Special Functions and Analysis on Smooth Metric Measure Spaces

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Abstract

In this thesis, we explore two distinct but closely linked avenues within the study of geometric PDEs. The first of these looks at the eigenfunctions of the Laplacian operator on rank one symmetric spaces. These eigenfunctions bridge the gap between classical and modern analysis. This is because these eigenfunctions are orthogonal polynomials (in particular the Jacobi polynomial, $P_n^{(\alpha,\beta)}(x)$, and the Gegenbauer polynomial, $C_n^{(\lambda)}(x)$.

Making use of the Faa di Bruno formula, we find differential identities involving these special functions and their matrix counterparts. We also find differential identities for the Hermite polynomial, $H_n(x)$. This includes the proof of the spectral identity:

$$\frac{d^{2m}}{dx^{2m}} H_{2n}(f(x))|_{x=0} = \sum_{j=1}^{m} \sum_{q=1}^{j} \mathfrak{C}(j,m,q) \frac{(-1)^{\lambda-j}}{2^{\lambda}} {}_{2\lambda}P_{\lambda} \lambda^q, \tag{0.0.1}$$

where λ is an eigenvalue of the Ornstein-Uhlenbeck operator.

The orthogonal polynomials also have representations using a hypergeometric series. From this, we produce Maclaurin expansions for composite hypergeometric series, and also identities using the differential operator:

$$\mathscr{L}_p = \sum P_N(d/d\theta) = \mathsf{p}_0 + \mathsf{p}_1 d/d\theta + ... + \mathsf{p}_N d^N/d\theta^N.$$

We then proceed to look at gradient estimates for geometric PDEs on Riemannian manifolds. These estimates were first introduced by Li and Yau [76, 130, 131], who looked at estimates for harmonic functions and the heat equation. Since these first estimates were established, similar results have also been proven for non-linear PDEs. Of particular interest is the case where one is working on a smooth metric measure space $(M^n, g, e^{-f}d\nu)$ with a diffusion operator $\Delta_f = \Delta - \langle \nabla f, \nabla \rangle$.

Within this thesis, we choose to look at the parabolic equation:

$$\Big(\Delta_f - \partial_t\Big) u(x,t) + A(x,t)u(x,t)\log u(x,t) + B(x,t)u(x,t)^p = 0$$

and its elliptic equivalent. We produce Li-Yau and Souplet-Zhang estimates, as well as Li-Yau estimates for the elliptic equation. As a fundamental part of the process of finding a gradient estimate, a cutoff function is used. This allows the use of the maximum principle, which lies at the heart of the proofs.

With these gradient estimates, we produce further results such as Liouville-type theorems, Harnack inequalities, and analysis with ancient solutions. These give further information about the PDE under consideration, and conditions under which certain solutions are obtained.

Finally, we find estimates under a time-evolving metric $g(t)$ for different geometric flows:

$$\frac{\partial}{\partial t} g_{ij}(x,t) = 2h_{ij}(x,t),$$

where h is a time dependent symmetric $(0,2)$ tensor. The flow above is a general flow, but we look at the specific cases when this is either Ricci, Yamabe, or Perelman-Ricci flow.

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another university for the award of any other degree.

Signed: Steven Lockwood

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Contents

Chapter 1

Introduction and preliminaries

In this study, we access a range of modern topics within differential geometry, partial differential equations, and classical special functions. The basis of our research is the study of geometric PDEs on Riemannian manifolds.

Our research takes two clear directions within this area. First, we work largely in the field of analysis, using combinatorial elements for special functions which arise from the eigenvalues of the Laplacian on rank one symmetric spaces. Symmetric spaces are Riemannian manifolds whose groups of symmetry contain an inverse symmetry for each point. We go into more detail on these in chapters 2 and 3. If we turn our attention to rank one spaces, we see that these are closely linked to the classical special functions such as the Jacobi and Gegenbauer polynomials (orthogonal polynomials); they are eigenfunctions of the Laplacian on these spaces. Importantly, they bridge the gap between these two areas of research, linking the classical with the modern. The spaces in question also have forms built from Lie groups, which in itself has a wealth of literature due to its importance within mathematics.

The second direction this thesis takes is to look at gradient estimates for elliptic and parabolic PDEs on smooth metric measure spaces, also known as weighted Riemannian manifolds. These were introduced by Li and Yau in the 1970s and provide extensive information about a given PDE in the absence of an explicit solution. This allows one to carry out useful analysis of the PDEs, even those with complicated non-linearities. Liouville-style theorems are easily calculated once an estimate is found. In addition to these, one can also rapidly establish Harnack inequalities. What often makes these PDEs interesting is their link to pre-existing problems or objects, such as the Yamabe problem or the logarithmic Sobolev inequality. Further to this, we see that much of the literature takes these estimates more generally by looking at PDEs involving a symmetric diffusion operator (or Witten-Laplacian)

$$\Delta_f u = \Delta u - \langle \nabla u, \nabla f \rangle.$$

We take a deeper look at a variety of these estimates in Chapters 4 and 5, before continuing to use them to gain additional information about PDEs in Chapter 7. Within our research, we consider a particular PDE in its elliptic and parabolic form with both a logarithmic and power non-linearity. Here we produce multiple estimates of different varieties and the subsequent analysis of these thereafter producing the Liouville style theorems and Harnack inequalities mentioned above. When looking at parabolic PDEs, we also delve into analysis involving ancient solutions. These are solutions that are valid for all negative time.

In Chapter 6, we also look at estimates that involve a time-evolving metric. This is an extremely new area of research within gradient estimates, and it takes its roots from the study of Ricci flow introduced by Hamilton. Ricci flow was also a fundamental component in Perelman's proof of the Poincare Conjecture.

1.1 Preliminaries on Riemannian geometry

In this chapter we give a basic account of the tools and results needed from Riemannian geometry. For a more thorough coverage the reader can consult [19, 20, 74].

Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$ and metric g. If $\{x^i\}$ is a positively oriented local coordinate system then the volume form is

$$dv_g = \sqrt{|g|} dx^1 \wedge ... \wedge dx^n \tag{1.1.1}$$

where $|g|$ denotes the determinant of the metric tensor g. Let $\exp_{x_0}$ denote the exponential map and $\{X^i\}$ be the standard Euclidean coordinates on the tangent bundle. Then we define the geodesic coordinates on M^n as

$$x^i = X^i \circ \exp_{x_0}^{-1} : M^n \backslash cut(x_0) \to \mathbb{R}. \tag{1.1.2}$$

In particular for geodesic coordinates it is easily seen that

$$g_{ij}(x_0) = \delta_{ij} \tag{1.1.3}$$

and

$$\frac{\partial}{\partial x^i} g_{jk}(x_0) = 0. \tag{1.1.4}$$

Let $\mathfrak{X}(M^n)$ be the space of smooth vector fields on M^n. We denote by ∇ the Levi-Civita connection on $\mathfrak{X}(M^n)$ which can be described in terms of the Christoffel symbols Γ^k_{ij} as

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}, \tag{1.1.5}$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \Big(\frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \Big). \tag{1.1.6}$$

(Here g^{kl} is the inverse of the metric g_{kl}.)

The Riemann curvature tensor is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.1.7}$$

where ∇ is the Levi-Civita connection, $X, Y, Z \in \mathfrak{X}(M^n)$, and $[X,Y]$ is the Lie bracket of vector fields X and Y. The Riemann curvature tensor can also be written in local coordinates as

$$R^l_{ijk} \frac{\partial}{\partial x^l} = R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \Big) \frac{\partial}{\partial x^k} \tag{1.1.8}$$

or more explicitly by using the Christoffel symbols as

$$R^l_{ijk} = \frac{\partial}{\partial x^i} \Gamma^l_{jk} - \frac{\partial}{\partial x^j} \Gamma^l_{ik} + \Gamma^p_{jk} \Gamma^l_{ip} - \Gamma^p_{ik} \Gamma^l_{jp}. \tag{1.1.9}$$

The Riemann curvature tensor is a difficult object to handle, so one often finds it much easier to deal with its trace, the Ricci curvature tensor. This is a $(0,2)$ tensor determined by the metric and, roughly, relates the metric tensor locally to that of Euclidean space. For X, Y, Z defined above

$$Ric(Y,Z) = tr(X \to R(X,Y)Z). \tag{1.1.10}$$

If $\{e_i\}_{i=1}^n$ is an orthonormal frame, such that $g(e_i, e_j) = \delta_{ij}$ then

$$Ric(Y,Z) = \sum_{i=1}^n \langle R(e_i, Y)Z, e_i \rangle \tag{1.1.11}$$

or

$$R_{jk} = \sum_{i=1}^n R^i_{ijk}. \tag{1.1.12}$$

As we are working on Riemannian manifolds, the Laplace-Beltrami operator is defined as

$$\begin{aligned} \Delta u =& \nabla \cdot \nabla u = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j} \Big(g^{jk} \sqrt{|g|} \frac{\partial u}{\partial x^k} \Big) \\ =& g^{jk} \frac{\partial^2 u}{\partial x^j \partial x^k} - g^{jk} \Gamma^l_{jk} \frac{\partial u}{\partial x^l} \end{aligned} \tag{1.1.13}$$

where $|g|$ is the determinant of g. Equation (1.1.13) is important when we discuss the time-evolving metrics as hidden inside the Laplace-Beltrami operator is the metric.

Let $u \in C^\infty(M^n)$ such that $u : M^n \to \mathbb{R}$ then the Bochner-Weitzenbock formula (sometimes refered to as just the Bochner formula) states:

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla^2 u|^2 + \langle \nabla \Delta u, \nabla u \rangle + Ric(\nabla u, \nabla u). \tag{1.1.14}$$

1.2 Evolution of various quantities under geometric flow

Various elements of the Riemannian manifold can evolve under an evolving metric.

Definition 1.2.1. *Let $g(t)$ be a smooth family of Riemmanian metrics on M^n where $t \in (0, T)$, $T \in \mathbb{R}^+$. If*

$$\frac{\partial}{\partial t} g_{ij}(x,t) = 2h_{ij}(x,t), \tag{1.2.1}$$

where h is a time dependent symmetric $(0,2)$ tensor then we say that $g_{ij}(x,t)$ is a solution to the generalised geometric flow.

Alternatively if we have the inverse to the metric we can find an expression for the time derivative.

Lemma 1.2.1. *Let g^{-1} be the metric inverse. Then*

$$\frac{\partial}{\partial t} g^{ij} = -2g^{ik}g^{jl}h_{kl}. \tag{1.2.2}$$

With this, we can find the variation of the Christoffel symbols.

Lemma 1.2.2. *Let $g(t)$ be a smooth family of metrics that solve (1.2.1). Then*

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}). \tag{1.2.3}$$

Proof. By using the standard formula for the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\Big(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij}\Big).$$

Then

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}\frac{\partial}{\partial t} g^{kl}\Big(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij}\Big) + \frac{1}{2} g^{kl}\Big(\frac{\partial}{\partial x_i}\frac{\partial}{\partial t} g_{jl} + \frac{\partial}{\partial x_j}\frac{\partial}{\partial t} g_{il} - \frac{\partial}{\partial x_l}\frac{\partial}{\partial t} g_{ij}\Big)$$

At an arbitrary point $z \in M^N$, $\Gamma^k_{ij}(z) = 0$. This gives $\frac{\partial}{\partial x_a} g_{bc}(z) = 0$. Then using (1.2.1) completes the proof. □

Remark 1.2.1. *Christoffel symbols are awkward to calculate. However, if our Riemannian manifold is a symmetric space, all the points look the same. This means that we can calculate the tensor anywhere. A choice of the origin gives the easiest way to calculate.*

The scalar curvature S, the Riemann curvature tensor R^l_{ijk}, and the volume elements all evolve in a similar way.

Lemma 1.2.3. *Let $g(t)$ be a smooth family of metrics that solve (1.2.1). Then*

$$\frac{\partial}{\partial t}S = -\Delta H + \nabla^p\nabla^q h_{pq} - \langle h, Ric\rangle \tag{1.2.4}$$

where $H = (Tr_g h) = g^{pq}h_{pq}$.

Lemma 1.2.4. *Let $g(t)$ be a smooth family of metrics that solve (1.2.1). Then*

$$\frac{\partial}{\partial t}R^l_{ijk} = g^{lp}\left[\nabla_i\Big(\nabla_j h_{kp} + \nabla_k h_{jp} - \nabla_p h_{jk}\Big) - \nabla_j\Big(\nabla_i h_{kp} + \nabla_k h_{ip} - \nabla_p h_{ik}\Big)\right]. \tag{1.2.5}$$

The volume element $d\mu$ evolves in the following way:

Lemma 1.2.5. *Let $g(t)$ be a smooth family of metrics that solve (1.2.1). Then*

$$\frac{\partial}{\partial t}d\mu = H d\mu. \tag{1.2.6}$$

In Riemannian manifolds, the inner product is defined as $|\cdot| : M^n \to \mathbb{R}$, where $|\mathbf{X}| = \sqrt{g(\mathbf{X},\mathbf{X})}$. Using this, one can compute the time derivative of the inner product.

Lemma 1.2.6. *Let $g(t)$ be a smooth family of metrics that solve (1.2.1), then*

$$\frac{\partial}{\partial t}|\nabla u|^2 = -2h\langle\nabla u, \nabla u\rangle + 2\langle\nabla u, \nabla u_t\rangle. \tag{1.2.7}$$

Proof. By straightforward computations:

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla u|^2 =& \frac{\partial}{\partial t}\Big(g^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)\\
=& -2h^{ij}\frac{\partial}{\partial x_i}u\frac{\partial}{\partial x_j}u + g^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial t}u\frac{\partial}{\partial x_j}u + g^{ij}\frac{\partial}{\partial x_i}u\frac{\partial}{\partial x_j}\frac{\partial}{\partial t}u\\
=& -2h_{ij}(\nabla u, \nabla u) + 2\langle\nabla u, \nabla u_t\rangle,
\end{aligned}$$

where we have used that the metric tensor is symmetric in the last step. ☐

Progressing from this, we can find an expression for the time derivative of the Laplace-Beltrami operator on a Riemannian manifold.

Lemma 1.2.7. *Let $g(t)$ be a smooth family of metrics that solve (1.2.1), then*

$$\frac{\partial}{\partial t}\Delta u = \Delta u_t - 2\langle\nabla\cdot h - \frac{1}{2}\nabla(\mathrm{Tr}_g\, h), \nabla u\rangle - 2\langle h, \nabla^2 u\rangle. \tag{1.2.8}$$

Proof.

$$
\begin{aligned}
\frac{\partial}{\partial t}\Delta u =& \frac{\partial}{\partial t}\Big[g^{ij}\Big(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial u}{\partial x_k}\Big)\Big] \\
=& \Delta u_t - 2\langle h, \nabla^2 u\rangle - g^{ij}\Big(\frac{\partial}{\partial t}\Gamma_{ij}^k\Big)\frac{\partial u}{\partial x_k} \\
=& \Delta u_t - 2\langle h, \nabla^2 u\rangle - g^{ij}g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})\nabla_k u \\
=& \Delta u_t - 2\langle h, \nabla^2 u\rangle - 2g^{kl}(g^{ij}\nabla_i h_{jl} - \frac{1}{2}\nabla_l(\mathrm{Tr}_g\, h))\nabla_k u \\
=& \Delta u_t - 2\langle h, \nabla^2 u\rangle - 2\langle \nabla \cdot h - \frac{1}{2}\nabla_l(\mathrm{Tr}_g\, h), \nabla u\rangle.
\end{aligned}
$$

□

More reading on this and other related material can be found in [35, 36].

1.3 Space forms and comparison theorems on Riemannian geometry

The distance sphere $S(x_0, r)$ is defined as

$$S(x_0, r) = \{x \in M^n : d(x, x_0) = r\} \tag{1.3.1}$$

where $d(x, x_0)$ is the distance function.

The second fundamental form $\mathfrak{h}$, which roughly measures how non-parallel the normal is, is defined as

$$\mathfrak{h}_{ij} = -\Gamma_{ij}^n = \frac{1}{2}\frac{\partial}{\partial r}g_{ij}, \tag{1.3.2}$$

where $\frac{\partial}{\partial r}$ is the unit normal to $S(x_0, r)$ and $g_{in} = g_{jn} = 0$. The Ricatti equation can be produce from $\mathfrak{h}$ by:

$$\frac{\partial}{\partial r}\mathfrak{h}_{ij} = -R_{nijn} + \mathfrak{h}_{ik}g^{kl}\mathfrak{h}_{lj}. \tag{1.3.3}$$

The mean curvature $\mathfrak{H}$, which is the trace of the second fundamental form of $S(x_0, r)$, is defined as

$$\mathfrak{H} = -g^{ij}\Gamma_{ij}^n. \tag{1.3.4}$$

It can be seen from the evolution of mean curvature for a hypersurface flow $\frac{\partial x}{\partial r} = B\nu$ that

$$\frac{\partial}{\partial r}\mathfrak{H} = \Delta B - Ric(\nu, \nu)B - B|\mathfrak{h}|^2. \tag{1.3.5}$$

The specific case when $B = 1$ yields

$$\frac{\partial}{\partial r}\mathfrak{H} = -Ric\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) - |\mathfrak{h}|^2 \tag{1.3.6}$$

which can be derived from (1.3.3) as $\frac{\partial}{\partial r}\mathfrak{H} = g^{ij}\left(\frac{\partial}{\partial r}\mathfrak{h}_{ij}\right) - \left(\frac{\partial}{\partial r}g_{ij}\right)\mathfrak{h}_{ij}$ and $\frac{\partial}{\partial r}g_{ij} = 2\mathfrak{h}_{ij}$.

Take a function ϕ and define the metric as

$$g = dr^2 + \phi(r)^2 g_{\mathbb{S}^{n-1}}. \tag{1.3.7}$$

This is called a rotationally symmetric metric and the sectional curvatures are

$$K_{\mathfrak{R}} = -\frac{\phi''}{\phi}, \tag{1.3.8}$$

$$K_{\mathfrak{S}} = \frac{1-(\phi')^2}{\phi^2}. \tag{1.3.9}$$

Here, $K_{\mathfrak{R}}$ is the sectional curvature containing the radial vector and $K_{\mathfrak{S}}$ is the sectional curvature perpendicular to the radial vector. This also gives the Ricci tensor as

$$Ric = -(n-1)\frac{\phi''}{\phi} + \Big[(n-2)(1-(\phi')^2) - \phi''\phi\Big]. \tag{1.3.10}$$

Space forms are complete Riemannian manifolds with constant sectional curvature. Most notable examples are the unit n-sphere ($\mathbb{S}^n$), Euclidean n-space ($\mathbb{R}^n$), and hyperbolic space (H^n). These have constant sectional curvature $K = 1$, $K = 0$, and $K = -1$ respectively.

The Killing-Hopf theorem states that if $\mathbf{U}$ is a universal cover for a Riemannian manifold (M^n, g) and the sectional curvature K is constant, then $\mathbf{U}$ is isometric to the n-sphere if $K = 1$, Euclidean n-space if $K = 0$, and hyperbolic space if $K = -1$. See [59, 70] for original documents.

Now let (M^n_K, g_K) be a space form with sectional curvature K. The metric is defined as

$$g_K = dr^2 + s_K(r)^2 g_{\mathbb{S}^{n-1}} \tag{1.3.11}$$

where

$$s_K = \begin{cases} \frac{1}{\sqrt{K}}\sin\sqrt{K}r & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}r & \text{if } K < 0. \end{cases} \tag{1.3.12}$$

If $K \leq 0$, then g_K is defined over $\mathbb{R}^n$, and if $K > 0$ then g_K defined on $B(0, \frac{\pi}{\sqrt{K}})$ extends to $\mathbb{S}^n$ by taking a 1-point compactification.

The mean curvature $\mathfrak{H}$ of $S(x_0, r)$ is

$$\mathfrak{H} = (n-1)\frac{\phi'}{\phi}. \tag{1.3.13}$$

If we have that the sectional curvature K is constant, then $\mathfrak{H}_K(r)$, the mean curvature of $S_K(x_0, r)$, is

$$\mathfrak{H}_K(r) = \begin{cases} (n-1)\sqrt{K}\cot(\sqrt{K}r) & \text{if } K > 0 \\ \frac{(n-1)}{r} & \text{if } K = 0 \\ \sqrt{|K|}\coth(\sqrt{|K|}r) & \text{if } K < 0. \end{cases} \tag{1.3.14}$$

Lemma 1.3.1. *Let (M^n, g) be a Riemannian manifold with lower bounded Ricci tensor $Ric \geq (n-1)K$ for $K \geq 0$. Then for points where r is smooth, the mean curvature of the distance sphere $S(x_0, r)$ satisfies*

$$\mathfrak{H}(r, \theta) \leq \mathfrak{H}_K(r). \tag{1.3.15}$$

This leads us onto the Laplacian comparison theorem. If

$$|\nabla r(x)|^2 = 1 \tag{1.3.16}$$

then $r(x) = d(x, x_0)$ is the generalised distance function such that $r : M^n \to [0, \infty)$.

The Laplacian comparison theorem, sometimes referred to as the mean curvature comparison theorem, compares the Laplacian of the distance function on a Riemannian manifold with the Laplacian of the distance function with constant curvature K. For K being positive, negative, or zero it states bounds.

Theorem 1.3.1 (Laplacian comparison theorem)**.** *Let (M^n, g) be a complete Riemannian manifold whose Ricci curvature is bounded from below $Ric \geq (n-1)K$ for $K \in \mathbb{R}$ and $x_0 \in M^n$. Then for any $x \in M^n \setminus (Cut\ locus\ of\ x)$ and r as described above*

$$\Delta r \leq \begin{cases} (n-1)\sqrt{K}\cot(\sqrt{K}r) & \textit{if } K > 0 \\ \frac{(n-1)}{r} & \textit{if } K = 0 \\ \sqrt{|K|}\coth(\sqrt{|K|}r) & \textit{if } K < 0. \end{cases} \tag{1.3.17}$$

Another comparison theorem of note is the Bishop volume comparison theorem, sometimes called the Bishop–Gromov inequality. This describes the relationship between the volume of the ball of a given radius on a Riemannian manifold and the volume of a geodesic ball.

Theorem 1.3.2 (Bishop volume comparison theorem)**.** *Let (M^n, g) be a complete Riemannian manifold with lower bounded Ricci curvature tensor $Ric \geq (n-1)K$ for $K \geq 0$. Then for $x_0 \in M^n$ and $R > 0$*

$$\frac{Vol(B(x_0, R))}{V(K, R)} \tag{1.3.18}$$

is non-increasing in R. Hence, as R goes to 0,

$$Vol(B(x_0, R)) \leq V(K, R). \tag{1.3.19}$$

$V(K, R)$ is the volume of the geodesic balls of radius R in the space form of M_K^n. This M_K^n is a simply connected space of constant sectional curvature K. See [103, 119, 120] for more information about comparison theorems.

1.4 Geometry on Smooth metric measure spaces

A complete smooth metric measure space, also known as a weighted Riemannian manifold, is a triple $(M^n, g, e^{-f}d\nu)$. Here (M^n, g) is a complete n-dimensional Riemannian manifold, f is a smooth real valued function on M^n, and $d\mu = e^{-f}d\nu$ is a weighted measure on M^n conformal to the Riemann volume measure, see [88]. This occurs when a collapsed measured Gromov-Hausdorff limit is taken on (1.4.2). Let $(M^n \times F^P, g_\epsilon)$ be a manifold with the metric

$$g_\epsilon = g_M + (\epsilon e^{-f})^2 g_F. \tag{1.4.1}$$

Then for a normalised measure $\tilde{d\nu_\epsilon}$:

$$(M^n \times F^P, g_\epsilon, \tilde{d\nu_\epsilon}) \xrightarrow{\epsilon \to 0} (M^n, g, e^{-Pf}d\nu_M). \tag{1.4.2}$$

This causes the Ricci tensor to be warped into the m-Bakry-Emery Ricci tensor.

The m-Bakry-Emery Ricci tensor is a natural extension of the Ricci tensor. It was initially studied by Bakry and Emery in [16, 17], see also [77, 98, 119]. We define this as

$$Ric_f^m = Ric + \nabla^2 f - \frac{df \otimes df}{m - n} \tag{1.4.3}$$

for $0 \leq n \leq m < \infty$, where Ric is the Ricci tensor of the manifold M^n and $\nabla^2 f$ is the Hessian operator acting on f. If $m = n$, then f must be constant. When $m = \infty$, the ∞-Bakry-Emery operator is obtained:

$$Ric_f = Ric + \nabla^2 f. \tag{1.4.4}$$

If f is a constant function then Ric_f becomes the standard Ricci tensor. For $m_1, m_2 < \infty$, such that $m_1 \leq m_2$, $Ric_f^{m_1} \leq Ric_f^{m_2}$.

The Bakry-Emery Ricci tensor shares many properties with the standard Ricci tensor, but also holds great importance in the study of Ricci solitons.

The Riemannian manifold can also be equipped with a variation of the Laplace-Beltrami operator called the Witten-Laplacian, which arises as a natural extension and is a symmetric diffusion operator, $\Delta_f = \Delta - \langle \nabla, \nabla f \rangle$. For any $u, v \in C_0^\infty(M^n)$, the integration by parts formula holds

$$\int_{M^n} \langle \nabla u, \nabla v \rangle d\mu = -\int_{M^n} u \Delta_f v d\mu = -\int_{M^n} v \Delta_f u d\mu \tag{1.4.5}$$

so the Witten-Laplacian is the infinitesimal generator of the Dirichlet form

$$\xi(u, v) = \int_{M^n} \langle \nabla u, \nabla v \rangle d\mu. \tag{1.4.6}$$

For more information of the above see [134].

The Bochner formula can be extended for use with the diffusion operator $\Delta_f u$. When $m = \infty$ we get

$$\frac{1}{2}\Delta_f(|\nabla u|^2) = |\nabla^2 u|^2 + \langle \nabla \Delta_f u, \nabla u \rangle + Ric_f(\nabla u, \nabla u) \qquad \text{for } m = \infty \tag{1.4.7}$$

Proposition 1.4.1. *Let (M^n, g) be a complete Riemannian manifold. Also let $u \in C^\infty(M^n)$ be a smooth function such that $u : M^n \to \mathbb{R}$. Then*

$$\frac{1}{2}\Delta_f(|\nabla u|^2) \geq \frac{(\Delta_f u)^2}{m} + \langle \nabla \Delta_f u, \nabla u \rangle + Ric_f^m(\nabla u, \nabla u) \qquad \textit{for } m < \infty \tag{1.4.8}$$

Proof. First we calculate

$$\begin{aligned}
\Delta_f |\nabla u|^2 =& \Delta |\nabla u|^2 - \langle \nabla f, \nabla |\nabla u|^2 \rangle \\
=& \Delta |\nabla u|^2 - 2\langle \nabla f, \Delta u \rangle \\
=& \Delta |\nabla u|^2 - 2 \operatorname{Hess}(u)(\nabla u, \nabla f)
\end{aligned}$$

and

$$\begin{aligned}
\langle \nabla u, \nabla(\Delta_f u) \rangle =& \langle \nabla u, \nabla(\Delta u - \langle \nabla u, \nabla f \rangle) \rangle \\
=& \langle u, \nabla(\Delta u) \rangle - \langle \nabla u, \langle \Delta u, \nabla f \rangle \rangle - \langle \nabla u, \langle \nabla u, \Delta f \rangle \rangle \\
=& \langle u, \nabla(\Delta u) \rangle - \operatorname{Hess}(u)(\nabla u, \nabla f) - \operatorname{Hess}(f)(\nabla u, \nabla u).
\end{aligned}$$

Then these combine with (1.4.3) to give

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla(\Delta_f u) \rangle + Ric_f^m(\nabla u, \nabla u) + \frac{|\langle \nabla u, \nabla f \rangle|^2}{m-n}.$$

Looking at the $|\nabla^2 u|^2$ we see

$$\begin{aligned}
|\nabla^2 u|^2 + \frac{|\langle \nabla u, \nabla f \rangle|^2}{m-n} \geq& \frac{(\Delta u)^2}{n} + \frac{|\langle \nabla u, \nabla f \rangle|^2}{m-n} \\
\geq& \frac{(\Delta_f u)^2}{m}.
\end{aligned}$$

Substitution of this gives the desired result. $\square$

Both the Laplacian comparison and volume comparison theorems have versions for smooth metric measure spaces. These can be seen in [119, 120] but we adopt the notation taken in [125, 126]. Firstly, we state the following for Ric_f:

Theorem 1.4.1 (Laplacian comparison theorem on smooth metric measure space)**.** *Let $(M^n, g, e^{-f}d\nu)$ be a complete smooth metric measure space with lower bound Ricci estimate $Ric_f \geq -(n-1)K$ for $K \geq 0$. Let $x_1 \in B(x_0, 2R)$. Then if $r(x_1, x_0) \geq 1$*

$$\Delta_f r(x_1) \leq \mu + (n-1)(2R-1)K \tag{1.4.9}$$

where $\mu := \max_{x|d(x,x_0)=1} \Delta_f r(x)$.

Similarly, for Ric_f^m we have the following:

Theorem 1.4.2 (Generalised Laplacian comparison theorem on smooth metric measure space)**.** *Let $(M^n, g, e^{-f}d\nu)$ be a complete smooth metric measure space with lower bound Ricci estimate $Ric_f^m \geq -(m-1)K$. Let $x_1 \in B(x_0, 2R)$. Then if $r(x_1, x_0) \geq 1$*

$$\Delta_f r(x_1) \leq (m+n-1)\sqrt{K}\coth(\sqrt{K}R). \tag{1.4.10}$$

Theorem 1.4.3 (Bishop volume comparison theorem on smooth metric measure space)**.** *Let $(M^n, g, e^{-f}d\nu)$ be a complete smooth metric measure space with $Ric_f \geq (n-1)K$. Fix a point $x_0 \in M^n$. Then*

1. *If $\partial_r f \geq -a$ along all minimal geodesic segments from x_0 then for $\frac{\pi}{2}\sqrt{K} \geq R \geq r > 0$, $K > 0$*
$$\frac{Vol_f(B(x_0,R))}{Vol_f(B(x_0,r))} \leq e^{aR}\frac{Vol_K^n(R)}{Vol_K^n(r)} \tag{1.4.11}$$

2. *If $|f(x)| \leq b$ then for $\frac{\pi}{4}\sqrt{K} \geq R \geq r > 0$, $K > 0$*
$$\frac{Vol_f(B(x_0,R))}{Vol_f(B(x_0,r))} \leq \frac{Vol_K^{n+4b}(R)}{Vol_K^{n+4b}(r)} \tag{1.4.12}$$

where $Vol_K^n(r)$ is the volume of the radius $r-$ball in the model space.

See [119, 120] for more information.

1.5 Some explicit examples of geometric flows on Riemannian manifolds

Ricci flow, first introduced by Hamilton [55], is the relation between the time derivative of the metric and the Ricci curvature tensor, and is a method of evolving the metric.

Definition 1.5.1. *Let $g(t)$ be a smooth family of Riemannian metrics on M^n where $t \in (0,T)$, $T \in \mathbb{R}^+$. If*

$$\frac{\partial}{\partial t} g(t) = -2Ric \tag{1.5.1}$$

then we say that $g_{ij}(x,t)$ is a solution to the Ricci flow.

If we normalise this then

$$\frac{\partial}{\partial t} g(t) = -2Ric + \frac{2\int_{M^N} Sd\mu}{n\int_{M^N} d\mu} g(t) \tag{1.5.2}$$

where S refers to the scalar curvature. Hamilton introduced this as an effort to find a proof for the geometrisation conjecture. However, it was the work of Perelman [90, 91, 92] that laid out the proof, finally confirmed by Kleiner and Lott in [71]. Perelman later built upon his own work to solve the Poincare conjecture.

If we look at the Ricci flow equation (1.1.10) in local coordinates and recall the Laplacian in these coordinates, where the metric is

$$ds^2 = g_{ij} dx^i dx^j, \tag{1.5.3}$$

it also follows that

$$\Delta g_{ij} = -2Ric. \tag{1.5.4}$$

From this, it is not hard to see the relationship with the heat equation and similarities therein.

Lemma 1.5.1. *Let $g(t)$ be a smooth family of metrics that solve (1.5.1), then*

$$g^{ij}\Big(\frac{\partial}{\partial t}\Gamma^k_{ij}\Big) = 0. \tag{1.5.5}$$

Proof. Through basic calculations

$$g^{ij}\Big(\frac{\partial}{\partial t}\Gamma^k_{ij}\Big) = -g^{kl}(2g^{ij}\nabla_i Ric_{jl} - \nabla_l S) \tag{1.5.6}$$

where S is the scalar curvature. Due to the contracted second Bianchi identity this is identically zero. □

Remark 1.5.1. *A similar statement can be made for Perelman-Ricci flow. We show this later.*

Lemma 1.5.2. *Let $g(t)$ be a smooth family of metrics that solve (1.5.1), then*

$$\frac{\partial}{\partial t}\Delta u = \Delta u_t + 2\langle Ric, \nabla^2 u\rangle. \tag{1.5.7}$$

Lemma 1.5.3. *Let (M^n, g) be an n-dimensional Riemannian manifold. If Ric is positive definite* *then the identity map*

$$id : (M^n, g) \to (M^n, \pm Ric) \tag{1.5.8}$$

is a harmonic map.

If we have a complete Riemannian manifold, such that

$$Ric = \lambda g - \frac{1}{2}\mathfrak{L}_V g, \tag{1.5.9}$$

where V is a smooth vector field, $\mathfrak{L}_v$ is the Lie derivative, and λ is some constant, then we can say that the metric g is a Ricci soliton. If f is a real valued function and $V = |\nabla f|$ then (1.5.9) becomes

$$Ric + \nabla^2 f = Ric_f = \lambda g, \tag{1.5.10}$$

and it is called a gradient Ricci soliton. Gradient solitons are important in the study of singularities for Ricci flow. See [56].

DeTurk formulated his own flow to construct a short-time solution $\tilde{g}(t)$ of the Ricci flow equation with initial condition $\tilde{g}(0) = g_0$. Here, the flow is defined as

$$\frac{\partial}{\partial t} g_{ij} = -2Ric_{ij} + \nabla_i W_j + \nabla_j W_i \tag{1.5.11}$$

where $W(t)$ is DeTurks vector field:

$$W_j = g_{jk} g^{pq} \left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq} \right), \tag{1.5.12}$$

Γ is the Levi-Civita connection and $\tilde{\Gamma}$ is the fixed background connection. Then by letting $\phi : M^n \to M^n$ such that

$$\frac{\partial}{\partial t}\phi(x_0) = -W(\phi(x_0), t), \tag{1.5.13}$$

$$\phi_0 = id^{M^n} \tag{1.5.14}$$

for $x_0 \in M^n$, and setting the metric of the pullback function ϕ, $\tilde{g} = \phi^* g(t)$ it can be shown that this is also a solution to the Ricci flow equation. For more background information and detail see [24, 36, 115].

Let g_0 be a conformal class of metrics on M^n, $n \geq 3$. Then for $g \in g_0$ the total scalar curvature is given by

$$\mathfrak{S} = V^{-\frac{n-2}{n}} \int_{M^n} S d\mu \tag{1.5.15}$$

for

$$V = \int_{M^n} d\mu, \tag{1.5.16}$$

and S the scalar curvature. Then by finding the gradient of $\mathfrak{S}$ the following equation is formed:

$$\frac{\partial g}{\partial t} = \frac{n-2}{2n} V^{-1} (\mathfrak{s} - S) g \tag{1.5.17}$$

where $\mathfrak{s}$ is the average scalar curvature. By this we mean that

$$\mathfrak{s} = \frac{\int_{M^n} S dvol}{\int_{M^n} dvol}. \tag{1.5.18}$$

Using this it then brings us to Yamabe flow.

Definition 1.5.2. *Let $g(t)$ be a smooth family of Riemannian metrics on M^n where $t \in (0, T)$, $T \in \mathbb{R}^+$. If*

$$\frac{\partial}{\partial t} g = -Sg. \tag{1.5.19}$$

then we say that $g(t)$ is a solution to the Yamabe flow.

The normalised version of this, which relates closely to (1.5.17), is

$$\frac{\partial g}{\partial t} = (\mathfrak{s} - S) g \tag{1.5.20}$$

Yamabe flow importantly conserves conformal structure. Let

$$g = u^{\frac{4}{n-2}} g_0 \tag{1.5.21}$$

and take a positive function $u > 0$ on M^n. Then the relationship between the scalar curvature on g and g_0 is given by the following:

$$S_g = -u^{\frac{n+2}{n-2}} \Big(\frac{4(n-1)}{n-2} \Delta_{g_0} u - S_{g_0} u \Big) \tag{1.5.22}$$

which can be used to find

$$\frac{\partial}{\partial t} u^{\frac{n+2}{n-2}} = \frac{n+2}{4} \Big(\frac{4(n-1)}{n-2)} \Delta_{g_0} u - S_{g_0} u + \mathfrak{s}_g u^{\frac{n+2}{n-2}} \Big). \tag{1.5.23}$$

Similarly if we let $g = e^u g_{\mathbb{S}^n}$, then

$$\frac{\partial}{\partial t} u = (n-1) e^{-u} \Big(\Delta u + \frac{n-2}{4} |\nabla u|^2 - n \Big) \tag{1.5.24}$$

which is similar to the heat equation. See [23, 37, 81, 132] for more information.

The variational formulas for Yamabe flow are as follows:

Lemma 1.5.4. *Let $g(t)$ be a smooth family of metrics that solve (1.5.19), then*

$$\frac{\partial}{\partial t}|\nabla u|^2 = S|\nabla u|^2 + 2\langle \nabla u_t, \nabla u\rangle. \tag{1.5.25}$$

Lemma 1.5.5. *Let $g(t)$ be a smooth family of metrics that solve (1.5.19), then*

$$\frac{\partial}{\partial t}\Delta u = -\frac{n-2}{2}\langle \nabla u, \nabla S\rangle + S\Delta u + \Delta u_t. \tag{1.5.26}$$

See [133] for more detail.

Alternatively, if we are working on a smooth metric measure space $(M^n, g(t), f(t))$, then (K, m) Perelman-Ricci flow is defined as:

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric_f^m(g) = -Kg, \tag{1.5.27}$$

$$\frac{\partial f}{\partial t} - \frac{1}{2}\operatorname{Tr}\left(\frac{\partial g}{\partial t}\right) = 0 \tag{1.5.28}$$

where Tr is the trace, K and m are fixed constants for $m \geq N$. The specific case

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric_f^m(g) \geq -Kg, \tag{1.5.29}$$

$$\frac{\partial f}{\partial t} - \frac{1}{2}\operatorname{Tr}\left(\frac{\partial g}{\partial t}\right) = 0 \tag{1.5.30}$$

is called super Perelman-Ricci flow. We discuss this more later.

1.6 The heat kernel on Riemannian manifolds and symmetric spaces

The heat kernel is a fundamental solution to the heat equation. In Euclidean space the heat kernel takes the form

$$H(x,y,t) = (4\pi t)^{-\frac{m}{2}} e^{-\frac{(r(x,y))^2}{4t}} \tag{1.6.1}$$

where $r(x,y)$ is the distance function. If we are looking at the heat equation on manifolds, then we note that since all Riemanian manifolds are locally Euclidean, we can use (1.6.1) as an approximation. We can state the following proposition for a Riemannian manifold with unstated boundary assumptions:

Proposition 1.6.1. *Let (M^n, g) be a complete Riemannian manifold without boundary. Then $\forall x \in M^n$, the heat kernel is approximated by*

$$H(x,y,t) \sim (4\pi t)^{-\frac{m}{2}} e^{-\frac{(r(x,y))^2}{4t}} \tag{1.6.2}$$

as $t \to 0$ and $r(x,y) \to 0$.

Theorem 1.6.1. *Let (M^n, g) be a complete, non-compact Riemannian manifold. Then there exists a positive symmetric heat kernel such that*

$$f(x,t) = \int_{M^n} H(x,y,t) f_0(y) dy \tag{1.6.3}$$

for $f_0 \in L^2(M^n)$, solves the heat equation on $M^n \times (0, \infty)$.

Instead, if we have that $H(x,y,t)$ is the minimal symmetric heat kernel then Li and Yau showed an upper bound for $H(t,x,y)$ on the ball and on (M^n, g) for positive time.

Remark 1.6.1. *A heat kernel $H(x,y,t)$ on $M \times M \times (0,\infty)$ is referred to as minimal if the following applies. Let $\overline{H}(x,y,t)$ be another heat kernel on $M \times M \times (0,\infty)$. Then*

$$H(x,y,t) \leq \overline{H}(x,y,t), \tag{1.6.4}$$

for all $x, y \in M$ and $t \in (0,\infty)$.

Theorem 1.6.2. *Let (M^n, g) be a complete Riemannian manifold with $Ric \geq (-n-1)K$, $K \in \mathbb{R}$, and $H(x,y,t)$ be the minimal symmetric heat kernel on $M \times M \times (0,\infty)$. Let $H(x,y,t)$ solve*

$$\Big(\Delta_y - \frac{\partial}{\partial t}\Big) H(x,y,t) = 0 \tag{1.6.5}$$

and

$$\lim_{t \to 0} H(x,y,t) = \delta_x(y). \tag{1.6.6}$$

Then for any $x_0 \in M^n$, $R, \epsilon > 0$, $t \leq \frac{R^2}{4}$, and $x, y \in B(x_0, R)$ the following inequality holds

$$H(x,y,t) \leq C V_x^{-\frac{1}{2}}(\sqrt{t}) e^{-\mu_1(M)t} \times V_y^{-\frac{1}{2}}(\sqrt{t}) e^{-\frac{r^2}{4(1+2\epsilon)t} + C\sqrt{(R^{-2}+K)t}} \tag{1.6.7}$$

where $\mu_1(M)$ is the first Dirichlet eigenvalue on M and $V_x(y) = Vol(B(x,y))$.

Corollary 1.6.1. *Let (M^n, g), $H(x,y,t)$, and Ric be defined as above. Then for $x, y \in M$ and $t \in (0,\infty)$, the following inequality holds*

$$H(x,y,t) \leq C V_x^{-\frac{1}{2}}(\sqrt{t}) e^{-\mu_1(M)t} \times V_x^{-\frac{1}{2}}(\sqrt{t}) e^{-\frac{r^2}{4(1+2\epsilon)t} + C\sqrt{Kt}}. \tag{1.6.8}$$

Behaviour for the heat kernel as $t \to \infty$ will differ according to whether our Riemannian manifold is a compact or complete and non-compact. If we have a compact Riemannian manifold with unspecified boundary and we let $t \to \infty$ the heat kernels for Dirichlet and Neumann boundary conditions, it can be expressed using the first eigenfunction and eigenvalue:

$$H(x,y,t) \sim e^{-\lambda_1} \phi_1(x) \phi_1(y). \tag{1.6.9}$$

If, however, the Riemannian manifold is complete and non-compact, then we observe the following:

Theorem 1.6.3. *Let (M^n, g) be an n-dimensional Riemannian manifold with non-negative Ricci curvature. Let $C > 0$ be a constant such that*

$$\liminf_{s \to \infty} \frac{V_{x_0}(s)}{s^n} = C \tag{1.6.10}$$

for some point $x_0 \in M^n$. Then

$$\lim_{t \to \infty} V_{x_0}(\sqrt{t}) H(x, y, t) = n^{-1} A (4\pi)^{-\frac{n}{2}} \tag{1.6.11}$$

where A is the area of the $(n-1)$-dimensional unit sphere.

See [50, 75] for more information.

Chapter 2

Special functions in the theory of symmetric spaces

Symmetric spaces are a special type of Riemannian manifolds whose groups of symmetries contain an inverse symmetry for each point. The study of these is still quite young in the mathematical world as its origins date back to Cartan and Weyl in the 1930s, [27, 121], with further study coming from Gel'fand and Chandra. For greater detail and work therein see [49, 58].

The rank of a Riemannian symmetric space refers to the rank of M, the maximal dimension of a flat, totally geodesic submanifold of M. In this context, a Riemannian manifold is said to be flat if its curvature tensor vanishes identically. If the rank is 1, then these turn out to be geodesics.

A Gel'fand pair $(\mathbf{G}, \mathbf{K})$, consists of a group $\mathbf{G}$ and a subgroup $\mathbf{K}$. The specific case when $\mathbf{G}$ is a Lie group and $\mathbf{K}$ a compact subgroup is of particular note. Further, when $\mathbf{X} = \mathbf{G}/\mathbf{K}$ is a symmetric space, we have that $\mathbf{G}$ is a semi-simple Lie group and a member of the Harish-Chandra class for which it is a reductive Lie group and $\mathbf{K}$ is the maximal compact subgroup. For a compact symmetric space where $\mathbf{G}$ a real simple Lie group, Cartan showed that there are seven varieties.

$\mathbf{G}$	$\mathbf{K}$	Rank
$\mathbf{SO}(p+q)$	$\mathbf{SO}(p) \times \mathbf{SO}(q)$	$min(p,q)$
$\mathbf{SO}(2n)$	$\mathbf{U}(n)$	$[n/2]$
$\mathbf{Sp}(n)$	$\mathbf{U}(n)$	n
$\mathbf{Sp}(p+q)$	$\mathbf{Sp}(p) \times \mathbf{Sp}(q)$	$min(p,q)$
$\mathbf{SU}(n)$	$\mathbf{SO}(n)$	$n-1$
$\mathbf{SU}(2n)$	$\mathbf{Sp}(n)$	$n-1$
$\mathbf{SU}(p+q)$	$S(\mathbf{U}(p) \times \mathbf{U}(q))$	$min(p,q)$

Table 2.1: The rank of Gel'fand pairs for certain choices of $\mathbf{G}$ and $\mathbf{K}$.

In table 2.1, $\mathbf{SO}(n)$, $\mathbf{Sp}(n)$, and $\mathbf{SU}(n)$ are the special orthogonal group, symplectic group, and special unity group respectively. Of particular note is that these groups are Lie groups. Lie groups have a rich body of research for them as they play an important role in both geometry and physics. For more information on Lie groups, see [15, 44, 48, 54, 104].

2.1 Non-compact rank one symmetric spaces

The table above consists only of compact symmetric spaces; however, these spaces can either be compact or non-compact. For non-compact rank one symmetric spaces, we have some important examples: the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$, the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, the quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{Q}}$, and the Cayley hyperbolic plane $\mathbb{H}^n_{\mathbb{C}a}$. It can be seen that

$$\mathbb{H}^n_{\mathbb{R}} = \mathbf{SO}(n,1)/\mathbf{SO}(n), \tag{2.1.1}$$

$$\mathbb{H}^n_{\mathbb{C}} = \mathbf{SU}(n,1)/\mathbf{U}(n), \tag{2.1.2}$$

$$\mathbb{H}^n_{\mathbb{Q}} = \mathbf{Sp}(n,1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1)), \tag{2.1.3}$$

$$\mathbb{H}^2_{\mathbb{C}a} = \mathbf{F}_4^*/\mathbf{Spin}(9) \tag{2.1.4}$$

where $\mathbf{F}_4^*$ is a compact form of an exceptional Lie group and **Spin** is the spin group.

These groups all have Beltrami-Klein models which project their spaces onto the unit disk.

The real hyperbolic space $\mathbf{SO}(n,1)/\mathbf{SO}(n)$

The real hyperbolic space $\mathbf{SO}(n,1)/\mathbf{SO}(n)$, where $n \in \mathbb{N}\setminus\{1\}$, can be shown to be isometric to the space $\mathbb{H}^n_{\mathbb{R}}$. To produce a Beltrami-Klein model for this space, we start by taking the open ball $B^n_{\mathbb{R}} = \{x \in \mathbb{R}^n : |x| < 1\}$ with Riemannian structure

$$ds^2 = \frac{|dx|^2}{1-|x|^2} + \frac{\langle x, dx\rangle^2_{\mathbb{R}}}{(1-|x|^2)^2}. \tag{2.1.5}$$

This then corresponds to

$$g_{ij} = \frac{\delta_{i,j}}{1-|x|^2} + \frac{x_i x_j}{(1-|x|^2)^2}. \tag{2.1.6}$$

By calculations we can show that

$$R_{ijkl} = -(g_{ik}g_{jl} - g_{jk}g_{il}) \tag{2.1.7}$$

which shows that $B^n_{\mathbb{R}}$ with the corresponding metric has constant sectional curvature of -1. As stated above, this space is isometric to $\mathbb{H}^n_{\mathbb{R}}$.

Proposition 2.1.1. *The Riemannian measure on* $\mathbb{H}^n_{\mathbb{R}}$ *is*

$$d\mu = \frac{dx}{(1-|x|^2)^{\frac{n+1}{2}}}. \tag{2.1.8}$$

The complex hyperbolic space $\mathbf{SU}(n,1)/\mathbf{U}(n)$

Like for the real hyperbolic space, the complex hyperbolic space has a similar isometry; holomorphically isometric to the space $\mathbb{H}^n_{\mathbb{C}}$. This is seen when the maximal sectional curvature is equal to 1.

Let $B^n_{\mathbb{C}} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball with standard complex structure $z = (z_1, ..., z_n) \in B^n_{\mathbb{C}}$. The metric for this is

$$ds^2 = 2\sum_{i,j=1}^{n} g_{i\bar{j}}(z)dz_i d\bar{z}_j \tag{2.1.9}$$

with metric tensor

$$g_{i\bar{j}}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left(\frac{1}{2}\log\frac{1}{(1-|z|^2)}\right). \tag{2.1.10}$$

With this we can find the Riemannian curvature tensor:

$$R_{ijkl} = 2(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}), \tag{2.1.11}$$

which gives that $B^n_{\mathbb{C}}$ with the corresponding metric is a space of constant holomorphic sectional curvature equal to -4. This space we define as $\mathbb{H}^n_{\mathbb{C}}$.

Proposition 2.1.2. *The Riemannian measure on* $\mathbb{H}^n_{\mathbb{C}}$ *is*

$$d\mu = \frac{dm_n(z)}{(1-|z|^2)^{n+1}} \tag{2.1.12}$$

where $dm_n(z)$ *is the Lebesgue measure on* $\mathbb{C}^n$*.*

The quaternionic hyperbolic space $\mathbf{Sp}(n,1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$

Similarly to what we have seen above, the quaternionic hyperbolic space of maximal sectional curvature equal to -1 is isometric to $\mathbb{H}^n_{\mathbb{Q}}$.

Let $B^n_{\mathbb{Q}} = \{q \in \mathbb{Q}^n : |q| < 1\}$, for $q = (q_1, ..., q_n) \in B^n_{\mathbb{Q}}$. Then because of the relation

$$q \leftrightarrow z = (z_1, ..., z_{2n}) \tag{2.1.13}$$

where

$$q_i = z_i + z_{n+i}\mathbf{i}_2 \tag{2.1.14}$$

for $1 \le i \le n$, we see that $B^n_{\mathbb{Q}} = z \in \mathbb{C}^{2n} : |z| < 1$. $\mathbf{i}$ is a basis for $\mathbb{Q}$ such that

$$\mathbf{i}_0 = 1, \qquad \mathbf{i}_1 = (\mathbf{i}_1^{\mathbb{C}}, 0),$$
$$\mathbf{i}_2 = (0, 1), \qquad \mathbf{i}_3 = \mathbf{i}_1\mathbf{i}_2.$$

Proposition 2.1.3. *The Riemannian measure on* $\mathbb{H}^n_{\mathbb{Q}}$ *is*

$$d\mu = \frac{dm_{2n}(z)}{(1-|z|^2)^{2n+2}}. \tag{2.1.15}$$

The Cayley hyperbolic plane $\mathbf{F}_4^*/\mathbf{Spin}(9)$

Despite the Cayley numbers not being associative, the Cayley hyperbolic plane with maximal sectional curvature equal to -1 is an isometry of $\mathbb{H}^2_{\mathbb{C}a}$.

Let $x = (x_1, ..., x_{16})$, $y = (y_1, ..., y_{16}) \in \mathbb{R}^{16}$. Also let

$$\Phi_{\mathbb{C}a}(a, b) = 2\sum_{k=1}^{8} p_k(x)p_k(y) + p_9(x)p_9(y) + p_{10}(x)p_{10}(y) \tag{2.1.16}$$

for $p_i(x)$ as defined in [116]. Then for $i, j \in (1, .., 16)$

$$a_{ij} = \delta_{i,j}(1-|x|^2)^2 + \frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_j}\Phi_{\mathbb{C}a}(x, y) \tag{2.1.17}$$

and

$$g_{ij} = \frac{a_{ij}}{(1-|x|^2)^2}. \tag{2.1.18}$$

It can then be shown that $||g_{ij}||^{16}_{i,j=1}$ causes a Riemannian structure on the open ball, which is denoted $\mathbb{H}^n_{\mathbb{C}a}$.

Proposition 2.1.4. *The Riemannian measure on* $\mathbb{H}^n_{\mathbb{C}a}$ *is*

$$d\mu = \frac{dx}{(1-|x|^2)^{12}}. \tag{2.1.19}$$

For further information of these spaces and for proofs of the propositions see [116].

2.2 Compact rank one symmetric spaces

Our main focus will be on the eigenfunctions of the Laplacian on compact rank one symmetric spaces. The choice of rank one symmetric spaces in particular is due to the fact that the eigenfunctions of the Laplacian on these spaces are the special functions (Jacobi and Gegenbauer polynomials). From these we can find valuable formulas.

There are also some quite notable examples of these spaces such as the sphere ($\mathbb{S}^n$), the real and complex projective spaces ($\mathbf{P}^{n+1}(\mathbb{R}), \mathbf{P}^{n+1}(\mathbb{C})$), the quaternionic projective space ($\mathbf{P}^{n+1}(\mathbb{H})$), and the Caley projective space ($\mathbf{P}^2(Cay)$). These are also notable for their links to Lie groups mentioned above; the sphere, real projective space, complex projective space, and quaternionic projective space are represented respectively by

$$\begin{aligned}
\mathbb{S}^n =& \mathbf{SO}(n+1)/\mathbf{SO}(n), && (2.2.1)\\
\mathbf{P}^{n+1}(\mathbb{R}) =& \mathbf{SO}(n+1)/\mathbf{O}(n), && (2.2.2)\\
\mathbf{P}^{n+1}(\mathbb{C}) =& \mathbf{SU}(n+1)/S(\mathbf{U}(n)\times\mathbf{U}(1)), && (2.2.3)\\
\mathbf{P}^{n+1}(\mathbb{H}) =& \mathbf{Sp}(n+1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1)). && (2.2.4)
\end{aligned}$$

In addition to these spaces having equivalent forms using Lie groups, there are also links to special functions - in particular, the orthogonal polynomials. We can see this through the involutive automorphism of compact Lie algebras.

The sphere $\mathbb{S}^n$

Let

$$ds^2 = dx_1^2 + ... + dx_{n+1}^2 \tag{2.2.5}$$

be the standard Riemannian metric on $\mathbb{S}^n$. In spherical coordinates

$$ds^2 = d\theta_n^2 + \sin^2\theta_n d\theta_{n-1}^2 + ... + \sin^2\theta_n ... \sin^2\theta_2 d\theta_1^2. \tag{2.2.6}$$

The Riemannian measure on $\mathbb{S}^n$ is defined by

$$d\mu = \sin^{n-1}\theta_n \sin^{n-2}\theta_{n-1} ... \sin\theta_2 d\theta_1 ... d\theta_n. \tag{2.2.7}$$

Instead, we can create a similar structure based off the Euclidean space. We denote $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. Now let $F_1 = \mathbb{R}^n$ and $F_2 = \overline{\mathbb{R}^n} \setminus \{0\}$. Now introduce the bijective mapping

$$\phi_k : F_k \to \mathbb{R}^n \tag{2.2.8}$$

for $k = 1, 2$ with

$$\phi_1(p) = p, \tag{2.2.9}$$

$$\phi_2(p) = \begin{cases} \frac{p}{|p|^2}, & p \in \mathbb{R}^n \setminus \{0\} \\ 0, & p = \infty. \end{cases} \tag{2.2.10}$$

It can be shown that the atlas (F_k, ϕ_k), for $k = 1, 2$, is a real analytic structure on $\overline{\mathbb{R}^n}$. Next, if we let

$$g_{ij} = \frac{4\delta_{i,j}}{(1+|x|^2)^2} \tag{2.2.11}$$

for $x \in \mathbb{R}^n$, $i, j \in (1, ..., n)$, then

$$d\mu = \frac{2^n dx}{(1 + |x|^2)^n} \tag{2.2.12}$$

for $x \in \mathbb{R}^n$. It then can be shown that

$$R_{ijkl} = g_{ik}g_{jl} - g_{jk}g_{il}. \tag{2.2.13}$$

Thus, the following can be stated:

Proposition 2.2.1. *The space $\mathbf{SO}(n+1)/\mathbf{SO}(n)$ of constant sectional curvature equal to 1 is isometric to the sphere $\mathbb{S}^n$ with the standard Riemannian metric on $\mathbb{S}^n$ or to $\overline{\mathbb{R}^n}$ with metric (2.2.11).*

The real projective space $\mathbf{SO}(n+1)/\mathbf{O}(n)$

We start by forming the projective space. Take the equivalence relation

$$x \sim \lambda x \tag{2.2.14}$$

for $\lambda \in \mathbb{R} \setminus \{0\}$ on $\mathbb{R}^{n+1} \setminus \{0\}$. We define the set of all equivalence classes to be the real projective space of dimension $n+1$, $\mathbf{P}^{n+1}(\mathbb{R})$. It is also useful to note that for $x \in \mathbb{R}^{n+1}$, there exists λ such that λx has a norm equal to 1.

It can be shown that this can be turned into a real analytic manifold. Let $f = \lambda x$ where $(f_0, ..., f_n) = f \in \mathbb{R}^{n+1} \setminus \{0\}$. Next, we define, similar to that in the sphere section,

$$F_k = \{[f] \in \mathbf{P}^{n+1}(\mathbb{R}) : f_k \neq 0\} \tag{2.2.15}$$

for $k = \{0, ..., n\}$. This is a cover for $\mathbf{P}^{n+1}(\mathbb{R}) = \bigcup_{k=0}^{n} F_k$. Also let

$$\phi_k : F_k \to \mathbb{R}^n \tag{2.2.16}$$

be a bijective mapping. It can then be shown that (F_k, ϕ_k) form an atlas, meaning that $\mathbf{P}^{n+1}(\mathbb{R})$ is a real analytic structure. We can then put a Riemannian structure onto $\mathbf{P}^{n+1}(\mathbb{R})$. Let

$$g_{ij} = \frac{\delta_{i,j}}{(1 + |x|^2)} - \frac{x_i x_j}{(1 + |x|^2)^2} \tag{2.2.17}$$

for $x \in \mathbb{R}^n$, $i, j \in \{1, ..., n\}$, and let the metric be defined as

$$\mathfrak{g}_{ij}^k(p) = g_{ij}(\phi_k(p)) \tag{2.2.18}$$

for $p \in F_k$, $k = \{0, ..., n\}$. The measure on this is

$$d\mu = \frac{dx}{(1 + |x|^2)^{\frac{n+1}{2}}} \tag{2.2.19}$$

for $x \in \mathbb{R}^n$.

By calculating the Riemann curvature tensor

$$R_{ijkl} = g_{ik}g_{jl} - g_{jk}g_{il} \tag{2.2.20}$$

it is seen that the sectional curvature is equal to 1. We also have that

$$\mathbf{P}^{n+1}(\mathbb{R}) = \mathbb{R}^n \cup \mathbf{P}^n(\mathbb{R}). \tag{2.2.21}$$

Proposition 2.2.2. *The real projective space $\mathbf{SO}(n+1)/\mathbf{O}(n)$ of constant sectional curvature equal to 1 is isometric to $\mathbf{P}^{n+1}(\mathbb{R})$ with metric (2.2.18).*

The complex projective space $\mathbf{SU}(n+1)/S(\mathbf{U}(n) \times \mathbf{U}(1))$

The complex projective space is very similar in its makeup to its real counterpart. Working on $\mathbb{C}^{n+1} \backslash \{0\}$, we take the equivalence relation (2.2.14) for $\lambda \in \mathbb{C} \backslash \{0\}$. The set of all these equivalence relations is $\mathbf{P}^{n+1}(\mathbb{C})$.

$\mathbf{P}^{n+1}(\mathbb{C})$ can now be made into a complex analytic manifold. Let $(f_0, ..., f_n) \in \mathbb{C}^{n+1}$ and define F_k as above on $\mathbf{P}^{n+1}(\mathbb{C})$. Also let

$$\phi_k : F_k \to \mathbb{C}^n \tag{2.2.22}$$

be a bijective mapping. Using this we see that (F_k, ϕ_k) form an atlas, and therefore applying that, it is seen that $\mathbf{P}^{n+1}(\mathbb{C})$ is a complex analytic manifold. A Hermitian metric can be applied to this giving

$$g_{i\bar{j}} = h_{ij}(z)/2 \tag{2.2.23}$$

where

$$h_{ij}(z) = \frac{\delta_{i,j}}{(1+|z|^2)} - \frac{\overline{z}_i z_j}{(1+|z|^2)^2} \tag{2.2.24}$$

for $z \in \mathbb{C}^n$. The Riemannian measure on F_0 is given by

$$d\mu = \frac{d_{m_n}(z)}{(1+|z|^2)^{n+1}}. \tag{2.2.25}$$

The Riemann curvature tensor is given by

$$R_{ijkl} = -2(g_{ij}g_{kl} + g_{il}g_{kj}) \tag{2.2.26}$$

which means that $\mathbf{P}^{n+1}(\mathbb{C})$ with Hermitian metric has constant holomorphic sectional curvature equal to 4. We also have that

$$\mathbf{P}^{n+1}(\mathbb{C}) = \mathbb{C}^n \cup \mathbf{P}^n(\mathbb{C}). \tag{2.2.27}$$

Proposition 2.2.3. *The complex projective space* $\mathbf{SU}(n+1)/S(\mathbf{U}(n)\times\mathbf{U}(1))$ *with minimal section curvature equal to* 1 *is holomorphically isometric to the space* $\mathbf{P}^{n+1}(\mathbb{C})$ *with Hermitian metric.*

The quaternionic projective space $\mathbf{Sp}(n+1)/(\mathbf{Sp}(n)\times\mathbf{Sp}(1))$

Working on $\mathbb{H}^{n+1}\setminus\{0\}$ with (2.1.13), we have the $(2n+2)$-tuples of $f = (f_0,...,f_{2n+1})$ which give rise to the class $[f] = [(f_0,...,f_{2n+1})]$. The set of all these classes is denoted $\mathbf{P}^{n+1}(\mathbb{H})$. In particular, two of these $(2n+2)$-tuples f and $\mathfrak{f}$ only belong to the same class if and only if for $(\lambda,\psi)\in\mathbb{C}^2\setminus\{0\}$

$$\lambda\mathfrak{f}_k - \psi\bar{\mathfrak{f}}_{n+1+k} = f_k, \tag{2.2.28}$$

$$\lambda\mathfrak{f}_{n+1+k} + \psi\bar{\mathfrak{f}}_k = f_{n+1+k} \tag{2.2.29}$$

for $k = \{0,...,n\}$.

Again, we can create a real analytic structure. Let

$$F_k = \{[f] : |f_k|^2 + |f_{n+1+k}|^2 \neq 0\} \tag{2.2.30}$$

for k above, and let

$$\phi_k : F_k \to \mathbb{C}^{2n} \tag{2.2.31}$$

be a bijective mapping. Then we can say that (F_k,ϕ_k) form an atlas so $\mathbf{P}^{n+1}(\mathbb{H})$ is a real analytic structure.

Let

$$h_{i+a,j+b}(z) = \frac{\theta_{a,b}(z)}{(1+|z|^2)^2} \tag{2.2.32}$$

where

$$\theta_{0,0}(z) = (1+|z|^2)\delta_{i,j} - \overline{z}_i z_j - z_{n+i}\overline{z}_{n+j},$$

$$\theta_{0,n}(z) = z_{n+i}\overline{z}_j - \overline{z}_i z_{n+j},$$

$$\theta_{n.0}(z) = z_i\overline{z}_{n+j} - \overline{z}_{n+i} z_j,$$

$$\theta_{n,n}(z) = (1+|z|^2)\delta_{i,j} - z_i\overline{z}_j - \overline{z}_{n+i} z_{n+j}.$$

Now we define the metric

$$\mathfrak{g}^k_{\mathrm{i},\mathrm{j}(p)} = h_{\mathrm{i},\mathrm{j}}(\phi(p)) \tag{2.2.33}$$

for $p \in F_k$, $\mathrm{i}, \mathrm{j} \in \{1, .., 2n\}$. The measure on $\mathbf{P}^{n+1}(\mathbb{H})$ with metric (2.2.33) is

$$d\mu = \frac{dm_{2n}(z)}{(1+|z|^2)^{2n+2}}. \tag{2.2.34}$$

Proposition 2.2.4. *The quaternionic projective space* $\mathbf{Sp}(n+1)/(\mathbf{Sp}(n) \times \mathbf{Sp}(1))$ *with minimal sectional curvature equal to* 1 *is isometric to* $\mathbf{P}^{n+1}(\mathbb{H})$ *with metric (2.2.33).*

For more detail and examples not linked to the spaces mentioned above, see [58, 116].

2.3 Orthogonal polynomials as eigenfunctions of the Laplacian on compact rank one symmetric spaces

The importance of compact rank one symmetric spaces is that the eigenfunctions of the Laplacian on these spaces are orthogonal polynomials. This creates a powerful link between classical and contemporary mathematics.

Let Δ be the Laplacian on a Riemannian manifold. Then there exists an orthonormal basis ϕ_i which also solves the eigenvalue problem

$$\Delta\phi_i = \lambda_i \phi_i, \tag{2.3.1}$$

where λ_i are the eigenvalues. Further to this, we define the heat semi-group

$$U(t) = e^{-t\Delta} \tag{2.3.2}$$

for $t > 0$. This produces a heat kernel that is symmetric and smooth. We can express this by the spectral sum

$$H(t,x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y) \tag{2.3.3}$$

for positive time, where $x, y \in M^n$. We note that 2.3.3 is explicitly for compact spaces, as for non-compact spaces there can be an uncountable number of eigenvalues.

Specifically for rank one symmetric spaces, the heat kernel is defined as

$$H(t,\theta) = \sum_{i=0}^{\infty} \frac{\mathfrak{M}_i^n}{Vol}\Phi_i(\theta)e^{-\lambda_i^n t} \tag{2.3.4}$$

where λ_i^n are distinct eigenvalues of Δ, $\mathfrak{M}_i^n$ is the multiplicity of the eigenvalue λ_i^n on M^n, Vol is the volume of M^n, Φ_i is the spherical function on M^n associated with the eigenvalues, and θ is the geodesic between two points $x, y \in M^n$.

The spherical functions mentioned above are in fact the special functions (orthogonal polynomials); the Jacobi polynomial (and by extension the Gegenbauer polynomial as this is a special case of the Jacobi polynomial). With certain choices of t, the differential equations of the same name also appear. We go into more detail about these polynomials later in chapter 2. Also, see [13] for more detail of the above discussion on rank one symmetric spaces of this kind.

For the specific choices of rank one symmetric spaces, we listed out their distinct eigenvalues λ_i^n, multiplicity $\mathfrak{M}_i^n$, and volume Vol below:

M^n	λ_i^n	$\mathfrak{M}_i^n$	Vol
$\mathbb{S}^n$	$i(i+n+1)$	$(2i+n-1)\frac{(i+n-2)!}{i!(n-1)!}$	$\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$
$\mathbf{P}^n(\mathbb{R})$	$2i(2i+n+1)$	$(4i+n-1)\frac{(2i+n-2)!}{(2i)!(n-1)!}$	$\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$
$\mathbf{P}^n(\mathbb{C})$	$i(i+n)$	$\frac{(2i+n)}{n}\left[\frac{\Gamma(i+n)}{\Gamma(n)i!}\right]^2$	$\frac{4^n\pi^n}{n!}$
$\mathbf{P}^n(\mathbb{H})$	$i(i+2n+1)$	$\frac{(2i+2n+1)(i+2n)}{2n(2n+1)(i+1)}\left[\frac{\Gamma(i+2n)}{\Gamma(2n)i!}\right]^2$	$\frac{(4\pi)^{2n}}{\Gamma(2n+1)}$

Table 2.2: Table of λ_i^n, $\mathfrak{M}_i^n$, and Vol for different choices of rank one symmetric spaces.

Example 2.3.1. *Let $M^n = \mathbb{S}^n$ be the n-dimensional sphere. For the sphere the heat kernel 2.3.4 has distinct eigenvalues λ_i^n, multiplicity $\mathfrak{M}_i^n$, volume Vol as defined in table 2.2. The spherical function associated with this is the Gegenbauer polynomial $C_i^{(\mathbf{X})}$. For this specific case we see that the polynomial in question has $\mathbf{X} = \frac{n-1}{2}$. These polynomials are eigenfunctions of the Laplacian on the sphere solving $\Delta\Phi_i = i(i+n+1)\Phi_i$. Exactly*

$$\Phi_i(\theta) = \frac{C_i^{\frac{n-1}{2}}(\cos\theta)}{C_i^{\frac{n-1}{2}}(1)} \tag{2.3.5}$$

where

$$\Phi_i(0) = 1, \quad C_i^{\frac{n-1}{2}}(1) = \frac{\Gamma(i+n-1)}{\Gamma(n-1)i!}. \tag{2.3.6}$$

Example 2.3.2. *Let $M^n = \mathbf{P}^n(\mathbb{R})$ be the n-dimensional real projective space. As seen above, the Lie group representation of this space is nearly the same as that for the sphere with its components being that of the orthogonal group. Due to this, the eigenvalues, multiplicity, and volume for the heat kernel are very closely related too; see table 2.2. Here the special function is also the Gegenbauer polynomial.*

Example 2.3.3. *Let $M^n = \mathbf{P}^n(\mathbb{C})$ be the n-dimensional complex projective space. It has distinct eigenvalues λ_i^n, multiplicity $\mathfrak{M}_i^n$, and volume Vol as defined in table 2.2. The special functions here take on the more general Jacobi polynomial. These take the form*

$$\Phi_i(\theta) = \frac{P_i^{(\alpha,\beta)}(\cos\theta)}{P_i^{(\alpha,\beta)}(1)} \tag{2.3.7}$$

with $\alpha = n-1$ *and* $\beta = 0$. *Also*

$$\Phi_i(0) = 1, \quad P_i^{\alpha,\beta}(1) = \frac{\Gamma(\alpha+i+1)}{\Gamma(\alpha+1)i!}. \tag{2.3.8}$$

Example 2.3.4. *Let* $M^n = \mathbf{P}^n(\mathbb{H})$ *be the* n*-dimensional quaternionic projective space. For this we have* λ_i^n, $\mathfrak{M}_i^n$, Vol *as depicted in table 2.2. Like with the complex projective space, the special functions are the Jacobi polynomials. This time* $\alpha = 2n-1$ *and* $\beta = 1$. $\Phi_i(\theta)$ *is the same and* $P_i^{(\alpha,\beta)}(1) = \frac{\Gamma(2n+i)}{\Gamma(2n)i!}$.

Authors such as Awonusika, Day, and Taheri have used this as a starting block for their calculations of differential spectral identities, which can be seen in [11, 12, 21, 40, 39]. Some of these use the incredibly useful Faa di Bruno formula, which finds the n^{th} derivative of a composite function. As part of these calculations the Bell polynomials appear.

2.4 The Faa di Bruno formula and explicit formulas for Bell polynomials

Within our studies, we will find Maclaurin expansions, as well as differential spectral identities on the symmetric spaces mentioned above. For this, we use the Faa di Bruno formula. This formula finds the n^{th} derivative of a composite function, $f(g(x))$. It is an alternative formula to the Leibniz formula and approaches it from a combinatorial prospective.

Proposition 2.4.1. *Let* f *and* g *be* n *times differentiable functions. Then*

$$\frac{d^n}{dx^n} f(g(x)) = \sum \frac{n!}{k_1!...k_n!} f^{(k)}(g(x)) \left(\frac{g'(x)}{1!}\right)^{k_1} ... \left(\frac{g^{(n)}(x)}{n!}\right)^{k_n} \tag{2.4.1}$$

where $k = k_1 + ... + k_n$ *and the sum is over all partitions of* n *such that*

$$k_1 + 2k_2 + ... + nk_n = n. \tag{2.4.2}$$

Here $f^{(k)}(g(x))$ is the k^{th} derivative of $f(x)$ at $g(x)$. Alternatively, there is an equivalent formula using Bell polynomials.

Proposition 2.4.2. *Let* f *and* g *be* n *times differentiable functions. Then*

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), ...g^{(n-k+1)}(x)). \tag{2.4.3}$$

Here, $B_{n,k}$ is the Bell polynomial. These are defined in (2.4.14) and (2.4.15). For more details and the original theory of the Faa di Bruno formula, see [46, 47].

As we can see from (2.4.3), Bell polynomials appear as a combinatorial element within the Faa di Bruno formula. This is due to their appearance in the study of set partitions; because of this, they have strong links with such objects as Bell and Stirling numbers.

The number of ways to partition a set of n elements into non-empty subsets is called a Bell number and denoted $B(n)$ or B_n. Let $B_0 = 1$. Then

$$B_{n+1} = \sum_{i=0}^{n} B_i \binom{n}{i}. \tag{2.4.4}$$

The rising and falling factorials are polynomials defined for a product. The falling factorials $\langle\alpha\rangle_n$ are defined as

$$\begin{aligned} \langle\alpha\rangle_n =& \alpha(\alpha-1)...(\alpha-n+1) \\ =& \prod_{k=0}^{n-1}(\alpha-k). \end{aligned} \tag{2.4.5}$$

for $n > 1$, with $\langle x\rangle_0 = 1$.

The rising factorials, also called Pochhammer symbols, are defined as

$$(x)_n = x(x+1)...(x+n-1) \tag{2.4.6}$$

$$= \prod_{k=0}^{n-1}(x-k). \tag{2.4.7}$$

for $n > 1$, with $(x)_0 = 1$. For further reading see [128].

Two key functions in the study of combinatorics and Bell numbers are the Stirling numbers of the first and second kind, see [38]. Stirling numbers of the first kind, $s(n,k)$, count permutations according to their number of cycles (where fixed points count as length one) multiplied by $(-1)^{n-k}$. These $s(n,k)$ are more precisely called the signed Stirling numbers of the first kind, and are defined by the coefficients of the falling factorials, so by the generating function

$$\langle x\rangle_n = \sum_{k=0}^{n} s(n,k)x^k. \tag{2.4.8}$$

Alternatively,

$$c(n,k) = (-1)^{n-k}s(n,k) \tag{2.4.9}$$

are the signless Stirling numbers of the first kind. Stirling numbers of the second kind are defined as the number of ways to partition a set of n elements into k number of non-empty subsets. They are denoted by $S(n,k)$. Stirling numbers of the second kind have a straightforward formula

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n. \tag{2.4.10}$$

This formula is easily calculated when knowing n and k. The generating function is defined as

$$e^{u(e^t-1)} = 1 + \sum_{1\le k\le n} S(n,k)\frac{t^n}{n!}u^k. \tag{2.4.11}$$

The link between Bell numbers and Stirling numbers of the second kind is

$$B_n = \sum_{k=0}^{n} S(n,k). \tag{2.4.12}$$

In addition to this, Guo and Qi in [53] found an alternative formula for the Bell numbers using both the Stirling numbers of the second kind and the hypergeometric series:

$$B_n = \frac{1}{e}\sum_{k=1}^{n}(-1)^{n-k}k!{}_1F_1(k+1;2;1)S(n,k). \tag{2.4.13}$$

We define the hypergeometric series in (2.5.5).

Bell polynomials appear in the study of combinatorics, and are used in the search for set partitions. A partial Bell polynomial (also called an incomplete Bell polynomial) $B_{n,k}(x_1, x_2, ..x_{n-k+1})$, shows us the way a set of n elements is split into k blocks. These appear in the Faa di Bruno formula and will be pivotal in calculations throughout this thesis. From this point on we will refer to these as just Bell polynomials.

Comtet [38], see also [9, 32], showed that Bell polynomials are a triangular array of polynomials given by

$$\begin{aligned}\phi(t,u) =& e^{\left(u\sum_{m\ge1} x_m \frac{t^m}{m!}\right)} \\ =& \sum_{n,k\ge0} B_{n,k}\frac{t^n}{n!}u^k \\ =& 1 + \sum_{n\ge1}\frac{t^n}{n!}\Big[\sum_{1\le k\le n} u^k B_{n,k}(x_1, x_2...x_{n-k+1})\Big]\end{aligned} \tag{2.4.14}$$

or

$$\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, ..., x_{n-k+1})\frac{t^n}{n!} = \frac{1}{k!}\Big(\sum_{m=1}^{\infty} x_m\frac{t^m}{m!}\Big)^k. \tag{2.4.15}$$

Comtet also proved in the same book that (2.4.15) could be written in what has become the more traditional definition:

Proposition 2.4.3. *The incomplete Bell polynomial also has the form*

$$B_{n,k}(x_1, x_2...x_{n-k+1}) = \sum \frac{n!}{j_1!j_2!...j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}...\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}} \tag{2.4.16}$$

where the summation is taken over all sequences $j_1, j_2, ...j_{n-k+1}$ of non-negative integers such that these two conditions are satisfied:

1. $j_1 + j_2 + ... + j_{n-k+1} = k$
2. $j_1 + 2j_2 + ... + (n-k+1)j_{n-k+1} = n.$

Proof. Taking (2.4.14) and using the series expansion of e^x we get

$$\phi(t,u) = \sum_{k\geq 0}\frac{u^k}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k.$$

Next, we note that

$$\begin{aligned}\left(\sum_{1\leq i\leq m} x_i\right)^n =&(x_1 + x_2 + .. + x_m)^n \\ =&\sum_{a_1+a_2+...=n}(a_1, a_2, ...a_m)x_1^{a_1}x_2^{a_2}...x_m^{a_m}\end{aligned}$$

where $a_1 + a_2 + ...a_m = n$ and

$$\begin{aligned}(a_1, a_2, ...a_m) =&\frac{(a_1 + a_2 + ...a_m)!}{a_1!a_2!...a_m!} \\ =&\frac{n!}{a_1!a_2!...a_m!}\end{aligned}$$

Combining the information above we get

$$\begin{aligned}\phi(t,u) &= \sum_{k\geq 0}\frac{u^k}{k!}\sum_{a_1+a_2+...=k}\frac{k!}{a_1!a_2!...a_m!}\left(\frac{x_1 t}{1!}\right)^{a_1}\left(\frac{x_2 t^2}{2!}\right)^{a_2}...\left(\frac{x_m t^m}{m!}\right)^{a_m} \\ &= \sum_{a_1+a_2+...=k}\frac{u^{a_1+a_2+...a_m}t^{a_1+2a_2+...ma_m}}{a_1!a_2!...a_m!}\left(\frac{x_1}{1!}\right)^{a_1}\left(\frac{x_2}{2!}\right)^{a_2}...\left(\frac{x_m}{m!}\right)^{m_1}.\end{aligned}$$

We see that $n!/(1!)^{a_1}(2!)^{a_2}...$ is the number of division into a_1 1-parts, a_2 2-parts and so on giving

$n = a_1 + 2a_2 + ...ma_m$. Therefore

$$\phi(t,u) = \sum_{a_1+a_2+...=k} \frac{u^k t^n}{a_1!a_2!...a_m!}\Big(\frac{x_1}{1!}\Big)^{a_1}\Big(\frac{x_2}{2!}\Big)^{a_2}...\Big(\frac{x_m}{m!}\Big)^{m_1}$$
$$= \sum_{a_1+a_2+...=k} \frac{u^k t^n}{n!}\frac{n!}{a_1!a_2!...a_m!}\Big(\frac{x_1}{1!}\Big)^{a_1}\Big(\frac{x_2}{2!}\Big)^{a_2}...\Big(\frac{x_m}{m!}\Big)^{m_1}$$

which then implies the equality required.

□

The complete Bell polynomial $B_n(x_1, ..., x_n)$ is the sum of the partial Bell polynomials over $1 \leq k \leq n$:

$$B_n(x_1, ..., x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, ..., x_{n-k+1}). \tag{2.4.17}$$

The Stirling numbers of the first and second kind are closely linked to Bell numbers which, in turn, means they are related to the Bell polynomials as seen in [8]. For Stirling numbers of the second kind, there exists the relationship between them and Bell polynomials:

Proposition 2.4.4. *Let $B_{n,k}(x_1, ...x_m)$ be a Bell polynomial and $S(n,k)$ stand for the Stirling numbers of the second kind. Then*

$$B_{n,k}(1, ...1) = S(n,k). \tag{2.4.18}$$

Proof. Take (2.4.14) and let $x_i = 1$. Then

$$\phi(t,u) = e^{\left(u\sum_{m\geq 1}\frac{t^m}{m!}\right)}.$$

The summation above is the series expansion of e^t without the first term (which is 1). Thus

$$\phi(t,u) = e^{u(e^t-1)}$$
$$= 1 + \sum_{n\geq 1}\frac{t^n}{n!}\Big[\sum_{1\leq k\leq n} u^k B_{n,k}(1, ...1)\Big].$$

We note that (2.4.11), giving the desired equality. □

If we instead have $c(n,k)$, the Stirling numbers of the first kind (signless) have their own relationship with Bell polynomials.

Proposition 2.4.5. *Let $B_{n,k}(x_1, ...x_m)$ be a Bell polynomial and $c(n,k)$ stand for the Stirling numbers of the first kind (signless). Then*

$$B_{n,k}(0!, 1!, 2!, ...(n-k)!) = c(n,k).$$

Proof. Again we start with (2.4.14) and substitute our values for x_i. This time $x_m = (m-1)!$.

$$\begin{aligned}\phi(t,u) =& e^{\left(u\sum_{n\geq 1}(m-1)!\frac{t^m}{m!}\right)}\\ =& e^{\left(u\sum_{n\geq 1}\frac{t^m}{m}\right)}\end{aligned}$$

The summation is the series expansion of $-\log(1-t)$.

$$\begin{aligned}\phi(t,u) =& e^{-u\log(1-t)}\\ =& e^{\log(1-t)^{-u}} = (1-t)^{-u}\end{aligned}$$

This is, however, not what we want, as by the definition of the Stirling numbers of the first kind we have $(1+t)^u$. We take the definition for the Stirling numbers and manipulate it: letting $q = -t$ and $p = -u$, we get

$$\begin{aligned}(1+q)^p =& 1 + \sum_{1\leq k\leq n} s(n,k)\frac{(-t)^n}{n!}(-u)^k\\ =& 1 + \sum_{1\leq k\leq n} s(n,k)(-1)^{n+k}\frac{t^n}{n!}u^k\\ =& 1 + \sum_{1\leq k\leq n} c(n,k)\frac{t^n}{n!}u^k\end{aligned}$$

where we have used the fact that for signed Stirling numbers $(-1)^{n+k} = (-1)^{n-k}$. This gives the desired result. ☐

Comtet also provided us with multiple useful properties of Bell polynomials which will be in vital future calculations, see [32],

$$B_{n,k}(abx_1, a^2bx_2, a^3bx_3, ...) = a^n b^k B_{n,k}(x_1, x_2, x_3, ...). \tag{2.4.19}$$

The power of (2.4.19) is particularly evident when looking for explicit formulas for the n^{th} derivative of functions like $f(e^x)$ and $f(\ln(1+x))$. Using the Faa di Bruno formula, we need to find an expression for the Bell polynomials involving e^x and $\ln(1+x)$. These are closely linked to the Stirling numbers of the first and second kind.

Proposition 2.4.6. *Let $g(x) = e^{\pm x}$. Then*

$$B_{n,k}(g'(x), g''(x), ..., g^{n-k+1}(x)) = (\pm 1)^n e^{\pm kx} S(n,k) \tag{2.4.20}$$

Proof. Using (2.4.19) and Proposition (2.4.4) we observe

$$\begin{aligned} B_{n,k}(g'(x), g''(x), ..., g^{n-k+1}(x)) =& B_{n,k}((\pm 1)e^{\pm x}, (\pm 1)^2 e^{\pm x}, ..., (\pm 1)^{n-k+1} e^{\pm x}) \\ =& (\pm 1)^n e^{\pm kx} B_{n,k}(1, 1, ...) \\ =& (\pm 1)^n e^{\pm kx} S(n,k). \end{aligned}$$

□

Similarly, we have a relationship for the logarithmic function.

Proposition 2.4.7. *Let $g(x) = \ln(1 + x)$. Then*

$$B_{n,k}(g'(x), g''(x), ..., g^{n-k+1}(x)) = \frac{c(n,k)}{(1+x)^n} \tag{2.4.21}$$

Proof. Using (2.4.19) and Proposition 2.4.5 we observe

$$\begin{aligned} B_{n,k}(g'(x), g''(x), ..., g^{n-k+1}(x)) =& B_{n,k}\left(\frac{1}{1+x}, -\frac{1}{(1+x)^2}, ..., \frac{(-1)^{n-k}(n-k)!}{(1+x)^{n-k+1}}\right) \\ =& \frac{(-1)^{n-k}}{(1+x)^n} B_{n,k}(0!, 1!, ...) \\ =& \frac{(-1)^{n-k} c(n,k)}{(1+x)^n}. \end{aligned}$$

□

Qi has done extensive work on Bell polynomials and the explicit formula for different sequences of x_m. For Bell polynomials, where x_m originate from the sine and cosine functions, we can use Qi's work [78, 93, 94].

Proposition 2.4.8. *Let $x_n = \big(\cos x, -\sin x, -\cos x, \sin x, ..., -\cos\left[x + (n-k)\frac{\pi}{2}\right]\big)$. Then*

$$\begin{aligned} B_{n,k}(x_n) =& \frac{(-1)^k \sin^k x}{k!} \sum_{l=0}^{k} \binom{k}{l} \frac{1}{(2\sin x)^l} \\ & \times \sum_{q=0}^{l} (-1)^q \binom{l}{q} (2q-l)^n \cos\left((2q-l)x + \frac{(n-l)\pi}{2}\right). \end{aligned} \tag{2.4.22}$$

Proposition 2.4.9. *Let* $x_n = \big(-\sin x, -\cos x, \sin x, \cos x, ..., -\sin\big[x + (n-k)\frac{\pi}{2}\big]\big)$. *Then*

$$\begin{aligned} B_{n,k}(x_n) =& \frac{(-1)^k \cos^k x}{k!} \sum_{l=0}^{k} \frac{(-1)^l}{(2\cos x)^l} \binom{k}{l} \sum_{q=0}^{l} \binom{l}{q} (2q-l)^n \\ & \times \cos\Big((2q-l)x + \frac{n\pi}{2} \Big). \end{aligned} \tag{2.4.23}$$

A useful result of Propositions 2.4.8 and 2.4.9 is when we let $x = 0$. This will be of particular use when finding Maclaurin expansions. In chapter 4 we do this for composite hypergeometric series.

Corollary 2.4.1. *Let* $x_n = (1, 0, -1, 0, ... \sin((n-k+1)\pi/2))$. *Then*

$$b_k^n[\sin x] = B_{n,k}(x_n) = \frac{(-1)^k}{k!2^k} \cos\left(\frac{(n-k)\pi}{2} \right) \sum_{q=0}^{k} (-1)^q \binom{k}{q} (2q-k)^n. \tag{2.4.24}$$

Corollary 2.4.2. *Let* $x_n = (0, -1, 0, 1, ... \cos(n-k+1)\pi/2)$. *Then*

$$b_k^n[\cos x] = B_{n,k}(x_n) = \frac{(-1)^k}{k!} \cos\left(\frac{n\pi}{2}\right) \sum_{l=0}^{k} \frac{(-1)^l}{2^l} \binom{k}{l} \sum_{q=0}^{l} \binom{l}{q} (2q-l)^n. \tag{2.4.25}$$

We have used the notation $b_k^n(x_n) = B_{n,k}(x_n)|_{x=0}$.

Another formula of interest is the Bell polynomial for $g(x) = x^\alpha$. A formula for this was given by Qi in [78, 95]. We start by noting that

$$B_{n,k}((x^\alpha)', (x^\alpha)'', .., (x^\alpha)^{(n-k+1)}) = x^{k\alpha - n} B_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, ...) \tag{2.4.26}$$

where $\langle\alpha\rangle_n$ are the falling factorials.

Proposition 2.4.10. *Let* $x_m = (x^\alpha)^{(m)}$. *Then*

$$B_{n,k}((x^\alpha)', (x^\alpha)'', .., (x^\alpha)^{(n-k+1)}) = x^{k\alpha - n} \frac{(-1)^k}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \langle\alpha l\rangle_n. \tag{2.4.27}$$

When calculating $b_k^n(x^\alpha) = B_{n,k}(x^\alpha)|_{x=0}$, we only get a value for the Bell polynomial when $n = \alpha k$. Hence, we come to the following Corollary.

Corollary 2.4.3. *Let* $x_m = (x^\alpha)^{(m)}$. *Then*

$$b_k^n(x^\alpha)|_{x=0} = \begin{cases} \frac{(-1)^k}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \langle\alpha l\rangle_n, & n = \alpha k \\ 0, & n \neq \alpha k. \end{cases} \tag{2.4.28}$$

2.5 Hypergeometric series and their matrix forms

The Gauss hypergeometric function (often just shortened to hypergeometric function) ${}_2F_1(a,b;c;z)$ is a series function defined on the unit disk. It is classed as a special function and we later see that it has links to the other special functions, namely the orthogonal polynomials, mentioned above. If $|z| < 1$ and $a,b,c \in \mathbb{C}$ with c not a non-positive integer then the hypergeometric function is defined as

$$
{}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{2.5.1}
$$

where $(a)_n$ are the rising factorials.

The hypergeometric function is also a solution to the differential equation

$$
z(1-z)w''(z) + [c - (a+b+1)z]w'(z) - abw(z) = 0. \tag{2.5.2}
$$

This equation, which is named after Euler, has three regular singular points at $0, 1, \infty$. Using the Frobenius Method with $w(z) = \sum_{n=0}^{\infty} \sigma_n z^n$, we can determine that the following relationship stands:

$$
\sigma_{n+1} = \frac{(n+a)(n+b)}{(n+1)(n+c)} \sigma_n. \tag{2.5.3}
$$

With this, (2.5.1) falls out immediately. The hypergeometric function also has a useful recursive formula:

$$
\frac{d^n}{dz^n} {}_2F_1(a,b;c;z) = \frac{(a)_n (b)_n}{(c)_n} {}_2F_1(a+n, b+n; c+n; z). \tag{2.5.4}
$$

The generalised hypergeometric series, of which the Gauss hypergeometric function is a special case (when $p = 2$ and $q = 1$), is defined for $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_q)$ as

$$
{}_pF_q(\mathbf{a}; \mathbf{b}; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}. \tag{2.5.5}
$$

In relation to convergence, it is seen that

- if $p < q+1$, then the series converges for z finite,
- if $p = q+1$, then the series converges for $|z| < 1$ and diverges for $|z| > 1$,
- if $p > q+1$, then the series diverges except for when $z = 0$,

- if $p = q + 1$ and

$$Re\Big(\sum b_i - \sum a_j\Big) > 0, \tag{2.5.6}$$

then the series converges for $z = 1$.

Also, if any a_i is non-positive then the series converges to a polynomial. If any b_j is non-positive then the series is undefined except for when $-b_j < a_i$. For more information about the hypergeometric series, and in particular convergence, see [73, 128].

Jodar and Cortes discuss the hypergeometric matrix function for $p = 2$ and $q = 1$ in [68]. This takes a similar form to 2.5.1:

$${}_2F_1(A, B; C; z) = \sum_{k=0}^{\infty} (A)_k (B)_k (C)_k^{-1} \frac{z^k}{k!} \tag{2.5.7}$$

for $A, B, C \in C^{r\times r}$ such that $C + nI$ is invertible for integers $n > 0$. They also showed that the series converges.

The Pochhammer symbols for matrix inputs take the form

$$(P)_n = P(P + I)...(P + (n - 1)I) \tag{2.5.8}$$

for integers $n > 0$ and $(P)_0 = I$.

Proposition 2.5.1. *Let $A, B, C \in \mathbb{C}^{r\times r}$ be positive stable matrices such that*

$$\beta(C) > \alpha(A) + \alpha(B) \tag{2.5.9}$$

where

$$\alpha(\mathbf{X}) = \max\{Re(z) : z \in \sigma(\mathbf{X})\}$$
$$\beta(\mathbf{X}) = \min\{Re(z) : z \in \sigma(\mathbf{X})\}.$$

Then (2.5.7) is convergent for $|z| < 1$ and absolutely convergent for $|z| = 1$.

A, B, and C are defined as being positive square matrices in Proposition 2.5.1. A square matrix is a positive stable matrix if every eigenvalue has positive real part. Here, $\sigma(A)$ for $A \in C^{n\times n}$ (the spectrum) denotes the set of all eigenvalues of A.

Proposition 2.5.2. *Let $C \in \mathbb{C}^{r\times r}$ be such that $C + nI$ is invertible for integer $n \geq 0$ and*

$CB = BC$. Then $F(A,B;C;z)$ is a solution to

$$z(1-z)W'' - zAW' + W'(C - z(B+I)) - AWB = O. \tag{2.5.10}$$

on $0 \leq |z| < 1$ with ${}_2F_1(A,B;C;0) = I$.

Remark 2.5.1. *Please note that within the paper by Jodar and Cortes [68], there is a typo in the introduction when stating (2.5.10) with the derivative on the $W'(C - z(B+I))$ not present. Within the proof this error does not appear.*

This can be extended to the hypergeometric series for $\mathbf{A} = (A_1, \ldots, A_p)$ and $\mathbf{B} = (B_1, \ldots, B_q)$:

$${}_pF_q(\mathbf{A};\mathbf{B};z) = \sum_{n=0}^{\infty} \prod_{i=0}^{p} (A_i)_n \prod_{j=0}^{q} [(B_j)_n]^{-1} \frac{z^n}{n!}. \tag{2.5.11}$$

The convergence of this series is closely connected to the scalar case. In [106] convergence was shown; this is also discussed in [1]:

- if $p \leq q$, then the series converges for z finite,
- if $p = q+1$, then the series converges for $|z| < 1$ and diverges for $|z| > 1$,
- if $p > q+1$, then the series diverges except for when $z = 0$,
- if $p = q+1$, and if

$$\sum \alpha(A_i) > \sum \beta(B_j) \tag{2.5.12}$$

  then the series is absolutely convergent for $|z| = 1$.

We next use a matrix argument X in place of the complex argument z and look into the resulting hypergeometric series. This appears frequently within the literature on statistical distributions of random matrices. These matrix arguments take the form of symmetric matrices parameterised over a single parameter $\alpha > 0$:

$${}_pF_q^{\alpha}(\mathbf{a};\mathbf{b};X) = \sum_{k=0}^{\infty} \sum_{\mathfrak{k}} \frac{(a_1)_{\mathfrak{k}}^{\alpha} \ldots (a_p)_{\mathfrak{k}}^{\alpha}}{(b_1)_{\mathfrak{k}}^{\alpha} \ldots (b_q)_{\mathfrak{k}}^{\alpha}} Z_{\mathfrak{k}}^{\alpha}(X). \tag{2.5.13}$$

The second summation is over the set partitions $\mathfrak{k}$ of k, the parameters a_i, b_j are complex numbers, $Z_{\mathfrak{k}}^{\alpha}(X)$ is the normalised Jack function, and $(a_i)_{\mathfrak{k}}^{\alpha}$ is the generalised Pochhammer function

$$(a_i)_{\mathfrak{k}}^{(\alpha)} = \prod_{\epsilon=1}^{m} \prod_{\eta=1}^{\mathfrak{k}_\epsilon} \left(a_i - \frac{\epsilon - 1}{\alpha} + \eta - 1 \right). \tag{2.5.14}$$

See [45] for more information on the generalised Pochhammer function. The Jack function is a family of orthogonal polynomials, where

$$J_{\mathfrak{k}}^{(\alpha)}(x_1) = x^{\mathfrak{k}}(1+\alpha)...(1+(\mathfrak{k}-1)\alpha) \tag{2.5.15}$$

and

$$J_{\mathfrak{k}}^{(\alpha)}(x_1, ..., x_m) = \sum_{\mu} J_{\mathfrak{k}}^{(\alpha)}(x_1, ..., x_{m-1}) x_m^{|\mathfrak{k}/\mu|} \beta_{\mathfrak{k}\mu} \tag{2.5.16}$$

where $\beta_{\mathfrak{k}\mu}$ is a function depending on the partitions of μ, skew partitions $\mathfrak{k}/\mu$, and the Young diagrams.

The function $Z_{\mathfrak{k}}^{\alpha}(X)$ is the normalised Jack function, which is related to the Jack function by

$$Z_{\mathfrak{k}}^{(\alpha)}(X) = \frac{\alpha^{|\mathfrak{k}|}(|\mathfrak{k}|)!}{j_{\mathfrak{k}}} J_{\mathfrak{k}}^{(\alpha)}(X) \tag{2.5.17}$$

where

$$j_{\mathfrak{k}} = \prod_{(i,j)\in\mathfrak{k}} (\mathfrak{k}' - i + \alpha(\mathfrak{k}_i - j + 1))(\mathfrak{k}'_j - i + 1 + \alpha(\mathfrak{k}_i - j)). \tag{2.5.18}$$

Here, the product $(i, j) \in \mathfrak{k}$ refers to the boxes of the Young diagram of the partition $\mathfrak{k}$, $\mathfrak{k}'$ being the conjugate partition to $\mathfrak{k}$.

The series converges for all X if $p \leq q$, and for $||X|| < 1$ if $p = q + 1$. Here we have $||X||$ denoting the maximum of the absolute values of the eigenvalues of X. When $m = 1$ and $\alpha = 2$ this reduces to the hypergeometric series.

The value α changes depending on the area of study. For example, in the theory of real random matrices, $\alpha = 2$. In some literature, β is also used instead of α. More information on this version of the hypergeometric function can be found in [51, 52, 87].

In the following sections, we make use of the Gamma function. For positive integers, the Gamma function is a factorial function $\Gamma(n) = (n-1)!$, but for complex entries the function can be written as an integral:

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t}\, dt, \qquad Re(n) > 0. \tag{2.5.19}$$

Jodar and Cortes defined the Gamma function for matrix inputs in [67]. Let $P \in \mathbb{C}^{r\times r}$ such that

$Re(z) > 0$ for all $z \in \sigma(P)$. Then

$$\Gamma(P) = \int_0^\infty t^{P-I} e^{-t}\, dt, \qquad t^{P-I} = \exp((P-I)\ln(t)). \tag{2.5.20}$$

The Beta function is a symmetric function, closely linked with the Gamma function. For complex scalar functions $x, y \in \mathbb{C}$, whose real part is positive, we define $B(x,y)$ via the following integral:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \tag{2.5.21}$$

for $x, y \in \mathbb{C}$ such that $Re(x), Re(y) > 0$. The Beta function and the Gamma function are connected by the following formula:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.5.22}$$

In [69], it is shown how one can extend the Beta function to accept matrix inputs. Let $A, B \in \mathbb{C}^{r\times r}$ such that $Re(z) > 0$ and $Re(w) > 0$ for all $z \in \sigma(A)$ and $w \in \sigma(B)$. Then

$$B(A,B) = \int_0^1 t^{A-I}(1-t)^{B-I}\, dt. \tag{2.5.23}$$

We can relate the matrix versions of the Beta and Gamma functions by

$$B(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q). \tag{2.5.24}$$

Jodar and Cortes also give the following relationship, linking the Pochhammer function and the Gamma function for matrix inputs:

$$P(P+I)...(P+(n-1)I)\Gamma^{-1}(P+nI) = \Gamma^{-1}(P). \tag{2.5.25}$$

Provided that $P+nI$ is invertible, the above can also be written as

$$P(P+I)...(P+(n-1)I) = \Gamma(P+nI)\Gamma^{-1}(P). \tag{2.5.26}$$

2.6 Orthogonal polynomials and their matrix forms

The study of orthogonal polynomials dates back to the work of the Russian mathematician Pafnuty Chebyshev. Further study continued throughout the 19th century by other noteworthy mathematicians of the time: Gauss, Jacobi, Hermite, and Markov, amongst others.

Orthogonal polynomials, $p(x)$, are a class of polynomials over a scalar range which obey an orthogonal relationship with a given weight. These follow the relationship:

$$\int_a^b w(x)p_m(x)p_n(x)dx = \delta_{m,n}C_n. \tag{2.6.1}$$

Here the weight function $w(x)$, the range $[a, b]$, and the constant C_n correspond to a given polynomial. Further reading on these can be found in [45, 109] but we do explain further in the following sections. We are primarily interested in the Jacobi, Gegenbauer, and Hermite polynomials. The integral interval, weight function and constant for (2.6.1) are listed in the table below.

Polynomial $p(x)$	Interval	$w(x)$	C_n
Gegenbauer	$[-1, 1]$	$(1-x^2)^{\lambda-\frac{1}{2}}$	$\begin{cases} \frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{n!(n+\lambda)(\Gamma(\lambda))^2} & \text{for} \quad \lambda \neq 0 \\ \frac{2\pi}{n^2} & \text{for} \quad \lambda = 0 \end{cases}$
Hermite	$(-\infty, \infty)$	e^{-x^2}	$\sqrt{\pi}2^n n!$
Jacobi	$(-1, 1)$	$(1-x)^\alpha(1+x)^\beta$	$\mathbf{X}$

Table 2.3: Table of interval, weight, and C_n for orthogonal polynomial.

Here $\mathbf{X}$ is:

$$\mathbf{X} = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}.$$

In particular, the Jacobi and Gegenbauer polynomials (where the Gegenbauer is a special case of the Jacobi for $\alpha = \beta = \lambda - \frac{1}{2}$) are eigenfunctions of the Laplacian on rank one symmetric spaces. With certain choices of the variable t, differential equations of the same name appear for which the solutions have representations using the hypergeometric function.

The Jacobi polynomial

The Jacobi polynomial was first introduced by Carl Gustav Jacob Jacobi and occurs within the study of rotational groups. Rotational groups consist of orthogonal matrices with determinant 1.

For n, α, β we define the Jacobi polynomial in the following ways:

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{(-1)^n}{(2^n n!)}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] \\
&= \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)}\left(\frac{x-1}{2}\right)^m \\
&= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \tfrac{1}{2}(1-x)\right). \qquad (2.6.2)
\end{aligned}$$

Moreover these polynomials appear as the eigenfunctions of the Laplacian on the rank one symmetric spaces. Specifically this is the case for the complex, quaternionic, and Cayley projective spaces when $\alpha = N-1$ and $\beta = 0$, $\alpha = 2N-1$ and $\beta = 1$, and $\alpha = 7$ and $\beta = 3$ respectively. Further to this the Jacobi differential equation

$$(1-x^2)y''(x) + [\beta - \alpha - (\alpha+\beta+2)x]y'(x) + n(n+\alpha+\beta+1)y(x) = 0 \qquad (2.6.3)$$

is obtained when $t = \cos\theta$ in the eigenvalue equation for the complex projective space. This has the solution

$$\begin{aligned}
y(x) = C_1{}_2F_1(-n, &n+1+\alpha+\beta; 1+\alpha; \frac{1}{2}(x-1)) \\
&+ C_2 2^{\alpha}(x-1)^{-\alpha}{}_2F_1(-n-\alpha, n+1+\beta; 1-\alpha; \frac{1}{2}(1-x)), \qquad (2.6.4)
\end{aligned}$$

for constants C_1 and C_2.

The formula and relationships above are for scalar inputs α, β, and variable x. Jodar et al. [43] took these definitions and found the relationships for matrices A and B.

Proposition 2.6.1. *Let A and B be matrices in $\mathbb{C}^{r\times r}$ satisfying $Re(x) > -1$, $\forall x \in \sigma(A)$, and $Re(x) > -1$, $\forall x \in \sigma(B)$. For any natural number $n \geq 0$, the n^{th} Jacobi matrix polynomial $P_n^{A,B}(x)$ is defined by*

$$\begin{aligned}
P_n^{A,B}(x) =& \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n+k}}{2^k n!}\Gamma(A+B+(n+k+1)I) \\
& \cdot \Gamma^{-1}(A+B+(n+1)I)\Gamma(B+(n+1)I)\Gamma^{-1}(B+(k+1)I)(1+x)^k \\
=& \frac{(-1)^n}{n!}F\Big(A+B+(n+1)I, -nI; B+I; \frac{1+x}{2}\Big) \\
& \cdot \Gamma^{-1}(B+I)\Gamma(B+(n+1)I) \\
=& \frac{1}{n!}F\Big(A+B+(n+1)I, -nI; B+I; \frac{1-x}{2}\Big) \\
& \cdot \Gamma^{-1}(A+I)\Gamma(A+(n+1)I). \qquad (2.6.5)
\end{aligned}$$

Then $P_n^{A,B}(x)$ satisfies the differential equation

$$
\begin{aligned}
(1-x^2)Y''(x)+2Y'(x)B-(A+B+&x(A+B+2I))Y'(x)\\
&+n(A+B+(n+1)I)Y(x)=0
\end{aligned}
\tag{2.6.6}
$$

for $-1<x<1$.

Jodar and Defez also presented the Rodrigues' formula for matrices A and B.

Proposition 2.6.2. *Let A and B be matrices in $\mathbb{C}^{r\times r}$ satisfying $Re(x)>-1$ $\forall x\in\sigma(A)$ and $Re(x)>-1$ $\forall x\in\sigma(B)$. Then for $n\in\mathbb{Z}^{0+}$ the Jacobi matrix polynomials are defined as*

$$
P_n^{A,B}(x)=\frac{(-1)^n}{2^n n!}(1-x)^{-A}(1+x)^{-B}\frac{d^n}{dx^n}\Big[(1-x)^{A+nI}(1+x)^{B+nI}\Big]. \tag{2.6.7}
$$

Before we can prove this outright we need first a lemma.

Lemma 2.6.1. *Let $C,D\in\mathbb{C}^{r\times r}$, with D positively stable, $BC=CB$, and*

$$
C-D+kI \quad \textit{and} \quad C+kI \tag{2.6.8}
$$

are invertible for all non-negative integers k. Then for $|t|<1$

$$
{}_2F_1(-nI,D;C;t)=(1-t)^n{}_2F_1\Big(-nI,C-D;C;\frac{-t}{1-t}\Big). \tag{2.6.9}
$$

Proof of this lemma can be found in [42]. Now we finish off the proof of Proposition 2.6.1 which was first stated in [43].

Proof of Proposition 2.6.2. By using Proposition 2.6.1 and Lemma 2.6.1

$$
\begin{aligned}
P_n^{(A,B)}(x)=&\frac{(-1)^n}{n!}F\Big(A+B+(n+1)I,-nI;B+I;\frac{1+x}{2}\Big)\\
&\times\Gamma^{-1}(B+I)\Gamma(B+(n+1)I)\\
=&\frac{(-1)^n}{2^n n!}(1-x)^n{}_2F_1\Big(-nI,-(A+nI);B+I;\frac{x+1}{x-1}\Big)\\
&\times\Gamma^{-1}(B+I)\Gamma(B+(n+1)I)\\
=&\frac{(-1)^n}{2^n n!}(1-x)^n\sum_{k=0}^{n}\frac{1}{k!}(-n)_k(-(A+nI))_k((B+I)_k)^{-1}\Big(\frac{x+1}{x-1}\Big)^k\\
&\times(B+I)_n\\
=&\frac{(-1)^n}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}(-(A+nI))_k((B+I)_k)^{-1}(1+x)^k(1-x)^{n-k}(B+I)_n
\end{aligned}
$$

where $(-n)_k$ only takes non-zero values for $k\leq n$. We also note that

$$
(-(A+nI))_k=(-1)^k(A+nI)_k.
$$

Then we have

$$\frac{d^k}{dx^k}(1-x)^{(A+nI)} = (-1)^k (A+nI)_k (1-x)^A (1-x)^{n-k}$$

and

$$\frac{d^{n-k}}{dx^{n-k}}(1+x)^{(B+nI)} = ((B+I)_k)^{-1}(B+I)_n(1+x)^B(1+x)^k.$$

Using these we can rewrite the Jacobi matrix as

$$\begin{aligned} P_n^{(A,B)}(x) =& \frac{(-1)^n}{2^n n!}(1-x)^{-A}(1+x)^{-B}\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}(1-x)^{(A+nI)} \\ & \times \frac{d^{n-k}}{dx^{n-k}}(1+x)^{(B+nI)} \end{aligned}$$

which by definition is the same as that of Proposition 2.6.1. □

The definitions of the Jacobi polynomials above were then extended to the multivariate case by Taşdelen, Çekim, and Aktaş [114].

The scalar recurrence relation for the derivatives of the Jacobi polynomial is known to be given by

$$\frac{d^k}{dx^k}P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\beta+n+1+k)}{2^k\Gamma(\alpha+\beta+n+1)}P_{n-k}^{(\alpha+k,\beta+k)}(x). \tag{2.6.10}$$

Çekim, Altin, and Aktaş also found a recurrence relation for the derivatives of the matrix polynomial in [29]:

$$\frac{d^k}{dx^k}P_n^{A,B}(x) = \frac{((n+1)I+A+B)_k}{2^k}P_{n-k}^{A+kI,B+kI}(x). \tag{2.6.11}$$

Jodar and Defez also show that the matrix Jacobi polynomial is an orthogonal polynomial. For this they set the matrix weight function to be $W(x) = (1-x)^A(1+x)^B$ and

$$\begin{aligned} C_n = & \frac{2^{A+B+I}}{n!}\Gamma(A+B+(2n+1))\Gamma^{-1}(A+B+(n+1)I), \\ & \times \Gamma(B+(n+1)I)\Gamma(A+(n+1)I)\Gamma^{-1}(A+B+(2n+2)I). \end{aligned} \tag{2.6.12}$$

To prove this they make use of Lemma 2.6.2.

Lemma 2.6.2. *Let A and B be defined as above with $AB = BA$. Let $Q(x)$ be an arbitary matrix polynomial. Then*

$$\lim_{x\to 1-}(1-x^2)(1-x)^A(1+x^2)^B Q(x) = 0 \tag{2.6.13}$$

$$\lim_{x\to -1+}(1-x^2)(1-x)^A(1+x^2)^B Q(x) = 0. \tag{2.6.14}$$

The Gegenbauer polynomial

The Gegenbauer polynomial is a specific case of the Jacobi polynomial when $\alpha = \beta = \lambda - \frac{1}{2}$. Under a normalisation condition, these are equivalent to Ultraspherical polynomials. For n, λ the Gegenbauer polynomials are defined as

$$\begin{aligned}
C_n^{(\lambda)}(x) =& \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x) \\
=& \frac{(-1)^n}{2^n n!} \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(\lambda+n+\frac{1}{2})}(1-x^2)^{-\lambda+1/2} \frac{d^n}{dx^n}\left[(1-x^2)^{n+\lambda-1/2}\right] \\
=& \frac{(2\lambda)_n}{n!} {}_2F_1\left(-n, 2\lambda+n; \lambda+\frac{1}{2}; \frac{1-x}{2}\right) \\
=& \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)k!(n-2k)!}(2x)^{n-2k}.
\end{aligned} \tag{2.6.15}$$

The differential recurrence relation for the Gegenbauer is defined by

$$\frac{d}{dx} C_n^{\lambda}(x) = 2\lambda C_{n-1}^{\lambda+1}(x), \tag{2.6.16}$$

which can be extended to

$$\frac{d^m}{dx^m} C_n^{\lambda}(x) = 2^m (\lambda)_m C_{n-m}^{\lambda+m}(x). \tag{2.6.17}$$

For more information on scalar Gegenbauer polynomials, see [8].

From the works [41, 102], we can find formulas similar to those in 2.6.15 but for matrix inputs. These hold a close resemblance to the scalar cases. For any complex matrix D, the Gegenbauer matrix polynomial is defined as

$$\begin{aligned}
C_n^D(x) &= \sum_{k=0}^{n/2} \frac{(-1)^k (D)_{n-k}}{k!(n-2k)!}(2x)^{n-2k} \\
&= \frac{(2D)_n}{n!} F\Big(-nI, 2D+nI; A+\frac{1}{2}I; \frac{1-x^2}{2}\Big) \\
&= (-1)^n \frac{(2D)_n}{n!} F\Big(-nI, 2D+nI; A+\frac{1}{2}I; \frac{1+x^2}{2}\Big).
\end{aligned} \tag{2.6.18}$$

The matrix case has an equivalent differential recurrence relation similar to that of (2.6.16) and (2.6.17):

$$\frac{d}{dx} C_n^D(x) = 2D C_{n-1}^{D+I}(x) \tag{2.6.19}$$

and for $0 \leq r \leq n$

$$\frac{d^r}{dx^r} C_n^D(x) = 2^r (D)_r C_{n-r}^{D+rI}(x). \tag{2.6.20}$$

The Rodrigues' formula for the matrix Gegenbauer polynomial is seen in [41].

Proposition 2.6.3. *Let $D \in \mathbb{C}^{r\times r}$ such that $k \in \sigma(D)$ for every integer $k \geq -1$ and $Re(z) < -1$*

$\forall z \in \sigma(D)$, *then*

$$C_n^D(x) = K_n^{-1}(1-x^2)^{\frac{D}{2}+1}\frac{d^n}{dx^n}\Big[(1-x^2)^{-\frac{D}{2}+(n-1)I}\Big], \tag{2.6.21}$$

for $n \in \mathbb{Z}^{0+}$ *and* $k_0 = I$ *or for* $n \geq 1$

$$K_n = \frac{(-1)^n n! 2^{3(n-1)}}{\sqrt{\pi}} 2^{-D}((2n-3)I - D)((2n-1)I - D)S_n \tag{2.6.22}$$

and

$$S_n = \Gamma^2\Big(-\frac{D}{2}+nI\Big)\Gamma^{-1}(-D+nI)\Big(-\frac{I+D}{2}\Big)_n \\ \times [(-D-I)_n]^{-1}\Gamma^{-1}\Big(-\frac{D}{2}\Big)\Gamma\Big(-\frac{I+D}{2}\Big). \tag{2.6.23}$$

Above we have used

$$\Gamma^2(x) = \Gamma(x) \times \Gamma(x).$$

The scalar case saw that the matrix Gegenbauer polynomial is a special case to the Jacobi polynomial. Defez showed that this is also the case for the matrix inputs. This happens when $A = B = -\frac{D}{2} - I$.

$$P_n^{-\frac{D}{2}-I,-\frac{D}{2}-I}(x) = \frac{(-1)^n}{2^n n!} K_n C_n^D(x) \tag{2.6.24}$$

The Hermite polynomial

The Hermite polynomial is a solution to the Hermite differential equation. The Hermite differential equation is obtained using the Ornstein-Uhlenbeck operator:

$$y''(x) - 2xy'(x) + 2\lambda y(x) = 0. \tag{2.6.25}$$

The solution is obtained using a series method and

$$y = a_{01}F_1\Big(-\frac{1}{4}\lambda; \frac{1}{2}; x^2\Big) + a_2 H_{\frac{\lambda}{2}}(x) \tag{2.6.26}$$

for

$$a_{n+2} = \frac{2n-\lambda}{(n+2)(n+1)} a_n. \tag{2.6.27}$$

See [10, 18, 72] for further detail. The Hermite polynomial $H_n(x)$ is expressed using the following way

$$H_n(x) =(-1)^n e^{x^2}\Big(\frac{d}{dx}\Big)^n e^{-x^2}$$
$$=\begin{cases} n!\displaystyle\sum_{l=0}^{\frac{n}{2}} \frac{(-1)^{\frac{n}{2}-l}}{(2l)!\left(\frac{n}{2}-l\right)!}(2x)^{2l} & \text{for even } n,\\ n!\displaystyle\sum_{l=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-l}}{(2l+1)!\left(\frac{n-1}{2}-l\right)!}(2x)^{2l+1} & \text{for odd } n. \end{cases} \tag{2.6.28}$$

We can also define the Hermite polynomial for odd and even terms separately by using the confluent hypergeometric function [26]:

$$H_{2n}(x) =(-1)^n\frac{(2n)!}{n!}{}_1F_1\Big(-n;\frac{1}{2};x^2\Big), \tag{2.6.29}$$
$$H_{2n+1}(x) =(-1)^n\frac{(2n+1)!}{n!}2x{}_1F_1\Big(-n;\frac{3}{2};x^2\Big). \tag{2.6.30}$$

The recurrence for the derivative of the Hermite polynomial $H_n'(x) = 2nH_{n-1}(x)$ can be extended to

$$\frac{d^m}{dx^m}H_n(x) = \frac{2^m n!}{(n-m)!}H_{n-m}(x). \tag{2.6.31}$$

Once again Jodar et al. extended the Hermite polynomial to cases involving matrices. These are second order and have a second dependency on a matrix A, see [66, 69]. Here they define the Hermite matrix polynomial of the second kind with dependencies on a variable x and matrix A.

Proposition 2.6.4. *Let $A \in \mathbb{C}^{r\times r}$ with $Re(x) > -1$ $\forall z \in \sigma(A)$ and $x \in \mathbb{C}$. For any natural number $n \geq 0$, the n^{th} second order Hermite polynomial $H_n(x, A)$ is defined by*

$$H_n(x,A) =\sum_{k=0}^{n/2}\frac{(-1)^k n!(x\sqrt{2A})^{n-2k}}{k!(n-2k)!}$$
$$=e^{x^2A/2}(-1)^n(A/2)^{-n/2}\Big[\frac{d^n}{dx^n}e^{-x^2A/2}\Big]. \tag{2.6.32}$$

Then $H_n(x, A)$ is a solution to the differential equation

$$Y''(x) - xAY'(x) + nAY(x) = 0. \tag{2.6.33}$$

It is important to note that we have used the definition of the square root of a matrix $2A$ as $\sqrt{2A} = exp(\frac{1}{2}log(2A))$. This follows from the definition given in [100]. This formula has a similar form to 2.6.25.

The Hermite matrix polynomial is an orthogonal matrix polynomial only when we let the matrix dependency A be Hermitian. This happens with the weight function $W(x) = e^{-\frac{Ax^2}{2}}$ with $C_n = 2^n n!(2\pi A^{-1})^{\frac{1}{2}}$.

As with the scalar case, there exists a recurrence relation for the derivatives of the Hermite

Matrix polynomial shown in Metwally et al. work [83]:

$$\frac{d}{dx}H_n(x,A) = n\sqrt{2A}H_{n-1}(x,A). \tag{2.6.34}$$

2.7 Formulas for orthogonal polynomials and their matrix forms using Bell polynomials

In this section we shall look at two different methods to calculate an explicit formula for the orthogonal polynomials; Gegenbauer, Legendre, and Hermite polynomials. Both methods use the Faa di Bruno formula to find these formula.

Qi and Guo in unpublished work, 'Some properties of the Hermite polynomials and their squares and generating functions', use the Faa di Bruno formula with the generating function of the Hermite polynomial to obtain a formula. In their method the important analytical step is the same as Qi used in finding the Bell polynomial for the sine and cosine functions mentioned above.

Our aim for this section is to find new formula using Qi's and Guo's work for the Gegenbauer and Legendre polynomials. We also introduce of our method which uses the Faa di Bruno formula on the Rodrigues' formula. We finish off this section by producing formula of this type for the matrix polynomials for the Hermite and Gegenbauer polynomial.

Before stating any theorems, we state the following by Qi and Zheng [96];

$$B_{n,k}(x,1,0,...) = \begin{cases} (2n-1)!! & \text{for } k = \frac{n}{2} \\ \frac{a_{n+1,2k-n}}{(2k-1)!!}x^{2k-n} & \text{for } n \geq k > \frac{1}{2}\left[n - \frac{1-(-1)^n}{n}\right] \\ 0 & \text{otherwise} \end{cases} \tag{2.7.1}$$

for

$$\begin{aligned} a_{2k-1,0} =& [(2k-3)!!]^2 \\ a_{2k,1} =& [(2k-1)!!]^2 \\ a_{k+1,k} =& (2k-1)!! \\ a_{n,k} =& \frac{(n+k-2)!!(n-1)!}{2^{n-k-2}k!}. \end{aligned} \tag{2.7.2}$$

This is a fairly unfriendly looking formula which was later refined by Qi and Guo in [94];

$$B_{n,k}(x,1,0,...) = \frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{2.7.3}$$

First we find the alternative formulas using the Rodrigues' formulas.

Proposition 2.7.1. *Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ then the Gegenbauer polynomial can be defined as*

$$C_n^{(\lambda)}(x) = A_n \sum_{k=0}^{n} \frac{(n+\lambda-\frac{1}{2})!(-2)^k}{(n+\lambda-\frac{1}{2}-k)!}(1-x^2)^{n-k}\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n} \tag{2.7.4}$$

where

$$A_n = \frac{(-1)^n}{2^n n!}\frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(\lambda+n+\frac{1}{2})} \tag{2.7.5}$$

Proof. Taking the Rodrigues' formula we note an n^{th} derivative. Focusing on this, we can use the Faa di Bruno formula:

$$\begin{aligned}\frac{d^n}{dx^n}(1-x^2)^\alpha &= \sum_{k=0}^{n} \frac{\alpha!}{(\alpha-k)!}(1-x^2)^{\alpha-k}B_{n,k}(-2x,-2,0,...)\\ &= \sum_{k=0}^{n} \frac{\alpha!}{(\alpha-k)!}(1-x^2)^{\alpha-k}(-2)^k B_{n,k}(x,1,0,...).\end{aligned}$$

We combine this with $\alpha = n+\lambda-\frac{1}{2}$ and (2.7.3). Then by substituting back into the Rodrigues' formula gives the desired result. □

The Gegenbauer polynomial is a specific case of the Jacobi polynomial when $\alpha = \beta = \lambda - \frac{1}{2}$. Another polynomial of interest is the Legendre polynomial. This polynomial is also a specific case of the Jacobi polynomial for α and β equal to zero. The Legendre polynomials also have a Rodrigues' formula (2.7.6), and we can produce a new formula for these in the way we have done for the Gegenbauer above.

$$P_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}(1-x^2)^n. \tag{2.7.6}$$

Proposition 2.7.2. *Let $n \in \mathbb{N}$. Then the Legendre polynomial can be defined as*

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^n(-2)^k}{2^n(n-k)!}(1-x^2)^{n-k}\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{2.7.7}$$

The proof is near identical to the Gegenbauer case so is omitted from the text.

Proposition 2.7.3. *Let $n \in \mathbb{N}$. Then the Hermite polynomial can be defined as*

$$H_n(x) = \sum_{k=0}^{n}(-1)^{n+k}(2)^k\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{2.7.8}$$

Proof. Taking the Rodrigues' formula for the Hermite polynomial we observe an n^{th} derivative.

$$\begin{aligned} H_n(x) =& (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2} \\ =& (-1)^n e^{x^2}\sum_{k=0}^{n} e^{-x^2} B_{n,k}(-2x,-2,0,...) \\ =& (-1)^n e^{x^2}\sum_{k=0}^{n} e^{-x^2}(-2)^k B_{n,k}(x,1,0,...) \\ =& \sum_{k=0}^{n}(-1)^{n+k}(2)^k B_{n,k}(x,1,0,...) \end{aligned}$$

Using formula (2.7.3) gives the desired result. □

Now we move on to the method used by Qi and Guo. This takes the generating function and finds the m^{th} derivative with respect to t. Then by sending $t \to 0$ we can rearrange to find a formula for the desired polynomial.

Proposition 2.7.4. *Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Then the Gegenbauer polynomial can be defined as*

$$C_n^{(\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k(k+\lambda)!}{k!\lambda!}\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{2.7.9}$$

Proof. We start by taking the n^{th} derivative of the generating function using the Faa di Bruno formula

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{d^m}{dt^m}C_n^{(\lambda)}(x)t^n =& \frac{d^m}{dt^m}(1-2xt+t^2)^{-\lambda} \\ \sum_{n=0}^{\infty}C_n^{(\lambda)}(x)\frac{n!}{(n-m)!}t^{n-m} =& \sum_{k=0}^{m} B_{m,k}(-2x+2t,2,0,...)(1-2xt+t^2)^{-\lambda-k} \\ & \times \frac{(-1)^k(k+\lambda)!}{\lambda!}. \end{aligned}$$

Sending $t \to 0$ we note that the only non-zero value on the left hand side is when $n = m$. Thus, with the assistance of (2.4.19) we can state that

$$n!C_n^{(\lambda)}(x) = \sum_{k=0}^{n} 2^j B_{n,k}(-x,1,0,...0)\frac{(-1)^k(k+\lambda)!}{\lambda!}.$$

Using (2.7.3) gives the desired result. □

Proposition 2.7.5. *Let $n \in \mathbb{N}$. Then the Legendre polynomial can be defined as*

$$P_n(x) = \sum_{k=0}^{n+1} \frac{(-1)^k\left(\frac{1}{2}\right)_k 2^k}{n!}\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{2.7.10}$$

Proof. Again we take the m^{th} derivative of the generating function with respect to t but to make the calculations easier we note that by first taking the integral, this produces a function that can adopt the method of using the Faa di Bruno formula.

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{d}{dt}\int \frac{x-t}{(1-2xt+t^2)^{\frac{3}{2}}}dt$$
$$= \frac{d}{dt}\frac{1}{(1-2xt+t^2)^{\frac{1}{2}}}$$

Then

$$\frac{d^m}{dt^m}\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = \frac{d^{m+1}}{dt^{m+1}}\frac{1}{(1-2xt+t^2)^{\frac{1}{2}}}$$
$$\sum_{n=1}^{\infty}\frac{n!}{(n-m)!}P_n(x)t^{n-m-1} = \sum_{k=0}^{m+1}\frac{(-1)^k\left(\frac{1}{2}\right)_k}{(1-2xt+t^2)^{\frac{1}{2}+k}}B_{m+1,k}(-2x+2t,2,0,...).$$

Sending $t \to 0$ it follows by the same logic as above that the only value other than zero is when $m = n$. So

$$n!P_n(x) = \sum_{k=0}^{n+1}(-1)^k\left(\frac{1}{2}\right)_k 2^k B_{n+1,k}(-x,1,0,...).$$

Using (2.7.3) and gives the desired result. □

The above method unfortunately doesn't work for the Jacobi polynomial due to it taking the form of $f(x)g(x)$ instead of $f(g(x))$. What we can do however is use the higher derivatives Leibniz rule. For a set of functions $f_i(x)$ for $1 \le i \le k$, the n^{th} derivative product of $f_i(x)$ is defined as

$$\left(\prod_{i=1}^{k} f_i(x)\right)^{(n)} = \sum_{j_1+j_2+...+j_k=n}\binom{n}{j_1,j_2,...,j_k}\prod_{i=1}^{k} f_i^{(n)}(x) \tag{2.7.11}$$

where

$$\binom{n}{j_1,j_2,...,j_k} = \frac{n!}{j_1!j_2!...j_k!}. \tag{2.7.12}$$

In our case with the Jacobi polynomial, $k = 2$. Specifically for this, we have

$$\frac{d^n}{dx^n}f(x)g(x) = \sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}(x)g^{(k)}(x). \tag{2.7.13}$$

Proposition 2.7.6. *Let $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$. Then the Jacobi polynomial can be defined as*

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{2n-k}}{(2^n n!)}\frac{(\alpha+n)!}{(n-k-1)!}(1-x)^k\frac{(\beta+n)!}{(k-1)!}(1+x)^{n-k}. \tag{2.7.14}$$

Proof. We start by looking at the derivatives:

$$\frac{d^n}{dx^n}\Big[(1-x)^{\alpha+n}(1+x)^{\beta+n)}\Big] = \sum_{k=0}^{n}\binom{n}{k}\Big[\frac{d^{n-k}}{dx^{n-k}}(1-x)^{\alpha+n}\Big]\Big[\frac{d^k}{dx^k}(1+x)^{\beta+n}\Big]$$
$$= \sum_{k=0}^{n}\binom{n}{k}\Big[(-1)^{n-k}\frac{(\alpha+n)!}{(n-k-1)!}(1-x)^{\alpha+k}\Big]\Big[\frac{(\beta+n)!}{(k-1)!}(1+x)^{\beta+n-k}\Big].$$

Placing this into the Rodrigues' formula we see that

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) =&\frac{(-1)^n}{(2^n n!)}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Big[(1-x)^{\alpha+n}(1+x)^{\beta+n}\Big] \\
=&\frac{(-1)^n}{(2^n n!)}(1-x)^{-\alpha}(1+x)^{-\beta}\sum_{k=0}^{n}\binom{n}{k}\Big[(-1)^{n-k}\frac{(\alpha+n)!}{(n-k-1)!}(1-x)^{\alpha+k}\Big] \\
&\times\Big[\frac{(\beta+n)!}{(k-1)!}(1+x)^{\beta+n-k}\Big] \\
=&\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{2n-k}}{(2^n n!)}\frac{(\alpha+n)!}{(n-k-1)!}(1-x)^k\frac{(\beta+n)!}{(k-1)!}(1+x)^{n-k}.
\end{aligned}
$$

□

Since there exists the Rodrigues' formulas for the matrix orthogonal polynomials for Gegenbauer and Hermite polynomials, thanks to the works of Jodar, Defez, and Company, we can use the ideas used for the scalar case to develop the same formulas. We note that even though the Faa di Bruno formula is for scalar x, this does not affect our use of it since we are taking the derivative to the variable x and the matrix arguments are constant. We do need to note two extra properties. The first is for use with the Gegenbauer matrix polynomial:

$$\frac{d^k}{dt^k}t^{(A+mI)} = (A+I)_m\Big[(A+I)_{m-k}\Big]^{-1}t^{A+(m-k)I}. \tag{2.7.15}$$

For the Hermite polynomial we use the properties of the matrix exponential

$$\frac{d}{dt}e^{t\mathbf{X}} = \mathbf{X}e^{t\mathbf{X}} = e^{t\mathbf{X}}\mathbf{X}, \tag{2.7.16}$$

for a scalar variable t and a square matrix $\mathbf{X}$.

If $f(x)$ and $g(x)$ are holomorphic functions on an open set Ω on the complex plane, $A, B \in \mathbb{C}^{r\times r}$ where $\sigma(A), \sigma(B) \in \Omega$, and $AB = BA$ then

$$f(A)g(B) = g(B)f(A). \tag{2.7.17}$$

Proposition 2.7.7. *Let $D \in \mathbb{C}^{r\times r}$ such that $\theta \in \sigma(D)$ for every integer $\theta \geq -1$ and $Re(z) < -1$ $\forall z \in \sigma(D)$. Then*

$$
\begin{aligned}
C_n^D(x) =&K_n^{-1}(1-x^2)^{\frac{D}{2}+I}\sum_{k=0}^{m}\Big(-\frac{D}{2}+I\Big)_{n-1}\Big[\Big(-\frac{D}{2}+I\Big)_{n-1-k}\Big]^{-1} \\
&\times(1-x^2)^{-\frac{D}{2}+(n-1-k)I}(-2)^k\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}.
\end{aligned} \tag{2.7.18}
$$

If $\Big(\frac{D}{2}+I\Big)$ and $\Big(-\frac{D}{2}+(n-1-k)I\Big)$ commute, then using (2.7.17) the following holds:

$$
\begin{aligned}
C_n^D(x) =&K_n^{-1}\sum_{k=0}^{m}(1-x^2)^{(n-k)I}\Big(-\frac{D}{2}+I\Big)_{n-1}\Big[\Big(-\frac{D}{2}+I\Big)_{n-1-k}\Big]^{-1} \\
&\times(-2)^k\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}.
\end{aligned} \tag{2.7.19}
$$

Proof. We start by focusing on the differential part of Rodrigues' formula for the matrix Gegenbauer. Using the Faa di Bruno formula with (2.7.15) we observe

$$\frac{d^n}{dx^n}(1-x^2)^{-\frac{D}{2}+(n-1)I} = \sum_{k=0}^{n}\Big(-\frac{D}{2}+I\Big)_{n-1}\Big[\Big(-\frac{D}{2}+I\Big)_{n-1-k}\Big]^{-1}$$
$$\times (1-x^2)^{-\frac{D}{2}+(n-1-k)I}B_{n,k}(-2x,-2,0,...).$$

By rearranging, this gives (2.7.18). Now if $-\frac{D}{2}+I$ and $-\frac{D}{2}+(n-1-k)I$ commute we can simplify the bracketed x terms to give (2.7.19) $\square$

Proposition 2.7.8. *Let $A \in \mathbb{C}^{r\times r}$ with $Re(x) > -1$ $\forall z \in \sigma(A)$ and $x \in \mathbb{C}$. Also let $\sigma(A) \subset \mathbb{C}$. Then for any natural number $n \geq 0$, the n^{th} second order Hermite polynomial $H_n(x, A)$ is defined by*

$$H_n(x,A) = \sum_{k=0}^{n}(-1)^{n+k}2^k\Big(\frac{A}{2}\Big)^{k-\frac{n}{2}}\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{2.7.20}$$

Proof. Taking the matrix Rodrigues' formula we observe

$$\begin{aligned}H_n(x,A) =& e^{x^2\frac{A}{2}}(-1)^n\Big(\frac{A}{2}\Big)^{-\frac{n}{2}}\Big[\frac{d^n}{dx^n}e^{-x^2\frac{A}{2}}\Big]\\ =& e^{x^2\frac{A}{2}}(-1)^n\Big(\frac{A}{2}\Big)^{-\frac{n}{2}}\sum_{k=0}^{n}\frac{d^k}{dz^k}e^{z\frac{A}{2}}|_{z=-x^2}B_{n,k}(-2x,-2,0,...)\\ =& e^{x^2\frac{A}{2}}(-1)^n\Big(\frac{A}{2}\Big)^{-\frac{n}{2}}\sum_{k=0}^{n}e^{-x^2\frac{A}{2}}\Big(\frac{A}{2}\Big)^k(-2)^kB_{n,k}(x,1,0,...).\end{aligned}$$

We next note that both $c\mathbf{X}^n$ and $e^{\mathbf{X}^2}$ are holomorphic functions and assuming that $\sigma(A) \subset \mathbb{C}$ we can use (2.7.3) and can rearrange the above to get the desired result. $\square$

Chapter 3

Differential spectral identities for orthogonal polynomials and hypergeometric functions

As we have seen already, the Jacobi polynomial (and by extension the Gegenbauer polynomial) are closely linked to rank one symmetric spaces due to themselves being the eigenfunctions of the Laplacian on these spaces. Awonusika and Taheri in [11, 12] showed differential spectral identities for the Jacobi and Gegenbauer polynomials. These look at even derivatives of the polynomials and find sums of integar powers of the eigenvalues of the respective operators. These identities were then computed using the Faa di Bruno formula with Bell polynomials in the end product by Day and Taheri [40]. Day and Taheri then introduced the differential operator

$$\mathscr{L}_p = \sum P_N(d/d\theta) = \mathsf{p}_0 + \mathsf{p}_1 d/d\theta + ... + \mathsf{p}_N d^N/d\theta^N \tag{3.0.1}$$

where

$$P_N(\mathbf{X}) = \mathsf{p}_0 + \mathsf{p}_1\mathbf{X} + ... + \mathsf{p}_N\mathbf{X}^N \tag{3.0.2}$$

for $(N \geq 2)$. With this they found an identity for the Gegenbauer polynomial [39]. Bond and Taheri then formulated a similar identity for the hypergeometric function [21].

In this chapter we follow the work of Taheri et al.. Our work consists of specialising their work for the Hermite polynomial, in particular, formulating a differential spectral identity for the even Hermite polynomial seen in Proposition 3.2.2. This is an important discussion as the Hermite function is an eigenfunction of the Fourier transform. Our next goal is to look at spectral identities for the hypergeometric form of the Hermite polynomial. This is achieved in Theorems 3.3.1 and

3.3.2.

The final part of this chapter is concerned with continuing the work of Taheri et al. and extending it to find a identity for the differential operator $\mathscr{L}_p$ on the matrix hypergeometric function. This is achieved in Theorem 3.3.3.

To start this section we look at Maclaurin expansions of composite hypergeometric series. This produces useful results linked to the ones mentioned already. We find Maclaurin expansions for the composite hypergeometric series in Proposition 3.1.1 before looking at the case when we have $A_1, ..A_p$ and $B_1, ..., B_q$, seen in Proposition 3.1.2. We also find a formula for the hypergeometric function for matrix input. We do this in Proposition 3.1.3.

3.1 Maclaurin expansion of composite hypergeometric series

As seen in (2.6.2), (2.6.15), (2.6.29), (2.6.30), the orthogonal polynomials (Jacobi, Gegenbauer, and Hermite) have hypergeometric series representations. Specifically, the Jacobi and Gegenbauer polynomials are hypergeometric functions, and the Hermite polynomial is a confluent hypergeometric function.

The hypergeometric series ${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; x)$ has the Maclaurin expansion (2.5.5). This is for a single variable x. Our aim in this section is to develop this into the more interesting case where in place of the single variable we have a function $f(x)$. Motivation for this comes from the hypergeometric series representation of the orthogonal polynomial.

Proposition 3.1.1. *Let $f(x) \in C^\infty(\mathbb{R})$ such that $f(x)$ is finite and $f(0) = 0$. Also let $p < q + 1$. Then*

$$ {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; f(x)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} \frac{x^n}{n!} b_k^n(\mathbf{X}), \tag{3.1.1} $$

where $\mathbf{X} = (f'(x), ...f^{(n-k+1)}(x))$ and $b_k^n(\mathbf{X}) = B_{n,k}(\mathbf{X})_{x=0}$.

Proof. Using the Faa di Bruno formula and the series expansion of the hypergeometric series:

$$ \begin{aligned} \frac{d^n}{dx^n} {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; f(x)) &= \sum_{k=1}^{n} {}_pF_q^{(k)}(a_1, ..., a_p; b_1, ..., b_q; f(x)) B_{n,k}(\mathbf{X}) \\ &= \sum_{k=1}^{n} \sum_{i=0}^{\infty} \frac{(a_1)_i, ..., (a_p)_i}{(b_1)_i, ..., (b_q)_i} \frac{(f(x))^{i-k}}{(i-k)!} B_{n,k}(\mathbf{X}), \end{aligned} $$

where $\mathbf{X} = (f'(x), ...f^{(n-k+1)}(x))$. When we let $x = 0$ the summation only takes a value when

$i = k$. So we observe

$$\frac{d^n}{dx^n}{}_pF_q(a_1,...,a_p;b_1,...,b_q;f(0)) = \sum_{k=1}^{n} \frac{(a_1)_k,...,(a_p)_k}{(b_1)_k,...,(b_q)_k} b_k^n(\mathbf{X}).$$

Then the Maclaurin series expansion is written as

$${}_pF_q(a_1,...,a_p;b_1,...,b_q;f(x)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{(a_1)_k,...,(a_p)_k}{(b_1)_k,...,(b_q)_k} b_k^n(\mathbf{X}) \frac{x^n}{n!}$$

□

Explicit examples and special cases

Providing we know the Bell polynomial for $f(x)$ when $x = 0$, we can find the series expansion for the hypergeometric series of a given function. The chosen functions have the added property that $f(0) = 0$. With Qi et al.'s and Comtet's work and Proposition 3.1.1, we can formulate the Maclaurin expansion for the hypergeometric series.

- $(e^x - 1)$

$$ {}_pF_q(\mathbf{a};\mathbf{b};e^x-1) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} S(n,k)\frac{x^n}{n!} \quad (3.1.2)$$

  where $S(n, k)$ are the Stirling numbers of the second kind seen in (2.4.10).

- $(\ln(1 + x))$

$$ {}_pF_q(\mathbf{a};\mathbf{b};\ln(1+x)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} (-1)^{n+k}c(n,k)\frac{x^n}{n!} \quad (3.1.3)$$

  where $c(n, k)$ are the signless Stirling numbers of the first kind seen in (2.4.9).

- $(\cos x - 1)$

$$\begin{aligned} {}_pF_q(\mathbf{a};\mathbf{b};\cos x-1) =& 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} \frac{(-1)^k}{k!} \cos\left(\frac{n\pi}{2}\right) \sum_{l=0}^{k} \frac{(-1)^l}{2^l}\binom{k}{l} \\ &\times \sum_{q=0}^{l}\binom{l}{q}(2q-l)^n\frac{x^n}{n!} \end{aligned} \quad (3.1.4)$$

- $(\sin x)$

$$ {}_pF_q(\mathbf{a};\mathbf{b};\sin x) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} \frac{(-1)^k}{k!2^k} \cos\left(\frac{(n-k)\pi}{2}\right) \sum_{q=0}^{k}(-1)^q\binom{k}{q}(2q-k)^n\frac{x^n}{n!} \quad (3.1.5)$$

- $(\cosh x - 1)$

$$ {}_pF_q(\mathbf{a};\mathbf{b};\cosh x-1) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} \frac{1}{k!2^k} \sum_{l=0}^{2k}(-1)^l\binom{2k}{l}(k-l)^n\frac{x^n}{n!} \quad (3.1.6)$$

- $(\sinh x)$

$$
{}_pF_q(\mathbf{a};\mathbf{b};\sinh x) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k}\frac{1}{k!2^k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}(k-2l)^n\frac{x^n}{n!} \tag{3.1.7}
$$

- (x^α)

$$
{}_pF_q(\mathbf{a};\mathbf{b};x^\alpha) = 1 + \sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{\frac{n}{2=\alpha}}}{\prod_{j=1}^{q}(b_j)_{\frac{n}{\alpha}}}\frac{(-1)^{\frac{n}{\alpha}}}{(\frac{n}{\alpha})!}\sum_{l=0}^{\frac{n}{\alpha}}(-1)^l\binom{\frac{n}{\alpha}}{l}\langle\alpha l\rangle_n\frac{x^n}{n!} \tag{3.1.8}
$$

Remark 3.1.1. *(3.1.8) is calculated over one less summation. This is due to that there is the additional requirement that $k = \frac{n}{\alpha}$.*

The Taylor expansion for the matrix hypergeometric function with scalar $f(x)$ similar to that of 2.5.7 and 2.5.11 will be procured in a very similar way. As the choice of A_j and B_j are fixed so the derivatives only sit on the function $f(x)$. This means we can use the Faa di Bruno formula the same way as in (3.1.1).

Proposition 3.1.2. *Let $f(x) \in C^\infty(\mathbb{R})$ such that $f(x)$ is finite and $f(0) = 0$. Also let $A_i, B_j \in \mathbb{C}^{r\times r}$ and $q \geq p$. Then*

$$
{}_pF_q(A_1,...,A_p;B_1,...,B_q;f(x)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\prod_{i=0}^{p}(A_i)_n\prod_{j=0}^{q}[(B_j)_n]^{-1}\frac{x^n}{n!}b_k^n(\mathbf{X}), \tag{3.1.9}
$$

where $\mathbf{X} = (f'(x),...f^{(n-k+1)}(x))$ and $b_k^n(\mathbf{X}) = B_{n,k}(\mathbf{X})_{x=0}$.

Remark 3.1.2. *Like with the case for scalar a_i, b_j the matrix case for A_i, B_j has expression for $f(x)$ being the functions seen in (3.1.2)-(3.1.8).*

Our final aim is to tackle the problem of a hypergeometric function for a matrix input $f(X)$, where X is a square matrix. Here a_i, b_j are scalars. Equation (2.5.13) will be used along with the fact that for matrix inputs as described above, the Jack function holds the following property

$$
J_{\mathfrak{k}}^{(\alpha)}(X) = J_{\mathfrak{k}}^{(\alpha)}(x_1,...,x_m), \tag{3.1.10}
$$

where $x_1,...,x_m$ are the eigenvalues of X.

Proposition 3.1.3. *Let $\mathbf{X}$ be an $n\times n$ complex matrix function such that $f(\mathbf{X}(x)) \in C^\infty(\mathbb{C}^{n\times n})$. Also let $\mathbf{X}$ only have y eigenvalues $g_1(x),...g_y(x)$ which are dependent on x. Then, for $\mathbf{a} = (a_1,\dots,a_p)$ and $\mathbf{b} = (b_1,\dots,b_q)$:*

$$
\begin{aligned}
{}_pF_q^\alpha(\mathbf{a};\mathbf{b};f(\mathbf{X})) = & 1 + \sum_{n=1}^{\infty}\mathfrak{C}'\sum_{\mu_\theta}\sum_{v_1+\ldots+v_y=n}\binom{n}{v_1,...,v_y}\prod_{w=1}^{y}\sum_{k=1}^{n}\sum_{q=1}^{k}\frac{|\frac{\mu_1}{\mu_{w+1}}|!}{(|\frac{\mu_1}{\mu_{w+1}}|-q-1)!} \\
& \times f(g_w(0))^{|\frac{\mu_1}{\mu_{w+1}}|-q}b_k^n(g_w(x))b_q^k(f(t))|_{t=g_w(0)}\beta_{\mu_1\mu_w}
\end{aligned} \tag{3.1.11}
$$

for

$$\mathfrak{C}' = \frac{(a_1)^{\alpha}_{\mu_1}...(a_p)^{\alpha}_{\mu_1}}{(b_1)^{\alpha}_{\mu_1}...(b_q)^{\alpha}_{\mu_1}}\Big(\frac{\alpha^{|\mu_1|}(|\mu_1|)!}{v_{\mu_1}}\Big)\Big[(1+\alpha)...(1+(\mu_1-1)\alpha)\Big]. \tag{3.1.12}$$

Also

$$\beta_{\mu_1\mu_w} = \frac{\prod_{(i,j)\in\mu_1} B^{\mu_1}_{\mu_1\mu_w}(i,j)}{\prod_{(i,j)\in\mu_w} B^{\mu_i}_{\mu_1\mu_w}(i,j)} \tag{3.1.13}$$

for

$$B^{\theta}_{\mu_1\mu_w}(i,j) = \begin{cases} \mu_1(j)' - i + \alpha(\mu_1(i) - j + 1), & \textit{if } \mu_1(j)' = \mu_w(j)' \\ \mu_1(j)' - i + 1 + \alpha(\mu_1(i) - j), & \textit{otherwise.} \end{cases} \tag{3.1.14}$$

The product over $(i,j)\in\mu_1$ means it is taken over all coordinates (i,j) of boxes in the Young diagram of the partition of μ_1.

Remark 3.1.3. *We use the notation of μ'_w as the conjugate partition of μ_w and $\mu_w(j)$ being the j^{th} element of the partition μ_i.*

Remark 3.1.4. *The second summation is actually a collection of summations over the set μ_w such that $\frac{\mu_1}{\mu_w}$ is a horizontal strip. The third summation extends over all m-tuples $(v_1,...,v_y)$ of non-negative integers with $\sum_{i=1}^{y} v_w = n$*

Proof. We immediately rewrite hypergeometric the matrix series for $f(X)$ using the standard Jack function

$$\begin{aligned}
{}_pF^{\alpha}_q(\mathbf{a};\mathbf{b};f(\mathbf{X})) &= \sum_{k=0}^{\infty}\sum_{\mu_1}\frac{(a_1)^{\alpha}_{\mu_1}...(a_p)^{\alpha}_{\mu_1}}{(b_1)^{\alpha}_{\mu_1}...(b_q)^{\alpha}_{\mu_1}}\Big(\frac{\alpha^{|\mu_1|}(|\mu_1|)!}{j_{\mu_1}}\Big)J^{\alpha}_{\mu_1}(f(\mathbf{X})) \\
&= \sum_{k=0}^{\infty}\sum_{\mu_1}\mathfrak{C}J^{\alpha}_{\mu_1}(f(\mathbf{X})) \\
&= \sum_{k=0}^{\infty}\sum_{\mu_1}\mathfrak{C}J^{\alpha}_{\mu_1}(f(g_1(x)),...,f(g_y(x))) \\
&= \sum_{k=0}^{\infty}\sum_{\mu_1}\mathfrak{C}\sum_{\mu_\theta}\prod_{w=1}^{y} f(g_w(x))^{|\frac{\mu_1}{\mu_{w+1}}|}\beta_{\mu_1\mu_w}
\end{aligned}$$

where $\mathfrak{C}' = \mathfrak{C}\Big[(1+\alpha)...(1+(\mu_y-1)\alpha)\Big]$. We have also used (3.1.10) between the second and third line. We also set $\frac{\mu_1}{\mu_{y+1}} = 1$. The third summation on the last line is taken over $\mu_\theta = \mu_2,...,\mu_y$. Next we focus on the derivatives of the product. For this we again use the general Leibniz rule and combine this with the Faa di Bruno formula twice.

$$\frac{d^n}{dx^n}\prod_{w=1}^{y} f(g_w(x))^{|\frac{\mu_1}{\mu_w+1}|} = \sum_{v_1+...+v_y=n}\binom{n}{v_1,...,v_y}\prod_{w=1}^{y}\frac{d^n}{dx^n}f(g_w(x))^{|\frac{\mu_1}{\mu_w+1}|}$$

$$= \sum_{v_1+...+v_y=n}\binom{n}{v_1,...,v_y}\prod_{w=1}^{y}\sum_{k=1}^{n}\frac{d^k}{dt^k}f(t)^{|\frac{\mu_1}{\mu_w+1}|}|B_{n,k}(g_w(x))$$

$$= \sum_{v_1+...+v_y=n}\binom{n}{v_1,...,v_y}\prod_{w=1}^{y}\sum_{k=1}^{n}\sum_{q=1}^{k}\frac{d^q}{dz^q}z^{|\frac{\mu_1}{\mu_w+1}|}|B_{n,k}(g_w(x))B_{k,p}(f(t))_{t=g_w(x)}$$

$$= \sum_{v_1+...+v_y=n}\binom{n}{v_1,...,v_y}\prod_{w=1}^{y}\sum_{k=1}^{n}\sum_{q=1}^{k}\frac{|\frac{\mu_1}{\mu_w+1}|!}{(|\frac{\mu_1}{\mu_w+1}|-q-1)!}f(g_w(x))^{|\frac{\mu_1}{\mu_w+1}|-q}$$

$$\times B_{n,k}(g_w(x))B_{k,q}(f(t))_{t=g_w(x)}$$

where $t = g_w(x)$ and $z = f(t)$. By letting $x = 0$ the proof's completion follows the normal path. □

Remark 3.1.5. *Similar expansions for selected $f(x)$ can be found provided the eigenvalues $g_1(x)$ are known and that the respective Bell polynomials for both $f(x)$ and $g_i(x)$ are known.*

3.2 Differential and spectral identities for the orthogonal polynomials as eigenfunctions of given spaces

It is known that the Maclaurin expansion of the heat kernel is given by

$$K(t,\theta) = \sum_{l=0}^{\infty}\frac{\theta^{2l}}{(2l)!}\frac{\partial^{2l}}{\partial\theta^{2l}}K(t,0). \tag{3.2.1}$$

The partial differential sits only on the spherical function in (3.2.1). Focusing on this, provided you know the spherical function, you can calculate it explicitly. Awonusika and Taheri in [12, 13] as well as Day and Taheri in [39] did just this when $\Phi_n(\theta) = P_n^{(\alpha,\beta)}(\cos x)$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial.

Proposition 3.2.1. *Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial for $\alpha, \beta > -1$ and $k \geq 0$. Then*

$$\frac{d^{2l}}{d\theta^{2l}}P_n^{(\alpha,\beta)}(\cos\theta)|_{\theta=0} = \sum_{j=1}^{l}c_j^l(\alpha,\beta)[\lambda_n^{(\alpha,\beta)}]^j \tag{3.2.2}$$

where $c_j^l(\alpha,\beta)$ is a constant dependent on α and β and $\lambda_n^{(\alpha,\beta)}$ are the eigenvalues of the Jacobi operator.

With this it was computed that

$$K(t,\theta) = \sum_{l=0}^{\infty}T_{2l}^n\frac{\theta^{2l}}{(2l)!} \tag{3.2.3}$$

for

$$
\begin{aligned}
T_{2l}^{n} =& \sum_{k=0}^{\infty} \frac{\mathfrak{M}_n^N}{Vol} e^{-t\lambda} \sum_{j=1}^{l} c_j^l [\lambda_k]^j \\
=& \frac{1}{Vol} tr\{\mathfrak{R}_l(-\Delta) e^{t\lambda}\}
\end{aligned}
$$

where

$$
\mathfrak{R}_l(\mathbf{X}) = \sum_{j=1}^{l} c_j^l \mathbf{X}^j. \tag{3.2.4}
$$

A similar method was used to find the spectral identity for the Gegenbauer polynomials in [11]. This coincides with the $n-$sphere.

As we have mentioned already, the Jacobi polynomial (and by extension the Gegenbauer polynomial) are eigenfunctions of the sphere, real projective space, complex projective space, and the quaternionic projective space. We have discussed both these polynomials in some detail alongside the Hermite polynomial. The Hermite polynomial is not an eigenfunction of a rank one symmetric space but in fact is an eigenfunction of the Fourier transform. More precisely the Hermite function (also known as Hermite-Gaussian function) are the eigenfunctions.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the Hermite function is defined as

$$
\psi_n(x) = \frac{1}{\left(2^n n! \sqrt{\pi}\right)^{\frac{1}{2}}} e^{-\frac{x^2}{2}} H_n(x). \tag{3.2.5}
$$

Orthogonality is seen in the usual way

$$
\int_{-\infty}^{\infty} \psi_n(x) \phi_m(x) dx = \delta_{n,m}. \tag{3.2.6}
$$

Remark 3.2.1. *The weight function $w(x)$ and constant C_n for the orthogonality for the Hermite function are the same as the Hermite polynomial as it is a scaling.*

The Hermite function is an eigenfunction of the Fourier transform in the following way:

$$
\begin{aligned}
\mathcal{F}[\psi_n](x) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \psi_n(t) dt \\
=& i^n \psi_n(x).
\end{aligned} \tag{3.2.7}
$$

For further discussion about the Hermite function as eigenfunction of the Fourier transform see [30, 31, 101].

To produce a spectral identity similar to that of Taheri et al. for the Hermite function we need to know the eigenvalues for the equivalent operator. For the Hermite polynomial this is the Ornstein-Uhlenbeck operator,

$$
D_u f(x) = \Delta f(x) - \langle x, \nabla f(x) \rangle. \tag{3.2.8}
$$

The eigenvalues for this operator acting on the Hermite polynomial are very simple, in fact for the n-dimensional case:

$$D_u H_{n_1,...,n_m}(x_1,...,x_m) = (n_1 + ... + n_m) H_{n_1,...,n_m}(x_1,...,x_m) \tag{3.2.9}$$

for $x \in \mathbb{R}^n$. This gives the eigenvalues $\lambda = (n_1 + ... + n_m)$.

Lemma 3.2.1. *Let $n, p, j \in \mathbb{N}$. Then*

$$\prod_{p=0}^{j-1}(n-p) = \sum_{q=1}^{j} C(j,q) n^q, \tag{3.2.10}$$

where

$$C(j,j) = 1 \tag{3.2.11}$$

$$C(j,1) = -qC(j-1,1) \tag{3.2.12}$$

$$C(j,q) = C(j-1,q-1) - jC(j-1,q) \tag{3.2.13}$$

for $j \geq 1$ and $2 \leq q \leq j$.

Proof. (3.2.11) and (3.2.12) are elementary. (3.2.13) follows from taking your result for a given $j-u$ and multiplying it by $(n-j+u-1)$. Equating coefficients gives the desired result. □

Proposition 3.2.2. *Let $f(x)$ be a function such that $f(0) = 0$ and $n \in \mathbb{N}$. Then the following differential identity holds*

$$\frac{d^{2m}}{dx^{2m}} H_{2n}(f(x))|_{x=0} = \sum_{j=1}^{m} \sum_{q=1}^{j} \mathfrak{C}(j,m,q) \frac{(-1)^{n-j}}{2^n} {}_{2n}P_n n^q \tag{3.2.14}$$

where $\mathfrak{C}(j,m,q)$ is defined as

$$\mathfrak{C}(j,m,q) = \frac{2^{3j} b_{2j}^{2m}(f(x))}{\sqrt{\pi}} C(j,q) \tag{3.2.15}$$

and $C(j,q)$ is defined as in Lemma 3.2.1.

${}_{2k}P_k$ in the above proposition refers to the k-permutation (also called partial-permutation) of $2k$. This is a permutation without repetition.

Corollary 3.2.1. *Let $f(x)$ be a function such that $f(0) = 0$ and $n \in \mathbb{N}$. Then the following differential identity holds*

$$\frac{d^{2m}}{dx^{2m}} H_{2n}(f(x))|_{x=0} = \sum_{j=1}^{m} \sum_{q=1}^{j} \mathfrak{C}(j,m,q) \frac{(-1)^{\lambda-j}}{2^\lambda} {}_{2\lambda}P_\lambda \lambda^q \tag{3.2.16}$$

where λ is the eigenvalue for the Ornstein-Uhlenbeck operator and $\mathfrak{C}(j,m,q)$ as defined above.

Proof of Proposition 3.2.2. Through elementary calculations:

$$
\begin{aligned}
\frac{d^{2m}}{dx^{2m}} H_{2n}(f(x))|_{x=0} = \frac{d^{\mathfrak{p}}}{dx^{\mathfrak{p}}} H_{\mathfrak{q}}(f(x))|_{x=0} &= \sum_{j=0}^{\mathfrak{p}} \frac{d^j}{dt^j}\Big(H_{\mathfrak{q}}(t)\Big)|_{t=0} b_j^{\mathfrak{p}}(f(x)) \\
&= \sum_{j=0}^{\mathfrak{p}} \frac{2^j \mathfrak{q}!}{(\mathfrak{q}-j)!} H_{\mathfrak{q}-j}(0) b_j^{\mathfrak{p}}(f(x)) \\
&= \sum_{j=0}^{2m} \frac{2^j (2n)!}{(2n-j)!} H_{2n-j}(0) b_j^{2m}(f(x)).
\end{aligned}
$$

The Hermite polynomial has a known value at 0 but is defined separately for odd and even dimensions of n:

$$
H_n(0) = \begin{cases} 0 & \text{for } n \text{ odd} \\ (-2)^{\frac{n}{2}}(n-1)!! & \text{for } n \text{ even.} \end{cases} \tag{3.2.17}
$$

Using this the odd terms vanish and thus we are left with just the even values. Thus

$$
\begin{aligned}
\frac{d^{2m}}{dx^{2m}} H_{2n}(f(x))|_{x=0} &= \sum_{j=1}^{m} \Bigg[\frac{2^{2j}(2n)!}{(2n-2j)!} H_{2n-2j}(0) b_{2j}^{2m}(f(x)) \\
&\quad + \frac{2^{2j-1}(2n)!}{(2n-(2j-1))!} H_{2n-(2j-1)}(0) b_{2j-1}^{2m}(f(x)) \Bigg] \\
&= \sum_{j=1}^{m} \frac{2^{2j}(2n)!}{(2n-2j)!} (-2)^{n-j}(2n-2j-1)!! b_{2j}^{2m}(f(x)) \\
&= \sum_{j=1}^{m} \frac{2^{4j-2n}(2n)!}{(n-j)!} (-2)^{n-j} b_{2j}^{2m}(f(x)) \\
&= \sum_{j=1}^{m} \frac{(2n)!}{(n-j)!} \frac{2^{3j}(-1)^{n-j}}{2^n} b_{2j}^{2m}(f(x))
\end{aligned}
$$

where we have used that for an odd double factorial

$$
(2T-1)!! = \frac{(2T)!}{2^T T!}. \tag{3.2.18}
$$

Looking at factorials using Legendre's duplication formula and Lemma 3.2.1:

$$
\begin{aligned}
\frac{(2n)!}{(n-j)!} &= \frac{(2n)!}{n!} \frac{n!}{(n-j)!} \\
&= {}_{2n}P_n \prod_{p=0}^{j-1} (n-p) \\
&= {}_{2n}P_n \sum_{q=1}^{j} C(j,q) n^q.
\end{aligned}
$$

Combining the above and simplifying gives the desired result. □

3.3 Spectral identities for hypergeometric series

Taheri et al. continued their work into spectral identities by defining the differential operator $\mathscr{L}_p$, which extends the identities on the Gegenbauer and Jacobi polynomials to the hypergeometric series. This operator is defined in [39], also in [21];

$$\mathscr{L}_p = P_N(d/d\theta) \tag{3.3.1}$$

where

$$P_N(\mathbf{X}) = \mathsf{p}_0 + \mathsf{p}_1\mathbf{X} + ... + \mathsf{p}_N\mathbf{X}^N \tag{3.3.2}$$

for $(N \geq 2)$. Taheri et al. represent the result with H_m which is an explicit form of $\mathfrak{R}_m$ found in the Gegenbauer and Jacobi identities. More precisely

$$\mathscr{L}_p[{}_2F_1(a,b,;c;\mathfrak{f}(x))]\Big|_{x=0} = \sum_{m=0}^{N} \mathsf{p}_m H_m(-ab), \tag{3.3.3}$$

for

$$H_m(\mathbf{X}) = \sum_{l=1}^{m}(-1)^l \sum_{j=l}^{m} \frac{b_j^m[\mathfrak{f}]s_l^j}{(c)_j} X^l. \tag{3.3.4}$$

With choices of $\mathfrak{f}(x) = \frac{1-\cos x}{2}$ then Taheri and Bond showed a very similar identity to (3.2.1) is obtained:

$$\begin{aligned}\mathscr{L}_p\Big[{}_2F_1\Big(a,b,;c;\frac{1-\cos x}{2}\Big)\Big] =& p_0 + \sum_{m=1}^{\frac{d}{2}} p_{2m} \sum_{j=1}^{m} c_j^m[-ab]^j \\ =& p_0 + \sum_{m=1}^{\frac{d}{2}} p_{2m}\mathfrak{R}_m(-ab),\end{aligned} \tag{3.3.5}$$

where $\mathfrak{R}_\mathfrak{m}$ is defined as in (3.2.4).

Remark 3.3.1. *The choice of $\mathfrak{f}(x)$ here links the hypergeometric function to the Jacobi polynomial and the eigenfunctions of the compact rank one symmetric spaces listed above.*

Above, s_l^j is a function dependent on the elementary symmetric polynomial. We will give a brief description of these here, however for more detail see [82].

The elementary symmetric polynomials $S_k(X_1, ...X_j)$ with j variables and $k = 0, 1, ...j$ are defined by

$$S_k(x_1, ...x_j) = \sum_{j_1<j_2<...<j_k} x_{j_1}x_{j_2}...x_{j_k}, \tag{3.3.6}$$

where $S_0 = 1$.

Now we will introduce the notation $s_l^j = S_{j-l}(Y_0, ...Y_{j-1})$ where $Y_k = k(z+k)$, with $0 \leq k \leq j-1$, specifically, $Y_0 = 0, Y_1 = (z+1), ..., Y_{j-1} = (j-1)^2 + z(j-1)$.

The hypergeometric function can be used to express the Jacobi and Gegenbauer polynomials. The Hermite polynomial is expressed using the confluent hypergeometric function. We shall now create a similar proposition to Taheri et al. for the confluent hypergeometric function before looking at the specific cases when $\mathfrak{f}(x)$ is a power function, similar to that found in the representation of the Hermite polynomial.

First we state an important lemma.

Lemma 3.3.1. *Let* $(a)_j = \prod_{k=0}^{j-1}(k+a)$ *such that* a *is a scalar, then*

$$(a)_j = \sum_{l=1}^{j} s_l^j(0,1,2,....,j-1)a^l, \tag{3.3.7}$$

where $s_l^j(\mathbf{X}) = S_{j-l}(\mathbf{X})$ *is the elementary symmetric polynomial.*

Proof. Using

$$Y_k = p(k) - \mathbf{X} = \sum_{l=1}^{p} S_{p-l}(a_1, ...a_p)k^l \tag{3.3.8}$$

with $p = 1$ we observe

$$Y_k = p(k) - \mathbf{X} = \sum_{l=1}^{1} S_{1-l}(a)k^1 = k.$$

We note that $\mathbf{X} = S_p(a) = a$ for the case where $p = 1$. Thus

$$\begin{aligned}
(a)_j &= \prod_{k=0}^{j-1}(k+a) = \prod_{k=0}^{j-1}(Y_k + \mathbf{X}) \\
&= \sum_{l=0}^{j} S_{j-l}(Y_0, Y_1, ...Y_{j-1})\mathbf{X}^l \\
&= \sum_{l=1}^{j} S_{j-l}(Y_0, Y_1, ...Y_{j-1})\mathbf{X}^l \\
&= \sum_{k=0}^{j} S_{j-l}(0, 1, 2, ...j-1)a^l.
\end{aligned}$$

□

Proposition 3.3.1. *Let* $\mathscr{L}_{\mathsf{P}}$ *be as defined above. Let* $\mathfrak{f}(x)$ *be a smooth function with* $\mathfrak{f}(0) = 0$*. Also let* $a, b \in \mathbb{C}$ *such that* $b \notin \{0, -1, -2, -3, ...\}$*. Then*

$$\mathscr{L}_{\mathsf{P}}[{}_1F_1(a; b; \mathfrak{f}(x))]|_{x=0} = \sum_{m=0}^{N} \mathsf{p}_m G_m(-a), \tag{3.3.9}$$

with

$$G_m(\mathbf{X}) = \sum_{l=0}^{m}(-1)^l \sum_{j=l}^{m} \frac{s_l^j b_j^m(\mathfrak{f}(x))}{(b)_j}\mathbf{X}^l \tag{3.3.10}$$

where $s_l^j = S_{j-l}(0, 1, 2, ..., j-1)$ *is the elementary symmetric polynomial and*

$b_j^m(\mathfrak{f}(x)) = B_{m,j}(\mathfrak{f}'(x), \mathfrak{f}''(x), ...\mathfrak{f}^{m-j+1}(x))|_{x=0}$.

Proof. We start by looking at the m^{th} derivative of the confluent hypergeometric function of the first kind and invoking the Faa di Bruno formula:

$$\frac{d^m}{dx^m}{}_1F_1(a;b;\mathfrak{f}(x))|_{x=0} = \sum_{j=1}^{m} \frac{d^j}{dx^j}{}_1F_1(a;b;z)|_{z=0} B_{m,j}(\mathfrak{f}', \mathfrak{f}'', ...)|_{x=0}.$$

Above we have used $B_{m,j}(\mathfrak{f}', \mathfrak{f}'', ...)|_{x=0} = b_j^m(\mathfrak{f}(x))$. The j^{th} derivative of the confluent hypergeometric function of the first kind is defined as $\frac{(a)_j}{(b)_j}{}_1F_1(a+j;b+j;z)$. However, we know that ${}_1F_1(a+j;b+j;0) = 1$. Thus we observe

$$\frac{d^m}{dx^m}{}_1F_1(a;b;\mathfrak{f}(x))|_{x=0} = \sum_{j=1}^{m} \frac{(a)_j}{(b)_j} b_j^m(\mathfrak{f}(x)).$$

Using Lemma 3.3.1 on $(a)_j$ gives

$$\frac{d^m}{dx^m}{}_1F_1(a;b;\mathfrak{f}(x))|_{x=0} = \sum_{j=1}^{m}\sum_{l=0}^{j} \frac{S_{j-l}(0,1,2,...,j-1)a^l}{(b)_j} b_j^m(\mathfrak{f}(x)).$$

Changing the limits of summation

$$\begin{aligned}\frac{d^m}{dx^m}{}_1F_1(a;b;\mathfrak{f}(x))|_{x=0} =& \sum_{l=0}^{m}\sum_{j=l}^{m} \frac{S_{j-l}(0,1,2,...,j-1)a^l}{(b)_j} b_j^m(\mathfrak{f}(x)) \\ =& \sum_{l=0}^{m}(-1)^l\sum_{j=l}^{m} \frac{S_{j-l}(0,1,2,...,j-1)(-a)^l}{(b)_j} b_j^m(\mathfrak{f}(x)) \\ =& G_m(-a)\end{aligned}$$

□

Motivated by the confluent hypergeometric function formula for the Hermite polynomial seen in (2.6.29) and (2.6.30), we choose $\mathfrak{f}(x) = x^\alpha$ in Proposition 3.3.1.

Corollary 3.3.1. *Let $\mathscr{L}_\mathsf{P}$ be as defined above. Also let $a, b \in \mathbb{C}$ such that $b \notin \{0, -1, -2, -3, ...\}$ and $\alpha \neq 0$, then*

$$\mathscr{L}_\mathsf{P}[{}_1F_1(a;b;x^\alpha)]|_{x=0} = \sum_{m=0}^{N} \mathsf{p}_m \sum_{l=0}^{m} \mathbf{X}, \tag{3.3.11}$$

with

$$\mathbf{X} = \begin{cases} \sum_{k=0}^{j}(-1)^{l+j+k}\binom{j}{k}\frac{\langle \alpha k\rangle_m}{j!}\frac{s_l^j}{(b)_j}(-a)^l, & \textit{for } m = \alpha j \\ 0, & \textit{otherwise} \end{cases} \tag{3.3.12}$$

where $s_l^j = S_{j-l}(0,1,2,...,j-1)$ is the elementary symmetric polynomial.

Here we have used (2.4.28) to define the Bell polynomial $b_j^m(\mathfrak{f}(x))$ in Proposition 3.3.1. Next we state a corollary for the specific case when $\alpha = 2$. This will use (2.7.3) for defining $b_j^m(\mathfrak{f}(x))$.

Corollary 3.3.2. *Let $\mathscr{L}_{\mathsf{P}}$ be as defined above. Also let $a, b \in \mathbb{C}$ such that $b \notin \{0, -1, -2, -3, ...\}$, then*

$$\mathscr{L}_{\mathsf{P}}[{}_1F_1(a; b; x^2)]|_{x=0} = \sum_{m=0}^{N} \mathsf{p}_m \sum_{l=0}^{m} (-1)^l \mathbf{X}, \tag{3.3.13}$$

with

$$\mathbf{X} = \begin{cases} \frac{s_l^j}{(b)_j}(-a)^l \Gamma\Big(j + \frac{1}{2}\Big) \frac{2^{2j}}{\sqrt{\pi}}, & \textit{for } m = 2j \\ 0, & \textit{otherwise} \end{cases} \tag{3.3.14}$$

where $s_l^j = S_{j-l}(0, 1, 2, ..., j-1)$ is the elementary symmetric polynomial.

The Hermite polynomial has two forms using the hypergeometric series, (2.6.29) and (2.6.30), one each for n odd and even. This leads to two important spectral differential theorems.

Theorem 3.3.1. *Let $\mathscr{L}_{\mathsf{P}}$ be as defined above. Also let λ be the eigenvalues for the Ornstein-Uhlenbeck operator. Then*

$$\mathscr{L}_{\mathsf{P}}\left[{}_1F_1\bigg(-n; \frac{1}{2}; x^2\bigg)\right]\bigg|_{x=0} = \sum_{m=0}^{N} \mathsf{p}_m \sum_{l=0}^{m} (-1)^l \mathbf{X}, \tag{3.3.15}$$

with

$$\mathbf{X} = \begin{cases} s_l^j (\lambda)^l 2^{2j+1}, & \textit{for } m = 2j \\ 0, & \textit{otherwise} \end{cases} \tag{3.3.16}$$

where $s_l^j = S_{j-l}(0, 1, 2, ..., j-1)$ is the elementary symmetric polynomial.

Theorem 3.3.2. *Let $\mathscr{L}_{\mathsf{P}}$ be as defined above. Also let λ be the eigenvalues for the Ornstein-Uhlenbeck operator. Then*

$$\mathscr{L}_{\mathsf{P}}\left[2x\,{}_1F_1\bigg(-n; \frac{3}{2}; x^2\bigg)\right]\bigg|_{x=0} = \sum_{m=0}^{N} \mathsf{p}_m \sum_{l=0}^{m-1} (-1)^l \mathbf{X}, \tag{3.3.17}$$

with

$$\mathbf{X} = \begin{cases} 2m\Big(2j + 1\Big)^{-1} s_l^j (\lambda)^l 2^{2j+1}, & \textit{for } m - 1 = 2j \\ 0, & \textit{otherwise} \end{cases} \tag{3.3.18}$$

where $s_l^j = S_{j-l}(0, 1, 2, ..., j-1)$ is the elementary symmetric polynomial.

Above we have looked at the case for the hypergeometric function and the confluent hypergeometric function. In unpublished work [22], Bond and Taheri discuss $H_m(\mathbf{X})$ when we have vectors $\mathbf{a}, \mathbf{b}$ with p elements. To return to the scalar case from the vector result set $\mathbf{a} = (a, b)$ and $\mathbf{b} = c$.

To be able to calculate $H_m(\mathbf{X})$ for the vector case of p elements we need to compute s_l^j. Above we have used $s_l^j = S_{j-l}(Y_0, ...Y_{j-1})$ with $Y_k = k(k+a+b)$, but this is specific to the case where $p \neq 2$.

We define s_1^j to be the coefficients of the factor $\mathbf{X} = \Pi_{i=1}^p a_i$

$$(a_1)_j(a_2)_j...(a_p)_j = \sum_{l=1}^{j} s_l^j(\mathbf{a}) \left[\prod_{i=1}^{p} a_i \right]^l . \tag{3.3.19}$$

We see that

$$Y_k = \sum_{l=1}^{p} z_l k^l \tag{3.3.20}$$

where $z_l = S_{j-l}(a_1, ...a_p)$. Like above $S_0(\mathbf{X}) = 1$ for any set $\mathbf{X}$. From this we can obtain values for s_l^j provided we know what p is. By putting $p = 2$ we obtain $Y_k = k(z_1 + z_2 + k)$.

A powerful ability of the hypergeometric function for parameters $\mathbf{a}$ and $\mathbf{b}$ which are vectors is that we can cancel out terms of the vectors $\mathbf{a}$ and $\mathbf{b}$. Thus, if we let $p < u$, $q < v$, ${}_pF_q(a_1, ...a_p; b_1, ...b_q; z)$ and ${}_uF_v(a_1, ...a_u; b_1, ...b_v; z)$ with $p - q = u - v$ then

$${}_pF_q(a_1, ...a_p; b_1, ...b_q; z) = {}_uF_v(a_1, ...a_u; b_1, ...b_v; z), \tag{3.3.21}$$

so long as $a_1, ...a_p, ...a_u$ and $b_1, ...b_q, ...b_v$ are chosen such that the additional terms of ${}_uF_v(a_1, ...a_u; b_1, ...b_v; z)$ cancel.

This property also is valid for the $H_m(\mathbf{X})$ polynomials as they are just the m^{th} derivatives of hypergeometric functions. So, for vectors of more elements we can reclaim the $H_m(\mathbf{X})$ polynomial for ${}_2F_1(a, b; c; z)$ so long as the above restriction is applied.

The next step to look at is when we have matrix inputs A_i, B_j. Also in [22], Bond and Taheri looked at the hypergeometric matrix function ${}_2F_1(A, B; C; z)$. Here A, B, C are square matrices. For matrices A, B, C it is seen that $s_l^j = S_{j-l}(Y_0, Y_1, \ldots, Y_{j-1})$ for $Y_k = k(A + B + k\mathbf{I})$.

Building on this we will look at a vector of matrices $\mathbf{A}$ and $\mathbf{B}$, which are vectors of p and q $n \times n$ matrices. We let $(\mathbf{B}_i + j\mathbf{I})$ be invertible for every $j \geq 0$ where $\mathbf{I}$ is the $n \times n$ identity matrix. Therefore, with the complex variable z, we have the infinite series for the hypergeometric function as

$${}_pF_q(\mathbf{A}; \mathbf{B}; z) = \sum_{k=0}^{\infty} (A_1)_k (A_2)_k ... (A_p)_k (B_1)_k^{-1} (B_2)_k^{-1} ... (B_q)_k^{-1} \frac{z^k}{k!}. \tag{3.3.22}$$

We note that the matrix extension for the rising factorial is defined by $(F)_0 = \mathbf{I}$ and for $k \geq 1$, $(\mathbf{X})_k = \mathbf{X}(\mathbf{X}+\mathbf{I})...(\mathbf{X}+(k-1)\mathbf{I})$. This means we can take derivatives for the matrix hypergeometric series in the usual way and the function that the Bell polynomial is taken of is still a scalar function so no additional analysis needs to be done there. It is useful to note that when taking the derivative of the hypergeometric series for matrix A, B then $(\mathbf{X})_{j+k} = (\mathbf{X})_m(\mathbf{X} + mI)_j$.

Theorem 3.3.3. *Let* $\mathbf{A} = (A_1, A_2, ...A_p)$ *and* $\mathbf{B} = (B_1, B_2, ...B_q)$, *where* $A_i, B_i \in \mathbb{C}^{n \times n}$ *and*

$(B_i + j\mathbf{I})$ *invertible. Then*

$$\mathscr{L}_p[{}_pF_q(\mathbf{A};\mathbf{B};\mathfrak{f})]\Big|_{x=0} = \sum_{m=0}^{N} \mathsf{p}_m \sum_{j=1}^{m} b_j^m(\mathfrak{f}) \prod_{s=0}^{p}(A_s)_j \left[\prod_{t=0}^{q}(B_t)_j\right]^{-1}. \tag{3.3.23}$$

If A_1 *to* A_p *commute and if* B_1 *to* B_q *commute then*

$$\mathscr{L}_p[{}_pF_q(\mathbf{A};\mathbf{B};\mathfrak{f})]\Big|_{x=0} = \sum_{m=0}^{N} \mathsf{p}_m \sum_{l=1}^{m}\sum_{j=l}^{m}(-1)^l b_j^m(\mathfrak{f}) s_l^j \left[-\prod_{i=1}^{p} A_i\right]^l \left[\prod_{t=0}^{q}(B_t)_j\right]^{-1}. \tag{3.3.24}$$

Proof. Using the differential operator defined above and the hypergeometric function for the vectors $\mathbf{A} = (A_1, A_2, ...A_p)$ and $\mathbf{B} = (B_1, B_2, ...B_q)$ where A_i and B_i are matrices we see that

$$\mathscr{L}_p[{}_pF_q(\mathbf{A};\mathbf{B};\mathfrak{f})]\Big|_{x=0} = \sum_{m=0}^{N} \mathsf{p}_m \frac{d^m}{dx^m}{}_pF_q(\mathbf{A};\mathbf{B};\mathfrak{f})\Big|_{x=0}.$$

Focusing on the derivative we make use of the Faa di Bruno formula. As mentioned above, we note that the Faa di Bruno formula is for scalar functions, thus here we take the hypergeometric function component wise:

$$\begin{aligned}\frac{d^m}{dx^m}{}_pF_q(\mathbf{A};\mathbf{B};\mathfrak{f})\Big|_{x=0} &= \sum_{j=1}^{m} \frac{d^j}{dz^j}{}_pF_q(\mathbf{A};\mathbf{B};z)\Big|_{z=0} B_{m,j}(\mathfrak{f}', \mathfrak{f}''...)\Big|_{x=0}\\ &= \sum_{j=1}^{m} b_j^m(\mathfrak{f}) \frac{d^j}{dz^j}{}_pF_q(\mathbf{A};\mathbf{B};z)\Big|_{z=0}.\end{aligned}$$

Looking at the derivative of the hypergeometric function of $\mathfrak{f}$:

$$\begin{aligned}\frac{d^j}{dz^j}{}_pF_q(\mathbf{A};\mathbf{B};z) &= \sum_{k=j}^{\infty}(A_1)_k...(A_p)_k(B_1)_k^{-1}...(B_q)_k^{-1}k(k-1)...(k-j+1)\frac{z^{k-j}}{k!}\\ &= \sum_{k=0}^{\infty}(A_1)_{j+k}...(A_p)_{j+k}(B_1)_{j+k}^{-1}...(B_q)_{j+k}^{-1}\frac{z^k}{k!}\\ &= \sum_{k=j}^{\infty}(A_1)_j(A_1+j\mathbf{I})_k...(A_p)_j(A_p+j\mathbf{I})_k(B_1+j\mathbf{I})_k^{-1}\times\\ &\quad (B_1)_j^{-1}...(B_q+j\mathbf{I})_k^{-1}(B_q)_j^{-1}\frac{z^{k-j}}{k!},\end{aligned}$$

and looking at when $z = 0$

$$\frac{d^j}{dz^j}{}_pF_q(\mathbf{A};\mathbf{B};z)|_{z=0} = (A_1)_j...(A_p)_j(B_1)_j^{-1}...(B_q)_j^{-1}.$$

Thus by substitution of this gives the first part of the proposition. Next, if A_1 to A_p commute and if B_1 to B_q commute then

$$\begin{aligned}(A_1)_j(A_2)_j...(A_p)_j &= \prod_{j=1}^{p}\prod_{k=0}^{j-1}(k + A_j)\\ &= \prod_{k=0}^{j-1}[\mathbf{X} + Y_k]\end{aligned}$$

where $Y_k = \sum_{l=1}^{p} z_l k^l$ for $z_l = S_{j-1}(A_1, ...A_p)$. Also,

$$
\begin{aligned}
(A_1)_j(A_2)_j...(A_p)_j &= \sum_{l=0}^{j} S_{j-1}(Y_0, ...Y_{j-1})\mathbf{X}^l \\
&= \sum_{l=1}^{j} s_l^j X^l = \sum_{l=1}^{j} s_l^j \left[\prod_{l=1}^{p} A_i\right]^l.
\end{aligned}
$$

Placing this into 3.3.23, we obtain

$$
\begin{aligned}
\mathscr{L}_p[{}_pF_q(\mathbf{A};\mathbf{B};\mathfrak{f})]\Big|_{x=0} &= \sum_{m=0}^{N} \mathsf{p}_m \sum_{j=1}^{m} b_j^m(\mathfrak{f}) \sum_{l=1}^{j} s_l^j \left[\prod_{l=1}^{p} A_i\right]^l \left[\prod_{t=0}^{q}(B_t)_j\right]^{-1} \\
&= \sum_{m=0}^{N} \mathsf{p}_m \sum_{j=1}^{m}\sum_{l=1}^{j} (-1)^l b_j^m(\mathfrak{f}) s_l^j \left[-\prod_{l=1}^{p} A_i\right]^l \left[\prod_{t=0}^{q}(B_t)_j\right]^{-1} \\
&= \sum_{m=0}^{N} \mathsf{p}_m \sum_{l=1}^{m}\sum_{j=l}^{m} (-1)^l b_j^m(\mathfrak{f}) s_l^j \left[-\prod_{l=1}^{p} A_i\right]^l \left[\prod_{t=0}^{q}(B_t)_j\right]^{-1}
\end{aligned}
$$

which is the desired result. □

We can now use this as we did above in the previous sections to compute the polynomial $H_m(\mathbf{X})$ for $\mathfrak{f}$ equalling $(1 - \cos x)/2$, $\sin x$, and x^d.

Proposition 3.3.2. *Let* $\mathbf{A} = (A_1, A_2, ...A_p)$ *and* $\mathbf{B} = (B_1, B_2, ...B_q)$*, where* $A_i, B_i \in \mathbb{C}^{n\times n}$ *and* $(B_i + j\mathbf{I})$ *invertible. Let* $\mathfrak{f} = (1 - \cos x)/2$*. If* A_1 *to* A_p *commute and if* B_1 *to* B_q *commute then the* $H_m(\mathbf{X})$ *polynomial is defined as*

$$
H_m(\mathbf{X}) = \sum_{l=1}^{m}\sum_{j=l}^{m} \cos\left(\frac{m\pi}{2}\right) \sum_{h=0}^{j} \frac{(-1)^{l+h}}{2^{j+h} j!}\binom{j}{h} \sum_{q=0}^{h}\binom{h}{q}(2q-l)^m s_l^j \left[\prod_{t=0}^{q}(B_t)_j\right]^{-1} \mathbf{X}^l \tag{3.3.25}
$$

where $\mathbf{X}$ *is a matrix of same dimensions as* A_i, B_i*.*

Proposition 3.3.3. *Let* $\mathbf{A} = (A_1, A_2, ...A_p)$ *and* $\mathbf{B} = (B_1, B_2, ...B_q)$ *where* $A_i, B_i \in \mathbb{C}^{n\times n}$ *and* $(B_i + j\mathbf{I})$ *invertible. Let* $\mathfrak{f} = \sin x$*. If* A_1 *to* A_p *commute and if* B_1 *to* B_q *commute then the* $H_m(\mathbf{X})$ *polynomial is defined as*

$$
H_m(\mathbf{X}) = \sum_{l=1}^{m}\sum_{j=l}^{m} \cos\left(\frac{(m-j)\pi}{2}\right) \sum_{q=0}^{j} \frac{(-1)^{j+l+q}}{j!2^j}\binom{j}{q}(2q-j)^m s_l^j \left[\prod_{t=0}^{q}(B_t)_j\right]^{-1} \mathbf{X}^l
$$

where $\mathbf{X}$ *is a matrix of same dimensions as* A_i, B_i*.*

Proposition 3.3.4. *Let* $\mathbf{A} = (A_1, A_2, ...A_p)$ *and* $\mathbf{B} = (B_1, B_2, ...B_q)$ *where* $A_i, B_i \in \mathbb{C}^{n\times n}$ *and* $(B_i + j\mathbf{I})$ *invertible. Let* $\mathfrak{f} = x^\alpha$*. If* A_1 *to* A_p *commute and if* B_1 *to* B_q *commute then the* $H_m(\mathbf{X})$ *polynomial is defined as*

$$
H_m(\mathbf{X}) = \begin{cases} \sum_{l=1}^{m}\sum_{k=0}^{j} \frac{(-1)^{j+k}}{j!}\binom{j}{k}\langle\alpha k\rangle_m s_l^j \left[\prod_{t=0}^{q}(B_t)_j\right]^{-1} \mathbf{X}^l, & \textit{for } m = \alpha j \\ 0, & \textit{otherwise} \end{cases}
$$

where $\mathbf{X}$ *is a matrix of same dimensions as* A_i, B_i.

Chapter 4

Li-Yau estimates for elliptic PDEs

Gradient estimates play an important role in understanding the behaviour of geometrical PDEs. Additionally, these gradient estimates can have useful bi-products that give further information under different conditions. This additional information can include, but is not limited to, analysis using ancient solutions, Liouville-type theorems, and Harnack inequalities. These bi-products are often fairly straightforward to calculate once an estimate is found, but give useful information and much greater detail about the systems they are in.

The gradient estimates that we see in the current literature are a fairly new area of study originating largely from Yau [131], which uses the maximum principle to find estimates for a harmonic function, see also [34, 130].

Specifically, Yau showed that if (M^n, g) is a Riemannian manifold with Ricci curvature satisfying $Ric \geq -(n-1)K$ for some constant $K \geq 0$, then if u is a positive harmonic function on M^n, the following inequality holds on a ball of radius $R/2$ in M^n:

$$\frac{|\nabla u|}{u} \leq C_n \left(\frac{1 + R\sqrt{K}}{R} \right), \tag{4.0.1}$$

where C_n is a constant dependent on the dimension n.

Remark 4.0.1. *A corollary to this is that if we have non-negative Ricci curvature, $Ric \geq 0$, then every positive harmonic function on M^n is a constant function. These are known as Liouville-type theorems and will be discussed later.*

Li and Yau in [76] extended these ideas to find a gradient estimate for parabolic PDEs. Their method has since been used to find estimates for both elliptic and parabolic linear and non-linear PDEs.

4.1 Cutoff functions and the maximum principle associated with Li-Yau gradient estimates

One of the fundamental steps in finding gradient estimates is the use of a cutoff function, which allows the maximum principle to be applied. For this, we choose a cutoff function that only takes non-zero values inside a ball of radius $2R$. The choice of cutoff function determines whether an estimate is considered to be a Li-Yau or Souplet-Zhang estimate. The difference between these two types of estimates is whether or not the cutoff function has a dependency on t.

Lemma 4.1.1. *Let $\tilde{\phi} : \mathbb{R}^+ \to \mathbb{R}$, $\tilde{\phi} \in C^2(\mathbb{R}^+)$ such that $\tilde{\phi}(s) = 1$ for $s \in [0, R]$, $\tilde{\phi}(s) = 0$ for $s \in [2R, \infty)$, and $\tilde{\phi}(s) \in [0, 1]$. Then*

$$0 \geq \frac{\tilde{\phi}'(s)}{\sqrt{\tilde{\phi}(s)}} \geq -\frac{C}{R}$$

and

$$|\tilde{\phi}''(s)| \leq \frac{C}{R^2}$$

for some positive constant C.

Now let $r(x) = d(x, x_0)$, where $d(x, x_0)$ is the distance function between x and x_0. We then define $\phi = \tilde{\phi}(r(x))$. Directly from [120] and using ∞-Bakry-Emery Ricci tensor, we observe that

$$\Delta_f r = \mu + (n-1)K(2R-1) \tag{4.1.1}$$

where $\mu := \max_{x|d(x,x_0)=1} \Delta_f r(x)$ and a constant $K \geq 0$. This therefore gives

$$\begin{aligned} \Delta_f \phi &= \frac{\phi' \Delta_f r}{R} + \frac{\phi'' |\nabla r|^2}{R^2} \\ &\geq -\frac{C}{R^2} - \frac{C[\mu + (n-1)K(2R-1)]}{R} \end{aligned} \tag{4.1.2}$$

We are also able to calculate $\Delta_f \phi$ in a similar way as above for the more general m-Bakry-Emery Ricci tensor, again using [120];

$$\Delta_f r \leq (m+n-1)\sqrt{K} \coth(\sqrt{K} r) \tag{4.1.3}$$

which gives

$$\begin{aligned} \Delta_f \phi &= \frac{\phi' \Delta_f r}{R} + \frac{\phi'' |\nabla r|^2}{R^2} \\ &\geq -\frac{C(m+n-1)\sqrt{K} \coth(\sqrt{K} r)}{R} - \frac{C}{R^2} \\ &\geq -\frac{C(m-1)(1 + R\sqrt{K})}{R^2}. \end{aligned} \tag{4.1.4}$$

With the help of this cutoff function the maximum principle is used. In this setting, the maximum principle takes the following form: if $u : M^n \to \mathbb{R}$ is maximised at p, then

$$\nabla u(p) = 0 \tag{4.1.5}$$

$$\Delta u(p) \leq 0. \tag{4.1.6}$$

Further discussion on cutoff functions and PDEs can be found in [110, 111].

4.2 Current literature and direction of research

Although the study of elliptic gradient estimates is a new and exciting area of mathematics, a rich body of literature is fast being developed.

Brighton, [25], took the ideas of Yau's work and applied them to the elliptic heat equation. Instead of using $h = \log u$ as a transformation (the transformation being a vital step in producing an inequality where the maximum principle is used), he opted for $h = u^\epsilon$ for $\epsilon \in (0, 1)$. This has now become the standard for estimates of elliptic equations. Here, Brighton finds both an estimate, and the subsequent Liouville-type theorem, for a positive weighted harmonic function. A similar estimate is found in [28].

Since the release of [25], the direction the research has taken is to find gradient estimates for elliptic equations with different non-linearities. Huang and Li in [62] found a gradient estimate for the elliptic heat equation with logarithmic non-linearity. This uses the Witten-Laplacian and Bakry-Emery Ricci tensor (see chapter 1 for more information). Other gradient estimates for this non-linearity are seen in [127], and [64, 97] for when f is a constant.

From these we see two distinct directions. The first is by Abolarinwa in [6] where he takes the non-linearity in [62] and raises the logarithmic term to the power of $\alpha \in \mathbb{R}$. Depending on the polarity of α, it produces two different estimates. The other direction was seen by Ma and Dong in [80], which adds an additional $bu(x)$ term for $b \in \mathbb{R}$.

4.3 Li-Yau gradient estimate for a varying coefficient elliptic PDE

The current literature and interest of research for elliptic estimates is built upon the work of Brighton, [25]. This found estimates and the resulting Liouville-type theorems for the elliptic heat equation. Since then, the study into estimates has looked at increasingly difficult non-linearities and further analysis on these results. We build on this work, mainly investigating logarithmic non-linearities such as [62, 80, 127]. However, we also move into a different direction by making our coefficient dependent on x. This is a very common occurrence for gradient estimates of parabolic

PDEs, but for elliptic PDEs the current literature is lacking. We will introduce two new methods to find estimates for this style of PDE.

Our research will focus on equations of the form:

$$\Delta_f u(x) + A(x)u(x)\log u(x) + B(x)u(x)^p = 0 \tag{4.3.1}$$

for functions $A(x), B(x)$, and constant $p \in \mathbb{R}$, where u is a positive solution to (4.3.1). Equation (4.3.1) exists on the smooth metric measure space $(M^n, g, e^{-f}d\nu)$, as described in more detail in chapter 1. When $A \equiv B \equiv 0$, we obtain the case studied by Brighton in [25], when $B \equiv 0$ and A is a constant we obtain the case in Huang and Ma [64], and when $p \equiv 1$ for A, B constants we obtain the case studied by Ma and Dong in [80].

As said above, in our study for gradient estimates for an equation of this kind, we introduce two methods which branch off from the work of Brighton. The first uses a maximal function whose outcome has a value greater than zero. This has the drawback of making Liouville-type theorems impossible to calculate. The other method we use employs the fact that $|\nabla h| \leq 1 + |\nabla h|^2$. Unlike the first method, this allows us to find Liouville-type theorems easily.

Theorem 4.3.1. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f \geq -(n-1)K$ for a constant $K \geq 0$ and $R > 1$. Suppose u is a positive smooth solution to (4.3.1) in $\overline{B(x_0, 2R)}$, and that $x_0 \in M^n$ is fixed. Then for any $x \in B(x_0, R)$, the following inequality holds:*

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} \leq \max\Bigg(&\frac{1}{\epsilon^2 u^{2\epsilon}}, \frac{C_7}{R^2} + C_8\Big((n-1)K + \sup_{B(x_0,2R)}[(A(\epsilon\log u + 1))_+] \\ &+ [(p-1+\epsilon)B]_+ \sup_{B(x_0,2R)} u^{p-1}\Big) + C_9\Big(|\nabla A| \sup_{B(x_0,2R)} (u^\epsilon|\epsilon\log u|) \\ &+ |\nabla B| \sup_{B(x_0,2R)} u^{p-1+\epsilon}\Big) + C_{10}\Big(\frac{\mu + (n-1)K(2R-1)}{R}\Big)\Bigg)\end{aligned} \tag{4.3.2}$$

where $C_n = C_n(\epsilon, \mu, n)$ are constants such that $\mu := \max_{x|d(x,x_0)=1} \Delta_f r(x)$ ($r(x)$ is the distance between x and x_0 on M^n), and $\epsilon \in (0,1)$ is such that

$$\frac{1}{n} + \frac{4(\epsilon - 1)}{\epsilon n^2} \geq 0. \tag{4.3.3}$$

Here we use the notation for the max function as $A_+ = \max(A, 0)$.

Theorem 4.3.1 is defined over the closed ball with radius $2R$. This can be extended to find the global estimate by sending $R \to \infty$:

Corollary 4.3.1. *Let $(M^n, g, e^{-f}d\nu)$ be a complete non-compact metric space with $Ric_f \geq -(n-1)K$ for a constant $K \geq 1$, $\epsilon \in (0,1)$, and (4.3.3) still apply. If u is a bounded solution to (4.3.1),* *the global estimate*

$$\frac{|\nabla u|^2}{u^2} \leq \max\left(\frac{1}{\epsilon^2 u^{2\epsilon}}, C_8\Big((N-1)K + \sup_{M^n}[(A(\epsilon \log u + 1))_+] + [(p-1+\epsilon)B]_+ \sup_{M^n} u^{p-1}\Big)\right.$$
$$\left. + C_9\Big(|\nabla A| \sup_{M^n}(u^\epsilon |\epsilon \log u|) + |\nabla B| \sup_{M^n} u^{p-1+\epsilon}\Big)\right) \tag{4.3.4}$$

exists for constants C_n and $\epsilon \in (0,1)$.

Additionally, we can formulate the following corollaries for when A, B are constants.

Corollary 4.3.2. *Take the same assumptions as in Theorem 4.3.1 and (4.3.3). Let A,B be constant functions. Then for any $x \in B(x_0, R)$ the following inequality holds*

$$\frac{|\nabla u|^2}{u^2} \leq \frac{C_7}{R^2} + C_8\Big((n-1)K + \sup_{B(x_0,2R)}[(A(\epsilon \log u + 1))_+] + [(p-1+\epsilon)B]_+ \sup_{B(x_0,2R)} u^{p-1}\Big)$$
$$+ C_{10}\Big(\frac{\mu + (n-1)K(2R-1)}{R}\Big) \tag{4.3.5}$$

for constants C_n and $\epsilon \in (0,1)$.

Corollary 4.3.3. *Let $(M^n, g, e^{-f}d\nu)$ be a complete non-compact metric space with $Ric_f \geq -(n-1)K$, and u be a positive bounded solution to (4.3.1) for A, B constant where (4.3.3) still applies. Then the following global estimate holds*

$$\frac{|\nabla u|^2}{u^2} \leq C_8\Big((n-1)K + \sup_{M^n}[(A(\epsilon \log u + 1))_+] + [(p-1+\epsilon)B]_+ \sup_{M^n} u^{p-1}\Big) \tag{4.3.6}$$

for constants C_n and $\epsilon \in (0,1)$.

Remark 4.3.1. *The proof of Corollaries (4.3.4), (4.3.2), and (4.3.3) is omitted as they are trivial.*

Remark 4.3.2. *The estimates (4.4.19), (4.3.4), (4.3.2), and (4.3.3) hold for the more general m-Bakry-Emery Ricci tensor, where the $-(n-1)K$ is replaced with $-(m-1)K$ and $\Delta_f \phi$ as defined in (4.1.4).*

In Theorem 4.3.1, we dealt with the issue caused by the gradient terms of the varying constants $A(x), B(x)$ by using a max function. This, however, comes with its own issues, principally that our inequality will now always have the right hand side greater than or equal to $1/(\epsilon^2 u^{2\epsilon})$. We aim for a sharper result.

If, instead of using max functions, we use that $|\nabla h| \leq 1 + |\nabla h|^2$, we produce a new estimate. This allows us to deal with the inequality like a quadratic in $G = \phi|\nabla h|^2$. This can then be solved straightforwardly.

Theorem 4.3.2. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f \geq -(n-1)K$ for a constant $K \geq 0$, $\epsilon \in (0,1)$, and $R > 1$. Suppose u is a positive* *smooth solution to (4.3.1) in $\overline{B(x_0,2R)}$ and that $x_0 \in M^n$ is fixed such that (4.3.3) hold. Then for any $x \in B(x_0,R)$ the following inequality holds:*

$$\frac{|\nabla u|^2}{u^2} \leq \frac{1}{\epsilon}\left(\frac{n}{2C_1^2-\gamma nC_2}\right)\Omega + \frac{2}{\epsilon}\left(\frac{n}{2C_1^2-nC_2}\right)^{\frac{1}{2}}\Theta^{\frac{1}{2}} \tag{4.3.7}$$

for

$$\begin{aligned}\Omega =&\Big((n-1)K + \sup_{B(x_0,2R)}([A(\epsilon^{-1}\log u+1)]_+) + [(p-1+\epsilon)B]_+\sup_{B(x_0,2R)} u^{p-1}\Big)\\ &+\Big(|\nabla A|\sup_{B(x_0,2R)}(u^\epsilon|\epsilon\log u|) + \epsilon|\nabla B|\sup_{B(x_0,2R)} u^{p-1+\epsilon}\Big) + \frac{C_4}{R^2}\\ &+\frac{C_5[\mu+(n-1)K(2R-1)]}{R},\end{aligned}$$

and

$$\Theta = \Big(|\nabla A|\sup_{B(x_0,2R)}(u^\epsilon|\epsilon\log u|) + \epsilon|\nabla B|\sup_{B(x_0,2R)} u^{p-1+\epsilon}\Big).$$

Corollary 4.3.4. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete non-compact smooth metric measure space with $Ric_f \geq -(n-1)K$ for a constant $K \geq 0$ and $\epsilon \in (0,1)$. Suppose u is a positive smooth solution to (4.3.1) on M^n, $x_0 \in M^n$ such that (4.3.3) hold. Then the following global estimate holds*

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{n}{2C_1^2-\gamma nC_2}\right)\frac{\Omega}{\epsilon} + \frac{1}{\epsilon}\left(4\left(\frac{n}{2C_1^2-\gamma nC_2}\right)\Theta\right)^{\frac{1}{2}} \tag{4.3.8}$$

for

$$\begin{aligned}\Omega =&\Big((n-1)K + \sup_{M^n}([A(\epsilon^{-1}\log u+1)]_+) + [(p-1+\epsilon)B]_+\sup_{M^n} u^{p-1}\Big)\\ &+\Big(|\nabla A|\sup_{M^n}(u^\epsilon|\epsilon\log u|) + \epsilon|\nabla B|\sup_{M^n} u^{p-1+\epsilon}\Big)\end{aligned}$$

and

$$\Theta = \Big(|\nabla A|\sup_{M^n}(u^\epsilon|\epsilon\log u|) + \epsilon|\nabla B|\sup_{M^n} u^{p-1+\epsilon}\Big).$$

Next, we produce an estimate similar to that of Yau [130], where the transformation is $h = \log u$. However, as mentioned in Brighton [25], with this choice a gap appears between the two cases on the closed ball; consequently, an extra set of parameters must be enforced in the proposition.

Proposition 4.3.1. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f \geq -(n-1)K$ for $K \geq 0$ and $R > 1$. Suppose u is a positive smooth solution to (4.3.1) such that*

$$\nabla f\nabla\log u - A\log u - Bu^{p-1} \leq \delta|\nabla\log u|^2 \quad \textit{for} \quad \delta \in \left(0,\frac{1}{2}\right) \tag{4.3.9}$$

and

$$\nabla f \nabla \log u - A \log u - Bu^{p-1} \geq 2|\nabla \log u|^2. \tag{4.3.10}$$

Then the following holds on $B(x_0, 2R)$:

$$\begin{aligned}|\nabla \log u|^2 \leq \max \Bigg(& 1, C_1\Big(|\nabla A| \sup_{B(x_0,2R)} (|\log u|) + |\nabla B| \sup_{B(x_0,2R)} u^{p-1+\epsilon}\Big) - C_2\Big((n-1)K \\ & + A_+ + \big[(p-1)B\big]_+ \sup_{B(x_0,2R)} u^{p-1}\Big) + \frac{C_3}{R}\Bigg) \end{aligned} \tag{4.3.11}$$

for a constant $C_n = C_n(\mu, n, \delta, D)$ and $\mu := \max_{x|d(x,x_0)=1} \Delta_f r(x)$.

4.4 Proof of results for elliptic PDEs

We now prove the propositions in section (4.3). As we defined our solution u to be a positive solution to 4.3.1, which means that $u : M^n \to \mathbb{R}^+$, we can use the Bochner formula. As in Brighton [25], we use the transformation $h = u^\epsilon$. Using this transform, equation (4.3.1) becomes

$$\Delta_f h + A(x) h \log h + \epsilon B h^{\epsilon^{-1}(p-1)+1} - \frac{(\epsilon - 1)}{\epsilon} \frac{|\nabla h|^2}{h} = 0. \tag{4.4.1}$$

Lemma 4.4.1. *Let $(M^n, g, e^{-f} d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f \geq -(n-1)K$ for a constant $K \geq 0$ and $R > 1$. Suppose u is a positive smooth solution to (4.3.1) in $\overline{B(x_0, 2R)}$, $x_0 \in M$ is fixed. Then there exists δ such that*

$$\frac{1}{n} + \frac{2(\epsilon - 1)}{\delta \epsilon n} \geq 0 \tag{4.4.2}$$

and

$$\frac{(\epsilon - 1)^2}{\epsilon^2 n} - \frac{(\epsilon - 1)}{\epsilon} + \frac{2\delta(\epsilon - 1)}{\epsilon n} > 0 \tag{4.4.3}$$

so that the following inequality holds:

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2) \geq & \frac{C_1^2}{n} \frac{|\nabla h|^4}{h^2} - C_1 \frac{|\nabla h|}{h} \nabla(|\nabla h|^2) - \Bigg((n-1)K + A(\log h + 1) \\ & + (p - 1 + \epsilon) B h^{\epsilon^{-1}(p-1)}\Bigg) |\nabla h|^2 - \Big(\nabla A h \log h + \epsilon \nabla B h^{\epsilon^{-1}(p-1)+1}\Big) \nabla h. \end{aligned} \tag{4.4.4}$$

Proof of Lemma 4.4.1. Let $u^\epsilon = h$, for $\epsilon \in (0, 1)$. We use the Bochner formula and $|\nabla^2 h| \geq \frac{1}{n}(\Delta h)^2$, which will give us an inequality that enables us to find a bound for the gradient.

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2) &= |\nabla^2 h| + \langle \nabla \Delta_f h, \nabla h \rangle + Ric_f(\nabla h, \nabla h) \\ &\geq \frac{1}{n}(\Delta h)^2 + \langle \nabla \Delta_f h, \nabla h \rangle + Ric_f(\nabla h, \nabla h)\end{aligned}$$

Taking each of the terms of the inequality above and rearranging, we get

$$\begin{aligned}\Delta_f h &= \Delta h - \langle \nabla f, \nabla h\rangle \\ &= \epsilon(u^{\epsilon-1})\Delta_f u + \epsilon(\epsilon-1)\frac{|\nabla u|^2}{u^2}u^{\epsilon} \\ &= \frac{(\epsilon-1)}{\epsilon}\frac{|\nabla h|^2}{h} - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}.\end{aligned}$$

We have used (4.3.1) in the final step. Using the above:

$$\begin{aligned}\langle \nabla h, \nabla(\Delta_f h\rangle =& \left\langle \nabla h, \nabla\left(\frac{(\epsilon-1)}{\epsilon}\frac{|\nabla h|^2}{h} - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}\right)\right\rangle \\ =& \frac{(\epsilon-1)}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - \frac{(\epsilon-1)}{\epsilon}\frac{|\nabla h|^4}{h^2} - \nabla A\nabla h h\log h - A|\nabla h|^2\log h - A|\nabla h|^2 \\ & - \epsilon\nabla B\nabla h h^{\epsilon^{-1}(p-1)+1} - (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}|\nabla h|^2\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n}(\Delta h)^2 =& \frac{1}{n}(\Delta_f h + \langle \nabla f, \nabla h\rangle)^2 \\ =& \frac{1}{n}\Bigg(\frac{(\epsilon-1)^2}{\epsilon^2}\frac{|\nabla h|^4}{h^2} - \frac{2(\epsilon-1)}{\epsilon}\frac{|\nabla h|^2}{h}(\langle \nabla f, \nabla h\rangle - Ah\log h - \\ & \epsilon B h^{\epsilon^{-1}(p-1)+1}) + \left(\langle \nabla f, \nabla h\rangle - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}\right)^2\Bigg).\end{aligned}$$

Using the equalities above and the lower bound Ricci estimate $Ric_f \geq -(n-1)K$, $K \geq 0$, we can change the Bochner inequality into

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2) \geq& \left(\frac{(\epsilon-1)^2}{\epsilon^2 n} - \frac{(\epsilon-1)}{\epsilon}\right)\frac{|\nabla h|^4}{h^2} + \frac{2(\epsilon-1)}{\epsilon n}\frac{|\nabla h|^2}{h}\Bigg(\langle \nabla f, \nabla h\rangle \\ & - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}\Bigg) + \frac{(\epsilon-1)}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) \\ & + \frac{1}{n}\left(\langle \nabla f, \nabla h\rangle - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}\right)^2 + \\ & \left(-(n-1)K - A(\log h + 1) - (p-1+\epsilon)B(x)h^{\epsilon^{-1}(p-1)}\right)|\nabla h|^2 \\ & - \left(\nabla A h\log h + \epsilon\nabla B h^{\epsilon^{-1}(p-1)+1}\right)\nabla h.\end{aligned}$$

Focusing on the second term of the inequality above, we will look at two cases on the closed ball $y \in \overline{B(x_0, 2R)}$. First, consider choosing y such that

$$\delta\frac{|\nabla h|^2}{h} \geq \left(\langle \nabla f, \nabla h\rangle - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}\right). \tag{4.4.5}$$

Noting that $(\epsilon - 1) < 0$, we can use this delta inequality to achieve

$$\frac{1}{2}\Delta_f(|\nabla h|^2) \geq \left(\frac{(\epsilon-1)^2}{e^2 n} - \frac{(\epsilon-1)}{\epsilon} + \frac{2\delta(\epsilon-1)}{\epsilon n}\right)\frac{|\nabla h|^4}{h^2} + \frac{(\epsilon-1)}{\epsilon}\frac{|\nabla h|}{h}\nabla(|\nabla h|^2)$$
$$+ \frac{1}{n}\left(\langle \nabla f, \nabla h\rangle - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}\right)^2 +$$
$$\left(-(n-1)K - A(\log h + 1) - (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\right)|\nabla h|^2$$
$$-\left(\nabla A h \log h + \epsilon \nabla B h^{\epsilon^{-1}(p-1)+1}\right)\nabla h. \tag{4.4.6}$$

The third term is always positive so we can remove this and still satisfy the inequality. Likewise, if we choose y such that

$$\delta\frac{|\nabla h|^2}{h} \leq \left(\langle \nabla f, \nabla h\rangle - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}\right), \tag{4.4.7}$$

then

$$\frac{1}{2}\Delta_f(|\nabla h|^2) \geq \left(\frac{(\epsilon-1)^2}{e^2 n} - \frac{(\epsilon-1)}{\epsilon}\right)\frac{|\nabla h|^4}{h^2} + \frac{2(\epsilon-1)}{\delta\epsilon n}\left(\langle \nabla f, \nabla h\rangle - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}\right)^2$$
$$+ \frac{(\epsilon-1)}{\epsilon}\frac{|\nabla h|}{h}\nabla(|\nabla h|^2) + \frac{1}{n}\left(\langle \nabla f, \nabla h\rangle - Ah\log h - \epsilon B h^{\epsilon^{-1}(p-1)+1}\right)^2$$
$$+\left(-(n-1)K - A(\log h + 1) - (p-1+\epsilon)B(h^{\epsilon^{-1}(p-1)}\right)|\nabla h|^2$$
$$-\left(\nabla A h \log h + \epsilon \nabla B h^{\epsilon^{-1}(p-1)+1}\right)\nabla h.$$

We are going to make the further restriction to keep the inner product term positive. For this we have

$$\frac{1}{n} + \frac{2(\epsilon-1)}{\delta\epsilon n} \geq 0. \tag{4.4.8}$$

Using this and $\delta\frac{|\nabla h|^2}{h} \geq 0$ gives

$$\frac{1}{2}\Delta_f(|\nabla h|^2) \geq \left(\frac{(\epsilon-1)^2}{\epsilon^2 n} - \frac{(\epsilon-1)}{\epsilon}\right)\frac{|\nabla h|^4}{h^2} + \frac{2(\epsilon-1)}{\delta\epsilon n}\left(\delta\frac{|\nabla h|^2}{h}\right)^2 + \frac{(\epsilon-1)}{\epsilon}\frac{|\nabla h|}{h}\nabla(|\nabla h|^2)$$
$$+\left(-(n-1)K - A(\log h + 1) - (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\right)|\nabla h|^2$$
$$-\left(\nabla A h \log h + \epsilon \nabla B h^{\epsilon^{-1}(p-1)+1}\right)\nabla h. \tag{4.4.9}$$

We also want the coefficient of $\frac{|\nabla h|^4}{h^2}$ to be positive. This gives

$$\frac{(\epsilon-1)^2}{\epsilon^2 n} - \frac{(\epsilon-1)}{\epsilon} + \frac{2\delta(\epsilon-1)}{\epsilon n} > 0. \tag{4.4.10}$$

Next, we introduce a cutoff function introduced in Lemma 4.1.1. This means it takes a positive value when inside the ball centre x_0, radius $2R$, but equivalently 0 outside of this. We also introduce a function G which obtains its maximum within this ball of centre x_0 and radius $2R$. This allows us to use the maximum principle on (4.4.9) for $|\nabla h|$. For convenience, we make the coefficient of $\frac{|\nabla h|^4}{h^2}$ positive. The choice of ϵ must obey (4.4.8) and (4.4.10). Specifically, we choose $\delta = \frac{n}{2}$ as this means that the last two terms of (4.4.10) cancel. We set $C_1 = \frac{(1-\epsilon)}{\epsilon} > 0$ for conciseness. Thus we have:

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2) \geq& \frac{C_1^2}{n}\frac{|\nabla h|^4}{h^2} - C_1\frac{|\nabla h|}{h}\nabla(|\nabla h|^2) - \Bigg((n-1)K + A(\log h + 1) \\ &+ (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\Bigg)|\nabla h|^2 - \Big(\nabla Ah\log h + \epsilon\nabla Bh^{\epsilon^{-1}(p-1)+1}\Big)\nabla h. \qquad (4.4.11)\end{aligned}$$

$\square$

Proof of Theorem 4.3.1. Now we will use a cutoff function $\phi(s)$ to allow us to use the maximum principle on equation (4.4.4).

Let $\phi = \phi(s)$ be as defined above in Lemma 4.1.1, and let $G = \phi|\nabla h|^2$. Due to the cutoff function, $G(x_1)$ achieves its maximum somewhere in the open ball $B(x_0, 2R)$ by the properties of ϕ. Taking $x_1 \in B(x_0, 2R)$. Let $G(x_1) > 0$, then

$$\nabla G = \phi\nabla(|\nabla h|^2) + \nabla\phi|\nabla h|^2 \qquad (4.4.12)$$

and $\Delta_f G \leq 0$. We then get

$$\begin{aligned}0 \geq& \phi\Delta_f(|\nabla h|^2) + |\nabla h|^2\Delta_f\phi + 2\nabla\phi\nabla(|\nabla h|^2) \\ \geq& 2\phi\Bigg[\frac{C_1^2}{n}\frac{|\nabla h|^4}{h^2} - C_1\frac{|\nabla h|}{h}\nabla(|\nabla h|^2) - \Bigg((n-1)K + A(\log h+1) + (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\Bigg)|\nabla h|^2 \\ &- \Big(\nabla Ah\log h + \epsilon\nabla Bh^{\epsilon^{-1}(p-1)+1}\Big)\nabla h\Bigg] + \frac{\Delta_f\phi}{\phi}G - 2\frac{|\nabla\phi|^2}{\phi^2}G \\ \geq& \frac{2C_1^2}{n}\phi\frac{|\nabla h|^4}{h^2} - 2C_1\phi\frac{|\nabla h|}{h}\nabla(|\nabla h|^2) - 2\Bigg((n-1)K + A(\log h+1) \\ &+ (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\Bigg)|\nabla h|^2\phi - 2\Big(\nabla Ah\log h + \epsilon\nabla Bh^{\epsilon^{-1}(p-1)+1}\Big)\nabla h\phi + \frac{\Delta_f\phi}{\phi}G - 2\frac{|\nabla\phi|^2}{\phi^2}G \\ \geq& \frac{2C_1^2}{n}\frac{G^2}{\phi h^2} + 2C_1\frac{|\nabla h|}{h}\nabla\phi\frac{G}{\phi} - 2\Bigg((n-1)K + A(\log h+1) + (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\Bigg)G \\ &- 2\Big(\nabla Ah\log h + \epsilon\nabla Bh^{\epsilon^{-1}(p-1)+1}\Big)\nabla h\phi + \frac{\Delta_f\phi}{\phi}G - 2\frac{|\nabla\phi|^2}{\phi^2}G. \qquad (4.4.13)\end{aligned}$$

Multiplying both sides by $\frac{\phi}{G}$ yields

$$\begin{aligned}0 \geq& \frac{2C_1^2}{n}\frac{G}{h^2} + 2C_1\frac{|\nabla h|}{h}\nabla\phi - 2\Bigg((n-1)K + A(\log h+1) + (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\Bigg)\phi \\ &- 2\Big(\nabla Ah\log h + \epsilon\nabla Bh^{\epsilon^{-1}(p-1)+1}\Big)\nabla h\frac{\phi}{G} + \Delta_f\phi - 2\frac{|\nabla\phi|^2}{\phi}. \qquad (4.4.14)\end{aligned}$$

We need to control the second, third, and fourth terms in the previous inequality. For the second term, we will impose Young's inequality:

$$-\frac{\nabla h}{h}\nabla\phi \le \frac{|\nabla h|}{h}|\nabla\phi| \le \frac{\gamma}{2}\frac{G}{h^2} + \frac{1}{2\gamma}\frac{|\nabla\phi|^2}{\phi}. \tag{4.4.15}$$

For the third term, we will use a max function to eliminate the case in which that term is less than 0. We use the notation $f(x)_+ = \max(f(x), 0)$:

$$\begin{aligned}(n-1)K + A(\log h + 1) + (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\\ \le (n-1)K + \sup_{B(x_0,2R)}[(A(\log h+1))_+] + [(p-1+\epsilon)B]_+ \sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)}.\end{aligned}$$

We will deal with the fourth term later in the proof. Now rearranging (4.4.13) and substituting these in, we obtain

$$\begin{aligned}\frac{(2C_1^2-\gamma nC_1)}{2n}\frac{G}{h^2} \le& 2\Big((n-1)K + \sup_{B(x_0,2R)}[(A(\log h+1))_+] + [(p-1+\epsilon)B]_+ \sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)}\Big)\phi\\ &+ 2\Big(\nabla Ah\log h + \epsilon\nabla Bh^{\epsilon^{-1}(p-1)+1}\Big)\nabla h\frac{\phi}{G} - \Delta_f\phi + \left(\frac{4\gamma + C_1}{2\gamma}\right)\frac{|\nabla\phi|^2}{\phi}.\end{aligned}$$

We select $\gamma = \frac{\epsilon C_1}{n}$. Then

$$\begin{aligned}C_2\frac{G}{h^2} \le& (n+2)\frac{|\nabla\phi|^2}{\phi} + 2\Bigg((n-1)K + \sup_{B(x_0,2R)}[(A(\log h+1))_+]\\ &+ [(p-1+\epsilon)B]_+ \sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)}\Bigg)\phi + 2\Big(\nabla Ah\log h + \epsilon\nabla Bh^{\epsilon^{-1}(p-1)+1}\Big)\nabla h\frac{\phi}{G}\\ &- \Delta_f\phi.\end{aligned} \tag{4.4.16}$$

We now need to eliminate the derivative terms of ϕ. This is achieved by using the work of [120] mentioned above; in particular, equations (4.1.1) and (4.1.2). Solving for G, (4.4.16) becomes

$$\begin{aligned}G \le& h^2\Bigg[\frac{C_3}{R^2} + C_4\Bigg((n-1)K + \sup_{B(x_0,2R)}[(A(\log h+1))_+] + [(p-1+\epsilon)B]_+ \sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)}\Bigg)\phi\\ &+ \frac{C_5[\mu + (n-1)K(2R-1)]}{R}\Bigg)\Bigg] + C_6h^2\Big(\nabla Ah\log h + \epsilon\nabla Bh^{\epsilon^{-1}(p-1)+1}\Big)\nabla h\frac{\phi}{G}.\end{aligned} \tag{4.4.17}$$

Now we deal with the fourth term. We aim to manipulate this term into one where it always positive but also to deal with the G and $|\nabla h|$. We have two cases to look at: the first is when $|\nabla h| \ge 1$; if this holds, then we can multiply the fourth term by $|\nabla h|$ and then cancel out the G.

$$\begin{aligned}C_6\Big(\nabla Ah\log h + \epsilon\nabla Bh^{\epsilon^{-1}(p-1)+1}\Big)\nabla h\frac{\phi}{G} &\le C_6\Big(|\nabla A|h\log h + \epsilon|\nabla B|h^{\epsilon^{-1}(p-1)+1}\Big)|\nabla h|^2\frac{\phi}{G}\\ &\le C_6\Big(|\nabla A|\sup_{B(x_0,2R)}(h|\log h|) + \epsilon|\nabla B|\sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)+1}\Big)\phi.\end{aligned} \tag{4.4.18}$$

When $0 \le |\nabla h| < 1$, this approach fails. However, we know that $|\nabla h|$ is bounded by 1. Using a max function combines these two scenarios into a single estimate.

Substituting back that $h = u^\epsilon$, $u \le \sup u$, $G = \phi\epsilon^2 h^2\frac{|\nabla u|^2}{u^2}$, and looking at each part of the max function individually, we find an estimate for u. Finally, we restrict $x \in B(x_0, R)$, since $G(x) \leq G(x_1)$. This results in $\phi = 1$ and gives our estimate as

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} \leq &\max\Bigg(\frac{1}{\epsilon^2 u^{2\epsilon}}, \frac{C_7}{R^2} + C_8\Big((n-1)K + \sup_{B(x_0,2R)}[(A(\epsilon\log u + 1))_+] \\ &+ [(p-1+\epsilon)B]_+ \sup_{B(x_0,2R)} u^{p-1}\Big) + C_9\Big(|\nabla A| \sup_{B(x_0,2R)}(u^\epsilon|\epsilon\log u|) \\ &+ |\nabla B| \sup_{B(x_0,2R)} u^{p-1+\epsilon}\Big) + C_{10}\Big(\frac{\mu + (n-1)K(2R-1)}{R}\Big)\Bigg). \end{aligned} \tag{4.4.19}$$

□

Next we prove Theorem 4.3.2. This follows a very similar path as Theorem 4.3.1 but varies when dealing with the $|\nabla h|$ term towards the end of the proof.

Proof of Theorem 4.3.2. The proof begins identically to that of Theorem 4.3.1 up to (4.4.13). Here, instead of multiplying by $\frac{\phi}{G}$, we multiply by ϕ. We also use that $|\nabla h| \leq 1 + |\nabla h|^2$.

$$\begin{aligned}0 \geq &\frac{2C_1^2}{n}\frac{G^2}{h^2} + 2C_1\frac{|\nabla h|}{h}\nabla\phi G - 2\Bigg((n-1)K + A(\log h + 1) + (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\Bigg)\phi G \\ &- 2\Big(\nabla A h\log h + \epsilon\nabla B h^{\epsilon^{-1}(p-1)+1}\Big)(1 + |\nabla h|^2)\phi^2 + \Delta_f\phi G - 2\frac{|\nabla\phi|^2}{\phi}G.\end{aligned}$$

Then by a similar process to (4.4.15),

$$\frac{\nabla h}{h}\nabla\phi \geq -\frac{|\nabla h|}{h}|\nabla\phi| \geq -\frac{\gamma G}{2h^2} - \frac{1}{2\gamma}\frac{|\nabla\phi|^2}{\phi}$$

which gives

$$\begin{aligned}0 \geq &G^2\Big(\frac{2C_1^2}{nh^2} - \frac{C_2}{h^2}\Big) + C_3 G\Bigg(\Big((n-1)K + A(\log h + 1) + (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)}\Big)\phi \\ &- \frac{1}{\gamma}\frac{|\nabla\phi|^2}{\phi} - \Big(\nabla A h\log h + \epsilon\nabla B h^{\epsilon^{-1}(p-1)+1}\Big)\phi + \Delta_f\phi - 2\frac{|\nabla\phi|^2}{\phi}\Bigg) \\ &- \Big(\nabla A h\log h + \epsilon\nabla B h^{\epsilon^{-1}(p-1)+1}\Big)\phi.\end{aligned}$$

Now we control the second and third terms:

$$\begin{aligned}&A(\log h + 1) + (p-1+\epsilon)Bh^{\epsilon^{-1}(p-1)} \\ &\qquad\qquad \leq \sup_{B(x_0,2R)}([A(\log h + 1)]_+) + [(p-1+\epsilon)]_+ \sup_{B(x_0,2R)} Bh^{\epsilon^{-1}(p-1)}\end{aligned}$$

and

$$\nabla A h\log h + \epsilon\nabla B h^{\epsilon^{-1}(p-1)} \leq |\nabla A| \sup_{B(x_0,2R)}(h|\log h|) + \epsilon|\nabla B| \sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)+1}.$$

Our inequality now becomes

$$\begin{aligned}
0 \geq & G^2\Big(\frac{2C_1^2}{nh^2} - \frac{C_2}{h^2}\Big) - C_3 G\Bigg(\Big((n-1)K + \sup_{B(x_0,2R)}([A(\log h + 1)]_+) \\
& + [(p-1+\epsilon)]_+ \sup_{B(x_0,2R)} Bh^{\epsilon^{-1}(p-1)}\Big)\phi + \Big(|\nabla A| \sup_{B(x_0,2R)}(h|\log h|) \\
& + \epsilon|\nabla B| \sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)+1}\Big)\phi + \frac{C_4}{R^2} + \frac{C_5[\mu + (n-1)K(2R-1)]}{R}\Bigg) \\
& - \Big(|\nabla A| \sup_{B(x_0,2R)}(h|\log h|) + \epsilon|\nabla B| \sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)+1}\Big)\phi.
\end{aligned}$$

This is a quadratic in G which can be solved straightforwardly. Setting

$$\Psi = \frac{2C_1^2}{nh^2} - \frac{C_2}{h^2},$$

$$\begin{aligned}
\tilde{\Omega} = & \Big((n-1)K + \sup_{B(x_0,2R)}([A(\log h + 1)]_+) + [(p-1+\epsilon)]_+ \sup_{B(x_0,2R)} Bh^{\epsilon^{-1}(p-1)}\Big)\phi \\
& + \Big(|\nabla A| \sup_{B(x_0,2R)}(h|\log h|) + \epsilon|\nabla B| \sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)+1}\Big)\phi + \frac{C_4}{R^2} \\
& + \frac{C_5[\mu + (n-1)K(2R-1)]}{R},
\end{aligned}$$

and

$$\tilde{\Theta} = \Big(|\nabla A| \sup_{B(x_0,2R)}(h|\log h|) + \epsilon|\nabla B| \sup_{B(x_0,2R)} h^{\epsilon^{-1}(p-1)+1}\Big)\phi.$$

Solving this gives

$$\begin{aligned}
G \leq & \frac{1}{2\Psi}\Big(\tilde{\Omega} + \Big(\tilde{\Omega}^2 + 4\Psi\tilde{\Theta}\Big)^{\frac{1}{2}}\Big) \\
\leq & \frac{\tilde{\Omega}}{\Psi} + \frac{1}{2\Psi}\Big(4\Psi\tilde{\Theta}\Big)^{\frac{1}{2}} \qquad (4.4.20)
\end{aligned}$$

Transforming back, simplifying, and restricting our radius as before, we observe the desired result.

□

Remark 4.4.1. *It is easy to produce similar corollaries to Theorem 4.3.1 here. For A, B constant, this method is unnecessary: the gradient terms do not appear, so we do not need to produce a quadratic in G. This means that for the case when A, B constant, it will give the same result as that of Theorem 4.3.1.*

Next we prove the Yau style estimate.

Proof of Proposition 4.3.1. Let $h = \log u$. Then

$$\begin{aligned}\Delta_f h &= \Delta(\log u) - \langle \nabla f, \nabla(\log u)\rangle \\ &= \frac{\Delta_f u}{u} - \frac{|\nabla u|^2}{u^2} \\ &= -(Ah + Be^{h(p-1)}) - |\nabla h|^2\end{aligned}$$

and

$$\begin{aligned}(\Delta h)^2 =&(\Delta_f h + \langle \nabla f, \nabla h\rangle)^2 \\ =&|\nabla h|^4 - 2|\nabla h|^2(Ah + Be^{h(p-1)} + \langle \nabla f, \nabla h\rangle) + (Ah + Be^{h(p-1)} \\ &+ \langle \nabla f, \nabla h\rangle)^2.\end{aligned}$$

If (4.3.9) then

$$\begin{aligned}(\Delta h)^2 \geq&|\nabla h|^4 - 2\delta|\nabla h|^4 + (Ah + Be^{h(p-1)} + \langle \nabla f, \nabla h\rangle)^2 \\ \geq&(1-2\delta)|\nabla h|^4.\end{aligned}$$

If (4.3.10) then

$$\begin{aligned}(\Delta h)^2 \geq&|\nabla h|^4 - (Ah + Be^{h(p-1)} + \langle \nabla f, \nabla h\rangle)^2 + (Ah + Be^{h(p-1)} \\ &+ \langle \nabla f, \nabla h\rangle)^2 \\ \geq&|\nabla h|^4 \\ \geq&(1-2\delta)|\nabla h|^4\end{aligned}$$

Then by use of the Bochner formula and following the proof of Theorem 4.3.1 we obtain the desired result. □

Chapter 5

Li-Yau and Souplet-Zhang gradient estimates for parabolic PDEs

In chapter 4 we focused on Li-Yau estimates for elliptic PDEs. A different type of estimate was introduced by Souplet and Zhang in [107]. These estimates are known as Souplet-Zhang estimates, and are very similar to Li-Yau estimate; the difference is that they use cutoff functions that have a dependency on time t. This means that they are specifically used for parabolic estimates.

Lemma 5.0.1. *Fix $t_0 \in \mathbb{R}$ and $T > 0$. We choose $\tau \in (t_0 - T, t_0]$. Let $\bar{\phi} : [0, \infty) \times [t_0 - T, t_0] \to \mathbb{R}$ such that*

$$0 \leq \bar{\phi}(r, t) \leq 1 \tag{5.0.1}$$

in $[0, R] \times [t_0 - T, t_0]$ with support, $\bar{\phi}(r, t) = 1$ in $[0, R/2] \times [\tau, t_0]$, and $\partial_r \bar{\phi}(r, t) = 0$ in $[0, R/2] \times [t_0 - T, t_0]$. Then

$$|\partial_t \bar{\phi}(r, t)| \leq \frac{C\bar{\phi}^{\frac{1}{2}}}{\tau - (t_0 - T)} \tag{5.0.2}$$

in $[0, \infty) \times [t_0 - T, t_0]$ for $C > 0$ and $\bar{\phi}(r, t_0 - T) = 0$ for all $r \in [0, \infty)$. Finally

$$-\frac{C_\epsilon}{R} \leq \frac{\partial_r \bar{\phi}}{\bar{\phi}^\epsilon} \leq 0 \quad \text{and} \quad \frac{|\partial_r^2 \bar{\phi}|}{\bar{\phi}^\epsilon} \leq \frac{C_\epsilon}{R^2} \tag{5.0.3}$$

in $[0, \infty) \times [t_0 - T, t_0]$, $C_\epsilon > 0$, and $\epsilon \in (0, 1)$.

As with our elliptic estimates, we define the specific case $\phi : M^n \times [t_0 - T, t_0] \to \mathbb{R}$ such that

$$\phi(r, t) = \bar{\phi}(d(x, x_0), t) \tag{5.0.4}$$

where $d(x, y)$ is the distance function.

Remark 5.0.1. *The Li-Yau estimate for an elliptic PDE and the Souplet-Zhang estimate for a parabolic PDE find an estimate for the gradient term only. However, if we find a Li-Yau estimate for a parabolic PDE, this estimate includes a time derivative. Examples of this can be seen in [3, 108]. We later produce one of these estimates.*

5.1 Souplet-Zhang estimate for parabolic PDE

Current research has taken the Souplet-Zhang estimate for the heat equation and, as with elliptic PDEs, found estimates for non-linear parabolic PDEs. Wu has produced several estimates for non-linear PDEs as well as different analytical procedures from the subsequent results. [124] looks at the parabolic heat equation, an extension of [107]. Wu also found estimates for non-linearities of the Yamabe type problem [126], similar to work by Abolarinwa [5]. See also [65] for a similar estimate for constant f.

Another non-linearity of interest is that of the logarithmic term, which had been looked at by a variety of authors [7, 33, 84, 125]. Wang and Zheng in [117] didn't specify their non-linearity and found estimates for this general parabolic equation.

Our interest is in finding estimates for

$$\Big(\Delta_f - \partial_t\Big)u(x,t) + A(x,t)u(x,t)\log u(x,t) + B(x,t)u(x,t)^p = 0 \tag{5.1.1}$$

which is the corresponding parabolic equation for (4.3.1). This is an extension to the nonlinearities mentioned above. For $B(x, t) \equiv 0$ and A constant, we obtain the equation studied in [125].

Our first objective is to find a Souplet-Zhang estimate for (5.1.1). Recall that for a parabolic PDE, the Souplet-Zhang estimate is a spatial-only estimate, focusing only on the gradient term for u.

Souplet-Zhang estimates are useful tools because they allow further analysis using ancient solutions. These are solutions taken over all negative time without singularities. We look at these further in chapter 7.

Theorem 5.1.1. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f \geq -(n-1)K$ for $K \geq 0$ and $R > 1$ in $B(x_0, R)$. Suppose u is a bounded positive smooth solution to (5.1.1) such that $0 < u \leq D$ in $Q_{2R,T} = B(x_0, 2R) \times [t_0 - T, t_0) \subset M^n \times (-\infty, \infty)$, where $x_0 \in M^n$ is fixed. Then there exist constants C_n such that the following inequality holds on*

$Q_{R,T}$:

$$\frac{|\nabla u|^2}{u^2} \leq \Big(1-\log\frac{u}{D}\Big)^2 \Bigg[C_1 \sup_{Q_{2R,T}} |\log u|^{\frac{2}{3}} \sup_{Q_{2R,T}} |\nabla A|^{\frac{2}{3}} + C_2 \sup_{Q_{2R,T}} |u^{p-1}|^{\frac{2}{3}} \sup_{Q_{2R,T}} |\nabla B|^{\frac{2}{3}} + C_3(A_+)$$
$$+ C_4\Big([(p-1)B]_+ + B_+\Big) \sup_{Q_{2R,T}} (u^{p-1}) + \frac{C_5}{R^2} + C_6(n-1)K + \frac{C_7 \sup \log(\frac{u}{D})^4}{R^4}$$
$$+ \frac{C_8}{(t-t_0+T)} + \frac{C_9}{R^4} + \frac{C_{10}(\mu_+)}{R} + (n-1)^{\frac{1}{2}} K \Bigg] \tag{5.1.2}$$

for $t \neq t_0 - T$.

We follow a similar path laid out in the work by Wu, who used Einstein notation for directional derivatives: specifically, an orthonormal frame with $e_1, ..., e_n$ at $x \in M^n$ with the covariant differentiations represented by the subscripts $1 \leq i, j, k \leq n$ in e_i, e_j, e_k. Note that $\nabla_i h = h_i$, $\Delta h = h_{ii}$, $\nabla_k \nabla_j \nabla_i h = h_{ijk}$.

We use the transform $h = \log\frac{u}{D}$, where the constant D is such that $0 < u(x,t) \leq D$ and $1 \leq D$. Then

$$\Delta_f u = De^h \Delta h + De^h |\nabla h|^2 - De^h \langle \nabla f, \nabla u \rangle$$
$$= De^h \Delta_f h + D|\nabla h|^2 e^h$$

which gives

$$\Delta_f h - h_t + A(x,t)(\log D + h) + B(x,t)(De^h)^{p-1} + |\nabla h|^2 = 0. \tag{5.1.3}$$

Lemma 5.1.1. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f \geq -(n-1)K$ for a constant $K \geq 0$ and $R > 1$ in $B(x_0, R)$. Suppose u is a bounded positive smooth solution to (5.1.1) such that $0 < u \leq D$ in $Q_{2R,T} = B(x_0, 2R) \times [t_0 - T, t_0) \subset M^n \times (-\infty, \infty)$, where $x_0 \in M^n$ is fixed. Let $h(x,t)$ be a non-positive function in $Q_{2R,T}$ satisfying (5.1.3). Then*

$$w = |\nabla \log(1-h)|^2 = \frac{|\nabla h|^2}{(1-h)^2} \tag{5.1.4}$$

satisfies

$$\Big(\Delta_f - \partial_t\Big) w \geq -2\nabla w \nabla h + \frac{2\nabla w \nabla h}{(1-h)} + 2(1-h)w^2 - \frac{2\nabla h(\mathbf{X})}{(1-h)^2}$$
$$- \frac{2|\nabla h|^2((\log D + h)A + B(De^h)^{p-1})}{(1-h)^3} + \frac{2Ric_f(\nabla h, \nabla h)}{(1-h)^2} \tag{5.1.5}$$

for $\mathbf{X} = (\log D + h)\nabla A + A\nabla h + (De^h)^{p-1}\nabla B + (p-1)B(De^h)^{p-1}\nabla h$.

Proof. Let

$$w = |\nabla \log(1-h)|^2 = \frac{|\nabla h|^2}{(1-h)^2}. \tag{5.1.6}$$

We now calculate

$$w_j = \frac{2h_i h_{ij}}{(1-h)^2} + \frac{2h_i^2 h_j}{(1-h)^3} \tag{5.1.7}$$

and

$$\Delta w = \frac{2|h_{ij}|^2}{(1-h)^2} + \frac{2h_i h_{ijj}}{(1-h)^2} + \frac{8h_i h_j h_{ij}}{(1-h)^3} + \frac{2h_i^2 h_{jj}}{(1-h)^3} + \frac{6h_i^2 h_j^2}{(1-h)^4}.$$

Using the Ricci identity $h_{ijj} = h_{jji} + R_{ij} h_j$ we first note that

$$\begin{aligned} h_{jji} - f_j h_{ij} =& (h_{jj} - f_j h_j)_i - f_{ij} h_i \\ =& (\Delta_f h)_i - f_{ij} h_i \end{aligned} \tag{5.1.8}$$

from which it follows that

$$\begin{aligned} \Delta_f w =& \Delta w - \langle \nabla f, \nabla w \rangle \\ =& \frac{2|h_{ij}|^2}{(1-h)^2} + \frac{2h_i(\Delta_f h)_i}{(1-h)^2} + \frac{2(R_{ij} + f_{ij}) h_i h_j}{(1-h)^2} + \frac{8h_i h_j h_{ij}}{(1-h)^3} + \frac{2h_i^2 \Delta_f h}{(1-h)^3} + \frac{6h_i^4}{(1-h)^4}. \end{aligned} \tag{5.1.9}$$

Also,

$$\begin{aligned} w_t =& \frac{2h_i (h_t)_i}{(1-h)^2} + \frac{2h_i^2 h_t}{(1-h)^3} \\ =& \frac{2\nabla h(\nabla \Delta_f h + \mathbf{X})}{(1-h)^2} + \frac{4h_i h_j h_{ij}}{(1-h)^2} + \frac{2|\nabla h|^4 + 2|\nabla h|^2 (\Delta_f h + Ah + Be^{h(p-1)})}{(1-h)^3} \end{aligned} \tag{5.1.10}$$

for $\mathbf{X} = (\log D + h)\nabla A + A\nabla h + (De^h)^{p-1}\nabla B + (p-1)B(De^h)^{p-1}\nabla h$. Using (5.1.9) and (5.1.10)

$$\begin{aligned} \Big(\Delta_f - \partial_t\Big) w =& \frac{2|h_{ij}|^2}{(1-h)^2} + \frac{8h_i h_j h_{ij}}{(1-h)^3} - \frac{4h_i h_j h_{ij}}{(1-h)^2} - \frac{2\nabla h(\mathbf{X})}{(1-h)^2} - \frac{2h_i^4}{(1-h)^3} \\ & - \frac{2h_i^2 (A(\log D + h) + B(De^h)^{(p-1)})}{(1-h)^3} + \frac{6h_i^4}{(1-h)^4} + \frac{2(R_{ij} + f_{ij}) h_i h_j}{(1-h)^2}. \end{aligned} \tag{5.1.11}$$

With (5.1.7) we have

$$0 = -2w_j h_j + \frac{4h_i h_j h_{ij}}{(1-h)^2} + \frac{4h_i^2}{(1-h)^3} \tag{5.1.12}$$

and

$$0 = \frac{2w_j h_j}{(1-h)} - \frac{4h_i h_j h_{ij}}{(1-h)^3} - \frac{4h_i^2}{(1-h)^4}. \tag{5.1.13}$$

Adding (5.1.12) and (5.1.13) to (5.1.11), we obtain

$$\begin{aligned} \Big(\Delta_f - \partial_t\Big) w =& \frac{2|h_{ij}|^2}{(1-h)^2} + \frac{4h_i h_j h_{ij}}{(1-h)^3} - 2w_j h_j + \frac{2w_j h_j}{(1-h)} - \frac{2\nabla h(\mathbf{X})}{(1-h)^2} + \frac{2h_i^4}{(1-h)^3} \\ & - \frac{2h_i^2((\log D + h)A + (De^h)^{(p-1)})}{(1-h)^3} + \frac{2h_i^4}{(1-h)^4} + \frac{2(R_{ij} + f_{ij}) h_i h_j}{(1-h)^2}. \end{aligned} \tag{5.1.14}$$

Next, we use that

$$\frac{2|h_{ij}|^2}{(1-h)^2} + \frac{4h_ih_jh_{ij}}{(1-h)^3} + \frac{2h_i^4}{(1-h)^4} = 2\left(\frac{h_{ij}}{(1-h)} + \frac{h_i^2}{(1-h)^2}\right)^2 \geq 0. \tag{5.1.15}$$

Combining the above, we have

$$\begin{aligned}\Big(\Delta_f - \partial_t\Big)w \geq & -2w_jh_j + \frac{2w_jh_j}{(1-h)} + 2(1-h)w^2 - \frac{2\mathbf{X}\nabla h}{(1-h)^2} - \frac{2h_i^2((\log D+h)A + B(De^h)^{p-1})}{(1-h)^3} \\ & + \frac{2(R_{ij}+f_{ij})h_ih_j}{(1-h)^2}.\end{aligned} \tag{5.1.16}$$

□

Proof of Theorem 5.1.1. We now introduce a cutoff function ϕ by using Lemma 5.0.1.

Let $t \in [t_0 - T, t_0]$, and let $\phi(r,t)$ be the cutoff function defined above. Specifically, let (x_1, t_1) be the maximum space-time point of ϕw. Then

$$\Delta_f(\phi w) \leq 0, \quad \nabla(\phi w) = 0, \quad (\phi w)_t \geq 0. \tag{5.1.17}$$

Basic calculations give

$$\Big(\Delta_f - \partial_t\Big)(\phi w) = \phi\Big(\Delta_f - \partial_t\Big)w + 2\nabla\phi\nabla w + w\Big(\Delta_f - \partial_t\Big)\phi. \tag{5.1.18}$$

Combining this with (5.1.17) and using $\nabla\phi\nabla w = \frac{\nabla\phi}{\phi}\nabla(\phi w) - \frac{|\nabla\phi|^2}{\phi}w$, we find that

$$\begin{aligned}\Big(\Delta_f - \partial_t\Big)(\phi w) \geq & \phi\Bigg[2(1-h)w^2 + \frac{2(R_{ij}+f_{ij})h_ih_j}{(1-h)^2} - \frac{2\mathbf{X}\nabla h}{(1-h)^2} \\ & - \frac{2h_i^2\big((\log D+h)A + B(De^h)^{p-1}\big)}{(1-h)^3}\Bigg] + 2\nabla(\phi w)\nabla h\frac{h}{1-h} \\ & - 2\nabla\phi\nabla hw\frac{h}{1-h} + 2\frac{\nabla\phi}{\phi}\nabla(\phi w) - 2\frac{|\nabla\phi|^2}{\phi}w + w\Big(\Delta_f - \partial_t\Big)\phi.\end{aligned} \tag{5.1.19}$$

Then, applying the Ricci identity $(R_{ij} + f_{ij}) \geq -(n-1)K$,

$$\begin{aligned}2\phi(1-h)w^2 \leq & 2\nabla\phi\nabla hw\frac{h}{1-h} + \frac{2\phi\mathbf{X}\nabla h}{(1-h)^2} + \frac{2\phi w\big((\log D+h)A + B(De^h)^{p-1}\big)}{(1-h)} \\ & + 2\phi(n-1)Kw + 2\frac{|\nabla\phi|^2}{\phi}w - w\Big(\Delta_f - \partial_t\Big)\phi.\end{aligned} \tag{5.1.20}$$

We now need to control the terms on the right hand side. For the first term, we observe that

$$\begin{aligned}2\nabla h\nabla\phi w\frac{h}{1-h} =& 2h\nabla\phi w^{\frac{3}{2}} \\ \leq & 2|h||\nabla\phi|w^{\frac{3}{2}} \\ \leq & 2(\phi w^2)^{\frac{3}{4}}\frac{|h||\nabla\phi|}{\phi^{\frac{3}{4}}} \\ \leq & \frac{1}{9}\phi w^2 + \frac{Ch^4}{R^4}.\end{aligned} \tag{5.1.21}$$

Note that by the definition of $\mathbf{X}$, it consists of A and B terms, and also contains the gradient terms of both of these functions. First, we look at the non-gradient terms and also include the third term in (5.1.20). We also note that $1-h \geq 1$. For the A terms, we have

$$\begin{aligned}
\frac{2\phi w(\log D + h)A}{1-h} + \frac{2\phi|\nabla h|^2 A}{(1-h)^2} \leq & \frac{2\phi w A(\log D + 1)}{(1-h)} \\
\leq & 2\phi w A(\log D + 1) \\
\leq & \frac{1}{9}\phi w^2 + C\phi(A_+)^2. \qquad (5.1.22)
\end{aligned}$$

For the B terms, we split the working into two. Firstly, we see that

$$\begin{aligned}
\Big((p-1) + \frac{1}{1-h}\Big)B &\leq (p-1)B + \frac{B_+}{1-h} \\
&\leq [(p-1)B]_+ + B_+. \qquad (5.1.23)
\end{aligned}$$

With this, we then get

$$\begin{aligned}
\frac{2\phi w B(De^h)^{p-1}}{1-h} + \frac{2\phi|\nabla h|^2 B(p-1)(De^h)^{p-1}}{(1-h)^2} \leq & \frac{2\phi w B(De^h)^{p-1}}{1-h} + 2\phi w B(p-1)(De^h)^{p-1} \\
\leq & 2\phi w\Big([(p-1)B]_+ + B_+\Big)(De^h)^{p-1} \\
\leq & \frac{1}{9}\phi w^2 + C\phi\Big([(p-1)B]_+ + B_+\Big)^2 \sup_{Q_{2R,T}}((De^h)^{2(p-1)}).
\end{aligned}$$
$$(5.1.24)$$

Now we examine the gradient terms for A and B which are within $\mathbf{X}$. We start with the ∇A terms:

$$\begin{aligned}
\frac{2\phi(\log D + h)\nabla h \nabla A}{(1-h)^2} \leq & \frac{2\phi|\log D + h||\nabla A||\nabla h|}{(1-h)^2} \\
\leq & 2\phi|\log D + h||\nabla A| w^{\frac{1}{2}} \\
\leq & 2\phi^{\frac{1}{4}} w^{\frac{1}{2}}(\phi^{\frac{3}{4}}|\log D + h||\nabla A|) \\
\leq & \frac{1}{9}\phi w^2 + C\phi \sup_{Q_{2R,T}} |\log D + h|^{\frac{4}{3}} \sup_{Q_{2R,T}} |\nabla A|^{\frac{4}{3}} \qquad (5.1.25)
\end{aligned}$$

Next, the ∇B terms:

$$\frac{2\phi\nabla h \nabla B(De^h)^{(p-1)}}{(1-h)^2} \leq \frac{1}{10}\phi w^2 + C\phi \sup_{Q_{2R,T}} |\nabla B|^{\frac{4}{3}} \sup_{Q_{2R,T}} |(De^h)^{(p-1)}|^{\frac{4}{3}}. \qquad (5.1.26)$$

For the fourth term, we have

$$\begin{aligned}
2\phi(N-1)Kw \leq & 2(n-1)K\phi^{\frac{1}{2}}\phi^{\frac{1}{2}}w \\
\leq & \frac{1}{9}\phi w^2 + C(n-1)^2\phi K^2. \qquad (5.1.27)
\end{aligned}$$

For the fifth term, we have

$$\begin{aligned}
2\frac{|\nabla\phi|^2}{\phi}w \leq& 2\phi^{\frac{1}{2}}w\frac{|\nabla\phi|^2}{\phi^{\frac{3}{2}}} \\
\leq& \frac{1}{9}\phi w^2 + C\Big(\frac{|\nabla\phi|^2}{\phi^{\frac{3}{2}}}\Big)^2 \\
\leq& \frac{1}{9}\phi w^2 + \frac{C}{R^4}.
\end{aligned} \tag{5.1.28}$$

Using that

$$\begin{aligned}
\Delta_f\phi =& \Delta\phi - \langle\nabla f, \nabla\phi\rangle \\
=& \phi_r\Delta r + |\nabla r|^2\phi_{rr} - \phi_r\langle\nabla f, \nabla r\rangle \\
=& \phi_r\Delta_f r + |\nabla r|^2\phi_{rr}
\end{aligned} \tag{5.1.29}$$

we can formulate

$$\begin{aligned}
-w\Delta_f\phi \leq& -[\phi_r\Delta_f r + \phi_{rr}|\nabla r|^2]w \\
\leq& [|\phi_{rr} + (\mu_+ + (n-1)K(R-1)|\phi_r|]w \\
\leq& \phi^{\frac{1}{2}}w\frac{|\phi_{rr}|}{\phi^{\frac{1}{2}}} + \phi^{\frac{1}{2}}w[\mu_+ + (n-1)K(R-1)]\frac{|\phi_r|}{\phi^{\frac{1}{2}}} \\
\leq& \frac{1}{6}\phi w^2 + \frac{C}{R^4} + \frac{C(\mu_+)^2}{R^2} + (n-1)K^2.
\end{aligned} \tag{5.1.30}$$

Lastly,

$$\begin{aligned}
w\partial_t(\phi) =& \phi^{\frac{1}{2}}w\frac{\phi_t}{\phi^{\frac{1}{2}}} \\
\leq& \frac{1}{18}\phi w^2 + C\Big(\frac{\phi_t}{\phi^{\frac{1}{2}}}\Big)^2 \\
\leq& \frac{1}{18}\phi w^2 + \frac{C}{(\tau - t_0 + T)^2}.
\end{aligned} \tag{5.1.31}$$

Combining the previous terms and using that $1 - h \geq 1$,

$$\begin{aligned}
\phi w^2 \leq& C_1\phi\sup_{Q_{2R,T}}|\log D + h|^{\frac{4}{3}}\sup_{Q_{2R,T}}|\nabla A|^{\frac{4}{3}} + C_2\phi\sup_{Q_{2R,T}}|(De^h)^{p-1}|^{\frac{4}{3}}\sup_{Q_{2R,T}}|\nabla B|^{\frac{4}{3}} + C_3\phi(A_+)^2 \\
&+ C_4\phi\Big([(p-1)B]_+ + B_+\Big)^2\sup_{Q_{2R,T}}((De^h)^{2(p-1)}) + \frac{C_5}{R^4} + C_6(n-1)^2\phi K^2 + \frac{C_7\sup h^4}{R^4} \\
&+ \frac{C_8}{(\tau - t_0 + T)^2} + \frac{C_9}{R^4} + \frac{C_{10}(\mu_+)^2}{R^2} + (n-1)K^2.
\end{aligned} \tag{5.1.32}$$

The inequality above holds at (x_1, t_1). Thus

$$\begin{aligned}
\phi^2 w^2(x,t) \leq& \phi^2 w^2(x_1,t_1) \leq \phi w^2(x_1,t_1) \\
\leq& C_1 \phi \sup_{Q_{2R,T}} |\log D + h|^{\frac{4}{3}} \sup_{Q_{2R,T}} |\nabla A|^{\frac{4}{3}} + C_2 \phi \sup_{Q_{2R,T}} |(De^h)^{p-1}|^{\frac{4}{3}} \sup_{Q_{2R,T}} |\nabla B|^{\frac{4}{3}} \\
&+ C_3 \phi (A_+)^2 + C_4 \phi \Big([(p-1)B]_+ + B_+\Big)^2 \sup_{Q_{2R,T}} ((De^h)^{2(p-1)}) + \frac{C_5 \phi}{R^4} \\
&+ C_6 \phi (n-1)^2 K^2 + \frac{C_7 \phi \sup h^4}{R^4} + \frac{C_8 \phi}{(\tau - t_0 + T)^2} + \frac{C_9 \phi}{R^4} + \frac{C_{10} \phi (\mu_+)^2}{R^2} + (n-1)K^2.
\end{aligned} \tag{5.1.33}$$

With $\phi(x,\tau) = 1$ when $d(x,x_0) < R$ and $\frac{h^4}{(1-h)^4} \leq 1$, it follows that

$$\begin{aligned}
w(x,\tau) \leq& \phi w(x,\tau) \leq w(x_1,t_1) \\
\leq& C_1 \sup_{Q_{2R,T}} |\log D + h|^{\frac{2}{3}} \sup_{Q_{2R,T}} |\nabla A|^{\frac{2}{3}} + C_2 \sup_{Q_{2R,T}} |(De^h)^{p-1}|^{\frac{2}{3}} \sup_{Q_{2R,T}} |\nabla B|^{\frac{2}{3}} + C_3(A_+) \\
&+ C_4 \Big([(p-1)B]_+ + B_+\Big) \sup_{Q_{2R,T}} ((De^h)^{p-1}) + \frac{C_5}{R^2} + C_6 (n-1)K + \frac{C_7 \sup h^4}{R^4} \\
&+ \frac{C_8}{(\tau - t_0 + T)} + \frac{C_9 \phi}{R^4} + \frac{C_{10}(\mu_+)}{R} + (n-1)^{\frac{1}{2}} K.
\end{aligned} \tag{5.1.34}$$

Transforming back and noting that our choice of τ was arbitrary,

$$\begin{aligned}
\frac{|\nabla u|^2}{u^2} \leq& \Big(1 - \log \frac{u}{D}\Big)^2 \Big[C_1 \sup_{Q_{2R,T}} |\log u|^{\frac{2}{3}} \sup_{Q_{2R,T}} |\nabla A|^{\frac{2}{3}} + C_2 \sup_{Q_{2R,T}} |u^{p-1}|^{\frac{2}{3}} \sup_{Q_{2R,T}} |\nabla B|^{\frac{2}{3}} + C_3(A_+) \\
&+ C_4 \Big([(p-1)B]_+ + B_+\Big) \sup_{Q_{2R,T}} (u^{p-1}) + \frac{C_5}{R^2} + C_6(n-1)K + \frac{C_7 \sup \log(\frac{u}{D})^4}{R^4} \\
&+ \frac{C_8}{(t - t_0 + T)} + \frac{C_9}{R^4} + \frac{C_{10}(\mu_+)}{R} + (n-1)^{\frac{1}{2}} K\Big].
\end{aligned} \tag{5.1.35}$$

□

5.2 Li-Yau gradient estimates for parabolic PDE

Our estimates so far have been space only gradient estimates; it is possible, though, to also establish space-time estimates. These estimates are actually Li-Yau estimates [76]. Recent examples can be found in [118, 129] for the standard Laplace-Beltrami and in [63, 123, 126] for the Witten-Laplacian. To achieve these estimates, a slightly different transform is used. As before, $h = \log \frac{u}{D}$ is the first transform; however, for the second we use $F = t[|\nabla h|^2 - \lambda(h_t - \mathbf{X})]$, where $\mathbf{X}$ is the non-linearity. Since the transform contains the time derivative of u, that element will also appear in our final estimate.

For our Li-Yau estimate, we will use the same scaling and bound as we did in Theorem (5.1.1). One thing also to note is that we are restricted to only using the more general m-Bakry-Emery tensor, as there are obstacles in the way of using the ∞-Bakry-Emery tensor - see [88] (also commented upon in [124, 126]).

Another thing to note is that unlike the Souplet-Zhang estimate for the parabolic equation, the Li-Yau estimate is only defined over positive time. This means unlike the Souplet-Zhang estimate, we will not be able to work with ancient solutions.

Theorem 5.2.1. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f^m \geq -(m-1)K$ for a constant $K \geq 0$, $m < \infty$ in $B(x_0, 2R)$ for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $H_{2R,T} = B(x_0, 2R) \times [0,T]$ such that $0 < u \leq D$ for $D \geq 1$ with $\lambda p > 1$ and $\epsilon \in (0,1)$. Also let*

$$|\nabla A| \leq a_1, \quad |\nabla B| \leq b_1,$$

$$\Delta_f A \geq a_2, \quad \Delta_f B \geq b_2$$

for positive constants a_1, a_2, b_1, b_2. Then the following gradient estimate holds on $H_{R,T}$:

$$\frac{|\nabla u|^2}{u^2} - \lambda\Big(\frac{u_t}{u} - A\log u - Bu^{(p-1)}\Big) \leq \frac{m\lambda^2}{2}\Omega + \left(\frac{m\lambda^2}{2}\Theta\right)^{\frac{1}{2}} \tag{5.2.1}$$

for

$$\Omega = \frac{C^2 + C(m-1)(1+R\sqrt{K})}{R^2} + \frac{1}{t} + \frac{tC\lambda^2}{(\lambda-1)R^2} + (\lambda A)_+ + (\lambda(p-1)B)_+ \sup_{H_{2R,T}} u^{p-1}, \tag{5.2.2}$$

$$\begin{aligned}\Theta =& C\Big(m^8\epsilon^{-1}\lambda^2(\lambda-1)^2\vartheta^4\Big)^{\frac{1}{3}} + \frac{m^2}{2}(1-\epsilon)^{-1}\lambda^2(\lambda-1)^2\tilde{K}^2 \\ &- a_2\Big(\log D + \sup_{H_{2R,T}} \log\frac{u}{D}\Big)_- - b_2\big(\inf_{H_{2R,T}} u^{p-1}\big)_-,\end{aligned} \tag{5.2.3}$$

$$\tilde{K} = (m-1)K - \frac{1}{2}[(\lambda-1)(2A-1)]_- - \frac{1}{2}[(p-1)(\lambda p-1)B]_- \sup_{H_{2R,T}} u^{p-1}, \tag{5.2.4}$$

and

$$\vartheta = a_1\big(\sup_{H_{2R,T}} \log u(\lambda-1) + \lambda\big)_+ + |\lambda p - 1| t b_1 \sup_{H_{2R,T}} u^{p-1}. \tag{5.2.5}$$

As before, we can find a global estimate from the proposition above.

Corollary 5.2.1. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete non-compact smooth metric measure space with $Ric_f^m \geq -(m-1)K$ for $K \geq 0$, $m < \infty$ in M^n and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $H_{M^n,T'} = M^n \times [0, T']$ for $T' \in \mathbb{R}^+ \setminus \infty$. Also let a_1, a_2, b_1, b_2, D, $\tilde{K}$, γ, Θ, λ, p, ϵ be as defined above. Then the following global gradient estimate holds:*

$$\frac{|\nabla u|^2}{u^2} - \lambda\Big(\frac{u_t}{u} - A\log u - Bu^{p-1}\Big) \leq \frac{m\lambda^2}{2}\left(\frac{1}{t} + (\lambda A)_+ + (\lambda(p-1)B)_+ \sup_{H_{M^n,T'}} u^{p-1}\right) + \left(\frac{m\lambda^2}{2}\Theta\right)^{\frac{1}{2}}. \tag{5.2.6}$$

We can also use Theorem (5.2.1) to form an estimate for the case where A and B are constants:

Corollary 5.2.2. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f^m \geq -(m-1)K$ for a constant $K \geq 0$, $m < \infty$ in $B(x_0, 2R)$ for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $H_{2R,T} = B(x_0, 2R) \times [0, T]$, where A and B are non-negative constants, $0 < u \leq D$ for $D \geq 1$, $\lambda, p > 1$, and $\epsilon \in (0, 1)$. Then the following gradient estimate holds on $B(x_0, R) \times (0, T]$:*

$$\frac{|\nabla u|^2}{u^2} - \lambda\Big(\frac{u_t}{u} - A\log u - Bu^{p-1}\Big) \leq \frac{m\lambda}{2}\hat{\Omega} + \left(\frac{m^2\lambda^2}{2}(1-\epsilon)^{-1}(\lambda-1)^2\hat{K}^2\right)^{\frac{1}{2}} \tag{5.2.7}$$

for

$$\hat{\Omega} = \Big[\frac{C^2 + C(m-1)(1+R\sqrt{K})}{R^2} + \frac{1}{t} + \frac{C\lambda^2}{(\lambda-1)R^2} + (\lambda A)_+ + (\lambda(p-1)B)_+ \sup_{H_{2R,T}} u^{p-1}\Big],$$

and

$$\hat{K} = (m-1)K - \frac{1}{2}[(\lambda-1)(2A-1)]_-.$$

Lemma 5.2.1. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f^m \geq -(m-1)K$ for $K \geq 0$, $m < \infty$ in $B(x_0, 2R)$ for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $H_{2R,T} = B(x_0, 2R) \times [0, T]$ such that $0 < u \leq D$ for $D \geq 1$ with $\lambda p > 1$ and $\epsilon \in (0, 1)$. Let $h = \log\frac{u}{D}$ be a non-positive solution to (5.1.3) and $F = t\left[|\nabla h|^2 - \lambda(h_t - A(\log D + h) - B(De^h)^{p-1})\right]$. Then the following inequality holds:*

$$\begin{aligned}
(\Delta_f - \partial_t) F \geq & \frac{2t}{m}\left(|\nabla h|^2 + Ah + Be^{h(p-1)} - h_t\right)^2 - \frac{F}{t} - 2\langle \nabla h, \nabla F\rangle - 2tRic_f^m\langle \nabla h, \nabla h\rangle \\
& + 2t[h(\lambda-1) + \log D(\lambda-1) + \lambda]\nabla A \nabla h + \lambda t(\log D + h)(\Delta_f A) \\
& + (\lambda-1)(2A-1)t|\nabla h|^2 - \lambda AF + 2(\lambda p - 1)t\nabla B\nabla h(De^h)^{p-1} \\
& + (p-1)(\lambda p - 1)tB|\nabla h|^2(De^h)^{p-1} + \lambda t(\Delta_f B)(De^h)^{p-1} \\
& - \lambda(p-1)BF(De^h)^{p-1}.
\end{aligned} \tag{5.2.8}$$

Proof. Let $h = \log\frac{u}{D}$ for $D \geq 1$ such that $0 < u \leq D$, and let F satisfy the Li-Yau transformation

$$F = t\left[|\nabla h|^2 - \lambda(h_t - A(\log D + h) - B(De^h)^{p-1})\right], \tag{5.2.9}$$

where $A = A(x, t)$ and $B = B(x, t)$ are functions of both time and space. By direct computations, we have that

$$\Delta_f F = t\left[\Delta_f(|\nabla h|^2) - \lambda\Delta_f(h_t) + \lambda\Delta_f \mathbf{X}\right] \tag{5.2.10}$$

where $\mathbf{X} = Ah + Be^{h(p-1)}$. Also,

$$\begin{aligned} F_t =&|\nabla h|^2 - \lambda\Big(h_t - A(\log D + h) - (De^h)^{p-1}\Big) + t\Big[|\nabla h|_t^2 - \lambda\Big(h_{tt} - A_t(\log D + h) - Ah_t \\ &- B_t(De^h)^{p-1} - B(p-1)h_t(De^h)^{p-1}\Big)\Big], \end{aligned} \tag{5.2.11}$$

and

$$\Delta_f h_t = h_{tt} - (|\nabla h|^2)_t - A_t(\log D + h) - Ah_t - B_t(De^h)^{p-1} - B(p-1)h_t(De^h)^{p-1}, \tag{5.2.12}$$

and finally,

$$\begin{aligned} \Delta_f h =& -|\nabla h|^2 + h_t + \mathbf{X} \\ =& -\frac{F}{\lambda t} - \left(1 - \frac{1}{\lambda}\right)|\nabla h|^2. \end{aligned} \tag{5.2.13}$$

Thus by (6.3.5), (5.2.11), (5.2.12), (5.2.13), and (5.1.3)

$$\begin{aligned} -\lambda\Delta_f h_t + 2\langle\nabla h, \nabla\Delta_f h\rangle =&\frac{F_t}{t} - \frac{F}{t^2} + 2(\lambda-1)\nabla h\nabla h_t + 2\langle\nabla h, \nabla\Delta_f h\rangle \\ =&\frac{F_t}{t} - \frac{F}{t^2} - \frac{2}{t}\langle\nabla h, \nabla F\rangle + 2(\lambda-1)\langle\nabla h, \nabla\mathbf{X}\rangle. \end{aligned} \tag{5.2.14}$$

Substituting these into (5.2.10) and using the Bochner formula (see (1.4.8)),

$$\begin{aligned} (\Delta_f - \partial_t)F \geq& t\Big[\frac{2(\Delta_f h)^2}{m} + 2\langle\nabla h, \nabla\Delta_f h\rangle + 2Ric_f^m(\nabla h, \nabla h) - \lambda\Delta_f h_t + \lambda\Delta_f\mathbf{X}\Big] \\ \geq& -\frac{F}{t} + \frac{2t}{m}(|\nabla h|^2 + \mathbf{X} - h_t)^2 + 2tRic_f^m(\nabla h, \nabla h) - 2\langle\nabla h, \nabla F\rangle \\ &+ 2(\lambda-1)t\langle\nabla h, \nabla\mathbf{X}\rangle + \lambda t\Delta_f\mathbf{X}. \end{aligned} \tag{5.2.15}$$

To find an expanded expression usable for the last two terms of (5.2.15) we first compute the gradient and Laplacian terms for the terms within $\mathbf{X}$:

$$\begin{aligned} \nabla(A(\log D + h)) =&\nabla A(\log D + h) + A\nabla h \\ \Delta(A(\log D + h)) =&\Delta A(\log D + h) + 2\nabla A\nabla h + A\Delta h \\ \Delta_f(A(\log D + h)) =&(\Delta_f A)(\log D + h) + (\Delta_f h)A + 2\nabla A\nabla h \end{aligned} \tag{5.2.16}$$

and

$$\begin{aligned}
\nabla(B(De^h)^{p-1}) =& \nabla B(De^h)^{p-1} + B(p-1)\nabla h(De^h)^{p-1} \\
\Delta(B(De^h)^{p-1}) =& \Delta B(De^h)^{p-1} + 2(p-1)\nabla B\nabla h(De^h)^{p-1} + B(p-1)\Delta h(De^h)^{p-1} \\
&+ B(p-1)^2|\nabla h|^2(De^h)^{p-1} \\
\Delta_f(B(De^h)^{p-1}) =& (\Delta_f B)(De^h)^{p-1} + (\Delta_f h)B(p-1)(De^h)^{p-1} \\
&+ 2(p-1)\nabla B\nabla h(De^h)^{p-1} + B(p-1)^2|\nabla h|^2(De^h)^{p-1}. \qquad (5.2.17)
\end{aligned}$$

Using these we compute an exact form for the last two terms:

$$\begin{aligned}
&2(\lambda-1)t\langle\nabla h, \nabla(A(\log D + h))\rangle + \lambda t\Delta_f(A(\log D + h)) \\
&= 2t[h(\lambda-1) + \log D(\lambda-1) + \lambda]\nabla A\nabla h + \lambda t(\log D + h)(\Delta_f A) - \lambda AF \\
&\quad + t(\lambda-1)(2A-1)|\nabla h|^2 \qquad (5.2.18)
\end{aligned}$$

and

$$\begin{aligned}
&2(\lambda-1)t\langle\nabla h, \nabla(B(De^h)^{p-1})\rangle + \lambda t\Delta_f(B(De^h)^{p-1}) \\
&= 2(\lambda p-1)t\nabla B\nabla h(De^h)^{p-1} + (p-1)(\lambda p-1)tB|\nabla h|^2(De^h)^{p-1} \\
&\quad + \lambda t(\Delta_f B)(De^h)^{p-1} - \lambda(p-1)BF(De^h)^{p-1}. \qquad (5.2.19)
\end{aligned}$$

Thus we can rewrite (5.2.15) as

$$\begin{aligned}
(\Delta_f - \partial_t)F \geq& \frac{2t}{m}\left(|\nabla h|^2 + \mathbf{X} - h_t\right)^2 - \frac{F}{t} - 2\langle\nabla h, \nabla F\rangle - 2tRic_f^m\langle\nabla h, \nabla h\rangle \\
&+ 2t[h(\lambda-1) + \log D(\lambda-1) + \lambda]\nabla A\nabla h + \lambda t(\log D + h)(\Delta_f A) \\
&+ (\lambda-1)(2A-1)t|\nabla h|^2 - \lambda AF + 2(\lambda p-1)t\nabla B\nabla h(De^h)^{p-1} \\
&+ (p-1)(\lambda p-1)tB|\nabla h|^2(De^h)^{p-1} + \lambda t(\Delta_f B)(De^h)^{p-1} \\
&- \lambda(p-1)BF(De^h)^{p-1}. \qquad (5.2.20)
\end{aligned}$$

□

Proof of Theorem 5.2.1. We now introduce a cutoff function. This will be as defined as in the elliptical case; that is, we let $\tilde{\phi}(s) \in C^2(\mathbb{R}^+)$ be such that $\tilde{\phi}(s) = 1$ for $s \in [0, R]$ and $\tilde{\phi(s)} = 0$ for $s \in [2R, \infty)$. The function's output is limited between $\tilde{\phi(s)} \in [0, 1]$. From [25] we also know that the function $\tilde{\phi}(s)$ obeys the following useful inequalities:

$$0 \geq \frac{|\nabla\tilde{\phi}(s)|}{\tilde{\phi}^{\frac{1}{2}}(s)} \geq -\frac{C}{R} \qquad (5.2.21)$$

and

$$\tilde{\phi}''(s) \leq \frac{C}{R^2}$$

for some positive constant C. We also let $r(x) = d(x, x_0)$, where $d(x, x_0)$ is the distance function. We then define $\phi = \tilde{\phi}(r(x))$. We then also get

$$\Delta_f r \leq (m + n - 1)\sqrt{K} \coth(\sqrt{K}R) \tag{5.2.22}$$

which gives

$$\begin{aligned} \Delta_f \phi &= \frac{\phi' \Delta_f r}{R} + \frac{\phi'' |\nabla r|^2}{R^2} \\ &\geq -\frac{C(m + n - 1)\sqrt{K} \coth(\sqrt{K}R)}{R} - \frac{C}{R^2} \\ &\geq -\frac{C(m - 1)(1 + R\sqrt{K})}{R^2}. \end{aligned} \tag{5.2.23}$$

Let $\tau \in (0, T]$. If $\phi F \leq 0$, then the proof is trivial, so we assume that $\max \phi F \geq 0$ for $(x, t) \in H_{2R,T}$. Now let (x_1, t_1) for $x_1 \in B(x_0, 2R)$ and $0 \leq t_1 \leq \tau$ be the maximum point of ϕF. We then get

$$\nabla(\phi F) = \phi \nabla F + \nabla \phi F = 0, \qquad F_t \geq 0, \qquad \Delta_f(\phi F) \leq 0.$$

Then

$$\Big(\Delta - \partial_t\Big)(\phi F) = \phi\Big(\Delta - \partial_t\Big)F + 2\nabla\phi\nabla F + F\Big(\Delta - \partial_t\Big)\phi. \tag{5.2.24}$$

As we are looking at a Li-Yau estimate, we have $\phi_t = 0$. Therefore, using the above, we obtain

$$\begin{aligned} 0 \geq & F\Delta_f\phi + 2\langle\nabla\phi, \nabla F\rangle + \phi\Big(\Delta_f - \partial_t\Big)F \\ \geq & F\Delta_f\phi - 2F\frac{|\nabla\phi|^2}{\phi^2} + \phi\Big[\frac{2t}{m}\left(|\nabla h|^2 + \mathbf{X} - h_t\right)^2 - \frac{F}{t} - 2\langle\nabla h, \nabla F\rangle \\ & - 2t(m-1)K|\nabla h|^2 + 2t[h(\lambda - 1) + \log D(\lambda - 1) + \lambda]\nabla A\nabla h - \lambda AF \\ & + \lambda t(\log D + h)(\Delta_f A) + (\lambda - 1)(2A - 1)t|\nabla h|^2 + 2(\lambda p - 1)t\nabla B\nabla h(De^h)^{p-1} \\ & + (p-1)(\lambda p - 1)tB|\nabla h|^2(De^h)^{p-1} + \lambda t(\Delta_f B)(De^h)^{p-1} \\ & - \lambda(p-1)BF(De^h)^{p-1}\Big]. \end{aligned} \tag{5.2.25}$$

As $\Delta_f\phi$ was stated previously for the m-Bakry-Emery tensor, we can say:

$$\begin{aligned} 0 \geq & F\left[-\frac{C(m-1)(1 + R\sqrt{K})}{R^2}\right] - 2F\frac{C^2}{R^2} + \phi\Big[\frac{2t}{m}\left(|\nabla h|^2 + \mathbf{X} - h_t\right)^2 - \frac{F}{t} - 2\langle\nabla h, \nabla F\rangle \\ & - 2t(m-1)K|\nabla h|^2 + 2t[h(\lambda - 1) + \log D(\lambda - 1) + \lambda]\nabla A\nabla h + \lambda t(\log D + h)(\Delta_f A) \\ & - \lambda AF + (\lambda - 1)(2A - 1)t|\nabla h|^2 + 2(\lambda p - 1)t\nabla B\nabla h(De^h)^{p-1} \\ & + (p-1)(\lambda p - 1)tB|\nabla h|^2(De^h)^{p-1} + \lambda t(\Delta_f B)(De^h)^{p-1} - \lambda(p-1)BF(De^h)^{p-1}\Big]. \end{aligned} \tag{5.2.26}$$

Multiplying by $t_1\phi$ and rearranging:

$$
\begin{aligned}
0 \geq & - t_1\phi F\left[\frac{C^2 + C(m-1)(1+R\sqrt{K})}{R^2} + \frac{1}{t_1}\right] - Ct_1R^{-1}F|\nabla h|\phi^{\frac{3}{2}} \\
& - \phi t_1 F\Big[\lambda A + \lambda(p-1)B(De^h)^{p-1}\Big] + \frac{2t_1^2\phi^2}{m}\Big[\big(|\nabla h|^2 + \mathbf{X} - h_t\big)^2 \\
& + \frac{m}{2}\Big((\lambda-1)(2A-1)t + (p-1)(\lambda p-1)tB(De^h)^{p-1} - 2(m-1)K\Big)|\nabla h|^2\Big] \\
& - 2t_1^2\phi\Big[[h(\lambda-1) + \log D(\lambda-1) + \lambda]\nabla A + (\lambda p - 1)t_1\nabla B(De^h)^{p-1}\Big]|\nabla h| \\
& + \phi^2\lambda t_1^2\Big[(\log D + h)(\Delta_f A) + (\Delta_f B)(De^h)^{p-1}\Big]. \qquad (5.2.27)
\end{aligned}
$$

Using the assumptions

$$
\begin{aligned}
&|\nabla A| \leq a_1, \quad |\nabla B| \leq b_1, \\
&\Delta_f A \geq a_2, \quad \Delta_f B \geq b_2
\end{aligned}
$$

for constants a_1, a_2, b_1, and b_2, we can write

$$
\begin{aligned}
0 \geq & - t_1\phi F\left[\frac{C^2 + C(m-1)(1+R\sqrt{K})}{R^2} + \frac{1}{t_1}\right] - Ct_1R^{-1}F|\nabla h|\phi^{\frac{3}{2}} \\
& - t_1\phi F\Big[(\lambda A)_+ + (\lambda(p-1)B)_+ \sup_{H_{2R,T}} u^{p-1}\Big] + \frac{2t_1^2\phi^2}{m}\Big[\big(|\nabla h|^2 + \mathbf{X} - h_t\big)^2 \\
& + \frac{m}{2}\Big(t[(\lambda-1)(2A-1)]_- + t[(p-1)(\lambda p-1)B]_- \sup_{H_{2R,T}} u^{p-1} - 2(m-1)K\Big)|\nabla h|^2\Big] \\
& - 2t_1^2\phi\Big[a_1(\sup_{H_{2R,T}} \log u(\lambda-1) + \lambda)_+ + |\lambda p - 1|t_1 b_1 \sup_{H_{2R,T}} u^{p-1}\Big]|\nabla h| \\
& + \phi^2\lambda t_1^2\Big[a_2(\log D + \sup_{H_{2R,T}} \log\frac{u}{D})_- + b_2\big(\inf_{H_{2R,T}} u^{p-1}\big)_-\Big]. \qquad (5.2.28)
\end{aligned}
$$

Using the work of Yau, we will now transform the second and third terms in the above inequality and make use of the calculations on page 161 and 162 of [76]. This allows us to form and solve a quadratic in (ϕF). See also [126]. First, we let

$$
y = \phi|\nabla h|^2, \qquad z = \phi\big(h_t - A\log u - Bu^{p-1}\big) \qquad (5.2.29)
$$

to give the third and fourth terms as

$$
\begin{aligned}
&(|\nabla h|^2 + \mathbf{X} - h_t)^2 - Ct_1 mR^{-1}F|\nabla h|\phi^{\frac{3}{2}} - \phi m\tilde{K}|\nabla h|^2 - \phi^{\frac{1}{2}}m|\nabla h|\vartheta \\
&\qquad = (y-z)^2 - \frac{Cm}{R}y^{\frac{1}{2}}(y - \lambda z) - (m)\tilde{K}y - (m)\vartheta y^{\frac{1}{2}} \qquad (5.2.30)
\end{aligned}
$$

for $\tilde{K}$ and ϑ as defined above. Next, we use Li and Yau's calculations:

$$\begin{aligned}
\mathbb{V} =&(y-z)^2 - \frac{Cm}{R}y^{\frac{1}{2}}(y-\lambda z) - m\tilde{K}y - m\vartheta y^{\frac{1}{2}} \\
=&(1-\epsilon-\delta)y^2 - (2-\epsilon\lambda)yz + z^2 + \left(\epsilon y - \frac{Cm}{R}y^{\frac{1}{2}}\right)(y-\lambda z) + \delta y^2 - m\tilde{K}y \\
&- m\vartheta y^{\frac{1}{2}} \\
=&\Big(\lambda - \frac{\epsilon}{2}\Big)(y-\lambda z)^2 + \Big(1-\epsilon-\delta-\lambda^{-1}+\frac{\epsilon}{2}\Big)y^2 + \Big(1-\lambda+\frac{\epsilon}{2}\lambda^2\Big)z^2 + \left(\epsilon y - \lambda\frac{C}{R}y^{\frac{1}{2}}\right)(y-\lambda z) \\
&+ \delta y^2 - m\tilde{K}y - m\vartheta y^{\frac{1}{2}}.
\end{aligned} \tag{5.2.31}$$

By a choice of $\delta = (\lambda^{-1}-1)^2$ and $\epsilon = 2 - 2\lambda^{-1} - 2(\lambda^{-1}-1)^2$ we can state that

$$\mathbb{V} \geq \lambda^{-2}(y-\lambda z)^2 - \frac{C}{R^2}\lambda^2(\lambda-1)^{-1}(y-\lambda z) + \lambda^{-2}(\lambda-1)^2y^2 - m\tilde{K}y - m(\lambda-1)\vartheta y^{\frac{1}{2}}, \tag{5.2.32}$$

where we have used

$$2\lambda^{-2}(\lambda-1)y - \frac{Cm}{R}y^{\frac{1}{2}} \geq -\frac{C}{R^2}\lambda^2(\lambda-1)^{-1}$$

for a different constant C.

Focusing on the last three terms:

$$\begin{aligned}
&\lambda^{-2}(\lambda-1)^2y^2 - m\tilde{K}y - m(\lambda-1)\vartheta y^{\frac{1}{2}} \\
&\geq \lambda^{-2}(\lambda-1)^2y^2 - (1-\epsilon)\lambda^{-2}(\lambda-1)^2y^2 - \frac{m^2}{2}(1-\epsilon)^{-1}\lambda^2(\lambda-1)^{-2}\tilde{K}^2 \\
&\quad - m(\lambda-1)\vartheta y^{\frac{1}{2}} \\
&\geq \epsilon\lambda^{-2}(\lambda-1)^2y^2 - \frac{1}{4}\Big(\epsilon^{\frac{1}{4}}\lambda^{-\frac{1}{2}}(\lambda-1)6\frac{1}{2}y^{\frac{1}{2}}\Big)^4 - \frac{3}{4}\Big(\epsilon^{-\frac{1}{4}}m^2\lambda^{\frac{1}{2}}(\lambda-1)^{\frac{1}{2}}\vartheta\Big)^{\frac{4}{3}} \\
&\quad - \frac{m^2}{2}(1-\epsilon)^{-1}\lambda^2(\lambda-1)^2\tilde{K}^2 \\
&\geq -\frac{m^2}{2}(1-\epsilon)^{-1}\lambda^2(\lambda-1)^2\tilde{K}^2 - C\Big(m^8\epsilon^{-1}\lambda^2(\lambda-1)^2\vartheta^4\Big)^{\frac{1}{3}}
\end{aligned} \tag{5.2.33}$$

for any $\epsilon \in (0,1)$.

So we rewrite $\mathbb{V}$ as

$$\begin{aligned}
\mathbb{V} \geq& \lambda^{-2}(\phi F)^2 - \frac{C}{R^2}\lambda^2(\lambda-1)^{-1}(\phi F) - \frac{m^2}{2}(1-\epsilon)^{-1}\lambda^2(\lambda-1)^2\tilde{K}^2 \\
&- C\Big(m^8\epsilon^{-1}\lambda^2(\lambda-1)^2\vartheta^4\Big)^{\frac{1}{3}}.
\end{aligned} \tag{5.2.34}$$

Hence, we now have a quadratic in ϕF that we can easily solve:

$$0 \geq \Psi(\phi F)^2 - t_1\Omega(\phi F) - t_1^2\Theta \tag{5.2.35}$$

for

$$\Psi = \frac{2}{m\lambda^2}, \tag{5.2.36}$$

$$\Omega = \frac{C^2 + C(m-1)(1+R\sqrt{K})}{R^2} + \frac{1}{t_1} + \frac{t_1 C\lambda^2}{(\lambda-1)R^2} + (\lambda A)_+ + (\lambda(p-1)B)_+ \sup_{H_{2R,T}} u^{p-1}, \tag{5.2.37}$$

and

$$\begin{aligned}\Theta =& C\Big(m^8\epsilon^{-1}\lambda^2(\lambda-1)^2\vartheta^4\Big)^{\frac{1}{3}} + \frac{m^2}{2}(1-\epsilon)^{-1}\lambda^2(\lambda-1)^2\tilde{K}^2 \\ &- a_2\Big(\log D + \sup_{H_{2R,T}} \log\frac{u}{D}\Big)_- - b_2\big(\inf_{H_{2R,T}} u^{p-1}\big)_-.\end{aligned} \tag{5.2.38}$$

Solving this gives the inequality

$$\begin{aligned}\phi F \le& \frac{t}{2\Psi}\Big(\Omega + \Big(\Omega^2 + 4\Psi\Theta\Big)^{\frac{1}{2}}\Big) \\ \le& \frac{t}{\Psi}\Omega + \frac{t}{2\Psi}\Big(4\Psi\Theta\Big)^{\frac{1}{2}}.\end{aligned} \tag{5.2.39}$$

Now we restrict our estimate onto $B(x_0, R) \times [0, \tau]$. This means our cutoff function $\phi \equiv 1$. Since we chose $t \in [0, \tau]$ then

$$\sup_{B(x_0,R)} \phi F(x, \tau) \le \phi F(x_1, t_1). \tag{5.2.40}$$

As our choice of τ was arbitrary, we transform back, and this completes the proof.

□

Chapter 6

Gradient estimates involving time-evolving metric

In the previous estimates, the metric is stationary, and so has no dependency on time. However, we can produce estimates for a time-evolving metric $g(t)$.

In chapter 1, we discussed two of these metrics: Ricci flow and Yamabe flow. These flows can be present on Riemannian manifolds. Another flow, which was briefly mentioned but not explored, is Perelman-Ricci flow. This is a flow specific to a smooth metric measure space and is analogous to Ricci flow.

Gradient estimates under time-evolving metrics are a new area of research. Most of the current focus has been on Li-Yau estimates under either Ricci or generalised geometric flow. In [7], Abolarinwa and Taheri studied the Souplet-Zhang estimates for a parabolic equation with a powered logarithmic non-linearity on manifolds with time-evolving metric. Here, they specifically looked at (K, m) Perelman-Ricci flow of the form

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric_f^m(g) = -Kg, \tag{6.0.1}$$

$$\frac{\partial f}{\partial t} - \frac{1}{2}\operatorname{Tr}\left(\frac{\partial g}{\partial t}\right) = 0, \tag{6.0.2}$$

where Tr is the trace and K and m are fixed constants for $m \geq N$. For $(0, m)$ super Perelman-Ricci flow satisfies

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric_f^m(g) \geq 0. \tag{6.0.3}$$

The key step that changes between finding Souplet-Zhang gradient estimates for stationary and time-evolving metrics is in the calculation of the time derivative of $|\nabla h|^2$:

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla h|^2 =& -\frac{\partial g}{\partial t}\nabla h \nabla h + 2\nabla h \nabla h_t \\ \leq& 2Ric_f^m(\nabla h, \nabla h) + 2\nabla h \nabla h_t + 2K|\nabla h|^2 \end{aligned} \tag{6.0.4}$$

As well as [7], Taheri has also produced more results for a general non-linearity $F(u)$ under Perelman-Ricci flow: see [112, 113].

In this chapter, we find a Souplet-Zhang estimate under Perelman-Ricci flow, before moving on to finding Li-Yau estimates under generalised geometric flow. We then look at the specific cases for Ricci, Yamabe, and Perelman-Ricci flow.

6.1 Souplet-Zhang gradient estimate for a time-evolving metric under Perelman-Ricci flow

Now we will produce a Li-Yau style estimate for (5.1.1) under Perelman-Ricci flow. This will be for a bounded solution $u \leq D$. It is preferable not to have this restriction, so we produce a global estimate afterward, similar to that found in [7].

Theorem 6.1.1. *Let $(M^n, g(t), f(t))_{t\in[0,T]}$ be a complete solution to (K, m) Perelman-Ricci flow, (6.0.1). Let u be a bounded solution in $H_{2R,T} = B(x_0, R) \times [0, T]$. Assuming the other conditions as Theorem 5.1.1 then the following estimate holds on $H_{R,T}$:*

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} \leq& \left(1 - \log\frac{u}{D}\right)^2 \Big[C_1 \sup_{H_{2R,T}} |\log u|^{\frac{2}{3}} \sup_{H_{2R,T}} |\nabla A|^{\frac{2}{3}} + C_2 \sup_{H_{2R,T}} |u^{p-1}|^{\frac{2}{3}} \sup_{H_{2R,T}} |\nabla B|^{\frac{2}{3}} + C_3(A_+) \\ &+ C_4\Big([(p-1)B]_+ + B_+\Big) \sup_{H_{2R,T}} (u^{p-1}) + \frac{C_5}{R^2} + \frac{C_6}{(t - t_0 + T)} + \frac{C_7(\mu_+)}{R} \\ &+ (n-1)^{\frac{1}{2}} K + C_9 K\Big] \end{aligned} \tag{6.1.1}$$

for $t \neq t_0 - T$

Proof. We follow a similar proof as (5.1.1). We calculate as above:

$$\Delta_f w = \frac{2|h_{ij}|^2}{(1-h)^2} + \frac{2h_i(\Delta_f h)_i}{(1-h)^2} + \frac{2(R_{ij} + f_{ij})h_i h_j}{(1-h)^2} + \frac{8h_i h_j h_{ij}}{(1-h)^3} + \frac{2h_i^2 \Delta_f h}{(1-h)^3} + \frac{6h_i^4}{(1-h)^4} \tag{6.1.2}$$

and

$$
\begin{aligned}
w_t =& -\frac{\partial_t g h_i h_j}{(1-h)^2} + \frac{2h_i(h_t)_i}{(1-h)^2} + \frac{2h_i^2 h_t}{(1-h)^3} \\
=& \frac{2(R_{ij}+f_{ij})}{(1-h)^2} + \frac{2kh_i^2}{(1-h)^2} + \frac{2h_i(h_t)_i}{(1-h)^2} + \frac{2h_i^2 h_t}{(1-h)^3} \\
=& \frac{2(R_{ij}+f_{ij})}{(1-h)^2} + \frac{2kh_i^2}{(1-h)^2} + \frac{2\nabla h(\nabla \Delta_f h + \mathbf{X})}{(1-h)^2} + \frac{4h_i h_j h_{ij}}{(1-h)^2} \\
&+ \frac{2|\nabla h|^4 + 2|\nabla h|^2(\Delta_f h + Ah + Be^{h(p-1)})}{(1-h)^3}.
\end{aligned}
\tag{6.1.3}
$$

Hence we can write the parabolic equation

$$
\begin{aligned}
\Big(\Delta_f - \partial_t\Big) w \geq& -2w_j h_j + \frac{2w_j h_j}{(1-h)} + 2(1-h)w^2 - \frac{2\mathbf{X}\nabla h}{(1-h)^2} \\
&- \frac{2h_i^2((\log D + h)A + B(De^h)^{p-1})}{(1-h)^3} - \frac{2kh_i^2}{(1-h)^2}
\end{aligned}
\tag{6.1.4}
$$

where we have used (5.1.12), (5.1.13), and (5.1.15) from above.

The rest of the proof follows as with a stationary metric, with the addition of the following inequality during the control stage of the proof:

$$
\begin{aligned}
\phi K w \leq& \phi^{\frac{1}{2}} K \phi^{\frac{1}{2}} w \\
\leq& \frac{1}{20}\phi w^2 + C\phi K^2.
\end{aligned}
$$

□

In our previous estimates, we have bounded $u \leq D$, so we are always able to expand our local estimate to a global one by sending $R \to \infty$. Abolarinwa and Taheri showed in [7] that we can compute a global estimate for the $(0,m)$ super Perelman-Ricci flow without the need for this bounding. This is done by creating an equation in $\Psi = t|\nabla u|^2 + \nu u^2$ upon which we apply the maximum principle.

Proposition 6.1.1. *Let $(M^n, g(t), f(t))_{t\in[0,T]}$ be a complete compact solution to $(0,m)$ super Perelman-Ricci flow, (6.0.3). Let u be a positive solution to (5.1.1) where A and B are non-positive constants and $p \geq 0$. Then if $u \geq 1$,*

$$
|u(x,t)|^2 + 2t|\nabla u(x,t)|^2 \leq \sup_{M^n} |u(x,0)|^2. \tag{6.1.5}
$$

Proof. Through basic calculations,

$$
\begin{aligned}
(|\nabla u|^2)_t \leq& 2Ric_f(\nabla u, \nabla u)2k|\nabla u|^2 + 2\nabla u \nabla(\Delta_f u + Au\log u + Bu^p) \\
\leq& 2Ric_f(\nabla u, \nabla u)2k|\nabla u|^2 + 2\nabla u\nabla \Delta_f u + 2\nabla u \nabla(Au\log u + Bu^p) \\
\leq& 2|\nabla u|^2 + 2\nabla u\nabla(Au\log u + Bu^p) + \Delta_f(|\nabla u|^2) - 2|\nabla^2 u|^2.
\end{aligned}
\tag{6.1.6}
$$

Above, we have used the Bochner formula to produce the third line. Using $\Delta_f(u^2) = 2|\nabla u|^2 + 2u\Delta_f u$:

$$
\begin{aligned}
(u^2)_t =& 2uu_t \\
=& 2u(\Delta_f u + (Au\log u + Bu^p)) \\
=& 2u(Au\log u + Bu^p) + \Delta_f(u^2) - 2|\nabla u|^2.
\end{aligned} \tag{6.1.7}
$$

Let $\Psi = t|\nabla u|^2 + \eta u^2$. Then we can compute:

$$
\begin{aligned}
\Psi_t \leq& t(2|\nabla u|^2 + 2\nabla u\nabla(Au\log u + Bu^p) + \Delta_f(|\nabla u|^2) - 2|\nabla^2 u|^2)|\nabla u|^2 \\
&+ \eta(2u(A\log u + bu^p) + \Delta_f(u^2) - 2|\nabla u|^2) \\
\leq& \Delta_f\Psi + (1 + 2kt - 2\eta)|\nabla u|^2 + 2t\nabla u\nabla(Au\log u + Bu^p) \\
&+ 2\eta u(Au\log u + Bu^p).
\end{aligned} \tag{6.1.8}
$$

Choosing $\eta = \frac{1}{2}$ and $k = 0$,

$$
\begin{aligned}
\Psi_t \leq& \Delta_f\Psi + 2t\nabla u\nabla(Au\log u + Bu^p) + u(Au\log u + Bu^p) \\
\leq& \Delta_f\Psi + 2t(A|\nabla u|^2\log u + A|\nabla u|^2 + Bp|\nabla u|^2 u^{p-1}) + u(Au\log u + Bu^p).
\end{aligned} \tag{6.1.9}
$$

If $u \geq 1$, then the logarithmic terms are positive. Hence with $p \geq 0$ and $A, B \leq 0$,

$$
\Psi_t \leq \Delta_f\Psi \tag{6.1.10}
$$

and we apply the maximum principle for Ψ. □

6.2 Li-Yau gradient estimate for geometric flow on a Riemannian manifold

Li-Yau estimates for time-evolving metrics for parabolic PDEs are also obtainable. Sun found an estimate for the heat equation in [108], and, later, Abolarinwa found an estimate for the heat equation with non-linearity of a function of time and space [3]. These estimates require that the flow is bounded. For this we set

$$
Ric \geq -C_1 g, \tag{6.2.1}
$$

$$
-C_2 g \leq \mathcal{R} \leq C_3 g, \tag{6.2.2}
$$

$$
|\nabla\mathcal{R}| \leq C_4, \tag{6.2.3}
$$

for constants C_n, where $\mathcal{R}$ is a $(0,2)$ tensor produced by the generalised geometric flow

$$\frac{\partial}{\partial t} g_{ij}(x,t) = 2\mathcal{R}_{ij}(x,t).$$

Even as the metric evolves, these bounds are preserved.

Remark 6.2.1. *The estimates where we have generalised geometric flow as well as Ricci and Yamabe flow are all done on a manifold opposed to a smooth metric measure space.*

Theorem 6.2.1. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the generalised geometric flow with Ric $\geq -C_1 g$ for $C_1 \geq 0$, $-C_2 g \leq \mathcal{R} \leq C_3 g$ and $|\nabla \mathcal{R}| \leq C_4$ for $C_1, C_2, C_3, C_4 \geq 0$, $m < \infty$ in $B(x_0, 2R)$ for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $H_{2R,T} = B(x_0, 2R) \times [0,T]$ such that $0 < u \leq D$ for $D \geq 1$. Also let*

$$|\nabla A| \geq a_1, \quad |\nabla B| \geq b_1,$$
$$\Delta A \leq a_2, \quad \Delta B \leq b_2$$

for positive constants a_1, a_2, b_1, b_2. Then the following gradient estimate holds on $B(x_0, R) \times (0,T]$ with $\lambda, p > 1$ and $\epsilon \in (0,1)$:

$$\frac{|\nabla u|^2}{u^2} - \lambda\Big(\frac{u_t}{u} - A\log u - Bu^{p-1}\Big) \leq \frac{n\lambda^2}{2}\tilde{\Omega} + \left(\frac{n\lambda^2}{2}\tilde{\Theta}\right)^{\frac{1}{2}} \tag{6.2.4}$$

for

$$\tilde{\Omega} = \frac{C^2 + Cn(1 + R\sqrt{K})}{R^2} + \frac{1}{t} + A_+ + [B(p-1)]_+ \sup_{H_{2R,T}} u^{p-1} + \frac{tC\lambda^2}{(\lambda-1)R^2}, \tag{6.2.5}$$

$$\begin{aligned}\tilde{\Theta} =& C\Big(n^8\epsilon^{-1}\lambda^2(\lambda-1)^2\tilde{\vartheta}^4\Big)^{\frac{1}{3}} + \frac{n^2}{2}(1-\epsilon)^{-1}\lambda^2(\lambda-1)^2\mathfrak{S}^2 + \lambda^2 n(C_2+C_3)^2 \\ &+ \lambda t a_2 \sup_{H_{2R,T}} (\log u)_+ + \lambda t b_2 \sup_{H_{2R,T}} u^{p-1},\end{aligned} \tag{6.2.6}$$

$$\begin{aligned}\mathfrak{S} =& 2n\Big((\lambda-1)C_3 + C_1 n\Big) + n\lambda\Big(1+\frac{1}{\lambda}\Big) + n\lambda\Big(1+\frac{1}{\lambda}\Big)[B(p-1)]_+ \sup_{H_{2R,T}} u^{(p-1)} \\ &+ n\lambda t B_+(p-1)^2 \sup_{H_{2R,T}} u^{p-1},\end{aligned} \tag{6.2.7}$$

and

$$\tilde{\vartheta} = 3\sqrt{n}C_4 + 2\lambda t a_1 + \lambda t b_1(p-1) \sup_{H_{2R,T}} u^{p-1}.$$

As generalised geometric flow is a general flow, Theorem 6.2.1 will later be used for specific choices of flow, including Ricci and Yamabe flow. Additionally, for a smooth metric measure space with $f(t)$ which is constant in x, this estimate can also be used for Perelman-Ricci flow.

Lemma 6.2.1. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the generalised geometric flow with*

$Ric \geq -nC_1 g$ for $C_1 \geq 0$, $-C_2 g \leq \mathcal{R} \leq C_3 g$ and $|\nabla \mathcal{R}| \leq C_4$ for $C_1, C_2, C_3, C_4 \geq 0$, $m < \infty$ in $B(x_0, 2R)$ for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $H_{2R,T} = B(x_0, 2R) \times [0, T]$ such that $0 < u \leq D$ for $D \geq 1$. Let $h = \log \frac{u}{D}$ be a non-positive solution to (5.1.3) and $F = t\left[|\nabla h|^2 - \lambda(h_t - A(\log D + h) - B(De^h)^{p-1})\right]$. Then the following inequality holds:

$$\begin{aligned}
\Big(\Delta - \frac{\partial}{\partial t}\Big)F \geq & -2\langle \nabla h, \nabla F\rangle + \frac{t}{n}(|\nabla h|^2 - h_t + \mathbf{X})^2 - 2((\lambda-1)C_3 + C_1 n)t|\nabla h|^2 \\
& - 3\sqrt{n}tC_4|\nabla h| + \lambda t\Bigg(\Big(1 + \frac{1}{\lambda}\Big)A|\nabla h|^2 - \frac{F}{\lambda t}A + \Big(1+\frac{1}{\lambda}\Big)B(p-1)|\nabla h|^2(De^h)^{p-1} \\
& - \frac{F}{\lambda t}B(p-1)(De^h)^{p-1} + \Delta A(\log D + h) + 2\nabla A \nabla h + \Delta B(De^h)^{p-1} \\
& + B(p-1)^2|\nabla h|^2(De^h)^{p-1} + \nabla B(p-1)|\nabla h|(De^h)^{p-1}\Bigg) \\
& - (|\nabla h|^2 - \lambda(h_t - \mathbf{X})) - \lambda^2 n(C_2 + C_3)^2.
\end{aligned}$$

Proof. Let $h = \log \frac{u}{D}$ for $D \geq 1$ such that $0 < u \leq D$, and let F satisfy the Li-Yau transformation

$$F = t\left[|\nabla h|^2 - \lambda(h_t - A(\log D + h) - B(De^h)^{p-1})\right], \tag{6.2.8}$$

where $A = A(x,t)$ and $B = B(x,t)$ are functions of both time and space. By direct computations,

$$\begin{aligned}
2t\langle \nabla \Delta h, \nabla h\rangle =& 2t\langle \nabla(h_t) - \nabla(|\nabla h|^2) - \nabla \mathbf{X}, \nabla h\rangle \\
=& -2\langle \nabla F, \nabla h\rangle + 2t\Big[\langle \nabla(h_t), \nabla h\rangle - \langle \nabla \mathbf{X}, \nabla h\rangle + \lambda\langle \nabla \mathbf{X}, \nabla h\rangle - \lambda\langle \nabla(h_t), \nabla h\rangle\Big] \\
=& -2\langle \nabla F, \nabla h\rangle + t\Big[(1-\lambda)(|\nabla h|^2)_t + (1-\lambda)\mathcal{R}(\nabla h, \nabla h)\Big].
\end{aligned}$$

Using this we calculate

$$\begin{aligned}
\Delta F =& t\Big[\Delta(|\nabla h|^2) - \lambda(\Delta(h_t) - \Delta \mathbf{X})\Big] \\
=& t\Big[2\langle \nabla \Delta h, \nabla h\rangle + 2|\nabla^2 h|^2 + 2Ric(\nabla h, \nabla h) - \lambda(\Delta(h_t) - \Delta \mathbf{X})\Big] \\
=& t\Big[2\langle \nabla(h_t - |\nabla h|^2 - \mathbf{X}), \nabla h\rangle + 2|\nabla^2 h|^2 + 2Ric(\nabla h, \nabla h) + \lambda\Big((|\nabla h|^2)_t - h_{tt} + \mathbf{X}_t\Big) \\
& - 2\lambda\langle \mathcal{R}, \nabla^2 h\rangle - 2\lambda\langle \nabla \cdot \mathcal{R} - \frac{1}{2}(\mathrm{Tr}_g\, \mathcal{R}), \nabla h\rangle + \lambda \Delta \mathbf{X}\Big] \\
=& -2\langle \nabla h, \nabla F\rangle + 2t\Big[|\nabla^2 h|^2 + Ric(\nabla h, \nabla h) + (1-\lambda)\mathcal{R}(\nabla h, \nabla h) - \lambda\langle \mathcal{R}, \nabla^2 h\rangle \\
& - \lambda\langle \nabla \cdot \mathcal{R} - \frac{1}{2}(\mathrm{Tr}_g\, \mathcal{R}), \nabla h\rangle\Big] + \lambda t \Delta \mathbf{X} + t(|\nabla h|^2)_t - \lambda t h_{tt} + \lambda t \mathbf{X}_t,
\end{aligned}$$

and

$$F_t = |\nabla h|^2 - \lambda[h_t - \mathbf{X}] + t\Big[(|\nabla h|^2)_t - \lambda h_{tt} + \lambda \mathbf{X}_t\Big],$$

where we have set $\mathbf{X} = A(\log D + h) + B(De^h)^{p-1}$ for convenience. Next we calculate:

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)F =& -2\langle\nabla h, \nabla F\rangle + 2t\Big[|\nabla^2 h|^2 + Ric(\nabla h, \nabla h) + (1-\lambda)\mathcal{R}(\nabla h, \nabla h) - \lambda\langle\mathcal{R}, \nabla^2 h\rangle \\ &- \lambda\langle\nabla\cdot\mathcal{R} - \frac{1}{2}(\mathrm{Tr}_g\,\mathcal{R}), \nabla h\rangle\Big] + \lambda t\Delta\mathbf{X} - (|\nabla h|^2 - \lambda(h_t - \mathbf{X})).\end{aligned}$$

From our assumptions

$$\begin{aligned}|\mathcal{R}|^2 \leq& |g|^2(C_2 + C_3)^2 \\ \leq& n(C_2 + C_3)^2.\end{aligned}$$

Also, by an application of Young's inequality

$$\begin{aligned}|\lambda\langle\mathcal{R}, |\nabla^2 h|\rangle| \leq& \frac{1}{2}|\nabla^2 h|^2 + \frac{1}{2}\lambda^2|\mathcal{R}|^2 \\ \leq& \frac{1}{2}|\nabla^2 h|^2 + \frac{1}{2}\lambda^2 n(C_2 + C_3)^2,\end{aligned}$$

and

$$\begin{aligned}|\nabla\cdot\mathcal{R} - \frac{1}{2}\nabla(\mathrm{Tr}_g\,\mathcal{R})| \leq& |g^{ij}(\nabla_i\mathcal{R}_{jl} - \frac{1}{2}\nabla_l\mathcal{R}_{ij})| \\ \leq& \frac{3}{2}|g||\nabla\mathcal{R}| \\ \leq& \frac{3}{2}\sqrt{n}C_4.\end{aligned}$$

Finally

$$\begin{aligned}|\nabla^2 h|^2 \geq& \frac{1}{n}(\Delta h)^2 \\ \geq& \frac{1}{n}(|\nabla h|^2 - h_t + \mathbf{X})^2.\end{aligned}$$

Hence we can write:

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)F \geq& -2\langle\nabla h, \nabla F\rangle + \frac{t}{n}(|\nabla h|^2 - h_t + \mathbf{X})^2 - 2((\lambda-1)C_3 + C_1 n)t|\nabla h|^2 \\ &- 3\sqrt{n}tC_4|\nabla h| + \lambda t\Delta\mathbf{X} - (|\nabla h|^2 - \lambda(h_t - \mathbf{X})) - \lambda^2 n(C_2 + C_3)^2.\end{aligned}$$

Before being able to use the cutoff function, we need to first access the $\Delta\mathbf{X}$ term, as this has the slightly obscured problem of containing Δh terms:

$$\begin{aligned}\Delta\mathbf{X} =& \Delta A(\log D + h) + 2\nabla A\nabla h + A\Delta h + \Delta B(De^h)^{p-1} + B(p-1)\Delta h(De^h)^{p-1} \\ &+ B(p-1)^2|\nabla h|^2(De^h)^{p-1} + \nabla B(p-1)|\nabla h|(De^h)^{p-1}.\end{aligned}$$

Focusing first on the $A\Delta h$ term:

$$
\begin{aligned}
A\Delta h =& A[|\nabla h|^2 - h_t + A(\log D + h) + B(De^h)^{p-1}] \\
=& A|\nabla h|^2 + A^2(\log D + h) + AB(De^h)^{p-1} - A\left(\frac{1}{\lambda}\Big(\frac{F}{t} - |\nabla h|^2\Big) + \mathbf{X}\right) \\
=& \Big(1 + \frac{1}{\lambda}\Big)A|\nabla h|^2 - \frac{F}{\lambda t}A.
\end{aligned}
$$

Similarly for the $B\Delta h$ term:

$$
\begin{aligned}
B(p-1)\Delta h(De^h)^{p-1} =& B(p-1)[|\nabla h|^2 - h_t + A(\log D + h) + (De^h)^{p-1}](De^h)^{p-1} \\
=& B(p-1)|\nabla h|^2(De^h)^{p-1} + AB(p-1)(\log D + h)(De^h)^{p-1} \\
&+ B^2(p-1)^2(De^h)^{2(p-1)} - B(p-1)(De^h)^{p-1}\left(\frac{1}{\lambda}\Big(\frac{F}{t} - |\nabla h|^2\Big) + \mathbf{X}\right) \\
=& \Big(1 + \frac{1}{\lambda}\Big)B(p-1)|\nabla h|^2(De^h)^{p-1} - \frac{F}{\lambda t}B(p-1)(De^h)^{(p-1)}.
\end{aligned}
$$

So we get

$$
\begin{aligned}
\Big(\Delta - \frac{\partial}{\partial t}\Big)F \geq& -2\langle \nabla h, \nabla F\rangle + \frac{t}{n}(|\nabla h|^2 - h_t + \mathbf{X})^2 - 2((\lambda-1)C_3 + C_1 n)t|\nabla h|^2 \\
&- 3\sqrt{n}tC_4|\nabla h| + \lambda t\left(\Big(1 + \frac{1}{\lambda}\Big)A|\nabla h|^2 - \frac{F}{\lambda t}A + \Big(1 + \frac{1}{\lambda}\Big)B(p-1)|\nabla h|^2(De^h)^{p-1}\right. \\
&- \frac{F}{\lambda t}B(p-1)(De^h)^{p-1} + \Delta A(\log D + h) + 2\nabla A\nabla h + \Delta B(De^h)^{p-1} \\
&\left. + B(p-1)^2|\nabla h|^2(De^h)^{p-1} + \nabla B(p-1)|\nabla h|(De^h)^{p-1}\right) - (|\nabla h|^2 - \lambda(h_t - \mathbf{X})) \\
&- \lambda^2 n(C_2 + C_3)^2.
\end{aligned}
$$

□

Now we can finish the proof of Theorem 6.2.1.

Proof of Theorem 6.2.1. Now we can introduce the cutoff function. Fix $T > 0$ and choose $\tau \in (0, T]$. Let $\bar{\phi} : [0, \infty) \times [0, T] \to \mathbb{R}$ be such that

$$0 \leq \bar{\phi}(r, t) \leq 1 \tag{6.2.9}$$

in $[0, R] \times [0, T]$ with support, $\bar{\phi}(r, t) = 1$ in $[0, R/2] \times [\tau, T]$ and $\partial_r \bar{\phi}(r, t) = 0$ in $[0, R/2] \times [0, T]$, and $\bar{\phi}(r, 0) = 0$ for all $r \in [0, \infty)$. Then

$$-\frac{C_\epsilon}{R} \leq \frac{\partial_r \bar{\phi}}{\bar{\phi}^\epsilon} \leq 0 \quad \text{and} \quad \frac{|\partial_r^2 \bar{\phi}|}{\bar{\phi}^\epsilon} \leq \frac{C_\epsilon}{R^2} \tag{6.2.10}$$

in $[0, \infty) \times [0, T]$, $C_\epsilon > 0$, and $\epsilon \in (0, 1)$. Let $\phi : M^n \times [0, T] \to \mathbb{R}$ be

$$\phi(r, t) = \bar{\phi}\Big(\frac{d(x, x_0, t)}{R}\Big), \tag{6.2.11}$$

where $d(x, y, t)$ is the distance function. As $d(x, x_0, t)$ is Lipschitz continuous, we have $|\nabla d| = 1$. Also

$$\frac{\partial}{\partial t} d(x, x_0, t) = \int_\gamma \mathcal{R}_{ij}(S, S) ds \tag{6.2.12}$$

so

$$\begin{aligned}\frac{\partial}{\partial t}\phi =& \frac{\bar{\phi}'}{R}\frac{\partial}{\partial t} d(x, x_0, t) \\ =& \frac{\bar{\phi}'}{R}\int_\gamma \mathcal{R}_{ij}(S, S) ds \\ \leq& \sqrt{C} C_3,\end{aligned} \tag{6.2.13}$$

where γ is the geodesic connecting x and x_0 under the metric at time t_1, S is the unit tangent vector to γ, and s is the arc length. See [56] section 12 for more detail. Then, by straightforward inequalities, we have

$$-\phi_t F \geq -\sqrt{C} C_3 F. \tag{6.2.14}$$

Further details can be found in [36].

As previously,

$$\begin{aligned}\Delta\phi =& \frac{\phi'\Delta r}{R} + \frac{\phi''|\nabla r|^2}{R^2} \\ \geq& -\frac{Cn\sqrt{K}\coth(\sqrt{K}r)}{R} - \frac{C}{R^2} \\ \geq& -\frac{Cn(1 + R\sqrt{K})}{R^2}.\end{aligned} \tag{6.2.15}$$

Let $t \in (0, T]$ and let $\phi(r, t)$ be the cutoff function defined above. Specifically, let (x_1, t_1) be the maximum space-time point of ϕF. Then

$$\Delta(\phi F) \leq 0, \quad \nabla(\phi F) = 0, \quad (\phi F)_t \geq 0. \tag{6.2.16}$$

Basic calculations give

$$\Big(\Delta - \partial_t\Big)(\phi F) = \phi\Big(\Delta - \partial_t\Big)F + 2\nabla\phi\nabla F + F\Big(\Delta - \partial_t\Big)\phi. \tag{6.2.17}$$

When $F(x_1, t_1) \leq 0$, the proof is trivial, so we consider the case when $F(x_1, t_1) > 0$ for $t_1 > 0$. In this case, we have

$$
\begin{aligned}
0 \geq & \phi \Bigg[-2\langle \nabla h, \nabla F\rangle + \frac{t_1}{n}(|\nabla h|^2 - h_t + \mathbf{X})^2 - 2((\lambda-1)C_3 + C_1 n)t_1|\nabla h|^2 \\
& - 3\sqrt{n}t_1 C_4|\nabla h| + \lambda t_1 \Bigg(\Big(1+\frac{1}{\lambda}\Big)A|\nabla h|^2 - \frac{F}{\lambda t_1}A + \Big(1+\frac{1}{\lambda}\Big)B(p-1)|\nabla h|^2(De^h)^{p-1} \\
& - \frac{F}{\lambda t_1}B(p-1)(De^h)^{p-1} + |\Delta A|(\log D + h) + 2|\nabla A||\nabla h| + |\Delta B|(De^h)^{p-1} \\
& + B(p-1)^2|\nabla h|^2(De^h)^{p-1} + |\nabla B|(p-1)|\nabla h|(De^h)^{p-1}\Bigg) \\
& - (|\nabla h|^2 - \lambda(h_t - \mathbf{X})) - \lambda^2 n(C_2+C_3)^2\Bigg] - 2F\frac{|\nabla\phi|}{\phi} - \sqrt{C}C_3F + F\Delta\phi.
\end{aligned}
$$

Multiplying both sides by (ϕt_1) and rearranging:

$$
\begin{aligned}
0 \geq & -\phi t_1\Bigg(\frac{C^2 + Cn(1+R\sqrt{K})}{R^2} + \frac{\phi}{t_1} + A + B(p-1)(De^h)^{p-1}\Bigg)F - \frac{2Ct_1}{R}F|\nabla h|\phi^{\frac{3}{2}} \\
& + \frac{\phi^2 t_1^2}{n}\Bigg((|\nabla h|^2 - h_t + \mathbf{X})^2 + \Bigg[-2n((\lambda-1)C_3 + C_1 n) + n\lambda\Big(1+\frac{1}{\lambda}\Big)A \\
& + n\lambda\Big(1+\frac{1}{\lambda}\Big)B(p-1)(De^h)^{p-1} + n\lambda t_1 B(p-1)^2(De^h)^{p-1}\Bigg]|\nabla h|^2\Bigg) \\
& - \phi^2 t_1^2\Bigg(3\sqrt{n}C_4 - 2\lambda|\nabla A| - \lambda|\nabla B|(p-1)(De^h)^{p-1}\Bigg)|\nabla h| \\
& - \phi^2 t_1\Big(\phi\lambda^2 n(C_2+C_3)^2 + \lambda t_1|\Delta A|(\log D + h) + \lambda t_1|\Delta B|(De^h)^{p-1}\Big).
\end{aligned}
$$

As with the non-evolving metric, we will let $y = \phi|\nabla h|^2$ and $z = \phi\big(h_t - A\log u - Bu^{(p-1)}\big)$. By basic inequalities,

$$
\begin{aligned}
&-2n((\lambda-1)C_3 + C_1 n) + n\lambda\Big(1+\frac{1}{\lambda}\Big) + n\lambda\Big(1+\frac{1}{\lambda}\Big)B(p-1)(De^h)^{p-1} + n\lambda t_1 B(p-1)^2(De^h)^{p-1} \\
&\geq -2n((\lambda-1)C_3 + C_1) - n\lambda\Big(1+\frac{1}{\lambda}\Big) - n\lambda\Big(1+\frac{1}{\lambda}\Big)[B(p-1)]_+ \sup_{H_{2R,T}} u^{p-1} \\
&\quad - n\lambda t_1 B_+(p-1)^2 \sup_{H_{2R,T}} u^{p-1} \\
&= -\mathfrak{S}
\end{aligned}
$$

and

$$
\begin{aligned}
-3\sqrt{n}C_4 + 2\lambda t_1|\nabla A| + \lambda t_1|\nabla B|(p-1)(De^h)^{p-1} \geq & -3\sqrt{n}C_4 - 2\lambda t_1 a_1 - \lambda t_1 b_1(p-1)\sup_{H_{2R,T}} u^{p-1} \\
= & -\vartheta.
\end{aligned}
$$

Following along the same lines as the proof for the time-independent metric, we conclude the proof.

□

6.3 Li-Yau gradient estimate for specific flows

Theorem 6.2.1 focuses on equation (5.1.1) on generalised geometric flow. There a limited amount of research on the area for Li-Yau gradient estimates on this flow for parabolic PDEs, see [3, 108]. Now we are going consider what happens when the flow in question is either Ricci flow or Yamabe flow. Estimates of this type can be seen in [2, 14, 79] and [133] respectively.

We start this section by looking at the gradient estimate for (5.1.1) under Ricci flow.

Theorem 6.3.1. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow equation with Ric $\geq -C_1 g$ for $C_1 \geq 0$ and $-C_2 g \leq$ Ric $\leq C_3 g$ for $C_1, C_2, C_3 \geq 0$, $m < \infty$ in $B(x_0, 2R)$ for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $H_{2R,T} = B(x_0, 2R) \times [0, T]$ such that $0 < u \leq D$ for some $D \geq 1$. Also let*

$$|\nabla A| \geq a_1, \quad |\nabla B| \geq b_1,$$
$$\Delta A \leq a_2, \quad \Delta B \leq b_2$$

for positive constants a_1, a_2, b_1, b_2. Then the estimate (6.3.1) holds on $B(x_0, R) \times (0, T]$ with $C_4 \equiv 0$.

Remark 6.3.1. *The lack of requirement for C_4 is due to Lemma 1.5.1, in particular the effect of the contracted second Bianchi identity.*

Next, we look at an estimate for Yamabe flow. Zhang in [133] produced a workable approach, but we instead alter the method for the generalised geometric flow for the specific case of Yamabe flow.

Theorem 6.3.2. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to Yamabe flow with Ric $\geq -C_1 g$, $-C_2 g \leq S \leq C_3 g$ and $|\nabla S| \leq C_4$ for $C_1, C_2, C_3, C_4 \geq 0$, $m < \infty$ in $B(x_0, 2R)$ for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $H_{2R,T} = B(x_0, 2R) \times [0, T]$ such that $0 < u \leq D$ for $D \geq 1$. Also let*

$$|\nabla A| \geq a_1, \quad |\nabla B| \geq b_1,$$
$$\Delta A \leq a_2, \quad \Delta B \leq b_2$$

for positive constants a_1, a_2, b_1, b_2. Then the following gradient estimate holds on $B(x_0, R) \times (0, T]$ with $\lambda, p > 1$ and $\epsilon \in (0, 1)$:

$$\frac{|\nabla u|^2}{u^2} - \lambda\Big(\frac{u_t}{u} - A\log u - Bu^{p-1}\Big) \leq \frac{n\lambda^2}{2}\tilde{\Omega} + \left(\frac{n\lambda^2}{2}\tilde{\Theta}\right)^{\frac{1}{2}} \tag{6.3.1}$$

for

$$\tilde{\Omega} = \frac{C^2 + Cn(1+R\sqrt{K})}{R^2} + \frac{(1+C_2)}{t} + A_+ + [B(p-1)]_+ \sup_{H_{2R,T}} u^{p-1} + \frac{tC\lambda^2}{(\lambda-1)R^2}, \tag{6.3.2}$$

$$\begin{aligned}\tilde{\Theta} =& C\Big(n^8\epsilon^{-1}\lambda^2(\lambda-1)^2\tilde{\vartheta}^4\Big)^{\frac{1}{3}} + \frac{n^2}{2}(1-\epsilon)^{-1}\lambda^2(\lambda-1)^2\mathfrak{S}^2 + \lambda^2 n(C_2+C_3)^2 + \frac{\lambda^2(n-2)^2}{2}C_4^2 \\ &+ \lambda t a_2 \sup_{H_{2R,T}} (\log u)_+ + \lambda t b_2 \sup_{H_{2R,T}} u^{p-1},\end{aligned} \tag{6.3.3}$$

$$\begin{aligned}\mathfrak{S} =& 2n\Big((\lambda-1)C_3 + C_1 n + \frac{1}{2}\Big) + n\lambda\Big(1+\frac{1}{\lambda}\Big) + n\lambda\Big(1+\frac{1}{\lambda}\Big)[B(p-1)]_+ \sup_{H_{2R,T}} u^{(p-1)} \\ &+ n\lambda t B_+(p-1)^2 \sup_{H_{2R,T}} u^{p-1},\end{aligned} \tag{6.3.4}$$

and

$$\tilde{\vartheta} = 2\lambda t a_1 + \lambda t b_1 (p-1) \sup_{H_{2R,T}} u^{p-1}.$$

Proof. Let $h = \log \frac{u}{D}$ for $D \geq 1$ such that $0 < u \leq D$. Let F satisfy the Li-Yau transformation

$$F = t\left[|\nabla h|^2 - \lambda(h_t - A(\log D + h) - B(De^h)^{p-1})\right] \tag{6.3.5}$$

where $A = A(x,t)$ and $B = B(x,t)$ are functions of both time and space. By direct computations,

$$\begin{aligned}2t\langle \nabla\Delta h, \nabla h\rangle =& 2t\langle \nabla(h_t) - \nabla(|\nabla h|^2) - \nabla\mathbf{X}, \nabla h\rangle \\ =& -2\langle \nabla F, \nabla h\rangle + 2t\Big[\langle \nabla(h_t), \nabla h\rangle - \langle \nabla\mathbf{X}, \nabla h\rangle + \lambda\langle \nabla\mathbf{X}, \nabla h\rangle - \lambda\langle \nabla(h_t), \nabla h\rangle\Big] \\ =& -2\langle \nabla F, \nabla h\rangle + t\Big[(1-\lambda)(|\nabla h|^2)_t + (1-\lambda)S|\nabla h|^2\Big].\end{aligned}$$

Using this, we can calculate

$$\begin{aligned}\Delta F =& t\Big[\Delta(|\nabla h|^2) - \lambda(\Delta(h_t) - \Delta\mathbf{X})\Big] \\ =& t\Big[2\langle \nabla\Delta h, \nabla h\rangle + 2|\nabla^2 h|^2 + 2Ric(\nabla h, \nabla h) - \lambda(\Delta(h_t) - \Delta\mathbf{X})\Big] \\ =& t\Big[2\langle \nabla(h_t - |\nabla h|^2 - \mathbf{X}), \nabla h\rangle + 2|\nabla^2 h|^2 + 2Ric(\nabla h, \nabla h) + \lambda\Big((|\nabla h|^2)_t - h_{tt} + \mathbf{X}_t\Big) \\ &- \frac{n-2}{2}\langle \nabla h, \nabla S\rangle + S\Delta h + \lambda\Delta\mathbf{X}\Big] \\ =& -2\langle \nabla h, \nabla F\rangle + 2t\Big[|\nabla^2 h|^2 + Ric(\nabla h, \nabla h) + (1-\lambda)S|\nabla h|^2 - \frac{\lambda(n-2)}{2}\langle \nabla h, \nabla S\rangle \\ &+ \lambda S\Delta h\Big] + \lambda t\Delta\mathbf{X} + t(|\nabla h|^2)_t - \lambda t h_{tt} + \lambda t\mathbf{X}_t\end{aligned}$$

and F_t as before. Next, we calculate

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)F =& -2\langle \nabla h, \nabla F\rangle + 2t\Big[|\nabla^2 h|^2 + Ric(\nabla h, \nabla h) + (1-\lambda)S|\nabla h|^2 \\ &- \frac{\lambda(n-2)}{2}\langle \nabla h, \nabla S\rangle + \lambda S\Delta h\Big] + \lambda t\Delta\mathbf{X} - (|\nabla h|^2 - \lambda(h_t - \mathbf{X})).\end{aligned}$$

Also, we can calculate that

$$\begin{aligned}|\lambda(n-2)\langle\nabla h,\nabla S\rangle \leq&\frac{1}{2}|\nabla h|^2+\frac{\lambda^2(n-2)^2}{2}|\nabla S|^2\\ \leq&\frac{1}{2}|\nabla h|^2+\frac{\lambda^2(n-2)^2}{2}C_4^2.\end{aligned}$$

Hence we can write

$$\begin{aligned}\Big(\Delta-\frac{\partial}{\partial t}\Big)F\geq&-2\langle\nabla h,\nabla F\rangle+\frac{t}{n}(|\nabla h|^2-h_t+\mathbf{X})^2-2((\lambda-1)C_3+C_1n-\frac{1}{2})t|\nabla h|^2+\lambda t\Delta\mathbf{X}\\ &-(1+C_2)(|\nabla h|^2-\lambda(h_t-\mathbf{X}))-\lambda^2n(C_2+C_3)^2-\frac{\lambda^2(n-2)^2}{2}C_4^2.\end{aligned}$$

The rest of the proof follows similarly to the proof of Theorem 6.2.1. □

Finally, we state an estimate for Perelman-Ricci flow. For this flow, we need to be working on a smooth metric measure space. As yet, there are no pre-existing estimates for a parabolic PDE under Perelman-Ricci flow. For our estimates, we have chosen $f(t)$ in our measure to be a function of t alone, with no dependency on x.

Remark 6.3.2. *Theorems 6.3.1 and 6.3.2 are also valid on a smooth metric measure space when $f(t)$ is a function of time alone.*

Lemma 6.3.1. *Let $g(t)$ be a smooth family of metrics that solve (6.0.1). Then*

$$g^{ij}\frac{\partial}{\partial t}\Gamma_{ij}^k=0. \tag{6.3.6}$$

Proof. As with the generalised geometric flow,

$$\frac{\partial}{\partial t}\Gamma_{ij}^k=\frac{1}{2}\frac{\partial}{\partial t}g^{kl}\Big(\frac{\partial}{\partial x_i}g_{jl}+\frac{\partial}{\partial x_j}g_{il}-\frac{\partial}{\partial x_l}g_{ij}\Big)+\frac{1}{2}g^{kl}\Big(\frac{\partial}{\partial x_i}\frac{\partial}{\partial t}g_{jl}+\frac{\partial}{\partial x_j}\frac{\partial}{\partial t}g_{il}-\frac{\partial}{\partial x_l}\frac{\partial}{\partial t}g_{ij}\Big).$$

Then at an arbitrary point $z\in M^N$, $\Gamma_{ij}^k(z)=0$. This gives $\frac{\partial}{\partial x_a}g_{bc}(z)=0$ and then

$$g^{ij}\frac{\partial}{\partial t}\Gamma_{ij}^k=\frac{1}{2}g^{ij}g^{kl}\Big(\frac{\partial}{\partial x_i}\frac{\partial}{\partial t}g_{jl}+\frac{\partial}{\partial x_j}\frac{\partial}{\partial t}g_{il}-\frac{\partial}{\partial x_l}\frac{\partial}{\partial t}g_{ij}\Big).$$

Using (6.0.1)

$$\begin{aligned}g^{ij}\frac{\partial}{\partial t}\Gamma_{ij}^k=&-\frac{1}{2}g^{ij}g^{kl}\Big(\frac{\partial}{\partial x_i}(Ric_{jl}+Kg_{jl})+\frac{\partial}{\partial x_j}(Ric_{il}+Kg_{il})-\frac{\partial}{\partial x_l}(Ric_{ij}+Kg_{ij})\Big)\\ =&-\frac{1}{2}g^{ij}g^{kl}\Big(\nabla_iRic_{jl}+\nabla_jRic_{il}-\nabla_lRic_{ij}\Big)-\frac{1}{2}Kg^{ij}g^{kl}(\nabla_ig_{jl}+\nabla_jg_{il}-\nabla_lg_{ij}\Big)\\ =&-\frac{1}{2}g^{kl}g^{ij}\Big(2\nabla_iRic_{jl}-\nabla_lS\Big)-\frac{1}{2}Kg^{ij}\Gamma_{ij}^k.\end{aligned}$$

Since z was arbitrary, we have $\Gamma_{ij}^k(z)=0$, and, by the contracted second Bianchi identity, the first term is also zero. □

Lemma 6.3.2. *Let $g(t)$ be a smooth family of metrics that solve (6.0.1). Then*

$$\frac{\partial}{\partial t}\Delta u = \Delta u_t + 2\langle Ric + K, \nabla^2 u\rangle. \tag{6.3.7}$$

Theorem 6.3.3. *Let $(M^n, g(t), e^{-f(t)}d\nu)_{t\in[0,T]}$ be a complete solution to the Perelman-Ricci flow with $Ric_f^m \geq -(m-1)C_1 g$ and $-C_2 g \leq Ric_f^m \leq C_3 g$ for $C_1, C_2, C_3 \geq 0$, $m < \infty$ in $B(x_0, 2R)$ for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $H_{2R,T} = B(x_0, 2R) \times [0,T]$ such that $0 < u \leq D$ for $D \geq 1$. Also suppose that $f(t)$ is a function of time alone. Now let*

$$|\nabla A| \geq a_1, \quad |\nabla B| \geq b_1,$$
$$\Delta A \leq a_2, \quad \Delta B \leq b_2$$

for positive constants a_1, a_2, b_1, b_2. Then the following gradient estimate holds on $B(x_0,R)\times(0,T]$ with $\lambda, p > 1$ and $\epsilon \in (0,1)$:

$$\frac{|\nabla u|^2}{u^2} - \lambda\Big(\frac{u_t}{u} - A\log u - Bu^{(p-1)}\Big) \leq \frac{m\lambda^2}{2}\tilde{\Omega} + \left(\frac{m\lambda^2}{2}\tilde{\Theta}\right)^{\frac{1}{2}} \tag{6.3.8}$$

for

$$\tilde{\Omega} = \frac{C^2 + C(m-1)(1+R\sqrt{K})}{R^2} + \frac{1}{t} + A_+ + [B(p-1)]_+ \sup_{H_{2R,T}} u^{p-1} + \frac{tC\lambda^2}{(\lambda-1)R^2}, \tag{6.3.9}$$

$$\begin{aligned}\tilde{\Theta} =& C\Big(m^8\epsilon^{-1}\lambda^2(\lambda-1)^2\tilde{\vartheta}^4\Big)^{\frac{1}{3}} + \frac{m^2}{2}(1-\epsilon)^{-1}\lambda^2(\lambda-1)^2\mathfrak{S}^2\\ &+ \lambda^2(n(C_2+C_3)^2 + K^2) + \lambda t a_2 \sup_{H_{2R,T}} (\log u)_+ + \lambda t b_2 \sup_{H_{2R,T}} u^{p-1}\end{aligned} \tag{6.3.10}$$

$$\begin{aligned}\mathfrak{S} =& 2m((\lambda-1)C_3 + C_1(m-1) + K) + m\lambda\Big(1+\frac{1}{\lambda}\Big)\\ &+ m\lambda\Big(1+\frac{1}{\lambda}\Big)[B(p-1)]_+ \sup_{H_{2R,T}} u^{p-1} + \lambda t B_+(p-1)^2 \sup_{H_{2R,T}} u^{p-1},\end{aligned} \tag{6.3.11}$$

and

$$\tilde{\vartheta} = 2\lambda t a_1 + \lambda t b_1(p-1) \sup_{H_{2R,T}} u^{p-1}.$$

Chapter 7

Applications of gradient estimates including Liouville type theorems, Harnack inequalities, and ancient solutions

Once a gradient estimate is formed, further analysis can give rise to plenty of additional information: this can be obtained through Liouville-type theorems, Harnack inequalities, and analysis using ancient solutions. These theorems and inequalities use the gradient estimates to look at conditions under which limits and solutions are found.

7.1 Liouville-type theorems for elliptic PDEs

Liouville-type theorems find the conditions under which the solutions to the PDEs are constant functions. Specifically for our analysis, Liouville-type theorems are looking for values of A and B which cause our gradient estimate to equal zero, meaning u is a constant function.

Yau in [130] showed that any positive or bounded harmonic function with a non-negative Ricci curvature must be a constant function. Together with the following corollary stated in Brighton's work [25], we are able to produce a Liouville-type theorem after obtaining the gradient estimate.

Corollary 7.1.1. *Let $(M^n, g, e^{-f}d\nu)$ be a complete smooth metric measure space with $Ric_f \geq 0$. If u is a bounded f-harmonic function defined on M^n, then u is constant.*

Further reading on Liouville theorems can be found in [77, 122]. For the finite dimensional case, see [60, 61, 99]

Proposition 7.1.1. *Let $(M^n, g, e^{-f}d\nu)$ be a complete non-compact metric space with non-negative Bakry-Emery curvature. Let $A \leq 0$ and $B \leq 0$, where A, B are constants. Also let $p > 1$. If u is a positive solution to (4.3.1) such that $u \geq \exp(-1/\epsilon)$, then u is a constant function. Moreover,*

$$u = \sqrt[p-1]{\frac{AW\left(\frac{B(p-1)}{A}\right)}{B(p-1)}}, \tag{7.1.1}$$

where $W(x)$ is the Lambert W function.

Proof. Taking the global estimate of Corollary 4.3.2, we note that when substituting $u \geq e^{-\frac{1}{\epsilon}}$ and using A,$B \leq 0$, the max term gives zero. Combining this with non-negative Bakry-Emery curvature, we see that u is a constant function. Thus

$$Au \log u + Bu^p = 0.$$

Collecting the u terms onto the left hand side gives

$$u^{1-p} \log u^{1-p} = \frac{B(p-1)}{A}$$

and manipulating the left hand side

$$e^{\log u^{1-p}} \log u^{1-p} = \frac{B(p-1)}{A}.$$

Now we make use of the Lambert W function for real values:

$$\log u^{1-p} = W\left(\frac{B(p-1)}{A}\right).$$

Note that if $\mathbf{Y}e^{\mathbf{Y}} = \mathbf{X}$, then $\mathbf{Y} = W(\mathbf{X})$ where $W(\mathbf{X})$ is the Lambert W function of $\mathbf{X}$. It is also known that $\exp W(x) = \frac{x}{W(x)}$. Then solving for u gives:

$$\begin{aligned} u &= \sqrt[1-p]{\frac{B(p-1)}{AW\left(\frac{B(p-1)}{A}\right)}} \\ &= \sqrt[p-1]{\frac{AW\left(\frac{B(p-1)}{A}\right)}{B(p-1)}}. \end{aligned}$$

□

From Proposition (7.1.1), if we take $u \geq e^{-\frac{1}{\epsilon}}$, then the value of A must be negative for our Liouville theorems. However, by choosing $u = \exp(-\epsilon^{-1})$, the lower bound of u from Proposition 7.1.1, then we can be more specific with our choices of A, B, and p.

Proposition 7.1.2. *If $u = \exp(-\epsilon^{-1})$, then for it to be a solution to (4.3.1)*

$$A = \epsilon B e^{\epsilon^{-1}(1-p)} \tag{7.1.2}$$

where $B \leq 0$ and $p > 1$.

Proof. Using Proposition (7.1.1), we see that when we impose the restriction within that proposition and taking u as its lower bound, u is constant. Now we can solve (4.3.1):

$$Ae^{-\epsilon^{-1}}\Big(-\frac{1}{\epsilon}\Big) + B(e^{-p\epsilon^{-1}}) = 0.$$

Solving for A gives

$$A = \epsilon B e^{\epsilon^{-1}(1-p)},$$

which is the desired result. □

Proposition 7.1.1 is for constant A and B, which means that we can use Corollary 4.3.2. If instead we desire to find Liouville-type theorems for varying coefficients, then we require the use of Theorem 4.3.2.

Proposition 7.1.3. *Let u be a positive solution to*

$$\Delta_f u(x) + B(x)u(x)^p = 0 \tag{7.1.3}$$

with non-negative Bakry-Ricci curvature as defined above. Also, let $p > 1$ and

$$B_+|_{B(x_0,R)} = o(R^{-\gamma(p-1)}), \qquad \sup_{B(x_0,R)} |\nabla B| = o(R^{-\gamma(p-1+\epsilon)}) \tag{7.1.4}$$

as $R \to \infty$, where $\gamma > 0$ and $\overline{\gamma} \in (0, \gamma)$. If $u(x) = o[r(x)^{\overline{\gamma}}]$, then u is a constant function.

Proof. Fix a point x_0 and apply Theorem 4.3.2 in $B(x_0, R)$. With this, we observe

$$\begin{aligned}\frac{|\nabla u(x_0)|^2}{u(x_0)^2} \leq & \frac{1}{\epsilon}\left(\frac{n}{2C_1^2 - \gamma n C_2}\right)(o(R^{(\overline{\gamma}-\gamma)(p-1)}) - o(R^{(\overline{\gamma}-\gamma)(p-1+\epsilon)})) \\ & + \frac{1}{\epsilon}\left(4\left(\frac{n}{2C_1^2 - \gamma n C_2}\right)o(R^{(\overline{\gamma}-\gamma)(p-1+\epsilon)})\right)^{\frac{1}{2}}.\end{aligned} \tag{7.1.5}$$

Then letting $R \to \infty$, we immediately observe that $|\nabla u(x_0, t_0)| = 0$, so, since x_0 was chosen arbitrarily, $u(x,t)$ must be a constant function in x .

□

7.2 Harnack inequalities and Harnack-type inequalities for elliptic and parabolic PDEs gradient estimates

Along with Liouville theorems, Harnack inequalities are obtainable from gradient estimates. These were first introduced by Harnack in [57] and give an upper bound for the supremum of a solution to a elliptic or parabolic PDE.

Proposition 7.2.1 (Classical Harnack inequality)**.** *Let (M^n, g) be a closed connected manifold. Let u be a solution to*

$$\partial_t u = \Delta u \tag{7.2.1}$$

for $u \in M^n \times [0, T]$. Then

$$\sup_{M^n} u(\cdot, t_1) \leq C \inf_{M^n} u(\cdot, t_2), \tag{7.2.2}$$

where C is a constant depending only on t_1 and t_2.

Further reading can be found in [85, 86, 89, 105]. For the unweighted Laplacian, examples of this can be seen in [4, 129]. We, however, look at the Harnack inequality for (4.3.1) with the Witten-Laplacian.

Proposition 7.2.2. *Let u be a bounded positive solution to (4.3.1) such that $u \leq D$ and $|A| \leq a_1$, $|\nabla A| \leq a_2$, $|B| \leq b_1$, and $|\nabla B| \leq b_2$ be constants. Then*

$$\frac{|\nabla u|^2}{u^2} \leq f(\mathbf{X}), \tag{7.2.3}$$

for a constant function $f(\mathbf{X}) = f(\epsilon, N, a_1, a_2, b_1, b_2, D)$.

Proof. The proof follows from the use of Theorem 4.3.1 with the restrictions above. □

Proposition 7.2.3. *Let the assumptions of Proposition 7.2.2 hold. Then for solutions $u(x)$ on $B(x_0, R)$, the following Harnack inequality holds:*

$$\sup_{B(x_0,R)} u \leq e^{2R\sqrt{f(\mathbf{X})}} \inf_{B(x_0,R)} u. \tag{7.2.4}$$

Proof. Let γ be the geodesic connecting x_1 and x_2. This, being the curve of shortest distance, is at most $2R$. Then

$$\begin{aligned}\log u(x_1) - \log u(x_2) &\leq \int_\gamma \frac{|\nabla u|}{u} \\ &\leq \int_\gamma \sqrt{f(\mathbf{X})} \\ &\leq 2R\sqrt{f(\mathbf{X})}.\end{aligned}$$

Choosing $u(x_1) = \sup_{B(x_0,R)} u$ and $u(x_2) = \inf_{B(x_0,R)} u$, taking the exponential, and rearranging yields the desired result. □

Remark 7.2.1. *When looking for a global estimate, Proposition 7.2.3 is not particularly useful due to the fact that when $R \to \infty$, the right hand side of the inequality also becomes unbounded.*

Proposition 7.2.4. *Let the assumptions of Proposition 7.2.2 hold. Then for solutions u on $B(x_0, R)$, the following Harnack inequality holds:*

$$\sup_{B(x_0,R)} u \leq e^{\mathfrak{d}\sqrt{f(\mathbf{X})}} \inf_{B(x_0,R)} u \tag{7.2.5}$$

where $\mathfrak{d}$ is the distance between any two points.

Similarly to the above, we can compute a Harnack-type inequality for the parabolic equation under a time-evolving metric. Examples of these can be found in [2, 3, 14, 108]. A similar argument to above is followed, and we again split our working into multiple propositions. First, we re-represent the findings in theorem 6.2.1 for a global estimate and upper bounds for the constants and functions. After this, we follow a classical method of integrating along a geodesic path on the complete manifold M^n. Before we start, we also define the following: given $x_1, x_2 \in M^n$ and $t_1, t_2 \in (0, T)$ such that $t_1 < t_2$, we write

$$\Gamma(x_1, x_2, t_1, t_2) = \int_{t_1}^{t_2} \left|\frac{d}{dt}\gamma(t)\right|^2 dt. \tag{7.2.6}$$

Proposition 7.2.5. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to generalised geometric flow with $Ric_f^m \geq -C_1 g$, $-C_2 g \leq \mathcal{R} \leq C_3 g$, and $|\nabla\mathcal{R}| \leq C_4$ for $C_1, C_2, C_3, C_4 \geq 0$ in M^n. Suppose that u is a bounded positive smooth solution to (5.1.1) in $M^n \times [0, T]$ such that $0 < u \leq D$ for some $D \geq 1$ with $\lambda, p > 1$. Also let*

$$\begin{aligned} |\nabla A| &\geq a_1, \quad |\nabla B| \geq b_1, \\ \Delta A &\leq a_2, \quad \Delta B \leq b_2, \\ A &\leq a_3, \quad B \leq b_3 \end{aligned}$$

for constants a_1, a_2, a_3, b_1, b_2, b_3. Then the following gradient estimate holds:

$$\frac{|\nabla u|^2}{u^2} - \lambda\frac{u_t}{u} \leq z_1(\mathbf{X}) + tz_2(\mathbf{X}) + \frac{1}{t}z_3(\mathbf{X}) + t^{\frac{j}{6}} z_i(\mathbf{X}), \tag{7.2.7}$$

for $z_\eta(\mathbf{X}) = z_\eta(a_1, a_2, a_3, b_1, b_2, b_3, D, C, \lambda, \epsilon)$.

Proposition 7.2.6. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to generalised geometric flow with $Ric_f^m \geq -C_1 g$ and $-C_2 g \leq \mathcal{R} \leq C_3 g$, and $|\nabla\mathcal{R}| \leq C_4$ for $C_1, C_2, C_3, C_4 \geq 0$ in M^n. Suppose that*

u is a bounded positive smooth solution to (5.1.1) in $M^n \times (0,T]$. Let $h = \log u$ and

$$\frac{\partial h}{\partial t} \geq \frac{1}{\alpha_1}\Big(|\nabla h|^2 - \alpha_2 - \alpha_3 t - \frac{\alpha_4}{t} - \alpha_{j+1} t^{\frac{j}{6}}\Big) \tag{7.2.8}$$

for $\alpha_\eta > 0$. Then

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \Big(\frac{t_2}{t_1}\Big)^{-\frac{\alpha_4}{\alpha_1}} \exp\Big(\Gamma(x_1,x_2,t_1,t_2) - \frac{\alpha_2}{\alpha_1}(t_2-t_1) - \frac{\alpha_3}{2\alpha_1}(t_2^2-t_1^2) - \frac{\alpha_{j+1}j}{6\alpha_1}(t_2^{\frac{j}{6}-1} - t_1^{\frac{j}{6}-1})\Big), \tag{7.2.9}$$

for $0 < t_1 < t_2 < T$.

Proof. Let $\gamma(t)$ be the path for $t \in [t_1, t_2]$, and $h = \log u$. Then the time derivative is

$$\begin{aligned}
\frac{d}{dt}h(\gamma(t),t) =& \nabla h(\gamma(t),t)\frac{d}{dt}\gamma(t) + \frac{\partial}{\partial s}h(\gamma(t),s)|_{s=t} \\
\geq& -|\nabla h(\gamma(t),t)|\Big|\frac{d}{dt}\gamma(t)\Big| + \frac{1}{\alpha_1}\Big(|\nabla h(\gamma(t),t)|^2 - \alpha_2 - t\alpha_3 - \frac{\alpha_4}{t} - \alpha_{j+1}t^{\frac{j}{6}}\Big) \\
\geq& -\frac{\alpha_1}{4}\Big|\frac{d}{dt}\gamma(t)\Big|^2 - \frac{1}{\alpha_1}\Big(\alpha_2 + t\alpha_3 + \frac{\alpha_4}{t} - \alpha_{j+1}t^{\frac{j}{6}}\Big),
\end{aligned}$$

where in the last line we use that for $ax^2 - bx \geq \frac{b^2}{4a}$, $a,b > 0$. Next, we integrate over the path from t_1 to t_2:

$$\begin{aligned}
h(x_2,t_2) - h(x_1,t_1) \geq& -\frac{\alpha_1}{4}\int_{t_1}^{t_2}\Big|\frac{d}{dt}\gamma(t)\Big|^2 dt - \int_{t_1}^{t_2}\frac{1}{\alpha_1}\Big(\alpha_2 + t\alpha_3 + \frac{\alpha_4}{t} - \alpha_{j+1}t^{\frac{j}{6}}\Big)dt \\
\geq& -\frac{\alpha_1}{4}\int_{t_1}^{t_2}\Big|\frac{d}{dt}\gamma(t)\Big|^2 dt - \frac{1}{\alpha_1}\Bigg(\alpha_2(t_2-t_1) - \frac{\alpha_3}{2}(t_2^2-t_1^2) - \alpha_4\log\Big(\frac{t_2}{t_1}\Big) \\
&- \frac{\alpha_{j+1}j}{6\alpha_1}\Big(t_2^{\frac{j}{6}-1} - t_1^{\frac{j}{6}-1}\Big)\Bigg).
\end{aligned}$$

By exponentiation, this gives the desired result. □

Proposition 7.2.7. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to generalised geometric flow with $Ric_f^m \geq -C_1 g$ and $-C_2 g \leq \mathcal{R} \leq C_3 g$, and $|\nabla\mathcal{R}| \leq C_4$ for $C_1, C_2, C_3, C_4 \geq 0$ in M^n for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $M^n \times [0,T]$ such that $0 < u \leq D$ for some $D \geq 1$. Also let*

$$|\nabla A| \geq a_1, \quad |\nabla B| \geq b_1,$$
$$\Delta A \leq a_2, \quad \Delta B \leq b_2$$

for constants a_1, a_2, b_1, b_2. Then the following estimate holds:

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \Big(\frac{t_2}{t_1}\Big)^{-\frac{z_3}{\lambda}} \exp\Bigg(\Gamma(x_1,x_2,t_1,t_2) - \frac{z_1}{\lambda}(t_2-t_1) - \frac{z_2}{2\lambda}(t_2^2-t_1^2) - \frac{z_{j+1}j}{6\lambda}\Big(t_2^{\frac{j}{6}-1} - t_1^{\frac{j}{6}-1}\Big)\Bigg), \tag{7.2.10}$$

for $0 < t_1 < t_2 < T$.

Proof. With the use of Proposition (7.2.5) then (7.2.6) the desired result is achieved. $\square$

Using this, we can find Harnack inequalities for specific geometric flows.

Corollary 7.2.1. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to Ricci flow with $Ric_f^m \geq -C_1 g$ and $-C_2 g \leq Ric_f^m \leq C_3 g$ for $C_1, C_2, C_3 \geq 0$ in M^n for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $M^n \times [0,T]$ such that $0 < u \leq D$ for some $D \geq 1$. Also let*

$$|\nabla A| \geq a_1, \quad |\nabla B| \geq b_1,$$
$$\Delta A \leq a_2, \quad \Delta B \leq b_2$$

for constants a_1, a_2, b_1, b_2. Then (7.2.10) holds for $0 < t_1 < t_2 < T$.

Corollary 7.2.2. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to Yamabe flow with $Ric_f^m \geq -C_1$ and $-C_2 \leq S \leq C_3$, and $|\nabla S| \leq C_4$ for $C_1, C_2, C_3, C_4 \geq 0$ in M^n for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $M^n \times [0,T]$ such that $0 < u \leq D$ for some $D \geq 1$. Also let*

$$|\nabla A| \geq a_1, \quad |\nabla B| \geq b_1,$$
$$\Delta A \leq a_2, \quad \Delta B \leq b_2$$

for constants a_1, a_2, b_1, b_2. Then (7.2.10) holds for $0 < t_1 < t_2 < T$.

Propositions (7.2.5) and (7.2.6) are defined on Riemannian manifolds, but similar results can easily be established on smooth metric measure spaces.

Corollary 7.2.3. *Let $(M^n, g(t), e^{-f(t)}d\nu)_{t\in[0,T]}$ be a complete solution to Perelman-Ricci flow with $Ric_f^m \geq -(m-1)C_1 g$ and $-C_2 g \leq Ric_f^m \leq C_3 g$ for $C_1, C_2, C_3 \geq 0$, and $m < \infty$ in M^n for $R > 0$ and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to (5.1.1) in $M^n \times [0,T]$ such that $0 < u \leq D$ for some $D \geq 1$. Also let*

$$|\nabla A| \geq a_1, \quad |\nabla B| \geq b_1,$$
$$\Delta A \leq a_2, \quad \Delta B \leq b_2$$

for constants a_1, a_2, b_1, b_2. Then (7.2.10) holds for $0 < t_1 < t_2 < T$.

Remark 7.2.2. *Here $f(t)$ is taken as a function of time alone, allowing the use of Theorem 6.3.3.*

7.3 Analysis using ancient solutions of parabolic PDEs with gradient estimates

As with the elliptic gradient estimates, we can find Liouville-type theorems; however, these are linked to ancient solutions. The term ancient solution was coined by Hamilton in [56] in his study of Ricci flow. These are solutions that can be taken backwards in time without singularities, $t \in (-\infty, T)$, $T \in \mathbb{R}$.

Within our study of gradient estimates, we use ancient solutions by letting $T = R$ and then taking $R \to \infty$. However, when doing this, we need to carefully analyse what happens as $t \to -\infty$ to make sure that these are acceptable solutions. We also make use of Landau symbols, which take care of the non-linearities. Further reading on this can be found in [124, 125, 126].

Proposition 7.3.1. *Let u be a positive ancient solution to*

$$\Big(\Delta_f - \partial_t\Big)u(x,t) + B(x)u(x,t)^p = 0 \tag{7.3.1}$$

with non-negative Bakry-Ricci curvature. Also, let $p > 1$ and

$$B_+|_{B(x_0,R)} = o(R^{-\gamma(p-1)}), \qquad \sup_{B(x_0,R)} |\nabla B| = o(R^{-\gamma(p-1)}) \tag{7.3.2}$$

as $R \to \infty$ where $\gamma > 0$, and $\overline{\gamma} \in (0,\gamma)$. If $u(x,t) = o[(r(x) + |t|)^{\overline{\gamma}}]$, then u does not exist.

A proof for this is seen in [126] for a similar equation but we repeat it here for clarity.

Proof. As described in the above, u is a positive ancient solution. Fix a point (x_0, t_0) and apply Theorem 5.1.1 for $R = T$ in $B(x_0, R) \times (t_0 - R, t_0]$. With $A(x) \equiv 0$, $B(x) \not\equiv 0$, we observe $u(x,t) = o[(r(x) + |t|)^{\overline{\gamma}}]$, and we get

$$\frac{|\nabla u(x_0,t_0)|^2}{u(x_0,t_0)^2} \leq C(1 + \log D - \log u(x_0,t_0))^2 \left[o(R^{\frac{2}{3}(\overline{\gamma}-\gamma)(p-1)}) + \frac{1+\mu_+}{R} + \frac{1}{R^2} \right].$$

We selected $R > 2$ such that $R \geq t_0$. Since $u(x,t) = o[(r(x) + |t|)^{\overline{\gamma}}]$, we have $D = o(R^{\overline{\gamma}})$ for D in $Q_{R,R}$. Then, letting $R \to \infty$, we immediately observe that $|\nabla u(x_0,t_0)| = 0$, so $u(x,t)$ is a constant function in x. Since (x_0,t_0) was arbitrary, we therefore have that $u(x,t) = u(t)$, a function of t only. This then gives us a renewed equation:

$$u'(t) = B(x)u(t)^p.$$

As the left hand side is independent of x, $B(x) = B$. Also, as $p > 1$ and $(Bp)_+ = o(R^{-\gamma(p-1)})$, we get that $B < 0$. Solving this gives

$$u(t)^{1-p} = B(1-p)t + u(0)^{1-p}. \tag{7.3.3}$$

Since u is a positive ancient solution we should be able to send $t \to -\infty$ but then (7.3.3) gives $u < 0$, which is a contradiction.

$\square$

Proposition 7.3.2. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with non-negative Bakry-Ricci curvature. Let u be an ancient solution to*

$$\Big(\Delta_f - \frac{\partial}{\partial t}\Big)u(x,t) + Au(x,t)\log u(x,t) + Bu(x,t) = 0. \tag{7.3.4}$$

If $A < 0$, $B < 0$ are negative constants and $u(x,t) = e^{o((r(x)+|t|)^{\frac{1}{2}})}$, then $u(x,t) = u(t)$ is constant in x, has a solution

$$u(t) = e^{A^{-1}(e^{A(t+c)}-B)}, \tag{7.3.5}$$

and is bounded $u \leq 1$.

Proof. Fix a point (x_0, t_0) and apply Theorem 5.1.1 for $A < 0$, $B < 0$ negative constants and $p = 1$ for $R = T$ in $B(x_0, R) \times (t_0 - R, t_0]$. As $u(x,t) = e^{o((r(x)+|t|)^{\frac{1}{2}})}$, $D = e^{o(R^{\frac{1}{2}})}$. This gives

$$\frac{|\nabla u(x_0,t_0)|^2}{u(x_0,t_0)^2} \leq C(1 + o(R^{\frac{1}{2}}) - \log u(x_0,t_0))^2 \left[\frac{1+\mu_+}{R} + \frac{1}{R^2}\right]$$

for large $R > 2$ depending on t_0 such that $R \geq |t_0|$. Now, letting $R \to \infty$, we see that $|\nabla u|^2 = 0$, so $u(x,t) = u(t)$ is constant in x. Using (7.3.4), we write the equation as a function of t alone:

$$u'(t) = Au(t)\log u(t) + Bu(t).$$

Solving this gives

$$u(t) = e^{A^{-1}(e^{A(c+t)}-B)}.$$

As above, $A < 0$ and $B < 0$, so $u(t) \leq e^0 = 1$. $\square$

7.4 Further analysis of solutions to PDEs using bounds

In this section we look at two final propositions using the gradient estimates obtained earlier. These do not fall under the umbrella of any of the previous Liouville-type theorems, Harnack inequalities, or ancient solutions. These two propositions use the estimates with bounds on the various elements that make up the PDEs to give bounds on solutions.

Proposition 7.4.1. *Let $(M^n, g, e^{-f}d\nu)$ be an n-dimensional complete smooth metric measure space with $Ric_f^m \geq -(m-1)K$ for $K \geq 0$, $m < \infty$ in $B(x_0, 2R)$ for $R > 0$ and $x_0 \in M^n$. Suppose that u is a positive bounded solution to (5.1.1) in $H_{2R,T'} = B(x_0, 2R) \times (0, T]$, where A and B are* *positive constants, such that $1 \leq u \leq D$ with $\lambda p > 1$ and $\epsilon \in (0,1)$. Also let $u_t \leq \mathcal{U}$ and $A \leq a$, $B \leq b$. Then the following gradient estimate holds on $H_{R,T}$:*

$$u \leq C_{\mathcal{U}} e^{A^{-1}(z_1 + \frac{z_2}{t})} \tag{7.4.1}$$

for $z_\eta(\mathbf{X}) = z_\eta(a, b, D, \mathcal{U}, C, \lambda, \epsilon)$ and $n = 1, 2$. $C_{\mathcal{U}} > 1$ is a constant dependent on the values of $D, \mathcal{U}, A$.

Proof. Using additional limits with Theorem (5.2.1) we can write the estimate as

$$\frac{|\nabla u|^2}{u^2} - \lambda \frac{\mathcal{U}}{D} + A \log u \leq z_1(\mathbf{X}) + \frac{1}{t} z_2(\mathbf{X}).$$

Then we can say

$$A \log u \leq \lambda \frac{\mathcal{U}}{D} + z_1(\mathbf{X}) + \frac{1}{t} z_2(\mathbf{X}).$$

We can then divide through by A, take the exponential, and rearrange to obtain the desired result. □

Remark 7.4.1. *We take A, B to be non-negative constants in order to ensure that when the rearrangements in the proof are carried out, we can be sure that z_1, z_2 are both positive. This makes sure that $\mathcal{U} > 0$, so that with $u \geq 1$ inequality (7.4.1) stays valid.*

Finally we present a bound for the solution u of the elliptical PDE using the Li-Yau gradient estimate. Estimates like this can be seen in [123, 129].

Proposition 7.4.2. *Let $(M^n, g, e^{-f} d\nu)$ be an n-dimensional complete non-compact smooth metric measure space with $Ric_f^m \geq -(m-1)K$ for $K \geq 0$, $m < \infty$ in M^n and $x_0 \in M^n$. Suppose that u is a bounded positive smooth solution to*

$$\Delta_f u(x) + Au(x) \log u(x) + Bu(x)^p = 0 \tag{7.4.2}$$

in M^n for A, B constants such that $B \geq 0$ and $A \neq 0$. Also let a_1, a_2, b_1, b_2, D, $\tilde{K}$, γ, Θ, λ, p, ϵ be as defined above. Then the following global gradient estimate holds:

$$\begin{aligned} u \leq \exp\Bigg[\frac{1}{\lambda A}\Bigg[&\frac{m\lambda^2}{2}\Big((\lambda A)_+ + (\lambda(p-1)B)_+ D^{p-1}\Big) \\ &+ \frac{m^{\frac{3}{2}}\lambda^2}{2(1-\epsilon)^{-\frac{1}{2}}}(\lambda - 1)\Big[(m-1)K - \frac{1}{2}(\lambda-1)(2A-1)_-\Big]\Bigg]\Bigg], \end{aligned} \tag{7.4.3}$$

for $A > 0$ and

$$\begin{aligned} u \geq \exp\Bigg[\frac{1}{\lambda A}\Bigg[&\frac{m\lambda^2}{2}\Big((\lambda(p-1)B)_+ D^{p-1}\Big) \\ &+ \frac{m^{\frac{3}{2}}\lambda^2}{2(1-\epsilon)^{-\frac{1}{2}}}(\lambda - 1)\Big[(m-1)K - \frac{1}{2}(\lambda-1)(2A-1)_-\Big]\Bigg]\Bigg], \end{aligned} \tag{7.4.4}$$

for $A < 0$.

Proof. With a combination of Corollary 5.2.1 and 5.2.2

$$\frac{|\nabla u|^2}{u^2} + \lambda(A\log u + Bu^{p-1}) \leq \frac{m\lambda^2}{2}\Big((\lambda A)_+ + (\lambda(p-1)B)_+ \sup_{M^n} u^{p-1}\Big)$$
$$+ \frac{m^{\frac{3}{2}}\lambda^2}{2(1-\epsilon)^{-\frac{1}{2}}}(\lambda-1)\Big[(m-1)K - \frac{1}{2}(\lambda-1)(2A-1)_-\Big].$$

As the first and last term on the left hand side are strictly positive, we can remove them:

$$\lambda A\log u \leq \frac{m\lambda^2}{2}\Big((\lambda A)_+ + (\lambda(p-1)B)_+ \sup_{M^n} u^{p-1}\Big)$$
$$+ \frac{m^{\frac{3}{2}}\lambda^2}{2(1-\epsilon)^{-\frac{1}{2}}}(\lambda-1)\Big[(m-1)K - \frac{1}{2}(\lambda-1)(2A-1)_-\Big].$$

Then by rearranging and exponentiating, and taking note of the sign of A, we achieve the desired results. $\square$

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