^*$.

Lemma 4.3.1 *The spaces $P_h^*(L_w^\infty(\Omega; \mathbf{M}^{\mathbb{P}}(S))$ and $L_w^\infty(\Omega; (S_h)^*)$ are isomorphic. In particular if*

$$P_h^*(\mathbf{Y}(\Omega, S)) \subset \mathbf{Y}(\Omega, S)$$

then

$$P_h^*(\mathbf{Y}(\Omega, S)) \cong \mathbf{Y}_h(\Omega, S).$$

This an important lemma since if we assume informally for a moment that $\mathbf{Y}_h(\Omega, S)$ is a space of computational measures which approximate measures belonging to $\mathbf{Y}(\Omega, S)$, then given an $\mu \in \mathbf{Y}(\Omega, S)$ there exist only one $\bar{\mu} \in \mathbf{Y}_h(\Omega, S)$ such that

$$\int_\Omega \langle \phi, \bar{\mu}_{x,t} \rangle dxdt = \int_\Omega \langle P_h \phi, \mu_{x,t} \rangle dxdt \tag{4.4}$$

for all $\phi \in L^1(\Omega; C(S))$. Property (4.4) leads to consistent and mathematically sound approximations of measures. In addition allows great flexibility, since different choices for S_h will yield different approximations to μ, in terms of the order of approximability as was as in terms of the structure of $\bar{\mu}$. We have the following result, [48],

Theorem 4.3.2 *Additionally if the projector has the following properties* *for all* $h \geq h' > 0 : P_h \circ P_{h'} = P_h$

and

for all $\phi \in L^1(\Omega; C(S)) : \lim_{h \to 0} \|\phi - P_h \phi\|_{L^1(\Omega; C(S))} = 0$

then it holds that

for all $h \geq h' > 0 : \mathbf{Y}_h(\Omega, S) \subset \mathbf{Y}_{h'}(\Omega, S) \subset \mathbf{Y}(\Omega, S)$

and

$\bar{\mu} \to \mu$ *weakly* for the measures in expression* (4.4)

$\big(w^* - cl \bigcup_{h>0} \mathbf{Y}_h(\Omega, S)\big)$

respectively.

Thus, if a projection meets the properties of Theorem 4.3.2 and Lemma 4.3.1 then we can think $\mathbf{Y}_h(\Omega, S)$ as a suitable space in which we can seek approximate measures. For instance if S_h is a finite element subspace of $C(S)$, then the interpolation operator of the form

$$P_h(\phi(x,t,\xi)) = \sum_{i=1}^{n} \phi(x,t,\xi_i)\upsilon_i(\xi)\,, \tag{4.5}$$

where $\{\upsilon_i\}_{i=1}^n$ is a basis of S_h and $\{\xi_i \in S\}_{i=1}^n$ are the mesh points, is such a projection(see [8]). In this work we will seek approximations of measure-valued solutions based on this operator, see (4.9).

Error Estimation

In order to estimate rate of convergence of computational measures of the form 4.10 we will consider the space $L^1(\Omega; C^{0,\gamma}(S))$ where $C^{0,\gamma}(S)$ is the space of Holder continuous functions with exponent γ, $0 < \gamma \leq 1$.The dual of this space is suitable for computing errors since $L^\infty_w(\Omega; \mathbf{M}^{\mathbb{P}}(S))$ constitutes a subset of it. Furthermore, using $L^1(\Omega; C^{0,\gamma}(S))$ we are able to employ standard error estimates of the interpolant 4.9. These estimates will subsequently lead to an error estimation of our approximate measures. We begin with the following lemma.

Lemma 4.3.3 *There exist a constant C such that*

$$\|\phi - P_h\phi\|_{L^1(\Omega;C(S))} \leq Ch\|\phi\|_{L^1(\Omega;C^{0,1}(S))} \tag{4.6}$$

for all $\phi \in L^1(\Omega; C^{0,1}(S))$.

(see [8])

Remark 4.3.4 *This can be seen directly from the fact that the spaces $C^{0,1}$ and W^1_∞ are equivalent.*

Theorem 4.3.5 *There exist a constant C such that*

$$\|\phi - P_h\phi\|_{L^1(\Omega;C(S))} \leq Ch^\gamma\|\phi\|_{L^1(\Omega;C^{0,\gamma}(S))} \tag{4.7}$$

for all $\phi \in L^1(\Omega; C^{0,\gamma}(S))$.

Proof: Let I_j be the j^{th} interval of a given uniform partition of S. For a $\phi \in L^1(\Omega; C^{0,\gamma}(S))$ it holds that

$$\|\phi - P_h\phi\|_{L^1(\Omega;C(S))} = \int_\Omega \|\phi(x,t) - P_h\phi(x,t)\|_{L^\infty(S)}dxdt$$

$$\leq Ch\int_\Omega |\phi(x,t)|_{W^1_\infty} = Ch\int_\Omega \sup_{\xi_1,\xi_2\in S}\frac{|\phi(x,t,\xi_1) - \phi(x,t,\xi_2)|}{|\xi_1 - \xi_2|}dxdt$$

$$= Ch\int_\Omega \max_j\{\sup_{\xi_1,\xi_2\in I_j}\frac{|\phi(x,t,\xi_1) - \phi(x,t,\xi_2)|}{|\xi_1 - \xi_2|}\}dxdt$$

$$\leq Ch\int_\Omega \max_j\{\sup_{\xi_1,\xi_2\in I_j}|\xi_1 - \xi_2|^{\gamma-1}\frac{|\phi(x,t,\xi_1) - \phi(x,t,\xi_2)|}{|\xi_1 - \xi_2|^\gamma}\}dxdt$$

$$\leq Ch\int_\Omega \max_j\{\sup_{\xi_1,\xi_2\in I_j}|\xi_1 - \xi_2|^{\gamma-1}\sup_{\xi_1,\xi_2\in I_j}\frac{|\phi(x,t,\xi_1) - \phi(x,t,\xi_2)|}{|\xi_1 - \xi_2|^\gamma}\}dxdt$$

$$= Ch^\gamma\int_\Omega \max_j\{\sup_{\xi_1,\xi_2\in I_j}\frac{|\phi(x,t,\xi_1) - \phi(x,t,\xi_2)|}{|\xi_1 - \xi_2|^\gamma}\}dxdt$$

$$\leq Ch^\gamma\|\phi\|_{L^1(\Omega;C^{0,\gamma}(S))}.$$

□

Corollary 4.3.6 *Given a measure $\mu \in \mathbf{Y}(\Omega, S)$ there exist a constant C and only one measure $\bar{\mu} \in \mathbf{Y}_h(\Omega, S)$ related to μ through expressions (4.4) and (4.9) , such that*

$$\|\mu - \bar{\mu}\|_{L^\infty_w(\Omega;(C^{0,\gamma}(S))^*)} \leq Ch^\gamma \|\mu\|_{L^\infty_w(\Omega;\mathbf{M}^{\mathbb{P}}(S))}$$

Proof:

$$\begin{aligned}
&\|\mu - \bar{\mu}\|_{L^\infty_w(\Omega;(C^{0,\gamma}(S))^*)} \leq \\
&\leq \sup_{\|\phi\|_{L^1(\Omega;C^{0,\gamma}(S))} \leq 1} \int_\Omega \langle \phi, \mu - \bar{\mu} \rangle dxdt \\
&= \sup_{\|\phi\|_{L^1(\Omega;C^{0,\gamma}(S))} \leq 1} \int_\Omega \langle \phi - P_h\phi, \mu \rangle \\
&\leq \sup_{\|\phi\|_{L^1(\Omega;C^{0,\gamma}(S))} \leq 1} \|\mu\|_{L^\infty_w(\Omega;\mathbf{M}^{\mathbb{P}}(S))} \|\phi - P_h\phi\|_{L^1(\Omega;C(S))}
\end{aligned}$$

Combining this result with Theorem 4.3.5 the proof is completed. □

4.4 Approximate discrete kinetic models

Suppose that $h > 0$ is a mesh discretisation parameter, let $S \subset \mathbb{R}^m$, and S_h is a finite dimensional subspace of $C(S)$. We assume that there exist a continuous linear projector $P_h : L^1(\Omega; C_0(S)) \to L^1(\Omega; S_h) = P_h(L^1(\Omega; C(S)))$. Let further $\mathbf{Y}_h(\Omega, S)$ be the set of all Young measures which map Ω into $(S_h)^*$. One can define $\mathbf{Y}_h(\Omega, S)$, the space of approximate Young measures, through the following procedure, see [48] for details. Given $\mu \in \mathbf{Y}(\Omega, S)$, μ is approximated by a $\bar{\mu} \in \mathbf{Y}_h(\Omega, S)$ defined as

$$\int_\Omega \langle \phi, \bar{\mu}_{x,t} \rangle dxdt = \int_\Omega \langle P_h\phi, \mu_{x,t} \rangle dxdt, \tag{4.8}$$

for all $\phi \in L^1(\Omega; C(S))$. To fix ideas, consider $m = 1$, S_h being the standard finite element space of continuous piecewise linear functions, and P_h the standard interpolation operator,

$$P_h(\phi(t,x,\xi)) = \sum_{i=1}^{n} \phi(x,t,\xi_i)\upsilon_i(\xi)\,. \tag{4.9}$$

Here, $\{\upsilon_i\}_{i=1}^n$ are the hat-basis elements of S_h and $\{\xi_i \in S\}_{i=1}^n$ are the mesh points. It is essential now to see the form of the approximate measure:

$$\begin{aligned}
&\int_\Omega \langle \phi, \bar{\mu}_{x,t}\rangle dxdt = \int_\Omega \langle \sum_{i=1}^{n} \phi(x,t,\xi_i)\upsilon_i(\xi), \mu_{x,t}\rangle dxdt \\
&= \sum_{i=1}^{n} \int_\Omega \phi(x,t,\xi_i)\langle \upsilon_i(\xi), \mu_{x,t}\rangle dxdt = \sum_{i=1}^{n} \int_\Omega \alpha_i(x,t) \int_S \phi(x,t,\lambda) d\delta_{\xi_i}(\lambda) dxdt \\
&= \int_\Omega \int_S \phi(x,t,\lambda) d[\sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i}(\lambda)] dxdt = \int_\Omega \langle \phi, \sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i}\rangle dxdt\,.,
\end{aligned}$$

for all $\phi \in L^1(\Omega; C(S))$ where $\alpha_i(x,t) = \langle \upsilon_i, \mu_{x,t}\rangle$ and δ_x is the Dirac measure at x.Thus we have proved,

Lemma 4.4.1 *Assume that for a given measure $\mu \in \mathbf{Y}(\Omega, S)$, we define $\bar{\mu} \in \mathbf{Y}_h(\Omega, S)$ through* (4.8) *and* (4.9)*, where $\{\upsilon_i\}_{i=1}^n$ are the hat-basis of S_h, consisting of standard piecewise linear finite element decomposition of S with nodes $\{\xi_i \in S\}_{i=1}^n$. Then, for $\alpha_i(x,t) = \langle \upsilon_i, \mu_{x,t}\rangle$,*

$$\bar{\mu}_{x,t} = \sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i}. \tag{4.10}$$

In other words, expression (4.10) indicates that such approximations of a Young measure μ is reduced to the evaluation of the action of μ on every basis function υ_i of the space S_h. As the functions α_i determine $\bar{\mu}$, the approximating schemes defined below will have as unknowns α_i, in a form of a PDE system.

We can now proceed to the computation of approximate measure-valued solutions. Substituting μ in expression (4.2) with $\bar{\mu}$ and supposing temporarily that $u_0 = 0$ one leads to the approximating scheme

$$\int_{\Omega} \big(\langle id, \bar{\mu}_{x,t}\rangle \cdot \phi_t + \langle A, \bar{\mu}_{x,t}\rangle \cdot \phi_x\big) dxdt = 0,$$

for all $\phi \in C_0^\infty(\overline{\Omega})$ hence,

$$\int_{\Omega} \big(\langle id, \sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i}\rangle \cdot \phi_t(x,t) + \langle A, \sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i}\rangle \cdot \phi_x(x,t)\big) dxdt = 0$$

for all $\phi \in C_0^\infty(\overline{\Omega})$. Thus, one may conclude that the evolution of α_i is dictated by the partial differential equation

$$\sum_{i=1}^{n} \xi_i \alpha_i(x,t)_t + \sum_{i=1}^{n} A(\xi_i)\alpha_i(x,t)_x = 0. \tag{4.11}$$

Expression (4.11) now will constitute the cornerstone of our approach for several reasons. However, some remarks are in order: although (4.11) has to be satisfied, this equation is under-determined since there are n unknown functions need to be determined but only one equation is in place and thus (4.11) does not constitute a complete PDE system. We need therefore to quest a system for α_i at another level which will be complete and will imply (4.11). Before proceeding further, at this point, we state the following consistency result, which justifies the reason of considering discrete equations of the above form.

Lemma 4.4.2 *Assume that for a given measure $\mu \in \mathbf{Y}(\Omega, S)$, $\bar{\mu} \in \mathbf{Y}_h(\Omega, S)$ is defined through* (4.8) *and* (4.9). *Let that*

$$E(\mu, \phi) := \int_{\Omega} \big(\langle \eta, \mu_{x,t}\rangle \cdot \phi_t + \langle Q, \mu_{x,t}\rangle \cdot \phi_x\big) dxdt \tag{4.12}$$

where $\phi \in C_0^\infty(\Omega)$ is given. Then, for $\alpha_i(x,t) = \langle \upsilon_i, \mu_{x,t}\rangle$,

$$E(\bar{\mu}, \phi) = \int_{\Omega} \Big[\sum_{i=1}^{n} \eta(\xi_i)\alpha_i(x,t) \cdot \phi_t + Q(\xi_i)\alpha_i(x,t) \cdot \phi_x\Big] dxdt\,. \tag{4.13}$$

If η, Q, are regular enough such that Corollary 4.3.6 is applicable, for a $\gamma > 0$, $\gamma \le 1$, then for a constant depending on η, Q,

$$\left| E(\bar{\mu}, \phi) - E(\mu, \phi) \right| \le C(\phi) h^{\gamma} . \tag{4.14}$$

4.4.1 A motivation from the kinetic formulation

Next we shall see how one can motivate the design of appropriate discrete (in ξ) models by considering the kinetic formulation of the conservation law. We first observe that an indicative such $n \times n$ system which can lead to (4.11) can be

$$\partial_t \alpha_i(x,t)\xi_i + \partial_x \alpha_i(x,t) A(\xi_i) = M_i, \quad i = 1, \dots, n, \tag{4.15}$$

where the source functions M_i are given and satisfy $\sum_i M_i = 0$. Such equations are reminiscent of discrete kinetic models, though one has to specify appropriately M_i. In a very rough analogy to Boltzmann equation, one might view (4.11) as the macroscopic expression, thus, we have to define appropriate microscopic equations in order to compute a meaningful solution o (4.2). This can be realised through the setting of the *kinetic formulation* of conservation laws, [40, 45]. In fact, to motivate the design of appropriate discrete kinetic models leading to (4.11) we will seek appropriate discretisations of functions $f(t,x,\xi)$ of the kinetic formulation: A function $f(t,x,\xi) \in L^{\infty}(0,+\infty; L^1(\mathbb{R}^2))$ is called a *generalised kinetic solution of the scalar conservation law,* [45], with initial data f_0, if for all $\phi \in C_c^{\infty}([0,+\infty) \times \mathbb{R} \times \mathbb{R})$ we have

$$\begin{aligned} &\int_0^{\infty} \int_{\mathbb{R}^2} f(t,x,\xi) \left[\frac{\partial \phi(t,x,\xi)}{\partial t} + A'(\xi) \frac{\partial \phi(t,x,\xi)}{\partial x} \right] dx d\xi dt \\ &\quad = \int_0^{\infty} \int_{\mathbb{R}^2} m(t,x,\xi) \frac{\partial \phi(t,x,\xi)}{\partial \xi} dx d\xi dt - \int_{\mathbb{R}^{m+1}} f_0(x,\xi)\phi(0,x,\xi) dx d\xi, \end{aligned} \tag{4.16}$$

where m is a bounded nonnegative measure on $((0,+\infty) \times \mathbb{R} \times \mathbb{R})$ and additionally it holds that

$$|f(t,x,\xi)| = sgn(\xi)f(x,t,\xi) \leq 1, \tag{4.17a}$$

$$f = \int_{\mathbb{R}} \chi_\lambda(\xi) d\nu_{x,t}(\lambda) \tag{4.17b}$$

where

$$sgn(\xi) = \begin{cases} 1 & \text{if} \quad 0 < \xi \\ -1 & \text{if} \quad \xi < 0. \end{cases}$$

Here, $\nu_{x,t}$ is a Young measure associated to f and χ_λ is given by

$$\chi_\lambda(\xi) = \begin{cases} 1 & \text{if} \quad 0 < \xi \leq \lambda \\ -1 & \text{if} \quad \lambda \leq \xi < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Kinetic functions and approximate Young measures. At this point, notice that the above formulation plays a key role. Specifically, the defect measure m might provide the additional information we are looking for leading to an appropriate measure-valued solution. Suppose now that the approximate Young measure $\bar{\nu}_{x,t} = \sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i}$ approximates $\nu_{x,t}$. We see then

$$\begin{aligned} f(x,t,\xi) \approx \bar{f}(x,t,\xi) &= \int_{\mathbb{R}} \chi_\lambda(\xi) d\bar{\nu}_{x,t} = \int_{\mathbb{R}} \chi_\lambda(\xi) d[\sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i}(\lambda)] \\ &= \sum_{i=1}^{n} \alpha_i(x,t) \int_{\mathbb{R}} \chi_\lambda(\xi) d\delta_{\xi_i}(\lambda) = \sum_{i=1}^{n} \alpha_i(x,t)\chi_{\xi_i}(\xi) = \sum_{i=1}^{n} \bar{f}_i, \end{aligned}$$

where $\bar{f}_i = \alpha_i(x,t)\chi_{\xi_i}(\xi)$. Therefore, one can define the approximate model

$$\sum_{i=1}^{n} \frac{\partial \bar{f}_i}{\partial_t} + \sum_{i=1}^{n} A'(\xi) \frac{\partial \bar{f}_i}{\partial_x} = \frac{\partial \bar{m}}{\partial \xi} \tag{4.18}$$

where $\bar{m}$ is an approximation of measure m. We observe now

$$\begin{aligned}\int_{\mathbb{R}} \frac{\partial \bar{f}_i}{\partial_t} + A'(\xi)\frac{\partial \bar{f}_i}{\partial_x}\, d\xi =& \partial_t \alpha_i(x,t) \int_{\mathbb{R}} \chi_{\xi_i}(\xi) d\xi + \partial_x \alpha_i(x,t) \int_{\mathbb{R}} A'(\xi)\chi_{\xi_i}(\xi) d\xi \\ =& \xi_i \partial_t \alpha_i(x,t) + A(\xi_i)\partial_x \alpha_i(x,t).\end{aligned} \tag{4.19}$$

Therefore, by integrating (4.18) we conclude that

$$\sum_{i=1}^{n} \partial_t \alpha_i(x,t)\xi_i + \sum_{i=1}^{n} \partial_x \alpha_i(x,t) A(\xi_i) = \int_{\mathbb{R}} \frac{\partial \bar{m}}{\partial \xi} d\xi = 0. \tag{4.20}$$

Let us now assume temporally and without loss of the generality that the nodes $\xi, > 0$ and $\xi_0 = 0$. We use the notation $\Phi_{I_j} = \mathbf{1}_{I_j}$, if $I_j \subset [0,\infty)$ and $\Phi_{I_j} = -\mathbf{1}_{I_j}$, if $I_j \subset (-\infty, 0]$, where $I_j = (\xi_{j-1}, \xi_j)$. Then,

$$\bar{f}(x,t,\xi) = \sum_{i=1}^{n} \bar{f}_i = \sum_{i=1}^{n} \alpha_i(x,t)\chi_{\xi_i}(\xi) = \sum_{i}^{n} \beta_j(x,t)\Phi_{I_j}(\xi)\,, \tag{4.21}$$

where

$$\begin{aligned}\beta_1 =& \alpha_n + \alpha_{n-1} + \cdots + \alpha_1 \\ \beta_2 =& \alpha_n + \cdots + \alpha_2 \\ & \dots \\ \beta_n =& \alpha_n\,,\end{aligned}$$

i.e.,

$$\beta_j = \alpha_n + \cdots + \alpha_j\,. \tag{4.22}$$

Similar relations hold for ξ_i which are allowed to take negative values. Notice that $\bar{f}(x,t,\xi)$ is piece-wise constant in the elements of S_h. A natural *finite volume* discretisation in ξ of (4.18) is obtained by integrating over I_i (4.18) for all i. This leads to the discrete kinetic model

$$\begin{aligned} h\beta_i(x,t)_t + \Big(A(\xi_i) - A(\xi_{i-1})\Big)\beta_i(x,t)_x = \\ \bar{m}(x,t,\xi_i) - \bar{m}(x,t,\xi_{i-1})\,. \end{aligned} \tag{4.23}$$

4.4.2 Entropic discrete kinetic models

We have the following

Proposition 4.4.3 *Consider approximate Young measures of the form*

$$\bar{\nu}_{x,t} = \sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i}\,.$$

and corresponding kinetic functions

$$\bar{f}(x,t,\xi) = \int_{\mathbb{R}} \chi_\lambda(\xi)d\bar{\nu}_{x,t} = \sum_{i=1}^{n} \alpha_i(x,t)\chi_{\xi_i}(\xi)\,.$$

Assume that we are given a positive measure $\bar{m}$ with compact support with respect to ξ, and let $\beta_i(t,x)$, $i = 1,\ldots,n$ satisfy the following discrete kinetic equations

$$\begin{aligned} h\beta_i(x,t)_t + \Big(A(\xi_i) - A(\xi_{i-1})\Big)\beta_i(x,t)_x = \\ \bar{m}(x,t,\xi_i) - \bar{m}(x,t,\xi_{i-1})\,. \end{aligned} \tag{4.24}$$

Then, if α_i and β_i are connected through (4.22) *the piecewise constant function $\bar{f}(x,t,\xi)$ satisfies* (4.21). *Furthermore for any convex function η the following discrete entropy inequality holds in the sense of distributions,*

$$\sum_{i=1}^{n} \eta(\xi_i)\alpha_i(x,t)_t + \sum_{i=1}^{n} Q_i\,\alpha_i(x,t)_x \leq 0. \tag{4.25}$$

Here the discrete entropy flux Q_i *is defined through,*

$$Q_j = \frac{1}{h}\sum_{k=1}^{j}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\Big(A(\xi_i) - A(\xi_{i-1})\Big)\,. \tag{4.26}$$

Proof: It remains to show the discrete entropy inequality (4.25). To this end, we multiply (4.24) by $\eta(\xi_i) - \eta(\xi_{i-1})$, and sum with respect to i to obtain

$$\begin{aligned} h\sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\beta_i(x,t)_t + \sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\Big(A(\xi_i) - A(\xi_{i-1})\Big)\beta_i(x,t)_x = \\ \sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\Big(\bar{m}(x,t,\xi_i) - \bar{m}(x,t,\xi_{i-1})\Big)\,. \end{aligned} \tag{4.27}$$

Now,

$$\begin{aligned} \sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\beta_i(x,t)_t &= \sum_{i=1}^{n}\eta(\xi_i)\beta_i(x,t)_t - \sum_{i=1}^{n}\eta(\xi_{i-1})\beta_i(x,t)_t \\ &= \sum_{i=1}^{n}\eta(\xi_i)\beta_i(x,t)_t - \sum_{i=0}^{n-1}\eta(\xi_i)\beta_{i+1}(x,t)_t \\ &= \sum_{i=1}^{n-1}\eta(\xi_i)\Big(\beta_i(x,t)_t - \beta_{i+1}(x,t)_t\Big) + \eta(\xi_n)\beta_n(x,t)_t - \eta(\xi_0)\beta_1(x,t)_t\,. \end{aligned}$$

Hence, since $\eta(\xi_0) = 0$, and using (4.22),

$$\sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\beta_i(x,t)_t = \sum_{i=1}^{n}\eta(\xi_i)\,\alpha_i(x,t)_t\,. \tag{4.28}$$

Next,

$$\begin{aligned} &\sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\Big(A(\xi_i) - A(\xi_{i-1})\Big)\beta_i(x,t)_x \\ &\qquad\qquad = h\sum_{i=1}^{n}\Big(Q_i - Q_{i-1}\Big)\beta_i(x,t)_x \end{aligned}$$

$$
\begin{aligned}
&= h\sum_{i=1}^{n} Q_i\beta_i(x,t)_x - \sum_{i=1}^{n} Q_{i-1}\beta_i(x,t)_x \\
&= h\sum_{i=1}^{n} Q_i\beta_i(x,t)_x - \sum_{i=0}^{n-1} Q_i\beta_{i+1}(x,t)_x \\
&= h\sum_{i=1}^{n-1} Q_i\Big(\beta_i(x,t)_x - \beta_{i+1}(x,t)_x\Big) \\
&\quad + hQ_n\beta_n(x,t)_x - hQ_0\beta_1(x,t)_x\,.
\end{aligned}
$$

Hence, since $\eta(\xi_0) = 0$, and using (4.22),

$$
\sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\Big(A(\xi_i) - A(\xi_{i-1})\Big)\beta_i(x,t)_x = h\sum_{i=1}^{n} Q_i\,\alpha_i(x,t)_x\,. \tag{4.29}
$$

For the term involving the measure, and using the fact that $\bar{m}$ is zero at the endpoints of S,

$$
\begin{aligned}
&\sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\Big(\bar{m}(x,t,\xi_i) - \bar{m}(x,t,\xi_{i-1})\Big) \\
&\qquad = \sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\bar{m}(x,t,\xi_i) - \sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\bar{m}(x,t,\xi_{i-1}) \\
&\qquad = \sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\bar{m}(x,t,\xi_i) - \sum_{i=0}^{n-1}\Big(\eta(\xi_{i+1}) - \eta(\xi_i)\Big)\bar{m}(x,t,\xi_i) \\
&\qquad = -\sum_{i=1}^{n-1}\Big(\eta(\xi_{i+1}) - 2\eta(\xi_i) + \eta(\xi_{i-1})\Big)\bar{m}(x,t,\xi_i).
\end{aligned}
$$

Hence, since η is convex,

$$
\begin{aligned}
&\sum_{i=1}^{n}\Big(\eta(\xi_i) - \eta(\xi_{i-1})\Big)\Big(\bar{m}(x,t,\xi_i) - \bar{m}(x,t,\xi_{i-1})\Big) \\
&\qquad = -\sum_{i=1}^{n-1}\Big(\eta(\xi_{i+1}) - 2\eta(\xi_i) + \eta(\xi_{i-1})\Big)\bar{m}(x,t,\xi_i) \le 0\,,
\end{aligned} \tag{4.30}
$$

and the proof is complete. □

4.4.3 Viscous discrete kinetic models

As it is obvious the implementation of the above system, (4.24), requires that the measure $\bar{m}$ is known, as well as discretisation with respect to x and t variables as well. There are many alternative ways to implement full discretisation and such methods will be subject of further research. As far as the defect measure is concerned, we claim that its choice plays a crucial role for the model design. We can observe now that in kinetic models for conservation laws with small diffusion the defect measure can be explicitly computed, [11]. Typically, it contains a diffusion term of the kinetic function $f(t, x, \xi)$ as well. In fact, [11, 41], the kinetic formulation of

$$\partial_t u + \partial_x A(u) = \epsilon u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{4.31}$$

is

$$\frac{\partial \chi_u(\xi)}{\partial t} + A'(\xi)\frac{\partial \chi_u(\xi)}{\partial x} - \epsilon\frac{\partial^2 \chi_u(\xi)}{\partial x^2} = \epsilon\left(\frac{\partial \delta(\xi - u)}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)^2\right) = \frac{\partial m^\epsilon}{\partial \xi} \tag{4.32}$$

where

$$\chi_u(\xi) = \begin{cases} 1 & \text{if} \quad 0 < \xi \leq u \\ -1 & \text{if} \quad u \leq \xi < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Motivated by the above discussion, and the fact that we would like to include in our approximate scheme artificial diffusion, we are led to the following approximating model: Let first $\widetilde{u}$ be defined through

$$\widetilde{u} = \int_{\mathbb{R}} \lambda d\bar{\nu}_{x,t}(\lambda) = \int_{\mathbb{R}} \lambda d\sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i} = \sum_{i=1}^{n} \alpha_i(x,t)\xi_i.$$

For fixed x_0, we consider $\bar{\delta}(\xi - x_0)$ to be a compactly supported smooth Gaussian-like approximation of $\delta(\xi - x_0)$. As before we consider approximate Young measures of the form

$$\bar{\nu}_{x,t} = \sum_{i=1}^{n} \alpha_i(x,t)\delta_{\xi_i}\,.$$

and corresponding kinetic functions

$$\bar{f}(x,t,\xi) = \int_{\mathbb{R}} \chi_\lambda(\xi) d\bar{\nu}_{x,t} = \sum_{i=1}^{n} \alpha_i(x,t)\chi_{\xi_i}(\xi)\,.$$

Assume that we are given a positive measure $\bar{m}^\epsilon$ with compact support with respect to ξ, and let $\beta_i(t,x)$, be such that α_i and β_i are connected through (4.22) and the piecewise constant function $\bar{f}(x,t,\xi)$ satisfies (4.21). We seek models where $\beta_i(t,x)$, $i = 1,\dots,n$ satisfy the following discrete kinetic equations

$$\begin{aligned} h\beta_i(x,t)_t + \Big(A(\xi_i) - A(\xi_{i-1})\Big)\beta_i(x,t)_x &= \epsilon\beta_i(x,t)_{xx} \\ &+ \bar{m}^\epsilon(x,t,\xi_i) - \bar{m}^\epsilon(x,t,\xi_{i-1})\,. \end{aligned} \tag{4.33}$$

where $\bar{m}^\epsilon(t,x,\xi) = \epsilon\left(\bar{\delta}(\xi - \widetilde{u})\,|\widetilde{u}_x|^2\right)$. We observe that at least formally, as $\epsilon \to 0$, the model (4.33) has the right form compatible with the kinetic formulation. The choice of the models is indicative and it is an open problem to find the schemes which will produce the most efficient approximations. Motivated by our analysis herein, Chapter 5 is devoted to the stability of viscous generalised kinetic solutions as $\epsilon \to 0$.

Chapter 5

Stability of Young measures through generalised kinetic solutions

5.1 Chapter overview

This chapter is devoted to stability analysis for generalised kinetic models including small diffusion terms and general initial data not necessarily restricted to $\chi_{u^0(x)}$, for some function u^0. We consider models associated to the scalar multidimensional conservation law. Generalised kinetic formulations were introduced by Perthame [44, 45] and are generalisations of kinetic formulations of conservation law [40]. We consider kinetic models of the form, see next section for precise definitions,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^2} f(t,x,\xi)\left[\frac{\partial}{\partial t}\phi(t,x,\xi) + \nabla_u A(\xi)\cdot\nabla_x\phi(t,x,\xi)\right] dx d\xi dt \\
&\quad = -\int_0^\infty \int_{\mathbb{R}^2} \epsilon \nabla_x f(t,x,\xi)\cdot\nabla_x\phi(t,x,\xi) dx d\xi dt \\
&\qquad + \int_0^\infty \int_{\mathbb{R}^2} m'(t,x,\xi)\frac{\partial}{\partial \xi}\phi(t,x,\xi) dx d\xi dt - \int_{\mathbb{R}^2} f_0(x,\xi)\phi(0,x,\xi) dx d\xi\,.
\end{aligned} \tag{5.1}$$

These formulations are generalisations of the kinetic formulation of viscous conservation laws, [11],

$$\frac{\partial \chi_u(\xi)}{\partial t} + \nabla_u A(\xi) \cdot \nabla_x \chi_u(\xi) = \epsilon (\nabla_x)^2 \chi_u(\xi) + \frac{\partial m^\epsilon}{\partial \xi}\,. \tag{5.2}$$

associated to

$$\partial_t u + \nabla_x A(u) = \epsilon \nabla_x^2 u. \tag{5.3}$$

As we have seen in Chapter 4, such models are relevant when one would like to approximate measure valued solutions through approximate Young measures. However, the study of generalised viscous kinetic formulations is of interest, even one considers alternative sampling approaches, such as Monte Carlo sampling, based on standard schemes for approximating the conservation law. In fact, since most of such schemes include a form of artificial diffusion, approximations can be modelled by variations of (5.3). In order to gain more understanding on the issue, consider different approximations u_j, $j = 1, \dots, J$, which correspond to different initial data u_j^0, $j = 1, \dots, J$. Assume that all u_j satisfy (5.3), then we would like to study the behaviour of the measure

$$\frac{1}{J} \sum_{j=1}^{J} \delta_{u_j}\,.$$

As we have seen to each δ_{u_j} corresponds the kinetic function χ_{u_j} and all these functions satisfy (5.2). Then, to the sample above, we associate the kinetic function,

$$f^J(t,x,\xi) = \frac{1}{J} \sum_{j=1}^{J} \chi_{u_j(t,x)}(\xi)\,. \tag{5.4}$$

Due to the linearity of the principal part of the viscous kinetic formulation, each such f^J satisfies (5.1), (here $B_\epsilon = I$), for an appropriate measure m' and for $f_0(x,\xi) = \frac{1}{J}\sum_{j=1}^{J} \chi_{u_j^0(x)}(\xi)\,.$

Next we give a precise definition of generalised viscous kinetic solutions and we consider the measure m' to be, in general, a function of f. Our main result, Theorem 5.2.2, implies *uniqueness within a* *class* under structural assumption hypotheses on the measure m'. This result essentially states that all viscous generalised kinetic functions have the same limit as soon as $\|B_\epsilon\|_{W_1^\infty(\mathbb{R}^d)} \to 0$, $\epsilon \to 0$, and the defect measures satisfy a dissipative structural assumption. These assumptions are to some extend generalisations of properties appearing in the analysis of [44, 45] for initial data $\chi_{u^0(x)}$. When is required, the analysis adapts arguments from [44, 45] and [11, 41] to our case.

5.2 A stability result for generalised viscous kinetic solutions

Although it is quite natural to design schemes which induce a form of artificial diffusion, a key question is, if it is possible to have some guarantees that we compute in the limit a unique measure. A partial result in this direction is stated below. We need first to extend the definition of generalised kinetic solutions to include small diffusion. To this end, a function $f(t,x,\xi) \in L^\infty(0,+\infty; L^1(\mathbb{R}^{d+1}))$ is called a *generalised kinetic solution of the viscous scalar conservation law* with *initial data* f_0, if for all $\phi \in C_c^\infty([0,+\infty) \times \mathbb{R}^d \times \mathbb{R})$ we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^2} f(t,x,\xi) \left[\frac{\partial}{\partial t}\phi(t,x,\xi) + \nabla_u A(\xi)\cdot \nabla_x \phi(t,x,\xi)\right] dxd\xi dt \\
&\quad = -\int_0^\infty \int_{\mathbb{R}^2} B_\varepsilon(x) \nabla_x f(t,x,\xi)\cdot \nabla_x \phi(t,x,\xi) dxd\xi dt \\
&\quad\quad + \int_0^\infty \int_{\mathbb{R}^2} m(t,x,\xi) \frac{\partial}{\partial \xi}\phi(t,x,\xi) dxd\xi dt - \int_{\mathbb{R}^2} f_0(x,\xi)\phi(0,x,\xi) dxd\xi
\end{aligned} \tag{5.5}$$

where m' is a given bounded nonnegative measure on $((0,+\infty)\times\mathbb{R}^d\times\mathbb{R})$, $B_\varepsilon(x)$ is a positive function belonging to the space $W^1_\infty(\mathbb{R}^d)$, which is the space of all $L^\infty(\mathbb{R}^d)$ functions with first order weak derivatives and

$$\int_0^{+\infty} \int_{\mathbb{R}^d} m(x,t,\xi) dxdt \le \mu(\xi) \in L_0^\infty(\mathbb{R})^1 \tag{5.6a}$$

$$|f(t,x,\xi)| = sgn(\xi) f(x,t,\xi) \le 1, \tag{5.6b}$$

[1] L_0^∞ bounded functions that vanish at infinity

$$f = \int_{\mathbb{R}} \chi_\lambda(\xi) d\nu_{x,t}(\lambda). \tag{5.6c}$$

Remark 5.2.1 *The function f is the distributional function corresponding to the measure $\nu_{x,t}$ for almost all x and t i.e.*

$$f(x,t,\xi) = \int_{-\infty}^{\xi} d\delta_0(\lambda) - \int_{-\infty}^{\xi} d\nu_{x,t}(\lambda). \tag{5.7}$$

Furthermore, the relation (5.6c) *is equivalent to the expression*

$$\frac{\partial f(x,t,\xi)}{\partial \xi} = \delta_0(\xi) - \nu(x,t,\xi) \tag{5.8}$$

in $\mathcal{M}(\mathbb{R})$ for almost all t and x. Indeed, From (5.7) it holds that

$$f(x,t,\xi) = \begin{cases} -\int_{-\infty}^{\xi} d\nu_{x,t} \text{ if } \xi < 0 \\ \\ \int_{\xi}^{+\infty} d\nu_{x,t} \text{ if } \xi > 0 \end{cases}$$

while on the other hand we also have

$$\int_{\mathbb{R}} \chi_\lambda(\xi) d\nu_{x,t}(\lambda) = \begin{cases} -\int_{-\infty}^{\xi} d\nu_{x,t} \text{ if } \xi < 0 \\ \\ \int_{\xi}^{+\infty} d\nu_{x,t} \text{ if } \xi > 0. \end{cases}$$

To show (5.8)*, consider a $\phi(\xi) \in C_c^\infty(\mathbb{R})$. Then we have*

$$\begin{aligned} &\int_{\mathbb{R}} \phi(\lambda) d\nu_{x,t}(\lambda) - \int_{\mathbb{R}} \phi(\lambda) d\delta_0(\lambda) = \int_{\mathbb{R}} \phi(\lambda) - \phi(0) d\nu_{x,t}(\lambda) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_\lambda(\xi) \frac{\partial \phi(\xi)}{\partial \xi} d\xi d\nu_{x,t} = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_\lambda(\xi) d\nu_{x,t} \frac{\partial \phi(\xi)}{\partial \xi} d\xi = \int_{\mathbb{R}} f(x,t,\xi) \frac{\partial \phi(\xi)}{\partial \xi} d\xi \\ &= -\int_{\mathbb{R}} \frac{\partial f(x,t,\xi)}{\partial \xi} \phi(\xi) d\xi. \end{aligned}$$

A direct consequence of (5.8) *is the fact that the mapping* $\nu_{x,t} \mapsto f$ *that is defined through* (5.6c) *is injective.*

The next result essentially states that all viscous generalised kinetic functions have the same limit as soon as $\|B_\epsilon\|_{W_1^\infty(\mathbb{R}^d)} \to 0$, $\epsilon \to 0$, and the defect measures satisfy a dissipative structural assumption. Note at this point the conceptual similarity with standard approaches for the scalar conservation law, where the unique entropy solution is characterised through the vanishing viscosity limit. Atomic measures satisfy such structural assumptions, however, it remains an open problem to investigate if such assumptions can be relaxed to a certain extend and in addition to characterise families of approximations which fulfil them. These are problems for future research. The assumption for the measures stated below is understood via regularisation, see the next section for details, and we are not precise regarding the smoothness assumptions on the viscosity coefficients B_ϵ which are assumed smooth enough. In Section 5.4 we present a variant of Theorem 5.2.2 with stronger hypotheses but with a simplified proof.

Theorem 5.2.2 *Assume that* f *is a solution of* (5.5) *and let* $\tilde{f}$ *a viscous generalised kinetic solution of* (5.5) *corresponding to* $\tilde{B}_\varepsilon(x), \tilde{m}$, *and* $\tilde{\nu}$. *Furthermore, suppose that the initial data and the solutions have compact support with* $\tilde{f}(0,x,\xi) = f(0,x,\xi)$. *In addition to these hypothesis, assume that the defect measures* m' *and* $\tilde{m}'$ *satisfy, up to regularisation,*

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} m - \tilde{m}\, d(\nu - \tilde{\nu}) dx\, d\xi\, dt \leq 0. \tag{5.9}$$

Then, as both $\|B\|_{W_1^\infty(\mathbb{R}^d)}, \|\tilde{B}\|_{W_1^\infty(\mathbb{R}^d)} \to 0$ *we have the limit*

$$\|f - \tilde{f}\|_{L^2} \to 0\,. \tag{5.10}$$

Note here that by Remark 5.2.1 when the limit (5.39) holds true then (5.8) implies that the measures $\nu_{x,t}$ and $\tilde{\nu}_{x,t}$ become equal at the limit.

5.3 L^1-based analysis

We shall need some preliminary results. First, as it is typical, [45], we need to introduce regularisations in order to handle operations on distributions. Let

$$\phi_\epsilon(t,x) = \frac{1}{\epsilon_1}\phi_1\left(\frac{t}{\epsilon_1}\right)\frac{1}{\epsilon_2^d}\phi_2\left(\frac{x}{\epsilon_2}\right)$$

where $\phi_1 \in C_c^\infty(\mathbb{R})$, $\phi_2 \in C_c^\infty(\mathbb{R}^d)$ and $\int\limits_{\mathbb{R}} \phi_1(x)dx = 1$, $\int\limits_{\mathbb{R}^d} \phi_2(x)dx = 1$. Here, ϵ_1, ϵ_2 are the parameters for time and space regularization respectively. In addition, we assume $supp(\phi_1) \subset (-1,0)$ for allow the time regularization. Furthermore for some constant C we assume $|\nabla_{x_i}\phi_j| < C$ for $i = 1,\ldots,d$. We now set

$$f_\epsilon(x,t,\xi) = [f \star \phi_\epsilon](x,t,\xi) = \int_0^{+\infty}\int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(t-s,x-y)dyds$$

and

$$m_\epsilon(x,t,\xi) = [m \star \phi_\epsilon](x,t,\xi) = \int_0^{+\infty}\int\limits_{\mathbb{R}^d} \phi_\epsilon(t-s,x-y)dm(s,y,\xi)dyds.$$

Accordingly we define the regularization corresponding to $\tilde{f}$ as

$$f_\epsilon(x,t,\xi) = [\tilde{f} \star \phi_\epsilon](x,t,\xi)$$

and

$$\tilde{m}_\epsilon(x,t,\xi) = [\tilde{m} \star \phi_\epsilon](x,t,\xi) = \int_0^{+\infty}\int\limits_{\mathbb{R}^d} \phi_\epsilon(t-s,x-y)d\tilde{m}(s,y,\xi)dyds.$$

We then have.

Lemma 5.3.1 *Assume $B_\epsilon(x) = k$ with $k > 0$ to be a constant. The functions $\xi \mapsto m_\epsilon(x,t,\xi)$ and $\xi \mapsto \tilde{m}_\epsilon(x,t,\xi)$ are Lipschitz continuous and for almost all ξ we have*

$$\frac{\partial}{\partial t} f_\epsilon(x,t,\xi) + \nabla_u A(\xi) \cdot \nabla_x f_\epsilon(x,t,\xi) = k\Delta_x f_\epsilon(x,t,\xi) + \frac{\partial}{\partial \xi} m_\epsilon(x,t,\xi) \tag{5.11}$$

$$\frac{\partial}{\partial t} \tilde{f}_\epsilon(x,t,\xi) + \nabla_u A(\xi) \cdot \nabla_x \tilde{f}_\epsilon(x,t,\xi) = -\nabla_x [\tilde{B}\nabla \tilde{f}]_\epsilon(x,t,\xi) + \frac{\partial}{\partial \xi} \tilde{m}_\epsilon(x,t,\xi) \tag{5.12}$$

for all $(x,t) \in [0,+\infty) \times \mathbb{R}^d$.

Proof: We first show (5.11). For some fixed parameters x and t we pick $\phi(y,s,\xi) = \psi(\xi)\phi_\epsilon(t-s, x-y)$ in (5.5) where $\psi(\xi) \in C_c^\infty(\mathbb{R})$. Thus, we have

$$\begin{aligned}
&\int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \psi(\xi) \left(f(y,s,\xi) \frac{\partial}{\partial s} \phi_\epsilon(t-s,x-y) + f(y,s,\xi) \nabla_u A(\xi) \cdot \nabla_y \phi_\epsilon(t-s,x-y) \right) d\xi dy ds \\
&= \int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \frac{\partial}{\partial \xi} \psi(\xi) \phi_\epsilon(t-s,x-y) dm(s,y,\xi) dy d\xi ds - \int\limits_{\mathbb{R}^{d+1}} \psi(\xi) f_0(y,\xi) \phi_\epsilon(t,x-y) d\xi dy \\
&\quad - k \int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \psi(\xi) \nabla_y f(y,s,\xi) \nabla_y \phi_\epsilon(t-s,x-y) d\xi dy ds.
\end{aligned}$$

Since $\phi_\epsilon(t,x-y) = 0$ for $t \geq 0$, the second integral of the right-hand side of the above equation equals to zero and therefore integration by parts in the last integral of the right-hand side yields

$$\begin{aligned}
&\int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \psi(\xi) \left(f(y,s,\xi) \frac{\partial}{\partial s} \phi_\epsilon(t-s,x-y) + f(y,s,\xi) \nabla_u A(\xi) \cdot \nabla_y \phi_\epsilon(t-s,x-y) \right) d\xi dy ds \\
&= \int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \frac{\partial}{\partial \xi} \psi(\xi) \phi_\epsilon(t-s,x-y) dm(s,y,\xi) dy d\xi ds \\
&\qquad\qquad - k \int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \psi(\xi) f(y,s,\xi) \Delta_y \phi_\epsilon(t-s,x-y) d\xi dy ds.
\end{aligned}$$

In consideration of

$$\frac{\partial}{\partial s} \phi_\epsilon(t-s,x-y) = -\frac{\partial}{\partial t} \phi_\epsilon(t-s,x-y)$$

and

$$\Delta_y \phi_\epsilon(t-s, x-y) = \Delta_x \phi_\epsilon(t-s, x-y)$$

we obtain

$$\int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \psi(\xi) \left(f(y,s,\xi) \frac{\partial}{\partial t} \phi_\epsilon(t-s, x-y) + f(y,s,\xi) \nabla_u A(\xi) \cdot \nabla_x \phi_\epsilon(t-s, x-y) \right) d\xi dy ds$$

$$= - \int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \frac{\partial}{\partial \xi} \psi(\xi) \phi_\epsilon(t-s, x-y) dm(s,y,\xi) dy d\xi ds$$

$$+ k \int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \psi(\xi) f(y,s,\xi) \Delta_x \phi_\epsilon(t-s, x-y) d\xi dy ds$$

and therefore

$$\int\limits_{\mathbb{R}} \psi(\xi) \left(\frac{\partial}{\partial t} [f \star \phi_\epsilon](x,t,\xi) + \nabla_u A(\xi) \cdot \nabla_x [f \star \phi_\epsilon](x,t,\xi) \right) d\xi$$

$$= - \int\limits_{\mathbb{R}} \frac{\partial}{\partial \xi} \psi(\xi) [m \star \phi_\epsilon](x,t,\xi) d\xi + k \int\limits_{\mathbb{R}} \psi(\xi) \Delta_x [f \star \phi_\epsilon](x,t,\xi) d\xi.$$

In order to prove (5.12) the same operations in (5.5) with $\tilde{f}, \tilde{B}(x), \tilde{m}$, and $\tilde{\nu}_{x,t}$ yield

$$\int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \psi(\xi) \left(\tilde{f}(y,s,\xi) \frac{\partial}{\partial s} \phi_\epsilon(t-s, x-y) + \tilde{f}(y,s,\xi) \nabla_u A(\xi) \cdot \nabla_y \phi_\epsilon(t-s, x-y) \right) d\xi dy ds$$

$$= \int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \frac{\partial}{\partial \xi} \psi(\xi) \phi_\epsilon(t-s, x-y) d\tilde{m}(s,y,\xi) dy d\xi ds - \int\limits_{\mathbb{R}^{d+1}} \psi(\xi) \tilde{f}_0(y,\xi) \phi_\epsilon(t, x-y) d\xi dy$$

$$- \int_0^{+\infty} \int\limits_{\mathbb{R}^{d+1}} \tilde{B}(y) \psi(\xi) \nabla_y \tilde{f}(y,s,\xi) \cdot \nabla_y \phi_\epsilon(t-s, x-y) d\xi dy ds.$$

Hence since,

$$\int_0^{+\infty}\int_{\mathbb{R}^d} \tilde{B}(y)\nabla_y f(s,y,\xi)\cdot\nabla_y\phi_\epsilon(t-s,x-y)dydt$$
$$= -\int_0^{+\infty}\int_{\mathbb{R}^d} \tilde{B}(y)\nabla_y f(s,y,\xi)\cdot\nabla_x\phi_\epsilon(t-s,x-y)dydt$$

we have

$$\int_0^{+\infty}\int_{\mathbb{R}^{d+1}} \psi(\xi)\left(\tilde{f}(y,s,\xi)\frac{\partial}{\partial t}\phi_\epsilon(t-s,x-y)+\tilde{f}(y,s,\xi)\nabla_u A(\xi)\cdot\nabla_x\phi_\epsilon(t-s,x-y)\right)d\xi dyds$$
$$= -\int_0^{+\infty}\int_{\mathbb{R}^{d+1}} \frac{\partial}{\partial\xi}\psi(\xi)\phi_\epsilon(t-s,x-y)d\tilde{m}(s,y,\xi)dyd\xi ds$$
$$-\int_0^{+\infty}\int_{\mathbb{R}^{d+1}} \psi(\xi)\tilde{B}(y)\nabla_y\tilde{f}(y,s,\xi)\cdot\nabla_x\phi_\epsilon(t-s,x-y)d\xi dyds.$$

Therefore

$$\int_{\mathbb{R}}\psi(\xi)\left(\frac{\partial}{\partial t}[f\star\phi_\epsilon](x,t,\xi)+\nabla_u A(\xi)\cdot\nabla_x[f\star\phi_\epsilon](x,t,\xi)\right)d\xi$$
$$= -\int_{\mathbb{R}}\frac{\partial}{\partial\xi}\psi(\xi)[m\star\phi_\epsilon](x,t,\xi)d\xi+\int_{\mathbb{R}}\nabla_x\cdot[\tilde{B}\nabla f\star\phi_\epsilon](x,t,\xi)d\xi.$$

$\square$

Lemma 5.3.2 *The regularized term $f_\epsilon(x,t,\xi)$ fulfils the relation*

$$sgn(\xi)f_\epsilon(x,t,\xi) = |f_\epsilon(x,t,\xi)|. \tag{5.13}$$

Proof: Relation (5.13) is a consequence of $(5.6b)$. To this end, we show first that

$$sgn(\xi)f_\epsilon(x,t,\xi) = [|f|]_\epsilon \tag{5.14}$$

where

$$[|f|]_\epsilon = \int_0^{+\infty} \int_{\mathbb{R}^d} |f(s,y,\xi)| \phi_\epsilon(t-s, x-y) dyds.$$

Indeed,

$$\begin{aligned} sgn(\xi) f_\epsilon(x,t,\xi) &= sgn(\xi) \int_0^{+\infty} \int_{\mathbb{R}^d} f(s,y,\xi) \phi_\epsilon(t-s, x-y) dsdy \\ &= \int_0^{+\infty} \int_{\mathbb{R}^d} sgn(\xi) f(s,y,\xi) \phi_\epsilon(t-s, x-y) dsdy \\ &= \int_0^{+\infty} \int_{\mathbb{R}^d} |f(s,y,\xi)| \phi_\epsilon(t-s, x-y) dsdy. \end{aligned}$$

On the other hand,

$$\begin{aligned} [|f|]_\epsilon = \big|[|f|]_\epsilon\big| &= \Big| \int_0^{+\infty} \int_{\mathbb{R}^d} |f(y,s,\xi)| \phi_\epsilon(t-s, x-y) dyds \Big| \\ &= \Big| \int_0^{+\infty} \int_{\mathbb{R}^d} sgn(\xi) f(y,s,\xi) \phi_\epsilon(t-s, x-y) dyds \Big| \\ &= \Big| sgn(\xi) \int_0^{+\infty} \int_{\mathbb{R}^d} f(y,s,\xi) \phi_\epsilon(t-s, x-y) dyds \Big| \\ &= |sgn(\xi)| \Big| \int_0^{+\infty} \int_{\mathbb{R}^d} f(y,s,\xi) \phi_\epsilon(t-s, x-y) dyds \Big| \\ &= |f_\epsilon(x,t,\xi)|. \end{aligned}$$

Hence from (5.14) we deduce that (5.13) holds true. □

Lemma 5.3.3 *Let $T > 0$ and $\phi(t,x) \in C_c^\infty([0,T] \times \mathbb{R}^d)$. Assume that m satisfies $(5.6a)$, then*

$$\int_0^T \int_{\mathbb{R}^d} m_\epsilon(x,t,\xi) \phi(x,t) dxdt \leq \mu(\xi) \|\phi(x,t)\|_{L^\infty([0,T]\times\mathbb{R}^d)} \tag{5.15}$$

for all $\epsilon > 0$.

Proof:

$$\int_0^T \int_{\mathbb{R}^d} m_\epsilon(x,t,\xi)\phi(x,t)dxdt \leq \|\phi(x,t)\|_{L^\infty} \int_0^T \int_{\mathbb{R}^d} m_\epsilon(x,t,\xi)dxdt$$

$$= \|\phi(x,t)\|_{L^\infty} \int_0^T \int_{\mathbb{R}^d} \int_0^{+\infty} \int_{\mathbb{R}^d} \phi_\epsilon(t-s,x-y)dm(s,y,\xi)dxdt$$

$$\leq \|\phi(x,t)\|_{L^\infty} \int_0^{+\infty} \int_{\mathbb{R}^d} \int_0^{+\infty} \int_{\mathbb{R}^d} \phi_\epsilon(t-s,x-y)dxdtdm(s,y,\xi)$$

$$= \|\phi(x,t)\|_{L^\infty} \int_0^{+\infty} \int_{\mathbb{R}^d} dm(s,y,\xi) \leq \mu(\xi)\|\phi(x,t)\|_{L^\infty}$$

where the last inequality comes from $(5.6a)$. □

Lemma 5.3.4 *If $f(t,x,\xi)$ is continuous at $t=0$ we have for all $T>0$*

$$\int_0^T \int_{\mathbb{R}^{d+1}} |f(x,t,\xi)|d\xi dxdt \leq T \int_{\mathbb{R}^{d+1}} |f(x,0,\xi)|d\xi dx. \tag{5.16}$$

Proof: Let $\phi(x) \in C_c^\infty(\mathbb{R}^d)$. In addition consider a family of convex functions $S_\delta(\xi) \in C^\infty(\mathbb{R})$ such that

$$S_\delta(\xi) \to |\xi| \text{ as } \delta \to 0,\ S(0)=0$$

with $\operatorname{supp} S''(\xi)$ uniformly bounded [44]. Subsequently (5.11) implies

$$\int_{\mathbb{R}^d} S'_\delta(\xi)\frac{\partial f_\epsilon(x,t,\xi)}{\partial t}\phi(x)d\xi dx - \int_{\mathbb{R}^d} S'_\delta(\xi)f_\epsilon(x,t,\xi)\nabla_u A(\xi)\cdot\nabla_x\phi(x)dxd\xi$$

$$= k\int_{\mathbb{R}^d} S'_\delta(\xi)f_\epsilon(x,t,\xi)\Delta_x\phi(x)dxd\xi + \int_{\mathbb{R}^d} S'_\delta(\xi)\frac{\partial}{\partial\xi}m_\epsilon(x,t,\xi)\phi(x)dxd\xi$$

At this point we make the choice

$$\phi(x) = \tilde{\phi}_R(x) \tag{5.17}$$

with $\tilde{\phi}_R(x) = \tilde{\phi}\left(\frac{x}{R}\right)$ and $\tilde{\phi} \in C_c^\infty(\mathbb{R}^d)$, $0 \leq \tilde{\phi} \leq 1$, $\tilde{\phi} \equiv 1$ on $\mathcal{B}(0,1)$, $\operatorname{supp} \tilde{\phi} \in \mathcal{B}(0,2)$ where by $\mathcal{B}(x,r)$ we denote the closed ball with center x and radius r. In view of

$$
\begin{aligned}
&S_\delta'(\xi) f_\epsilon(x,t,\xi) \Delta_x \tilde{\phi}_R(x) \to 0 \\
&S_\delta'(\xi) \frac{\partial}{\partial \xi} m_\epsilon(x,t,\xi) \tilde{\phi}_R(x) \to S_\delta'(\xi) \frac{\partial}{\partial \xi} m_\epsilon(x,t,\xi) \\
&S_\delta'(\xi) \frac{\partial}{\partial t} f_\epsilon(x,t,\xi) \tilde{\phi}_R(x) \to S_\delta'(\xi) \frac{\partial f_\epsilon(x,t,\xi)}{\partial t} \\
&S_\delta'(\xi) f_\epsilon(x,t,\xi) \nabla_u A(\xi) \cdot \nabla_x \tilde{\phi}_R(x) \to 0
\end{aligned}
$$

(see [45], [40]) a.e. in x, t and ξ as $R \to +\infty$ we conclude that

$$
\begin{aligned}
\int\limits_{\mathbb{R}^{d+1}} S_\delta'(\xi) \frac{\partial f_\epsilon(x,t,\xi)}{\partial t} d\xi dx &= \int\limits_{\mathbb{R}^{d+1}} S_\delta'(\xi) \frac{\partial}{\partial \xi} m_\epsilon(x,t,\xi) dx d\xi \\
&= -\int\limits_{\mathbb{R}^{d}} S_\delta''(\xi) m_\epsilon(x,t,\xi) dx d\xi.
\end{aligned}
\tag{5.18}
$$

We now integrate w.r.t. t and hence

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}} S_\delta'(\xi) f_\epsilon(x,t,\xi) d\xi dx - \int\limits_{\mathbb{R}^{d+1}} S_\delta'(\xi) f_\epsilon(x,0,\xi) d\xi dx \\
&\qquad\qquad = -\int_0^t \int\limits_{\mathbb{R}^{d+1}} S_\delta''(\xi) m_\epsilon(x,t,\xi) dx d\xi dt \leq 0.
\end{aligned}
$$

Subsequently we notice that for all $\delta > 0$ we have

$$
-\int_0^t \int\limits_{\mathbb{R}^{d+1}} S_\delta''(\xi) m_\epsilon(x,t,\xi) dx d\xi dt \leq 0 \tag{5.19}
$$

since $m_\epsilon > 0$ and $S''_\delta > 0$. Passing to the limit as $\delta \to 0$ while in addition taking into account the limits

$$\int\limits_{\mathbb{R}^d} S'_\delta(\xi) f_\epsilon(x,t,\xi)dx \to \int\limits_{\mathbb{R}^d} sgn(\xi) f_\epsilon(x,t,\xi)dx, \qquad \delta \to 0$$

$$\int\limits_{\mathbb{R}^d} S'_\delta(\xi) f_\epsilon(x,0,\xi)dx \to \int\limits_{\mathbb{R}^d} sgn(\xi) f_\epsilon(x,0,\xi)dx, \qquad \delta \to 0$$

a.e. in ξ we obtain

$$\int\limits_{\mathbb{R}^{d+1}} |f_\epsilon(x,t,\xi)|d\xi dx - \int\limits_{\mathbb{R}^{d+1}} |f_\epsilon(x,0,\xi)|d\xi dx = -\int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,0)dxdt$$

where we have also used (5.13). The limit in right-hand side is understood in the sense of weak star convergence in $\mathbf{M}^+(\mathbb{R})$ i.e.

$$\int\limits_{\mathbb{R}} S''_\delta(\xi) \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\xi)dxdtd\xi \to \int\limits_{\mathbb{R}} 2\delta_0(\xi) \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\xi)dxdtd\xi \geq 0,\, \delta \to 0.$$

Therefore we conclude from (5.19) that

$$\int\limits_{\mathbb{R}^{d+1}} |f_\epsilon(x,t,\xi)|d\xi dx - \int\limits_{\mathbb{R}^{d+1}} |f_\epsilon(x,0,\xi)|d\xi dx \leq 0. \tag{5.20}$$

Now integration w.r.t t from 0 to T in (5.20) yields

$$\int_0^T \int\limits_{\mathbb{R}^{d+1}} |f_\epsilon(x,t,\xi)|d\xi dxdt \leq T \int\limits_{\mathbb{R}^{d+1}} |f_\epsilon(x,0,\xi)|d\xi dx. \tag{5.21}$$

Letting $\epsilon \to 0$ in the above inequality we have

$$\int_0^T \int\limits_{\mathbb{R}^d} |f_\epsilon(x,t,\xi)|dxdt \to \int_0^T \int\limits_{\mathbb{R}^d} |f(x,t,\xi)|dxdt$$

$$\int\limits_{\mathbb{R}^d} |f_\epsilon(x,0,\xi)|dx \to \int\limits_{\mathbb{R}^d} |f(x,0,\xi)|dx$$

a.e. in ξ. The first limit holds true due to standard L^p approximation theory. To see the second limit we observe that since $f(t,x,\xi)$ is continuous at $t=0$ we have again from well known results that

$$sgn(\xi)f_{\epsilon_1,\epsilon_2}(x,0,\xi) \to sgn(\xi)f_{\epsilon_2}(x,0,\xi)$$

as $\epsilon_1 \to 0$ a.e. in ξ and x. Thus, by the D.C.T we have

$$\int_{\mathbb{R}^d} |f_{\epsilon_1,\epsilon_2}(x,0,\xi)|dx \to \int_{\mathbb{R}^d} |f_{\epsilon_2}(x,0,\xi)|dx.$$

But

$$\int_{\mathbb{R}^d} |f_{\epsilon_2}(x,0,\xi)|dx \to \int_{\mathbb{R}^d} |f(x,0,\xi)|dx$$

a.e. in ξ. Therefore

$$\int_{\mathbb{R}^d} |f_{\epsilon}(x,0,\xi)|dx \to \int_{\mathbb{R}^d} |f(x,0,\xi)|dx.$$

as $\epsilon \to 0$ a.e. in ξ and thus using again the D.C.T. we pass to the limit in (5.21) and obtain (5.16). $\square$

Next we have also two supplementary results of the previous lemma.

Lemma 5.3.5 *If $f(t,x,\xi)$ is continuous at $t=0$ we have for all $T>0$*

$$\int_0^T \int_{\mathbb{R}^d} \int_{\xi}^{+\infty} |f(x,t,\eta)|d\eta dx dt \leq T \int_{\mathbb{R}^d} \int_{\xi}^{+\infty} |f(x,0,\eta)|d\eta dx. \tag{5.22}$$

for all $\xi \geq 0$.

Proof: We arguing here as in the previous lemma. Take $\phi(x) \in C_c^\infty(\mathbb{R}^d)$ and consider a family of convex functions $S_\delta(\xi) \in C^\infty(\mathbb{R})$ such that

$$S_\delta(\xi) \to \xi^+ \text{ as } \delta \to 0$$

where

$$\xi^{+}=\begin{cases}\xi, & \xi>0\\ \\ 0, & \xi<0\end{cases}$$

and supp $S''(\xi)$ uniformly bounded . Therefore (5.11) implies

$$\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} S'_{\delta}(\eta)\frac{\partial f_{\epsilon}(x,t,\eta)}{\partial t}\phi(x)d\eta dx-\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} S'_{\delta}(\eta)f_{\epsilon}(x,t,\eta)\nabla_u A(\eta)\cdot\nabla_x\phi(x)d\eta dx$$
$$=k\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} S'_{\delta}(\eta)f_{\epsilon}(x,t,\eta)\Delta_x\phi(x)d\eta dx+\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} S'_{\delta}(\eta)\frac{\partial}{\partial\eta}m_{\epsilon}(x,t,\eta)\phi(x)d\eta dx$$

for all $\xi>0$. At this point the choice

$$\phi(x)=\tilde{\phi}_R(x) \tag{5.23}$$

as in the previous lemma implies

$$\begin{aligned}
&S'_{\delta}(\eta)f_{\epsilon}(x,t,\eta)\Delta_x\tilde{\phi}_R(x)\to 0\\
&S'_{\delta}(\eta)\frac{\partial}{\partial\xi}m_{\epsilon}(x,t,\eta)\tilde{\phi}_R(x)\to S'_{\delta}(\eta)\frac{\partial}{\partial\eta}m_{\epsilon}(x,t,\eta)\\
&S'_{\delta}(\eta)\frac{\partial}{\partial t}f_{\epsilon}(x,t,\eta)\tilde{\phi}_R(x)\to S'_{\delta}(\eta)\frac{\partial f_{\epsilon}(x,t,\eta)}{\partial t}\\
&S'_{\delta}(\eta)f_{\epsilon}(x,t,\eta)\nabla_u A(\eta)\cdot\nabla_x\tilde{\phi}_R(x)\to 0
\end{aligned}$$

(see [45], [40]) a.e. in x, t and η as $R\to+\infty$ we conclude that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} S'_{\delta}(\eta)\frac{\partial f_{\epsilon}(x,t,\eta)}{\partial t}d\eta dx=\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} S'_{\delta}(\eta)\frac{\partial}{\partial\eta}m_{\epsilon}(x,t,\eta)d\eta dx\\
&=-\int\limits_{\mathbb{R}^d} S'_{\delta}(\xi)m_{\epsilon}(x,t,\xi)dx-\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} S''_{\delta}(\eta)m_{\epsilon}(x,t,\eta)d\eta dx.
\end{aligned} \tag{5.24}$$

We now integrate w.r.t. t and hence

$$\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} S'_\delta(\eta) f_\epsilon(x,t,\eta)d\eta dx - \int\limits_{\mathbb{R}^d} S'_\delta(\xi) f_\epsilon(x,0,\eta)d\eta dx$$
$$= -\int_0^t\int\limits_{\mathbb{R}^d} S'_\delta(\xi) m_\epsilon(x,t,\xi)dxdt - \int_0^t\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} S''_\delta(\eta) m_\epsilon(x,t,\eta)dxd\eta dt \le 0.$$

since $m_\epsilon, S''_\delta > 0$ and $S'_\delta > 0$. Passing to the limit as $\delta \to 0$, since for almost all positive η and ξ we have

$$\int\limits_{\mathbb{R}^d} S'_\delta(\eta) f_\epsilon(x,t,\eta)dx \to \int\limits_{\mathbb{R}^d} f_\epsilon(x,t,\eta)dx, \qquad \delta \to 0$$
$$\int\limits_{\mathbb{R}^d} S'_\delta(\eta) f_\epsilon(x,0,\eta)dx \to \int\limits_{\mathbb{R}^d} f_\epsilon(x,0,\eta)dx, \qquad \delta \to 0$$
$$\int\limits_{\mathbb{R}^d} S'_\delta(\xi) m_\epsilon(x,t,\xi)dx \to \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\xi)dx, \qquad \delta \to 0$$

we obtain

$$\int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} |f_\epsilon(x,t,\eta)|d\eta dx - \int\limits_{\mathbb{R}^d}\int_{\xi}^{+\infty} |f_\epsilon(x,0,\eta)|d\eta dx$$
$$= -\int_0^t\int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\xi)dxdt - \int_{\xi}^{+\infty} \delta_0(\eta)\int_0^t\int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta)dxdtd\eta$$

where we have also used (5.13). The last limit in right-hand side is understood in the sense of weak star convergence in $\mathbf{M}^+(\mathbb{R})$ i.e.

$$\int_{\xi}^{+\infty} S''_\delta(\eta)\int_0^t\int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta)dxdtd\eta \to \int_{\xi}^{+\infty}\int_0^t\int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta)dxdtd\delta_0(\eta) \ge 0, \delta \to 0.$$

To show this we first consider $\xi > 0$. There holds,

$$\int_{\xi}^{+\infty} S''_\delta(\eta)\int_0^t\int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta)dxdtd\eta \to 0, \delta \to 0.$$

Next if $\xi = 0$ we have

$$\int_0^{+\infty} S''_\delta(\eta) \int_0^t \int_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\eta$$
$$= \int_{-\infty}^{+\infty} S''_\delta(\eta) \int_0^t \int_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\eta \to \int_{-\infty}^{+\infty} \int_0^t \int_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\delta_0(\eta)$$
$$= \int_0^{+\infty} \int_0^t \int_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\delta_0(\eta).$$

Therefore we conclude that

$$\int_{\mathbb{R}^d} \int_\xi^{+\infty} |f_\epsilon(x,t,\eta)| d\eta dx - \int_{\mathbb{R}^d} \int_\xi^{+\infty} |f_\epsilon(x,0,\eta)| d\eta dx \leq 0. \tag{5.25}$$

Now integration w.r.t t from 0 to T in (5.25) yields

$$\int_0^T \int_{\mathbb{R}^d} \int_\xi^{+\infty} |f_\epsilon(x,t,\eta)| d\eta dx dt \leq T \int_{\mathbb{R}^d} \int_\xi^{+\infty} |f_\epsilon(x,0,\eta)| d\eta dx. \tag{5.26}$$

Letting $\epsilon \to 0$ in the above inequality we have

$$\int_0^T \int_{\mathbb{R}^d} |f_\epsilon(x,t,\eta)| dx dt \to \int_0^T \int_{\mathbb{R}^d} |f(x,t,\eta)| dx dt$$
$$\int_{\mathbb{R}^d} |f_\epsilon(x,0,\eta)| dx \to \int_{\mathbb{R}^d} |f(x,0,\eta)| dx$$

a.e. in ξ. To see the second limit we observe that since $f(t,x,\xi)$ is continuous at $t = 0$ we have again from well known results that

$$sgn(\xi) f_{\epsilon_1,\epsilon_2}(x,0,\xi) \to sgn(\xi) f_{\epsilon_2}(x,0,\xi)$$

as $\epsilon_1 \to 0$ a.e. in ξ and x. Thus, by the D.C.T we have

$$\int_{\mathbb{R}^d} |f_{\epsilon_1,\epsilon_2}(x,0,\xi)|dx \to \int_{\mathbb{R}^d} |f_{\epsilon_2}(x,0,\xi)|dx.$$

But

$$\int_{\mathbb{R}^d} |f_{\epsilon_2}(x,0,\xi)|dx \to \int_{\mathbb{R}^d} |f(x,0,\xi)|dx$$

a.e. in ξ. Therefore

$$\int_{\mathbb{R}^d} |f_{\epsilon}(x,0,\xi)|dx \to \int_{\mathbb{R}^d} |f(x,0,\xi)|dx.$$

as $\epsilon \to 0$ a.e. in ξ and thus using again the D.C.T. we pass to the limit in (5.26) and obtain (5.16). $\square$

Lemma 5.3.6 *If $f(t,x,\xi)$ is continuous at $t = 0$ we have for all $T > 0$*

$$\int_0^T \int_{\mathbb{R}^d} \int_{-\infty}^{\xi} |f(x,t,\eta)|d\eta dx dt \le T \int_{\mathbb{R}^d} \int_{-\infty}^{\xi} |f(x,0,\eta)|d\eta dx. \tag{5.27}$$

for all $\xi \le 0$.

Proof: We arguing here as in the previous lemma. Take $\phi(x) \in C_c^\infty(\mathbb{R}^d)$ and consider a family of convex functions $S_\delta(\xi) \in C^\infty(\mathbb{R})$ such that

$$S_\delta(\xi) \to \xi^- \text{ as } \delta \to 0$$

where

$$\xi^- = \begin{cases} 0, & \xi > 0 \\ -\xi, & \xi < 0 \end{cases}$$

and supp $S''(\xi)$ uniformly bounded . Therefore (5.11) implies

$$\int\limits_{\mathbb{R}^d}\int_{-\infty}^{\xi} S'_\delta(\eta)\frac{\partial f_\epsilon(x,t,\eta)}{\partial t}\phi(x)d\eta dx - \int\limits_{\mathbb{R}^d}\int_{-\infty}^{\xi} S'_\delta(\eta)f_\epsilon(x,t,\eta)\nabla_u A(\eta)\cdot\nabla_x\phi(x)d\eta dx$$
$$= k\int\limits_{\mathbb{R}^d}\int_{-\infty}^{\xi} S'_\delta(\eta)f_\epsilon(x,t,\eta)\Delta_x\phi(x)d\eta dx + \int\limits_{\mathbb{R}^d}\int_{-\infty}^{\xi} S'_\delta(\eta)\frac{\partial}{\partial\eta}m_\epsilon(x,t,\eta)\phi(x)d\eta dx$$

for all $\xi < 0$. Arguing as before we pick

$$\phi(x) = \tilde{\phi}_R(x) \tag{5.28}$$

and thus

$$S'_\delta(\eta)f_\epsilon(x,t,\eta)\Delta_x\tilde{\phi}_R(x) \to 0$$
$$S'_\delta(\eta)\frac{\partial}{\partial\xi}m_\epsilon(x,t,\eta)\tilde{\phi}_R(x) \to S'_\delta(\eta)\frac{\partial}{\partial\eta}m_\epsilon(x,t,\eta)$$
$$S'_\delta(\eta)\frac{\partial}{\partial t}f_\epsilon(x,t,\eta)\tilde{\phi}_R(x) \to S'_\delta(\eta)\frac{\partial f_\epsilon(x,t,\eta)}{\partial t}$$
$$S'_\delta(\eta)f_\epsilon(x,t,\eta)\nabla_u A(\eta)\cdot\nabla_x\tilde{\phi}_R(x) \to 0$$

a.e. in x, t and η as $R \to +\infty$ we conclude that

$$\begin{aligned}\int\limits_{\mathbb{R}^d}\int_{-\infty}^{\xi} S'_\delta(\eta)\frac{\partial f_\epsilon(x,t,\eta)}{\partial t}d\eta dx &= \int\limits_{\mathbb{R}^d}\int_{-\infty}^{\xi} S'_\delta(\eta)\frac{\partial}{\partial\eta}m_\epsilon(x,t,\eta)d\eta dx\\ &= \int\limits_{\mathbb{R}^d} S'_\delta(\xi)m_\epsilon(x,t,\xi)dx - \int\limits_{\mathbb{R}^d}\int_{-\infty}^{\xi} S''_\delta(\eta)m_\epsilon(x,t,\eta)d\eta dx.\end{aligned} \tag{5.29}$$

We now integrate w.r.t. t and hence

$$\int\limits_{\mathbb{R}^d}\int_{-\infty}^{\xi} S'_\delta(\eta)f_\epsilon(x,t,\eta)d\eta dx - \int\limits_{\mathbb{R}^d} S'_\delta(\xi)f_\epsilon(x,0,\eta)d\eta dx$$
$$= \int_0^t\int\limits_{\mathbb{R}^d} S'_\delta(\xi)m_\epsilon(x,t,\xi)dxdt - \int_0^t\int\limits_{\mathbb{R}^d}\int_{-\infty}^{\xi} S''_\delta(\eta)m_\epsilon(x,t,\eta)dxd\eta dt \le 0.$$

since $m_\epsilon, S''_\delta > 0$ and $S'_\delta < 0$. Passing to the limit as $\delta \to 0$, since for almost all negative η and ξ we have

$$\int\limits_{\mathbb{R}^d} S'_\delta(\eta) f_\epsilon(x,t,\eta)dx \to -\int\limits_{\mathbb{R}^d} f_\epsilon(x,t,\eta)dx, \qquad \delta \to 0$$

$$\int\limits_{\mathbb{R}^d} S'_\delta(\eta) f_\epsilon(x,0,\eta)dx \to -\int\limits_{\mathbb{R}^d} f_\epsilon(x,0,\eta)dx, \qquad \delta \to 0$$

$$\int\limits_{\mathbb{R}^d} S'_\delta(\xi) m_\epsilon(x,t,\xi)dx \to -\int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\xi)dx, \qquad \delta \to 0$$

we obtain

$$\int\limits_{\mathbb{R}^d} \int_{-\infty}^{\xi} |f_\epsilon(x,t,\eta)| d\eta dx - \int\limits_{\mathbb{R}^d} \int_{-\infty}^{\xi} |f_\epsilon(x,0,\eta)| d\eta dx$$
$$= -\int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\xi) dx dt - \int_{-\infty}^{\xi} \delta_0(\eta) \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\eta$$

where we have also used (5.13). The last limit in right-hand side is understood in the sense of weak star convergence in $\mathbf{M}^+(\mathbb{R})$ i.e.

$$\int_{-\infty}^{\xi} S''_\delta(\eta) \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\eta \to \int_{-\infty}^{\xi} \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\delta_0(\eta) \geq 0,\, \delta \to 0.$$

To show this we first consider $\xi < 0$. There holds,

$$\int_{-\infty}^{\xi} S''_\delta(\eta) \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\eta \to 0,\, \delta \to 0.$$

Next if $\xi = 0$ we have

$$\int_{-\infty}^{0} S''_\delta(\eta) \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\eta$$
$$= \int_{-\infty}^{+\infty} S''_\delta(\eta) \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\eta \to \int_{-\infty}^{+\infty} \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\delta_0(\eta)$$
$$= \int_{-\infty}^{0} \int_0^t \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\eta) dx dt d\delta_0(\eta).$$

Therefore we conclude that

$$\int\limits_{\mathbb{R}^d} \int_{-\infty}^{\xi} |f_\epsilon(x,t,\eta)| d\eta dx - \int\limits_{\mathbb{R}^d} \int_{-\infty}^{\xi} |f_\epsilon(x,0,\eta)| d\eta dx \leq 0. \tag{5.30}$$

Now integration w.r.t t from 0 to T in (5.30) yields

$$\int_0^T \int\limits_{\mathbb{R}^d} \int_{-\infty}^{\xi} |f_\epsilon(x,t,\eta)| d\eta dx dt \leq T \int\limits_{\mathbb{R}^d} \int_{-\infty}^{\xi} |f_\epsilon(x,0,\eta)| d\eta dx. \tag{5.31}$$

Letting $\epsilon \to 0$ in the above inequality we have

$$\int_0^T \int\limits_{\mathbb{R}^d} |f_\epsilon(x,t,\eta)| dx dt \to \int_0^T \int\limits_{\mathbb{R}^d} |f(x,t,\eta)| dx dt$$

$$\int\limits_{\mathbb{R}^d} |f_\epsilon(x,0,\eta)| dx \to \int\limits_{\mathbb{R}^d} |f(x,0,\eta)| dx$$

a.e. in ξ. To see the second limit we observe that since $f(t,x,\xi)$ is continuous at $t = 0$ we have again from well known results that

$$sgn(\xi) f_{\epsilon_1,\epsilon_2}(x,0,\xi) \to sgn(\xi) f_{\epsilon_2}(x,0,\xi)$$

as $\epsilon_1 \to 0$ a.e. in ξ and x. Thus, by the D.C.T we have

$$\int\limits_{\mathbb{R}^d} |f_{\epsilon_1,\epsilon_2}(x,0,\xi)| dx \to \int\limits_{\mathbb{R}^d} |f_{\epsilon_2}(x,0,\xi)| dx.$$

But

$$\int\limits_{\mathbb{R}^d} |f_{\epsilon_2}(x,0,\xi)| dx \to \int\limits_{\mathbb{R}^d} |f(x,0,\xi)| dx$$

a.e. in ξ. Therefore

$$\int_{\mathbb{R}^d} |f_\epsilon(x,0,\xi)|dx \to \int_{\mathbb{R}^d} |f(x,0,\xi)|dx.$$

as $\epsilon \to 0$ a.e. in ξ and thus using again the D.C.T. we pass to the limit in (5.31) and obtain 5.16. $\square$

Lemma 5.3.7 *Suppose that function $\tilde{f}$ is continuous at $t=0$. We then have the L_1-stability estimate*

$$\int_0^T \int_{\mathbb{R}^{d+1}} |\tilde{f}(x,t,\xi)|d\xi dxdt \le T \int_{\mathbb{R}^{d+1}} |\tilde{f}(x,0,\xi)|d\xi dx. \tag{5.32}$$

for all $T>0$.

Proof: Arguing as in the previous lemma we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\frac{\partial \tilde{f}_\epsilon(x,t,\xi)}{\partial t}\phi(x)d\xi dx - \int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\tilde{f}_\epsilon(x,t,\xi)\nabla_u A(\xi)\cdot\nabla_x\phi(x)dxd\xi \\
&= \int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\nabla_x[\tilde{B}\nabla\tilde{f}]_\epsilon\phi(x)dxd\xi + \int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\frac{\partial}{\partial\xi}m_\epsilon(x,t,\xi)\phi(x)dxd\xi \\
&= -\int_{\mathbb{R}^{d+1}} S'_\delta(\xi)[\tilde{B}\nabla\tilde{f}]_\epsilon\cdot\nabla_x\phi(x)dxd\xi + \int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\frac{\partial}{\partial\xi}m_\epsilon(x,t,\xi)\phi(x)dxd\xi \\
&= -\int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\int_0^{+\infty}\int_{\mathbb{R}^d}\tilde{B}(y)\nabla_y\tilde{f}(s,y,\xi)\phi_\epsilon(t-s,x-y)dsdy\cdot\nabla_x\phi(x)dxd\xi \\
&\quad + \int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\frac{\partial}{\partial\xi}m_\epsilon(x,t,\xi)\phi(x)dxd\xi \\
&\quad = \int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\int_0^{+\infty}\int_{\mathbb{R}^d}\nabla_y\tilde{B}(y)\tilde{f}(s,y,\xi)\phi_\epsilon(t-s,x-y)dsdy\cdot\nabla_x\phi(x)dxd\xi \\
&\quad + \int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\int_0^{+\infty}\int_{\mathbb{R}^d}\tilde{B}(y)\tilde{f}(s,y,\xi)\nabla_y\phi_\epsilon(t-s,x-y)dsdy\cdot\nabla_x\phi(x)dxd\xi \\
&\quad + \int_{\mathbb{R}^{d+1}} S'_\delta(\xi)\frac{\partial}{\partial\xi}m_\epsilon(x,t,\xi)\phi(x)dxd\xi
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\mathbb{R}^{d+1}} S'_\delta(\xi) \int_0^{+\infty} \int\limits_{\mathbb{R}^d} \nabla_y \tilde{B}(y) \tilde{f}(s,y,\xi) \phi_\epsilon(t-s,x-y) ds dy \cdot \nabla_x \phi(x) dx d\xi \\
&- \int\limits_{\mathbb{R}^{d+1}} S'_\delta(\xi) \int_0^{+\infty} \int\limits_{\mathbb{R}^d} \tilde{B}(y) \tilde{f}(s,y,\xi) \nabla_x \phi_\epsilon(t-s,x-y) ds dy \cdot \nabla_x \phi(x) dx d\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} S'_\delta(\xi) \frac{\partial}{\partial \xi} m_\epsilon(x,t,\xi) \phi(x) dx d\xi \\
&= \int\limits_{\mathbb{R}^{d+1}} S'_\delta(\xi) \int_0^{+\infty} \int\limits_{\mathbb{R}^d} \nabla_y \tilde{B}(y) \tilde{f}(s,y,\xi) \phi_\epsilon(t-s,x-y) ds dy \cdot \nabla_x \phi(x) dx d\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} S'_\delta(\xi) \int_0^{+\infty} \int\limits_{\mathbb{R}^d} \tilde{B}(y) \tilde{f}(s,y,\xi) \phi_\epsilon(t-s,x-y) ds dy \Delta_x \phi(x) dx d\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} S'_\delta(\xi) \nabla_\xi m_\epsilon(x,t,\xi) \phi(x) dx d\xi
\end{aligned}
$$

As before we are choosing $\phi(x,t)$ as in (5.17) and taking $R \to 0$, to obtain the analogue of (5.29) for $\tilde{f}_\epsilon$. The rest of the proof is the same as in the previous lemma. $\square$

The following lemma is the extension of [45, Lemma 4.2.1] to the diffusion problem and for general initial data. The proof is similar but we include it below for clarity.

Lemma 5.3.8 *For the measure $\mu(\xi)$ in* (5.15) *we can choose the formula*

$$
\mu(\xi) = \mathcal{X}_{\{\xi \geq 0\}} T \int\limits_{\mathbb{R}^d} \int_\xi^{+\infty} |f(y,0,\eta)| d\eta dy + \mathcal{X}_{\{\xi \leq 0\}} T \int\limits_{\mathbb{R}^d} \int_{-\infty}^{\xi} |f(y,0,\eta)| d\eta dy
$$

Proof: We multiply (5.11) by a test function $\phi(t,x) \in C_c^\infty([0,T] \times \mathbb{R}^d)$ and integrate w.r.t x and t. There holds,

$$
\begin{aligned}
&\frac{\partial}{\partial \xi} \int_0^T \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\xi) \phi(x,t) dx dt - k \int_0^T \int\limits_{\mathbb{R}^d} \nabla_x f_\epsilon(x,t,\xi) \cdot \nabla_x \phi(x,t) dx dt \\
&= - \int_0^T \int\limits_{\mathbb{R}^d} f_\epsilon(x,t,\xi) \left[\frac{\partial \phi(x,t)}{\partial t} + \nabla_u A(\xi) \cdot \nabla_x \phi(x,t) \right] dx dt \qquad (5.33) \\
&\qquad + \int\limits_{\mathbb{R}^d} f_\epsilon(x,T,\xi) \phi(T,x) dx - \int\limits_{\mathbb{R}^d} f_\epsilon(x,0,\xi) \phi(0,x) dx
\end{aligned}
$$

At this point we make the choice

$$\phi(x,t) = \tilde{\phi}_R(x)\psi(t)$$

with $\tilde{\phi}_R(x) = \tilde{\phi}\left(\frac{x}{R}\right)$ as in (5.17) and $\psi(t) \in C_c^\infty([0,T])$. In view of

$$\nabla_x f_\epsilon(x,t,\xi)\cdot\nabla_x\tilde{\phi}_R(x)\psi(t) \to 0$$

$$m_\epsilon(x,t,\xi)\tilde{\phi}_R(x)\psi(t) \to m_\epsilon(x,t,\xi)\psi(t)$$

$$f_\epsilon(x,t,\xi)\frac{\partial\tilde{\phi}_R(x)\psi(t)}{\partial t} \to f_\epsilon(x,t,\xi)\frac{\partial\psi(t)}{\partial t}$$

$$f_\epsilon(x,t,\xi)\nabla_u A(\xi)\cdot\nabla_x\tilde{\phi}_R(x)\psi(t) \to 0$$

$$f_\epsilon(x,0,\xi)\tilde{\phi}_R(x)\psi(0) \to f_\epsilon(x,0,\xi)\psi(0)$$

$$f_\epsilon(x,T,\xi)\tilde{\phi}_R(x)\psi(T) \to f_\epsilon(x,T,\xi)\psi(T)$$

for every x,t and a.e. in ξ as $R\to+\infty$, integrating w.r.t. ξ in (5.33) while considering $\xi \geq 0$ and subsequently passing to the limit we get

$$\begin{aligned}\int_\xi^{+\infty}\frac{\partial}{\partial\eta}\int_0^T\int_{\mathbb{R}^d} m_\epsilon(x,t,\eta)\psi(t)dxdtd\eta &= -\int_\xi^{+\infty}\int_0^T\int_{\mathbb{R}^d} f_\epsilon(x,t,\eta)\frac{\partial\psi(t)}{\partial t}dxdtd\eta \\ &- \int_\xi^{+\infty}\int_{\mathbb{R}^d} f_\epsilon(x,0,\eta)\psi(0)dxd\eta + \int_\xi^{+\infty}\int_{\mathbb{R}^d} f_\epsilon(x,T,\eta)\psi(T)dxd\eta.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d} m_\epsilon(x,t,\xi)\psi(t)dxdt &= \int_\xi^{+\infty}\int_0^T\int_{\mathbb{R}^d} f_\epsilon(x,t,\eta)\frac{\partial\psi(t)}{\partial t}dxdtd\eta \\ &+ \int_\xi^{+\infty}\int_{\mathbb{R}^d} f_\epsilon(x,0,\eta)\psi(0)dxd\eta - \int_\xi^{+\infty}\int_{\mathbb{R}^d} f_\epsilon(x,T,\eta)\psi(T)dxd\eta.\end{aligned}$$

We now pick $0 \leq \psi \leq 1, \psi(0) = 0$, $-1 \leq \frac{\partial \psi(t)}{\partial t} \leq 1$. Using the fact that $f_\epsilon(x,t,\xi) \geq 0$ for $\xi \geq 0$, we get,

$$\begin{aligned}
&\int_0^T \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\xi)\psi(t)dxdt \leq \int_\xi^{+\infty} \int_0^T \int\limits_{\mathbb{R}^d} f_\epsilon(x,t,\eta)dxdtd\eta \\
&= \int_\xi^{+\infty} \int_0^T \int\limits_{\mathbb{R}^{2d}} \int_0^{+\infty} f(y,s,\eta)\phi^1_{\epsilon_1}(t-s)\phi^2_{\epsilon_2}(x-y)dsdydxdtd\eta \\
&= \int\limits_{\mathbb{R}^{2d}} \int_0^T \int_0^{+\infty} \int_\xi^{+\infty} f(y,s,\eta)d\eta\phi^1_{\epsilon_1}(t-s)dsdt\phi^2_{\epsilon_2}(x-y)dydx \\
&\leq \int_0^T \int_0^{+\infty} \int\limits_{\mathbb{R}^d} \int_\xi^{+\infty} f(y,s,\eta)d\eta dy\phi^1_{\epsilon_1}(t-s)dsdt \\
&\leq \int_0^T \int\limits_{\mathbb{R}^d} \int_\xi^{+\infty} f(y,s,\eta)d\eta dyds \leq T \int\limits_{\mathbb{R}^d} \int_\xi^{+\infty} f(y,0,\eta)d\eta dy
\end{aligned}$$

Similarly we can show for $\xi \leq 0$,

$$\int_0^T \int\limits_{\mathbb{R}^d} m_\epsilon(x,t,\xi)\psi(t)dxdt \leq T \int\limits_{\mathbb{R}^d} \int_{-\infty}^\xi |f(y,0,\eta)|d\eta dy.$$

Finally the continuity of μ is implied by the continuity of Lebesque integral. $\square$

Lemma 5.3.9 *Assume $\kappa = \|\tilde{B}\|_{W_1^\infty(\mathbb{R}^d)}$ and fix $\epsilon_2 = O(\kappa^{\frac{1}{3}})$. Then if the conditions of* (5.3.4) *hold the regularised viscosity terms satisfy the limit*

$$\mathcal{G} := \int_0^T \int_{\mathbb{R}^{d+1}} (-\nabla_x[\tilde{B}\nabla\tilde{f}]_\epsilon + \kappa\Delta_x f_\epsilon)(\tilde{f}_\epsilon - f_\epsilon)dxd\xi dt \to 0 \tag{5.34}$$

as ϵ vanishes.

Proof: From $(5.6b)$ it follows that $|\tilde{f}_\epsilon - f_\epsilon| \leq |f_\epsilon| + |\tilde{f}_\epsilon| \leq 2$. Therefore,

$$
\begin{aligned}
|\mathcal{G}| = \bigg| \int_0^T \int_{\mathbb{R}^{d+1}} (-\nabla_x [\tilde{B}\nabla\tilde{f}]_\epsilon + \kappa\Delta_x f_\epsilon)(\tilde{f}_\epsilon - f_\epsilon) dxd\xi dt \bigg| \\
=& \left| \int_0^T \int_{\mathbb{R}^{d+1}} (-\nabla_x \cdot [\tilde{B}\nabla\tilde{f}]_\epsilon (\tilde{f}_\epsilon - f_\epsilon) + \kappa\Delta_x f_\epsilon (\tilde{f}_\epsilon - f_\epsilon)) dxd\xi dt \right| \\
\leq& \left| \int_0^T \int_{\mathbb{R}^{d+1}} -\nabla_x \cdot [\tilde{B}\nabla\tilde{f}]_\epsilon (\tilde{f}_\epsilon - f_\epsilon) dxd\xi dt \right| + \left| \int_0^T \int_{\mathbb{R}^{d+1}} \kappa\Delta_x f_\epsilon (\tilde{f}_\epsilon - f_\epsilon) dxd\xi dt \right| \\
\leq& \int_0^T \int_{\mathbb{R}^{d+1}} |\nabla_x \cdot [\tilde{B}\nabla\tilde{f}]_\epsilon| |(\tilde{f}_\epsilon - f_\epsilon)| dxd\xi dt + \int_0^T \int_{\mathbb{R}^{d+1}} \kappa |\Delta_x f_\epsilon| |(\tilde{f}_\epsilon - f_\epsilon)| dxd\xi dt \\
\leq& 2\int_0^T \int_{\mathbb{R}^{d+1}} |\nabla_x \cdot [\tilde{B}\nabla\tilde{f}]_\epsilon| dxd\xi dt + 2\int_0^T \int_{\mathbb{R}^{d+1}} \kappa |\Delta_x f_\epsilon| dxd\xi dt \\
\leq& 2\int_0^T \int_{\mathbb{R}^{d+1}} \left| \int_0^{+\infty} \int_{\mathbb{R}^d} \tilde{B}(y)\nabla_y \tilde{f}(s,y,\xi) \cdot \nabla_y \phi_\epsilon(t-s, x-y) dsdy \right| dxd\xi dt \\
+& 2\int_0^T \int_{\mathbb{R}^{d+1}} \int_0^{+\infty} \int_{\mathbb{R}^d} |\kappa f(s,y,\xi) \Delta_y \phi_\epsilon(t-s, x-y)| \, dsdydxd\xi dt \\
\leq& 2\int_0^T \int_{\mathbb{R}^{d+1}} \int_0^{+\infty} \int_{\mathbb{R}^d} \left| \tilde{f}(s,y,\xi) \nabla_y \tilde{B}(y) \cdot \nabla_y \phi_\epsilon(t-s, x-y) \right| dsdydxd\xi dt \\
+& 2\int_0^T \int_{\mathbb{R}^{d+1}} \int_0^{+\infty} \int_{\mathbb{R}^d} \left| \tilde{B}(y) \tilde{f}(s,y,\xi) \Delta_y \phi_\epsilon(t-s, x-y) \right| dsdydxd\xi dt \\
+& 2\int_0^T \int_{\mathbb{R}^{d+1}} \int_0^{+\infty} \int_{\mathbb{R}^d} |\kappa f(s,y,\xi) \Delta_y \phi_\epsilon(t-s, x-y)| \, dsdydxd\xi dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\mathcal{G}| \leq & 4\kappa \int_0^T \int_{\mathbb{R}^{d+1}} \int_0^{+\infty} \int_{\mathbb{R}^d} \left| \tilde{f}(s,y,\xi) \Delta_y \phi_\epsilon(t-s,x-y) \right| dsdydxd\xi dt \\
& +2\kappa \int_0^T \int_{\mathbb{R}^{d+1}} \int_0^{+\infty} \int_{\mathbb{R}^d} \left| f(s,y,\xi) \nabla_y \phi_\epsilon(t-s,x-y) \right| dsdydxd\xi dt \\
\leq & 4\kappa \int_0^T \int_{\mathbb{R}^{2d}} \int_0^{+\infty} \int_{\mathbb{R}} |\tilde{f}(s,y,\xi)| d\xi |\Delta_y \phi_\epsilon(t-s,x-y)| dsdydxdt \\
& +2\kappa \int_0^T \int_{\mathbb{R}^{2d}} \int_0^{+\infty} \int_{\mathbb{R}} |f(s,y,\xi)| d\xi |\nabla_y \phi_\epsilon(t-s,x-y)| dsdydxdt \\
= & 4\kappa \int_{\mathbb{R}^{2d}} \int_0^T \int_0^{+\infty} \int_{\mathbb{R}} |\tilde{f}(s,y,\xi)| d\xi \phi^1_{\epsilon_1}(t-s) dsdt |\Delta_y \phi^2_{\epsilon_2}(x-y)| dydx \\
& +2\kappa \int_{\mathbb{R}^{2d}} \int_0^T \int_0^{+\infty} \int_{\mathbb{R}} |f(s,y,\xi)| d\xi \phi^1_{\epsilon_1}(t-s) dsdt |\nabla_y \phi^2_{\epsilon_2}(x-y)| dydx \\
\leq & 4\kappa \int_0^T \int_0^{+\infty} \int_{\mathbb{R}^{d+1}} |\tilde{f}(s,y,\xi)| d\xi dy \phi^1_{\epsilon_1}(t-s) dsdt \int_{\mathbb{R}^d} |\Delta_{\tilde{y}} \phi^2_{\epsilon_2}(\tilde{y})| d\tilde{y} \\
& +2\kappa \int_0^T \int_0^{+\infty} \int_{\mathbb{R}^{d+1}} |f(s,y,\xi)| d\xi dy \phi^1_{\epsilon_1}(t-s) dsdt \int_{\mathbb{R}^d} |\nabla_{\tilde{y}} \phi^2_{\epsilon_2}(\tilde{y})| d\tilde{y}.
\end{aligned}
$$

Now, to treat the integrals with respect to t and s we observe that for $g \geq 0$ we have

$$
\begin{aligned}
|A| &= \int_0^T \int_0^{+\infty} g(s)\phi^1_{\epsilon_1}(t-s) dsdt = \int_0^T \int_{-\infty}^{+\infty} g(s)\phi^1_{\epsilon_1}(t-s) dsdt \\
&= \int_0^T \int_{-\infty}^{+\infty} g(t-\kappa)\phi^1_{\epsilon_1}(\kappa) d\kappa dt
\end{aligned}
$$

where in the inner integral we made a change of variables by setting $t - s = \kappa$. Thus,

$$
|A| = \int_{-\infty}^{+\infty} \phi^1_{\epsilon_1}(\kappa) \int_0^T g(t-\kappa) dt d\kappa = \int_{-1}^{0} \phi^1_{\epsilon_1}(\kappa) \int_0^T g(t-\kappa) dt d\kappa
$$

and by setting $t-\kappa=t'$ in the inner integral we get

$$
\begin{aligned}
|A| &= \int_{-1}^{0} \phi^1_{\epsilon_1}(\kappa) \int_{-\kappa}^{T-\kappa} g(t')dt'd\kappa \leq \int_{0}^{T+1} g(t')dt' \int_{-1}^{0} \phi^1_{\epsilon_1}(\kappa)d\kappa \\
&= \int_{0}^{T+1} g(t')dt'.
\end{aligned}
$$

We now have using Lemma 5.3.4, $|\nabla_y \phi^2_{\epsilon_2}(y)| \leq \frac{C}{\epsilon_2^d}$ and $|\Delta_y \phi^2_{\epsilon_2}(y)| \leq \frac{C}{\epsilon_2^{2d}}$

$$
\begin{aligned}
|\mathcal{G}| \leq & 4\kappa \int_{0}^{T+1} \int\limits_{\mathbb{R}^{d+1}} |\tilde{f}(s,y,\xi)|d\xi dyds \int\limits_{\mathbb{R}^{d}} |\Delta_y \phi^2_{\epsilon_2}(x-y)|dy \\
& + 2\kappa \int_{0}^{T+1} \int\limits_{\mathbb{R}^{d+1}} |f(s,y,\xi)|d\xi dyds \int\limits_{\mathbb{R}^{d}} |\nabla_y \phi^2_{\epsilon_2}(x-y)|dy \\
& \leq \frac{4\kappa C}{\epsilon_2^{2d}} \int_{0}^{T+1} \int\limits_{\mathbb{R}^{d+1}} |\tilde{f}(s,y,\xi)|d\xi dyds + \frac{2\kappa C}{\epsilon_2^{d}} \int_{0}^{T+1} \int\limits_{\mathbb{R}^{d+1}} |f(s,y,\xi)|d\xi dyds \\
& \leq \frac{4\kappa C(T+1)}{\epsilon_2^{2d}} \int\limits_{\mathbb{R}^{d+1}} |\tilde{f}(0,y,\xi)|d\xi dy + \frac{2\kappa C(T+1)}{\epsilon_2^{d}} \int\limits_{\mathbb{R}^{d+1}} |f(0,y,\xi)|d\xi dy.
\end{aligned}
$$

Hence $\mathcal{G} \to 0$ if we take $\epsilon_2 = O(\kappa^{\frac{1}{3}})$ as $\kappa \to 0$ and the proof is completed. □

Lemma 5.3.10 *Assume that functions f and $\tilde{f}$ are continuous at $t=0$ for all $\kappa > 0$ and $f_0(x,\xi) = \tilde{f}_0(x,\xi)$ a.e. in $\mathbb{R}^{d+1}$. Then the regularisations f_ϵ and $\tilde{f}_\epsilon$ satisfy the limit*

$$
\int\limits_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(0,x,\xi) - f_\epsilon(0,x,\xi)\right)^2 dxd\xi \to 0 \tag{5.35}
$$

as ϵ approaches to zero.

Proof: It is enough to prove that

$$
\int\limits_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(0,x,\xi) - \tilde{f}(0,x,\xi)\right)^2 dxd\xi \to 0
$$

or equivalently

$$\int\limits_{\mathbb{R}^{d+1}} \left(f_\epsilon(0,x,\xi) - f(0,x,\xi)\right)^2 dxd\xi \to 0 \tag{5.36}$$

since

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(0,x,\xi) - f_\epsilon(0,x,\xi)\right)^2 dxd\xi \\
&= \int\limits_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(0,x,\xi) - f_\epsilon(0,x,\xi) - \tilde{f}(0,x,\xi) + \tilde{f}(0,x,\xi)\right)^2 dxd\xi \\
&\leq 2\int\limits_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(0,x,\xi) - \tilde{f}(0,x,\xi)\right)^2 dxd\xi + 2\int\limits_{\mathbb{R}^{d+1}} \left(\tilde{f}(0,x,\xi) - f_\epsilon(0,x,\xi)\right)^2 dxd\xi \\
&\leq 2\int\limits_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(0,x,\xi) - \tilde{f}(0,x,\xi)\right)^2 dxd\xi + 4\int\limits_{\mathbb{R}^{d+1}} \left(\tilde{f}(0,x,\xi) - f(0,x,\xi)\right)^2 dxd\xi \\
&\qquad + 4\int\limits_{\mathbb{R}^{d+1}} \left(f(0,x,\xi) - f_\epsilon(0,x,\xi)\right)^2 dxd\xi \\
&= 2\int\limits_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(0,x,\xi) - \tilde{f}(0,x,\xi)\right)^2 dxd\xi + 4\int\limits_{\mathbb{R}^{d+1}} \left(f(0,x,\xi) - f_\epsilon(0,x,\xi)\right)^2 dxd\xi.
\end{aligned}$$

We now claim that a.e. in x and ξ as $\epsilon_1 \to 0$ we have the following limit

$$\int_0^{+\infty} \int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(-s,x-y)dyds \to \int\limits_{\mathbb{R}^d} f(0,y,\xi)\phi^2_{\epsilon_2}(x-y)dy.$$

We conclude this by the following calculations.

$$\begin{aligned}
&\left|\int_0^{+\infty} \int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(-s,x-y)dyds - \int\limits_{\mathbb{R}^d} f(0,y,\xi)\phi^2_{\epsilon_2}(x-y)dy\right| \\
&= \left|\int_0^{+\infty} \int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(-s,x-y)dyds - \int_0^{+\infty} \int\limits_{\mathbb{R}^d} f(0,y,\xi)\phi^2_{\epsilon_2}(x-y)dy\phi^1_{\epsilon_1}(-s)ds\right| \\
&\leq \int_0^{+\infty} \int\limits_{\mathbb{R}^d} |f(s,y,\xi) - f(0,y,\xi)|\phi^2_{\epsilon_2}(x-y)dy\phi^1_{\epsilon_1}(-s)ds.
\end{aligned}$$

Recall at this point that f is continuous at $s=0$ so given $\delta>0$ there exist an η such that for $|s|<\delta$

$$|f(s,y,\xi)-f(0,y,\xi)|<\eta$$

and hence

$$\begin{aligned}
&\left|\int_0^{+\infty}\int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(-s,x-y)dyds-\int\limits_{\mathbb{R}^d} f(0,y,\xi)\phi^2_{\epsilon_2}(x-y)dy\right|\\
\leq&\int\limits_{s<\delta}\int\limits_{\mathbb{R}^d}\eta\phi^2_{\epsilon_2}(x-y)dy\phi^1_{\epsilon_1}(-s)ds+2\|f\|_{L^\infty}\int\limits_{s>\delta}\phi^1_{\epsilon_1}(-s)ds\int\limits_{\mathbb{R}^d}\phi^2_{\epsilon_2}(x-y)dy\\
\leq&\eta+2\|f\|_{L^\infty}\int\limits_{s>\delta}\phi^1_{\epsilon_1}(-s)ds.
\end{aligned}$$

Thus for a given $\delta>0$ there exist η such that letting $\epsilon_1\to 0$ it holds that

$$\left|\int_0^{+\infty}\int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(-s,x-y)dyds-\int\limits_{\mathbb{R}^d} f(0,y,\xi)\phi^2_{\epsilon_2}(x-y)dy\right|<\eta$$

for $s<0$. Subsequently notice that

$$\begin{aligned}
&\left|\int_0^{+\infty}\int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(-s,x-y)dyds-f(0,x,\xi)\right|<\\
\leq&\int_0^{+\infty}\int\limits_{\mathbb{R}^d}|f(s,y,\xi)|\phi_\epsilon(-s,x-y)dyds+|f(0,x,\xi)|<2.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}} \left(\frac{1}{2}(f_\epsilon(0,x,\xi) - f(0,x,\xi))\right)^2 dxd\xi \\
&= \int\limits_{\mathbb{R}^{d+1}} \left(\frac{1}{2}\left(\int_0^{+\infty}\int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(-s,x-y)dyds - f(0,x,\xi)\right)\right)^2 dxd\xi \\
&\leq \int\limits_{\mathbb{R}^{d+1}} \frac{1}{2}\left|\int_0^{+\infty}\int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(-s,x-y)dyds - f(0,x,\xi)\right| dxd\xi.
\end{aligned}
$$

In view of

$$
f_\epsilon(0,x,\xi) - f(0,x,\xi) \to \int\limits_{\mathbb{R}^d} f(0,y,\xi)\phi^2_{\epsilon_2}(x-y)dyd\xi - f(0,x,\xi)
$$

as $\epsilon_1 \to 0$. a.e. in x and ξ we obtain by the D.C.T.

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}} \frac{1}{2}\left|\int_0^{+\infty}\int\limits_{\mathbb{R}^d} f(s,y,\xi)\phi_\epsilon(-s,x-y)dyds - f(0,x,\xi)\right| dxd\xi \\
&\qquad\qquad \to \int\limits_{\mathbb{R}^{d+1}} \frac{1}{2}\left|\int\limits_{\mathbb{R}^d} f(0,y,\xi)\phi^2_{\epsilon_2}(x-y)dy - f(0,x,\xi)\right| dxd\xi.
\end{aligned}
$$

But

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}} \frac{1}{2}\left|\int\limits_{\mathbb{R}^d} f(0,y,\xi)\phi^2_{\epsilon_2}(x-y)dy - f(0,x,\xi)\right| dxd\xi \\
&\leq \int\limits_{\mathbb{R}^{d+1}} \frac{1}{2}\int\limits_{\mathbb{R}^d} |f(0,y,\xi) - f(0,x,\xi)|\phi^2_{\epsilon_2}(x-y)dydxd\xi \\
&= \int\limits_{\mathbb{R}^{d+1}} \frac{1}{2}\int\limits_{\mathbb{R}^d} |f(0,x-y,\xi) - f(0,x,\xi)|\phi^2_{\epsilon_2}(y)dydxd\xi \\
&= \int\limits_{\mathbb{R}^d} \frac{1}{2}\int\limits_{\mathbb{R}^{d+1}} |f(0,x-y,\xi) - f(0,x,\xi)|d\xi dx\phi^2_{\epsilon_2}(y)dy.
\end{aligned}
$$

Thus given a $\delta > 0$ by L_1 continuity there exist an η such that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}} \frac{1}{2}\left|\int\limits_{\mathbb{R}^d} f(0,y,\xi)\phi_{\epsilon_2}^2(x-y)dy - f(0,x,\xi)\right| dxd\xi \\
\leq &\int\limits_{\mathbb{R}^d} \frac{1}{2} \int\limits_{\mathbb{R}^{d+1}} |f(0,x-y,\xi) - f(0,x,\xi)| d\xi dx \phi_{\epsilon_2}^2(y)dy \\
\leq &\frac{1}{2}\eta \int\limits_{y<\delta} \phi_{\epsilon_2}^2(y)dy + \int\limits_{\mathbb{R}^{d+1}} |f(0,x,\xi)|dxd\xi \int\limits_{y>\delta} \phi_{\epsilon_2}^2(y)dy
\end{aligned}$$

and therefore 5.36 is proved. □

We have the following

Lemma 5.3.11 *The quantity* $\int_0^T \int_{\mathbb{R}^{d+1}} m_\epsilon(x,t,\lambda) - \tilde{m}_\epsilon(x,t,\lambda) d(\nu_{x,t}^\epsilon - \tilde{\nu}_{x,t}^\epsilon)(\lambda) dxdt$ *satisfies*

$$\mathcal{J}_\epsilon(T) := \int_0^T \int_{\mathbb{R}^{d+1}} m_\epsilon(x,t,\lambda) - \tilde{m}_\epsilon(x,t,\lambda) d(\nu_{x,t}^\epsilon - \tilde{\nu}_{x,t}^\epsilon)(\lambda) dxd = \int_0^T \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon)\, dxd\xi dt.$$

Proof:

$$\begin{aligned}
&\int_0^T \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon)\, dxd\xi dt \\
&\quad = \int_0^T \int\limits_{\mathbb{R}^{d+2}} \chi_\lambda(\xi) d(\nu_{x,t}^\epsilon - \tilde{\nu}_{x,t}^\epsilon)(\lambda) \frac{\partial}{\partial \xi} \left(m_\epsilon(x,t,\xi) - \tilde{m}_\epsilon(x,t,\xi)\right) d\xi dxdt \\
&\quad = \int_0^T \int\limits_{\mathbb{R}^{d+2}} \chi_\lambda(\xi) \frac{\partial}{\partial \xi} \left(m_\epsilon(x,t,\xi) - \tilde{m}_\epsilon(x,t,\xi)\right) d(\nu_{x,t}^\epsilon - \tilde{\nu}_{x,t}^\epsilon)(\lambda) d\xi dxdt \\
&\quad = \int_0^T \int\limits_{\mathbb{R}^{d+2}} \chi_\lambda(\xi) \frac{\partial}{\partial \xi} \left(m_\epsilon(x,t,\xi) - \tilde{m}_\epsilon(x,t,\xi)\right) d\xi d(\nu_{x,t}^\epsilon - \tilde{\nu}_{x,t}^\epsilon)(\lambda) dxdt \\
&\quad = \int_0^T \int_{\mathbb{R}^{d+1}} \left(m_\epsilon(x,t,\lambda) - \tilde{m}_\epsilon(x,t,\lambda) + \tilde{m}_\epsilon(x,t,0) - m_\epsilon(x,t,0)\right) d(\nu_{x,t}^\epsilon - \tilde{\nu}_{x,t}^\epsilon)(\lambda) dxdt \\
&\quad = \int_0^T \int_{\mathbb{R}^{d+1}} \left(m_\epsilon(x,t,\lambda) - \tilde{m}_\epsilon(x,t,\lambda)\right) d(\nu_{x,t}^\epsilon - \tilde{\nu}_{x,t}^\epsilon)(\lambda) dxdt \\
&\qquad\qquad + \int_0^T \int\limits_{\mathbb{R}^d} \left(\tilde{m}_\epsilon(x,t,0) - m_\epsilon(x,t,0)\right) \int\limits_{\mathbb{R}} d(\nu_{x,t}^\epsilon - \tilde{\nu}_{x,t}^\epsilon)(\lambda) dxdt \\
&\quad = \int_0^T \int_{\mathbb{R}^{d+1}} \left(m_\epsilon(x,t,\lambda) - \tilde{m}_\epsilon(x,t,\lambda)\right) d(\nu_{x,t}^\epsilon - \tilde{\nu}_{x,t}^\epsilon)(\lambda) dxdt.
\end{aligned}$$

$\square$

Proof of Theorem 5.2.2

Using the estimates established earlier, we conclude this section by completing the proof of Theorem 5.2.2:

Proof: By subtracting (5.11) from (5.12) and multiplying by $2(\tilde{f}_\epsilon - f_\epsilon)$ we have for almost all ξ

$$\partial_t(\tilde{f}_\epsilon - f_\epsilon)^2 + \nabla_u A(\xi) \cdot \nabla_x(\tilde{f}_\epsilon - f_\epsilon)^2 + 2[-\nabla_x[\tilde{B}\nabla\tilde{f}]_\epsilon + \kappa\Delta_x f_\epsilon](\tilde{f}_\epsilon - f_\epsilon)$$
$$= 2(\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial\xi}\left(\tilde{m}_\epsilon - m_\epsilon\right)$$

Integration with respect to x, t, ξ now gives

$$\int_0^T \int_{\mathbb{R}^{d+1}} \partial_t(\tilde{f}_\epsilon - f_\epsilon)^2 + \nabla_u A(\xi) \cdot \nabla_x(\tilde{f}_\epsilon - f_\epsilon)^2 dxdtd\xi + 2\mathcal{G}$$
$$= \int_0^T \int_{\mathbb{R}^{d+1}} 2(\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial\xi}\left(\tilde{m}_\epsilon - m_\epsilon\right) dxdtd\xi$$

and hence,

$$\int_{\mathbb{R}^{d+1}} \left(\left(\tilde{f}_\epsilon(T) - f_\epsilon(T)\right)^2 - \left(\tilde{f}_\epsilon(0) - f_\epsilon(0)\right)^2\right) dxd\xi + 2\mathcal{G}$$
$$= \int_0^T \int_{\mathbb{R}^{d+1}} 2(\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial\xi}\left(\tilde{m}_\epsilon - m_\epsilon\right) dxdtd\xi$$

or

$$\int_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(T) - f_\epsilon(T)\right)^2 dxd\xi - \int_0^T \int_{\mathbb{R}^{d+1}} 2(\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial\xi}\left(\tilde{m}_\epsilon - m_\epsilon\right) dxdtd\xi$$
$$= \int_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(0) - f_\epsilon(0)\right)^2 dxd\xi - 2\mathcal{G}. \tag{5.37}$$

From Lemmata 5.3.10, and 5.3.9 the right-hand side of (5.37) tends to zero as $\epsilon \to 0$. Therefore, the left-hand side also approaches zero. But from assumption (5.9) the limit of the term

$$-\int_0^T \int_{\mathbb{R}^{d+1}} 2(\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial \xi}\left(\tilde{m}_\epsilon - m_\epsilon\right) dxdtd\xi$$

is greater than or equal to zero. Hence, as $\epsilon \to 0$ we have

$$\int_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(T) - f_\epsilon(T)\right)^2 dxd\xi \to 0$$

for almost all $T > 0$. Therefore,

$$\begin{aligned}
\|\tilde{f} - f\|^2_{L^2((0,T)\times\mathbb{R}^2)} &= \int_0^{T_1} \int_{\mathbb{R}^{d+1}} \left(\tilde{f}(t,x,\xi) - f(t,x,\xi)\right)^2 dxd\xi dt \\
\leq 4 \int_0^{T_1} \int_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(t,x,\xi) - \tilde{f}(t,x,\xi)\right)^2 dxd\xi dt &+ 4 \int_0^{T_1} \int_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(t,x,\xi) - f_\epsilon(t,x,\xi)\right)^2 dxd\xi dt \\
&+ 2 \int_0^{T_1} \int_{\mathbb{R}^{d+1}} \left(f(t,x,\xi) - f_\epsilon(t,x,\xi)\right)^2 dxd\xi dt
\end{aligned}$$

From the D.C.T. letting $\epsilon \to 0$ we have that

$$\int_0^{T_1} \int_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(t,x,\xi) - f_\epsilon(t,x,\xi)\right)^2 dxd\xi dt \to 0.$$

Thus, since both limits

$$\begin{aligned}
&\int_0^{T_1} \int_{\mathbb{R}^{d+1}} \left(f(t,x,\xi) - f_\epsilon(t,x,\xi)\right)^2 dxd\xi dt \to 0 \\
&\int_0^{T_1} \int_{\mathbb{R}^{d+1}} \left(\tilde{f}_\epsilon(t,x,\xi) - \tilde{f}(t,x,\xi)\right)^2 dxd\xi dt \to 0
\end{aligned}$$

we see that the assertion is proved. $\square$

5.4 L^2-based analysis

In this section we briefly discuss how one can use an L^2- based argument and simplify the proof considerably. However, this approach will require stronger assumptions, i.e., we need to assume $f(x,t,\xi) \in L^\infty(0,+\infty; H^1(\mathbb{R}^{d+1}))$ with compact support with respect to x and ξ. In practice, when considering computational methods, it is reasonable to expect that our approximations will satisfy such restrictions, and thus the proof below might be useful (with appropriate modifications) in applications to numerical methods. The plan of the analysis follows[41]. We present the main steps of the proof.

Theorem 5.4.1 *Assume that $f \in L^\infty(0,+\infty; H^1(\mathbb{R}^{d+1}))$ is a solution of* (5.5) *with $B_\epsilon(x) = \kappa$ and let $\tilde{f} \in L^\infty(0,+\infty; H^1(\mathbb{R}^{d+1}))$ a viscous generalised kinetic solution of* (5.5) *corresponding to $\tilde{a}(x), \tilde{m}$, and $\tilde{\nu}$ with $\kappa = \|\tilde{a}(x)\|_{L^\infty(\mathbb{R})}$. Furthermore, suppose that the initial data are continuous at $t = 0$, for almost all x and ξ, $\tilde{f}(0,x,\xi) = f(0,x,\xi)$, and the solutions are supported in compact set $\mathcal{D} \subset \mathbb{R}^{d+1}$. In addition to these hypothesis, assume that the defect measures m and $\tilde{m}$ satisfy, up to regularisation and as the regularisation parameter tends to zero,*

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} m - \tilde{m}\, d(\nu - \tilde{\nu}) dx\, d\xi\, dt \leq 0, \tag{5.38}$$

and $m = 0$ ($\tilde{m} = 0$) if $f = 0$ ($\tilde{f} = 0$). Then, as $\kappa \to 0$ we have the limit

$$\|f - \tilde{f}\|_{L^2} \to 0. \tag{5.39}$$

Proof: We write again the regularized equations

$$\frac{\partial f_\epsilon(x,t,\xi)}{\partial t} + \nabla_u A(\xi) \cdot \nabla_x f_\epsilon(x,t,\xi) = \kappa \Delta_x f_\epsilon(x,t,\xi) + \frac{\partial}{\partial \xi} m^\epsilon(x,t,\xi) \tag{5.40}$$

$$\frac{\partial \tilde{f}_\epsilon(x,t,\xi)}{\partial t} + \nabla_u A(\xi) \cdot \nabla_x \tilde{f}_\epsilon(x,t,\xi) = \nabla_x \cdot (\tilde{a}(x) \nabla_x \tilde{f}_\epsilon(x,t,\xi)) + \frac{\partial}{\partial \xi} \tilde{m}_\epsilon(x,t,\xi) \tag{5.41}$$

where $f_\epsilon = f(x,t,\xi)\star\phi_\epsilon$, $m_\epsilon(x,t,\xi) = [m\star\phi_\epsilon](x,t,\xi)$ with $\phi_\epsilon = \frac{1}{\epsilon_1}\phi_1(\frac{t}{\epsilon_1})\frac{1}{\epsilon_2^d}\phi_2(\frac{x}{\epsilon_2})$ as in the previous section. We also consider $\kappa = \|\tilde{a}(x)\|_{L^\infty(\mathbb{R})}$. Define

$$Q(t) := \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon(x,t,\xi) - f_\epsilon(x,t,\xi))^2 dx d\xi.$$

It will be sufficient to prove that $Q(t) \to 0$ as $\epsilon \to 0$ in order to show limit (5.39) as we have already seen in the previous section. The derivative of $Q(t)$ w.r.t. t is

$$\frac{d}{dt}Q(t) = \int\limits_{\mathbb{R}^{d+1}} \frac{\partial}{\partial t}(\tilde{f}_\epsilon - f_\epsilon)^2 dx d\xi = \int\limits_{\mathbb{R}^{d+1}} 2(\tilde{f}_\epsilon - f_\epsilon)(\frac{\partial \tilde{f}_\epsilon}{\partial t} - \frac{\partial f_\epsilon}{\partial t}) dx d\xi\,.$$

Using (5.40), (5.41) we have

$$\begin{aligned}
\frac{d}{dt}Q(t) &= \int\limits_{\mathbb{R}^{d+1}} 2[(\nabla_x\cdot(\tilde{a}(x)\nabla_x\tilde{f}_\epsilon) - \nabla_u A(\xi)\cdot\nabla_x\tilde{f}_\epsilon) - (\kappa\Delta_x f_\epsilon - \nabla_u A(\xi)\cdot\nabla_x f_\epsilon)](\tilde{f}_\epsilon - f_\epsilon) dx d\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial\xi}(\tilde{m}_\epsilon - m_\epsilon)\, dx d\xi \\
&= \int\limits_{\mathbb{R}^{d+1}} 2(\tilde{f}_\epsilon - f_\epsilon)[\nabla_x\cdot(\tilde{a}(x)\nabla_x\tilde{f}_\epsilon) - \nabla_u A(\xi)\cdot\nabla_x\tilde{f}_\epsilon - \kappa\Delta_x f_\epsilon + \nabla_u A(\xi)\cdot\nabla_x f_\epsilon] dx d\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial\xi}(\tilde{m}_\epsilon - m_\epsilon)\, dx d\xi \\
&= \int\limits_{\mathbb{R}^{d+1}} 2(\tilde{f}_\epsilon - f_\epsilon)[\nabla_x\cdot(\tilde{a}(x)\nabla_x\tilde{f}_\epsilon) + \nabla_u A(\xi)\cdot\nabla_x[f_\epsilon - \tilde{f}_\epsilon] - \kappa\Delta_x f_\epsilon] dx d\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial\xi}(\tilde{m}_\epsilon - m_\epsilon)\, dx d\xi \\
&= \int\limits_{\mathbb{R}^{d+1}} \nabla_u A(\xi)\cdot\nabla_x[f_\epsilon - \tilde{f}_\epsilon]^2 + 2(\tilde{f}_\epsilon - f_\epsilon)\nabla_x\cdot(\tilde{a}(x)\nabla_x\tilde{f}_\epsilon - \kappa\nabla_x f_\epsilon) dx d\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial\xi}(\tilde{m}_\epsilon - m_\epsilon)\, dx d\xi \\
&= \int\limits_{\mathbb{R}^{d+1}} 2(\tilde{f}_\epsilon - f_\epsilon)\nabla_x\cdot(\tilde{a}(x)\nabla_x\tilde{f}_\epsilon - \kappa\nabla_x f_\epsilon) dx d\xi + \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial\xi}(\tilde{m}_\epsilon - m_\epsilon)\, dx d\xi
\end{aligned}$$

By integrating by parts we have,

$$\frac{d}{dt}Q(t) = \int\limits_{\mathbb{R}^{d+1}} 2\kappa \nabla_x \tilde{f}_\epsilon \cdot \nabla_x f_\epsilon - 2\tilde{a}(x)(\nabla_x \tilde{f}_\epsilon)^2 - 2\kappa(\nabla_x f_\epsilon)^2 + 2\tilde{a}(x)\nabla_x f_\epsilon \cdot \nabla_x \tilde{f}_\epsilon dxd\xi \\ + \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial \xi}(\tilde{m}_\epsilon - m_\epsilon)\, dxd\xi. \tag{5.42}$$

Since $\tilde{a}(x) > 0$ and $|\tilde{a}(x)| \leq \kappa$, a.e. in $\mathbb{R}^d$, we get

$$\begin{aligned}
&\frac{d}{dt}Q(t) \leq \\
&\leq \int\limits_{\mathbb{R}^{d+1}} 2\kappa \nabla_x \tilde{f}_\epsilon \cdot \nabla_x f_\epsilon dxd\xi - \int\limits_{\mathbb{R}^{d+1}} 2\tilde{a}(x)(\nabla_x \tilde{f}_\epsilon)^2 dxd\xi \\
&- \int\limits_{\mathbb{R}^{d+1}} 2\kappa(\nabla_x f_\epsilon)^2 dxd\xi + \int\limits_{\mathbb{R}^{d+1}} |2\tilde{a}(x)\nabla_x f_\epsilon \cdot \nabla_x \tilde{f}_\epsilon| dxd\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial \xi}(\tilde{m}_\epsilon - m_\epsilon)\, dxd\xi \\
&\leq \int\limits_{\mathbb{R}^{d+1}} 2\kappa \nabla_x \tilde{f}_\epsilon \cdot \nabla_x f_\epsilon dxd\xi - \int\limits_{\mathbb{R}^{d+1}} 2\tilde{a}(x)(\nabla_x \tilde{f}_\epsilon)^2 dxd\xi - \int\limits_{\mathbb{R}^{d+1}} 2\kappa(\nabla_x f_\epsilon)^2 dxd\xi \\
&+ 2\left(\int\limits_{\mathbb{R}^{d+1}} \tilde{a}(x)(\nabla_x f_\epsilon)^2 dxd\xi\right)^{1/2} \left(\int\limits_{\mathbb{R}^{d+1}} \tilde{a}(x)(\nabla_x \tilde{f}_\epsilon)^2 dxd\xi\right)^{1/2} + \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial \xi}(\tilde{m}_\epsilon - m_\epsilon)\, dxd\xi \\
&\leq \int\limits_{\mathbb{R}^{d+1}} 2\kappa \nabla_x \tilde{f}_\epsilon \cdot \nabla_x f_\epsilon dxd\xi - \int\limits_{\mathbb{R}^{d+1}} 2\tilde{a}(x)(\nabla_x \tilde{f}_\epsilon)^2 dxd\xi - \int\limits_{\mathbb{R}^{d+1}} 2\kappa(\nabla_x f_\epsilon)^2 dxd\xi + \\
&+ \int\limits_{\mathbb{R}^{d+1}} \tilde{a}(x)(\nabla_x f_\epsilon)^2 dxd\xi + \int\limits_{\mathbb{R}^{d+1}} \tilde{a}(x)(\nabla_x \tilde{f}_\epsilon)^2 dxd\xi + \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial \xi}(\tilde{m}_\epsilon - m_\epsilon)\, dxd\xi \\
&\leq \int\limits_{\mathbb{R}^{d+1}} 2\kappa \nabla_x \tilde{f}_\epsilon \cdot \nabla_x f_\epsilon dxd\xi - \int\limits_{\mathbb{R}^{d+1}} \tilde{a}(x)(\nabla_x \tilde{f}_\epsilon)^2 dxd\xi - \int\limits_{\mathbb{R}^{d+1}} \kappa(\nabla_x f_\epsilon)^2 dxd\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial \xi}(\tilde{m}_\epsilon - m_\epsilon)\, dxd\xi.
\end{aligned}$$

Hence, using the fact $|\tilde{f}_\epsilon| \leq 1$, we conclude that

$$\begin{aligned}
&\frac{d}{dt}Q(t) + \int\limits_{\mathbb{R}^{d+1}} \tilde{a}(x)(\nabla_x \tilde{f}_\epsilon)^2 dx d\xi + \int\limits_{\mathbb{R}^{d+1}} \kappa(\nabla_x f_\epsilon)^2 dx d\xi \\
&\quad \leq \int\limits_{\mathbb{R}^{d+1}} 2\kappa \nabla_x \tilde{f}_\epsilon \cdot \nabla_x f_\epsilon dx d\xi + \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon) \, dx d\xi \\
&\quad = -2\kappa \int\limits_{\mathbb{R}^{d+1}} \tilde{f}_\epsilon \Delta_x f_\epsilon dx d\xi + \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon) \, dx d\xi \\
&\quad \leq 2\kappa \int\limits_{\mathbb{R}^{d+1}} |\Delta_x f_\epsilon| dx d\xi + \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon) \, dx d\xi \,.
\end{aligned} \tag{5.43}$$

But,

$$\begin{aligned}
|\Delta f_\epsilon| &= |\int\limits_{\mathbb{R}^{d+1}} \Delta f(s,y,\xi) \phi_\epsilon(t-s, x-y) dy ds| = |\int\limits_{\mathbb{R}^{d+1}} \nabla f(s,y,\xi) \cdot \nabla \phi_\epsilon(t-s, x-y) dy ds| \\
&\leq \int\limits_{\mathbb{R}^{d+1}} |\nabla f(s,y,\xi)| |\nabla \phi_\epsilon(t-s, x-y)| dy ds.
\end{aligned} \tag{5.44}$$

Therefore

$$\begin{aligned}
&\frac{d}{dt}Q(t) + \int\limits_{\mathbb{R}^{d+1}} \tilde{a}(x)(\nabla_x \tilde{f}_\epsilon)^2 dx d\xi + \int\limits_{\mathbb{R}^{d+1}} \kappa(\nabla_x f_\epsilon)^2 dx d\xi \\
&\leq 2\kappa \int\limits_{\mathbb{R}^{d+1}} \int\limits_{\mathbb{R}^{d+1}} |\nabla f(s,y,\xi)| |\nabla \phi_\epsilon(t-s, x-y)| dy ds dx d\xi \\
&+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon) \, dx d\xi \\
&= 2\kappa \int_0^t \int\limits_{\mathbb{R}^{2d}} \int\limits_{\mathbb{R}} |\nabla f(s,y,\xi)| d\xi \, |\nabla \phi_\epsilon(t-s, x-y)| dx dy ds \\
&+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon) \, dx d\xi
\end{aligned}$$

$$= 2\kappa \int\limits_{\mathbb{R}^{2d}} \int_0^t \int\limits_{\mathbb{R}} |\nabla f(s,y,\xi)| d\xi |\phi_\epsilon^1(t-s)| ds |\nabla \phi_\epsilon^2(x-y)| dy dx$$

$$+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon) \, dx d\xi$$

$$\leq \left(\int_0^t \left(\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}} |\nabla f(s,y,\xi)| d\xi dy \right)^2 ds \right)^{\frac{1}{2}} \left(\int_0^t |\phi_\epsilon^1(t-s)| ds \right)^{\frac{1}{2}} \frac{1}{\epsilon_2^d} \int\limits_{\mathbb{R}^d} |\nabla_x \phi^2 \left(\frac{x-y}{\epsilon_2} \right) | dy$$

$$+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon) \, dx d\xi$$

Moreover since we have assumed here that ∇f has compact support w.r.t. ξ and y, and using the change of variables, $z = \frac{x-y}{\epsilon_2}$, we have

$$\frac{d}{dt} Q(t) + \int\limits_{\mathbb{R}^{d+1}} \tilde{a}(x) (\nabla_x \tilde{f}_\epsilon)^2 dx d\xi + \int\limits_{\mathbb{R}^{d+1}} \kappa (\nabla_x f_\epsilon)^2 dx d\xi$$

$$\leq 2\kappa \left(supp_{\xi,y}(\nabla f) \int_0^t \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}} |\nabla f(s,y,\xi)|^2 d\xi dy ds \right)^{1/2} \frac{1}{\epsilon_2^d} \int\limits_{\mathbb{R}^d} |\nabla_x \phi^2 \left(\frac{x-y}{\epsilon_2} \right) | dy$$

$$+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon) \, dx d\xi$$

$$\leq 2\kappa t^{1/2} |\mathcal{D}|^{1/2} \left(\int_0^t \int\limits_{\mathbb{R}^{d+1}} |\nabla f(s,y,\xi)|^2 d\xi dy ds \right)^{1/2} \frac{1}{\epsilon_2} \int\limits_{\mathbb{R}^d} |\nabla_z \phi^2 (z)| dz$$

$$+ \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon) \frac{\partial}{\partial \xi} (\tilde{m}_\epsilon - m_\epsilon) \, dx d\xi$$

In order to find a bound for the first term on the right hand side, we observe that multiplying (5.40) by f_ϵ and assuming that f has the required regularity and we can show that

$$\frac{1}{2} \int\limits_{\mathbb{R}^{d+1}} (f(x,t,\xi))^2 d\xi dx + \kappa \int_0^t \int\limits_{\mathbb{R}^d} (\nabla f(x,t,\xi))^2 d\xi dx \leq C \int\limits_{\mathbb{R}^{d+1}} (f(x,0,\xi))^2 d\xi dx + Z_\epsilon(t)$$

where $Z_\epsilon \to 0$ as $\epsilon \to 0$, by our hypothesis on the measures. We conclude therefore that

$$\frac{d}{dt}Q(t) \leq \frac{C\kappa^{1/2}}{\epsilon_2}\|f(0)\|_{L^2(\mathbb{R}^{d+1})} + \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial \xi}(\tilde{m}_\epsilon - m_\epsilon)\, dx d\xi$$

and

$$Q(t) \leq Q(0) + \frac{C\kappa^{1/2}}{\epsilon_2}\|f(0)\|_{L^2(\mathbb{R}^{d+1})} + \int_0^t \int\limits_{\mathbb{R}^{d+1}} (\tilde{f}_\epsilon - f_\epsilon)\frac{\partial}{\partial \xi}(\tilde{m}_\epsilon - m_\epsilon)\, dx d\xi$$

From Lemma 5.3.10 we have already seen that as $\epsilon \to 0$ then $Q(0) \to 0$. Therefore as $\epsilon \to 0$ with $\kappa = o(\epsilon_2^2)$, then $Q(t) \to 0$. This completes the proof. □

Chapter 6

Future work

In this chapter we briefly discuss how the research outlined in this thesis could be further developed. A potential future work could be devoted to the further development of the research in Chapter 3, where we introduced a new approach for computing entropy solutions of HCL which has as a starting point a new mixed reformulation of the hyperbolic system which retains the original variables but still allows for conservative discretisation. For instance, we can extend our study to discontinuous Galerkin spatial discretisations, to finite difference-finite volume methods, in the spirit of [49, 50], and also to important systems, such as quasiconvex elastodynamics, where schemes based on entropy variables cannot be used. Furthermore, as we have mentioned in Section 3.4, the scheme (3.9) is quite simplified. Thus, one could try to add more refined stabilisation terms, such as shock capturing, aiming at improved computational results. Also another possible extension, could be the addition of nonlinear artificial viscosity terms. In this way we may be able to ensure L^p boundness of the approximate solution under weaker assumptions on the entropy function $\eta(u)$ apart from convexity.

There are several open questions relater to Chapter 4 and the numerical computation of measure-valued solutions of HCL. As far as the mathematical theory is concerned, there are many emerging questions for future research related to the analysis of such problems, such as uniqueness and stability issues mainly for kinetic (and systems thereof) approximations to continuum macroscopic models considered. Also, it is interesting to investigate the preservation of qualitative properties of the models, such as positivity and compatibility with the measure structure of approximations. As a first step in this direction, in Chapter 5 we have studied stability issues of generalised viscus kinetic formulations of conservation laws. There are open issues, as well, related to the efficiency and the computational justification of our approach. The choice of the numerical model in addition to its implementation, can be a really challenging task. In Section 4.4.3 we have described possible choices of numerical schemes that may be successful in capturing meaningful measures. Nevertheless, it is to be noted, that specific approximate defect measures may affect also the computational results drastically. The numerical/computational investigation of such issues is quite interesting. Furthermore, interesting future research could be to further study the problem of the stability of the computational measure with respect to the choice of the numerical model. A source of interesting questions is the investigation of possible connections of discrete kinetic models to the study and computation of *statistical solutions,* and the corresponding probability measures on function spaces, [22]. Most of the numerical algorithms for the computation of measure valued and statistical solutions for HCL are, to date, mainly based on Monte Carlo sampling, i.e., on solving several deterministic problems and sampling the results. In Section 5.1 we have underlined the compatibility relation between the Monte Carlo sampling method based on viscosity approximating models and the generalised viscous kinetic formulation. This observation gives rise to the question whether is possible to show uniqueness for the computational measure obtained by this method in the scalar case. Finally, it could be very interesting to make a systematic comparison on a variety of problems between the approximate measure-valued solutions of HCL which potentially can be computed through the approach presented in this thesis and the approximate measure-valued solutions which are obtained by the Monte Carlo approach.

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