School of Mathematical and Physical Sciences

Thesis submitted for the degree of Doctor of Philosophy

The Road to Quantum Gravity

From Elementary Considerations to Universal Predictions

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27 May 2022

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another university for the award of any other degree.

The work in this thesis has been completed in collaboration with Prof. Xavier Calmet, Prof. Roberto Casadio and Prof. Stephen Hsu, and has been published in the following papers:

- X. Calmet, R. Casadio and F. Kuipers,
  Quantum gravitational corrections to a star metric and the black hole limit,
  Phys. Rev. D **100**, no. 08, p. 6010 (2019).
  doi:10.1103/PhysRevD.100.086010; arXiv:1909.13277 [hep-th].

- F. Kuipers and X. Calmet,
  Singularity theorems in the effective field theory for quantum gravity at second order in curvature,
  Universe **6**, no. 10, p. 0171 (2020).
  doi:10.3390/universe6100171; arXiv:1911.05571 [gr-qc].

- X. Calmet, R. Casadio and F. Kuipers,
  Singularities in quantum corrected space-times,
  Phys. Lett. B **807**, p. 135605 (2020).
  doi:10.1016/j.physletb.2020.135605; arXiv:2003.04220 [hep-th].

- X. Calmet, R. Casadio and F. Kuipers,
  Quantum corrected equations of motion in the interior and exterior Schwarzschild spacetime,
  Phys. Rev. D **102**, no. 02, p. 6018 (2020).
  doi:10.1103/PhysRevD.102.026018; arXiv:2007.05416 [hep-th].

- X. Calmet and F. Kuipers,
  Bounds on very weakly interacting ultra light scalar and pseudoscalar dark matter from quantum gravity,
  Eur. Phys. J. C **80**, no. 08, p. 0781 (2020).
  doi:10.1140/epjc/s10052-020-8350-7; arXiv:2008.06243 [hep-ph].

- X. Calmet and F. Kuipers,
  Theoretical bounds on dark matter masses,
  Phys. Lett. B **814**, p. 136068 (2021).
  doi:10.1016/j.physletb.2021.136068; arXiv:2009.11575 [hep-ph].

- F. Kuipers,
  Stochastic Quantization on Lorentzian Manifolds,
  JHEP **05**, a. 028 (2021).
  doi:10.1007/JHEP05(2021)028; arXiv:2101.12552 [hep-th].

- F. Kuipers,
  Stochastic Quantization of Relativistic Theories,
  J. Math. Phys. **62**, no. 12, p. 2301 (2021).
  doi:10.1063/5.0057720; arXiv:2103.02501 [gr-qc].

- X. Calmet and F. Kuipers,
  Quantum gravitational corrections to the entropy of a Schwarzschild black hole,
  Phys. Rev. D **104**, no. 06, p. 6012 (2021).
  doi:10.1103/PhysRevD.104.066012; arXiv:2108.06824 [hep-th].

- X. Calmet, R. Casadio, S. D. H. Hsu and F. Kuipers,
  Quantum Hair from Gravity,
  Phys. Rev. Lett. **128**, no. 11, p. 1301 (2022).
  10.1103/PhysRevLett.128.111301; arXiv:2110.09386 [hep-th].

- F. Kuipers,
  Analytic Continuation of Stochastic Mechanics,
  J. Math. Phys. **63**, no. 04, p. 2301 (2022).
  doi: 10.1063/5.0073096; arXiv:2109.10710 [math-ph].

I have made significant contributions to all of these papers.

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Summary

General relativity and quantum theory are two cornerstones of modern physics, which, despite their huge individual successes, have so far failed to work together in a complete and consistent way. As a consequence, the unification of the two theories in a consistent formulation of quantum field theory on curved spacetimes and eventually in a theory of quantum gravity has become one of the holy grails of modern physics. In this thesis, some of the many aspects of such a theory of quantum gravity are explored.

The first part of the thesis is devoted to the construction of diffeomorphism invariant theories on a spacetime that is itself fluctuating. We show how this can be achieved for basic theories using second order geometry, which is an extension of the geometrical framework applied in general relativity.

After these elementary considerations, we move on to study predictions from quantum gravity at sub-Planckian energy scales, where quantum field theory and general relativity can be combined in the framework of effective field theory. The resulting effective field theory of gravity allows to make model independent predictions in quantum gravity.

In the second part of the thesis, we discuss perturbative predictions following from the unique effective action for quantum gravity. Here, we particularly focus on predictions from this formalism for compact stars, black holes and the fate of singularities in quantum gravity.

Finally, in the third part, we use effective field theory and the universality of the gravitational coupling to study quantum gravitational effects that lie within the reach of current experiments. We then discuss the implications for beyond the Standard Model physics and dark matter models in particular.

Acknowledgements

I would like to express my sincere gratitude to Xavier for supervising me throughout the Ph.D. journey and, in particular, for the many stimulating discussions we have had over the years. I would also like to thank Roberto and Steve for many interesting discussions and for collaborating on various projects, and express a special word of thanks to Roberto for hosting me at the University of Bologna.

Many thanks goes also to my fellow Ph.D. students, the postdocs and the staff in the group for the many entertaining conversations and for making my Ph.D. such a joyful experience. Finally, I would like to thank my family and friends for their continuous support.

Contents

Part I

Introduction

Chapter 1

Prelude

Quantum theory and general relativity are among the most successful theories in physics and are regarded as two of the corner stones of modern physics. Both theories were developed early in the 20th century, and managed to resolve several issues that were much discussed at the end of the 19th century. However, despite all successes of both quantum theory and general relativity, the theories seem to be at odds with each other. This realization led to the search for consistent quantum theories in gravitational backgrounds and the idea halfway the 20th century that gravity itself must be quantized in a theory of quantum gravity.

General relativity, on the one hand, is an extremely successful theory that provides a precise description of the gravitational force and relativistic effects. It has, for example, successfully predicted the existence of gravitational waves and black holes. Moreover, the Standard Model of cosmology or ΛCDM Model, which provides our best understanding of the evolution of the universe, has been formulated within this framework.

An important feature of general relativity is that the theory is mathematically consistent. Nevertheless, it predicts the existence of singular spacetimes, which indicates physical incompleteness of the theory. Moreover, by construction, general relativity does not incorporate any quantum effects, and is therefore only valid in the limit $\hbar \to 0$.

On a phenomenological level, the ΛCDM Model has several shortcomings. For example, the model requires a vast amount of dark matter and dark energy to be consistent with observations, but it is still a mystery what comprises this dark matter and dark energy.

Furthermore, the ΛCDM Model provides a good description of the evolution of the universe from the cosmic microwave background, which indicates the decoupling of photons in the early universe, up to our current epoch. Combined with insights from particle physics one can extrapolate the theory back to the end of a hypothetical inflationary era at which point matter begins to form. However, little is known about what happened during the first picosecond after the hypothetical Big Bang.

Quantum field theory, on the other hand, is arguably even more successful, as it lies at the heart of our understanding of both particle physics and the theory of condensed matter systems. As such, it underpins a large fraction of all technological advancement in the 20th and 21st century. Moreover, the Standard Model of particle physics is formulated in the language of quantum field theory, and has been tested to extreme precision in experiments.

However, whereas general relativity is a mathematically consistent theory, the mathematical underpinnings of quantum field theory are not yet completely understood. Although the Euclidean approach in quantum field theory has allowed to give a proper mathematical treatment of many quantum field theories in dimensions $d < 4$, and no go theorems have been formulated for various theories in $d > 4$ dimensions, there are still many open questions about the mathematical foundations of quantum field theory in any number dimensions, and, in particular, in the critical case $d = 4$.

Furthermore, even though the Standard Model of particle physics has been tested to extreme precisions, there are still many outstanding issues within the Standard Model. For example, the model has a large number of free parameters whose values can be fixed by experiment, but not on the basis of theoretical considerations. A more fundamental theory that fixes a subset of these parameters by theoretical arguments is desirable.

A major shortcoming of both the Standard Model and general relativity is that they have not yet been combined in one framework. This is problematic, as it implies that the corner stones of modern physics can only be applied in the regime where either quantum effects or gravitational effects are negligible. Although this allows to describe most of the physical phenomena, it also prevents from developing a proper understanding of the nature of black holes and the very early universe. It is hoped that a theory of quantum gravity that combines general relativity and quantum field theory in a consistent framework will be able to answer such questions. Furthermore, a theory of quantum gravity could provide handles towards the resolution of the aforementioned issues encountered in both the ΛCDM Model and the Standard Model of particle physics.

The search for this theory of quantum gravity has gained a vast amount of attention during the last decades. However, despite the formulation of many approaches to the problem and despite a huge amount of research within these approaches, there still is no widely accepted complete and consistent theory of quantum gravity. Nevertheless, using the various approaches, progress has been made towards a resolution of the problem, and many hints have been provided towards both the mathematical and phenomenological properties of such an illustrious theory of quantum gravity.

Early attempts to quantize gravity focus on the quantization of the fluctuations of the metric as a spin-2 graviton field. However, as was soon realized such a theory is not renormalizable. Later, it was realized that renormalizability of the theory could be regained, if the gravitational action is modified with higher derivative terms. However, this comes at the price of introducing a ghost in the theory. Nowadays, the modification of the gravitational action lies at the heart of the higher derivative gravity, the non-local gravity and the asymptotic safety approach. These approaches aim to obtain a renormalizable and ghost free theory of quantum gravity by adding a finite number of terms to the gravitational action.

The path of quantizing fluctuations around a fixed background metric is also followed in string theory and string inspired approaches such as the holographic approaches. However, in these approaches the fundamental point-like degrees of freedom encountered in ordinary quantum theories are substituted by the excitations of strings.

Another class of approaches aims to quantize gravity in a background independent way. Whereas string theory and higher derivative gravity quantize fluctuations of the metric as a gauge force around a fixed background spacetime, the background independent approaches aim to quantize spacetime itself without fixing a particular background metric. As there is little guidance on how to do so, this leads to a variety of ideas. Notable approaches include Loop Quantum Gravity, Causal Dynamical Triangulations, Group Field Theory and Causal Set Theory.

This thesis discusses two conservative frameworks at two extremes of the spectrum of approaches to quantum gravity. In part II, the elementary considerations, we will follow the line of thought that a final theory of quantum gravity must also provide a mathematical consistent framework of quantum field theory. As stochastic techniques have been very successful in constructive quantum field theory, we will explore what stochastic analysis can teach us about the interplay between gravity and quantum theories. Here, we will use methods from stochastic mechanics and stochastic quantization, i.e. combine ideas from the foundations of quantum mechanics and constructive quantum field theory, to obtain a consistent quantum theory in curved spacetimes.

As this framework has not been explored much in the literature, we won't be able to go beyond any elementary considerations. However, we will see that stochastic analysis provides a strong clue about the type of extensions of differential geometry, which is the mathematical foundation of general relativity, that are necessary to incorporate quantum effects in the theory of general relativity.

In parts III and IV, we jump to the other side of the spectrum of theories of quantum gravity, and discuss the low energy effective field theory of quantum gravity. Although general relativity with a quantized graviton is not renormalizable, any theory of quantum gravity that contains a graviton in its spectrum, respects general covariance, and reduces to general relativity at low energy scales, can be studied at sub-Planckian energy scales in a model independent way using methods from effective field theory.

In part III, we focus on predictions from a purely quantum gravitational theory using the unique effective action of quantum gravity, which allows to study quantum corrections to general relativity. In part IV, we will introduce the coupling of quantum gravity to matter, which allows to systematically study a wide range of effective interactions generated by quantum gravity.

Chapter 2

General Relativity

General relativity is undoubtedly the most successful theory of the gravitational force. It describes the gravitational force to great accuracy, and has been verified in many experiments. It correctly predicts the existence of gravitational waves and serves as the mathematical framework of the ΛCDM Model. In this chapter, we review some basic aspects of this theory.

2.1 Construction of general relativity

In this section, we review some of the essential ideas of the theory of general relativity by constructing the theory step by step, starting from the theory of classical Newtonian gravity for a massive point particle. In the Newtonian theory of gravity the motion of such a particle is given by a trajectory

$$x : \mathcal{T} \to \mathbb{R}^d, \tag{2.1}$$

where $t \in \mathcal{T} \subseteq \mathbb{R}$ labels the universal time, and $\mathbb{R}^d$ is the d-dimensional space through which the particle propagates. The motion of such a particle is governed by Newton's second law

$$m \frac{d^2 x^i}{dt^2} = F^i_{\text{grav}}(x) + F^i_{\text{other}}(x, \dot{x}, t), \tag{2.2}$$

where

$$F^i_{\text{grav}}(x) = -m \, \nabla^i V_{\text{grav}}(x) \tag{2.3}$$

is the gravitational force described in terms of the gravitational potential V_{grav}, and F_{other} represents other non-gravitational forces.

The first step towards a theory of general relativity is the realization that the gravitational force is induced by the geometry of the space through which the particle propagates. For this description, we promote the configuration space to a d-dimensional Riemannian manifold $(\mathcal{M}, g)$, where the metric g encodes the geometry that induces the gravitational force. We now consider trajectories

$$x : \mathcal{T} \to \mathcal{M} \tag{2.4}$$

and the motion of the particle is described by a generalization of Newton's second law

$$m\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}(x)\,\frac{dx^j}{dt}\frac{dx^k}{dt}\right) = F^i(x,\dot{x},t), \tag{2.5}$$

where m is the mass of the particle, and Γ is the Christoffel connection. Moreover, F represents the non-gravitational forces, that can generically be described in terms of vector and scalar potentials, i.e.,

$$F_i(x,\dot{x},t) = q\left[\nabla_i A_j(x,t) - \nabla_j A_i(x,t)\right]\frac{dx^j}{dt} - q\,\partial_t A_i(x,t) - \nabla_i \mathfrak{U}(x,t) \tag{2.6}$$

with q the charge of the particle. We notice that in this geometrical description the gravitational force is described through a modification of the acceleration term rather than an external force.

For $F_i(x,\dot{x},t) = 0$, eq. (2.5) is the geodesic equation and its solutions are geodesics. These are paths that minimize both the length

$$L[x(t)] = \int_{\mathcal{T}} ds = \int_{\mathcal{T}} \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\,dt \tag{2.7}$$

and the energy

$$E[x(t)] = \frac{m}{2}\int_{\mathcal{T}} g_{ij}\dot{x}^i\dot{x}^j\,dt, \tag{2.8}$$

where the line element ds is defined by

$$ds^2 = g_{ij}dx^i dx^j. \tag{2.9}$$

More generally, for a non-vanishing force eq. (2.5) minimizes the action

$$S[x(t)] = \int_{\mathcal{T}} L(x,\dot{x},t)\,dt \tag{2.10}$$

with the Lagrangian $L : \mathcal{T}\times T\mathcal{M} \to \mathbb{R}$ given by

$$L(x,\dot{x},t) = \frac{m}{2}g_{ij}(x)\,\dot{x}^i\dot{x}^j + q\,A_i(x,t)\,\dot{x}^j - \mathfrak{U}(x,t). \tag{2.11}$$

It is straightforward to show that minimization of the action with respect to the trajectory $x(t)$ indeed leads to eq. (2.5) with a force given in eq. (2.6).

The guiding principle of general relativity is that any Lagrangian defined on the manifold $\mathcal{M}$ should satisfy the principle of general covariance, i.e. the physical theory is coordinate invariant and there exists no preferred frame of reference. In the described non-relativistic framework, this is reflected by the fact that the action must be invariant under transformations generated by the inhomogeneous Galilean group.

Up to this point, we have described a non-relativistic test particle subjected to a gravitational force induced by the geometry. However, it is well known that the world is relativistic. This must be included in our description. In order to do so, we must promote the d-dimensional Riemannian manifold, which looks locally like the Euclidean space $\mathbb{R}^d$

to a n-dimensional Lorentzian manifold with $n = d + 1$, which looks locally like the n-dimensional Minkowski space $\mathbb{R}^{d,1}$. Furthermore, we must impose that any theory defined on this space satisfies general covariance. For this, the invariance under the inhomogeneous Galilean group is promoted to an invariance under the Poincaré group.

We thus study trajectories

$$x : \mathcal{T} \to \mathcal{M}, \tag{2.12}$$

where $\lambda \in \mathcal{T} \subset \mathbb{R}$ labels an affine parameter along the trajectory of the particle, and $\mathcal{M}$ is a Lorentzian manifold. As we consider a Lorentzian manifold, the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.13}$$

is no longer positive definite. Therefore, in order to obtain a well-defined variational principle, one must separate the tangent spaces into three sections:

$$ds^2 \begin{cases} < 0 & \text{timelike,} \\ = 0 & \text{lightlike,} \\ > 0 & \text{spacelike,} \end{cases} \tag{2.14}$$

where we use a $(- + ...+)$ metric signature. By restricting the tangent spaces to any of these sections, one can define a norm and construct a variational principle.

In addition, the construction of a relativistic theory requires invariance under reparametrizations of the affine parameter. It is easy to see that the length[1]

$$L[x(\lambda)] = \int_{\mathcal{T}} ds = \int_{\mathcal{T}} \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\lambda \tag{2.15}$$

is indeed invariant under reparametrization of the affine parameter λ, but the energy

$$E[x(\lambda)] = \frac{m}{2} \int_{\mathcal{T}} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \, d\lambda \tag{2.16}$$

is not. We note, however, that reparametrization invariance of the length leads to a secondary constraint

$$p_\mu p^\mu + m^2 = 0, \tag{2.17}$$

which can be implemented in the energy functional by introducing a gauge fixing term.

The equations of motion can then be derived by minimizing the action

$$S[x(\tau)] = \int_{\mathcal{T}} L(x, \dot{x}) \tag{2.18}$$

with the Lagrangian $L : T\mathcal{M} \to \mathbb{R}$ given by

$$L(x, \dot{x}) = \frac{1}{2\,e} g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu - \frac{e\, m^2}{2} + q A_\mu(x)\, \dot{x}^\mu, \tag{2.19}$$

[1]We consider timelike or lightlike paths satisfying $ds^2 \leq 0$

where $e(\lambda)$ is an einbein field along the wordline of the particle. The equations of motion are given by

$$g_{\mu\nu}\left(\frac{d^2x^\nu}{d\tau^2}+\Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right)=e\,q\left(\nabla_\mu A_\nu-\nabla_\nu A_\mu\right)\frac{dx^\nu}{d\tau}. \tag{2.20}$$

The relativistic constraint (2.17) can be written as

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau}=-e^2m^2 \tag{2.21}$$

and follows from minimization of the action with respect to e. Finally, we gauge fix

$$e=\begin{cases}m^{-1} & \text{if } m>0,\\ 1 & \text{if } m=0.\end{cases} \tag{2.22}$$

For timelike particles, this fixes the affine parameter to be the proper time $\lambda=\tau$.

We have discussed the motion of a test particle moving on a Riemannian or Lorentzian manifold $(\mathcal{M},g)$. Although this provides a correct description of a test particle, it is not the theory of general relativity. Indeed, general relativity describes the interaction between matter and the geometry through which the matter propagates. Since the particle gravitates, a complete theory of gravity must include the change of the geometry due to the presence of a particle. In order to treat this interaction, we must leave our description of point particles and move towards a classical field theory.

Before discussing a field theory, we review some basic definitions. Given a point $x\in\mathcal{M}$, a real (k,l)-*tensor* is a map $T(x)\in T^{(k,l)}(T_x\mathcal{M})=(T_x\mathcal{M})^{\otimes k}\otimes(T^*_x\mathcal{M})^{\otimes l}$, i.e

$$T(x):(T^*_x\mathcal{M})^k\times(T_x\mathcal{M})^l\to\mathbb{R}. \tag{2.23}$$

The *bundle of* (k,l)*-tensors* is

$$T^{(k,l)}(T\mathcal{M})=\bigsqcup_{x\in\mathcal{M}}T^{(k,l)}(T_x\mathcal{M}) \tag{2.24}$$

and a *tensor field* T is a smooth section $T\in\Gamma\big(T^{(k,l)}(T\mathcal{M})\big)$ of this bundle, i.e. it is a smooth map

$$T:\mathcal{M}\to T^{(k,l)}(T\mathcal{M}). \tag{2.25}$$

We have encountered three examples: the real scalar field

$$\mathfrak{U}:\mathcal{M}\to\mathcal{M}\times\mathbb{R} \tag{2.26}$$

with $(k,l)=(0,0)$; the covector field

$$A:\mathcal{M}\to T^*\mathcal{M}, \tag{2.27}$$

with $(k,l)=(0,1)$; and the metric

$$g:\mathcal{M}\to T^2(T^*\mathcal{M}) \tag{2.28}$$

with $(k,l) = (0,2)$.

More generally, in classical field theory one considers a *fiber bundle* over the manifold $\mathcal{M}$, i.e. a tuple $(E, \pi, \mathcal{M})$, where $\pi : E \to \mathcal{M}$ is a projection, that looks locally like a product space $\mathcal{M} \times F$ with fibers F. A *field* ϕ is then defined as a *section* of this bundle. This is a map $\phi : \mathcal{M} \to E$ such that $\pi(\phi(x)) = x$ for all $x \in \mathcal{M}$. We note that the tensor bundle $T^{(k,l)}(T\mathcal{M})$ is an example of a fiber bundle with fibers $T^{(k,l)}(T_x\mathcal{M})$.

In classical field theory, one considers k-dimensional fields $\Phi = \{\phi^a\}_{a\in\{1,\dots,k\}}$ as sections of some fiber bundle E, which may be equipped with internal symmetry groups. The typical example is classical electromagnetism, where the fibers are invariant under the action of the $U(1)$ symmetry group.

We note that fields cannot always be defined globally, i.e. global sections may not exist. Therefore, one considers *local sections* $\phi \in \Gamma_x(E)$ instead. These are fields $\phi : U \to E$ such that $U \subset \mathcal{M}$ is open and $x \in U$. We say that two local fields $\Phi, \Psi \in \Gamma_x(E)$ are *1-equivalent at* x, if

$$\phi^a(x) = \psi^a(x) \quad \text{and} \quad \partial_\mu \phi^a(x) = \partial_\mu \psi^a(x) \quad \forall a, \mu. \tag{2.29}$$

This defines an equivalence relation on the field space $\Gamma_x(E)$ whose equivalence classes are called *first order jets at* x. The first order jet containing the field ϕ is called the *first order jet of* ϕ *at* x and denoted by $j^1_x\phi$. The set of all first order jets

$$J^1\pi := \{j^1_x\phi : x \in \mathcal{M}, \phi \in \Gamma_x(\pi)\} \tag{2.30}$$

is called the jet manifold and can be endowed with a $(n + k + nk)$-dimensional manifold structure. Moreover, it is a fiber bundle over $\mathcal{M}$ with *source projection*

$$\pi_1 : J^1\pi \to \mathcal{M} \quad \text{s.t.} \quad j^1_x\phi \mapsto x \tag{2.31}$$

and a fiber bundle over E with *target projection*

$$\pi_{1,0} : J^1\pi \to E \quad \text{s.t.} \quad j^1_x\phi \mapsto \phi(x). \tag{2.32}$$

The jet manifold is the configuration space for a classical field theory. A Lagrangian is thus a function $\mathcal{L} : J^1\pi \to \mathbb{R}$, and the action is given by

$$S = \int \sqrt{|g|}\, \mathcal{L}\, d^n x. \tag{2.33}$$

After this brief review of classical field theory, we can discuss the action of general relativity. This action contains two parts

$$S = S_{\mathrm{EH}} + S_{\mathrm{M}} \tag{2.34}$$

The second part is the action of all matter defined by a Lagrangian $\mathcal{L}_{\mathrm{M}}(\Phi, \nabla\Phi, g)$ that depends on the metric, the matter fields and derivatives of the matter fields. The first part is the Einstein-Hilbert action, and it depends on the metric and derivatives of the metric. It is given by

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int \sqrt{|g|}\ (\mathcal{R} - 2\Lambda)\ d^n x, \tag{2.35}$$

where $\mathcal{R}$ is the Ricci scalar, and where we have introduced a cosmological constant Λ.

Minimizing the action (2.34) with respect to the metric tensor then leads to the Einstein equation

$$\mathcal{R}_{\mu\nu} + \left(\Lambda - \frac{1}{2}\mathcal{R}\right) g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.36}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}} \tag{2.37}$$

is the energy-momentum tensor.

The Einstein equation is the fundamental equation of general relativity and it describes the interaction between matter and geometry. Here, the kinetics of the geometry, as derived from the Einstein action is written on the left hand side and it is sourced by the presence of matter given on the right hand side.

2.2 Singularities in general relativity

In previous section we described the basic ingredients of the theory of general relativity, which despite its apparent simplicity is a very rich theory. The rich structure of the theory appears when one solves the Einstein equation in the presence of various matter fields. In general, this is a hard task due to the non-linearity of the equations. However, it leads to a plethora of physically interesting properties. In this thesis, we will only discuss a few aspects of this theory. For more detail we refer to the large body of literature on the topic and to introductory textbooks such as Refs. [106, 342].

One particular aspect of relevance for this thesis is the presence of singularities in the theory of general relativity. Singularities are characterized by the divergence of curvature invariants. These are scalar quantities constructed from products of the Riemann tensor. The presence of singularities is often associated to the notion of geodesic incompleteness of the theory.

It is this property of geodesic incompleteness that is problematic from the physical point of view, as it means that not all geodesics can be extended infinitely far into the past and the future. Instead, some geodesics have a starting or endpoint. If the geodesic incompleteness is due to a curvature singularity, these starting and end points are located at the curvature singularity.

The geodesic incompleteness implies that particles moving along such geodesics hit the singularity within a finite proper time towards the future or the past. The singularities thus act as sinks or sources for physical information. From the physicist's perspective this is problematic, as the predictivity of the theory breaks down at these singularities.

As an example of a singular spacetime, we consider the Einstein equation in vacuum such that $T_{\mu\nu} = 0$. It is well known that the unique time independent and spherically symmetric solution of this equation is the Schwarzschild solution, whose existence was first derived by Schwarzschild [312], and whose uniqueness is provided by the Birkhoff theorem [47, 216]. In the coordinate frame of an asymptotic observer, the solution can be represented by the line element

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2 d\Omega^2, \tag{2.38}$$

where

$$d\Omega^2 = d\theta^2 + \sin(\theta)^2 d\phi^2 \tag{2.39}$$

is the metric on a unit 2-sphere, and

$$f(r) = 1 - \frac{2GM}{r}. \tag{2.40}$$

with M the mass of the solution. This representation of the solution is well known to have a coordinate singularity at the horizon $r = 2GM$. However, this coordinate singularity is harmless, as it can be resolved by choosing a different set of coordinates. In addition, the solution has a true curvature singularity at $r = 0$, which is characterized by the divergence of the Kretschmann scalar

$$\lim_{r\to 0} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \lim_{r\to 0} \frac{48G^2M^2}{r^6} = \infty. \tag{2.41}$$

Another example of a singular solution in general relativity is the Reissner-Nordström solution [217, 276, 300, 352], which describes a charged black hole. In the frame of an asymptotic observer, the line element can be represented by (2.38) with

$$f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}, \tag{2.42}$$

where Q is the total charge. This solution has a curvature singularity at $r = 0$ and horizons at

$$r_\pm = GM \pm \sqrt{G^2\,M^2 - G\,Q^2}. \tag{2.43}$$

Another textbook example of a singular spacetime is the Kerr solution [222], which describes a rotating black hole. It can be generalized into the Kerr-Newman solution [273, 274] that describes a rotating charged black hole. The line element for the Kerr-Newman solution in Boyer-Lindquist coordinates is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr - GQ^2}{\rho(r,\theta)^2}\right) dt^2 - \frac{G\,(2Mr - Q^2)\,a\sin(\theta)^2}{\rho(r,\theta)^2}\,(dt\,d\phi + d\phi\,dt) \\ &+ \frac{\rho(r,\theta)^2}{\Delta(r)}\,dr^2 + \rho(r,\theta)^2\,d\theta^2 + \frac{\sin(\theta)^2}{\rho(r,\theta)^2}\left[(r^2+a^2)^2 - a^2\Delta(r)\sin(\theta)^2\right]d\phi^2 \end{aligned} \tag{2.44}$$

with $a = J/M$ and J the total angular momentum. Furthermore,

$$\rho(r,\theta)^2 = r^2 + a^2\cos(\theta)^2, \tag{2.45}$$

$$\Delta(r) = r^2 - 2GMr + GQ^2 + a^2. \tag{2.46}$$

The Kerr-Newman solution has a curvature singularity at $(r, \theta) = (0, \frac{\pi}{2})$. We note that this is a circle at $r = 0$ with radius a. In addition, the solution has horizons at

$$r_{\pm} = GM \pm \sqrt{G^2 M^2 - G\, Q^2 - a^2}. \tag{2.47}$$

The mass M, charge Q and angular momentum J fully characterize all classical black hole solutions. This is conjectured in the no-hair theorem [107, 212, 213, 262], which states that all stationary asymptotically flat black hole solutions of general relativity coupled to electromagnetism that are non-singular outside the event horizon are completely characterized by the mass, charge and angular momentum.

In this classical setting gravitational mass, electromagnetic charge and angular momentum is the only hair of the black hole. All other information contained in matter falling into the black hole is lost to outside observers, once it crosses the event horizon. In the classical theory this can be explained by the fact that all geodesics that cross the horizon end up at the singularity within finite proper time. All particles that fall into the black hole will thus end up at the singularity, which acts as a sink for the physical information contained in the particle, while leaving an imprint of their mass, charge and angular momentum in the geometry.

Although one could hope that the singular solutions of the Einstein equations are only spurious effects and do not occur in reality, there are several reasons why they have to be taken seriously. On theoretical grounds, they are expected to be the endpoints of the gravitational collapse of massive stars. This is due to the fact that there are no known forces that can counteract the gravitational force at the core of collapsing massive stars.

The critical point in such a collapse is the formation of a closed trapped surface. Any future directed light ray that is sent from such a surface will be directed towards one side of the surface. Such a surface thus splits spacetime into two regions: one that can be reached by future directed light rays and another region that cannot be reached by future directed light rays. Consequently, the region contained by the trapped surface is hidden from outside observers. The typical example of a closed timelike surface is the event horizon encountered in black hole solutions.

Early models of the collapse of massive objects up to the formation of a horizon were considered in Ref. [280]. Later, the idea that such a collapse indeed leads to black hole solutions was strengthened by the formulation of singularity theorems first developed by Hawking and Penrose [191, 291]. Typically these theorems prove geodesic incompleteness of a spacetime under generic assumptions about the global structure of the spacetime and energy conditions that are satisfied by the matter defined on the spacetime.

On top of these theoretical considerations, there are strong empirical arguments in favor of the existence of black holes. Astronomical observations provide strong evidence for the existence of dark compact objects that induce a Kerr-like metric. Evidence is for example provided by the observation of dark supermassive compact objects at the center of galaxies [59, 169, 176, 231], the observation of massive compact objects in other regions of galaxies [54, 114, 117, 348, 355], the analysis of gravitational waves from black hole like sources [2–5], and an image of a black hole like object [12].

2.3 Dark matter and dark energy

Another issue arising from general relativity is that, assuming general relativity and the Standard Model of particle physics, a large part of the energy content of the universe is unknown. Indeed, within the best current model of cosmology, it is estimated that the energy content of the universe consists for about 68% out of dark energy, 27% dark matter and only 5% ordinary matter [8,9]. Therefore, the Standard Model of cosmology is referred to as the ΛCDM Model, where Λ is the cosmological constant indicating the presence of dark energy, and CDM stands for cold dark matter.

Dark matter refers to the presence of matter that interacts gravitationally with other matter, but not or very weakly through the Standard Model interactions. In short, the presence of dark matter is necessary to account for the imbalance between the amount of matter observed through gravitational interactions, and the much smaller amount of matter observed through electromagnetic interactions. Concrete evidence for the presence of dark matter comes from a wide variety of sources such as galaxy rotation curves [122], velocity dispersions in galaxies [152], gravitational lensing [361], the cosmic microwave background [8, 9], and observation of the bullet cluster [118].

Despite the wide range of cosmological evidence for the existence of dark matter, little is known about the nature of dark matter, as it has not been observed directly in laboratory experiments. As a consequence, there exists a large variety of models explaining dark matter. Most popular are Standard Model extensions, but alternatives exist and include modifications of gravity, neutrinos and massive composite objects, such as primordial black holes and MaCHOs. However, it is expected that the latter two can only make up few percent of the total dark matter [158]. Modifications of gravity, on the other hand, are not always compatible with data from the bullet cluster [118]. Moreover, classes of these models can be mapped to Einstein-Hilbert gravity with a Standard Model extension, as will be discussed in the next section.

If dark matter is a type of matter, not much is known about its fundamental properties. Indeed, it can have any spin and its mass must roughly be in the range

$$10^{-22}\,\mathrm{eV} \lesssim m \lesssim 10^{70}\,\mathrm{eV}, \tag{2.48}$$

where the upper bound comes from the absence of observed tidal disruptions in galaxies, and the lower bound from the fact that dark matter must be bound within the smallest galaxies, i.e. its de Broglie wavelength should not be larger than the size of the smallest galaxies. In addition, as dark matter has not been detected in laboratory experiments, its non-gravitational interactions with Standard Model matter must be extremely weak. Furthermore, cosmological data implies that any non-gravitational interactions between dark matter particles themselves must be very weak and that the dark matter is cold or non-relativistic.

Dark energy is necessary to explain the accelerated expansion of the universe. Evidence for this acceleration follows from a variety of observations such as the observation of supernovae [292], the cosmic microwave background [8, 9], large scale structure [50], and the integrated Sachs-Wolfe effect [123, 172, 198].

Dark energy can be explained by a non-vanishing cosmological constant

$$\Lambda \approx 10^{-122}\, l_p^{-2} \tag{2.49}$$

where $l_p = \sqrt{\hbar G/c^3}$ is the Planck length. However, there exists no explanation for this particular value within the ΛCDM Model nor in the Standard Model of particle physics. Moreover, there is a small tension between the value of Λ as determined from the cosmic microwave background, i.e. early universe measurements [8,9], and the value as determined from standard candles, i.e. late universe measurements [301]. There are various models that attempt to explain the presence of dark energy. Examples are modifications of the gravitational theory and the introduction of a dynamical quintessence field that satisfies unusual energy conditions.

2.4 Modifications of gravity

As discussed in previous sections, general relativity is a mathematically clean theory that provides an extremely well tested description of the gravitational force. Nevertheless, there are several issues, such as the existence of singularities and the presence of dark matter, that cannot be described by general relativity. This tension provides a strong indication that classical general relativity is an incomplete theory of gravity that should be regarded as a limit of a more complete theory of gravity.

In the next chapter, we will discuss the construction of such a more complete theory by incorporating quantum effects into the gravitational theory. However, before moving to such quantum theories of gravity, it is important to point out that the classical theory of gravity can also be modified in several ways. Here, we discuss some general classes of modifications.

A first natural way to modify general relativity is to extend the geometrical description of spacetime. In the construction of general relativity as a theory on pseudo-Riemannian geometry, it is assumed that the connection of spacetime is the Levi-Civita connection, which is the unique metric compatible and torsion free connection on a pseudo-Riemannian manifold. However, generalizations of general relativity exist that use a more general connection.

An example of such an extension is the Palatini formalism in which one considers the Einstein-Hilbert action with the metric and connection as independent degrees of freedom. Variation of the action with respect to the connection then leads to new field equations for the connection. If the connection is assumed to be torsion-free, this field equation is solved for the Levi-Civita connection, implying that the non-metricity tensor given by

$$Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu} \tag{2.50}$$

vanishes.

If, on the other hand, the torsion is non-vanishing, then, under the assumption that the non-metricity vanishes, one obtains the Einstein-Cartan theory of gravity, which is characterized by the torsion field

$$T^{\rho}_{\mu\nu} = 2\,\Gamma^{\rho}_{[\mu\nu]} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} \tag{2.51}$$

or equivalently by the contorsion tensor

$$K_{\rho\mu\nu} = \frac{1}{2}\left(T_{\rho\mu\nu} + T_{\mu\nu\rho} - T_{\nu\rho\mu}\right) \tag{2.52}$$

with $T_{\rho\mu\nu} = g_{\rho\sigma}T^{\sigma}_{\mu\nu}$. The two extreme cases of this theory are given by general relativity, in which the torsion vanishes but the curvature does not, and the teleparallel theory of gravity, in which the curvature vanishes but torsion does not.

More general theories that include both torsion and non-metricity are referred to as metric-affine theories of gravity. In this case the connection takes the form

$$\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + K^{\rho}_{\mu\nu} + \frac{1}{2}Q^{\rho}_{\mu\nu} \tag{2.53}$$

with $\bar{\Gamma}^{\mu}_{\nu\rho}$ the Levi-Civita connection.

Notice, however, that there exist equivalences between the Einstein-Hilbert theory of gravity with additional matter fields and theories of gravity that include torsion or non-metricity. Indeed, instead of referring to K and Q as contorsion and non-metricity, one could include these fields in the matter Lagrangian and thus treat them as new matter degrees of freedom.

A second natural modification of general relativity can be obtained by modifying the Einstein-Hilbert action. This generically leads to a higher derivative theory of gravity, as such theories often incorporate higher powers of curvature scalars or derivatives of curvature scalars. A typical example of such theories is an $f(\mathcal{R})$ theory for which the gravitational action is of the form

$$S_G = \frac{1}{16\pi G}\int \sqrt{|g|}\, f(\mathcal{R})\, d^n x, \tag{2.54}$$

with f a smooth function of the scalar curvature. More generally, one can consider smooth functions $f(\mathcal{R}_{\mu\nu\rho\sigma}, g_{\mu\nu})$ or even smooth functions containing derivatives of the Riemann tensor. Finally, non-analytic functions can be considered, but this generically leads to a non-local theory.

A third modification of general relativity can be obtained by introducing an extra term in the action S_{int} that explicitly couples certain matter fields to curvature. The standard example of such a term is provided by a non-minimal coupling of a scalar field ϕ to gravity, which introduces a term of the form

$$S_{\text{int}} = -\frac{\xi}{2}\int \sqrt{|g|}\,\mathcal{R}\,\phi^2\, d^n x. \tag{2.55}$$

In this case, there exists two special cases: $\xi = 0$ is called minimal-coupling and

$$\xi = \frac{n-2}{4(n-1)} \tag{2.56}$$

is called conformal coupling, since for this value the massless scalar field theory

$$S = -\frac{1}{2} \int \sqrt{|g|} \left(\nabla_\mu \phi \, \nabla^\mu \phi + \xi \, \mathcal{R} \, \phi^2 \right) d^n x. \tag{2.57}$$

is invariant under conformal transformations

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \omega^2(x) \, g_{\mu\nu}(x). \tag{2.58}$$

A more general class of theories that contain an interaction between gravity and scalar fields is the Horndenski theory [201]. In four dimensions, this is the most general theory that includes an interaction between gravity and scalar fields, and yields second order equations of motion.

Finally, we point out that there exist equivalences between certain classes of the second and third modifications of general relativity and Einstein-Hilbert gravity with additional matter fields. If this is the case, the representation with a modified gravitational action is called the Jordan frame representation, while the Einstein-Hilbert action with additional matter fields is the Einstein frame representation. We will encounter an example of such an equivalence between modified gravity and classical gravity with additional matter fields in chapter 11.

Chapter 3

Towards a Quantum Theory of Gravity

In previous chapter, we have discussed general relativity, which is a geometric theory of gravity. In doing so, we have treated both spacetime and the matter defined on the spacetime as classical fields. However, it is well known that all matter in our universe must respect the laws of quantum theory. In this chapter, we discuss the quantization of the theory described in the previous chapter.

3.1 Quantum field theory

Quantum field theory is the most advanced theoretical framework that combines quantum theory with classical field theory and general relativity. This framework has been hugely successful, as both the Standard Model of particle physics and many condensed matter models are formulated within this framework.

The starting point of a quantum field theory is a classical field theory defined on some fiber bundle with an underlying manifold given by the real space $\mathbb{R}^d$ for a non-relativistic field theory and the Minkowski space $\mathbb{R}^{d,1}$ for a relativistic theory. For example, the Standard Model is formulated on the Minkowski space $\mathbb{R}^{3,1}$, and the fibers are invariant under the action of the group $SU(3) \times SU(2) \times U(1)$.

In order to obtain a quantum field theory, one must introduce the quantum fluctuations into the theory. This is called the quantization of the theory and the two most used procedures to quantize a theory are canonical quantization and functional integral quantization. A thorough treatment of the various quantization procedures is beyond the scope of this thesis, but can be found in any introductory book on quantum field theory. Here, we will simply highlight a few basics of the quantization procedures.

In a canonical quantization procedure, one quantizes a theory by promoting all variables and their conjugates to operators. One then imposes commutation relations between the variables and their conjugates. As discussed in previous chapter, the variables of a field theory are the fields, i.e. smooth local sections of the fiber bundle. In a canonical quantization procedure, one must thus determine the conjugate momenta of the fields, which are given by

$$\pi(\vec{x},t) = \frac{\partial \mathcal{L}(\phi, \nabla\phi)}{\partial \dot{\phi}(\vec{x},t)}, \tag{3.1}$$

and impose canonical quantization conditions of the form

$$[\phi(\vec{x},t), \pi(\vec{x}',t)]_{t=t'} = i\,\hbar\,\delta^d(\vec{x}-\vec{x}'). \tag{3.2}$$

As is the case in ordinary quantum mechanics, it is convenient to rewrite the fields in terms of creation and annihilation operators, which in the case of a field theory corresponds to a Fourier expansion of the fields given by

$$\begin{aligned} \phi(\vec{x},t) &= \sqrt{\frac{\hbar}{(2\pi)^d}} \int d^dk \sqrt{\frac{1}{2\omega}} \left[a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}-i\omega t} + a^\dagger(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}+i\omega t} \right], \\ \pi(\vec{x},t) &= -i\sqrt{\frac{\hbar}{(2\pi)^d}} \int d^dk \sqrt{\frac{\omega}{2}} \left[a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}-i\omega t} - a^\dagger(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}+i\omega t} \right]. \end{aligned} \tag{3.3}$$

The creation and annihilation operators, $a^\dagger$ and a, then satisfy the commutation relation

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta^d(\vec{k}-\vec{k}'), \tag{3.4}$$

and they can be used to construct excited states of the lowest state or quantum vacuum $|0\rangle$ of the Fock space corresponding to the theory. The observables of a quantum field theory are correlation functions. In a canonically quantized theory these can simply be calculated by sandwiching the operators between the vacuum states. For example, the expectation value of an observable $f(\phi)$ is given by

$$\langle f(\phi)\rangle = \langle 0|f(\phi)|0\rangle. \tag{3.5}$$

Another widely used quantization approach, which is more relevant for this thesis, is functional integral quantization or quantization by path integrals. The central object in this procedure is the path integral, which can formally be written as

$$\int D\phi\, e^{\frac{i}{\hbar}S(\phi)}, \tag{3.6}$$

and should be interpreted as an integral over all possible field configurations, where every field configuration has a weight given by $e^{\frac{i}{\hbar}S(\phi)}$. The path integral allows for a straightforward calculation of expectation values:

$$\langle f(\phi)\rangle = \frac{\int D\phi\, f(\phi)\, e^{\frac{i}{\hbar}S(\phi)}}{\int D\phi\, e^{\frac{i}{\hbar}S(\phi)}}. \tag{3.7}$$

For this purpose, it is convenient to define a generating functional by

$$Z[J] = \int D\phi\, e^{\frac{i}{\hbar}[S(\phi)+\langle J,\phi\rangle]}, \tag{3.8}$$

where $\langle .,. \rangle$ denotes the standard L^2 inner product. This allows to calculate n-point correlation functions as

$$\langle \phi^k \rangle = (-i\,\hbar)^k \frac{\delta^k}{\delta J^k} \ln Z[J]\Big|_{J=0}, \tag{3.9}$$

which allows to compute $\langle f(\phi) \rangle$ for any analytic function f.

Although these basic ideas of quantum field theory are rather straightforward, the actual calculation of correlation functions in quantum field theory is a difficult task. This is partially due to the complications in defining and evaluating path integrals. The path integral for a free field theory can, however, be evaluated. The result is an ordinary integral over space or equivalently over the dual momentum space. Any interaction term in the Lagrangian can then be introduced using perturbative methods. This can neatly be represented using Feynman diagrams. These provide a diagrammatic language for the remaining ordinary integrals that need to be evaluated.

These ordinary integrals are often divergent, but can be regularized in various regularization schemes. After regularization of the divergences, they can be absorbed into the theory by a redefinition of the original fields and coupling constants. This process is called renormalization of the theory. Any theory where this process can be applied, while adding only a finite number of new terms into the Lagrangian is called renormalizable. Theories for which this cannot be done are non-renormalizable.

In order to study quantum field theory in gravitational backgrounds, a framework of quantum field theory in curved spacetimes is necessary. Following the above construction, it can be obtained by simply replacing the Minkowski spacetime $\mathbb{R}^{d,1}$ with an arbitrary smooth Lorentzian manifold $\mathcal{M}$.

An issue that arises in such a construction is that the vacuum of the Fock space becomes an observer dependent notion. In other words, a quantum system that is in its ground state according to one observer can appear excited in the frame of another observer. In fact, any observer that accelerates through a Minkowski vacuum will observe a thermal spectrum of particles, which is known as the Unruh effect [128, 162, 339].

This effect does not only apply to accelerating observers, but is also observed in gravitational fields. Indeed, the equivalence principle states that observers cannot distinguish between accelerating frames and frames in a gravitational background field. This implies that an effect similar to the Unruh effect must be present in the neighborhood of gravitating objects. This leads to the prediction of Hawking radiation [192] and the idea that black holes are thermal objects with a radiation spectrum and entropy [41]. These predictions are among the most fascinating ideas of quantum field theory in curved spacetime.

3.2 Quantizing general relativity

In previous section, we have discussed quantum field theory in curved spacetime, which treats quantum theories on a static spacetime. However, as discussed in previous chapter, spacetime is dynamical in general relativity and its dynamics is governed by the Einstein equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{3.10}$$

When the quantum properties of matter are introduced, the energy-momentum tensor on the right hand side of this equation is quantized. If the left hand side is kept classical, this leads to an inconsistency in the equation, as the classical and quantum configuration spaces are different.

A first approach to resolve this inconsistency is provided by semiclassical gravity. In semiclassical gravity the right hand side of the Einstein equation is classicalized by taking the expectation value of the energy momentum tensor:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle. \tag{3.11}$$

This semiclassical approach is consistent and incorporates the quantum behavior of matter. However, it does not provide a complete picture of the interplay between quantum field theory and gravity, as it neglects the back-reaction of the gravitational field on the quantum fluctuations of matter. Therefore, semiclassical gravity is only an approximation to a more complete theory that can treat both quantum field theory and its interplay with gravity. Such a more complete theory requires the quantization of the gravitational field on the left hand side of the Einstein equation and is therefore called a theory of quantum gravity.

A second reason for the quantization of gravity is that there still exist many unanswered questions within both the Standard Model and the ΛCDM Model. Although it is not expected that a theory of quantum gravity will resolve all open questions in particle physics and cosmology, it is hoped that the quantization of gravity will shine a new light on several of these issues. In particular, quantum gravity could provide solutions to cosmological issues related to dark matter, dark energy, and the resolution of singularities.

A consistent and complete formulation of quantum gravity is still absent, but many approaches to its formulation have been suggested. A first approach to the quantization of gravity is the background field method in which fluctuations of the metric are quantized using standard methods from quantum field theory. In this approach, one uses the fact that the the tangent bundle $T\mathcal{M}$ is itself a fiber bundle over the manifold, where the fibers are the tangent spaces $T_x\mathcal{M}$ with the Lorentz group $SO(3,1)$ as its symmetry group. One then interprets gravity as a gauge force associated to this symmetry group by splitting the metric g into a background metric $\bar{g}$ and a perturbation h:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa\, h_{\mu\nu}, \tag{3.12}$$

where $\kappa^2 = 8\pi\hbar G$. The background can be chosen arbitrarily, but the most common choice is the Minkowski background $\bar{g} = \eta$. One then defines a new fiber bundle, where the underlying manifold with metric g is replaced by a manifold with metric $\bar{g}$. Moreover, the fibers are extended such that they include the new $(2,0)$-tensor field h that is associated to the $SO(3,1)$ symmetry of the fibers. The gauge field h is a massless spin-2 particle, it is called the graviton, and its wavelike dual is the gravitational wave.

The background metric is kept classical, while the gravitons can be quantized similar to the gauge forces. However, the resulting theory is non-renormalizable, which can easily be checked by power counting from the fact that the mass dimension of the coupling $[\kappa^2] = 2$

is positive.

If one checks the non-renormalizability explicitly, one finds that the one-loop divergences of pure gravity can in 4 dimensions be canceled [200] using the Chern-Gauss-Bonnet identity

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2} \int_{\mathcal{M}} d^4x \sqrt{|g|} \left(\mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}\right), \tag{3.13}$$

where $\chi(\mathcal{M})$ is the Euler characteristic of the manifold $\mathcal{M}$. However, this is no longer true for the divergences appearing in 2-loop diagrams [179]. In order to cancel such divergences one would require an infinite number of counterterms, and thus an action of the form

$$S = \int d^4x \sqrt{|g|} \sum_{k=1}^{\infty} a_k \, \kappa^{2(k-2)} \, \mathfrak{R}^k, \tag{3.14}$$

where a_k are dimensionless coefficients and $\mathfrak{R}^k$ denotes any scalar contraction of a product of k Riemann tensors. As this would require to fix an infinite number of coefficients a_k by experiment, such a theory is not predictive.

3.3 Quadratic gravity

The fact that the non-renormalizability of the Einstein-Hilbert action can be deduced from power counting arguments suggests that an action must be constructed where the coupling constants have non-positive mass dimensions. A natural candidate for such an extension is quadratic gravity with an action given by

$$S = \int d^4x \sqrt{|g|} \left(\frac{\mathcal{R}}{2\kappa^2} + c_1 \, \mathcal{R}^2 + c_2 \, \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \right). \tag{3.15}$$

Here, we have not included the quadratic curvature invariant $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$, as it can be rewritten in terms of $\mathcal{R}^2$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ and a boundary term, that does not affect the equations of motion, using the Chern-Gauss-Bonnet identity. The couplings for the quadratic interactions in this theory are dimensionless, which suggests renormalizability of the action by power counting.

It was shown by Stelle [329] that this action of quadratic gravity is indeed renormalizable. However, the renormalizability of the quadratic action comes at the price of introducing a ghost field in the action, i.e. a field with negative energy excitations. The presence of this ghost can be deduced from the propagator of quadratic gravity. This propagator can be derived from linearizing the action around flat spacetime [330], and is given by

$$\frac{\delta^\rho_\mu \delta^\sigma_\nu + \delta^\sigma_\mu \delta^\rho_\nu - \eta_{\mu\nu}\eta^{\rho\sigma}}{2\,k^2} - \frac{P^{(2)\rho\sigma}_{\mu\nu}}{k^2 - m_2^2} + \frac{P^{(0)\rho\sigma}_{\mu\nu}}{2\,(k^2 - m_0^2)} \tag{3.16}$$

with

$$
\begin{aligned}
m_0^2 &= \frac{1}{4\,\kappa^2\,(3c_1 + c_2)}, \\
m_2^2 &= -\frac{1}{2\,\kappa^2\,c_2},
\end{aligned} \tag{3.17}
$$

and where the Nieuwenhuizen operators are given by

$$
\begin{aligned}
P^{(0)\rho\sigma}_{\mu\nu} &= \frac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma} \\
P^{(2)\rho\sigma}_{\mu\nu} &= \frac{1}{2}\left(\theta^{\rho}_{\mu}\theta^{\sigma}_{\nu} + \theta^{\sigma}_{\mu}\theta^{\rho}_{\nu}\right) - \frac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma}
\end{aligned} \tag{3.18}
$$

with

$$
\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \tag{3.19}
$$

The first term is the same as the propagator encountered in Einstein-Hilbert gravity. This term corresponds to a massless spin-2 field, which is the graviton. The second term represents the propagator of a massive spin-2 field, and the third the propagator of a massive spin-0 field. Both these fields are not present in the Einstein-Hilbert action of gravity. Moreover, the negative sign in front of the second term indicates that the massive spin-2 field carries a negative kinetic energy, and is thus a ghost field. In addition, the new degrees of freedom could be tachyonic. However, this can be avoided by simply imposing constraints on the coefficients c_i such that $m_0^2, m_2^2 \geq 0$.

As a final remark, we note that the definition of the masses allows to rewrite the quadratic gravity action as [197]

$$
S = \frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}\left(\mathcal{R} + \frac{1}{6\,m_0^2}\,\mathcal{R}^2 - \frac{1}{2\,m_2^2}\,\mathcal{C}_{\mu\nu\rho\sigma}\mathcal{C}^{\mu\nu\rho\sigma}\right), \tag{3.20}
$$

where $\mathcal{C}_{\mu\nu\rho\sigma}$ is the Weyl tensor.

3.4 Gravity and gauge forces

In the next chapter, we will build an effective field theory for quantum gravity using the background field method. As we will see in parts III and IV, this allows to study quantum gravity at sub-Planckian energy scales in a way that is independent of the details of the ultraviolet completion of quantum gravity.

Before moving on to this framework, we point out that the failure of constructing a renormalizable quantum theory of gravity in the background field method does not come as a complete surprise. Indeed, in the previous chapter we have seen that the configuration space of classical general relativity is a fiber bundle over a manifold. Moreover, gravity is associated to the general covariance of the underlying manifold, while gauge forces are associated to symmetries in the fibers. Even though there are many similarities between gravity and gauge forces, this makes that they are fundamentally different on a classical level.

In the background field approach we quantize gravity by reformulating gravity as a gauge force. An opposite approach is to reformulate the gauge forces as a gravitational type of force by equipping the fiber bundle with a higher dimensional manifold structure. Ideas along those lines go back to the works of Kaluza [220] and Klein [225], who attempted to unify gravity with the electromagnetic force in a geometric formulation, and are now prominent in string theoretic approaches.

In part II of this thesis, we will explore how to incorporate quantum effects in the theory of general relativity, while keeping the fiber bundle structure from classical general relativity intact. We will do this by exploring stochastic formulations of quantum theories, and in particular stochastic mechanics using the stochastic quantization procedure as developed by Edward Nelson [270].

Stochastic quantization in the sense of Nelson[1] finds its roots in two closely related ideas from the 1950s. First, the development of stochastic mechanics [153,267] showed that quantum fluctuations can mathematically be described using extensions of the theory of Brownian motion, and can thus be interpreted as a special kind of stochastic fluctuations. This idea has since then become important in the foundations of quantum mechanics, as it provides a solution to some of the outstanding issues in this field.

The second root for Nelson's stochastic quantization approach lies in the mathematical foundation of quantum field theory. Despite the elegant construction of path integrals introduced by Feynman, their mathematical construction is still not properly understood [14]. Nevertheless, it was realized shortly after the introduction of the path integral that one can make sense of the path integral for the free field using a Wick rotation by mapping it to a stochastic integral [218]. Since then, stochastic analysis has become foundational to constructive quantum field theory [177, 185, 269].

In part II, we will find that the extension of a stochastic framework to manifolds requires second order geometry [150], which is an extension of ordinary first order differential geometry. The latter provides the mathematical foundation of general relativity. Both Nelson's theory and the second order geometry framework will be introduced extensively in chapter 6 and will therefore not be discussed in more detail in this introductory part.

[1] There exists a different stochastic quantization procedure developed by Parisi and Wu [284]. This approach will not be covered in this thesis.

Chapter 4

Effective Action of Quantum Gravity

In previous chapter, we have discussed the fact that a quantum field theory of general relativity, in which the fluctuations of the metric are quantized as a massless spin-2 field, is not renormalizable. This non-renormalizability is reflected by the fact that, in order to absorb all divergences in the coupling constants of the quantum theory, one must add an infinite number of counterterms. Therefore, the quantum action for the gravitational interaction will be of the form

$$S = \int d^4x \sqrt{|g|} \left(\frac{M_p^2}{2} \mathcal{R} + a_1 \mathfrak{R}^2 + \frac{a_2}{M_p^2} \mathfrak{R}^3 + \frac{a_3}{M_p^4} \mathfrak{R}^4 + ... \right), \tag{4.1}$$

where M_p is the reduced Planck mass[1], a_i are dimensionless coefficients and $\mathfrak{R}^n$ denotes any contraction of the product of n Riemann tensors. Since there is an infinite number of coefficients that must be fixed by experiment this theory is not predictive.

It is important to notice that this action is perturbative in the Planck mass, i.e. at scales $\mathfrak{R} \ll M_p^2$ one can safely truncate the action, and neglect higher order terms. Therefore, the quantum field theory of general relativity can still provide predictions that are under perturbative control as long as the curvature does not grow beyond the Planck scale, i.e. as long as $\mathfrak{R} < M_p^2$.

In practice, super-Planckian curvatures are only reached in the very early universe, and at the center of black holes. Therefore, an action as described in eq. (4.1) applies to all visible scales and all scales that are achievable by current and near future experiments. For comparison, the LHC reaches an energy scale of the order 1.3×10^4 GeV, whereas $M_p = 2.4 \times 10^{18}$ GeV.

4.1 Construction of the effective action

The idea sketched above can be made more precise in the language of effective field theories. In this framework, one constructs a quantum field theory around a low energy limit of

[1] $M_p = \kappa^{-1}$ in units where $c, \hbar = 1$.

the theory. The resulting effective field theory is valid from this low energy limit up to a certain cutoff scale of the theory. The main philosophy behind this approach is that the high energy modes of a theory are not always the relevant modes for physical processes taking place at low energy scales. This is due to the fact that high energy modes cannot be excited at low energy scales.

The first step in constructing an effective action is to set the cutoff scale of the theory and split all modes of the theory in high energy modes, i.e. modes with energies above the given energy scale, and low energy modes, i.e. modes with energies below the given energy scale. One can then obtain the effective action by integrating out all the high energy modes in the path integral formalism.

If the microscopic theory is known, one can perform this integration explicitly and obtain an exact effective action. On the other hand, if the microscopic theory is unknown, one cannot perform the integration explicitly. Nevertheless, if one knows the symmetries of the underlying microscopic theory, one can make predictions about the type of interactions that are present in the effective action. This is due to the fact that any effective interaction that is added to the Lagrangian by integrating out high energy modes must respect the symmetries of the underlying theory.

The calculation of the effective action by integrating out high energy degrees of freedom has been performed in the literature in a variety of models, cf. e.g. [65, 66, 167] for introductions to the topic. For some of these models the ultraviolet completion is known. This allows to compare the effective field theory obtained by integrating out fields with and without assuming the properties of the ultraviolet completion. An example of a model where the ultraviolet complete theory is known and such integration has been performed explicitly is the linear sigma model. The effective field theory for this model and its similarity to quantum gravity is discussed in more detail in Refs. [64, 141, 142].

In the case of gravity, the low energy limit is given by general relativity, but the microscopic theory is unknown. General relativity respects general covariance, and it is expected that this symmetry is respected by a fundamental theory of quantum gravity. Therefore, all operators appearing in an effective action of quantum gravity must respect this symmetry, and will be of the form $\mathfrak{R}^n$. Furthermore, using dimensional analysis, one can set the cut-off scale of the effective theory to be equal to the reduced Planck mass

$$M_p = \sqrt{\frac{\hbar\, c}{8\,\pi\, G}} = 2.435 \times 10^{18}\,\mathrm{GeV}. \tag{4.2}$$

Knowing the cut-off scale, the fundamental symmetries and the low energy limit, one can qualitatively write down an effective action for quantum gravity. Indeed, after integrating out any unknown heavy degrees of freedom of the microscopic theory, one will obtain corrections to the Einstein-Hilbert action that respect general covariance and are Planck scale suppressed. One thus arrives at an action as given in eq. (4.1). We note that the dimensionless Wilson coefficients, a_i, in this action are unknown, and can only be derived in a specific ultraviolet complete model of quantum gravity, where the heavy degrees of freedom are known and can be integrated out explicitly.

Additionally, one can integrate out the gravitons, as these are expected to be present in any fundamental theory of quantum gravity and are excited at the Planck scale, cf. eq. (3.12). This integration can be performed explicitly by considering a microscopic graviton action of the form

$$S[g] = \frac{M_p^2}{2} \int d^4x \sqrt{|g|}\, \mathcal{L}(g). \tag{4.3}$$

The moment generating functional for this theory is defined as the path integral

$$W[J] = -i\,\hbar\, \ln \left[\mathcal{N} \left(\int Dg\, e^{\frac{i}{\hbar}\left(S[g] + \int d^4x \sqrt{|g|}\, g_{\mu\nu} J^{\mu\nu}\right)} \right) \right], \tag{4.4}$$

where $\mathcal{N}$ is a normalization factor. The effective action can then be obtained by taking the Legendre transform of W:

$$\Gamma[\bar{g}] = W[J] - \int d^4x \sqrt{|g|}\, \bar{g}_{\mu\nu} J^{\mu\nu}, \tag{4.5}$$

where

$$\bar{g}_{\mu\nu} = \frac{\delta W}{\delta J^{\mu\nu}} = \langle g_{\mu\nu} \rangle \tag{4.6}$$

is the vacuum expectation value of the metric. One can then plug in the expression for the metric (3.12):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + M_P^{-1}\, h_{\mu\nu}. \tag{4.7}$$

This yields

$$\Gamma[\bar{g}] = S[\bar{g}] + \frac{i}{2\,M_p^2} \mathrm{Tr} \left[\ln \left(\frac{\delta^2 S(\bar{g})}{\delta g^2} \right) \right] + \mathcal{O}(M_p^{-2}), \tag{4.8}$$

where we note the $\bar{g}$ is the classical background metric. Unlike the original microscopic action (4.3), this action no longer depends on the quantum fluctuations $h_{\mu\nu}$, as these have been integrated out of the path integral.

The next step in obtaining an effective action is the evaluation of the corrections to the classical action. In the language of Feynman diagrams, the expansion (4.8) can be regarded as a loop expansion of the gravitons: the leading order term is the on-shell graviton action; the next term represents the diagrams containing one graviton loop; and higher order represent Feynman diagrams with multiple graviton loops. The corrections to the on-shell action can thus be obtained by evaluating the loop diagrams order by order.

Calculation of the loop diagrams gives rise to two types of operators in the effective action. The first are local or analytic operators that were already obtained in eq. (4.1). These originate in the UV-divergences of the graviton loops. The second type are non-local or non-analytic operators that originate in the infrared divergences of the graviton loops, which result from the masslessness of the graviton. Within the effective field theory approach, the local and non-local terms do not impact each other at any finite order in the energy expansion, as they have a different origin.

The calculation of these terms requires to fix the gauge for the graviton field $h_{\mu\nu}$. However, fixing the gauge for $h_{\mu\nu}$, will generically also fix the gauge of the background field $\bar{g}_{\mu\nu}$. Therefore, the resulting effective action will be gauge dependent. The gauge freedom for the background field can, however, be preserved using DeWitt's method of background or mean-field gauges. This method allows to fix the gauge of $h_{\mu\nu}$ without fixing the gauge of $\bar{g}_{\mu\nu}$ [341,356–358], and leads to the Vilkovisky-DeWitt unique effective action.

The calculation of the unique effective action for gravity can be done using a method called covariant perturbation theory or the generalized Schwinger-DeWitt technique, which has been developed in Refs. [34–39]. The resulting unique effective action of quantum gravity is given by

$$\Gamma_{\mathrm{EQG}} = \Gamma_{\mathrm{L}} + \Gamma_{\mathrm{NL}} + S_{\mathrm{M}}, \tag{4.9}$$

where we Γ_{L} and Γ_{NL} denote the effective gravitational action and S_{M} is the matter action. The local effective action is given by

$$\begin{aligned}\Gamma_{\mathrm{L}} = \int d^4x\,\sqrt{|g|}\,\bigg[&\frac{M_P^2}{2}(\mathcal{R}-2\Lambda) + c_1(\mu)\,\mathcal{R}^2 + c_2(\mu)\,\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\\ &+ c_3(\mu)\,\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + c_4(\mu)\,\Box\mathcal{R} + \mathcal{O}(M_P^{-2})\bigg]\end{aligned} \tag{4.10}$$

and the non-local action is given by

$$\begin{aligned}\Gamma_{\mathrm{NL}} = -\int d^4x\,\sqrt{|g|}\,\bigg[&\alpha\,\mathcal{R}\ln\left(\frac{\Box}{\mu^2}\right)\mathcal{R} + \beta\,\mathcal{R}_{\mu\nu}\ln\left(\frac{\Box}{\mu^2}\right)\mathcal{R}^{\mu\nu}\\ &+\gamma\,\mathcal{R}_{\mu\nu\rho\sigma}\ln\left(\frac{\Box}{\mu^2}\right)\mathcal{R}^{\mu\nu\rho\sigma} + \mathcal{O}(M_P^{-2})\bigg]\,.\end{aligned} \tag{4.11}$$

We note that $\mathcal{O}(M_P^{-2})$ denotes the higher curvature terms that appear in multiple graviton loop diagrams. In this thesis, we will neglect such terms and focus on the linear and quadratic parts of the effective action, i.e. the on-shell and one loop contributions. Additionally, we will ignore the cosmological constant. Furthermore, we note that the $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$ terms can be rewritten in terms of $\mathcal{R}^2$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ and a boundary term using the Chern-Gauss-Bonnet theorem. This boundary term and the boundary term with Wilson coefficient c_4 do not affect the equations of motion and will therefore be ignored in the remainder of the thesis.

The local coefficients c_i are associated with UV-divergences in the Feynman diagrams, and can therefore not be calculated within the effective field theory approach. They are, however, calculable in ultraviolet complete theories of quantum gravity, cf. e.g. Ref. [15] for an example in string theory. Moreover, the graviton contribution to the running of these Wilson coefficients can be determined within any renormalization scheme. This introduces the dependence on the renormalization scale μ.

The fact that the Wilson coefficients cannot be calculated within the effective field theory approach renders the local theory unpredictive. Nevertheless, the coefficients are expected to be non vanishing, unless some unknown symmetry in the ultra-violet complete theory is present or if fine tuning occurs. Therefore, the presence of higher order local interaction can be regarded as a qualitative prediction of the effective field theory.

The non-local coefficients, on the other hand, are associated with IR-divergences in the Feynman diagrams and can be calculated within the effective field theory approach.

	α	β	γ
Scalar	$5(6\xi - 1)^2$	-2	2
Fermion	-5	8	7
Vector	-50	176	-26
Graviton	250	-244	424

Table 4.1: Non-local Wilson coefficients for various fields. All numbers should be divided by $11520\pi^2$. Furthermore, ξ denotes the value of the non-minimal coupling for a scalar theory. The values for the scalar, fermion and vector field have been calculated in Refs. [48, 132]. The values for the graviton can be gauge dependent due to the graviton self-interaction diagrams [219]. However, it is possible to define a unique effective action with gauge independent coefficients leading to the gauge independent results quoted here [34, 35, 341].

The values of these coefficients depend on the type of matter that the graviton couples to, and are given for several types of matter in Table 4.1. For any theory, the value of the non-local coefficients is then given by

$$\alpha = N_s\,\alpha_s + N_f\,\alpha_f + N_v\,\alpha_v + N_g\,\alpha_g, \tag{4.12}$$

where N_s, N_f, N_v, N_g denote the number of scalar, fermionic, vector, and graviton fields in the theory.

We conclude this section with a remark on the chosen cut-off scale. We have used dimensional analysis to set this scale equal to the Planck scale. However, a priori, there is no reason for the scale of quantum gravity to be exactly equal to the Planck scale. One expects

$$M_{\mathrm{QG}} = a\,M_P, \tag{4.13}$$

where a is an unknown numerical factor.

Despite the fact that a is unknown, one can set rough bounds on its value. Indeed, since quantum gravity effects have not been observed in the lab, the scale of quantum gravity is expected to be several order of magnitude above the TeV scale that can be reached in collider experiments. This provides a lower bound on the value of a. An upper bound can be obtained from the violation of tree unitarity in the effective field theory of quantum gravity [190]:

$$a \leq \sqrt{\frac{480\,\pi}{N_S + 3N_F + 12N_V}}\,, \tag{4.14}$$

where N_S, N_F and N_V denote the number of scalar fields, Weyl fermions and vector bosons in the theory. This bound can roughly be translated into $a \lesssim 1$.

By constructing the effective action with an arbitrary value of a, one can easily see that the larger the scale of quantum gravity, the larger the range of validity of the effective field theory, and the larger the suppression of quantum gravitational effects. In this thesis, we use the conservative estimate $a = 1$. This provides a maximal suppression of quantum gravitational effects.

4.2 Comparisons to quadratic gravity

The effective field theory of quantum gravity allows to calculate quantum gravitational corrections to any gravitational observable. Perhaps, one of its most interesting predictions is the computation of corrections to the Newtonian potential. The corrections due to the local terms were first calculated by Stelle in the framework of quadratic gravity [329, 330], while the corrections due to the non-local terms were first obtained in the works of Donoghue [49,139,140]. More recently, the two types of corrections have been put together yielding a quantum corrected potential of the form [93, 94]

$$\Phi(r) = -\frac{G\,M}{r}\left(1 + \frac{1}{3}e^{-\mathrm{Re}(m_0)\,r} - \frac{4}{3}e^{-\mathrm{Re}(m_2)\,r}\right) \tag{4.15}$$

with complex masses given by

$$\begin{aligned} m_0^2 &= -\frac{M_P^2}{4\,(3\alpha+\beta+\gamma)\,W\left(\frac{M_P^2}{4\mu^2(3\alpha+\beta+\gamma)}\exp\left[-\frac{3c_1+c_2+c_3}{3\alpha+\beta+\gamma}\right]\right)}\,, \\ m_2^2 &= \frac{M_P^2}{2\,(\beta+4\gamma)\,W\left(-\frac{M_P^2}{2\mu^2(\beta+4\gamma)}\exp\left[-\frac{c_2+4c_3}{\beta+4\gamma}\right]\right)}\,, \end{aligned} \tag{4.16}$$

where $W(z)$ denotes the principal branch of the Lambert-W function, which is defined as the solution $w = W(z)$ of the equation

$$w\,e^w = z. \tag{4.17}$$

We note that these complex masses are generalizations of the masses of the spin-0 and spin-2 field in quadratic gravity that were given in eq. (3.17). The major difference between the masses given in eq. (4.16) and those given in eq. (3.17) is that the masses of the spin-0 and spin-2 fields in the effective action are complex and include corrections due to the non-local part of the action. The complex parts of the masses represent the decay width for the decay of the spin-0 and spin-2 fields into the classical graviton and into Standard Model fields [94].

There are many other similarities between quadratic gravity and the effective action for quantum gravity, which is due to the fact that the quadratic action is contained in the effective action of quantum gravity. As we truncate the effective action at quadratic order, the only difference resides in the presence of the non-local terms in the effective action.

Given these similarities, it is important to stress some fundamental differences between the two frameworks. In quadratic gravity, the fundamental classical action is the quadratic action, which contains three classical degrees of freedom: a massless spin-2, a massive spin-0, and a massive spin-2 particle. When quantized, the theory is renormalizable, implying that the quantum action is also the quadratic action. However, quantization of the theory requires to quantize all three degrees of freedom. This is problematic for the massive spin-2 field, as it is a ghost field.

In contrast, in the effective field theory of quantum gravity the fundamental classical action is the Einstein-Hilbert action. This action contains only one degree of freedom, which is the massless spin-2 graviton. When quantized, there is no ghost problem, as only this particle is quantized. However, the theory is non-renormalizable implying that an infinite number of counterterms must be added.

Since the counterterms are perturbatively under control up to the Planck scale, the resulting quantum action can be truncated at some order and be treated as an effective field theory. From the effective field theory perspective, the resulting action is again classical, since the quantum degrees of freedom have been integrated out. If the truncation is made after the quadratic order, one will find three degrees of freedom. If, on the other hand, the truncation is made at a higher order, one can find other new degrees of freedom appearing in the action [95].

By the Ostragradsky theorem [148, 282, 360], at least one of these new degrees of freedom will be ghostlike. Therefore, when treated as a fundamental theory, such a theory will suffer from instabilities. However, in the effective field theory approach, these instabilities are harmless, as they lie at the Planck scale just beyond the range of validity of the effective field theory. They result from truncating an expansion of the unknown ultraviolet complete theory around the known infrared limit and do not necessarily correspond to poles in the ultraviolet complete theory [64, 238, 315]. However, the presence of these new poles also modifies the infrared regime of the theory. These modifications at sub-Planckian energy scales can be studied reliably within the effective field theory framework.

The new massive spin-0 and spin-2 fields thus provide perturbative quantum corrections to the gravitational field that are obtained by integrating out the quantum fluctuations of the graviton. The massive spin-0 provides an attractive correction, while the massive spin-2 provides a repulsive correction to the gravitational force.

4.3 Coupling quantum gravity to matter

In the previous sections, we have focused on the effective action of quantum gravity for a purely gravitational theory. However, all matter interacts gravitationally, and therefore quantum gravitational effects will also generate new effective interactions in the matter Lagrangian $\mathcal{L}_{\mathrm{M}}$.

In order to see this, let us recall the effective action (4.9)

$$\Gamma_{\mathrm{EQG}} = \Gamma_{\mathrm{L}} + \Gamma_{\mathrm{NL}} + S_{\mathrm{M}}, \tag{4.18}$$

where the local and non-local action are given by eqs. (4.10) and (4.11). Linearizing this action around Minkowski space in the Einstein frame yields [93]

$$
\begin{aligned}
\Gamma = \int d^4x \Big[& -\frac{1}{2} h_{\mu\nu}\partial^2 h^{\mu\nu} + \frac{1}{2} h^\mu_\mu \partial^2 h^\nu_\nu - h^{\mu\nu}\partial_\mu\partial_\nu h^\rho_\rho + h^{\mu\nu}\partial_\rho\partial_\nu h^\rho_\mu \\
& + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{m_0^2}{2}\sigma^2 \\
& + \frac{1}{2} k_{\mu\nu}\partial^2 k^{\mu\nu} - \frac{1}{2} k^\mu_\mu \partial^2 k^\nu_\nu + k^{\mu\nu}\partial_\mu\partial_\nu k^\rho_\rho - k^{\mu\nu}\partial_\rho\partial_\nu k^\rho_\mu \\
& - \frac{m_2^2}{2}\left(k_{\mu\nu}k^{\mu\nu} - k^\mu_\mu k^\nu_\nu\right) \\
& - \frac{1}{M_P}\left(h_{\mu\nu} - k_{\mu\nu} + \frac{1}{\sqrt{3}}\sigma\eta_{\mu\nu}\right) T^{\mu\nu} + \mathcal{L}_{\mathrm{M}}(\eta) + \mathcal{O}(M_p^{-2}) \Big] ,
\end{aligned}
\tag{4.19}
$$

where η is the Minkowski metric, h is the graviton, k is the massive spin-2 particle and σ is the massive spin-0 particle associated to the effective quantum gravity action. Here, the first lines result from the effective gravitational action and the last line follows from linearizing the matter Lagrangian S_{M}. We see that the last line generates an explicit interaction between the gravitational degrees of freedom and the energy momentum tensor. As the gravitational degrees are heavy, this leads to effective contact interactions of the form

$$
O_{\mathrm{EFT}} = \frac{8\pi}{M_P^2\, M_i^2} O_{\mathrm{M}}\, O'_{\mathrm{M}}, \tag{4.20}
$$

where M_i is the mass of the massive spin-0 and spin-2 degrees of freedom, i.e. the real part of eq. (4.16), and $O_{\mathrm{M}}, O'_{\mathrm{M}} \in \mathcal{L}_{\mathrm{M}}$ are matter interactions.

As $M_i \sim M_P$, these interaction are heavily suppressed, and lie beyond experimental reach. However, in constructing these interactions, we have only used the quantum gravitational effective action generated by graviton loops, i.e. perturbative effects. Non-perturbative quantum gravitational effects, such as virtual black holes, will also generate new interactions between matter.

Little is known about these non-perturbative quantum gravitational effects. However, following the standard reasoning from effective field theory, one should expect any type of operator that respects the symmetries of the underlying theory. We should thus expect any interaction that respects the symmetries of the Standard Model and general covariance, as long as the interaction is turned off in the limit $M_P \to \infty$, where quantum gravitational effects vanish. As an example, consider a matter Lagrangian that contains a real scalar field ϕ and a vector field A_μ. For such a Lagrangian quantum gravity will generate an effective interaction of the form

$$
\frac{c}{M_P}\, \phi\, F_{\mu\nu}F^{\mu\nu}, \tag{4.21}
$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength and c is a dimensionless constant. For a more general classification of the possible effective interactions we refer to Ref. [72].

Chapter 5

Outlook for the Thesis

The remainder of the thesis consists of 4 more parts. In part II, we discuss the construction of diffeomorphism invariant theories on fluctuating spacetimes. This will be done in the framework of Nelson's stochastic quantization. Here, we will only discuss the most elementary theory in such a framework, which is the stochastic quantization of a single relativistic scalar particle in an arbitrary gravitational background. As we do not consider field theoretic extensions, we stick to a first quantization approach in which we quantize the particle but do not quantize gravity or the gauge forces.

In chapter 6, we provide a review of stochastic mechanics, and discuss in depth the framework of second order geometry, which is key in constructing diffeomorphism invariant stochastic theories on pseudo-Riemannian manifolds. In chapter 7, we employ the results from chapter 6 to consider the physically relevant case of a relativistic particle on Lorentzian manifolds. Finally, in chapter 8, we show that the Nelson process, used in stochastic quantization, can be redefined as a complex Brownian motion.

In part III, we move away from the elementary considerations of part II, and discuss model independent predictions of quantum gravity. This will be done in the framework of the unique effective action of quantum gravity.

In chapter 9 we discuss quantum gravitational corrections to the metric of a compact star and chapter 10 we discuss the consequence of these corrections for the motion of scalar test particles and scalar fields in this geometry.

In chapters 11 and 12, we discuss the existence of singularities and secularities in the effective action of quantum gravity. Chapter 11 discusses the avoidance of classical singularities due to the quantum gravitational effects, while chapter 12 discusses how the effective action generates new singular and secular solutions.

Finally, chapters 13 and 14 deal with issues related to black hole information. In chapter 13, we employ the effective action of quantum gravity to calculate the leading quantum gravitational corrections to the entropy of a black hole, and, in chapter 14, we show that quantum gravitational effects generate hair for compact objects.

In part IV, we move away from the unique effective action for a purely quantum gravitational theory and consider effective matter interactions generated by quantum gravity. Here, we discuss how quantum gravity generates new interaction terms between the Standard Model and a hidden sector. In chapter 15, we discuss how these new effective interactions can be combined with experimental results to set new lower bounds on the masses of dark matter fields. In chapter 16, we then combine these results with upper bounds on dark matter masses.

Finally, in part V, the results obtained in this thesis will be summarized and we will provide an outlook for future research directions.

Part II

Construction of Diffeomorphism Invariant Theories on Fluctuating Spacetimes

Chapter 6

Stochastic Quantization on Lorentzian Manifolds

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Abstract
We embed Nelson's theory of stochastic quantization in the Schwartz-Meyer second order geometry framework. The result is a non-perturbative theory of quantum mechanics on (pseudo-)Riemannian manifolds. Within this approach, we derive stochastic differential equations for massive spin-0 test particles charged under scalar potentials, vector potentials and gravity. Furthermore, we derive the associated Schrödinger equation. The resulting equations show that massive scalar particles must be conformally coupled to gravity in a theory of quantum gravity. We conclude with a discussion of some prospects of the stochastic framework.

This chapter has been published in the Journal of High Energy Physics **05**, a. 028 (2021). A preprint of this chapter can be found at arXiv:2101.12552 [hep-th].

6.1 Introduction

The construction of a theory of quantum gravity is one of the main open issues in theoretical high energy physics. One of the reasons why such a theory is desirable is that general relativity is unable to completely describe physical aspects of gravity at extremely high energy scales. This feature is most prominent in the fact that singularities seem to be unavoidable in general relativity, when natural assumptions are made [191, 280, 291, 312].

From a physical perspective, the formation of such singularities would require the continuous collapse of a matter distribution to a delta distribution located at the singularity. On $\mathbb{R}^n$ one can make sense of such a collapse, as one can construct a family of smooth distributions that converges to the delta distribution. In general relativity, on the other hand, point-like sources cannot be obtained as a continuous limit of matter distributions defined on manifolds with smooth metrics, as the Einstein equations must be satisfied during the collapse [168].

It is expected that this paradox will be resolved, when general relativity is embedded into a quantum theory such that gravity is quantized. However, when one attempts such an embedding using standard quantum field theory methods, one runs into the problem that the resulting quantum theory is non-renormalizable [200]. Up to the Planck scale, one can still make predictions regarding quantum gravity using effective field theory methods, since the ultraviolet divergences responsible for the non-renormalizability of the theory can be kept under control perturbatively. However, beyond the Planck scale this is no longer true, which renders the theory incomplete.

Over the last decades many approaches to an ultraviolet complete theory of quantum gravity have been developed, and many interesting insights have been obtained within these approaches. In this paper, we argue that Nelson's stochastic quantization framework could help gain further insight in theories of quantum gravity. We will motivate this by showing that stochastic quantization allows to construct a well defined non-perturbative theory of quantum mechanics on (pseudo-)Riemannian manifolds.

We will adopt the framework of stochastic mechanics, also known as Nelsonian stochastic quantization[1], that was proposed by Fényes [153] and Kershaw [223], rederived by Nelson [267, 268, 270] and further developed by many others. The main idea governing stochastic mechanics is that quantum mechanics can be derived from a stochastic theory. In this more fundamental theory all particles follow trajectories through a randomly fluctuating background field. Due to the interactions with this background field all matter behaves quantum mechanically. An equivalent way[2] to state this idea is that all particles and fields are defined on a randomly fluctuating space-time.

We focus in this paper on ordinary quantum mechanics. We will thus work with point-like particles instead of fields. Moreover, we work on a fixed Lorentzian manifold. Therefore, the metric is not considered to be a dynamical field. We leave extensions to

[1]In this paper, we use the terms stochastic mechanics and stochastic quantization interchangeably. We emphasize that the framework is related to, but different from the Parisi-Wu formulation of stochastic quantization.

[2]One could call this a 'passive' description of stochastic quantization, since the space-time fluctuates, while in the previous 'active' description the matter defined on the space-time fluctuates.

a field theory framework and to dynamical geometries for future work. In the stochastic quantization framework such extensions lead to a theory of quantum gravity.

6.1.1 Stochastic quantization

Since the quantization procedure in stochastic quantization is different from more commonly used quantization procedures, we will compare the main steps to canonical quantization. In a canonical quantization procedure one starts with a classical Hamiltonian $H(p,x)$ and promotes the variables p,x to operators P,X such that

$$H(p,x) \to \hat{H}(P,X).$$

One then imposes canonical commutation relations

$$[X^\nu, P_\mu] = i\,\hbar\,\delta^\nu_\mu. \tag{6.1}$$

Moreover, one postulates the existence of a wave function Ψ, which is an element of a complex Hilbert space with L^2-norm, that can be used to calculate observables, i.e.,

$$\langle\Psi|\hat{O}|\Psi\rangle = O. \tag{6.2}$$

In stochastic quantization, one starts with a classical Lagrangian $L_c(x,v,\tau)$, and promotes the position of a particle x to a stochastic process $X(\tau)$. Since the stochastic process is not differentiable, one can define two velocities $v_\pm$ using conditional expectations:

$$\begin{aligned} v_+(X(\tau),\tau) &= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}\left[X(\tau+h)-X(\tau)|X(\tau)\right], \\ v_-(X(\tau),\tau) &= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}\left[X(\tau)-X(\tau-h)|X(\tau)\right]. \end{aligned} \tag{6.3}$$

One can then introduce a stochastic Lagrangian

$$L_c(x,v,\tau) \to L(X,V_+,V_-,\tau) = \frac{1}{2}\left[L_c(X,V_+,\tau)+L_c(X,V_-,\tau)\right] \tag{6.4}$$

Moreover, one fixes the quadratic variation[3] of the process X by the background hypothesis:

$$[[X^\nu, X_\mu]](\tau) = \frac{\hbar}{m}\,\delta^\nu_\mu\,\tau. \tag{6.5}$$

We remind the reader that the joint quadratic variation of two processes X,Y is itself a stochastic process and can be written as

$$[[X,Y]](\tau) = X(\tau)Y(\tau) - X(0)Y(0) - \int_0^\tau X(s)dY(s) - \int_0^\tau Y(s)dX(s). \tag{6.6}$$

[3]More commonly used notations for $d[[X^i,X^j]]$ are $d[X^i,X^j]$ or $dX^i dX^j$. We use the double brackets instead to avoid confusion with the commutator, first order bilinear tensors and second order vectors that will be introduced in section 6.2.

The Itô integral used in this expression is defined by

$$\underline{\int}_{\tau_i}^{\tau_f} f(X,\tau)\,dX := \lim_{k\to\infty} \sum_{[\tau_j,\tau_{j+1}]\in\pi_k} f(X(\tau_j),\tau_j)\,[X(\tau_{j+1})-X(\tau_j)]\,, \tag{6.7}$$

where π_k is a partition of $[\tau_i, \tau_f]$.

Observables in stochastic quantization can be calculated using the expectation $\mathbb{E}$, which is defined on a filtered probability space, and evaluated as a Lebesgue integral in the L^2-space of stochastic processes. The construction of expectation values in modern probability theory as founded by Kolmogorov [230] requires the existence of a probability measure $\mathbb{P}$ in the probability space, and a measure μ in the L^2-space, but not the existence of a probability density.[4] Therefore, the wave function Ψ no longer needs to be postulated in stochastic quantization.

Since the wave function is no longer fundamental to the theory, the interpretation of quantum mechanics in the stochastic quantization framework is different from the standard Copenhagen interpretation. In stochastic quantization, one assumes that particles follow well defined trajectories through space-time. However it is assumed that all matter moves through a fluctuating background field, which is sometimes called the aether, but can also be regarded as a fluctuating space-time or as a diffeomorphism invariant quantum vacuum.

Due to the fluctuating background field, the motion of massive particles[5] will become stochastic and comparable to a frictionless Brownian motion.[6] This Brownian motion is imposed to be time-reversible. This additional assumption introduces an important distinction from Brownian motion processes that are more commonly studied in statistical physics.

Most stochastic diffusion processes that are studied in physics, such as for example the Ornstein-Uhlenbeck process, are dissipative diffusions. These processes are not time reversible, and energy is transferred from the system to the environment until an equilibrium is reached. The processes studied in stochastic mechanics are conservative diffusion processes. These processes are time-reversible and the expected energy transfer between the system and environment is 0 at all times.

The fact that the wave function is no longer fundamental in stochastic quantization has two further important consequences. First, constructing normalized wave functions on Riemannian manifolds is a difficult task, that complicates extensions of ordinary quantum mechanics to manifolds. This problem is circumvented in the stochastic approach, as the wave function no longer needs to exist globally.

Secondly, due to the secondary role of the wave function, there is no measurement problem in stochastic mechanics. The wave function and probability density in stochastic mechanics have the same status as in standard probability theory. A theoretically perfect measurement in stochastic mechanics thus corresponds to conditioning of the process. Conditioning is a mathematical operation that still leads to collapse of the wave function, but since the wave function is only a mathematical construct and not a physical object,

[4] If a probability density $\rho(x)$ exists, one has the familiar relation $d\mu(x) = \rho(x)d^n x$.

[5] Stochastic quantization has yet to be extended to massless particles.

[6] Notice that eq. (6.5) characterizes a scaled Brownian motion [242].

this does not correspond to a physical interaction.

6.1.2 Successes of stochastic quantization

The success of stochastic quantization relies on the relation between probability density functions associated to stochastic processes and partial differential equations. In the case of dissipative diffusions, the probability density associated to the solution of a stochastic differential equation evolves according to a parabolic differential equation. This result is known as the Feynman-Kac formula [218]. An example of this relation is the fact that the probability density of a dissipative Brownian motion evolves according to the heat equation, which is a real diffusion equation.

A similar relation exists for conservative diffusion processes. For example, the probability density of a conservative Brownian motion evolves according to the Schrödinger equation, which is a complex diffusion equation. This result is closely related to the Feynman-Itô formula [14, 214]. Before this latter relation was formally established, it was discovered independently by Fényes, Kershaw and Nelson [153, 223, 267, 268, 270] that the Schrödinger equation can be derived from a stochastic theory, if one assumes that particles follow a time-reversible stochastic process, governed by a stochastic version of Newton's second law, where the force is derived from a potential.

The theory that was developed in this way is called stochastic mechanics. The immediate consequence of this discovery is that all predictions of quantum mechanics that follow from the Schrödinger equation, are also predictions of stochastic mechanics. Later it was shown that the same result can be formulated in terms of Lagrangian dynamics using the stochastic variational calculus developed by Yasue [362–364]. This Lagrangian approach goes by the name of stochastic quantization.

The theory of stochastic mechanics and stochastic quantization has been extended to Riemannian manifolds, see e.g. Refs. [127, 135–137, 187, 270]. Moreover, extensions of stochastic quantization to bosonic field theory have been developed, cf. e.g. Refs. [165, 184, 186, 188, 189, 227, 251, 265, 289]. Furthermore, the notion of spin has been discussed in this framework, cf. e.g. Refs. [127, 159, 270].

It is worth noticing that in the dissipative field theoretic stochastic framework that was later developed by Parisi and Wu [125, 284], and also goes by the name of stochastic quantization, extensions to fermionic field theories have been developed, cf. e.g. Ref. [126]. Although this framework is different from the stochastic quantization as developed by Nelson and others, there exist many similarities. It is also worth mentioning that several authors have incorporated stochastic mechanics into models of quantum gravity, cf. e.g. Refs. [151, 249].

Many basic results from quantum mechanics such as the commutation relations, the uncertainty principle, the double slit experiment and the motion of particles in various potentials have been discussed within the stochastic framework, see e.g. Refs. [164, 184, 270, 278, 290, 293, 364]. We emphasize that the interpretation of these results radically changes in the stochastic quantization framework, as the particle follows a well defined trajectory. For example, in the double slit experiment, a particle always goes through one slit. One still obtains an interference pattern, as this is the unique solution of the time-reversible diffusion process.[7]

6.1.3 Criticism on stochastic quantization

Despite the successes described above, stochastic quantization has never been widely studied. We will therefore review some of the main concerns that have been raised against stochastic quantization.

Historically, one of the more prominent confusions arose from the idea that a diffusion process is necessarily dissipative, and cannot give rise to quantum mechanics. As argued before, this is not the case, when the diffusion is time-reversible. This point has been well explained by Nelson in section 14 of Ref. [270], where an analogy is made with the difference between Aristotelean and Galilean dynamics. It should be noted that in order to describe entanglement in stochastic quantization, the background field has to be non-local. This particular feature was disliked by Nelson, cf. e.g. Ref. [271]. We stress that this non-locality is merely a feature of quantum mechanics, and not specific to stochastic quantization. Moreover, it is an open question, whether the non-locality of the background can be avoided, if one considers non-Markovian diffusion processes.

Another concern that may be raised against stochastic mechanics is that it can be regarded as a hidden variable theory, as it is assumed that a background field exists that is responsible for the quantum fluctuations. One could thus expect that stochastic mechanics satisfies the Bell inequalities, which would distinguish it from quantum mechanics. We will avoid this issue by assuming that the background field is fundamentally random, in the sense that the fluctuations cannot be derived from a more fundamental theory. Under this assumption there are no deterministic hidden variables. This assumption distinguishes the framework from for example the Brownian motion of a colloid suspended in a liquid, where the trajectory of the colloid can in theory be derived by solving the equations of motion of all the molecules in the liquid.

A more pressing issue for stochastic quantization is Wallstrom's criticism [346, 347], which states that the 2π periodicity of the wave function has to be imposed as an additional assumption. Such an assumption must be made ad hoc, since the wave function is not a fundamental object in the theory. Several responses against this criticism have been given, such as for example the incorporation of zitterbewegung [130, 131], adding a postulate regarding the boundedness of the Laplace operator acting on the probability density [309] or by adding the assumption of unitarity of superpositions of wave functions [159]. It is also worth mentioning that it was pointed out in Ref. [11] that the stochastic processes should be lifted to the universal cover of the configuration space, as the configuration space itself might not be simply connected. When this is done, the wave function obtains periodicity factors that are related to the winding numbers around the holes in the configuration

[7] Let us be a bit more precise, as the process is slightly more complicated in stochastic quantization: after passing through one of the slits, the particle will diffuse according to a one slit diffusion process. However, due to the imposed time-reversibility of the motion, it will transition into a double slit diffusion process. The length scale associated to this transition is the width of the slit, cf. e.g. sections 16 and 17 in Ref. [270].

space, which could resolve Wallstrom's criticism.

Since no consensus yet exists about the solution of Wallstrom's criticism, we will take a more pragmatic approach: we accept this ad hoc constraint and remain agnostic about its solution. The reason for this is that imposing such a constraint is only problematic at a foundational level. Even if Wallstrom's criticism cannot be resolved within stochastic quantization, the theory can still be used as an alternative mathematical model of quantum theory, and can thus be used to make predictions about quantum systems. As we will show in this paper, a particular advantage of the stochastic model is that it can be formulated on (pseudo-)Riemannian manifolds, which could help guide the way towards a theory of quantum gravity.

A more practical concern regarding stochastic quantization is that analytical calculations require to solve stochastic differential equations. This is notoriously difficult. In fact, an important solution method relies on the mapping stochastic differential equations to path integral problems and to partial differential equations, as established by the Feynman-Kac formula. It is thus expected that many calculations can more easily be performed in ordinary quantum theory. This would render stochastic mechanics as an alternative mathematical model unnecessary. Despite this fact, it is expected that stochastic quantization could prove to be useful in numerical calculations, and a small number of analytical calculations. More interesting, however, is the potential of stochastic quantization on a more formal level. In particular, it could prove to be useful in mathematically rigorous definitions of the path integral, which is expected to be essential for constructing a theory of quantum gravity. We note here that stochastic approaches already serve as one of the stepping stones of the Euclidean approach in quantum field theory, see e.g. [269, 313, 332, 354].

6.1.4 Postulates of the theory

Before moving on, let us summarize the fundamental assumptions of stochastic quantization: we assume that all particles follow well defined trajectories through a diffeomorphism invariant background field. This background field induces stochastic fluctuations such that the motion of particles resembles a conservative Brownian motion. Moreover, the quadratic variation of this process scales with the Planck constant according to the background hypothesis. We have the following postulates:

- All observables are invariant under a change of coordinate system.
- The stochastic motion of a particle with mass m is Markovian.
- The stochastic motion of a particle with mass m is time-reversible.
- The stochastic motion obeys the structure equation $[[X_\mu, X^\nu]](\tau) = \frac{\hbar}{m}\delta^\nu_\mu \tau$.

We note that the classical limit of the theory can be obtained straightforwardly by taking the limit $\hbar \to 0$.

6.1.5 Main results of the paper

In this paper, we work in the $(-+++)$ signature with a Riemann tensor defined by $\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\kappa}\Gamma^\kappa_{\nu\sigma} - \Gamma^\rho_{\nu\kappa}\Gamma^\kappa_{\mu\sigma}$ and Ricci tensor $\mathcal{R}_{\mu\nu} = \mathcal{R}^{\rho}{}_{\mu\rho\nu}$. In addition, we set $c = 1$ throughout the paper.

The main result we present in this paper is the following: in the stochastic quantization framework, a massive scalar particle moving on a Lorentzian manifold and governed by the stochastic Lagrangian

$$L(X, V_+, V_-, \tau) = \frac{1}{2} L_c(X, V_+, \tau) + \frac{1}{2} L_c(X, V_-, \tau) \tag{6.8}$$

where the classical Lagrangian is given by

$$L_c(x, v, \tau) = \frac{m}{2}\, g_{\mu\nu}(x)\, v^\mu\, v^\nu - \hbar\, A_\mu(x,\tau)\, v^\mu - \mathfrak{U}(x,\tau) \tag{6.9}$$

with $x = (t, \vec{x})$ and τ is the proper time, evolves according to the Stratonovich stochastic differential equation

$$\begin{aligned} m\, g_{\mu\nu} \left(d^2 X^\nu + \Gamma^\nu_{\rho\sigma}\, dX^\rho dX^\sigma \right) = &\left(\hbar\, \partial_\tau A_\mu - \nabla_\mu \mathfrak{U} - \frac{\hbar^2}{12m} \nabla_\mu \mathcal{R} \right) d\tau^2 \\ &- \hbar \left(\nabla_\mu A_\nu - \nabla_\nu A_\mu \right) dX^\nu d\tau. \end{aligned} \tag{6.10}$$

Furthermore, if the probability density $\rho(x, \tau)$ associated to the probability measure $\mu = \mathbb{P} \circ X^{-1}$ exists, one can construct the wave function

$$\Psi(x,\tau) = \sqrt{\rho(x,\tau)}\, \exp\left\{ \frac{i}{\hbar}\, \mathbb{E}\left[\int_{\tau_i}^{\tau} L\big(X(t), V_+(t), V_-(t), t\big)\, dt \Big| X(\tau) = x \right] \right\} \tag{6.11}$$

that evolves according to a generalization of the Schrödinger equation given by

$$i\hbar \frac{\partial}{\partial \tau} \Psi = \left[-\frac{\hbar^2}{2m} \left(\Big[\nabla_\mu + i A_\mu \Big] \Big[\nabla^\mu + i A^\mu \Big] - \frac{1}{6} \mathcal{R} \right) + \mathfrak{U} \right] \Psi. \tag{6.12}$$

This wave function obeys the Born rule

$$|\Psi(x,\tau)|^2 = \rho(x,\tau). \tag{6.13}$$

If there is no explicit proper time dependence in A_μ or $\mathfrak{U}$, one can solve by separation of variables such that

$$\Psi(x,\tau) = \sum_k \phi_k(x)\, \exp\left(\frac{i\, m\, \lambda_k}{2\, \hbar} \tau \right), \tag{6.14}$$

where $\phi_k(x)$ solves the generalization of the Klein-Gordon equation given by

$$\hbar^2 \left(\Big[\nabla_\mu + i A_\mu \Big] \Big[\nabla^\mu + i A^\mu \Big] - \frac{1}{6} \mathcal{R} \right) \phi_k = m^2\, \lambda_k\, \phi_k + 2\, m\, \mathfrak{U}\, \phi_k. \tag{6.15}$$

We note that the derivation of eqs. (6.11), (6.12) and (6.13) is a well established result on $\mathbb{R}^n$, see e.g. Refs. [184, 223, 267, 268, 270, 364]. Moreover, partial extensions to

Riemannian manifolds have been known for some time, cf. Refs. [127, 135–137, 187, 270].

In this paper, we show that these results can be generalized to pseudo-Riemannian manifolds. An important ingredient for these extensions is the second order geometry as developed by Schwartz and Meyer [150, 257, 311]. This is an extension of ordinary differential geometry that allows to describe stochastic processes on manifolds. In addition to the extension of stochastic quantization to pseudo-Riemannian manifolds, we will give some new interpretations of stochastic quantization.

This paper is organized as follows: in the next section 6.3, we introduce the relevant semi-martingale processes for quantum mechanics; section 6.4 discusses integration along semi-martingales on manifolds; in section 6.5, we discuss stochastic variational calculus; in section 6.6, we discuss the shape of the stochastic action; in section 6.7, we put everything together and derive the stochastic differential equations for quantum mechanical scalar test particles on pseudo-Riemannian manifolds, and the associated Schrödinger equation. Finally, in section 6.8, we conclude and summarize some future perspectives of the stochastic approach.

6.2 Second order geometry

In this section, we review the theory of Schwartz-Meyer second order geometry, that can be used to extend the theory of stochastic calculus to manifolds. The first three subsections are loosely based on Ref. [150]. The later subsections contain new material and extend some important concepts from first order geometry into second order geometry. For more detail we refer to the work of Emery [150] and the original works by Schwartz [311] and Meyer [257].

6.2.1 Second order vectors and forms

We consider a $(n = d + 1)$-dimensional pseudo-Riemannian manifold $\mathcal{M}$ with the usual first order tangent and cotangent spaces $T_x\mathcal{M}$, $T_x^*\mathcal{M}$. For every $x \in \mathcal{M}$ and any coordinate chart containing x one can write down bases for the tangent and cotangent space respectively given by $\{\partial_\mu | \mu \in \{0, 1, ..., d\}\}$ and $\{dx^\mu | \mu \in \{0, 1, ..., d\}\}$. In particular for $v \in T_x\mathcal{M}$ and $\omega \in T_x^*\mathcal{M}$ we have

$$\begin{aligned} v &= v^\mu \, \partial_\mu, \\ \omega &= \omega_\mu \, dx^\mu. \end{aligned} \tag{6.16}$$

Furthermore, a form $\omega \in T_x^*\mathcal{M}$ can often be written as the differential form of some function $f : \mathcal{M} \to \mathbb{R}$ i.e.

$$\omega = df = \partial_\mu f \, dx^\mu. \tag{6.17}$$

The product rule for such differential forms is given by

$$d(fg) = f \, dg + g \, df. \tag{6.18}$$

In addition, there exists a metric associated to the tangent space that is given by

$$g : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R} \quad \text{s.t.} \quad (v, w) \mapsto \langle v|w\rangle = g_{\mu\nu}v^\mu w^\nu, \tag{6.19}$$

and is bilinear, symmetric and non-degenerate. Moreover the metric induces an isomorphism $g^\flat : T_x\mathcal{M} \to T_x^*\mathcal{M}$ between the tangent and cotangent space, that is defined by

$$\langle g^\flat(v), w\rangle = \langle v, g^\flat(w)\rangle = \langle v|w\rangle \tag{6.20}$$

We define a similar bracket for two forms $\alpha, \beta \in T_x^*\mathcal{M}$ by

$$\langle \alpha|\beta\rangle = \langle \alpha, g^\sharp(\beta)\rangle = \langle g^\sharp(\alpha), \beta\rangle. \tag{6.21}$$

We will now define a second order tangent space and cotangent space $\tilde{T}_x\mathcal{M}$, $\tilde{T}_x^*\mathcal{M}$. For every $x \in \mathcal{M}$ and any coordinate chart containing x one can write down bases for the tangent and cotangent space respectively given by $\{\partial_\mu, \partial_{\mu\nu} | \mu \le \nu \in \{0, 1, ..., d\}\}$ and $\{d_2x^\mu, dx^\mu \cdot dx^\nu | \mu \le \nu \in \{0, 1, ..., d\}\}$.[8] In particular, for $V \in \tilde{T}_x\mathcal{M}$ and $\Omega \in \tilde{T}_x^*\mathcal{M}$ we have

$$\begin{aligned} V &= v^\mu\, \partial_\mu + v^{\mu\nu}\, \partial_{\mu\nu}, \\ \Omega &= \omega_\mu\, d_2x^\mu + \omega_{\mu\nu}\, dx^\mu \cdot dx^\nu. \end{aligned} \tag{6.22}$$

Notice that $T_x\mathcal{M} \subset \tilde{T}_x\mathcal{M}$ and $T_x^*\mathcal{M} \subset \tilde{T}_x^*\mathcal{M}$. Furthermore, $\partial_{\mu\nu} := \partial_\mu\partial_\nu$ is a symmetric object, which implies that $v^{\mu\nu}$ must be symmetric. Moreover, we choose the basis of the cotangent space dual to the basis of the tangent space. This imposes $dx^\mu \cdot dx^\nu$, and $\omega_{\mu\nu}$ to be symmetric as well.

We have a duality pairing between the bases of the tangent and cotangent space such that:

$$\begin{aligned} \langle \partial_\mu, d_2x^\rho\rangle &= \delta_\mu^\rho, \\ \langle \partial_\mu, dx^\rho \cdot dx^\sigma\rangle &= 0, \\ \langle \partial_{\mu\nu}, dx^\rho\rangle &= 0, \\ \langle \partial_{\mu\nu}, dx^\rho \cdot dx^\sigma\rangle &= \frac{1}{2}\left(\delta_\mu^\rho\, \delta_\nu^\sigma + \delta_\mu^\sigma\, \delta_\nu^\rho\right). \end{aligned} \tag{6.23}$$

The duality pairing of an arbitrary vector and covector is then given by

$$\langle V, \Omega\rangle = v^\mu\omega_\mu + v^{\mu\nu}\omega_{\mu\nu}. \tag{6.24}$$

As in the classical case, forms $\Omega \in \tilde{T}_x^*\mathcal{M}$ can often be written as a differential form of some function $f : \mathcal{M} \to \mathbb{R}$:

$$\Omega = d_2f = \partial_\mu f\, d_2x^\mu + \partial_{\mu\nu} f\, dx^\mu \cdot dx^\nu. \tag{6.25}$$

[8] Notice that $dx^\mu \cdot dx^\nu \neq dx^\mu \otimes dx^\mu$.

The product rule for differential forms is given by

$$d_2(fg) = f\, d_2 g + g\, d_2 f + 2\, df \cdot dg \tag{6.26}$$

where the product of first order forms[9] $\omega, \theta \in T_x\mathcal{M}$ is defined by

$$\begin{aligned} \omega \cdot \theta &:= \frac{1}{2}\left(\omega_\mu\, \theta_\nu + \omega_\nu\, \theta_\mu\right) dx^\mu \cdot dx^\nu \\ &= \omega_\mu\, \theta_\nu\, dx^\mu \cdot dx^\nu. \end{aligned} \tag{6.27}$$

Therefore, the product for two first order differential forms can be written as

$$df \cdot dg = \partial_\mu f\, \partial_\nu g\, dx^\mu \cdot dx^\nu. \tag{6.28}$$

It will be useful to define mappings between the first order and second order tangent spaces. The projection map[10] can be defined as:

$$\mathcal{P} : \tilde{T}^*_x\mathcal{M} \to T^*_x\mathcal{M} \quad \text{s.t.} \quad \begin{cases} \mathcal{P}(d_2 f) = df, \\ \mathcal{P}(\omega \cdot \theta) = 0. \end{cases} \tag{6.29}$$

Furthermore, there exists a unique smooth and invertible linear map $\mathcal{H}$ from bilinear first order forms to second order forms, such that $\mathcal{P} \circ \mathcal{H} = 0$, given by[11]

$$\mathcal{H} : T^*_x\mathcal{M} \times T^*_x\mathcal{M} \to \tilde{T}^*_x\mathcal{M} \quad \text{s.t.} \quad (\omega, \theta) \mapsto \omega \cdot \theta, \tag{6.30}$$

The adjoint of this map is denoted by $\mathcal{H}^* : \tilde{T}_x\mathcal{M} \to T_x\mathcal{M} \otimes T_x\mathcal{M}$. In addition there exists a unique linear map[12] $\underline{d} : T^*_x\mathcal{M} \to \tilde{T}^*_x\mathcal{M}$ such that for any $f \in C^\infty(\mathcal{M}, \mathbb{R})$, $\omega \in T^*_x\mathcal{M}$ and $u, v \in T_x\mathcal{M}$

$$\begin{aligned} \underline{d}(df) &= d_2 f, \\ \underline{d}(f\omega) &= f\underline{d}\omega + df \cdot \omega, \\ \langle \underline{d}\omega, [u,v] \rangle &= \langle \omega, [u,v] \rangle, \\ \langle \underline{d}\omega, \{u,v\} \rangle &= u\langle \omega, v \rangle + v\langle \omega, u \rangle, \end{aligned} \tag{6.31}$$

where $[u, v]$ is the commutator, $\{u, v\}$ the anti-commutator and $[[u, v]]$ the joint quadratic variation of u and v.

Finally,[13] one can define maps $\mathcal{F} : \tilde{T}_x\mathcal{M} \to T_x\mathcal{M}$ and $\mathcal{G} : T^*_x\mathcal{M} \to \tilde{T}^*_x\mathcal{M}$ such that for any affine connection[14] $\Gamma : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ the following relations define a

[9] More generally, one often defines the *carré du champ* operator or the *squared field operator* associated to a linear mapping L for two functions f, g by $\Gamma(f,g) := \frac{1}{2}\left[L(fg) - f\, Lg - g\, Lf\right]$. Cf. e.g. Lemma 6.1 in [150]. We can then interpret $df \cdot dg$ as the squared field operator associated to the second order differential operator d_2 acting on f, g.

[10] In Ref. [150] this map is called the restriction R.

[11] cf. Proposition 6.13 in Ref. [150].

[12] cf. Theorem 7.1 in Ref. [150]. We use an underlined $\underline{d}$ to avoid confusion with the exterior derivative.

[13] cf. Proposition 7.28 in Ref. [150].

[14] $\mathfrak{X}(\mathcal{M})$ is the space of all smooth vector fields on $\mathcal{M}$, i.e. the space of all smooth sections of the tangent

bijection between $\mathcal{F}$ and Γ

$$
\begin{aligned}
(\mathcal{F}\,V)f &= V\,f - \langle \mathcal{H}\,\Gamma^*(df), V\rangle,\\
\Gamma(u,v)f &= u\,v\,f - \mathcal{F}(u\,v)f,
\end{aligned}
\tag{6.32}
$$

where V is a second order vector and u, v are first order vector fields. A bijection between $\mathcal{G}$ and Γ is then defined by

$$
\begin{aligned}
\mathcal{G}(df) &= d_2 f - \mathcal{H}\,\Gamma^*(df),\\
\Gamma(u,v)f &= u\,v\,f - \langle \mathcal{G}(df), u\,v\rangle.
\end{aligned}
\tag{6.33}
$$

Moreover, $\mathcal{F}$ and $\mathcal{G}$ are each others adjoint.[15]

6.2.2 Coordinate transformations

In this section, we investigate the change of vectors and covectors under coordinate transformations. For a vector field V we find:

$$
\begin{aligned}
V\,f &= \left(v^\mu \partial_\mu + v^{\mu\nu}\partial_{\mu\nu}\right) f\\
&= \left(v^\mu \frac{\partial \tilde{x}^\rho}{\partial x^\mu}\tilde{\partial}_\rho + v^{\mu\nu}\partial_\mu\left[\frac{\partial \tilde{x}^\rho}{\partial x^\nu}\tilde{\partial}_\rho\right]\right) f\\
&= \left(v^\mu \frac{\partial \tilde{x}^\rho}{\partial x^\mu}\tilde{\partial}_\rho + v^{\mu\nu}\frac{\partial^2 \tilde{x}^\rho}{\partial x^\mu \partial x^\nu}\tilde{\partial}_\rho + v^{\mu\nu}\frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\frac{\partial \tilde{x}^\rho}{\partial x^\nu}\tilde{\partial}_{\sigma\rho}\right) f.
\end{aligned}
\tag{6.34}
$$

Hence, we find the active transformations laws

$$
\begin{aligned}
v^\mu \to \tilde{v}^\mu &= v^\rho \frac{\partial \tilde{x}^\mu}{\partial x^\rho} + v^{\rho\sigma}\frac{\partial^2 \tilde{x}^\mu}{\partial x^\rho \partial x^\sigma},\\
v^{\mu\nu} \to \tilde{v}^{\mu\nu} &= v^{\rho\sigma}\frac{\partial \tilde{x}^\mu}{\partial x^\rho}\frac{\partial \tilde{x}^\nu}{\partial x^\sigma},
\end{aligned}
\tag{6.35}
$$

or equivalently the passive transformation laws

$$
\begin{aligned}
\partial_\mu \to \tilde{\partial}_\mu &= \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\partial_\rho,\\
\partial_{\mu\nu} \to \tilde{\partial}_{\mu\nu} &= \frac{\partial^2 x^\rho}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}\partial_\rho + \frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\frac{\partial x^\rho}{\partial \tilde{x}^\nu}\partial_{\rho\sigma}.
\end{aligned}
\tag{6.36}
$$

A form Ω transforms as

$$
\begin{aligned}
\Omega(V\,f) &= \left(\omega_\mu d_2 x^\mu + \omega_{\mu\nu} dx^\mu \cdot dx^\nu\right)(V\,f)\\
&= \left(\omega_\mu \frac{\partial x^\mu}{\partial \tilde{x}^\rho} d_2\tilde{x}^\rho + \omega_\mu \frac{\partial^2 x^\mu}{\partial \tilde{x}^\rho \partial \tilde{x}^\sigma} d\tilde{x}^\rho \cdot d\tilde{x}^\sigma + \omega_{\mu\nu}\frac{\partial x^\mu}{\partial \tilde{x}^\rho}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma} d\tilde{x}^\rho \cdot d\tilde{x}^\sigma\right)(V\,f).
\end{aligned}
\tag{6.37}
$$

bundle $T\mathcal{M}$.

[15]It is possible to take a connection in the defining relation for $\mathcal{G}$ that is different from $\mathcal{F}$. If such a choice is made, $\mathcal{F}$ and $\mathcal{G}$ are no longer each others adjoint. In this paper, we will not make such a choice, as we will restrict ourselves to the Levi-Civita connection.

Therefore, the active transformation laws are given by

$$\begin{aligned}\omega_\mu \to \tilde{\omega}_\mu &= \omega_\rho \frac{\partial x^\rho}{\partial \tilde{x}^\mu},\\ \omega_{\mu\nu} \to \tilde{\omega}_{\mu\nu} &= \omega_\rho \frac{\partial^2 x^\rho}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} + \omega_{\rho\sigma} \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu},\end{aligned} \tag{6.38}$$

and the passive transformation law is

$$\begin{aligned}d_2 x^\mu \to d_2 \tilde{x}^\mu &= \frac{\partial \tilde{x}^\mu}{\partial x^\rho} d_2 x^\rho + \frac{\partial^2 \tilde{x}^\mu}{\partial x^\rho \partial x^\sigma} dx^\rho \cdot dx^\sigma,\\ dx^\mu \cdot dx^\nu \to d\tilde{x}^\mu \cdot d\tilde{x}^\nu &= \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\frac{\partial \tilde{x}^\nu}{\partial x^\sigma} dx^\rho \cdot dx^\sigma.\end{aligned} \tag{6.39}$$

The transformation laws should leave the duality pairing (6.24) invariant. Indeed we find[16]

$$\begin{aligned}\langle V, \Omega\rangle &= v^\mu \omega_\mu + v^{\mu\nu}\omega_{\mu\nu}\\ &= \tilde{v}^\rho \frac{\partial x^\mu}{\partial \tilde{x}^\rho}\frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\tilde{\omega}_\sigma + \tilde{v}^{\rho\sigma}\frac{\partial^2 x^\mu}{\partial \tilde{x}^\rho \partial \tilde{x}^\sigma}\frac{\partial \tilde{x}^\kappa}{\partial x^\mu}\tilde{\omega}_\kappa + \tilde{v}^{\rho\sigma}\frac{\partial x^\mu}{\partial \tilde{x}^\rho}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma}\frac{\partial^2 \tilde{x}^\kappa}{\partial x^\mu \partial x^\nu}\tilde{\omega}_\kappa + \tilde{v}^{\rho\sigma}\frac{\partial x^\mu}{\partial \tilde{x}^\rho}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma}\frac{\partial \tilde{x}^\kappa}{\partial x^\mu}\frac{\partial \tilde{x}^\lambda}{\partial x^\nu}\tilde{\omega}_{\kappa\lambda}\\ &= \tilde{v}^\mu \tilde{\omega}_\mu + \tilde{v}^{\mu\nu}\tilde{\omega}_{\mu\nu}.\end{aligned} \tag{6.40}$$

6.2.3 Covariance

In previous subsection, we found that vectors and forms in second order geometry transform in an affine but not contravariant/covariant way. This can be fixed by introducing a covariant basis $\{\hat{\partial}_\mu, \hat{\partial}_{\mu\nu}\}$ for $\tilde{T}_x\mathcal{M}$ such that

$$V = \hat{v}^{\overset{\mu}{\nu\rho}}\hat{\partial}_{\underset{\nu\rho}{\mu}} = \hat{v}^\mu \hat{\partial}_\mu + \hat{v}^{\nu\rho}\,\hat{\partial}_{\nu\rho}, \tag{6.41}$$

and

$$\begin{aligned}\hat{\partial}_\mu &:= \partial_\mu,\\ \hat{\partial}_{\mu\nu} &:= \partial_{\mu\nu} - \Gamma^\rho_{\mu\nu}\partial_\rho,\\ \hat{v}^\mu &:= v^\mu + v^{\rho\sigma}\Gamma^\mu_{\rho\sigma},\\ \hat{v}^{\mu\nu} &:= v^{\mu\nu}.\end{aligned} \tag{6.42}$$

In a similar way, we can introduce a contravariant basis for the cotangent space $\tilde{T}^*_x\mathcal{M}$, such that

$$\Omega = \hat{\omega}_{\underset{\nu\rho}{\mu}}\, d_2\hat{x}^{\overset{\mu}{\nu\rho}} = \hat{\omega}_\mu\, d_2\hat{x}^\mu + \hat{\omega}_{\nu\rho}\, d\hat{x}^\nu \cdot d\hat{x}^\rho \tag{6.43}$$

[16] One can use the Christoffel symbols to make the second term in the second line vanish.

with

$$
\begin{aligned}
d_2\hat{x}^\mu &:= d_2x^\mu + \Gamma^\mu_{\nu\rho} dx^\nu \cdot dx^\rho, \\
d\hat{x}^\mu \cdot d\hat{x}^\nu &:= dx^\mu \cdot dx^\nu, \\
\hat{\omega}_\mu &:= \omega_\mu, \\
\hat{\omega}_{\mu\nu} &:= \omega_{\mu\nu} - \omega_\rho \Gamma^\rho_{\mu\nu}.
\end{aligned} \tag{6.44}
$$

It is possible to extend the notion of vector fields and forms to arbitrary (k, l)-tensors. Indeed, one can construct mappings

$$
T(x) : (\tilde{T}^*_x\mathcal{M})^k \times (\tilde{T}_x\mathcal{M})^l \to \mathbb{R}. \tag{6.45}
$$

In local coordinates such a tensor will be given by

$$
\begin{aligned}
T = {} & T^{\binom{\mu}{\nu\rho}_1 \dots \binom{\mu}{\nu\rho}_k}_{\binom{\sigma}{\kappa\lambda}_1 \dots \binom{\sigma}{\kappa\lambda}_l} \quad \partial_{\binom{\mu}{\nu\rho}_1} \otimes \dots \otimes \partial_{\binom{\mu}{\nu\rho}_k} \otimes d_2x^{\binom{\sigma}{\kappa\lambda}_1} \otimes \dots \otimes d_2x^{\binom{\sigma}{\kappa\lambda}_l} \\
= {} & T^{\mu_1 \dots \mu_k}_{\sigma_1 \dots \sigma_l} \quad \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes d_2x^{\sigma_1} \otimes \dots \otimes d_2x^{\sigma_l} \\
& + T^{(\nu\rho)_1 \mu_2 \dots \mu_k}_{\sigma_1 \dots \sigma_l} \quad \partial_{\nu_1\rho_1} \otimes \partial_{\mu_2} \otimes \dots \otimes \partial_{\mu_k} \otimes d_2x^{\sigma_1} \otimes \dots \otimes d_2x^{\sigma_l} \\
& \quad + T^{\mu_1 (\nu\rho)_2 \mu_3 \dots \mu_k}_{\sigma_1 \dots \sigma_l} \quad \partial_{\mu_1} \otimes \partial_{\nu_2\rho_2} \otimes \partial_{\mu_3} \otimes \dots \otimes \partial_{\mu_k} \otimes d_2x^{\sigma_1} \otimes \dots \otimes d_2x^{\sigma_l} \\
& \quad + \dots \\
& \quad + T^{\mu_1 \dots \mu_k}_{\sigma_1 \dots \sigma_{l-1} (\kappa\lambda)_l} \quad \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes d_2x^{\sigma_1} \otimes \dots \otimes d_2x^{\sigma_{l-1}} \otimes dx^{\kappa_l} \cdot dx^{\lambda_l} \\
& + T^{(\nu\rho)_1 (\nu\rho)_2 \mu_3 \dots \mu_k}_{\sigma_1 \dots \sigma_l} \quad \partial_{\nu_1\rho_1} \otimes \partial_{\nu_2\rho_2} \otimes \partial_{\mu_3} \otimes \dots \otimes \partial_{\mu_k} \otimes d_2x^{\sigma_1} \otimes \dots \otimes d_2x^{\sigma_l} \\
& + \dots \\
& + T^{(\nu\rho)_1 \dots (\nu\rho)_k}_{(\kappa\lambda)_1 \dots (\kappa\lambda)_l} \quad \partial_{\nu_1\rho_1} \otimes \dots \otimes \partial_{\nu_k\rho_k} \otimes dx^{\kappa_1} \cdot dx^{\lambda_1} \otimes \dots \otimes dx^{\kappa_l} \cdot dx^{\lambda_l}.
\end{aligned} \tag{6.46}
$$

The components of T do not transform in a covariant/contravariant way. However, one can construct a representation with components $\hat{T}$ such that

$$
T = \hat{T}^{\binom{\mu}{\nu\rho}_1 \dots \binom{\mu}{\nu\rho}_k}_{\binom{\sigma}{\kappa\lambda}_1 \dots \binom{\sigma}{\kappa\lambda}_l} \quad \hat{\partial}_{\binom{\mu}{\nu\rho}_1} \otimes \dots \otimes \hat{\partial}_{\binom{\mu}{\nu\rho}_k} \otimes d_2\hat{x}^{\binom{\sigma}{\kappa\lambda}_1} \otimes \dots \otimes d_2\hat{x}^{\binom{\sigma}{\kappa\lambda}_l}. \tag{6.47}
$$

If expanded as in eq. (6.47), the coefficients $\hat{T}$ do transform covariantly/contravariantly. The relation between components T and $\hat{T}$ for a general (k, l)-tensor can then be derived from the transformation laws for $(1, 0)$- and $(0, 1)$-tensors.

Finally, we note that there exists a relation between the second order contravariant vectors and covariant forms and the maps $\mathcal{F}, \mathcal{G}, \mathcal{H}$. For $V \in \tilde{T}_x\mathcal{M}$ we have

$$
\begin{aligned}
\mathcal{F}(V) &= \left(v^\mu + v^{\rho\sigma}\Gamma^\mu_{\rho\sigma}\right) \partial_\mu \\
&= \hat{v}^\mu \, \hat{\partial}_\mu,
\end{aligned} \tag{6.48}
$$

$$
\begin{aligned}
\mathcal{H}^*(V) &= v^{\mu\nu} \, \partial_\mu \otimes \partial_\nu \\
&= \hat{v}^{\mu\nu} \, \hat{\partial}_\mu \otimes \hat{\partial}_\nu,
\end{aligned} \tag{6.49}
$$

and for $\alpha, \beta \in T_x^*\mathcal{M}$

$$\begin{aligned}\mathcal{G}(\alpha) &= \alpha_\mu \left(d_2x^\mu + \Gamma^\mu_{\rho\sigma} dx^\rho \cdot dx^\sigma\right) \\ &= \hat{\alpha}_\mu \, d_2\hat{x}^\mu, \qquad (6.50) \\ \mathcal{H}(\alpha \otimes \beta) &= \alpha_\mu \beta_\nu \, dx^\mu \cdot dx^\nu \\ &= \hat{\alpha}_\mu \hat{\beta}_\nu \, d\hat{x}^\mu \cdot d\hat{x}^\nu. \qquad (6.51)\end{aligned}$$

Therefore, all second order vectors and forms can be decomposed into first order vectors, forms and symmetric bilinear tensor products of first order vectors and forms. More generally, any second order (k,l)-tensor can be decomposed into first order tensors of degree (κ, λ) with $k \leq \kappa \leq 2k$ and $l \leq \lambda \leq 2l$.

6.2.4 Second order metric

In this subsection, we extend the notion of a metric to the second order geometry framework. We can define a symmetric bilinear function $\tilde{g} : \tilde{T}_x\mathcal{M} \times \tilde{T}_x\mathcal{M} \to \mathbb{R}$, that we call the second order metric tensor. Analogously to the first order metric, it acts on two second order vectors $V, W \in \tilde{T}_x\mathcal{M}$, such that

$$\tilde{g}(V, W) = \langle V|W\rangle. \qquad (6.52)$$

Moreover, it induces an isomorphism between vectors and forms

$$\tilde{g}^\flat : \tilde{T}_x\mathcal{M} \to \tilde{T}^*_x\mathcal{M} \qquad \text{s.t.} \qquad \begin{cases} \langle V|W\rangle = \langle \tilde{g}^\flat(V), W\rangle, \\ \langle \Omega, \Theta\rangle = \langle \Omega|\tilde{g}^\sharp(\Theta)\rangle. \end{cases} \qquad (6.53)$$

In a local coordinate chart the metric tensor $\tilde{g}$ can be written as

$$\begin{aligned}\tilde{g} &= \tilde{g}_{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}} \, d_2x^{\binom{\mu}{\rho\sigma}} \otimes d_2x^{\binom{\nu}{\kappa\lambda}} \\ &= \tilde{g}_{\mu\nu} \, d_2x^\mu \otimes d_2x^\nu + \tilde{g}_{\mu(\kappa\lambda)} \, d_2x^\mu \otimes dx^\kappa \cdot dx^\lambda \\ &\quad + \tilde{g}_{(\rho\sigma)\nu} \, dx^\rho \cdot dx^\sigma \otimes d_2x^\nu + \tilde{g}_{(\rho\sigma)(\kappa\lambda)} \, dx^\rho \cdot dx^\sigma \otimes dx^\kappa \cdot dx^\lambda. \qquad (6.54)\end{aligned}$$

Using the defining isomorphism (6.53) and the duality pairing, eq. (6.24), we find the rules for transforming second order vectors into second order forms:

$$\begin{aligned}\tilde{g}_{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}} \, v^{\overset{\nu}{\kappa\lambda}} &= v_{\underset{\rho\sigma}{\mu}}, \\ \tilde{g}_{\mu\nu} \, v^\nu + \tilde{g}_{\mu(\kappa\lambda)} \, v^{\kappa\lambda} &= v_\mu, \\ \tilde{g}_{(\rho\sigma)\nu} \, v^\nu + \tilde{g}_{(\rho\sigma)(\kappa\lambda)} \, v^{\kappa\lambda} &= v_{\rho\sigma}. \qquad (6.55)\end{aligned}$$

Furthermore, the inverse $\tilde{g}^{-1}$ can be used to transform second order forms into second order vectors:

$$\begin{aligned}\tilde{g}^{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}}\ \omega_{\substack{\nu\\ \kappa\lambda}} &= \omega^{\substack{\mu\\ \rho\sigma}},\\ \tilde{g}^{\mu\nu}\ \omega_\nu + \tilde{g}^{\mu(\kappa\lambda)}\ \omega_{\kappa\lambda} &= \omega^\mu,\\ \tilde{g}^{(\rho\sigma)\nu}\ \omega_\nu + \tilde{g}^{(\rho\sigma)(\kappa\lambda)}\ \omega_{\kappa\lambda} &= \omega^{\rho\sigma}.\end{aligned} \tag{6.56}$$

The components of the metric tensor do not transform covariantly. Therefore, we define a covariant representation of the second order metric:

$$\begin{aligned}\tilde{g} &= \tilde{g}_{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}}\ d_2x^{\substack{\mu\\ \rho\sigma}} \otimes d_2x^{\substack{\nu\\ \kappa\lambda}}\\ &= \tilde{g}_{\mu\nu}\ d_2\hat{x}^\mu \otimes d_2\hat{x}^\nu\\ &\quad + \left(\tilde{g}_{\mu(\kappa\lambda)} - \tilde{g}_{\mu\nu}\,\Gamma^\nu_{\kappa\lambda}\right)\ d_2\hat{x}^\mu \otimes d\hat{x}^\kappa \cdot d\hat{x}^\lambda\\ &\quad + \left(\tilde{g}_{(\rho\sigma)\nu} - \tilde{g}_{\mu\nu}\,\Gamma^\mu_{\rho\sigma}\right)\ d\hat{x}^\rho \cdot d\hat{x}^\sigma \otimes d_2\hat{x}^\nu\\ &\quad + \left(\tilde{g}_{(\rho\sigma)(\kappa\lambda)} + \tilde{g}_{\mu\nu}\,\Gamma^\mu_{\rho\sigma}\,\Gamma^\nu_{\kappa\lambda} - \tilde{g}_{\mu(\kappa\lambda)}\,\Gamma^\mu_{\rho\sigma} - \tilde{g}_{(\rho\sigma)\nu}\,\Gamma^\nu_{\kappa\lambda}\right)\ d\hat{x}^\rho \cdot d\hat{x}^\sigma \otimes d\hat{x}^\kappa \cdot d\hat{x}^\lambda\\ &= \hat{g}_{\mu\nu}\ d_2\hat{x}^\mu \otimes d_2\hat{x}^\nu + \hat{g}_{\mu(\kappa\lambda)}\ d_2\hat{x}^\mu \otimes d\hat{x}^\kappa \cdot d\hat{x}^\lambda\\ &\qquad + \hat{g}_{(\rho\sigma)\nu}\ d\hat{x}^\rho \cdot d\hat{x}^\sigma \otimes d_2\hat{x}^\nu + \hat{g}_{(\rho\sigma)(\kappa\lambda)}\ d\hat{x}^\rho \cdot d\hat{x}^\sigma \otimes d\hat{x}^\kappa \cdot d\hat{x}^\lambda\\ &= \hat{g}_{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}}\ d_2\hat{x}^{\substack{\mu\\ \rho\sigma}} \otimes d_2\hat{x}^{\substack{\nu\\ \kappa\lambda}}.\end{aligned} \tag{6.57}$$

We notice that a second order vector can be uniquely decomposed in a first order vector and a bilinear first order tensor. We will therefore impose

$$\tilde{g}^\flat = \begin{pmatrix} \mathcal{G} \circ g^\flat \circ \mathcal{F} \\ \mathcal{H} \circ \left(g^\flat \otimes g^\flat\right) \circ \mathcal{H}^* \end{pmatrix} \tag{6.58}$$

We can then write in a local coordinate system

$$\begin{aligned}\hat{g}_{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}} &= \begin{pmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu(\kappa\lambda)} \\ \hat{g}_{(\rho\sigma)\nu} & \hat{g}_{(\rho\sigma)(\kappa\lambda)} \end{pmatrix}\\ &= \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \frac{1}{2}\left(g_{\rho\kappa}g_{\sigma\lambda} + g_{\rho\lambda}g_{\sigma\kappa}\right) \end{pmatrix}\end{aligned} \tag{6.59}$$

where we have suppressed the maps $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{H}^*$ in the second line and where $g_{\mu\nu}$ are the components of the first order metric. The inverse can be written as

$$\begin{aligned}\hat{g}^{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}} &= \begin{pmatrix} \hat{g}^{\mu\nu} & \hat{g}^{\mu(\kappa\lambda)} \\ \hat{g}^{(\rho\sigma)\nu} & \hat{g}^{(\rho\sigma)(\kappa\lambda)} \end{pmatrix}\\ &= \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \frac{1}{2}\left(g^{\rho\kappa}g^{\sigma\lambda} + g^{\rho\lambda}g^{\sigma\kappa}\right) \end{pmatrix}\end{aligned} \tag{6.60}$$

We can now raise and lower indices on covariant forms and contravariant vectors in the usual way

$$
\begin{aligned}
\hat{g}_{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}}\ \hat{v}^{\overset{\nu}{\kappa\lambda}} &= \hat{v}_{\underset{\rho\sigma}{\mu}}, \\
\hat{g}_{\mu\nu}\hat{v}^{\nu} &= \hat{v}_{\mu}, \\
\hat{g}_{(\rho\sigma)(\kappa\lambda)}\hat{v}^{\kappa\lambda} &= \hat{v}_{\rho\sigma}, \\
\hat{g}^{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}}\ \hat{\omega}_{\underset{\kappa\lambda}{\nu}} &= \hat{\omega}^{\overset{\mu}{\rho\sigma}}, \\
\hat{g}^{\mu\nu}\hat{\omega}_{\nu} &= \hat{\omega}^{\mu}, \\
\hat{g}^{(\rho\sigma)(\kappa\lambda)}\hat{\omega}_{\kappa\lambda} &= \hat{\omega}^{\rho\sigma},
\end{aligned} \tag{6.61}
$$

where we used the symmetry of $v^{\mu\nu}$ and $\omega_{\mu\nu}$. Finally we can express the second order metric components $\tilde{g}$ in terms of the first order metric:

$$
\begin{aligned}
\tilde{g}_{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}} &= \begin{pmatrix} \tilde{g}_{\mu\nu} & \tilde{g}_{\mu(\kappa\lambda)} \\ \tilde{g}_{(\rho\sigma)\nu} & \tilde{g}_{(\rho\sigma)(\kappa\lambda)} \end{pmatrix} \\
&= \begin{pmatrix} g_{\mu\nu} & g_{\mu\alpha}\,\Gamma^{\alpha}_{\kappa\lambda} \\ g_{\alpha\nu}\,\Gamma^{\alpha}_{\rho\sigma} & \frac{1}{2}\left(g_{\rho\kappa}g_{\sigma\lambda} + g_{\rho\lambda}g_{\sigma\kappa}\right) + g_{\alpha\beta}\,\Gamma^{\alpha}_{\rho\sigma}\,\Gamma^{\beta}_{\kappa\lambda} \end{pmatrix}
\end{aligned} \tag{6.62}
$$

Its inverse is given by

$$
\begin{aligned}
\tilde{g}^{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}} &= \begin{pmatrix} \tilde{g}^{\mu\nu} & \tilde{g}^{\mu(\kappa\lambda)} \\ \tilde{g}^{(\rho\sigma)\nu} & \tilde{g}^{(\rho\sigma)(\kappa\lambda)} \end{pmatrix} \\
&= \begin{pmatrix} g^{\mu\nu} + g^{\alpha\eta}g^{\beta\xi}\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\eta\xi} & -g^{\alpha\kappa}g^{\beta\lambda}\Gamma^{\mu}_{\alpha\beta} \\ -g^{\rho\alpha}g^{\sigma\beta}\Gamma^{\nu}_{\alpha\beta} & \frac{1}{2}\left(g^{\rho\kappa}g^{\sigma\lambda} + g^{\rho\lambda}g^{\sigma\kappa}\right) \end{pmatrix}
\end{aligned} \tag{6.63}
$$

6.2.5 k-forms

In this subsection, we extend the notion of k-forms to the second order geometry framework. As usual, we denote the bundle of covariant k-tensors by $T^k(T^*\mathcal{M})$ and the subbundle of alternating k-tensors by $\Lambda^k(T^*\mathcal{M})$. The rank of the latter bundle is $\binom{n}{k}$ and a k-form $\omega \in \Lambda^k(T^*\mathcal{M})$ can be written as

$$
\omega = \omega_{\mu_1 \ldots \mu_k}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \tag{6.64}
$$

where we assume $\mu_1 < \ldots < \mu_k$. Similarly, we construct a bundle of second order k-tensors $T^k(\tilde{T}^*\mathcal{M})$ and a subbundle $\Lambda^k(\tilde{T}^*\mathcal{M})$ of rank $\binom{N}{k}$ with $N = \frac{1}{2}n(n+3)$. A second order

k-form $\Omega \in \Lambda^k(T^*\mathcal{M})$ can be written as

$$
\begin{aligned}
\Omega = {} & \omega_{\binom{\mu}{\nu\rho}_1 \cdots \binom{\mu}{\nu\rho}_k} \, d_2 x^{\binom{\mu}{\nu\rho}_1} \wedge \ldots \wedge d_2 x^{\binom{\mu}{\nu\rho}_k} \\
= {} & \omega_{\mu_1 \ldots \mu_k} \, d_2 x^{\mu_1} \wedge \ldots \wedge d_2 x^{\mu_k} \\
& + \omega_{(\nu\rho)_1 \mu_2 \ldots \mu_k} \, dx^{\nu_1} \cdot dx^{\rho_1} \wedge d_2 x^{\mu_2} \wedge \ldots \wedge d_2 x^{\mu_k} \\
& \quad + \omega_{\mu_1 (\nu\rho)_2 \mu_3 \ldots \mu_k} \, d_2 x^{\mu_1} \wedge dx^{\nu_2} \cdot dx^{\rho_2} \wedge d_2 x^{\mu_3} \wedge \ldots \wedge d_2 x^{\mu_k} \\
& \quad + \ldots \\
& \quad + \omega_{\mu_1 \mu_2 \ldots \mu_{k-1} (\nu\rho)_k} \, d_2 x^{\mu_1} \wedge d_2 x^{\mu_2} \wedge \ldots \wedge d_2 x^{\mu_{k-1}} \wedge dx^{\nu_k} \cdot dx^{\rho_k} \\
& + \omega_{(\nu\rho)_1 (\nu\rho)_2 \mu_3 \ldots \mu_k} \, dx^{\nu_1} \cdot dx^{\rho_1} \wedge dx^{\nu_2} \cdot dx^{\rho_2} \wedge d_2 x^{\mu_3} \wedge \ldots \wedge d_2 x^{\mu_k} \\
& + \ldots \\
& + \omega_{(\nu\rho)_1 \ldots (\nu\rho)_k} \, dx^{\nu_1} \cdot dx^{\rho_1} \wedge \ldots \wedge dx^{\nu_k} \cdot dx^{\rho_k}.
\end{aligned}
\tag{6.65}
$$

6.2.6 Exterior derivatives

In this subsection, we extend the notion of the exterior derivative to the second order geometry framework. The first order exterior derivative is a map $d : \Lambda^k(T^*\mathcal{M}) \to \Lambda^{k+1}(T^*\mathcal{M})$ such that

$$
d\omega = \partial_\nu \omega_{\mu_1 \ldots \mu_k} \, dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}, \tag{6.66}
$$

which is linear:

$$
\begin{aligned}
d(\omega + \theta) &= d\omega + d\theta \qquad \forall\, \omega, \theta \in \Lambda^k(T^*\mathcal{M}), \\
d(c\,\omega) &= c\, d\omega \qquad \forall\, \omega \in \Lambda^k(T^*\mathcal{M}),\ c \in \mathbb{R};
\end{aligned}
\tag{6.67}
$$

satisfies the modified Leibniz rule:

$$
d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta \qquad \forall\, \omega \in \Lambda^k(T^*\mathcal{M}),\ \theta \in \Lambda^l(T^*\mathcal{M}); \tag{6.68}
$$

satisfies the closure condition

$$
d(d(\omega)) = 0 \qquad \forall \omega \in \Lambda^k(T^*\mathcal{M}); \tag{6.69}
$$

and commutes with pullbacks:

$$
\phi^*(d\omega) = d(\phi^*(\omega)) \qquad \forall\, \omega \in \Lambda^k(T^*\mathcal{M}),\ \phi \in C^\infty(\mathcal{M}, \mathbb{R}). \tag{6.70}
$$

Analogously we define a a second order exterior derivative $d_2 : \Lambda^k(\tilde{T}^*\mathcal{M}) \to \Lambda^{k+1}(\tilde{T}^*\mathcal{M})$ such that

$$
\begin{aligned}
d_2\, \Omega &= \partial^{\nu}_{\kappa\lambda}\, \omega_{\binom{\mu}{\rho\sigma}_1 \cdots \binom{\mu}{\rho\sigma}_k} \, d_2 x^{\nu}_{\kappa\lambda} \wedge d_2 x^{\binom{\mu}{\rho\sigma}_1} \wedge \ldots \wedge d_2 x^{\binom{\mu}{\rho\sigma}_k} \\
&= \partial_\nu\, \omega_{\binom{\mu}{\rho\sigma}_1 \cdots \binom{\mu}{\rho\sigma}_k} \, d_2 x^\nu \wedge d_2 x^{\binom{\mu}{\rho\sigma}_1} \wedge \ldots \wedge d_2 x^{\binom{\mu}{\rho\sigma}_k} \\
&\quad + \partial_\kappa \partial_\lambda\, \omega_{\binom{\mu}{\rho\sigma}_1 \cdots \binom{\mu}{\rho\sigma}_k} \, dx^\kappa \cdot dx^\lambda \wedge d_2 x^{\binom{\mu}{\rho\sigma}_1} \wedge \ldots \wedge d_2 x^{\binom{\mu}{\rho\sigma}_k}.
\end{aligned}
\tag{6.71}
$$

This second order exterior derivative is also linear and commutes withs pullbacks. Furthermore, it obeys the closure condition

$$d_2(d_2(\Omega)) = 0 \qquad \forall\, \Omega \in \Lambda^k(\tilde{T}^*\mathcal{M}); \tag{6.72}$$

and a new modified Leibniz rule

$$d_2(\Omega\wedge\Theta) = d_2\Omega\wedge\Theta+(-1)^k\Omega\wedge d_2\Theta+2d\Omega\cdot d\Theta \qquad \forall\, \Omega \in \Lambda^k(\tilde{T}^*\mathcal{M}),\ \Theta \in \Lambda^l(\tilde{T}^*\mathcal{M}), \tag{6.73}$$

where

$$d\Omega\cdot d\Theta = \partial_\alpha\, \omega_{\binom{\mu}{\rho\sigma}_1\cdots\binom{\mu}{\rho\sigma}_k}\, \partial_\beta\, \omega_{\binom{\nu}{\kappa\lambda}_1\cdots\binom{\nu}{\kappa\lambda}_l}\, dx^\alpha\cdot dx^\beta\wedge d_2x^{\binom{\mu}{\rho\sigma}_1}\wedge\ldots\wedge d_2x^{\binom{\mu}{\rho\sigma}_k}\wedge d_2x^{\binom{\nu}{\kappa\lambda}_1}\wedge\ldots\wedge d_2x^{\binom{\nu}{\kappa\lambda}_l}. \tag{6.74}$$

The proof for these properties is similar to the proof for the corresponding properties in first order geometry, and is therefore omitted.

6.2.7 Interior products

In this subsection, we extend the notion of the interior product to the second order geometry framework. The first order interior product is a map $\iota_v : \Lambda^k(T^*\mathcal{M}) \to \Lambda^{k-1}(T^*\mathcal{M})$ such that

$$\iota_v\, \omega = \sum_{l=1}^{k} (-1)^{l-1}\, v^{\mu_l}\, \omega_{\mu_1\ldots\mu_k}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{l-1}}\wedge dx^{\mu_{l+1}}\wedge\ldots\wedge dx^{\mu_k}. \tag{6.75}$$

This map is linear, commutes with pullbacks, satisfies the modified Leibniz rule and satisfies the anti-symmetry property

$$\{\iota_u, \iota_v\}\, \omega = 0. \tag{6.76}$$

Similarly, one can define a second order interior product $\iota_V : \Lambda^k(T^*\mathcal{M}) \to \Lambda^{k-1}(T^*\mathcal{M})$, such that

$$\iota_V\, \Omega = \sum_{l=1}^{k} (-1)^{l-1}\, v^{\binom{\mu}{\rho\sigma}_l}\, \omega_{\binom{\mu}{\rho\sigma}_1\cdots\binom{\mu}{\rho\sigma}_k}\, d_2x^{\binom{\mu}{\rho\sigma}_1}\wedge\ldots\wedge d_2x^{\binom{\mu}{\rho\sigma}_{l-1}}\wedge d_2x^{\binom{\mu}{\rho\sigma}_{l+1}}\wedge\ldots\wedge d_2x^{\binom{\mu}{\rho\sigma}_k}, \tag{6.77}$$

which satisfies the same properties with the modified Leibniz rule replaced by a new modified Leibniz rule as in previous subsection.

6.2.8 Lie derivatives

Using the results from previous subsections, we can extend the notion of a Lie derivative to the second order geometry framework. A family of diffeomorphisms $\phi_\lambda := \mathbb{R}\times\mathcal{M} \to \mathcal{M}$ satisfying the usual (semi-)group properties can be thought of as a vector field $v \in \mathfrak{X}(\mathcal{M})$ that generates a set of integral curves $\gamma_v : \mathbb{R} \to \mathcal{M}$ along the vector field. Along any such integral curve parametrized by λ, one can define the first order derivative of a function

$f \in C^\infty(\mathcal{M}, \mathbb{R})$ by

$$\frac{d}{d\lambda} f = \frac{dx^\mu}{d\lambda} \partial_\mu f = v^\mu \partial_\mu f = v\, f. \tag{6.78}$$

This derivative is equivalent to the Lie derivative along the vector field v

$$\mathcal{L}_v f = v\, f, \tag{6.79}$$

which can be generalized to a Lie derivative acting on vectors and forms given by

$$\begin{aligned} \mathcal{L}_v u &= [v, u], \\ \mathcal{L}_v \omega &= \{\iota_v, d\}\omega. \end{aligned} \tag{6.80}$$

In a local coordinate chart, these expressions can be written as

$$\mathcal{L}_v u^\mu = v^\nu \partial_\nu u^\mu - u^\nu \partial_\nu v^\mu, \tag{6.81}$$

$$\mathcal{L}_v \omega_\mu = v^\nu \partial_\nu \omega_\mu + (\partial_\mu v^\nu)\omega_\nu. \tag{6.82}$$

Furthermore, using the Leibniz rule one can construct Lie derivatives acting on arbitrary tensor fields.

We can analogously define a notion of a Lie derivative of second order tensors along a second order vector field $V \in \tilde{\mathfrak{X}}(\mathcal{M})$. As defining relations for derivatives of vectors $U \in \tilde{T}_x\mathcal{M}$ and forms $\Omega \in \tilde{T}^*_x\mathcal{M}$ we take

$$\begin{aligned} \mathcal{L}_V f &= V f, \\ \mathcal{L}_V U &= [V, U], \\ \mathcal{L}_V \Omega &= \{\iota_V, d_2\}\Omega. \end{aligned} \tag{6.83}$$

In order to make these expressions well defined, we impose

$$v^{\mu\sigma} \partial_\sigma u^{\nu\rho} = u^{\mu\sigma} \partial_\sigma v^{\nu\rho}, \tag{6.84}$$

$$\omega_{\mu\nu} = \partial_\mu \omega_\nu. \tag{6.85}$$

In order to satisfy the first condition, we impose $u^{\mu\nu} = k\, v^{\mu\nu}$ with $k \in \mathbb{R}$ and define $W \in T\mathcal{M} \subset \tilde{T}\mathcal{M}$ such that

$$W = kV - U = \begin{pmatrix} kv^\mu - u^\mu \\ 0 \end{pmatrix} \tag{6.86}$$

In local coordinates we then find

$$
\begin{aligned}
\mathcal{L}_V f &= \left(v^\sigma\partial_\sigma + v^{\sigma\kappa}\partial_\sigma\partial_\kappa\right) f,\\
\mathcal{L}_V\, U^\mu &= \left(v^\sigma\partial_\sigma + v^{\sigma\kappa}\partial_\sigma\partial_\kappa\right) u^\mu - u^\sigma\partial_\sigma v^\mu - u^{\sigma\kappa}\partial_\sigma\partial_\kappa v^\mu,\\
\mathcal{L}_V\, U^{\nu\rho} &= w^\sigma\partial_\sigma v^{\nu\rho} - v^{\nu\sigma}\partial_\sigma w^\rho - v^{\rho\sigma}\partial_\sigma w^\nu,\\
\mathcal{L}_V\, \Omega_\mu &= \left(v^\sigma\partial_\sigma + v^{\sigma\kappa}\partial_\sigma\partial_\kappa\right)\omega_\mu + \omega_\sigma\partial_\mu v^\sigma + \omega_{\sigma\kappa}\partial_\mu v^{\sigma\kappa},\\
\mathcal{L}_V\, \Omega_{\nu\rho} &= \left(v^\sigma\partial_\sigma + v^{\sigma\kappa}\partial_\sigma\partial_\kappa\right)\omega_{\nu\rho} + \omega_\sigma\partial_\nu\partial_\rho v^\sigma + \omega_{\sigma\kappa}\partial_\nu\partial_\rho v^{\sigma\kappa}\\
&\qquad + 2\partial_{(\nu}v^\sigma\partial_{\rho)}\omega_\sigma + 2\partial_{(\nu}v^{\sigma\kappa}\partial_{\rho)}\omega_{\sigma\kappa}.
\end{aligned}
\tag{6.87}
$$

or equivalently with respect to the covariant bases

$$
\begin{aligned}
\mathcal{L}_V f &= \left(\hat{v}^\sigma\nabla_\sigma + \hat{v}^{\sigma\kappa}\nabla_\sigma\nabla_\kappa\right) f,\\
\mathcal{L}_V\, \hat{U}^\mu &= \left(\hat{v}^\sigma\nabla_\sigma + \hat{v}^{\sigma\kappa}\nabla_\sigma\nabla_\kappa\right)\hat{u}^\mu - \hat{u}^\sigma\nabla_\sigma\hat{v}^\mu - \hat{u}^{\sigma\kappa}\nabla_\sigma\nabla_\kappa\hat{v}^\mu + \mathcal{R}^\mu{}_{\sigma\kappa\lambda}\hat{v}^{\sigma\kappa}\hat{w}^\lambda,\\
\mathcal{L}_V\, \hat{U}^{\nu\rho} &= \hat{w}^\sigma\nabla_\sigma\hat{v}^{\nu\rho} - \hat{v}^{\nu\sigma}\nabla_\sigma\hat{w}^\rho - \hat{v}^{\rho\sigma}\nabla_\sigma\hat{w}^\nu,\\
\mathcal{L}_V\, \hat{\Omega}_\mu &= \left(\hat{v}^\sigma\nabla_\sigma + \hat{v}^{\sigma\kappa}\nabla_\sigma\nabla_\kappa\right)\hat{\omega}_\mu + \hat{\omega}_\sigma\nabla_\mu\hat{v}^\sigma + \hat{\omega}_{\sigma\kappa}\nabla_\mu\hat{v}^{\sigma\kappa} + \mathcal{R}^\sigma{}_{\kappa\lambda\mu}\hat{v}^{\kappa\lambda}\hat{\omega}_\sigma,\\
\mathcal{L}_V\, \hat{\Omega}_{\nu\rho} &= \left(\hat{v}^\sigma\nabla_\sigma + \hat{v}^{\sigma\kappa}\nabla_\sigma\nabla_\kappa\right)\hat{\omega}_{\nu\rho} + \hat{\omega}_\sigma\nabla_{(\nu}\nabla_{\rho)}\hat{v}^\sigma + \hat{\omega}_{\sigma\kappa}\nabla_{(\nu}\nabla_{\rho)}\hat{v}^{\sigma\kappa} + 2\nabla_{(\nu|}\hat{v}^\sigma\nabla_{|\rho)}\hat{\omega}_\sigma\\
&\quad + 2\nabla_{(\nu|}\hat{v}^{\sigma\kappa}\nabla_{|\rho)}\hat{\omega}_{\sigma\kappa} - \mathcal{R}^\kappa{}_{(\nu\rho)\sigma}\hat{v}^\sigma\hat{\omega}_\kappa + 2\hat{v}^{\sigma\kappa}\left(\mathcal{R}^\lambda{}_{\sigma\kappa(\nu}\hat{\omega}_{\rho)\lambda} - \mathcal{R}^\lambda{}_{(\nu\rho)\sigma}\hat{\omega}_{\kappa\lambda}\right)\\
&\quad - \hat{v}^{\sigma\kappa}\hat{\omega}_\lambda\left(\nabla_\sigma\mathcal{R}^\lambda{}_{(\nu\rho)\kappa} + \nabla_{(\nu|}\mathcal{R}^\lambda{}_{\sigma|\rho)\kappa}\right).
\end{aligned}
\tag{6.88}
$$

The Lie derivatives for first order vectors and forms and along first order vector fields can easily be obtained from these formulae by taking the appropriate limit. Only the Lie derivative of a second order vector field along a first order vector field cannot be derived as a limit from these formulae. This one can be obtained by replacing $v^{\mu\nu} \to u^{\mu\nu}$ and $w^\mu \to v^\mu$ in the above formulae.

6.2.9 Parallel transport

In this subsection, we discuss the notion of parallel transport along second order vector fields. This notion is similar to the notion of stochastic parallel transport along semi-martingales as developed by Dohrn and Guerra [135, 136]. It is different from first order parallel transport, as the second order part of the vector fields generate geodesic deviation. Here, we closely follow the presentation of stochastic parallel transport by Nelson, cf. section 10 in Ref. [270].

Let $X(\tau)$ be a path in $\mathcal{M}$, passing through the points $x, y \in \mathcal{M}$ at times τ_1, τ_2. We will assume that there exists a convex coordinate chart (U, χ) such that $x, y \in U$. Moreover, let $V \in \tilde{T}_x\mathcal{M}$ be a second order tangent vector at x with $\hat{v} = \mathcal{F}(V)$ its contravariant first order projection, such that in $\chi(U)$ we have $y^\mu = x^\mu + \hat{v}^\mu$.

Let $d_2\hat{X}(\tau) \in \mathcal{F}(T\mathcal{M})$ be a transport and let $d_2\hat{x}^\mu = d_2\hat{X}(\tau_1)$ and $d_2\hat{y}^\mu = d_2\hat{X}(\tau_2)$ be its values when passing through x and y respectively. Then, using the standard notion of parallel transport, $d_2\hat{X}(\tau)$ is said to be a parallel transport, if

$$
d_2\hat{y}^\mu = d_2\hat{x}^\mu - \Gamma^\mu_{\rho\sigma}(x)\,\hat{v}^\rho\, d_2\hat{x}^\sigma. \tag{6.89}
$$

In order to extend this notion to second order vector fields, we define the difference vector

$$d_2\hat{v}^\mu := d_2 y^\mu - d_2 x^\mu. \tag{6.90}$$

Using the parallel transport equation (6.89), the relations

$$\begin{aligned} d_2\hat{x}^\mu &= d_2 x^\mu + \Gamma^\mu_{\rho\sigma}(x)\, d\hat{x}^\rho \cdot d\hat{x}^\sigma, \\ d_2\hat{y}^\mu &= d_2 y^\mu + \Gamma^\mu_{\rho\sigma}(y)\, d\hat{y}^\rho \cdot d\hat{y}^\sigma \end{aligned} \tag{6.91}$$

and the Taylor expansion

$$\Gamma^\mu_{\rho\sigma}(y) = \Gamma^\mu_{\rho\sigma}(x) + \partial_\nu \Gamma^\mu_{\rho\sigma}(x)\hat{v}^\nu + \mathcal{O}(\hat{v}^2), \tag{6.92}$$

we find

$$\begin{aligned} d_2\hat{v}^\mu &= -\Gamma^\mu_{\rho\sigma}\hat{v}^\rho d_2 x^\sigma - \left(\partial_\nu \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\nu\kappa}\Gamma^\kappa_{\rho\sigma} - 2\Gamma^\mu_{\rho\kappa}\Gamma^\kappa_{\nu\sigma}\right)\hat{v}^\nu\, dx^\rho \cdot dx^\sigma \\ &= -\Gamma^\mu_{\rho\sigma}\hat{v}^\rho d_2 \hat{x}^\sigma - \left(\partial_\nu \Gamma^\mu_{\rho\sigma} - 2\Gamma^\mu_{\rho\kappa}\Gamma^\kappa_{\nu\sigma}\right)\hat{v}^\nu\, d\hat{x}^\rho \cdot d\hat{x}^\sigma \end{aligned} \tag{6.93}$$

where $\Gamma^\mu_{\rho\sigma} = \Gamma^\mu_{\rho\sigma}(x)$. We will call this the *equation of second order parallel transport*. Notice that the equation of first order parallel transport is obtained if $d\hat{X} \in T\mathcal{M}$ is a first order transport and $V \in T\mathcal{M}$ is a first order vector, as this implies $dx^\rho \cdot dx^\sigma = 0$ and $\hat{v} = v$ respectively.

The equation of second order parallel transport is linear in $\hat{v}$ and has a solution of the form

$$\hat{v}^\mu(\tau_2) = P^\mu{}_\nu(\tau_2, \tau_1)\, \hat{v}^\nu(\tau_1), \tag{6.94}$$

where $P^\mu{}_\nu(\tau_2,\tau_1)$ is the *second order parallel propagator*. Using this propagator, we can define the *second order directional covariant derivative* $\hat{d}$ by

$$\begin{aligned} \hat{d}_2\hat{v}^\mu &= P^\mu{}_\nu(\tau_1,\tau_2)\, \hat{v}^\nu(\tau_2) - \hat{v}^\mu(\tau_1) \\ &= d_2\hat{v}^\mu + \Gamma^\mu_{\rho\sigma}\hat{v}^\rho d_2\hat{x}^\sigma + \left(\partial_\nu \Gamma^\mu_{\rho\sigma} - 2\Gamma^\mu_{\rho\kappa}\Gamma^\kappa_{\nu\sigma}\right)\hat{v}^\nu\, d\hat{x}^\rho \cdot d\hat{x}^\sigma. \end{aligned} \tag{6.95}$$

6.2.10 Embeddings into higher dimensions

As an aside, we discuss the relation between second order geometry and first order geometry on higher dimensional manifolds. One can embed a n-dimesional pseudo-Riemannian manifold with signature[17] $(d,1,0)$ into a N-dimensional pseudo-Riemannian manifold $\tilde{M}$ with signature[18] $(D,n,0)$ with $N = \frac{1}{2}n(n+3)$ and $D = \frac{1}{2}n(n+1)$. We can for example take the trivial embedding

$$\iota : \mathcal{M} \hookrightarrow \tilde{\mathcal{M}} \qquad \text{s.t.} \qquad \begin{cases} \iota^\alpha(x) = x^\alpha, & \text{if } \alpha \le d; \\ \iota^\alpha(x) = 0, & \text{if } \alpha > d. \end{cases} \tag{6.96}$$

[17] We denote the signature by (+,-,0). i.e. $(d,1,0)$ corresponds to a $(-+\ldots+)$ metric.

[18] More generally, if $\mathcal{M}$ has signature (k,l,m), then $\tilde{\mathcal{M}}$ has signature (K,L,M) with $K = \frac{1}{2}\left[k(k+3) + l(l+1)\right]$, $L = l(k+1)$ and $M = \frac{m}{2}(2k+2l+m+3)$.

The pushforward ι_* of this embedding defines for every $x \in \mathcal{M}$ a bijection between the second order tangent space $\tilde{T}_x\mathcal{M}$ and the first order tangent space $T_{\iota(x)}\tilde{\mathcal{M}}$. Additionally, the pullback ι^* defines a bijection between the cotangent spaces $\tilde{T}^*_x\mathcal{M}$ and $T^*_{\iota(x)}\tilde{\mathcal{M}}$. This bijection ι^* acts on the basis vectors as[19]

$$\begin{aligned} d_2x^\mu &\mapsto dx^\mu, \\ dx^\rho \cdot dx^\sigma &\mapsto dx^{n+\frac{1}{2}\rho(2n-\rho-1)+\sigma}. \end{aligned} \tag{6.97}$$

Moreover, this induces a bijection between the second order metric on $\mathcal{M}$ and the first order metric on $\tilde{\mathcal{M}}$:

$$\tilde{g}_{\binom{\mu}{\rho\sigma}\binom{\nu}{\kappa\lambda}} \mapsto \tilde{g}_{\alpha\beta} \tag{6.98}$$

with $\alpha, \beta \in \{0, 1, ..., N\}$. One can thus describe the second order geometry framework using the first order formalism on a N-dimensional manifold $\tilde{\mathcal{M}}$ instead of the original n-dimensional manifold $\mathcal{M}$. However, the support of functions defined on $\tilde{\mathcal{M}}$ must be restricted to the subspace $\mathcal{M} \subset \tilde{\mathcal{M}}$.

6.3 Manifold valued semi-martingales

In this section, we discuss stochastic motion on a manifold. Classically, a particle follows a trajectory or path on the manifold, that is parametrized by its proper time. In other words a trajectory is a map $\gamma : T \to \mathcal{M}$, where $T = [\tau_i, \tau_f] \subset \mathbb{R}$.

We make this notion stochastic by promoting the manifold to a measurable space $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, where $\mathcal{B}(\mathcal{M})$ is the Borel sigma algebra of $\mathcal{M}$. Furthermore, we introduce the probability space $(\Omega, \Sigma, \mathbb{P})$, and the random variable $X : (\Omega, \Sigma, \mathbb{P}) \to (\mathcal{M}, \mathcal{B}(\mathcal{M}))$. Given $T = [\tau_i, \tau_f] \subset \mathbb{R}$ we can introduce a filtration $\{\mathcal{P}_\tau\}_{\tau \in T}$, which is by definition an ordered set such that $\mathcal{P}_{\tau_i} \subseteq \mathcal{P}_s \subseteq \mathcal{P}_t \subseteq \Sigma \; \forall \, s < t \in T$. In addition, we assume the filtration to be right-continuous, i.e. $\mathcal{P}_\tau = \cap_{\epsilon > 0} \mathcal{P}_{\tau+\epsilon}$.

We can then introduce a stochastic process adapted to this filtration as a family of random variables $\{X(\tau) : \tau \in T\}$. We will restrict the set of stochastic processes to the continuous manifold valued semi-martingales. These are the continuous manifold valued stochastic processes $\{X(\tau)\}_{\tau \in T}$ such that $f(X)$ is a semi-martingale for every smooth function $f \in C^\infty(\mathcal{M}, \mathbb{R}^n)$. In particular, for a coordinate chart $\chi : U \to V$ with $U \subset \mathcal{M}$ and $V \subset \mathbb{R}^{d+1}$ the coordinates $X^\mu = \chi^\mu(X)$ are semi-martingales. A semi-martingale is a process $X(\tau)$ that can be decomposed as

$$X(\tau) = x_i + C_+(\tau) + W_+(\tau), \tag{6.99}$$

where $x_i := X(\tau_i)$ is the initial value of the process, $C_+(\tau)$ is a local càdlàg process with finite variation, such that $C_+(\tau_i) = 0$, and $W_+(\tau)$ is a local martingale process, such that $W_+(\tau_i) = 0$, satisfying the martingale property

$$\mathbb{E}_{t^+}[W_+(\tau)] := \mathbb{E}[W_+(\tau) | \{\mathcal{P}_s\}_{\tau_i \leq s \leq t}] = W_+(t) \qquad \forall \, t < \tau \in T. \tag{6.100}$$

[19]Notice that $\mu \in 0, 1, ..., d$, and that $\rho \leq \sigma$.

We will make the additional assumption that the time-reversed process is also a semi-martingale. Hence, we can construct a time reversed filtration $\{\mathcal{F}_\tau\}_{\tau\in T}$, which is a left-continuous and decreasing set of sigma algebras, i.e. $\mathcal{F}_\tau = \cap_{\epsilon>0}\mathcal{F}_{\tau-\epsilon}$ and $\mathcal{F}_{\tau_f} \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \Sigma \;\forall\, s > t \in T$. Moreover, X is adapted to this filtration and can be decomposed as

$$X(\tau) = x_f + C_-(\tau) + W_-(\tau), \tag{6.101}$$

where $X(\tau_f) = x_f$, $C_-(\tau_f) = 0$ and $W_-(\tau_f) = 0$. Furthermore, W_- satisfies the backward martingale property

$$\mathbb{E}_{t^-}[W_-(\tau)] := \mathbb{E}[W_-(\tau)|\{\mathcal{F}_s\}_{t\leq s\leq\tau_f}] = W_-(t) \qquad \forall\, t > \tau \in T. \tag{6.102}$$

For obvious reasons, we will call $\{\mathcal{P}_\tau\}_{\tau\in T}$ the past filtration and $\{\mathcal{F}_\tau\}_{\tau\in T}$ the future filtration. The intersection of the two $\mathfrak{P}_\tau = \mathcal{P}_\tau \cap \mathcal{F}_\tau$, will be called the present sigma algebra, and we denote conditional expectations with respect to this sigma algebra by

$$\mathbb{E}_t[X(\tau)] := \mathbb{E}[X(\tau)|\mathfrak{P}_t]. \tag{6.103}$$

Furthermore, we will assume Markovianness of both the forward and backward process, i.e.

$$\mathbb{E}_{t^+}[X(\tau)] = \mathbb{E}_t[X(\tau)] \qquad \text{and} \qquad \mathbb{E}_{t^-}[X(\tau)] = \mathbb{E}_t[X(\tau)]. \tag{6.104}$$

Finally, one can define a sample path for every $\omega \in \Omega$ as the set $\gamma(\omega) := \{X(\tau,\omega) : \tau \in T\}$. The measurable space of sample paths is the cylinder $\big(\mathcal{M}^T, \mathrm{Cyl}(\mathcal{M}^T)\big)$, where we take the cylinder sigma algebra on $\mathcal{M}^T$. This construction allows to interpret the stochastic process as a single random variable $\gamma : (\Omega, \Sigma, \mathbb{P}) \to \big(\mathcal{M}^T, \mathrm{Cyl}(\mathcal{M}^T)\big)$.

6.3.1 Time derivatives

Stochastic motions are not differentiable, and therefore the notion of velocity is not well defined. However, one can define the conditional velocities for the forward and backward process:

$$\begin{aligned} v_f^\mu\left[X(\tau),\tau\right] &:= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}_{\tau^+}\left[X^\mu(\tau+h) - X^\mu(\tau)\right], \\ v_b^\mu\left[X(\tau),\tau\right] &:= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}_{\tau^-}\left[X^\mu(\tau-h) - X^\mu(\tau)\right], \end{aligned} \tag{6.105}$$

Using these velocities, we can construct the compensators $C_\pm(\tau)$. These càdlàg processes are given by

$$\begin{aligned} C_+^\mu(\tau) &= \int_{\tau_i}^{\tau} v_f^\mu(X(s),s)\, ds, \\ C_-^\mu(\tau) &= \int_{\tau}^{\tau_f} v_b^\mu(X(s),s)\, ds. \end{aligned} \tag{6.106}$$

Since we are dealing with stochastic processes with non-zero quadratic variation, we can also define

$$
\begin{aligned}
v_f^{\mu\nu}\left[X(\tau),\tau\right] &:= \lim_{h\downarrow 0}\frac{1}{2h}\mathbb{E}_{\tau^+}\Big\{[X^\mu(\tau+h)-X^\mu(\tau)][X^\nu(\tau+h)-X^\nu(\tau)]\Big\},\\
v_b^{\mu\nu}\left[X(\tau),\tau\right] &:= \lim_{h\downarrow 0}\frac{1}{2h}\mathbb{E}_{\tau^-}\Big\{[X^\mu(\tau-h)-X^\mu(\tau)][X^\nu(\tau-h)-X^\nu(\tau)]\Big\}.
\end{aligned}
\tag{6.107}
$$

This can be used to construct the compensator[20] $C^{\mu\nu}(\tau)$ of the quadratic variation process $[[X^\mu, X^\nu]]$, which is given by

$$
\begin{aligned}
C_+^{\mu\nu}(\tau) &= 2\int_{\tau_i}^{\tau} v_f^{\mu\nu}(X(s),s)ds,\\
C_-^{\mu\nu}(\tau) &= 2\int_{\tau}^{\tau_f} v_b^{\mu\nu}(X(s),s)ds.
\end{aligned}
\tag{6.108}
$$

In practice, we choose the direction of time. We will therefore introduce a slightly modified notion of velocity and define a *forward velocity* and *backward velocity* by

$$
\begin{aligned}
v_+(X,\tau) &= v_f(X,\tau),\\
v_-(X,\tau) &= -v_b(X,\tau).
\end{aligned}
\tag{6.109}
$$

Using the Markov property, these velocities can be defined by[21]

$$
\begin{aligned}
v_+^{\mu}\left[X(\tau),\tau\right] &:= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}_{\tau}\left[X^\mu(\tau+h)-X^\mu(\tau)\right],\\
v_-^{\mu}\left[X(\tau),\tau\right] &:= \lim_{h\uparrow 0}\frac{1}{h}\mathbb{E}_{\tau}\left[X^\mu(\tau+h)-X^\mu(\tau)\right],
\end{aligned}
\tag{6.110}
$$

and

$$
\begin{aligned}
v_+^{\mu\nu}\left[X(\tau),\tau\right] &:= \lim_{h\downarrow 0}\frac{1}{2h}\mathbb{E}_{\tau^+}\Big\{[X^\mu(\tau+h)-X^\mu(\tau)][X^\nu(\tau+h)-X^\nu(\tau)]\Big\},\\
v_-^{\mu\nu}\left[X(\tau),\tau\right] &:= \lim_{h\uparrow 0}\frac{1}{2h}\mathbb{E}_{\tau^-}\Big\{[X^\mu(\tau+h)-X^\mu(\tau)][X^\nu(\tau+h)-X^\nu(\tau)]\Big\}.
\end{aligned}
\tag{6.111}
$$

Reversibility of the process imposes

$$
v_b^{\mu\nu}(\tau) = v_f^{\mu\nu}(\tau),
\tag{6.112}
$$

and therefore

$$
v_+^{\mu\nu}(\tau) = -v_-^{\mu\nu}(\tau).
\tag{6.113}
$$

Moreover, the background hypothesis imposes

$$
[[X_\mu, X^\nu]](\tau) = \frac{\hbar}{m}\,\delta_\mu^\nu\,\tau.
\tag{6.114}
$$

[20]The compensator of the quadratic variation process is often denoted by the angle bracket $\langle X^\mu, X^\nu\rangle$. We will use $C^{\mu\nu}(\tau)$ instead to avoid confusion with the duality pairing.

[21]Note that the backward velocity can equivalently be defined as $v_-^\mu[X(\tau),\tau] := \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}_\tau[X^\mu(\tau) - X^\mu(\tau-h)]$.

Hence,

$$d[[X_\mu, X^\nu]] = \frac{\hbar}{m}\,\delta_\mu^\nu\,d\tau. \tag{6.115}$$

Consequently,

$$\begin{aligned} v_+^{\mu\nu}[X(\tau),\tau] &= \frac{1}{2\,d\tau}\mathbb{E}_\tau\Big[g^{\mu\rho}(X(\tau))\,d[[X_\rho(\tau), X^\nu(\tau)]]\Big] \\ &= \frac{\hbar}{2m}g^{\mu\nu}(X(\tau)). \end{aligned} \tag{6.116}$$

$v_\pm[X(\tau),\tau]$ has the structure of a second order vector, i.e. $v_\pm(x) \in \tilde{T}_x\mathcal{M}$. If the metric is fixed[22], the second order parts $v_\pm^{\mu\nu}(x)$ are also fixed. The vectors then live in n-dimensional subspaces $v_\pm^\mu \in T_x^\pm\mathcal{M} \subset \tilde{T}_x\mathcal{M}$. Since these slices are not invariant under coordinate transformations, we will consider $(\hat{v}_+, \hat{v}_-) \in \hat{T}_x^+\mathcal{M} \oplus \hat{T}_x^-\mathcal{M}$ instead.

Finally, we define a *current velocity* by

$$v := \frac{1}{2}\left(v_+ + v_-\right) \tag{6.117}$$

and an *osmotic velocity* by

$$u := \frac{1}{2}\left(v_+ - v_-\right). \tag{6.118}$$

Notice that $v \in T_x\mathcal{M}$ is a first order vector, while $u \in \tilde{T}_x\mathcal{M}$ has the structure of a second order vector.

6.3.2 Diffeomorphism invariance

In classical physics, one imposes a theory to be invariant under diffeomorphisms: general relativity should be invariant under the action of any diffeomorphism $\phi \in C^\infty(\mathcal{M},\mathcal{N})$. The diffeomorphism ϕ induces associated maps on the tangent and cotangent spaces, which are the pullback $\phi^* : T_y^*\mathcal{N} \to T_x^*\mathcal{M}$ and the pushforward $\phi_* : T_x\mathcal{M} \to T_y\mathcal{N}$, where $y = \phi(x)$. The tangent space and cotangent space are invariant under respectively the pullback and the pushforward.

In quantum physics, we would like to impose the same invariance under diffeomorphisms. However, it is not immediately clear that the n-dimensional tangent subspace $\hat{T}_x\mathcal{M} \subset \tilde{T}_x\mathcal{M}$ and cotangent subspace $\hat{T}_x^*\mathcal{M} \subset \tilde{T}_x^*\mathcal{M}$ with fixed second order parts are invariant spaces under the the pullback $\tilde{\phi}^* : \tilde{T}_y^*\mathcal{N} \to \tilde{T}_x^*\mathcal{M}$ and pushforward $\tilde{\phi}_* : \tilde{T}_x\mathcal{M} \to \tilde{T}_y\mathcal{N}$ of a diffeomorphism ϕ. In order to establish this invariance, we require the notion of a Schwartz morphism:[23]

Definition. *Given two manifolds $\mathcal{M}$,$\mathcal{N}$ and points $x \in \mathcal{M}$, $y \in \mathcal{N}$, a linear mapping $f : \tilde{T}_x\mathcal{M} \to \tilde{T}_y\mathcal{N}$ is called a* Schwartz morphism, *if*

1. $f(T_x\mathcal{M}) \subset T_y\mathcal{N}$,

2. $\forall\, L \in \tilde{T}_x\mathcal{M},\ \mathcal{H}^*(f(L)) = (f^\circ \otimes f^\circ)\mathcal{H}^*(L)$,

[22] In this paper, we only consider test particles in a fixed geometry.
[23] cf. Definition 6.22 in Ref. [150].

where f° is the restriction of f to $T_x\mathcal{M}$.

A Schwartz morphism is thus a morphism that leaves the slices $\hat{T}_x\mathcal{M}$ invariant. Furthermore, it can be shown[24] that a mapping $f : \tilde{T}_x\mathcal{M} \to \tilde{T}_y\mathcal{N}$ is a Schwartz morphism if and only if $f = \tilde{T}_x\phi$ for a smooth $\phi : \mathcal{M} \to \mathcal{N}$ with $\phi(x) = y$. It immediately follows that the pushforward $\tilde{\phi}_*$ of a diffeomorphism ϕ is a Schwartz morphism. Therefore, all slices $\hat{T}\mathcal{M} \subset \tilde{T}\mathcal{M}$ are invariant under the pushforward $\tilde{\phi}_* : \tilde{T}_x\mathcal{M} \to \tilde{T}_{\phi(x)}\mathcal{N}$ induced by a diffeomorphism $\phi : \mathcal{M} \to \mathcal{N}$. Moreover, all slices $\hat{T}^*\mathcal{M} \subset \tilde{T}^*\mathcal{M}$ are invariant under the pullback $\tilde{\phi}^* : \tilde{T}^*_{\phi(x)}\mathcal{N} \to \tilde{T}^*_x\mathcal{M}$ of the diffeomorphism ϕ. We note that this invariance is a consequence of the construction of the 'covariant slices' $\hat{T}_x\mathcal{M}$.

6.4 Integration along semi-martingales

In the previous sections, we have introduced manifold valued semi-martingales and second order geometry. This allows us to construct a notion of integration along semi-martingales on manifolds. This section is loosely based on the review by Emery [150]. For mathematical detail we refer to this work by Emery [150] or the original works by Schwartz [311] and Meyer [257].

In first order geometry, one defines integrals using forms $\omega \in T^*\mathcal{M}$. The integral of a form along a curve $\gamma : I \to \mathcal{M}$ with $I \subset \mathbb{R}$ is given by

$$\int_\gamma : T^*\mathcal{M} \to \mathbb{R} \qquad \text{s.t.} \qquad \omega \mapsto \int_\gamma \omega(x), \tag{6.119}$$

which can be written as

$$\int_\gamma \omega = \int_{\tau_i}^{\tau_f} \omega_\mu \, d\gamma^\mu = \int_{\tau_i}^{\tau_f} \omega_\mu \dot{\gamma}^\mu \, d\tau, \tag{6.120}$$

where $d\gamma = \gamma^*(\omega)$. If we assume that the form can be written as a differential form $\omega = dF$ for a function $F \in C^\infty(\mathcal{M}, \mathbb{R})$ we find

$$\int_\gamma dF(x) = \int_{\tau_i}^{\tau_f} \partial_\mu F(\gamma) \, d\gamma^\mu = \int_{\tau_i}^{\tau_f} \partial_\mu F(\gamma) \dot{\gamma}^\mu \, d\tau. \tag{6.121}$$

Moreover, the fundamental theorem for line integrals states

$$\int_\gamma dF(x) = F[\gamma(\tau_f)] - F[\gamma(\tau_i)]. \tag{6.122}$$

One can analogously construct an integral of second order forms $\Omega \in \tilde{T}^*\mathcal{M}$. The integral of a second order form along a semi-martingale X can be written as

$$\int_X : \tilde{T}^*\mathcal{M} \to \mathbb{R} \qquad \text{s.t.} \qquad \Omega \mapsto \int_X \Omega(x) \tag{6.123}$$

[24] cf. Exercise 6.23 in Ref. [150]

with

$$\begin{aligned}\int_X \Omega &= \int_{\tau_i}^{\tau_f} \omega_\mu \, d_2X^\mu + \int_{\tau_i}^{\tau_f} \omega_{\mu\nu} \, dX^\mu \cdot dX^\nu \\ &= \int_{\tau_i}^{\tau_f} \hat{\omega}_\mu \, d_2\hat{X}^\mu + \int_{\tau_i}^{\tau_f} \hat{\omega}_{\mu\nu} \, d\hat{X}^\mu \cdot d\hat{X}^\nu, \end{aligned} \tag{6.124}$$

where $(d_2X \quad dXdX) = X^*(\Omega)$. If we assume that the form can be written as a differential form $\Omega = d_2F$ for a function $F \in C^\infty(\mathcal{M}, \mathbb{R})$, we find

$$\begin{aligned}\int_X d_2F(x) &= \int_{\tau_i}^{\tau_f} \partial_\mu F(X) \, d_2X^\mu + \int_{\tau_i}^{\tau_f} \partial_\mu \partial_\nu F(X) \, dX^\mu \cdot dX^\nu \\ &= \int_{\tau_i}^{\tau_f} \nabla_\mu F(X) \, d_2\hat{X}^\mu + \int_{\tau_i}^{\tau_f} \nabla_\mu \nabla_\nu F(X) \, d\hat{X}^\mu \cdot d\hat{X}^\nu. \end{aligned} \tag{6.125}$$

The fundamental theorem for line integrals can be extended to the second order context, such that[25]

$$\int_X d_2F(x) = F[X(\tau_f)] - F[X(\tau_i)]. \tag{6.126}$$

Moreover, one can relate the second order integral to first order order integrals. For this we consider a form $\omega \in T\mathcal{M} \subset \tilde{T}\mathcal{M}$. We can then construct two second order integrals, that are manifestly invariant under coordinate transformations, using the maps $\underline{d}$ and $\mathcal{G}$ respectively:

$$\begin{aligned}⨍_X \omega &= \int_X \underline{d}\omega \\ &= \int_{\tau_i}^{\tau_f} \omega_\mu \, d_2X^\mu + \int_{\tau_i}^{\tau_f} \partial_\nu \omega_\mu \, dX^\mu \cdot dX^\nu \\ &= \int_{\tau_i}^{\tau_f} \hat{\omega}_\mu \, d_2\hat{X}^\mu + \int_{\tau_i}^{\tau_f} \nabla_\nu \omega_\mu \, d\hat{X}^\mu \cdot d\hat{X}^\nu \end{aligned} \tag{6.127}$$

and

$$\begin{aligned}\underline{\int}_X \omega &= \int_X \mathcal{G}(\omega) \\ &= \int_{\tau_i}^{\tau_f} \omega_\mu d_2X^\mu + \int_{\tau_i}^{\tau_f} \omega_\mu \Gamma^\mu_{\nu\rho} \, dX^\nu \cdot dX^\rho \\ &= \int_{\tau_i}^{\tau_f} \hat{\omega}_\mu \, d_2\hat{X}^\mu. \end{aligned} \tag{6.128}$$

The first of these integrals is a Stratonovich integral,[26] while the second is an Itô integral.[27] We immediately find a relation between the two

$$⨍_X \omega = \underline{\int}_X \omega + \int_{\tau_i}^{\tau_f} \nabla_\nu \hat{\omega}_\mu \, d\hat{X}^\mu \cdot d\hat{X}^\nu. \tag{6.129}$$

In order to evaluate the integral over the second order part we use that the integral over

[25] cf. Theorem 6.24 in Ref. [150].
[26] cf. Definition 7.3 and Proposition 7.4 in Ref. [150].
[27] cf. Definition 7.33 and Proposition 7.34 in Ref. [150].

a bilinear form is given by[28]

$$\int_{\tau_i}^{\tau_f} f_{\mu\nu}(X,\tau)\, dX^\mu \otimes dX^\nu = \int_{\tau_i}^{\tau_f} f_{\mu\nu}(X,\tau)\, d[[X^\mu, X^\nu]]. \tag{6.130}$$

Using the map $\mathcal{H}$ one can then map the integral over the second order part to an integral over a bilinear form. This yields[29]

$$\int_{\tau_i}^{\tau_f} f_{\mu\nu}(X,\tau)\, dX^\mu \cdot dX^\nu = \frac{1}{2}\int_{\tau_i}^{\tau_f} f_{\mu\nu}(X,\tau)\, d[[X^\mu, X^\nu]] = \int_{\tau_i}^{\tau_f} f_{\mu\nu}(X,\tau)\, v^{\mu\nu}(X,\tau)\, d\tau. \tag{6.131}$$

Moreover, if ω can be written as a differential form $\omega = dF$, the two first order integrals can be written as[30]

$$\begin{aligned} \int_X d_2F(x) &= \fint_{\tau_i}^{\tau_f} \partial_\mu F(X)\, dX^\mu, \\ \int_X d_2F(x) &= \underline{\int}_{\tau_i}^{\tau_f} \nabla_\mu F(X)\, d_+\hat{X}^\mu + \int_{\tau_i}^{\tau_f} \nabla_\mu\nabla_\nu F(X)\, d\hat{X}^\mu \cdot d\hat{X}^\nu. \end{aligned} \tag{6.132}$$

Using the decomposition of the semi-martingale, we can then write

$$\int_X d_2F(x) = \int_{\tau_i}^{\tau_f} v^\mu(X,\tau)\partial_\mu F(X)\, d\tau + \fint_{\tau_i}^{\tau_f} \partial_\mu F(X)\, dW^\mu, \tag{6.133}$$

$$\int_X d_2F(x) = \int_{\tau_i}^{\tau_f} \hat{v}_+^\mu(X,\tau)\nabla_\mu F(X)\, d\tau + \underline{\int}_{\tau_i}^{\tau_f} \nabla_\mu F(X)\, dW_+^\mu + \int_{\tau_i}^{\tau_f} \hat{v}_+^{\mu\nu}(X,\tau)\nabla_\mu\nabla_\nu F(X)\, d\tau.$$

Notice that all integrals are manifestly invariant under coordinate transformations. Furthermore, the Itô integral is a local martingale, i.e.

$$\mathbb{E}_{\tau_i^+}\left[\underline{\int}_{\tau_i}^{\tau} \nabla_\mu F(X)\, dW_+^\mu\right] = 0. \tag{6.134}$$

In addition, we will construct a backward Itô integral such that

$$\begin{aligned} \int_X d_2F(x) &= \overline{\int}_{\tau_i}^{\tau_f} \nabla_\mu F(X)\, d_-\hat{X}^\mu - \int_{\tau_i}^{\tau_f} \nabla_\mu\nabla_\nu F(X)\, d\hat{X}^\mu \cdot d\hat{X}^\nu \\ &= \int_{\tau_i}^{\tau_f} \hat{v}_-^\mu(X,\tau)\nabla_\mu F(X)\, d\tau + \overline{\int}_{\tau_i}^{\tau_f} \nabla_\mu F(X)\, dW_-^\mu + \int_{\tau_i}^{\tau_f} \hat{v}_-^{\mu\nu}(X,\tau)\nabla_\mu\nabla_\nu F(X)\, d\tau. \end{aligned} \tag{6.135}$$

The backward integral is a local backward martingale, i.e.

$$\mathbb{E}_{\tau_f^-}\left[\overline{\int}_{\tau}^{\tau_f} \nabla_\mu F(X)\, dW_-^\mu\right] = 0. \tag{6.136}$$

[28] cf. Theorem 3.8 in Ref. [150].

[29] cf. Proposition 6.31 in Ref. [150].

[30] We use the notation $\underline{\int} f_\mu(X)\, d_+\hat{X}^\mu$ instead of $\underline{\int} f_\mu(X)\, dX^\mu$ to make the covariance of the expression explicit.

We note that the three integrals are related by

$$\fint_X dF(x) = \frac{1}{2}\left(\underline{\int}_X dF(x) + \overline{\int}_X dF(x)\right). \tag{6.137}$$

Let us now relate the Stratonovich and Itô integral to their well known definitions in $\mathbb{R}^n$. If there exists a coordinate chart $\chi : U \to \mathbb{R}^n$ such that $f([\tau_i, \tau_f]) \subset U$, we have[31]

$$\begin{aligned}
\fint_{\tau_i}^{\tau_f} f_\mu(X,\tau)\, dX^\mu &:= \lim_{k\to\infty} \sum_{[\tau_j,\tau_{j+1}]\in\pi_k} \frac{1}{2}\Big[f_\mu\big(X(\tau_j),\tau_j\big) + f_\mu\big(X(\tau_{j+1}),\tau_{j+1}\big)\Big] \\
&\qquad\qquad \times \Big[X^\mu(\tau_{j+1}) - X^\mu(\tau_j)\Big], \\
\underline{\int}_{\tau_i}^{\tau_f} f_\mu(X,\tau)\, d_+X^\mu &:= \lim_{k\to\infty} \sum_{[\tau_j,\tau_{j+1}]\in\pi_k} f_\mu\big(X(\tau_j),\tau_j\big)\Big[X^\mu(\tau_{j+1}) - X^\mu(\tau_j)\Big], \\
\overline{\int}_{\tau_i}^{\tau_f} f_\mu(X,\tau)\, d_-X^\mu &:= \lim_{k\to\infty} \sum_{[\tau_j,\tau_{j+1}]\in\pi_k} f_\mu\big(X(\tau_{j+1}),\tau_{j+1}\big)\Big[X^\mu(\tau_{j+1}) - X^\mu(\tau_j)\Big], \\
\int_{\tau_i}^{\tau_f} f_{\mu\nu}(X,\tau)\, d[[X^\mu, X^\nu]] &:= \lim_{k\to\infty} \sum_{[\tau_j,\tau_{j+1}]\in\pi_k} f_{\mu\nu}\big(X(\tau_j),\tau_j\big)\Big[X^\mu(\tau_{j+1}) - X^\mu(\tau_j)\Big] \\
&\qquad\qquad \times \Big[X^\nu(\tau_{j+1}) - X^\nu(\tau_j)\Big],
\end{aligned} \tag{6.138}$$

where π_k is a partition of $[\tau_i, \tau_f]$, $f_\mu = (\chi \circ f)_\mu$ and $X^\mu = (\chi \circ X)^\mu$. We thus have

$$\fint_{\tau_i}^{\tau_f} f_\mu(X,\tau)\, dX^\mu = \frac{1}{2}\left(\underline{\int}_{\tau_i}^{\tau_f} f_\mu(X,\tau)\, d_+X^\mu + \overline{\int}_{\tau_i}^{\tau_f} f_\mu(X,\tau)\, d_-X^\mu \right), \tag{6.139}$$

and we will define an osmotic integral by

$$\fint_{\tau_i}^{\tau_f} f_\mu(X,\tau)\, d_\circ X^\mu := \frac{1}{2}\left(\underline{\int}_{\tau_i}^{\tau_f} f_\mu(X,\tau)\, d_+X^\mu - \overline{\int}_{\tau_i}^{\tau_f} f_\mu(X,\tau)\, d_-X^\mu \right). \tag{6.140}$$

6.4.1 Integration by parts

In this subsection, we state two integration by parts formulae, that will be useful for stochastic variational calculus. The first is given by

$$\begin{aligned}
\int_{\tau_i}^{\tau_f} d\left[f_\mu(\tau) g^\mu(\tau)\right] &= f_\mu(\tau_f)\, g^\mu(\tau_f) - f_\mu(\tau_i)\, g^\mu(\tau_i) \\
&= \fint_{\tau_i}^{\tau_f} f_\mu(\tau)\, dg^\mu(\tau) + \fint_{\tau_i}^{\tau_f} g^\mu(\tau)\, df_\mu(\tau) \\
&= \underline{\int}_{\tau_i}^{\tau_f} f_\mu(\tau)\, d_+g^\mu(\tau) + \underline{\int}_{\tau_i}^{\tau_f} g^\mu(\tau)\, d_+f_\mu(\tau) + 2\int_{\tau_i}^{\tau_f} df_\mu(\tau)\cdot dg^\mu(\tau) \\
&= \overline{\int}_{\tau_i}^{\tau_f} f_\mu(\tau)\, d_-g^\mu(\tau) + \overline{\int}_{\tau_i}^{\tau_f} g^\mu(\tau)\, d_-f_\mu(\tau) - 2\int_{\tau_i}^{\tau_f} df_\mu(\tau)\cdot dg^\mu(\tau),
\end{aligned} \tag{6.141}$$

[31] This is a consequence of Theorem 7.14 and Theorem 7.37 in Ref. [150].

where we write $f_\mu(\tau) = f_\mu(X(\tau), \tau)$, $g^\mu(\tau) = g^\mu(X(\tau), \tau)$. We immediately find

$$\int f_\mu(\tau)\, d_\circ g^\mu(\tau) + \int g^\mu(\tau)\, d_\circ f_\mu(\tau) = -2 \int df_\mu(\tau) \cdot dg^\mu(\tau), \tag{6.142}$$

where we recall

$$\int df_\mu(\tau) \cdot dg^\mu(\tau) = \frac{1}{2} \int d[[f_\mu, g^\mu]](\tau). \tag{6.143}$$

There exists another integration by parts formula, which can be derived from eq. (6.138) and is given by[32]

$$\begin{aligned} \int_{\tau_i}^{\tau_f} d\left[f_\mu(\tau) g^\mu(\tau)\right] &= \underline{\int} f_\mu(\tau)\, d_+ g^\mu(\tau) + \overline{\int} g^\mu(\tau)\, d_- f_\mu(\tau) \\ &= \overline{\int} f_\mu(\tau)\, d_- g^\mu(\tau) + \underline{\int} g^\mu(\tau)\, d_+ f_\mu(\tau). \end{aligned} \tag{6.144}$$

Combining eqs. (6.142) and (6.144) then yields

$$\int f_\mu(\tau)\, d_\circ g^\mu(\tau) = \int g^\mu(\tau)\, d_\circ f_\mu(\tau) = - \int df_\mu(\tau) \cdot dg^\mu(\tau). \tag{6.145}$$

6.5 Stochastic variational calculus

In this section, we discuss stochastic variational calculus as developed by Yasue [362–364]. We will consider the tangent bundle

$$\hat{T}\mathcal{M} = \bigsqcup_{x \in \mathcal{M}} \left(\hat{T}_x^+ \mathcal{M} \oplus \hat{T}_x^- \mathcal{M} \right), \tag{6.146}$$

which can be endowed with a $(3n)$-dimensional manifold structure with coordinates $(x^\mu, v_+^\mu, v_-^\mu)$. We define the Lagrangian as a map

$$L : \hat{T}\mathcal{M} \to \mathbb{R}, \tag{6.147}$$

and the action as the integral

$$S = \mathbb{E}\left[\int_{\tau_i}^{\tau_f} L(X, V_+, V_-)\, d\tau \right]. \tag{6.148}$$

Equivalently the action can be expressed as a function of the processes X, $V(V_+, V_-)$ and $U(V_+, V_-)$, which we will use later on. We emphasize that $V_\pm(\tau)$ are processes on the tangent bundle, while $v_\pm(X, \tau)$ are second order vector fields. The two are related as follows

$$\begin{aligned} \lim_{s \to \tau} \mathbb{E}_\tau \left[V_+^\mu(s) \right] &= v_+^\mu(X, \tau), \\ \lim_{s \to \tau} \mathbb{E}_\tau \left[V_-^\mu(s) \right] &= v_-^\mu(X, \tau). \end{aligned} \tag{6.149}$$

[32] See also e.g. Refs. [270, 364] for a derivation of this formula

As we intend to do variational calculus, we require the notion of a norm on the space of manifold valued time-reversible semi-martingales. In order to construct such a norm, we would like to split the space of all processes into spaces of time-like, space-like, and null-like processes. For this, we need to define the notion of a time-like process. We will call the process $X = X(\tau)$ *time-like*, if

$$g_{\mu\nu}(X)\, v^{\mu}(X,\tau)\, v^{\nu}(X,\tau) < 0 \qquad \forall\, \tau \in T. \tag{6.150}$$

Moreover, we call the process *space-like*, if

$$g_{\mu\nu}(X)\, v^{\mu}(X,\tau)\, v^{\nu}(X,\tau) > 0 \qquad \forall\, \tau \in T, \tag{6.151}$$

and *light-like* or *null-like*, if

$$g_{\mu\nu}(X)\, v^{\mu}(X,\tau)\, v^{\nu}(X,\tau) = 0 \qquad \forall\, \tau \in T. \tag{6.152}$$

Note that sample paths of a time-like process are not necessarily time-like. Indeed, for a time-like process we have

$$\mathbb{E}\Big[g_{\mu\nu}(X(\tau))\, dX^{\mu}(\tau) \otimes dX^{\nu}(\tau)\Big] < 0 \qquad \forall\, \tau \in T. \tag{6.153}$$

However, this relation does not hold without the expectation value. Therefore, sample paths can contain segments that are not time-like. A similar remark holds for space-like and light-like processes.

We will now restrict the semi-martingales on $\mathcal{M}$ to those that are time-like. After a Wick rotation, the space of these time-like processes can be equipped with the L^2-norm

$$||X|| = \sqrt{\mathbb{E}\left[\int \big|X_{\mu}(\tau) X^{\mu}(\tau)\big|\, d\tau\right]}, \tag{6.154}$$

which is the conventional choice in quantum mechanics.

6.5.1 Euler-Lagrange equations

The stochastic Euler Lagrange equations can be derived similar to the classical Euler-Lagrange equations. We vary the action with respect to a semi-martingale δX independent of X that satisfies

$$\delta X(\tau_i) = \delta X(\tau_f) = 0. \tag{6.155}$$

This leads to

$$
\begin{aligned}
\delta S(X) &:= S(X+\delta X) - S(X)\\
&= \mathbb{E}\left[\int_{\tau_i}^{\tau_f} L\left(X+\delta X, V_+ + \delta V_+, V_- + \delta V_-\right)\, d\tau\right] - \mathbb{E}\left[\int_{\tau_i}^{\tau_f} L\left(X, V_+, V_-\right)\, d\tau\right]\\
&= \mathbb{E}\left[\int_{\tau_i}^{\tau_f} \left\{\frac{\partial L(X,V_+,V_-)}{\partial X^\mu}\delta X^\mu + \frac{\partial L(X,V_+,V_-)}{\partial V_+^\mu}\delta V_+^\mu\right.\right.\\
&\qquad\qquad\left.\left. + \frac{\partial L(X,V_+,V_-)}{\partial V_-^\mu}\delta V_-^\mu\right\} d\tau\right] + \mathcal{O}(||\delta X||^2)\\
&= \mathbb{E}\left[\int_{\tau_i}^{\tau_f} \left\{\frac{\partial L(X,V_+,V_-)}{\partial X^\mu}\delta X^\mu d\tau + \frac{\partial L(X,V_+,V_-)}{\partial V_+^\mu} d_+\delta X^\mu\right.\right.\\
&\qquad\qquad\left.\left. + \frac{\partial L(X,V_+,V_-)}{\partial V_-^\mu} d_-\delta X^\mu\right\}\right] + \mathcal{O}(||\delta X||^2)\\
&= \mathbb{E}\left[\int_{\tau_i}^{\tau_f} \delta X^\mu \left\{\frac{\partial L(X,V_+,V_-)}{\partial X^\mu} d\tau - d_- \frac{\partial L(X,V_+,V_-)}{\partial V_+^\mu}\right.\right.\\
&\qquad\qquad\left.\left. - d_+ \frac{\partial L(X,V_+,V_-)}{\partial V_-^\mu}\right\}\right] + \mathcal{O}(||\delta X||^2),
\end{aligned} \tag{6.156}
$$

where we used the partial integration formula (6.144). We find a system of stochastic differential equations given by

$$
\int_{\tau_i}^{\tau_f} \frac{\partial}{\partial X^\mu} L(X,V_+,V_-) d\tau = \int_{\tau_i}^{\tau_f} \left\{ d_- \frac{\partial}{\partial V_+^\mu} L(X,V_+,V_-) + d_+ \frac{\partial}{\partial V_-^\mu} L(X,V_+,V_-)\right\} \tag{6.157}
$$

or equivalently

$$
\int_{\tau_i}^{\tau_f} \frac{\partial}{\partial X^\mu} L(X,V,U) d\tau = \int_{\tau_i}^{\tau_f} \left\{ d \frac{\partial}{\partial V^\mu} L(X,V,U) - d_\circ \frac{\partial}{\partial U^\mu} L(X,V,U)\right\}. \tag{6.158}
$$

Since $\delta X \perp\!\!\!\perp X$, the osmotic integral vanishes, and we obtain

$$
\int_{\tau_i}^{\tau_f} \frac{\partial}{\partial X^\mu} L(X,V,U) d\tau = \int_{\tau_i}^{\tau_f} d \frac{\partial}{\partial V^\mu} L(X,V,U). \tag{6.159}
$$

6.5.2 Hamilton equations

As in classical physics, one can define an Hamiltonian picture. We define the generalized momenta by

$$
\begin{aligned}
P_\mu^+(\tau) &= \frac{\partial L}{\partial V_+^\mu},\\
P_\mu^-(\tau) &= \frac{\partial L}{\partial V_-^\mu}.
\end{aligned} \tag{6.160}
$$

and the Hamiltonian as the Legendre transform

$$
H(X,P^+,P^-) = P_\mu^+ V_+^\mu + P_\mu^- V_-^\mu - L(X,V_+,V_-). \tag{6.161}
$$

We can take a first order total derivative. This yields

$$dH = \frac{\partial H}{\partial X^\mu} dX^\mu + \frac{\partial H}{\partial P^+_\mu} dP^+_\mu + \frac{\partial H}{\partial P^-_\mu} dP^-_\mu \tag{6.162}$$

and

$$\begin{aligned} dH &= P^+_\mu dV^\mu_+ + V^\mu_+ dP^+_\mu + P^-_\mu dV^\mu_- + V^\mu_- dP^-_\mu - \frac{\partial L}{\partial X^\mu} dX^\mu - \frac{\partial L}{\partial V^\mu_+} dV^\mu_+ - \frac{\partial L}{\partial V^\mu_-} dV^\mu_- \\ &= V^\mu_+ dP^+_\mu + V^\mu_- dP^-_\mu - \left(\frac{d_-}{d\tau} P^+_\mu + \frac{d_+}{d\tau} P^-_\mu \right) dX^\mu. \end{aligned} \tag{6.163}$$

One can then read off the Hamilton equations:

$$\begin{aligned} V^\mu_+(\tau) &= \frac{\partial H}{\partial P^+_\mu}, \\ V^\mu_-(\tau) &= \frac{\partial H}{\partial P^-_\mu}, \end{aligned} \tag{6.164}$$

and

$$\int \left(d_+ P^-_\mu + d_- P^+_\mu \right) = - \int \frac{\partial H}{\partial X^\mu} d\tau. \tag{6.165}$$

Furthermore, if an explicit proper time dependence is introduced, one finds

$$\frac{\partial}{\partial \tau} H(X, P^+, P^-, \tau) = - \frac{\partial}{\partial \tau} L(X, V_+, V_-, \tau). \tag{6.166}$$

As is the case for the Lagrangian, one can express the Hamiltonian in terms of *current* and *osmotic momenta*. These can be defined as

$$\begin{aligned} P_\mu(\tau) &= \frac{\partial}{\partial V^\mu} L(X, V, U), \\ Q_\mu(\tau) &= \frac{\partial}{\partial U^\mu} L(X, V, U). \end{aligned} \tag{6.167}$$

The Hamiltonian is then given by

$$H(X, P, Q) = P_\mu V^\mu + Q_\mu U^\mu - L(X, V, U). \tag{6.168}$$

This leads to the Hamilton equations

$$\begin{aligned} V^\mu(\tau) &= \frac{\partial H}{\partial P_\mu}, \\ U^\mu(\tau) &= \frac{\partial H}{\partial Q_\mu}. \end{aligned} \tag{6.169}$$

and

$$\int dP_\mu = - \int \frac{\partial H}{\partial X^\mu} d\tau. \tag{6.170}$$

Let us summarize the relation between U, V, V_+, V_-:

$$V = \frac{1}{2}\left(V_+ + V_-\right), \qquad V_+ = V + U,$$
$$U = \frac{1}{2}\left(V_+ - V_-\right), \qquad V_- = V - U. \tag{6.171}$$

Furthermore, for P, Q, P_+, P_- we have

$$P = P_+ + P_-, \qquad P_+ = \frac{1}{2}\left(P + Q\right),$$
$$Q = P_+ - P_-, \qquad P_- = \frac{1}{2}\left(P - Q\right). \tag{6.172}$$

6.5.3 Hamilton-Jacobi equations

The Hamilton-Jacobi equations play an important role in the derivation of the Schrödinger equation in stochastic quantization. We will therefore review the derivation of these equations. We define Hamilton's principal function as the action conditioned on its end point

$$S(X,\tau) = \mathbb{E}\left[\int_{\tau_i}^{\tau} L(X, V_+, V_-)\, ds \middle| X(\tau)\right], \tag{6.173}$$

such that the Euler-Lagrange equations are satisfied.

We consider the variation of the principal function under a variation of the end point. This yields

$$\begin{aligned}
\delta S(X,\tau) &= S(X+\delta X,\tau) - S(X,\tau) \\
&= \mathbb{E}\left[\int_{\tau_i}^{\tau} L(X,V_+,V_-)\, ds \middle| X(\tau) + \delta X(\tau)\right] - \mathbb{E}\left[\int_{\tau_i}^{\tau} L(X,V_+,V_-)\, ds \middle| X(\tau)\right] \\
&= \mathbb{E}\left[\int_{\tau_i}^{\tau} L(X+\delta X, V_+ + \delta V_+, V_- + \delta V_-)\, ds - \int_{\tau_i}^{\tau} L(X,V_+,V_-)\, ds \middle| X(\tau), \delta X(\tau)\right] \\
&= \mathbb{E}\Bigg[\int_{\tau_i}^{\tau} \bigg\{ \frac{\partial}{\partial X^\mu} L(X,V_+,V_-)\, \delta X^\mu + \frac{\partial}{\partial V_+^\mu} L(X,V_+,V_-)\, \delta V_+^\mu \\
&\qquad + \frac{\partial}{\partial V_-^\mu} L(X,V_+,V_-)\, \delta V_-^\mu \bigg\}\, ds \bigg| X(\tau), \delta X(\tau)\Bigg] + \mathcal{O}\left(||\delta X||^2\right) \\
&= \mathbb{E}\Bigg[\int_{\tau_i}^{\tau} \bigg\{ \delta X^\mu\, d_- \frac{\partial L}{\partial V_+^\mu} + \delta X^\mu\, d_+ \frac{\partial L}{\partial V_-^\mu} \\
&\qquad + \frac{\partial L}{\partial V_+^\mu}\, d_+ \delta X^\mu + \frac{\partial L}{\partial V_-^\mu}\, d_- \delta X^\mu \bigg\} \bigg| X(\tau), \delta X(\tau)\Bigg] + \mathcal{O}\left(||\delta X||^2\right) \\
&= \mathbb{E}\left[\int_{\tau_i}^{\tau} d\left[\left(\frac{\partial L}{\partial V_+^\mu} + \frac{\partial L}{\partial V_-^\mu}\right)\delta X^\mu\right] + \mathcal{O}\left(||\delta X||^2\right) \middle| X(\tau), \delta X(\tau)\right] \\
&= \Big(p_\mu^+(X,\tau) + p_\mu^-(X,\tau)\Big)\delta X^\mu + \mathcal{O}\Big(||\delta X||^2\Big),
\end{aligned} \tag{6.174}$$

where we used the Euler-Lagrange equations in the fifth line. Furthermore, in the third line, we have rewritten the original trajectory which is the minimal path between (τ_i, x_i) and $(\tau, X(\tau) + \delta X(\tau))$ as two independent trajectories $X, \delta X$, which are the minimal paths between (τ_i, x_i) and $(\tau, X(\tau))$ and between $(\tau_i, 0)$ and $(\tau, \delta X(\tau))$ respectively.

We conclude with the first Hamilton-Jacobi equation

$$\nabla_\mu S(X,\tau) = p^+_\mu(X,\tau) + p^-_\mu(X,\tau) = p_\mu(X,\tau). \tag{6.175}$$

Moreover, taking a first order total derivative of Hamilton's principal function yields

$$\begin{aligned} dS &= \mathbb{E}_\tau\left[L\, d\tau\right], \\ dS &= \mathbb{E}_\tau\left[\frac{\partial S}{\partial x^\mu} dX^\mu + \frac{\partial S}{\partial \tau} d\tau\right]. \end{aligned} \tag{6.176}$$

This leads to the second Hamilton-Jacobi equation

$$\frac{\partial}{\partial \tau} S(X,\tau) = \mathbb{E}_\tau\left[L(X,V,U)\right] - p_\mu v^\mu. \tag{6.177}$$

6.5.4 Kolmogorov equations

In this section, we derive the Kolmogorov equations. Although these do not follow from a variational principle, they are another crucial ingredient for the derivation of the Schrödinger equation.

Let $\mu(x,\tau)$ be a probability measure on $\mathcal{M} \times T$, such that

$$\int_{\mathcal{M}\times T} f(x,\tau)\, d\mu(x,\tau) = \int_T \mathbb{E}\left[f(X(\tau),\tau)\right] d\tau \tag{6.178}$$

for any smooth function f compactly supported on $\mathcal{M} \times \mathrm{int(T)}$, where $\mathrm{int(T)}$ is the interior of T. We will assume that the probability density ρ associated to the measure μ exists, such that $d\mu(x,\tau) = \sqrt{|g|}\rho(x,\tau) d^n x d\tau$. Then

$$\begin{aligned} 0 &= \mathbb{E}[f(X(\tau_f),\tau_f)] - \mathbb{E}[[f(X(\tau_i),\tau_i)] \\ &= \int_T \frac{d_2}{d\tau}\mathbb{E}[f(X(\tau),\tau)]d\tau \\ &= \int_T \mathbb{E}\left[\frac{d_2}{d\tau} f(X(\tau),\tau)\right] d\tau \\ &= \int_T \mathbb{E}\left[\mathbb{E}_\tau\left[\frac{d_2}{d\tau} f(X(\tau),\tau)\right]\right] d\tau \\ &= \int_T \mathbb{E}\left[\left(\frac{\partial}{\partial\tau} + \hat{v}^\mu(X,\tau)\nabla_\mu + \hat{v}^{\mu\nu}(X,\tau)\nabla_\mu\nabla_\nu\right) f\left(X,t\right)\right] d\tau \\ &= \int_{\mathcal{M}\times T} \left(\frac{\partial}{\partial\tau} + \hat{v}^\mu(x,\tau)\nabla_\mu + \hat{v}^{\mu\nu}(x,\tau)\nabla_\mu\nabla_\nu\right) f(x,\tau)\, d\mu(x,\tau) \\ &= \int_{\mathcal{M}\times T} \sqrt{|g|}\,\rho(x,\tau)\left(\frac{\partial}{\partial\tau} + \hat{v}^\mu(x,\tau)\nabla_\mu + \hat{v}^{\mu\nu}(x,\tau)\nabla_\mu\nabla_\nu\right) f(x,\tau)\, d^n x\, d\tau \\ &= \int_{\mathcal{M}\times T} \sqrt{|g|}\, f(x,\tau)\left(-\frac{\partial}{\partial\tau}\rho(x,\tau) - \nabla_\mu\left[\hat{v}^\mu(x,\tau)\,\rho(x,\tau)\right] + \nabla_\mu\nabla_\nu\left[\hat{v}^{\mu\nu}(x,\tau)\,\rho(x,\tau)\right]\right) d^n x\, d\tau \end{aligned} \tag{6.179}$$

for all compactly supported functions f. We can choose $v = v_\pm$, and plug in the background hypothesis

$$\hat{v}^{\mu\nu}_{\pm} = \pm\frac{\hbar}{2m} g^{\mu\nu}. \tag{6.180}$$

This leads to the Kolmogorov forward and backward equations or equivalently the Fokker-Planck equations associated to the forward and backward process:

$$\begin{aligned}\frac{\partial}{\partial\tau}\rho(x,\tau) &= -\nabla_\mu\left[\hat{v}^\mu_+(x,\tau)\rho(x,\tau)\right] + \frac{\hbar}{2m}\nabla^2\rho(x,\tau),\\ \frac{\partial}{\partial\tau}\rho(x,\tau) &= -\nabla_\mu\left[\hat{v}^\mu_-(x,\tau)\rho(x,\tau)\right] - \frac{\hbar}{2m}\nabla^2\rho(x,\tau).\end{aligned} \tag{6.181}$$

Adding and subtracting the two equations leads to the continuity and osmotic equations

$$\frac{\partial}{\partial\tau}\rho(x,\tau) = -\nabla_\mu\left[v^\mu(x,\tau)\rho(x,\tau)\right], \tag{6.182}$$

$$\hat{u}^\mu(x,\tau) = \frac{\hbar}{2m}\nabla^\mu \ln\left[\rho(x,\tau)\right]. \tag{6.183}$$

6.6 The stochastic Lagrangian

In classical physics a Lagrangian is a function of the form $L(X,V,\tau)$. In stochastic quantization on the other hand the Lagrangian is a function of the form $L(X,V_+,V_-,\tau)$. Due to the existence of two different velocities, it is not immediately clear how the classical Lagrangian should be generalized to the stochastic framework. However, it was shown by Zambrini, cf. Ref. [364] that for any classical Lagrangian of the form

$$L_c(x,v,\tau) = \frac{m}{2}T_{\mu\nu}(x,\tau)v^\mu v^\nu - \hbar A_\mu(x,\tau)v^\mu - \mathfrak{U}(x,\tau) \tag{6.184}$$

the minimal stochastic extension that is compatible with gauge invariance and Maupertuis' principle is given by

$$L(X,V_+,V_-,\tau) = \frac{1}{2}L_c(X,V_+,\tau) + \frac{1}{2}L_c(X,V_-,\tau). \tag{6.185}$$

We note that this form of the Lagrangian was also assumed by Yasue [362, 363]. In the remainder of this paper, we will assume that gravity is the only spin-2 field, i.e.

$$T_{\mu\nu}(x,\tau) = g_{\mu\nu}(x). \tag{6.186}$$

The stochastic Lagrangian corresponding to the classical Lagrangian (6.184) is then given by

$$L\left(X,V_+,V_-\right) = \frac{m}{4}g_{\mu\nu}\left(V^\mu_+V^\nu_+ + V^\mu_-V^\nu_-\right) - \frac{\hbar}{2}A_\mu(X)\left(V^\mu_+ + V^\mu_-\right) - \mathfrak{U}(X) \tag{6.187}$$

or equivalently

$$L\left(X,V,U\right) = \frac{m}{2}g_{\mu\nu}\left(V^\mu V^\nu + U^\mu U^\nu\right) - \hbar\, A_\mu(X)V^\mu - \mathfrak{U}(X). \tag{6.188}$$

Compared to the classical Lagrangian there is an additional energy contribution:

$$\frac{m}{2} g_{\mu\nu} U^\mu U^\nu. \tag{6.189}$$

This is the osmotic energy and can be interpreted as the kinetic energy of the background field.

There also exists a Hamiltonian description. The momenta for this Lagrangian are

$$\begin{aligned} P^+_\mu(\tau) &= \frac{m}{2} g_{\mu\nu} V^\nu_+(\tau) - \frac{\hbar}{2} A_\mu(X), \\ P^-_\mu(\tau) &= \frac{m}{2} g_{\mu\nu} V^\nu_-(\tau) - \frac{\hbar}{2} A_\mu(X), \\ P_\mu(\tau) &= m\, g_{\mu\nu} V^\nu(\tau) - \hbar\, A_\mu(X), \\ Q_\mu(\tau) &= m\, g_{\mu\nu} U^\nu(\tau). \end{aligned} \tag{6.190}$$

The Hamiltonian is then given by

$$H\left(X, P^+, P^-\right) = \frac{1}{m} g^{\mu\nu} \left(P^+_\mu P^+_\nu + P^-_\mu P^-_\nu + \hbar \left(P^+_\mu + P^-_\mu\right) A_\nu(X) + \frac{\hbar^2}{2} A_\mu(X) A_\nu(X) \right) + \mathfrak{U}(X) \tag{6.191}$$

or equivalently

$$H\left(X, P, Q\right) = \frac{1}{2m} g^{\mu\nu} \left(P_\mu P_\nu + Q_\mu Q_\nu + 2\hbar\, P_\mu A_\nu(X) + \hbar^2\, A_\mu(X) A_\nu(X) \right) + \mathfrak{U}(X). \tag{6.192}$$

6.6.1 Conditional expectations

In section 6.5.3, we derived the Hamilton-Jacobi equations and obtained expressions that contained the conditional expectation of the Lagrangian $\mathbb{E}_\tau\left[L(X, V, U, \tau)\right]$. We can calculate this expression for the Lagrangian (6.187) obtained in previous subsection. For this we notice that for any smooth function $\mathfrak{U} : T \times \mathcal{M} \to \mathbb{R}$

$$\mathbb{E}_\tau\left[\mathfrak{U}(X(\tau), \tau)\right] = \lim_{s \to \tau} \mathbb{E}_\tau\left[\mathfrak{U}(X(s), s)\right] = \mathfrak{U}(X(\tau), \tau). \tag{6.193}$$

For the terms that depend on the velocity process, we need to make sense of the processes $V_\pm$. This can be done by performing an integration over $d\tau$. At linear order we have

$$\begin{aligned} A_\mu(X(\tau), \tau) V^\mu_+(\tau) &= \lim_{h \to 0} \frac{1}{h} \int_\tau^{\tau+h} A_\mu(X(s), s) V^\mu_+(s)\, ds \\ &= \lim_{h \to 0} \frac{1}{h} \left[\int_\tau^{\tau+h} \Big(A_\mu(X(s), s)\, d_+ X^\mu(s) + \partial_\nu A_\mu(X(s), s)\, dX^\mu \cdot dX^\nu(s) \Big) \right] \\ &= \lim_{h \to 0} \frac{1}{h} \left[\int_\tau^{\tau+h} \Big(A_\mu(X(s), s)\, d_+ \hat{X}^\mu(s) + \nabla_\nu A_\mu(X(s), s)\, d\hat{X}^\mu \cdot d\hat{X}^\nu(s) \Big) \right]. \end{aligned} \tag{6.194}$$

By a similar calculation, we obtain

$$A_\mu(X(\tau),\tau)V_-^\mu(\tau) = \lim_{h\to 0}\frac{1}{h}\left[\int_\tau^{\tau+h}\Big(A_\mu(X(s),s)\,d_-\hat{X}^\mu(s) - \nabla_\nu A_\mu(X(s),s)\,d\hat{X}^\mu\cdot d\hat{X}^\nu(s)\Big)\right]. \tag{6.195}$$

We note that we can write these expressions in differential notation as

$$A_\mu V_\pm^\mu\,d\tau = A_\mu\,d_\pm\hat{X}^\mu \pm \nabla_\nu A_\mu\,d\hat{X}^\mu\cdot d\hat{X}^\nu \tag{6.196}$$

Taking the expectation value of these expressions yields

$$\begin{aligned}\mathbb{E}_\tau\left[A_\mu(X(\tau),\tau)V_+^\mu(\tau)\right] &= \lim_{h\to 0}\frac{1}{h}\mathbb{E}_\tau\bigg[\int_\tau^{\tau+h}A_\mu(X(s),s)\,\hat{v}_+^\mu(X(s),s)\,ds\\ &\qquad + \int_\tau^{\tau+h}A_\mu(X(s),s)\,dW_+^\mu(s)\\ &\qquad + \int_\tau^{\tau+h}\nabla_\nu A_\mu(X(s),s)\,\hat{v}_+^{\mu\nu}(X(s),s)\,ds\bigg]\\ &= A_\mu(X(\tau),\tau)\,\hat{v}_+^\mu(X,\tau) + \frac{\hbar}{2m}\nabla_\mu A^\mu(X(\tau),\tau),\end{aligned} \tag{6.197}$$

where we used the martingale property (6.134). Moreover,

$$\mathbb{E}_\tau\left[A_\mu(X(\tau),\tau)\,V_-^\mu(\tau)\right] = A_\mu(X(\tau),\tau)\,\hat{v}_-^\mu(X,\tau) - \frac{\hbar}{2m}\nabla_\mu A^\mu(X(\tau),\tau). \tag{6.198}$$

Consequently,

$$\mathbb{E}_\tau\left[A_\mu(X(\tau),\tau)\,V^\mu(\tau)\right] = A_\mu(X(\tau),\tau)\,v^\mu(X,\tau), \tag{6.199}$$

$$\mathbb{E}_\tau\left[A_\mu(X(\tau),\tau)\,U^\mu(\tau)\right] = A_\mu(X(\tau),\tau)\,\hat{u}^\mu(X,\tau) + \frac{\hbar}{2m}\nabla_\mu A^\mu(X(\tau),\tau). \tag{6.200}$$

For the terms quadratic in velocity we will perform a double integral over $d\tau$. In differential notation we have[33]

$$\begin{aligned}g_{\mu\nu}V_+^\mu V_+^\nu\,d\tau^2 &= g_{\mu\nu}\,d_+\hat{X}^\mu\otimes d_+\hat{X}^\nu + g_{\mu\nu}\,\nabla_\rho\left(d_+\hat{X}^\mu\right)\otimes d\hat{X}^\nu\cdot d\hat{X}^\rho\\ &\quad + g_{\mu\nu}\,d\hat{X}^\mu\cdot d\hat{X}^\rho\otimes\nabla_\rho\left(d_+\hat{X}^\nu\right) - \frac{2}{3}\mathcal{R}_{\mu\nu\rho\sigma}\,d\hat{X}^\mu\cdot d\hat{X}^\rho\otimes d\hat{X}^\nu\cdot d\hat{X}^\sigma,\\ g_{\mu\nu}V_-^\mu V_-^\nu\,d\tau^2 &= g_{\mu\nu}\,d_-\hat{X}^\mu\otimes d_-\hat{X}^\nu - g_{\mu\nu}\,\nabla_\rho\left(d_-\hat{X}^\mu\right)\otimes d\hat{X}^\nu\cdot d\hat{X}^\rho\\ &\quad - g_{\mu\nu}\,d\hat{X}^\mu\cdot d\hat{X}^\rho\otimes\nabla_\rho\left(d_-\hat{X}^\nu\right) - \frac{2}{3}\mathcal{R}_{\mu\nu\rho\sigma}\,d\hat{X}^\mu\cdot d\hat{X}^\rho\otimes d\hat{X}^\nu\cdot d\hat{X}^\sigma,\\ g_{\mu\nu}V_+^\mu V_-^\nu\,d\tau^2 &= g_{\mu\nu}\,d_+\hat{X}^\mu\otimes d_-\hat{X}^\nu - g_{\mu\nu}\,\nabla_\rho\left(d_+\hat{X}^\mu\right)\otimes d\hat{X}^\nu\cdot d\hat{X}^\rho\\ &\quad + g_{\mu\nu}\,d\hat{X}^\mu\cdot d\hat{X}^\rho\otimes\nabla_\rho\left(d_-\hat{X}^\nu\right) + \frac{2}{3}\mathcal{R}_{\mu\nu\rho\sigma}\,d\hat{X}^\mu\cdot d\hat{X}^\rho\otimes d\hat{X}^\nu\cdot d\hat{X}^\sigma.\end{aligned} \tag{6.201}$$

[33]cf. section 9 in Ref. [270].

We can take the expectation values of these expressions. This yields

$$\begin{aligned}
\mathbb{E}_\tau\Big[g_{\mu\nu}\, d_+\hat{X}^\mu \otimes d_+\hat{X}^\nu\Big] &= \mathbb{E}_\tau\Big[g_{\mu\nu}\Big(\hat{v}_+^\mu \hat{v}_+^\nu\, d\tau^2 + \hat{v}_+^\mu\, dW_+^\nu\, d\tau + \hat{v}_+^\nu\, dW_+^\nu\, d\tau + dW_+^\mu \otimes dW_+^\nu\Big)\Big] \\
&= g_{\mu\nu}\Big(\hat{v}_+^\mu \hat{v}_+^\nu\, d\tau^2 + 2\,\hat{v}_+^{\mu\nu}\, d\tau\Big) \\
&= \frac{n\,\hbar}{m}\, d\tau + g_{\mu\nu}\hat{v}_+^\mu \hat{v}_+^\nu\, d\tau^2 \qquad (6.202)
\end{aligned}$$

where we used that the expectation value of the terms linear in dW_+ vanishes, due to the martingale property of W_+. Moreover, we used eq. (6.130) to evaluate the term $dW_+^\mu dW_+^\nu = dW_+^\mu \otimes dW_+^\nu$. By a similar calculation we obtain

$$\mathbb{E}_\tau\Big[g_{\mu\nu}\, d_-\hat{X}^\mu \otimes d_-\hat{X}^\nu\Big] = -\frac{n\,\hbar}{m}\, d\tau + g_{\mu\nu}\hat{v}_-^\mu \hat{v}_-^\nu\, d\tau^2 \qquad (6.203)$$

$$\mathbb{E}_\tau\Big[g_{\mu\nu}\, d_+\hat{X}^\mu \otimes d_-\hat{X}^\nu\Big] = g_{\mu\nu}\hat{v}_+^\mu \hat{v}_-^\nu\, d\tau^2. \qquad (6.204)$$

Furthermore,

$$\begin{aligned}
\mathbb{E}_\tau\Big[g_{\mu\nu}\nabla_\rho\Big(d_+\hat{X}^\mu\Big) \otimes d\hat{X}^\nu \cdot d\hat{X}^\rho\Big] &= \mathbb{E}_\tau\Big[g_{\mu\nu}\hat{v}_+^{\nu\rho}\nabla_\rho\left(\hat{v}_+^\mu d\tau + dW_+^\mu\right) d\tau\Big] \\
&= \frac{\hbar}{2m}\nabla_\mu \hat{v}_+^\mu\, d\tau^2. \qquad (6.205)
\end{aligned}$$

Similarly,

$$\mathbb{E}_\tau\Big[g_{\mu\nu}\nabla_\rho\Big(d_-\hat{X}^\mu\Big) \otimes d\hat{X}^\nu \cdot d\hat{X}^\rho\Big] = \frac{\hbar}{2m}\nabla_\mu \hat{v}_-^\mu\, d\tau^2. \qquad (6.206)$$

For the remaining term we find

$$\begin{aligned}
\mathbb{E}_\tau\Big[\mathcal{R}_{\mu\nu\rho\sigma}\, d\hat{X}^\mu \cdot d\hat{X}^\rho \otimes d\hat{X}^\nu \cdot d\hat{X}^\sigma\Big] &= \mathbb{E}_\tau\left[\mathcal{R}_{\mu\nu\rho\sigma}\,\hat{v}^{\mu\rho}\hat{v}^{\nu\sigma} d\tau^2\right] \\
&= \frac{\hbar^2}{4m^2}\mathcal{R}\, d\tau^2. \qquad (6.207)
\end{aligned}$$

We conclude,

$$\begin{aligned}
\mathbb{E}_\tau\left[g_{\mu\nu}\, V_+^\mu V_+^\nu\right] &= g_{\mu\nu}\hat{v}_+^\mu \hat{v}_+^\nu + \frac{\hbar}{m}\nabla_\mu \hat{v}_+^\mu - \frac{\hbar^2}{6m^2}\mathcal{R} + \frac{n\,\hbar}{m\,d\tau}, \\
\mathbb{E}_\tau\left[g_{\mu\nu}\, V_-^\mu V_-^\nu\right] &= g_{\mu\nu}\hat{v}_-^\mu \hat{v}_-^\nu - \frac{\hbar}{m}\nabla_\mu \hat{v}_-^\mu - \frac{\hbar^2}{6m^2}\mathcal{R} - \frac{n\,\hbar}{m\,d\tau}, \\
\mathbb{E}_\tau\left[g_{\mu\nu}\, V_+^\mu V_-^\nu\right] &= g_{\mu\nu}\hat{v}_+^\mu \hat{v}_-^\nu - \frac{\hbar}{2m}\nabla_\mu \hat{v}_+^\mu + \frac{\hbar}{2m}\nabla_\mu \hat{v}_-^\mu + \frac{\hbar^2}{6m^2}\mathcal{R} \qquad (6.208)
\end{aligned}$$

or equivalently

$$\begin{aligned}
\mathbb{E}_\tau\Big[g_{\mu\nu}\, V^\mu V^\nu\Big] &= g_{\mu\nu} v^\mu v^\nu, \\
\mathbb{E}_\tau\left[g_{\mu\nu}\, U^\mu U^\nu\right] &= g_{\mu\nu}\hat{u}^\mu \hat{u}^\nu + \frac{\hbar}{m}\nabla_\mu \hat{u}^\mu - \frac{\hbar^2}{6m^2}\mathcal{R}, \\
\mathbb{E}_\tau\left[g_{\mu\nu}\, V^\mu U^\nu\right] &= g_{\mu\nu} v^\mu \hat{u}^\nu + \frac{\hbar}{2m}\nabla_\mu v^\mu + \frac{n\,\hbar}{2\,m\,d\tau}. \qquad (6.209)
\end{aligned}$$

The conditional expectation of the Lagrangian (6.188) is thus given by[34]

$$\mathbb{E}_\tau\left[L(X,V,U,\tau)\right] = \frac{m}{2} g_{\mu\nu}\left(v^\mu v^\nu + \hat{u}^\mu \hat{u}^\nu\right) + \frac{\hbar}{2}\nabla_\mu \hat{u}^\mu - \frac{\hbar^2}{12m}\mathcal{R} - \hbar\, A_\mu v^\mu - \mathfrak{U}. \tag{6.210}$$

6.6.2 Correlation functions

Observables in quantum mechanics can be constructed from correlation functions computed in the path integral formalism. Since this computation is slightly different in stochastic quantization, we review the main steps.

In order to compute correlation functions within the stochastic quantization, one must first solve the stochastic equations of motion derived from the action. The solution is a stochastic process $\{X(\tau)|\tau \in T\}$. For this stochastic process one can define a characteristic functional $\Phi_X(J)$, and a moment generating functional $M_X(J)$:

$$\Phi_X(J) = \mathbb{E}\left[e^{\frac{i}{\hbar}\int_{\tau_i}^{\tau_f} J_\mu(\tau) X^\mu(\tau) d\tau}\right], \tag{6.211}$$

$$M_X(J) = \mathbb{E}\left[e^{\frac{1}{\hbar}\int_{\tau_i}^{\tau_f} J_\mu(\tau) X^\mu(\tau) d\tau}\right], \tag{6.212}$$

where $J(\tau)$ is a bounded process of finite variation that corresponds to the source in the path integral formulation. We emphasize that one no longer averages over the action, as this is essentially done in the first step, where the stochastic differential equation is solved.

Using the characteristic and moment generating functionals for the process $X(\tau)$, one can calculate all moments of the theory. For example, the two-point correlation function is given by

$$\mathbb{E}\left[X^\mu(s) X^\nu(r)\right] = \lim_{||J||\to 0} \frac{\partial}{\partial J_\mu(s)} \frac{\partial}{\partial J_\nu(r)} M_X(J). \tag{6.213}$$

We emphasize that the integrals that need to be evaluated in the path integral formalism and stochastic quantization are constructed in different ways. Due to this different construction, theories that require renormalization in the path integral formalism can be finite in stochastic quantization.

6.6.3 Uncertainty principle

Due to the relevance of the uncertainty principle in quantum mechanics, we will derive it in stochastic quantization, which can be done using the results from section 6.6.1.

[34]Note that the divergent term $\frac{n\hbar}{m d\tau}$ does not appear in the Lagrangian.

For $s > \tau$ we find

$$\begin{aligned}
\mathrm{Cov}_\tau\left[X_\mu(s), X^\nu(s)\right] &= \mathbb{E}_\tau\left[X_\mu(s)X^\nu(s)\right] - \mathbb{E}_\tau\left[X_\mu(s)\right]\mathbb{E}_\tau\left[X^\nu(s)\right] \\
&= \mathbb{E}_\tau\left[\left(X_\mu(\tau) + \int_\tau^s V_{+\mu}(r)\,dr\right)\left(X^\mu(\tau) + \int_\tau^s V_+^\mu(r)\,dr\right)\right] \\
&\quad - \mathbb{E}_\tau\left[\left(X_\mu(\tau) + \int_\tau^s V_\mu(r)\,dr\right)\right]\mathbb{E}_\tau\left[\left(X^\mu(\tau) + \int_\tau^s V^\mu(r)\,dr\right)\right] \\
&= \frac{\hbar}{m}\delta_\mu^\nu(s-\tau) + \frac{\hbar}{2m}\left(\nabla_\mu \hat{v}_+^\nu + \nabla^\nu \hat{v}_{+\mu} - \frac{\hbar}{3m}\mathcal{R}_\mu^\nu\right)(s-\tau)^2 + o(s-\tau)^2.
\end{aligned} \tag{6.214}$$

Furthermore, the covariance for the momenta is given by

$$\begin{aligned}
\mathrm{Cov}_\tau\left[P_\mu^+(s), P^{+\nu}(s)\right] &= \frac{m^2}{4}\Big\{\mathbb{E}_\tau\left[V_{+\mu}(s)V_+^\nu(s)\right] - \mathbb{E}_\tau\left[V_{+\mu}(s)\right]\mathbb{E}_\tau\left[V_+^\nu(s)\right]\Big\} \\
&\quad - \frac{m\,\hbar}{4}\Big\{\mathbb{E}_\tau\left[V_{+\mu}(s)A^\nu(s)\right] - \mathbb{E}_\tau\left[V_{+\mu}(s)\right]\mathbb{E}_\tau\left[A^\nu(s)\right]\Big\} \\
&\quad - \frac{m\,\hbar}{4}\Big\{\mathbb{E}_\tau\left[A_\mu(s)V_+^\nu(s)\right] - \mathbb{E}_\tau\left[A_\mu(s)\right]\mathbb{E}_\tau\left[V_+^\nu(s)\right]\Big\} \\
&\quad - \frac{\hbar^2}{4}\Big\{\mathbb{E}_\tau\left[A_\mu(s)A^\nu(s)\right] - \mathbb{E}_\tau\left[A_\mu(s)\right]\mathbb{E}_\tau\left[A^\nu(s)\right]\Big\} \\
&= \frac{m\,\hbar}{4}\delta_\mu^\nu(s-\tau)^{-1} + \frac{m\,\hbar}{8}\left(\nabla_\mu \hat{v}_+^\nu + \nabla^\nu \hat{v}_{+\mu}\right) \\
&\quad - \frac{\hbar^2}{8}\left(\nabla_\mu A^\nu + \nabla^\nu A_\mu\right) - \frac{\hbar^2}{24}\mathcal{R}_\mu^\nu + o(1).
\end{aligned} \tag{6.215}$$

If we take the limit $s \to \tau$, we find

$$\lim_{s\to\tau} \mathrm{Cov}_\tau\left[X_\mu(s), X^\nu(s)\right] = 0, \tag{6.216}$$

$$\lim_{s\to\tau} \mathrm{Cov}_\tau\left[P_\mu^+(s), P^{+\nu}(s)\right] = \infty. \tag{6.217}$$

This reflects the fact that we have constructed the stochastic theory in a position representation, i.e. the process (X, P_+, P_-) is adapted to the filtration generated by the process X.

We can calculate the product of the two variances. For this we fix the indices $\mu = \nu = \bar{\mu}$, and obtain

$$\mathrm{Var}_\tau\left[X^{\bar{\mu}}(s)\right]\mathrm{Var}_\tau\left[P_{\bar{\mu}}^+(s)\right] = \frac{\hbar^2}{4} + \frac{\hbar^2}{2}\left(\nabla_{\bar{\mu}}\hat{v}_+^{\bar{\mu}} - \frac{\hbar}{2m}\nabla_{\bar{\mu}}A^{\bar{\mu}} - \frac{\hbar}{6m}\mathcal{R}_{\bar{\mu}}^{\bar{\mu}}\right)(s-\tau) + o(s-\tau). \tag{6.218}$$

If we then take the limit $s \to \tau$, we find

$$\lim_{s\to\tau}\mathrm{Var}_\tau\left[X^{\bar{\mu}}(s)\right]\mathrm{Var}_\tau\left[P_{\bar{\mu}}^+(s)\right] = \frac{\hbar^2}{4}. \tag{6.219}$$

This corresponds to the lower bound given by the Heisenberg uncertainty principle.

6.7 Scalar test particles

In this section, we derive the equations of motion that govern a quantum mechanical spin-0 test particle on a pseudo-Riemannian manifold subjected to the Lagrangian (6.188).

6.7.1 Stochastic equation of motion

We consider the Lagrangian (6.188):

$$L\,(X,V,U) = \frac{m}{2} g_{\mu\nu}\,(V^\mu V^\nu + U^\mu U^\nu) - \hbar\, A_\mu V^\mu - \mathfrak{U}. \tag{6.220}$$

After integrating this expression twice over τ we obtain, cf. eq. (6.201),

$$\begin{aligned}\mathbb{E}\left[L\, d\tau^2\right] = \mathbb{E}\Big[&\frac{m}{2} g_{\mu\nu}\Big\{dX^\mu dX^\nu + d_\circ \hat{X}^\mu d_\circ \hat{X}^\nu + \nabla_\rho \left(d_\circ \hat{X}^\mu\right) d[[X^\nu, X^\rho]] \\ &\qquad - \frac{\hbar^2}{6m^2} \mathcal{R}^\mu{}_{\rho\kappa\sigma}\, d[[X^\nu, X^\kappa]]\, d[[X^\rho, X^\sigma]]\Big\} \\ &- \hbar\, A_\mu\, dX^\mu\, d\tau - \mathfrak{U}\, d\tau^2\Big] \\ = \mathbb{E}\Big[&\frac{m}{2} g_{\mu\nu}\, dX^\mu dX^\nu - \hbar\, A_\mu dX^\mu d\tau - \left(\mathfrak{U} + \frac{\hbar^2}{12m}\mathcal{R}\right) d\tau^2\Big],\end{aligned} \tag{6.221}$$

where we used

$$\mathbb{E}\left[g_{\mu\nu}\left\{d_\circ \hat{X}^\mu d_\circ \hat{X}^\nu + \nabla_\rho \left(d_\circ \hat{X}^\mu\right) d[[X^\nu, X^\rho]]\right\}\right] = 0, \tag{6.222}$$

which follows from eq. (6.145) and the metric compatibility. If we vary this expression with respect to a stochastically independent deviation process δX, we obtain the stochastic Euler-Lagrange equations (6.159) that take the form

$$m\,\left(g_{\mu\nu} d^2 X^\nu + g_{\mu\nu}\Gamma^\nu_{\rho\sigma} dX^\rho dX^\sigma\right) = \left(\hbar\, \partial_\tau A_\mu - \nabla_\mu \mathfrak{U} - \frac{\hbar^2}{12m}\nabla_\mu \mathcal{R}\right) d\tau^2 - \hbar\, H_{\mu\nu}\, dX^\nu d\tau, \tag{6.223}$$

where

$$H_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu. \tag{6.224}$$

In the classical limit $\hbar \to 0$, the quadratic variation vanishes. This gives[35]

$$m\left(g_{\mu\nu}\frac{d^2 X^\nu}{d\tau^2} + g_{\mu\nu}\Gamma^\nu_{\rho\sigma}\frac{dX^\rho}{d\tau}\frac{dX^\sigma}{d\tau}\right) = -\lim_{\hbar\to 0}\left\{\nabla_\mu \mathfrak{U} + \hbar\left[-\partial_\tau A_\mu + H_{\mu\nu}\frac{dX^\nu}{d\tau}\right]\right\}, \tag{6.225}$$

which is consistent with general relativity. On the other hand, taking the flat space-time limit $G_{\rm N} \to 0$ gives $g_{\mu\nu} = \eta_{\mu\nu}$, and therefore

$$m\,\eta_{\mu\nu}\, d^2 X^\nu = \left(\hbar\, \partial_\tau A_\mu - \partial_\mu \mathfrak{U}\right) d\tau^2 - \hbar\, H_{\mu\nu}\, dX^\nu d\tau. \tag{6.226}$$

If we then take the non-relativistic limit $c \to \infty$, we identify $t = \tau$ and replace $\eta_{\mu\nu} \to \delta_{ij}$.

[35]Note that $\mathfrak{U}$ and A_μ could contain an additional $\hbar$ dependence.

The resulting equation is consistent with stochastic quantization in flat spaces [184, 223, 267, 268, 270, 364].

The stochastic differential equation (6.223) is the fundamental equation of motion in stochastic quantization. The solutions describe the stochastic trajectories of quantum mechanical spin-0 test particles in any geometry. In section 6.7.3, we will show that probability density function associated to the solution $X(\tau)$ of this equation evolves according to the Schrödinger equation.

6.7.2 Stochastic Newton equation

The stochastic differential equation derived in previous section can be rewritten as a diffusion equation for the vector fields $v_\pm(x,\tau)$. This representation is known as the stochastic Newton equation, see e.g. Ref. [270]. In order to derive it, we define a function

$$R(x,\tau) := \frac{\hbar}{2} \ln\left[\rho(x,\tau)\right]. \tag{6.227}$$

The osmotic (6.183) and continuity equation (6.182) can then be rewritten as

$$\nabla^\mu R(x,\tau) = m\, \hat{u}^\mu, \tag{6.228}$$

$$\frac{\partial}{\partial\tau} R(x,\tau) = -\left(m\, g_{\mu\nu}\hat{u}^\nu + \frac{\hbar}{2}\nabla_\mu\right)\hat{v}^\mu. \tag{6.229}$$

Furthermore, we recall that the Hamilton Jacobi equations (6.175) and (6.177) are given by

$$\nabla_\mu S(x,\tau) = p_\mu, \tag{6.230}$$

$$\frac{\partial}{\partial\tau} S(x,\tau) = E_\tau\left[L(X,V,U,\tau)\right] - p_\mu v^\mu. \tag{6.231}$$

We consider the Lagrangian (6.188)

$$L(X,V,U,\tau) = \frac{m}{2} g_{\mu\nu}\left(V^\mu V^\mu + U^\mu U^\mu\right) - \hbar\, A_\mu V^\mu - \mathfrak{U} \tag{6.232}$$

with momenta

$$\begin{aligned} P_\mu(\tau) &= m\, g_{\mu\nu} V^\nu - \hbar\, A_\mu, \\ Q_\mu(\tau) &= m\, g_{\mu\nu} U^\nu. \end{aligned} \tag{6.233}$$

Therefore,

$$\begin{aligned} p_\mu(x,\tau) &= \mathbb{E}_\tau\left[P_\mu(\tau)\right] = m\, g_{\mu\nu} v^\nu - \hbar\, A_\mu, \\ \hat{q}_\mu(x,\tau) &= \mathbb{E}_\tau\left[Q_\mu(\tau)\right] = m\, g_{\mu\nu}\hat{u}^\nu. \end{aligned} \tag{6.234}$$

Moreover, in eq. (6.210), we found

$$\mathbb{E}_\tau\left[L(X,V,U,\tau)\right] = \frac{m}{2} g_{\mu\nu}\left(v^\mu v^\nu + \hat{u}^\mu \hat{u}^\nu\right) + \frac{\hbar}{2}\nabla_\mu \hat{u}^\mu - \frac{\hbar^2}{12m}\mathcal{R} - \hbar\, A_\mu v^\mu - \mathfrak{U}. \tag{6.235}$$

Putting everything together yields

$$\nabla_\mu S(x,\tau) = p_\mu = m\, g_{\mu\nu} v^\nu - \hbar\, A_\mu, \tag{6.236}$$
$$\nabla_\mu R(x,\tau) = \hat{q}_\mu = m\, g_{\mu\nu} \hat{u}^\nu \tag{6.237}$$

and

$$\frac{\partial}{\partial\tau} S(x,\tau) = -\frac{m}{2} g_{\mu\nu} \left(v^\mu v^\nu - \hat{u}^\mu \hat{u}^\nu \right) + \frac{\hbar}{2} \nabla_\mu \hat{u}^\mu - \frac{\hbar^2}{12m} \mathcal{R} - \mathfrak{U}, \tag{6.238}$$
$$\frac{\partial}{\partial\tau} R(x,\tau) = -m\, g_{\mu\nu} v^\mu \hat{u}^\mu - \frac{\hbar}{2} \nabla_\mu v^\mu. \tag{6.239}$$

We take the covariant derivative of eq. (6.238). This yields

$$m \frac{\partial v_\mu}{\partial\tau} - \hbar \frac{\partial A_\mu}{\partial \tau} = -m\, v^\nu \nabla_\mu v_\nu + m\, \hat{u}^\nu \nabla_\mu \hat{u}_\nu + \frac{\hbar}{2} \nabla_\mu \nabla_\nu \hat{u}^\nu - \frac{\hbar^2}{12m} \nabla_\mu \mathcal{R} - \nabla_\mu \mathfrak{U}. \tag{6.240}$$

Using eqs. (6.236) and (6.237), we find

$$\begin{aligned} \nabla_\mu \hat{u}_\nu &= \nabla_\nu \hat{u}_\mu, \\ \nabla_\mu v_\nu &= \nabla_\nu v_\mu + \frac{\hbar}{m} H_{\mu\nu}, \\ \nabla_\mu \nabla_\nu \hat{u}^\nu &= \Box \hat{u}_\mu - \mathcal{R}_{\mu\nu} \hat{u}^\nu. \end{aligned} \tag{6.241}$$

Therefore,

$$\begin{aligned} \hbar \left(\frac{\partial A_\mu}{\partial \tau} - H_{\mu\nu} \hat{v}^\nu \right) - \frac{\hbar^2}{12m} \nabla_\mu \mathcal{R} - \nabla_\mu \mathfrak{U} = m \bigg(& \frac{\partial \hat{v}_\mu}{\partial\tau} + v^\nu \nabla_\nu \hat{v}_\mu - \hat{u}^\nu \nabla_\nu \hat{u}_\mu \\ & - \hat{u}^{\rho\sigma} \nabla_\rho \nabla_\sigma \hat{u}_\mu + \hat{u}^{\rho\sigma} \mathcal{R}^\nu{}_{\rho\sigma\mu} \hat{u}_\nu \bigg). \end{aligned} \tag{6.242}$$

We will associate the left hand side with a force, i.e.

$$F_\mu := \hbar \left(\frac{\partial A_\mu}{\partial\tau} - H_{\mu\nu} \hat{v}^\nu \right) - \frac{\hbar^2}{12m} \nabla_\mu \mathcal{R} - \nabla_\mu \mathfrak{U}. \tag{6.243}$$

Moreover, we rewrite the left hand side in terms of the forward and backward velocity. We find

$$\begin{aligned} F^\mu = \frac{m}{2} \Bigg[& \left(\frac{\partial}{\partial\tau} + \hat{v}_+^\nu \nabla_\nu + \hat{v}_+^{\rho\sigma} \nabla_\rho \nabla_\sigma \right) \hat{v}_-^\mu - \mathcal{R}^\mu{}_{\rho\sigma\nu} \hat{v}_+^{\rho\sigma} \hat{v}_-^\nu \\ & + \left(\frac{\partial}{\partial\tau} + \hat{v}_-^\nu \nabla_\nu + \hat{v}_-^{\nu\rho} \nabla_\nu \nabla_\rho \right) \hat{v}_+^\mu - \mathcal{R}^\mu{}_{\rho\sigma\nu} \hat{v}_-^{\rho\sigma} \hat{v}_+^\nu \Bigg]. \end{aligned} \tag{6.244}$$

As we would like to associate the right hand side with an acceleration, we define second order acceleration vectors $a_{\pm\pm}$ by

$$\begin{aligned} a^\mu_{+\pm}(x,\tau) &:= \lim_{h\to 0} \frac{1}{h} \mathbb{E}_\tau \big[V^\mu_\pm(\tau+h) - V^\mu_\pm(\tau) \big], \\ a^\mu_{-\pm}(x,\tau) &:= \lim_{h\to 0} \frac{1}{h} \mathbb{E}_\tau \big[V^\mu_\pm(\tau) - V^\mu_\pm(\tau-h) \big], \end{aligned} \tag{6.245}$$

and

$$
\begin{aligned}
a^{\rho\sigma}_{+\pm}(x,\tau) &:= \lim_{h\to 0}\frac{1}{2h}\mathbb{E}_\tau\Big\{\big[V^\rho_\pm(\tau+h)-V^\rho_\pm(\tau)\big]\big[X^\sigma(\tau+h)-X^\sigma(\tau)\big]\Big\}\\
&\qquad + \frac{1}{2h}\mathbb{E}_\tau\Big\{\big[X^\rho(\tau+h)-X^\rho(\tau)\big]\big[V^\sigma_\pm(\tau+h)-V^\sigma_\pm(\tau)\big]\Big\},\\
a^{\rho\sigma}_{-\pm}(x,\tau) &:= \lim_{h\to 0}\frac{1}{2h}\mathbb{E}_\tau\Big\{\big[V^\rho_\pm(\tau)-V^\rho_\pm(\tau-h)\big]\big[X^\sigma(\tau)-X^\sigma(\tau-h)\big]\Big\}\\
&\qquad + \frac{1}{2h}\mathbb{E}_\tau\Big\{\big[X^\rho(\tau)-X^\rho(\tau-h)\big]\big[V^\sigma_\pm(\tau)-V^\sigma_\pm(\tau-h)\big]\Big\}.
\end{aligned}
\tag{6.246}
$$

Using the parallel transport equation (6.95), we then find

$$
\begin{aligned}
a^\mu_{+\pm} &= \lim_{d\tau\to 0}\frac{1}{d\tau}\mathbb{E}_\tau\Big[d_+\hat{v}^\mu_\pm + \Gamma^\mu_{\nu\rho}\hat{v}^\nu_\pm d_+x^\rho + \left(\partial_\nu\Gamma^\mu_{\rho\sigma}+\Gamma^\mu_{\nu\kappa}\Gamma^\kappa_{\rho\sigma}-2\Gamma^\mu_{\rho\kappa}\Gamma^\kappa_{\nu\sigma}\right)\hat{v}^\nu_\pm dx^\rho\cdot dx^\sigma + o(d\tau)\Big]\\
&= \partial_\tau\hat{v}^\mu_\pm + v^\nu_+\partial_\nu\hat{v}^\mu_\pm + v^{\rho\sigma}_+\partial_\rho\partial_\sigma\hat{v}^\mu_\pm + \Gamma^\mu_{\nu\rho}\hat{v}^\nu_\pm v^\rho_+ + \left(\partial_\nu\Gamma^\mu_{\rho\sigma}+\Gamma^\mu_{\nu\kappa}\Gamma^\kappa_{\rho\sigma}-2\Gamma^\mu_{\rho\kappa}\Gamma^\kappa_{\nu\sigma}\right)\hat{v}^\nu_\pm v^{\rho\sigma}_+\\
&= \partial_\tau\hat{v}^\mu_\pm + \hat{v}^\nu_+\nabla_\nu\hat{v}^\mu_\pm + \hat{v}^{\rho\sigma}_+\nabla_\rho\nabla_\sigma\hat{v}^\mu_\pm - 2\Gamma^\mu_{\nu\rho}\hat{v}^{\rho\sigma}_+\nabla_\sigma\hat{v}^\nu_\pm - \mathcal{R}^\mu{}_{\rho\sigma\nu}\hat{v}^{\rho\sigma}_+\hat{v}^\nu_\pm
\end{aligned}
\tag{6.247}
$$

and

$$
\begin{aligned}
a^\mu_{-\pm} &= \lim_{d\tau\to 0}\frac{1}{d\tau}\mathbb{E}_\tau\Big[d_-\hat{v}^\mu_\pm + \Gamma^\mu_{\nu\rho}\hat{v}^\nu_\pm d_-x^\rho - \left(\partial_\nu\Gamma^\mu_{\rho\sigma}+\Gamma^\mu_{\nu\kappa}\Gamma^\kappa_{\rho\sigma}-2\Gamma^\mu_{\rho\kappa}\Gamma^\kappa_{\nu\sigma}\right)\hat{v}^\nu_\pm dx^\rho\cdot dx^\sigma + o(d\tau)\Big]\\
&= \partial_\tau\hat{v}^\mu_\pm + \hat{v}^\nu_-\nabla_\nu\hat{v}^\mu_\pm + \hat{v}^{\rho\sigma}_-\nabla_\rho\nabla_\sigma\hat{v}^\mu_\pm - 2\Gamma^\mu_{\nu\rho}\hat{v}^{\rho\sigma}_-\nabla_\sigma\hat{v}^\nu_\pm - \mathcal{R}^\mu{}_{\rho\sigma\nu}\hat{v}^{\rho\sigma}_-\hat{v}^\nu_\pm,
\end{aligned}
\tag{6.248}
$$

where we allow for an explicit proper-time dependence of the velocity $v_\pm(X,\tau)$. For the second order parts we find

$$
\begin{aligned}
a^{\rho\sigma}_{+\pm} &= \lim_{d\tau\to 0}\frac{2}{d\tau}\mathbb{E}_\tau\left[d\hat{v}^{(\rho}_\pm\cdot dx^{\sigma)} + \Gamma^{(\rho|}_{\kappa\lambda}\hat{v}^\kappa_\pm\, dx^\lambda\cdot dx^{|\sigma)} + o(d\tau)\right]\\
&= \hat{v}^{\rho\kappa}_+\nabla_\kappa\hat{v}^\sigma_\pm + \hat{v}^{\kappa\sigma}_+\nabla_\kappa\hat{v}^\rho_\pm
\end{aligned}
\tag{6.249}
$$

and

$$
\begin{aligned}
a^{\rho\sigma}_{-\pm} &= \lim_{d\tau\to 0}\frac{2}{d\tau}\mathbb{E}_\tau\left[-d\hat{v}^{(\rho}_\pm\cdot dx^{\sigma)} - \Gamma^{(\rho|}_{\kappa\lambda}\hat{v}^\kappa_\pm\, dx^\lambda\cdot dx^{|\sigma)} + o(d\tau)\right]\\
&= \hat{v}^{\rho\kappa}_-\nabla_\kappa\hat{v}^\sigma_\pm + \hat{v}^{\kappa\sigma}_-\nabla_\kappa\hat{v}^\rho_\pm.
\end{aligned}
\tag{6.250}
$$

Eq. (6.244) can now be rewritten as the stochastic Newton equation

$$
F^\mu(X,\tau) = \frac{1}{2}m\left[\hat{a}^\mu_{+-}(X,\tau)+\hat{a}^\mu_{-+}(X,\tau)\right],
\tag{6.251}
$$

where $\hat{a}^\mu = a^\mu + \Gamma^\mu_{\rho\sigma}a^{\rho\sigma}$ is the covariant form of a^μ and F^μ is a first order vector.

There exists another representation of the stochastic Newton equation that is given by

$$
F^\mu(X,\tau) = \frac{1}{2}m\left(D_+D_- + D_-D_+\right)X^\mu,
\tag{6.252}
$$

where the *covariant diffusion operators* $D_\pm$ act on an arbitrary first order (k,l)-tensor field $A(X,\tau)$ as, cf. Refs. [135, 136, 270],

$$D_{\pm}A = \left[\frac{\partial}{\partial\tau} + \hat{v}^{\mu}_{\pm}\nabla_{\mu} + \hat{v}^{\mu\nu}_{\pm}\left(\nabla_{\mu}\nabla_{\nu} + \mathcal{R}^{\cdot}{}_{\mu\cdot\nu}\right)\right] A, \tag{6.253}$$

where

$$\mathcal{R}^{\cdot}{}_{\alpha\cdot\beta}A^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_l} = \sum_{i=1}^{k}\mathcal{R}^{\mu_i}{}_{\alpha\lambda\beta}A^{\mu_1\dots\mu_{i-1}\lambda\mu_{i+1}\dots\mu_k}_{\nu_1\dots\nu_l} - \sum_{j=1}^{l}\mathcal{R}^{\lambda}{}_{\alpha\nu_j\beta}A^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_{j-1}\lambda\mu_{j+1}\dots\nu_l}. \tag{6.254}$$

Using that $v^{\mu\nu}_{\pm} = \pm\frac{\hbar}{2m}g^{\mu\nu}$, eq. (6.253) can be rewritten as

$$D_{\pm}A = \left(\frac{\partial}{\partial\tau} + \hat{v}^{\mu}_{\pm}\nabla_{\mu} \pm \frac{\hbar}{2m}\Box_{\mathrm{DG}}\right) A, \tag{6.255}$$

where the *Dohrn-Guerra Laplacian* is defined by

$$\Box_{\mathrm{DG}} := g^{\mu\nu}\left(\nabla_{\mu}\nabla_{\nu} + \mathcal{R}^{\cdot}{}_{\mu\cdot\nu}\right). \tag{6.256}$$

6.7.3 Schrödinger equation

The solutions of the stochastic differential equation (6.223) are stochastic processes. One can associate a probability density to these stochastic processes, and derive a partial differential equation for the evolution of this probability density. As argued in the introduction, the equation governing this evolution is the Schrödinger equation. Here, we present an explicit derivation.

Using eqs. (6.236) and (6.237), we can rewrite eqs. (6.238) and (6.239) as

$$\frac{\partial}{\partial\tau}S(x,\tau) = -\frac{1}{2m}\left(\nabla_{\mu}S\nabla^{\mu}S - \nabla_{\mu}R\,\nabla^{\mu}R - \hbar\Box R + 2\hbar A^{\mu}\nabla_{\mu}S + \hbar^2 A_{\mu}A^{\mu} + \frac{\hbar^2}{6}\mathcal{R}\right) - \mathfrak{U}, \tag{6.257}$$

$$\frac{\partial}{\partial\tau}R(x,\tau) = -\frac{1}{m}\left(\nabla_{\mu}S\,\nabla^{\mu}R + A^{\mu}\nabla_{\mu}R + \frac{\hbar}{2}\Box S + \frac{\hbar^2}{2}\nabla_{\mu}A^{\mu}\right). \tag{6.258}$$

If we define the wave function

$$\Psi(x,\tau) = e^{\frac{1}{\hbar}(R+iS)}, \tag{6.259}$$

we find that these equations are equivalent to the equation

$$i\hbar\frac{\partial}{\partial\tau}\Psi = \left\{-\frac{\hbar^2}{2m}\left[\left(\nabla_{\mu} + iA_{\mu}\right)\left(\nabla^{\mu} + iA^{\mu}\right) - \frac{1}{6}\mathcal{R}\right] + \mathfrak{U}\right\}\Psi. \tag{6.260}$$

This is a generalization of the Schrödinger equation to pseudo-Riemannian geometry.[36] We note that the Born rule is an immediate consequence:

$$|\Psi(x,\tau)|^2 = e^{\frac{2}{\hbar}R(x,\tau)} = \rho(x,\tau) \tag{6.261}$$

[36] Note that for flat space-times $\mathcal{R} = 0$. Moreover, in the non-relativistic limit one replaces $x^{\mu} \to x^{i}$ and identifies $\tau = t$. Therefore, in the flat non-relativistic limit we obtain the standard Schrödinger equation.

by the definition of R in eq. (6.227).

6.7.4 Conformal coupling

In this section, we show that the generalization of the Schrödinger equation (6.260) imposes a conformal coupling of massive scalar particles to gravity. For this, we consider the Lagrangian of a free scalar field non-minimally coupled to gravity

$$\mathcal{L}(\phi, \nabla\phi) = -\frac{1}{2}\left(\nabla_\mu\phi\,\nabla^\mu\phi + \frac{m^2}{\hbar^2}\,\phi^2 + \xi\,\mathcal{R}\,\phi^2\right). \tag{6.262}$$

The field equation is given by the Klein-Gordon equation

$$\Box\,\phi = \frac{m^2}{\hbar^2}\,\phi + \xi\,\mathcal{R}\,\phi. \tag{6.263}$$

We can construct an explicitly proper time dependent field Φ defined on $\mathcal{M}\times T$, such that

$$\Phi(x,\tau) = \phi(x)\,e^{\frac{im}{2\hbar}\tau}, \tag{6.264}$$

where $x = (t, \vec{x})$ is a four-vector. Then Φ satisfies the generalized Schrödinger equation (6.260) with $A_\mu = 0$, $\mathfrak{U} = 0$ and conformal coupling $\xi = \frac{1}{6}$. This result can be generalized in a straightforward manner to the cases $A_\mu \neq 0$ and $\mathfrak{U} \neq 0$.

We conclude that stochastic quantization predicts that any scalar test particle must be conformally coupled to gravity. It is expected that this result can be generalized to arbitrary scalar fields. However, proof of this latter statement can only be achieved within a field theory description of stochastic quantization.

6.8 Discussion

In this paper, we have reviewed some aspects of second order geometry and stochastic quantization, and shown that the combination of the two leads to a consistent quantum theory on manifolds. In addition, we have further developed second order geometry, and constructed the notion of a Lie derivative in this framework. Furthermore, we have provided new results within stochastic quantization. In particular, we have shown that a diffeomorphism invariant framework of stochastic quantization imposes a conformal coupling of massive spin-0 test particles. It is expected that this result can be generalized to arbitrary scalar fields, but a proof of such a generalization requires further study of a field theory framework.

Since stochastic quantization can be formulated on (pseudo-)Riemannian manifolds, it is a natural approach to explore quantum gravity. However, in order to do so, a major hurdle must still be overcome, which is a consistent extension to both bosonic and fermionic field theories. Until now only a few specific bosonic examples have been studied in this framework, see for example Refs. [165, 184, 186, 188, 189, 227, 251, 265, 289], but no general formalism has yet been developed. The embedding of stochastic quantization into second order geometry, as developed in this paper could help guide the way towards such an extension. Particularly interesting in this respect are recent developments in the study of Lagrangian dynamics on higher order jet bundles, see e.g. Refs. [102, 103], as this is the natural extension of second order geometry to a field theory setting.

There are several studies that can be performed within the stochastic quantization framework without going to a field theory description or to dynamical backgrounds. The stochastic differential equation (6.223) allows to solve and simulate the motion of quantum mechanical spin-0 test particles charged under scalar and vector potentials in any geometry. Such a study would be particularly interesting when performed in black hole geometries. One can then calculate the probability that a particle hits the singularity[37] or escapes the black hole. Furthermore, one can calculate the expected proper time until one of these events occurs. Also, higher moments such as the variance for these events can be calculated. Such calculations could provide microscopic insights into Hawking radiation and black hole thermodynamics.

In this paper, we have restricted ourselves to time-like processes with positive mass. A formulation for space-like processes can be obtained by considering imaginary masses and by replacing the proper time with the proper distance. However, a theory for massless particles on null-like surfaces is not easily obtained from the theory presented in this paper, and deserves further study.

There are many other issues that deserve further exploration within the stochastic framework. For example, as discussed in the introduction, there is no consensus yet on the resolution of Wallstrom's criticism. Moreover, the notion of spin in stochastic quantization is only partially understood, see e.g. Refs. [127, 159, 270]. In this paper, we have focused on scalar particles, in the presence of commuting spin-0 and spin-1 fields and gravity. Extensions to fermions, non-commuting potentials and higher spin fields would be interesting to investigate.

Furthermore, the formulation of stochastic quantization presented here was entirely in a position representation. Investigation of the dual picture in terms of momenta deserves further exploration. Early considerations along these lines can for example be found in Ref. [314].

Another open question is whether stochastic quantization can be formulated on complex manifolds instead of real manifolds. An argument for such a construction is that the wave function resembles the probability density of a complex random variable $Z = X + iY$ with $dZ = (V + iU)d\tau$. Discussions along these lines can also be found in Ref. [285]. Related to this is the question whether the function R can be interpreted as an action for the background field in a Wick rotated version of the theory. The action S would then be related to the probability density for the coordinates Y.

Finally, the presence of an osmotic velocity in stochastic quantization could provide new insights in the nature of dark matter. In this respect, it is worth noticing that the kinetic energy in stochastic quantization does not only contain the classical kinetic energy given by $\frac{m}{2} g_{\mu\nu} v^\mu v^\nu$, but also the osmotic energy of the background field given

[37] In stochastic quantization, geodesic incompleteness of the space-time does not imply that the particle ends up at the singularity. One should study the Brownian completeness of the geometry instead, see e.g. Section 5 in Ref. [150].

by $\frac{m}{2} g_{\mu\nu} \hat{u}^{\mu} \hat{u}^{\nu}$. It is expected that the notion of osmotic energy is also present in a field theoretical extension of stochastic quantization. In such an extension it will take the shape of the kinetic term of additional fields that only interact gravitationally with other fields. This suggests that the osmotic energy could be interpreted as dark matter.

We conclude that stochastic quantization is an interesting framework, that deserves further exploration. We are currently investigating several aspects of the theory along the lines mentioned above, and hope to report on it elsewhere.

Acknowledgments

This work is supported by a doctoral studentship of the Science and Technology Facilities Council. I would like to thank Joshua Erlich for interesting discussions on stochastic quantization. Furthermore, I would like to thank Xavier Calmet for helpful comments on the manuscript.

Chapter 7

Stochastic Quantization of Relativistic Theories

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Abstract
It was shown recently that stochastic quantization can be made into a well defined quantization scheme on (pseudo-)Riemannian manifolds using second order differential geometry, which is an extension of the commonly used first order differential geometry. In this letter, we show that restrictions to relativistic theories can be obtained from this theory by imposing a stochastic energy-momentum relation. In the process, we derive non-perturbative quantum corrections to the line element as measured by scalar particles. Furthermore, we extend the framework of stochastic quantization to massless scalar particles.

This chapter has been published in the Journal of Mathematical Physics **62**, no. 12, p. 2301 (2021).
A preprint of this chapter can be found at arXiv:2103.02501 [gr-qc].

7.1 Introduction

Stochastic quantization is a quantization scheme comparable to canonical quantization and path integral quantization that is employed in the theory of stochastic mechanics [153,184,187,223,267,268,270,362–364]. Stochastic mechanics is a theory of Newtonian mechanics coupled to a fluctuating Gaussian background field. Due to the coupling to this background field, particles follow stochastic processes instead of deterministic trajectories. The evolution of the probability density of these processes is governed by complex diffusion equations.

Processes described by complex diffusion equations generically have a single well defined position, but two independent well defined velocities. If one imposes there to be a single well defined velocity, one obtains a real diffusion equation that is better known as the heat equation. The process described by the heat equation is the well known dissipative Brownian motion. This dissipative Brownian motion breaks time reversal symmetry. If, on the other hand, time reversibility is imposed as a constraint, the governing complex diffusion equation is the Schrödinger equation. The resulting process is often called a conservative Brownian motion or a Nelson process.

The derivation of the Schrödinger equation for a Newtonian system coupled to a time reversible Gaussian background field is the central result of stochastic mechanics. The stochastic quantization scheme that is employed in stochastic mechanics is build upon five fundamental principles: diffeomorphism invariance, gauge invariance, time reversal symmetry, the principle of least action and the background hypothesis.

The background hypothesis states that all variables in the theory must be promoted to random variables and the trajectories to time reversible semi-martingale processes. The quadratic variation for these stochastic processes is fixed by the background hypothesis. For massive scalar particles the condition on the quadratic variation takes the form[1]

$$d[[X^\mu, X^\nu]] = \frac{\hbar}{m} h^{\mu\nu}\, d\tau, \tag{7.1}$$

where h is a positive definite metric tensor, obtained from the metric tensor g with Lorentzian signature by a Wick rotation. The construction of this positive definite tensor is discussed in more detail in Ref. [137] and reviewed in appendix 7.A. We note that this condition imposes the stochastic part of X to be a scaled Brownian motion by the Lévy characterization. Furthermore, we remark that this relation is the equivalent of the canonical commutation relation imposed in the canonical quantization scheme.

Stochastic quantization is closely related to the path integral formulation, as it can be regarded as a local construction scheme for path integrals. In imaginary time, this is achieved by the Feynman-Kac theorem [218], which maps the path integral formulation to the stochastic formulation. Stochastic quantization extends this stochastic formulation to a real time description. An extension of the Feynman-Kac theorem to the real time path integral is given by the Feynman-Itô theorem [14, 214]. Although this theorem does not

[1] In order to avoid confusion with the commutator, we denote the quadratic variation with a double bracket $[[X^\mu, X^\nu]]$ instead of a single bracket, which is the more common notation.

have an immediate stochastic interpretation, the real time path integral has been related explicitly to the stochastic quantization framework [288].

The mathematical advantage of the stochastic quantization scheme over the path integral formulation resides in the fact that stochastic integrals are better understood than path integrals. This is an important motivation for the study of stochastic quantization. For similar reasons, the framework is used as an important tool in constructive approaches to quantum field theory [14, 270]. The study of constructive approaches to quantum field theory bears relevance, as the absence of a mathematically rigorous framework of relativistic quantum field theory lies at the heart of several issues in quantum field theory. One of which is the non-renormalizability of gravity as a quantum theory.

A second motivation for the study of stochastic mechanics is of a foundational nature. The philosophy governing stochastic quantization is closely related to the quantum foam introduced by Wheeler [353]. However, in stochastic quantization the quantum foam is considered to be the source rather than the consequence of quantum mechanics.

Stochastic mechanics is a classical[2] probabilistic[3] interpretation of quantum theory. In this framework, the physical configuration space is a measurable covering space of the classical configuration space. The L^2-space containing the wave functions is built on top of this. Although this L^2-space is crucial for mathematical analysis, global existence of the wave functions is not required in a stochastic formulation. The wave function represents the best possible prediction of a system given the measurements of the system at earlier times, but is not a physical object. Measuring a system amounts to conditioning the stochastic process.[4] Collapse of the wave function thus occurs due to updating the filtration to which the process is adapted.[5]

Finally, stochastic quantization has received attention, since it can be used as a computational framework in quantum field theory. Stochastic quantization provides an alter-

[2]We call the theory classical, as the quantum configuration space is a covering space over the classical configuration space. The covering is crucial for the treatment of intrinsically quantum properties such as spin and discretized spectra, cf. e.g. Ref. [270].

[3]We call a theory probabilistic, if there is a structure of a probability space $(\Omega, \Sigma, \mathbb{P})$, a measurable configuration space $(\mathcal{M}, \mathcal{B}(\mathcal{M}), \mu)$ and random variables $X : (\Omega, \Sigma, \mathbb{P}) \to (\mathcal{M}, \mathcal{B}(\mathcal{M}))$ such that $\mu = \mathbb{P} \circ X^{-1}$. The random variables are elements of an L^p-space. As usual in quantum mechanics, we consider the L^2-space, which has the important properties that it is a Hilbert space and that it is self-dual.

[4]We consider measurements where the interaction between the measurement device and the system is negligible. For microscopic systems, such measurements are unachievable. However, these interactions are unrelated to the wave function collapse in the stochastic interpretation.

[5]Let us add a clarification by making a comparison to stock markets: the shares in a stock market have a well defined value at any point in time. However, if we do not observe the value for a certain amount of time, we can only give a probabilistic description of the value of the stock, which is modeled by a probability distribution. Once we decide to observe the market this probability distribution collapses to a delta distribution. According to stochastic mechanics the situation in quantum mechanics is similar. A difference between the two pictures is that quantum mechanics is governed by a time-reversible Brownian motion, while stock markets are usually modeled by a dissipative Brownian motion. As a consequence, quantum mechanics is modeled by a complex wave function, while the probability distributions in stock markets often take the shape of a Gaussian profile. We should stress that the picture is not in conflict with the superposition principle. The superposition principle holds in the stochastic interpretation as particles move between different layers in the covering space. Before measuring a particle, the observers can only give a probabilistic prediction on which layer they will measure the particle, and thus what values of spin or other discretized spectra they will measure. This leads to the superposition principle in the description given by the observer. Furthermore, we emphasize that stochastic mechanics is agnostic about the question whether the quantum fluctuations are fundamental or can be derived from a more fundamental deterministic theory. However, the Bell experiments suggest that the stochasticity is fundamental.

native mathematical model that can be used to calculate observables in quantum theories. For certain problems this could simplify the calculations, while other problems are more easily solved using standard quantum field theory methods. Stochastic quantization should therefore be regarded as complementary to other approaches. In this respect, the reformulation due to Parisi and Wu [125, 126, 284] has achieved considerable success in numerical calculations. This reformulation has also been related to quantum gravity inspired theories [134, 248, 281].

The general formalism of stochastic quantization is a well defined approach to quantum mechanics for non-relativistic scalar particles on $\mathbb{R}^n$ charged under scalar and vector potentials. Extensions have been made to Riemannian manifolds [127, 135–137, 187, 270]. In addition, particles with spin have been discussed in this framework, cf. e.g. Refs. [127, 159, 270]. Furthermore, field theoretic extensions have been developed, see e.g. [165, 184, 186, 188, 189, 227, 251, 265, 289]. We note that the field theoretic framework is more evolved in the Parisi-Wu formulation. Furthermore, it is worth noticing that many standard quantum mechanics problems have been discussed in the stochastic quantization framework, see e.g. Refs. [164, 178, 184, 270, 278, 290, 293, 364]. Finally, the ideas governing stochastic quantization have been incorporated in models of quantum gravity [151, 249]. For a more complete review of stochastic quantization we refer to Refs. [184, 233, 270, 364].

Most successes of stochastic quantization are of a non-relativistic nature. Although a relativistic version has been treated in the literature, cf. e.g. Refs. [137, 165, 184, 188, 189, 251, 265, 289], it is not as well established as the non-relativistic theory. In this letter, we remedy this and show that stochastic quantization can be made into a relativistic quantization scheme. Here, we build on our previous work [233], where stochastic quantization was extended to (pseudo-)Riemannian geometry. In this letter, we restrict this general framework to a special class of theories, namely the relativistic theories defined on Lorentzian manifolds. More concretely, we discuss the stochastic quantization of a single relativistic spinless particle on a curved space-time charged under scalar and vector potentials.

A difficulty that arises, when one tries to extend stochastic quantization to (pseudo-)Riemannian manifolds is that there exists a single well defined position X, but two independent well defined velocities[6]

$$\begin{aligned} v_+(X(\tau),\tau) &= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}\left[X(\tau+h)-X(\tau)|X(\tau)\right], \\ v_-(X(\tau),\tau) &= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}\left[X(\tau)-X(\tau-h)|X(\tau)\right], \end{aligned} \tag{7.2}$$

which are often re-expressed as $v=\frac{1}{2}(v_+ + v_-)$ and $u=\frac{1}{2}(v_+ - v_-)$. These velocity vectors are not vectors in the usual geometrical sense, i.e. they do not transform as vectors under coordinate transformations. Therefore, stochastic quantization cannot be easily embedded in differential geometry, which is the mathematical corner stone of general relativity. This issue was resolved for semi-martingale processes on smooth manifolds with a connection

[6]Note that the definition requires to take conditional expectations. Without this conditional expectation there is no notion of velocity, as the stochastic process is almost surely nowhere differentiable.

by extending the ordinary first order geometry to a second order geometry, cf. Refs. [150, 257, 311]. In second order geometry the (co)tangent spaces are extended to second order (co)tangent spaces. This allows to interpret $v_\pm$ as vectors in these second order spaces. Consequently, the stochastic processes discussed in this paper are diffeomorphism invariant. We refer to Ref. [233] for a more detailed exposition of stochastic quantization in the context of second order geometry.

This paper is organized as follows: in the next section 7.2, we discuss relativistic massive theories. In section 7.3, we extend stochastic quantization to massless theories. In section 7.4, we discuss the notion of off-shellness in stochastic quantization, and in section 7.5 we conclude.

7.2 Massive scalar particles

We consider the classical relativistic action

$$S(x) = -\left[\int m\sqrt{-g_{\mu\nu}(x)\, v^\mu\, v^\nu} + q\, A_\mu(x)\, v^\mu\right] d\tau \tag{7.3}$$

defined on an $(n = (d+1))$-dimensional Lorentzian manifold $\mathcal{M}$. Following standard procedures we rewrite this action in the form

$$S(x) = \int \left[\frac{e}{2}\left(e^{-2} g_{\mu\nu}(x)\, v^\mu\, v^\nu - m^2\right) - q\, A_\mu(x)\, v^\mu\right] d\tau, \tag{7.4}$$

where e is an einbein field along the worldline of the particle. As we will consider the equations of motion of massive particles under the gauge fixing condition $e = m^{-1}$, this action is equivalent to the action

$$S(x) = \int \left[\frac{m+\lambda}{2}\, g_{\mu\nu}(x)\, v^\mu\, v^\nu + \frac{\lambda}{2} - q\, A_\mu(x)\, v^\mu\right] d\tau, \tag{7.5}$$

where λ is a Lagrange multiplier that must be gauge fixed to $\lambda = 0$ in the equations of motion. Its equation of motion is algebraic and reproduces the energy-momentum relation

$$g_{\mu\nu} v^\mu v^\nu = -1. \tag{7.6}$$

We will thus consider the classical Lagrangian

$$L_c(x, v) = \frac{m+\lambda}{2}\, g_{\mu\nu}(x)\, v^\mu\, v^\nu + \frac{\lambda}{2} - q\, A_\mu(x)\, v^\mu. \tag{7.7}$$

If the gauge symmetries of the classical action are to be preserved, the stochastic quantization of this Lagrangian is given by, cf. Ref. [233, 364],

$$L(X, V, U) = \frac{m+\lambda}{2}\, g_{\mu\nu}(X)\, (V^\mu\, V^\nu + U^\mu\, U^\nu) + \frac{\lambda}{2} - q\, A_\mu(X)\, V^\mu, \tag{7.8}$$

where (X, V, U) is a stochastic process on the second order tangent bundle $\hat{T}\mathcal{M}$. X represents the position, V the current velocity and U the osmotic velocity. The corresponding action is given by

$$S(X) = \mathbb{E}\left[\int L(X,V,U)\, d\tau\right], \tag{7.9}$$

where τ is the proper time. The equation of motion for λ yields the stochastic energy-momentum relation

$$\mathbb{E}\left[g_{\mu\nu}\left(V^\mu V^\nu + U^\mu U^\nu\right)\right] = -1 \tag{7.10}$$

or equivalently, cf. Ref. [233],

$$\mathbb{E}\Big[g_{\mu\nu}\left(dX^\mu dX^\nu + d_\circ \hat{X}^\mu d_\circ \hat{X}^\nu\right) + \frac{\hbar}{m}\nabla_\mu\left(d_\circ \hat{X}^\mu\right) d\tau - \frac{\hbar^2}{6m^2}\mathcal{R}\, d\tau^2\Big] = -d\tau^2. \tag{7.11}$$

We note that the geometrical line element remains $g_{\mu\nu}dx^\mu dx^\nu = -d\tau^2$. However, a quantum particle traveling through this geometry does not measure the same length, as it fluctuates around its classical path. Due to these quantum fluctuations, the line element as measured by a quantum particle obtains a quantum correction as given in eq. (7.11). For a single scalar particle adapted to its own natural filtration the osmotic integral vanishes, cf. Ref. [233]. This allows to re-express the quantized energy-momentum relation as

$$\mathbb{E}\left[g_{\mu\nu}\, dX^\mu dX^\nu + \left(1 - \frac{\hbar^2}{6m^2}\mathcal{R}\right) d\tau^2\right] = 0. \tag{7.12}$$

It follows that scalar quantum particles fluctuate around a quantum corrected path, where the quantum correction is given by the term $\frac{\hbar^2}{6m^2}\mathcal{R}$.

Minimizing the action leads to the stochastic differential equations in the sense of Stratonovich, cf. Ref. [233],

$$m\, g_{\mu\nu}\left(d^2 X^\nu + \Gamma^\nu_{\rho\sigma}\, dX^\rho dX^\sigma\right) = -\frac{\hbar^2}{12m}\nabla_\mu \mathcal{R}\, d\tau^2 - q\left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right) dX^\nu d\tau \tag{7.13}$$

and the condition (7.12). When supplemented with the background hypothesis

$$d[[X^\mu, X^\nu]] = \frac{\hbar}{m} h^{\mu\nu}(X)\, d\tau, \tag{7.14}$$

these equations can be solved for the appropriate boundary conditions. The result is a stochastic process $X(\tau)$ parametrized by the proper time. Observables of the theory can be determined from this stochastic process using the standard definitions of the characteristic and moment generating functional

$$\Phi_X(J) = \mathbb{E}\left[e^{\frac{i}{\hbar}\int J_\mu(\tau) X^\mu(\tau)\, d\tau}\right], \tag{7.15}$$

$$M_X(J) = \mathbb{E}\left[e^{\frac{1}{\hbar}\int J_\mu(\tau) X^\mu(\tau)\, d\tau}\right]. \tag{7.16}$$

We remark that in contrast to the path integral framework, these expressions do not average over the action. The averaging over the action effectively takes place when the system of equations (7.12), (7.13) and (7.14) is solved.

If a probability density $\rho(x,\tau)$ associated to the stochastic process X exists, one can construct the wave function[7]

$$\Psi(x,\tau) = \sqrt{\rho(x,\tau)}\, e^{\frac{i}{\hbar} S(x,\tau)} \tag{7.17}$$

with Hamilton's principal function defined by

$$S(x,\tau) = \mathbb{E}\left[\int_{\tau_i}^{\tau} L\big(X,V,U\big)\, dt \Big| X(\tau) = x \right]. \tag{7.18}$$

One can then show that this wave function must evolve according to a generalization of the Schrödinger equation, cf. Ref. [233] and references therein,

$$i\hbar \frac{\partial}{\partial \tau} \Psi = -\frac{\hbar^2}{2m} \left[\Big(\nabla_\mu + i\,\frac{q}{\hbar}\, A_\mu\Big) \Big(\nabla^\mu + i\,\frac{q}{\hbar}\, A^\mu\Big) - \frac{1}{6}\mathcal{R} \right] \Psi. \tag{7.19}$$

As there is no explicit dependence on the affine parameter τ, one can solve eq. (7.19) by separation of variables such that

$$\Psi(x,\tau) = \Phi_\alpha(x) \exp\left(\frac{i\, m\, \alpha}{2\, \hbar} \tau \right), \tag{7.20}$$

where α is a dimensionless parameter. If we gauge fix τ to be the proper time, we impose the condition (7.10). Under this constraint the expectation of the energy becomes $-\frac{m}{2}$, which implies $\alpha = 1$. We conclude that

$$\Psi(x,\tau) = \Phi(x) \exp\left(\frac{i\, m}{2\, \hbar} \tau \right), \tag{7.21}$$

where $\Phi(x)$ solves the generalization of the Klein-Gordon equation given by

$$\left[\Big(\nabla_\mu + i\,\frac{q}{\hbar}\, A_\mu\Big) \Big(\nabla^\mu + i\,\frac{q}{\hbar}\, A^\mu\Big) - \frac{1}{6}\mathcal{R} - \frac{m^2}{\hbar^2} \right] \Phi = 0. \tag{7.22}$$

We remark that the function $\Psi(x,\tau)$ and the relativistic Schrödinger equation (7.19) are not constructed in the traditional approaches to the quantization of relativistic theories. However, their construction is not forbidden in these approaches, while their construction seems necessary in the stochastic approach. The reason for this is that the probability density is defined on the space $\mathcal{M} \times \mathcal{T}$, where $\mathcal{M}$ is the space-time manifold and $\mathcal{T}$ is the proper time monoid. Moreover, the wave function $\Psi(x,\tau)$ is defined on the universal cover[8] of $\mathcal{M} \times \mathcal{T}$.

An important feature of relativistic theories is that the theory is invariant under proper time reparametrizations. Therefore, one can always perform separation of variables. Consequently, it is sufficient to consider the Klein-Gordon equation for the wave function $\Phi(x)$ in any relativistic quantum theory, as it determines the dynamics of the function $\Psi(x,\tau)$

[7]Note that the wave function is not always well defined on the configuration space, as this space might not be simply connected. This is the essence of Wallstrom's criticism [346, 347]. However, if the process is lifted to the universal cover of the configuration space, the wave function Ψ becomes well defined, cf. Ref. [270].

[8]See previous footnote

completely up to a phase factor. This phase factor is given in eq. (7.21) and is a genuine prediction of stochastic quantization.

7.3 Massless scalar particles

Following similar arguments as in previous section using the gauge fixing $e = 1$, we obtain the stochastic Lagrangian

$$L(X, V, U) = \frac{\lambda}{2}\, g_{\mu\nu}(X)\,(V^\mu\, V^\nu + U^\mu\, U^\nu) - q\, A_\mu(X)\, V^\mu, \tag{7.23}$$

where the Lagrange multiplier must be gauge fixed to $\lambda = 1$ in the equations of motion. The equation of motion for the Lagrange multiplier yields the stochastic energy-momentum relation

$$\mathbb{E}\Big[g_{\mu\nu}\,(V^\mu V^\nu + U^\mu U^\nu)\Big] = 0, \tag{7.24}$$

which can be rewritten as

$$\mathbb{E}\left[g_{\mu\nu}\, dX^\mu dX^\nu - \frac{\hbar^2}{6}\mathcal{R}\, d\tau^2\right] = 0. \tag{7.25}$$

Minimizing the action leads to stochastic differential equations in the sense of Stratonovich given by

$$g_{\mu\nu}\left(d^2X^\nu + \Gamma^\nu_{\rho\sigma}\, dX^\rho dX^\sigma\right) = -\frac{\hbar^2}{12m}\nabla_\mu \mathcal{R}\, d\eta^2 - q\,(\nabla_\mu A_\nu - \nabla_\nu A_\mu)\, dX^\nu d\eta \tag{7.26}$$

and the constraint (7.25). We note that η is an affine parameter that has the dimension of time per unit mass. The background hypothesis in the massless case under the gauge fixing $\lambda = 1$ takes the shape

$$d[[X^\mu, X^\nu]] = \hbar\, h^{\mu\nu}(X)\, d\eta. \tag{7.27}$$

The system of equations (7.25), (7.26) and (7.27) can be solved for the appropriate boundary conditions. The result is a stochastic process $X(\eta)$ parametrized by the parameter η. Observables of the theory can be determined from this stochastic process using the characteristic and moment generating functional.

The derivation of the Schrödinger equation in the massless case is similar to the derivation in the massive case, which can be found in Ref. [233] and references therein. If a probability density $\rho(x, \eta)$ associated to the stochastic process X exists, one can construct the wave function

$$\Psi(x, \eta) = \sqrt{\rho(x, \eta)}\, \exp\left\{\frac{i}{\hbar}\,\mathbb{E}\left[\int_{\eta_i}^{\eta} L\big(X(t), V(t), U(t), t\big)\, dt\,\middle|\, X(\eta) = x\right]\right\} \tag{7.28}$$

that evolves according to a generalization of the Schrödinger equation

$$i\hbar\,\frac{\partial}{\partial\eta}\Psi = -\frac{\hbar^2}{2}\left[\left(\nabla_\mu + i\,\frac{q}{\hbar}\,A_\mu\right)\left(\nabla^\mu + i\,\frac{q}{\hbar}\,A^\mu\right) - \frac{1}{6}\mathcal{R}\right]\Psi. \tag{7.29}$$

As there is no explicit dependence on the affine parameter η, one can solve by separation of variables, such that

$$\Psi(x,\eta) = \Phi_\alpha(x)\exp\left(\frac{i\,\hbar\,\alpha}{2}\,\eta\right), \tag{7.30}$$

where α has the dimension of inverse length squared. If we impose the condition (7.24), the kinetic energy becomes 0. This imposes $\alpha = 0$. We conclude that

$$\Psi(x,\eta) = \Phi(x), \tag{7.31}$$

where $\Phi(x)$ solves the generalization of the Klein-Gordon equation given by

$$\left(\left[\nabla_\mu + i\,\frac{q}{\hbar}\,A_\mu\right]\left[\nabla^\mu + i\,\frac{q}{\hbar}\,A^\mu\right] - \frac{1}{6}\mathcal{R}\right)\Phi = 0. \tag{7.32}$$

We remark that the vanishing phase factor in eq. (7.31) is expected, as massless particles are restricted to d-dimensional submanifolds of $\mathcal{M}$.

7.4 Off-shell motion

Let us consider the Lagrangian (7.7) for a massive particle in the simple case that $g_{\mu\nu} = \eta_{\mu\nu}$ and $q = 0$. The on-shell condition (7.12) is then given by

$$\mathbb{E}\Big[\eta_{\mu\nu}\,dX^\mu dX^\nu\Big] = -d\tau^2. \tag{7.33}$$

Moreover,

$$dX^\mu(\tau) = v^\mu(\tau)\,d\tau + \frac{1}{2}\Big(dW_+^\mu(\tau) + dW_-^\mu(\tau)\Big), \tag{7.34}$$

where $W_\pm$ are independent Brownian motions, cf. e.g. Ref. [233] and references therein. Consequently,

$$\begin{aligned}\mathbb{E}\Big[\eta_{\mu\nu}\,dX^\mu dX^\nu\Big] &= \mathbb{E}\Big[\eta_{\mu\nu}\Big(v^\mu v^\nu d\tau^2 + v^\mu\big(dW_+^\nu + dW_-^\nu\big)d\tau \\ &\qquad + \frac{1}{4}\big(dW_+^\mu(\tau) + dW_-^\mu(\tau)\big)\big(dW_+^\nu + dW_-^\nu\big)\Big)\Big] \\ &= \eta_{\mu\nu}\,v^\mu v^\nu\,d\tau^2,\end{aligned} \tag{7.35}$$

where we used that

$$\mathbb{E}\left[dW_\pm^\mu\right] = 0, \tag{7.36}$$

$$\mathbb{E}\left[dW_+^\mu dW_-^\nu\right] = \mathbb{E}\left[dW_+^\mu\right]\mathbb{E}\left[dW_-^\nu\right], \tag{7.37}$$

$$\mathbb{E}\left[dW_+^\mu dW_+^\nu\right] = -\mathbb{E}\left[dW_-^\mu dW_-^\nu\right]. \tag{7.38}$$

The first equation follows from the fact that $W_\pm$ is a martingale, the second from the stochastic independence of the forward and backward processes W_+ and W_-, and the third from the time reversibility of the semi-martingale X.

Under the expectation value the particle moves on-shell, i.e.

$$\eta_{\mu\nu}\, v^\mu v^\nu = -1. \tag{7.39}$$

However, without the expectation value this relation is not satisfied. Therefore, the expected trajectory of a particle is on-shell, but the actual trajectory of a particle can be off-shell. As $dW_\pm^\mu(\tau) \sim \mathcal{N}\left(0, \frac{\hbar}{m} d\tau\right)$, it is easy to see that the quantum fluctuations dominate in the regime

$$c\, d\tau \lesssim \frac{\hbar}{m\, c}, \tag{7.40}$$

which corresponds to length scales less than the de Broglie wavelength. On these length scales the event $\{\eta_{\mu\nu}\, dX^\mu dX^\nu \geq 0\}$ becomes very likely. Therefore, according to stochastic mechanics, particles have a high probability of traveling faster than light on length scales less than the de Broglie wavelength, while the probability of traveling faster than light over length scales larger than the de Broglie wavelength quickly decays to 0. According to the stochastic interpretation, this is the reason why particles are not localized within their de Broglie wavelength. We remark that this interpretation is given in a position representation, where the process (X, V, U) is adapted to the natural filtration of X. In other words we perform position measurements only. As is the case in other quantization schemes, stochastic quantization predicts an uncertainty relation between position and momentum measurements.

We note that this result is similar in the path integral approach. However, there is a difference in the interpretation: in the stochastic approach there is a single well defined stochastic trajectory, while the path integral approach considers the statistical ensemble of the sample paths of the stochastic trajectory. These sample paths are virtual and in this approach there is no notion of the real trajectory. From the perspective of modern probability theory, the path integral can in principle be derived from the stochastic integral, if both are well defined. Consequently, it is unlikely that the two interpretations can be distinguished experimentally, as their physical predictions are equivalent.

7.5 Conclusion

In this letter, we have shown that stochastic quantization can be made into a well defined quantization scheme for relativistic theories. Furthermore, we have extended the framework such that it includes massless particles. We point out that stochastic quantization is a local quantization scheme and that the motion of particles in this framework is governed by stochastic differential equations. In this framework, the Schrödinger equation and Klein-Gordon equation are derived from first principles. Finally, we have discussed the interpretation of off-shellness in the stochastic framework. We conclude that stochastic quantization is an interesting framework with important implications for the mathematical and philosophical foundations of quantum theory.

Acknowledgments

This work is supported by a doctoral studentship of the Science and Technology Facilities Council. I am grateful to Xavier Calmet for helpful comments on the manuscript.

7.A Construction of the Brownian metric

The background field hypothesis was introduced in Ref. [270] for massive particles as

$$d[[X^\mu, X^\nu]] = \frac{\hbar}{m} g^{\mu\nu}(X)\, d\tau. \tag{7.41}$$

If $g^{\mu\nu}$ has a definite signature this condition has a semi-martingale solution, but for indefinite signature there exist no semi-martingale satisfying this condition. The extension of this condition to manifolds with indefinite signatures, and in particular with a Lorentzian signature has been the subject of several studies, see e.g. Refs. [137, 165, 184, 188, 189, 251, 265, 289]. In this paper, we adopt the approach discussed in Ref. [137].

We reformulate the background hypothesis as

$$d[[X^\mu, X^\nu]] = \frac{\hbar}{m} h^{\mu\nu}(X)\, d\tau, \tag{7.42}$$

where $h^{\mu\nu}$ is a positive definite tensor that is sometimes called the *Brownian metric*. Its inverse $h_{\mu\nu}$ is defined by the relation

$$h_{\mu\nu} h^{\nu\rho} = \delta^\rho_\mu. \tag{7.43}$$

Moreover, it is related to the kinetic metric $g_{\mu\nu}$ through the compatibility condition

$$g_{\mu\nu} h^{\mu\rho} h^{\nu\sigma} = g^{\rho\sigma}. \tag{7.44}$$

where $g^{\mu\nu}$ is the inverse of the *kinetic metric* $g_{\mu\nu}$. If the kinetic metric $g_{\mu\nu}$ has a definite signature, the compatibility condition yields a unique solution for the Brownian metric $g^{\mu\nu} = h^{\mu\nu}$, but for a Lorentzian signature there is a family of positive definite solutions $h^{\mu\nu}$. In this paper, we work in the $(- + ... +)$ convention and set

$$h^{\mu\nu} = g^{\mu\nu} + 2\, u^\mu u^\nu \tag{7.45}$$

with time-like vector $u^\mu = (1, 0, ..., 0)$, which is uniquely defined and satisfies the given conditions.

We remark that in order to obtain a covariant stochastic theory, we have adopted the Schwartz-Meyer second order geometry framework discussed in Refs. [150, 257, 311]. In a local coordinate system second order vectors can be expressed as

$$V = v^\mu\, \partial_\mu + v^{\mu\nu}\, \partial_\mu \partial_\nu, \tag{7.46}$$

where $v^{\mu\nu}$ is the second order part of the vector v. As discussed in Ref. [233], the background hypothesis fixes the second order part of the velocity vectors such that

$$v_{\pm}^{\mu\nu} = \pm \frac{\hbar}{2m} \, g^{\mu\nu} \, d\tau, \tag{7.47}$$

which is defined in terms of the kinetic metric. Consequently, the kinetic equations (7.12), (7.13) and (7.19) are independent of the Brownian metric, as was already observed in Ref. [137].

Finally, we notice that the constructions in this appendix can be generalized straightforwardly to the massless case discussed in section 7.3.

Chapter 8

Analytic Continuation of Stochastic Mechanics

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Abstract
We study a (relativistic) Brownian motion on a complexified (pseudo-)Riemannian manifold. Using Nelson's stochastic quantization, we derive three equivalent descriptions for this problem. If the process has a purely real pseudo-variance, we obtain the ordinary dissipative Brownian motion. In this case, the result coincides with the Feynman-Kac formula. On the other hand, for a purely imaginary pseudo-variance, we obtain a conservative Brownian motion, which provides a description of a quantum particle on a curved spacetime.

This chapter has been published in the Journal of Mathematical Physics **63**, no. 04, p. 2301 (2022).
A preprint of this chapter can be found at arXiv:2109.10710 [math-ph].

8.1 Introduction

Brownian motion has been at the forefront of physics research ever since the phenomenon, first observed by Brown, was described by Einstein. Moreover, it has attracted much attention in the mathematics literature, since the early works of Wiener, Kolmogorov and Lévy, and it plays a major role in the stochastic calculus developed by Itô and Stratonovich.

Nowadays the literature on Brownian motion is rich and extends far beyond its original purpose of describing the motion of pollen suspended in water. In particular, since the introduction of the path integral by Feynman, it has become an important tool in quantum physics. This is mainly due to the Feynman-Kac theorem [218], which made use of the relation between the Euclidean path integral and the Wiener integral. This result became one of the cornerstones of the mathematical foundations of Euclidean quantum field theory, and has been used by several authors as a starting point in attempts to develop a mathematically consistent formulation of Lorentzian quantum field theory, cf. e.g. Refs. [14, 177] for reviews.

Later, Parisi and Wu exploited the relation between Brownian motion and Euclidean quantum field theory to develop a framework called stochastic quantization [125, 284], which became a very useful computational tool in Euclidean quantum field theory. In recent years, this framework has also been used to relate various string theory inspired models [134, 196, 248, 281].

Before the work of Parisi and Wu, the notion of stochastic quantization was used by Nelson in the theory of stochastic mechanics [267]. This theory, originally proposed by Fényes [153], serves as an interpretation of quantum mechanics in which quantum mechanics is generated by a time reversible Nelson process. However, later studies of stochastic mechanics were also motivated by the fact that it can be used as an computational framework in quantum theories [184] or as a mathematical tool in constructive quantum field theory [270].

Both the Nelsonian and the Parisi-Wu framework quantize a theory by bringing it in contact with a stochastic background field. However in the Parisi-Wu framework this is done using an ordinary dissipative Brownian motion, while the Nelsonian approach makes use of a Nelson process, which can be regarded as a time-reversible or conservative Brownian motion. As the Parisi-Wu framework uses an ordinary Brownian motion, it can only establish an equivalence between a Euclidean quantum theory and the equilibrium limit of the stochastic theory. The Nelsonian approach, on the other hand, allows to establish a full equivalence between quantum theories and stochastic theories. However, as the Nelson process is more complicated than the ordinary Brownian motion, its field theoretic formulation is not as far evolved as the Parisi-Wu formalism.

Since the early work by Fényes and Nelson, the theory of stochastic mechanics and its associated stochastic quantization scheme have been extended to include spin [127, 159, 270], to describe processes on (pseudo-)Riemannian manifolds [127, 135, 136, 187, 229, 233, 270], and to relativistic theories [137, 165, 184, 188, 189, 234, 251, 265, 289]. Furthermore, field theoretic extensions have been made [151,165,184,186,188,227,228,265,269,272], but this theory is still in its infancy.

In this paper, we focus on stochastic mechanics of a single particle on a manifold. However, we will do this by reformulating the Nelson process as a complexified Brownian motion. We will thus study a Brownian motion of a relativistic[1] spinless test particle on a complexified (pseudo-)Riemannian manifold. We then find that for a real pseudo-variance the ordinary real Brownian motion is obtained, while for a purely imaginary pseudo-variance the Nelson process is obtained.

This paper is organized as follows: in the next section 8.2, we review the ideas governing stochastic mechanics and the complex generalization discussed in this paper; in section 8.3, we review the connections between stochastic mechanics and generalizations of the Feynman-Kac theorem; in sections 8.4 and 8.5, we introduce the relativistic stochastic process considered in this paper and the manifold on which this process is defined; in section 8.6, we discuss the variational equations that govern the stochastic process; in sections 8.7, 8.8 and 8.9, we then derive three different formulations for the diffusion problem; finally, in section 8.10, we conclude. Furthermore, in appendix 8.A, we summarize our results for the non-relativistic case; in appendix 8.B, we review the basics of stochastic integration and appendix 8.C contains calculations of conditional expectations that are necessary to derive our results. Throughout the paper we work in Planck units, i.e. $\hbar = 1$, $c = 1$, $G = 1$ and $k_B = 1$. Moreover, we work in the $(- + ++)$ signature convention.

8.2 Stochastic mechanics

In order to illustrate the ideas governing stochastic mechanics, we will start with the discussion of a single scalar non-relativistic particle with mass m moving on $\mathbb{R}^n$.

In classical mechanics, the motion of such a particle is governed by the Euler-Lagrange equations that can be derived using a variational principle from an action

$$S = \int_0^T L(x, v)\, dt \tag{8.1}$$

with Lagrangian L. Given some initial conditions $(x, v)(0) = (x_0, v_0)$, one then obtains a unique solution $x(t)$ on $\mathcal{T} = [0, T]$.

We will now make the additional assumption that the particle moves through some randomly fluctuating background field. In order to introduce the stochastic fluctuations from this background field, we must promote the trajectory $\{x(t) : t \in \mathcal{T}\}$ to a semi-martingale process $\{X_t : t \in \mathcal{T}\}$. We then impose, as before, that this process must satisfy the Euler-Lagrange equations, which should now be interpreted as stochastic differential equations. On top of this, we will fix the stochastic law of the background field. The simplest way to do so is by imposing a condition on the quadratic variation. For example, a Brownian motion satisfies the structure relation

$$d[[X^i, X^j]]_t = \alpha\, \delta^{ij}\, dt, \tag{8.2}$$

where α is a positive definite constant. If, on the other hand, we were to consider a Poisson

[1]We implement relativity following Refs. [137, 234].

process, the quadratic variation would be given by

$$d[[X^i, X^j]]_t = \alpha\, \delta^{ij}\, dt + \delta^{ij}\, c_k\, dX_t^k, \tag{8.3}$$

where c_k is a constant covector.

Many stochastic processes are completely determined by such a structure relation. It is thus expected that, if quantum mechanics allows for a stochastic description, there exists a quantum structure relation. In fact, the canonical commutation relations

$$[X^i, P_j] = i\, \delta^i_j \tag{8.4}$$

suggest that such a structure relation must be of the form

$$m\, d[[X^i, X^j]]_t = i\, \delta^{ij}\, d\tau. \tag{8.5}$$

However, since the right hand side is not a positive definite tensor, there does not exist a real semi-martingale X satisfying this relation.

In stochastic mechanics this issue is circumvented by promoting the process X to a bidirectional stochastic process that is adapted to both a future and a past filtration. One then introduces two time generators d_+ and d_-. The first is the forward generator, which is adapted to the past filtration and generates the future. The second is the backward generator, is adapted to the future and generates the past. This allows to construct the Nelson process X that satisfies the condition

$$\begin{aligned} m\, d_+[[X^i, X^j]]_t &= \delta^{ij}\, dt, \\ m\, d_-[[X^i, X^j]]_t &= \delta^{ij}\, dt. \end{aligned} \tag{8.6}$$

The stochastic Euler-Lagrange equations supplemented with these structure relation can then be solved within the framework of stochastic optimal control theory. It is well-established that this process indeed generates quantum mechanics of a spin-0 particle with mass m, cf. e.g. [184, 270, 364] for reviews.

In this paper, we advocate a slightly different route. Instead of introducing the notion of a bidirectional process and two time generators $d_\pm$, we will analytically continue our space $\mathbb{R}^n$ to the complex space $\mathbb{C}^n$. We can then impose a condition on the quadratic variation of the form

$$m\, d[[Z^i, Z^j]]_t = \alpha\, \delta^{ij}\, dt. \tag{8.7}$$

where $\alpha \in \mathbb{C}$. As the right hand side now determines a pseudo-variance, it no longer has to be positive definite. Similarly, for the complex conjugate process $\bar{Z}$, we impose

$$m\, d[[\bar{Z}^i, \bar{Z}^j]]_t = \bar{\alpha}\, \delta^{ij}\, dt, \tag{8.8}$$

$$m\, d[[Z^i, \bar{Z}^j]]_t = (|\alpha| + \beta)\, \delta^{ij}\, dt \tag{8.9}$$

with $\beta \in [0, \infty)$. We note that this last expression is positive definite as required for the existence of Z_t. Furthermore, we remark that β determines the conformal part of the process, i.e., if $\alpha = 0$, Z is a conformal martingale.

If we use polar coordinates $\alpha = \rho\, e^{i\phi}$ and set $Z = X + i\, Y$, we find

$$m\, d[[X^i, X^j]]_t = \frac{\beta + \rho\,(1 + \cos\phi)}{2}\, \delta^{ij}\, dt, \tag{8.10}$$

$$m\, d[[Y^i, Y^j]]_t = \frac{\beta + \rho\,(1 - \cos\phi)}{2}\, \delta^{ij}\, dt, \tag{8.11}$$

$$m\, d[[X^i, Y^j]]_t = \frac{\rho\, \sin\phi}{2}\, \delta^{ij}\, dt. \tag{8.12}$$

It is then easy to see that for $(\alpha, \beta) = (1, 0)$, we recover the Gaussian process defined by eq. (8.2). Moreover, the case $(\alpha, \beta) = (i, 0)$ suggested in eq. (8.5) is now well-defined.

We will set $\beta = 0$ and in analogy with the ordinary Brownian motion defined by eq. (8.2), we call the process defined by eq. (8.7) a complex Brownian motion. Furthermore, we refer to the special case $\alpha = \pm i$ as a conservative Brownian motion. This is due to fact that its stochastic dynamics is symmetric under the time reversal operation. As a consequence the conservative Brownian motion is at all times in a statistical equilibrium with the background field. This is in stark contrast with the real Brownian motion obtained for $\alpha = 1$, which is known to be a dissipative process.

In the remainder of the paper, we study the complex Brownian motion defined by the structure relation (8.7) for a general $\alpha \in \mathbb{C}$ in more detail. We will do this in the more complicated setting where the particle is relativistic and moves on a curved spacetime.

8.3 Stochastic mechanics and the Feynman-Kac theorem

Using Nelson's stochastic quantization, we will in section 8.9 derive a complex diffusion equation that governs the process described in the previous section. This result is closely tied to the Feynman-Kac theorem [218], which we review in this section.

The Feynman-Kac theorem states[2] that given the real diffusion equation

$$\frac{\partial}{\partial t}\Psi(x,t) = -\left[\frac{\alpha}{2}\,\delta^{ij}\,\partial_i\partial_j + v^i(x,t)\,\partial_i - \mathfrak{U}(x,t)\right]\Psi(x,t) \tag{8.13}$$

with $x \in \mathbb{R}^n$ and $t \in [0, T]$ subjected to the terminal condition

$$\Psi(x, T) = u(x) \tag{8.14}$$

The solution can be written as the conditional expectation

$$\Psi(x,t) = \mathbb{E}\left[\exp\left(-\int_t^T \mathfrak{U}(X_s, s)\, ds\right) u(X_T)\middle| X_t = x\right] \tag{8.15}$$

[2]We present an elementary form of the Theorem. Extensions beyond the formula presented here are known.

for the Itô process defined by

$$\begin{aligned} dX^i_t &= v^i(X_t)\,dt + dM^i_t, \\ d[[X^i, X^j]]_t &= \alpha\,\delta^{ij}\,dt, \end{aligned} \tag{8.16}$$

with $\alpha \geq 0$, M a local martingale and $[[X, X]]$ denotes the quadratic variation. The process X thus describes a real Brownian motion with drift.

It was suggested by by Gelfand and Yaglom [166], that a similar relation could exist for the Schrödinger equation

$$i\,\frac{\partial}{\partial t}\Psi(x,t) = -\left[\frac{\alpha}{2}\,\delta^{ij}\,\partial_i\partial_j + v^i(x,t)\,\partial_i - \mathfrak{U}(x,t)\right]\Psi(x,t). \tag{8.17}$$

However, soon after, it was pointed out by Cameron and Daletskii [14, 101, 124] that a straightforward generalization does not exist, as the complex measure necessary to construct such an equivalence will have an infinite total variation.

Later, Pavon [288] showed that, if one considers, instead of the process (8.16), a bidirectional Nelson process defined by

$$\begin{aligned} d_+X^i_t &= v^i_+(X_t)\,dt + d_+M^i_t, \\ d_+[[X^i, X^j]]_t &= \alpha\,\delta^{ij}\,dt, \end{aligned} \tag{8.18}$$

and

$$\begin{aligned} d_-X^i_t &= v^i_-(X_t)\,dt + d_-M^i_t, \\ d_-[[X^i, X^j]]_t &= \alpha\,\delta^{ij}\,dt. \end{aligned} \tag{8.19}$$

such a relation could still be established.

In deriving this result Pavon build on earlier work [285–287], where the forward and backward velocity were combined into a single complex velocity

$$v_q = v - i\,u, \tag{8.20}$$

with the current velocity v and osmotic velocity u given by

$$v = \frac{1}{2}\big(v_+ + v_-\big), \tag{8.21}$$

$$u = \frac{1}{2}\big(v_+ - v_-\big). \tag{8.22}$$

In contrast to the the earlier works [101, 124, 166], Pavon did not only complexify the measure, but also the underlying degrees of freedom. In this case, the velocity of the process. In this work, we go one step further and also complexify the position of the process. As pointed out in the previous section, we can then replace the bidirectional

Nelson process by a complex process satisfying

$$\begin{aligned} dZ^i_t &= w^i(Z_t)\,dt + dM^i_t, \\ d[[Z^i, Z^j]]_t &= \alpha\,\delta^{ij}\,dt \end{aligned} \tag{8.23}$$

with $\alpha \in \mathbb{C}$.

8.4 The geometry

We will generalize the discussion in previous sections to the context of relativistic particles on Lorentzian manifolds. For this, we consider a set $\mathcal{T} = [0, T]$, a real $(n = d+1)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$, and trajectories $x(\tau) : \mathcal{T} \to \mathcal{M}$.

We intend to superpose stochastic dynamics on these trajectories. However, stochastic dynamics violates the Leibniz rule, as stochastic processes have a non-vanishing quadratic variation. As a consequence, diffeomorphism invariance of stochastic theories defined on this manifold is broken. In this paper, we resolve this issue using the second order geometry framework as developed by Schwartz, Meyer and Emery [150, 257, 311].

The most important aspect of the second order geometry framework is that all tangent spaces $T_x\mathcal{M}$ are extended to second order tangent spaces $T_{2,x}\mathcal{M}$. In a local coordinate chart, second order vectors can be expressed as[3]

$$v = v^\mu\,\partial_\mu + \frac{1}{2}\,v^{\mu\nu}\,\partial_\mu\partial_\nu, \tag{8.24}$$

where $v^\mu\partial_\mu \in T_x\mathcal{M} \subset T_{2,x}\mathcal{M}$ represents the first order part and $v^{\mu\nu}\partial_\mu\partial_\nu$ the second order part. This second order part can be mapped bijectively onto a symmetric bilinear first order tensor, which in turn can be mapped bijectively onto the quadratic variation of the process X_t.[4]

When regarded as part of a second order vector, the first order vector $v^\mu\partial_\mu \in T_x\mathcal{M}$ no longer transforms in a covariant manner. However, one can construct the objects

$$\hat{v}^\mu = v^\mu + \frac{1}{2}\,\Gamma^\mu_{\sigma\kappa}v^{\sigma\kappa}, \tag{8.25}$$

$$\hat{v}^{\nu\rho} = v^{\nu\rho}, \tag{8.26}$$

which both transform covariantly. Diffeomorphism invariance of the physical theory can then be restored by replacing all vectors v^μ with their covariant expression $\hat{v}^\mu$.

For a more complete exposition of the material, we refer to the works of Schwartz, Meyer and Emery [150, 257, 311]. We note that the construction of a diffeomorphism invariant theory of stochastic mechanics was already studied extensively, cf. e.g. Refs. [135, 136, 270]. Recently, we have translated and extended these results into the second order geometry language [233].

[3]We slightly deviate from Refs. [150, 233], as we have introduced a factor $\frac{1}{2}$ in the second order part of the vector.

[4]cf. Theorem 3.8 and Proposition 6.13 in Ref. [150]

As a final step, we will need to analytically continue the manifold to the complexified manifold $\mathcal{M}^{\mathbb{C}}$. Similarly the tangent spaces are analytically continued, such that we obtain a first and second order tangent bundle

$$(T\mathcal{M})^{\mathbb{C}} = T\mathcal{M} \otimes \mathbb{C} = T^{1,0}\mathcal{M} \oplus T^{0,1}\mathcal{M}, \tag{8.27}$$

$$(T_2\mathcal{M})^{\mathbb{C}} = T_2\mathcal{M} \otimes \mathbb{C} = T_2^{1,0}\mathcal{M} \oplus T_2^{0,1}\mathcal{M}. \tag{8.28}$$

8.5 The stochastic process

In order to introduce stochastic dynamics, we must promote the complex manifold to a measurable space $\big(\mathcal{M}^{\mathbb{C}}, \mathcal{B}(\mathcal{M}^{\mathbb{C}})\big)$ with Borel sigma algebra. Moreover, we introduce the probability space $(\Omega, \Sigma, \mathbb{P})$ and study random variables $Z : (\Omega, \Sigma, \mathbb{P}) \to \big(\mathcal{M}^{\mathbb{C}}, \mathcal{B}(\mathcal{M}^{\mathbb{C}}), \mu\big)$ with $\mu = \mathbb{P} \circ Z^{-1}$. More precisely, we study stochastic processes, i.e. families of random variables $\{Z_\tau : \tau \in \mathcal{T}\}$. We will therefore introduce a filtration $\{\mathcal{F}_\tau\}_{\tau \in \mathcal{T}}$, which is by definition an ordered set that is increasing, i.e. $\emptyset \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \Sigma \quad \forall s < t \in \mathcal{T}$, and right-continuous, i.e. $\mathcal{F}_\tau = \cap_{\epsilon > 0} \mathcal{F}_{\tau+\epsilon}$.

We assume the stochastic processes Z_τ to be continuous semi-martingale processes adapted to a filtration. We remind the reader that continuous manifold valued semi-martingales are processes such that for every every coordinate chart $\chi : U \to V$ with $U \subset \mathcal{M}^{\mathbb{C}}$ and $V \subset \mathbb{C}^n$ the processes $Z^\mu = \chi^\mu(Z)$ are continuous semi-martingales, i.e they can locally be decomposed uniquely as

$$Z_\tau = C_\tau + M_\tau, \tag{8.29}$$

where C_τ is a continuous local càdlàg process with finite variation and M_τ is a continuous local martingale satisfying the martingale property

$$\mathbb{E}\big[M_\tau \big| \mathcal{F}_t\big] = M_t \qquad \forall\, t < \tau \in \mathcal{T}. \tag{8.30}$$

In the remainder of the paper, we will also use a shorthand notation for conditional expectation values:

$$\mathbb{E}_t\big[Z_\tau\big] := \mathbb{E}\big[Z_\tau \big| \mathcal{F}_t\big]. \tag{8.31}$$

Using this notation, one can define the second order velocity of the process by the conditional expectation[5]

$$w^\mu(Z_\tau, \tau) = \lim_{h \to 0} \frac{1}{h} \mathbb{E}_\tau \left[Z^\mu_{\tau+h} - Z^\mu_\tau \right], \tag{8.32}$$

$$w^{\mu\nu}(Z_\tau, \tau) = \lim_{h \to 0} \frac{1}{h} \mathbb{E}_\tau \Big[\Big(Z^\nu_{\tau+h} - Z^\nu_\tau \Big) \Big(Z^\rho_{\tau+h} - Z^\rho_\tau \Big) \Big]. \tag{8.33}$$

The object $(w^\mu, w^{\nu\rho})$ is a second order vector field. As discussed in the previous section,

[5] We note that the definition for the second order part deviates by a factor 2 compared to the definition used in our previous works [233, 234], see also previous footnote.

these objects are not covariant. However, one can obtain a covariant formulation given by

$$\hat{w}^{\mu} = w^{\mu} + \frac{1}{2}\Gamma^{\mu}_{\sigma\kappa} w^{\sigma\kappa}, \tag{8.34}$$

$$\hat{w}^{\nu\rho} = w^{\nu\rho}. \tag{8.35}$$

Moreover, using these velocities, one can reconstruct the Càdlàg process as

$$C^{\mu}_{\tau} - C^{\mu}_{0} = \lim_{h\to 0} \int_0^{\tau} \frac{1}{h}\, \mathbb{E}_s \left[Z^{\mu}_{s+h} - Z^{\mu}_{s} \right] ds \tag{8.36}$$

and the angle bracket process

$$\begin{aligned} C^{\mu\nu}_{\tau} - C^{\mu\nu}_{0} &:= \langle\langle Z^{\mu}, Z^{\nu} \rangle\rangle_{\tau} - \langle\langle Z^{\mu}, Z^{\nu} \rangle\rangle_{0} \\ &= \lim_{h\to 0} \int_0^{\tau} \frac{1}{h}\, \mathbb{E}_s \Big[\Big(Z^{\mu}_{s+h} - Z^{\mu}_{s} \Big) \Big(Z^{\nu}_{s+h} - Z^{\nu}_{s} \Big) \Big] ds, \end{aligned} \tag{8.37}$$

which is the compensator for the quadratic variation, i.e. the process

$$M^{\mu\nu}_{\tau} := [[Z^{\mu}, Z^{\nu}]]_{\tau} - \langle\langle Z^{\mu}, Z^{\nu} \rangle\rangle_{\tau} \tag{8.38}$$

is a local martingale.

The process Z can be lifted to the tangent bundle, yielding a process (Z_τ, W_τ) that is a continuous semi-martingale on the second order holomorphic tangent bundle $T^{1,0}_{2}\mathcal{M}$. This process can be decomposed into the real processes X_τ, Y_τ and V_τ, U_τ such that

$$Z_\tau = X_\tau + i\, Y_\tau, \tag{8.39}$$

$$W_\tau = V_\tau + i\, U_\tau. \tag{8.40}$$

As discussed in section 2, we will fix the quadratic variation of the processes by

$$d[[Z^{\mu}, Z^{\nu}]]_{\tau} = W^{\mu\nu}_{\tau} = \alpha\, \lambda\, g^{\mu\nu}(Z_\tau)\, d\tau, \tag{8.41}$$

where $\alpha \in \mathbb{C}$ and λ is a dimensionful constant characterizing the particle. Moreover,[6]

$$\hat{w}^{\mu\nu}(Z_\tau, \tau) = w^{\mu\nu}(Z_\tau, \tau) = \mathbb{E}_\tau \left[W^{\mu\nu}_{\tau} \right] = \alpha\, \lambda\, g^{\mu\nu}(Z_\tau). \tag{8.42}$$

8.6 Variational equations

Having specified the geometry and the stochastic dynamics, we can derive equations of motion for the stochastic particle. For this, we assume the geometry to be non-dynamical, and thus the metric to be a fixed symmetric bilinear form $g_{\mu\nu}(z)$. Consequently, the processes $(Z^{\mu}_{\tau}, W^{\nu}_{\tau}, W^{\rho\sigma}_{\tau})$ defined on the $\frac{n(n+5)}{2}$-dimensional second order holomorphic tangent bun-

[6] Note that $\hat{w}^{\mu\nu}$ is the second order part of a second order vector field, while $g^{\mu\nu}$ is a bilinear first order tensor. Therefore, the two cannot be equated straightforwardly. However, there exists a unique smooth and invertible linear map $\mathbf{H}$ from bilinear first order forms to second order forms, cf. Proposition 6.13 in Ref. [150] or eq. (2.15) in Ref. [233]. Using this mapping, which is notationally suppressed in equation (8.42), one can equate the two objects.

dle $T_2^{1,0}\mathcal{M}$ are restricted to (Z^μ_τ, W^ν_τ) defined on the $2n$-dimensional slice $T^{1,0}\mathcal{M} \subset T_2^{1,0}\mathcal{M}$. The Lagrangian for these processes is a complex function on the holomorphic tangent bundle, i.e.

$$L : T^{1,0}\mathcal{M} \to \mathbb{C}, \tag{8.43}$$

and the action is given by

$$S = \mathbb{E}\left[\int L(Z,W)\, d\tau\right]. \tag{8.44}$$

The stochastic Euler-Lagrange equations are given by

$$\int \frac{\partial}{\partial Z^\mu} L(Z,W)\, d\tau = \int \circ\, d\frac{\partial}{\partial W^\mu} L(Z,W), \tag{8.45}$$

which is a stochastic differential equation in the sense of Stratonovich. One can also construct a stochastic Hamiltonian function

$$H(Z,P) = P_\mu W^\mu - L(Z,W), \tag{8.46}$$

where P_μ is the conjugate momentum process, i.e.

$$P_\mu = \frac{\partial}{\partial W^\mu} L(Z,W). \tag{8.47}$$

In addition, we define Hamilton's principal function by

$$S(z,\tau) = \mathbb{E}\left[\int_0^\tau L(Z,W)\, ds \,\Big|\, Z_\tau = z\right]. \tag{8.48}$$

The corresponding stochastic Hamilton-Jacobi equations are given by

$$\nabla_\mu S(z,\tau) = \mathbb{E}_\tau\left[P_\mu\right], \tag{8.49}$$

$$\frac{\partial}{\partial \tau} S(z,\tau) = \mathbb{E}_\tau\left[-H(Z,P)\right]. \tag{8.50}$$

Finally, we remark that our relativistic theory is invariant under rescalings of the proper time parameter, which imposes

$$\frac{\partial}{\partial \tau} S(z,\tau) = 0. \tag{8.51}$$

8.7 Stochastic Euler-Lagrange equations

We consider a classical real Lagrangian $L : T\mathcal{M} \to \mathbb{R}$ of the form

$$L(x,v) = \frac{1}{2\lambda} g_{\mu\nu}(x)\, v^\mu v^\nu - \frac{\lambda\, m^2}{2} + q\, A_\mu(x)\, v^\mu, \tag{8.52}$$

where λ is an einbein field along the worldline of the particle with charge q. For massive theories we gauge fix $\lambda = m^{-1}$, while for massless theories we gauge fix $\lambda = 1$ in the equations of motion. We consider the stochastic analytic continuation of this Lagrangian given by $L : T^{1,0}\mathcal{M} \to \mathbb{C}$ such that

$$L(Z, W) = \frac{1}{2\lambda} g_{\mu\nu}(Z)\, W^\mu W^\nu - \frac{\lambda\, m^2}{2} + q\, A_\mu(Z)\, W^\mu. \tag{8.53}$$

The Euler-Lagrange equations for this Lagrangian become

$$\fint\!\!\fint g_{\mu\nu} \circ \left(d^2 Z^\nu + \Gamma^\nu_{\rho\sigma} dZ^\rho dZ^\sigma\right) = \int\!\!\fint \lambda\, q\, (\nabla_\mu A_\nu - \nabla_\nu A_\mu) \circ dZ^\nu d\tau, \tag{8.54}$$

which is a complex second order stochastic differential equation in the sense of Stratonovich. This equation must be supplemented with the relativistic constraint equation

$$\mathbb{E}_\tau \left[g_{\mu\nu} \circ dZ^\mu dZ^\nu + \lambda^2\, m^2\, d\tau^2\right] = 0 \tag{8.55}$$

that follows from the variation of the action with respect to λ. In addition, it must be supplemented with the condition on the quadratic variation

$$d[[Z^\mu, Z^\nu]] = \alpha\, \lambda\, g^{\mu\nu}(Z)\, d\tau. \tag{8.56}$$

We note that in the limit $\alpha \to 0$, one obtains the classical results: the Euler-Lagrange equations become ordinary differential equations

$$g_{\mu\nu} \left(\frac{d^2 Z^\nu}{d\tau^2} + \Gamma^\nu_{\rho\sigma} \frac{dZ^\rho}{d\tau} \frac{dZ^\sigma}{d\tau}\right) = \lambda\, q \left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right) \frac{dZ^\nu}{d\tau} \tag{8.57}$$

with constraint

$$g_{\mu\nu} \frac{dZ^\mu}{d\tau} \frac{dZ^\nu}{d\tau} = -\lambda^2\, m^2 \tag{8.58}$$

and the quadratic variation vanishes.

8.8 Field equations

Although the equations of motion derived in the previous section can be written down formally, for practical purposes it may be easier to solve a system of first order stochastic differential equations in the sense of Itô. In this section, we will therefore derive an equivalent system of stochastic differential equations in the Itô formulation using the Hamilton-Jacobi formalism.

The Hamilton-Jacobi equations for the Lagrangian introduced in previous section yield

$$\nabla_\mu S(z, \tau) = \lambda^{-1}\, g_{\mu\nu} \hat{w}^\nu + q\, A_\mu. \tag{8.59}$$

and

$$\frac{\partial}{\partial \tau} S(z,\tau) = -\mathbb{E}_\tau \left[\frac{1}{2\lambda} g_{\mu\nu}(Z)\, W^\mu W^\nu + \frac{\lambda\, m^2}{2} \right]$$
$$= -\frac{1}{2\lambda} g_{\mu\nu} \hat{w}^\mu \hat{w}^\nu - \frac{\alpha}{2} \nabla_\mu \hat{w}^\mu + \frac{\alpha^2 \lambda}{12} \mathcal{R} - \frac{\lambda\, m^2}{2}, \tag{8.60}$$

where we used the results from appendix 8.C. We can combine these two equations by taking a covariant derivative of the second equation and plugging in the first equation. This yields

$$\frac{1}{\lambda}\, \hat{w}^\nu\, \nabla_\mu \hat{w}_\nu + \frac{\alpha}{2} \nabla_\mu \nabla_\nu \hat{w}^\nu - \frac{\alpha^2 \lambda}{12}\, \nabla_\mu \mathcal{R} = 0, \tag{8.61}$$

where we applied the relativistic constraint $\partial_\tau S = 0$. Then using that

$$\nabla_\mu \hat{w}_\nu = \nabla_\nu \hat{w}_\mu - \lambda\, q\, H_{\mu\nu}, \tag{8.62}$$
$$\nabla_\mu \nabla_\nu \hat{w}^\nu = \Box\, \hat{w}_\mu - \lambda\, q\, \nabla^\nu H_{\mu\nu} - \mathcal{R}_{\mu\nu} \hat{w}^\nu \tag{8.63}$$

with the field strength defined by

$$H_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{8.64}$$

we find

$$\left[\frac{1}{\lambda}\, g_{\mu\nu}\, \hat{w}^\rho\, \nabla_\rho - q\, H_{\mu\nu} + \frac{\alpha}{2} \Big(g_{\mu\nu}\, \Box - \mathcal{R}_{\mu\nu} \Big) \right] \hat{w}^\nu = \frac{\alpha\, \lambda}{2} \left(q\, \nabla^\nu H_{\mu\nu} + \frac{\alpha}{6}\, \nabla_\mu \mathcal{R} \right), \tag{8.65}$$

which can be solved for the velocity field $\hat{w}^\mu(z)$ under the relativistic constraint

$$g_{\mu\nu} \hat{w}^\mu \hat{w}^\nu + \alpha\, \lambda \nabla_\mu \hat{w}^\mu - \frac{\alpha^2 \lambda^2}{6} \mathcal{R} = -\lambda^2\, m^2. \tag{8.66}$$

The solution can then be plugged into the first order stochastic differential equation in the sense of Itô

$$dZ^\mu_\tau = w^\mu(Z_\tau)\, d\tau + dM^\mu_\tau, \tag{8.67}$$
$$d[[Z^\mu, Z^\nu]]_\tau = \alpha\, \lambda\, g^{\mu\nu}(Z_\tau)\, d\tau. \tag{8.68}$$

where we note that $\hat{w}^\mu = w^\mu + \frac{\alpha\,\lambda}{2}\, \Gamma^\mu$. This system can be solved for the appropriate boundary conditions, yielding a stochastic process Z_τ. The moments of this process can be calculated using the characteristic and moment generating functional

$$\Phi_Z(J) = \mathbb{E} \left[e^{i \int J_\mu Z^\mu d\tau} \right], \tag{8.69}$$
$$M_Z(J) = \mathbb{E} \left[e^{\int J_\mu Z^\mu\, d\tau} \right]. \tag{8.70}$$

8.9 Diffusion equation

In this section, we derive a diffusion equation governing the stochastic process described in previous sections.

The Hamilton-Jacobi equations (8.59) and (8.60) can be combined such that

$$\frac{\partial}{\partial\tau}S = -\frac{\lambda}{2}\left(\nabla_\mu S\,\nabla^\mu S + \alpha\,\Box S - 2\,q\,A_\mu\,\nabla^\mu S - \alpha\,q\,\nabla_\mu A^\mu + q^2 A_\mu A^\mu - \frac{\alpha^2}{6}\mathcal{R} + m^2\right). \tag{8.71}$$

If we then define the wave function

$$\Psi(z,\tau) = \exp\left\{\frac{1}{\alpha}\left[S(z,\tau) + \frac{\lambda\,m^2}{2}\,\tau\right]\right\}, \tag{8.72}$$

we find that eq. (8.71) is equivalent to the diffusion equation

$$\frac{\partial}{\partial\tau}\Psi = -\frac{\alpha\,\lambda}{2}\left[\left(\nabla_\mu - \frac{q}{\alpha}\,A_\mu\right)\left(\nabla^\mu - \frac{q}{\alpha}\,A^\mu\right) - \frac{1}{6}\mathcal{R}\right]\Psi. \tag{8.73}$$

Moreover, we have

$$\left|\Psi(z,\tau)\right|^2 = \exp\left[\frac{2}{\rho}\left(\cos(\phi)\left\{\mathrm{Re}\big[S(z,\tau)\big] + \frac{\lambda\,m^2}{2}\tau\right\} + \sin(\phi)\,\mathrm{Im}\big[S(z,\tau)\big]\right)\right]. \tag{8.74}$$

We note that this equation should be interpreted as a backward equation, i.e. subjected to a terminal condition.

We will now set $\rho = |\alpha| = 1$ and consider several special cases. As anticipated in sections 8.2 and 8.3, for $\phi \in \{0,\pi\}$ we obtain the heat equation

$$\frac{\partial}{\partial\tau}\Psi = \mp\frac{\lambda}{2}\left[\left(\nabla_\mu \mp q\,A_\mu\right)\left(\nabla^\mu \mp q\,A^\mu\right) - \frac{1}{6}\mathcal{R}\right]\Psi \tag{8.75}$$

with

$$\Psi(z,\tau) = \exp\left\{\pm\left[S(z,\tau) + \frac{\lambda\,m^2}{2}\tau\right]\right\}, \tag{8.76}$$

$$\left|\Psi(z,\tau)\right|^2 = \exp\left(\pm\left\{2\,\mathrm{Re}\big[S(z,\tau)\big] + \lambda\,m^2\,\tau\right\}\right). \tag{8.77}$$

On the other hand for $\phi \in \{-\frac{\pi}{2},\frac{\pi}{2}\}$, we obtain the Schrödinger equation

$$i\,\frac{\partial}{\partial\tau}\Psi = \mp\frac{\lambda}{2}\left[\left(\nabla_\mu \mp i\,q\,A_\mu\right)\left(\nabla^\mu \mp i\,q\,A^\mu\right) - \frac{1}{6}\mathcal{R}\right]\Psi \tag{8.78}$$

with

$$\Psi(z,\tau) = \exp\left\{\pm i\left[S(z,\tau) + \frac{\lambda\,m^2}{2}\tau\right]\right\}, \tag{8.79}$$

$$\left|\Psi(z,\tau)\right|^2 = \exp\left\{\mp 2\,\mathrm{Im}\big[S(z,\tau)\big]\right\}. \tag{8.80}$$

Furthermore, we note that the relativistic constraint imposes $S(z,\tau) = S(z)$, which allows to solve eq. (8.73) by separation of variables. We then obtain

$$\Psi(z,\tau) = \Phi(z) \exp\left(\frac{m\,\tau}{2\,\alpha}\right), \tag{8.81}$$

where $\Phi(z) = \exp\left[\alpha^{-1} S(z)\right]$ solves the Klein-Gordon equation

$$\left[\left(\nabla_\mu - \frac{q}{\alpha}\,A_\mu\right)\left(\nabla^\mu - \frac{q}{\alpha}\,A^\mu\right) - \frac{1}{6}\mathcal{R} + \frac{m^2}{\alpha^2}\right]\Phi = 0. \tag{8.82}$$

8.10 Conclusion

In this paper, we have derived three equivalent descriptions for the diffusion of a single scalar relativistic particle on a complexified Lorentzian manifold charged under a vector potential. The first is as a second order stochastic differential equation in the sense of Stratonovich; the second is a system of first order stochastic differential equations in the sense of Itô and the third is as the Kolmogorov backward equation associated to the process. In addition, we have presented the results for the non-relativistic particle in appendix 8.A.

In fact, this result is well known for non-relativistic diffusion processes on $\mathbb{R}^n$ with a real variance, and is given by the Feynman-Kac formula. In this paper, we have used Nelson's stochastic quantization scheme to generalize this result to the case of (relativistic) diffusion processes on (pseudo-)Riemannian manifolds with a complex pseudo-variance. We should emphasize, however, that we have derived our results under the assumption of the existence of unique solutions to the given formulations. A mathematically rigorous proof of our results will be left for future work.

It is worth pointing out the similarities and differences of the rotation of the pseudo-variance around the angle ϕ studied in this paper and the Wick rotation. Both rotations transform a heat-type equation into a Schrödinger-type equation. This is due to the fact that both rotations act on the proper time parameter. However, there is also an important difference, as the rotation discussed in this paper also acts on all coordinates. As a consequence it preserves the (k,l,m) signature of the (pseudo-)Riemannian manifold. In contrast the Wick rotation only acts on the time-like coordinates, and therefore transforms a pseudo-Riemannian manifold with (k,l,m) signature into a Riemannian manifold with $(k+l,0,m)$ signature.

Furthermore, it is worth noticing that the diffusion equation (8.73) contains a term proportional to the Ricci scalar. This term comes with a prefactor $\frac{1}{6}$ that results from a Taylor expansion, cf. appendix 8.C. On the other hand, it is well known that for a prefactor given by $\frac{n-2}{4(n-1)}$ the diffusion equation is conformally invariant. Interestingly, the two prefactors coincide in 4 dimensions.

Finally, as the description given in this paper requires the complexification of space-time, we are forced to give a physical interpretation to the imaginary part of position and velocity vectors. Here, Pavon's formulation in terms of bidirectional processes and complex velocities as discussed in section 8.2 might provide an answer: Pavon considered a velocity $v_q = v - i\,u$, where v is the current velocity associated to the particle itself and u is the osmotic velocity associated to the motion of the background field through which the particle propagates. In our formulation in terms of unidirectional complex processes, we can give the same interpretation to the velocity field.

We will thus interpret $\mathrm{Re}(W) = V$ as the velocity of the particle and $\mathrm{Im}(W) = U$ as the velocity of the background field. Consequently, we must also associate $\mathrm{Re}(Z) = X$ to the position of the particle and $\mathrm{Im}(Z) = Y$ to the position of an associated particle in the background field. We conclude that both matter and the background field move under evolution of the proper time. Interestingly, for $\alpha \in \mathbb{C} \setminus \mathbb{R}$ the stochastic dynamics of the particle and the background field are coupled and therefore they cannot be treated independently. This is particularly true for pure quantum systems where $\alpha \in i\,\mathbb{R}$, but is in stark contrast with the real Brownian motion with $\alpha \in \mathbb{R}$. In this latter case, the motion of the background field and matter are completely decoupled, which allows to neglect the motion of the background field.

We conclude that our results further illustrate the close connection between Brownian motion and quantum physics and open up new avenues to tackle quantum problems using the theory of stochastic differential equations. In addition, our results reaffirm the central result of stochastic mechanics that quantum physics can be understood in terms of stochastic processes.

Acknowledgments

This work is supported by a doctoral studentship of the Science and Technology Facilities Council, award number: 2131791.

8.A Non-relativistic theories

In this paper, we have presented a stochastic formulation of relativistic diffusion processes. In this appendix, we present the results for non-relativistic diffusion processes, which can be derived in a similar fashion.

We consider a set $\mathcal{T} = [0, T]$, a real $(n = d)$-dimensional Riemannian manifold and trajectories $x(t) : \mathcal{T} \to \mathcal{M}$. We consider a classical non-relativistic theory of the form

$$L(x, v, t) = \frac{m}{2}\, g_{ij}(x)\, v^i v^j + q\, A_i(x, t)\, v^i - \mathfrak{U}(x, t). \tag{8.83}$$

The stochastic analytic continuation is then given by

$$L(Z, W, t) = \frac{m}{2}\, g_{ij}(Z)\, W^i W^j + q\, A_i(Z, t)\, W^i - \mathfrak{U}(Z, t) \tag{8.84}$$

and the stochastic Euler-Lagrange equations are

$$m\, g_{ij} \circ \left(d^2 Z^j + \Gamma^j_{kl}\, dZ^k dZ^l \right) = q \left(\nabla_i A_j - \nabla_j A_i \right) \circ dZ^j dt - \left(q\, \partial_t A_i + \nabla_i \mathfrak{U} \right) dt^2, \tag{8.85}$$

which must be supplemented with the condition on the quadratic variation

$$d[[Z^i, Z^j]] = \frac{\alpha}{m}\, g^{ij}(Z)\, dt. \tag{8.86}$$

On te other hand, in the Itô formulation, we find that the velocity field is governed by the equation

$$\Big[m\, g_{ij}\left(\partial_t + \hat{w}^k \nabla_k\right) - q\, H_{ij} + \frac{\alpha}{2}\left(g_{ij}\,\Box - \mathcal{R}_{ij}\right)\Big]\, \hat{w}^j = \frac{\alpha\, q}{2m}\, \nabla^j H_{ij} - q\, \partial_t A_i - \nabla_i \mathfrak{U} + \frac{\alpha^2}{12m}\, \nabla_i \mathcal{R}. \tag{8.87}$$

As in the relativistic case the solution $w^i(z,t)$ can be plugged into the first order stochastic differential equation in the sense of Itô:

$$dZ_t^i = w^i(Z_t, t)\, dt + dM_t^i, \tag{8.88}$$

$$d[[Z^i, Z^j]]_t = \frac{\alpha}{m} g^{ij}(Z_t)\, dt, \tag{8.89}$$

where we note that $\hat{w}^i = w^i + \frac{\alpha}{2m}\Gamma^i$.

Furthermore, we can define the wave function

$$\Psi(z,t) = \exp\left[\frac{S(z,t)}{\alpha}\right] \tag{8.90}$$

for which we find that eq. (8.71) is equivalent to the complex diffusion equation

$$\alpha\, \frac{\partial}{\partial t}\Psi = -\left\{\frac{\alpha^2}{2m}\left[\left(\nabla_i - \frac{q}{\alpha}\, A_i\right)\left(\nabla^i - \frac{q}{\alpha}\, A^i\right) - \frac{1}{6}\mathcal{R}\right] + \mathfrak{U}\right\}\Psi. \tag{8.91}$$

If there is no explicit time dependence, i.e. $\mathfrak{U}(x,t) = \mathfrak{U}(x)$ and $A_i(x,t) = A_i(x)$, this can be solved by separation of variables, such that

$$\Psi(z,t) = \sum_k \Phi_k(z)\, \exp\left[\frac{E_k}{\alpha}\, t\right], \tag{8.92}$$

where $\Phi_k(z)$ solves the wave equation

$$\left\{\frac{\alpha^2}{2m}\left[\left(\nabla_i - \frac{q}{\alpha}\, A_i\right)\left(\nabla^i - \frac{q}{\alpha}\, A^i\right) - \frac{1}{6}\mathcal{R}\right] + \mathfrak{U} + E_k\right\}\Psi_k = 0. \tag{8.93}$$

8.B Stochastic integration

In this appendix, we review some notions from stochastic integration on manifolds. Let us first review the definition of stochastic integrals on $\mathbb{R}^n$. The *Stratonovich integral* is defined as

$$\int_0^T f(X_\tau) \circ dX_\tau^\mu := \lim_{k\to\infty} \sum_{[\tau_i, \tau_{i+1}] \in \pi_k} \frac{1}{2}\big[f(X_{\tau_i}) + f(X_{\tau_{i+1}})\big]\big[X_{\tau_{i+1}}^\mu - X_{\tau_i}^\mu\big], \tag{8.94}$$

where π_k is a partition of $[0,T]$. The *Itô integral* is defined by

$$\underline{\int}_0^T f(X_\tau)\, dX^\mu_\tau := \lim_{k\to\infty} \sum_{[\tau_i,\tau_{i+1}]\in\pi_k} f(X_{\tau_i}) \left[X^\mu_{\tau_{i+1}} - X^\mu_{\tau_i}\right] \tag{8.95}$$

and the integral over the quadratic variation is given by

$$\int f(X_\tau)\, d[[X^\mu, X^\nu]]_\tau := \lim_{k\to\infty} \sum_{[\tau_i,\tau_{i+1}]\in\pi_k} f(X_{\tau_i}) \left[X^\mu_{\tau_{i+1}} - X^\mu_{\tau_i}\right]\left[X^\nu_{\tau_{i+1}} - X^\nu_{\tau_i}\right]. \tag{8.96}$$

By a straightforward calculation, one can then derive a relation between the three integrals:

$$\fint_0^T f(X_\tau)\, dX^\mu_\tau = \underline{\int}_0^T f(X_\tau)\, dX^\mu_\tau + \frac{1}{2}\int \partial_\nu f(X_\tau)\, d[[X^\mu, X^\nu]]_\tau \tag{8.97}$$

The Stratonovich integral has the advantage that it obeys the Leibniz rule:

$$\circ\, d(X^\mu Y^\nu) = X^\mu \circ dY^\nu + Y^\nu \circ dX^\mu, \tag{8.98}$$

while the Itô integral satisfies a modified Leibniz rule given by

$$d(X^\mu Y^\nu) = X^\mu\, dY^\nu + Y^\nu dX^\mu + d[[X^\mu, Y^\nu]]. \tag{8.99}$$

On the other hand, the Itô integral has the advantage that for any martingale M_τ

$$\mathbb{E}_\tau\left[\underline{\int}_\tau^T f(X_s)\, dM^\mu_s\right] = 0. \tag{8.100}$$

All these integrals can be extended to smooth manifolds with a connection. As usual this must be done using differential forms. We will express a first order form $\omega \in T^*\mathcal{M}$ in a local coordinate chart as

$$\omega = \omega_\mu \circ dx^\mu. \tag{8.101}$$

The Stratonovich integral is then defined by

$$\fint_{X_\tau} \omega := \fint_0^T \omega_\mu(X_\tau) \circ dX^\mu_\tau. \tag{8.102}$$

The right hand side can be calculated using the definition (8.94) in a local coordinate chart.

The construction of the Itô integral on the other hand, requires the construction of second order forms $\Omega \in T^*_2\mathcal{M}$. These can be expressed in a local coordinate chart as[7]

$$\omega = \omega_\mu\, dx^\mu + \frac{1}{2}\,\partial_\nu\omega_\mu\, d[[x^\mu, x^\nu]] \tag{8.103}$$

[7]Note that we deviate here from the notation used in Refs. [150, 233], where first order forms are expressed as $\omega = \omega_\mu dx^\mu$ and second order forms as $\omega = \omega_\mu d_2 x^\mu + \omega_{\mu\nu}\, dx^\mu \cdot dx^\nu$. The notation used in Refs. [150, 233] is the standard notation in the geometry literature, while the notation adapted in this paper is closer to the stochastics literature.

Expressions of the form

$$\underline{\int}_0^T \omega_\mu(X_\tau)\, dX_\tau^\mu \qquad \text{and} \qquad \int_0^T \omega_{\mu\nu}(X_\tau)\, d[[X^\mu, X^\nu]]_\tau$$

can then be calculated in a local coordinate chart using definitions (8.95) and (8.96) respectively. Moreover, the second expression represents the integral over the quadratic variation on a manifold. The first, however, does not define an Itô integral on manifolds, as it is not covariant. Instead, the Itô integral is defined by the covariant expression

$$\begin{aligned}\underline{\int}_{X_\tau} \omega &:= \underline{\int}_0^T \omega_\mu(X_\tau)\, d\hat{X}_\tau^\mu \\ &:= \underline{\int}_0^T \omega_\mu(X_\tau)\, dX_\tau^\mu + \frac{1}{2}\int_0^T \omega_\mu(X_\tau)\, \Gamma^\mu_{\nu\rho}(X_\tau)\, d[[X^\nu, X^\rho]]_\tau .\end{aligned} \tag{8.104}$$

The relation between the Stratonovich and Itô integral on a manifold is then given by

$$\oint_0^T \omega_\mu(X_\tau) \circ dX_\tau^\mu = \underline{\int}_{X_\tau} \omega_\mu(X_\tau)\, d\hat{X}_\tau^\mu + \frac{1}{2}\int_0^T \nabla_\nu \omega_\mu(X_\tau)\, d[[X^\mu, X^\nu]]_\tau . \tag{8.105}$$

8.C Calculation of conditional expectations

In this appendix we derive the following expressions

$$\mathbb{E}_\tau\big[\mathfrak{U}\big] = \mathfrak{U}, \tag{8.106}$$

$$\mathbb{E}_\tau\big[g_{\mu\nu} W^{\mu\nu}\big] = n\,\alpha\,\lambda, \tag{8.107}$$

$$\mathbb{E}_\tau\big[A_\mu W^\mu\big] = A_\mu \hat{w}^\mu + \frac{\alpha\,\lambda}{2}\,\nabla_\mu A^\mu, \tag{8.108}$$

$$\mathbb{E}_\tau\big[g_{\mu\nu}\, W^\mu W^\nu\big] = g_{\mu\nu}\hat{w}^\mu \hat{w}^\nu + \alpha\,\lambda\,\nabla_\mu \hat{w}^\mu - \frac{\alpha^2\lambda^2}{6}\,\mathcal{R}. \tag{8.109}$$

The proof of the first equality is immediate by "taking out what is known":

$$\mathbb{E}_\tau\left[\mathfrak{U}(Z_\tau)\right] = \mathfrak{U}(z). \tag{8.110}$$

For the second equality we find

$$\begin{aligned}\mathbb{E}_\tau\left[\int_\tau^{\tau+d\tau} g_{\mu\nu}(Z_s)\, W_s^{\mu\nu}\, ds\right] &= \mathbb{E}_\tau\left[\int g_{\mu\nu}(Z_s)\, d[[Z^\mu, Z^\nu]]_s\right] \\ &= \mathbb{E}_\tau\left[\alpha\,\lambda \int g_{\mu\nu}(Z_s)\, g^{\mu\nu}(Z_s)\, ds\right] \\ &= \mathbb{E}_\tau\big[n\,\alpha\,\lambda\, d\tau\big] \\ &= n\,\alpha\,\lambda\, d\tau,\end{aligned} \tag{8.111}$$

In the limit $d\tau \to 0$ we then obtain the result (8.107).

For the third equality, we find

$$
\begin{aligned}
\mathbb{E}_\tau\left[\int_\tau^{\tau+d\tau} A_\mu(Z_s)\, W_s^\mu\, ds\right] &= \mathbb{E}_\tau\left[\fint A_\mu(Z_s)\circ dZ_s^\mu\right] \\
&= \mathbb{E}_\tau\left[\underline{\int} A_\mu(Z_s)\, dZ_s^\mu + \frac{1}{2}\int \partial_\nu A_\mu(Z_s)\, d[[Z^\mu, Z^\nu]]_s\right] \\
&= \mathbb{E}_\tau\left[\int \Big(A_\mu(Z_s)\, w^\mu(Z_s) + w^{\mu\nu}(Z_s)\,\partial_\nu A_\mu(Z_s)\Big) ds + \underline{\int} A_\mu(Z_s)\, dM_s^\mu\right] \\
&= \mathbb{E}_\tau\left[\Big(A_\mu(Z_\tau)\, w^\mu(Z_\tau) + w^{\mu\nu}(Z_\tau)\,\partial_\nu A_\mu(Z_\tau)\Big) d\tau + o(d\tau)\right] \\
&= \mathbb{E}_\tau\left[\Big(A_\mu(Z_\tau)\, \hat{w}^\mu(Z_\tau) + \hat{w}^{\mu\nu}(Z_\tau)\,\nabla_\nu A_\mu(Z_\tau)\Big) d\tau + o(d\tau)\right] \\
&= \left(A_\mu\, \hat{w}^\mu + \frac{\alpha\,\lambda}{2}\,\nabla_\mu A^\mu\right) d\tau + o(d\tau),
\end{aligned}
\tag{8.112}
$$

where we rewrote the Stratonovich integral as an Itô integral, such that the martingale property (8.100) can be applied on the stochastic integral dM. In the limit $d\tau \to 0$, we then obtain eq. (8.108).

8.C.1 Quadratic in velocity

The calculation of the conditional expectation of a term quadratic in the velocity process is slightly more involved. This calculation was first performed by Guerra and Nelson in Ref. [270]. Here, we reproduce their result using a slightly different presentation.

We first notice that

$$
g_{\mu\nu}(Z_\tau)\circ dZ_\tau^\mu\, dZ_\tau^\nu = g_{\mu\nu}(Z_\tau)\, W_\tau^{\mu\nu}\, d\tau + g_{\mu\nu}(Z_\tau)\, W_\tau^\mu W_\tau^\nu\, d\tau^2 + o(d\tau^2),
\tag{8.113}
$$

where the left hand side is a Stratonovich integral. In order to calculate the conditional expectation of this expression, we will need to rewrite this into an Itô integral. For this, we note that[8]

$$
\begin{aligned}
d^2 f &= d\left(\partial_\mu f\, dZ^\mu + \frac{1}{2}\partial_\mu\partial_\nu f\, d[[Z^\mu, Z^\nu]]\right) \\
&= \partial_\mu f\, d^2 Z^\mu + \partial_\nu\partial_\mu f\, dZ^\mu dZ^\nu + \partial_\rho\partial_\nu\partial_\mu f\, dZ^\mu\, d[[Z^\nu, Z^\rho]] \\
&\quad + \frac{1}{4}\,\partial_\sigma\partial_\rho\partial_\nu\partial_\mu f\, d[[Z^\mu, Z^\nu]]\, d[[Z^\rho, Z^\sigma]] \\
&= \partial_\mu f\, d^2 Z^\mu + \partial_\nu\partial_\mu f\, dZ^\mu dZ^\nu + \frac{1}{3}\,\partial_\rho\partial_\nu\partial_\mu f\, dZ^\mu dZ^\nu dZ^\rho \\
&\quad + \frac{1}{12}\,\partial_\sigma\partial_\rho\partial_\nu\partial_\mu f\, dZ^\mu dZ^\nu dZ^\rho dZ^\sigma,
\end{aligned}
\tag{8.114}
$$

[8] We make use of the that Brownian motion is completely determined by its quadratic moment: all even moment can be expressed in terms of the quadratic moment and all odd moments vanish.

where we introduced the notation

$$dZ^\mu dZ^\nu dZ^\rho = dZ^\mu\, d[[Z^\nu, Z^\rho]] + dZ^\nu\, d[[Z^\mu, Z^\rho]] + dZ^\rho\, d[[Z^\mu, Z^\nu]], \tag{8.115}$$

$$dZ^\mu dZ^\nu dZ^\rho dZ^\sigma = d[[Z^\mu, Z^\nu]]\, d[[Z^\rho, Z^\sigma]] + d[[Z^\mu, Z^\rho]]\, d[[Z^\nu, Z^\sigma]] + d[[Z^\mu, Z^\sigma]]\, d[[Z^\nu, Z^\rho]]. \tag{8.116}$$

This expression can be rewritten into an explicitly covariant form:

$$\begin{aligned} d^2 f = \nabla_\mu f \Big[& dZ^\mu + \Gamma^\mu_{\nu\rho}\, dZ^\nu dZ^\rho + \frac{1}{3}\Big(\partial_\nu \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\nu\kappa}\Gamma^\kappa_{\rho\sigma}\Big) dZ^\nu dZ^\rho dZ^\sigma \\ & + \frac{1}{12}\,\partial_\kappa\Big(\partial_\nu \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}\Big) dZ^\nu dZ^\rho dZ^\sigma dZ^\kappa \\ & + \frac{1}{12}\,\Gamma^\mu_{\kappa\lambda}\Big(\partial_\nu \Gamma^\lambda_{\rho\sigma} + \Gamma^\lambda_{\nu\alpha}\Gamma^\alpha_{\rho\sigma}\Big) dZ^\nu dZ^\rho dZ^\sigma dZ^\kappa \Big] \\ + \nabla_\nu \nabla_\mu f \Big[& dZ^\mu dZ^\nu + \frac{2}{3}\,\Gamma^\mu_{\rho\sigma}\, dZ^\nu dZ^\rho dZ^\sigma + \frac{1}{3}\,\Gamma^\nu_{\rho\sigma}\, dZ^\mu dZ^\rho dZ^\sigma \\ & + \frac{1}{4}\,\Gamma^\mu_{\rho\sigma}\Gamma^\nu_{\kappa\lambda}\, dZ^\rho dZ^\sigma dZ^\kappa dZ^\lambda + \frac{1}{4}\Big(\partial_\kappa \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\kappa\lambda}\Gamma^\lambda_{\rho\sigma}\Big) dZ^\nu dZ^\rho dZ^\sigma dZ^\kappa \\ & + \frac{1}{12}\Big(\partial_\kappa \Gamma^\nu_{\rho\sigma} + \Gamma^\nu_{\kappa\lambda}\Gamma^\lambda_{\rho\sigma}\Big) dZ^\mu dZ^\rho dZ^\sigma dZ^\kappa \Big] \\ + \frac{1}{3}\,\nabla_\rho \nabla_\nu \nabla_\mu f \Big(& dZ^\mu dZ^\nu dZ^\rho + \frac{3}{4}\,\Gamma^\mu_{\sigma\kappa}\, dZ^\nu dZ^\rho dZ^\sigma dZ^\kappa \\ & + \frac{1}{2}\,\Gamma^\nu_{\sigma\kappa}\, dZ^\mu dZ^\rho dZ^\sigma dZ^\kappa + \frac{1}{4}\,\Gamma^\rho_{\sigma\kappa}\, dZ^\mu dZ^\nu dZ^\sigma dZ^\kappa \Big) \\ + \frac{1}{12}\,\nabla_\sigma \nabla_\rho \nabla_\nu \nabla_\mu f\, & dZ^\mu dZ^\nu dZ^\rho dZ^\sigma, \end{aligned} \tag{8.117}$$

and therefore

$$\begin{aligned} d^2 f = \nabla_\mu f \Big[& dZ^\mu + \Gamma^\mu_{\nu\rho}\, dZ^\nu dZ^\rho + \frac{1}{3}\Big(\partial_\nu \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\nu\kappa}\Gamma^\kappa_{\rho\sigma}\Big) dZ^\nu dZ^\rho dZ^\sigma \\ & + \frac{1}{12}\,\partial_\kappa\Big(\partial_\nu \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}\Big) dZ^\nu dZ^\rho dZ^\sigma dZ^\kappa \\ & + \frac{1}{12}\,\Gamma^\mu_{\kappa\lambda}\Big(\partial_\nu \Gamma^\lambda_{\rho\sigma} + \Gamma^\lambda_{\nu\alpha}\Gamma^\alpha_{\rho\sigma}\Big) dZ^\nu dZ^\rho dZ^\sigma dZ^\kappa \\ & + \frac{1}{12}\,\Gamma^\lambda_{\rho\sigma}\mathcal{R}^\mu{}_{\nu\lambda\kappa} dZ^\nu dZ^\rho dZ^\sigma dZ^\kappa \Big] \\ + \nabla_{(\nu} \nabla_{\mu)} f \Big[& dZ^\mu dZ^\nu + \frac{1}{2}\,\Gamma^\mu_{\rho\sigma}\, dZ^\nu dZ^\rho dZ^\sigma + \frac{1}{2}\,\Gamma^\nu_{\rho\sigma}\, dZ^\mu dZ^\rho dZ^\sigma \\ & + \frac{1}{4}\,\Gamma^\mu_{\rho\sigma}\Gamma^\nu_{\kappa\lambda}\, dZ^\rho dZ^\sigma dZ^\kappa dZ^\lambda + \frac{1}{6}\Big(\partial_\kappa \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\kappa\lambda}\Gamma^\lambda_{\rho\sigma}\Big) dZ^\nu dZ^\rho dZ^\sigma dZ^\kappa \\ & + \frac{1}{6}\Big(\partial_\kappa \Gamma^\nu_{\rho\sigma} + \Gamma^\nu_{\kappa\lambda}\Gamma^\lambda_{\rho\sigma}\Big) dZ^\mu dZ^\rho dZ^\sigma dZ^\kappa \Big] \\ + \frac{1}{3}\,\nabla_{(\rho} \nabla_\nu \nabla_{\mu)} f \Big(& dZ^\mu dZ^\nu dZ^\rho + \frac{1}{2}\,\Gamma^\mu_{\sigma\kappa}\, dZ^\nu dZ^\rho dZ^\sigma dZ^\kappa \\ & + \frac{1}{2}\,\Gamma^\nu_{\sigma\kappa}\, dZ^\mu dZ^\rho dZ^\sigma dZ^\kappa + \frac{1}{2}\,\Gamma^\rho_{\sigma\kappa}\, dZ^\mu dZ^\nu dZ^\sigma dZ^\kappa \Big) \\ + \frac{1}{12}\,\nabla_{(\sigma} \nabla_\rho \nabla_\nu \nabla_{\mu)} f\, & dZ^\mu dZ^\nu dZ^\rho dZ^\sigma. \end{aligned} \tag{8.118}$$

By reading of the term proportional to $\nabla_\mu \nabla_\nu f$, we conclude

$$g_{\mu\nu} \circ dZ^\mu_\tau dZ^\nu_\tau = g_{\mu\nu} \Big[dZ^\mu_\tau \, dZ^\nu_\tau + \Gamma^\mu_{\rho\sigma} \, dZ^\nu_\tau dZ^\rho_\tau dZ^\sigma_\tau + \frac{1}{4} \Gamma^\mu_{\rho\sigma} \Gamma^\nu_{\kappa\lambda} \, dZ^\rho_\tau dZ^\sigma_\tau dZ^\kappa_\tau dZ^\lambda_\tau \\ + \frac{1}{3} \left(\partial_\kappa \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\kappa\lambda} \Gamma^\lambda_{\rho\sigma} \right) dZ^\nu_\tau dZ^\rho_\tau dZ^\sigma_\tau dZ^\kappa_\tau \Big] . \tag{8.119}$$

where the Itô differential is given by

$$\begin{aligned} dZ^\mu_\tau &= Z^\mu_{\tau + d\tau} - Z^\mu_\tau \\ &= \int_\tau^{d\tau} w^\mu(Z_s) \, ds + dM^\mu_\tau \end{aligned} \tag{8.120}$$

We can now calculate the conditional expectation of this expression. We find

$$
\begin{aligned}
\mathbb{E}_\tau\Big[dZ^\mu_\tau\, dZ^\nu_\tau\Big] &= \mathbb{E}_\tau\left[dM^\mu_\tau dM^\nu_\tau + dM^\mu_\tau \int_\tau^{\tau+d\tau} w^\nu(Z_s)\, ds + dM^\nu_\tau \int_\tau^{\tau+d\tau} w^\mu(Z_s)\, ds \right.\\
&\qquad \left. + \int_\tau^{\tau+d\tau} w^\mu(Z_s)\, ds \int_\tau^{\tau+d\tau} w^\nu(Z_r)\, dr + o(d\tau^2)\right]\\
&= \mathbb{E}_\tau\left[\int_\tau^{\tau+d\tau} w^{\mu\nu}(Z_s)\, ds + dM^\mu_\tau \int_\tau^{\tau+d\tau} w^\nu(Z_s)\, ds + dM^\nu_\tau \int_\tau^{\tau+d\tau} w^\mu(Z_s)\, ds \right.\\
&\qquad \left. + \int_\tau^{\tau+d\tau} w^\mu(Z_s)\, ds \int_\tau^{\tau+d\tau} w^\nu(Z_r)\, dr + o(d\tau^2)\right]\\
&= \mathbb{E}_\tau\left[w^{\mu\nu}(Z_\tau) \int_\tau^{\tau+d\tau} ds + \partial_\rho w^{\mu\nu}(Z_\tau) \int_\tau^{\tau+d\tau} \left(M^\rho_s - M^\rho_\tau\right)\, ds \right.\\
&\qquad + \partial_\rho w^{\mu\nu}(Z_\tau) \int_\tau^{\tau+d\tau} \int_\tau^s w^\rho(Z_r)\, dr\, ds\\
&\qquad + \frac{1}{2}\, \partial_\rho \partial_\sigma w^{\mu\nu}(Z_\tau) \int_\tau^{\tau+d\tau} \left(M^\rho_s - M^\rho_\tau\right) \left(M^\sigma_s - M^\sigma_\tau\right)\, ds\\
&\qquad + w^\nu(Z_\tau)\, dM^\mu_\tau \int_\tau^{\tau+d\tau} ds + \partial_\rho w^\nu(Z_\tau)\, dM^\mu_\tau \int_\tau^{\tau+d\tau} \left(M^\rho_s - M^\rho_\tau\right) ds\\
&\qquad + w^\mu(Z_\tau)\, dM^\nu_\tau \int_\tau^{\tau+d\tau} ds + \partial_\rho w^\mu(Z_\tau)\, dM^\nu_\tau \int_\tau^{\tau+d\tau} \left(M^\rho_s - M^\rho_\tau\right) ds\\
&\qquad \left. + w^\mu(Z_\tau)\, w^\nu(Z_\tau) \int_\tau^{\tau+d\tau} ds \int_\tau^{\tau+d\tau} dr + o(d\tau^2)\right]\\
&= \mathbb{E}_\tau\Big[w^{\mu\nu}(Z_\tau)\, d\tau + w^\rho(Z_\tau)\, \partial_\rho w^{\mu\nu}(Z_\tau) \int_\tau^{\tau+d\tau} (s-\tau)\, ds\\
&\qquad + \frac{1}{2}\, \partial_\rho \partial_\sigma w^{\mu\nu}(Z_\tau) \int_\tau^{\tau+d\tau} \int_\tau^s w^{\rho\sigma}(Z_r)\, dr\, ds\\
&\qquad + \partial_\rho w^\nu(Z_\tau) \int_\tau^{\tau+d\tau} \int_\tau^s w^{\mu\rho}(Z_r)\, dr\, ds + \partial_\rho w^\mu(Z_\tau) \int_\tau^{\tau+d\tau} \int_\tau^s w^{\nu\rho}(Z_r)\, dr\, ds\\
&\qquad + w^\mu(Z_\tau)\, w^\nu(Z_\tau)\, d\tau^2 + o(d\tau^2)\Big]\\
&= w^{\mu\nu}(Z_\tau)\, d\tau + \frac{1}{2}\, w^\rho(Z_\tau)\, \partial_\rho w^{\mu\nu}(Z_\tau)\, d\tau^2 + \frac{1}{4}\, w^{\rho\sigma}(Z_\tau)\, \partial_\rho \partial_\sigma w^{\mu\nu}(Z_\tau)\, d\tau^2\\
&\quad + \frac{1}{2}\, w^{\mu\rho}(Z_\tau)\, \partial_\rho w^\nu(Z_\tau)\, d\tau^2 + \frac{1}{2}\, w^{\nu\rho}(Z_\tau)\, \partial_\rho w^\mu(Z_\tau)\, d\tau^2\\
&\quad + w^\mu(Z_\tau)\, w^\nu(Z_\tau)\, d\tau^2 + o(d\tau^2),
\end{aligned}
\tag{8.121}
$$

$$
\begin{aligned}
\mathbb{E}_\tau\Big[dZ^\nu_\tau dZ^\rho_\tau dZ^\sigma_\tau\Big] &= \mathbb{E}_\tau\left[dM^\nu_\tau dM^\rho_\tau \int_\tau^{\tau+d\tau} w^\sigma(Z_s)\,ds + dM^\nu_\tau dM^\sigma_\tau \int_\tau^{\tau+d\tau} w^\rho(Z_s)\,ds\right.\\
&\qquad \left. + dM^\rho_\tau dM^\sigma_\tau \int_\tau^{\tau+d\tau} w^\nu(Z_s)\,ds + dM^\nu_\tau dM^\rho_\tau dM^\sigma_\tau + o(d\tau^2)\right]\\
&= \mathbb{E}_\tau\left[\int_\tau^{\tau+d\tau} w^\nu(Z_s)\,ds \int_\tau^{\tau+d\tau} w^{\rho\sigma}(Z_r)\,dr + dM^\nu_\tau \int_\tau^{\tau+d\tau} w^{\rho\sigma}(Z_s)\,ds\right.\\
&\qquad + \int_\tau^{\tau+d\tau} w^\rho(Z_s)\,ds \int_\tau^{\tau+d\tau} w^{\nu\sigma}(Z_r)\,dr + dM^\rho_\tau \int_\tau^{\tau+d\tau} w^{\nu\sigma}(Z_s)\,ds\\
&\qquad \left. + \int_\tau^{\tau+d\tau} w^\sigma(Z_s)\,ds \int_\tau^{\tau+d\tau} w^{\nu\rho}(Z_r)\,dr + dM^\sigma_\tau \int_\tau^{\tau+d\tau} w^{\nu\rho}(Z_s)\,ds\right] + o(d\tau^2)\\
&= \mathbb{E}_\tau\Big[w^\nu(Z_\tau)\,w^{\rho\sigma}(Z_\tau)\,d\tau^2 + w^\rho(Z_\tau)\,w^{\nu\sigma}(Z_\tau)\,d\tau^2 + w^\sigma(Z_\tau)\,w^{\nu\rho}(Z_\tau)\,d\tau^2\\
&\qquad + w^{\rho\sigma}(Z_\tau)\,dM^\nu_\tau \int_\tau^{\tau+d\tau} ds + \partial_\kappa w^{\rho\sigma}(Z_\tau)\,dM^\nu_\tau \int_\tau^{\tau+d\tau} (M^\kappa_s - M^\kappa_\tau)\,ds\\
&\qquad + w^{\nu\sigma}(Z_\tau)\,dM^\rho_\tau \int_\tau^{\tau+d\tau} ds + \partial_\kappa w^{\nu\sigma}(Z_\tau)\,dM^\rho_\tau \int_\tau^{\tau+d\tau} (M^\kappa_s - M^\kappa_\tau)\,ds\\
&\qquad + w^{\nu\rho}(Z_\tau)\,dM^\sigma_\tau \int_\tau^{\tau+d\tau} ds + \partial_\kappa w^{\nu\rho}(Z_\tau)\,dM^\sigma_\tau \int_\tau^{\tau+d\tau} (M^\kappa_s - M^\kappa_\tau)\,ds\Big] + o(d\tau^2)\\
&= \mathbb{E}_\tau\Big[w^\nu(Z_\tau)\,w^{\rho\sigma}(Z_\tau)\,d\tau^2 + w^\rho(Z_\tau)\,w^{\nu\sigma}(Z_\tau)\,d\tau^2 + w^\sigma(Z_\tau)\,w^{\nu\rho}(Z_\tau)\,d\tau^2\\
&\qquad + \partial_\kappa w^{\rho\sigma}(Z_\tau) \int_\tau^{\tau+d\tau}\int_\tau^s w^{\nu\kappa}(Z_r)\,dr ds + \partial_\kappa w^{\nu\sigma}(Z_\tau) \int_\tau^{\tau+d\tau}\int_\tau^s w^{\rho\kappa}(Z_r)\,dr ds\\
&\qquad + \partial_\kappa w^{\nu\rho}(Z_\tau) \int_\tau^{\tau+d\tau}\int_\tau^s w^{\sigma\kappa}(Z_r)\,dr ds\Big] + o(d\tau^2)\\
&= \frac{1}{2}\Big[w^{\nu\kappa}(Z_\tau)\,\partial_\kappa w^{\rho\sigma}(Z_\tau) + w^{\rho\kappa}(Z_\tau)\,\partial_\kappa w^{\nu\sigma}(Z_\tau) + w^{\sigma\kappa}(Z_\tau)\,\partial_\kappa w^{\nu\rho}(Z_\tau)\Big]\,d\tau^2\\
&\quad + \Big[w^\nu(Z_\tau)\,w^{\rho\sigma}(Z_\tau) + w^\rho(Z_\tau)\,w^{\nu\sigma}(Z_\tau) + w^\sigma(Z_\tau)\,w^{\nu\rho}(Z_\tau)\Big]\,d\tau^2 + o(d\tau^2)
\end{aligned} \tag{8.122}
$$

and

$$
\begin{aligned}
\mathbb{E}_\tau\Big[dZ^\mu_\tau dZ^\nu_\tau dZ^\rho_\tau dZ^\sigma_\tau\Big] &= \mathbb{E}_\tau\Big[dM^\mu_\tau dM^\nu_\tau dM^\rho_\tau dM^\sigma_\tau + o(d\tau^2)\Big]\\
&= \mathbb{E}_\tau\left[\int_\tau^{\tau+d\tau} w^{\mu\nu}(Z_s)\,ds \int_\tau^{\tau+d\tau} w^{\rho\sigma}(Z_r)\,dr\right.\\
&\qquad + \int_\tau^{\tau+d\tau} w^{\mu\rho}(Z_s)\,ds \int_\tau^{\tau+d\tau} w^{\nu\sigma}(Z_r)\,dr\\
&\qquad \left. + \int_\tau^{\tau+d\tau} w^{\mu\sigma}(Z_s)\,ds \int_\tau^{\tau+d\tau} w^{\nu\rho}(Z_r)\,dr\right] + o(d\tau^2)\\
&= \Big[w^{\mu\nu}(Z_\tau)\,w^{\rho\sigma}(Z_\tau) + w^{\mu\rho}(Z_\tau)\,w^{\nu\sigma}(Z_\tau) + w^{\mu\sigma}(Z_\tau)\,w^{\nu\rho}(Z_\tau)\Big]\,d\tau^2\\
&\quad + o(d\tau^2).
\end{aligned} \tag{8.123}
$$

If we then use that $w^{\mu\nu} = \alpha\,\lambda\, g^{\mu\nu}$, we find

$$
\begin{aligned}
\mathbb{E}_\tau\Big[g_{\mu\nu}\circ dZ^\mu_\tau dZ^\nu_\tau\Big] &= g_{\mu\nu} w^{\mu\nu}\, d\tau \\
&\quad + g_{\mu\nu}\left(w^\mu w^\nu + \frac{1}{2}\, w^\rho\,\partial_\rho w^{\mu\nu} + \frac{1}{2}\, w^{\mu\rho}\,\partial_\rho w^\nu + \frac{1}{2}\, w^{\nu\rho}\,\partial_\rho w^\mu + \frac{1}{4}\, w^{\rho\sigma}\,\partial_\rho\partial_\sigma w^{\mu\nu}\right) d\tau^2 \\
&\quad + g_{\mu\nu}\Gamma^\mu_{\rho\sigma}\Big(w^\nu w^{\rho\sigma} + w^\rho w^{\nu\sigma} + w^\sigma w^{\nu\rho}\Big)\, d\tau^2 \\
&\quad + \frac{1}{2} g_{\mu\nu}\Gamma^\mu_{\rho\sigma}\Big(w^{\nu\kappa}\partial_\kappa w^{\rho\sigma} + w^{\rho\kappa}\partial_\kappa w^{\nu\sigma} + w^{\sigma\kappa}\partial_\kappa w^{\nu\rho}\Big)\, d\tau^2 \\
&\quad + \frac{1}{4} g_{\mu\nu}\Gamma^\mu_{\rho\sigma}\Gamma^\nu_{\kappa\lambda}\Big(w^{\rho\sigma}w^{\kappa\lambda} + w^{\rho\kappa}w^{\sigma\lambda} + w^{\rho\lambda}w^{\sigma\kappa}\Big)\, d\tau^2 \\
&\quad + \frac{1}{3} g_{\mu\nu}\Big(\partial_\kappa\Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\kappa\lambda}\Gamma^\lambda_{\rho\sigma}\Big)\Big(w^{\nu\kappa}w^{\rho\sigma} + w^{\nu\rho}w^{\sigma\kappa} + w^{\nu\sigma}w^{\rho\kappa}\Big)\, d\tau^2 + o(d\tau^2) \\
&= n\,\alpha\,\lambda\, d\tau + g_{\mu\nu} w^\mu w^\nu\, d\tau^2 + \alpha\,\lambda\,\Big(\partial_\mu w^\mu - \Gamma^\mu_{\mu\nu} w^\nu\Big) d\tau^2 \\
&\quad + \frac{\alpha^2\lambda^2}{2}\, g^{\rho\sigma}\Big(g_{\mu\nu}g^{\kappa\lambda}\Gamma^\mu_{\rho\kappa}\Gamma^\nu_{\sigma\lambda} + \Gamma^\mu_{\rho\nu}\Gamma^\nu_{\mu\sigma} - \partial_\rho\Gamma^\mu_{\mu\sigma}\Big)\, d\tau^2 \\
&\quad + \alpha\,\lambda\,\Big(g_{\mu\nu}g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} w^\nu + 2\,\Gamma^\mu_{\mu\nu} w^\nu\Big)\, d\tau^2 \\
&\quad - \alpha^2\lambda^2 g^{\rho\sigma}\Big(g_{\mu\nu}g^{\kappa\lambda}\Gamma^\mu_{\rho\kappa}\Gamma^\nu_{\sigma\lambda} + 2\,\Gamma^\mu_{\rho\nu}\Gamma^\nu_{\mu\sigma}\Big)\, d\tau^2 \\
&\quad + \frac{\alpha^2\lambda^2}{4} g_{\mu\nu} g^{\rho\sigma} g^{\kappa\lambda}\Big(\Gamma^\mu_{\rho\sigma}\Gamma^\nu_{\kappa\lambda} + 2\,\Gamma^\mu_{\rho\kappa}\Gamma^\nu_{\sigma\lambda}\Big)\, d\tau^2 \\
&\quad + \frac{\alpha^2\lambda^2}{3} g^{\rho\sigma}\Big(\partial_\mu\Gamma^\mu_{\rho\sigma} + 2\,\partial_\rho\Gamma^\mu_{\mu\sigma} + \Gamma^\mu_{\mu\nu}\Gamma^\nu_{\rho\sigma} + 2\,\Gamma^\mu_{\rho\nu}\Gamma^\nu_{\mu\sigma}\Big)\, d\tau^2 + o(d\tau^2) \\
&= n\,\alpha\,\lambda\, d\tau + g_{\mu\nu}\hat{w}^\mu\hat{w}^\nu\, d\tau^2 + \alpha\,\lambda\,\nabla_\mu\hat{w}^\mu d\tau^2 - \frac{\alpha^2\lambda^2}{6}\,\mathcal{R}\, d\tau^2 + o(d\tau^2).
\end{aligned}
\tag{8.124}
$$

Plugging this result into eq. (8.113) then yields eq. (8.109).

Part III

Effective Field Theory of Quantum Gravity: Perturbative Effects

Chapter 9

Quantum Gravitational Corrections to a Star Metric and the Black Hole Limit

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Abstract

In this paper we consider the full set of quantum gravitational corrections to a star metric to second order in curvature. As we use an effective field theoretical approach, these corrections apply to any model of quantum gravity that is based on general coordinate invariance. We then discuss the black hole limit and identify an interesting phenomenon which could shed some light on the nature of astrophysical black holes: while star metrics receive corrections at second order in curvature, vacuum solutions such as black hole metrics do not. What happens to these corrections when a star collapses?

This chapter has been published in Physical Review D **100**, no. 08, p. 6010 (2019).
A preprint of this chapter can be found at arXiv:1909.13277 [hep-th].

9.1 Introduction

Since the seminal work of Weinberg in 1979 [351], much progress has been made in quantum gravity using effective field theory methods [34–37, 61, 70, 140]. While finding a consistent theory of quantum gravity valid at all energy scales remains an elusive goal, effective field theory methods can be applied at energies below the Planck mass which might be all that is ever needed as physics is an empirical science. This approach enables calculations in quantum gravity which are model independent, see e.g. [49, 68, 69, 73–76, 81–83, 91, 93, 94, 97, 98, 111, 143]. The model independence only applies to models that assume that general coordinate invariance is also the correct symmetry of quantum gravity. Obviously in fundamental models with e.g. Lorentz violation, the effective field theory could be different. One of the important results recently obtained is that there are no quantum gravitational corrections to vacuum solutions of general relativity [70] to second order in curvature. This in particular applies to eternal black hole metrics which are static vacuum solutions [97]. On the other hand, real astrophysical black holes are clearly not in vacuum and they undergo a time evolution as they are formed out of some time dependent astrophysical process such as during the collapse of a heavy star.

Understanding the transition from a star to a black hole state could help to understand the nature of astrophysical black holes better. The aim of this work is to do a first step in that direction by calculating quantum gravitational corrections to the metric of a star in stable equilibrium, as described by the Tolman-Oppenheimer-Volkoff equation. In general relativity, the metric outside non-rotating black holes and stars is given in both cases by the vacuum Schwarzschild solution. Our aim is to compare the quantum gravitational corrections to a star metric and black hole metric as seen by an observer who is far away from both objects. While it is known that in the black hole case there are no corrections to the metric at second order in curvature, we will show that there is a correction at this order in the case of a star. This phenomenon is intriguing as a distant observer could in principle differentiate a star that is collapsing from an eternal black hole (i.e. a vacuum solution) by measuring the correction at order $G_{\rm N}^2$ to Newton's potential. The collapsing star would have a potential that deviates from $1/r$ by corrections of order $G_{\rm N}^2$ while the black hole vacuum solution does not have such corrections.

We then consider the limit when the mass and the radius of the star are taken towards respectively the Planck mass and the Planck length and discuss whether the metric obtained in that limit could be used to describe the metric of a quantum black hole, i.e. the lightest black holes that could have masses of the order of the Planck mass and a Schwarzschild radius of the order of the Planck length. We argue that as quantum black holes cannot be described as a classical vacuum, the quantum corrected star metric should be a better model for the metric of a quantum black hole than the Schwarzschild vacuum solution.

This paper is organized as follows. In Section 9.2, we introduce the effective quantum gravitational action and calculate the leading order corrections to the metric for a homogeneous isotropic star. In Section 9.3, we discuss the validity of our results close to the surface of the star. In Section 9.4, we discuss the differences with an eternal Schwarzschild black hole metric and argue that quantum black holes might be better described by the star metric. Finally, we conclude with some outlooks in Section 9.5.

9.2 Quantum corrections to a star metric

Aim of this section is to calculate the leading order quantum gravitational corrections to the metric of a stable star satisfying the Tolman-Oppenheimer-Volkoff equation. This investigation was started in [97], but that paper only considered the contribution of the term $R \log \Box R$. Here we consider the full set of corrections at second order in curvature. We also take this opportunity to fix a calculational mistake in [97].

We work within the framework of the effective quantum gravitational action given by [34–37, 61, 70, 140, 351]

$$\Gamma[g] = \Gamma_{\mathrm{L}}[g] + \Gamma_{\mathrm{NL}}[g], \tag{9.1}$$

where the local part of the action is given by [1]

$$\Gamma_{\mathrm{L}} = \int d^4x \sqrt{g} \left[\frac{R}{16\,\pi\, G_{\mathrm{N}}} + c_1(\mu)\, R^2 + c_2(\mu)\, R_{\mu\nu}\, R^{\mu\nu} + c_3(\mu)\, R_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta} \right] \tag{9.2}$$

and the non-local part of the action by

$$\Gamma_{\mathrm{NL}} = - \int d^4x \sqrt{g} \left[\alpha\, R\, \ln\!\left(\frac{\Box}{\mu^2}\right) R + \beta\, R_{\mu\nu}\, \ln\!\left(\frac{\Box}{\mu^2}\right) R^{\mu\nu} + \gamma\, R_{\mu\nu\alpha\beta}\, \ln\!\left(\frac{\Box}{\mu^2}\right) R^{\mu\nu\alpha\beta} \right]. \tag{9.3}$$

This effective action is obtained by integrating out the fluctuations of the graviton and potentially other massless matter fields. While the Wilson coefficients of the local part of the action are not calculable from first principles as we do not specify the ultraviolet theory of quantum gravity, those of the non-local part are calculable and model independent quantum gravitational predictions. We reproduce these coefficients, which have been derived by many different authors, see e.g. [34, 35, 48, 140, 143, 149, 190, 219, 260, 261], in Table 9.1.

The equations of motion obtained from varying the effective action which respect to the metric are given by

$$G_{\mu\nu} + 16\,\pi\, G_{\mathrm{N}} \left(H^{\mathrm{L}}_{\mu\nu} + H^{\mathrm{NL}}_{\mu\nu} \right) = 0, \tag{9.4}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} \tag{9.5}$$

is the usual Einstein tensor. The local part of the equation of motion is given by

$$\begin{aligned} H^{\mathrm{L}}_{\mu\nu} =& \bar{c}_1 \left(2\, R\, R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R^2 + 2\, g_{\mu\nu}\, \Box R - 2 \nabla_\mu \nabla_\nu R \right) \\ &+ \bar{c}_2 \left(2\, R^{\alpha}_{\ \mu}\, R_{\nu\alpha} - \frac{1}{2}\, g_{\mu\nu}\, R_{\alpha\beta}\, R^{\alpha\beta} + \Box R_{\mu\nu} + \frac{1}{2}\, g_{\mu\nu}\, \Box R - \nabla_\alpha \nabla_\mu R^{\alpha}_{\ \nu} - \nabla_\alpha \nabla_\nu R^{\alpha}_{\ \mu} \right), \end{aligned} \tag{9.6}$$

[1] In this paper we work in the $(+ - - -)$ signature and use the convention where the Riemann tensor is defined by $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \ldots$ and the Ricci tensor by $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$

	α	β	γ
Scalar	$5(6\xi-1)^2$	-2	2
Fermion	-5	8	7
Vector	-50	176	-26
Graviton	250	-244	424

Table 9.1: Non-local Wilson coefficients for different fields. All numbers should be divided by $11520\pi^2$. Here, ξ denotes the value of the non-minimal coupling for a scalar theory. All these coefficients including those for the graviton are gauge invariant. It is well known that one needs to be careful with the graviton self-interaction diagrams and that the coefficients α and β can be gauge dependent, see [219], if the effective action is defined in a naive way. For example, the numbers $\alpha = 430/(11520\pi^2)$ and $\beta = -1444/(11520\pi^2)$ for the graviton quoted in [143] are obtained using the Feynman gauge. However, there is a well-established procedure to derive a unique effective action which leads to gauge independent results [34, 35]. Here we are quoting the values of α and β for the graviton obtained using this formalism as it guaranties the gauge independence of observables.

with $\bar{c}_1 = c_1 - c_3$ and $\bar{c}_2 = c_2 + 4\,c_3$. Finally, the non-local part reads

$$\begin{aligned} H^{\mathrm{NL}}_{\mu\nu} =& -2\,\alpha\left(R_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu}\,R + g_{\mu\nu}\,\Box - \nabla_\mu\nabla_\nu\right)\ln\left(\frac{\Box}{\mu^2}\right)\,R \\ & -\beta\Big(2\,\delta^\alpha_{(\mu}\,R_{\nu)\beta} - \frac{1}{2}\,g_{\mu\nu}\,R^\alpha{}_\beta + \delta^\alpha_\mu\,g_{\nu\beta}\,\Box + g_{\mu\nu}\,\nabla^\alpha\nabla_\beta \\ & -\delta^\alpha_\mu\,\nabla_\beta\nabla_\nu - \delta^\alpha_\nu\,\nabla_\beta\nabla_\mu\Big)\ln\left(\frac{\Box}{\mu^2}\right)\,R^\beta{}_\alpha \\ & -2\,\gamma\left(\delta^\alpha_{(\mu}\,R_{\nu)}{}^\beta{}_{\sigma\tau} - \frac{1}{4}\,g_{\mu\nu}\,R^{\alpha\beta}{}_{\sigma\tau} + \left(\delta^\alpha_\mu\,g_{\nu\sigma} + \delta^\alpha_\nu\,g_{\mu\sigma}\right)\nabla^\beta\nabla_\tau\right)\ln\left(\frac{\Box}{\mu^2}\right)\,R_{\alpha\beta}{}^{\sigma\tau}. \end{aligned} \tag{9.7}$$

Note that the variation of the $\ln\Box$ term yields terms of higher order in curvature and can thus safely be ignored at second order in curvature.

We consider a stationary homogeneous and isotropic star with density

$$\rho(r) = \rho_0\,\Theta(R_{\mathrm{s}} - r) = \begin{cases} \rho_0 & \text{if } r < R_{\mathrm{s}} \\ 0 & \text{if } r > R_{\mathrm{s}}, \end{cases} \tag{9.8}$$

where $\rho_0 > 0$ is a constant and $\Theta(x)$ is Heaviside's step function. The solution to the Einstein equation inside this star (for $r \le R_{\mathrm{s}}$) is the well-known interior Schwarzschild metric [312, 342]

$$\begin{aligned} ds^2 =& \left(3\sqrt{1 - \frac{2\,G_{\mathrm{N}}\,M}{R_{\mathrm{s}}}} - \sqrt{1 - \frac{2\,G_{\mathrm{N}}\,M\,r^2}{R_{\mathrm{s}}^3}}\right)^2\frac{dt^2}{4} - \left(1 - \frac{2\,G_{\mathrm{N}}\,Mr^2}{R_{\mathrm{s}}^3}\right)^{-1}dr^2 - r^2\,d\Omega^2 \\ \equiv& \, g^{\mathrm{int}}_{\mu\nu}\,dx^\mu\,dx^\nu, \end{aligned} \tag{9.9}$$

where

$$M = 4\,\pi\int_0^{R_{\mathrm{s}}}\rho\,r^2\,dr = \frac{4\,\pi}{3}\,R_{\mathrm{s}}^3\,\rho_0 \tag{9.10}$$

is the total Misner-Sharp mass of the source. The corresponding pressure is given by

$$P(r) = \rho_0 \, \frac{\sqrt{1 - \frac{2\,G_\mathrm{N}\,M}{R_\mathrm{s}}} - \sqrt{1 - \frac{2\,G_\mathrm{N}\,M\,r^2}{R_\mathrm{s}^3}}}{\sqrt{1 - \frac{2\,G_\mathrm{N}\,M\,r^2}{R_\mathrm{s}^3}} - 3\sqrt{1 - \frac{2\,G_\mathrm{N}\,M}{R_\mathrm{s}}}} = \mathcal{O}(G_\mathrm{N}), \tag{9.11}$$

and is of order G_N in agreement with the fact that the pressure does not gravitate in Newtonian physics. Of course, the metric outside the star (for $r > R_\mathrm{s}$) is the usual vacuum Schwarzschild metric [312, 342]

$$ds^2 = \left(1 - \frac{2\,G_\mathrm{N}\,M}{r}\right) dt^2 - \left(1 - \frac{2\,G_\mathrm{N}\,M}{r}\right)^{-1} dr^2 - r^2\, d\Omega^2 \equiv g_{\mu\nu}^{\mathrm{ext}}\, dx^\mu\, dx^\nu, \tag{9.12}$$

from which one can see that M is also the Arnowitt-Deser-Misner (ADM) mass [16] of the system.

We now perturb the above metrics,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + g_{\mu\nu}^{\mathrm{q}}, \tag{9.13}$$

and take the perturbation $g_{\mu\nu}^{\mathrm{q}}$ to be $\mathcal{O}(G_\mathrm{N})$. The equations of motion then become

$$G_{\mu\nu}^{\mathrm{L}}[g^{\mathrm{q}}] + 16\,\pi\,G_\mathrm{N}\left(H_{\mu\nu}^{\mathrm{L}}[g] + H_{\mu\nu}^{\mathrm{NL}}[g]\right) = 0, \tag{9.14}$$

where the linearised Einstein tensor is given by

$$\begin{aligned} 2\,G_{\mu\nu}^{\mathrm{L}} = & \Box g_{\mu\nu}^{\mathrm{q}} - g_{\mu\nu}\,\Box g^{\mathrm{q}} + \nabla_\mu\nabla_\nu g^{\mathrm{q}} + 2\,R^{\alpha\ \beta}_{\ \mu\ \nu}\, g_{\alpha\beta}^{\mathrm{q}} - \nabla_\mu\nabla^\beta g_{\nu\beta}^{\mathrm{q}} - \nabla_\nu\nabla^\beta g_{\mu\beta}^{\mathrm{q}} \\ & + g_{\mu\nu}\,\nabla^\alpha\nabla^\beta g_{\alpha\beta}^{\mathrm{q}}. \end{aligned} \tag{9.15}$$

We first calculate solutions to equation (9.14) due to the local corrections. Outside the star, where the unperturbed metric equals the Schwarzschild vacuum solution (9.12) with $R = R_{\mu\nu} = 0$, these corrections are trivially 0. Inside the star this is not the case. However, these corrections turn out to be $\mathcal{O}(G_\mathrm{N}^3)$, and thus sub-leading. Therefore the local part in the equations of motion (9.6) does not contribute.

In order to calculate corrections due to the non-local corrections of the equation of motion (9.6) we use the fact that the Ricci Scalar, Ricci tensor and Riemann tensor are all $\mathcal{O}(G_\mathrm{N})$. We thus obtain

$$\begin{aligned} \frac{G_{\mu\nu}^{\mathrm{L}}}{16\,\pi\,G_\mathrm{N}} = & 2\,\alpha\left(g_{\mu\nu}\,\Box - \nabla_\mu\nabla_\nu\right)\ln\!\left(\frac{\Box}{\mu^2}\right) R \\ & + \beta\left(\delta_\mu^\alpha\, g_{\nu\beta}\,\Box + g_{\mu\nu}\,\nabla^\alpha\nabla_\beta - \delta_\mu^\alpha\,\nabla_\beta\nabla_\nu - \delta_\nu^\alpha\,\nabla_\beta\nabla_\mu\right)\ln\!\left(\frac{\Box}{\mu^2}\right) R^\beta_{\ \alpha} \\ & + 2\gamma\left(\delta_\mu^\alpha\, g_{\nu\sigma} + \delta_\nu^\alpha\, g_{\mu\sigma}\right)\nabla^\beta\nabla_\tau \ln\!\left(\frac{\Box}{\mu^2}\right) R_{\alpha\beta}^{\ \ \sigma\tau} + \mathcal{O}(G_\mathrm{N}^3). \end{aligned} \tag{9.16}$$

We will solve this equation perturbatively in G_N. We use Einstein equations to rewrite the Ricci scalar and tensor in terms of the energy-momentum tensor of the source,

$$R = -8\,\pi\, G_{\rm N} T \tag{9.17}$$

$$R_{\mu\nu} = 8\,\pi\, G_{\rm N} \left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T \right), \tag{9.18}$$

where, for a perfect isotropic fluid like our star, we have

$$T = \rho_0 + \mathcal{O}(G_{\rm N}) \tag{9.19}$$

$$T_{\mu\nu} = \delta^0_\mu\, \delta^0_\nu\, \rho_0 + \mathcal{O}(G_{\rm N}), \tag{9.20}$$

where ρ_0 is the energy density.

By applying the results from Appendix 9.5 to the homogeneous distribution (9.8), we find

$$8\,\pi\, G_{\rm N}\, \ln\!\left(\frac{\Box}{\mu^2}\right) \rho = \frac{6 G_{\rm N} M}{R_{\rm s}^3}\, f(r) + \mathcal{O}(G_{\rm N}^2), \tag{9.21}$$

with

$$f(r) = \begin{cases} -2\left[\gamma_E - 1 + \ln\left(\mu\sqrt{R_{\rm s}^2 - r^2}\right)\right] & \text{if } r < R_{\rm s}, \\[2ex] 2\dfrac{R_{\rm s}}{r} - \ln\left(\dfrac{r+R_{\rm s}}{r-R_{\rm s}}\right) & \text{if } r > R_{\rm s}. \end{cases} \tag{9.22}$$

Note that the function f in equation (9.22) is not defined at $r = R_{\rm s}$. In fact, one can verify that the results should be taken with some care in a small region around $R_{\rm s}$, as we discuss in more detail in Section 9.3.

Furthermore, we emphasize that equation (9.22) is the main source of the discrepancy between the results reported here and those obtained in [97], where the calculation was only done for $r > R_{\rm s}$. In equation (31) of [97] a factor of 2 is missing in front of the term $R_{\rm s}/r$ and a factor of -1 is missing in front of the log term.

In order to obtain the contribution proportional to γ in equation (9.16), we first rewrite it in terms of those proportional to α and β using the non-local Gauss-Bonnet theorem [37–39, 70], which holds for the non-local part up to second order in curvature (hence $\mathcal{O}(G_{\rm N}^2)$). We then evaluate equation (9.16) using $\alpha' = \alpha - \gamma$ and $\beta' = \beta + 4\,\gamma$. We thus have to solve

$$\begin{aligned} G^{\rm L}_{\mu\nu} =& \,192\,\pi\,(\alpha-\gamma)\,\frac{G_{\rm N}^2\,M}{R_{\rm s}^3}\,(\nabla_\mu\nabla_\nu - g_{\mu\nu}\,\Box)\,f(r) \\ &+ 96\,\pi\,(\beta+4\,\gamma)\,\frac{G_{\rm N}^2\,M}{R_{\rm s}^3}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\,\Box + \delta^0_\mu\, g_{\nu 0}\,\Box\right) f(r) + \mathcal{O}(G_{\rm N}^3), \end{aligned} \tag{9.23}$$

where we used that

$$\left(g_{\mu\nu}\,\nabla^0\nabla_0 - \delta^0_\mu\,\nabla_0\nabla_\nu - \delta^0_\nu\,\nabla_0\nabla_\mu\right) f(r) = \mathcal{O}(G_{\rm N}). \tag{9.24}$$

We solve this equation, imposing the solution to be spherically symmetric and time independent. In addition we fix the gauge freedom by setting $g^{\rm q}_{\theta\theta} = 0$. Doing so, we obtain the quantum corrections $g^{\rm q}_{\mu\nu} = \delta g^{\rm ext}_{\mu\nu}$ to the Schwarzschild metric (9.12) outside the star. The corrections are given by [2]

$$\begin{aligned}
\delta g^{\rm ext}_{tt} &= (\alpha+\beta+3\,\gamma)\,\frac{192\,\pi\,G_{\rm N}^2\,M}{R_{\rm s}^3}\left[2\,\frac{R_{\rm s}}{r}+\ln\!\left(\frac{r-R_{\rm s}}{r+R_{\rm s}}\right)\right]+\frac{C_1}{r}+C_2+\mathcal{O}(G_{\rm N}^3)\\
\delta g^{\rm ext}_{rr} &= (\alpha-\gamma)\,\frac{384\,\pi\,G_{\rm N}^2\,M}{r\,(r^2-R_{\rm s}^2)}+\frac{C_1}{r}+\mathcal{O}(G_{\rm N}^3),
\end{aligned} \tag{9.25}$$

where C_i are integration constants which must be set to zero, if we require asymptotic flatness, that is $\lim_{r\to\infty}\delta g_{\mu\nu} = \lim_{r\to\infty} r\,\delta g_{\mu\nu} = 0$ [3].

In a similar way, using the same gauge condition, one can find the corrections $g^{\rm q}_{\mu\nu} = \delta g^{\rm int}_{\mu\nu}$ to the metric (9.9) inside the star. These are given by

$$\begin{aligned}
\delta g^{\rm int}_{tt} &= (\alpha+\beta+3\,\gamma)\,\frac{192\,\pi\,G_{\rm N}^2\,M}{R_{\rm s}^3}\,\ln\!\left(\frac{R_{\rm s}^2}{R_{\rm s}^2-r^2}\right)+\frac{C_3}{r}+C_4+\mathcal{O}(G_{\rm N}^3)\\
\delta g^{\rm int}_{rr} &= (\alpha-\gamma)\,\frac{384\,\pi\,G_{\rm N}^2\,M\,r^2}{R_{\rm s}^3\,(R_{\rm s}^2-r^2)}+\frac{C_3}{r}+\mathcal{O}(G_{\rm N}^3),
\end{aligned} \tag{9.26}$$

where C_i are integration constants, which we will set to 0 by requiring regularity in the origin $r = 0$.

In the limit $r \to R_{\rm s}$ we find that the corrections diverge, but it is easy to explain that these divergences are generated, because we assumed a model for the star described by a discontinuous density at $r = R_{\rm s}$, which is not realistic for an astrophysical star. This discontinuity leads to a discontinuity in the first derivative of the pressure (9.11), in the second derivative of the g_{tt} component and in the first derivative of the g_{rr} component. We thus do not expect that our star model and hence the quantum corrections apply to a real star in a small region around $R_{\rm s}$. We shall discuss this observation in more details as well as how to cure these divergences in the next section.

We can now consider our result in different limits. Far away from the star (for $r \gg R_{\rm s}$), the leading behavior of the metric corrections (9.25) is given by

$$\begin{aligned}
\delta g^{\rm ext}_{tt} &= -(\alpha+\beta+3\,\gamma)\,\frac{128\,\pi\,G_{\rm N}^2\,M}{r^3}+\mathcal{O}(G_{\rm N}^3)\\
\delta g^{\rm ext}_{rr} &= (\alpha-\gamma)\,\frac{384\,\pi\,G_{\rm N}^2\,M}{r^3}+\mathcal{O}(G_{\rm N}^3),
\end{aligned} \tag{9.27}$$

whereas, to the same order in $G_{\rm N}$, the corrections (9.26) for the metric inside the star far away from the star radius (for $r \ll R_{\rm s}$) vanish,

$$\delta g^{\rm int}_{tt} = \delta g^{\rm int}_{rr} = \mathcal{O}(G_{\rm N}^3). \tag{9.28}$$

It is important to realize that the correction to the components of a metric are gauge

[2]Note that we take the metric with signature $(+\,-\,-\,-)$. With signature $(-\,+\,+\,+)$ the corrections obtain an extra minus sign.

[3]These conditions ensure that we recover the classical weak field limit with ADM mass M as $r \to \infty$, which is the usual boundary condition for the classical Schwarzschild black hole.

dependent. As such components are not observables, this is not an issue. For example, one could calculate the metric corrections in the harmonic gauge. In this case one finds the asymptotic $r \gg R_{\rm s}$ expressions

$$\begin{aligned}
g_{tt} =& 1 - \frac{2\,G_{\rm N}\,M}{r} + \frac{2\,G_{\rm N}^2\,M^2}{r^2} - (\alpha+\beta+3\,\gamma)\,\frac{128\,\pi\,G_{\rm N}^2\,M}{r^3} + \mathcal{O}(G_{\rm N}^3)\\
g_{ti} =& 0\\
g_{ij} =& -\delta_{ij}\left\{1 + \frac{2\,G_{\rm N}M}{r} + \frac{G_{\rm N}^2\,M^2}{r^2} - (2\,\alpha+\beta+2\,\gamma)\,\frac{128\,\pi\,G_{\rm N}^2\,M}{r^3}\left[\frac{1}{3} + \ln\!\left(\frac{C\,r}{R_{\rm s}}\right)\right]\right\}\\
&- \frac{x_i x_j}{r^2}\left[\frac{G_{\rm N}^2\,M^2}{r^2} - (\alpha-\gamma)\,\frac{384\,\pi\,G_{\rm N}^2\,M}{r^3} + (2\,\alpha+\beta+2\,\gamma)\,\frac{384\,\pi\,G_{\rm N}^2\,M}{r^3}\ln\!\left(\frac{C\,r}{R_{\rm s}}\right)\right]\\
&+ \mathcal{O}(G_{\rm N}^3), \qquad (9.29)
\end{aligned}$$

where C is a dimensionless integration constant [4]. We derived this result using the expression for the Schwarzschild metric outside a star in the harmonic gauge, which can, for example, be found in [349]. Furthermore, we imposed the solutions to be spherically symmetric and time independent and imposed the harmonic (De Donder) gauge condition instead of setting $\delta g_{\theta\theta} = 0$.

Taking the graviton values for α, β and γ from [143], one can set the scale $C/R_{\rm s} = \mu \exp(-173/132)$, to recover the quantum correction due to the vacuum polarization diagram found in [49]. It should be emphasized that the graviton values for α and β presented in [143] are not gauge invariant [219] and do not correspond to the values obtained when the unique effective action formalism [34] is used, which are presented in Table 9.1. The results in [49, 143] are thus dependent on the gauge in which the effective action is obtained. The results presented in this paper on the other hand do not suffer from this gauge dependence. Naturally, both the results presented in this paper and those in [49, 143] depend on the gauge (that is, the reference frame) in which the field equations are solved. This gauge dependence cannot be removed, as the metric components are not gauge invariant quantities.

Let us emphasize that the results presented in this section are interesting: we have shown that although the metric outside an eternal static black hole and of a static star are given at the classical level by the Schwarzschild solution, quantum gravity makes a difference between the two objects due to its non-local nature. The star metric receives a quantum correction at second order in curvature, while there is no such correction for an eternal black hole [70]. A distant observer can in principle monitor the gravitational collapse of a star by studying the quantum gravitational corrections to Newton's potential to second order in curvature. This raises the question whether astrophysical black holes should really be described by metrics corresponding to vacuum solutions of general relativity. Note that our argument does not rely on the limit $R_s \to 0$, but rather on a comparison of the initial state (e.g. collapsing star or star before it has even started to collapse) and the final state which is an eternal black hole.

[4] As in previous results, one obtains a couple more integration constants, which can be set to 0 by requiring that one recovers the classical weak field limit as $r \to \infty$.

9.3 Divergence at the surface

The explicit calculation shown in Appendix 9.5 makes it clear that the non-local function $\ln\left(\frac{\Box}{\mu^2}\right)$ must be treated as a distribution in order to allow for the various exchanges of limits and integrations. This in turn implies that the functions f upon which it can act must belong to a suitable set of regular test functions. Clearly, the density profile (9.8) does not satisfy this requirement, the Heaviside function Θ being a distribution itself. It therefore comes as no surprise that $\ln\left(\frac{\Box}{\mu^2}\right)\rho$ is not well defined around $r = R_\mathrm{s}$, unless the density (9.8) is replaced with a function that falls to zero smoothly.

It is important to remark that, although the density (9.8) generating the classical Schwarzschild interior metric (9.9) drops to zero within a vanishingly short length, it causes no issues in general relativity despite the fact that the manifold is not smooth at the star surface. Instead, it conjures with the non-local terms of the effective action (9.3) to give rise to divergences. The divergence thus purely arises due to inclusion of higher order derivatives of the metric, while the metric is only once continuously differentiable. However, it is obvious that the density profile of any realistic matter distribution will go to zero in a finite width $\epsilon > 0$. For instance, we could replace (9.8) with the infinitely smooth

$$\rho(r) = \begin{cases} \rho_0 \exp\left(\dfrac{\epsilon^2}{R_\mathrm{s}^2} - \dfrac{\epsilon^2}{R_\mathrm{s}^2 - r^2}\right) & \text{for } 0 \le r \le R_\mathrm{s} \\ 0 & \text{for } R_\mathrm{s} < r, \end{cases} \tag{9.30}$$

where we can safely assume that $\epsilon \gtrsim \ell_\mathrm{p}$. This implies that our solutions (9.25) and (9.26) should only be considered outside a layer of thickness ϵ around R_s. On the other hand, it is important to remark that the size of the corrections does not depend on ϵ explicitly (only the region of space excluded in our results does).

In some more details, Eqs. (9.25) and (9.26) contain divergences for $\epsilon \equiv |r - R_\mathrm{s}| \to 0^+$, namely

$$\begin{aligned} \delta g_{tt}^{\mathrm{int}} &\simeq -(\alpha + \beta + 3\,\gamma)\,\frac{192\,\pi\,G_\mathrm{N}^2\,M}{R_\mathrm{s}^3}\,\ln\left(\frac{2\,\epsilon}{R_\mathrm{s}}\right) \\ \delta g_{tt}^{\mathrm{ext}} &\simeq (\alpha + \beta + 3\gamma)\,\frac{192\,\pi\,G_\mathrm{N}^2\,M}{R_\mathrm{s}^3}\left[2 + \ln\left(\frac{\epsilon}{2\,R_\mathrm{s}}\right)\right] \\ \delta g_{rr}^{\mathrm{int}} &\simeq (\alpha - \gamma)\,\frac{192\,\pi\,G_\mathrm{N}^2\,M}{R_\mathrm{s}^3}\left(\frac{R_\mathrm{s}}{\epsilon} - \frac{3}{2}\right) \\ \delta g_{rr}^{\mathrm{ext}} &\simeq (\alpha - \gamma)\,\frac{192\,\pi\,G_\mathrm{N}^2\,M}{R_\mathrm{s}^3}\left(\frac{R_\mathrm{s}}{\epsilon} - \frac{1}{2}\right), \end{aligned} \tag{9.31}$$

which appear in two forms, namely

$$d_1 \sim \frac{G_\mathrm{N}^2\,M}{R_\mathrm{s}^3}\,\ln\left(\frac{|r - R_\mathrm{s}|}{r + R_\mathrm{s}}\right), \tag{9.32}$$

or

$$d_2 \sim \frac{G_\mathrm{N}^2\,M}{r\,|r^2 - R_\mathrm{s}^2|}. \tag{9.33}$$

Since we obtained the corrections in a "weak" field approximation, such terms should be small compared to the unperturbed metric coefficients, that is

$$d_i \lesssim V \sim \frac{G_{\rm N}\,M}{r}. \tag{9.34}$$

By recalling that $G_{\rm N} = \ell_{\rm p}^2$ in our units, this means that $d_1 \ll V$ provided

$$\frac{\ell_{\rm p}^2}{R_{\rm s}^2}\,\ln\left(\frac{|r-R_{\rm s}|}{R_{\rm s}}\right) \lesssim 1 \tag{9.35}$$

and $d_2 \ll V$ if

$$\frac{\ell_{\rm p}^2}{R_{\rm s}\,|r-R_{\rm s}|} \lesssim 1. \tag{9.36}$$

The above two conditions are clearly satisfied if $\epsilon \equiv |r - R_{\rm s}| \lesssim \ell_{\rm p}$, since $R_{\rm s} \gg \ell_{\rm p}$ is the radius of a macroscopic matter source. To illustrate this, one can derive numerical estimates on the size of ϵ for various values of $R_{\rm s}$. In particular, we find for a typical neutron star with radius $R_{\rm s} \simeq 10\,{\rm km}$, that $\epsilon \gtrsim 10^{-78}\,R_{\rm s}$, while for objects of the order of the Planck length $R_{\rm s} \approx 10^{-35}\,{\rm m}$, we find $\epsilon \approx R_{\rm s}$. As expected, our approximation fails for sub-Planckian objects, and we must therefore restrict our analysis to $M \gtrsim 1/\sqrt{G_N} = M_P$ where M_P is the Planck scale. Moreover for Planck sized objects these restrictions are of major importance, and must be considered in any further analysis.

9.4 Model for quantum black holes?

While it is remarkable to be able to calculate model independent quantum gravitational corrections to the metric of a star or vacuum solutions of general relativity, it is clear that these corrections are tiny and probably of little empirical value from an astrophysical perspective. However, quantum gravitational corrections could be important for objects such as Planckian quantum black holes [68, 69, 74–76, 82, 83, 111], i.e. hypothetical objects with a mass close to the Planck scale and size of the order of the Planck length, which could have played an important role during the big bang. We have seen that quantum gravity makes a difference between a static star metric and an eternal black hole solution, the latter being described by a vacuum solution of Einstein equations. In this section we investigate which of the two external metrics would be better suited to model a Planckian quantum black hole. In order to address this question, we need to extrapolate our star model into the quantum regime.

In Section 9.2 we derived quantum corrections to the metric generated by a homogeneous ball of dust with density (9.8) and isotropic pressure (9.11). According to general relativity, this unperturbed classical configuration is stable only provided the size of the source does not violate the Buchdahl limit [62, 342], so that its radius must satisfy

$$R_{\rm s} > \frac{9}{8}\,R_{\rm M} \equiv \frac{9}{8}\,(2\,G_{\rm N}\,M)\,, \tag{9.37}$$

where R_M is the gravitational radius of the ball and would be the horizon radius of the outer Schwarzschild metric. While this is the classical limit, it may not hold for quantum black holes as can be seen by taking $R_{\rm s} \sim \ell_{\rm p} \sim \sqrt{G_N}$ and $M \sim M_P \sim 1/\sqrt{G_N}$ [5]. Quantum black holes are not expected to be stable objects anyway, but one expects them to decay very quickly within a time of the order of the Planck time $\tau_P \simeq \sqrt{G_N}$. We thus do not expect Planckian black holes to be well described by vacuum solutions. The inside of Planckian black holes is certainly not in vacuum as the fluctuations of space-time are expected to be large and space-time could lose its meaning altogether on such short distances. A better approximation might thus be to describe such objects might with a quantum corrected star metric.

In fact, even if we accept the general relativistic prediction that the collapsed matter giving rise to a black hole geometry must end in a very small region of extremely high density [6], it is not *a priori* clear that the size of this region remains negligible when the black hole mass M approaches the Planck scale.

In particular, the external metric (9.12) receives the quantum corrections (9.25) in the regime $|r - R_{\rm s}| \gg \ell_{\rm p}$ (as we explained in Section 9.3). For $r \gg R_{\rm s}$, the corrected metric can therefore be written as

$$ds^2 = g_{tt}\, dt^2 - g_{rr}\, dr^2 - r^2\, d\Omega^2, \tag{9.38}$$

with

$$\begin{aligned} g_{tt} &\simeq 1 - \frac{2\, G_{\rm N}\, M}{r} - \frac{\hat{\alpha}\, \hbar\, G_{\rm N}^2\, M}{r^3} \\ &\simeq 1 - \frac{2\, \ell_{\rm p}\, M}{M_P\, r} - \frac{\hat{\alpha}\, \ell_{\rm p}^3\, M}{M_P\, r^3}, \end{aligned} \tag{9.39}$$

and

$$\begin{aligned} g_{rr} &\simeq -\left(1 - \frac{2\, G_{\rm N}\, M}{r}\right)^{-1} + \frac{\hat{\beta}\, \hbar\, G_{\rm N}^2\, M}{r^3} \\ &\simeq -\left(1 - \frac{2\, \ell_{\rm p}\, M}{M_P\, r}\right)^{-1} + \frac{\hat{\beta}\, \ell_{\rm p}^3\, M}{M_P\, r^3}, \end{aligned} \tag{9.40}$$

where $\hat{\alpha} = 128\, \pi\, (\alpha + \beta + 3\, \gamma)$ and $\hat{\beta} = 384\, \pi\, (\alpha - \gamma)$. Note that $\hat{\alpha} > 0$ for scalar and vector particles as well as for fermions and gravitons, while $\hat{\beta} < 0$ for vectors, fermions and gravitons, and can be both positive and negative for scalars depending on the value of the non-minimal coupling ξ (see Table 9.1). On considering the particle content of the Standard Model and minimal coupling $\xi = 0$, one would then find $\hat{\beta} < 0$.

The gravitational radius $R_{\rm H}$ of the system is then determined by the condition $g^{rr}(R_{\rm H}) =$

[5] In this section we shall use units with $c = 1$, $G_{\rm N} = \ell_{\rm p}/M_P$ and $\hbar = \ell_{\rm p}\, M_P$.

[6] It is worth recalling that delta-like sources in general relativity are not mathematically consistent [168].

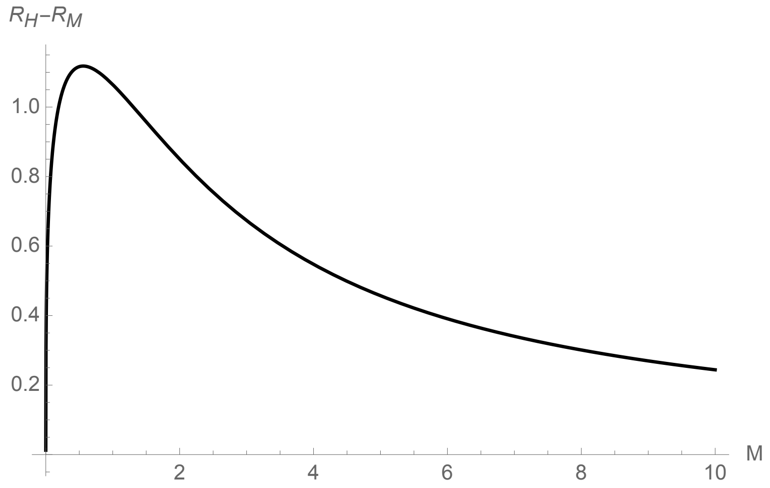

Figure 9.1: Difference $R_H - R_M$ for $M > 0$ and $\hat{\beta} = -10$ in Planck units.

0. For $\hat{\beta} < 0$, one finds

$$\frac{R_{\rm H}}{\ell_{\rm p}} = \frac{2\,M}{3\,M_P} + \left\{ -\frac{M}{2M_P} \left[\hat{\beta} - \frac{16M^2}{27M_P^2} + \sqrt{\hat{\beta}\left(\hat{\beta} - \frac{32M^2}{27M_P^2}\right)} \right] \right\}^{1/3} + \left\{ -\frac{2M}{M_P} \left[\hat{\beta} - \frac{16\,M^2}{27\,M_P^2} - \sqrt{\hat{\beta}\left(\hat{\beta} - \frac{32M^2}{27M_P^2}\right)} \right] \right\}^{1/3} . \quad (9.41)$$

and it follows that $R_{\rm H} > R_M$ for any values of $M > 0$ (see Figure 9.1). If we push the above description to values of the mass $M \gtrsim M_P$, this implies that, if the matter which sources the metric is not confined in a singularity, but occupies a finite volume [112] of size, say $R_{\rm s} \sim \ell_{\rm p}$, its gravitational radius is significantly larger than it would be in the vacuum Schwarzschild geometry. Consequently, the probability of this system of size $R_{\rm s}$ to be a black hole would be larger according to the Horizon Quantum Mechanics [108, 113]. Moreover, this is qualitatively similar to what was found in [110], namely that the horizon area would also be larger than in general relativity. However, one has to be careful interpreting the results obtained in Figure 9.1, since $R_H - R_M$ doesn't exceed l_p, which is precisely the region where our approach breaks down, as discussed in the previous section.

Ideally, for sufficiently large $\hat{\beta}$ and small mass M, one could have

$$R_{\rm H} \gtrsim \frac{9}{8}\,R_M, \quad (9.42)$$

which implies that the classical Buchdahl limit will not survive in this quantum realm as anticipated. These considerations indicate that the metric of a Planckian quantum black hole might be better described by our quantum corrected star model rather than by a Schwarzschild metric.

9.5 Conclusions

In this paper we have calculated the full set of quantum gravitational corrections to the metric of a star in stable equilibrium, as described by the Tolman-Oppenheimer-Volkoff equation, to second order in curvature. We have found a remarkable result. While eternal black holes, which are static vacuum solutions of general relativity, and stars have the same outside metric in general relativity, namely the famous Schwarzschild vacuum metric, quantum gravity makes a difference between black holes and stars at second order in curvature. Star solutions receive a quantum gravitational correction at this order, while vacuum black holes do not. It raises a deep question, namely what happens to this correction if we were to follow the gravitational collapse of a ball of dust? According to our results, a distant observer would be able to monitor the collapse of the star by measuring the quantum gravitational corrections to Newton's gravitational potential. If he followed the process, he would have an operational procedure to determine that an eternal black hole has formed.

It is usually argued that astrophysical black holes are well described by a Kerr metric (as they rotate), however it is a vacuum solution and there are thus no quantum gravitational corrections to second order in curvature. Our calculations thus raise deep questions about the nature of astrophysical black holes. Are they truly vacuum solutions?

Clearly answering these questions is beyond the scope of this paper. It would require to follow precisely quantum gravitational corrections during the dynamical process of a star collapsing into a black hole.

From a technical point of view, we have obtained an interesting result showing that the standard textbook metric for a star [312, 342] is too naive when it is assuming that matter is distributed according to a step function at the boundary of the star. Quantum gravity forces us to consider stars with a smooth matter profile at their surfaces.

Our results also have interesting consequences for quantum black holes. We have argued that the quantum corrected star metric could be used as an effective metric for a quantum black holes which, if they exist, are clearly not vacuum solutions.

In conclusion, quantum gravity corrections have deep implications for black holes and stars. Even though these corrections might be too tiny to be observable, they demonstrate that black holes are even more mysterious than usually assumed.

Acknowledgments

The work of X.C. and F.K. is supported in part by the Science and Technology Facilities Council (grant number ST/P000819/1). R.C. is partially supported by the INFN grant FLAG and works in the framework of activities of the National Group of Mathematical Physics (GNFM, INdAM) and SIGRAV of Italy.

9.A Derivation of the non-local term in equation (9.22)

We will here show how to calculate the expression

$$\ln\left(\frac{\Box}{\mu^2}\right) f(t,\vec{x}), \tag{9.43}$$

for time-independent and spherically symmetric functions $f(t,\vec{x}) = f(r)$, where $r = |\vec{x}|$. In particular, we will consider the following two cases: a) if $\exists\, \epsilon > 0$ such that $f(r') = 0$ for $|r' - r| < \epsilon$, we will find equation (9.43) can be computed rather straightforwardly and it yields

$$\ln\left(\frac{\Box}{\mu^2}\right) f(r) = \frac{1}{r}\int_0^\infty \left(\frac{r'}{r+r'} - \frac{r'}{|r-r'|}\right) f(r')\, dr'; \tag{9.44}$$

b) otherwise, if $r > 0$, $f(r) \neq 0$ and $\exists\, \epsilon > 0$ such that $f(r')$ is smooth for $|r' - r| \leq \epsilon$, equation (9.43) requires some care to make sense and yields

$$\begin{aligned}\ln\left(\frac{\Box}{\mu^2}\right) f(r) = \frac{1}{r}\int_0^\infty \frac{r'}{r+r'}\, f(r')\, dr' - \lim_{\epsilon\to 0^+}\Bigg\{ &\frac{1}{r}\int_0^{r-\epsilon} \frac{r'}{r-r'}\, f(r')\, dr' \\ &+ \frac{1}{r}\int_{r+\epsilon}^\infty \frac{r'}{r'-r}\, f(r')\, dr' \\ &+ 2\, f(r)\, [\gamma_E + \ln(\mu\epsilon)]\Bigg\},\end{aligned} \tag{9.45}$$

which contains a Cauchy principal value integral, as was found in [97].

As a first step, we use time independence to express the function f in terms of its Fourier transform $\hat{f}$ and write

$$\ln\left(\frac{\Box}{\mu^2}\right) f(\vec{x}) = \int \frac{d^3k}{(2\,\pi)^3}\, \ln\left(\frac{k^2}{\mu^2}\right) e^{i\,\vec{k}\cdot\vec{x}}\, \hat{f}(\vec{k}), \tag{9.46}$$

where $k = |\vec{k}|$. Next, we use the spherical symmetry of f (and $\hat{f}$) and assume that $\vec{x} = (0,0,r)$ without loss of generality, so that

$$\begin{aligned}\ln\left(\frac{\Box}{\mu^2}\right) f(r) &= \frac{1}{(2\,\pi)^2}\int_0^\infty k^2\, dk \int_{-1}^{+1} d(\cos\theta)\, \ln\left(\frac{k^2}{\mu^2}\right) e^{i\,k\,r\,\cos\phi}\, \hat{f}(k) \\ &= \frac{1}{2\,\pi^2\, r}\int_0^\infty dk\, k\, \ln\left(\frac{k^2}{\mu^2}\right) \sin(k\,r)\, \hat{f}(k) \\ &= \frac{1}{\pi^2\, r}\int_0^\infty dk\, k\, \ln\left(\frac{k}{\mu}\right) \sin(k\,r)\, \hat{f}(k).\end{aligned} \tag{9.47}$$

We can now Fourier transform back to coordinate space by making use of the relation between the Fourier and the Hankel transforms for spherically symmetric functions in 3 dimensions, namely

$$k^{1/2}\, \hat{f}(k) = (2\,\pi)^{3/2}\int_0^\infty r^{3/2}\, f(r)\, J_{1/2}(k\,r)\, dr, \tag{9.48}$$

where $J_{1/2}(k\,r) = \sqrt{\frac{2}{\pi\,k\,r}}\,\sin(k\,r)$. Therefore, we obtain

$$
\begin{aligned}
\ln\left(\frac{\Box}{\mu^2}\right)\, f(r) &= \frac{4}{\pi\,r}\int_0^\infty dk \int_0^\infty dr'\, \ln\left(\frac{k}{\mu}\right) \sin(kr)\, \sin(kr')\, r'\, f(r') \\
&= \frac{1}{\pi\,r}\int_0^\infty dk \int_0^\infty dr' \lim_{\delta\to 0^+} \left\{ f(r')\, r'\, \ln\left(\frac{k}{\mu}\right) e^{-\delta\,k} \right. \\
&\qquad \left. \times \left[e^{i\,k\,(r-r')} + e^{-i\,k\,(r-r')} - e^{i\,k\,(r+r')} - e^{-i\,k\,(r+r')} \right] \right\} \\
&= \frac{\mu}{\pi\,r}\int_0^\infty dr' \lim_{\delta\to 0^+} \int_0^\infty dq\, f(r')\, r'\, \ln(q)\, e^{-\delta\,\mu\,q} \\
&\qquad \times \left[e^{i\,\mu\,q\,(r-r')} + e^{-i\,\mu\,q\,(r-r')} - e^{i\,\mu\,q\,(r+r')} - e^{-i\,\mu\,q\,(r+r')} \right] ,
\end{aligned}
\tag{9.49}
$$

where we rescaled the momentum variable and swapped the limit with momentum integration in the last line. For $\mathrm{Re}(\alpha) > 0$, we have

$$
\int_0^\infty dq\, \ln(q)\, e^{-\alpha\,q} = -\frac{1}{\alpha}\left[\gamma_E + \ln(\alpha)\right] , \tag{9.50}
$$

which allows us to get

$$
\begin{aligned}
\ln\left(\frac{\Box}{\mu^2}\right)\, f(r) = \frac{1}{\pi\,r}\int_0^\infty dr'\, f(r')\, r' \lim_{\delta\to 0^+} &\left[\frac{\gamma_E + \ln(\mu\,R_+) + i\,\phi_+}{\delta + i\,(r+r')} + \frac{\gamma_E + \ln(\mu\,R_+) - i\,\phi_+}{\delta - i\,(r+r')} \right. \\
&\left. - \frac{\gamma_E + \ln(\mu\,R_-) + i\,\phi_-}{\delta + i\,(r-r')} - \frac{\gamma_E + \ln(\mu\,R_-) - i\,\phi_-}{\delta - i\,(r-r')} \right] ,
\end{aligned}
\tag{9.51}
$$

where $R_\pm = \sqrt{\delta^2 + (r \pm r')^2}$ and $\phi_\pm = \arctan[(r \pm r')/\delta]$. The first two terms are regular and we can take the limit $\delta \to 0$ straightforwardly, whereas the last two terms may contain a pole at $r' = r$. Here is where the two cases mentioned above occur:
Case a): since $f(r') = 0$ around r, there is no pole in equation (9.51), which immediately yields the result (9.44).
Case b): for $f(r) \neq 0$ but bounded and sufficiently smooth, we can rewrite equation (9.51) as

$$
\begin{aligned}
\ln\left(\frac{\Box}{\mu^2}\right)\, f(r) =& \frac{1}{r}\int_0^\infty dr'\, \frac{r'\, f(r')}{r+r'} - \lim_{\epsilon\to 0^+} \frac{1}{r}\left\{ \int_0^{r-\epsilon} dr'\, \frac{r'\, f(r')}{|r-r'|} + \int_{r+\epsilon}^\infty dr'\, \frac{r'\, f(r')}{|r-r'|} \right. \\
& \left. + \frac{1}{\pi}\int_{r-\epsilon}^{r+\epsilon} dr'\, f(r')\, r' \lim_{\delta\to 0^+} \left[\frac{\gamma_E + \ln(\mu\,R_-) + i\,\phi_-}{\delta + i\,(r-r')} + \frac{\gamma_E + \ln(\mu R_-) - i\,\phi_-}{\delta - i\,(r-r')} \right] \right\} \\
=& \frac{1}{r}\int_0^\infty \frac{r'}{r+r'}\, f(r')\, dr' - \frac{1}{r}\lim_{\epsilon\to 0^+} \left[\int_0^{r-\epsilon} \frac{r'}{r-r'}\, f(r')\, dr' + \frac{1}{r}\int_{r+\epsilon}^\infty \frac{r'}{r'-r}\, f(r')\, dr' \right] \\
& + L_1 ,
\end{aligned}
\tag{9.52}
$$

where it is understood that $0 < \delta < \epsilon$ before the limits are taken. The first line in equation (9.52) already reproduces the first line in the result (9.45), and we need only compute

$$L_1 \equiv -\frac{1}{\pi\, r} \lim_{\epsilon\to 0^+} \int_{r-\epsilon}^{r+\epsilon} dr'\, f(r')\, r' \lim_{\delta\to 0^+} \left[\frac{\gamma_E + \ln(\mu\, R_-) + i\,\phi_-}{\delta + i\,(r-r')} + \frac{\gamma_E + \ln(\mu R_-) - i\,\phi_-}{\delta - i\,(r-r')}\right].$$

By swapping the limit with the integral and defining a contour around the pole at $r' = r$, we get

$$\begin{aligned} L_1 = -\frac{1}{\pi\, r} \lim_{\epsilon\to 0^+} \Bigg\{ \lim_{\delta\to 0^+} \int_{\pi}^{2\pi} & i\,\epsilon\, e^{i\,t}\, dt\, (r + \epsilon\, e^{i\,t})\, f(r + \epsilon\, e^{i\,t}) \\ & \times \left[\frac{\gamma_E + \ln\!\left(\mu\sqrt{\delta^2 + \epsilon^2\, e^{2\,i\,t}}\right) - i \arctan\!\left(\frac{\epsilon\, e^{i\,t}}{\delta}\right)}{\delta - i\,\epsilon\, e^{i\,t}} \right. \\ & \left. + \frac{\gamma_E + \ln\!\left(\mu\sqrt{\delta^2 + \epsilon^2\, e^{2\,i\,t}}\right) + i \arctan\!\left(\frac{\epsilon\, e^{i\,t}}{\delta}\right)}{\delta + i\,\epsilon\, e^{i\,t}} \right] \Bigg\}. \end{aligned} \tag{9.53}$$

We can finally use the fact that f is locally smooth and Taylor expand it as $f(r + \epsilon\, e^{i\,t}) = f(r) + \mathcal{O}(\epsilon)$. Hence,

$$\begin{aligned} L_1 =& -\frac{f(r)}{\pi} \lim_{\epsilon\to 0^+} \left[\lim_{\delta\to 0} \int_{\pi}^{2\pi} i\,\epsilon e^{i\,t}\, dt\, \frac{\gamma_E + \ln\!\left(\mu\sqrt{\delta^2 + \epsilon^2\, e^{2\,i\,t}}\right) - i\, \arctan\!\left(\frac{\epsilon\, e^{i\,t}}{\delta}\right)}{\delta - i\,\epsilon\, e^{i\,t}} + \mathcal{O}(\epsilon) \right] \\ & -\frac{f(r)}{\pi} \lim_{\epsilon\to 0^+} \left[\lim_{\delta\to 0} \int_{\pi}^{2\pi} i\,\epsilon e^{i\,t}\, dt\, \frac{\gamma_E + \ln\!\left(\mu\sqrt{\delta^2 + \epsilon^2\, e^{2\,i\,t}}\right) + i\, \arctan\!\left(\frac{\epsilon e^{i\,t}}{\delta}\right)}{\delta + i\,\epsilon\, e^{i\,t}} + \mathcal{O}(\epsilon) \right] \\ =& -\frac{4\, f(r)}{\pi} \lim_{\epsilon\to 0^+} \left\{ \lim_{\delta\to 0^+} \arctan\!\left(\frac{\epsilon}{\delta}\right) \left[\gamma_E + \ln\!\left(\mu\sqrt{\delta^2 + \epsilon^2}\right)\right] + \mathcal{O}(\epsilon) \right\} \\ =& -2\, f(r)\, [\gamma_E + \ln(\mu\, \epsilon)]\,, \end{aligned} \tag{9.54}$$

which completes the result presented in equation (9.45).

Chapter 10

Quantum Corrected Equations of Motion in the Interior and Exterior Schwarzschild Spacetime

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Abstract

In this paper we derive the leading quantum gravitational corrections to the geodesics and the equations of motion for a scalar field in the spacetime containing a constant density star. It is shown that these corrections can be calculated in quantum gravity reliably and in a model independent way. Furthermore, we find that quantum gravity gives rise to an additional redshift that results from the gradient instead of the amplitude of the density profile.

This chapter has been published in Physical Review D **102**, no. 02, p. 6018 (2020).
A preprint of this chapter can be found at arXiv:2007.05416 [hep-th].

10.1 Introduction

In earlier work [77] we derived the leading quantum corrections to the interior and exterior region of the spacetime containing a constant and uniform density star, which are classically described by the well-known interior and vacuum Schwarzschild solutions. These calculations were done in the framework of the effective field theory for quantum gravity [34–37, 61, 140, 351]. Corrections obtained in this way are the result of integrating out the quantum fluctuations of the graviton.

Remarkably, despite the fact that quantum general relativity is not renormalizable, it is possible to make predictions in quantum gravity. These predictions apply to any model for which Lorentz invariance is a fundamental symmetry, general relativity is the correct low energy limit, and for which quantum field theory methods remain applicable up to the Planck scale. The quantum gravitational effective action contains two parts consisting of local and nonlocal operators. While the Wilson coefficients of the local part are non-calculable without knowing the ultraviolet complete theory of quantum gravity, the Wilson coefficients of the nonlocal part of the action are calculable from first principles and depend only on the infrared physics which is very well understood as we know general relativity.

Any unknown physics coming from an ultraviolet complete theory, would give rise to extra quantum corrections in the form of local operators. However, such physics only gives rise to contact interactions below the Planck scale. For example, integrating out Kaluza-Klein interactions would give rise to contact interactions. Furthermore, it was shown in [77] that corrections due to such contact interactions are subleading in the case of a star, assuming higher order curvature terms are not unnaturally large. The leading order corrections to the metric describing the spacetime around a star only depend on the nonlocal physics which is calculable from first principles and in a model independent way, without a detailed knowledge of the ultraviolet complete theory of quantum gravity.

In this paper we will use the results from Ref. [77] to derive the leading quantum corrections to the geodesics and the scalar waves in such a quantum corrected spacetime. A complication in these calculations may arise, since the metric corrections and curvature invariants, such as the Ricci scalar, diverge when the surface of the star is approached. These secularities indicate a breakdown of the perturbative approach that is used, and result from the fact that the interior Schwarzschild solution of general relativity contains a step-like discontinuity in the energy density at the star surface.

Since the Einstein equations in general relativity only involve second order derivatives of the metric, step-like discontinuities result in acceptable C^1 metrics. [1] Quantum gravity in the effective field theory approach, on the other hand, is an infinite derivative theory and it therefore requires C^∞ sources in order to produce continuous metrics. In other words, one should also determine a quantum correction to the matter source which makes it compatible with the effective quantum equations for the metric. However, quantum corrections to the uniform matter source appear really necessary only within a layer of

[1] Even Dirac delta-like discontinuities in the energy density produce continuous metrics, which leads to the well-known case of shell-like sources.

thickness of the order of the Planck length around the surface, and are expected to remain phenomenologically negligible.

In any case, and although the non-smooth solutions of general relativity are not expected to be physical, they can still serve as important toy models. This is particularly true for the Schwarzschild interior, as it is an analytical solution of the Einstein equations, that could approximate compact objects.

While it seems difficult to find practical applications for our results, they are a further demonstration that model independent calculations are possible in quantum gravity at energies below the Planck scale. This is in sharp contrast to the standard lore which states that quantum gravity is a mystery: we do not have a theory of quantum gravity and thus quantum gravitational calculations are not possible. This is simply not true and our results help to reinforce this point. As such, our findings are very important as they further demonstrate that quantum gravitational calculations are possible at energies below the Planck scale.

This paper is organized as follows: in the next section we state the results derived in [77]; in section 10.3, we solve the radial geodesics perturbatively and derive the leading quantum corrections; in section 10.4 we turn to the radial modes of the scalar field and solve their equations of motion perturbatively to derive the leading quantum corrections; finally in section 10.5 we conclude.

10.2 The quantum corrected metric

We here consider the quantum corrected metric derived in [77], which is static and spherically symmetric and can therefore be written as

$$ds^2 = -f(r)\, dt^2 + g(r)\, dr^2 + r^2\, d\Omega^2. \tag{10.1}$$

where $d\Omega^2 = d\theta^2 + (\sin\theta)^2\, d\phi^2$. Outside the star of radius R_s (that is, for $r > R_\mathrm{s}$), the metric functions are given by

$$f(r) = 1 - \frac{2\, G_\mathrm{N} M}{r} + \alpha_\mathrm{e}(r), \tag{10.2}$$

$$g(r) = \left(1 - \frac{2\, G_\mathrm{N} M}{r}\right)^{-1} + \beta_\mathrm{e}(r), \tag{10.3}$$

where

$$\begin{aligned}\alpha_\mathrm{e}(r) &= \tilde{\alpha}\, \frac{2\, G_\mathrm{N}\, \ell_\mathrm{p}^2\, M}{R_\mathrm{s}^3} \left[2\, \frac{R_\mathrm{s}}{r} + \ln\left(\frac{r - R_\mathrm{s}}{r + R_\mathrm{s}}\right)\right] + \mathcal{O}\left(G_\mathrm{N}^3\right), \\ \beta_\mathrm{e}(r) &= \tilde{\beta}\, \frac{2\, G_\mathrm{N}\, \ell_\mathrm{p}^2\, M}{r\,(r^2 - R_\mathrm{s}^2)} + \mathcal{O}\left(G_\mathrm{N}^3\right),\end{aligned} \tag{10.4}$$

with [2]

$$\tilde{\alpha} = 96\,\pi\,(\alpha + \beta + 3\gamma) \tag{10.5}$$
$$\tilde{\beta} = 192\,\pi\,(\gamma - \alpha). \tag{10.6}$$

In the stellar interior (given by $0 \leq r < R_{\rm s}$), we likewise have

$$f(r) = \frac{1}{4}\left(3\,\sqrt{1 - \frac{2\,G_{\rm N} M}{R_{\rm s}}} - \sqrt{1 - \frac{2\,G_{\rm N} M r^2}{R_{\rm s}^3}}\right)^2 + \alpha_{\rm i}(r), \tag{10.7}$$
$$g(r) = \left(1 - \frac{2\,G_{\rm N} M r^2}{R_{\rm s}^3}\right)^{-1} + \beta_{\rm i}(r), \tag{10.8}$$

where now

$$\alpha_{\rm i}(r) = \tilde{\alpha}\,\frac{2\,G_{\rm N}\,\ell_{\rm p}^2\,M}{R_{\rm s}^3}\,\ln\left(\frac{R_{\rm s}^2}{R_{\rm s}^2 - r^2}\right) + \mathcal{O}(G_{\rm N}^3),$$
$$\beta_{\rm i}(r) = \tilde{\beta}\,\frac{2\,G_{\rm N}\,\ell_{\rm p}^2\,M\,r^2}{R_{\rm s}^3\,(R_{\rm s}^2 - r^2)} + \mathcal{O}(G_{\rm N}^3). \tag{10.9}$$

Moreover, we assume throughout the paper that the Buchdahl limit [62] is satisfied, so that

$$R_{\rm s} \geq \frac{9}{8}\,(2\,G_{\rm N}\,M). \tag{10.10}$$

Let us remark that the Newton constant $G_{\rm N}$ is dimensionful and the displayed perturbation expansion is therefore a shorthand notation for two contributions, which are different in nature. In particular,

$$\mathcal{O}\left(G_{\rm N}^3\right) = \ell_{\rm p}^2\,\mathcal{R}\,\mathcal{O}\left([2G_{\rm N}M/R_{\rm s}]^2\right) + \mathcal{O}\left(\ell_{\rm p}^4\,\mathcal{R}^2\right), \tag{10.11}$$

where $\ell_{\rm p}$ is the Planck length, and $\mathcal{R}$ is the curvature scalar. The true perturbation parameters are thus the inverse of the radius of curvature in units of the Planck length and the compactness of the star, which are dimensionless as they should.

Furthermore, the quantum corrections become secular when $r \sim R_{\rm s}$. This secularity can be avoided, if the layer

$$(1 - \delta)\,R_{\rm s} < r < (1 + \delta)\,R_{\rm s} \qquad \text{with} \quad \delta \sim \left(\frac{2G_{\rm N}M}{R_{\rm s}}\right)\left(\frac{\ell_{\rm p}}{R_{\rm s}}\right)^2 \tag{10.12}$$

is excluded, as discussed in [77].

Finally, we recall that the metric can be rewritten as

$$ds^2 = f(r)(-dt^2 + dr_*^2) + r^2 d\Omega^2 \tag{10.13}$$

[2] The values for α, β and γ can be found in [77].

by introducing the tortoise coordinate

$$r_* = \int^r \sqrt{\frac{g(r')}{f(r')}}\, dr'. \tag{10.14}$$

This form is particularly useful fo studying waves and will be employed in section 10.4.

10.3 Geodesics

Geodesic equations can be derived in a way similar to the derivation in a Schwarzschild metric. The quantum corrected star metric has four Killing vectors. Three of those are due to the spherical symmetry, and one due to time-invariance. We use two of these Killing vectors to fix the direction of the angular momentum along the polar axis by setting

$$\theta = \frac{\pi}{2}. \tag{10.15}$$

The remaining two Killing vectors can then be written as

$$K^\mu = (\partial_t)^\mu, \tag{10.16}$$

$$R^\mu = (\partial_\phi)^\mu, \tag{10.17}$$

and can be used to define a conserved energy

$$E = -K_\mu \frac{dx^\mu}{d\lambda} = f(r)\,\frac{dt}{d\lambda} \tag{10.18}$$

and a conserved angular momentum

$$L = R_\mu \frac{dx^\mu}{d\lambda} = r^2\,\frac{d\phi}{d\lambda}. \tag{10.19}$$

Furthermore, along geodesics the quantity

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{10.20}$$

is also conserved. For massive particles we can set $\epsilon = 1$, as long as we identify $\lambda = \tau$ as the proper time along the geodesic. For massless particles $\epsilon = 0$ with λ an arbitrary affine parameter. By making use of the conserved quantities, we can rewrite Eq. (10.20) as

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{g(r)}\left(\frac{L^2}{r^2} + \epsilon\right) = \frac{E^2}{f(r)\,g(r)}. \tag{10.21}$$

Compatibly with the quantum corrections described in section 10.2, we will solve this equation perturbatively in the Planck length and the star compactness, by writing

$$r(\lambda) = r_{\rm c}(\lambda) + r_{\rm q}(\lambda), \tag{10.22}$$

where

$$r_{\rm c}(\lambda) = \sum_{m=0}^{\infty} r_{0,m}(\lambda) \left(\frac{2\,G_{\rm N} M}{R_{\rm s}}\right)^m \tag{10.23}$$

represents the classical trajectory, and

$$r_{\rm q}(\lambda) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} r_{n,m}(\lambda) \left(\frac{\ell_{\rm p}}{R_{\rm s}}\right)^{2n} \left(\frac{2\,G_{\rm N} M}{R_{\rm s}}\right)^m \tag{10.24}$$

is the quantum correction.

10.3.1 Exterior region

In the exterior region, $r > R_{\rm s}$, we can write

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} - \frac{2\,G_{\rm N} M}{R_{\rm s}} \left(\frac{L^2}{r^2} + \epsilon\right) \frac{R_{\rm s}}{r} + V_{\rm q}(r) = \eta^2, \tag{10.25}$$

where $\eta = \sqrt{E^2 - \epsilon}$ and the effective quantum potential is given by

$$V_{\rm q}(r) = E^2\,\alpha_{\rm e}(r) + \eta^2\,\beta_{\rm e}(r) - \frac{L^2}{r^2}\,\beta_{\rm e}(r). \tag{10.26}$$

We notice that the term proportional to E^2 signals a violation of the equivalence principle, since the acceleration undergone by the particle following the geodesic depends on its energy. However, $\alpha_{\rm e} = \mathcal{O}(G_{\rm N}^2) \sim (\ell_{\rm p}/R_{\rm s})^2$ in the sense explained after Eq. (10.11), and the size of this violation remains negligibly small throughout space. The quantum corrections to the metric outside the star are larger near the surface. In order to study geodesics for which the quantum corrections are expected to be the largest, we impose the boundary conditions

$$r_{\rm c}(0) = R_{\rm s}, \tag{10.27}$$

$$r_{\rm q}\left(\lambda_0 \to \infty\right) = 0. \tag{10.28}$$

This somewhat unconventional choice of specifying the boundary conditions at two different points is motivated by the fact that one cannot set $r_{\rm q}(\lambda = 0) = 0$, as the quantum corrections diverge at the surface of the star. Instead one can impose any boundary condition on $r_{\rm q}(\lambda_0)$ for any $\lambda_0 > 0$, as this boundary condition does not impact the cumulative quantum corrections along a particular segment of the geodesic. For this one has to evaluate the difference $r_{\rm q}(\lambda_2) - r_{\rm q}(\lambda_1)$, for specified values λ_1 and λ_2, and any such difference is independent of the specific choice of λ_0.

For $L = 0$ one finds the leading classical solutions for an outgoing radial geodesic $(\lambda \geq 0)$

$$r_{0,0}(\lambda) = \eta\,\lambda + R_{\rm s}, \tag{10.29}$$

$$r_{0,1}(\lambda) = \frac{\epsilon\,R_{\rm s}}{2\,\eta^2}\,\ln\left(1 + \frac{\eta\,\lambda}{R_{\rm s}}\right), \tag{10.30}$$

and the leading quantum corrections

$$r_{1,0}(\lambda) = 0 \tag{10.31}$$

$$\begin{aligned} r_{1,1}(\lambda) = & \frac{\tilde{\alpha} E^2 R_\mathrm{s}}{2\,\eta^2} \left[2 \ln \left(\frac{2R_\mathrm{s} + \eta\lambda}{R_\mathrm{s} + \eta\lambda} \right) - 2 + \frac{\eta\lambda}{R_\mathrm{s}} \ln \left(1 + \frac{2R_\mathrm{s}}{\eta\lambda} \right) \right] \\ & + \frac{\tilde{\beta} R_\mathrm{s}}{4} \ln \left[\frac{\eta\lambda(2R_\mathrm{s} + \eta\lambda)}{(R_\mathrm{s} + \eta\lambda)^2} \right] . \end{aligned} \tag{10.32}$$

Notice that $r_{11}(\lambda)$ contains a secular term proportional to $\tilde{\beta}$ for $\lambda \to 0$, which was expected, and occurs within the interval of Eq. (10.12). However, the term proportional to E^2 never grows large even for $\lambda \sim 0$, and the violation to the equivalence principle therefore remains of order $(\ell_\mathrm{p}/R_\mathrm{s})^2$ everywhere in $r > R_\mathrm{s}$.

10.3.2 Interior region

In the interior region we can write

$$\begin{aligned} \left(\frac{dr}{d\lambda} \right)^2 + \frac{L^2}{r^2} - & \left[\left(\frac{L^2}{r^2} + \epsilon \right) \left(\frac{r}{R_\mathrm{s}} \right)^2 + \frac{3E^2(R_\mathrm{s}^2 - r^2)}{2R_\mathrm{s}^2} \right] \frac{2G_\mathrm{N}M}{R_\mathrm{s}} \\ & - \frac{3E^2(11R_\mathrm{s}^4 - 14R_\mathrm{s}^2 r^2 + 3r^4)}{16R_\mathrm{s}^4} \left(\frac{2G_\mathrm{N}M}{R_\mathrm{s}} \right)^2 + V_q(r) = \eta^2 , \end{aligned} \tag{10.33}$$

where we again set $\eta = \sqrt{E^2 - \epsilon}$ and the effective quantum potential now reads

$$V_\mathrm{q}(r) = E^2\, \alpha_\mathrm{i}(r) + \eta^2\, \beta_\mathrm{i}(r) - \frac{L^2}{r^2}\, \beta_\mathrm{i}(r). \tag{10.34}$$

Like in the exterior, we impose initial conditions suitable for studying radial geodesics near the surface, that is

$$r_\mathrm{c}(0) = R_\mathrm{s}\ , \tag{10.35}$$

$$r_\mathrm{q} \left(-\frac{R_\mathrm{s}}{\eta} \right) = A, \tag{10.36}$$

where we will fix the value of A at a later stage.

For $L = 0$ one finds the leading classical solution for an outgoing radial geodesic ($\lambda \leq 0$) is given by

$$r_{0,0}(\lambda) = \eta\, \lambda + R_\mathrm{s}, \tag{10.37}$$

$$r_{0,1}(\lambda) = \frac{\epsilon\, \lambda}{2\, \eta} - (3\, E^2 - 2\, \epsilon)\, (3\, R_\mathrm{s} + \eta\, \lambda)\, \frac{\lambda^2}{12\, R_\mathrm{s}^2}, \tag{10.38}$$

and the leading quantum corrections read

$$r_{1,0}(\lambda) = 0 \tag{10.39}$$

$$\begin{aligned} r_{1,1}(\lambda;x) = & \frac{\tilde{\alpha} E^2 R_{\rm s}}{2\eta^2}\left[2\ln\left(2+\frac{\eta\lambda}{R_{\rm s}}\right) - 2 + \frac{\eta\lambda}{R_{\rm s}}\left\{\ln\left[-\frac{\eta\lambda}{R_{\rm s}}\left(2+\frac{\eta\lambda}{R_{\rm s}}\right)\right]-2\right\}\right] \\ & + \frac{\tilde{\beta} R_{\rm s}}{4}\left\{2 + 2\frac{\eta\lambda}{R_{\rm s}} - \ln\left[-\left(1+\frac{2R_{\rm s}}{\eta\lambda}\right)\right]\right\} + A. \end{aligned} \tag{10.40}$$

Notice that $r_{11}(\lambda)$ also contains a secular term proportional to $\tilde{\beta}$ for $\lambda \to 0$ which, like for the exterior expression (10.32), occurs within the interval given in Eq. (10.12).

10.3.3 Crossing the surface

By means of the previous results, we can analyze the discontinuities (of quantum origin) that the radial geodesics would encounter across $r = R_{\rm s}$. Since we assumed initial conditions such that the classical radial geodesics $r_{\rm c}$ can be joined continuously across $r = R_{\rm s}$, we just need to calculate the difference between the non-vanishing quantum exterior correction in Eq. (10.32) and the interior analogue in Eq. (10.40) at $r = R_{\rm s}$, which yields

$$\lim_{\lambda\to 0}\left[r_{1,1}^{\rm ext}(\lambda) - r_{1,1}^{\rm int}(\lambda)\right] = \frac{\tilde{\beta}\, R_{\rm s}}{2}\left[\ln(2) - 1\right] - A. \tag{10.41}$$

We then notice that the interior and exterior geodesics can be continuously connected by fixing A such that the boundary condition for the interior solution is given by

$$r_{\rm q}^{\rm int}\left(-\frac{R_{\rm s}}{\eta}\right) = \frac{\tilde{\beta}\, R_{\rm s}}{2}\left[\ln(2) - 1\right], \tag{10.42}$$

provided for the exterior solution one employs the condition

$$r_{\rm q}^{\rm ext}(\infty) = 0, \tag{10.43}$$

which was used to determine Eq. (10.32).

One could go further and check the smoothness of the solution, and find that there is a discontinuity in the first derivative that cannot be removed. However, this is not a physical effect, as it occurs in the interval (10.12), and is thus expected to be regularized once the interior Schwarzschild solution is smoothened like we wrote in the Introduction.

10.4 Scalar fields

The equation of motion for a free scalar field Φ with mass μ is given by

$$\Box\,\Phi = \mu^2\,\Phi. \tag{10.44}$$

Since our metric (10.1) has spherical symmetry, we can separate the angular variables from the other coordinates and write $\Phi(t,r,\theta,\phi) = \Phi(t,r)\,S(\theta,\phi)$, where S can be decomposed in the usual spherical harmonics satisfying

$$\left(\partial_\theta^2 + \frac{\cos\theta}{\sin\theta}\,\partial_\theta + \frac{1}{(\sin\theta)^2}\,\partial_\phi^2\right) Y(\theta,\phi) = -\,l(l+1)\,Y(\theta,\phi). \tag{10.45}$$

It is then convenient to consider one mode at a time and further separate time from the radial coordinate, to wit $\Phi(t,r) = \Psi(t)\,\Phi(r)$, where $\Psi \sim e^{i\,\omega\,t}$ and satisfies

$$\ddot{\Psi}(t) = -\omega^2\,\Psi(t). \tag{10.46}$$

Furthermore, using the metric (10.13) with the tortoise-like coordinate r_* yields the radial equation

$$\left[\partial_{r_*}^2 + \eta^2 - \frac{l(l+1)}{r^2}\right] u(r) = (V_{\rm c} + V_{\rm q})\,u(r), \tag{10.47}$$

where r_* is given as a function of r in Eq. (10.14), $\eta^2 = \omega^2 - \mu^2 > 0$ and

$$u(r) = r_*(r)\,\Phi(r). \tag{10.48}$$

Notice that we have explicitly separated the effective potential into a classical part,

$$V_{\rm c}(r) = [f(r) - \alpha(r) - 1]\left[\mu^2 + \frac{l\,(l+1)}{r^2}\right] \tag{10.49}$$

and a quantum contribution

$$V_{\rm q}(r) = \alpha(r)\left[\mu^2 + \frac{l\,(l+1)}{r^2}\right]. \tag{10.50}$$

Like for the geodesics, we can expand the radial function in the same perturbative parameters of the quantum corrections to the metric and write

$$\begin{aligned} u(r) &= u_{\rm c}(r) + u_{\rm q}(r) \\ &= \sum_{n,m=0}^{\infty} u_{n,m}(r)\left(\frac{\ell_{\rm p}}{R_{\rm s}}\right)^{2n}\left(\frac{2G_{\rm N}M}{R_{\rm s}}\right)^{m}, \end{aligned} \tag{10.51}$$

where $u_{\rm c}$ contains all the terms with $n = 0$.

We are particularly interested in how quantum corrections to the metric affect the s-waves with $l = 0$ originating near the surface of the star. On using the fact that $V_{\rm c}$ and $V_{\rm q}$ are of order (at least) $G_{\rm N}$, we immediately obtain

$$u_{0,0}(r) = A\cos\left[\eta\,(r_* - R_{\rm s}^*)\right], \tag{10.52}$$

$$u_{1,0}(r) = 0, \tag{10.53}$$

where A and $R_{\rm s}^*$ are integration constants which we will suitably set in the following subsections. The effect of the potentials (10.49) and (10.50) can then be determined perturbatively by treating them as sources acting on the unperturbed solutions defined by Eqs. (10.52) and (10.53).

10.4.1 Exterior region

In the exterior region, the tortoise coordinate is given by

$$r_*(r) = r + 2\,G_{\rm N} M \ln\left(\frac{r}{2\,G_{\rm N} M} - 1\right) + \frac{1}{2}\int_r^\infty \left[\alpha_{\rm e}(r') - \beta_{\rm e}(r')\right] dr' + C, \tag{10.54}$$

where we set the integration constant C so that

$$R_{\rm s}^* = R_{\rm s} + 2\,G_{\rm N}\,M \ln\left(\frac{R_{\rm s}}{2\,G_{\rm N} M} - 1\right). \tag{10.55}$$

In order to determine the radial function in such a way that all corrections to the unperturbed solutions (10.52) and (10.53) vanish at some $r = (1+\delta)\,R_{\rm s} > R_{\rm s}$, we impose the boundary condition

$$u[(1+\delta)\,R_{\rm s}] = A, \tag{10.56}$$

where A is the same constant as in Eq. (10.52) and δ is the same parameter that defines the excluded layer in Eq. (10.12).

We want to see how these modes behave for values of $r > (1+\delta)\,R_{\rm s}$. The radial equation (10.47) can then be rewritten as the integral equation

$$u(r_*) = A\,\cos[\eta\,(r_* - R_{\rm s}^*)] + \int_{(1+\delta)\,R_{\rm s}^*}^\infty G(r_*, r_*')\left[V_{\rm c}(r_*') + V_{\rm q}(r_*')\right] u(r_*')\,dr_*', \tag{10.57}$$

where the Green's function is given by

$$G(r_*, r_*') = \begin{cases} \frac{1}{\eta}\sin\left[\eta(r_* - r_*')\right] & \text{if} \quad r_*' \le r_*, \\ 0 & \text{if} \quad r_*' > r_*. \end{cases} \tag{10.58}$$

In order to solve the integral equation, one needs to invert Eq. (10.54), which can be done perturbatively using

$$r_* - R_{\rm s}^* = r - R_{\rm s} + \mathcal{O}\left(2\,G_{\rm N} M/R_{\rm s}\right). \tag{10.59}$$

This is valid if the secularity is avoided, which is the case for $r > (1+\delta)\,R_{\rm s}$. The leading classical solution is then found to be [3]

$$\begin{aligned} u_{0,1}(r) = \frac{\mu^2 R_{\rm s} A}{2\eta}\Big\{ &\ln(R_{\rm s}/r)\sin\left[\eta(r_* - R_{\rm s}^*)\right] + \left[\mathrm{Si}(2\eta r) - \mathrm{Si}(2\eta R_{\rm s})\right]\cos\left[\eta(r + R_{\rm s})\right] \\ &- \left[\mathrm{Ci}(2\eta r) - \mathrm{Ci}(2\eta R_{\rm s})\right]\sin\left[\eta(r + R_{\rm s})\right]\Big\}, \end{aligned} \tag{10.60}$$

[3]Note we make use of Eq. (10.59) also in order to express the result in the coordinate r.

and the leading quantum correction

$$
\begin{aligned}
u_{1,1}(r) = \frac{\tilde{\alpha}\mu^2 A}{4\eta^2} \Bigg(& \left\{ \gamma_{\mathrm{E}} + \ln\left[\frac{4\eta R_{\mathrm{s}}(r-R_{\mathrm{s}})}{r+R_{\mathrm{s}}}\right] - \mathrm{Ci}(2\eta|r-R_{\mathrm{s}}|) \right\} \cos[\eta(r_* - R_{\mathrm{s}}^*)] \\
& + \left[4\eta R_{\mathrm{s}} \ln\left(\frac{2r}{r+R_{\mathrm{s}}}\right) + 2\eta(r-R_{\mathrm{s}}) \ln\left(\frac{r-R_{\mathrm{s}}}{r+R_{\mathrm{s}}}\right) \right. \\
& \qquad\qquad \left. - \mathrm{Si}(2\eta|r-R_{\mathrm{s}}|) \right] \sin[\eta(r_* - R_{\mathrm{s}}^*)] \\
& - 4\eta R_{\mathrm{s}} \left[\mathrm{Si}(2\eta r) - \mathrm{Si}(2\eta R_{\mathrm{s}})\right] \cos[\eta(r+R_{\mathrm{s}})] \\
& + 4\eta R_{\mathrm{s}} \left[\mathrm{Ci}(2\eta r) - \mathrm{Ci}(2\eta R_{\mathrm{s}})\right] \sin[\eta(r+R_{\mathrm{s}})] \\
& + \left\{\mathrm{Ci}[2\eta(r+R_{\mathrm{s}})] - \mathrm{Ci}(4\eta R_{\mathrm{s}})\right\} \cos[\eta(r+3R_{\mathrm{s}})] \\
& + \left\{\mathrm{Si}[2\eta(r+R_{\mathrm{s}})] - \mathrm{Si}(4\eta R_{\mathrm{s}})\right\} \sin[\eta(r+3R_{\mathrm{s}})] \Bigg),
\end{aligned} \tag{10.61}
$$

where Ci and Si are cosine and sine integrals. Notice that the results are independent of δ, since corrections due to δ are subleading by its definition in Eq. (10.12). Furthermore, at this order in perturbation theory, the divergences of the metric corrections (10.4) can be absorbed in the phase

$$
\begin{aligned}
r_* - R_{\mathrm{s}}^* &= r - R_{\mathrm{s}} + 2G_{\mathrm{N}}M \ln(r/R_{\mathrm{s}}) + \frac{1}{2}\int_r^\infty \left[\alpha_e(r') - \beta_e(r')\right] dr' \\
&= (r-R_{\mathrm{s}})\left(1 + \frac{2G_{\mathrm{N}}M}{R_{\mathrm{s}}}\right) + \left(\left\{ \tilde{\alpha}\left[\ln(2) - 1\right] + \frac{\tilde{\beta}}{4} \ln\left[\frac{2(r-R_{\mathrm{s}})}{R_{\mathrm{s}}}\right] \right\} R_{\mathrm{s}} \right. \\
&\qquad \left. - \frac{1}{2}\left\{ \tilde{\alpha}\left[1 + \ln\left(\frac{r-R_{\mathrm{s}}}{2R_{\mathrm{s}}}\right)\right] + \frac{3}{4}\tilde{\beta} \right\} (r-R_{\mathrm{s}}) \right) \frac{2G_{\mathrm{N}}M}{R_{\mathrm{s}}} \frac{\ell_{\mathrm{p}}^2}{R_{\mathrm{s}}^2} \\
&\quad + \mathcal{O}\left(r-R_{\mathrm{s}}\right)^2 + \mathcal{O}\left(\frac{2G_{\mathrm{N}}M}{R_{\mathrm{s}}}\right)^2 + \mathcal{O}\left(\frac{\ell_{\mathrm{p}}}{R_{\mathrm{s}}}\right)^4 .
\end{aligned} \tag{10.62}
$$

10.4.2 Interior region

In the interior region the tortoise coordinate is given by

$$
r_*(r) = r + \frac{r}{4}\left(3 - \frac{r^2}{R_{\mathrm{s}}^2}\right)\frac{2G_{\mathrm{N}}M}{R_{\mathrm{s}}} - \frac{1}{2}\int_0^r \left[\alpha_i(r') - \beta_i(r')\right] dr' + D, \tag{10.63}
$$

and D is chosen so that

$$
R_{\mathrm{s}}^* = R_{\mathrm{s}} + G_{\mathrm{N}}M. \tag{10.64}
$$

We again impose a boundary condition, in order to fix the wave mode this time at $r = (1-\delta)\,R_{\mathrm{s}}$, to wit

$$
u[(1-\delta)\,R_{\mathrm{s}}] = A. \tag{10.65}
$$

Like in the exterior, Eq. (10.47) yields the integral equation

$$
u(r) = A\,\cos[\eta\,(R_{\mathrm{s}}^* - r_*)] + \int_0^{(1-\delta)\,R_{\mathrm{s}}^*} G(r_*, r'_*)\left[V_c(r') + V_q(r')\right] u(r')\,dr'_*, \tag{10.66}
$$

with the Green's function here given by

$$G(r_*, r'_*) = \begin{cases} 0 & \text{if} \quad r'_* < r_*, \\ \frac{1}{\eta}\sin\left[\eta(r'_* - r_*)\right] & \text{if} \quad r'_* \geq r_*. \end{cases} \tag{10.67}$$

Eq. (10.63) can again be inverted perturbatively using

$$R_{\mathrm{s}}^* - r_* = R_{\mathrm{s}} - r + \mathcal{O}\left(\frac{2G_{\mathrm{N}}M}{R_{\mathrm{s}}}\right), \tag{10.68}$$

which is valid inside the ball $0 \leq r < (1-\delta)\,R_{\mathrm{s}}$. The leading classical solution is then found to be [4]

$$\begin{aligned} u_{0,1}(r) = \frac{m^2 A}{24\eta^3 R_{\mathrm{s}}^2}\Big\{ & 3\eta\,(r^2 - R_{\mathrm{s}}^2)\cos[\eta(r_* - R_{\mathrm{s}}^*)] \\ & + \Big[2\eta^2(r - R_{\mathrm{s}})(r^2 + rR_{\mathrm{s}} - 8R_{\mathrm{s}}^2) - 3(r + R_{\mathrm{s}})\Big]\sin[\eta(r_* - R_{\mathrm{s}}^*)]\Big\}, \end{aligned} \tag{10.69}$$

and the leading quantum correction

$$\begin{aligned} u_{1,1}(r) = \frac{\tilde{\alpha} m^2 A}{4\eta^2}\Bigg(& -\left[\gamma_{\mathrm{E}} + \ln\left(\frac{\eta|r^2 - R_{\mathrm{s}}^2|}{R_{\mathrm{s}}}\right) - \mathrm{Ci}(2\eta|r - R_{\mathrm{s}}|)\right]\cos[\eta(r_* - R_{\mathrm{s}}^*)] \\ & + \Bigg\{4\eta R_{\mathrm{s}}\ln\left(\frac{2R_{\mathrm{s}}}{r + R_{\mathrm{s}}}\right) + 2\eta(r - R_{\mathrm{s}})\left[2 - \ln\left(\frac{|r^2 - R_{\mathrm{s}}^2|}{R_{\mathrm{s}}^2}\right)\right] \\ & \qquad - \mathrm{Si}(2\eta|r - R_{\mathrm{s}}|)\Bigg\}\sin[\eta(r_* - R_{\mathrm{s}}^*)] \\ & + \{\mathrm{Ci}[2\eta(r + R_{\mathrm{s}})] - \mathrm{Ci}(4\eta R_{\mathrm{s}})\}\cos[\eta(r + 3R_{\mathrm{s}})] \\ & + \{\mathrm{Si}[2\eta(r + R_{\mathrm{s}})] - \mathrm{Si}(4\eta R_{\mathrm{s}})\}\sin[\eta(r + 3R_{\mathrm{s}})]\Bigg). \end{aligned} \tag{10.70}$$

The results in the interior are again independent of δ, to leading order, and the divergences of the metric corrections (10.9) can also be absorbed in the phase

$$\begin{aligned} r_* - R_{\mathrm{s}}^* &= r - R_{\mathrm{s}} + \left[\frac{r}{4}\left(3 - \frac{r^2}{R_{\mathrm{s}}^2}\right) - \frac{R_{\mathrm{s}}}{2}\right]\frac{2G_{\mathrm{N}}M}{R_{\mathrm{s}}} - \frac{1}{2}\int_0^r \alpha_i(r') - \beta_i(r')dr' \\ &= r - R_{\mathrm{s}} - \Bigg(R_{\mathrm{s}}\left\{\tilde{\alpha}[1 - \ln(2)] + \frac{\tilde{\beta}}{4}\left[2 + \ln\left(\frac{|r - R_{\mathrm{s}}|}{2R_{\mathrm{s}}}\right)\right]\right\} \\ &\qquad + \left\{\frac{\tilde{\alpha}}{2}\left[1 - \ln\left(\frac{2|r - R_{\mathrm{s}}|}{R_{\mathrm{s}}}\right) + \frac{3\,\tilde{\beta}}{8}\right]\right\}(r - R_{\mathrm{s}})\Bigg)\frac{2G_{\mathrm{N}}M}{R_{\mathrm{s}}}\frac{\ell_{\mathrm{p}}^2}{R_{\mathrm{s}}^2} \\ &\quad + \mathcal{O}\,(r - R_{\mathrm{s}})^2 + \mathcal{O}\left(\frac{2G_{\mathrm{N}}M}{R_{\mathrm{s}}}\right)^2 + \mathcal{O}\left(\frac{\ell_{\mathrm{p}}}{R_{\mathrm{s}}}\right)^4. \end{aligned} \tag{10.71}$$

10.5 Discussion

In this work we calculated the leading quantum corrections to the geodesics and the scalar waves in a spacetime containing a constant and uniform density star. We have shown as a proof of principle that such calculations can be done in quantum gravity. Furthermore,

[4] We make use of Eq. (10.68) to revert to the coordinate r.

we have found that the divergences at the surface of the star found in Ref. [77], do not cause serious issues for such calculations. In fact, these divergences can be kept well under control, if a Planck length layer around the surface of the star is excluded from the analysis. It is then possible to connect the interior and exterior solutions in a continuous, but not differentiable way, between the boundaries of such a layer.

In the case of geodesics the quantum corrections only affect the velocity with respect to the proper time for a particle following the geodesic. For scalar waves on the other hand the quantum corrections give rise to both wavelike perturbations to the classical wave solution and to a phase shift of the classical solution. The latter could in principle lead to a measurable blueshift when the star surface is approached. However, this would require compact objects to have density profiles that are smoothened out within a Planck length interval around the surface of the star, and thus derivatives of the energy density that exceed the Planck scale. For any realistic matter distribution one would expect that all derivatives of the energy density remain below the Planck scale.

We conclude that neither the perturbations nor the phase shift are expected to be measurable for realistic density profiles with current or near future experiments. However, the latter effect is in fact very interesting, as it shows that quantum gravity introduces a redshift due to the gradient of the density profile, while the redshift in general relativity results only from the presence of mass.

Acknowledgments

The work of X.C. is supported in part by the Science and Technology Facilities Council (grant number ST/P000819/1). R.C. is partially supported by the INFN grant FLAG and his work has also been carried out in the framework of activities of the National Group of Mathematical Physics (GNFM, INdAM) and COST action *Cantata*. The work of F.K. is supported by a doctoral studentship of the Science and Technology Facilities Council. F.K. is grateful for the hospitality of the Università di Bologna, where most of this work was carried out.

Chapter 11

Singularity Theorems in the Effective Field Theory for Quantum Gravity at Second Order in Curvature

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Abstract

In this paper we discuss singularity theorems in quantum gravity using effective field theory methods. To second order in curvature, the effective field theory contains two new degrees of freedom which have important implications for the derivation of these theorems: a massive spin-2 field and a massive spin-0 field. Using an explicit mapping of this theory from the Jordan frame to the Einstein frame, we show that the massive spin-2 field violates the null energy condition, while the massive spin-0 field satisfies the null energy condition, but violates the strong energy condition. Due to this violation classical singularity theorems are no longer applicable, indicating that singularities can be avoided, if the leading quantum correction are taken into account.

This chapter has been published in Universe **6**, no. 10, p. 0171 (2020).
A preprint of the chapter can be found at arXiv:1911.05571 [gr-qc].

11.1 Introduction

The significance of singularity theorems in general relativity first presented in the seminal papers of Penrose and Hawking [191, 291] cannot be overemphasized. Since these foundational works several adaptions and refinements of the singularity theorems have been developed (see e.g. [55, 60, 155, 302, 344]). In general, all these theorems boil down to the same principle: the assumption of some energy condition together with some global statement about space-time leads to the prediction of geodesic incompleteness somewhere in the space-time. Geodesic incompleteness is then often taken as equivalent to the existence of a singularity, although the latter is a slightly stronger statement (see e.g. [359]).

A crucial ingredient for the proof of most of singularity theorems is the Raychaudhuri equation[1], that can be derived from the Einstein field equations. It is therefore crucial to assume classical general relativity for singularity theorems to hold, and for any deviations of general relativity one would have to reassess the derivation of singularity theorems, as was done, for example, for $f(R)$ gravity [13].

It is clear that general relativity needs to be embedded in a gravitational theory which can be quantized, i.e. a theory of quantum gravity, if one accounts for the quantum properties of matter and space-time. Such a theory of quantum gravity is not known yet, but many different approaches to such a theory have been formulated. Furthermore any theory of quantum gravity should in the infrared limit reduce to general relativity. Despite the lack of a unique theory of quantum gravity, quantum corrections to general relativity solutions can be calculated using effective field theory methods [34–37, 61, 140, 351]. Calculations done in this framework apply to any ultraviolet complete theory of quantum gravity and are valid at energies scales up to the Planck mass, and thus in the entire spectrum that can potentially be probed experimentally.

It is expected that in a theory for quantum gravity singularities will be resolved, since singularities lead to pathologies both in general relativity and quantum field theory. However, singularities cannot be avoided as long as singularity theorems hold. It is therefore an important question whether the assumptions of the singularity theorems break down in a theory for quantum gravity. A discussion of possible quantum loop holes for the singularity theorems can for example be found in [157].

In this work we discuss the validity of the singularity theorems in the framework of the effective field theory approach to quantum gravity. A drawback of this approach is that the theory is not valid at energy scales larger than the Planck mass which corresponds to regions of large curvature, where singularities are expected to form. We shall assume that the physics responsible for the avoidance of singularities becomes relevant at energies below the Planck scale and can thus be described within our mathematical framework, an example would be, e.g., a bounce solution in a stellar collapse to a black hole [161] or in FLRW cosmology which would avoid a Big Crunch solution [143]. We note that this approach goes beyond general relativity and it is applicable to any theory of quantum gravity that does not break diffeomorphism invariance.

This paper is organized as follows: in the next section we derive the action for effective

[1]However, see [156] for a recent example that doesn't make use of this equation

	α	β	γ
Scalar	$5(6\xi-1)^2$	-2	2
Fermion	-5	8	7
Vector	-50	176	-26
Graviton	250	-244	424

Table 11.1: Non-local Wilson coefficients of various fields. All numbers should be divided by $11520\pi^2$. ξ denotes the value of the non-minimal coupling for a scalar theory. We refer to [48, 132] for the calculation of the values for the scalar, fermion and vector field. It is known that the graviton self interactions [219] make the form factors ill-defined, as the Wilson coefficients become gauge dependent. However, there is a well defined procedure to resolve these ambiguities [34, 35].

quantum gravity in the Einstein frame. In section 11.3 we discuss singularity theorems in effective quantum gravity using this action. In section 11.4 we then conclude. Furthermore, in appendix 11.A we discuss the classical Hawking and Penrose singularity theorems, and in appendix 11.B we discuss a refined statement of Hawking's theorem using weakened energy conditions.

In this paper we work in the $(+---)$ metric and use the conventions $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - ...$, $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$, $T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}}$. Furthermore $\kappa^2 = 8\pi G_{\rm N}$.

11.2 Effective quantum gravity in the Einstein frame

While an ultraviolet complete theory of quantum gravity is still elusive, it has been shown [34–37, 61, 140, 351], that quantum gravity can be well described by an effective field theory as long as one considers physical effects taking place at energies below the Planck scale. The effective field theory is obtained by integrating out the graviton fluctuations and potentially other massless degrees of freedom. After the various low energy fields have been integrated out, one finds the following action

$$S = \int d^4x\sqrt{|g|}\left\{-\frac{R}{2\kappa^2} + c_1(\mu)R^2 + c_2(\mu)R_{\mu\nu}R^{\mu\nu} + c_3(\mu)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \alpha R\ln\left(\frac{\Box}{\mu^2}\right)R \right.$$
$$\left. +\beta R_{\mu\nu}\ln\left(\frac{\Box}{\mu^2}\right)R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}\ln\left(\frac{\Box}{\mu^2}\right)R^{\mu\nu\rho\sigma} + \mathcal{O}(\kappa^2)\right\} + S_{\rm m}, \qquad (11.1)$$

where μ is the renormalization scale. The action is given up to second order in curvature and higher order corrections are suppressed in the $\mathcal{O}(\kappa^2)$ term. The effective action of any ultraviolet complete theory of quantum gravity that respects diffeomorphism invariance can be written in this form when expanded to second order in curvature. We emphasize that the Wilson coefficients c_i depend on the UV-completion of the theory and are only calculable within a specific UV-complete model of quantum gravity. Nevertheless it is expected that these coefficients are non-zero unless some undiscovered symmetry protects them or if fine tuning occurs. Moreover the values are bounded by the Eöt-Wash experiment [202] to $c_i \lesssim 10^{61}$. The non-local Wilson coefficients α, β, γ are calculable and independent of such a specific UV-completion. The values of these coefficients are given in Table 11.1.

We will now map this theory to the Einstein frame, in which the theory is represented as standard general relativity with additional matter fields. After this frame transformation, the usual singularity theorems are applicable, if the new fields satisfy the given energy conditions. Mappings to the Einstein frame for R and $R_{\mu\nu}$ theories have been discussed in [92, 100, 154, 215, 245, 246]. Furthermore, the case of higher derivative gravity without non-local interactions has been discussed in [197]. Here we follow the same approach but include the non-local terms in the effective quantum gravity formalism.

Using the Gauss-Bonnet theorem the effective action can be rewritten as[2]

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|g|} \left\{ R - \kappa^2 R \hat{L}_1 R - \kappa^2 C_{\mu\nu\rho\sigma} \hat{L}_2 C^{\mu\nu\rho\sigma} + \mathcal{O}(\kappa^4) \right\} + S_{\mathrm{m}}, \tag{11.2}$$

where C is the Weyl tensor and

$$\hat{L}_1 = \frac{2}{3} \left[3c_1(\mu) + c_2(\mu) + c_3(\mu) + (3\alpha + \beta + \gamma) \ln\left(\frac{\Box}{\mu^2}\right) \right], \tag{11.3}$$

$$\hat{L}_2 = \left[c_2(\mu) + 4c_3(\mu) + (\beta + 4\gamma) \ln\left(\frac{\Box}{\mu^2}\right) \right]. \tag{11.4}$$

We apply a Legendre transform to the function

$$f_1(R) = R - \kappa^2 R \hat{L}_1 R, \tag{11.5}$$

and find

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|g|} \left\{ \phi R - V_1(\phi) - \kappa^2 C_{\mu\nu\rho\sigma} \hat{L}_2 C^{\mu\nu\rho\sigma} + \mathcal{O}(\kappa^4) \right\} + S_{\mathrm{m}}, \tag{11.6}$$

where

$$R = \frac{\partial V_1(\phi)}{\partial \phi}, \tag{11.7}$$

$$\phi = \frac{\partial f_1(R)}{\partial R}. \tag{11.8}$$

We integrate the first equation and fix the integration constant such that

$$V_1(\phi) = -\frac{1}{4\kappa^2}(\phi - 1)\hat{L}_1^{-1}(\phi - 1), \tag{11.9}$$

where we use the notation $\hat{L}_1^{-1}$ to denote the Green's function of the operator $\hat{L}_1$. If we apply a conformal transformation to the metric

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = |\phi| g_{\mu\nu} = \exp\left(\sqrt{\frac{2\kappa^2}{3}} \chi \right) g_{\mu\nu}, \tag{11.10}$$

[2] Due to the presence of a $\ln(\Box)$ term in $\hat{L}_2$, the Gauss-Bonnet theorem does not hold in full generality. However, it is valid up to this order in κ [37–39, 70]

where we have introduced a new field χ, we can rewrite the action as

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{|\bar{g}|}\left\{\bar{R} + \sqrt{6}\kappa\bar{\Box}\chi - \kappa^2\bar{\nabla}^\mu\chi\bar{\nabla}_\mu\chi - \frac{V_1[\phi(\chi)]}{\phi(\chi)^2} - \kappa^2\bar{C}_{\mu\nu\rho\sigma}\hat{L}_2\bar{C}^{\mu\nu\rho\sigma}\right.$$
$$\left. - \frac{2\kappa^2\mathcal{L}_{\rm m}\left(X, g^{\mu\nu}\right)}{\phi(\chi)^2} + \mathcal{O}(\kappa^4)\right\}, \tag{11.11}$$

where we have used that the Weyl tensor does not transform under a conformal rescaling of the metric. Furthermore, X represents all matter fields.

We can drop the total divergence term, since it does not affect the equations of motion, and apply the Gauss-Bonnet theorem to rewrite the Weyl tensor. We then find

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{|\bar{g}|}\left\{\bar{R} - \kappa^2\bar{\nabla}^\mu\chi\bar{\nabla}_\mu\chi - \frac{V_1[\phi(\chi)]}{\phi(\chi)^2} - 2\kappa^2\bar{R}_{\mu\nu}\hat{L}_2\bar{R}^{\mu\nu} + \frac{2\kappa^2}{3}\bar{R}\hat{L}_2\bar{R}\right.$$
$$\left. - \frac{2\kappa^2\mathcal{L}_{\rm m}\left(X, g^{\mu\nu}\right)}{\phi(\chi)^2} + \mathcal{O}(\kappa^4)\right\}. \tag{11.12}$$

We consider the function

$$f_2(\bar{R}_{\mu\nu}) = \bar{R} - 2\kappa^2\bar{R}_{\mu\nu}\hat{L}_2\bar{R}^{\mu\nu} + \frac{2\kappa^2}{3}\bar{R}\hat{L}_2\bar{R}, \tag{11.13}$$

and apply a Legendre transform to this part of the action, which results in

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{|\bar{g}|}\left\{\psi^{\mu\nu}\bar{R}_{\mu\nu} - V_2(\psi^{\mu\nu}) - \kappa^2\bar{\nabla}^\mu\chi\bar{\nabla}_\mu\chi - \frac{V_1[\phi(\chi)]}{\phi(\chi)^2}\right.$$
$$\left. - \frac{2\kappa^2\mathcal{L}_{\rm m}\left(X, g^{\mu\nu}\right)}{\phi(\chi)^2} + \mathcal{O}(\kappa^4)\right\}, \tag{11.14}$$

where[3]

$$\bar{R}_{\mu\nu} = \frac{\partial V_2(\psi^{\mu\nu})}{\partial\psi^{\mu\nu}}, \tag{11.15}$$

$$\psi^{\mu\nu} = \frac{\partial f_2(\bar{R}_{\mu\nu})}{\partial\bar{R}_{\mu\nu}}. \tag{11.16}$$

We integrate the first equation and fix the integration constant such that[4]

$$V_2(\psi^{\mu\nu}) = -\frac{1}{8\kappa^2}\left(\psi_{\mu\nu} - \frac{1\mp i\sqrt{3}}{4}\psi\,\bar{g}_{\mu\nu} \mp i\sqrt{3}\,\bar{g}_{\mu\nu}\right)\hat{L}_2^{-1}\left(\psi^{\mu\nu} - \frac{1\mp i\sqrt{3}}{4}\psi\,\bar{g}^{\mu\nu} \mp i\sqrt{3}\,\bar{g}^{\mu\nu}\right). \tag{11.17}$$

We perform another metric transformation such that

$$\bar{g}_{\mu\nu} \to \tilde{g}_{\mu\nu} = \sqrt{|\psi|}\,\bar{g}_{\mu\rho}\left(\psi^{-1}\right)^\rho{}_\nu\,, \tag{11.18}$$

[3] Note that the spin-2 field is symmetric in its indices, since $R_{\mu\nu}$ is symmetric.

[4] The potential V_2 is real, which can easily be shown by evaluating the expression.

where we define the determinants

$$|g| = \det(g_{\mu\nu}), \tag{11.19}$$

$$|\psi| = \det(\pi^{\mu}{}_{\nu}), \tag{11.20}$$

and we write

$$\tilde{\psi}^{\mu}{}_{\nu} = \psi^{\mu}{}_{\nu}, \tag{11.21}$$

$$\tilde{\psi}^{\mu\nu} = \tilde{\psi}^{\mu}{}_{\rho}\tilde{g}^{\rho\nu}, \tag{11.22}$$

$$\tilde{\psi}_{\mu\nu} = \tilde{g}_{\mu\rho}\tilde{\psi}^{\rho}{}_{\nu}. \tag{11.23}$$

We obtain the transformed action

$$\begin{aligned} S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{|\tilde{g}|}\Big\{&\tilde{R} - \kappa^2\left(\psi^{-1}\right)^{\mu}{}_{\nu}\tilde{\nabla}^{\nu}\chi\tilde{\nabla}_{\mu}\chi \\ &+ \tilde{g}^{\mu\nu}\left(\tilde{\nabla}_{\rho}Q^{\rho}{}_{\mu\nu} - \tilde{\nabla}_{\nu}Q^{\rho}{}_{\rho\mu} + Q^{\rho}{}_{\rho\sigma}Q^{\sigma}{}_{\mu\nu} - Q^{\rho}{}_{\sigma\mu}Q^{\sigma}{}_{\rho\nu}\right) \\ &- \frac{V_1[\phi(\chi)]}{\phi(\chi)^2\sqrt{|\psi|}} - \frac{V_2(\psi^{\mu\nu})}{\sqrt{|\psi|}} - \frac{2\kappa^2\mathcal{L}_{\mathrm{m}}(X, g^{\mu\nu})}{\phi(\chi)^2\sqrt{|\psi|}} + \mathcal{O}(\kappa^4)\Big\}, \end{aligned} \tag{11.24}$$

where

$$Q^{\rho}{}_{\mu\nu}(\psi^{\alpha}{}_{\beta}) = \frac{1}{2}\bar{g}^{\rho\sigma}(\psi^{\alpha}{}_{\beta})\left(\tilde{\nabla}_{\mu}\bar{g}_{\nu\sigma}(\psi^{\alpha}{}_{\beta}) + \tilde{\nabla}_{\nu}\bar{g}_{\sigma\mu}(\psi^{\alpha}{}_{\beta}) - \tilde{\nabla}_{\sigma}\bar{g}_{\mu\nu}(\psi^{\alpha}{}_{\beta})\right). \tag{11.25}$$

We again drop the total derivative terms, and we define a new spin-2 field ξ such that

$$\psi^{\mu}{}_{\nu} = \left(1 + \frac{\kappa}{2}\xi\right)\delta^{\mu}_{\nu} - \kappa\xi^{\mu}{}_{\nu} \tag{11.26}$$

with $\xi = \xi^{\mu}{}_{\mu}$. We find

$$V_2(\psi^{\mu\nu}) = -\frac{1}{8}\left(\xi^{\mu}{}_{\nu}\hat{L}_2^{-1}\xi^{\nu}{}_{\mu} - \xi\hat{L}_2^{-1}\xi\right). \tag{11.27}$$

After this transformation the action becomes

$$\begin{aligned} S = \int d^4x\sqrt{|\tilde{g}|}\Bigg\{&-\frac{\tilde{R}}{2\kappa^2} + \frac{1}{2}\tilde{\nabla}^{\nu}\chi\tilde{\nabla}_{\mu}\chi + \frac{V_1[\phi(\chi)]}{2\kappa^2\phi(\chi)^2\sqrt{|\psi(\xi)|}} \\ &- \left[\frac{1}{2}\xi\tilde{\Box}\xi - \frac{1}{2}\xi^{\mu\nu}\tilde{\Box}\xi_{\mu\nu} - \xi^{\mu\nu}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\xi + \xi^{\mu\nu}\tilde{\nabla}_{\rho}\tilde{\nabla}_{\nu}\xi^{\rho}{}_{\mu}\right] \\ &+ \frac{V_2(\psi^{\mu\nu}(\xi))}{2\kappa^2\sqrt{|\psi(\xi)|}} + \mathcal{L}_{\mathrm{m}}(X, g^{\mu\nu})\Bigg\} + \mathcal{O}(\kappa), \end{aligned} \tag{11.28}$$

where we used that $\phi(\chi) = 1 + \mathcal{O}(\kappa)$, $\psi^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \mathcal{O}(\kappa)$. In addition, we expand the terms containing a potential using $\hat{L} = \hat{\tilde{L}} + \mathcal{O}(\kappa)$ and find

$$S = \int d^4x\sqrt{|\tilde{g}|}\Bigg\{-\frac{\tilde{R}}{2\kappa^2} + \frac{1}{2}\tilde{\nabla}^\mu\chi\tilde{\nabla}_\mu\chi - \chi(12\kappa^2\hat{\tilde{L}}_1)^{-1}\chi$$
$$- \left[\frac{1}{2}\xi\tilde{\Box}\xi - \frac{1}{2}\xi^{\mu\nu}\tilde{\Box}\xi_{\mu\nu} - \xi^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\xi + \xi^{\mu\nu}\tilde{\nabla}_\rho\tilde{\nabla}_\nu\xi^\rho{}_\mu\right]$$
$$- \left[\xi^{\mu\nu}(16\kappa^2\hat{\tilde{L}}_2)^{-1}\xi_{\mu\nu} - \xi(16\kappa^2\hat{\tilde{L}}_2)^{-1}\xi\right] + \mathcal{L}_{\mathrm{m}}\left(X, g^{\mu\nu}\right)\Bigg\} + \mathcal{O}(\kappa), \quad (11.29)$$

where indices on ξ are raised and lowered with $\tilde{g}$. We then find the equations of motion for the scalar field:

$$\tilde{\Box}\chi = -(6\kappa^2\hat{\tilde{L}}_1)^{-1}\chi + \mathcal{O}(\kappa). \quad (11.30)$$

We can solve the equation of motion for the Green's function $(6\kappa^2\hat{\tilde{L}}_1)^{-1}$ by Fourier transformation:

$$\int d^4k\left\{-k^2 + \frac{1}{4\kappa^2\left[3c_1(\mu) + c_2(\mu) + c_3(\mu) + (3\alpha+\beta+\gamma)\ln\left(\frac{-k^2}{\mu^2}\right)\right]}\right\}\chi(k) = \mathcal{O}(\kappa). \quad (11.31)$$

This results in the mass of the scalar field given by

$$m_0^2 = \frac{1}{4\kappa^2(3\alpha+\beta+\gamma)W\left(-\frac{1}{4\mu^2\kappa^2(3\alpha+\beta+\gamma)}\exp\left[\frac{3c_1(\mu)+c_2(\mu)+c_3(\mu)}{3\alpha+\beta+\gamma}\right]\right)}, \quad (11.32)$$

which corresponds to earlier results (see e.g. [93]). We can do a similar analysis for the tensor field, which yields (cf. [93])

$$m_2^2 = \frac{1}{2\kappa^2(\beta+4\gamma)W\left(-\frac{1}{2\mu^2\kappa^2(\beta+4\gamma)}\exp\left[\frac{c_2(\mu)+4c_3(\mu)}{\beta+4\gamma}\right]\right)}. \quad (11.33)$$

This resulting action is

$$S = \int d^4x\sqrt{|\tilde{g}|}\Bigg\{-\frac{\tilde{R}}{2\kappa^2} + \frac{1}{2}\tilde{\nabla}^\mu\chi\tilde{\nabla}_\mu\chi - \frac{1}{2}m_0^2\chi^2$$
$$- \left[\frac{1}{2}\xi\tilde{\Box}\xi - \frac{1}{2}\xi^{\mu\nu}\tilde{\Box}\xi_{\mu\nu} - \xi^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\xi + \xi^{\mu\nu}\tilde{\nabla}_\rho\tilde{\nabla}_\nu\xi^\rho{}_\mu\right]$$
$$-\frac{1}{2}m_2^2\left[\xi^{\mu\nu}\xi_{\mu\nu} - \xi\xi\right] + \mathcal{L}_{\mathrm{m}}\left(X, g^{\mu\nu}\right)\Bigg\} + \mathcal{O}(\kappa). \quad (11.34)$$

We can then find the equation of motion for the metric

$$\begin{aligned}\left(\tilde{R}_{\mu\nu}-\frac{1}{2}\tilde{R}\,\tilde{g}_{\mu\nu}\right)=&\kappa^2\left\{\tilde{T}_{\mu\nu}+\tilde{\nabla}_\mu\chi\tilde{\nabla}_\nu\chi-\frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\rho\chi\tilde{\nabla}_\rho\chi+\frac{1}{2}m_0^2\tilde{g}_{\mu\nu}\chi^2\right.\\&-2\xi_{\mu\nu}\tilde{\Box}\xi-\xi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\xi+2\xi_{\mu\rho}\tilde{\Box}\xi^\rho{}_\nu+\xi^{\rho\sigma}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\xi_{\rho\sigma}\\&+2\xi^\rho{}_\mu\tilde{\nabla}_\nu\tilde{\nabla}_\rho\xi+2\xi^\rho{}_\mu\tilde{\nabla}_\rho\tilde{\nabla}_\nu\xi+2\xi^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\xi_{\mu\nu}\\&-2\xi^\rho{}_\mu\tilde{\nabla}_\sigma\tilde{\nabla}_\rho\xi^\sigma{}_\nu-2\xi^\rho{}_\mu\tilde{\nabla}_\sigma\tilde{\nabla}_\nu\xi^\sigma{}_\rho-2\xi^{\rho\sigma}\tilde{\nabla}_\mu\tilde{\nabla}_\sigma\xi_{\nu\rho}\\&+\tilde{g}_{\mu\nu}\left[\frac{1}{2}\xi\tilde{\Box}\xi-\frac{1}{2}\xi^{\rho\sigma}\tilde{\Box}\xi_{\rho\sigma}-\xi^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\xi+\xi^{\rho\sigma}\tilde{\nabla}_\lambda\tilde{\nabla}_\sigma\xi^\lambda{}_\rho\right]\\&\left.-2m_2^2\left[\xi^\rho{}_\mu\xi_{\nu\rho}-\xi_{\mu\nu}\xi\right]+\frac{1}{2}m_2^2\tilde{g}_{\mu\nu}\left[\xi^{\rho\sigma}\xi_{\rho\sigma}-\xi\xi\right]\right\}\\&+\mathcal{O}(\kappa^3).\end{aligned}\tag{11.35}$$

This can be rewritten in the form

$$\begin{aligned}\tilde{R}_{\mu\nu}=&\kappa^2\left\{\tilde{T}_{\mu\nu}-\frac{1}{2}\tilde{T}\tilde{g}_{\mu\nu}+\tilde{\nabla}_\mu\chi\tilde{\nabla}_\nu\chi-\frac{1}{2}m_0^2\tilde{g}_{\mu\nu}\chi^2\right.\\&-2\xi_{\mu\nu}\tilde{\Box}\xi-\xi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\xi+2\xi_{\mu\rho}\tilde{\Box}\xi^\rho{}_\nu+\xi^{\rho\sigma}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\xi_{\rho\sigma}\\&+2\xi^\rho{}_\mu\tilde{\nabla}_\nu\tilde{\nabla}_\rho\xi+2\xi^\rho{}_\mu\tilde{\nabla}_\rho\tilde{\nabla}_\nu\xi+2\xi^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\xi_{\mu\nu}\\&-2\xi^\rho{}_\mu\tilde{\nabla}_\sigma\tilde{\nabla}_\rho\xi^\sigma{}_\nu-2\xi^\rho{}_\mu\tilde{\nabla}_\sigma\tilde{\nabla}_\nu\xi^\sigma{}_\rho-2\xi^{\rho\sigma}\tilde{\nabla}_\mu\tilde{\nabla}_\sigma\xi_{\nu\rho}\\&+\tilde{g}_{\mu\nu}\left[\xi\tilde{\Box}\xi-\xi^{\rho\sigma}\tilde{\Box}\xi_{\rho\sigma}-2\xi^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\xi+2\xi^{\rho\sigma}\tilde{\nabla}_\lambda\tilde{\nabla}_\sigma\xi^\lambda{}_\rho\right]\\&\left.-2m_2^2\left[\xi^\rho{}_\mu\xi_{\nu\rho}-\xi_{\mu\nu}\xi\right]+\frac{1}{2}m_2^2\tilde{g}_{\mu\nu}\left[\xi^{\rho\sigma}\xi_{\rho\sigma}-\xi\xi\right]\right\}+\mathcal{O}(\kappa^3).\end{aligned}\tag{11.36}$$

11.3 Singularity theorems in effective quantum gravity

11.3.1 Massive scalar field

It is known that a massive scalar field always satisfies the null energy condition, but can easily violate the strong condition (cf. [42, 193]). The energy momentum tensor is given by

$$T_{\mu\nu}=\nabla_\mu\chi\nabla_\nu\chi-\frac{1}{2}g_{\mu\nu}\left(\nabla^\rho\chi\nabla_\rho\chi+m^2\chi_0^2\right).\tag{11.37}$$

Hence,

$$T_{\mu\nu}v^\mu v^\nu=\left(v^\mu\nabla_\mu\chi\right)^2\geq 0,\tag{11.38}$$

where v is an arbitrary null vector. We conclude that the null energy condition is satisfied. However,

$$T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T=\nabla_\mu\chi\nabla_\nu\chi-\frac{1}{2}g_{\mu\nu}m_0^2\chi^2\tag{11.39}$$

which leads to

$$\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)t^\mu t^\nu=\left(t^\mu\nabla_\mu\chi\right)^2-\frac{1}{2}m_0^2\chi^2,\tag{11.40}$$

where t is an arbitrary normalized time-like vector. We see that this expression could be both larger and smaller to 0. Consequently the strong energy condition does not necessarily hold. We conclude that the scalar field arising in effective quantum gravity could resolve cosmological singularities, but not black hole singularities.

11.3.2 Bounds on the mass of the massive scalar field

Using the results from appendix 11.B we can derive a bound on the mass of the scalar field for which the cosmological singularity theorem still holds. First consider the action (11.34) containing only the massive scalar. Eq. (11.36) then reduces to

$$\tilde{R}_{\mu\nu} = \kappa^2 \left\{ \tilde{T}_{\mu\nu} - \frac{1}{2}\tilde{T}\tilde{g}_{\mu\nu} + \tilde{\nabla}_\mu \chi \tilde{\nabla}_\nu \chi - \frac{1}{2} m_0^2 \tilde{g}_{\mu\nu} \chi^2 \right\} + \mathcal{O}(\kappa^3). \tag{11.41}$$

Let us consider a globally hyperbolic 4-dimensional space-time with compact Cauchy hypersurface S, and assume $|\chi| < \chi_{\max}$ is bounded towards te past of S. Then

$$\begin{aligned} \int_0^T e^{-\frac{2C\tau}{n-1}} R_{\mu\nu}(\tau)\hat{\gamma}^\mu \hat{\gamma}^\nu(\tau) d\tau &\geq -\frac{1}{2}\kappa^2 m_0^2 \chi_{\max}^2 \int_0^T e^{-\frac{2C\tau}{n-1}} d\tau \\ &\geq -\frac{3\kappa^2}{4C} m_0^2 \chi_{\max}^2, \end{aligned} \tag{11.42}$$

where $\hat{\gamma}$ is a normalized tangent vector to a past directed time-like geodesic and where we have used the strong energy condition in the first line. We find

$$-\frac{C}{2} + \int_{-T}^0 e^{\frac{2C\tau}{n-1}} R_{\mu\nu}(\tau)\hat{\gamma}^\mu(\tau)\hat{\gamma}^\nu(\tau) d\tau \geq -\frac{C}{2} - \frac{3\kappa^2}{4C} m_0^2 \chi_{\max}^2 \tag{11.43}$$

for any $C > 0$. The right hand side is maximized for $C = \sqrt{\frac{3}{2}}\kappa m_0 \chi_{\max}$. By Theorem 11.4 we then find that $\mathcal{M}$ is past geodesically incomplete, if

$$\theta > \sqrt{\frac{3}{2}}\, \kappa\, m_0\, \chi_{\max} \tag{11.44}$$

everywhere on S. Hence for

$$m_0 < \sqrt{\frac{2}{3}} \frac{\theta_{\min}}{\kappa\, \chi_{\max}} \tag{11.45}$$

the singularity theorem still holds.

We can use the expression for the mass of the scalar (11.32) to find a condition for the Wilson coefficients. Let us first ignore the nonlocal terms α, β, γ. We then find

$$m_0^2 = \frac{1}{4\kappa^2 \left[3c_1(\mu) + c_2(\mu) + c_3(\mu)\right]}. \tag{11.46}$$

We thus find that the singularity theorem holds for

$$3c_1(\mu) + c_2(\mu) + c_3(\mu) > \frac{3\chi_{\max}^2}{8\theta_{\min}^2}, \tag{11.47}$$

where we have assumed $3c_1(\mu) + c_2(\mu) + c_3(\mu) > 0$, as the opposite would imply that the scalar field is tachyonic.[5] If we include the non-local contributions, we find instead

$$3c_1(\mu) + c_2(\mu) + c_3(\mu) > \mathrm{Re}\left(\frac{3\chi_{\mathrm{max}}^2}{8\theta_{\mathrm{min}}^2} + (3\alpha+\beta+\gamma)\ln\left[-\frac{3\mu^2\kappa^2\chi_{\mathrm{max}}^2}{2\theta_{\mathrm{min}}^2}\right]\right), \tag{11.48}$$

where only the logarithm has a complex part that accounts for the decay width of the field [68, 75, 76].

We can make an estimate of the expansion parameter for our universe, by assuming the FLRW-metric, and by assuming that we live on a compact Cauchy hypersurface with a Hubble parameter that is constant along the surface. We find

$$\theta_{\mathrm{min}} = \frac{1}{3}H \approx 10^{-18}\,\mathrm{s}^{-1}, \tag{11.49}$$

where the Hubble parameter is fixed by experiment[6]. In addition, we require an estimate for χ_{max}, which will rely on theoretical prejudice. However, for the effective action to be consistent one would expect that both the scalar and tensor fields arising in the Einstein frame do not exceed the Planck scale. We thus make the rough estimate

$$\chi_{\mathrm{max}} = \sqrt{\frac{c^5}{8\pi G_{\mathrm{N}}\hbar}} = 10^{42}\,\mathrm{s}^{-1}. \tag{11.50}$$

Hence,

$$\frac{3\chi_{\mathrm{max}}^2}{8\theta_{\mathrm{min}}^2} = 10^{121}. \tag{11.51}$$

Furthermore, the non-local part leads to a correction given by

$$(3\alpha+\beta+\gamma)\ln\left[-\frac{3\mu^2\kappa^2\chi_{\mathrm{max}}^2}{2\theta_{\mathrm{min}}^2}\right] \approx 10^2, \tag{11.52}$$

where we have used the known values for α, β, γ assuming only Standard Model fields. Furthermore, we have set the cutoff scale $\mu \approx \kappa^{-1}$. These non-local corrections are thus negligible compared to the local contributions. We conclude that the singularity theorem holds, if the local Wilson coefficients satisfy the condition

$$3c_1(\mu) + c_2(\mu) + c_3(\mu) \gtrsim 10^{121} \tag{11.53}$$

or equivalently

$$m_0 \lesssim 10^{-34}\,\mathrm{eV}/c^2. \tag{11.54}$$

The singularity theorem can thus be violated for a large range of values.

The scalar and spin-2 particles give rise to corrections to the Newtonian potential according to the formula

$$\Phi(r) = -\frac{G_{\mathrm{N}}m}{r}\left(1 + \frac{1}{3}e^{-\mathrm{Re(m_0)r}} - \frac{4}{3}e^{-\mathrm{Re(m_2)r}}\right) \tag{11.55}$$

[5] We do not consider the tachyonic case, as it is unphysical. It can be shown, however, using eq. (11.40) that in this case the strong energy condition is satisfied.

[6] We take $H_0 \approx 70\mathrm{km\,s^{-1}\,Mpc^{-1}}$

The Eöt-Wash experiment [202] sets bounds on deviations from this potential. Assuming that the corrections do not cancel each other, both corrections should satisfy these experimental bounds, i.e.

$$m_0, m_2 \geq 10^{-3}\,\mathrm{eV/c^2}. \tag{11.56}$$

Hence, the singularity theorem can be violated for all feasible values of the Wilson coefficients.

It might seem counterintuitive that tiny Wilson coefficients already lead to a breakdown of the assumptions of the singularity theorems, while large Wilson coefficients do not. In particular, since the smaller the Wilson coefficients the closer the action is to the Einstein-Hilbert action. However, small Wilson coefficients lead to very massive scalar fields, which can violate the strong energy condition, as can be seen in eq. (11.40). Furthermore, the Einstein equation is a second order differential equation, while the introduction of the terms quadratic in the Ricci scalar and tensor make it a fourth order equation. As is well known solutions of differential equations are generically not stable against perturbations that change the class of the differential equation (cf. [33] for a discussion of this fact in the context of general relativity).

11.3.3 Spin-2 massive ghost

Let us now turn to the massive spin-2 field. Since this field is a ghost, one would expect it to violate the null energy condition. Indeed we can write the energy momentum tensor explicitly

$$\begin{aligned}
T_{\mu\nu} = &-2\xi_{\mu\nu}\tilde{\Box}\xi - \xi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\xi + 2\xi_{\mu\rho}\tilde{\Box}\xi^\rho{}_\nu + \xi^{\rho\sigma}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\xi_{\rho\sigma}\\
&+ 2\xi^\rho{}_\mu\tilde{\nabla}_\nu\tilde{\nabla}_\rho\xi + 2\xi^\rho{}_\mu\tilde{\nabla}_\rho\tilde{\nabla}_\nu\xi + 2\xi^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\xi_{\mu\nu}\\
&- 2\xi^\rho{}_\mu\tilde{\nabla}_\sigma\tilde{\nabla}_\rho\xi^\sigma{}_\nu - 2\xi^\rho{}_\mu\tilde{\nabla}_\sigma\tilde{\nabla}_\nu\xi^\sigma{}_\rho - 2\xi^{\rho\sigma}\tilde{\nabla}_\mu\tilde{\nabla}_\sigma\xi_{\nu\rho}\\
&+ \tilde{g}_{\mu\nu}\left[\frac{1}{2}\xi\tilde{\Box}\xi - \frac{1}{2}\xi^{\rho\sigma}\tilde{\Box}\xi_{\rho\sigma} - \xi^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\xi + \xi^{\rho\sigma}\tilde{\nabla}_\lambda\tilde{\nabla}_\sigma\xi^\lambda{}_\rho\right]\\
&- 2m_2^2\left[\xi^\rho{}_\mu\xi_{\nu\rho} - \xi_{\mu\nu}\xi\right] + \frac{1}{2}m_2^2\tilde{g}_{\mu\nu}\left[\xi^{\rho\sigma}\xi_{\rho\sigma} - \xi\xi\right].
\end{aligned} \tag{11.57}$$

In order to show that the field can violate the null energy condition, we construct a counterexample. We consider the special case in which the tensor field is aligned with the metric:

$$\xi_{\mu\nu} = \frac{1}{4}g_{\mu\nu}\xi. \tag{11.58}$$

This results in an energy momentum tensor given by

$$T_{\mu\nu} = -\frac{1}{16}\left(g_{\mu\nu}\xi\Box\xi + \xi\nabla_\mu\nabla_\nu\xi + \xi\nabla_\nu\nabla_\mu\xi\right). \tag{11.59}$$

Hence,

$$
\begin{aligned}
T_{\mu\nu} v^\mu v^\nu &= -\frac{1}{8} \xi \nabla_\mu \nabla_\nu \xi v^\mu v^\nu \\
&= -\frac{1}{8} (k_\mu v^\mu \xi)^2 \\
&\leq 0,
\end{aligned}
\tag{11.60}
$$

where v is an arbitrary null-like vector and where we assumed the field ξ to be an eigenvector of $\nabla_\mu \nabla_\nu$ with eigenvector $k_\mu k_\nu$, as is the case if the field exhibits sinusoidal behavior with wave vector k.

Since the spin-2 field can violate the null energy condition, it can violate the strong energy condition as well. We conclude that the massive spin-2 field can resolve both kinds of singularities, since it does not satisfy any of the required energy conditions.

The fact that the ghost field can resolve singularities is less of a surprise, if one takes into account that the ghost field leads to a repulsive contribution to Newton's potential [73,95], and could thus result in a effective repulsive force at small distances.

11.4 Conclusion and outlook

It is well known that the classical singularity theorems by Penrose and Hawking [191,291] only hold if general relativity is assumed. Quantum gravity, however, leads to deviations from general relativity, as can easily be shown using effective field theory methods. Furthermore, one of the main objectives of quantum gravity theories is to resolve singularities. In this work, we have discussed the validity of the classical singularity theorems in the context of the unique effective field theory for quantum gravity at second order in curvature.

We have considered singularity theorems by making an explicit mapping to the Einstein frame. The local terms in this theory give rise to an additional scalar and tensor field at second order in curvature. Moreover, the inclusion of the nonlocal terms at this order only gives rise to a shift in the mass of these fields.

We have shown that the massive spin-2 ghost field can violate the null energy condition and thus the strong energy condition as well. This is independent of its unknown mass. Although this is expected from a ghost field, it shows that the ghost field can be useful for resolving singularities in quantum gravity. We emphasize that the ghost field in effective theories for quantum gravity is not problematic, since it results from integrating out the low energy quantum degrees of freedom. In this framework, it must thus be treated as a classical field, and not be quantized again [95].

Furthermore, we have shown that the scalar field satisfies the null energy condition, but may violate the strong energy condition. The latter is a necessary assumption of Hawking's original theorem. For the entire mass range that is allowed by experiment the scalar field poses troubles for a singularity theorem with weakened energy conditions derived by Fewster and Galloway in [155]. It should be noted that singularity avoidance in our framework has already been found in [143, 161]. Moreover, other examples of singularity resolution in various theories such as higher derivative gravity [170,171], string theory [337] and polynomial gravity models [6] have been found. The topic has also extensively been discussed within many ultraviolet complete approaches to quantum gravity.

It is important to notice that the breakdown of the assumptions of Hawking's and Penrose's singularity theorem does not imply the non-existence of singularities. However, it does imply that singularities can potentially be avoided. If the assumptions for the singularity theorems hold, the singular solutions of general relativity are the necessary endpoint of a collapsing star or universe. When considering perturbative corrections in the effective field theory approach, it is expected that these singular solutions such as the Kerr black hole remain to be viable solutions. However, it is possible that new solutions such as the ones discussed in [243, 244, 330] are present when the higher order curvature corrections are taken into account. If the singularity theorems are no longer applicable such non-singular solutions can become the physically relevant solutions.

Finally, we should notice that the results discussed in this paper only hold up to second order in curvature. Inclusion of higher orders might force us back into a regime where the singularity theorems hold or might draw us further away from this regime. The effects of these higher order terms are sub-leading but not negligible, as singularities form in highly curved regions of space-time. It is interesting, however, that singularities can potentially already be resolved at second order in curvature without making assumptions about the correct UV-complete theory of quantum gravity. This fact may help guide the way to singularity resolution in ultraviolet complete theories of quantum gravity.

Acknowledgments

This work is supported in part by the Science and Technology Facilities Council (grant number ST/P000819/1).

11.A Classical singularity theorems

11.A.1 Hawking's cosmological singularity theorem

In this appendix we state and proof Hawking's singularity theorem [191].

Theorem. *Let $\mathcal{M}$ be a globally hyperbolic n-dimensional space-time with $n \geq 2$ and a Cauchy surface S. Assume that $\exists C > 0$ such that $\theta_x < -C \ \forall x \in S$, where $\theta = \frac{1}{2} g^{\mu\nu}\partial_\tau g_{\nu\mu}$ is the expansion parameter. Furthermore, assume that matter within this space-time satisfies the strong energy condition*

$$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) t^\mu t^\nu \geq 0 \tag{11.61}$$

for every normalized time-like vector t^μ everywhere in the future of the Cauchy surface S. Then the space-time $\mathcal{M}$ is geodesically incomplete towards the future of S. Moreover, if $\theta_x > C \ \forall x \in S$ and the strong energy condition is satisfied everywhere in the past of S, then $\mathcal{M}$ is geodesically incomplete towards the past of S.

Proof. Consider an n-dimensional globally hyperbolic space-time $\mathcal{M}$ with Cauchy surface S. Then we can find an open neighborhood $\hat{S} \supset S$ and a coordinate system on $\hat{S}$ such that the metric is given by

$$ds^2 = -dt^2 + g_{ij}(t, \vec{x}) dx^i dx^j. \tag{11.62}$$

In order to proof Hawking's singularity theorem [191], we can write down the Raychaudhuri equation [299]:

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{n-1} - \sigma_{\mu\nu}\sigma^{\nu\mu} - R_{\mu\nu}t^\mu t^\nu, \tag{11.63}$$

where the expansion θ and shear $\sigma_{\mu\nu}$ are given by

$$\theta = \frac{1}{2} g^{\mu\nu}\partial_\tau g_{\nu\mu} = \frac{\dot{V}}{V}, \tag{11.64}$$

$$\sigma^\mu_\nu = \frac{1}{2}\left(g^{\mu\rho}\partial_\tau g_{\rho\nu} - \frac{1}{n-1}\delta^\mu_\nu g^{\rho\sigma}\partial_\tau g_{\sigma\rho}\right), \tag{11.65}$$

where we defined

$$V = \sqrt{\det(g)} \tag{11.66}$$

and the time-derivative by $\dot{V} = \partial_\tau V$. Furthermore, θ and $\sigma_{\mu\nu}$ are taken along a time-like path γ parametrized by τ with normalized tangent vectors t^μ, and $\gamma(0) \in S$.

If we use the Einstein field equation, we can rewrite the Raychaudhuri equation to

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{n-1} - \sigma_{\mu\nu}\sigma^{\nu\mu} - \kappa^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)t^\mu t^\nu. \tag{11.67}$$

Assuming the strong energy condition

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)t^\mu t^\nu \geq 0, \tag{11.68}$$

we find

$$\frac{d\theta}{d\tau} \leq -\frac{\theta^2}{n-1}. \tag{11.69}$$

Hence,

$$\frac{d}{d\tau}\theta^{-1} \geq \frac{1}{n-1}. \tag{11.70}$$

Assume $\exists\, C > 0$ such that $\theta_x(0) < -C \;\forall\, x \in S$, then we can integrate (11.70) and obtain

$$\frac{1}{\theta(\tau)} \geq \frac{\tau}{n-1} - \frac{1}{C}. \tag{11.71}$$

Hence for $\tau \in \left(-\infty, \frac{n-1}{C}\right)$

$$\theta(\tau) \leq -\left(\frac{1}{C} - \frac{\tau}{n-1}\right)^{-1}. \tag{11.72}$$

We can rewrite in terms of V and integrate to find

$$0 \leq V(\tau) \leq V(0)\left(1 - \frac{C\tau}{n-1}\right)^{n-1}. \tag{11.73}$$

Therefore

$$\lim_{\tau \to \frac{n-1}{C}} V(\tau) = 0. \tag{11.74}$$

We thus conclude that any geodesic emanating from the Cauchy surface will develop a focal point for $0 < \tau \leq \frac{n-1}{C}$. Furthermore, since S is a Cauchy surface and $\mathcal{M}$ is globally hyperbolic, any point $y \in \mathcal{M}$ is connected to a point $x \in S$ through a causal path of maximal proper time. We thus conclude that no geodesic $\gamma(\tau)$ can be extended to $\tau \geq \frac{n-1}{C}$. Therefore, the space-time is geodesically incomplete towards the future. This proves the future version of the theorem. The past version immediately follows by inverting the time direction in the proof. □

We conclude this subsection by mentioning an immediate result of the theorem: if there exists a Cauchy surface S such that the Hubble parameter $H \geq H_0 > 0$ on the entire surface S, and the strong energy condition is expected to hold anywhere in the past of this surface, then the space-time is geodesically incomplete towards the past. More precisely no geodesic can be extended beyond $\tau = H_0^{-1}$ towards the past. To see this, we recall that the Hubble constant given by

$$H = \frac{\dot{a}}{a} = (n-1)\frac{\dot{V}}{V} \tag{11.75}$$

for the FLRW-metric

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2. \tag{11.76}$$

11.A.2 Penrose's black hole singularity theorem

In this appendix we state and prove Penrose's singularity theorem [291]. Here we closely follow the proof provided in [359].

Theorem. *Let $\mathcal{M}$ be a globally hyperbolic n-dimensional space-time with $n \geq 3$ and a non-compact Cauchy surface S. Assume that $\mathcal{M}$ contains a compact trapped surface*[7] *U. Furthermore, assume that matter within this space-time satisfies the null energy condition*

$$T_{\mu\nu}v^\mu v^\nu \geq 0 \tag{11.77}$$

for every null-like vector v^μ everywhere in the future of the trapped surface U. Then the space-time $\mathcal{M}$ is null-geodesically incomplete towards the future of U.

Proof. Consider a globally hyperbolic n-dimensional space-time with non-compact Cauchy surface S, and a compact trapped surface U. Then we can find an open neighborhood

[7] A codimension 2 spacelike and achronal submanifold such that the null expansion parameter is negative everywhere on U for each family of orthogonal future going null geodesics.

$\hat{U} \supset U$ and a coordinate system on $\hat{U}$ such that the metric is given by (cf. [304, 359])

$$ds^2 = -2e^q dv du + g_{AB}(dx^A + c^A dv)(dx^B + c^B dv), \tag{11.78}$$

where x^A is an arbitrary but fixed local coordinate system on the $(n-2)$-dimensional surface U. Furthermore, q and c are respectively a scalar and vector function of the coordinates. In this metric we can evaluate the Ricci tensor and find

$$R_{uu} = -\frac{1}{2}\partial_u \left(g^{AB}\partial_u g_{AB}\right) - \frac{1}{4}\left(g^{AC}\partial_u g_{BC}\right)\left(g^{BD}\partial_u g_{DA}\right). \tag{11.79}$$

We can define the area of a bundle of orthogonal null geodesics locally by

$$A = \sqrt{\det(g_{AB})}, \tag{11.80}$$

which allows us to define the null expansion as

$$\theta = \frac{\dot{A}}{A} = \frac{1}{2} g^{AB}\partial_u g_{BA}, \tag{11.81}$$

where the dot represents a derivative with respect to u. Furthermore, we can define the null shear by

$$\sigma^A_B = \frac{1}{2}\left(g^{AC}\partial_u g_{CB} - \frac{1}{n-2}\delta^A_B g^{CD}\partial_u g_{DC}\right). \tag{11.82}$$

We then find the null Raychaudhuri equation given by

$$\frac{d\theta}{du} = -\frac{\theta^2}{n-2} - \sigma_{AB}\sigma^{BA} - R_{uu}. \tag{11.83}$$

Furthermore, we can use the Einstein equation and the fact $g_{uu} = 0$ to write

$$R_{uu} = \kappa^2 T_{uu}. \tag{11.84}$$

Imposing the null energy condition results in

$$\frac{d}{du}\theta^{-1} \geq \frac{1}{n-2}. \tag{11.85}$$

Using that U is a trapped surface $\exists\, C > 0$ such that $\theta_x < -C\ \forall x \in U$, one can integrate this equation in a similar way as was done in the proof of theorem 11.4. One obtains

$$\lim_{u \to \frac{n-2}{C}} A(u) = 0. \tag{11.86}$$

Therefore, all future going null like geodesics develop a focal point for an affine distance $0 < u \leq \frac{n-2}{C}$.

Let us now assume that all null-geodesics can be extended beyond this focal point, and let us pick such a geodesic l arbitrarily. Then at least a small segment of this geodesic is prompt, and lies in the lightcone $\partial J^+(U)$. Furthermore, the part of l that lies in $\partial J^+(U)$ is connected, and the part beyond its first focal point cannot be in $\partial J^+(U)$, since it is not prompt. Therefore $l \cap \partial J^+(U)$ is a finite non-empty interval, which has to be closed, since $\partial J^+(U)$ is closed in $\mathcal{M}$.

If we take an arbitrary point $p \in \partial J^+(U)$, then this point can be reached by a null geodesic originating from U. This point is thus determined by the point $q \in U$, where the geodesic emanates, the value of the affine parameter u measured along the geodesic and the direction (i.e. ingoing or outgoing) of the geodesic. Since U is compact and since the affine parameters measured along the geodesics range over a compact interval, we find that $\partial J^+(U)$ is compact.

However, by construction $\partial J^+(U)$ is an achronal codimension 1 submanifold of $\mathcal{M}$. Furthermore, by assumption $\mathcal{M}$ is a globally hyperbolic manifold with noncompact Cauchy hypersurface S, and thus does not allow for an achronal codimension 1 submanifold (see e.g. [359]). Hence, we arrive at a contradiction and conclude that at least one of the future going null geodesics orthogonal to U cannot be extended beyond an affine distance $(n-2)/C$, which proves the theorem. □

11.B Singularity theorems for weakened energy conditions

In this section, we state a theorem and its proof from [155]. The theorem is similar to Hakwing's cosmological singularity theorem, but uses relaxed conditions on the energy momentum tensor.

Theorem. *Let $\mathcal{M}$ be a globally hyperbolic n-dimensional space-time ($n \geq 2$) with a compact Cauchy surface S. Assume that $\exists\, C \geq 0$ such that along every future directed geodesic γ issuing orthogonally from S we have*

$$\liminf_{T\to\infty} \int_0^T e^{-\frac{2C\tau}{n-1}} R_{\mu\nu}(\tau)\hat{\gamma}^\mu(\tau)\hat{\gamma}^\nu(\tau)d\tau > \theta(x_0) + \frac{C}{2}, \tag{11.87}$$

where $x_0 = \gamma(0) \in S$, $\theta(x_0)$ is the expansion at x_0, and $\hat{\gamma}(\tau)$ is a normalized time-like tangent vector of $\gamma(\tau)$. Then $\mathcal{M}$ is geodesically incomplete towards the future of S. Moreover, if

$$\liminf_{T\to\infty} \int_0^T e^{-\frac{2C\tau}{n-1}} R_{\mu\nu}(\tau)\hat{\gamma}^\mu(\tau)\hat{\gamma}^\nu(\tau)d\tau > -\theta(x_0) + \frac{C}{2} \tag{11.88}$$

with γ a past directed geodesic, then $\mathcal{M}$ is geodesically incomplete towards the past of S.

For the proof we will use the following lemma which is proved in [155].

Lemma 1. *Consider the initial value problem*

$$\begin{cases} \dot{x}(t) = \frac{x(t)^2}{q(t)} + p(t), \\ x(0) = x_0, \end{cases} \tag{11.89}$$

where $q(t)$ and $p(t)$ are continuous on $[0,\infty)$ and $q(t) > 0\ \forall t \in [0,\infty)$. If

$$\int_0^\infty q(t)^{-1}dt = \infty, \tag{11.90}$$

$$\liminf_{T\to\infty} \int_0^T p(t)dt > -x_0, \tag{11.91}$$

eq. (11.89) *has no solution on* $[0,\infty)$. *Moreover it implies that* $\lim_{t\to t_c} x(t) \to \infty$ *for* $t_c \in (0,\infty)$.

Proof of Theorem 11.4. We follow the same argument as in the proof of Theorem 11.4 and find the Raychaudhuri equation

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{n-1} - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}t^\mu t^\nu, \tag{11.92}$$

which can be rewritten to

$$\frac{dx(\tau)}{d\tau} = \frac{x(\tau)^2}{q(\tau)} + p(\tau) \tag{11.93}$$

with

$$x(\tau) = -(\theta + C)e^{-\frac{2C\tau}{n-1}}, \tag{11.94}$$

$$p(\tau) = \left(\frac{C^2}{n-1} + \sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}t^\mu t^\nu\right)e^{-\frac{2C\tau}{n-1}}, \tag{11.95}$$

$$q(\tau) = (n-1)e^{-\frac{2C\tau}{n-1}}. \tag{11.96}$$

Then $q(\tau)$ satisfies condition (11.90), while $p(\tau)$ satisfies condition (11.91), if

$$\liminf_{T\to\infty} \int_0^T \left(\frac{C^2}{n-1} + \sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}t^\mu t^\nu\right)e^{-\frac{2C\tau}{n-1}}d\tau > \theta(0) + C, \tag{11.97}$$

which is satisfied, if

$$\liminf_{T\to\infty} \int_0^T e^{-\frac{2C\tau}{n-1}} R_{\mu\nu}t^\mu t^\nu d\tau > \theta(0) + \frac{C}{2}. \tag{11.98}$$

By assumption (11.87), this holds for all geodesics emanating from the Cauchy surface S. Thus $\lim_{\tau\to\tau_\gamma} x(\tau) \to \infty$ for some $\tau_\gamma \in (0,\infty)$, which immediately implies that $\lim_{\tau\to\tau_\gamma} \theta(\tau) \to -\infty$. Hence

$$\forall\gamma : [0,\infty) \to \mathcal{M} \quad \text{with} \quad \gamma(0) \in S \quad \exists\tau_\gamma \in (0,\infty) \quad \text{s.t.} \quad \lim_{\tau\to\tau_\gamma} V(\tau) \to 0. \tag{11.99}$$

By compactness of S, $\sup\{\tau_\gamma | \gamma : [0,\infty) \to \mathcal{M}, \gamma(0) \in S\} < \infty$. Furthermore, since $\mathcal{M}$ is globally hyperbolic every point $y \in J^+(S)$ can be connected through a geodesic γ with maximal proper time. The past version can be obtained with a similar proof by inverting the direction of time. □

Let us finally note that one can derive a similar theorem for the black hole case [155].

Chapter 12

Singularities in Quantum Corrected Spacetimes

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Abstract

In this paper we consider a static and regular fluid generating a locally spherically symmetric and time-independent space-time and calculate the leading quantum corrections to the metric to first order in curvature. Starting from a singularity free classical solution of general relativity, we show that singularities can be introduced in the curvature invariants by quantum gravitational corrections calculated using an effective field theory approach to quantum gravity. We identify non-trivial conditions that ensure that curvature invariants remain singularity free to leading order in the curvature expansion of the effective action.

This chapter has been published in Physics Letters B **807**, p. 135605 (2020).
A preprint of this chapter can be found at arXiv:2003.04220 [hep-th].

12.1 Introduction

Black holes are stationary vacuum solutions of Einstein's field equations. Despite the simplicity of these solutions, rotating black holes which are described by the Kerr metric appear to account for the numerous observations of astrophysical black holes with a remarkable accuracy. Furthermore, it is known that massive stars collapse at the end of their lifetime and form black holes. The idea that gravitational collapse leads to a black hole is strengthened by singularity theorems that prove geodesic incompleteness when a trapped surface is formed [191, 291]. However, the collapse picture may not be fully realistic. Solutions like the Kerr metric are in fact vacuum solutions, and delta-like sources are not well defined in general relativity [168]. The collapse of a star to a Kerr black hole thus requires the destruction of the matter that made up the star, while some information like the mass and angular momentum is conserved. Indeed this is the reason why general relativity is considered to break down at singularities.

It is usually assumed that a theory of quantum gravity will provide a physical mechanism to resolve these classical curvature singularities. Although many candidate theories of quantum gravity have been developed, an ultraviolet complete theory of quantum gravity is still illusive. There exists however a unique infrared theory of quantum gravity [34–37, 61, 140, 351] that is valid up to the scale where the new physics necessary for an ultraviolet completion kicks in. This scale is known to be far beyond the reach of current experiments and is assumed to be at the Planck scale unless there is a very large number of fields in the model.

An important prediction of the effective field theory of quantum gravity is the leading quantum correction to the Newtonian potential [93, 139, 330]. This correction can equivalently be described by the introduction of two classical fields. Recently it has been shown that these fields can lead to the violation of assumptions of the singularity theorems and that singularities can therefore be avoided before Planck scale energies are reached [237, 239]. Moreover the possibility of the singularity avoidance in a gravitational collapse [161] and a hypothetical big crunch [143] was shown earlier in the same framework.

It is thus possible that a space-time that is classically singular becomes regular, when perturbative quantum gravitational effects are taken into account. In the specific cases studied in Refs. [143, 161], singularities avoidance happens at energy densities that do not exceed the Planck scale and can thus be described by the effective action approach to quantum gravity.

In this paper we investigate the opposite scenario in which curvature singularities can be introduced on regular space-times when the quantum gravitational corrections are taken into account. Since we are working in an effective field theory framework, this question should be rephrased as: can quantum corrections to the curvature invariants reach the Planck scale, if the classical curvature is well behaved, i.e. singularity free? It is not possible within the effective field theory approach to draw strong conclusions about the fate of such new singularities, as the logic of any perturbative approach dictates to dismiss the results in regimes where the perturbation theory is no longer under control. Nevertheless this is an intriguing question, as non-perturbative quantum gravity is not yet well-understood. Moreover, from an effective field theory perspective singularities arising at any order in perturbation theory should be treated at the same level as the classical singularities. Indeed in the effective field theory framework general relativity is the zeroth order approximation of a theory of quantum gravity, and any higher order theory is considered to be an improvement over this low energy approximation.

In this work we derive non-trivial conditions for which classical regular space-times remain regular in a first order effective field theory approach. Furthermore, we provide an explicit example of a space-time that is classically regular but contains a singularity at first order.

This paper is organized as follows: in the next section we introduce the general form for the metric considered in this paper; in section 12.3 we discuss some properties of this metric, related to the pressure and density of its matter source. These properties can be used to put further constraints on the metric; section 12.4 discusses the leading quantum corrections to this metric and derives conditions for which a classical regular metric remains regular, if the leading quantum corrections are taken into account; in section 12.5 we conclude and appendix 12.A discusses the Bardeen metric as an explicit example.

12.2 A general metric

We consider a Lorentzian $(3+1)$-dimensional space-time containing a regular fluid. We choose the origin of our local coordinate system at a local maximum in the density of the fluid and assume the space-time to be locally spherically symmetric and time-independent around the origin. The line element can then be written as [1]

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{g(r)} + r^2\left(d\theta^2 + \sin\theta^2\,d\phi^2\right), \tag{12.1}$$

in which we employ the usual areal radial coordinate r. Consistently with the regular matter source, we assume the space-time to be smooth and regular everywhere, which we define in this paper by $|R|$, $|R_{\mu\nu}\,R^{\mu\nu}|$, $|R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}| < \infty$. [2] If we impose these conditions, we can expand f and g around $r = 0$ in the following way:

$$f(r) = 1 + \sum_{n=0}^{\infty} a_{2n}\,r^{2n} + f_\infty(r), \tag{12.2}$$

$$g(r) = 1 + \sum_{n=1}^{\infty} b_{2n}\,r^{2n} + g_\infty(r), \tag{12.3}$$

[1]Since we assume only local spherical symmetry and time-independence, this line element needs only be valid in a small region around $r = 0$.

[2]Of course, $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, $R_{\mu\nu}$ the Ricci tensor and R the Ricci scalar. The above conditions immediately imply for the Weyl tensor $|C_{\mu\nu\rho\sigma}\,C^{\mu\nu\rho\sigma}| < \infty$.

where the Lorentzian signature requires $a_0 > -1$. Furthermore, the non-analytic parts f_∞ and g_∞ have the property

$$\lim_{r\to 0}\frac{f_\infty(r)}{r^n} = \lim_{r\to 0}\frac{g_\infty(r)}{r^n} = 0, \qquad \forall\, n \in \mathbb{N}. \tag{12.4}$$

and

$$\exists\, \epsilon > 0 \quad \text{s.t.} \quad f_\infty(r) = f_\infty(-r) \quad \text{and} \quad g_\infty(r) = g_\infty(-r), \quad \forall\, r \in [0,\epsilon). \tag{12.5}$$

Since we only want to perform a local analysis, we can truncate the series so that

$$f(r) = 1 + a_0 + a_m\, r^m + \mathcal{O}\left(r^{m+2}\right), \tag{12.6}$$

$$g(r) = 1 + b_n\, r^n + \mathcal{O}\left(r^{n+2}\right), \tag{12.7}$$

where $m, n \geq 2$, $a_0 > -1$, $a_m, b_n \neq 0$. Furthermore, we say that $m = \infty$, $n = \infty$, if $f(r) = 1 + a_0 + f_\infty(r)$, $g(r) = 1 + g_\infty(r)$ respectively.

12.3 Energy conditions

For the line element introduced in the previous section, the regular energy density, radial and transversal pressure generating the space-time metric (12.1) with f and g given in Eqs. (12.2) and (12.3), respectively, can be calculated from the Einstein tensor and read

$$\rho = -\frac{1}{8\,\pi\,G_{\rm N}}\, b_n\,(n+1)\, r^{n-2} + \mathcal{O}(r^{n-1}), \tag{12.8}$$

$$p_\parallel = \frac{1}{8\,\pi\,G_{\rm N}}\left(\frac{a_m}{1+a_0}\, m\, r^{m-2} + b_n\, r^{n-2}\right) + \mathcal{O}(r^{m-1}, r^{n-1}), \tag{12.9}$$

$$p_\perp = \frac{1}{8\,\pi\,G_{\rm N}}\left(\frac{a_m}{1+a_0}\,\frac{m^2}{2}\, r^{m-2} + b_n\,\frac{n}{2}\, r^{n-2}\right) + \mathcal{O}(r^{m-1}, r^{n-1}). \tag{12.10}$$

A positive energy density thus requires $b_n < 0$. Moreover, a non-zero energy density and pressure at $r = 0$ requires $n = 2$. Furthermore, $|p_\perp| \geq |p_\parallel|$, and a necessary but not sufficient requirement for equality (hence isotropy) is that one of the following is true:

- $(m = 2 \vee m = \infty)\ \wedge\ (n = 2 \vee n = \infty)$,
- $m = n\ \wedge\ a_m\, m + b_n\,(1 + a_0) = 0$.

We can write,

$$p_\parallel = -\frac{\rho}{n+1}\left(1 + \frac{a_m\, m}{b_n\,(1+a_0)}\, r^{m-n}\right) + \mathcal{O}(r^{m-n-1}, r^{-1}) \tag{12.11}$$

$$\equiv w\,\rho. \tag{12.12}$$

Depending on the parameter of the model one can find various values for w. It follows that a de Sitter core with $w=-1$ requires

$$m = n \;\wedge\; \frac{a_m}{b_n\,(1+a_0)} = 1, \tag{12.13}$$

a non-relativistic matter (dust) core with $w=0$ requires

$$m = n \;\wedge\; \frac{a_m}{b_n\,(1+a_0)} = -\frac{1}{n}, \tag{12.14}$$

and an ultra-relativistic (radiation) core with $w=\frac{1}{3}$ requires

$$m = n \;\wedge\; \frac{a_m}{b_n\,(1+a_0)} = -\frac{n+4}{3\,n}. \tag{12.15}$$

Finally an asymptotically Minkowski core with $\rho=0$ and $|w|<\infty$ requires $m\geq n>2$.

12.4 Quantum corrections to the metric

We shall here use the same approach as discussed in Refs. [77, 97], for which we review the main steps. [3] The effective action is given by

$$\Gamma = \Gamma_{\mathrm{L}}[g] - \Gamma_{\mathrm{NL}}[g] + S_{\mathrm{M}} + \mathcal{O}(G_{\mathrm{N}}) \tag{12.16}$$

with S_{M} the matter action and

$$\Gamma_{\mathrm{L}}[g] = \int d^4x\,\sqrt{g}\left[\frac{R}{16\,\pi\,G_{\mathrm{N}}} + \tilde{c}_1(\mu)\,R^2 + \tilde{c}_2(\mu)\,R_{\mu\nu}\,R^{\mu\nu}\right], \tag{12.17}$$

$$\Gamma_{\mathrm{NL}}[g] = \int d^4x\,\sqrt{g}\left[\tilde{\alpha}\,R\ln\!\left(-\frac{\Box}{\mu^2}\right)R + \tilde{\beta}\,R_{\mu\nu}\ln\left(-\frac{\Box}{\mu^2}\right)R^{\mu\nu}\right], \tag{12.18}$$

where $\tilde{c}_i(\mu)$ are renormalization scale dependent coefficients that follow from matching with an ultraviolet complete theory and experiment. Furthermore, $\tilde{\alpha}=\alpha-\gamma$, $\tilde{\beta}=\beta+4\,\gamma$ with the values of α, β, γ given in Table 12.1. The equation of motion for the metric can then be solved perturbatively in $\ell_{\mathrm{p}}^2=\hbar\,G_{\mathrm{N}}$ (and we will set $\hbar=1$ when no ambiguity can arise). The zeroth order equation is the Einstein equation

$$G_{\mu\nu} = 8\,\pi\,G_{\mathrm{N}}\,T_{\mu\nu}, \tag{12.19}$$

and the first order equation is given by

$$G^{\mathrm{L}}_{\mu\nu} + 16\,\pi\,G_{\mathrm{N}}\left(H^{\mathrm{L}}_{\mu\nu} - H^{\mathrm{NL}}_{\mu\nu}\right) = 0, \tag{12.20}$$

[3] Note that the signature conventions in this work differ from those in [77, 97].

	α	β	γ
Scalar	$5(6\xi-1)^2$	-2	2
Fermion	-5	8	7
Vector	-50	176	-26
Graviton	250	-244	424

Table 12.1: Non-local Wilson coefficients for different fields. All numbers should be divided by $11520\pi^2$. Furthermore, ξ denotes the value of the non-minimal coupling for a scalar theory. The values for the scalar, fermion and vector field have been calculated in Refs. [48, 132]. The values for the graviton can be gauge dependent due to the graviton self-interaction diagrams [219]. However, it is possible to define a unique effective action with gauge independent coefficients leading to the gauge independent results quoted here [34, 35, 341].

where

$$\begin{aligned}
G^{\mathrm{L}}_{\mu\nu} &= -\frac{1}{2}\Big(\Box g^{\mathrm{q}}_{\mu\nu} - g_{\mu\nu}\Box g^{\mathrm{q}} + \nabla_\mu\nabla_\nu g^{\mathrm{q}} + 2R^{\alpha\ \beta}_{\ \mu\ \nu}g^{\mathrm{q}}_{\alpha\beta} \\
&\qquad -\nabla_\mu\nabla^\beta g^{\mathrm{q}}_{\nu\beta} - \nabla_\nu\nabla^\beta g^{\mathrm{q}}_{\mu\beta} + g_{\mu\nu}\nabla^\alpha\nabla^\beta g^{\mathrm{q}}_{\alpha\beta}\Big)\,, \qquad (12.21)\\
H^{\mathrm{L}}_{\mu\nu} &= 2\,\tilde{c}_1\left(R\,R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^2 + g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R\right) \\
&\quad + \tilde{c}_2\left(2R^{\alpha}_{\ \mu}R_{\nu\alpha} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \Box R_{\mu\nu}\right. \\
&\qquad \left. +\frac{1}{2}g_{\mu\nu}\Box R - \nabla_\alpha\nabla_\mu R^{\alpha}_{\ \nu} - \nabla_\alpha\nabla_\nu R^{\alpha}_{\ \mu}\right)\,, \qquad (12.22)\\
H^{\mathrm{NL}}_{\mu\nu} &= 2\tilde{\alpha}\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\ln\left(-\frac{\Box}{\mu^2}\right)R \\
&\quad + \tilde{\beta}\left(\delta^\alpha_\mu\,R_{\nu\beta} + \delta^\alpha_\nu\,R_{\mu\beta} - \frac{1}{2}g_{\mu\nu}R^{\alpha}_{\ \beta} + \delta^\alpha_\mu g_{\nu\beta}\Box\right. \\
&\qquad \left. + g_{\mu\nu}\nabla^\alpha\nabla_\beta - \delta^\alpha_\mu\nabla_\beta\nabla_\nu - \delta^\alpha_\nu\nabla_\beta\nabla_\mu\right)\ln\left(-\frac{\Box}{\mu^2}\right)R^{\beta}_{\ \alpha}. \qquad (12.23)
\end{aligned}$$

Equation (12.20) can now be solved for the leading quantum corrections to the metric such that

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \hbar\, G_{\mathrm{N}}\,\delta g_{\mu\nu}, \qquad (12.24)$$

where $\tilde{g}_{\mu\nu}$ is the quantum corrected metric, $g_{\mu\nu}$ is the solution of Eq. (12.19) and $\delta g_{\mu\nu} \equiv g^q_{\mu\nu}$.

12.4.1 Local corrections

The leading local quantum corrections are found to be

$$\begin{aligned}\delta g^{\mathrm{L}}_{tt} =& -32\,\pi\,G_{\mathrm{N}}\left[\tilde{c}_1\,a_m\,m\,(m+1)\,r^{m-2}+(2\tilde{c}_1+\tilde{c}_2)\,b_n\,(1+a_0)\,(n+1)\,r^{n-2}\right]\\ &+\mathcal{O}\left(r^{m-1},r^{n-1}\right),\end{aligned} \tag{12.25}$$

$$\begin{aligned}\delta g^{\mathrm{L}}_{rr} =& -16\,\pi\,G_{\mathrm{N}}\left[(2\tilde{c}_1+\tilde{c}_2)\,\frac{a_m}{1+a_0}\,m\,(m^2-m-2)\,r^{m-2}\right.\\ &\left.+(4\tilde{c}_1+\tilde{c}_2)\,b_n\,(n^2-n-2)\,r^{n-2}\right]+\mathcal{O}\left(r^{m-1},r^{n-1}\right).\end{aligned} \tag{12.26}$$

Hence, the leading local corrections are of the order r^{m-2} and r^{n-2}. For $m \leq 3$ or $n \leq 3$ these corrections would make the space-time singular at $r = 0$. Interestingly for the special cases $m, n = 2$ we find an exact cancellation keeping the space-time regular. Moreover, due to the assumptions of local spherical symmetry, regularity and smoothness of the classical space-time, we imposed the conditions $m, n \geq 2$ and m, n even from the onset. Therefore the local corrections do not pose any threats to the regularity and smoothness of the space-time at this order.

12.4.2 Non-local corrections

Corrections due to the non-local terms in Eq. (12.23) are more difficult to calculate, if the line element is only known locally, since the $\ln\Box$ is an infinite derivative operator. However, for a smooth time-independent and spherically symmetric function f, one can derive an expansion in a flat background given by [4] [5]

$$\ln\left(-\frac{\Box}{\mu^2}\right)\,f(x)=\sum_{k=0}^{\infty}c_{2k}\,r^{2k} \tag{12.27}$$

with

$$c_0=-2\lim_{\epsilon\to 0}\left\{\left[\gamma_{\mathrm{E}}-1+\ln(\mu\,\epsilon)\right]f(0)+\int_{\epsilon}^{\infty}\frac{f(r)}{r}\,dr\right\}. \tag{12.28}$$

Therefore, we find

$$\ln\left(-\frac{\Box}{\mu^2}\right)R=d_0+d_2\,r^2+\mathcal{O}\left(r^4\right), \tag{12.29}$$

$$\ln\left(-\frac{\Box}{\mu^2}\right)R^t_{\ t}=x_0+x_2\,r^2+\mathcal{O}\left(r^4\right), \tag{12.30}$$

$$\ln\left(-\frac{\Box}{\mu^2}\right)R^r_{\ r}=y_0+y_2\,r^2+\mathcal{O}\left(r^4\right), \tag{12.31}$$

$$\ln\left(-\frac{\Box}{\mu^2}\right)R^\theta_{\ \theta}=z_0+z_2\,r^2+\mathcal{O}\left(r^4\right). \tag{12.32}$$

[4] Since we assume a regular geometry, $\ln\Box$ can be expanded as a power series in G_{N}. The leading term in such an expansion is determined by the flat space kernel of $\ln\Box$. Corrections due to curvature are subleading.

[5] Calculation of such expressions has a long history in both the mathematics and physics literature. For the calculation of this particular expression we follow the same steps as presented in appendix A of Ref. [77]. Similar calculations have been performed in e.g. Refs. [119, 143].

These expressions lead to the corrections

$$\delta g_{tt}^{\mathrm{NL}} = -16\,\pi\,G_{\mathrm{N}}\,(1+a_0)\left[2\,\tilde{\alpha}\,d_2 - \tilde{\beta}\,\left(x_2 - y_2 - 2\,z_2\right)\right]\,r^2 + \mathcal{O}\left(r^4\right), \tag{12.33}$$

$$\delta g_{rr}^{\mathrm{NL}} = -32\,\pi\,G_{\mathrm{N}}\,\left\{\tilde{\beta}\,(y_0 - z_0) + \left[2\,\tilde{\alpha}\,d_2 + \tilde{\beta}\,(x_2 + 2\,y_2 - z_2)\right]\,r^2\right\} + \mathcal{O}\left(r^4\right). \tag{12.34}$$

Hence, if singularities are to be avoided, it is necessary to impose

$$y_0 = z_0. \tag{12.35}$$

Interestingly, this condition can be translated into the following condition for the pressure anisotropy: [6]

$$\lim_{r\to 0} \ln\left(-\frac{\Box}{\mu^2}\right)\left(p_\perp - p_\parallel\right) = 0. \tag{12.36}$$

If the time-independence and spherical symmetry holds globally, this is equivalent to

$$\int_0^\infty dr\,\frac{p_\perp(r) - p_\parallel(r)}{r} = 0. \tag{12.37}$$

Moreover, if $p_\parallel$ is differentiable, conservation of the energy momentum tensor, $\nabla^\mu T_{\mu\nu} = 0$, implies that the above condition is equivalent to [7]

$$p_\parallel(0) = \int_0^\infty dr\,\frac{f'(r)}{2\,f(r)}\,\left[\rho(r) + p_\parallel(r)\right]. \tag{12.38}$$

These identities are clearly satisfied for isotropic fluids, but cause trouble for many anisotropic fluids. For instance, it can easily be verified that the condition is not satisfied for the Bardeen [28] (see Appendix 12.A), Hayward [194] and Frolov [160] metrics, but it is satisfied for the Simpson-Visser metric [316] and for a constant density star (cf. Ref. [77] for the explicit calculation.).

12.5 Discussion

We have shown that perturbative quantum gravity can introduce singularities to geometries that are classically regular. Furthermore, for a locally spherically symmetric and time-independent space-time, we have found a condition, given in Eq. (12.36), on the pressure anisotropy for which this scenario occurs. Matter distributions that violate the condition contain a singularity at this order in perturbation theory.

It should be stressed that the employed methods break down at and close to the emerging singularity, as the truncation of the effective field theory becomes invalid. Therefore it cannot be concluded that the singularity is physical. In fact it is not unlikely that the newly found singularity is a spurious effect, and that it disappears, when the full expansion of the effective action is taken into account and/or when the non-local terms are evaluated using the complete curvature expansion. Nevertheless such an effect is interesting, as it

[6]Notice that only smoothness, along with local time-independence and spherical symmetry are required.
[7]We assume $p_\parallel(\infty) = 0$.

points out that for certain geometries naive application of perturbation theory fails when calculating the quantum corrections at first order, even when the classical energy density is close to the vacuum. On the other hand, one cannot exclude the possibility that the singularity will persist, when the effective action is considered up to infinite order.

It would be interesting to investigate the higher order corrections within this framework. As the higher order corrections will come with different power of the Planck length, it is not expected that the terms at 2-loop or at any finite order will cancel the new singularities. [8] However, such a higher order analysis could generate new singular terms and thus provide new constraints such as the one in Eq. (12.36), for which classical matter distributions are safe in the sense that quantum corrections do not generate secular terms. Moreover, a higher order analysis could provide indication whether resummation effects can occur, which would indicate that the singularities are spurious, as they would be resolved at infinite order in perturbation theory.

We will leave such an higher order analysis for future research. Let us note however that higher order terms in the effective action are potentially more dangerous than the first order terms analyzed in this paper. Indeed, from dimensional analysis, one expects that local terms in the effective action at order ℓ_{p}^{2k} generate corrections to the metric at order $\min\{r^{m-2k}, r^{n-2k}\}$. This was indeed found in Eqs. (12.25) and (12.26). For $2k \geq \min\{m-2, n-2\}$, this generates terms that make the metric singular. For $k = 1$ we found that these dangerous terms are not present, as their coefficients vanish. It is expected that the dangerous terms also vanish for $k > 1$. If such a cancellation mechanism were not present, this would pose new challenges for the use of perturbative methods in quantum gravity. Note that a priori there is no reason to expect that non-perturbative quantum gravity should be invoked, as the classical densities and pressures we consider here are well below the Planck scale. For the same reason, we do not expect that the singularity we found can be removed by any matter rearrangements which keep density and pressure in the sub-Planckian range.

Furthermore, following the same reasoning, we find that, if $2 < m, n < \infty$, the quantum corrections will always generate corrections with smaller powers of r. In particular, the non-local terms are expected to generate corrections at order r^2. Therefore, a byproduct of our analysis is that a regular and smooth quantum space-time, that is locally spherically symmetric and time-independent should always have the form of Eq. (12.1) with f and g given by Eqs. (12.2) and (12.3), with the extra assumption that all coefficients a_{2n} and b_{2n} are non-zero unless some kind of fine-tuning occurs. In addition, $a_0 > -1$, and using the analysis in section 12.3, one can impose $b_2 < 0$, and $|a_2| \leq (1 + a_0)|b_2|$.

Finally, one could try to generalize these results to space-times that do not have local spherical symmetry or are time-dependent, as it could lead to similar conditions on the expansion of the metric components. We will leave this for a future paper.

[8] We might mention in passing the famous result by Goroff and Sagnotti that 2-loop corrections diverge for pure gravity [179].

Acknowledgments

The work of X.C. is supported in part by the Science and Technology Facilities Council (grant number ST/P000819/1). R.C. is partially supported by the INFN grant FLAG and his work has also been carried out in the framework of activities of the National Group of Mathematical Physics (GNFM, INdAM) and COST action *Cantata*. The work of F.K. is supported by a doctoral studentship of the Science and Technology Facilities Council. F.K. is grateful for the hospitality of the Università di Bologna, where most of this work was carried out.

12.A The Bardeen metric

Let us consider the Bardeen metric [28] as an example of regular space-time where one encounters singularities of the type discussed in this work. The Bardeen space-time has no central singularity but a de Sitter core (see Eq. (12.13)), and can be motivated by coupling Einstein gravity to a non-linear electrodynamic field [22]. The Bardeen line element is of the form in Eq. (12.1) with

$$f(r) = g(r) = 1 - \frac{2\,G_{\rm N}\,M\,r^2}{(r^2+l^2)^{3/2}}, \tag{12.39}$$

where $l > 0$ is some length scale. For sufficiently small values of l, this space-time contains a horizon and, in fact, it reduces to the Schwarzschild geometry in the limit $l \to 0$ and to the Minkowski space-time in the limit $l \to \infty$.

Using the procedure outlined in section 12.4, one can calculate the metric corrections. Up to $\mathcal{O}(G_{\rm N}^3)$ the local corrections are given by

$$\delta g_{tt}^{\rm L} = -\frac{192\,\pi\,\hbar\,G_{\rm N}^2\,M\,l^2}{(r^2+l^2)^{7/2}}\left[\tilde{c}_1(\mu)\left(r^2-4\,l^2\right)-\tilde{c}_2(\mu)\left(r^2+l^2\right)\right], \tag{12.40}$$

$$\delta g_{rr}^{\rm L} = \frac{480\,\pi\,\hbar\,G_{\rm N}^2\,M\,l^2\,r^2}{(r^2+l^2)^{9/2}}\left[2\,\tilde{c}_1(\mu)\left(r^2-6\,l^2\right)+\tilde{c}_2(\mu)\left(2\,r^2-5\,l^2\right)\right], \tag{12.41}$$

and the non-local corrections are given by

$$\begin{aligned}\delta g_{tt}^{\rm NL} = \frac{128\,\pi\,\hbar\,G_{\rm N}^2\,M}{(r^2+l^2)^{7/2}}&\left\{\tilde{\alpha}\left[r^4+16\,l^2\,r^2-31\,l^4-3\,l^2\left(r^2-4\,l^2\right)\left(\gamma_{\rm E}+\ln\left[\frac{2\,\mu\,(r^2+l^2)}{l}\right]\right)\right]\right.\\ &\left.+\tilde{\beta}\left[r^4-6\,l^2\,r^2-7\,l^4+3\,l^2\left(r^2+l^2\right)\left(\gamma_{\rm E}+\ln\left[\frac{2\,\mu\,(r^2+l^2)}{l}\right]\right)\right]\right\},\end{aligned} \tag{12.42}$$

$$\begin{aligned}\delta g_{rr}^{\rm NL} = -\frac{64\,\pi\,\hbar\,G_{\rm N}^2\,M}{(r^2+l^2)^{9/2}}&\left\{2\,\tilde{\alpha}\,r^2\left[3\,r^4+82\,l^2\,r^2-273\,l^4-15\,l^2\left(r^2-6\,l^2\right)\left(\gamma_{\rm E}+\ln\left[\frac{2\,\mu\,(r^2+l^2)}{l}\right]\right)\right]\right.\\ &\left.+\tilde{\beta}\,l^2\left[125\,r^4-224\,l^2\,r^2+3\,l^4-15\,r^2\left(2\,r^2-5\,l^2\right)\left(\gamma_{\rm E}+\ln\left[\frac{2\,\mu\,(r^2+l^2)}{l}\right]\right)\right]\right\},\end{aligned} \tag{12.43}$$

where $\gamma_{\rm E}$ is Euler constant. Using these expressions, one can calculate the quantum corrected Ricci scalar, which we split into the classical Ricci scalar and a quantum correction as

$$R = R^{\rm c} + R^{\rm q}_{\rm fin} + R^{\rm q}_{\rm div}. \tag{12.44}$$

The classical part $R^{\rm c}$ is finite everywhere, while the quantum correction contains both a finite contribution $R^{\rm q}_{\rm fin}$ and a contribution

$$R^{\rm q}_{\rm div} = -\frac{384\,\pi\,\tilde{\beta}\,\hbar\,G_{\rm N}^2\,M\,l^8}{r^2\,(r^2+l^2)^{11/2}} \tag{12.45}$$

which diverges for $r \to 0$. This divergence cannot be canceled within perturbation theory, since corrections only appear at $\mathcal{O}(G_{\rm N}^3)$. Resolution of this singularity can thus only be achieved in a non-perturbative way. Notice, however, that this singularity is integrable in the sense that radial geodesics can be extended through the singularity. Furthermore, the expansion in $G_{\rm N}$ consists of both a classical expansion in $G_{\rm N}M/l$ and a quantum expansion in $\hbar\,G_{\rm N}/l^2$. To keep the classical expansion under control we will thus assume $l > G_{\rm N}M$. For this choice the space-time does not contain a horizon, implying that the singularity is naked.

Chapter 13

Quantum Gravitational Corrections to the Entropy of a Schwarzschild Black Hole

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Abstract
We calculate quantum gravitational corrections to the entropy of black holes using the Wald entropy formula within an effective field theory approach to quantum gravity. The corrections to the entropy are calculated to second order in curvature and we calculate a subset of those at third order. We show that, at third order in curvature, interesting issues appear that had not been considered previously in the literature. The fact that the Schwarzschild metric receives corrections at this order in the curvature expansion has important implications for the entropy calculation. Indeed, the horizon radius and the temperature receive corrections. These corrections need to be carefully considered when calculating the Wald entropy.

This chapter has been published in Physical Review D **104**, no. 06, p. 6012 (2021).
A preprint of the chapter can be found at arXiv:2108.06824 [hep-th].

Black holes are fascinating objects for many different reasons. Hawking's groundbreaking intuition that black holes are not black but have a radiation spectrum that is very similar to that of a black body makes black holes an ideal laboratory to investigate the interplay between quantum mechanics, gravity and thermodynamics. This has led to the notion of Bekenstein-Hawking entropy or black hole entropy which has attracted much attention over the last almost 50 years. The calculation of quantum corrections to this entropy has been the subject of many publications, see e.g. [319, 345] for reviews.

In this work we revisit the calculation of the entropy of a Schwarzschild black hole in quantum gravity and identify new important subtleties that have been overlooked in previous calculations. To be very specific, we use effective field theoretical methods to calculate quantum gravitational corrections to the entropy of this black hole using the Wald entropy formula [343]. We highlight new intriguing relations between the quantum corrections to the entropy, the Euler characteristic and quantum corrections to the metric of the Schwarzschild black hole. Previous calculations within the effective theory approach to quantum gravity [163, 255, 256] have used the Euclidean path integral formulation of the entropy. We present a systematic approach that can easily be extended to any order in perturbation theory or to any black hole metric.

The Wald approach to the calculation of a black hole entropy is very elegant and does not involve the Wick rotation to Euclidean time which is known to be tricky in quantum gravity. The Wald entropy formula reads [343]

$$S_{Wald} = -2\pi \int d\Sigma \; \epsilon_{\mu\nu}\epsilon_{\rho\sigma} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\Big|_{r=r_H}, \tag{13.1}$$

where $d\Sigma = r^2 \sin\theta d\theta d\phi$, L is the Lagrangian of the model, $R^{\mu\nu\rho\sigma}$ is the Riemann tensor and r_H is the horizon radius. Furthermore, $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$, $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. The integral is over the perimeter of the horizon of the black hole and we thus need to determine the location of the horizon with radius r_H. This is our first observation: to calculate the entropy of the black hole, we do not only need the Lagrangian of the gravitational action, but we also need to verify whether the metric receives quantum corrections as these could impact the position of the horizon. This important point had simply been overlooked in previous calculations for Schwarzschild black holes.

As explained before, we are using the effective action to quantum gravity [34–37, 61, 140, 351]. At second order in curvature, one has

$$S_{\text{EFT}} = \int \sqrt{|g|} d^4x \left(\frac{R}{16\pi G_N} + c_1(\mu)R^2 + c_2(\mu)R_{\mu\nu}R^{\mu\nu} + c_3(\mu)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \mathcal{L}_m \right) , \tag{13.2}$$

for the local part of the action and the nonlocal part is given by

$$\Gamma^{(2)}_{\text{NL}} = -\int \sqrt{|g|} d^4x \left[\alpha R \ln\left(\frac{\Box}{\mu^2}\right) R + \beta R_{\mu\nu} \ln\left(\frac{\Box}{\mu^2}\right) R^{\mu\nu} + \gamma R_{\mu\nu\alpha\beta} \ln\left(\frac{\Box}{\mu^2}\right) R^{\mu\nu\alpha\beta} \right], \tag{13.3}$$

where $\Box := g^{\mu\nu}\nabla_\mu\nabla_\nu$.

It is straightforward to show [70, 97] that there are no corrections up to second order in curvature to the Schwarzschild metric using the non-local Gauss-Bonnet identity [70]

$$\begin{aligned}\int \sqrt{|g|}d^4x R_{\mu\nu\alpha\beta}\left(c_3(\mu)-\gamma\ln\left(\frac{\Box}{\mu^2}\right)\right)R^{\mu\nu\alpha\beta} &= +4\int \sqrt{|g|}d^4x R_{\mu\nu}\left(c_3(\mu)-\gamma\ln\left(\frac{\Box}{\mu^2}\right)\right)R^{\mu\nu} \\ &\quad -\int \sqrt{|g|}d^4x R\left(c_3(\mu)-\gamma\ln\left(\frac{\Box}{\mu^2}\right)\right)R \\ &\quad +\mathcal{O}(R^3)+\text{boundary terms.}\end{aligned} \tag{13.4}$$

This identity can be proven using [144, 226]

$$\log\frac{\Box}{\mu^2} = \int_0^\infty ds \frac{e^{-s}-e^{-s\frac{\Box}{\mu^2}}}{s} \tag{13.5}$$

and [37]

$$\begin{aligned}\Box R^{\alpha\beta\mu\nu} &= \nabla^\mu\nabla^\alpha R^{\nu\beta} - \nabla^\nu\nabla^\alpha R^{\mu\beta} - \nabla^\mu\nabla^\beta R^{\nu\alpha} + +\nabla^\nu\nabla^\beta R^{\mu\alpha} \\ &\quad -4R^{\alpha\ \ [\mu}_{\ \sigma\ \ \lambda}R^{\beta\sigma\nu]\lambda} + 2R^{[\mu}_{\ \ \lambda}R^{\alpha\beta\lambda\nu]} - R^{\alpha\beta}_{\ \ \ \sigma\lambda}R^{\mu\nu\sigma\lambda},\end{aligned} \tag{13.6}$$

which follows from the Bianchi identity. One obtains [20, 133, 226, 334]

$$R_{\alpha\beta\mu\nu}\Box R^{\alpha\beta\mu\nu} = 4R_{\alpha\beta\mu\nu}\nabla^\alpha\nabla^\mu R^{\beta\nu} + \mathcal{O}(R^3). \tag{13.7}$$

It is straightforward to generalize this result to higher power of the Laplacian. Inserting this relation into the Lagrangian and using partial integrations and the contracted Bianchi identity, we obtain the non-local Gauss-Bonnet identity. As the Riemann tensor can be eliminated from the dynamical part of the action at second order in curvature, we find that there are no corrections to the field equations at this order for vacuum solutions of general relativity [70].

As there are no corrections to the metric, the horizon radius is unchanged and we can calculate the Wald entropy at second order in a straightforward manner using (13.2) and (13.3)[1]

$$S^{(2)}_{Wald} = \frac{A}{4G_N} + 64\pi^2 c_3(\mu) + 64\pi^2\gamma\left(\log\left(4G_N^2M^2\mu^2\right) - 2 + 2\gamma_E\right) \tag{13.8}$$

where $A = 16\pi(G_N M)^2$ is the area of the black hole. A similar answer was obtained using the Euclidean path integral formulation. Note that the entropy is renormalization group invariant and finite. As there are no corrections to the metric, the temperature remains unchanged and the classical relation $TdS = dM$ receives a quantum correction. Indeed we find $TdS = (1+\gamma 16\pi/(G_N M^2))dM$.

A possible interpretation of this result is that the nonlocal quantum effects generate a

[1] Note that we need to use this basis for the calculation of the entropy, as we have not calculated the boundary term generated by Gauss-Bonnet identity explicitly.

pressure for the black hole. The first law of thermodynamics is then given by

$$TdS - PdV = \left(1 + \gamma\frac{16\pi}{G_N M^2}\right) dM = dM + \gamma\frac{16\pi}{G_N M^2} dM, \tag{13.9}$$

where P is the pressure of the black hole. Its volume is given by $V = 4/3\pi r_H^3$, where $r_H = 2G_N M$ is the horizon radius. We can then identify $TdS = dM$ and $\gamma 16\pi/(G_N M^2)dM = -PdV$ with $dV = 32\pi G_N^3 M^2 dM$. We thus obtain

$$P = -\gamma\frac{1}{2G_N^4 M^4}, \tag{13.10}$$

which can be negative as γ is positive for spin 0, 1/2 and 2 fields or positive as γ is negative for spin 1 fields. Indeed, one finds $\gamma_0 = 2/(11520\pi^2)$ [132], $\gamma_{1/2} = 7/(11520\pi^2)$ [132], $\gamma_1 = -26/(11520\pi^2)$ [132] and $\gamma_2 = 424/(11520\pi^2)$ [34]. We note that Dolan had discussed the possibility that black holes would have a pressure [138] in the context of gravitational models with a cosmological constant. It is remarkable that quantum gravity leads to a pressure for Schwarzschild black holes. Note that this is the main difference with previous results [163,255,256] who did not study quantum corrections to the metric. Because there is no dynamical correction to the metric at this order in curvature, the interpretation of the correction to the entropy as a pressure term is forced upon us.

At third order in curvature, we need to add the following operators to the effective action

$$\mathcal{L}^{(3)} = c_6 G_N R^{\mu\nu}{}_{\alpha\sigma} R^{\alpha\sigma}{}_{\delta\gamma} R^{\delta\gamma}{}_{\mu\nu}\ , \tag{13.11}$$

where c_6 is dimensionless. As pointed out by Goroff and Sagnotti [180], there is only one invariant involving only Riemann tensors in vacuum, as $R_{\alpha\beta\gamma\delta} R^{\alpha}{}_{\epsilon}{}^{\gamma}{}_{\zeta} R^{\beta\epsilon\delta\zeta}$ can be rewritten in terms of $R^{\mu\nu}{}_{\alpha\sigma} R^{\alpha\sigma}{}_{\delta\gamma} R^{\delta\gamma}{}_{\mu\nu}$ and terms involving the Ricci scalar or Ricci tensors which both vanish in vacuum. There is a corresponding non-local operator $R^{\mu\nu}{}_{\alpha\sigma} \log\Box R^{\alpha\sigma}{}_{\delta\gamma} R^{\delta\gamma}{}_{\mu\nu}$. While the Wilson coefficient is known in a specific gauge [180], it is not known for the unique effective action and we will thus neglect this term.

The dimension six local operator leads to a correction to the metric. We find

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\Omega^2 \tag{13.12}$$

with

$$\begin{aligned} d\Omega^2 &= d\theta^2 + \sin(\theta)^2 d\phi^2, && (13.13)\\ f(r) &= 1 - \frac{2G_N M}{r} + 640\pi c_6 \frac{G_N^5 M^3}{r^7}, && (13.14)\\ g(r) &= 1 - \frac{2G_N M}{r} + 128\pi c_6 \frac{G_N^4 M^2}{r^6}\left(27 - 49\frac{G_N M}{r}\right). && (13.15) \end{aligned}$$

The corrections to the metric implies a shift of the horizon radius

$$r_H = 2G_N M\left(1 - c_6\frac{5\pi}{G_N^2 M^4}\right). \tag{13.16}$$

Clearly for astrophysical black holes the correction to the classical Schwarzschild radius goes to zero very quickly but it can be an order one correction for quantum black holes with masses of the order of the Planck scale.

The $\epsilon_{\mu\nu}$ tensors also need to be redefined. We have

$$\epsilon_{\mu\nu} = \begin{cases} \sqrt{f(r)/g(r)} & \text{if} \quad (\mu,\nu)=(t,r), \\ -\sqrt{f(r)/g(r)} & \text{if} \quad (\mu,\nu)=(r,t), \\ 0 & \text{otherwise.} \end{cases} \tag{13.17}$$

One can easily verify that $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$, $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$, and $\epsilon_{\mu\nu} = 0$, if $\mu, \nu \neq t, r$.

At third order in curvature, we thus obtain the following correction to the entropy:

$$S^{(3)}_{Wald} \;=\; S^{(2)}_{Wald} + 128\pi^3 c_6 \frac{G_N}{A_{tot}} \;, \tag{13.18}$$

where we neglect third order non-local terms which would compensate for the scale dependence of c_6 [2]. Note that while the dimension six operator has been considered before [320], our result differs from that paper as the metric corrections were not taken into account in that work.

With corrections to the metric that deviate from the Schwarzschild solution, one may wonder whether the Euler characteristic given by

$$\chi = \frac{1}{32\pi^2}\int_0^{1/T} dt_E \int_{r_H}^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sqrt{|g|}\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \tag{13.19}$$

remains 2 for black holes. It is however easy to see that this is the case, because there is also a correction to the temperature which is given by

$$T = \frac{\sqrt{f'(r_H)g'(r_H)}}{4\pi} = \frac{1}{8\pi G_N M}\left[1 + 2\pi c_6\left(\frac{1}{G_N^2 M^4}\right)\right]. \tag{13.20}$$

With this in mind, it is easy to verify that $\chi = 2$ is fulfilled, which is required for our results to be consistent. One can also easily verify that the thermodynamic law $TdS = dM$ holds at order of $\mathcal{O}(c_6)$ with the modified temperature and entropy. The non-local correction to the action at third order in curvature would lead to a contribution to the pressure which is much smaller than the seconder order correction obtained in eq. (13.10). A back of the

[2] We can estimate the magnitude of the non-local correction of the entropy (albeit in the de Donder gauge, the actual calculation in the unique effective action would be much more involved) using the result in [180] for the two-loop divergences of Einstein gravity $\Gamma_\infty = \frac{209}{2880(4\pi)^4}\frac{1}{\epsilon}\int d^4x\sqrt{-g}R^{\mu\nu}{}_{\alpha\sigma}R^{\alpha\sigma}{}_{\delta\gamma}R^{\delta\gamma}{}_{\mu\nu}$. This divergent term fixes the renormalization group equation for c_6 and thus the Wilson coefficient of the term $R^{\mu\nu}{}_{\alpha\sigma}\log\Box R^{\alpha\sigma}{}_{\delta\gamma}R^{\delta\gamma}{}_{\mu\nu}$. For the entropy to be renormalization group invariant at third order in curvature, the non-local correction to the entropy must go as $\frac{209}{2880(4\pi)^4}G_N/A_{tot}\log(4G_N^2M^2\mu^2)$. These corrections are thus very small in comparison to those obtained in eq. (13.8).

envelop calculation shows that, as expected, the third order curvature nonlocal correction to the pressure is suppressed by a factor $(G_N M^2)^{-1}$ in comparison to the leading second order term that we have calculated.

Our work has interesting implications for quantum black holes. The temperature of black holes can be seen as an indicator of how quantum a black hole is. A black hole with a mass of the order of ten times the reduced Planck mass $\bar{M}_P$ would still be a very good approximation and have a temperature close to its classical value

$$T_{QBH} = \frac{1}{8\pi G_N \bar{M}_P}\left[1 + 128\pi^3 c_6 \frac{\bar{M}_P^4}{M_{QBH}^4}\right]. \tag{13.21}$$

Assuming that c_6 is of order unity, we see that the classical temperature receives an order one correction from the third order curvature term in the action for $M_{QBH} = \bar{M}_P$, but these corrections are very tiny for quantum black holes with masses of the order of $M_{QBH} \sim 10\bar{M}_P$. This justifies the geometrical cross-section adopted for quantum black holes in the framework of low scale quantum gravity at colliders [83, 174, 203, 254]. The semi-classical approximation appears to be an excellent one. Describing quantum black holes with the classical Schwarzschild metric is clearly a good approximation as well as long as their masses are larger than $\mathcal{O}(10\bar{M}_P)$.

In this work we have calculated quantum gravitational corrections to the entropy of black holes using the Wald entropy formula within an effective theory approach to quantum gravity at third order in curvature. We first have revisited the calculation of the entropy of black holes at second order in curvature and have found that the quantum gravitational correction to the entropy can be interpreted as a pressure term in the first law of thermodynamics for black holes. This pressure can be positive or negative depending on the field content of the theory. Furthermore, we have shown that at third order in curvature, there are interesting issues that had not been considered previously in the literature. The fact that the Schwarzschild metric receives corrections at this order in the curvature expansion has important implications for the entropy calculation. Indeed, the horizon radius and the temperature receive corrections. These corrections need to be carefully considered when calculating the Wald entropy, knowing the corrections to the Lagrangian is not enough. The reason why previous entropy calculations at second order in curvature match our results is that there are no correction to the Schwarzschild metric at that order. We can actually justify this result with our approach. Finally, our results have interesting consequences for the lightest black holes of Planckian masses [68,75] which are much more classical than naively expected.

Acknowledgments: We would like to thank Yong Xiao for very useful discussions. The work of X.C. is supported in part by the Science and Technology Facilities Council (grants numbers ST/T00102X/1, ST/T006048/1 and ST/S002227/1). The work of F.K. is supported by a doctoral studentship of the Science and Technology Facilities Council.

Chapter 14

Quantum Hair from Gravity

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Abstract

We explore the relationship between the quantum state of a compact matter source and of its asymptotic graviton field. For a matter source in an energy eigenstate, the graviton state is determined at leading order by the energy eigenvalue. Insofar as there are no accidental energy degeneracies there is a one to one map between graviton states on the boundary of spacetime and the matter source states. A typical semiclassical matter source results in an entangled asymptotic graviton state. We exhibit a purely quantum gravitational effect which causes the subleading asymptotic behavior of the graviton state to depend on the internal structure of the source. These observations establish the existence of ubiquitous quantum hair due to gravitational effects.

This chapter has been published in Physical Review Letters **128**, no. 11, p. 1301 (2022). A preprint of this chapter can be found at arXiv:2110.09386 [hep-th].

14.1 Introduction

Classical no-hair theorems limit the information that can be obtained about the internal state of a black hole by outside observers [262]. External features ("hair") of black hole solutions in general relativity are determined by specific conserved quantities such as mass, angular momentum, and charge. In this letter we investigate how the situation changes when both the matter source (black hole interior state) and the gravitational field itself are quantized.

We begin by showing that the graviton state associated with an energy eigenstate source is determined, at leading order, by the energy eigenvalue of the source. These graviton states can be expressed as coherent states of non-propagating graviton modes, with explicit dependence on the source energy eigenvalue. Semiclassical matter sources (e.g., a star or black hole) are superpositions of energy eigenstates with support in some band of energies, and produce graviton states that are superpositions of the coherent states. Next, we consider quantum gravitational corrections which lead to r^{-3} and r^{-5} corrections to the r^{-1} Newtonian potential. We show that the r^{-5} corrections are sensitive to the internal structure of the matter source. That is, two matter sources with the same semiclassical mass M can produce different r^{-5} terms in the metric. These observations imply that information about the interior state of a black hole exists outside the classical horizon. This could, in principle, affect the Hawking radiation states produced as the hole evaporates. We discuss implications for black hole information and holography in the conclusions.

14.2 Asymptotic quantum states of the graviton field

General relativity relates the spacetime metric to the energy-momentum distribution of matter, but only applies when both the metric (equivalently, the gravitational field) and matter sources are semiclassical. A theory of quantum gravity is necessary to relate the quantum state of the gravitational field to the quantum state of the matter source.

A semiclassical matter configuration S is a superposition of energy eigenstates with support concentrated in some narrow band of energies

$$\psi_S = \sum_n c_n \psi_n, \tag{14.1}$$

where ψ_n are energy eigenstates with eigenvalues E_n. S produces a gravitational field (metric) governed by the Einstein equations $G_{\mu\nu} = 8\pi G_{\rm N} T_{\mu\nu}$, where the energy momentum tensor is itself semiclassical. Here we assume that S is compact – localized in some spatial region of an otherwise empty universe – and consider the gravitational field asymptotically far away.

What can be said about the quantum state of the graviton field given the exact quantum state of the matter source? This question extends beyond the realm of classical general relativity, but we show below that the properties of semiclassical gravity constrain the result in an interesting way.

We find that the quantum state of the asymptotic gravitational field of a matter source which is an energy eigenstate is controlled by the energy eigenvalue E_n. In particular, energy eigenstate sources with different eigenvalues produce distinct graviton states. This immediately implies that the asymptotic graviton field of a typical semiclassical matter source is a superposition state of the form

$$\psi_g(S) = \sum_n c_n \psi_g(E_n), \tag{14.2}$$

where $\psi_g(E) \neq \psi_g(E')$ when $E \neq E'$.

It is typically assumed in many body physics that there are no accidental degeneracies – i.e., that the eigenvalues E_n of a complex matter system are distinct (barring exact symmetries of the Hamiltonian; note even these may be violated by quantum gravity effects), although energy level splittings might be exponentially small in the size of the system. If this is the case, then the above results imply that the state of the matter system can, in principle, be reconstructed from the asymptotic graviton state. The quantum information encoded in the matter system is also stored, via entanglement, in the spacetime metric at infinity.

To obtain the desired result we use the following gedanken construction. In brief, we want to show that the matter source energy eigenstate ψ_n produces a different asymptotic graviton state than another state $\psi_{n'}$ of the system with $E_{n'} \neq E_n$. The problem is that the energy splitting $E_{n'} - E_n$ could be exponentially small in the size of S and as far as the classical Einstein equations are concerned the corresponding sources T and T' are effectively identical.

However, we can imagine configurations made of N identical copies of the original system S, which we take to be an energy eigenstate ($\psi_S = \psi_n$), and the same number N of identical copies of the system S' with the source in the eigenstate $\psi_{S'} = \psi_{n'}$. For sufficiently large N the difference in source terms T and T' becomes macroscopic, and the difference between the corresponding metrics is governed by the classical Einstein equations. The asymptotic behaviors of these metrics are equivalent to the additive Newtonian gravitational potentials resulting from each of the N copies of S and S', respectively. Hence the asymptotic graviton state $\psi_g(NE_n)$ of the system S cannot be identical to $\psi_g(NE_{n'})$ of the system S', otherwise the resulting sums would also be identical.[1]

This analysis does not determine the graviton states $\psi_g(E)$, but does establish that different energies E correspond to different (albeit possibly very similar) states ψ_g.

We can obtain the same result via quantum field theory using the property that the spin-2 graviton $h_{\mu\nu}$ couples to the operator $T_{\mu\nu}$. The gravitational potential is generated by graviton exchange between the source "particle" S and a test mass. At long wave-

[1] Some details of the gedanken construction: 1. Place the N copies of the system S at distances r apart, where r is much larger than the size of S. We can stabilize the copies against their mutual gravitational attraction by assuming a repulsive force mediated by a boson with mass $\sim 1/r$. This finite range interaction is negligible at asymptotically large distances. 2. Consider the graviton field at distances R much larger than $N^{1/3}r$ (i.e., far from all of the matter sources). 3. Take the limits $R, N, r \to \infty$ such that the leading contribution to the Newtonian potential at large R is given by the total energy $E = N\,E_n$, up to corrections that can be made as small as desired.

lengths, we can treat the composite state S as a single particle, analogous to a nucleon which is composite and has its own complex substructure. The Feynman amplitude for graviton emission from an incoming source particle S has a vertex factor which is simply its energy eigenvalue E. States S with different energies E have different graviton emission amplitudes, and hence produce different asymptotic states of the $h_{\mu\nu}$ field.

The graviton quantum state $\psi_g(E)$ is exactly analogous to the quantum state of the $U(1)$ vector field (Coulomb potential) created by a charge Q [29, 109, 266]. This can be constructed explicitly as a coherent state

$$|0\rangle_Q = \exp\left[Q \int d^3k\, q(k) b(k)\right] |0\rangle_{Q=0}\,, \tag{14.3}$$

where $b(k)$ is a linear combination of annihilation operators of the non-propagating (temporal and longitudinal, depending on choice of gauge) modes of the photon. The factor of Q in the exponent shows how the photon state depends on the source charge. In the gravitational case Q is replaced by the energy eigenvalue of the source state and the coherent state modes are temporal and longitudinal graviton modes. In both gauge theory and gravity the manner in which the charge or energy control the asymptotic quantum state is determined by the Gauss law via constrained quantization. The direct connection between the gravitational field (Schwarzschild metric) and the Coulomb potential can also be seen as a consequence of the double copy relationship [264]. For our purposes the most important point is that $\psi_g(E)$ depends explicitly on E and for each distinct energy eigenstate of the compact source there is a different graviton quantum state.

The evaporation of a black hole takes place over a timescale $\sim M^3$ so its evolution from a matter configuration to outgoing radiation is confined to a finite region of spacetime. Hence the asymptotic gravitational field at $r \gg M^3$ remains unchanged, in the form (14.2), throughout the entire process. However, near the horizon the gravitational quantum state presumably reflects the changing internal state of the hole. The internal state is itself dependent on the previously emitted Hawking radiation – e.g., due to conservation of energy, angular momentum, etc. This provides a mechanism connecting the region just outside the horizon, where the *next* quantum of Hawking radiation originates, to the internal state of the black hole and the radiation quanta emitted in the past. Once we go beyond the semiclassical approximation the amplitude for radiation emission is a function of $\psi_g(E)$ which itself depends on the internal state of the hole. We discuss this further in the conclusions.

14.3 Leading corrections from quantum gravity

In general relativity, Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat. In other words, the exterior solution must be given by the Schwarzschild metric. It has been shown that this is not the case in quantum gravity [77, 97]: the asymptotic gravitational potential of a compact object received quantum gravitational corrections [77, 79] which are not present for an eternal black hole [70,77]. Quantum gravitational corrections depend on the composition of the compact object. This quantum memory effect has also been observed in FLRW cosmology [143]. In this section, we show that compact objects are hairy in quantum gravity. We work within the framework of the effective quantum gravitational action at second order in curvature [34–37, 61, 70, 140, 351]: $\Gamma[g] = \Gamma_{\rm L}[g] + \Gamma_{\rm NL}[g]$, where the local part of the action is given by

$$\Gamma_{\rm L} = \int d^4x\,\sqrt{g}\left[\frac{\mathcal{R}}{16\,\pi\,G_{\rm N}} + c_1(\mu)\,\mathcal{R}^2 + c_2(\mu)\,\mathcal{R}_{\mu\nu}\,\mathcal{R}^{\mu\nu} + c_3(\mu)\,\mathcal{R}_{\mu\nu\alpha\beta}\,\mathcal{R}^{\mu\nu\alpha\beta}\right] \tag{14.4}$$

and the non-local part of the action by

$$\begin{aligned}\Gamma_{\rm NL} = -\int d^4x\,\sqrt{g}\,\Bigg[&\alpha\,\mathcal{R}\,\ln\!\left(\frac{\Box}{\mu^2}\right)\mathcal{R} + \beta\,\mathcal{R}_{\mu\nu}\,\ln\!\left(\frac{\Box}{\mu^2}\right)\mathcal{R}^{\mu\nu}\\ &+\gamma\,\mathcal{R}_{\mu\nu\alpha\beta}\,\ln\!\left(\frac{\Box}{\mu^2}\right)\mathcal{R}^{\mu\nu\alpha\beta}\Bigg].\end{aligned} \tag{14.5}$$

This effective action is obtained by integrating out the fluctuations of the graviton and potentially other massless matter fields. The Wilson coefficients of the local part of the action are not calculable from first principles, as we do not specify the ultraviolet theory of quantum gravity. However, those of the non-local part are calculable and model independent quantum gravitational predictions. These non-local coefficients can be found in e.g. [77]. The equations of motion obtained from varying the effective action with respect to the metric are given by

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\,\mathcal{R}\,g_{\mu\nu} + 16\,\pi\,G_{\rm N}\left(H^{\rm L}_{\mu\nu} + H^{\rm NL}_{\mu\nu}\right) = 8\,\pi\,G_N T_{\mu\nu}. \tag{14.6}$$

The local part of the equation of motion is given by

$$\begin{aligned}H^{\rm L}_{\mu\nu} = \bar{c}_1 &\left(2\,\mathcal{R}\,\mathcal{R}_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,\mathcal{R}^2 + 2\,g_{\mu\nu}\,\Box\mathcal{R} - 2\nabla_\mu\nabla_\nu\mathcal{R}\right)\\ &+\bar{c}_2\left(2\,\mathcal{R}^{\alpha}_{\ \mu}\,\mathcal{R}_{\nu\alpha} - \frac{1}{2}\,g_{\mu\nu}\,\mathcal{R}_{\alpha\beta}\,\mathcal{R}^{\alpha\beta} + \Box\mathcal{R}_{\mu\nu} + \frac{1}{2}\,g_{\mu\nu}\,\Box\mathcal{R} - \nabla_\alpha\nabla_\mu\mathcal{R}^{\alpha}_{\ \nu} - \nabla_\alpha\nabla_\nu\mathcal{R}^{\alpha}_{\ \mu}\right)\end{aligned} \tag{14.7}$$

with $\bar{c}_1 = c_1 - c_3$ and $\bar{c}_2 = c_2 + 4\,c_3$. Finally, the non-local part reads

$$\begin{aligned}H^{\rm NL}_{\mu\nu} =& -2\,\alpha\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu}\,\mathcal{R} + g_{\mu\nu}\,\Box - \nabla_\mu\nabla_\nu\right)\ln\left(\frac{\Box}{\mu^2}\right)\mathcal{R}\\ &-\beta\Bigg(2\,\delta^{\alpha}_{(\mu}\,\mathcal{R}_{\nu)\beta} - \frac{1}{2}\,g_{\mu\nu}\,\mathcal{R}^{\alpha}_{\ \beta} + \delta^{\alpha}_{\mu}\,g_{\nu\beta}\,\Box + g_{\mu\nu}\,\nabla^{\alpha}\nabla_{\beta}\\ &\quad-\delta^{\alpha}_{\mu}\,\nabla_\beta\nabla_\nu - \delta^{\alpha}_{\nu}\,\nabla_\beta\nabla_\mu\Bigg)\ln\left(\frac{\Box}{\mu^2}\right)\mathcal{R}^{\beta}_{\ \alpha}\\ &-2\,\gamma\left(\delta^{\alpha}_{(\mu}\,\mathcal{R}_{\nu)\ \ \sigma\tau}^{\ \ \beta} - \frac{1}{4}\,g_{\mu\nu}\,\mathcal{R}^{\alpha\beta}_{\ \ \sigma\tau} + \left(\delta^{\alpha}_{\mu}\,g_{\nu\sigma} + \delta^{\alpha}_{\nu}\,g_{\mu\sigma}\right)\nabla^{\beta}\nabla_{\tau}\right)\ln\left(\frac{\Box}{\mu^2}\right)\mathcal{R}_{\alpha\beta}^{\ \ \ \sigma\tau}.\end{aligned} \tag{14.8}$$

Note that the variation of the $\ln\Box$ term yields terms of higher order in curvature and can thus safely be ignored at second order in curvature. The non-local parts of the field equations are responsible for the memory effect. We can easily illustrate this by considering the corrections to the metric of a stationary homogeneous and isotropic star with radius R_s and density

$$\rho(r) = \rho_0\,\Theta(R_\mathrm{s} - r) = \begin{cases} \rho_0 & \text{if } r < R_\mathrm{s} \\ 0 & \text{if } r > R_\mathrm{s}, \end{cases} \tag{14.9}$$

where $\rho_0 > 0$ is a constant and $\Theta(x)$ is Heaviside's step function. The solution to the Einstein equation inside this star (for $r \le R_\mathrm{s}$) is the well-known interior Schwarzschild metric

$$\begin{aligned} ds^2 &= \left(3\sqrt{1 - \frac{2\,G_\mathrm{N}\,M}{R_\mathrm{s}}} - \sqrt{1 - \frac{2\,G_\mathrm{N}\,M\,r^2}{R_\mathrm{s}^3}}\right)^2 \frac{dt^2}{4} - \left(1 - \frac{2\,G_\mathrm{N}\,M r^2}{R_\mathrm{s}^3}\right)^{-1} dr^2 - r^2\,d\Omega^2 \\ &\equiv g_{\mu\nu}^{\mathrm{int}}\,dx^\mu\,dx^\nu, \end{aligned} \tag{14.10}$$

where

$$M = 4\,\pi \int_0^{R_\mathrm{s}} \rho\, r^2\,dr = \frac{4\,\pi}{3}\,R_\mathrm{s}^3\,\rho_0 \tag{14.11}$$

is the total Misner-Sharp mass of the source. The corresponding pressure is of order G_N [77] in agreement with the fact that the pressure does not gravitate in Newtonian physics. Of course, the metric outside the star (for $r > R_\mathrm{s}$) is the usual vacuum Schwarzschild metric

$$ds^2 = \left(1 - \frac{2\,G_\mathrm{N}\,M}{r}\right) dt^2 - \left(1 - \frac{2\,G_\mathrm{N}\,M}{r}\right)^{-1} dr^2 - r^2\,d\Omega^2 \equiv g_{\mu\nu}^{\mathrm{ext}}\,dx^\mu\,dx^\nu, \tag{14.12}$$

from which one can see that M is also the Arnowitt-Deser-Misner (ADM) mass of the system.

We now perturb the above metrics: $\tilde{g}_{\mu\nu} = g_{\mu\nu} + g_{\mu\nu}^{\mathrm{q}}$, and take the perturbation $g_{\mu\nu}^{\mathrm{q}}$ to be $\mathcal{O}(G_\mathrm{N})$. We solve this equation, imposing the solution to be spherically symmetric and time independent. In addition we fix the gauge freedom by setting $g_{\theta\theta}^{\mathrm{q}} = 0$. Doing so, we obtain the quantum corrections $g_{\mu\nu}^{\mathrm{q}} = \delta g_{\mu\nu}^{\mathrm{ext}}$ to the Schwarzschild metric (14.12) outside the star. The corrections are given in [77]:

$$\begin{aligned} \delta g_{tt}^{\mathrm{ext}} &= (\alpha + \beta + 3\,\gamma)\,\frac{192\,\pi\,G_\mathrm{N}^2\,M}{R_\mathrm{s}^3}\left[2\,\frac{R_\mathrm{s}}{r} + \ln\!\left(\frac{r - R_\mathrm{s}}{r + R_\mathrm{s}}\right)\right] + \frac{C_1}{r} + C_2 + \mathcal{O}(G_\mathrm{N}^3) \\ \delta g_{rr}^{\mathrm{ext}} &= (\alpha - \gamma)\,\frac{384\,\pi\,G_\mathrm{N}^2\,M}{r\,(r^2 - R_\mathrm{s}^2)} + \frac{C_1}{r} + \mathcal{O}(G_\mathrm{N}^3), \end{aligned} \tag{14.13}$$

where C_i are integration constants which can be set to zero. We work with the metric with signature $(+---)$, in the signature $(-+++)$ case, the corrections obtain an extra minus sign. Note the two terms in large brackets, when combined, give rise to the r^{-3} and r^{-5} corrections mentioned in the introduction. The coefficient of this term is proportional to $G_\mathrm{N}^2 M R_\mathrm{s}^{-3}$: i.e., it is a quantum gravitational effect proportional to the density of the source object. Two source objects with the same mass M but different densities give rise to different metric perturbations.

Now compare the result to that generated by two nested (one inside the other) dust balls with densities

$$\rho_i(r) = \rho_{0,i}\,\Theta(R_i - r) = \begin{cases} \rho_{0,i} & \text{if } r < R_i \\ 0 & \text{if } r > R_i, \end{cases} \tag{14.14}$$

and masses M_1 and M_2

$$M_i = 4\,\pi \int_0^{R_i} \rho_{0,i}\, r^2\, dr = \frac{4\,\pi}{3}\, R_i^3\, \rho_{0,i} \tag{14.15}$$

with $i \in \{1,2\}$, such that, e.g., $R_1 < R_s$, $R_2 = R_s$ and $M = M_1 + M_2$. In other words, the star built from two nested dust balls has total mass equal to M and the same outer radius R_s as the star composed of only one component.

It is straightforward to show that a solution in general relativity exists. In the region $r \in [R_2, \infty)$, the metric is the exterior Schwarzschild solution with mass M. In the region $r \in [0, R_1)$ (the most inner one), the metric is the interior Schwarzschild solution with radius R_1 and mass $M_1 + M_2(R_1/R_2)^3$. In the region $r \in [R_1, R_2)$, the metric is the interior Schwarzschild solution with radius R_2 and mass M_2.

In general relativity, an external observer cannot differentiate a star with radius R_s and mass M from the star with two different components but same external radius and same total mass M. However, we will show that the quantum gravitational corrections are different for the two matter distributions and there is thus a memory effect. Repeating the same calculation as in [77], using the fact that at this order in $G_{\rm N}$ the equations are linearized, we find a correction

$$\begin{aligned} \delta g_{tt}^{\rm ext} &= (\alpha+\beta+3\,\gamma)\,\frac{192\,\pi\, G_{\rm N}^2\, M_1}{R_1^3}\left[2\,\frac{R_1}{r} + \ln\!\left(\frac{r-R_1}{r+R_1}\right)\right] \\ &\quad + (\alpha+\beta+3\,\gamma)\,\frac{192\,\pi\, G_{\rm N}^2\, M_2}{R_2^3}\left[2\,\frac{R_2}{r} + \ln\!\left(\frac{r-R_2}{r+R_2}\right)\right] + \mathcal{O}(G_{\rm N}^3) \\ \delta g_{rr}^{\rm ext} &= (\alpha-\gamma)\,\frac{384\,\pi\, G_{\rm N}^2\, M_1}{r\,(r^2 - R_{s,1}^2)} + (\alpha-\gamma)\,\frac{384\,\pi\, G_{\rm N}^2\, M_2}{r\,(r^2 - R_{s,2}^2)} + \mathcal{O}(G_{\rm N}^3). \end{aligned} \tag{14.16}$$

While the classical part of the metric cannot distinguish between the one ball of dust with mass M and two concentric dust balls with masses M_1, M_2 and $M_1 + M_2 = M$, the quantum gravitational corrections depend on the matter distribution of the nested balls.

For the one-layer star we obtain

$$\begin{aligned} g_{tt} \;=\; & 1 - \frac{2G_N M}{r} \\ & -128\pi^2(\alpha+\beta+3\gamma)\frac{l_p^2}{r^2}\left[\frac{G_N M}{r}\left(1 + \frac{3R_s^2}{5r^2} + \mathcal{O}(R_s/r)^4\right) + \mathcal{O}(G_N M/r)^2\right] \\ & + \mathcal{O}(l_p/r)^4, \end{aligned} \tag{14.17}$$

where $l_p = \sqrt{\hbar G}$ is the Planck length, and for two layers we obtain

$$
\begin{aligned}
g_{tt} &= 1 - \frac{2G_N M}{r} \\
&\quad -128\pi^2(\alpha+\beta+3\gamma)\frac{l_p^2}{r^2}\left[\frac{G_N M}{r}\left(1+\frac{3(M_1R_1^2+M_2R_s^2)}{5Mr^2}+\mathcal{O}(R_s/r)^4\right)+\mathcal{O}(G_N M/r)^2\right] \\
&\quad +\mathcal{O}(l_p/r)^4.
\end{aligned}
\tag{14.18}
$$

Clearly, the quantum gravitational corrections are different for the two stars. Here we made explicit the different expansion parameters. The series in l_p/r reflects the truncation of the effective action at second order in curvature. The series in $G_N M/r$ is due to the linearization of the field equations and the expansion in R_s/r corresponds to the asymptotic limit. In this limit we see that potentials generated by the two stars are composition dependent at order r^{-5}.

In this case we have considered a two-layered star and shown that the result can differ from a single-layered star. However, the above argument can easily be extended to show that any n- and m-layered stars with $n \neq m$ can be distinguished by an outside observer due to quantum gravitational effects, although their classical external gravity fields are identical. The quantum memory effect leads to hairy stars.

To extend the above discussion, consider two homogeneous stars both with initial mass M_i and radius R_i. We assume that at a certain time both stars run out of fuel and collapse towards a new equilibrium state with mass M_f and radius R_f. Let us furthermore assume that the first star remains homogeneous, while the second collapses to a two-layered state as described above. The initial configurations are gravitationally indistinguishable in terms of classical effects. Moreover, due to Birkhoff's theorem the two final states are classically indistinguishable. However, due the quantum gravitational memory effect the two final states are distinguishable at the quantum level.

While earlier we assumed a time-independent static star, we could consider a collapsing dust ball which can form a black hole. We introduce time-dependence via the radius of the star $R_s(t)$. For a distant observer, $r \gg R_s(t)$ at all times, we can expand the correction to the metric in Eq. (14.13), and it seems likely that the r^{-5} dependence remains during the totality of the collapse.

Eventually, $R_s(t)$ will reach $2G_N M$ and a closed trapped surface will form indicating the formation of a black hole. An observer could in principle measure the coefficient of the r^{-5} correction to the metric. This correction contains information about the matter distribution that collapsed and could thus enable the observer to differentiate between black holes formed by different matter distributions.

The r^{-5} correction shifts the location of the horizon slightly and modifies the metric near the horizon. This presumably has an effect on Hawking radiation. A fully quantum mechanical treatment of the metric g, as opposed to the semiclassical perturbation analysis above, would yield the detailed quantum state of the graviton field (analogous to (14.3)) in place of the r^{-5} correction we obtained.

We find that quantum gravity produces a new kind of hair on black holes.

14.4 Conclusions: holography and black hole information

The existence of a one to one map between the quantum states of compact matter sources and of their asymptotic gravitational fields is clearly suggestive of holography and area bounds on entropy. We emphasize that the appearance of the charge or energy in results like (14.3) originates in Gauss law constraints which play an important role in the quantization of gauge theories and gravity. The recovery of bulk information from asymptotic gravitational fields at the boundary is also discussed in [115, 116, 240, 250].

In a fully quantum mechanical treatment the evolution of the matter source cannot be considered independently from that of its gravitational field. This contrasts sharply with the usual approximation of a fixed spacetime background in which matter fields evolve. For example, Hawking radiation from a black hole is computed in this approximation, whereas our analysis shows that a precise treatment (e.g., one which hopes to examine the unitarity of black hole evaporation) must consider that the metric outside the horizon depends on the state of the interior. The evaporation process takes the form

$$|B_0, g_0\rangle \;\rightarrow\; |B_1, g_1, \gamma_1\rangle \;\rightarrow\; |B_2, g_2, \gamma_2, \gamma_1\rangle \;\rightarrow\; |B_3, g_3, \gamma_3, \gamma_2, \gamma_1\rangle \cdots \tag{14.19}$$

where B is the black hole internal state, g the quantum state of the (external) graviton field or metric, and γ the emitted radiation which originates at the horizon. The radiation state γ_{n+1} depends on the metric state g_n, and each g_n depends on, and is entangled with, B_n. From this perspective it is clear that the Hawking radiation state is connected to the internal state of the black hole.

We can give some idea of the complexity of this process through the following schematic description. Consider the semiclassical superposition state in (14.2),

$$\psi_g(S) = \sum_n c_n \psi_g(E_n), \tag{14.20}$$

and suppose that each graviton state $\psi_g(E_n)$ (describing the exterior metric) has amplitude $\alpha(E_n, \Delta)$ to produce a Hawking radiation quantum γ with energy Δ. Then the exterior state evolves to

$$\psi \approx \sum_n c_n \left[\psi_g(E_n) + \alpha(E_n, \Delta)\psi_g(E_n - \Delta)\gamma(\Delta) + \cdots\right]. \tag{14.21}$$

The state after radiation emission (from second term in the sum, above) is a different semiclassical state constructed from ψ_g corresponding to energies shifted by Δ. Through $\alpha(E_n, \Delta)$ and $\psi_g(E_n - \Delta)$ the detailed form of this quantum state depends on the emitted radiation, including on quantum numbers we have suppressed such as momentum, spin, charge, etc. Even if the deviation of $\alpha(E_n, \Delta)$ from the semiclassical amplitude is exponentially small, the aggregate effect on the process of evaporation could be significant. It is plausible that each initial black hole state, specified by coefficients c_n, evolves into a different final quantum state – i.e., the evolution is unitary.

For each history of radiation quanta $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ there is a corresponding quantum spacetime $\{g_1, g_2, \cdots, g_n\}$. A black hole with entropy A can produce $\sim \exp A$ distinct evaporation states and corresponding quantum spacetimes. Schrödinger evolution of the initial state will produce a superposition of these radiation states and spacetimes [63, 204, 207]. It has been conjectured that black hole evaporation is unitary when all of these branches of the wavefunction are taken into account [27, 205, 206, 298].

Acknowledgments: The work of X.C. is supported in part by the Science and Technology Facilities Council (grants numbers ST/T00102X/1, ST/T006048/1 and ST/S002227/1). The work of R.C. is partially supported by the INFN grant FLAG, it has also been carried out in the framework of activities of the National Group of Mathematical Physics (GNFM, INdAM).

Part IV

Effective Field Theories for Quantum Gravity: Non-Perturbative Effects

Chapter 15

Bounds on Very Weakly Interacting Ultra Light Scalar and Pseudoscalar Dark Matter from Quantum Gravity

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Abstract
In this paper we consider very weakly interacting and ultra light scalar and pseudoscalar dark matter candidates. We show that quantum gravity has important implications for such models and that the masses of the singlet scalar and pseudoscalar fields must be heavier than 3×10^{-3} eV. However, if they are gauged, their masses could be much lighter and as light as 10^{-22} eV. The existence of new gauge forces in the dark matter sector can thus be probed by atomic clocks or quantum sensors experiments.

This chapter has been published in the European Physics Journal C **80**, no. 08, p. 781 (2020).
A preprint of the chapter can be found at arXiv:2008.06243 [hep-ph].

15.1 Introduction

A strong evidence for physics beyond the Standard Model of particle physics comes from the observation that 75% of the matter balance of our universe cannot be accounted for by the Standard Model. Some form of non-luminous matter must exist. Despite being the most abundant form of matter, embarrassingly little is known about dark matter and a wide range of masses and couplings to the Standard Model particles are still possible. In this paper, we focus on ultralight and very weakly coupled scalar and pseudoscalar dark matter models which have recently received a fair share of attention and for which a large part of the parameter space can now be probed experimentally [44–46, 51, 67, 208, 210, 224, 252, 294, 295, 297, 306, 340, 350].

In particular experiments that search for oscillations in the fundamental constants resulting from the coupling of scalar or pseudoscalar dark matter with the Standard Model [1,17,19,182,318,322,323,326,327] have a great potential of testing such models in the mass range $m_\phi \in [10^{-16}, 10^{-23}]$eV. The optimal sensitivity of such experiments typically lies around 10^{-22} eV, and the bounds on the sensitivity are set by the fact that the oscillation frequency is proportional to the mass of the scalar field. Masses of the order $m_\phi \sim 10^{-16}$ eV correspond to oscillation times of the order $T \sim 10\,\mathrm{s}$, while masses of the order $m_\phi \sim 10^{-23}$ eV correspond to oscillation times of the order $T \sim 10\,\mathrm{yr}$.

In this paper we follow the line of arguments put forward in refs. [71, 72] based on quantum gravity to put further theoretical bounds on such searches. In particular, we exploit the fact that dark matter will always couple gravitationally to the Standard Model. Therefore quantum gravity will generate effective interactions between the Standard Model and the hidden sector. This fact together with current experimental bounds restricts the mass range for such weakly interacting light particles considerably. While this is the case for singlet scalar fields, we show that this is not the case if there are new forces in the dark matter sector.

15.2 Interactions generated by quantum gravity

For any dark matter model we can write the following effective action.

$$S = S_{\mathrm{EH}} + \int \sqrt{|g|}\,(\mathcal{L}_{\mathrm{SM}} + \mathcal{L}_{\mathrm{DM}} + \mathcal{L}_{\mathrm{int}})\, d^4x, \tag{15.1}$$

where the Standard Model Lagrangian and the dark matter sector Lagrangian can be written as

$$\mathcal{L}_{\mathrm{SM}} = \sum_i c_i\, \mathcal{O}_{\mathrm{SM},i}, \tag{15.2}$$

$$\mathcal{L}_{\mathrm{DM}} = \sum_j c_j\, \mathcal{O}_{\mathrm{DM},j}, \tag{15.3}$$

where c_i, c_j are dimensionless Wilson coefficients. Interactions between the Standard Model particles and those of the dark matter section can be introduced via a Lagrangian

$$\mathcal{L}_{\text{int}} = \sum_k c_k \, \mathcal{O}_{\text{int},k}, \tag{15.4}$$

where again c_k are dimensionless Wilson coefficients.

Besides the "particle physics" interactions induced by the operator $\mathcal{O}_{\text{int,k}}$, there will be some gravitational interaction between the two sectors. Indeed, since both the Standard Model and the hidden sector couple to gravity, gravity will generate operators connecting the two sectors whether there is an interaction operator $\mathcal{O}_{\text{int,k}}$ at tree level or not.

For every $O_{\text{SM},i}$ and $O_{\text{DM},j}$, perturbative quantum gravity will generate the additional interactions $M_{\text{P}}^{-4} O_{\text{SM},i} \, O_{\text{DM},j}$. We thus have

$$\mathcal{L}_{\text{int}} = \sum_k c_k \, \mathcal{O}_{\text{int,k}} + \sum_{i,j} \frac{c_{i,j}}{M_{\text{P}}^4} \, \mathcal{O}_{\text{SM},i} \, \mathcal{O}_{\text{DM},j}, \tag{15.5}$$

where M_{P} is the reduced Planck scale, which is the scale of quantum gravity and where $c_{i,j}$ are Wilson coefficients of order unity. It is clear from eq. (15.5) that the interactions generated by perturbative quantum gravity are suppressed by the reduced Planck scale to the fourth power. Therefore these interactions are not expected to be measurable in any contemporary or near future experiment. Hence, perturbative quantum gravity cannot yet provide any constraints to dark matter models.

Non-perturbative quantum gravity, on the other hand, can constrain dark matter models. Using the same argument, namely that everything couples to gravity as it is universal, one can deduce that non-perturbative quantum gravity effects could generate effective operators of any dimension. However any such operator must be suppressed by the scale of quantum gravity as such interactions must vanish in the limit where $M_{\text{P}} \to \infty$, i.e. when gravity decouples. We thus expect quantum gravity induced effective interactions to be of the form

$$\sum_{n\geq 0} \sum_k \tilde{c}_{n,k} \, \mathcal{O}_{\text{QG},n,k} = \sum_{n\geq 0} \sum_k \frac{\tilde{c}_{n,k}}{M_{\text{P}}^n} \, O_{\text{QG},n,k}, \tag{15.6}$$

where $\mathcal{O}_{\text{QG},n}$ has mass-dimension 4 and $O_{\text{QG},n}$ has mass-dimension $n + 4$.

As the Wilson coefficients $\tilde{c}_{d,k}$ depend on the ultraviolet completion of quantum gravity, one might be inclined to conclude that no predictions can be made until such a theory is known. However, experience with effective field theories, see discussion in [71, 72], shows that sensible predictions on the order of magnitude of the Wilson coefficients can be made. Quite generically, Wilson coefficients are expected to be of order one, if the scale of the physics generating the interaction is known and properly normalized. In particular, there is no reason to expect an exponential suppression as it is sometimes claimed. For example, it has been shown that there is no exponential suppression in the production of quantum black holes in high energy collisions of particles [203].

In the case of quantum gravity, it is known that the scale of quantum gravity is dynamical. Naively, one might expect that the scale is the reduced Planck scale $M_{\text{P}} =$

2.435×10^{18} GeV. However it is now well understood that the scale at which quantum gravitational interactions become relevant is $M_{\mathrm{P}}\sqrt{160\pi/N}$ with $N = 1/3N_S + N_F + 4N_V$ where N_S, N_F and N_V are respectively the number of real scalar fields, Weyl fermions and vector bosons in the model [21, 68, 84, 190]. For the Standard Model, this is very close to the naive reduced Planck scale. Once the suppression scale for these operators has been properly defined there is no reason to expect a further suppression via smaller than unity Wilson coefficients. Furthermore, as we are considering non-perturbative physics, the Wilson coefficients will not be suppressed by loop factors or small coupling constants to some power. Note that the scale of quantum gravity cannot be larger than the reduced Planck scale as adding more fields to the theory can only lead to a lower scale of quantum gravity. We are thus being as conservative as possible by taking the scale of quantum gravity to be the reduced Planck scale.

We can now combine the quantum gravitational effective interactions with the non-gravitational interactions between the Standard Model and the dark matter sector. These can be written as

$$\sum_k c_k\, \mathcal{O}_{\mathrm{int},k} = \sum_{n\geq 0}\sum_k \frac{c_{n,k}}{\Lambda_{n,k}^n}\, O_{\mathrm{int},n,k}, \tag{15.7}$$

where $\Lambda_{n,k}$ is the energy scale associated with this effective operator. Comparing these two we find that non-gravitationally induced effective operators between the Standard Model and the hidden sector are corrected by gravitationally induced operators. Therefore, excluding all operators of dimension less than 4, we can write down an interaction Lagrangian of the form

$$\begin{aligned} \mathcal{L}_{\mathrm{int}} &= \sum_{n\geq 0}\sum_k \left(\frac{c_{n,k}}{\Lambda_{n,k}^n} + \frac{\tilde{c}_{n,k}}{M_{\mathrm{P}}^n}\right) O_{\mathrm{int},n,k} \\ &= \sum_{n\geq 0}\sum_k \frac{c_{n,k}}{\Lambda_{n,k}^n}\left[1 + \frac{\tilde{c}_{n,k}}{c_{n,k}}\left(\frac{\Lambda_{n,k}}{M_{\mathrm{P}}}\right)^n\right] O_{\mathrm{int},n,k}. \end{aligned} \tag{15.8}$$

As both $\tilde{c}_{n,k}$ and $c_{n,k}$ are expected to be of order 1, we find that the quantum gravitational interactions dominate, if $\Lambda_{n,k} > M_{\mathrm{P}}$. Note that $c_{n,k}$ could contain further loop suppression factors if the corresponding operators are generated perturbatively, but this does not change our analysis, the important point is that as we are considering nonperturbative quantum gravitational effects, there are no loop suppression factors in $\tilde{c}_{n,k}$.

Experiments looking for weakly interacting dark matter put bounds on the interaction strength $c_{n,k}/\Lambda_{n,k}^n$. For some operators with $n \leq 2$ these bounds have reached the Planck scale, i.e. $c_{n,k}M_{\mathrm{P}}^n \gtrsim \Lambda_{n,k}^n$. Therefore, since $c_{n,k}, \tilde{c}_{n,k} = \mathcal{O}(1)$, it is possible to exclude various models without probing more feeble interactions. In particular, if one operator can be excluded up to the Planck scale for a certain mass range, quantum gravity will exclude the existence of the scalar or pseudoscalar field for this mass range. This follows from the fact that quantum gravity will generate all possible, i.e. allowed by gauge symmetries, operators at the Planck scale.

15.3 Scalar and pseudoscalar dark matter

In this section we discuss the consequences of the argument from the previous section for some specific scalar and pseudoscalar dark matter models. The most relevant models involving spinless dark matter are dimension 4 operators. However, it is expected that the Wilson coefficients of dimension four operators must be exponentially suppressed by a factor $e^{-M_\mathrm{P}/\mu}$, as such quantum gravity induced operators should vanish in the limit $M_\mathrm{P} \to \infty$, i.e., when gravity decouples. Here μ is a renormalization scale.

The next most relevant operators for a spinless dark matter boson coupling to the Standard Model are dimension 5 operators. An example is an operator of the form

$$O_1 = \frac{c_1}{\Lambda_1}\,\phi\, F_{\mu\nu}F^{\mu\nu}, \tag{15.9}$$

where ϕ is the scalar dark matter field, and $F_{\mu\nu}$ is the electromagnetic field tensor. The results from the Eöt-Wash torsion pendulum experiment that searches for fifth forces [10, 11, 58, 202, 221, 241, 308, 317, 365] lead to the following bound[1]

$$\frac{c_1}{\Lambda_1} \lesssim M_\mathrm{P}^{-1} \qquad \text{if} \qquad m_\phi \lesssim 3 \cdot 10^{-3}\,\mathrm{eV} \tag{15.10}$$

and slightly stronger bounds for lower masses. Moreover atomic spectroscopy measurements [195,335] put even tighter bounds on such an interaction for masses $m_\phi \lesssim 10^{-18}$ eV, however these bounds rely on the assumption that the scalar field is the unique component of the dark matter sector.

As argued above, quantum gravity will lead to an additional contribution

$$O_{1,\mathrm{QG}} = \left(\frac{c_1}{\Lambda_1} + \frac{\tilde{c}_1}{M_\mathrm{P}}\right)\phi\, F_{\mu\nu}F^{\mu\nu}, \tag{15.11}$$

with $\tilde{c}_1 \sim \mathcal{O}(1)$ as argued before. Therefore the current bounds exclude this interaction for all masses $m_\phi \leq 3 \times 10^{-3}$eV. The resulting bounds on this interaction are summarized in figure 15.1, which can be compared[2] to figure 31.1 in ref. [328].

Moreover, since quantum gravity generates interactions between all the particles of the Standard Model and the scalar field. Any scalar field with a mass below 3×10^{-3}eV would generate a Planck scale gravitational operator, which has not been detected by the Eöt-Wash experiment. Therefore the derived bound does not exclusively apply to models containing the non-gravitationally induced interaction (15.9). In fact, any dark matter model containing scalar dark matter fields of masses $m_\phi \lesssim 3 \times 10^{-3}$ eV is excluded. A similar analysis can be done for a pseudoscalar field a. The interaction between an axion-

[1] Bounds in the Eöt-Wash experiments are usually presented in terms of the coupling strength α and the length scale of the Yukawa interaction λ. Such bounds can be translated into a mass-bound using the fact that $\alpha = \mathcal{O}(1)$ as discussed before and by noticing that $m_\phi c^2 = \frac{\hbar c}{\lambda}$.

[2] Note that there is a factor 4 difference: $g_\phi = \frac{g^s_\gamma}{4}$, where g_ϕ is the dimensionful coupling in this paper, and g^s_γ is the dimensionful coupling in ref. [328].

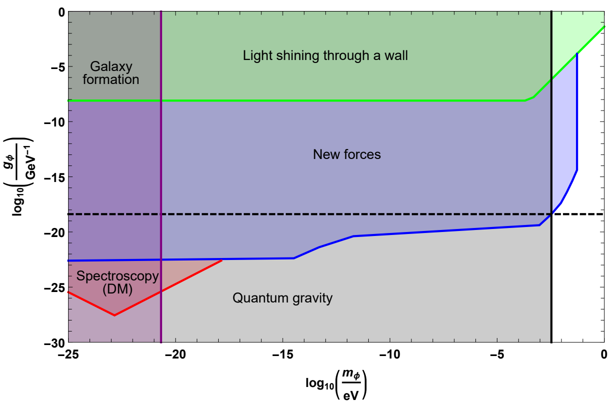

Figure 15.1: Limits on the linear scalar interaction $g_\phi = c_1/\Lambda_1$ as a function of the mass of the scalar m_ϕ. Green: limits from light shining through a wall experiments [25, 146]. Blue: limits from torsion experiments [10, 11, 58, 202, 221, 241, 308, 317, 365]. Red: limits from atomic spectroscopy experiments [195, 328, 335]. Purple: limits from galaxy formation, quasar lensing and stellar streams [57, 121, 211, 277, 307, 310]. Black: limits from quantum gravity as discussed in this paper. Dashed black line: reduced Planck scale.

like-particle a and gluons will receive a quantum gravitational correction

$$O_{2,\mathrm{QG}} = \left(\frac{c_2}{\Lambda_2} + \frac{\tilde{c}_2}{M_\mathrm{P}}\right) a\, G_{\mu\nu}\tilde{G}^{\mu\nu}, \tag{15.12}$$

where $\tilde{c}_2 \sim \mathcal{O}(1)$ and $G_{\mu\nu}$ is the usual gluonic field strength and $\tilde{G}^{\mu\nu}$ its dual. Magnetometry measurements [1] constrain the strength of this interaction by

$$\frac{c_2}{\Lambda_2} + \frac{\tilde{c}_2}{M_\mathrm{P}} \lesssim M_\mathrm{P}^{-1} \qquad \text{if} \qquad m_a \lesssim 5 \cdot 10^{-21}\,\mathrm{eV}. \tag{15.13}$$

Therefore, any dark matter model containing scalar axion-like fields of masses $m_a \lesssim 10^{-21}$ eV is excluded. The result for this particular interaction are summarized in figure 15.2, which can be compared to figure 4 in ref. [1] and figure 31.5 in ref. [328]. Note that this bound assumes that all of dark matter is described by the axion-like-particle a. It is possible to relax this bound if dark matter has multiple components.

On the other hand, for interactions of the form

$$O_{3,\mathrm{QG}} = \left(\frac{c_3}{\Lambda_3} + \frac{\tilde{c}_3}{M_\mathrm{P}}\right) a\, F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{15.14}$$

with $\tilde{c}_3 \sim \mathcal{O}(1)$, the bounds are much weaker[3]. Therefore, there is still a large parameter space to explore. However, the bound (15.13) excludes axion like particles with masses below 10^{-21} eV, because of the universality of gravity: one cannot have the interaction $aF_{\mu\nu}\tilde{F}^{\mu\nu}$ without the interaction $aG_{\mu\nu}\tilde{G}^{\mu\nu}$.

Furthermore, there is no reason why parity symmetry would be preserved by quantum gravitational interactions, see e.g. [32, 199]. Indeed, it is not a gauge interaction. In this

[3] cf. Figure 31.4 in ref. [328].

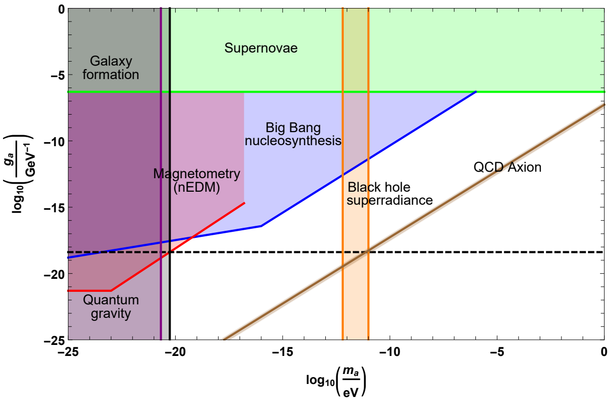

Figure 15.2: Parity conserving quantum gravity. Limits on the linear axion interaction $g_a = c_3/\Lambda_3$ as a function of the mass of the axion m_a. Green: limits from supernovae measurements [181]. Blue: limits from the big bang nucleosynthesis [52,321,323–325]. Red: limits from magnetometry experiments [1, 328]. Purple: limits from galaxy formation, quasar lensing and stellar streams [57, 121, 211, 277, 307, 310]. Orange: limits from the superradiance instability of black holes [105], however note that these bounds can be avoided, if the self-interaction of the axion-like particle is sufficiently strong [18]. Brown: predicted value of the QCD axion [296, 333]. Black: axion masses below $m_a \lesssim 10^{-21}$ eV are excluded by parity conserving quantum gravity as discussed in this paper. Dashed black line: reduced Planck scale.

case, the operators

$$O_4 = \frac{\tilde{c}_4}{M_\text{P}}\, a\, G_{\mu\nu}G^{\mu\nu}, \tag{15.15}$$

and

$$O_5 = \frac{\tilde{c}_5}{M_\text{P}}\, a\, F_{\mu\nu}F^{\mu\nu}, \tag{15.16}$$

which are parity violating will be generated. As before we expect $\tilde{c}_4 \sim \mathcal{O}(1)$ and $\tilde{c}_5 \sim \mathcal{O}(1)$. These operators lead to a Yukawa-type interaction and thus to a fifth force. Therefore, if quantum gravity violates parity, axion-like-particle with masses $m_a \lesssim 3 \times 10^{-3}$ eV are excluded. As shown in figure 15.3, this reduces the parameter space for axion models massively.

Another possible interaction of a spinless dark matter boson coupling to the Standard Model is a dimension 6 interaction of the form

$$O_{6,\text{QG}} = \left(\frac{c_6}{\Lambda_6^2} + \frac{\tilde{c}_6}{M_\text{P}^2}\right) \phi^2\, F_{\mu\nu}F^{\mu\nu}, \tag{15.17}$$

which does not distinguish between scalars and pseudoscalars, as parity is automatically conserved. Again we have $\tilde{c}_6 \sim \mathcal{O}(1)$. Atomic spectroscopy measurements [323, 327] constrain the strength of this interaction by

$$\frac{c_6}{\Lambda_6^2} + \frac{\tilde{c}_6}{M_\text{P}^2} \lesssim M_\text{P}^{-2} \qquad \text{if} \qquad m_\phi \lesssim 2 \cdot 10^{-22}\,\text{eV}. \tag{15.18}$$

Therefore, any dark matter model containing scalar dark matter fields of masses $m_\phi \lesssim 10^{-22}$ eV that couple to the Standard Model in this way are excluded. Note that bounds

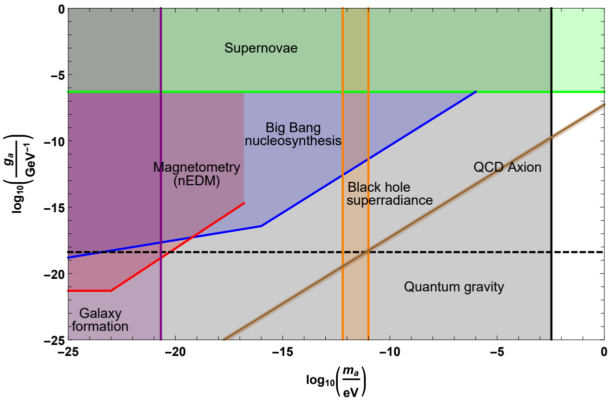

Figure 15.3: Parity violating quantum gravity. Limits on the linear axion interaction $g_a = c_3/\Lambda_3$ as a function of the mass of the axion m_a. Green: limits from supernovae measurements [181]. Blue: limits from the big bang nucleosynthesis [52,321,323–325]. Red: limits from magnetometry experiments [1, 328]. Purple: limits from galaxy formation, quasar lensing and stellar streams [57, 121, 211, 277, 307, 310]. Orange: limits from the superradiance instability of black holes [105], however note that these bounds can be avoided, if the self-interaction of the axion-like particle is sufficiently strong [18]. Brown: predicted value of the QCD axion [296,333]. Black: axion masses below $m_a \lesssim 3 \times 10^{-3}$ eV are excluded by parity violating quantum gravity as discussed in this paper. Dashed black line: reduced Planck scale.

from galaxy formation, quasar lensing and stellar streams are slightly more stringent and lead to $m_\phi \lesssim 10^{-21}$ eV but they have a larger uncertainty. Quantum gravity will however also generate operators of the type $M_\mathrm{P}^{-1}\phi F_{\mu\nu}F^{\mu\nu}$ and $M_\mathrm{P}^{-1}\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$ even if these operators are not introduced in the interaction Lagrangian and we can thus rule out masses below 3×10^{-3} eV. In the case of axions, this bound applies if parity is violated by quantum gravity which we argued is to be expected. The results are summarized in figure 15.4, which can be compared to figure 31.6 in ref. [328].

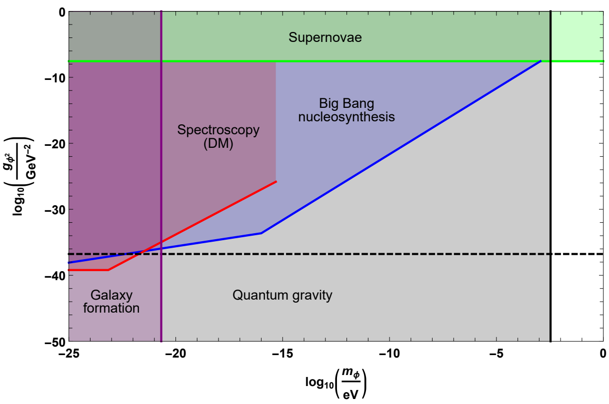

Figure 15.4: Limits on the quadratic scalar interaction $g_{\phi^2} = c_6/\Lambda_6$ as a function of the mass of the scalar m_ϕ. Green: limits from supernovae measurements [279]. Blue: limits from the big bang nucleosynthesis [323]. Red: limits from atomic spectroscopy [323, 327, 328]. Purple: limits from galaxy formation, quasar lensing and stellar streams [57, 121, 211, 277, 307, 310]. Black: limits from quantum gravity as discussed in this paper. Dashed black line: reduced Planck scale.

Our results rule out most of the parameter range for ultralight and very weakly coupled singlet scalar dark matter models. It is worth mentioning that our bound applies as well to the quintessence type models which are often advocated to generate a cosmological time evolution of fundamental constant. A change of the hyperfine constant within the last Hubble time, implies the existence of a scalar field with a very light mass of the order of the present Hubble scale $H = 10^{-33}$ eV [145]. This is ruled out because of quantum gravity. If a time variation of the hyperfine constant is observed, we can safely conclude that it is not due to such a scalar field or dark matter.

Also, it had already been pointed out that the axion is not a valid solution to the strong CP problem of quantum chromodynamics because quantum gravitational effects would destabilize its potential [32, 199], our results imply that the quantum chromodynamics axion is ruled out for most of its parameter range because of quantum gravity if parity is, as expected, violated by quantum gravitational effects.

Obviously there is a well known mechanism to avoid the bound from the Eöt-Wash experiment namely the screening mechanism. However, if the masses of light scalar fields were screened by the matter density on Earth thereby increasing their masses on Earth, they would also be heavy for atomic clocks and quantum sensor experiments based on Earth and would thus not lead to the usual signatures mimicking a time variation of fundamental constants. Interestingly, this could be probed by putting atomic clocks or quantum sensor experiments on a satellite where the screening mechanism would be inefficient.

While we focused thus far on scalar and pseudoscalar fields which are singlets under gauge symmetries, it is possible to avoid some of the bounds from quantum gravity discussed above if we consider scalar or pseudoscalar fields that are gauged under some new gauge group, as gauge symmetries are preserved by quantum gravity. In that case, the only relevant operators are dimension 6 ones of the type

$$O_{7,\mathrm{QG}} = \left(\frac{c_7}{\Lambda_7^2} + \frac{\tilde{c}_7}{M_\mathrm{P}^2} \right) \Phi \cdot \Phi \, F_{\mu\nu} F^{\mu\nu}, \tag{15.19}$$

where Φ is a scalar or pseudoscalar field gauged under some new gauge group of the dark matter sector and $\Phi \cdot \Phi$ is a scalar under that gauge symmetry. We find

$$\frac{c_7}{\Lambda_7^2} + \frac{\tilde{c}_7}{M_\mathrm{P}^2} \lesssim M_\mathrm{P}^{-2} \qquad \text{if} \qquad m_\Phi \lesssim 2 \cdot 10^{-22}\,\mathrm{eV}. \tag{15.20}$$

in which case we can only exclude masses $m_\Phi \lesssim 10^{-22}$ eV for scalar and pseudoscalar fields (or $m_\Phi \lesssim 10^{-21}$ eV if we use the bound from galaxy formation, quasar lensing and stellar streams [57, 121, 211, 277, 307, 310]). If atomic clocks or quantum sensor experiments were to discover such scalar or pseudoscalar fields, they would not only have discovered dark matter but also proven the existence of a new gauge force in the dark matter sector. The results are summarized in figure 15.5. For quintessence fields, the effect would be of order $(\Delta\phi/M_\mathrm{P})^2$ and thus more suppressed than usually assumed.

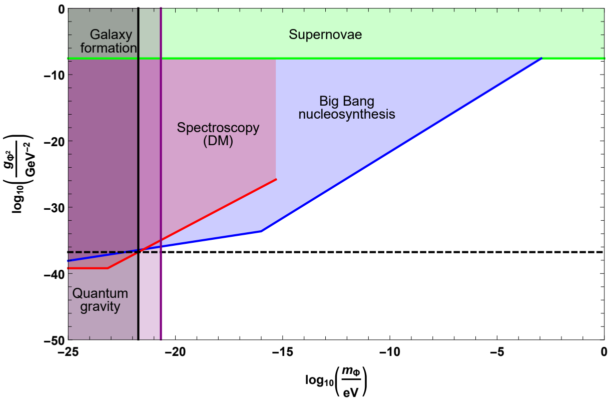

Figure 15.5: Limits on the quadratic gauged scalar interaction $g_{\Phi^2} = c_7/\Lambda_7$ as a function of the mass of the scalar m_Φ. Green: limits from supernovae measurements [279]. Blue: limits from the big bang nucleosynthesis [323]. Red: limits from atomic spectroscopy [323, 327, 328]. Purple: limits from galaxy formation, quasar lensing and stellar streams [57, 121, 211, 277, 307, 310]. Black: limits from quantum gravity as discussed in this paper. Dashed black line: reduced Planck scale.

Let us finally emphasize that the bounds on quantum gravity shown in figures 15.1, 15.2, 15.3, 15.4 and 15.5 carry a small theoretical uncertainty, as the Wilson coefficients are not exactly known. We argued that we know the scale of quantum gravity and that it can be calculated given the number of fields introduced in the model. While the scale of quantum gravity incorporates any suppression for the operators generated by quantum gravity, it is conceivable that the Wilson coefficients could take values between 10^{-1} and 10. Smaller than unity Wilson coefficients could still decrease the bounds by about a factor of 10, which would bring the bound from $g = 4 \times 10^{-19}\text{GeV}^{-1}$ to $g = 4 \times 10^{-20}\text{GeV}^{-1}$ in figures 15.1, 15.2 and 15.3, and from $g = 2 \times 10^{-37}\text{GeV}^{-1}$ to $g = 2 \times 10^{-39}\text{GeV}^{-1}$ in figure 15.4. If the Wilson coefficients were order 10^{-1}, we could only exclude masses below 1×10^{-4}eV.

Moreover, the bounds derived from spectroscopy experiments (red lines) and from models of galaxy formation, are based on the assumption that the scalar field accounts for the total observed local dark matter density $\rho = 0.4\text{GeV}/\text{cm}^3$. Multicomponent dark matter models would loosen the bounds shown in figures 15.1, 15.2, 15.3 and 15.4.

15.4 Conclusions

In this paper we have considered models of dark matter with ultra-light scalar or pseudoscalar fields which have received a lot of attention as they could be discovered with tabletop experiments looking for dark matter using modern quantum sensors or atomic clocks. These particles are usually assumed to be extremely light and very weakly coupled to the particles of the Standard Model.

We have argued that quantum gravity will induce interactions between scalar or pseudoscalar dark matter particles and those of the Standard Model. These quantum gravitational interactions often dominate over the strength of the interaction posited in these models. We have shown that these quantum gravitational interactions are of the fifth force type for scalar dark matter and also for pseudoscalar dark matter if quantum gravity violates parity symmetry. Such interactions are constrained by torsion pendulum experiments such as the Eöt-Wash experiment. Scalar dark matter must be heavier than 3×10^{-3} eV and the same bound applies to pseudoscalar particles assuming that quantum gravity violates parity symmetry. If quantum gravity does not violate parity, pseudoscalar particles are only constrained to have masses larger than 10^{-21} eV. We stress that these bounds are universal and applicable to any scalar dark matter models including models of fuzzy dark matter as discussed for example in [208, 253, 306].

While singlet scalar or pseudoscalar fields are constrained to be heavier than 3×10^{-3} eV, gauged fields could be much lighter. They could be as light as $m_\Phi \sim 10^{-22}$ eV and thus very much relevant to current experiments using atomic clocks or quantum sensors. A positive signal would not only be potentially the sign of dark matter but also a sign that the dark matter sector is very rich and contains new forces. Another way to look at our results is that very low energy tabletop experiments such as atomic clocks and other experiments based on quantum sensors are directly probing quantum gravitational effects.

Acknowledgments

The authors would like to thank Yevgeny Stadnik for very helpful discussions and comments on the draft of this paper. The work of X.C. is supported in part by the Science and Technology Facilities Council (grant number ST/P000819/1). The work of F.K. is supported by a doctoral studentship of the Science and Technology Facilities Council.

Chapter 16

Theoretical Bounds on Dark Matter Masses

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Abstract
In this letter, we show that quantum gravity leads to lower and upper bounds on the masses of dark matter candidates. These bounds depend on the spins of the dark matter candidates and the nature of interactions in the dark matter sector. For example, for singlet scalar dark matter, we find a mass range $10^{-3}\text{eV} \lesssim m_\phi \lesssim 10^{7}\text{eV}$. The lower bound comes from limits on fifth force type interactions and the upper bound from the lifetime of the dark matter candidate.

This chapter has been published in Physics Letters B **814**, p. 136068 (2021).
A preprint of the chapter can be found at arXiv:2009.11575 [hep-ph].

There is overwhelming evidence that most of the matter in our universe is dark and cannot be described by the Standard Model of particle physics. The case for the existence of dark matter is strong because it comes from astrophysical and cosmological observations made on different scales and times in our universe. For example, the cosmic microwave background or galaxy rotation curves involve very different physics and eras in the evolution of our universe but they both require that about 75% of the matter content of the universe consists of cold, non-baryonic, dark matter.

From a theoretical point of view, very little is known of the nature of dark matter. We know that there is no viable candidate in the Standard Model of particle physics. There are basically three different approaches. The first approach consists in introducing a new particle stable enough over the lifetime of the universe which couples at most extremely weakly to the photon so that it remains dark enough. A typical example of such a particle would be a weakly interacting massive particle (WIMP), see e.g. [303] for a review. The second one consists in modifying gravity see e.g. [43, 104, 258, 263], but it is difficult to construct a proper model and even when that is case, it has been argued [92] that this approach is identical to the first one with the caveat that the new field is only coupled gravitationally to the Standard Model particles. Finally, one could hope that some massive astrophysical compact halo objects (MACHOs) such as primordial black holes [183] could explain the missing matter without having to modify the Standard Model or general relativity. Alas, this solution to the dark matter problem, while beautifully simple and minimalistic as it does not require new physics beyond the Standard Model or general relativity, does not appear to be relevant to Nature, see e.g. [158].

If we accept that new physics is required to address the missing matter problem, we are faced with a huge theoretical challenge as we have very little information about the nature of the dark matter particle or particles. We do not know their spins, masses, self-interactions or couplings to the Standard Model particles. Galaxy formation simulations seem to prefer cold, i.e. non-relativistic, dark matter. The interactions of dark matter particles with that of Standard Model or dark matter self-interactions must be weak see e.g. [338] for a review.

Fortunately, quantum gravity can provide some guidance on the allowed parameter range for a given dark matter candidate. The reason for this is simple. In general, quantum gravitational effects will lead to a decay of any dark matter candidate that is not protected by Lorentz invariance or a gauge symmetry from decaying. Furthermore, gravity is universal, it will thus couple to all forms of matter and it will create portals between the Standard Model and any hidden sector. While these decays will be suppressed by powers of the Planck mass, they will still lead to an upper bound on dark matter particles given the large age of our universe. Furthermore, if the dark matter particles are light, the same quantum gravitational effects will lead to fifth force type interactions and these interactions are bounded by limits coming from the Eöt-Wash experiment [10, 11, 58, 202, 221, 241, 308, 317, 365]. Finally, there is a well known lower bound coming from quantum mechanics and more specifically the spin-statistics theorem which applies to fermionic dark matter candidate. This last bound depends on the dark matter profile. Putting all these bounds together, we obtain tight mass ranges for scalar, pseudo-scalar, spin 1/2 and spin 2 dark matter particles which are gauge singlets. These bounds can be relaxed if the fields describing these particles are gauged, we however note that there are fairly tight constraints on the strength of the interactions in the dark matter sector. Finally, we argue that spin-1 vector dark matter particles are less constrained by quantum gravity, because of the chiral nature of the fermions in the Standard Model.

We consider local operators that are generated by non-perturbative quantum gravity effects (see e.g. [32, 71, 72, 86, 99, 120, 173, 175, 199]):

$$O_1 = \frac{c_\phi}{M_{\rm P}}\,\phi\, F_{\mu\nu}F^{\mu\nu}, \tag{16.1}$$

where $M_{\rm P} = 2.4 \times 10^{18}$ GeV is the reduced Planck scale, ϕ is the scalar dark matter field, and $F_{\mu\nu}$ is the electromagnetic field tensor. We note that there are solid arguments showing that the Wilson coefficient c_1 is of order one [86].

The results from the Eöt-Wash torsion pendulum experiment that searches for fifth forces [10, 11, 58, 202, 221, 241, 308, 317, 365] imply that $m_\phi \gtrsim 10^{-3}\,\text{eV}$ [71, 72, 86]. The same operator can lead to the decay of the dark matter scalar [96, 247] with a decay width $\Gamma \sim m_\phi^3/(4\pi M_P^2)$ and lead to an upper bound $m_\phi \lesssim 10^7\text{eV}$ from the requirement that the dark matter candidate lives long enough to still be present in today's universe. Quantum gravity thus enables to restrict the mass of any singlet scalar particle to be in the range:

$$10^{-3}\text{eV} \lesssim m_\phi \lesssim 10^7\text{eV}, \tag{16.2}$$

independently of its potential non-gravitational couplings to Standard Model particles or self-interactions. Note that these bounds would not apply to a gauged scalar field as only dimension six operators would be generated by quantum gravity. In that case, one has $m_\phi \gtrsim 10^{-22}\text{eV}$ [86], and the upper bound disappears [1].

The same bound applies to the mass of a pseudo-scalar dark matter candidate, an axion like particle, a if quantum gravity violates parity (and time reversal invariance) [86]

$$10^{-3}\text{eV} \lesssim m_a \lesssim 10^7\text{eV}. \tag{16.3}$$

On the other hand, if quantum gravity preserves parity, we have to consider the operator

$$O_a = \frac{c_a}{M_{\rm P}}\,a\,\tilde{F}_{\mu\nu}F^{\mu\nu}. \tag{16.4}$$

For an axion-like-particle, we then find [86, 247]

$$10^{-21}\text{eV} \lesssim m_a \lesssim 10^7\text{eV}, \tag{16.5}$$

for parity conserving quantum gravity. The upper bound comes from the requirement that the particle is long-lived in comparison to the age of the universe and the lower bound is

[1] Note that some readers may be worried about the naturalness of very light scalars. We take an agnostic approach and simply derive bounds from quantum gravity assuming that such light scalars exist.

derived from magnetometry searches [1, 86].

For spin 1/2 fermions ψ, quantum gravity leads to an upper bound on the mass of the dark matter candidate [56, 96, 247] as it could decay to the Standard Model fields, while a lower bound comes from the Pauli exclusion principle. We consider the operator [96, 247]:

$$O_\psi = \frac{c_\psi}{M_\mathrm{P}}\,\bar{\psi}\tilde{H}^\dagger \not{D} L, \tag{16.6}$$

where H is the Higgs doublet of the Standard Model with $\tilde{H} = -i\sigma_2 H^*$. This operator implies that the singlet right-handed fermion ψ can decay to an off-shell Z boson and a neutrino, the Z boson then decays to two light fermions. Requiring that the fermion singlet lives long enough to still be present today imposes an upper bound on its mass. One finds $m_\psi < 10^{10}$eV using $\Gamma = v^2 G_F^2 m_\psi^5/(192\pi^3 M_P^2)$ where G_F is the Fermi constant and $v = 246$ GeV the electroweak vacuum expectation value.

Since fermions cannot be in the same state, only a limited amount of fermions can be present in a galaxy with momenta below the escape velocity. Together with the assumption that the fermions must account for the observed dark matter density in a typical galaxy this leads to a lower bound on the mass of the fermions [283, 305, 336]. The bounds on the mass of the dark fermion are then given by

$$10^2\mathrm{eV} \lesssim m_\psi \lesssim 10^{10}\mathrm{eV}. \tag{16.7}$$

The lower bound holds for the Standard Model, but it can be relaxed by assuming multi-component dark matter [129].

We now consider a vector boson dark matter V^μ. The well studied dimension four operator $F^{\mu\nu}B_{\mu\nu}$, where $F^{\mu\nu}$ is the field strength of the hypercharge photon of the Standard Model and $B_{\mu\nu}$ that of the dark photon, while generated by quantum gravity, is expected to be exponentially suppressed [86, 96]. Within the Standard Model, the only dimension five gauge invariant operator is given by $c_{V,5} M_\mathrm{P}^{-1}\, V^\mu(\bar{\psi}_R i\tilde{H}^\dagger\gamma_\mu L)$ but after electroweak symmetry breaking, this simply accounts for a shift of the photon field. The next operators are of mass dimension 6 $c_{V,6} M_\mathrm{P}^{-2} V_\mu (H^\dagger D_\nu H) F^{\mu\nu}$ or $M_\mathrm{P}^{-2}(\bar{\psi}\sigma_{\mu\nu}\tilde{H}^\dagger \not{D} L)B^{\mu\nu}$. These operators lead to dimension five operators after electroweak symmetry breaking but there is a chiral suppression v/M_P. The only useful dimension five operator involves the production of a graviton $h_{\mu\nu}$

$$O_V = \frac{c_V}{M_\mathrm{P}} h_{\mu\alpha} F^\mu_{\ \nu} B^{\nu\alpha}\,, \tag{16.8}$$

which enables the decay of a vector dark matter to a photon and a graviton. This operator exists in the Standard Model with the vector boson replaced by a Z-boson [275]. It is straightforward to estimate the decay width of the V boson, one finds $\Gamma \sim c_V^2 m_V^3/M_P^2$ and we can thus find an upper bound on the mass of a vector dark matter particle from the requirement that it is still around in today's universe. We find $m_V < 10^7$ eV. We can get a lower bound on its mass if we assume that all of dark matter is described by a vector particle. As for a scalar field, see e.g. [328] for a recent review, the requirement that the boson's de Broglie wavelength does not exceed the dark matter halo size of the smallest dwarf galaxies gives a lower bound on its mass $m_V > 10^{-22}$ eV. We thus find

$$10^{-22}\text{eV} \lesssim m_V \lesssim 10^7\text{eV}. \tag{16.9}$$

Using the results developed in [72], it is straightforward to see that for a massive spin-2 field dark matter field, one obtains similar bounds for its mass to that of a singlet scalar field dark matter candidate:

$$10^{-3}\text{eV} \lesssim m_2 \lesssim 10^7\text{eV}. \tag{16.10}$$

In this letter, we have shown that a few very well motivated theoretical concepts based on quantum gravity and the spin-statistics theorem enable to constrain the masses of low spin dark matter candidates. Quantum gravity generates operators that will lead to a decay of all dark matter candidates that are represented by fields that are not gauged or prevented by Lorentz invariance from decaying to Standard Model particles. This lead to an upper bound on their masses. If these dark matter candidates are bosons, they will mediate a fifth force and we can apply bounds from the Eöt-Wash experiment which provide a lower bound on their masses. In the case of fermion dark matter candidates, the lower bound comes from the spin-statistics theorem.

Our bounds are derived assuming the worst case scenario for quantum gravity, namely that it has only one scale and that this scale is the traditional reduced Planck scale i.e. 2.4×10^{18} GeV. In other words, we assumed that quantum gravity is as weak as possible. Our bounds become much more stringent if the effective scale of quantum gravity is below 2.4×10^{18} GeV as it is the case in models with large extra-dimensions where it could be in the TeV region or if there is another infrared cutoff that is below the reduced Planck mass as it is the case in some specific models of quantum gravity see, e.g., [147, 259, 331].

We would like to stress that our bounds are orders of magnitude estimates. We argue that because we are dealing with non-perturbative quantum gravity, the only relevant coupling constant should be the Planck mass. It is however conceivable that there is a further suppression of some of the Wilson coefficients which could involve coupling constants of the Standard Model. For example, c_ϕ could contain a factor $g^2/(4\pi)$ where g is the hyperfine coupling constant of the $U(1)$ group of the Standard Model or c_ψ could be proportional to the electron Yukawa coupling which is of the order of 10^{-5}. Clearly, this would impact our bounds. Here, we made the strong assumption that the dimension five operators are of pure quantum gravitational origin.

Finally, as explained already, we emphasize that these bounds will not apply to hidden sector fields that are gauged under some gauge symmetry whether this is a continuous or discrete gauge symmetry [26, 232]. For gauged fields, dimension 5 operators will not be generated directly, one would expect dimension 6 or higher operators. For a gauged scalar field Φ for example, one has $M_{\rm P}^{-2}\Phi \cdot \Phi\, F_{\mu\nu}F^{\mu\nu}$ in which case we can only exclude masses $m_\Phi \lesssim 10^{-22}$ eV. Dimension 5 operators, if they exist, would have a further suppression if they are generated from a higher dimensional operators. For example, if Φ has some none-vanishing expectation value v_Φ in the TeV region, the resulting dimension five operator $v_\Phi M_{\rm P}^{-2}\phi\, F_{\mu\nu}F^{\mu\nu}$ would be suppressed by a factor $v_\Phi/M_{\rm P} \sim 10^{-16}$. A similar suppression would be generated in models with a discrete gauge symmetry. Such a suppression would open up the allowed mass range for dark matter candidates. The situation is similar for complex scalar dark matter, see e.g. [53], which carries a charge: quantum gravity would form operators of the type $M_{\mathrm{P}}^{-2}\ \phi^{\star}\phi\, O_{SM}$ (where O_{SM} are operators build with fields of the Standard Model) which would be at least of dimension 6, if the complex scalars are gauged. If it is a discrete symmetry, one would expect that quantum gravity breaks the global U(1) symmetry. One would then obtain operators of the type $M_{\mathrm{P}}^{-1}\ \phi\, O_{SM}$ and our bounds would apply. This is particularly important in the case of WIMPs, which are largely excluded by our bounds, if the WIMP is a gauge singlet. Our bounds can be avoided if one gauges WIMPs. Clearly the origin of the dimension five operators that we have discussed in this letter is model dependent and one needs to verify on the case-by-case whether such operators will be generated in a specific dark matter model.

Acknowledgments: We would like to thank Stephen Barr and Mark Wise for useful suggestions. The work of X.C. is supported in part by the Science and Technology Facilities Council (grants numbers ST/T00102X/1, ST/T006048/1 and ST/S002227/1). The work of F.K. is supported by a doctoral studentship of the Science and Technology Facilities Council.

Part V

Discussion

Chapter 17

Conclusions

The formulation of a consistent theory of quantum gravity that is valid at all energy scales is one of the major open questions in theoretical physics. Although direct probes of quantum gravity are beyond the reach of contemporary experiments, quantum gravity could be probed indirectly and future technologies could allow to directly probe regimes, where both quantum effects and gravity are relevant. This necessitates a theory of quantum gravity. Moreover, even when effects of the theory cannot be probed directly in experiments, such a theory is desirable for aesthetic reasons and is necessary to understand extreme regimes such as the early universe and black holes. In addition, a theory of quantum gravity could help to solve some of the outstanding issues in both cosmology and particle physics.

The search for a theory of quantum gravity has been the subject of many studies over several decades and has led to a variety of approaches tackling the issues involved in constructing such a theory. Within these approaches, many insights about the properties of a final theory of quantum gravity have been obtained. Nevertheless, the correct ultraviolet complete theory of quantum gravity is still unknown. Furthermore, there is little experimental guidance directing towards a theory of quantum gravity. As a consequence, favoring one approach over the other strongly relies on theoretical prejudice.

In this thesis, we have discussed a few aspects of a theory of quantum gravity. Here, we have taken a conservative approach to the problem, as we have not introduced any new physics that has not been verified experimentally. Instead, we have investigated what our state of the art theories of quantum and gravitational physics teach us about quantum gravity. We have done this by covering two sides of the spectrum of quantum gravity theories.

First, we have exploited the tight connection between quantum theories and stochastic theories and discussed the formulation of elementary physical theories from the stochastic viewpoint. Moreover, we have shown that preservation of general covariance in such theories requires extensions of ordinary differential geometry due to the modifications of Leibniz' rule.

After this, we studied effective field theories of quantum gravity. Here, we used that a theory of quantum gravity must reduce to quantum field theory in flat spacetime, when the gravitational interaction is turned off, and to general relativity, when quantum effects are turned off. Under these assumptions, it is possible to construct an effective action of quantum gravity containing features that are independent of the ultraviolet completion of quantum gravity. Using this effective action, we were able to make model independent quantum gravity predictions at sub-Planckian energy scales.

In part II of this thesis, we have discussed stochastic dynamics of a single particle on manifolds and the connections of the stochastic framework to quantum theories. Moreover, we have discussed the necessary geometric extensions that allow to preserve general covariance in these theories.

In chapter 6 and Ref. [233], we reviewed the theory of stochastic mechanics. This stochastic theory is known to be equivalent to the first quantization of single scalar particles in standard approaches to quantum mechanics. Moreover, building on the success of stochastic analysis in constructive quantum field theory, it is expected that this equivalence can be extended beyond this simple case. In addition, we have reviewed the second order geometry framework, which allows to describe stochastic dynamics in a consistent and diffeomorphism invariant setting on smooth manifolds with a connection. This chapter then brings the two frameworks together and shows that this provides a consistent extension of stochastic mechanics to pseudo-Riemannian manifolds.

In chapter 7 and Ref. [234], we used the results from chapter 6 to formulate a relativistic version of stochastic mechanics on Lorentzian manifolds. The key ingredient here was the introduction of a stochastic energy-momentum relation. Furthermore, this formulation required the construction of a Brownian metric on the manifold, which can be obtained by a Wick rotation from the kinetic Lorentzian metric.

In chapter 8 and Ref. [235], we have reformulated the Nelson process as the real projection of a complex Wiener process. This simplifies the cumbersome original formulation of stochastic mechanics, as this original formulation is makes use of the Nelson process, which is constructed using two instead of one differential operators: one going forward in time and one going backward in time.

In part III of this thesis, we have considered the unique effective action of quantum gravity and used this action to obtain model independent quantum gravity predictions.

In chapter 9 and Ref. [77], we have used this framework to calculate the leading quantum gravitational corrections to the metric of a constant density star. Although the corrections for this object at infinity were known and reproduce the leading quantum gravitational corrections to the Newtonian potential, the analysis in chapter 9 allowed to extend the calculations to the entire spacetime, i.e., both the exterior region and the interior region.

In chapter 10 and Ref. [79], we have used the results from chapter 9 to calculate the corrections to the trajectories of test particles and the propagation of scalar fields in this geometry.

In chapter 11 and Ref. [237], we have discussed the breakdown of singularity theorems in quantum gravity. Although extensions of the classical singularity theorems have been formulated that include the quantum nature of matter, we have shown that the introduction of the quantum nature of gravity allows to violate the sufficient conditions for singularities to occur. This can be ascribed to the introduction of higher order terms in the gravitational action that change the order of the Einstein equations.

In chapter 12 and Ref. [78], we have shown that non-local corrections to the Einstein-Hilbert action can generate singularities in spacetimes that are classically singularity free. Moreover, we have derived conditions for which such a scenario can be avoided. From the effective field theory point of view these singularities indicate regions in spacetime where the perturbative approach breaks down.

In chapter 13 and Ref. [89], we have calculated corrections to the entropy of a black hole within the effective field theory formalism. Here, we have uncovered various subtleties in such calculations. Moreover, we have seen that the non-local terms in the effective action generate a topological correction, which can be interpreted as a pressure term.

In chapter 14 and Ref. [80], we showed that quantum gravity induces quantum hair in the external geometry of compact objects. Whereas in semiclassical gravity only the mass, charge, and angular momentum of an object are stored in the external geometry, quantum gravity allows to store more information such as the density profile in the external geometry.

In part IV of this thesis, we have considered more general effective field theories that include the quantum gravitational interactions in the matter section. Here, we used that all terms that respect the symmetry of the underlying high energy theory are expected to be present in a low energy effective field theory, as long as they are suppressed by the relevant scale.

In chapter 15 and Ref. [86], we have applied this idea and have considered dimension 5 and dimension 6 operators generated by the quantum gravitational interaction. We have then compared the resulting effective action to various experiments looking for weakly coupled light particles. This comparison allowed us to obtain a set of lower bounds for various types of dark matter particles using only assumptions about the symmetries of quantum gravity.

In chapter 16 and Refs. [87, 88], we have complemented the results from chapter 15 with upper bounds on the masses. Here, we used that effective interactions lead to the decay of massive dark matter particles into ordinary matter. Moreover, the time scale of such a decay must be larger than the age of the universe for the dark matter particles to still be present.

Chapter 18

Outlook

The results discussed in this thesis show that research into a theory of quantum gravity remains relevant and can be fruitful. We have particularly showed that progress can be made both on a phenomenological level and on a more formal level.

In part II, we have seen that quantum theories of gravity likely require a geometrical framework beyond Riemannian geometry. Second order geometry is a good candidate for such a framework. However, whether second order geometry or something else, such as non-commutative geometry [40], is the correct geometrical framework of quantum gravity remains an open question that must be answered by future research.

Any extension of Riemannian geometry to a quantum framework will require extensions of many notions encountered in differential geometry. Such extensions will play a crucial role in determining which spacetime symmetries are broken and which are unbroken in a quantum theory of gravity. For example, in chapter 6 and Refs. [209,233,236], extensions of Lie derivatives and Killing vectors are constructed in a second order geometry framework. It is then found that the second order modifications of first order Killing vectors lead to the breaking of spacetime symmetries according to classical observers.

Moreover, part II shows that stochastic quantization can lead to important insights in the interplay between quantum theories and gravity. However, in order to obtain such insights in a Lorentzian framework, the Nelsonian formulation of stochastic quantization must be further developed in a field theory context as was initiated in Ref. [184]. As discussed in chapter 6 and Refs. [233, 236], such a stochastic field theory formulated in curved spacetime will likely build on developments in the study of classical field theories in higher order jet bundles, as studied in Refs. [102,103]. Future research in this direction is desirable and could help to tackle outstanding issues in the formulation of quantum theories of gravity.

The results from part III show that model independent predictions of quantum gravity are possible, due to the fact that all theories of quantum gravity must reproduce general relativity at low energy scales. In this thesis, we have discussed several predictions at second order in curvature. In particular, we have seen that the unique effective action allows to calculate quantum gravitational corrections to classical solutions of the Einstein equation. An interesting extension of our results would be to go beyond the second order framework, and calculate corrections at third order.

Moreover, in chapter 11, we have seen that singularity theorems no longer apply in a quantum gravity context. This result was obtained by considering higher derivative terms in the action, which arise in a perturbative treatment. This chapter discusses the fact that singularity theorems can be formulated for classical gravity interacting with quantum matter, but also illustrates the difficulty in making any rigorous statements about the fate of singularities when gravity is quantized.

In order to understand the fate of singularities in a non perturbative treatment of quantum gravity, it will be necessary to generalize the intrinsically classical concept of geodesic completeness to a quantum context. Using the strong similarities between quantum and stochastic theories as discussed in part II, one could argue that in the context of quantum gravity one should study stochastic completeness instead of geodesic completeness. Stochastic completeness theorems, as for example discussed in Refs. [23, 24], can then provide strong guidance in the formulation of spacetime completeness theorems in a quantum context.

In chapter 13, we have discussed corrections to the entropy of a Schwarzschild black hole. The fact that the effective action allows to calculate such corrections raises the question whether the framework can be used to improve our understanding of questions related to the information stored in black holes and the black hole information paradox. Combining this idea with the calculations presented in chapter 9, it was found in chapter 14 that quantum corrections obtained from the effective action of quantum gravity induce a new type of hair for black holes. This presents a loophole in the formulation of the black hole information paradox.

Indeed, the black hole information paradox is usually formulated for classical spacetimes interacting with quantum matter. It is then often assumed that quantum gravitational effects can be ignored at the horizon of astronomical black holes, as the effects are hugely suppressed. However, as is the case for singularity theorems discussed in chapter 11, even tiny effects of quantum gravity can invalidate such classical reasoning [90]. This line of thought has been worked out and has been combined with recent insights from quantum information theory in Ref. [85]. Here, it is shown that the black hole information paradox is a semiclassical problem that does not persist in quantum gravity.

Another interesting possibility for future research is quantum gravity phenomenology and the connection to experiments. Although all the quantum gravity effects found in part III are far beyond the range of current experiments, certain effects may build up on cosmological timescales, which could bring them within experimental reach. In order to explore such effects, one should go beyond the quantum corrections studied in this thesis, as these are all obtained for static solutions of the Einstein equations, and consider time-dependent solutions of the Einstein equations instead.

Another path towards measuring quantum gravity is discussed in part IV. Here, we found that general arguments about the symmetries of an underlying UV-complete theory of quantum gravity provide an effective action that incorporates quantum gravity effects and is largely independent of the ultra violet completion. This allows to categorize and test ultraviolet completions of quantum gravity based on their symmetries.

As shown in part IV, some effective interactions lie within reach of current and near future experiments. For example, quantum gravity could induce an effective variation in some of the fundamental constants of the Standard Model. Such variations can be measured, as for example discussed in Refs. [30, 31]. Related ideas are also discussed in Ref. [7] for a variety of other experiments. Current and near future experiments can thus provide guidance towards the formulation of a theory of quantum gravity.

Bibliography

[1] C. Abel *et al.*, "Search for Axionlike Dark Matter through Nuclear Spin Precession in Electric and Magnetic Fields," Phys. Rev. X **7**, no. 4, 041034 (2017).

[2] B. P. Abbott *et al.* [LIGO Scientific and Virgo], "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116**, no.6, 061102 (2016).

[3] B. P. Abbott *et al.* [LIGO Scientific and Virgo], "GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs," Phys. Rev. X **9**, no.3, 031040 (2019).

[4] R. Abbott *et al.* [LIGO Scientific and Virgo], "GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run," Phys. Rev. X **11**, 021053 (2021).

[5] R. Abbott *et al.* [LIGO Scientific, VIRGO and KAGRA], "GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo During the Second Part of the Third Observing Run," arXiv:2111.03606 [gr-qc] (2021).

[6] A. Accioly, B. L. Giacchini and I. L. Shapiro, "Low-energy effects in a higher-derivative gravity model with real and complex massive poles," Phys. Rev. D **96**, no. 10, 104004 (2017).

[7] A. Addazi *et al.*, "Quantum gravity phenomenology at the dawn of the multi-messenger era – A review," arXiv:2111.05659 [hep-ph] (2021).

[8] P. A. R. Ade *et al.* [Planck], "Planck 2013 results. I. Overview of products and scientific results," Astron. Astrophys. **571**, A1 (2014).

[9] P. A. R. Ade *et al.* [Planck], "Planck 2015 results. XIII. Cosmological parameters," Astron. Astrophys. **594**, A13 (2016).

[10] E. G. Adelberger, B. R. Heckel, S. A. Hoedl, C. D. Hoyle, D. J. Kapner and A. Upadhye, "Particle Physics Implications of a Recent Test of the Gravitational Inverse Sqaure Law," Phys. Rev. Lett. **98**, 131104 (2007).

[11] E. G. Adelberger, J. H. Gundlach, B. R. Heckel, S. Hoedl and S. Schlamminger, "Torsion balance experiments: A low-energy frontier of particle physics," Prog. Part. Nucl. Phys. **62**, 102 (2009).

[12] K. Akiyama *et al.* [Event Horizon Telescope], "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole," Astrophys. J. Lett. **875**, L1 (2019).

[13] I. Alani and O. Santillan, "Cosmological singularity theorems for $f(R)$ gravity theories," JCAP **05**, 023 (2016).

[14] S.A. Albeverio, R.J. Høegh-Krohn and S. Mazzucchi, "Mathematical Theory of Feynman Path Integrals," Lecture Notes in Mathematics **523**, Springer, Berlin (2008).

[15] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lüst and A. Riotto, "Aspects of Quadratic Gravity," Fortsch. Phys. **64**, no.2-3, 176-189 (2016).

[16] R. Arnowitt, S. Deser and C. W. Misner, "Finite Self-Energy of Classical Point Particles," Phys. Rev. Lett. **4**, 375-377 (1960).

[17] A. Arvanitaki, J. Huang and K. Van Tilburg, "Searching for dilaton dark matter with atomic clocks," Phys. Rev. D **91**, no. 1, 015015 (2015).

[18] A. Arvanitaki, M. Baryakhtar and X. Huang, "Discovering the QCD Axion with Black Holes and Gravitational Waves," Phys. Rev. D **91**, no.8, 084011 (2015).

[19] A. Arvanitaki, S. Dimopoulos and K. Van Tilburg, "Sound of Dark Matter: Searching for Light Scalars with Resonant-Mass Detectors," Phys. Rev. Lett. **116**, no. 3, 031102 (2016).

[20] M. Asorey, J. L. Lopez and I. L. Shapiro, "Some remarks on high derivative quantum gravity," Int. J. Mod. Phys. A **12**, 5711-5734 (1997).

[21] M. Atkins and X. Calmet, "On the unitarity of linearized General Relativity coupled to matter," Phys. Lett. B **695**, 298 (2011).

[22] E. Ayon-Beato and A. Garcia, "The Bardeen model as a nonlinear magnetic monopole," Phys. Lett. B **493**, 149-152 (2000).

[23] I. Bailleul, "A Probabilistic View on Singularities" J. Math. Phys. **52**, no. 2, p.023520 (2011).

[24] I. Bailleul and J. Franchi, "Non-explosion Criteria for Relativistic Diffusions" Ann. Probab. **40**, no. 5, p.2168-2196 (2010).

[25] R. Ballou *et al.* [OSQAR], "New exclusion limits on scalar and pseudoscalar axionlike particles from light shining through a wall," Phys. Rev. D **92**, no. 9, 092002 (2015).

[26] T. Banks and N. Seiberg, "Symmetries and Strings in Field Theory and Gravity," Phys. Rev. D **83**, 084019 (2011).

[27] N. Bao, S. M. Carroll, A. Chatwin-Davies, J. Pollack and G. N. Remmen, "Branches of the Black Hole Wave Function Need Not Contain Firewalls," Phys. Rev. D **97**, no.12, 126014 (2018).

[28] J. M. Bardeen "Non-singular general-relativistic gravitational collapse" in *Proceedings of International Conference GR5*, Tbilisi, 174 (1968).

[29] G. Barnich, "The Coulomb solution as a coherent state of unphysical photons," Gen. Rel. Grav. **43**, 2527-2530 (2011).

[30] G. Barontini, V. Boyer, X. Calmet, N. J. Fitch, E. M. Forgan, R. M. Godun, J. Goldwin, V. Guarrera, I. R. Hill and M. Jeong, *et al.* "QSNET, a network of clocks for measuring the stability of fundamental constants," Proc. SPIE Int. Soc. Opt. Eng. **11881**, 63 (2021).

[31] G. Barontini, L. Blackburn, V. Boyer, F. Butuc-Mayer, X. Calmet, J. R. Crespo Lopez-Urrutia, E. A. Curtis, B. Darquie, J. Dunningham and N. J. Fitch, *et al.* "Measuring the stability of fundamental constants with a network of clocks," EPJ Quant. Technol. **9**, no.1, 12 (2022).

[32] S. M. Barr and D. Seckel, "Planck scale corrections to axion models," Phys. Rev. D **46**, 539 (1992).

[33] J. D. Barrow and A. C. Ottewill, "The Stability of General Relativistic Cosmological Theory," J. Phys. A **16**, 2757 (1983).

[34] A. O. Barvinsky and G. A. Vilkovisky, "The Generalized Schwinger-de Witt Technique And The Unique Effective Action In Quantum Gravity," Phys. Lett. **131B**, 313 (1983).

[35] A. O. Barvinsky and G. A. Vilkovisky, "The Generalized Schwinger-Dewitt Technique in Gauge Theories and Quantum Gravity," Phys. Rept. **119**, 1 (1985).

[36] A. O. Barvinsky and G. A. Vilkovisky, "Beyond the Schwinger-Dewitt Technique: Converting Loops Into Trees and In-In Currents," Nucl. Phys. B **282**, 163 (1987).

[37] A. O. Barvinsky and G. A. Vilkovisky, "Covariant perturbation theory. 2: Second order in the curvature. General algorithms," Nucl. Phys. B **333**, 471 (1990).

[38] A. O. Barvinsky, Y. V. Gusev, V. V. Zhytnikov and G. A. Vilkovisky, "Covariant perturbation theory. 4. Third order in the curvature," arXiv:0911.1168 [hep-th] (1993).

[39] A. O. Barvinsky, Y. V. Gusev, G. A. Vilkovisky and V. V. Zhytnikov, "The Basis of nonlocal curvature invariants in quantum gravity theory. (Third order.)," J. Math. Phys. **35**, 3525-3542 (1994).

[40] E. J. Beggs and S. Majid "Quantum Riemannian Geometry," Springer, Cham (2020).

[41] J. D. Bekenstein, "Black holes and the second law," Lett. Nuovo Cim. **4**, 737-740 (1972).

[42] J. D. Bekenstein, "Nonsingular General Relativistic Cosmologies," Phys. Rev. D **11**, 2072 (1975).

[43] J. D. Bekenstein, “Relativistic gravitation theory for the MOND paradigm,” Phys. Rev. D **70**, 083509 (2004).

[44] T. Bernal, V. H. Robles and T. Matos, “Scalar field dark matter in clusters of galaxies,” Mon. Not. Roy. Astron. Soc. **468**, no. 3, 3135 (2017).

[45] T. Bernal, L. M. Fernández-Hernández, T. Matos and M. A. Rodríguez-Meza, “Rotation curves of high-resolution LSB and SPARC galaxies with fuzzy and multistate (ultralight boson) scalar field dark matter,” Mon. Not. Roy. Astron. Soc. **475**, no. 2, 1447 (2018).

[46] J. Beyer and C. Wetterich, “Small scale structures in coupled scalar field dark matter,” Phys. Lett. B **738**, 418 (2014).

[47] G.D. Birkhoff, “Relativity and Modern Physics,” Harvard University Press, Cambridge (1923).

[48] N. D. Birrell and P. C. W. Davies, “Quantum Fields in Curved Space,” Cambridge University Press, Cambridge (1982).

[49] N. E. J. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, “Quantum corrections to the Schwarzschild and Kerr metrics,” Phys. Rev. D **68**, 084005 (2003). [erratum: Phys. Rev. D **71**, 069904 (2005)].

[50] C. Blake *et al.* “The WiggleZ Dark Energy Survey: mapping the distance-redshift relation with baryon acoustic oscillations,” Mon. Not. Roy. Astron. Soc. **418**, 1707-1724 (2011).

[51] D. Blas, D. L. Nacir and S. Sibiryakov, “Ultralight Dark Matter Resonates with Binary Pulsars,” Phys. Rev. Lett. **118**, no. 26, 261102 (2017).

[52] K. Blum, R. T. D’Agnolo, M. Lisanti and B. R. Safdi, “Constraining Axion Dark Matter with Big Bang Nucleosynthesis,” Phys. Lett. B **737**, 30 (2014).

[53] C. Böhm, X. Chu, J. L. Kuo and J. Pradler, “Scalar dark matter candidates revisited,” Phys. Rev. D **103**, no.7, 075005 (2021).

[54] C. T.. Bolton, “Identification of Cygnus X-1 with HDE 226868,” Nature **235**, 271-273 (1972).

[55] A. Borde, “Geodesic focusing, energy conditions and singularities,” Class. Quant. Grav. **4**, 343 (1987).

[56] M. S. Boucenna, R. A. Lineros and J. W. F. Valle, “Planck-scale effects on WIMP dark matter,” Front. in Phys. **1**, 34 (2013).

[57] B. Bozek, D. J. E. Marsh, J. Silk and R. F. G. Wyse, “Galaxy UV-luminosity function and reionization constraints on axion dark matter,” Mon. Not. Roy. Astron. Soc. **450**, no. 1, 209 (2015).

[58] V. B. Braginsky and V. I. Panov, "Verification of the equivalence of inertial and gravitation mass," JETP **34**, no. 3, 463 (1972).

[59] A. E. Broderick, A. Loeb and R. Narayan, "The Event Horizon of Sagittarius A*," Astrophys. J. **701**, 1357-1366 (2009).

[60] P. J. Brown, C. J. Fewster and E. A. Kontou, "A singularity theorem for Einstein-Klein-Gordon theory," Gen. Rel. Grav. **50**, no. 10, 121 (2018).

[61] I. L. Buchbinder, S. D. Odintsov and I. L. Shapiro, "Effective action in quantum gravity," CRC Press, Bristol, (1992).

[62] H. A. Buchdahl, "General Relativistic Fluid Spheres," Phys. Rev. **116**, 1027 (1959).

[63] R. V. Buniy and S. D. H. Hsu, "Macroscopic Superposition States in Isolated Quantum Systems," Found. Phys. **51**, no.4, 85 (2021).

[64] C. P. Burgess, "Quantum gravity in everyday life: General relativity as an effective field theory," Living Rev. Rel. **7**, 5-56 (2004).

[65] C. P. Burgess, "Introduction to Effective Field Theory," Ann. Rev. Nucl. Part. Sci. **57**, 329-362 (2007).

[66] C. P. Burgess and M. Williams, "Who You Gonna Call? Runaway Ghosts, Higher Derivatives and Time-Dependence in EFTs," JHEP **08**, 074 (2014).

[67] E. Calabrese and D. N. Spergel, "Ultra-Light Dark Matter in Ultra-Faint Dwarf Galaxies," Mon. Not. Roy. Astron. Soc. **460**, no. 4, 4397 (2016).

[68] X. Calmet, "The Lightest of Black Holes," Mod. Phys. Lett. A **29**, no.38, 1450204 (2014).

[69] X. Calmet, "Virtual Black Holes, Remnants and the Information Paradox," Class. Quant. Grav. **32**, no.4, 045007 (2015).

[70] X. Calmet, "Vanishing of Quantum Gravitational Corrections to Vacuum Solutions of General Relativity at Second Order in Curvature," Phys. Lett. B **787**, 36 (2018).

[71] X. Calmet, "On Searches for Gravitational Dark Matter with Quantum Sensors," Eur. Phys. J. Plus **134**, no. 10, 503 (2019).

[72] X. Calmet, "Hidden Sector and Gravity," Phys. Lett. B **801**, 135152 (2020).

[73] X. Calmet, S. Capozziello and D. Pryer, "Gravitational Effective Action at Second Order in Curvature and Gravitational Waves," Eur. Phys. J. C **77**, 589 (2017).

[74] X. Calmet, B. Carr and E. Winstanley, "Quantum Black Holes," Springer, Berlin (2014).

[75] X. Calmet and R. Casadio, "The horizon of the lightest black hole," Eur. Phys. J. C **75**, no.9, 445 (2015).

[76] X. Calmet, R. Casadio, A. Y. Kamenshchik and O. V. Teryaev, "Graviton propagator, renormalization scale and black-hole like states," Phys. Lett. B **774**, 332-337 (2017).

[77] X. Calmet, R. Casadio and F. Kuipers, "Quantum Gravitational Corrections to a Star Metric and the Black Hole Limit," Phys. Rev. D **100**, no.8, 086010 (2019).

[78] X. Calmet, R. Casadio and F. Kuipers, "Singularities in quantum corrected spacetimes," Phys. Lett. B **807**, 135605 (2020).

[79] X. Calmet, R. Casadio and F. Kuipers, "Quantum corrected equations of motion in the interior and exterior Schwarzschild spacetime," Phys. Rev. D **102**, no.2, 026018 (2020).

[80] X. Calmet, R. Casadio, S. D. H. Hsu and F. Kuipers, "Quantum Hair from Gravity," Phys. Rev. Lett. **128**, no.11, 111301 (2022).

[81] X. Calmet, D. Croon and C. Fritz, "Non-locality in Quantum Field Theory due to General Relativity," Eur. Phys. J. C **75**, 605 (2015).

[82] X. Calmet, D. Fragkakis and N. Gausmann, "The flavor of quantum gravity," Eur. Phys. J. C **71**, 1781 (2011).

[83] X. Calmet, W. Gong and S. D. H. Hsu, "Colorful quantum black holes at the LHC," Phys. Lett. B **668**, 20 (2008).

[84] X. Calmet, S. D. H. Hsu and D. Reeb, "Quantum gravity at a TeV and the renormalization of Newton's constant," Phys. Rev. D **77**, 125015 (2008).

[85] X. Calmet and S. D. H. Hsu, "Quantum Hair and Black Hole Information," arXiv:2112.05171 [hep-th] (2021).

[86] X. Calmet and F. Kuipers, "Bounds on very weakly interacting ultra light scalar and pseudoscalar dark matter from quantum gravity," Eur. Phys. J. C **80**, no.8, 781 (2020).

[87] X. Calmet and F. Kuipers, "Theoretical bounds on dark matter masses," Phys. Lett. B **814**, 136068 (2021).

[88] X. Calmet and F. Kuipers, "Implications of quantum gravity for dark matter," Int. J. Mod. Phys. D **30**, no.14, 2142004 (2021).

[89] X. Calmet and F. Kuipers, "Quantum gravitational corrections to the entropy of a Schwarzschild black hole," Phys. Rev. D **104**, no.6, 066012 (2021).

[90] X. Calmet and F. Kuipers, "Black Holes in Quantum Gravity," Il Nuovo Cimento **45 C**, 037 (2022).

[91] X. Calmet and I. Kuntz, "Higgs Starobinsky Inflation," Eur. Phys. J. C **76**, 289 (2016).

[92] X. Calmet and I. Kuntz, "What is modified gravity and how to differentiate it from particle dark matter?," Eur. Phys. J. C **77**, no. 2, 132 (2017).

[93] X. Calmet and B. Latosh, "Three Waves for Quantum Gravity," Eur. Phys. J. C **78**, 205 (2018).

[94] X. Calmet and B. Latosh, "Dark Matter in Quantum Gravity," Eur. Phys. J. C **78**, 520 (2018).

[95] X. Calmet and B. Latosh, "The Spectrum of Quantum Gravity," Phys. Part. Nucl. Lett. **16**, no. 6, 656 (2019).

[96] X. Calmet and S. K. Majee, "Effective Theory for Dark Matter and a New Force in the Dark Matter Sector," Phys. Lett. B **679**, 267-269 (2009).

[97] X. Calmet and B. K. El-Menoufi, "Quantum Corrections to Schwarzschild Black Hole," Eur. Phys. J. C **77**, no.4, 243 (2017).

[98] X. Calmet, B. K. El-Menoufi, B. Latosh and S. Mohapatra, "Gravitational Radiation in Quantum Gravity," Eur. Phys. J. C **78**, no.9, 780 (2018).

[99] X. Calmet and V. Sanz, "Excursion into Quantum Gravity via Inflation," Phys. Lett. B **737**, 12-15 (2014).

[100] X. Calmet and T. C. Yang, "Frame Transformations of Gravitational Theories," Int. J. Mod. Phys. A **28**, 1350042 (2013).

[101] R.H. Cameron, "A Family of Integrals Serving to Connect the Wiener and Feynman Integrals," J. Math. and Phys. **39**, p. 126-140 (1960).

[102] C. M. Campos, M. de Leon, D. M. de Diego and J. Vankerschaver, "Unambiguous Formalism for Higher-Order Lagrangian Field Theories," J. Phys. A **42**, 475207 (2009).

[103] C. M. Campos, M. de Leon and D. M. de Diego, "Constrained Variational Calculus for Higher Order Classical Field Theories," J. Phys. A **43**, no.45, 455206 (2010).

[104] S. Capozziello and M. De Laurentis, "Extended Theories of Gravity," Phys. Rept. **509**, 167-321 (2011).

[105] V. Cardoso, Ó. J. C. Dias, G. S. Hartnett, M. Middleton, P. Pani and J. E. Santos, "Constraining the mass of dark photons and axion-like particles through black-hole superradiance," JCAP **1803**, 043 (2018).

[106] S. M. Carroll, "Spacetime and Geometry," Pearson, Harlow (2014).

[107] B. Carter, "Axisymmetric Black Hole Has Only Two Degrees of Freedom," Phys. Rev. Lett. **26**, 331-333 (1971).

[108] R. Casadio, "Localised particles and fuzzy horizons: A tool for probing Quantum Black Holes," arXiv:1305.3195 [gr-qc] (2013).

[109] R. Casadio, “Quantum black holes and resolution of the singularity,” arXiv:2103.00183 [gr-qc] (2021).

[110] R. Casadio, A. Giugno and A. Orlandi, “Thermal corpuscular black holes,” Phys. Rev. D **91**, 124069 (2015).

[111] R. Casadio, A. Y. Kamenshchik and O. V. Teryaev, “Hawking radiation and the Bloom–Gilman duality,” Class. Quant. Grav. **35**, no.15, 155001 (2018).

[112] R. Casadio, M. Lenzi and O. Micu, “Bootstrapping Newtonian gravity,” Phys. Rev. D **98**, no.10, 104016 (2018).

[113] R. Casadio and F. Scardigli, “Horizon wave-function for single localized particles: GUP and quantum black hole decay,” Eur. Phys. J. C **74**, no.1, 2685 (2014).

[114] A. Celotti, J. C. Miller and D. W. Sciama, “Astrophysical evidence for the existence of black holes: Topical review,” Class. Quant. Grav. **16**, A3 (1999).

[115] C. Chowdhury, O. Papadoulaki and S. Raju, “A physical protocol for observers near the boundary to obtain bulk information in quantum gravity,” SciPost Phys. **10**, no.5, 106 (2021).

[116] C. Chowdhury, V. Godet, O. Papadoulaki and S. Raju, “Holography from the Wheeler-DeWitt equation,” arXiv:2107.14802 [hep-th] (2021).

[117] J. E. McClintock and R. A. Remillard, “Black hole binaries” arXiv:astro-ph/0306213 [astro-ph] (2003).

[118] D. Clowe, M. Bradac, A. H. Gonzalez, M. Markevitch, S. W. Randall, C. Jones and D. Zaritsky, “A direct empirical proof of the existence of dark matter,” Astrophys. J. Lett. **648**, L109-L113 (2006).

[119] A. Codello and R. K. Jain, “On the covariant formalism of the effective field theory of gravity and its cosmological implications,” Class. Quant. Grav. **34**, no.3, 035015 (2017).

[120] S. R. Coleman, “Why There Is Nothing Rather Than Something: A Theory of the Cosmological Constant,” Nucl. Phys. B **310**, 643-668 (1988).

[121] P. S. Corasaniti, S. Agarwal, D. J. E. Marsh and S. Das, “Constraints on dark matter scenarios from measurements of the galaxy luminosity function at high redshifts,” Phys. Rev. D **95**, no. 8, 083512 (2017).

[122] E. Corbelli and P. Salucci, “The Extended Rotation Curve and the Dark Matter Halo of M33,” Mon. Not. Roy. Astron. Soc. **311**, 441-447 (2000).

[123] R. G. Crittenden and N. Turok, “Looking for Lambda with the Rees-Sciama effect,” Phys. Rev. Lett. **76**, 575 (1996).

[124] Yu. L. Daletskii, "Functional integrals connected with operator evolution equations" Russ. Math. Surv. **17**, no. 5, 1-107 (1962).

[125] P. H. Damgaard and H. Huffel, "Stochastic Quantization," Phys. Rept. **152**, 227 (1987).

[126] P. H. Damgaard and K. Tsokos, "Stochastic Quantization With Fermions," Nucl. Phys. B **235**, 75-92 (1984).

[127] T. G. Dankel, "Mechanics on manifolds and the incorporation of spin into Nelson's stochastic mechanics," Arch. Rational. Mech. Anal. **37**, 192 (1971).

[128] P. C. W. Davies, "Scalar particle production in Schwarzschild and Rindler metrics," J. Phys. A **8**, 609-616 (1975).

[129] H. Davoudiasl, P. B. Denton and D. A. McGady, "Ultralight fermionic dark matter," Phys. Rev. D **103**, no.5, 055014 (2021).

[130] M. Derakhshani, "A Suggested Answer To Wallstrom's Criticism: Zitterbewegung Stochastic Mechanics I," arXiv:1510.06391 [quant-ph] (2015).

[131] M. Derakhshani, "A Suggested Answer To Wallstrom's Criticism: Zitterbewegung Stochastic Mechanics II," arXiv:1607.08838 [quant-ph] (2016).

[132] S. Deser and P. van Nieuwenhuizen, "One Loop Divergences of Quantized Einstein-Maxwell Fields," Phys. Rev. D **10**, 401 (1974).

[133] S. Deser and A. N. Redlich, "String Induced Gravity and Ghost Freedom," Phys. Lett. B **176**, 350 (1986). [erratum: Phys. Lett. B **186**, 461 (1987)].

[134] R. Dijkgraaf, D. Orlando and S. Reffert, "Relating Field Theories via Stochastic Quantization," Nucl. Phys. B **824**, 365-386 (2010).

[135] D. Dohrn and F. Guerra, "Nelson's stochastic mechanics on Riemannian manifolds," Lett. Nuovo Cimento **22**, no.4, 121 (1978).

[136] D. Dohrn and F. Guerra, "Geodesic correction to stochastic parallel displacement of tensors" in "Stochastic Behavior in Classical and quantum Hamiltonian Systems," Lecture Notes in Physics **93**, 241-249, Springer, Berlin (1979).

[137] D. Dohrn and F. Guerra, "Compatibility between the Brownian metric and the kinetic metric in Nelson stochastic quantization," Phys. Rev. D **31**, 2521-2524 (1985).

[138] B. P. Dolan, "Pressure and volume in the first law of black hole thermodynamics," Class. Quant. Grav. **28**, 235017 (2011).

[139] J. F. Donoghue, "Leading quantum correction to the Newtonian potential," Phys. Rev. Lett. **72**, 2996-2999 (1994).

[140] J. F. Donoghue, "General relativity as an effective field theory: The leading quantum corrections," Phys. Rev. D **50**, 3874 (1994).

[141] J. F. Donoghue, "Introduction to the effective field theory description of gravity," arXiv:gr-qc/9512024 [gr-qc] (1995).

[142] J. F. Donoghue, "The effective field theory treatment of quantum gravity," AIP Conf. Proc. **1483**, no.1, 73-94 (2012).

[143] J. F. Donoghue and B. K. El-Menoufi, "Nonlocal quantum effects in cosmology: Quantum memory, nonlocal FLRW equations, and singularity avoidance," Phys. Rev. D **89**, no.10, 104062 (2014).

[144] J. F. Donoghue and B. K. El-Menoufi, "Covariant non-local action for massless QED and the curvature expansion," JHEP **10**, 044 (2015).

[145] G. R. Dvali and M. Zaldarriaga, "Changing alpha with time: Implications for fifth force type experiments and quintessence," Phys. Rev. Lett. **88**, 091303 (2002).

[146] K. Ehret *et al.*, "New ALPS Results on Hidden-Sector Lightweights," Phys. Lett. B **689**, 149 (2010).

[147] A. Eichhorn, "Asymptotically safe gravity," arXiv:2003.00044 [gr-qc] (2020).

[148] D. A. Eliezer and R. P. Woodard, "The Problem of Nonlocality in String Theory," Nucl. Phys. B **325**, 389 (1989).

[149] E. Elizalde and S. D. Odintsov, "A Renormalization group improved nonlocal gravitational effective Lagrangian," Mod. Phys. Lett. A **10**, 1821 (1995).

[150] M. Emery, "Stochastic Calculus in Manifolds," Springer, Berlin (1989).

[151] J. Erlich, "Stochastic Emergent Quantum Gravity," Class. Quant. Grav. **35**, no.24, 245005 (2018).

[152] S. M. Faber and R. E. Jackson, "Velocity dispersions and mass to light ratios for elliptical galaxies," Astrophys. J. **204**, 668 (1976).

[153] I. Fényes, "Eine Wahrscheinlichkeitstheoretische Begründung und Interpretation der Quantenmechanik," Zeitschrift für Physik **132**, 81 (1952).

[154] M. Ferraris, M. Francaviglia and G. Magnano, "Do non-linear metric theories of gravitation really exist?," Class. Quant. Grav. **5**, L95 (1988).

[155] C. J. Fewster and G. J. Galloway, "Singularity theorems from weakened energy conditions," Class. Quant. Grav. **28**, 125009 (2011).

[156] C. J. Fewster and E. A. Kontou, "A new derivation of singularity theorems with weakened energy hypotheses," Class. Quant. Grav. **37**, no.6, 065010 (2020).

[157] L. H. Ford, "The Classical singularity theorems and their quantum loop holes," Int. J. Theor. Phys. **42**, 1219-1227 (2003).

[158] K. Freese, B. Fields and D. Graff, “Limits on stellar objects as the dark matter of our halo: nonbaryonic dark matter seems to be required,” Nucl. Phys. B Proc. Suppl. **80**, 0305 (2000).

[159] L. Fritsche and M. Haugk, “Stochastic Foundation of Quantum Mechanics and the Origin of Particle Spin,” arXiv:0912.3442 [physics.gen-ph] (2009).

[160] V. P. Frolov, “Information loss problem and a ’black hole‘ model with a closed apparent horizon,” JHEP **05**, 049 (2014).

[161] V. P. Frolov and G. Vilkovisky, “Spherically Symmetric Collapse in Quantum Gravity,” Phys. Lett. B **106**, no. 4, 307 (1981).

[162] S. A. Fulling, “Nonuniqueness of canonical field quantization in Riemannian spacetime,” Phys. Rev. D **7**, 2850-2862 (1973).

[163] D. V. Fursaev, “Temperature and entropy of a quantum black hole and conformal anomaly,” Phys. Rev. D **51**, 5352-5355 (1995).

[164] G. Gaeta, “EPR and stochastic mechanics,” Phys. Lett. A **175**, no.5, 267 (1993).

[165] P. Garbaczewski, J. R. Klauder and R. Olkiewicz, “The Schrodinger problem, Levy processes and all that noise in relativistic quantum mechanics,” Phys. Rev. E **51**, 4114-4131 (1995).

[166] I.M. Gelfand and A.M. Yaglom, “Integration in Functional Spaces and its Applications in Quantum Physics,” J. Math. Phys. **1**, 48 (1960).

[167] H. Georgi, “Effective field theory,” Ann. Rev. Nucl. Part. Sci. **43**, 209-252 (1993).

[168] R. P. Geroch and J. H. Traschen, “Strings and Other Distributional Sources in General Relativity,” Phys. Rev. D **36**, 1017 (1987).

[169] A. M. Ghez, B. L. Klein, M. Morris and E. E. Becklin, “High proper motion stars in the vicinity of Sgr A*: Evidence for a supermassive black hole at the center of our galaxy,” Astrophys. J. **509**, 678-686 (1998).

[170] B. L. Giacchini and T. de Paula Netto, “Weak-field limit and regular solutions in polynomial higher-derivative gravities,” Eur. Phys. J. C **79**, no. 3, 217 (2019).

[171] B. L. Giacchini and T. de Paula Netto, “Effective delta sources and regularity in higher-derivative and ghost-free gravity,” JCAP **1907**, 013 (2019).

[172] T. Giannantonio, R. Scranton, R. G. Crittenden, R. C. Nichol, S. P. Boughn, A. D. Myers and G. T. Richards, “Combined analysis of the integrated Sachs-Wolfe effect and cosmological implications,” Phys. Rev. D **77**, 123520 (2008).

[173] S. B. Giddings and A. Strominger, “Loss of Incoherence and Determination of Coupling Constants in Quantum Gravity,” Nucl. Phys. B **307**, 854-866 (1988).

[174] S. B. Giddings and S. D. Thomas, "High-energy colliders as black hole factories: The End of short distance physics," Phys. Rev. D **65**, 056010 (2002).

[175] G. Gilbert, "Wormhole Induced Proton Decay," Nucl. Phys. B **328**, 159-170 (1989).

[176] S. Gillessen, F. Eisenhauer, S. Trippe, T. Alexander, R. Genzel, F. Martins and T. Ott, "Monitoring stellar orbits around the Massive Black Hole in the Galactic Center," Astrophys. J. **692**, 1075-1109 (2009).

[177] J. Glimm and A. M. Jaffe, "Quantum Physics: a functional integral point of view," Springer-Verlag, New York (1987).

[178] C. Gökler "Equivalence of quantum harmonic oscillators and classical oscillators subject to random forces," arXiv:2010.14930 [quant-ph] (2020).

[179] M. H. Goroff and A. Sagnotti, "Quantum Gravity at Two Loops," Phys. Lett. B **160**, 81-86 (1985).

[180] M. H. Goroff and A. Sagnotti, "The Ultraviolet Behavior of Einstein Gravity," Nucl. Phys. B **266**, 709-736 (1986).

[181] P. W. Graham and S. Rajendran, "New Observables for Direct Detection of Axion Dark Matter," Phys. Rev. D **88**, 035023 (2013).

[182] P. W. Graham, D. E. Kaplan, J. Mardon, S. Rajendran and W. A. Terrano, "Dark Matter Direct Detection with Accelerometers," Phys. Rev. D **93**, no. 7, 075029 (2016).

[183] A. M. Green and B. J. Kavanagh, "Primordial Black Holes as a dark matter candidate," J. Phys. G **48**, no.4, 043001 (2021).

[184] F. Guerra, "Structural Aspects of Stochastic Mechanics and Stochastic Field Theory," Phys. Rept. **77**, 263-312 (1981).

[185] F. Guerra, "Euclidean field theory," arXiv:math-ph/0510087 [math-ph] (2005).

[186] F. Guerra and M. I. Loffredo, "Stochastic equations for the Maxwell field," Lett. Nuovo Cimento **27**, 41-45 (1980).

[187] F. Guerra and L. M. Morato, "Quantization of Dynamical Systems and Stochastic Control Theory," Phys. Rev. D **27**, 1774 (1983).

[188] F. Guerra and P. Ruggiero, "New interpretation of the Euclidean-Markov field in the framework of physical Minkowski space-time," Phys. Rev. Lett. **31**, 1022-1025 (1973).

[189] F. Guerra and P. Ruggiero, "A note on relativistic Markov processes," Lett. Nuovo Cimento **23**, 528 (1978).

[190] T. Han and S. Willenbrock, "Scale of quantum gravity," Phys. Lett. B **616**, 215 (2005).

[191] S. W. Hawking, “Singularities in the universe,” Phys. Rev. Lett. **17**, 444-445 (1966).

[192] S. W. Hawking, “Particle Creation by Black Holes,” Commun. Math. Phys. **43**, 199-220 (1975). [erratum: Commun. Math. Phys. **46**, 206 (1976)].

[193] S. W. Hawking and G. F. R. Ellis, “The Large Scale Structure of Space-Time,” Cambridge University Press, Cambridge (1973).

[194] S. A. Hayward, “Formation and evaporation of regular black holes,” Phys. Rev. Lett. **96**, 031103 (2006).

[195] A. Hees, J. Guéna, M. Abgrall, S. Bize and P. Wolf, “Searching for an oscillating massive scalar field as a dark matter candidate using atomic hyperfine frequency comparisons,” Phys. Rev. Lett. **117**, no. 6, 061301 (2016).

[196] N. Heller and V. Sahakian, “On Strongly Coupled Matrix Theory and Stochastic Quantization: A New Approach to Holographic Dualities,” arXiv:2108.08872 [hep-th] (2021).

[197] A. Hindawi, B. A. Ovrut and D. Waldram, “Consistent spin two coupling and quadratic gravitation,” Phys. Rev. D **53**, 5583 (1996).

[198] S. Ho, C. Hirata, N. Padmanabhan, U. Seljak and N. Bahcall, “Correlation of CMB with large-scale structure: I. ISW Tomography and Cosmological Implications,” Phys. Rev. D **78**, 043519 (2008).

[199] R. Holman, S. D. H. Hsu, T. W. Kephart, E. W. Kolb, R. Watkins and L. M. Widrow, “Solutions to the strong CP problem in a world with gravity,” Phys. Lett. B **282**, 132 (1992).

[200] G. ’t Hooft and M. J. G. Veltman, “One loop divergencies in the theory of gravitation,” Ann. Inst. H. Poincare Phys. Theor. A **20**, 69-94 (1974).

[201] G. W. Horndeski, “Second-order scalar-tensor field equations in a four-dimensional space,” Int. J. Theor. Phys. **10**, 363-384 (1974).

[202] C. D. Hoyle, D. J. Kapner, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, U. Schmidt and H. E. Swanson, “Sub-millimeter tests of the gravitational inverse-square law,” Phys. Rev. D **70**, 042004 (2004).

[203] S. D. H. Hsu, “Quantum production of black holes,” Phys. Lett. B **555**, 92-98 (2003).

[204] S. D. H. Hsu, “The Black hole information paradox and macroscopic superpositions,” J. Phys. Conf. Ser. **222**, 012037 (2010).

[205] S. D. H. Hsu, “Macroscopic superpositions and black hole unitarity,” arXiv:1302.0451 [hep-th] (2013).

[206] S. D. H. Hsu, “Factorization of unitarity and black hole firewalls,” arXiv:1308.5686 [hep-th] (2013).

[207] S. D. H. Hsu and D. Reeb, “Black holes, information and decoherence,” Phys. Rev. D **79**, 124037 (2009).

[208] W. Hu, R. Barkana and A. Gruzinov, “Cold and fuzzy dark matter,” Phys. Rev. Lett. **85**, 1158-1161 (2000).

[209] Q. Huang and J. C. Zambrini, “From Second-order Differential Geometry to Stochastic Geometric Mechanics,” arXiv:2201.03706 [math-ph] (2022).

[210] L. Hui, J. P. Ostriker, S. Tremaine and E. Witten, “Ultralight scalars as cosmological dark matter,” Phys. Rev. D **95**, no. 4, 043541 (2017).

[211] V. Iršič, M. Viel, M. G. Haehnelt, J. S. Bolton and G. D. Becker, “First constraints on fuzzy dark matter from Lyman-α forest data and hydrodynamical simulations,” Phys. Rev. Lett. **119**, no.3, 031302 (2017).

[212] W. Israel, “Event horizons in static vacuum space-times,” Phys. Rev. **164**, 1776-1779 (1967).

[213] W. Israel, “Event horizons in static electrovac space-times,” Commun. Math. Phys. **8**, 245-260 (1968).

[214] K. Itô, “Generalized uniform complex measures in the Hilbertian metric space with their application to the Feynman path integral,” Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability **II**, no.1, 145-161 (1967).

[215] A. Jakubiec and J. Kijowski, “On Theories of Gravitation With Nonlinear Lagrangians,” Phys. Rev. D **37**, 1406 (1988).

[216] J.T. Jebsen, “Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinischen Gravitationsgleichungen im Vakuum,” Arkiv för Matematik, Astronomi och Fysik **15**, 1-9 (1921).

[217] G.B. Jeffery, “The field of an electron on Einstein’s theory of gravitation,” Proc. Roy. Soc. Lond. A **99**, 697, 123-134 (1921).

[218] M. Kac, “On Distribution of Certain Wiener Functionals,” Trans. Amer. Math. Soc. **65**, 1-13 (1949).

[219] R. E. Kallosh, O. V. Tarasov and I. V. Tyutin, “One Loop Finiteness Of Quantum Gravity Off Mass Shell,” Nucl. Phys. B **137**, 145 (1978).

[220] T. Kaluza, “Zum Unitätsproblem der Physik,” Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966-972 (1921).

[221] D. J. Kapner, T. S. Cook, E. G. Adelberger, J. H. Gundlach, B. R. Heckel, C. D. Hoyle and H. E. Swanson, “Tests of the gravitational inverse-square law below the dark-energy length scale,” Phys. Rev. Lett. **98**, 021101 (2007).

[222] R. P. Kerr, "Gravitational field of a spinning mass as an example of algebraically special metrics," Phys. Rev. Lett. **11**, 237-238 (1963).

[223] D. Kershaw, "Theory of Hidden Variables," Phys. Rev. **136**, no.6B, B1850 (1964).

[224] A. Khmelnitsky and V. Rubakov, "Pulsar timing signal from ultralight scalar dark matter," JCAP **1402**, 019 (2014).

[225] O. Klein, "Quantum Theory and Five-Dimensional Theory of Relativity. (In German and English)," Z. Phys. **37**, 895-906 (1926).

[226] B. Knorr, C. Ripken and F. Saueressig, "Form Factors in Asymptotic Safety: conceptual ideas and computational toolbox," Class. Quant. Grav. **36**, no.23, 234001 (2019).

[227] T. Kodama and T. Koide, "Variational Principle of Hydrodynamics and Quantization by Stochastic Process," arXiv:1412.6472 [quant-ph] (2014).

[228] T. Koide and T. Kodama "Stochastic variational method as quantization scheme: Field quantization of the complex Klein–Gordon equation," Prog. Theor. Exp. Phys. **2015**, 9, 093A03 (2015).

[229] T. Koide and T. Kodama "Novel effect induced by spacetime curvature in quantum hydrodynamics," Phys. Lett. A **383**, 2713-2718 (2019).

[230] A. Kolmogorov, "Grundbegriffe der Wahrscheinlichkeitsrechnung," Springer, Berlin (1933).

[231] J. Kormendy and D. Richstone, "Inward bound: The Search for supermassive black holes in galactic nuclei," Ann. Rev. Astron. Astrophys. **33**, 581 (1995).

[232] L. M. Krauss and F. Wilczek, "Discrete Gauge Symmetry in Continuum Theories," Phys. Rev. Lett. **62**, 1221 (1989).

[233] F. Kuipers, "Stochastic Quantization on Lorentzian Manifolds," JHEP **05**, 028 (2021).

[234] F. Kuipers, "Stochastic Quantization of Relativistic Theories," J. Math. Phys. **62**, no. 12, 122301 (2021).

[235] F. Kuipers, "Analytic Continuation of Stochastic Mechanics," J. Math. Phys. **63**, no. 04, 042301 (2022).

[236] F. Kuipers, "Spacetime Stochasticity and Second Order Geometry," arXiv:2203.16399 [gr-qc] (2022).

[237] F. Kuipers and X. Calmet, "Singularity theorems in the effective field theory for quantum gravity at second order in curvature," Universe **6**, no.10, 171 (2020).

[238] I. Kuntz, "Exorcising ghosts in quantum gravity," Eur. Phys. J. Plus **135**, no.10, 859 (2020).

[239] I. Kuntz and R. Casadio, "Singularity avoidance in quantum gravity," Phys. Lett. B **802**, 135219 (2020).

[240] A. Laddha, S. G. Prabhu, S. Raju and P. Shrivastava, "The Holographic Nature of Null Infinity," SciPost Phys. **10**, no.2, 041 (2021).

[241] N. Leefer, A. Gerhardus, D. Budker, V. V. Flambaum and Y. V. Stadnik, "Search for the effect of massive bodies on atomic spectra and constraints on Yukawa-type interactions of scalar particles," Phys. Rev. Lett. **117**, no. 27, 271601 (2016).

[242] P. Lévy "Processus stochastiques et mouvement brownien," Gauthier-Villars, Paris (1948).

[243] H. Lü, A. Perkins, C. N. Pope and K. S. Stelle, "Black Holes in Higher-Derivative Gravity," Phys. Rev. Lett. **114**, no. 17, 171601 (2015).

[244] H. Lü, A. Perkins, C. N. Pope and K. S. Stelle, "Spherically Symmetric Solutions in Higher-Derivative Gravity," Phys. Rev. D **92**, no. 12, 124019 (2015).

[245] G. Magnano, M. Ferraris and M. Francaviglia, "Nonlinear gravitational Lagrangians," Gen. Rel. Grav. **19**, 465 (1987).

[246] G. Magnano, M. Ferraris and M. Francaviglia, "Legendre transformation and dynamical structure of higher derivative gravity," Class. Quant. Grav. **7**, 557 (1990).

[247] Y. Mambrini, S. Profumo and F. S. Queiroz, "Dark Matter and Global Symmetries," Phys. Lett. B **760**, 807-815 (2016).

[248] D. S. Mansi, A. Mauri and A. C. Petkou, "Stochastic Quantization and AdS/CFT," Phys. Lett. B **685**, 215-221 (2010).

[249] F. Markopoulou and L. Smolin, "Quantum theory from quantum gravity," Phys. Rev. D **70**, 124029 (2004).

[250] D. Marolf, "Unitarity and Holography in Gravitational Physics," Phys. Rev. D **79**, 044010 (2009).

[251] R. Marra and M. Serva, "Variational principles for a relativistic stochastic mechanics," Ann. Inst. H. Poincare Phys. Theor. **53**, 97-108 (1990).

[252] D. J. E. Marsh, "Axion Cosmology," Phys. Rept. **643**, 1 (2016).

[253] D. J. E. Marsh and J. Silk, "A Model For Halo Formation With Axion Mixed Dark Matter," Mon. Not. Roy. Astron. Soc. **437**, no.3, 2652-2663 (2014).

[254] P. Meade and L. Randall, "Black Holes and Quantum Gravity at the LHC," JHEP **05**, 003 (2008).

[255] B. K. El-Menoufi, "Quantum gravity of Kerr-Schild spacetimes and the logarithmic correction to Schwarzschild black hole entropy," JHEP **05**, 035 (2016).

[256] B. K. El-Menoufi, “Quantum gravity effects on the thermodynamic stability of 4D Schwarzschild black hole,” JHEP **08**, 068 (2017).

[257] P. A. Meyer, “A differential geometric formalism for the Itô calculus” in Stochastic Integrals. Lecture Notes in Mathematics **851**, pp. 256-270, Springer, Berlin (1981).

[258] M. Milgrom, “A Modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis,” Astrophys. J. **270**, 365-370 (1983).

[259] D. Minic, “From quantum foundations of quantum field theory, string theory and quantum gravity to dark matter and dark energy,” arXiv:2003.00318 [hep-th] (2020).

[260] A. G. Mirzabekian and G. A. Vilkovisky, “The Model independent approach to quantum gravity: Implications of asymptotic flatness,” Phys. Lett. B **317**, 517 (1993).

[261] A. G. Mirzabekian and G. A. Vilkovisky, “Particle creation in the effective action method,” Annals Phys. **270**, 391 (1998).

[262] C. W. Misner, K. S. Thorne and J. A. Wheeler, ”Gravitation,” W.H. Freeman, San Francisco (1973).

[263] J. W. Moffat, “Scalar-tensor-vector gravity theory,” JCAP **03**, 004 (2006).

[264] R. Monteiro, D. O’Connell and C. D. White, “Black holes and the double copy,” JHEP **12**, 056 (2014).

[265] L. M. Morato and L. Viola, “Markov diffusions in comoving coordinates and stochastic quantization of the free relativistic spinless particle,” J. Math. Phys. **36**, 4691-4710 (1995). [erratum: J. Math. Phys. **37**, 4769 (1996)].

[266] W. Mück, “Counting Photons in Static Electric and Magnetic Fields,” Eur. Phys. J. C **73**, no.12, 2679 (2013).

[267] E. Nelson, “Derivation of the Schrodinger equation from Newtonian mechanics,” Phys. Rev. **150**, 1079-1085 (1966).

[268] E. Nelson, “Dynamical Theories of Brownian Motion,” Princeton University Press, Princeton (1967).

[269] E. Nelson, “Construction of quantum fields from Markoff fields,” J. Funct. Anal. **12**, no.1, 97 (1973).

[270] E. Nelson, “Quantum Fluctuations,” Princeton University Press, Princeton (1985).

[271] E. Nelson, “Review of stochastic mechanics,” Journal of Physics: Conference Series **361**, 012011 (2012).

[272] E. Nelson, “Stochastic mechanics of relativistic fields,” J. Phys. Conf. Ser. **504**, 012013 (2014).

[273] E. T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash and R. Torrence, "Metric of a Rotating, Charged Mass," J. Math. Phys. **6**, 918-919 (1965).

[274] E. T. Newman and A. I. Janis, "Note on the Kerr spinning particle metric," J. Math. Phys. **6**, 915-917 (1965).

[275] J. F. Nieves and P. B. Pal, "Gravitational decay of the Z-boson," Phys. Rev. D **72**, 093006 (2005).

[276] G. Nordström, "On the Energy of the Gravitational Field in Einstein's Theory," Verhandl. Konikl. Ned. Akad. Wetenschap, Afdel. Natuurk. **26**, 1201-1208 (1918).

[277] M. Nori, R. Murgia, V. Iršič, M. Baldi and M. Viel, "Lyman α forest and non-linear structure characterization in Fuzzy Dark Matter cosmologies," Mon. Not. Roy. Astron. Soc. **482**, no.3, 3227-3243 (2019).

[278] L.S.F. Olavo, L.C. Lapas and A. Figueiredo, "Foundations of quantum mechanics: The Langevin equations for QM," Ann. Phys. **327**, no.5, 1391 (2012).

[279] K. A. Olive and M. Pospelov, "Environmental dependence of masses and coupling constants," Phys. Rev. D **77**, 043524 (2008).

[280] J. R. Oppenheimer and H. Snyder, "On Continued gravitational contraction," Phys. Rev. **56**, 455-459 (1939).

[281] D. Orlando and S. Reffert, "On the Renormalizability of Horava-Lifshitz-type Gravities," Class. Quant. Grav. **26**, 155021 (2009).

[282] M. Ostrogradsky, "Mémoires sur les équations différentielles, relatives au problème des isopérimètres," Mem. Acad. St. Petersbourg **6**, no.4, 385-517 (1850).

[283] C. Di Paolo, F. Nesti and F. L. Villante, "Phase space mass bound for fermionic dark matter from dwarf spheroidal galaxies," Mon. Not. Roy. Astron. Soc. **475**, no.4, 5385-5397 (2018).

[284] G. Parisi and Y.S. Wu, "Perturbation Theory Without Gauge Fixing," Sci. Sin. **24**, 483 (1981).

[285] M. Pavon, "A new formulation of stochastic mechanics," Phys. Lett. A **209**, 143-149 (1995).

[286] M. Pavon, "Hamilton's principle in stochastic mechanics," J. Math. Phys. **36**, 6774 (1995).

[287] M. Pavon, "Lagrangian dynamics for classical, Brownian and quantum mechanical particles," J. Math. Phys. **37**, 3375 (1996).

[288] M. Pavon, "Stochastic Mechanics and the Feynman Integral," J. Math. Phys. **41**, 6060 (2000).

[289] M. Pavon, "On the stochastic mechanics of the free relativistic particle," J. Math. Phys. **42**, 4846-4856 (2001).

[290] P. de la Pena, A.M. Cetto and A. Valdes Hernandez, "The Emerging Quantum," Springer, Cham (2015).

[291] R. Penrose, "Gravitational collapse and space-time singularities," Phys. Rev. Lett. **14**, 57-59 (1965).

[292] S. Perlmutter *et al.* [Supernova Cosmology Project], "Measurements of Ω and Λ from 42 high redshift supernovae," Astrophys. J. **517**, 565-586 (1999).

[293] N.C. Petroni and L.M. Morato, "Entangled states in stochastic mechanics," J. Phys. A **33**, no.33, 5833-5848 (2000).

[294] F. Piazza and M. Pospelov, "Sub-eV scalar dark matter through the super-renormalizable Higgs portal," Phys. Rev. D **82**, 043533 (2010).

[295] N. K. Porayko and K. A. Postnov, "Constraints on ultralight scalar dark matter from pulsar timing," Phys. Rev. D **90**, no. 6, 062008 (2014).

[296] M. Pospelov and A. Ritz, "Theta vacua, QCD sum rules, and the neutron electric dipole moment," Nucl. Phys. B **573**, 177 (2000).

[297] J. Preskill, M. B. Wise and F. Wilczek, "Cosmology of the Invisible Axion," Phys. Lett. **120B**, 127 (1983).

[298] S. Raju, "Lessons from the Information Paradox," arXiv:2012.05770 [hep-th] (2020).

[299] A. Raychaudhuri, "Relativistic cosmology. 1.," Phys. Rev. **98**, 1123 (1955).

[300] H. Reissner, "Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theory," Ann. Phys. **50**, 9, 106-120 (1916).

[301] A. G. Riess *et al.* "A Comprehensive Measurement of the Local Value of the Hubble Constant with 1 km/s/Mpc Uncertainty from the Hubble Space Telescope and the SH0ES Team," arXiv:2112.04510 [astro-ph.CO] (2021).

[302] T. A. Roman, "On the 'Averaged Weak Energy Condition' and Penrose's Singularity Theorem," Phys. Rev. D **37**, 546 (1988).

[303] L. Roszkowski, E. M. Sessolo and S. Trojanowski, "WIMP dark matter candidates and searches-current status and future prospects," Rept. Prog. Phys. **81**, no.6, 066201 (2018).

[304] R. K. Sachs, "On the characteristic initial value problem in gravitational theory," J. Math. Phys. **3**, 908 (1962).

[305] D. Savchenko and A. Rudakovskyi, "New mass bound on fermionic dark matter from a combined analysis of classical dSphs," Mon. Not. Roy. Astron. Soc. **487**, no.4, 5711-5720 (2019).

[306] H. Y. Schive, T. Chiueh and T. Broadhurst, "Cosmic Structure as the Quantum Interference of a Coherent Dark Wave," Nature Phys. **10**, 496 (2014).

[307] H. Y. Schive, T. Chiueh, T. Broadhurst and K. W. Huang, "Contrasting Galaxy Formation from Quantum Wave Dark Matter, ψDM, with ΛCDM, using Planck and Hubble Data," Astrophys. J. **818**, no. 1, 89 (2016).

[308] S. Schlamminger, K.-Y. Choi, T. A. Wagner, J. H. Gundlach and E. G. Adelberger, "Test of the equivalence principle using a rotating torsion balance," Phys. Rev. Lett. **100**, 041101 (2008).

[309] I. Schmelzer, "An answer to the Wallstrom objection against Nelsonian stochastics," arXiv:1101.5774 [quant-ph] (2011).

[310] K. Schutz, "Subhalo mass function and ultralight bosonic dark matter," Phys. Rev. D **101**, no.12, 123026 (2020).

[311] L. Schwartz, "Semi-Martingales and their Stochastic Calculus on Manifolds," Presses de l'Université de Montréal, Montréal (1984).

[312] K. Schwarzschild, "On the gravitational field of a mass point according to Einstein's theory," Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1916**, 189-196 (1916).

[313] J. Schwinger, "Four-dimensional Euclidean formulation of quantum field theory," Proc. 8th Annual International Conference on High Energy Physics, 134-140 (1958).

[314] D.S. Shucker, "Stochastic Mechanics of Systems with Zero Potential," J. Funct. Anal. **38**, 146 (1980).

[315] J. Z. Simon, "The Stability of flat space, semiclassical gravity, and higher derivatives," Phys. Rev. D **43**, 3308-3316 (1991).

[316] A. Simpson and M. Visser, "Regular black holes with asymptotically Minkowski cores," Universe **6**, no.1, 8 (2019).

[317] G. L. Smith, C. D. Hoyle, J. H. Gundlach, E. G. Adelberger, B. R. Heckel and H. E. Swanson, "Short range tests of the equivalence principle," Phys. Rev. D **61**, 022001 (2000).

[318] C. Smorra *et al.*, "Direct limits on the interaction of antiprotons with axion-like dark matter," Nature **575**, no.7782, 310-314 (2019).

[319] S. N. Solodukhin, "Entanglement entropy of black holes," Living Rev. Rel. **14**, 8 (2011).

[320] S. N. Solodukhin, "Logarithmic terms in entropy of Schwarzschild black holes in higher loops," Phys. Lett. B **802**, 135235 (2020).

[321] Y. Stadnik, "Manifestations of Dark Matter and Variations of the Fundamental Constants of Nature in Atoms and Astrophysical Phenomena," Springer, Cham (2017).

[322] Y. V. Stadnik and V. V. Flambaum, “Searching for dark matter and variation of fundamental constants with laser and maser interferometry,” Phys. Rev. Lett. **114**, 161301 (2015).

[323] Y. V. Stadnik and V. V. Flambaum, “Can dark matter induce cosmological evolution of the fundamental constants of Nature?,” Phys. Rev. Lett. **115**, no. 20, 201301 (2015).

[324] Y. V. Stadnik and V. V. Flambaum, “Constraining scalar dark matter with Big Bang nucleosynthesis and atomic spectroscopy,” arXiv:1504.01798 [astro-ph.CO] (2015).

[325] Y. V. Stadnik and V. V. Flambaum, “Manifestations of dark matter and variations of fundamental constants in atoms and astrophysical phenomena,” arXiv:1509.00966 [physics.atom-ph] (2015).

[326] Y. V. Stadnik and V. V. Flambaum, “Enhanced effects of variation of the fundamental constants in laser interferometers and application to dark matter detection,” Phys. Rev. A **93**, no. 6, 063630 (2016).

[327] Y. V. Stadnik and V. V. Flambaum, “Improved limits on interactions of low-mass spin-0 dark matter from atomic clock spectroscopy,” Phys. Rev. A **94**, no. 2, 022111 (2016).

[328] Y. V. Stadnik and V. V. Flambaum, “Searches for New Particles Including Dark Matter with Atomic, Molecular and Optical Systems,” arXiv:1806.03115 [physics.atom-ph] (2018).

[329] K. S. Stelle, “Renormalization of Higher Derivative Quantum Gravity,” Phys. Rev. D **16**, 953-969 (1977).

[330] K. S. Stelle, “Classical Gravity with Higher Derivatives,” Gen. Rel. Grav. **9**, 353-371 (1978).

[331] A. Strominger, “Lectures on the Infrared Structure of Gravity and Gauge Theory,” arXiv:1703.05448 [hep-th] (2017).

[332] K. Symanzik, “Euclidean Quantum Field Theory. I. Equations for a Scalar Model,” J. Math. Phys. **7**, 510 (1966).

[333] M. Tanabashi *et al.* [Particle Data Group], “Review of Particle Physics,” Phys. Rev. D **98**, no. 3, 030001 (2018).

[334] P. D. Teixeira, I. L. Shapiro and T. G. Ribeiro, “One-loop effective action: nonlocal form factors and renormalization group,” Grav. Cosmol. **26**, no.3, 185-199 (2020).

[335] K. Van Tilburg, N. Leefer, L. Bougas and D. Budker, “Search for ultralight scalar dark matter with atomic spectroscopy,” Phys. Rev. Lett. **115**, no. 1, 011802 (2015).

[336] S. Tremaine and J. E. Gunn, “Dynamical Role of Light Neutral Leptons in Cosmology,” Phys. Rev. Lett. **42**, 407-410 (1979).

[337] A. A. Tseytlin, "On singularities of spherically symmetric backgrounds in string theory," Phys. Lett. B **363**, 223 (1995).

[338] S. Tulin and H. B. Yu, "Dark Matter Self-interactions and Small Scale Structure," Phys. Rept. **730**, 1-57 (2018).

[339] W. G. Unruh, "Notes on black hole evaporation," Phys. Rev. D **14**, 870 (1976).

[340] L. A. Ureña-López and A. X. Gonzalez-Morales, "Towards accurate cosmological predictions for rapidly oscillating scalar fields as dark matter," JCAP **1607**, 048 (2016).

[341] G. A. Vilkovisky, "The Unique Effective Action in Quantum Field Theory," Nucl. Phys. B **234**, 125-137 (1984).

[342] R. M. Wald, "General Relativity," Chicago University Press, Chicago (1984).

[343] R. M. Wald, "Black hole entropy is the Noether charge," Phys. Rev. D **48** no.8, 3427-3431 (1993).

[344] R. M. Wald and U. Yurtsever, "General proof of the averaged null energy condition for a massless scalar field in two-dimensional curved space-time," Phys. Rev. D **44**, 403 (1991).

[345] A. C. Wall, "A Survey of Black Hole Thermodynamics," arXiv:1804.10610 [gr-qc] (2018).

[346] T. C. Wallstrom, "On the derivation of the Schrödinger equation from Stochastic Mechanics," Found. Phys. Lett. **2**, no.2, 113 (1988).

[347] T.C. Wallstrom, "Inequivalence between the Schrödinger equation and the Madelung hydrodynamic equations," Phys. Rev. **A49**, no.3, 1613 (1993).

[348] B. L. Webster and P. Murdin, "Cygnus X-1-a Spectroscopic Binary with a Heavy Companion?," Nature **235**, 37-38 (1972).

[349] S. Weinberg, "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity," Wiley and Sons, New York (1972).

[350] S. Weinberg, "A New Light Boson?," Phys. Rev. Lett. **40**, 223 (1978).

[351] S. Weinberg, "Ultraviolet Divergences In Quantum Theories Of Gravitation," *General Relativity: An Einstein Centenery Survey*, Cambridge, 790 (1980).

[352] H. Weyl, "Zur Gravitationstheory," Ann. Phys. **54**, 18, 117-145 (1917).

[353] J. A. Wheeler, "Geons," Phys. Rev. **97**, 511-536 (1955).

[354] G. C. Wick, "Properties of Bethe-Salpeter Wave Functions," Phys. Rev. **96**, 1124-1134 (1954).

[355] L. M. Winter, R. F. Mushotzky and C. S. Reynolds, "Xmm-newton archival study of the ulx population in nearby galaxies," Astrophys. J. **649**, 730-752 (2006).

[356] B. S. DeWitt, "Quantum Theory of Gravity. 1. The Canonical Theory," Phys. Rev. **160**, 1113-1148 (1967).

[357] B. S. DeWitt, "Quantum Theory of Gravity. 2. The Manifestly Covariant Theory," Phys. Rev. **162**, 1195-1239 (1967).

[358] B. S. DeWitt, "Quantum Theory of Gravity. 3. Applications of the Covariant Theory," Phys. Rev. **162**, 1239-1256 (1967).

[359] E. Witten, "Light Rays, Singularities, and All That," Rev. Mod. Phys. **92**, no.4, 045004 (2020).

[360] R. P. Woodard, "Avoiding dark energy with 1/r modifications of gravity," Lect. Notes Phys. **720**, 403-433 (2007).

[361] X. P. Wu, T. Chiueh, L. Z. Fang and Y. J. Xue, "A comparison of different cluster mass estimates: consistency or discrepancy ?," Mon. Not. Roy. Astron. Soc. **301**, 861 (1998).

[362] K. Yasue, "Quantum Mechanics and Stochastic Control Theory," J. Math. Phys. **22**, 1010-1020 (1981).

[363] K. Yasue, "Stochastic calculus of variations," J. Funct. Anal. **41**, no.3, 327 (1981).

[364] J. C. Zambrini, "Stochastic Dynamics: A Review of Stochastic Calculus," Int. J. Theor. Phys. **24**, no.3, 277 (1985).

[365] L. Zhou *et al.*, "Test of Equivalence Principle at 10^{-8} Level by a Dual-species Double-diffraction Raman Atom Interferometer," Phys. Rev. Lett. **115**, no. 1, 013004 (2015).